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DEVELOPMENT OF ALGEBRAIC REASONING IN CHILDREN AND ADOLESCENTS: CULTURAL, CURRICULAR, AND AGE-RELATED EFFECTS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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Anne Krislov Morris
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TO MY DAUGHTERS—JENNIFER, GENIE, AND GWEN
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CHAPTER I
INTRODUCTION

Goals of the study

This study was designed to provide a clearer understanding of the development, and mechanisms of algebraic reasoning by (1) identifying sources of variation affecting algebraic reasoning; (2) linking the sources of variation to specific cognitive processes and cognitive outcomes involved in algebraic reasoning; (3) determining whether sources of variation affect algebraic reasoning independently or jointly; and (4) assessing the relative contributions of those factors, and/or their interactions, to observed variation in algebraic reasoning.

Introduction

Comparative and developmental studies consistently reveal cross-cultural and cross-sectional differences in mathematical reasoning. Differences in the development and utilization of mathematical reasoning have been observed across countries (Arbeiter, 1984; Crosswhite, Dossey, Swafford, McKnight, & Cooney, 1985; Lapointe, Mead, & Phillips, 1989; Robitaille & Garden, 1989; Stanley, Huang, & Zu, 1986; Song & Ginsburg, 1987; Stevenson, Stigler, & Lee, 1986); across socioeconomic lines and ethnic lines within a given country (Reyes & Stanic, 1988; Secada, 1992; Yando, Seitz, & Zigler, 1979); and across ages (Inhelder & Piaget, 1958; Küchemann, 1981; Piaget, 1983).
While cross-cultural variability in mathematical reasoning has been established, specific cultural and social variables accountable for the variation (i.e., factors contributing to the development of mathematical reasoning) remain unclear. Candidate factors include educational variables (e.g., curriculum and instruction) (Davydov, 1975; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987; Stigler & Perry, 1988); family processes (e.g., children's and parents' educational values, beliefs, and practices) (Huntsinger, Jose, Liaw, & Ching, 1995); cultural belief systems and practices (Stevenson & Stigler, 1992); semiotic systems such as language, literacy, and numerical systems (Miura, Kim, Chang, & Okamoto, 1988; Song & Ginsburg, 1988); social factors (e.g., poverty rates, divorce rates, etc.) (Jaeger, 1992); and interactions between these variables (Bronfenbrenner, 1979).

Sources of within-cultural, cross-sectional variability in mathematical reasoning also remain unclear. Age has been firmly established as an important contributing factor, and cross-sectional differences in mathematical reasoning have been primarily attributed to cognitive developmental stages (e.g., Almy, 1970; Beilin, 1970; Bell, 1979; Bitner, 1991; Carpenter, Kepner, Corbitt, Lindquist, & Reys, 1982; Collis, 1975; Forman, 1980; Küchemann, 1978, 1981; Lovell, Mitchell, & Everett, 1968; Piaget, 1983), or to an interaction between developmental and socio-cultural factors (e.g., curriculum and instruction, cognitive tools, SES) (e.g., Davydov, 1975).

The mechanisms via which socio-cultural and developmental variables affect mathematical reasoning also need to be specified, including: (a) the relative contributions of specific socio-cultural and developmental variables; (b) the relationships among contributing variables (i.e., whether these factors contribute independently or interactively); and (c) the links between specific sources of variation
and specific cognitive outcomes and cognitive processes involved in mathematical reasoning.

There are inherent difficulties in addressing both problems, i.e., in identifying explanatory variables, and establishing mechanisms via which those factors affect mathematical reasoning. First, sources of variation are usually examined in isolation from one another. Consequently, the critical sources of group differences, and the interrelationships among contributing variables remain unclear (e.g., Reyes & Stanic, 1988; Saxe, 1991; Stedman, 1994). For example, within-cultural cross-sectional studies have attributed observed variation in algebraic reasoning to cognitive developmental factors without an accompanying analysis of the effects of sociocultural and curricular variables (see, e.g., Küchemann, 1978, 1981). To develop a cohesive model of mathematical reasoning, multiple sources of variation must be examined within a single design (e.g., Stedman, 1994). Second, to establish relationships between specific socio-cultural variables (e.g., family practices, curricular variables) and specific cognitive outcomes and cognitive processes involved in mathematical reasoning, sufficient variability has to be obtained in both sets of variables (Osborne, 1993).

This study attempted to address both problems—to identify explanatory variables (to detect and measure effects, and to point to likely candidate factors), and to establish mechanisms via which those factors affect mathematical reasoning. This was attempted, first, by examining multiple sources of variation in a single research design, and secondly, by obtaining sufficient variability in socio-cultural contexts (viewed as a preliminary move toward establishing links between variability in specific socio-cultural factors and variability in reasoning).
This study specifically attempted to identify factors affecting the development and utilization of algebraic reasoning in children and adolescents. Three explanatory variables were included in the design: (1) culture (Russia and England), (2) curriculum (experimental and non-experimental), and (3) age (10–16 years). The outcome measures included students' (1) algebraic and logical deductive reasoning, (2) concepts of algebraic structure, and (3) concepts of algebraic letter interpretations.

To obtain variability in sociocultural contexts robust enough to ensure sufficient contrast, the curricular contrasts included a comparison of opposed, theoretically driven curricula. For example, in Davydov's (1975, 1990) experimental curriculum in Russia, algebraic operations are developed in primary school to engender abstract mathematical thought/reasoning in children, while National Mathematics Project (1987) in England assumes abstract algebraic operations are inaccessible until children have attained formal operational thought. Since curricular variables were nested within different cultures, cultural and curricular variables could be examined within a single design. Where similarities existed in the development of algebraic reasoning despite cultural, linguistic, and curricular variation, the study would provide evidence for the influence of developmental factors.

Theoretical framework

Theoretical model of reasoning

Reasoning in general and algebraic reasoning in particular (as particular kinds of cognitive activity and cognitive performance), are modeled as the resultant of a working functional system that has three component subsystems (Naglieri & Sloutsky, 1994): (a) a biological subsystem that includes brain, sensory, and motor systems and processes; (b) a cognitive subsystem that includes all representational processes; and (c)
a socio-cultural subsystem that includes cognitive tools and externally existing socio-cultural knowledge. There is evidence that the subsystems affect cognitive performance independently and jointly (e.g., Anderson, 1983; Baddeley, 1986; Flavell, 1976; Fuson & Hall, 1983; Goswami, 1990; Greenfield, 1991; Hansen & Bowey, 1994; Luria, 1973, 1976; Metcalfe & Shimamura, 1994; Saxe & Posner, 1983; Schacter, 1987; Shimamura, 1994; Siegler, 1991). For example, the socio-cultural subsystem can profoundly change not only the products, but also the process of cognition (e.g., generalization, abstraction, reasoning) through the provision of cognitive tools and a knowledge base for representing, processing, and acquiring new information (Luria, 1976; Vygotsky, 1978; Wertsch, 1991a, 1991b). Cognitive tools are means of representation (e.g., signs, symbols, and semiotic systems) that initially exist externally; in the course of development they undergo the process of internalization, becoming an essential part of human cognition. In the sections to follow, we consider (a) effects of the socio-cultural subsystem on reasoning, and (b) cognitive mechanisms of reasoning.

**Effects of the socio-cultural subsystem on reasoning**

Cognitive tools affect reasoning—e.g., semiotic systems, such as language, numerical systems, and formal mathematical symbol systems. In addition, potentially important factors include educational variables (e.g., curriculum and instruction), family processes, cultural belief systems and practices, and social factors. Chapter 2 will examine these variables in more depth.

*Cognitive aspects of reasoning: Hypothetical models and mechanisms (everyday versus algebraic deductive reasoning)*
Main theoretical approaches underlying research on cognitive aspects of reasoning include (a) the competence approach, stemming from work of Piaget (Inhelder & Piaget, 1958), and (b) the pragmatic–heuristic approach. The competence approach suggests that ideally, human reasoning approximates deductive inference. This approach includes the following models:


(2) Mental models: Subjects reason by (a) constructing models representing states of the world in which the given premises are true, and (b) constructing conclusions by generating a parsimonious description of their models (e.g., Johnson–Laird, 1983; Johnson–Laird & Byrne, 1991).

The pragmatic–heuristic approach assumes human reasoning employs content–specific heuristics that are not necessarily consistent with the rules of deduction. This approach includes the following models:

(1) Domain–sensitive rules or schemas: Reasoning in real world domains is achieved with the aid of domain specific rules or context–sensitive schemas that are retrieved and applied. Schemas include rules of reasoning in particular pragmatic contexts (Cheng & Holyoak, 1985).

(2) Heuristics and biases approach: This approach attributes biases to the operation of non–logical heuristics that are separate from the processes responsible for competent reasoning. These heuristics are often used to explain systematic errors or "biases" in reasoning tasks (Kahneman, Slovic, & Tversky, 1982; Tversky & Kahneman, 1983).

It has been shown (Cheng & Holyoak, 1985; Kahneman, Slovic, & Tversky, 1982; Tversky & Kahneman, 1983) that a pragmatic–heuristic approach more
adequately models content-specific reasoning (e.g., everyday reasoning or practical reasoning where rules of reasoning are implicit). However, the normative competence approach seems more applicable to abstract mathematical reasoning, where rules of reasoning are relatively explicit.

Moshman (1995) distinguishes types of reasoning on the basis of their defining constraints—i.e., the kinds of constraints the thinker is attempting to apply or respect (e.g., case-based, law-based, coherence-based, dialectical). Mathematical deductive reasoning can be characterized as law-based reasoning—i.e., thinking constrained by the purposeful application of abstract laws, rules, principles, or theories. The reasoner believes laws provide a satisfactory basis for reaching conclusions. This may occur, for example, when solving problems of formal logic; calculating probabilities; interpreting theories; or applying moral, legal, or social-conventional rules or principles. Conversely, in empirical reasoning, data are purposely generated to provide evidence relevant to a hypothesis. If the hypothesis and the generated evidence conflict, the evidence is regarded as having priority. Empirical reasoning, then, is reasoning consciously constrained by (what is taken to be) evidence (Moshman, 1995).

Application of a given set of constraints may be more or less appropriate in a given realm. For example, learners often use empirical reasoning in mathematics where deductive proofs are required. Learners' preference for, and inability to distinguish between, empirical versus deductive argumentation in mathematics has been documented in Balacheff (1988), Bell (1976), Chazan (1993), Fischbein and Kedem (1982), Lee and Wheeler (1987), Martin and Harel (1989), Porteous (1986, 1991), Schoenfeld (1989), Vinner (1983), and Williams (1979). In a review of the literature, Chazan (1993) characterizes two sets of student beliefs about argumentation
in mathematics: (a) Evidence is proof: Learners believe that by measuring specific geometrical figures, or by generating numerical examples, one can reach conclusions that are certain, and applicable to infinite sets (e.g., Balacheff, 1988; Martin & Harel, 1989; Williams, 1979). For example, Balacheff (1988) describes "naive empirical arguments"; in naive empiricism, learners assert the truth of a result after verifying several cases—mere observation is sufficient proof. Students often examine a number of kinds or types of cases (e.g., types of triangles, types of numbers); establish whether a generalization holds for all types; and make a probabilistic assessment of the validity of a generalization based on the generated empirical evidence (e.g., Chazan, 1993). (b) Deductive proof is simply evidence: Students view deductive proofs in geometry and algebra as proofs for a single case. The deductive proof applies to a particular diagram, a particular kind of number; or the general deductive argument may be regarded as a method for examining and verifying particular cases (e.g., Martin & Harel, 1989; Vinner, 1983). Learners are skeptical that a deductive proof can guarantee that no counterexamples will exist. They may reapply the deductive argument to other kinds of objects—e.g., reapply a deductive proof to a variety of kinds of triangles or numbers (e.g., Lee & Wheeler, 1987). Both sets of beliefs (evidence is proof, proof is evidence) lead to empirical reasoning, where deductive argumentation is required.

In everyday reasoning, it is plausible to suggest two component cognitive processes in deductive reasoning. An individual (1) uses inductive reasoning to generate, or to evaluate the validity of a given set of premises that require hypothetico-deductive reasoning; and (2) applies rules of transformation if necessary. In evaluating the validity of a given set of premises, an individual makes a probabilistic assessment of the truth or falsity of the premises based on past experiences (e.g.,
Anderson, 1985). The individual appeals, for example, to prior knowledge (more concrete, more abstract, physical experience), generated empirical evidence, higher principles, rules, values, beliefs, etc.

We suggest the following normative mechanism in mathematical deductive reasoning (proof), however:

1. creation of abstract objects (i.e., givens);
2. evaluation of the truth or falsity of a given or developed set of statements;
3. application of rules of transformation.

Conjecture of three separate cognitive processes in deductive reasoning in mathematics has some empirical support (e.g., Balacheff, 1988; Hill, 1961; Lester, 1975). Hill, for example, found mathematical content significantly influences children's ability to make valid inferences (i.e., (2) affects (3)). Studies suggest students' acceptance of, or request for a deductive proof (versus an empirical argument) increases when content is more familiar (Chazan, 1993; Williams, 1979).

For purposes of analysis, we model algebraic reasoning (as a specific kind of mathematical reasoning) as a cognitive process that requires (1) component understandings, including (a) concepts of algebraic letter interpretations, (b) concepts of mathematical structure, and (c) concepts of real number; and (2) inductive and deductive reasoning. In algebraic deductive reasoning, the reasoner (1) creates abstract objects as givens, that are then operated upon; (2) evaluates the truth or falsity of a given or developed set of statements that require hypothetico–deductive reasoning, or assumes the truth of premises that in fact he/she knows to be false (e.g., *reductio ad absurdum*); and (3) applies rules of transformation. Rules of transformation include propositional operations (abstract rules of deductive logic), and operations specific to algebraic reasoning such as the ability to manipulate algebraic symbols (e.g., algebraic
simplification, factoring). In assessing the truth or falsity of statements and in applying rules of transformation, the reasoner adheres to a defined set of rules: assessing the validity of statements and transformations based on properties of quantities, relationships between quantities, principles, syntactic conventions, etc.

Evaluation of the validity of statements and transformations, and the ability to conceive of the fully developed proof as a generalized argument, assumes the ability to create and operate on abstract objects. Van Dormolen (1977) described levels in understanding of mathematical proof: from operating on particular single objects to conceiving of particular objects merely as an illustration of a general system of defined constructs, and of propositions describing their relationships. Balacheff (1988), for example, found learners often do not discern the generic aspect of diagrams in geometric proofs—i.e., that a generic example proof makes "explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class" (Balacheff, 1988, p. 219).

It is reasonable to assume that if learners are unable to create and operate on abstract objects, they inappropriately apply everyday reasoning in mathematical argumentation. For example, learners use algebraic statements in a proof to create numerical examples, and then make a probabilistic assessment of the validity of the generalization and deductive proof based on numerical (empirical) cases (see, e.g., Lee & Wheeler, 1987; Martin & Harel, 1989; Vinner, 1983).

In order to identify sources of variation affecting algebraic reasoning, and to link sources of variation to specific cognitive processes and cognitive outcomes involved in algebraic reasoning, two curricular settings were identified that
incorporated profoundly different models for developing algebraic reasoning. The first mathematics curriculum, *National Mathematics Project* in England, emphasizes inductive, case-based (Moshman, 1995) reasoning—the investigation of a number of particular instances to formulate and to assess the validity of algebraic generalizations, emphasizing the importance of empirical checks. The second, Davydov's experimental mathematics curriculum in Russia, emphasizes deductive, law-based (Moshman, 1995) reasoning—the logical derivation of particular (e.g., numerical) cases from general mathematical principles and relationships where those principles and relationships are first expressed algebraically. This study examined children's notions as to the logical necessity of deductive conclusions.

With respect to developing children's ability to create and operate on abstract objects, to recognize and use mathematical structure, and to perform algebraic transformations, *National Mathematics Project* uses a procedural-to-structural curricular model (Kieran, 1992); Davydov uses a structural-to-procedural model. With respect to developing concepts of algebraic letter interpretations, *National Mathematics Project* incorporates Harper and Küchemann's Piagetian-based curricular model, emphasizing the gradual development of abstraction, with a move from less abstract interpretations (e.g., specific unknowns) to more general interpretations in Years 7 and 8. In contrast, Davydov suggests formal abstract symbols are essentially related to mathematical reasoning, to abstraction and generalization and deduction; algebraic symbols are introduced in primary school. Davydov contends "the abstract as an element of thought should be introduced into instruction as early as it can be accessible to the child, who must not be held too long at the stage of sensory impressions, in any case" (1990, p. 319). In order to allow children to examine relationships among objects, to discern particular empirical cases as concrete
manifestations of generalized relationships and principles, Davydov begins with more
general interpretations of letters (general quantity), and moves to less abstract
interpretations (specific unknowns).

Curricular variables across cultures were confounded with other potentially
important variables such as language, family and cultural beliefs and practices, etc. To
control potential confounds, and to investigate alternative sources of variation, two
non–experimental schools were selected in the same countries. Schools were in the
same geographical area, and had comparable student and teacher populations;
however, they did not have curricula that were designed to develop specific kinds of
algebraic reasoning.

Research questions

This study examined the effects of socio–cultural variables (e.g., culture,
language, and curriculum) and developmental variables on mathematical reasoning.
Based on previous research findings and theoretical considerations, differences in
mathematical reasoning across socio–cultural contexts were predicted. It was
predicted that (1) cultural and curricular variables would affect specific components of
algebraic reasoning (e.g., emphasis on induction versus deduction, understanding of
logical necessity as a component of deduction), resulting in between–group differences
in algebraic reasoning; and (2) cultural and curricular effects would be more
pronounced than age–related effects.

The study examined the following research questions: (1) Do students exhibit
differences in acquiring algebraic letter interpretations, structure, and reasoning across
cultures, curricula, and developmental lines?; (2) If socio–cultural and developmental
variables affect algebraic reasoning, do they affect it independently or jointly?
Significance

In previous studies, the relative effects of developmental, cultural, and curricular variables on mathematics performance have not been examined within a single design. This study integrated different sources of variation in mathematics performance within a single theoretical perspective (Naglieri & Sloutsky, 1994), combined different explanatory variables within a single research design, and assessed the relative contribution of those variables to observed variation in mathematical reasoning. This study will contribute to the understanding of ontogenetic and microgenetic mechanisms of (a) abstract algebraic reasoning, and (b) abstract reasoning in general.

The study can contribute to curriculum theory since it compares two theoretically opposed curricula (Davydov's experimental curriculum in Russia and National Mathematics Project in England), where both curricula are designed to develop algebraic reasoning.
CHAPTER II
THEORETICAL ANALYSIS: CULTURE AND CURRICULUM AS CONTRIBUTING FACTORS TO ALGEBRAIC REASONING

This chapter examines factors that potentially affect mathematical reasoning. This chapter will review strands of research on (a) characteristics of a culture that may affect the development and utilization of mathematical reasoning; and (b) characteristics of a curriculum (curricular variables) that may affect the development of algebraic reasoning. Characterizations of, and theoretical underpinnings for the experimental curricula will be presented.

Characteristics of a culture that may affect mathematical and logical reasoning
I. Semiotic systems
Language

Language characteristics may affect the development of particular mathematics skills/concepts, and mathematical reasoning in general. Language may affect recognition of structure; ability to represent and understand quantitative relationships; and logic competence.

a. Recognition of structure

There is some evidence that the cognitive representation of number is influenced by the linguistic structure of the number–naming system. These differences in number representation may affect numerical reasoning. For example, there is
evidence that the structural regularity of the Chinese number system, and the similar Japanese and Korean systems, facilitates the acquisition of abstract counting skills and the induction of rules for generating numbers (Miller & Stigler, 1987; Miura, 1987; Song & Ginsburg, 1988); place value concepts and base 10 structure (Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Song & Ginsburg, 1987); and flexibility in mental number manipulation and addition and subtraction of multidigit numbers (Hatano, 1982; Miura, Kim, Chang, & Okamoto, 1988).

b. Ability to represent and understand quantitative relationships

Studies have revealed inappropriate transfers from natural language processing to algebra. Tall and Thomas (1991), for example, identify the "parsing obstacle"—written English is processed from left to right, and learners may inappropriately process all algebraic expressions in the same manner. For example, $2 + 3\%$ is simplified as $5\%$.

The linguistic features of natural language may affect the translation of quantitative relationships from natural language into algebraic statements. For example, Clement, Lochhead, and Monk (1981) posed the following problem to 150 freshman engineering students:

Write an equation using the variables $S$ and $P$ to represent the following statement: "There are six times as many students as professors at this university." Use $S$ for the number of students and $P$ for the number of professors.

Thirty-seven percent answered incorrectly. In a sample of 47 non-science majors in a college algebra course, 57% answered incorrectly. In both cases, 68% of the errors involved variable reversal errors; students wrote $6S=P$, rather than $S=6P$. Clement
(1982) suggests one major source of the variable-reversal error stems from the linguistic structure of the English statement: students use a syntactic, left-to-right, word-for-symbol mapping in formulating an algebraic equation (see also, Fisher, 1988; Hegarty, Mayer, & Green, 1992; Lewis & Mayer, 1987; Lochhead, 1980; Mestre, 1988; Rosnick, 1981; Rosnick & Clement, 1980; Schoenfeld, 1985; Spanos, Rhodes, Dale, & Crandall, 1988). Rosnick (1981) suggests the reversal error is not only common, but deeply entrenched. In a study involving Spanish speakers, Niaz (1989) asked 54 freshman science majors at Universidad de Oriente, Venezuela to respond to the "student-professor problem": 3% wrote correct equations, while 78% wrote reversed equations.

There is some evidence that language differences may affect the ability to express and understand quantitative relationships. For example, Orr (1987) suggests Black English Vernacular (BEV) does not contain certain constructions available in Standard English—particularly prepositional constructions used to express certain quantitative relationships—limiting speakers' ability to generate valid mathematical formulations of the quantitative relationships involved. Wilson (1981) explains that, while mathematics requires the precise expression of a wide range of relationships (e.g. $A$ is greater than $B$; if $p$, then $q$; some $C$'s are $D$'s; for each $X$ such that $x$, there is a $Y$ such that $y$), a medium language (e.g., Bahasa, Kiswaheli) does not always allow the expression of the necessary mathematical structures.

c. Language effects on reasoning

Language differences may affect logic competence. For example, Bloom (1981) found Chinese speakers less likely than English speakers to give counterfactual interpretations to a counterfactual story. This finding—combined with the existence of
a distinct counterfactual marker (the subjunctive) in English versus the absence of a linguistic construction for the counterfactual *per se* in Chinese—was interpreted as evidence for the weak form of the Sapir–Whorf hypothesis (i.e., the structure of a language is related to speakers' thought). Bloom concludes there is a national–linguistic difference in counterfactual logic competence. That is, the absence of a distinct marker hinders Chinese speakers from reasoning counterfactually.

Comparing Chinese speakers in Hong Kong and Taiwan with English speakers in the U.S., only 7% of the Chinese–speaking subjects gave counterfactual interpretations to the Chinese version of a story, while 98% of the English speakers gave counterfactual interpretations in response to the English version (Bloom, 1981). In a subsequent study, Bloom found Taiwanese non–student bilinguals more inclined to reason counterfactually in response to an English version of a counterfactual story (86% counterfactual responses) than in response to a Chinese version (50% counterfactual responses). Bloom (1981) concluded that "for many, if not most, of the bilinguals in the study, the counterfactual mode of thought remains associated in their minds with the English linguistic world, activated more readily when cognitive processing is elicited by that linguistic world rather than by their native Chinese" (Bloom, 1981, pp. 31–32).

Au (1983) attempted to replicate Bloom's findings, using both Chinese and English versions of a different counterfactual story, as well as the story used by Bloom rewritten in more idiomatic Chinese. Au's study yielded no support for the Sapir–Whorf hypothesis: Chinese subjects showed little difficulty in understanding stories in either language. Nearly monolingual Chinese who did not know the English subjunctive gave primarily counterfactual responses. Au suggests the difficulty Bloom's subjects had with the counterfactual story was probably due to the
unidiomatic Chinese used in the story, rather than to the counterfactual logic of the story *per se*. Au suggests mastery of the English subjunctive is probably tangential to counterfactual reasoning in Chinese speakers.

There is some evidence that language differences may affect the ability to generalize. For example, Greenfield (1966) interviewed Wolof children in Senegal; the Wolof language lexicon is at a single level of generality, sufficient for the analysis of a given domain into its component parts, but inadequate for the synthesis of several domains into a single superordinate category. When children who were bilingual in Wolof and French were questioned in French, they formed more abstract equivalence groupings based on form and function, and fewer merely perceptual groupings such as groupings by color, than when questioned in Wolof. The use, rather than the extent of knowledge, of the French language was found to be the crucial variable.

*Literacy, writing*

Scribner and Cole (1978a, 1978b) examined the effects of literacy *per se* on performance on logical and classification tasks. The Vai of Liberia use a phonetic writing system; some Vai are also literate in Arabic or English. English is used in Western style schools, while Arabic is used and learned in traditional Qur'anic schools. Qur'anic schools emphasize rote memorization, or reading aloud of religious passages. The Vai script has diverse practical uses, but Vai literacy does not lead to learning of new knowledge, nor to expository writing concerned with examination/analysis of ideas; Scribner and Cole (1978a) predicted this kind of literacy would not have the general intellectual consequences that have been associated with high levels of school–based literacy (Rogoff, 1981).
Scribner and Cole found little difference between nonschooled Vai literates and nonliterate in performance on logical and classification tasks. Additional types of comparisons suggested components of reading and writing may promote very specific language-processing and cognitive skills (e.g., memory for certain kinds of information, integration of syllables into meaningful linguistic units). However, skills in logical and classification tasks are not the inevitable outcome of learning to use alphabetic scripts or to write any kind of text (Rogoff, 1981).

Olson (1976) suggests writing leads to emphasis on self-contained, internally consistent general arguments, arguments that are examined in isolation for logical meaning. Some kinds of statements (definitions, logical principles) that are not easily memorized appear with literacy. Olson suggests cultures without written prose are less concerned with the formulation of general statements than cultures with widespread literacy. This difference may be responsible for cultural differences in the tendency to search for generalizations.

Nonlinguistic conventional means of representation

Culturally developed representational systems structure intellectual activity, e.g., reasoning processes in mathematics (e.g., Nunes, Light, & Mason, 1993). For example, in Hatano, Miyake, and Binks (1977) and Stigler (1984), children develop a mental abacus after gaining some expertise with the abacus; i.e., they do mental calculations by forming a visual image of an abacus, often performing extremely rapid and accurate mental calculations. Abacus training may result in increased digit span, and greater understanding of the numeration system and number concepts (Hatano, Misake, & Binks, 1977; Miller, 1988; Stigler, Chalip, & Miller, 1986).
II. Cultural belief systems (e.g., values, attributions, expectations, attitudes)

*Expectations and beliefs regarding mathematics achievement*

The well-documented Asian--American disparity in mathematics achievement suggests factors, other than school-related variables, influence mathematical reasoning (e.g., cognitive factors such as language; parental and cultural beliefs, values, practices): (a) Superior performance of Asian students in numerical reasoning, and in mathematics achievement are already apparent by first grade, before teaching effectiveness and other school-related factors can account for such large variations (Geary, Fan, & Bow-Thomas, 1992; Stevenson, Stigler, & Lee, 1986). (b) Huntsinger, Jose, Liaw, and Ching (1995) report a disparity in achievement among Chinese--American and Euro--American children attending U.S. schools with similar instructional methods. (c) Stevenson, Stigler, Lee, and Lucker (1985) tested 240 first graders and 240 fifth graders from 40 classrooms in 10 schools from each of three cities: Taipei, (Taiwan), Sendai (Japan), and Minneapolis (U.S.). Japanese and Chinese children outperformed American children on mathematics tests for Grades 1 and 5. However, Asian and American children did not differ in scores on tests of basic cognitive ability (tests of verbal ability and non-verbal ability). Stevenson et al. (1985) conclude the mathematics achievement of Chinese and Japanese children cannot be attributed to higher intellectual abilities.

A positive association between students' academic achievement, and students' perceptions of parents' or teachers' expectations and beliefs has been reported for both Western and Asian children (e.g., Au & Harackiewica, 1986; Brophy, 1982; Good, 1987). Studies reveal cross-cultural differences in parents' beliefs regarding the value of mathematics achievement, the value of effort, and attributions of success and failure in mathematics. Greater cultural value is attached to mathematics in the Chinese
Asian cultures place a greater emphasis on attaining academic excellence in mathematics, as compared to the U.S. (Stevenson & Lee, 1990). Stevenson, Chen, and Lee (1993) found Chinese parents tended to have higher standards for their children's academic performance, and tended to rate their children more modestly against those standards as compared to American parents. Uttal, Lummis, and Stevenson (1988) reported dramatic differences: American mothers were more satisfied with their children's performance than Asian mothers; and the levels of satisfaction of mothers of low-achieving American children were nearly as high as those of mothers of high-achieving Japanese children. In interviews with American, Japanese, and Chinese mothers, Crystal and Stevenson (1991) found that, although American children received significantly lower scores than their Asian peers on the mathematics tests used in the study, fewer American mothers stated that their children had experienced problems in mathematics. American mothers viewed these problems as less serious and more transitory. American parents tend to be nativists in comparison with Japanese parents, believing innate ability rather than effort, is the key to successful learning (Holloway, Kashiwagi, Hess, & Azuma, 1986; Stigler & Baranes, 1988).

Cultural beliefs regarding children's capabilities also differ. For example, Perry, VanderStoep, and Yu (1993) suggest Asian and American teachers behave as though they hold different beliefs about the level of thought primary schoolchildren are capable of. Asian teachers appear to believe first-graders are capable of being engaged in conceptual and abstract mathematical thought, while American teachers do not.

Substantial cross-national variation in the size and direction of gender differences in mathematics performance (e.g., Baker & Jones, 1993; Finn, 1980) may also be linked to variability in cultural beliefs and attitudes.
Cultural emphasis on various forms of reasoning, argumentation

Disposition to form generalizations, to search for reasons, to generate alternative explanations, to search for inconsistencies, and to engage in rational thinking have been characterized as cultural values (e.g., Cole, 1972; Goodnow, 1976). Cultural groups may differ in their assumptions as to (a) whether it is reasonable, necessary, or possible to search for complete answers in a given content area (i.e., answers covering present and possible events, versus the more pragmatic goal of identifying only immediately needed information); (b) whether particular processes can, or should be extended to other contexts; and (c) what is regarded as a good or correct answer, or as a good, elegant, or clumsy method (Glick, 1975; Goodnow, 1976). For example, Americans tend to emphasize similarity as the basis for classifying; in Maccoby and Modiano (1966), rural Mexicans value sensitivity to difference, are uncomfortable with generalities. Gladwin (1970) describes a navigational system developed by a Pacific group, the Puluwa, that satisfies many of the criteria for abstract thought (e.g., containing a large number of variables and involving the use of an imagined reference point). However, the system is internally inconsistent. By Western standards, it is incomplete—yet it achieves the Pulawats' goal of reaching their destination (Goodnow, 1976).

III. Cultural practices: Family and school

Family processes

Cross-cultural and cross-national differences in mathematical reasoning may be linked to cross-cultural differences in parental practices. For example, Chinese-American parents are more direct in their teaching as compared to Euro-American parents, systematically "preteaching" formal mathematics to their children (Huntsinger,
Chinese-American parents are more likely to provide regular, formal instruction at home for their preschoolers, as compared to Euro-American parents (Steward & Steward, 1974); and substantially increase their child's homework/practice time from preschool to kindergarten, while Euro-Americans maintain a low level (Huntsinger et al., 1995). Stigler, Lee, Lucker, and Stevenson (1982) found American children had much less help from their family when doing their homework, as compared to Asian children. Crystal and Stevenson (1991) found Asian mothers of fifth graders (64%) asked other members of the family for assistance with children's problems in mathematics significantly more often than did American mothers of fifth graders (29%).

American parents and teachers do not consider homework of great value (Stevenson, Stigler, & Lee, 1986). This attitude is in marked contrast with that of Japanese parents and teachers. Children in Japan spend between two and three times as much time engaged in the study and practice of mathematics as compared to their American cohorts (Stevenson, Lee, & Stigler, 1986; Stevenson, Lee, Stigler, et al., 1986). Japanese students may receive intensive tutoring; study in the juku; and prepare for an intense exam system that determines senior high school, as well as university admission (Becker, Silver, Kantowski, Travers, & Wilson, 1990; Feiler, 1991; Stedman, 1993; U.S. Department of Education, 1986).

Carr, Kurtz, Schneider, Turner, and Borkowski (1989) examined the role of parents in the development of children's strategic behavior and metacognition. German parents reported more direct instruction of learning strategies in the home than American parents. Reported strategy instruction in the home was related to cross-national differences in second- and third-graders' use of a clustering-rehearsal strategy on a sort-recall task, and associated metacognitive knowledge (e.g.,
children's understanding of the usefulness of clustering words into taxonomic groups as a memory aid, knowledge of other encoding and retrieval strategies).

Schooling

a. Schooling per se

Schooling as a source of variance has been debated: (a) Substantial impacts of formal schooling on a variety of cognitive processes and outcomes have been reported (e.g., Husen & Tuijnman, 1991; Stevenson, Parker, Wilkinson, Bonnevaux, & Gonzalez, 1978). (b) Studies in the Piagetian tradition (e.g., Kiminyo, 1977) find no important effects of schooling. (c) Inconsistent effects of schooling are reported; schooling has positive effects in some areas, and seems to make no difference in other areas (e.g., Ginsburg, Posner, & Russell, 1981; Sharp, Cole, & Lave, 1979).

Hypothesized mechanisms via which schooling affects cognition include:

(a) An emphasis on searching for, and using, explicitly stated general rules, universals, and principles via which specific instances can be understood: Practice applying common operations (e.g., counting, classifying), general solution rules, formal symbol systems (e.g., mathematics, logic), concepts, etc. in varied kinds of tasks, and extending known techniques or concepts to new contexts, may contribute to the observed tendency of schooled subjects to generalize rules across situations (Cole, 1972; Cole, Gay, Glick, & Sharp, 1971; Goodnow, 1976; Greenfield & Bruner, 1966; Scribner & Cole, 1973).

(b) Instruction in the verbal mode, out of context: Learning concepts starting from general definitions rather than from exemplars, and emphases on defining, organizing, and classifying may be related to the following findings: schooled subjects are more likely to use taxonomic categories, while nonschooled subjects use more functional
categories; schooled subjects find it easier to explain their reasoning and solution processes than nonschooled subjects; and schooled subjects can communicate with each other more easily than nonschooled subjects when the immediate context is removed (e.g., Cole & Scribner, 1974; Luria, 1976; Vygotsky, 1962).

(c) *Specific skills taught in school:* Schooled subjects are exposed to specific kinds of problems. For example, Scribner (1977) points out that the logical syllogism fits a specific genre of problem. Judgments should be made, purely on the basis of information presented in the problem; this genre is common in schools (e.g., arithmetic story problems).

Rogoff (1981) summarizes studies examining effects of schooling:

(a) *Memory tasks:* In free recall memory tasks, schooled subjects show striking superiority in recall, as well as more clustering of items by semantic category (Cole et al., 1971; Sharpe, Cole, & Lave, 1979). Nonliterates (and those with a small amount of schooling) are unlikely to engage in strategies (e.g., semantic categorization) that provide structure for apparently unrelated material (e.g., Cole et al., 1971). Wagner (1982) provided evidence that use of mnemonic strategies, such as verbal rehearsal, seem to be linked to schooling.

(b) *Classification:* Populations differ with respect to preferred dimensions of classification. Use of taxonomic categorization has consistently been found in subjects with more than a fourth-grade education (Cole et al., 1971; Scribner, 1974; Sharp, Cole, & Lave, 1979). Subjects with less schooling use some taxonomic categorization, but often group objects functionally (e.g., Luria, 1976).

(c) *Deductive reasoning:* Schooled and non-schooled subjects handle verbal syllogisms differently. For example, in Luria (1976), illiterate Central Asian subjects refused to accept the premises in verbal syllogisms as a point of departure for
subsequent reasoning—treating premises as descriptions of particular phenomena rather than as "universal" givens, and treating the syllogism as a collection of independent particular statements rather than as a self-contained logical problem from which a conclusion may be drawn. Luria's findings have been replicated by Cole et al. (1971), Scribner (1975), Fobih (1979), and Sharp et al. (1979). Scribner characterized schooled subjects' reasoning as "theoretic"—explicitly relating the conclusions to the premises—and the reasoning of nonschooled subjects as "empiric"—justifying the conclusion on the basis of personal experience or personal belief. There is no evidence that there are differences in logical reasoning per se among schooled and non-schooled subjects; rather, the difference lies in non-schooled subjects' willingness to use deduction and inference without reliance on direct experience, i.e., when they cannot verify the premises (Rogoff, 1981).

b. Curriculum

Cross-national differences in curriculum may contribute to cross-cultural and cross-sectional differences in the development and utilization of mathematical reasoning.

i. Development of abstract relationships in arithmetic word problems

Stigler, Fuson, Ham, and Kim (1986) analyzed the presentation of addition and subtraction word problems in primary school mathematics textbooks (first, second, and third grade levels), comparing four American textbook series, and a Soviet textbook series. Stigler et al. used Carpenter and Moser's (1983) coding scheme for problem types; the scheme categorizes problems according to (a) semantic structure (i.e., type of story action), and (b) position of the unknown in the equation representing the story. They identified three major categories of addition and
subtraction word problems—change, combine, and compare problems—with 15 problem types.

American texts resembled each other, but differed markedly from the Soviet text series. (1) Distribution of word problems across various problem types was extremely uneven in the American texts. The four most numerous problem types represented 91%, 82%, 80%, and 75% of the problems in the four American series, and these were of the simplest types: single-step problems that have semantic structure equations identical to their solution procedure equations (i.e., the arithmetic solution procedure directly parallels the semantic structure of the problem). Of the 15 types of one-step problems, these types are the easiest for American children to solve (e.g., Carpenter & Moser, 1983, 1984; Riley et al., 1983). Data clearly indicate that American children entering first grade can competently solve the simple kinds of addition and subtraction word problems that American texts stress in grades one through three. In the Soviet text, Stigler et al. found an even distribution of problems according to type; between three and ten times more problems as compared to the four American texts; and more complex two-step problems. (2) Across all American text series, two-step problems comprised only 7% of the total number of problems, whereas 44% of the Soviet problems were two-step problems. Stigler et al. found a rich range of two-step problem types in the Soviet text, while American texts primarily included combine–combine problems. (3) Soviet texts presented students with a more varied sequence of word problems, using more problem types in a given problem set, with more frequent changes in problem types.

In an analysis of German mathematics textbooks, Stern (1995) found only 3% of word problems in Grade 2 and Grade 3 texts were comparison problems. In Eastern Europe, as well as East Asia (particularly Japan), elementary schoolchildren
are frequently exposed to comparison problems (Fuson, 1992). In an analysis of addition and subtraction word problems in first-grade Belgian textbooks, DeCorte, Verschaffel, Janssens, and Joillet (1984) found that the range of problem types was quite restricted, and usually limited to the easiest types.

\textit{ii. Developmental notions and timing of topics}

Cultural beliefs regarding children’s capabilities are reflected in cross-national differences in curriculum—e.g., in the inclusion and timing of topics, level of abstraction, relative emphasis on inductive versus deductive reasoning. In England, for example, the National Curriculum has reduced the demands in the algebra content area for 16 year-olds (Tall & Thomas, 1991). Bell (1976) writes: "The present French syllabus adopts an axiomatic treatment of geometry from the third secondary school year (age 14) . . . . In England, proofs of geometrical theorems have been steadily disappearing from O-level syllabuses for thirty years, and 'it continues to be the policy of the SMP to argue the likelihood of a general result from particular cases' (Preface to Book 5). Underlying this divergence in practice lies the tension between the awareness that deduction is essential to mathematics, and the fact that generally only the ablest school pupils have achieved understanding of it" (p. 23).

In an analysis of Soviet vs. American first grade texts, Stigler, Fuson, Ham, and Kim (1986) observe that the inclusion of numerous one-step and two-step problems in a Soviet text "that seem quite difficult to the American eye must imply that children are capable of solving more difficult problems than we typically believe they can solve" (pp. 169–170).

Textbooks in Japan, China, Taiwan, and the Soviet Union exhibit a high degree of uniformity in the grade placement of topics in addition and subtraction. Both the simplest and most difficult multi-digit addition and subtraction problems appear
from one to three years earlier than in U.S. texts (Fuson, Stigler, & Bartsch, 1988). Sugiyama (1987) found word problems in Japanese texts for grades 7 and 8 were more difficult than those found in U.S. texts; problems in U.S. texts for grades 7 and 8 were found in grade 5 in Japan. For secondary texts, concepts tend to be introduced up to a year earlier in Japan; and more complex problems are included in Japanese texts, with a great deal of repetition of concepts in American texts (Stevenson and Bartsch, in press).

iii. Opportunity to learn

Across-group differences in mathematical reasoning may be linked to differences in the amount, and kind of exposure to mathematics. In a review of the literature, Secada (1992) concludes that two of the most powerful predictors of student achievement in large-scale mathematics assessments have been increased time on mathematics, and taking of advanced coursework (e.g., Welch, Anderson, & Harris, 1982). For example, in the 1978 National Assessment data for 17 year-olds, groups reporting more advanced mathematics coursework scored higher on measures of algebraic skills and understandings (Carpenter, Kepner, Corbitt, Lindquist, & Reys, 1982).

In the Second International Mathematics Study, pretest scores were a principal determinant of eighth grade posttest achievement (McLean, 1989), with an impact in the U.S. twice that of Japan (Schaub & Baker, 1991). This suggests American eighth grade mathematics teachers are more hampered by students' prior, inadequate preparation in elementary school (Stedman, 1994).

In their analysis of Soviet vs. American texts, Stigler, Fuson, Ham, and Kim (1986) cite frequency of exposure to various types of word problems as an explanation for American students' inability to represent quantitative relationships in more complex
word problem types; i.e., Americans have limited opportunities for learning to
approach, and to practice on, non-trivial forms of word problems.

Stigler, Lee, Lucker, and Stevenson (1982) analyzed the national textbook
series used in Japan and Taiwan, and a popular textbook series used in the U.S.
Significantly more topics were covered by the Japanese text, as compared to the
Chinese and American texts. The American text was similar to the Japanese text until
some point beyond first grade, but fell behind by fifth grade. By fifth grade, the
Japanese text had covered 86% of the topics covered by at least one country, while the
American texts had covered only 66%.

Mayer, Tajika, and Stanley (1991) provide evidence for an exposure
hypothesis, i.e., that international differences in mathematical performance are caused
by differences in the amount and kind of exposure to mathematics. Fifth graders in
Japan and the U.S. were given a test of mathematics achievement that evaluated
computational skill, and a test of mathematical problem solving that evaluated
problem-representation and solution-planning skills. Japanese children scored
highest on both tests. However, American fifth-graders performed relatively better on
problem solving than on computation, while Japanese children showed the reverse
trend. Mayer et al. selected American and Japanese children who possessed equivalent
amounts of basic mathematical knowledge (as measured by the mathematics
achievement test), to assess whether they differed with respect to problem-translation,
problem-integration, and solution-planning skills (as measured by the problem-
solving test). Using the mathematics achievement test scores, students in each sample
were partitioned into five achievement levels, with at least five students at each level.
Americans scored significantly higher than Japanese children on problem integration at
all five achievement levels; and scored higher on problem translation at four of the five
achievement levels, and on solution planning at all five achievement levels, though
differences failed to reach statistical significance. Mayer et al. suggest the results are
consistent with an exposure hypothesis: Japanese students receive more exposure to
basic mathematics in elementary school than American students, while the relative
emphasis on language based reasoning skills versus mathematically based quantitative
reasoning skills is higher for elementary school students in the U.S. than in Japan
(McKnight et al., 1987; Stevenson, Lee, & Stigler, 1986; Stevenson, Lee, Stigler, et
al., 1986). Japanese educators have expressed concern that lack of emphasis on
language–based aspects of problem–solving, and pressures to conform in Japanese
schools may limit students' learning of creative problem–solving skills, as measured
by the test of translation, integration, and planning (Stevenson, Azuma, & Hakuta,
1986). Interpretation of Mayer et al.'s results is problematic, however; in partitioning
the sample based on mathematics achievement test scores, Mayer et al. excluded the
data for 59 Americans who scored below 7, since only one Japanese child scored
below 7—resulting in the comparison of 109 of the original 110 Japanese subjects
with 63 of the original 132 American children.

Iben (1988, 1991) tested seventh and eighth graders in public schools in
Australia (n=979), Japan (n=216), and the U.S. (n=549). "Abstract mathematical
thought" was measured by the Iowa Algebra Aptitude Test (IAAT). Iben claims the
test contains problem types not normally studied in grades 7 and 8 in these countries—
that it is not a computational test, but a test of associating abstract operations in an
example with abstract operations on subsequent problems. The American sample
mean for each grade was the highest, followed by Japan and Australia, and the
differences between countries were significant (p<.01).
c. Classroom variables

Classroom variables are potentially related to cross-national differences in mathematical reasoning.

i. Classroom organization

In elementary schools, Chinese and Japanese children spend more time in focused academic activities. A greater percentage of time is devoted to teaching mathematics; and Chinese and Japanese classrooms are more centrally organized and teacher-led (Stigler & Perry, 1988; Stigler, Lee, & Stevenson, 1987).

ii. Lesson organization and classroom discourse

Japanese lessons have a coherent structure, designed to achieve a single, integrated instructional goal; U.S. lessons appear disjointed by comparison, with shifts in topics or activities that often seem unrelated to each other, or to any identifiable goal (Stigler & Stevenson, 1991).

Perry, VanderStoep, and Yu (1993) examined the kinds of questions asked by teachers in first grade addition and subtraction lessons in Japan, Taiwan, and the U.S. Japanese and Chinese teachers asked significantly more problem-solving strategies questions, and more "conceptual knowledge questions" than American teachers. Evidence suggests that students who are asked to respond to questions that require higher order thinking perform better than students who do not have to answer such questions, and this holds even when students do not actually answer the questions (see, e.g., Pressley, Symons, McDaniel, Snyder, & Turnure, 1988).

Cross-national differences have also been observed in the percentage of time spent in direct instruction—for example, in the introduction and explanation of new mathematical concepts. Stigler et al. (1987) found Chinese teachers used direct instruction 63% of the time, Japanese teachers 33%, and American teachers 25%. In
Japanese classrooms, there is a greater amount of verbal explanation by either the teacher or students during mathematics instruction. There is prolonged attention to one problem with direct, and explicit explanation. Japanese teachers tend to use public evaluation, allowing all children to examine alternative approaches, to reflect upon errors, and to examine processes that led to errors. Evaluation tends to be private in American classrooms (Stigler & Perry, 1988).

iii. Manipulatives

Japanese and Chinese teachers use more manipulatives and applied problem solving scenarios than American teachers. In Japanese classrooms, the frequency of verbal explanation increases while manipulatives are being used; in the U.S., the frequency remains the same.

iv. Tracking

In an attempt to "identify variables that significantly contribute to development of abstract mathematical thought," Iben (1988) tested seventh and eighth graders in public schools in Australia (n=979), Japan (n=216), and the U.S. (n=549). Independent variables included age, ability grouping, ethnicity, gender, mathematics attitudes, and mathematics classroom behaviors. The outcome variable was "abstract mathematical thought," as measured by students' scores on the Iowa Algebra Aptitude Test (IAAT). For schools that grouped by ability, accelerated mathematics ability group membership had the largest significant predictive regression weight for the U.S. and Australia on abstract mathematical thought, and membership in the lowest ability group had a negative regression weight for U.S. students.

v. Teachers' load

American teachers face a heavy bureaucratic workload. In the Second International Mathematics Study, American secondary school mathematics teachers
reported handling 5 classes daily—spending approximately 23 hours per week teaching, with meetings dominated by administrative issues. Japanese teachers reported handling three to four classes daily, and teaching 16 to 17 hours per week; meetings focused on improving instruction.

Characteristics of a curriculum (curricular variables) that may affect the development of algebraic reasoning

The curricular contrasts in this study included a comparison of opposed, theoretically driven ("experimental") curricula: *National Mathematics Project* in England, a five–book secondary school mathematics curriculum for Years 7–11 developed by Harper, Küchemann, Mahoney, Marshall, Martir, McLeay, Reed, and Russell ('987); and Davydov's elementary school mathematics curriculum in Russia. In this section, characterizations of, and theoretical underpinnings for each curricular model will be presented. Theoretical underpinnings and curricular models for developing each of the outcome variables in the experimental curricula are outlined in Tables 1–3.
Table 1. Experimental curricular models for developing algebraic reasoning

<table>
<thead>
<tr>
<th><strong>National Mathematics Project</strong></th>
<th><strong>Davydov</strong></th>
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<tr>
<td>• Developmentally oriented curriculum: Suggests learning follows cognitive development.</td>
<td>• Developing curriculum: Seeks to amplify cognitive development through curricular content and provision of cognitive tools.</td>
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<tr>
<td>• Empirical learning: Students compare a number of objects, identify their common observable characteristics, and then, by an act of generalization, formulate a general concept about the class of objects.</td>
<td>• Theoretical learning: Students are supplied with general and &quot;optimal&quot; methods for handling certain classes of problems, that direct them toward essential (not simply shared) characteristics of the problems of each class. These general methods are then used to solve concrete problems.</td>
</tr>
<tr>
<td>• Very gradual increase in the level of abstraction from Years 7 to 11. Visual, numerical, and geometrical supports for algebraic reasoning are continually provided throughout the curriculum.</td>
<td>• Abstract to concrete progression. As rapidly as is feasible, beginning in first grade, the curriculum (a) introduces formal, abstract properties of quantity, and properties of equality and inequality relationships; and (b) moves from visual and physical supports for reasoning, to a purely verbal and symbolic mode.</td>
</tr>
<tr>
<td>• The ability to reason, and the ability to symbolically represent the reasoning process are differentiated from one another, and are sometimes developed separately. In Years 7–11, concepts, principles, and reasoning processes are developed in a concrete context first; the corresponding algebraic representation is introduced later.</td>
<td>• Contends formal symbols are essentially related to the reasoning process, to abstraction and generalization and deduction. Introduces algebraic symbolism in first grade.</td>
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<tr>
<td>• Emphasis on inductive, case-based reasoning—the investigation of a number of particular instances to formulate and to assess the validity of algebraic generalizations, emphasizing the importance of empirical checks. Algebraic symbolism is presented as a means for writing general instructions and rules where the instruction or rule has been induced from a number of concrete numerical and/or geometrical exemplars. The activities &quot;provide valuable exercises in making (symbolic) generalizations from a small number of exemplars. (Warning needs to be given, of course, that erroneous false generalizations are easy to arrive at, and so any generalization should be checked and rechecked with examples.)&quot; (Teachers’ File, Book 4, p. 34).</td>
<td>• Emphasis on deductive, law-based reasoning—the logical derivation of particular cases from general mathematical principles and relationships where those principles and relationships are first expressed algebraically. Algebraic symbolism is presented as a means for representing quantitative relationships or structure. Beginning in grade 1, the curriculum develops notions of quantity, quantitative relations, and algebraic structure in &quot;pure form&quot;—not as a derivative of their concrete numerical manifestations—through actions on real objects, graphic models of underlying quantitative relationships, and algebraic symbols. Algebraic symbolism is initially used to represent particular relationships among concrete objects—then to represent general quantitative relationships and properties of those relationships in the absence of concrete objects. Students then identify this structure in numerical contexts.</td>
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Table 2. Experimental curricular models for developing concepts of algebraic structure

<table>
<thead>
<tr>
<th>National Mathematics Project</th>
<th>Davydov</th>
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<tbody>
<tr>
<td>• Procedural-to-structural model</td>
<td>• Structural-to-procedural model</td>
</tr>
<tr>
<td>• Emphasis on numerical input-output interpretations (&quot;procedural interpretations&quot;) of algebraic constructs precedes emphasis on structural interpretations, and precedes emphasis on algebraic manipulations and transformations. Procedural interpretations develop meaning for algebraic constructs.</td>
<td>• Emphasis on algebraic structure precedes emphasis on algebraic manipulations and transformations.</td>
</tr>
<tr>
<td>• Structural conceptions developed via inductive generalization. For example, notions of equivalence are hypothetically induced as students use a test and check method (i.e., a numerical input-output, procedural interpretation) to solve algebraic equations. Very gradual move from purely numerical reasoning, to numerical input-output interpretations of algebraic symbolism, to structural interpretations of algebraic symbolism from Years 7 to 11.</td>
<td>• Rapid and direct development of concepts of mathematical structure in a pre-arithmetic curriculum in first grade. Children represent quantitative relationships with an algebraic formula first; then generate specific numerical cases by substituting numerical values for variables (i.e., use a procedural interpretation to generate specific cases).</td>
</tr>
<tr>
<td>• Recurring emphasis on number concepts and number operations.</td>
<td>• Replaces number as a foundational concept with quantity.</td>
</tr>
<tr>
<td>• Concepts of mathematical structure and ability to operate on algebraic entities as abstract objects derived from actions on numbers, computational processes. Actions are reified and become mental mathematical objects that can themselves be operated upon (Kieran, 1992; Sfard, 1991).</td>
<td>• Concepts of mathematical structure derived from specified external actions on real objects (real quantities) and graphic models of particular quantitative relationships among objects. These specific actions are the starting points for the child's understanding of the meaning of arithmetic operations, relationships between arithmetic operations, quantitative relationships, and properties of quantitative relationships. Derivation of algebraic concepts from actions on concrete quantities and graphic models seems to develop ability to interpret algebraic entities as objects.</td>
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Table 3. Experimental curricular models for developing algebraic letter concepts

<table>
<thead>
<tr>
<th>National Mathematics Project</th>
<th>Davydov</th>
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<tbody>
<tr>
<td>• Critical curricular variable: Age (i.e., suggests younger children cannot acquire certain letter interpretations. Therefore the curriculum delays the introduction of various letter interpretations until children are 13 to 15 years-old).</td>
<td>• Critical curricular variable: Age (i.e., contends mathematical knowledge is abstract and generalized; therefore, students have to be given material whose mastery, <em>from the outset</em>, assures the development of content-based abstractions, generalizations, and concepts. Davydov contends &quot;the abstract as an element of thought should be introduced into instruction as early as it can be accessible to the child, who must not be held too long at the stage of sensory impressions, in any case&quot;; and suggests formal abstract symbols are essentially related to mathematical reasoning, to abstraction and generalization and deduction. Therefore provides abstract tools of reasoning, such as literal symbols, in primary school.</td>
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<tr>
<td>• Piagetian-based curricular model. Gradual development of abstraction, with a move from less abstract interpretations (specific unknowns) to more general interpretations (generalized numbers, variables).</td>
<td>• In order to allow children to examine <em>relationships</em> among objects, to discern empirical cases as concrete manifestations of generalized relationships, Davydov begins with more general interpretations of letters (general quantity), and moves to less abstract interpretations (specific unknowns).</td>
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<tr>
<td>• Letter concepts induced from concepts of number.</td>
<td>• Letter concepts derived from concepts of quantity using concrete objects (real quantities) and graphic models. Letter interpretations are gradually detached from their original object sources.</td>
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<tr>
<td>• Curriculum is designed to reflect the proposed &quot;naturally occurring&quot; order of acquisition of various letter interpretations. It is inferred from students' performances in empirical studies that 1) certain letter interpretations are inaccessible until children have attained formal operational thought; and 2) letters-as-unknowns are more cognitively accessible than letters as generalized numbers, variables, and givens (Harper, Küchemann, Booth, Collins). Letters-as-unknowns are introduced in Year 7, generalized numbers and variables in Year 8. Letter interpretations are carefully developed from Year 7 through Year 11.</td>
<td>• Letters assume a different pedagogical function: Introduced prior to numbers in given content areas, letters are used to isolate and to analyze and to orient children toward structural relationships. Literal expressions, equations, etc. are not initially generalized statements of numerical activity. To develop notions of algebraic structure, various interpretations of letters are introduced and developed in grade 1: letters as specific quantities, unknown quantities, given known quantities, specific unknowns, and generalized numbers.</td>
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Development of algebraic reasoning and concepts of structure: National Mathematics Project

National Mathematics Project emphasizes inductive, case-based (Moshman, 1995) reasoning—the investigation of a number of particular instances to formulate and to assess the validity of algebraic generalizations, emphasizing the importance of empirical checks. Numerical and geometrical supports for algebraic reasoning are continually provided throughout the series. Algebraic symbolism is presented as a means for expressing generalizations, writing "general instructions," and writing general rules where the generalization, instruction, or rule has been induced from a number of concrete numerical and/or geometrical exemplars (e.g., as a means of describing relationships between numbers of different objects in pictured collections and patterns).

With respect to developing children's ability to create and operate on abstract objects, to recognize and use mathematical structure, and to perform algebraic transformations, National Mathematics Project uses a procedural-to-structural curricular model (Kieran, 1992). Concepts and skills are developed via prolonged and increased emphasis on numbers and number operations; and procedural (numerical input–output) interpretations of algebraic symbolism. For example, notions of equivalence are hypothetically induced as students use a test and check method to solve algebraic equations. There is a very gradual move from purely numerical reasoning, to procedural interpretations of algebraic symbolism, to formal structural interpretations of algebraic symbolism from Years 7 to 11.

The ability to reason, and the ability to symbolically represent the reasoning process are differentiated from one another, and are sometimes developed separately. For example, in Year 10, Pythagoras' rule for the areas of squares surrounding right-
angled triangles is examined: "The work is intentionally non-algebraic so that pupils can concentrate more clearly upon the principles and interpretations involved" (Teachers' File, Red Track Book 4, p. 17).

**Theoretical underpinnings**

*National Mathematics Project* uses a procedural-to-structural model for developing algebraic concepts, as described by Kieran (1992). While *NMP* was developed prior to formulation of Kieran's curricular model, Kieran (1992) suggests *National Mathematics Project* exemplifies a procedural-to-structural approach. "However, to my knowledge, the long-term cognitive impact of these textbooks has not yet begun to be researched" (Kieran, 1992, p. 413).

Kieran applies Sfard's (1991) operational-to-structural model of mathematical concept acquisition to school algebra. Sfard characterizes mathematical knowledge in terms of processes and objects. She suggests most mathematical notions may be interpreted as processes or objects, and describes stages in concept development: A mathematics concept is initially conceived as a process (i.e., operationally), and then as an object (i.e., structurally).

Closely related models have been developed by Dubinsky (1991), Harel and Kaput (1991), and Thompson (1985). Harel and Kaput's notions are related to Greeno's notion of conceptual entity (Greeno, 1983). Dubinsky and Thompson appeal to notions of Piaget, particularly reflective abstraction. In Thompson's model, for example, mathematical objects are connected by relationships, and processes are operations on these objects. Thompson suggests most mathematical notions may be interpreted as processes or objects, and describes stages in concept development: 1) The operations of a process are carried out. 2) With familiarity, the process takes the
form of a series of operations that can be carried out in thought; the "learner has achieved operational thought with respect to this concept." The learner constructs a mental process that transforms mental objects. 3) The mental picture of this process crystallizes into a single entity, a new object. The learner can now think of this notion either dynamically as a process, or statically as an object. An essential step in mathematics learning is therefore objectification: making an object out of a process.

Similarly, Sfard (1991) and Kieran (1992) describe two possible interpretations of algebraic constructs. In a process (or "procedural" (Kieran, 1992) or "operational" (Sfard, 1991)) conception, an algebraic entity (e.g., an expression, equation, function) is interpreted as a concise description of a computational process, an algorithm, an action. For example, $2(x + 8) + 1$ is interpreted as a series of arithmetic operations on some number. In a procedural interpretation, numbers are substituted for variables and computations are performed, yielding a numerical result. The objects that are operated upon are numbers.

In a structural interpretation, an algebraic construct is interpreted as an object, an entity that can be operated upon. For example, an algebraic equation is interpreted as a static relation between two magnitudes. In a structural interpretation, literal symbolism represents general quantitative relationships and mathematical structures, and operations on those structures. The objects that are operated upon are algebraic constructs, rather than numbers. Operations include simplifying expressions, factoring expressions, solving equations by performing the same operation on both sides, etc., typically yielding algebraic expressions. Structural conceptions require (a) interpretation of algebraic letters, expressions, equations, etc. as objects that can be operated upon (Kieran, 1992; Sfard, 1991); and (b) recognition and use of mathematical structure (i.e., recognition of properties of arithmetic operations,
relationships between arithmetic operations, quantitative relationships, properties of quantitative relationships) (Booth, 1989; Kieran, 1989). Properties of arithmetic operations (e.g., commutativity, associativity), relationships between arithmetic operations (e.g., distributivity, inverse relation between addition and subtraction), quantitative relationships (e.g., equality), and properties of quantitative relationships (e.g., symmetric and transitive character of equality, addition property of equality) underlie, and provide the basis for, algebraic transformations.

Sfard and Kieran's basic theoretical theses are (1) There is an invariant developmental progression in concept acquisition that is "immune to changes in external stimuli," including effects of curriculum. The procedural (operational) conception is the first stage of concept formation, and the structural conception evolves from the operational conception. The structural interpretation represents a higher level of abstraction and generality. Sfard appeals to Piaget's notion of reflective abstraction, and suggests ontogenesis might mirror the historical evolution of mathematical concepts: most contemporary structural definitions evolved from operationally conceived notions. (2) Both interpretations are needed. (3) The transition from procedural to structural conceptions is a long and inherently difficult process, accomplished in three hierarchical stages: interiorization, condensation, and reification. In the third stage, reification, a process solidifies into a static structure. The learner can now investigate properties of this new object, and various relations between its representatives; and can perform processes where the new object is an input.

Given the hypothesized invariant progression in concept development, Sfard and Kieran support a curricular progression that fosters an understanding of processes and algorithms before translating them into structural definitions. That is, procedural,
numerical input–output interpretations of algebraic constructs should be emphasized prior to structural interpretations. Reinterpreting the algebra learning research, Kieran (1992) attributes learners' inability to acquire algebraic concepts and reasoning to an overly rapid transition from numerical arithmetical activity in elementary school to a purely structural approach in school algebra. Research supports Kieran's claim that procedural interpretations are cognitively accessible; yet in school algebra courses algebraic expressions and equations are often only very briefly interpreted as arithmetic operations upon some number. Kieran suggests a lengthy period of experience with procedural interpretations would serve as a kind of transition period, developing meaning for algebraic constructs. Algebraic symbolism would be interpreted as generalized statements of the operations carried out in arithmetic, i.e., treated in procedural terms with numerical values substituted into algebraic expressions to yield specific output values. For example, students would use various arithmetical techniques to solve algebraic equations.

*Examples from the National Mathematics Project curriculum*

*National Mathematics Project* emphasizes inductive, empirical reasoning. With respect to developing children's ability to create and operate on abstract objects, to recognize and use mathematical structure, and to perform algebraic transformations, *National Mathematics Project* uses a procedural–to–structural curricular model (Kieran, 1992). *National Mathematics Project* includes five texts for Years 7–11, with two separate tracks. The five "Red Track" texts are intended for higher achieving students, while "Blue Track" texts are intended for lower achieving students in the same Year levels. The following examples are from the Red Track texts.
1. DEVELOPMENT OF CONCEPTS OF STRUCTURE.

Concepts of structure are developed via concrete examples, and procedural interpretations of algebraic symbolism. For example, in Year 7, children examine basic properties of arithmetic operations, and relationships between the four operations, using various methods:

(1) students examine numerical cases (27+4=4+27; 35–23≠23–35; "Write this multiplication statement as an addition in two ways: 2×5");

(2) students formulate two or more open sentences that represent the same quantitative situation, using inverse operations (e.g., 'There were 46 pigeons in a loft. 36 were male. How many were female?' is represented by 46=7+36 and 46–?=36);

(3) arrow diagrams (diagrams with number facts that use forward-pointing and backward-pointing arrows to illustrate the inverse relationship between addition and subtraction, and between multiplication and division);

(4) dot patterns that illustrate the commutative property (e.g., Write each of these dot patterns as a multiplication in two ways);

(5) students use principles to solve computational problems (e.g., "Look at these number sentences: 1149+567=1716, 2409–777=1632, 777+1149=1926, 1632–1149=483. Without doing any calculations, write down the answer to these: 1716–567; 777+1632; 1149+483; 1926–777" (Teachers' Handbook, Book 1, p. 45)).

The ability to write equations that represent quantitative relationships is developed throughout the NMP curriculum. In Year 7, situations are modeled (mathematized) using number sentences. Equations gradually increase in complexity and intermediary kinds of supports are offered—e.g., "skeleton number sentences" that students complete. For example, in Year 8, the following problem is posed:
Jim used 60 g of flour for a bread roll. He used what was left for two rock cakes both the same size. He started with 150 g of flour. How many grams did he use for each rock cake?

A "skeleton sentence" is provided for students: \( V - V - ? \times V = V \) where ? is defined as the number of grams of flour used for each rock cake:

Replace each [triangle] by a number. Make a number sentence for the problem. Write down the number which ? replaces. Check that your answer fits the problem and the sentence. (Teachers' Handbook, Book 2, p. 35)

2. PROCEDURAL-TO-STRUCTURAL PROGRESSION.

In NMP, there is a very gradual move from purely numerical reasoning, to procedural (numerical input–output) interpretations of algebraic symbolism, to formal structural interpretations of algebraic symbolism from Years 7 to 11.

For example, in Year 7, students solve equations using an intuitive 'guess and check' technique that develops notions of equivalence, as a precursor to more formal methods. For example, the following task is posed:

Find the number which the letter replaces in each of these sentences. Guess and check until you find the correct number: \( r + t = 5 + f \), \( c + c = 12 - c \), \( k \times k = 30 + k \), \( 2 \times n + 4 = 36 \). (Teachers' Handbook, Book 1, p. 187)

In Year 8, activities demonstrate how seemingly different number sentences can model the same quantitative situation. Formal methods of manipulation and solution of equations "are delayed until later, when pupils will have a firm foundation and appreciation of the role and meaning of number sentences in relation to mathematical problem situations" (Teachers' Handbook, Book 2, p. 33).
In Year 8, generalized numbers are presented as a means for writing "general instructions"—i.e., a procedural interpretation is developed. For example, the following problem is posed:

Here are some [verbal] instructions for drawing triangles. For each set of instructions, (i) draw a possible triangle accurately; (ii) rewrite the instructions using letters; (iii) under each drawing, write down the number you chose for each letter. *(Teachers' Handbook, Book 2, p. 110)*

Numerical input-output activities emphasize the distinction between an identity and an equation; students consider pairs of expressions that are always, sometimes, or never equal for selected substituted values of the variables, developing students' ability to use identities in simplifying algebraic expressions. Year 8 introduces some elementary algebraic manipulation (with strong geometrical support). Sample problems include:

Here are some more pairs of expressions [e.g., $k^2 + 2$ and $2^k$; $n \times 15$ and $15 \times n$]. For each pair, choose values for the letter. Try to make the two expressions stand for the same number. For some you will find only one possibility. For some you will find none. For others you will find many. Write down what you find out for each pair. *(Teachers' Handbook, Book 2, p. 139)*

What value of $k$ will produce a square for these instructions? $k$ and $k + 2$. What does this tell us about the expression $k = k + 2$? *(Teachers' Handbook, Book 2, p. 138)*

In Year 9, activities extend work on modeling and solving equations, including equations in two unknowns. Solutions by formal methods are not required, although a formal approach is offered in a section of the enrichment material. The teachers' handbook advises: "This section introduces a formal approach to solving equations. You might wish to ignore it in favor of maintaining an 'open', 'intuitive' approach,
and move straight to section E" *(Teachers' Handbook, Book 3, p. 58).* The only required manipulation of equations is the collection of like terms.

The Year 10 text introduces a formal method of solving algebraic equations—performing the same operation to both sides of an equation. The text encourages students to use intuitive 'think and test' methods with formal methods. The importance of checking a solution is emphasized throughout. The text shows the power of formal methods by considering complex equations that are difficult to solve by more intuitive means, and simultaneously encourages pupils to assess whether the "more 'direct' or 'intuitive' methods ('think and test' and 'think it out')" might be more applicable "before spending what might be wasted time and labour on a more formal method. The most complex—appearing equation might often have a solution which can be 'spotted'" *(Teachers' File, Book 4, p. 56).*

Year 10 students solve simultaneous equations using 'think and test' methods and graphical methods of solution, providing an "intuitive base" for more formal methods in Year 11. In Year 11, graphical methods of solving quadratic equations are considered; algebraic factorization is presented as a means of solving equations when factors are readily identifiable; and the use of 'calculator methods' (i.e., 'trial and improvement' methods) are encouraged for more complex situations. The formula for the roots of a quadratic equation is introduced. Simultaneous equations are reconsidered; the text introduces more complex pairs of equations to motivate the need for more formal algebraic solving methods, rather than graphing or listing of solution pairs. The aim is "both to present formal methods of approach and also to leave open more intuitive 'trial and improvement' methods" *(Teachers' File, Book 5, p. 149).* The text "presents five formal methods for solving simultaneous equations, related to the complexity of the equations: listing solutions; finding the value of the unknowns..."
by adding/subtracting; multiplying one equation by a number; multiplying both equations by a number; substitution" (Teachers' File, Book 5, p. 149).

3. DEVELOPMENT OF ALGEBRAIC REASONING.

National Mathematics Project emphasizes induction. Children observe a number of particular instances of a principle or generalization, and then, by an act of generalization, arrive at the principle or generalization. When children express generalizations algebraically, the algebraic rule is tested empirically (by substituting numbers) until children are "certain" that the generalization is valid for all numbers in a given set—finite or infinite. The texts emphasize "the importance of checking a generalization with more than one example" (Teachers' Handbook, Book 2, p. 190). Numerical and geometrical supports for algebraic reasoning are continually provided throughout the series. The ability to reason, and the ability to symbolically represent the reasoning process are differentiated from one another, and are sometimes developed separately.

For example, in Year 8, the text presents a series of numerical and geometrical situations designed to develop children's ability to make generalizations, and to express them in the form of verbal rules (for example, 'To find the number of white tiles, subtract 1 from the number of colored tiles, then multiply by 2'). In Years 9 and 10, this activity is linked to the concept of variable; verbal rules are used to generate symbolic rules, formulas, and functions (e.g., $d=2(n-1)$). "However, [in Year 8] this is postponed so that pupils can concentrate exclusively upon the act of generalizing at this stage" (Teachers' Handbook, Book 2, p. 191). The Year 8 text suggests a method for formulating generalizations: students should use a systematic approach
when making and checking generalizations; they should predict the pattern; and they should check their prediction with examples until they are sure they are correct.

In Year 9, the text introduces the use of letters to express rules (or 'formulas'). Reasoning is supported with visual illustrations of the situations that are to be 'algebraicised', or students induce a generalization from numerical examples. Students are advised:

Take note: Don't be satisfied with checking [an algebraic] rule by using just one or two [numerical] replacements. Always ask yourself if the rule works for all the numbers you want it to work for. (*Book 3, p. 177*)

This table gives the lengths of shoes for various British shoe sizes. (a) Copy and complete this graph . . . . (d) There is a rule which connects the length of shoe and British shoe size. Find it, and write it in words and in letters. (e) Use your rule to check that the length of a size 4 shoe is $9\frac{2}{3}$ inches. If your rule does not give this, it is not correct: go back to (d) and try again. (*Book 3, p. 207*)

The idea of a rule is developed from the notion of an instruction. For example, the text notes that "by inserting the sign '=' between $2x$ and $p+2$ for this instruction [the text shows a rectangle having dimensions $2x$ and $p+2$] we immediately change the status of each letter $p$ and $t$ to that of a 'correlated variable' . . ., where the value of one depends upon the value of the other (if the relation is to be satisfied)"
(*Teachers' Handbook, Book 3, p. 108*).

Simple transformations (rewriting) of algebraic formulas are guided by visual representations of the quantitative situations. Students try to "think a formula into a new form," using drawings and illustrations if necessary "so that an intuitive approach is developed, rather than a formal rule bound approach (e.g., inverse elements or inverse operation approaches)" (*Teachers' Handbook, Book 3, p. 107*).
In Year 10, activities provide additional experience in investigating and developing generalizations in concrete contexts, and writing generalizations with letters. For example, the text shows a sequence of patterns of crosses, and asks:

How many crosses does the twentieth pattern have? Use the expression $2n - 1$. Choose $n$ to be 1, 2, 3, 4, . . . What do your results tell you about the patterns of crosses? One of the patterns has 131 crosses. What position is the pattern in the sequence? What value of $n$ gives this number of crosses? (Book 4, p. 151)

The teachers' handbook states: The activities "provide valuable exercises in making (symbolic) generalizations from a small number of exemplars. (Warning needs to be given, of course, that erroneous false generalizations are easy to arrive at, and so any generalization should be checked and rechecked with examples.)" (Teachers' File, Book 4, p. 34).

4. SUMMARY AND POSSIBLE IMPLICATIONS.

While the inductive procedural-to-structural model has some empirical support, it remains unclear whether prolonged and increased emphasis on procedural interpretations will assist or hinder learners (a) in moving from procedural to structural conceptions (Sfard, 1989); (b) in developing an understanding of the relation between the two interpretations (Kieran, 1992); or (c) in discerning the deductive nature of algebraic argumentation. That is, it is unclear whether attempts to invite inductive heuristics are not, in fact, misleading.
Development of algebraic reasoning and concepts of structure: Davydov

Davydov emphasizes deductive, law-based (Moshman, 1995) reasoning—the logical derivation of particular (e.g., numerical) cases from general mathematical principles and relationships where those principles and relationships are first expressed algebraically. Designed to develop theoretical concepts and reasoning, Davydov's curriculum stresses a particular method of generalization and concept formation: theoretical generalization that moves from the abstract to the "intellectually concrete" and then on to an enriched abstraction. Students first master the organizing ideas of a discipline, the principles underlying diversified particular phenomena.

The curriculum emphasizes theoretical learning, rather than empirical learning. In empirical learning, students compare a number of objects, identify their common observable characteristics, and then, by an act of generalization, formulate a general concept about the class of objects. In theoretical learning, the student is supplied with general and "optimal" methods for handling certain classes of problems, that direct him/her toward essential (not simply common) characteristics of the problems of each class (Karpov & Bransford, 1995). These general methods are then used to solve concrete problems.

Davydov argues (a) formal, abstract understanding (e.g., of mathematical structures) should precede the use of that understanding in practical, concrete (e.g., numerical) contexts; (b) the mathematics course, at the various stages of learning, should be unified; and (c) the traditional break between the arithmetic and algebra courses can be avoided by developing the concept of number (and its various manifestations—whole numbers, rational numbers, irrational numbers) from the notion of quantity. That is, the common notion underlying various manifestations of number is quantity. Thus in keeping with (a), (b), and (c), the curriculum first
develops notions of quantity, quantitative relations, and algebraic structure in "pure form"—not as a derivative of their concrete numerical manifestations—through actions on real objects, graphic models of underlying quantitative relationships, and algebraic symbols. Prior to numerical activity, first graders develop some understanding of (a) some aspects of algebraic structure (e.g., properties of order, properties of equivalence, properties of operations); and (b) use of this structure as a basis for very elementary techniques of algebra. Students then identify this structure in numerical contexts, and solve arithmetic word problems by using general algebraic solution methods beginning in grade one.

As rapidly as is feasible, the curriculum (a) introduces formal, abstract properties of quantity, and properties of equality and inequality relationships; and (b) moves from visual and physical supports for reasoning, to a purely verbal and symbolic mode. Algebraic symbolism is initially used to represent particular relationships among concrete objects—then to represent general quantitative relationships and properties of those relationships in the absence of concrete objects. Formal symbols are viewed as essentially related to the reasoning process, to abstraction and generalization and deduction.

Theoretical underpinnings and curricular design

Davydov draws on Vygotsky's distinction between everyday and scientific concepts. For Vygotsky, the greatest strides in developing the intellect—becoming aware of mental processes and mastering them—were made through the acquisition of scientific concepts. Vygotsky (1987) contrasted learning by preschoolers and learning at school. Preschoolers' concepts are primarily unsystematic, empirical, and unconscious—developing as a generalization of their concrete experiences. In contrast
to these "spontaneous concepts" of everyday-life thinking, "scientific concepts" represent the essence of some class of phenomena; and must be acquired consciously and in a certain system. Having been acquired, systems of scientific concepts transform the child's everyday knowledge. Spontaneous concepts become structured and conscious (Karpov & Bransford, 1995). Thus, for Vygotsky, substantive progress in development was associated primarily with the content of instruction, particularly scientific content (El'konin, 1975).

Davydov distinguishes further—between rational–empirical concepts and theoretical concepts. Davydov suggests the primary factor differentiating scientific and spontaneous concepts is their content; content is theoretical for the first group, and empirical for the second. Acquisition of spontaneous and scientific concepts are the result of fundamentally different types of learning activity. Everyday concepts are developed through empirical generalization, involving the abstraction of similarities from collections of concrete objects and observable phenomena that may in themselves represent varied functions and structures (Schmittau, 1993a). Empirical generalization is rooted in observation, involving the categorization of objects via a formal comparison of their external properties, while scientific theoretical concepts require a theoretical abstraction since their essences (i.e., necessary and sufficient attributes) cannot be discerned at the surface level of appearance. Theoretical knowledge is concerned with essential internal relationships among objects.

Davydov criticizes the inductive approach of what he calls traditional empirical school programs that require students to appropriate theoretical generalizations (characteristic of mathematics) through empirical abstraction; that attempt to move from "sensory–concrete diversity" to a discovery of general principles or forms; and argues that the "ability to solve practical problems does not necessarily imply children's
knowledge or understanding of . . . deeper [mathematical] principles" (Davydov, 1982, p. 225). "It is known that scientific knowledge is not a simple extension, intensification, and expansion of people's everyday experience. It requires the cultivation of particular means of abstracting, a particular analysis, and generalization, which permits the internal connections of things, their essence, and particular ways of idealizing the objects of cognition to be established" (Davydov, 1990, p. 86).

According to Davydov, if the basic curricular content remains empirical knowledge, with the perpetuation of inductive/empirical generalization in schooling, then instruction only exercises, and thereby improves the mental processes involved in the mastery of empirical knowledge. Since the development of empirical knowledge is characteristic of the preceding period of development (i.e. during the preschool years), this kind of schooling demands only previously formed intellectual processes for solution (Davydov, 1990; El'konin, 1975). Rather, schooling should develop theoretical reasoning, analysis, conscious planning of actions, and mental reflection.

Davydov (1990) cites the example of "roundness" to illustrate the distinction. Roundness can be empirically abstracted from a dish, a wheel, and a heavenly body. Developing the notion of circularity in this manner, however, does not reveal the essential properties of circularity, since the essential properties are not apparent in the mere appearance of roundness. Further, an empirical abstraction of this kind reflects the particular perspective of the individual constructing it, the perception of the individual; it is a mental construction. In contrast, Spinoza has described the objective content (i.e., existing outside of the mental construction of any particular individual) of the scientific concept of a circle as "a figure described by a line one end of which is fastened and the other end of which is movable." This expresses the "essence" of circularity and is independent of particular instances embodying the concept. Further,
it discloses its objective origins through a scheme of activity (Confrey, 1991; Schmittau, 1993a). Following Hegel, who held that children should not be held very long at the level of empirical thinking, Davydov contends elementary schoolchildren are capable of constructing a circle and hence, of appropriating its theoretical essence through activity. Then, in an "ascent from the abstract to the concrete," this scientific understanding can enrich their experience of the variety of objects embodying the concept of circularity (Schmittau, 1993a).

Davydov argues that the development of formal, abstract understanding should precede the use of that understanding in concrete contexts, and since the overriding task of education is to ensure the appropriation of the objective content of scientific concepts, this requires that children be taught theoretical concepts/generalization from the very beginning. Davydov (1990) cites Kolmogorov's characterization of the objects of mathematics study: "Mathematics studies the material world from a particular point of view and its immediate object is the spatial forms and quantitative relationships of the real world. These forms and their relationships in their pure form, rather than concrete material bodies, are the reality which mathematics studies" (p. 87). Davydov sets up the elementary mathematics course (a) to develop concepts of "relation or structure"; and (b) to lay the groundwork for the notion of real number by using the "common root of whole and real numbers," i.e., the notion of quantity. Children's introduction to quantity and mathematical structure is made a special section of the elementary mathematics course, prior to arithmetic activity.

Davydov (1975) suggests three uses of this particular knowledge. First, by acquainting the child with properties of quantities, this section lays the foundation for the detailed introduction of whole numbers; allows a smooth transition to real numbers; makes it possible to soften the sharp opposition teaching traditionally sets up
between these types of numbers; and thereby allows the algebraization of the regular elementary school mathematics course. The notion of quantity and certain axioms related to quantity are used to develop the system of relationships that serve as the basis for subsequent mathematical transformations. Natural numbers, rational numbers, and real numbers can be represented as quantities; certain essential characteristics apply to all (whatever $a$ and $b$ are, one and only one of the three relationships $a=b$, $a<b$, $a>b$ holds true; if $a<b$ and $b<c$, then $a<c$; for any two quantities $a$ and $b$, there exists a single particular quantity $c$ to which $a+b$ is equal; commutativity of addition, associativity of addition, monotony of addition; if $a > b$, then there exists one and only one quantity $c$ for which $b+c=a$ (the possibility of subtraction)).

Second, if children understand the basic characteristics of equality and inequality, and the means of moving from one to the other, the emphasis in mathematics instruction can shift from "techniques of calculation" to the study of the structural relationships that regulate these calculations, to the study of the structural characteristics of mathematical "objects" (e.g., equalities).

Third, from the very beginning, working with quantities and abstracting their properties involves letter symbols, via which children can examine particular relationships among objects—essential for all subsequent progress in mathematics.

Davydov (1990) claims this section increases the child's ability to evaluate abstract relations in objects, a quality detected in the study of subsequent sections of the curriculum. For example, during familiarization with number, first graders transfer their knowledge of general properties of quantities (transitivity, monotonicity, the "equilibrium feature" of equality, etc.) to the number series.
General characteristics of the curriculum can be cited. First, the curriculum emphasizes theoretical learning, rather than empirical learning. In empirical learning, students compare a number of objects, identify their common observable characteristics, and then, by an act of generalization, formulate a general concept about the class of objects. In theoretical learning, the student is supplied with general and optimal methods for handling certain classes of problems, that direct him/her toward essential (not simply common) characteristics of the problems of each class (Karpov & Bransford, 1995). These general methods are then used to solve concrete problems. In Davydov's mathematics curriculum, it is the general quantitative relationships of the subject matter under study that serve as the basis for these general solution processes (Cobb, Perlwitz, & Underwood, in press). Children are trained to apply certain methods of analyses, consistently and systematically (Bodanskii, 1991). "The effectual nature of this mode [or general solution process] is verified during the resolution of discrete particular tasks, when schoolchildren approach those tasks as variants of the initial learning tasks and immediately—'on the spot,' as it were—identify the general relationship in each of them. Once oriented toward that general relationship, they will be able to apply the general mode of resolution that they have previously assimilated" (Davydov, 1988, p. 32). Through specially organized activity, students master and internalize the processes of use of these methods (Karpov & Bransford, 1995).

Second, Davydov attempts to unify mathematics learning at the various stages of learning by avoiding the need for massive restructuring of the child's conceptual schema. The introductory course is designed to develop initial concepts that are broad and flexible enough to allow future expansions of the conceptual system(s). For example, traditional school practice builds on children's spontaneously constructed
counting structures. This narrow restriction of the initial concept of number creates difficulties when the notion is expanded (e.g., to include rational numbers). In Davydov's curriculum, the concept of number is developed from the notion of quantity; the concept of natural number is introduced as a relationship of quantities, via measurement activity. This leads naturally to rational and irrational numbers.

Third, Davydov attempts to expose the original material content of concepts, to expose the sources of concepts. Davydov cites Kolmogorov: "[D]ivorcing mathematics concepts from their origins, in teaching, results in a course with a complete absence of principles and with defective logic" (in Davydov, 1975, p. 120).

Fourth, Davydov begins with concrete objects. They serve as the source of appropriate concepts. The child has to learn to operate in this realm before he can make the transition to full-fledged concepts. The use of visual aids is limited, however. According to Davydov, the exploration of relationships between physical objects should not be unduly extensive because "one must make extensive use, even at the elementary school level, of [mathematical] generalizations shaped on the basis of a minimum number of appropriately organized observations." This can be achieved by ensuring that the actions children perform on physical objects are those that "permit an object or situation to be transformed in such a way that a person can immediately single out in them the relationship that has a universal character" (Davydov, 1990, p. 346) (Cobb, Perlwitz, & Underwood, in press).

Fifth, initial material actions are specified that allow the child to isolate and master properties of quantity and mathematical structure: e.g., comparing various attributes of physical objects; selecting an object of "the same" length, volume, or color as a model; making a new object that matches the model in a particular attribute (e.g., making a piece of plasticine of the same volume as a block); using concrete
objects (e.g., paper strips) to represent transformations that correspond to pairs of expressions such as \(a=b\) and \(a<b\). Acquisition of formulae, concepts, symbols, etc. requires mastery of the processes underlying these tools. Actions are the starting points for the child's understanding. "In doing certain object-related operations [i.e., physical actions] as indicated by the instructor, the pupils detect and establish essential features of objects such that orientation in them permits the solution of any problems in a given class that are connected with some similar situation. These operations are initially performed in a material or materialized form, and then are converted step by step into mental operations that are done with symbolic substitutes for the material objects" (Davydov, 1990, p. 349).

Sixth, Davydov contends this process of isolating essential properties and relationships depends on the use of intermediate means of depicting and describing the results of concrete actions on objects (e.g., "copied" and "abstract" sketches to help isolate relationships involving equality-inequality comparisons). Graphic models are used to represent abstract relationships. They include features that reflect the structure of the mathematical objects they represent in a sensory-visual form, allowing them to be analyzed further. (Davydov draws on Leont'ev and Galperin's learning theory, arguing visual aids and models can play a crucial role in making ideal mathematical forms accessible (Cobb, Perlwitz, & Underwood, in press).)

Finally, Davydov emphasizes the critical role that symbol use plays in the internalization process, in psychological development. In a curricular sequence, the use of concrete objects and graphic models is reduced; emphasis given to symbolic formulas is increased. Letters assume a different pedagogical function. Letters are used to isolate and to analyze (and to orient children toward) mathematical relationships (e.g., \(a=b\), \(a>b\), \(a<b\), \(a=b+x\), part-whole relationships, number as a
particular case of the representation of a general relationship of quantities where one of the quantities is taken as a measure of the other, isolation of the pivotal relation in a word problem). Literal expressions, equations, etc. are not initially generalized statements of numerical activity. Letter formulas are introduced prior to numbers in a given content area to assist students in acquiring a *general* concept of underlying quantitative relationships, to allow an analysis of the properties of mathematical relationships. With the introduction of numbers, numerals are substituted into general literal formulas. Thus numerical instances appear as particular instantiations of a general relationship between quantities (i.e., a structural to procedural progression).

*Examples from Davydov’s curriculum*

1. **PRE–NUMERICAL SECTION OF THE ELEMENTARY MATHEMATICS COURSE.**

   In a pre–arithmetic curriculum in the first semester of first grade, prior to introduction of numbers, children abstract the property of quantity using physical objects (beakers of water, strips of paper, etc.). They compare the objects with respect to a certain quantity (weight, volume, area, length, loudness of sounds, duration of sounds, etc.); learn to identify the equality or inequality of, say, the length of the physical objects (forming the abstraction "more than" and "less than"); solve problems involving a change in the criteria of comparison for the same objects; and make drawings to represent the results of an equality–inequality comparison (first a "copy" of the real objects, and then an "abstraction," e.g., using lines or triangles).

   For example, given two weights on a scale with the heavier weight on the right, students represent the relationship by drawing two line segments, drawing a short line on the left and a long line on the right. The relationship between the length of the lines corresponds to the relationship between the objects, where the objects are
compared with respect to any criteria. These signs are intended to develop the understanding that when any quantities are compared, only their relationships are singled out and taken into consideration (Davydov, 1990). That is, it is the relationship itself, its type, rather than the objects through which the relationship manifests itself, that is of interest.

They then move from signs to letters. Quantities are designated by letters (e.g., \(a, b\)), and the result of a comparison is written as a formula \((a=b, a>b, a<b)\). By using letters, the need for concrete, verbal descriptions of quantities is gradually eliminated. Students refer to "the quantity \(a\)" instead of "the length of this paper strip", "the number of cubes in this group", etc. Using literal formulas, it becomes possible to study the actual properties of the equality–inequality relationship itself (Davydov, 1975).

Properties are studied using physical objects and literal formulas: the disjunction of equality and inequality; reversibility and reflexivity of equality; connections between the relationships "more than" and "less than" (if \(a>b\), then \(b<a\), and so forth); and transitivity as a property of equality and inequality. Various problems that require a knowledge of these properties are solved (e.g., Given \(a>b\), and \(b=c\), find the relationship between \(a\) and \(c\)) (Davydov, 1975). Instructional activities are designed to move children away from direct observation, to solving problems only through analysis of the letter formulas with no dependence on objects as visual aids—shifting to verbal and logical evaluations (constructions of the type, "if . . . and . . ., then . . .") (Davydov, 1975).

Children then learn to observe changes in the values of physical quantities (e.g., the amount of water in a flask, the force of a push); to describe the changes as increases or decreases; and to represent increases or decreases in quantities using plus
and minus signs. They learn to coordinate changes in the values of quantities with the properties of equality and inequality, and to pass from inequalities to equalities by means of addition and subtraction. Children are led to discover that an inequality between quantities may be "taken away" by determining the specific difference between them. For example, given two objects, say, strips of paper, where one is longer than the other, the children make a comparison, representing the relationship as \(a < b\). They are then challenged to change this inequality into an equality. With the objects before them, the children, for example, suggest making \(b\) smaller or \(a\) larger. Using the children's suggestion, and demonstrating with strips of paper or other visual aids, the teacher shows that it is not known beforehand how much needs to be added or subtracted in order to make the two quantities equivalent; the teacher proposes the use of the special symbol \(x\) to designate the unknown quantity that needs to be added or subtracted to obtain equal quantities. The child is thus introduced to a very simple form of an equation. For example, children write: If \(a < b\), then \(a + x = b\) where \(x\) is used to designate the unknown difference; or If \(a > b\), then \(a - x = b\). Here, too, a deeper understanding of the relationship between addition and subtraction is acquired. They then determine the value of \(x\) (e.g., writing formulas of the type \(a < b, a + x = b, x = b - a, a + (b - a) = b\)). The process of formulating and solving an equation is done with the aid of objects.

In the next stage, there is a move toward solving equations using only literal formulas. Children solve problems, including word problems with literal data, that require the operations of addition and subtraction. For example, problems are posed: "There are \(A\) kilograms of apples in one box and \(B\) kilograms in another. We know that \(A\) is less than \(B\). What needs to be done so the apples in the first box weigh the same as those in the second?" (Davydov, 1975, p. 190). Children respond "Some
apples need to be added to the first box. $A < B, A + x = B$," and recognize that they need to solve for $x$. The teacher introduces graphic models to assist children in determining $x$. Children draw lines to depict the weight of the apples. Line $A$ is superimposed on line $B$, and the remainder, expressed as $B - A$, is defined as being equal to $x$. Initially, simultaneous operations are performed with objects (e.g., weighing of apples). Gradually children can determine $x$ without relying on objects or graphic models (Davydov, 1975).

To synthesize previously presented material, and to inculcate elementary reasoning based on properties of relationships, children are introduced to addition and subtraction of equalities and inequalities (e.g., if $A = B$ and $M = D$, then $A + M = B + D$). Various problems are posed that require simultaneous consideration of several properties: for example,

If $A = B$ and $K > M$, then $A - K \_\_ B - M$.
If $E < B$ and $M < G$, then $E + M \_\_ B + G$.

Students learn that the specific value of a quantity may be expressed in multiple forms, laying the groundwork for commutativity and associativity of addition. For example, students are asked to rewrite the inequality $A > B$, given that $A = K + M + N$. Students also master the following properties in general form: 1) if $a = b$, then $a + c > b$; and 2) if $a = b$, then $a + c = b + c$. Davydov (1990) writes: "This knowledge is learned well by the children, and thus they are allowed to solve highly diversified problems related to the need to consider the "equilibrium" feature and the conditions for preserving it . . . Experience in experimental mathematics instruction from grade 1 to grade 5 shows that the child's earlier mastery of . . . [the] property [of monotonicity] in general form substantially simplifies the subsequent mastery and understanding of its particular manifestations and applied significance" (p. 360).
Children are then familiarized with commutativity and associativity of addition. They demonstrate the properties while operating with physical objects, and later with numbers.

In this phase of instruction, concepts of mathematical structure are being developed prior to arithmetic activity. Davydov is developing an understanding of quantitative relationships (e.g., equivalence), properties of quantitative relationships (e.g., transitivity), properties of operations on quantities, and relationships between operations. He is also developing skill in making mathematical models, in describing changes in physical quantities. Various letter interpretations are also being developed.

In the next stage of instruction, the child is introduced to number, as the expression of a relationship between the whole of some object and a part of it. A number is obtained by the general formula, \( A/K = N \), where \( N \) is any number, \( A \) is any object represented as a quantity, and \( K \) is any measure. Concrete numbers are initially obtained by measuring real objects, and by changing the measure one changes the number pertaining to the same object. From the very beginning, the child must keep in mind that measuring or counting may yield a remainder. Davydov thus introduces a general form for obtaining any integer or fraction. In this stage of instruction, first-graders transfer the basic properties of quantity (the disjunction of equality and inequality, transitivity, and reversibility) and of the operation of addition (commutativity, associativity, monotony, and the possibility of subtraction) to whole numbers.

Since the child has been introduced to certain structural features of equality, the relationships between addition and subtraction can be approached differently. For example, tasks are posed: "Given that \( 8+1=6+3 \) and \( 4>2 \), find the relationship between \( 8+1-4 \) and \( 6+3-2 \). If this expression is unequal, make it equal." (First the
symbol for "less than" needs to be put in, and then a "2" added to $8+1-^.) Thus, if
the number series is treated as quantity, the skills of addition and subtraction (and later
multiplication and division) can be developed in a new manner (Davydov, 1975).

2. PART–WHOLE RELATIONSHIPS.

In the second semester of first grade, letter symbols anticipate numeric
symbols in instruction on word problems in addition and subtraction. To eliminate the
difficulties encountered by children in solving indirect arithmetic problems, instruction
is based on children's mastery of part–whole relationships. It is argued that, since
simple general relations between parts and a whole are the basis for solving problems
in addition and subtraction (Minskaya, 1991), children should master a general method
of analysis that will allow them to solve all such problems, applicable to both indirect
and direct problems. As in other content areas, the general relationships of the subject
matter under study serve as the basis for general solution processes.

Instruction has two basic stages. First, children acquire a concept of the part–
whole relationship when it is demonstrated in general form, i.e., as a particular
relationship between quantities independent of their concrete numerical value. The
relationship is abstracted, made the subject of special analysis, and described with
graphic models and formulas. Second, previously developed concepts are made
concrete as children learn to isolate, and use this relationship in solving word
problems.

The part–whole relationship is first presented as a relation between real
quantities. Children paste several strips of paper together to make a new strip, then do
the reverse, cutting up one strip of paper into several new ones. Similarly, they
distinguish a part of a certain volume of water or of a weight of grain, and so forth. In
the exercises, they discover that any quantity can be characterized as a whole in one case and as a part in another, and, that a certain characterization can be given to a quantity only when it is in a certain relationship with other quantities (Mikulina, 1991).

Children abstract this relationship by constructing graphic models. Students graphically portray the part–whole relationship as indicated by the physical objects in "drawings" and "diagrams," where the relationship between parts and a whole is represented in the form of a relationship between geometric elements (segments, rectangles, etc., with preference being given to segments). Wholes and parts are indicated by letters in drawings and diagrams (see Figure 1).

![Figure 1. Drawing (left) and diagram (right) of a whole, k, composed of three parts, a, b, and c.](image)

Assignments require a translation from one representation to another (from a drawing to a diagram, from a diagram to a drawing, from a drawing to an object model). The concept of the part–whole relationship is made more precise in the context of work with models, and various properties of the relationship are studied.

Wholes and parts are then related in equations with plus, minus, and equal signs. Students learn to describe a part–whole relation with a system of letter formulas. The relationship is described by as many equalities as there are quantities, and all quantities are used in each equality. For example, given \(a\) as a whole and \(b, c\) as its parts, three formulas are written: \(a = b + c; b = a - c; c = a - b\). Children learn
to describe the part–whole relationship in the form of letter formulas by solving a series of instructional problems (e.g., passing from a formula to a drawing, making a change in a formula to comply with a change in a drawing).

"Texts" with literal data are then introduced. Texts have no unknowns and no questions (e.g., "There were $a$ red pencils and $b$ blue pencils in a box, and the total number of pencils in the box was $c$"). Texts direct attention to the analysis of internal relationships. Students isolate this relationship in the content of the text. Children describe the texts by all methods: a drawing, diagram, and formulas. Students then compose their own texts, retaining the relation that has been singled out.

Numerical values are then introduced by using concrete objects. A part–whole relationship is represented with objects. Students first represent the part–whole relation that is indicated by the objects with drawings and literal formulas, then obtain concrete numerical values for the data; they measure or count one quantity, then the others, and establish that they can obtain the numerical value of the final quantity by substituting numerical values in the formula in which the unknown is isolated, i.e., without actually measuring or counting. The preliminary expression of a part–whole relation in a drawing, and the gradual replacement of letter values with concrete numbers obtained by working with objects, enables students to discern part–whole relations in the relationships of concrete numbers (Mikulina, 1991). They are required to form the literal equations first. Literal formulas become plans for solutions, procedures for computing specific numerical outputs.

In the next stage, texts are again used. A text with letter data is represented in a drawing, and children choose a quantity to be calculated (i.e., the "unknown quantity"). The original letter designation of this quantity is crossed out on the drawing, and replaced by an $x$. Thus the graphic model of the text is transformed into
a problem. The teacher asks the children to formulate a question for the problem in the words of the text; students alter the text into a problem. A constant change of the unknown within a single problem compels students to formulate all possible problem types.

In other word problem activities, children suggest (reasonable) numerical values for quantities other than the unknown. Thus, letters are divided into knowns and unknowns, and the notion of generalized number is seemingly developed.

Mikulina (1991) contends students' mastery of the material depends on the objects, drawings, diagrams, and letter formulas that children use to model and describe the relationship between the whole and its parts in general form. The model-making serves as a means of isolating relationships in an analysis of the conditions of concrete problems; letter symbols, related to the model of an indicated relationship, are used to establish and describe the relationship.

3. SUMMARY AND POSSIBLE IMPLICATIONS.

In Davydov's curriculum, children operate on real objects and visual models that embody the quantitative relationships that underlie structural conceptions. The models allow children to represent implicit and explicit relationships visually. Visual images, being compact and integrative, seem to support structural conceptions (Sfard, 1991). For example, using a graphic model of a part–whole relationship, the child can simultaneously view all component quantities and the relationship between them as he/she formulates an algebraic equation, can treat the relationship almost as if it were a material entity.

Properties of abstract relations and properties of arithmetic operations are studied using real objects, models, symbols, word problems with literal data, etc. The
child's understanding of quantitative relations (e.g., understanding of equivalence), and of the properties and results of an operation, guides her operations on letters/expressions/equations as objects. That is, the child determines permissible operations on algebraic constructs based on her understanding of abstract relationships and properties, and does not interpret literal representation as a generalization of her experience with numbers.

With the introduction of numbers in a given content area, a prepared algebraic formula becomes a plan for a solution (Mikulina, 1991), a procedure for computing specific numerical outputs. Numerals are substituted into general literal formulas—thus numerical instances appear as particular instantiations of a general relationship between quantities. Procedural conceptions of algebraic entities are ostensibly derived from structural conceptions as a "special case"; i.e., in the case of numerical substitution in literal formulas describing general relationships between quantities, a particular instance is generated and calculation is often involved. The level of generality implied by a literal formula, as opposed to a particular instance generated by numerical substitution, is ostensibly established—establishing the relation between the two interpretations, and developing children's ability to utilize algebraic symbolism as a tool for generalization and deduction. Similarly, relationship signs (=, >) and operation signs (+, −) signify a relationship between quantities or a change in a quantity, respectively; in the case of numerical substitution, a particular instance of a general relationship is revealed and a change in a quantity can be expressed as a calculated result. The gap between arithmetic and algebraic interpretations of operation and relationship signs is narrowed (see, e.g., Behr, Erlwanger, & Nichols, 1976; Kieran, 1979; Lee & Wheeler, 1989).
It is unclear whether Davydov's curriculum "generalizes" to the more complex operations and uses of algebra involved in/conferred by structural perspectives of algebraic entities.

There is some direct and related empirical support for both an inductive procedural-to-structural curricular model (see, e.g., Booth, 1988; Demana & Leitzel, 1988; Dreyfus & Vinner, 1982; Kieran, 1992; Sfard, 1987; Sfard, 1988; Sfard, 1989; Sfard & Linchevski, 1994; Soloway, Lochhead, & Clement, 1982; Vinner & Dreyfus, 1989) and a deductive structural-to-procedural model (see, e.g., Bodanskii, 1991; Booth, 1981; Byers & Herscovics, 1977; Cauzinille–Marmeche, Mathieu, & Resnick, 1984; Chaiklin & Lesgold, 1984; Davydov, 1975; Fischer, 1990; Greeno, 1982; Karpov & Bransford, 1995; Lindvall & Ibarra, 1978; Mevarech & Yitschak, 1983; Mikulina, 1991; Minskaya, 1991; Resnick & Greeno, 1990; Riley & Greeno, 1978; Schmittau, 1993a, 1993b; Stern, 1994; Talyzina, 1981; Wolters, 1978). However, there have been relatively few cross-curricular comparisons. When cross-curricular comparisons have been conducted, the experimental curricular models have been compared with traditional curricula, rather than with other theoretically driven curricula.

*Development of algebraic letter concepts: National Mathematics Project and Davydov*

Use of algebra as a tool for reasoning requires concepts of various letter interpretations (specific unknowns, generalized numbers, variables, givens, etc.). Davydov suggests formal abstract symbols are essentially related to mathematical reasoning, to abstraction and generalization and deduction; algebraic symbols are introduced in primary school. Davydov asserts "the abstract as an element of thought
should be introduced into instruction as early as it can be accessible to the child, who must not be held too long at the stage of sensory impressions, in any case" (1990, p. 319). In order to allow children to examine relationships among objects, to discern empirical cases as concrete manifestations of generalized relationships, Davydov begins with more general interpretations of letters (general quantity), and moves to less abstract interpretations (specific unknowns).

National Mathematics Project incorporates Harper and Küchemann's Piagetian–based curricular model for developing letter concepts, emphasizing the gradual development of abstraction, with a move from less abstract interpretations (e.g., specific unknowns) to more general interpretations in Years 7 and 8. The model suggests there are age–related, hierarchical levels in students' understanding of various letter interpretations. Introduction of letters as unknowns in Year 7, and as generalized numbers and variables in Year 8 is

. . . advised by the finding of the CSMS enquiry into school algebra . . ., and in particular by the conclusion that the majority of 13 to 15–year-olds are not able to cope consistently with algebraic items that require an interpretation of the letter as anything other than an unknown number. (Harper, Küchemann, et al., 1987, Teachers' Handbook, Book 2, p. 137)

The curricular model is supported by empirical studies that suggest (1) certain letter interpretations are inaccessible until children have attained formal operational thought; and 2) letters–as–unknowns are more cognitively accessible than letters as generalized numbers, variables, and givens (e.g., Booth, 1984; Collis, 1969, 1971, 1975; Harper, 1987; Küchemann, 1978, 1981). Analyzing the CSMS data, Küchemann (1981) concluded:
On the [CSMS] algebra test the majority of 13, 14, and 15 year olds were at Levels 1 and 2 (73, 59, and 53 per cent respectively) and were not able to cope consistently with items that can properly be called algebra at all, i.e. items where the use of letters as unknown numbers cannot be avoided. In Piagetian terms these children would seem to be at the stage of concrete operations, which means that for most children the teaching should be firmly rooted in this level whether the aim is to consolidate their understanding or to ease the transition to formal operational thought. (Küchemann, 1981, p. 118)

Booth suggests understanding of literal notation may proceed in stages. In a teaching experiment specifically designed to develop the notion of generalized number, Booth (1982, 1983) encountered strong resistance from students:

[Some of the difficulty which children have appears to be related more to a 'cognitive readiness' factor. This is particularly so with regard to the apprehension of letters as representing generalized number rather than specific unknowns . . . . One . . . [explanation has] been suggested by Collis (1969, 1971, 1975). Discussing the distinction drawn by Piaget between concrete and formal operational thinking in terms of the child's degree of reliance on 'reality' (see Collis, 1973), Collis suggested that this distinction would be reflected in the child's perception of the nature of algebraic elements. While the concrete operational thinker could deal with the notion of letters as representing particular unique, if currently unknown, values, therefore, it would not be until the child had attained formal operational thinking that the generalized nature of the values represented by letters would be appreciated. The observation that children typically experience considerable difficulty in coming to terms with the generalized nature of algebraic representation, and that there appears to be a maturation-linked factor in children's ability to assimilate this notion (as indicated by the teaching trials in the present study, in which noticeable gains in this direction were made only by the second-year [age 13] top ability groups and the fourth-year [age 15] middle ability groups—the two third-year classes [age 14] involved being of lower mathematical ability), is at least not inconsistent with the suggestion that the attainment of this level of conceptualization [i.e., understanding of generalized number] is related to the development of 'higher-order' cognitive structures. (pp. 87-88)

Harper (1987) also proposes a stage model in the acquisition of various letter interpretations. Harper interviewed 144 secondary school pupils from Years 7 to 12, using an open-ended generalization problem ("If you are given the sum and the
difference of any two numbers, show that you can always find out what the numbers are"). In Years 7, 8, and 9, rhetorical solutions predominated. From Year 10 onward, the balance shifted in favor of, first, Diophantine solutions (requiring use of letters as unknowns) and then Vietan solutions (requiring use of letters as both unknowns and givens). Although students were expected to use letters as unknowns in problem solving from Year 7, and were using letters as givens in the context of functions, and as a tool to make generalizations from Year 8, only 28 of the 144 subjects used letters as givens (i.e., gave a Vietan response). Twenty of the 28 were in Year 12. Harper compares the findings to the 8% success rate among Year 10 students in a CSMS task involving letters as variables: Which is larger: $2n$ or $n+2$? (Küchermann, 1978). Harper suggests letters as "Diophantine unknowns" are more cognitively accessible than letters as "givens," with the latter interpretation acquired by only a minority of more able pupils.

Age-related models are countered by Mikulina (1991), Minskaya (1991), Bodanskii (1991), and Sutherland's (1993) descriptions of 7 to 11 year-old schoolchildren's use of letters as general quantities, general numbers, knowns, unknowns, and variables.

**Examples of the development of letter concepts**

The development of letter concepts in Davydov's curriculum was previously discussed. Letter concepts are derived from concepts of quantity using actions on concrete objects (real quantities), graphic models, and literal symbolism. Letter interpretations are gradually detached from their original object sources. To develop concepts of general quantitative relationships and general solution processes, various interpretations of letters are introduced and developed in Grade 1: letters as specific
quantities, unknown quantities, given known quantities, specific unknowns, and generalized numbers.

As instruction progresses in Davydov's curriculum, a letter that initially designated a specific physical quantity (e.g., 'a') presumably becomes any given known quantity as relationships and properties of relationships are analyzed with letters and models in the absence of real objects; 'x' designates the size of an unknown change in a quantity, a function of given known quantities (e.g., \( x = a - b \)). With the introduction of numbers, 'x' becomes a specific unknown and the notion that 'a' can take on any numerical value that makes sense in a given context is developed (i.e., 'a' becomes a generalized number). Children presumably understand that a can assume a set of values as they are allowed to think up appropriate numerical values for substitution in letter formulas.

National Mathematics Project incorporates Harper and Küchemann's Piagetian-based curricular model for developing letter concepts, emphasizing the gradual development of abstraction, with a move from less abstract interpretations (e.g., specific unknowns) to more general interpretations in Years 7 and 8. The following examples are from National Mathematics Project Red Track texts (Harper, Küchemann, et al., 1987).

In Year 7, students are introduced to use of letters as temporary 'stand-ins' for numbers. Students construct number sentences using letters as specific unknowns, and invent stories for such sentences. For example, problems of the following kind are included:

Horace walks to school and back home every day. "I walk \( p \) kilometers every week," he says. Each day, Horace walks 7 km. What number does \( p \) stand for? (Book 1, p. 306)
This stamp is drawn full size. Its perimeter is $t$ cm. What number does $t$ stand for? (Book 1, p. 306)

Write a story for this number sentence: $14+k = 27$ (Book 1, p. 310)

A tree is 21 metres tall. Five years ago it was $t$ metres tall. During the past five years it has grown $t$ cm each year. What number does $t$ replace? How tall was the tree five years ago? How many centimetres has the tree grown in the past five years? (Book 1, p. 313)

In Year 8, students are introduced to generalized numbers and variables in algebraic expressions through practical drawing activities. The notion that letters are not merely 'replacements for numbers' is presented. Generalized numbers are presented as a means for writing "general instructions" and expressing general rules.

Geometrical drawings (e.g., "ripple diagrams") are used (1) to provide a representation of a variable; (2) as an illustration of an algebraic expression; (3) to develop the notion that a geometrical drawing can be interpreted in both a 'general' and 'specific' sense; and (4) to develop the notion that a letter is often used to help make a general statement. For example, a circle with radius marked '$k$ cm' can either be intended as a specific circle so that $k$ has an 'unknown value' that can be determined; or as a general figure, in which case $k$ is a shorthand way of writing 'any radius'. Activities reveal how figures change shape as different numbers are chosen for the letters in their instructions.

In Year 9, there is continued development of the concepts of specific unknown and generalized number. Variables are used to model geometrical and physical/dynamically changing situations.
CHAPTER III
METHODOLOGY AND INSTRUMENTATION

Design

Three independent variables and three clusters of dependent variables were included in the design. Theoretical foundations for the study led to the selection of three potential sources of variation in students' acquisition of algebraic representation and reasoning; the independent variables were culture (Russia and England), curriculum (experimental and non-experimental), and age (10 to 16 years). Three groups of dependent variables were measured: algebraic and logical deductive reasoning, use/interpretation of algebraic structure, and use/interpretation of algebraic letters. The dependent variables were measured through written open-ended problems and follow-up interviews.

Subjects

For purposes of comparison, four groups were included in the sample: (1) students and graduates of Davydov’s experimental elementary school mathematics curriculum (implemented in Moscow School #91 in Moscow, Russia); (2) students in a non-experimental school in Moscow, Russia; (3) students in an upper school in England that had used the National Mathematics Project curriculum for seven years; and (4) students in an upper school in England with a "non-experimental" curriculum. The sample is described in Table 4.
### Table 4. Sample

<table>
<thead>
<tr>
<th>COUNTRY</th>
<th>CURRICULUM</th>
<th>YEAR OR GRADE</th>
<th>MEAN AGE; STANDARD DEVIATION</th>
<th>NUMBER OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENGLAND</td>
<td>NMP</td>
<td>YEAR 9</td>
<td>13.8; 0.29</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>YEAR 10</td>
<td>14.8; 0.27</td>
<td>40</td>
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<tr>
<td></td>
<td></td>
<td>YEAR 11</td>
<td>15.8; 0.28</td>
<td>40</td>
</tr>
<tr>
<td>ENGLAND</td>
<td>NON-EXPERIMENTAL</td>
<td>YEAR 9</td>
<td>13.7; 0.23</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>YEAR 10</td>
<td>14.8; 0.28</td>
<td>40</td>
</tr>
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<td></td>
<td></td>
<td>YEAR 11</td>
<td>15.7; 0.25</td>
<td>40</td>
</tr>
<tr>
<td>RUSSIA</td>
<td>DAVYDOV</td>
<td>GRADE 5</td>
<td>11.4; 0.45</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 6</td>
<td>12.4; 0.41</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 8</td>
<td>14.7; 0.35</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 9</td>
<td>15.7; 0.37</td>
<td>30</td>
</tr>
<tr>
<td>RUSSIA</td>
<td>NON-EXPERIMENTAL</td>
<td>GRADE 6</td>
<td>11.8; 0.41</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 7</td>
<td>12.4; 0.47</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 9</td>
<td>14.7; 0.28</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GRADE 10</td>
<td>15.4; 0.32</td>
<td>13</td>
</tr>
</tbody>
</table>

Selections of grade levels for the four groups were based on the following considerations: (1) to obtain as much overlap as possible with respect to age; and (2) to select grade levels that were appropriate for the given curriculum in order to assess the effects of the curriculum on concept development and reasoning.

The data were analyzed with respect to curricular groups (Russian experimental, Russian non-experimental, English experimental, English non-experimental) and age levels. Subjects in each of the four curricular groups were divided into two age groups: 10 to 14 years, and 14 to 16 years. Descriptive statistics for each of the age groups from the four curricula are given in Table 5. Table 6 gives the numbers of males and females in the total sample.

Sample selection in Russia was arranged through the International Laboratory for Comparative Social Research, established by Ohio State University and the Russian Academy of Education. In Moscow School #91, two intact classes in each grade level were tested. For each grade level, students from each of the two classrooms were then...
randomly selected for inclusion in the study. Students in the sample had experienced Davydov's curriculum for at least three years. For the Russian non-experimental group, one intact classroom per grade level was included in the study.

Both schools were in Moscow. Student populations were comparable with respect to SES.

Table 5. Sample described in terms of two age groups, 10–14 years and 14–16 years

<table>
<thead>
<tr>
<th>CURRICULUM</th>
<th>10 TO 14 YEAR–OLD AGE GROUP</th>
<th>14 TO 16 YEAR–OLD AGE GROUP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NUMBER IN AGE GROUP</td>
<td>MEAN AGE, STANDARD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>DEVIATION</td>
</tr>
<tr>
<td>NMP</td>
<td>32</td>
<td>13.7; 0.26</td>
</tr>
<tr>
<td>English non–experimental</td>
<td>35</td>
<td>13.6; 0.18</td>
</tr>
<tr>
<td>Davydov</td>
<td>62</td>
<td>12.0; 0.70</td>
</tr>
<tr>
<td>Russian non–experimental</td>
<td>55</td>
<td>12.1; 0.55</td>
</tr>
</tbody>
</table>

Table 6. Numbers of boys and girls in the sample

<table>
<thead>
<tr>
<th>CURRICULUM</th>
<th>MALES</th>
<th>FEMALES</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMP</td>
<td>56</td>
<td>64</td>
</tr>
<tr>
<td>English non–experimental</td>
<td>56</td>
<td>64</td>
</tr>
<tr>
<td>Davydov</td>
<td>68</td>
<td>53</td>
</tr>
<tr>
<td>Russian non–experimental</td>
<td>40</td>
<td>49</td>
</tr>
</tbody>
</table>

NON–EXPERIMENTAL CURRICULUM IN RUSSIA. Study of arithmetic is concentrated in the elementary mathematics course. Concentrated study of algebra begins in seventh grade. The breakdown by topics is as follows: (a) Primary school: Study of the four arithmetic operations; dimension and measurement; geometric figures; simple
equations. (b) Mathematics, grades 5 and 6: Positive and negative numbers; the concept of number as a result of measurement; rational numbers; decimals; proportions; using letters to record the characteristics of arithmetical operations; formation of algebraic expressions; rudimentary algebraic transformations (expansion of parentheses, reduction of like terms, cancellation of factors); formation of linear equations from information given in textual problems; solution of linear equations with one unknown; geometrical concepts. (c) Algebra, grades 7 through 9: Irrational numbers; real numbers; transformations of integral and rational expressions (cancellation of factors and reduction of like terms, factorization of polynomials, manipulation of algebraic fractions, etc.); solution of linear and quadratic equations, and systems of equations; solving textual problems by forming equations and systems of equations; inequalities; elementary functions. (d) Geometry, grades 7 through 9: Deductive proof; coordinates; vectors; trigonometry. (e) Algebra and Beginning Analysis, grades 10 and 11: Functions; solution of trigonometric, exponential, and logarithmic equations and inequalities; limits; derivatives; integrals; differential equations. (f) Geometry, grades 10 and 11: Three-dimensional geometrical solids; proof.

The fifth and sixth grade course "is structured upon induction, but does have recourse to elements of deductive reasoning. The theoretical material . . . is presented in graphically intuitive terms; mathematical methods and laws are formulated as rules" (USSR Academy of Pedagogical Sciences' Scientific Research Institute of Curriculum and Teaching Methods, 1987, p. 65). The algebra course for grades 7–9 is "characterized by the enhancement of the theoretical level of instruction and by the stronger emphasis that is gradually placed upon the role of theoretical generalizations and deductions" (USSR Academy of Pedagogical Sciences' Scientific Research
Institute of Curriculum and Teaching Methods, 1987, p. 67). In the geometry course for grades 7–9, both the role of deduction and the degree of abstraction in the material under study are augmented. Students "master the techniques of analytic and synthetic (deductive) activity by proving theorems and solving problems" (USSR Academy of Pedagogical Sciences' Scientific Research Institute of Curriculum and Teaching Methods, 1987, p. 68).

Sample selection in England was carried out by the author. For the English experimental group, students were selected from two tracks. "Red Track" National Mathematics Project materials are designed for higher achieving students, and "Blue Track" materials for lower achieving students. Twenty Year 9 students, twenty Year 10 students, and twenty Year 11 students were selected from each of the two tracks (i.e., a total of 40 students in each year group). Year groups were receiving instruction from Books 3, 4, and 5 of the National Mathematics Project curriculum, respectively. Thirty–seven of the 40 Year 9 students in the sample had received instruction in Books 1, 2, and 3 of the NMP curriculum (i.e., they received instruction in Books 1 and 2 of the NMP curriculum before entering the upper school); three had received instruction in Books 2 and 3. Thirty–two of the 40 Year 10 students in the sample had received instruction in Books 1, 2, 3, and 4 of the NMP curriculum; eight had received instruction in Books 2, 3, and 4. Twenty–four of the 40 Year 11 students had received instruction in Books 1, 2, 3, 4, and 5 of the NMP curriculum; 16 had received instruction in Books 2, 3, 4, and 5.

Teachers had selected students for various tracks in the upper school on the following bases. Year 9 included 320 students, Year 10 included 350 students, and Year 11 included 315 students. Teachers divided each year group (Year 9, Year 10,
Year 11) into two half year groups. Each half year group was divided into seven sets based on ability level. Each set constituted a single mathematics class. The seven sets within each half year group were determined as follows:

1) Year 9: Based on internal tests and middle school records, the students in each half year group in Year 9 were divided into seven sets. Set 1 and Set 2 received instruction from the National Mathematics Project Red Track materials. Sets 3, 4, 5, and 6 received instruction from the National Mathematics Project Blue Track materials. Set 7 was remedial. Sets were arranged hierarchically according to achievement level. Set 1 included the highest achieving Red Track students. Set 2 included the lower achieving Red Track students. Set 3 included the highest achieving Blue Track students, Set 4 included the next highest achieving group in the Blue Track, and so forth.

2) Year 10: Each year approximately 50 Year 10 students were moved from their Year 9 placement to a higher or lower set based on their Year 9 performance; approximately 25 students were moved to a higher set, and 25 to a lower set. Students were moved when they consistently performed at the top or the bottom of their class and were shifted into the adjacent hierarchical set.

3) Year 11: During Year 10 and Year 11, there was very little movement from one set to another. The sets were very stable in terms of student make-up throughout Year 10 and Year 11.

Sampling in Years 9, 10, and 11 was done as follows. Students were selected from a single half year group. For each Year, Set 1 and Set 3, i.e., top achieving Red Track and top achieving Blue Track students, were tested. Each set consisted of an intact classroom of 30 students. Twenty students from each set were then randomly selected for inclusion in the study.
For the English "non-experimental curriculum," the sample included 40 Year 9, 40 Year 10, and 40 Year 11 students selected from full grade groups. The upper school selected students for various tracks as follows. Year 9 included 285 students, Year 10 290 students, and Year 11 240 students. Teachers divided each year group (Year 9, Year 10, Year 11) into two half year groups. For Years 9 and 10, each half year group was divided into six sets based on achievement level. For Year 11, each half year group was divided into five sets based on achievement level. Each set constituted a single mathematics class. The sets within each half year group were determined as follows:

1) Year 9: Based on internal tests and middle school records, the students in each half year group in Year 9 were divided into six sets. Set 1 included the highest achieving students, Set 2 the next highest, and so forth. There was some movement at October half term (approximately 15 of the 285 students were moved during the year of the study), and occasional changes were expected to occur later in the year.

2) Year 10: Each year, approximately 50 Year 10 students were moved from their Year 9 placement to a higher or lower set based on their Year 9 performance; approximately 25 students were moved to a higher set, and 25 to a lower set. Students were moved when they consistently performed at the top or the bottom of their class and were shifted into the adjacent hierarchical set.

3) Year 11: During Year 10 and Year 11, there was virtually no movement from one set to another. The sets were very stable in terms of student make-up throughout Year 10 and Year 11.

Sampling in Years 9, 10, and 11 was done as follows. For each Year, Set 1 and Set 3 from one half year group were tested. Each set consisted of an intact
classroom of 23 to 27 students. Twenty students from each set were then randomly selected for inclusion in the study.

The following discussion gives some indication of the comparability of the English experimental and non-experimental samples:

(1) ABILITY LEVEL.

(i) The selected sets were similar with respect to ability level; i.e., Set 1 students in the non-experimental group were roughly equivalent in ability to Set 1 Red Track students in the experimental group, and Set 3 students in the non-experimental group were roughly equivalent in ability to Set 3 Blue Track students in the experimental group.

(ii) England's National Curriculum includes five attainment targets: Using and Applying, Number, Algebra, Shape and Space, and Handling Data. For all attainment targets, students progress through levels (Levels 1 through 10) during Year 1 through Year 11, with the maximum level usually attained being Level 7 or 8. In all schools, regardless of the curriculum, students work progressively through levels; topics from the five attainment targets are intermingled during each school year. For the two British samples, students in the selected sets were working at the following levels in algebra (see Table 7).

Table 7. Levels attained by the two British samples with respect to the National Curriculum in algebra

<table>
<thead>
<tr>
<th>YEAR</th>
<th>SET 1 Experimental</th>
<th>SET 1 Non-experimental</th>
<th>SET 2 Experimental</th>
<th>SET 2 Non-experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>6/7</td>
<td>6</td>
<td>5/6</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>7/8</td>
<td>6/7</td>
<td>6/7</td>
<td>6/7</td>
</tr>
<tr>
<td>11</td>
<td>8/9</td>
<td>8/9</td>
<td>7/8</td>
<td>7</td>
</tr>
</tbody>
</table>
The schools were approximately 8 miles apart, located in the same county. Both schools were in middle and working class communities. Approximately 25–30% of the pupils' families were in Social Classes I (professional) or II (managerial and technical).

**NON-EXPERIMENTAL CURRICULUM IN ENGLAND.** In Year 9, students used booklets developed by the school's mathematics department. Booklets covered the five attainment targets in England's National Curriculum (Using and Applying, Number, Algebra, Shape and Space, and Handling Data). Each booklet required approximately three to six weeks to complete. In Years 10 and 11, the following textbooks were used: *Everyday Maths Practice, Maths in Focus, GCSE Maths* (Rayner), *Higher GCSE Maths* (Rayner), *Maths for GCSE* (Bindley), and *Task Maths*.

**Measurement and Instruments**

I. Explanatory variables

Three explanatory variables were included in the design: culture (Russia and England), curriculum (experimental and non-experimental), and age (10–16 years). Levels of curriculum were nested within each culture, and ages were nested within curricular levels.

II. Outcome variables

Three clusters of outcome variables were included in the design: use of algebraic and logical deductive reasoning, use/interpretation of algebraic structure, and use/interpretation of algebraic letters. The dependent variables were measured with the following instruments.
A. **WRITTEN TEST.**

A 24-item written test was designed that was intended to measure students' acquisition of algebraic concepts and reasoning (Appendix A). Two criteria were used to select and construct items: (1) items measured the dependent variables; and (2) items from related studies were selected if possible to allow comparison with previous findings, with priority given to tasks from studies supporting Küchemann (1981), Harper (1987), Kieran (1992), and Sfard's (1991) notions of age-related capabilities for learning and/or a procedural-to-structural progression in concept acquisition. Seventeen tasks were selected from previous research studies that examined students' use of algebra as a tool for deductive reasoning; students' notions as to the logical necessity of deductive conclusions; students' logical deductive reasoning; and students' notions of mathematical structure and letter concepts. The remaining items were tailored by the investigator. Items were selected from the *Concepts in Secondary Mathematics and Science* project's written algebra test (Küchemann, 1978, 1981); Lee and Wheeler (1987, 1989); Chevallard and Conne (1984); Fischbein and Kedem (1982); Vinner (1983); Harper (1987); Clement (1982); Cauzinille-Marmache, Mathieu, and Resnick (1984); Chaiklin and Lesgold (1984); and the National Longitudinal Study of Mathematical Abilities (NLSMA) test battery (Wilson, Cahen, & Begle, 1966). The test allows researchers (a) to compare the acquisition of algebraic symbolism and reasoning by the studied groups, and (b) to compare the current findings with previous studies.

The test was designed to measure three clusters of dependent variables: use of algebraic and logical deductive reasoning, use/interpretation of algebraic structure, and use/interpretation of algebraic letters.
For the first cluster of variables, *algebraic and logical deductive reasoning*, test items measured students’ ability to formulate deductive algebraic arguments; students’ notions as to the logical necessity of deductive conclusions derived from algebraic arguments; and logical reasoning. Test items designed to examine students’ reasoning included the following:

(a) A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right.

(Lee & Wheeler, 1989)

(b) Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . Now do it with another three consecutive numbers . . . . What happens? Can you prove it will always work?

(Adapted from Chevallard & Conne, 1984)

(c) In an algebra class the teacher proved that every whole number of the form $n^3 - n$ is divisible by 6 (that is, if you divide $n^3 - n$ by 6, there will be no remainder). The proof was as follows:

We can write:

$$n^3 - n = n(n^2 - 1)$$

But we can rewrite the expression on the right:

$$n(n^2 - 1) = n(n - 1)(n + 1)$$

So:

$$n^3 - n = n(n - 1)(n + 1) = (n - 1)(n)(n + 1)$$

But $(n - 1)(n)(n + 1)$ is a product of three consecutive whole numbers. Therefore, one of them should be divisible by 2, and one of them (not necessarily a different one) should be divisible by 3. Thus their product should be divisible by $2 \times 3$, that is, by 6.

Please answer the following:

(A) I understand all the details of the proof and the proof seems correct to me.
Yes / No (Circle one)

(B) There are some details in the proof that I do not understand. They are the following:

(C) If you think the teacher has given a correct proof for the theorem "every whole number of the form \(n^3 - n\) is divisible by 6", then answer the following question:

Do you think that further checks (by substituting numbers) are necessary in order to verify the validity of the theorem?

Yes / No (Circle one)

Explain.

(D) Victor is a doubter. He thinks that we have to check at least a hundred numbers in order to be sure that the theorem is correct. What is your opinion? Explain your answer.

(Adapted from Fischbein & Kedem, 1982; and Vinner, 1983)

(d) For the following problem:
Assume the first two sentences (in bold) are true. Make a conclusion from the assumptions. (Choose a, b, c, or d.)

All fahmooth numbers can be divided evenly by 8.
26 is a fahmooth number.

Therefore . . .
a) 26 must not be a fahmooth number.
b) 26 is an exception to the rule.
c) It is probably true that fahmooth numbers cannot be divided evenly by 8.
d) 26 can be divided evenly by 8.

(e) In this question you just have to tell whether or not the sentences show correct reasoning. All of the sentences are really nonsense, but you are to think only about the reasoning. Circling Yes means the reasoning is good; circling No means the reasoning is not good. Please choose Yes or No for each sentence:
(1) If all birds have purple tails and all cats are birds, then all cats have purple tails.
(2) If all cars have sails and some swimming pools are cars, then some swimming pools have sails.
(3) If no skunks have green toes and all skunks are pigs, then no pig has green toes.
(4) If all horses have wings and no turtle has wings, then no turtle is a horse.
(5) If some men are purple and everything which is purple is a horse, then some horses are men.
For the second cluster of variables, *use interpretation of algebraic structure*, test items measure (1) students' ability to formulate algebraic equations that represent quantitative relationships; (2) students' use of arithmetic structure; (3) students' ability to manipulate and simplify algebraic symbolism; and (4) students' ability to interpret algebraic symbolism procedurally and structurally. Test items designed to examine students' use/interpretation of algebraic structure included the following:

(a) Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If $b$ is the number of blue pencils bought and if $r$ is the number of red pencils bought, write an equation involving $b$ and $r$.

(Küchemann, 1981)

(b) Write an equation using the letters $S$ and $T$ to represent the following statement: "There are six times as many students as teachers at this school." Use $S$ for the number of students and $T$ for the number of teachers.

(Adapted from Clement, Lochhead, and Monk, 1981)

(c) Which of the following expressions are equivalent to $5 \times (6 + 3)$? (Circle them.)

- A) $5 \times 6 + 5 \times 3$
- B) $5 \times (3 + 6)$
- C) $5 + 6 \times 3$
- D) $5 \times 3 + 5 \times 6$
- E) $(5 \times 6) + 3$

Explain what you did to figure this out.

(d) What can $x$ equal in the following equation?

$$\frac{12x - 2x + 16}{2} = 2(x + 4) + 3x$$

Explain.
(e) Which of the following expressions are always equivalent? Sometimes equivalent?

A) \( b - a + c \)
B) \( c + a - b \)
C) \( c - b + a \)
D) \( c - a + b \)

Explain.

For the third cluster of variables, *use/interpretation of algebraic letters*, test items were designed to measure students' ability to (1) use/interpret letters as specific unknowns, (2) use/interpret letters as generalized numbers, (3) use/interpret letters as variables, and (4) use/interpret letters as givens. Test items designed to examine students' ability to use/interpret algebraic letters included the following:

(a) Decide whether the following statement is always true, sometimes true, or never true. Put a circle around the right answer. If you put a circle around 'sometimes true' explain when this statement is true.

\[ m + n + q = m + p + q \]

(Küchemann, 1981)

(b) What can you say about \( c \) if \( c + d = 10 \) and \( c \) is less than \( d \)?

(Küchemann, 1981)

(c) Which is larger, \( 2n \) or \( n+2? \) Explain.

(Küchemann, 1981)

(d) If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

(Harper, 1987)
B. INTERVIEW GUIDE.

The test was followed by interviews that clarified students' written responses (letter interpretation, order and steps of their arguments, bases for their conclusions, etc.). Since interviewers clarified students' reasoning, a standardized set of questions was inappropriate. A general interview guide was developed; the guide served as a basic checklist during the interviews (Appendix C). After clarifying students' written responses, interviewers returned to four proof tasks in the test. If the student had not used algebra in those items, he/she was asked to do so by the interviewer. Students' tendency to use algebra as a tool for reasoning could therefore be examined, as well as their ability to do so.

The written test and interview guide were originally prepared in English. Two translators were involved in the preparation of the Russian versions of the test and interview guide (Appendix B and Appendix D). A native speaker of Russian prepared the original translation, and a native English speaker then back-translated the Russian version into English (see Werner & Campbell, 1973). A three-way conference between the two translators and the investigator was conducted for each item; the adequacy of the first English back-translation was assessed, errors and inadequacies in the original Russian version were corrected, and the revised Russian version was again back-translated. Final corrections were made in the Russian version. Pilot studies led to further revisions.

Procedures

Interviewing in Russia was arranged through the International Laboratory for Comparative Social Research. Interviews were conducted by three graduate students
from Moscow State University; Dr. Vladimir Sloutsky, professor in the College of Education at Ohio State University, supervised the data collection in Russia. Interviews in England were conducted by the author.

In each school, students had 90 minutes to complete the test, and the test was given in group settings. Face-to-face individual interviews were conducted in a separate room. Interviews were audio-taped and transcribed.

**Data analysis**

The author and a research assistant encoded the data. Data were encoded in accordance with a coding catalogue that was developed by grouping individual responses on each question into more general categories (Appendix E). Inter-rater reliability (Cohen's kappa) was established. Cohen's kappas ranged from 0.96 to 1. Log-linear analysis was used to analyze the data.

The data were analyzed with respect to Group and Age. Four groups were used in the log-linear analyses: Russian non-experimental curriculum ("R-NEX"), Russian experimental curriculum ("DV"), English non-experimental curriculum ("E-NEX"), and English experimental curriculum ("NMP"). Subjects within each of the four curricular groups were divided into two age groups: 10 to 14 years, and 14 to 16 years.

Log-linear models were selected as follows. First, all legitimate log-linear models were fit. Then these models were screened. Models that did not fit the data well (yielded significant differences between fitted and observed cell frequencies) were excluded from further analyses. For the remaining models, the following criteria were applied: maximization of both parsimony and component chi squares. Thus, in order to be selected, the model had to (1) fit the data well, (2) have fewer terms than other
well fitting models, and (3) have component chi squares larger than other well fitting models. When the data did not allow the selection of a parsimonious model, the fully saturated model was selected.

Since four curricular groups were used in the analyses, cultural and curricular effects could not be distinguished. The following approach was used. If significant differences existed in contrasting the two Russian groups with the two English groups on some measure, it was interpreted as a cultural effect. If a group significantly differed from the other three groups on some measure, with no accompanying cross-cultural difference (i.e., no significant difference between Russian and English groups on that measure), then it was interpreted as a curricular effect. If a group significantly differed from the other three groups on some measure, with an accompanying difference between English and Russian groups, findings were interpreted as being due to a combination of curricular and cultural effects.
CHAPTER IV
RESULTS AND DISCUSSION

Descriptions and analyses of children's solutions to selected tasks are presented in this chapter. ¹

Algebraic reasoning

Algebraic deductive reasoning

Analyses revealed profound cross-cultural and cross-curricular differences in students' reasoning on proof tasks. Russian children were more likely to acquire and use algebraic, deductive reasoning than English children; and the Russian experimental group was more likely to acquire and use algebraic deductive reasoning than other groups. Numerical, inductive reasoning was predominant among the English samples.

Analyses for selected proof tasks follow. Problem 17 examined students' tendency, and ability to formulate an algebraic proof:

Problem 17: Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . Now do it

¹ In the figures and tables, the following abbreviations are used: DV represents Davydov's curriculum (the Russian experimental curriculum); R–NEX represents the Russian non-experimental curriculum; NMP represents the National Mathematics Project curriculum (the English experimental curriculum); and E–NEX represents the English non-experimental curriculum. Numbers following the curricular abbreviations indicate the grade or year level. For example, DV 6 represents the sixth grade sample from Davydov's curriculum; R–NEX 9/10 represents the aggregated ninth and tenth grade samples from the Russian non-experimental curriculum; NMP 11 represents the Year 11 sample from the National Mathematics Project curriculum; and so forth.
with another three consecutive numbers . . . . What happens? Can you prove it will always work? (Adapted from Chevallard & Conne, 1984)

Figures 2, 3, and 4 compare the percentages of students formulating a correct algebraic proof in response to this task, versus an "empirical proof." Figure 2 compares the percentages for the various curricular groups and year/grade levels.

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**Figure 2.** Percentages independently formulating a correct algebraic proof in response to problem 17; versus percentages concluding a generalization held for an infinite set on the basis of numerical examples only.

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2 In Figure 2, groups are arranged along the horizontal axis according to this criterion: from groups showing the highest percentages of use of empirical proof, to those using empirical proof least often. All figures that show percentages for the various curricular groups and year/grade levels will have one variable ordered monotonically. The order of the groups along the horizontal axis will therefore vary.
In an "empirical proof," children formulated inductive, probabilistic numerical arguments—concluding a generalization held for an infinite set after verifying that a generalization held for particular numerical cases.

Figures 3 and 4 show students' reasoning on problem 17 for the various curricular groups and age groups.

Figure 3. Percentages independently formulating a correct algebraic proof in response to problem 17

For use of algebraic proof (Figure 3), parsimonious log-linear models did not fit the data well, requiring introduction of the interaction term. The fully saturated model fit the data. Effects of Group, Age and interaction between Group and Age were significant (see Table 8). The effect size of Group was moderate (0.48); the effect size of Age was large (0.66); and the effect size of the interaction between
Group and Age (0.13) was small. Russian groups formulated algebraic proofs more often than English groups (p<.0001), and the Russian experimental group formulated algebraic proofs more often than other groups (p<.01). Ten to fourteen year-olds formulated proofs less often than 14 to 16 year-olds (p<.001). In addition, the interaction term suggested that in the Russian experimental curriculum, 10 to 14 year-olds formulated proofs less often than 14 to 16 year-olds (p<.0001); whereas in the English non-experimental curriculum, the proportions for both age groups were equivalent (p<.0001).

![Figure 4](image)

Figure 4. Percentages concluding a generalization held for an infinite set on the basis of numerical examples only in response to problem 17

For use of "empirical proof" (Figure 4), the model with main effects of Group fit the data. Effects due to Group were significant (see Table 8). The effect size of
Group was large (0.74). English groups formulated "empirical proofs" more often than Russian groups (p<.0001).

The written test examined children's tendency to use algebraic proof. During interviews, interviewers returned to proof tasks in the test. If the student had not used algebra in those items, he/she was asked to do so by the interviewer. Students' ability to use algebra as a tool for reasoning could therefore be examined, as well as their tendency to do so. To provide some indication of students' ability to formulate an algebraic proof, results will be reported for high-achieving ("Red Track") students in the English experimental curriculum. When prompted to use algebra on problem 17 during the interview, 0% of Red Track Year 9 students, 5% of Red Track Year 10 students, and 20% of Red Track Year 11 students formulated an algebraic proof.

Similar findings were obtained for Problem 4:

Problem 4: A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right. (Lee & Wheeler, 1989)

For the written test, Figures 5, 6, and 7 compare the percentages of students formulating an algebraic proof in response to this task, versus an "empirical proof."

Figure 5 compares the percentages for the various curricular groups and year/grade levels.
Figure 5. Percentages formulating an algebraic proof in response to problem 4; versus percentages concluding a generalization held for an infinite set on the basis of numerical examples only.

Figures 6 and 7 show students' reasoning on problem 4 for the various curricular groups and age groups.
Figure 6. Percentages independently formulating a correct algebraic proof in response to problem 4

For use of algebraic proof (Figure 6), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 9). Effect sizes of Group (0.5) and Age (0.67) were large. Russian groups formulated algebraic proofs significantly more often than English groups (p<.0001), and the Russian experimental group formulated algebraic proofs more often than other groups (p<.0001). Ten to fourteen year-olds formulated proofs less often than 14 to 16 year-olds (p<.0001).
Figure 7. Percentages concluding a generalization held for an infinite set on the basis of numerical examples only in response to problem 4

For use of "empirical proof" (Figure 7), parsimonious models did not fit the data well, requiring introduction of the interaction term. The fully saturated model fit the data. Effects of Group, Age and interaction between Group and Age were significant (see Table 9). Effect sizes of Group (0.3) and Age (0.45) were moderate; and the effect size of the interaction between Group and Age was small (0.17). The Russian experimental group used "empirical proofs" less often than other groups (p<.0001), whereas NMP students used "empirical proofs" more often than other groups (p<.0001). Ten to fourteen year-olds used empirical arguments more often than 14 to 16 year-olds (p<.0001). In addition, the interaction term suggested that in both the English experimental (p<.0001) and Russian experimental (p<.0001) groups,
10 to 14 year-olds used "empirical proofs" more often than 14 to 16 year-olds; whereas in the Russian non-experimental group, 14 to 16 year-olds used empirical proofs more often than 10 to 14 year-olds (p<.0001). This last finding is probably attributable to differences in response rates for the two age groups in the Russian non-experimental group. In the Russian non-experimental group, 42% of the 10 to 14 year-olds did not respond to this task, while only 18% of the 14 to 16 year-olds failed to respond. (Twenty-seven percent of the 10 to 14 year-olds in the Russian experimental group gave no response. For the remaining age and curricular groups, non-response rates ranged from 3% to 14%.)

When prompted to use algebra on problem 4 during the interview, 0% of NMP Red Track Year 9 students, 25% of Red Track Year 10 students, and 25% of Red Track Year 11 students formulated an algebraic proof. Sixty-five percent of Red Track Year 9 students, 20% of Red Track Year 10 students, and 30% of Red Track Year 11 students used the following approach: interviewees formulated an algebraic equation or expression; substituted numbers for variables; and concluded a generalization held for an infinite set on the basis of the numerical examples.

Figures 8–10 compare the percentages of students using some algebra in response to problem 4 during the written test, versus percentages of students using numerical reasoning only. Figure 8 compares the percentages for the various curricular groups and year/grade levels.
Figure 8. Use of algebraic versus numerical reasoning: Percentages independently using some algebra in response to problem 4; versus percentages using numerical examples only.

For problem 4, Figures 9 and 10 compare students' use of algebra versus numerical reasoning for the various curricular groups and age groups.
For use of algebra (Figure 9), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 9). Effect sizes of Group (0.47) and Age (0.44) were moderate. Russian groups used algebra significantly more often than English groups (p<.0001), and the Russian experimental group used algebra significantly more often than other groups (p<.0001). Ten to fourteen year-olds used algebra less often than 14 to 16 year-olds (p<.0001).
Figure 10. Percentages using numerical examples only in response to problem 4

For use of "numerical examples only" (Figure 10), parsimonious models did not fit the data well, requiring introduction of the interaction term. The fully saturated model fit the data. Effects of Group, Age and interaction between Group and Age were significant (see Table 9). However, effect sizes of Age (0.14), and interactions between Group and Age (0.17) were small, while the effect size of Group was large (0.5). Russian groups used "numerical examples only" less often than English groups (p<.0001). The Russian experimental group used "numerical examples only" less often than other groups (p<.0001), while the English experimental group used "numerical examples only" more often than other groups (p<.0001). Ten to fourteen year-olds used "numerical examples only" more often than 14 to 16 year-olds (p<.0001). The interaction term suggested that in the Russian experimental group, 10
to 14 year-olds used "numerical examples only" more often than 14 to 16 year-olds (p<.0001); whereas in the Russian non-experimental group, 14 to 16 year-olds used "numerical examples only" more often than 10 to 14 year-olds (p<.0001). This last finding is probably attributable to the low response rate (58%) among 10 to 14 year-olds in the Russian non-experimental group (see the discussion above).

Comparisons of use of algebra versus numerical reasoning for problem 17 can be found in Table 8.
Table 8. Reasoning on a generalization task: Problem 17

<table>
<thead>
<tr>
<th>Use of algebraic proof (Problem 17)</th>
<th>Use of &quot;empirical proof&quot; (Problem 17)</th>
<th>Use of some algebra (Problem 17)</th>
<th>Use of numerical examples only (Problem 17)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model fit</td>
<td>Main effects of Group and Age.</td>
<td>Main effects of Group</td>
<td>Main effects of Group and Age.</td>
</tr>
<tr>
<td>Goodness of fit</td>
<td>$\chi^2 = 0$</td>
<td>$\chi^2(4) = 5.42, p=0.247$</td>
<td>$\chi^2(3) = 4.18, p=0.242$</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>$\chi^2(3) = 104.36, p&lt;.0001$</td>
<td>$\chi^2(4) = 249, p&lt;.0001$</td>
<td>$\chi^2(3) = 138.4, p&lt;.0001$</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.48</td>
<td>0.74</td>
<td>0.44</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>N.S.</td>
<td>$\chi^2(2) = 218, p&lt;.0001$</td>
<td>$\chi^2(2) = 21.4, p&lt;.001$</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.66</td>
<td>N.S.</td>
<td>0.70</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>$\chi^2(3) = 8.01, p&lt;.01$</td>
<td>N.S.</td>
<td>$\chi^2(3) = 8.01, p&lt;.01$</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>0.13</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>(1) R&gt;E (p&lt;.0001)</td>
<td>E&gt;R (p&lt;.0001)</td>
<td>(1) E&gt;R (p&lt;.0001)</td>
</tr>
<tr>
<td>Direction of Age contrasts and p-values</td>
<td>14 to 16 years &gt; 10 to 14 years (p&lt;.0001)</td>
<td>N.S.</td>
<td>14 to 16 years &lt; 10 to 14 years (N.S.)</td>
</tr>
<tr>
<td>Direction of Interaction contrasts and p-values</td>
<td>(1) DV(10-14)&lt;DV(14-16) (p&lt;.0001)</td>
<td>N.S.</td>
<td>(1) DV(10-14)&gt;DV(14-16) (p&lt;.01)</td>
</tr>
</tbody>
</table>

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.
Table 9. Reasoning on a generalization task: Problem 4

<table>
<thead>
<tr>
<th>Model fit</th>
<th>Use of algebraic proof (Problem 4)</th>
<th>Use of &quot;empirical proof&quot; (Problem 4)</th>
<th>Use of some algebra (Problem 4)</th>
<th>Use of numerical examples only (Problem 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit</td>
<td>( \chi^2(3) = 0.85 ) \text{ p=0.8375}</td>
<td>( \chi^2 = 0 ) \text{ p=0.168}</td>
<td>( \chi^2(3) = 5.05 ) \text{ p=0.168}</td>
<td>( \chi^2 = 0 ) \text{ p=0.168}</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>( \chi^2(3) = 111, p&lt;0.0001 )</td>
<td>( \chi^2(3) = 41.11, p&lt;0.0001 )</td>
<td>( \chi^2(3) = 90.5, p&lt;0.0001 )</td>
<td>( \chi^2(3) = 112.0, p&lt;0.0001 )</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.5</td>
<td>0.3</td>
<td>0.47</td>
<td>0.5</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>( \chi^2(2) = 203.8, p&lt;0.0001 )</td>
<td>( \chi^2(2) = 92.2, p&lt;0.0001 )</td>
<td>( \chi^2(2) = 86.3, p&lt;0.0001 )</td>
<td>( \chi^2(2) = 9.3, p&lt;0.0001 )</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.67</td>
<td>0.45</td>
<td>0.44</td>
<td>0.14</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>N.S.</td>
<td>( \chi^2(3) = 12.24, p&lt;0.01 )</td>
<td>N.S.</td>
<td>( \chi^2(3) = 11.48, p&lt;0.01 )</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>N.S.</td>
<td>0.17</td>
<td>N.S.</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Direction of Group contrasts and p-values:
1. R>E (p<0.0001)
2. DV>ALL (p<0.0001)
3. NMP>ALL (p<0.0001)
4. R-NEX (p<0.0001)
5. NMP>ALL (p<0.0001)

Direction of Age contrasts and p-values:
1. 14 to 16 years > 10 to 14 years (p<0.0001)
2. 14 to 16 years < 10 to 14 years (p<0.0001)
3. 14 to 16 years > 10 to 14 years (p<0.0001)
4. 14 to 16 years < 10 to 14 years (p<0.0001)

Direction of Interaction contrasts and p-values:
1. DV(10-14)>DV(14-16) (p<0.0001)
2. R-NEX(10-14)<R-NEX(14-16) (p<0.0001)
3. NMP(10-14)>NMP(14-16) (p<0.0001)
4. DV(10-14)>DV(14-16) (p<0.0001)
5. R-NEX(10-14)<R-NEX(14-16) (p<0.0001)

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.
DISCUSSION.

For acquisition of algebraic deductive reasoning, Age and Group appeared to have independent effects. In the four analyses for use of algebraic proof and use of algebra as a tool for reasoning, there were main effects of Group and Age; while an interaction term was present in one of the four fitted models, the effect size of the interaction was small (0.13). For the four analyses, the effect size of Group was moderate to large (approximately 0.5). The effect size of Age was moderate to large (ranging from 0.44 to 0.7).

Analyses revealed cultural effects, and a combination of cultural and curricular effects on algebraic deductive reasoning. In the four analyses, Russian groups used algebraic proof and algebra significantly more often than English groups, and the Russian experimental group was more likely to use algebraic proof and algebra than other groups. For these measures, differences between the Russian experimental group and other groups tended to increase with children's age—suggesting effects of instruction tend to increase with age (see Figures 3, 6, 9).

When prompted to use algebra during the interviews, 23% of top achieving 14 to 16 year-olds in the NMP group (mean age of 15.1 years) formulated an algebraic proof in response to problem 4. Since 19% of the 10 to 14 year-olds in the Russian experimental curriculum (mean age of 12.0 years) formulated an algebraic proof in response to this task without prompting, this suggests instruction can amplify development of algebraic deductive reasoning—over-shadowing effects of age.

For use of numerical reasoning, there were main effects of Group and Age, and interactions between Group and Age. However, effect sizes of the interactions were small (approximately 0.15), and could be attributed to differential within-group
response rates for the two age levels. For the two analyses, the effect size of Group was large (approximately 0.5), while the effect size of Age was small.

Data analysis revealed effects of culture, and a combination of cultural and curricular effects. In both analyses, English groups were more likely than Russian groups to use numerical reasoning.

For use of "empirical proof," the analysis for problem 17 suggested main effects of Group. The effect size of Group was large (0.74). The analysis suggested cultural effects, with English groups using empirical proofs more often than Russian groups.

The corresponding analysis for problem 4 included main effects of Group and Age, and interactions between Group and Age. The effect size of the interaction was small (0.17), and could be attributed to differential within-group response rates for the two age levels. Effect sizes of Group (0.3) and Age (0.45) were moderate. The analysis suggested curricular effects, with Davydov's group using empirical proof less often than other groups, and the NMP group using empirical proof more often than other groups.

Logical necessity of deductive conclusions derived from algebraic arguments

Problem 11(D) examined students' ability to understand logical necessity as a component of deduction:

Problem 11: In an algebra class the teacher proved that every whole number of the form \(n^3 - n\) is divisible by 6 (that is, if you divide \(n^3 - n\) by 6, there will be no remainder). The proof was as follows:

We can write:

\[ n^3 - n = n(n^2 - 1) \]

But we can rewrite the expression on the right:
\[ n(n^2 - 1) = n(n - 1)(n + 1) \]

So:

\[ n^3 - n = n(n - 1)(n + 1) = (n - 1)n(n + 1) \]

But \((n - 1)n(n + 1)\) is a product of three consecutive whole numbers. Therefore, one of them should be divisible by 2, and one of them (not necessarily a different one) should be divisible by 3. Thus their product should be divisible by \(2 \times 3\), that is, by 6.

Please answer the following:

(A) I understand all the details of the proof and the proof seems correct to me.

Yes / No (Circle one)

(B) There are some details in the proof that I do not understand. They are the following:

(C) If you think the teacher has given a correct proof for the theorem "every whole number of the form \(n^3 - n\) is divisible by 6", then answer the following question:

Do you think that further checks (by substituting numbers) are necessary in order to verify the validity of the theorem?

Yes / No (Circle one)

Explain.

(D) Victor [Petya in the Russian version of the test] is a doubter. He thinks that we have to check at least a hundred numbers in order to be sure that the theorem is correct. What is your opinion? Explain your answer.

(Adapted from Fischbein & Kedem, 1982; and Vinner, 1983)

For Part (D), Figures 11, 12, and 13 compare percentages of students claiming (a) empirical checks are needed to verify the validity of the theorem (deductive proof cannot guarantee "universal validity" for an infinite set); versus (b) the algebraic proof establishes the validity of the theorem (no empirical support is needed). Figure 11
compares the percentages for the various curricular groups and year/grade levels. Russian groups in grades 5, 6, and 7 were not included in the figure, as their response rates were low for this task.

![Graph showing percentages]

Figure 11. Percentages claiming empirical checks are needed to verify the validity of the theorem, versus percentages claiming the algebraic proof establishes the validity of the theorem in response to problem 11(D).

Figures 12 and 13 show students' responses for the various curricular groups and age groups.
Figure 12. Percentages claiming the algebraic proof establishes the validity of the theorem in response to problem 11(D)

For the data in Figure 12, the log-linear model with main effects of Group and Age was selected. Effects of Group and Age were significant (see Table 10). The effect size of Group was moderate (0.33), while the effect size of Age was large (0.87). The Russian experimental group claimed deductive proof established "universal validity" (i.e., no empirical support was needed) significantly more often than other groups (p<.001). Fourteen to sixteen year-olds claimed the algebraic proof established the validity of the theorem more often than 10 to 14 year-olds (p<.0001).
Figure 13. Percentages claiming empirical checks are needed to verify the validity of the theorem in response to problem 11(D)

For the data in Figure 13, the model with main effects of Group fit the data. Effects due to Group were significant (see Table 10). The effect size of Group was large (0.55). English groups claimed empirical checks were needed to verify the validity of the theorem significantly more often than Russian groups (p<.0001). The English experimental group claimed empirical checks were needed significantly more often than other groups (p<.0001).
Table 10. Reasoning: Notions as to the logical necessity of deductive conclusions derived from algebraic arguments

<table>
<thead>
<tr>
<th>Model fit</th>
<th>Claimed deductive proof cannot guarantee &quot;universal validity.&quot; Empirical support is needed. (Problem 11D)</th>
<th>Claimed deductive proof establishes &quot;universal validity.&quot; No empirical support is needed. (Problem 11D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit</td>
<td>( \chi^2(4) = 3.89 \quad \text{p}=0.4215 )</td>
<td>( \chi^2(3) = 4.62 \quad \text{p}=0.2018 )</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>( \chi^2(4) = 133.1 \quad \text{p}&lt;0.001 )</td>
<td>( \chi^2(3) = 50.5 \quad \text{p}&lt;0.001 )</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.55</td>
<td>0.33</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>N.S.</td>
<td>( \chi^2(2) = 342.4 \quad \text{p}&lt;0.001 )</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>N.S.</td>
<td>0.87</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>(1) E&gt;R ( \text{p}&lt;0.001 )</td>
<td>DV&gt;ALL ( \text{p}&lt;0.001 )</td>
</tr>
<tr>
<td></td>
<td>(2) NMP&gt;ALL ( \text{p}&lt;0.001 )</td>
<td></td>
</tr>
<tr>
<td>Direction of Age contrasts and p-values</td>
<td>N.S.</td>
<td>14 to 16 years &gt; 10 to 14 years ( \text{p}&lt;0.001 )</td>
</tr>
<tr>
<td>Direction of Interaction contrasts and p-values</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
</tbody>
</table>

**NOTE:** E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.

Some representative responses to problem 11(D) follow:

**DAVYDOV’S CURRICULUM, GRADE 5:**

- I disagree with Petya. I don't understand why \( n^3 - n = n(n^2 - 1) \) [a step in the supplied proof]. It's enough to understand the equation \( n^3 - n = n(n^2 - 1) \), and the substitution of even 100 numbers doesn't help.
DAVYDOV'S CURRICULUM, GRADE 6:
• Yes, I think that Petya is right because it's hard to realize right away if the teacher is right or not. He can also make a mistake. It has to be checked and reasoned out and then it will be easier to find the right answer.

DAVYDOV'S CURRICULUM, GRADE 8:
• If we check the theorem on numbers, we won't be able to be sure it's correct. There are an infinite amount of numbers, and you can't substitute an infinite number of numbers in. What if after substituting 100 numbers, the 101st number will be contradictory? If you are a skeptic, be one to the end.
• Petya is a fool. Because the letter can equal any number.
• Bad method of checking the theorem, because all 100 numbers can satisfy the theorem, but the theorem (or proof) could still have mistakes.
• I don't think Petya is right because the theorem is proven analytically.
• I think it's stupid to check it like this because there are no mistakes in the proof.
• The theorem is proved correctly, so it's correct for any number so there is no reason to check it.
• For 100 numbers the theorem might be correct. It can be incorrect for one number.
• I think it's unnecessary because the teacher proved the theorem for any number.
• I disagree. To prove that the theorem is wrong, we have to find one number that doesn't fit the definition. To prove that the theorem is correct, all numbers must fit it.
• No, I trust the algebraic method of proof more.
• If it's correct for at least one number, then it's correct for all the following numbers.
• Yes, I don't understand the proof, but understand during substitution.
DAVYDOV'S CURRICULUM, GRADE 9:
• There is no point because there are an infinite number of numbers and it can't be checked.
• It is my opinion that Petya doesn't know math. In this case, I don't think that you need to check on numbers.
• A check I think is not necessary because the proof is obvious.
• I think it's wrong, I mean pointless. Because if the theorem is wrong, it's possible there is a 101st number for which the theorem is wrong.
• I think you just have to think well about the proof and check the theorem on a few numbers.
• I think that it's not that important. We can do it on a couple of numbers, but 100 numbers is too much. And I think that "n" fits very well as the number, except we have to check in case of zero and negative numbers.
• Petya has an inferiority complex. He should consult a psychotherapist.

RUSSIAN NON-EXPERIMENTAL CURRICULUM, GRADE 6:
• I think that 100 numbers is too much, about 10 to 20 is enough. You'll die before you check 100 numbers.

RUSSIAN NON-EXPERIMENTAL CURRICULUM, GRADE 7:
• You can do that, but it will take very long.
• Using substitution of numbers, we check the correctness of the equation.
• Without substituting numbers you can't check anything, ever.

RUSSIAN NON-EXPERIMENTAL CURRICULUM, GRADE 9:
• Petya has too much free time.
• You don't have to check the theorem on 100 numbers. If we know the theorem, we can check it on fewer numbers.

RUSSIAN NON-EXPERIMENTAL CURRICULUM, GRADE 10:
• Yes, a theory is just a theory before it's been checked in real life.

NMP, YEAR 9:
• Yes, I agree with Victor, because \( n^3 - n = n(n^2 - 1) \) [a step in the supplied proof] might not work. [Writes \( n=3: \ 9 \times 9 = 81 \times 9 = 729 - 9 = 720 \).] I think we need to check it several times.
• I think Victor is right because then we know if there is a pattern or not.
• No, you may see a pattern before 100 numbers.
• I think that you do have to check a large amount of numbers to make sure you are correct but a hundred is a bit too much.
• I think that you should check 100 numbers just in case there is an exception.
• I think Victor is right to be doubtful. Just because it works for some numbers doesn't mean it works for others.
• You should experiment with different numbers.
• Yes, it may well work for letters but does it work for numbers, e.g., odd numbers, even numbers, numbers more than 10, prime numbers?

NMP, YEAR 10:
• I think Victor is stupid because he must get very bored. After about four have worked he should know for certain, as if the theorem was incorrect it would go wrong on about the third.
• I think you have to check things but not 100 times. I think 10 times max is enough.
• I think that you need to check the theorem a hundred times so you make sure it works and have enough numbers to prove it.

• I think that the more you check a theorem (or anything else) the surer you can be that it's right.

• I think you definitely have to replace $n$ by at least 3 or 4 different numbers because (like in a past question [on this test]), $2n > n+2$ was true for all numbers over 1 but not for numbers under 1.

• Yes, you need to use more numbers to experiment on formulas.

• Yes, so you are 100% sure that it will work every time.

NMP, YEAR 11:

• I think a theorem should be checked until a correct pattern is found. If the theorem has been checked 100 times and there is no pattern, then the theorem is not correct.

• If the formula was made from the studying of a set of numbers, it should still be tested once.

• I think test as many as you want until you feel ok about it.

• Yes, you prove something by showing evidence, so you should do more examples.

• 100 is a bit excessive, but certainly it needs checking a lot. A check with all different types of numbers is required—positive, negative, fractions, small, large, etc. . . as well as some other random figures just to clinch the proof.

• No, of course not. Three or maybe four random checks will establish just as much.

• Yes, it's easier to work with a given number that you can get a grip on than to work with any number [means variable stands for any number] which is a bit vague. You see by the numbers why or what's going on.

• Yes, you are more likely to get a correct result.

• Yes, it is essential to use further data to validate it.
• Yes, the only way to check your theory is correct is by substituting numbers.
• Yes, the logic of the statement may be correct but you have to use example numbers in order to prove conclusively.
• Yes, because the logic the teacher has worked through is sound, although it may well be worth testing it, just in case.

ENGLISH NON-EXPERIMENTAL CURRICULUM, YEAR 9:
• Yes, because you don't know until you try it.

ENGLISH NON-EXPERIMENTAL CURRICULUM, YEAR 10:
• When you think that you have gone far enough to see if it is right then leave it at that.
• I think we should check at least 100 numbers because it may be right for 99 but then wrong for 1, and that would muck up the whole theorem.
• Victor is correct. Because this would give the amount of testing that I would be happy with.
• I believe checking one hundred numbers would be adequate, perhaps you could even go so far as to check a thousand to be sure.
• Checking 100 numbers would still not prove it was right. It is impossible to prove a formula is right because you would have to check every number. But there are an infinite number of numbers.
• I agree. A theory can always be proved wrong. That is why it is a theory and not a fact.
• Yes, to get concrete evidence that the rule is correct.
• Yes, although this may work with algebra it is always best to test out a theory such as this using numbers.
• Yes, because it might not work for some numbers and also you will know for certain that it is true.

ENGLISH NON-EXPERIMENTAL CURRICULUM, YEAR 11:
• Yes, I think that too, just to be on the safe side.
• Five will be enough, then we have a fair test.
• Yes, Victor is right as this would be a fair and accurate way of being sure.
• In an equation you need only check a couple of numbers because the pattern will just be repeated with larger numbers.
• Yes, to make sure it is correct even more.
• Yes, because there is a large chance someone is likely to find a number that it doesn't work with.

DISCUSSION.

Data analysis revealed curricular effects on deductive law–based reasoning. Davydov's group claimed algebraic proof can guarantee "universal validity" (i.e., no empirical support is needed) significantly more often than other groups. Group and Age had independent effects. The effect size of age was large.

For empirical reasoning, data analysis revealed cultural effects, and a combination of curricular and cultural effects. English groups claimed empirical checks are needed (i.e., that algebraic proof cannot guarantee the absolute validity of a statement) significantly more often than Russian groups. The NMP group claimed empirical evidence is needed more often than other groups. Age–related effects were not significant.

National Mathematics Project emphasizes inductive, case–based (Moshman, 1995) reasoning—the investigation of a number of particular empirical cases to
formulate, and to assess the validity of algebraic generalizations. Davydov's curriculum emphasizes deductive, law-based (Moshman, 1995) reasoning—the logical derivation of particular (e.g., numerical) cases from general mathematical principles and relationships where those principles and relationships are first expressed algebraically. Emphases of the experimental curricula were reflected in students' reasoning. Curricular emphasis on inductive, empirical reasoning appears to promote empirical reasoning in mathematics. The English experimental group formulated empirical "proofs," and claimed empirical checks of an algebraic proof are needed significantly more often than other groups (see Figures 7 and 13). The Russian experimental group formulated algebraic proofs, and claimed algebraic proof can guarantee "universal validity" significantly more often than other groups.

Students' comments supported the statistical analyses. Davydov's group tended to operate at the level of structure or relationship in proof tasks, making empirical support superfluous. NMP students tended to use process interpretations of algebraic constructs; empirical checks provided empirical evidence. Differences are, perhaps, best captured by comparing students' reasoning in the following quotes:

DAVYDOV'S CURRICULUM, GRADE 5:

• I disagree with Petya. I don't understand why $n^3 - n = n(n^2 - 1)$ [a step in the supplied proof]. It's enough to understand the equation $n^3 - n = n(n^2 - 1)$, and the substitution of even 100 numbers doesn't help.

NMP, YEAR 9:

• Yes, I agree with Victor, because $n^3 - n = n(n^2 - 1)$ might not work. [Writes $n=3: 9 \times 9 = 81 \times 9 = 729 - 9 = 720$.] I think we need to check it several times.
Problems 19 and 24 were verbal reasoning tasks:

**Problem 19:** For the following problem: Assume the first two sentences (in bold) are true. Make a conclusion from the assumptions. (Choose a, b, c, or d.)

**All fahmooth numbers can be divided evenly by 8.**
**26 is a fahmooth number.**

Therefore . . .

a) 26 must not be a fahmooth number.
b) 26 is an exception to the rule.
c) It is probably true that fahmooth numbers cannot be divided evenly by 8.
d) 26 can be divided evenly by 8.

Problem 24(1): In this question you just have to tell whether or not the sentences show correct reasoning. All of the sentences are really nonsense, but you are to think only about the reasoning. . . . You have two choices, Yes or No. The first choice means yes, the reasoning is good. The second choice means no, the reasoning is not good. . . .

**Please choose Yes or No for each sentence:**

(1) If all birds have purple tails and all cats are birds, then all cats have purple tails.

Yes

No

(Adapted from National Longitudinal Study of Mathematical Abilities (NLSMA) test battery (Wilson, Cahen, & Begle, 1966))

Problem 19 seems to require both meta–components and transformational components of deductive reasoning; while problem 24(1) seems to require only transformational components (see e.g., Braine, 1990; Moshman, 1990). Figures 14 and 15 show the percentages of correct responses to these tasks for the various curricular groups and age groups.
Figure 14. Percentages giving a correct response to a logical reasoning task (problem 19)

For the data shown in Figure 14, parsimonious models did not fit the data well, requiring introduction of the interaction term. The fully saturated model fit the data. Effects of Age and interaction between Group and Age were significant (see Table 11). Effect sizes of Group and interactions between Group and Age were small; while the effect size of Age was large (0.75). Ten to fourteen year-olds gave correct responses less often than 14 to 16 year-olds (p<.01). The interaction term suggested that in the Russian experimental group, 14 to 16 year-olds were more likely than 10 to 14 year-olds to respond correctly (p<.0001); whereas within the other groups, there were no significant differences for the two age levels (p<.0001).
Figure 15. Percentages giving a correct response to a logical reasoning task (problem 24(1))

For the data shown in Figure 15, the log-linear model with main effects of Group and Age was selected. Effects of Group and Age were significant (see Table 11). The effect size of Group was small (0.24), and the effect size of Age was moderate (0.36). English groups answered correctly significantly more often than Russian groups (p<.05). The Russian non-experimental group answered correctly less often than other groups (p<.0001). Fourteen to sixteen year-olds answered correctly more often than 10 to 14 year-olds (p<.0001).
Table 11. Logical reasoning: Correct solutions to two problems with essentially the same logical structure

<table>
<thead>
<tr>
<th>Model fit</th>
<th>Correct solution (Problem 19)</th>
<th>Correct solution (Problem 24(1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit</td>
<td>$\chi^2 = 0$</td>
<td>$\chi^2(3) = 5.61$ p=0.1319</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>$\chi^2(3) = 4.3$, p&lt;.1</td>
<td>$\chi^2(3) = 27.0$, p&lt;.0001</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.098</td>
<td>0.24</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>$\chi^2(2) = 252$, p&lt;.0001</td>
<td>$\chi^2(2) = 58.7$, p&lt;.0001</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.75</td>
<td>0.36</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>$\chi^2(3) = 13.61$, p&lt;.005</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>0.17</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>N.S.</td>
<td>(1) E&gt;R (p&lt;.05)</td>
</tr>
<tr>
<td>Direction of Age contrasts and p-values</td>
<td>14 to 16 years &gt; 10 to 14 years (p&lt;.01)</td>
<td>(2) R-NEX&lt;ALL (p&lt;.0001)</td>
</tr>
<tr>
<td>Direction of Interaction contrasts and p-values</td>
<td>(1) DV(10-14)&lt;DV(14-16) (p&lt;.0001)</td>
<td>N.S.</td>
</tr>
<tr>
<td></td>
<td>(2)ALL(10-14)=ALL(14-16) (p&lt;.0001)</td>
<td></td>
</tr>
</tbody>
</table>

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.

DISCUSSION.

Comparison of performances on problems 19 and 24(1) suggested that, given two tasks with essentially the same logical structure, Russian and English children were more likely to apply abstract logical principles in a verbal reasoning task that seemed to require only transformational components of deductive reasoning, than in a verbal reasoning task that appeared to require both meta–components (e.g.,
understanding of logical necessity) and transformational components of deductive reasoning (see Sloutsky & Morris, 1995). Braine (1990) and Moshman (1990) theorized that meta-components of reasoning ontogenetically develop later than transformational components: children who do not develop meta-components and do not understand logical necessity are still able to formulate some deductive arguments. Though differences failed to reach statistical significance (p=.065), Davydov's older group responded correctly to problem 19 more often than other older groups; in addition, in Davydov's group, 14 to 16 year-olds were more likely than 10 to 14 year-olds to respond correctly (p<.0001), whereas within other curricular groups, there were no significant differences for the two age levels (p<.0001). This tentatively suggests that curriculum may affect the development of meta-components of reasoning.

**Algebraic structure**

*Formulation of algebraic equations representing quantitative relationships*

Problems 7 and 12 measured children's ability to formulate algebraic equations that represented verbally described quantitative relationships:

**Problem 7:** Write an equation using the letters $S$ and $T$ to represent the following statement:
"There are six times as many students as teachers at this school."
Use $S$ for the number of students and $T$ for the number of teachers.

(Adapted from Clement, Lochhead, and Monk, 1981)

**Problem 12:** Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If $b$ is the number of blue pencils bought and if $r$ is the number of red pencils bought, write an equation involving $b$ and $r$.

(Küchemann, 1981)
For the various curricular groups and year/grade levels, Figure 16 shows the percentages of students formulating a correct algebraic equation in response to problem 7 (the "student-professor problem").

Figure 16. Percentages formulating a correct algebraic equation in response to problem 7 (the "student-professor problem")

Figure 17 shows the percentages of students formulating a correct algebraic equation for the various curricular groups and age groups.
Figure 17. Percentages formulating a correct algebraic equation ($S=6T$) in response to problem 7

For formulation of a correct algebraic equation (Figure 17), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 12). The effect size of Group was moderate (0.42) and the effect size of Age was small (0.14). The Russian experimental group formulated correct algebraic equations significantly more often than other groups ($p<.0001$). The experimental groups formulated correct equations more often than the non-experimental groups ($p<.0001$). The Russian non-experimental group had a higher proportion of correct responses than the English non-experimental group ($p<.0001$). Ten to fourteen year-olds formulated correct equations less often than 14 to 16 year-olds ($p<.0001$).
Figure 18 examines students' formulation of reversed equations ($T=6S$). For each group, the number of reversed equations was divided by the total number of attempted responses.

![Figure 18. Proportions of reversed equations ($T=6S$) for problem 7](image)

The proportion of reversed equations increases with age. The increase with age probably reflects an increase in the numbers of students able to formulate equations of any kind. In the older age group, the similarity in the proportions of reversals for three of the four curricular groups did not support a "linguistic hypothesis": i.e., that the variable-reversal error primarily stems from the linguistic structure of the English statement (see Clement, 1982). The performance of the Russian experimental group suggests reversals are related to children's understanding
of quantitative relationships, and/or the representation of quantitative relationships: (1) the percentage of correct equations for the Russian experimental group was very high, even for fifth and sixth graders; (2) the percentage of correct equations for the Russian experimental group was significantly higher than the percentages for other groups, while the proportion of reversed equations among the older age group was significantly lower than the corresponding proportions for the other groups; and (3) students in the younger age group used models to represent the relevant quantitative relationships before formulating correct equations. Graphic models of quantitative relationships are used in Davydov's curriculum.

For the various curricular groups and year/grade levels, Figure 19 shows the percentages of students formulating correct and incorrect algebraic equations in response to problem 12.
Figure 19. Percentages formulating a correct algebraic equation ($5b + 6r = 90$), versus percentages writing incorrect equations in response to problem 12.

Figure 20 shows the percentages of students formulating a correct algebraic equation in response to problem 12 for the various curricular groups and age groups.
For formulation of a correct algebraic equation (Figure 20), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 12). Effect sizes of Group (0.48) and Age (0.4) were moderate. Russian groups formulated correct algebraic equations more often than English groups (p<.0001). The Russian experimental group formulated correct algebraic equations more often than other groups (p<.0001). Ten to fourteen year-olds formulated correct equations less often than 14 to 16 year-olds (p<.0001).
Table 12. Formulation of algebraic equations representing quantitative relationships in verbal problems

<table>
<thead>
<tr>
<th>Model fit</th>
<th>Formulated correct algebraic equation: ( S=6T ) (Problem 7)</th>
<th>Formulated correct algebraic equation: ( 5b + 6r = 90 ) (Problem 12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit</td>
<td>( \chi^2(3) = 7.49 ) ( p = 0.0579 )</td>
<td>( \chi^2(3) = 3.83 ) ( p = 0.2801 )</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>( \chi^2(3) = 78.7 ) ( p &lt; 0.001 )</td>
<td>( \chi^2(3) = 102.6 ) ( p &lt; 0.001 )</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.42</td>
<td>0.48</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>( \chi^2(2) = 9.4 ) ( p &lt; 0.01 )</td>
<td>( \chi^2(2) = 72.5 ) ( p &lt; 0.001 )</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.14</td>
<td>0.4</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>(1) DV&gt;ALL ( p &lt; 0.0001 )</td>
<td>(1) R&gt;E ( p &lt; 0.0001 )</td>
</tr>
<tr>
<td></td>
<td>(2) NMP&gt;R-NEX ( p &lt; 0.0001 )</td>
<td>(2) DV&gt;ALL ( p &lt; 0.0001 )</td>
</tr>
<tr>
<td></td>
<td>(3) NMP&gt;E-NEX ( p &lt; 0.0001 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4) R-NEX&gt;E-NEX ( p &lt; 0.0001 )</td>
<td></td>
</tr>
<tr>
<td>Direction of Age contrasts and p-values</td>
<td>14 to 16 years &gt; 10 to 14 years ( p &lt; 0.0001 )</td>
<td>14 to 16 years &gt; 10 to 14 years ( p &lt; 0.0001 )</td>
</tr>
<tr>
<td>Direction of Interaction contrasts and p-values</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
</tbody>
</table>

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.

DISCUSSION.

Group and Age independently affected students' abilities to formulate algebraic equations. In the two analyses for formulation of correct algebraic equations, the effect size of Group was moderate to large. The effect size of Age was small to moderate.
The analysis for problem 7 revealed curricular effects; whereas the analysis for problem 12 suggested cultural effects, and a combination of curricular and cultural effects. In both cases, the Russian experimental group formulated correct equations more often than other groups.

*Use of arithmetic structure, ability to manipulate and simplify algebraic expressions, and ability to interpret algebraic constructs procedurally and structurally*

Problem 3 examined students' use of arithmetic structure:

**Problem 3:** Which of the following expressions are equivalent to \( 5 \times (6 + 3) \)?

(Circle them.)

A) \( 5 \times 6 + 5 \times 3 \)  B) \( 5 \times (3 + 6) \)  C) \( 5 + 6 \times 3 \)  D) \( 5 \times 3 + 5 \times 6 \)  E) \( (5 \times 6) + 3 \)

Explain what you did to figure this out.

For the various curricular groups and year/grade levels, Figure 21 shows the percentages of students using one of two "extreme approaches" in response to problem 3: (a) students correctly selected all equivalent expressions, using only arithmetical principles with no calculation; or (b) students calculated the prompt, and calculated the five expressions (A–E) to determine whether expressions were equivalent, and made no reference to, or use of, arithmetical principles. The latter group included both incorrect and correct responses; i.e., students did not always calculate correctly, did not select the correct expressions, etc.
Figure 21. Percentages using only arithmetical principles to correctly determine equivalent expressions, versus percentages calculating all expressions to determine equivalence in response to problem 3.

Figures 22 and 23 show the corresponding percentages for the various curricular groups and age groups.
For use of arithmetical principles (Figure 22), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 13). Effect sizes of Group (0.5) and Age (0.63) were large. Russian groups used arithmetical principles to determine equivalence more often than English groups (p<.0001). The Russian experimental group used arithmetical principles to determine equivalence more often than other groups (p<.0001). Ten to fourteen year-olds used arithmetical principles less often than 14 to 16 year-olds (p<.0001).
Figure 23. Percentages calculating all expressions to determine equivalence in response to problem 3

For the data in Figure 23, the log-linear model with main effects of Group and Age was selected. Effects of Group and Age were significant (see Table 13). The effect size of Group was moderate (0.45) and the effect size of Age was small (0.14). English groups calculated all expressions to determine equivalence more often than Russian groups (p<.0001). The Russian experimental group used this approach less often than other groups (p<.01); whereas the English experimental group used the approach more often than other groups (p<.001). Ten to fourteen year-olds used "calculation only" more often than 14 to 16 year-olds (p<.001).

Problem 9 examined (a) students' ability to manipulate algebraic symbols; and (b) whether students would adopt a structural or procedural interpretation in their
solutions. Solution requires a structural interpretation since $x$ can equal any real number.

Problem 9: What can $x$ equal in the following equation?

$$\frac{12x - 2x + 16}{2} = 2(x + 4) + 3x$$

Explain.

For the various curricular groups and year/grade levels, Figure 24 shows (a) the percentages of students using only a procedural interpretation of algebraic constructs in their solutions (e.g., using guess and test methods); versus (b) percentages of students using a structural interpretation, correctly simplifying and manipulating algebraic expressions, and concluding $x$ can equal any real number.
Figure 24. Percentages using only a procedural approach to find solutions; versus percentages using a structural approach with correct symbolic manipulation and correct conclusion in response to problem 9.

Figures 25 and 26 show corresponding percentages for the various curricular groups and age groups.
Figure 25. Percentages using a structural approach, with correct symbolic manipulation and correct conclusion (i.e., concluding $x$ can be any real number) in response to problem 9.

For the data in Figure 25, the log-linear model with main effects of Group and Age was selected. Effects of Group and Age were significant (see Table 13). The effect size of Group was moderate (0.37), while the effect size of Age was large (0.76). Russian groups used a correct structural approach more often than English groups ($p<.001$). The Russian experimental group used a correct structural approach more often than other groups ($p<.001$), and the English experimental group used a correct structural approach more often than the English non-experimental group ($p<.001$). Ten to fourteen year-olds used a correct structural approach less often than 14 to 16 year-olds ($p<.0001$).
Figure 26. Percentages using only a procedural approach (e.g., guess and test methods) to find solutions in response to problem 9

For use of a "procedural approach only" (Figure 26), the model with main effects of Group fit the data. Effects due to Group were significant (see Table 13). The effect size of Group was large (0.85). English groups used a "procedural approach only" more often than Russian groups (p<.0001). The English experimental group used a "procedural approach only" more often than other groups (p<.0001).

Figure 27 shows percentages of responses where students (a) used only a procedural approach, substituting numbers for $x$ and performing the indicated calculations; and (b) concluded the solution set consisted of the substituted numbers, or some infinite set.
For the data in Figure 27, the model with main effects of Group was selected. Effects due to Group were significant (see Table 13). The effect size of Group was large (0.97). English groups used the approach more often than Russian groups (p<.01). The English experimental group used this approach more often than other groups (p<.01).

The English experimental group relied heavily on procedural interpretations of algebraic constructs. In interviews, NMP students preferred to use procedural interpretations. The difficulty in moving from process (procedural) interpretations, to structural object-oriented interpretations is illustrated in the following exchange with a Red Track Year 9 student:

![Figure 27. Percentages using only a procedural approach to find solutions, and concluding solution set consisted of substituted numbers or an infinite set in response to problem 9](image-url)
I: Can you use algebra on this task?

Choose any number between 1 and 10. Write your number in this box:

\[ \underline{ } \]

Add the number in the box to 10 and write down the answer here: \[ \underline{ } \].

Now take the number in the box, subtract it from 10, and write down the answer here:

\[ \underline{ } \]

Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.

M: \((N + 10) + (10 - N) = \). Thought I had it for a minute. \(N\) is any number.

I: Keep looking at it, you're on to something there.

M: Is there any way of calling just \((N + 10)\) something? Then I could probably do it.

I haven't quite got it yet. I don't reckon it can be proved. If you start with a number there (points to \(N\) on the left), then subtract it there (points to \(N\) on the right), then it would depend on the number you started with. Like if you used a number greater than 10, it wouldn't work because then you would have a negative number.

I: What if you removed the brackets?

M: Any number plus 10, plus 10 again, then minus the number you started with from the sum of all these.

I: What are you trying to prove: that it always works, or how or why it works?

M: I was trying to prove it always works, then I realized it wouldn't because it depends on what number you started with.

I: What if you removed the brackets and rearranged everything? Would that be permissible?

M: No, it still wouldn't work because at the end when you try to subtract it. Whatever way you try to rearrange it, you're still subtracting at the end.

I: What about \(10 - N + N + 10 = \) ?
M: I think that would work. It would minus $N$ from that number, add $N$, then it would add 10.
I: Would anything go away there? All sorts of calculations are going on, but would anything cancel anything out?
M: I don't know.
I: [Interviewer circles $-N + N$ in the equation $10 - N + N + 10 =$.] What happens here?
M: That would cancel that out.
I: What would you be left with?
M: $10 + 10$ which would still work. Seems to work for 5, but doesn't work for any other number.
I: This cancellation, $-N + N$, would work for 5, but not for any other number.
M: This would work for every number, actually, thinking about it.
I: Could the algebra tell you that no matter what number you use in "$10 - N + N + 10 = "$, $-N + N$ falls away and I'm left with $10 + 10 = 20$?
M: There is no need for it because in "$(N + 10) + (10 - N) =$", this has the brackets, do each bit separately, it gives you the answer that you need.
I: Which is what?
M: Well if $N$ is 5, it would be $(N + 10) + (10 - N) = (5 + 10) + (10 - 5) = 20$. It would work for any other number as well. Yes, it could be proved.
I: What would a complete proof look like?
M: $(N + 10) + (10 - N) =$; $N = 5$: $(5 + 10) + (10 - 5) = 20$; $N = 4$: $(4 + 10) + (10 - 4) = 20$.
I: The proof in black doesn't do it? [Note: All references to the proof in black refer to $10 - N + N + 10 = 20$ where it was noted that $N$ and $-N$ canceled.]
M: The black proof does it, but it does it in a different way. It just adds 10 to 10, cancels out the number you put in there. The red proof is preferable. [Note: The "red proof" refers to M's proof: \((N + 10) + (10 - N) = ; N = 5: (5 + 10) + (10 - 5) = 20; N = 4: (4 + 10) + (10 - 4) = 20.\) It has more logic in it, because it's not canceling anything out. I can see exactly what's going on in terms of the calculations in the red proof. The black proof, it's not really including the number in it; it's just adding 10 to 10.

I: What does the red proof tell you?

M: The red proof shows you how it works — the calculations. The black proof tells you why it works.

I: Is it necessary in a proof to demonstrate why it works?

M: Yes it is.

I: Could the black proof be preferable then since it tells you why it works?

M: I'm not sure anymore.
Table 13. Use of arithmetic structure, and use of procedural and structural interpretations of algebraic constructs

<table>
<thead>
<tr>
<th>Model fit</th>
<th>Used only principles and correctly selected all equivalent expressions (Problem 3)</th>
<th>Calculated all expressions to determine equivalence (Problem 3)</th>
<th>Used a correct structural approach to find solutions, correct conclusion (Problem 9)</th>
<th>Used only a procedural approach to find solutions (Problem 9)</th>
<th>Used only a procedural approach to find solutions and concluded found all solutions (Problem 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goodness of fit</td>
<td>$\chi^2(3) = 2.11$ p=0.5507</td>
<td>$\chi^2(3) = 4.57$ p=0.2061</td>
<td>$\chi^2(3) = 1.06$ p=0.7856</td>
<td>$\chi^2(4) = 4.31$ p=0.3656</td>
<td>$\chi^2(4) = 1.44$ p=0.8379</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>$\chi^2(3) = 105.65$ p&lt;.0001</td>
<td>$\chi^2(3) = 89.6$ p&lt;.0001</td>
<td>$\chi^2(3) = 61.4$ p&lt;.0001</td>
<td>$\chi^2(4) = 326.04$ p&lt;.0001</td>
<td>$\chi^2(4) = 421.6$ p&lt;.0001</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.5</td>
<td>0.45</td>
<td>0.37</td>
<td>0.85</td>
<td>0.97</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>$\chi^2(2) = 181.6$ p&lt;.0001</td>
<td>$\chi^2(2) = 7.01$ p&lt;.05</td>
<td>$\chi^2(2) = 260.1$ p&lt;.0001</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.63</td>
<td>0.14</td>
<td>0.76</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>(1) R&gt;E (p&lt;.0001)</td>
<td>(1) E&gt;R (p&lt;.0001)</td>
<td>(1) R&gt;E (p&lt;.0001)</td>
<td>(1) E&gt;R (p&lt;.0001)</td>
<td>(1) E&gt;R (p&lt;.0001)</td>
</tr>
<tr>
<td></td>
<td>(2) DV&gt;ALL (p&lt;.0001)</td>
<td>(2) DV&lt;ALL (p&lt;.01)</td>
<td>(2) DV&gt;ALL (p&lt;.0001)</td>
<td>(2) NMP&gt;ALL (p&lt;.0001)</td>
<td>2) NMP&gt;ALL (p&lt;.0001)</td>
</tr>
<tr>
<td></td>
<td>(3) NMP&gt;ALL (p&lt;.001)</td>
<td>(3) NMP&gt;E-NEX (p&lt;.001)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Direction of Age contrasts and p-values</td>
<td>14 to 16 years &gt; 10 to 14 years (p&lt;.0001)</td>
<td>14 to 16 years &lt; 10 to 14 years (p&lt;.0001)</td>
<td>14 to 16 years &gt; 10 to 14 years (p&lt;.0001)</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
</tbody>
</table>

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.
Problem 10 examined students’ ability to interpret algebraic symbolism both structurally and procedurally:

Problem 10: Which of the following expressions are always equivalent? sometimes equivalent? (A) \( b - a + c \); (B) \( c + a - b \); (C) \( c - b + a \); (D) \( c - a + \)

In the English sample, 4% of students in the experimental curriculum, and 1% of students in the non-experimental curriculum gave a correct conditional response. In the Russian sample, 14% of students in the experimental curriculum, and 4% of students in the non-experimental curriculum gave a correct conditional response.

DISCUSSION.

For recognition/use of arithmetical structure, Group and Age had independent effects. Effect sizes of Group and Age were large. Data analysis revealed cultural effects, and a combination of cultural and curricular effects. Cross-cultural differences were profound: Russian groups tended to use mathematical structure, while English groups emphasized calculation (see Figures 21, 22, and 23). The Russian experimental group recognized and/or used arithmetical principles significantly more often than other groups.

The English experimental group relied heavily on computation (see Table 13): the National Mathematics Project group calculated all expressions to determine equivalence (problem 3), and used a procedural approach in an algebraic task (problem 9) significantly more often than other groups.

For use of a correct structural approach in problem 9, Group and Age had independent effects. The effect size of Group was moderate, and the effect size of Age
was large. Analyses revealed cultural effects, and a combination of curricular and cultural effects. Russian groups used a correct structural approach more often than English groups. For within-cultural comparisons, experimental groups gave correct structural responses more often than non-experimental groups.

Analyses did not support Kieran and Sfard's theoretical theses that (a) there is an invariant procedural-to-structural developmental progression in concept acquisition that is "immune to changes in external stimuli," including effects of curriculum; or that (b) structural conceptions of algebraic constructs are rarely achieved. First, the analyses examining students' use of a procedural approach (Figures 26 and 27) show (a) an absence of age effects, and (b) large effect sizes of Group. Second, in Figure 24, irrespective of grade/year level, Russian groups used structural interpretations as often, or more often than procedural interpretations, while the reverse was true for English groups. Third, Figures 6 and 25 suggest structural conceptions are not uncommon.

On the other hand, there was some support for a procedural-to-structural curriculum: NMP students used a correct structural approach on problem 9 more often than the English non-experimental group. There were apparent difficulties, however, in moving NMP students from process interpretations to object-oriented interpretations of algebraic constructs, as illustrated in the interview transcript above.

Algebraic letter interpretation

Problem 1 examined children's ability to interpret letters as generalized numbers:
Problem 1: Decide whether the following statement is always true, sometimes true, or never true. Put a circle around the right answer. If you put a circle around 'sometimes true' explain when this statement is true.

\[ m + n + q = m + p + q \]

(Küchemann, 1981)

Figure 28 compares the percentages of correct responses (i.e., "sometimes true, when \( n=p \)") for the various curricular groups and year/grade levels.

**Figure 28.** Percentages giving a correct response to a task requiring interpretation of letters as generalized numbers (problem 1)
Figure 29 shows the percentages of correct responses for the various curricular groups and age groups.

For correct responses (Figure 29), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 14). The effect size of Group was moderate (0.43) and the effect size of Age was small (0.14). Russian groups gave correct responses significantly more often than English groups (p<.0001). The Russian experimental group gave correct responses significantly more often than other groups (p<.0001), and the English experimental group responded correctly more often than the English non-experimental group.
(p<.0001). Ten to fourteen year-olds responded correctly less often than 14 to 16 year-olds (p<.0001).

Problem 8 examined children's ability to interpret letters as variables:

Problem 8: Which is larger, \(2n\) or \(n+2?\) Explain.

(Küchemann, 1981)

Figure 30 compares the percentages of correct conditional responses (e.g., \(2n\), when \(n>2\)) for the various curricular groups and year/grade levels.

![Figure 30. Percentages giving a correct conditional response to a task requiring interpretation of letters as variables (problem 8)](image_url)
Figure 31 shows the percentages of correct conditional responses for the various curricular groups and age groups.

![Graph showing percentages of correct responses for different groups and age ranges.](image)

Figure 31. Percentages giving correct conditional responses to a task requiring interpretation of letters as variables

For correct conditional responses (Figure 31), the log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 14). The effect size of Group was large (0.53), while the effect size of Age was moderate (0.3). The Russian experimental group gave correct conditional responses significantly more often than other groups (p<.0001), and the English non-experimental group gave correct conditional responses less often than other groups (p<.0001). Ten to fourteen year-olds gave correct conditional responses less often than 14 to 16 year-olds (p<.0001).
Problem 18 examined children's ability to use letters as givens and unknowns:

Problem 18: If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

(Harper, 1987)

For the various curricular groups and age groups, Figure 32 compares the percentages of students setting up a system of two linear equations with use of letters as unknowns and givens in response to this task.

![Graph showing percentages](image)

Figure 32. Percentages setting up a system of two linear equations with use of letters as unknowns and givens in response to problem 18

The log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 14). The effect size of Group was
moderate (0.45), while the effect size of Age was large (0.83). Russian groups formulated a system of linear equations with use of letters as unknowns and givens more often than English groups (p<.0001). The Russian experimental group formulated a system of linear equations with use of letters as unknowns and givens more often than other groups (p<.001). Ten to fourteen year-olds used letters as unknowns and givens less often than 14 to 16 year-olds (p<.0001).

Problem 2 examined (a) children's ability to interpret letters as generalized numbers, and (b) whether children would give whole number responses, or non-whole number solutions as well:

Problem 2: What can you say about \( c + f + c = 10 \) and \( c \) is less than \( d \)?

(Küchemann, 1981)

For the various curricular groups and year/grade levels, Figure 33 compares the percentages of students giving continuous solution sets for \( c \) (e.g., "c<5"), versus percentages giving "whole number responses" (e.g., "c=1, 2, 3, 4").
Figure 33. Percentages giving continuous versus discrete solution sets for $c$ in response to problem 2

For the various curricular groups and age groups, Figure 34 shows the percentages of responses giving a continuous solution set.
Figure 34. Percentages giving a continuous solution set (e.g., $c<5$) in response to problem 2

The log-linear model with main effects of Group and Age fit the data. Effects of Group and Age were significant (see Table 14). The effect size of Group was moderate (0.37), while the effect size of Age was large (0.53). Russian groups responded in terms of continuous solution sets more often than English groups ($p<.0001$). The Russian experimental group gave continuous solution sets more often than other groups ($p<.0001$). Ten to fourteen year-olds' responses involved continuous intervals less often than 14 to 16 year-olds' ($p<.0001$). Whole number versus real number thinking has important implications for algebraic thought.
Table 14. Algebraic letter interpretation

<table>
<thead>
<tr>
<th></th>
<th>Generalized number: correct response (Problem 1)</th>
<th>Generalized number: Continuous solution set (Problem 2)</th>
<th>Variable: correct conditional response (Problem 8)</th>
<th>Use of letters as unknowns and givens (Problem 18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model fit</td>
<td>Main effects of Group and Age</td>
<td>Main effects of Group and Age</td>
<td>Main effects of Group and Age</td>
<td>Main effects of Group and Age</td>
</tr>
<tr>
<td>Goodness of fit</td>
<td>$\chi^2(3) = 2.93$, $p=0.4029$</td>
<td>$\chi^2(3) = 3.63$, $p=0.2800$</td>
<td>$\chi^2(3) = 2.26$, $p=0.52$</td>
<td>$\chi^2(3) = 3.31$, $p=0.3459$</td>
</tr>
<tr>
<td>Effects due to Group</td>
<td>$\chi^2(3) = 76.6$, $p&lt;0.0001$</td>
<td>$\chi^2(3) = 61.01$, $p&lt;0.0001$</td>
<td>$\chi^2(3) = 127.2$, $p&lt;0.0001$</td>
<td>$\chi^2(3) = 91.6$, $p&lt;0.0001$</td>
</tr>
<tr>
<td>Effect size of Group</td>
<td>0.43</td>
<td>0.37</td>
<td>0.53</td>
<td>0.45</td>
</tr>
<tr>
<td>Effects due to Age</td>
<td>$\chi^2(2) = 8.46$, $p&lt;0.025$</td>
<td>$\chi^2(2) = 125.7$, $p&lt;0.0001$</td>
<td>$\chi^2(2) = 40.4$, $p&lt;0.0001$</td>
<td>$\chi^2(2) = 313.6$, $p&lt;0.0001$</td>
</tr>
<tr>
<td>Effect size of Age</td>
<td>0.14</td>
<td>0.53</td>
<td>0.3</td>
<td>0.83</td>
</tr>
<tr>
<td>Effects due to interaction</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Effect size of interaction</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
<td>N.S.</td>
</tr>
<tr>
<td>Direction of Group contrasts and p-values</td>
<td>(1) R&gt;E $,(p&lt;0.01)$</td>
<td>(1) R&gt;E $,(p&lt;0.0001)$</td>
<td>(1) DV&gt;ALL $,(p&lt;0.0001)$</td>
<td>(1) R&gt;E $,(p&lt;0.0001)$</td>
</tr>
<tr>
<td></td>
<td>(2) DV&gt;ALL $,(p&lt;0.0001)$</td>
<td>(2) DV&gt;ALL $,(p&lt;0.0001)$</td>
<td>(2) E-NEX&lt;ALL $,(p&lt;0.0001)$</td>
<td>(2) DV&gt;ALL $,(p&lt;0.001)$</td>
</tr>
<tr>
<td></td>
<td>(3) NMP&gt;E-NEX $,(p&lt;0.0001)$</td>
<td></td>
<td></td>
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<tr>
<td>Direction of Age contrasts and p-values</td>
<td>14 to 16 years &gt; 10 to 14 years $,(p&lt;0.0001)$</td>
<td>14 to 16 years &gt; 10 to 14 years $,(p&lt;0.0001)$</td>
<td>14 to 16 years &gt; 10 to 14 years $,(p&lt;0.0001)$</td>
<td>14 to 16 years &gt; 10 to 14 years $,(p&lt;0.0001)$</td>
</tr>
</tbody>
</table>

NOTE: E = both English groups (NMP and E-NEX). R = both Russian groups (DV and R-NEX). ALL = all other curricular groups.

DISCUSSION.

For use of letters as generalized numbers, variables, and givens, Group and Age had independent effects. The effect size of Group was moderate for use of letters as generalized numbers and givens, and large for variables. For generalized number,
the effect size of Age was small (0.14). For interpretation of letters as variables, the
effect size of Age was moderate. For givens, the effect size of Age was large.

For interpretation of letters as generalized numbers and givens, analyses
suggested cultural effects, and a combination of curricular and cultural effects. For
interpretation of letters as variables, the analysis suggested curricular effects. Russian
groups interpreted/used letters as generalized numbers and givens significantly more
often than English groups. The Russian experimental group was more likely to
acquire/use letters as generalized numbers, variables, and givens than other groups.
The English experimental group was more likely to acquire/use letters as generalized
numbers and variables than the English non–experimental group.
CHAPTER V
CONCLUSIONS AND GENERAL DISCUSSION

This study was designed to provide a clearer understanding of the development, and mechanisms of algebraic reasoning by (1) identifying sources of variation affecting algebraic reasoning; (2) linking the sources of variation to specific cognitive processes and cognitive outcomes involved in algebraic reasoning; (3) determining whether sources of variation affect algebraic reasoning independently or jointly; and (4) assessing the relative contributions of those factors, and/or their interactions, to observed variation in algebraic reasoning.

The study examined the following research questions: (1) Do students exhibit differences in acquiring algebraic letter interpretations, structure, and reasoning across cultures, curricula, and developmental lines?; (2) If socio-cultural and developmental variables affect algebraic reasoning, do they affect it independently or jointly?

Cultural, curricular, age, and contextual effects

As discussed in the data analysis section in Chapter 3, the data were analyzed with respect to Group and Age. Four groups were used in the log-linear analyses: Russian non-experimental curriculum, Davydov's curriculum, English non-experimental curriculum, and the National Mathematics Project curriculum. Subjects within each of the four curricular groups were divided into two age groups: 10 to 14 years, and 14 to 16 years.
Since four curricular groups were used in the log-linear analyses, cultural and curricular effects could not be distinguished. The following approach was used. If significant differences existed in contrasting the two Russian groups with the two English groups on some measure, it was interpreted as a cultural effect. If a group significantly differed from the other three groups on some measure, with no accompanying cross-cultural difference (i.e., no significant difference between Russian and English groups on that measure), then it was interpreted as a curricular effect. If a group significantly differed from the other three groups on some measure, with an accompanying difference between English and Russian groups, findings were interpreted as being due to a combination of curricular and cultural effects.

Analyses revealed profound cross-cultural differences. Russian groups were more likely to use algebraic deductive arguments; to use algebra as a tool for reasoning; to formulate correct algebraic equations; to use arithmetical structure; to simplify and manipulate algebraic expressions correctly; and to acquire concepts of generalized numbers and givens. English groups were more likely to use inductive, probabilistic numerical arguments when reasoning independently on proof tasks; to believe deductive algebraic proof cannot establish "universal validity" (to believe algebraic proof requires empirical support); to compute, rather than use arithmetical structure; and to use only procedural (numerical input–output) interpretations of algebraic constructs.

Data analyses revealed profound cross-curricular differences. Some of the following curricular effects were coupled with cultural effects (see discussion sections in Chapter 4). In comparison with other curricular groups, Davydov's group was more likely to use algebraic deductive arguments; to use algebra as a tool for reasoning; to believe algebraic proof establishes "universal validity"; to formulate
correct algebraic equations; to use arithmetical structure; to simplify and manipulate algebraic expressions correctly; and to acquire concepts of generalized numbers, variables, and givens. The group was less likely to use inductive, probabilistic numerical arguments on proof tasks. Differences between Davydov's group and other groups tended to increase with children's age—suggesting effects of instruction tend to increase with age (see Figures 3, 6, 9, 12). Though age is an important contributing factor in development of algebraic reasoning, comparison of younger and older children's responses across groups suggests socio-cultural factors can amplify development of algebraic reasoning—over-shielding effects of age.

In comparison with other curricular groups, the NMP group was more likely to use inductive, probabilistic numerical arguments on proof tasks; to believe algebraic proof requires empirical support; to compute, rather than use arithmetical structure; and to use only procedural interpretations of algebraic constructs.

In comparison with the English non-experimental group, the NMP group was more likely to acquire concepts of generalized numbers and variables; to formulate correct algebraic equations; and to simplify and manipulate algebraic expressions correctly.

Abstract reasoning differed across tasks. Given tasks with essentially the same logical structure, Russian and English children were more likely to apply abstract logical principles in a verbal reasoning task that seemed to require only transformational components of deductive reasoning, than in a verbal reasoning task that appeared to require both meta-components (e.g., understanding of logical necessity) and transformational components of deductive reasoning (see Sloutsky & Morris, 1995).
Algebraic reasoning measures included the following: use of algebraic proof; use of algebra as a tool for reasoning in proof tasks; belief that algebraic proof establishes "universal validity"; formulation of algebraic equations; use of arithmetical structure; correct simplification and manipulation of algebraic expressions; and acquisition of generalized numbers, variables, and givens. For all algebraic reasoning measures, Age and Group appeared to have independent effects. (An interaction term was present in one of the two fitted models for use of algebraic proof; however, the effect size of the interaction was small (0.13).)

Logical deductive reasoning was significantly affected by age; cultural and curricular effects were less pronounced as the effect size of Group was low. For the age range of 10–16 years, acquisition of generalized number and formulation of algebraic equations appears to be primarily affected by cultural and/or curricular variables. Effect sizes of Group and effect sizes of Age were moderate to large for use of algebraic proof; use of algebra as a tool for reasoning; belief that algebraic proof establishes "universal validity"; use of arithmetical structure; correct simplification and manipulation of algebraic expressions; and acquisition of concepts of variables and givens. Age-related models for acquisition of algebraic letter concepts were partially supported: the effect size of age was low for acquisition of generalized number, moderate for variable, and high for givens.

Testing the models of cognitive performance and algebraic reasoning

Cultural and curricular effects on cognition, in general, and algebraic reasoning, in particular, were predicted on the basis of Naglieri and Sloutsky's (1994) model of cognitive performance. Specifically, the model led to the following
prediction: cultural and curricular effects on algebraic reasoning can overshadow effects of age. Findings supported the validity of this prediction.

In order to examine specific aspects of algebraic reasoning, we modeled algebraic reasoning as a cognitive process that requires (1) component understandings (e.g., concepts of individual–class relationships, allowing interpretation of algebraic letters; mathematical structure that remains invariant when elements change; real number); and (2) inductive and deductive reasoning. In algebraic deductive reasoning, the reasoner (a) creates abstract objects as givens, that are then operated upon; (b) evaluates the truth or falsity of a given or developed set of statements that require hypothetico–deductive reasoning, or assumes the truth of premises that in fact he/she knows to be false (e.g., reductio ad absurdum); and (c) applies rules of transformation. Rules of transformation include propositional operations (abstract rules of deductive logic), and operations specific to algebraic reasoning such as the ability to manipulate algebraic symbols (e.g., algebraic simplification, factoring). In assessing the truth or falsity of statements and in applying rules of transformation, the reasoner adheres to a defined set of rules: assessing the validity of statements and transformations based on properties of quantities, relationships between quantities, principles, syntactic conventions, etc.

The two models led to the following prediction: Cultural and curricular variables would affect specific components of algebraic reasoning (e.g., emphasis on induction versus deduction, understanding of logical necessity), resulting in between-group differences in algebraic reasoning. In fact, English groups tended to use inductive, empirical reasoning in tasks requiring deductive mathematical arguments; operated on particular single objects in proof tasks; and believed deductive algebraic
proof was evidence and numerical evidence was proof. Russian groups tended to use deductive algebraic reasoning.

The theoretical models suggest candidate factors that may have affected this preference for different modes of reasoning. They include: (1) curricular variables; (2) socio-cultural variables, including children's beliefs (e.g., children's trust in deduction; beliefs as to what constitutes evidence in mathematics); cultural variables (e.g., emphases on inductive versus deductive modes of reasoning; different cognitive tools and representational systems); parents' and teachers' expectations and beliefs (e.g., expectations related to children's ability to engage in deductive reasoning; expectations and beliefs regarding mathematics achievement; types of mathematics competencies desired and fostered); family processes (e.g., parental teaching of formal mathematics); and (3) formation of the various components of algebraic reasoning.

This study did not specifically examine the socio-cultural variables mentioned above. However, curricular variables, and the formation of various components of algebraic reasoning, were closely examined.

One potential explanation for between-group differences is the following: curricula in England tend to emphasize inductive case-based reasoning (investigating a number of particular instances to formulate, and to assess the validity of generalizations), while curricula in Russia tend to emphasize deductive law-based reasoning (specifying laws and regularities, and deriving particular cases as specific instances of those laws and regularities). This difference was even more pronounced in the experimental curricula.

For example, Bell (1976) writes: "In England, proofs of geometrical theorems have been steadily disappearing from O-level syllabuses for thirty years, and 'it continues to be the policy of the SMP to argue the likelihood of a general result from
particular cases' (Preface to Book 5). Underlying this divergence in practice lies the
tension between the awareness that deduction is essential to mathematics, and the fact
that generally only the ablest school pupils have achieved understanding of it" (Bell,
1976, p. 23). Tall and Thomas (1991) write: "In the UK, the National Curriculum
now makes fewer demands on algebra for 16 year olds than was previously the case,
and there are moves elsewhere to reduce formal algebra by using more numerical
problem-solving" (p. 127).

In the Russian non-experimental curriculum, the sixth grade course "is
structured upon induction, but . . . [has] recourse to elements of deductive
reasoning", while the courses for grades 7–9 are "characterized by the enhancement of
the theoretical level of instruction and by the stronger emphasis . . . gradually placed
upon the role of theoretical generalizations and deductions" (USSR Academy of
Pedagogical Sciences' Scientific Research Institute of Curriculum and Teaching

Across-group differences in acquisition of algebraic deductive reasoning
reflected the curricular emphases: English groups tended to use inductive arguments;
Russian groups tended to use deductive arguments. In comparison with other
curricular groups, Davydov's students were more likely to use deductive, law–based
reasoning, and NMP students were more likely to use inductive, empirical reasoning.
Responses among the Russian non-experimental group showed more of a
"developmental mix" (see Figures 5, 6, and 7).

Emphasis on numerical, inductive reasoning was highest among the NMP
group. In comparing the NMP and Russian non-experimental groups, 14 to 16 year–
olds were similar in terms of their abilities to (a) formulate algebraic equations, and (b)
interpret algebraic letters as variables and generalized numbers. Yet, they were
dissimilar in their use of (a) algebraic proof; and (b) arithmetic structure. This suggests that for many NMP students, emphasis/reliance on empirical reasoning stems, not from a lack of component understandings of algebraic letters and equations, but rather from (1) deeply ingrained process interpretations of algebraic constructs (i.e., numerical input–output interpretations); and (2) inability to operate at the level of structure or relationship in proof tasks. Given (1) and (2), algebraic expressions are, at best, templates for generating numerical inductive arguments.

In comparison with the English non–experimental group, the English experimental group was more likely to acquire concepts of generalized numbers and variables; to formulate correct algebraic equations; and to simplify and manipulate algebraic expressions correctly. Hence, the approach developed meaning for algebraic letters, expressions, and equations. However, there was little evidence that prolonged emphasis on numerical input–output interpretations of algebraic constructs develops (a) algebraic deductive reasoning; or (b) use of structure. Percentages of NMP students using numerical, empirical reasoning were similar for both age levels, or increased with age—i.e., with exposure to this approach (see Figures 4, 7, 10, 13, 23, 26, 27). Prolonged emphasis on empirical reasoning—if not accompanied by some kind of systematic movement toward discerning and using structure—promotes empiric, rather than theoretic thought (Hatano, Morita, & Inagaki, 1995). Using numerical reasoning, children attempted to establish "whether a generalization worked," rather than "why it worked."

To evaluate the truth or falsity of a given or developed set of statements that required hypothetico–deductive reasoning, the NMP group substituted numbers to make sense of algebraic statements, i.e., tested particular cases. Russian groups operated at a different level in evaluating the truth or falsity of statements—operating at
the level of relationship or structure. This was particularly evident among Davydov's group.

Differences between Davydov's group and other groups were most pronounced for 14 to 16 year-olds. To identify candidate curricular variables that might account for the pronounced difference in performance between Davydov's graduates and other older groups, it seemed reasonable to search for "precursors" among 10 to 14 year-olds currently experiencing Davydov's curriculum. That is, were there any qualities/characteristics/skills that might (a) distinguish Davydov's younger group, and (b) predict the later gap in performance? Performances of the two Russian 10 to 14 year-old samples were therefore compared.

For 10 to 14 year-olds, comparisons revealed no differences between the Russian experimental and non-experimental groups in use of proof. A similar proportion of younger children acquired the ability to formulate algebraic proofs. However, there were significant differences for the two younger samples for the following measures: acquisition of generalized numbers; formulation of algebraic equations; acquisition of variables; and use of arithmetical structure. Since 10 to 14 year-olds in the Russian non-experimental group had received less instruction in algebra, differences in the first two measures are, perhaps, not surprising. Performance on two measures, however, cannot be explained merely in terms of amount of exposure to algebra. Davydov's fifth and sixth graders (1) tended to use arithmetic structure to determine the equivalence of numerical expressions, rather than computing; and (2) demonstrated an ability to consider a second order relationship in a variable task (i.e., on Küchemann's task, Which is larger, $2n$ or $n+2$?). What seemed to differentiate Davydov's younger group, then, was their ability to consider relationship or structure.
Assuming groups were of comparable intelligence, this suggests direct and systematic development of notions of *structure* and *relationship* (rather than developing these notions as a derivative of numerical work) is an important candidate variable. Approximately 70% of Davydov's graduates operated at the level of structure—writing proofs, demonstrating an understanding of the nature of algebraic proof—while only a small elite group of *NMP* students acquired this ability. Findings supported Davydov's contention that (the majority of) children are unable to appropriate theoretical generalizations (characteristic of mathematics) through empirical abstraction—to abstract essential, i.e., necessary and sufficient attributes by observing particular phenomena. "It is known that scientific knowledge is not a simple extension, intensification, and expansion of people's everyday experience. It requires the cultivation of particular means of abstracting, a particular analysis, and generalization, which permits the internal connections of things, their essence, and particular ways of idealizing the objects of cognition to be established" (Davydov, 1990, p. 86).

Candidate curricular variables, potentially contributing to between-group differences, include the following:

1) Theoretical versus empirical learning: Davydov's approach emphasizes theoretical learning, while *NMP* emphasizes empirical learning. In empirical learning, students observe and compare a number of objects, identify their common observable characteristics, and then, by an act of generalization, formulate a general concept about the class of objects. In theoretical learning, the student is supplied with general and "optimal" methods for handling certain classes of problems, that direct him/her toward essential (not simply common) characteristics of the problems of each class. These general methods are then used to solve concrete problems. Through specially
organized activity, students master and internalize the processes of use of these methods (Karpov & Bransford, 1995).

(2) Order, and means of acquiring concepts: In Davydov's curriculum, concepts of "relation or structure" (e.g., of the structural characteristics of mathematical objects, such as equalities) are developed and emphasized (a) prior to numerical work, and (b) prior to emphasis on algebraic transformations, such as algebraic simplification and factoring. Students then identify this structure in numerical contexts—recognize that properties of quantities, operations, and quantitative relationships hold in all particular cases. In NMP, numerical work and numerical input–output interpretations of algebraic constructs are used to develop (a) concepts of structure, and (b) meaning for algebraic transformations.

(3) Actions on objects, graphic models, and algebraic symbols: To assist children in isolating essential properties and relationships in the analysis of concrete problems, Davydov uses actions on objects; graphic models that depict abstract relationships in sensory–visual form, allowing further analysis; and algebraic letters. Objects, models, and symbols are used to develop the following notion: It is the isolation, and specification of the underlying quantitative relationship that is of interest—rather than the objects or particular instances through which the relationship manifests itself.

(4) Age of introduction, and amount of exposure to algebra: Davydov (1990) asserts "the abstract as an element of thought should be introduced into instruction as early as it can be accessible to the child, who must not be held too long at the stage of sensory impressions, in any case" (p. 319); and suggests formal abstract symbols are essentially related to mathematical reasoning, to abstraction and generalization and deduction. Hence, algebraic symbols are introduced in primary school, when children
are approximately 7 years old. In the other curricula, algebraic symbols are introduced when children are approximately 10–13 years old.

With respect to applying rules of transformation, English groups were able to apply abstract rules of deductive logic: on a deductive reasoning task, English groups gave correct responses more often than Russian groups (see Figure 15). However, English groups were not as skilled in performing transformations specific to algebraic reasoning (e.g., algebraic simplification). Moreover, the function, or role, of algebraic manipulations and simplifications in proof remained unclear—children often performed computations to generate particular cases, rather than isolating and clarifying underlying relationships. This may be related to two factors: (1) English groups did not seem to attend to/use properties of quantities, relationships between quantities, principles, etc.; and (2) \( NMP \) students tended to interpret algebraic constructs procedurally.

Skill in manipulating algebraic symbols—prerequisite for using algebra as a tool for deduction—was more evident among Russian groups. Candidate curricular variables include the following: extensive development of concepts, and demanding arithmetic training in Russian curricula, preparing the majority of students for algebra (see, e.g., Stigler, Fuson, Ham, & Kim, 1986).

In sum, analyses revealed the following:

1. Culture, curriculum, and age affect cognitive outcomes and cognitive processes involved in algebraic reasoning.
2. Across-group differences in algebraic reasoning reflected the curricular emphases.
3. These across-group differences in reasoning tended to increase with age.
4. There are separate components in algebraic reasoning, including use of various letter interpretations, use of structure, use of proof, and understanding of the logical
necessity of deductive conclusions derived from algebraic arguments. The curricula had different effects on the development of the various components of algebraic reasoning.

(5) Although age effects are important, curricular effects on algebraic reasoning can overshadow effects of age.

**Future studies**

This study represented the first phase of a long-term program of research. This phase was intended to establish and specify theoretically predicted cultural, curricular, and age-related effects. To be attributed to specific variables, these findings have to be replicated with rigorous measurement and control of candidate variables.

The next phase will examine whether cross-cultural, cross-curricular, and cross-contextual differences in algebraic reasoning are replicable phenomena; and will simultaneously attempt to eliminate potential confounds. It seems particularly important to control the SES status and intelligence levels of children in all curricular groups. Future studies will attempt to identify (1) specific factors affecting algebraic reasoning, and (2) the relative contributions of these factors.
APPENDIX A

WRITTEN TEST INSTRUMENT (ENGLISH VERSION)
Complete all the items that you know how to do.

1) Decide whether the following statement is always true, sometimes true, or never true. Put a circle around the right answer. If you put a circle around 'sometimes true' explain when this statement is true.

\[ m + n + q = m + p + q \]

Always true

Never true

Sometimes true that is when ____________________________

2) What can you say about \( c \) if \( c + d = 10 \) and \( c \) is less than \( d \)?

3) Which of the following expressions are equivalent to \( 5 \times (6 + 3) \)? (Circle them.)

A) \( 5 \times 6 + 5 \times 3 \)  
B) \( 5 \times (3 + 6) \)  
C) \( 5 + 6 \times 3 \)  
D) \( 5 \times 3 + 5 \times 6 \)  
E) \( (5 \times 6) + 3 \)

Explain what you did to figure this out.
4) A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right.

5) Is the following statement always true? sometimes true? never true? Say how you know.

\[(a^2 + b^2)^3 = a^6 + b^6\]
6) If \( e + f = 8 \), then \( e + f + g = \) _______? \\

7) Write an equation using the letters \( S \) and \( T \) to represent the following statement: 
"There are six times as many students as teachers at this school." 
Use \( S \) for the number of students and \( T \) for the number of teachers.

8) Which is larger, \( 2n \) or \( n + 2 \)? Explain.

9) What can \( x \) equal in the following equation?

\[
\frac{12x - 2x + 16}{2} = 2(x + 4) + 3x
\]

Explain.

10) Which of the following expressions are always equivalent? sometimes equivalent?

A) \( b - a + c \)  
B) \( c + a - b \)  
C) \( c - b + a \)  
D) \( c - a + b \)  

Explain.
11) In an algebra class the teacher proved that every whole number of the form \( n^3 - n \) is divisible by 6 (that is, if you divide \( n^3 - n \) by 6, there will be no remainder). The proof was as follows:

We can write:

\[ n^3 - n = n(n^2 - 1) \]

But we can rewrite the expression on the right:

\[ n(n^2 - 1) = n(n - 1)(n + 1) \]

So:

\[ n^3 - n = n(n - 1)(n + 1) = (n - 1)n(n + 1) \]

But \( (n - 1)n(n + 1) \) is a product of three consecutive whole numbers. Therefore, one of them should be divisible by 2, and one of them (not necessarily a different one) should be divisible by 3. Thus their product should be divisible by \( 2 \times 3 \), that is, by 6.

Please answer the following:

(A) I understand all the details of the proof and the proof seems correct to me.

Yes / No (Circle one)

(B) There are some details in the proof that I do not understand. They are the following:

(C) If you think the teacher has given a correct proof for the theorem "every whole number of the form \( n^3 - n \) is divisible by 6", then answer the following question:

Do you think that further checks (by substituting numbers) are necessary in order to verify the validity of the theorem?

Yes / No (Circle one)

Explain.
(D) Victor is a doubter. He thinks that we have to check at least a hundred numbers in order to be sure that the theorem is correct. What is your opinion? Explain your answer.

12) Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If \( b \) is the number of blue pencils bought and if \( r \) is the number of red pencils bought, write an equation involving \( b \) and \( r \).

13) Pick a number, but do not tell me what it is. Now add your number to 8, then subtract your starting number from the total. What did you get? Why? Do you think it would work for any starting number? Use algebra to show it.

14) Mary's basic wage is 20 dollars per week. She is also paid another 2 dollars for each hour of overtime that she works. If \( h \) stands for the number of hours of overtime that she works, and if \( W \) stands for her total wage (in dollars), write down an equation connecting \( W \) and \( h \).

15) Which of the following expressions are equivalent to \( w \times (k + q) \)? (Circle them.)

A) \( w \times k + w \times q \)  B) \( w \times (q + k) \)  C) \( (w \times k) + q \)  D) \( w \times q + w \times k \)  E) \( k \times w + q \)

Explain how you figured this out.
16) For the rectangle below, think about increasing the number for $\kappa$. What happens to the shape of the rectangle as $\kappa$ increases from zero to very big numbers? For example, does the rectangle become tall and thin? Does it become square-shaped? What happens? Draw a picture that shows what happens to the shape of the rectangle as $\kappa$ increases from zero to very big numbers.

17) Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . Now do it with another three consecutive numbers . . . What happens? Can you prove it will always work?
18) If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

19) For the following problem:

Assume the first two sentences (in bold) are true. Make a conclusion from the assumptions. (Choose a, b, c, or d.)

**All fahmooth numbers can be divided evenly by 8.**

26 is a fahmooth number.

Therefore . . .

a) 26 must not be a fahmooth number.
b) 26 is an exception to the rule.
c) It is probably true that fahmooth numbers cannot be divided evenly by 8.
d) 26 can be divided evenly by 8.

20) Which of the following expressions are equivalent?

A) 685 – 492 + 947  
B) 947 + 492 – 685  
C) 947 – 685 + 492  
D) 947 – 492 + 685

Explain how you figured this out.
21) Cakes cost \( c \) pence each and buns cost \( b \) pence each. If I buy 4 cakes and 3 buns, what does \( 4c + 3b \) stand for?

22) Part of this figure is not drawn. There are \( n \) sides altogether. All sides have length 2. What is the perimeter of the figure? ________________

23) Choose any number between 1 and 10. Write your number in this box:

Add the number in the box to 10 and write down the answer here: ________.
Now take the number in the box, subtract it from 10, and write down the answer here: ________.
Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.
24) In this question you just have to tell whether or not the sentences show correct reasoning. All of the sentences are really nonsense, but you are to think only about the reasoning. Here is a practice question:

(a) If all trees are fish and all fish are horses, then all trees are horses.  
   \[ \text{YES} \quad \text{NO} \]
   This sentence shows good reasoning even though what it says is nonsense. You have two choices, Yes or No. The first choice means yes, the reasoning is good. The second choice means no, the reasoning is not good. Since sentence (a) shows good reasoning, the first choice, Yes, has been underlined for sentence (a).

Here is another practice question:

(b) If all trees are fish and all fish are horses, then all horses are trees.
   \[ \text{YES} \quad \text{NO} \]
   This sentence does not show good reasoning. If all trees are fish and all fish are horses, then all trees are horses — but that does not mean that all horses are trees. There could be some horses that are not trees. Since sentence (b) shows poor reasoning, the answer No has been underlined for sentence (b).

Remember, underlining Yes means the reasoning is good; underlining No means the reasoning is not good.

Please choose Yes or No for each sentence:

(1) If all birds have purple tails and all cats are birds, then all cats have purple tails.
   Yes \quad \text{No}

(2) If all cars have sails and some swimming pools are cars, then some swimming pools have sails.
   Yes \quad \text{No}

(3) If no skunks have green toes and all skunks are pigs, then no pig has green toes.
   Yes \quad \text{No}

(4) If all horses have wings and no turtle has wings, then no turtle is a horse.
   Yes \quad \text{No}

(5) If some men are purple and everything which is purple is a horse, then some horses are men.
   Yes \quad \text{No}
APPENDIX B

WRITTEN TEST INSTRUMENT (RUSSIAN VERSION)
Ответьте, пожалуйста, на следующие вопросы.

(1) Подумайте над следующим уравнением:

\[ m + n + q = m + p + q \]

Всегда ли оно верно, только иногда верно или никогда не верно? Подчеркните правильный ответ. Если вы считаете, что уравнение хотя бы иногда верно, объясните, когда именно оно верно.

А. Всегда верно
Б. Никогда не верно
В. Верно иногда, когда __________________________________________________________________________________________

(2) Что можно сказать про с, если \( c + d = 10 \), но с меньше, чем \( d \)?

(3) Какие из нижеследующих выражений эквивалентны выражению \( 5 \times (6 + 3) \)? (Подчеркните их).

А. \( 5 \times 6 + 5 \times 3 \)  Б. \( 5 \times (3 + 6) \)  В. \( 5 + 6 \times 3 \)  Г. \( 5 \times 3 + 5 \times 6 \)  Д. \( (5 \times 6) + 3 \)

Объясните, как Вы рассуждали при решении этой задачи.
(4) Девочка умножает число на 5, а потом прибавляет к произведению 12. После этого она вычитает от суммы первоначальное число и делит результат на 4. Она обнаруживает, что полученное число на 3 больше, чем число, с которого она начала. Она говорит, что так получится всегда, с какого числа я бы не начала. Права ли она? Почему Вы считаете, что Ваш ответ верен?

(5) Подумайте над следующим уравнением:

\[(a^2 + b^5)^3 = a^6 + b^8\]

Всегда ли оно верно, только иногда верно или никогда не верно? Объясните, как вы получили ответ.
(6) Если $e + f = 8$, то $e + f + g =$________. 

(7) Напишите уравнение, используя буквы $Ш$ и $У$, представляющее следующее предложение: "В этой школе в шесть раз больше школьников, чем учителей". Используйте $Ш$ для обозначения числа школьников, а $У$ для обозначения числа учителей.

(8) Что больше: $2n$ или $n + 2$? Объясните.

(9) Чему может равняться $x$ в следующем уравнении?

\[
12x - 2x + 16 = 2(x + 4) + 3x
\]

\[
2
\]

Объясните.
(10) Какие из следующих выражений всегда эквивалентны? Иногда эквивалентны?
A) $b - a + c$
B) $c + a - b$
B) $c - b + a$
Г) $c - a + b$
Объясните.

(11) На уроке алгебры учитель доказал, что любое целое число, которое можно представить в виде $n^3 - n$, делится на 6 без остатка. Доказательство было следующее:
Мы можем записать, что:
$$n^3 - n = n(n^2 - 1)$$
Теперь мы можем записать правую часть уравнения в виде:
$$n(n^2 - 1) = n(n - 1)(n + 1), \text{ так что } n^3 - n = n(n - 1)(n + 1) = (n - 1)n(n + 1)$$
Но $(n - 1)n(n + 1)$ -- произведение трех последовательных целых чисел. Следовательно одно из них должно делиться на 2, а другое (или то же самое) -- должно делиться на 3. Соответственно их произведение должно делиться на $2 \times 3$, т.е. на 6.
Пожалуйста ответьте на следующие вопросы:
(A) Мне всё понятно в доказательстве, и оно мне кажется верным.  
Да  \ Нет  (обведите)

(B) В доказательстве мне не понятно следующее:

(B) Если вы считаете, что учитель верно доказал теорему о том, что "любое целое число, которое можно представить в виде $n^3 - n$, делиться на 6", то ответьте на следующий вопрос:
Считаете ли вы, что справедливость теоремы необходимо проверить (подстановкой чисел)?  
Да  \ Нет  (обведите)
(Г) Петя всегда и во всем сомневается. Он считает, что мы должны проверить теорему, по крайней мере, на 100 числах, чтобы быть уверенным в ее правильности. Что вы думаете по этому поводу? Объясните свой ответ.

(12) Синие карандаши стоят по 5 рублей каждый, красные - по 6 рублей каждый. Я покупаю по несколько синих и красных карандашей, и всё вместе стоит 90 рублей.
Если М - количество синих карандашей, а N - количество красных карандашей, купленных мною, составьте уравнение с М и N.

(13) Загадайте число, но не называйте его мне. Теперь прибавьте задуманное число к 8, потом отнимите от суммы задуманное число. Что у вас получилось? Почему? Считаете ли вы, что так будет при любом задуманном числе? Покажите это с помощью алгебраического уравнения.
(14) Зарплата Марии 20 долларов в неделю. Кроме того, она получает по 2 долларов за каждый час сверхурочного времени. Если M - количество сверхурочных часов, которые она работала, а N - её заработок в целом, запишите уравнение с M и N.

(15) Какие из следующих выражений эквивалентны \( w \times (k + q) \)? (Обведите их).

A) \( w \times k + w \times q \)  Б) \( w \times (q + k) \)  В) \( (w \times k) + q \)  Г) \( w \times q + w \times k \)  Д) \( k \times w + q \)

Объясните, как вы рассуждали при решении этой задачи.

(16) Имея в виду прямоугольник, изображённый внизу, представьте себе, что число \( k \) увеличивается. Как изменяется форма прямоугольника с увеличением \( k \) от нуля до очень больших величин? Например, становится ли прямоугольник высоким и узким? Становится ли он квадратом? Что с ним происходит? Сделайте рисунок, показывающий, что происходит с формой прямоугольника по мере роста \( k \) от нуля до очень больших величин.
(17) Возьмите три последовательных числа. Теперь вычислите квадрат среднего из них и отнимите от него произведение крайних.... Теперь сделайте то же самое для трёх других последовательных чисел. Что получилось? Можете ли вы доказать это в общем виде?

(18) Если вы знаете сумму и разность любых двух чисел, докажите, что вы всегда можете найти эти числа.
(19) При решении следующей задачи считайте, что первые два (выделенные) утверждения верны. Сделайте вывод из этих утверждений. (Выберите А, Б, В, или Г.)
Все композитные числа делятся на 8 без остатка.
26 - композитное число.
Следовательно:
А. Должно быть 26 -- не композитное число
Б. 26 -- исключение из правила
В. Наверное не все композитные числа делятся без остатка на 8
Г. 26 делится без остатка на 8.

(20) Какие из следующих выражений эквивалентны?
А. 685 - 492 + 947
Б. 947 + 492 - 685
В. 947 - 685 + 492
Г. 947 - 492 + 685
Объясните, как Вы рассуждали при решении этой задачи.
(21) Пирожные стоят М рублей за штуку, а булочки - Н рублей за штуку. Если купить 4 пирожных и 3 булочки, то что будет означать выражение 4M + 3N?

(22) Часть этой фигуры не нарисована. Всего в фигуре N сторон. Каждая из сторон имеет длину 2.

Чему равен ее периметр?

(23) Выберите любое число между 1 и 10. Впишите число в рамку:

Прибавьте 10 к числу в рамке и запишите ответ _____.
Теперь возьмите число в рамке, вычтите его из 10 и запишите ответ _____.
Сложите оба полученных ответа. Сколько у Вас получилось? Будет ли ответ один и тот же, какое бы число ни было в рамке? Приведите доказательство своего ответа.
(24) В этом вопросе Ваша задача -- определить правильность приведенных рассуждений. Каждое предложение утверждает бессмыслицу. Но попытайтесь думать только о правильности рассуждений, а не о смысле. Например, утверждение

Если все деревья -- рыбы, а все рыбы -- лошади, то все деревья -- лошади

содержит правильное рассуждение, в то время, как утверждение

Если все деревья -- рыбы, а все рыбы -- лошади, то все лошади -- деревья

содержит неправильное рассуждение. Действительно, давайте применим то же рассуждение к другому материалу. "Если все ученики Вашей школы -- школьники, а все школьники -- люди, то все люди -- ученики Вашей школы" -- пример неправильного рассуждения.

Выберите Да (для правильного рассуждения) или Нет (для неправильного рассуждения) для каждого из нижеследующих утверждений:

(1) Если у всех птиц розовые хвосты, а все кошки -- птицы, то у всех кошек -- розовые хвосты.

ДА    НЕТ

(2) Если у всех машин есть паруса, а некоторые бассейны -- машины, то у некоторых бассейнов есть паруса.

ДА    НЕТ

(3) Если у коров не бывает зеленых копыт, а все коровы -- свиньи, то у сvinей не бывает зеленых копыт.

ДА    НЕТ

(4) Если у всех лошадей есть крылья, и нет ни одной черепахи с крыльями, то ни одна черепаха -- не лошадь.

ДА    НЕТ

(5) Если некоторые люди розового цвета, а все, что имеет розовый цвет является лошадью, то некоторые лошади -- люди.

ДА    НЕТ
APPENDIX C

INTERVIEW GUIDE (ENGLISH VERSION)
Interviewer's Guide

This study is concerned with children's interpretations of algebraic symbolism. The study is devoted to identifying effective ways to teach algebra and algebraic operations.

Students will first take the written test. Students taking the written test should be instructed to do all items that they know how to do. Reassure them that they can skip items if they are unable to do them. However, they should continue to look carefully at each item to decide whether they can solve it or not. The items are not ordered according to the level of difficulty, and so they may find that they are able to solve a problem after being unable to solve several previous problems. Students may have 90 minutes to complete the test.

During the interview, students will explain their answers to problems in the written test to you. As you interview the children, please try to clarify each step of their solutions. Try to clarify how students are interpreting and using algebraic letters, algebraic expressions, and algebraic equations.

Specific interview guidelines for each problem in the test can be found in the table below. Questions #1–24 in the table are the items in the written test. Questions 25, 26, 27, and 28 in the table refer to test items #4, 17, 18, and 23 respectively and should only be included in the interview if the child has given a purely numerical argument or no argument for those items. (Please read items #4, 17, 18, 23, 25, 26, 27, and 28 in the table below; this will clarify the process for you.)

Students can learn from the interview. To spread the effect uniformly across the items, I would like to use a rotated forms process. Please use the following procedure:
The first interviewed child should start at question #1. Do questions #1–24 in order. Then do #25–28 below if applicable.
The second interviewed child starts at #2, explains his/her answers to you for all items in order through #24, then explains #1. Then proceed to #25–28 below if applicable.
The third interviewed child starts at #3, explains his/her answers to you for all items in order through #24, then explains #1, then explains #2. Then proceed to #25–28 below if applicable.
The fourth child starts at #4, explains his/her answers to you for all items in order through #24, then explains #1, then explains #2, then explains #3. Then proceed to #25–28 below if applicable.

Etc.
<table>
<thead>
<tr>
<th>Question</th>
<th>Interview instructions</th>
</tr>
</thead>
</table>
| (1) Decide whether the following statement is always true, sometimes true, or never true. Put a circle around the right answer. If you put a circle around 'sometimes true' explain when this statement is true. \( m + n + q = m + p + q \)  
Always true  
Never true  
Sometimes true that is when | Ask the student to explain his/her answer in full. |
| (2) What can you say about \( c + d = 10 \) and \( c \) is less than \( d \)? | Ask the student to explain his/her answer in full. |
| (3) Which of the following expressions are equivalent to \( 5 \times (6 + 3) \)? (Circle them.)  
A) \( 5 \times 6 + 5 \times 3 \)  
B) \( 5 \times (3 + 6) \)  
C) \( 5 + 6 \times 3 \)  
D) \( 5 \times 3 + 5 \times 6 \)  
E) \( (5 \times 6) + 3 \)  
Explain what you did to figure this out. | Clarify that the child understands the meaning of 'equivalent.' Ask her to explain her answer in full. How did she make a decision as to which expressions were equivalent? What precisely did she do? Did she perform any calculations? |
| (4) A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right. | Find out his entire argument and clarify the order of the steps of his argument. What precisely did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on? Please choose one of the following for each interviewed student:  
The student's argument in this problem was algebraic _____ purely numerical _____ no argument _____  
If the student's argument involved the use of algebraic letters, ask him to clarify what the algebraic letters in his argument represented. Did he do any algebraic simplification/manipulation? Ask him to explain how his algebraic work proved it, or whether it proved it. |
| (5) Is the following statement always true? sometimes true? never true? Say how you know. \( (a^2 + b^2)^2 = a^6 + b^6 \) | Find out his entire argument and clarify the order of the steps of his argument. What precisely did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on? Ask all students what \( a \) and \( b \) represent. |
| (6) If \( e + f = 8 \), then \( e + f + g = \) _______ ? | Ask the student to explain his/her answer. |
| (7) Write an equation using the letters \( S \) and \( T \) to represent the following statement: "There are six times as many students as teachers at this school." Use \( S \) for the number of students and \( T \) for the number of teachers. | Ask the student to explain his/her answer. |
| (8) Which is larger, \( 2n \) or \( n + 27 \)? Explain. | Ask the student to explain his/her answer in full. What is his conclusion? How did he reach that conclusion; i.e., what is his conclusion based on? |
(9) What can \( x \) equal in the following equation?

\[
\frac{12x - 2x + 16}{2} = 2(x + 4) + 3x
\]

Explain. Ask the student to explain his/her answer in full. Clarify the steps in his/her solution. How did he/she decide what values \( x \) could be?

(10) Which of the following expressions are always equivalent? sometimes equivalent?

A) \( b - a + c \)
B) \( c + a - b \)
C) \( c - b + a \)
D) \( c - a + b \)

Explain. Clarify that the child understands the meaning of 'equivalent.' Ask her to explain her answer in full. How did she make a decision as to which expressions were always or sometimes equivalent? What precisely did she do? Ask her to explain what the letters represent.

(11) In an algebra class the teacher proved that every whole number of the form \( n^3 - n \) is divisible by 6 (that is, if you divide \( n^3 - n \) by 6, there will be no remainder). The proof was as follows . . . .

(A) I understand all the details of the proof and the proof seems correct to me.

Yes / No (Circle one)

(B) There are some details in the proof that I do not understand. They are the following:

(C) If you think the teacher has given a correct proof for the theorem "every whole number of the form \( n^3 - n \) is divisible by 6," then answer the following question:

Do you think that further checks (by substituting numbers) are necessary in order to verify the validity of the theorem?

Yes / No (Circle one)

Explain.

(D) Victor is a doubter. He thinks that we have to check at least a hundred numbers in order to be sure that the theorem is correct. What is your opinion? Explain your answer.

Ask the student to explain his/her answers to B, C, and D in full.

(12) Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence.

If \( b \) is the number of blue pencils bought and if \( r \) is the number of red pencils bought, write an equation involving \( b \) and \( r \).

Ask the student to explain his/her answer.

(13) Pick a number, but do not tell me what it is. Now add your number to 8, then subtract your starting number from the total. What did you get? Why? Do you think it would work for any starting number? Use algebra to show it.

Ask the student to explain his answer in full. If the student uses algebraic letters, ask him what the letters stand for.

(14) Mary's basic wage is 20 dollars per week. She is also paid another 2 dollars for each hour of overtime that she works.

If \( h \) stands for the number of hours of overtime that she works, and if \( W \) stands for her total wage (in dollars), write down an equation connecting \( W \) and \( h \).

Ask the student to explain his/her answer.
(15) Which of the following expressions are equivalent to $w \times (k + q)$? (Circle them.)
A) $w \times k + w \times q$
B) $w \times (q + k)$
C) $(w \times k) + q$
D) $w \times q + w \times k$
E) $k \times w + q$
Explain how you figured this out.

(16) For the rectangle below, think about increasing the number for $k$. What happens to the shape of the rectangle as $k$ increases from zero to very big numbers? For example, does the rectangle become tall and thin? Does it become square-shaped? What happens? Draw a picture that shows what happens to the shape of the rectangle as $k$ increases from zero to very big numbers.

(17) Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . . Now do it with another three consecutive numbers . . . . What happens? Can you prove it will always work?

<table>
<thead>
<tr>
<th>(15) Which of the following expressions are equivalent to $w \times (k + q)$? (Circle them.)</th>
<th>Clarify that the child understands the meaning of 'equivalent.' Ask her to explain her answer in full. How did she make a decision as to which expressions were equivalent? What precisely did she do? Ask her to explain what the letters represent.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A) $w \times k + w \times q$</td>
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<tr>
<td>B) $w \times (q + k)$</td>
<td></td>
</tr>
<tr>
<td>C) $(w \times k) + q$</td>
<td></td>
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<tr>
<td>D) $w \times q + w \times k$</td>
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<tr>
<td>E) $k \times w + q$</td>
<td></td>
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<tr>
<td>Explain how you figured this out.</td>
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<tr>
<th>(16) For the rectangle below, think about increasing the number for $k$. What happens to the shape of the rectangle as $k$ increases from zero to very big numbers? For example, does the rectangle become tall and thin? Does it become square-shaped? What happens? Draw a picture that shows what happens to the shape of the rectangle as $k$ increases from zero to very big numbers.</th>
<th>Ask the student to explain his/her answer in full.</th>
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<tr>
<td></td>
<td>2 $\times \kappa$</td>
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<tr>
<td>K $\times$ K</td>
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</table>

| (17) Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . . Now do it with another three consecutive numbers . . . . What happens? Can you prove it will always work? | If the student does not know the meaning of any of the terms (consecutive, square, product), provide a definition. Find out his entire argument and clarify the order of the steps of his argument. What exactly did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on? Please choose one of the following for each interviewed student:
The student's argument in this problem was algebraic _______
purely numerical _______
no argument _______
If the student's argument involved the use of algebraic letters, ask him to clarify what the algebraic letters in his argument represented. Ask him to explain how his algebraic work proved it, or whether it proved it. |
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</table>
(18) If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

Provide definitions for 'sum' and 'difference' if the student needs them.

Find out his entire argument and clarify the order of the steps of his argument. What precisely did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on?

Please choose one of the following for each interviewed student:
The student's argument in this problem was algebraic ______ purely numerical ______ no argument ______

If the student's argument involved the use of algebraic letters, ask him to clarify what each of the different algebraic letters in his argument represented. Ask him to explain how his algebraic work proved it, or whether it proved it.

(19) For the following problem:
Assume the first two sentences (in bold) are true. Make a conclusion from the assumptions. (Choose a, b, c, or d.)
All fahmooth numbers can be divided evenly by 8.
26 is a fahmooth number.
Therefore . . .
a) 26 must not be a fahmooth number.
b) 26 is an exception to the rule.
c) It is probably true that fahmooth numbers cannot be divided evenly by eight.
d) 26 can be divided evenly by 8.

Ask the student to explain his/her answer in full.

(20) Which of the following expressions are equivalent?
A) 685 – 492 + 947
B) 947 + 492 – 685
C) 947 – 685 + 492
D) 947 – 492 + 685

Explain how you figured this out.

Clarify that the student understands the meaning of 'equivalent.' Ask her to explain her answer in full. How did she make a decision as to which expressions were equivalent? What precisely did she do? Did she perform any calculations?

(21) Cakes cost c pence each and buns cost b pence each. If I buy 4 cakes and 3 buns, what does 4c + 3b stand for?

Ask the student to explain his/her answer in full.

(22) Part of this figure is not drawn. There are n sides altogether. All sides have length 2. What is the perimeter of the figure?

Ask the student to explain his/her answer. Provide the definition of perimeter if the child needs it.
(23) Choose any number between 1 and 10. Write your number in this box:

Add the number in the box to 10 and write down the answer here: ________.
Now take the number in the box, subtract it from 10, and write down the answer here: ________.
Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.

(24) Please choose Yes or No for each sentence:

<table>
<thead>
<tr>
<th>Sentence</th>
<th>Choice</th>
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</thead>
<tbody>
<tr>
<td>(1) If all birds have purple tails and all cats are birds, then all cats have purple tails.</td>
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<tr>
<td>(2) If all cars have sails and some swimming pools are cars, then some swimming pools have sails.</td>
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<tr>
<td>(3) If no skunks have green toes and all skunks are pigs, then no pig has green toes.</td>
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<tr>
<td>(4) If all horses have wings and no turtle has wings, then no turtle is a horse.</td>
<td></td>
</tr>
<tr>
<td>(5) If some men are purple and everything which is purple is a horse, then some horses are men.</td>
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</tbody>
</table>

Make sure the student's answers are clearly marked.

(25) To the interviewer: If you chose "purely numerical argument" or "no argument" in question #4, return to question #4 and ask, "Can you use algebra to show this? Prove it with algebra?"
If the student can't get started, try prompting: "Start with x. Could you start with x and do the problem again using algebra?" Continue to provide minimal prompts if necessary; i.e., try to lead him into an algebraic argument if possible.

**Question 4:** A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right.

For this second argument, find out his entire proof and clarify the order of the steps of his proof. What precisely did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on?
If he sets up an algebraic expression or equation and then does not use it as the basis for his conclusions, point to the equation/expression after he is completed with his argument and ask, "Could this equation (expression) be used in your proof?"
If the student uses algebraic letters, ask him what the letters stand for. Ask him to explain how his algebraic work proved it, or whether it proved it.

Please indicate that this is his second argument; I want to see both his first and his second argument.
<table>
<thead>
<tr>
<th>Question 17</th>
<th>Question 18</th>
<th>Question 23</th>
</tr>
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<tbody>
<tr>
<td>Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . Now do it with another three consecutive numbers . . . What happens? Can you prove it will always work?</td>
<td>If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.</td>
<td>Choose any number between 1 and 10. Write your number in this box: ___ Add the number in the box to 10 and write down the answer here: ___ Now take the number in the box, subtract it from 10, and write down the answer here: ___ Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.</td>
</tr>
</tbody>
</table>

For this second argument, find out his entire proof and clarify the order of the steps of his proof. What precisely did he do, what were his conclusions, how did he reach those conclusions, what did he base his conclusions on? If he sets up an algebraic expression or equation and then does not use it as the basis for his conclusions, point to the equation/expression after he is completed with his argument and ask, "Could this equation (expression) be used in your proof?" If the student uses algebraic letters, ask him what the letters stand for. Ask him to explain how his algebraic work proved it, or whether it proved it. Please indicate that this is his second argument; I want to see both his first and his second argument. | For this second argument, find out her entire proof and clarify the order of the steps of her proof. What precisely did she do, what were her conclusions, how did she reach those conclusions, what did she base her conclusions on? If she sets up an algebraic expression or equation and then does not use it as the basis for her conclusions, point to the equation/expression after she is completed with her argument and ask, "Could this equation (expression) be used in your proof?" If the student uses algebraic letters, ask her what the letters stand for. Ask her to explain how her algebraic work proved it, or whether it proved it. Please indicate that this is her second argument; I want to see both her first and her second argument. | For this second argument, find out her entire proof and clarify the order of the steps of her proof. What precisely did she do, what were her conclusions, how did she reach those conclusions, what did she base her conclusions on? If she sets up an algebraic expression or equation and then does not use it as the basis for her conclusions, point to the equation/expression after she is completed with her argument and ask, "Could this equation (expression) be used in your proof?" If the student uses algebraic letters, ask her what the letters stand for. Ask her to explain how her algebraic work proved it, or whether it proved it. Please indicate that this is her second argument; I want to see both her first and her second argument. |
Инструкция Интервьюеру
Данное исследование посвящено изучению того, как школьники используют алгебру. Результаты исследования помогут лучше понять когнитивные механизмы использования алгебры и разработать более эффективные пути преподавания алгебры. Исследование состоит из двух частей — (1) письменное решение задач и (2) интервью.

Сначала школьники дадут письменные ответы. Пожалуйста, не забудьте на каждого школьника собрать следующие данные: дату рождения, дату тестирования, образование родителей, и уровень владения математикой (школьные оценки за 3 четверти или за первое полугодие + качественная характеристика учителя).

Перед письменным заданием предуприте учеников, что у них есть 90 минут для того, чтобы решить задачи. Объясните им, что, если они не знают как решить ту или иную задачу, лучше ее пропустить и перейти к следующей. Вместе с тем, если у них останется время, то следует вернуться к нерешенным задачам.

Во время интервью ученик должен объяснить Вам, как они решили письменные задачи. Постарайтесь выяснить последовательность их рассуждений при решении задач. Постарайтесь выяснить, как они используют алгебраические обозначения, выражения и уравнения.

Инструкции к каждой отдельной задаче Вы найдете в приложенной ниже таблице. Пункты таблицы №№ 1-24 относятся к письменным задачам. Пункты таблицы №№25-28 относятся соответственно к письменным задачам №№ 4, 17, 18 и 23. Вы обращаетесь к №№ 25-28 только если в соответствующих письменных задачах ученик использовал "чисто числовые аргументы" или не привел вообще никаких аргументов (пожалуйста, внимательно прочитайте пункты таблицы №№4,17,18,23,25,26,27 и 28, содержащие более подробную информацию). Чисто числовые аргументы -- это аргументы, в которых используются только числа и не используются алгебраические обозначения.

<table>
<thead>
<tr>
<th>Вопрос</th>
<th>Инструкции интервьюеру</th>
</tr>
</thead>
</table>
| (1) Подумайте над следующим уравнением: \( m + n + q = m + p + q \)  
Всегда ли оно верно, только иногда верно или никогда не верно? Подчеркните  
правильный ответ. Если вы считаете, что  
уравнение хотя бы иногда верно, объясните,  
когда именно оно верно.  
А. Всегда верно  
Б. Никогда не верно  
В. Верно иногда, когда  
| Попросите ученика подробно объяснить свой ответ. |
| (2) Что можно сказать про \( c \), если \( c + d = 10 \),  
но \( c \) меньше, чем \( d \)?  
|  |
| (3) Какие из нижеследующих выражений эквивалентны выражению \( 5 \times (6 + 3) \)?  
(Подчеркните их).  
А. \( 5 \times 6 + 5 \times 3 \)  
Б. \( 5 \times (3 + 6) \)  
В. \( 5 + 6 \times 3 \)  
Г. \( 5 \times 6 + 5 \times 3 \)  
Д. \( (6 \times 6) + 3 \)  
Obъясните, как Вы рассуждали при решении  
этой задачи.  
| Выясните, понимает ли ребенок, что значит "эквиваленты". Попросите дать как можно  
более подробное объяснение. Постарайтесь  
узнать, как именно ребенок пришел к тому,  
что два выражения эквивалентны.  
Производил ли ребенок какие-нибудь  
вычисления? |
| (4) Девочка умножает число на 5, а потом  
прибавляет к произведению 12. После этого  
she вычитает от суммы первоначальное число  
и дает результат на 4. Она обнаруживает,  
что полученное число на 3 больше, чем число,  
s которого она начала. Она говорит, что так  
получается всегда, с какого числа я бы не  
начала. Права ли она? Почему Вы считаете,  
что Вис ответ верен?  
| Выясните характер и порядок приведенных  
arгументов. Что именно сделал ученик, к  
каким выводам он пришел, как он  
obснововал свои выводы?  
Выберите одно из следующих для каждого  
ученика:  
Обоснование решения, данное учеником,  
было  
алгебраическим_______  
чisto чиловым_______  
ученик не привел аргументов_______  
Если доводы ученика включали в себя  
alгебраические буквы, спросите его, что они  
обозначают в его решении. Использовал ли  
ученик алгебраическое преобразования/упрощения. Спросите  
ученика, почему проделанная  
alгебраическая работа доказывает его  
решение.  |
| (5) Подумайте над следующим уравнением:  
\( (a + b^2)^3 = a^6 + b^6 \)  
Всегда ли оно верно, только иногда верно  
или никогда не верно? Объясните, как вы  
получили ответ.  
| Попытайтесь выяснить аргументы ученика и  
то в каком порядке они сформулированы: (1)  
что именно сделал ученик; (2) каковы были  
egro выводы; (3) как он к ним пришел и на  
чем они основаны.  
Спросите ученика, что означает буквы \( a \) и \( b \).  |
| (6) Если \( e + f = 8 \), то \( e + f + g = \)_______  
| Попросите ученика подробно объяснить свой  
ответ. |
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<th>№</th>
<th>Тема</th>
<th>Задание</th>
<th>Комментарий</th>
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<tbody>
<tr>
<td>(7)</td>
<td>Напишите уравнение, используя буквы И и Ю, представляющие следующее предложение: &quot;В этой школе в шесть раз больше школьников, чем учителей&quot;. Используйте И для обозначения числа школьников, а Ю для обозначения числа учителей.</td>
<td>Попросите ученика объяснить ответ.</td>
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<td>(8)</td>
<td>Что больше: 2л или п + 2? Объясните.</td>
<td>Попросите ученика дать подробное объяснение. Каковы были его выводы? Как он к ним пришёл и на чем они основаны?</td>
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<tr>
<td>(9)</td>
<td>Чему может равняться х в следующем уравнении? [12x - 2x + 16 = 2(x + 4) + 3x]</td>
<td>Попросите ученика дать подробное объяснение. Выяснили, каковы были этапы его решения. Как он решил, чему равняется х?</td>
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<tr>
<td>(11)</td>
<td>На уроке алгебры учитель доказал, что любое целое число, которое можно представить в виде (n^2 - n), делится на 6 без остатка. Доказательство было следующее ...</td>
<td>Попросите ученика подробно объяснить его ответы на пункты В, В и Г.</td>
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<tr>
<td></td>
<td>(A) Мне всё понятно в доказательстве, и оно мне кажется верным. Да \ Нет (обведите)</td>
<td>Спросите учителя, что он думает о доказательстве. Если ученик отвечает &quot;да&quot; в пункте В, выполните, хочет ли он подставить числовые значения, т.к. ему не вполне ясно доказательство, или он понимает доказательство, но ему кажется, что числовые подстановки необходимы для установления справедливости теоремы.</td>
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<td></td>
<td>(B) В доказательстве мне не понятно следующее:</td>
<td>Если ученик отвечает &quot;да&quot; в пункте В, выполните, хочет ли он подставить числовые значения, т.к. ему не вполне ясно доказательство, или он понимает доказательство, но ему кажется, что числовые подстановки необходимы для установления справедливости теоремы.</td>
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<td>(В) Если вы считаете, что учитель верно доказал теорему о том, что &quot;любое целое число, которое можно представить в виде (n^2 - n), делится на 6&quot;, то ответьте на следующий вопрос:</td>
<td>Если ученик отвечает &quot;да&quot; в пункте В, выполните, хочет ли он подставить числовые значения, т.к. ему не вполне ясно доказательство, или он понимает доказательство, но ему кажется, что числовые подстановки необходимы для установления справедливости теоремы.</td>
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<td></td>
<td>Считаете ли вы, что справедливость теоремы необходимо проверить (подстановкой чисел)? Да \ Нет (обведите)</td>
<td>Если ученик отвечает &quot;да&quot; в пункте В, выполните, хочет ли он подставить числовые значения, т.к. ему не вполне ясно доказательство, или он понимает доказательство, но ему кажется, что числовые подстановки необходимы для установления справедливости теоремы.</td>
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<td></td>
<td>Объясните.</td>
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</table>
(12) Синие карандаши стоят по 5 рублей каждый, красные - по 6 рублей каждый. Я покупаю по несколько синих и красных карандашей, и всё вместе стоит 90 рублей. Если M - количество синих карандашей, а N - количество красных карандашей, запишите уравнение с M и N.

Попросите ученика объяснить свой ответ.

(13) Загадайте число, но не называйте его мне. Теперь прибавьте задуманное число к 8, потом отнимите от суммы задуманное число. Что у вас получилось? Почему? Считаете ли вы, что так будет при любом задуманном числе? Покажите это с помощью алгебраического уравнения.

Попросите ученика подробно объяснить ответ. Если ученик использует буквенное обозначение переменных, спросите его, что обозначают буквы.

(14) Зарплата Марии 20 долларов в неделю. Кроме того, она получает по 2 доллара за каждый час сверхурочного времени. Если M - количество сверхурочных часов, которые она работает, а N - её зарплату в целом, запишите уравнение с M и N.

Попросите ученика объяснить свой ответ.

(15) Какие из следующих выражений эквивалентны w x (k + q) (Обведите их) A) w x k + w x q B) w x (q + k) B) w x k + q

Поскольку вы рассуждали при решении этой задачи, объясните, как Вы рассуждали при решении этой задачи.

(16) Имея в виде прямоугольника, изображённый внизу, представьте себе, что число k увеличивается. Как изменяется форма прямоугольника с увеличением k от кула до очень больших величин? Например, становится ли прямоугольник высоким и узким? Становится ли он квадратом? Что с ним происходит? Сделайте рисунок, показывающий, что происходит с формой прямоугольника по мере роста k от кула до очень больших величин.

Попросите ученика подробно объяснить свой ответ.

<table>
<thead>
<tr>
<th>2 x k</th>
</tr>
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<tbody>
<tr>
<td>k x k</td>
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</table>
(17) Возьмите три последовательных числа. Теперь вычислите квадрат среднего из них и отнимите от него произведение крайних.... Теперь сделайте то же самое для трёх других последовательных чисел. Что получилось? Можете ли вы доказать это в общем виде?

Если ученик не знает значения какого-либо из терминов (последовательные, квадрат, произведение), дайте определение. Выясните полностью, как он рассуждал, какова была последовательность его рассуждения. Что конкретно он делал, каковы были его выводы, как он к ним пришёл, на чём он основывал свои выводы?

Выберите одно из следующих для каждого ученика:

- Обоснование решения, данное учеником, было
- алгебраическим
- чисто чиловым
- ученик не привёл аргументов

Если доводы ученика включали в себя алгебраические буквы, спросите его, что они обозначают в его решении. Использовал ли ученик алгебраические преобразования/упрощения. Спросите ученика, почему проделанная алгебраическая работа доказывает его решение.

(18) Если вы знаете сумму и разность любых двух чисел, докажите, что вы всегда можете найти эти числа.

Если необходимо, дайте определения "суммы" и "разности".

Выясните полностью, как рассуждал ученик, какова была последовательность решения. Что конкретно он делал, каковы были его выводы, как он к ним пришёл, на чём он их основывал?

Выберите одно из следующих для каждого ученика:

- Обоснование решения, данное учеником, было
- алгебраическим
- чисто чиловым
- ученик не привёл аргументов

Если доводы ученика включали в себя алгебраические буквы, спросите его, что они обозначают в его решении. Использовал ли ученик алгебраические преобразования/упрощения. Спросите ученика, почему проделанная алгебраическая работа доказывает его решение.
| (19) | При решении следующей задачи считайте, что первые два (выделенные) утверждения верны. Сделайте вывод из этих утверждений. (Выберите А, Б, В, или Г.) Все композитные числа делятся на 8 без остатка.  
26 - композитное число. Следовательно:  
А. Должно быть 26 — не композитное число  
Б. 26 — исключение из правила  
В. Назерное из все композитные числа делятся без остатка на 8  
Г. 26 делится без остатка на 8. | Попросите ученика подробно объяснить свой ответ. |
|---|---|---|
| (20) | Какие из следующих выражений эквивалентны?  
А. 635 - 492 + 947  
Б. 947 + 492 - 655  
В. 947 - 492 + 655  
Г. 947 - 492 + 685  
Объясните, как Вы рассуждали при решении этой задачи. | Убедитесь, что ребёнок понимает значение термина "эквивалентны". Попросите подробно объяснить свой ответ. Как он пришёл к решению о том, какие выражения эквивалентны? Что конкретно он делал? Производил ли ученик какие-нибудь подсчеты. |
| (21) | Пирожные стоят 3 рубля за штуку, а булочки — 2 рубля за штуку. Если купить 4 пирожных и 3 булочки, то что будет означать выражение 4С + 3В? | Попросите ученика подробно объяснить свой ответ. |
| (22) | Часть этой фигуры не нарисована. Всего в фигуре N сторон. Каждая из сторон имеет длину 2. Чему равен её периметр? | Попросите ученика объяснить ответ. Дайте, если это необходимо, определение периметра (периметр равен сумме длин сторон). |
| № 23 | Выберите любое число между 1 и 10. Впишите число в рамку:  
Прибавьте 10 к числу в рамке и запишите ответ ____.  
Теперь вычитите число из рамки, вычтите его из 10 и запишите ответ _______.  
Сложите оба полученных ответа. Сколько у Вас получилось? Будет ли ответ один и тот же, какое бы число ни было в рамке?  
Приведите доказательство своего ответа. |
| № 24 | Выберите Да (для правильного рассуждения) или Нет (для неправильного рассуждения) для каждого из нижеследующих утверждений:  
(1) Если у всех птиц розовые хвости, а все кошки — птицы, то у всех кошек — розовые хвости.  
(2) Если у всех машин есть паруса, а некоторые бассейны — машины, то у некоторых бассейнов есть паруса.  
(3) Если у коров не бывает зеленых копыт, а все коровы — свиньи, то у свиней не бывает зеленых копыт.  
(4) Если у всех лошадей есть крылья, и нет ни одной черепахи с крыльями, то ни одна черепаха — не лошадь.  
(5) Если некоторые люди розового цвета, а все, что имеет розовый цвет является лошадью, то некоторые лошади — люди. | Зафиксируйте какие аргументы приводит ученик и в каком порядке. Что именно он сделал и к каким выводам пришел. На чем были основаны его выводы?  
Выберите одно из следующих для каждого ученика:  
Обоснование решения, данное учеником, было  
алгебраическим ______  
чисто числовым _______  
ученик не привел аргументов _______  
Если доводы ученика включали в себя алгебраические буквы, спросите его, что они обозначают в его решении. Использовал ли ученик алгебраические преобразования/упрощения. Спросите ученика, почему проделанная алгебраическая работа доказывает его решение.  
Проследите, чтобы ученик дал ответ на каждое утверждение. |
(25) Интервьюеру: Если Вы считаете, что в ответе на вопрос №4 ученик привел чисто числовые аргументы или не привел никаких аргументов, вернитесь к вопросу №4 и спросите: "А можем ли ты использовать алгебру для доказательства своей правоты?" Если ученик не знает с чего начать, помогите ему, сказав: "Может быть можно начать с X и применить алгебру?"

Вопрос 4: Девочка умножает число на 5, а потом прибавляет к произведению 12. После этого она вычитает от суммы первоначальное число и делит результат на 4. Она обнаруживает, что полученное число на 3 больше, чем число, с которого она начала. Она говорит, что так получится всегда, а какого числа я бы не начала. Права ли она? Почему Вы считаете, что Ваш ответ верен?

Простукая к этому вопросу во второй раз, постараетесь выяснить каждый шаг в рассуждении ребенка. Что именно сделал ребенок, к каким выводам он пришел и как он пришел к этим выводам. Если ученик составляет алгебраическое выражение или уравнение и не использует его в своих рассуждениях, укажите на это уравнение/выражение в своем доказательстве.

Если ученик использует алгебраические обозначения, спросите, что именно они обозначают. Попросите ученика объяснить, как использование алгебры помогло ему в построении доказательства.

Не забудьте, что это -- второе рассуждение (т.е. возврат к уже сформулированному рассуждению). Как первое рассуждение, так и второе, необходимы для исследования.

(26) Интервьюеру: Если Вы считаете, что в ответе на вопрос №17 ученик привел чисто числовые аргументы или не привел никаких аргументов, вернитесь к вопросу №17 и спросите: "А можем ли ты использовать алгебру для доказательства своей правоты?" Если ученик не знает с чего начать, помогите ему, сказав: "Может быть можно начать с X и применить алгебру?"

Вопрос 17: Возьмите три последовательных числа. Теперь вычислите квадрат среднего из них и отнимите от него произведение крайних... Теперь сделайте то же самое для трёх других последовательных чисел. Что получилось? Можете ли вы доказать это в общем виде?

Простукая к этому вопросу во второй раз, постараетесь выяснить каждый шаг в рассуждении ребенка. Что именно сделал ребенок, к каким выводам он пришел и как он пришел к этим выводам. Если ученик составляет алгебраическое выражение или уравнение и не использует его в своих рассуждениях, укажите на это уравнение/выражение в своем доказательстве.

Если ученик использует алгебраические обозначения, спросите, что именно они обозначают. Попросите ученика объяснить, как использование алгебры помогло ему в построении доказательства.

Не забудьте, что это -- второе рассуждение (т.е. возврат к уже сформулированному рассуждению). Как первое рассуждение, так и второе, необходимы для исследования.
(27) Интервьюер: Если Вы считаете, что в ответе на вопрос №18 ученик привел чисто числовые аргументы или не привел никаких аргументов, вернитесь к вопросу №18 и спросите: "А можешь ли ты использовать алгебру для доказательства своей правоты?"
Если ученик не знает с чего начать, помогите ему, сказав: "Можно быть можно начать с X и применить алгебру?"
Вопрос 18: Если вы знаете сумму и разность любых двух чисел, докажите, что вы всегда можете найти эти числа.

(28) Интервьюер: Если вы считаете, что в ответе на вопрос №23 ученик привел чисто числовые аргументы или не привел никаких аргументов, вернитесь к вопросу №23 и спросите: "А можешь ли ты использовать алгебру для доказательства своей правоты?"
Если ученик не знает с чего начать, помогите ему, сказав: "Можно быть можно начать с X и применить алгебру?"
Вопрос 23: Выберите любое число между 1 и 10. Впишите число в рамке:

Прибавьте 10 к числу в рамке и запишите ответ _______.
Теперь вычтите число из 10 и запишите ответ _______.
Сложите оба полученных ответа. Сколько у вас получилось? Будет ли ответ один и тот же, какое бы число ни было в рамке?
Приведите доказательство своего ответа.

Приступая к этому вопросу во второй раз, постараитесь выяснить каждый шаг в рассуждении ребенка. Что именно сделал ребенок, к каким выводам он пришел и как он пришел к этим выводам. Если ученик составляет алгебраическое выражение или уравнение и не использует его в своих рассуждениях, укажите на это уравнение/выражение и спросите: "А можешь ли ты использовать это уравнение/выражение в своем доказательстве?"
Если ученик использует алгебраические обозначения, спросите, что именно они обозначают. Попросите ученика объяснить, как использование алгебры помогло ему в построении доказательства.
Не забудьте, что это -- второе рассуждение (т.е. возврат к уже сформулированному рассуждению). Как первое рассуждение, так и второе, необходимы для исследования.

Приступая к этому вопросу во второй раз, постараитесь выяснить каждый шаг в рассуждении ребенка. Что именно сделал ребенок, к каким выводам он пришел и как он пришел к этим выводам. Если ученик составляет алгебраическое выражение или уравнение и не использует его в своих рассуждениях, укажите на это уравнение/выражение и спросите: "А можешь ли ты использовать это уравнение/выражение в своем доказательстве?"
Если ученик использует алгебраические обозначения, спросите, что именно они обозначают. Попросите ученика объяснить, как использование алгебры помогло ему в построении доказательства.
Не забудьте, что это -- второе рассуждение (т.е. возврат к уже сформулированному рассуждению). Как первое рассуждение, так и второе, необходимы для исследования.
APPENDIX E

CODING CATALOGUE
**Coding Catalogue**

**Problem 1:** Select only one category per student.

Decide whether the following statement is always true, sometimes true, or never true. Put a circle around the right answer. If you put a circle around ‘sometimes true’ explain when this statement is true.

\[ m + n + q = m + p + q \]

- Always true
- Never true
- Sometimes true that is when \[ ____________________________ \]

**CODING:**
(0) No response
(1) Always true
(2) Never true
(3) Sometimes true
(4) Sometimes true, when \( n = p \)

**Problem 2:** Select only one category per student.

What can you say about c if \( c + 4 = 10 \) and \( c \) is less than \( d \)?

**CODING:**
(0) No response
(1) Gives only one value for \( a \)
(2) \( c = 1, 2, 3, 4 \) OR \( c = 2, 3, 4 \) OR \( c = 0, 1, 2, 3, 4 \) OR \( c < 4 \) OR some variant
(3) \( c = 4.9 \) or less, \( c = 4.99 \) or less
(4) \( c < 5, c \) is an element of \( (-\infty, 5) \), \( 0 \leq c < 5 \)
(5) Other response

**Problem 3:**

Which of the following expressions are equivalent to \( 5 \times (6 + 3) \)? (Circle them.)

A) \( 5 \times 6 + 5 \times 3 \)  B) \( 5 \times (3 + 6) \)  C) \( 5 + 6 \times 3 \)  D) \( 5 \times 3 + 5 \times 6 \)  E) \( (5 \times 6) + 3 \)

Explain what you did to figure this out.

**CODING:**
(1) Part A: Yes, it is equivalent to the original expression (0, 1)
(2) Part A: Mention or use distributive property (0, 1)
(3) Part A: Calculated (0, 1)
(4) Part B: Yes, it is equivalent to the original expression (0, 1)
(5) Part B: Mention or use commutative property (0, 1)
(6) Part B: Calculated (0, 1)
(7) Part C: Calculated (0, 1)
(8) Part D: Yes, it is equivalent to the original expression (0, 1)
(9) Part D: Mention or use distributive property; or commutative property is applied to option A (0, 1)
(10) Part D: Calculated (0, 1)
(11) Part E: Calculated (0, 1)
(12) Calculated all expressions and used no principles (0, 1)
(13) Calculated all expressions, used no principles, and didn't select A, B, and D (0, 1)
(14) No response (0, 1)

Problem 4A: INDEPENDENT REASONING ONLY. Use written test (i.e., independent written work on the test prior to he interview) combined with interviewer's clarification of the original work only.

A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right.

CODING:
(1) Valid verbal argument with no use of algebra (0, 1)
(2) Used numerical examples only (0, 1)
(3) Used only numerical examples and decided some generalization will always hold on the basis of numerical examples only. (Note: Something will always occur for R or for some infinite set because it worked for a few examples.) (0, 1)
(4) Used numerical examples only and decided something is not always true on the basis of numerical counterexamples, or decided there is no pattern from numerical examples. (0, 1)
(5) Used some algebraic work (0, 1)
(6) Reasoning: Used some algebra, but final judgment rested on numerical examples. (Note: If their argument seems to be primarily inductive and numerically based, then give them a 1; if their argument seems to be primarily deductive and algebraically based, then give them a 0. If they use a procedural interpretation only to decide, and never refer to simplification, give them a 1 here.) (0, 1)
(7) Interpreted algebraic expression(s)/equation(s) ONLY procedurally. (Note: Applies if person uses algebra as a formula; i.e., substitutes numbers for letters, or uses letters to show how to calculate.) (0, 1)
(8) Expressed the belief that numerical substitution is a necessary part of the algebraic proof. (0, 1)
(9) Reasoning: Independently completed a correct algebraic proof. (Note: They make a conclusion and justify the conclusion by referring to algebraic work.) (0, 1)
(10) Used letters as generalized numbers (0, 1)
(11) Formulated correct algebraic equation or expression (0, 1)
(213)

For example: (a) \( \frac{5x + 12 - x}{4} = x + 3 \); or (b) \( \frac{5x + 12 - x}{4} \)

(12) Simplified equation or expression (0, 1)
(13) Simplified equation or expression correctly (0, 1)
(14) Simplified correctly and interpreted algebraic expression or equation(s) structurally. (Note: They accept the expression/equation and the simplification as evidence that the generalization will always hold.) (0, 1)
(15) Simplified incorrectly with a minor error and interpreted algebraic expression(s) or equation(s) structurally. (Note: They accept the expression/equation and the simplification as evidence that a generalization will always hold.) (0, 1)
(16) \( x + 3 \) is obtained from reading the problem. (0, 1)
(17) No response (0, 1)

Problem 4(B): PROMPTED REASONING. Code response after interviewer asks student to use algebra.

A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Is she right? Explain carefully why your answer is right.

CODING:
(1) Used some algebraic work (0, 1)
(2) Reasoning: Used some algebra, but final judgment rested on numerical examples. (Note: If their argument seems to be primarily inductive and numerically based, then give them a 1; if their argument seems to be primarily deductive and algebraically based, then give them a 0. If they use a procedural interpretation only to decide, and never refer to simplification here, give them a 1.)
(3) Interpreted algebraic equation(s)/expression(s) ONLY procedurally. (Note: Applies if person uses algebra as a formula; i.e., substitutes numbers for letters, or uses letters to show how to calculate.) (0, 1)
(4) Used the following "proof": Wrote algebraic equation/expression; substituted numbers; and claimed proof consists of combination of algebra and numerical examples, or numerical examples generated through substitution (0, 1)
(5) Reasoning: Independently completed correct algebraic proof (0, 1) (Note: They make a conclusion and justify the conclusion by referring to algebraic work.)
(6) Used letters as generalized numbers (0, 1)
(7) Formulated correct algebraic equation or expression (0, 1)
For example: (a) \( \frac{5x + 12 - x}{4} = x + 3 \); or (b) \( \frac{5x + 12 - x}{4} \)
(8) Formulated multiple algebraic equations/expressions (0, 1)
(9) Independently simplified equation or expression (0, 1)
(10) Independently simplified equation or expression correctly (0, 1)
(11) Independently formulated a correct equation/expression; independently simplified correctly; interpreted algebraic expression or equation(s) structurally with prompting. (Note: They accept the expression/equation and the simplification as evidence that the generalization will always hold.) (0, 1)
(12) Independently formulated a correct equation/expression; independently simplified and needed correction in one step; interpreted algebraic expression or equation(s) structurally without prompting. (Note: They accept the expression/equation and the simplification as evidence that the generalization will always hold.) (0, 1)

(13) Independently formulated a correct equation/expression; independently simplified incorrectly with minor error; and interpreted algebraic expression or equation(s) structurally. (Note: They accept the expression/equation and the simplification as evidence that a generalization will always hold.) (0, 1)

(14) x + 3 is obtained from reading the problem. (0, 1)

(15) Asked to simplify; simplified independently; and interpreted structurally. (0, 1)

**Problem 5: Select only one category per student based on "furthest" level attained.**

Is the following statement always true? sometimes true? never true? Say how you know.

\[(a^2 + b^2)^3 = a^6 + b^6\]

**CODING:**

(0) No response

(1) Method: Student uses procedural interpretation only (i.e., substituted numbers)

(2) Method: Student uses structural interpretation only: some clear verbal reference to algebraic rules, or use of algebraic simplification and manipulation

(3) Method: Student uses procedural and structural interpretation

(4) Student uses procedural interpretation only; uses procedural interpretation to find exceptions; and states: The statement is sometimes true when (mentions at least one of the following) \(a = b = 0\); or when \(a = 0\) OR \(b = 0\). (Possibly refers to structural aspect, e.g., referring to, or writing out expanded form of left side "from memory", but does no algebraic manipulation and simplification.)

(5) Student uses structural interpretation; performs correct algebraic manipulations and simplifications of some kind to find exceptions; and states: The statement is sometimes true when (mentions at least one of the following possibilities) \(a = b = 0\); or when \(a = 0\) OR \(b = 0\).

(6) Student uses structural interpretation; performs incorrect algebraic simplification and manipulation; and states "sometimes when (cites various exceptions)".

(7) Other response

**Problem 7: Select only one category per student.**

Write an equation using the letters \(S\) and \(T\) to represent the following statement:

"There are six times as many students as teachers at this school."

Use \(S\) for the number of students and \(T\) for the number of teachers.

**CODING:**

(0) No response

(1) Student writes \(S = 6T\)

(2) Student writes \(T = 6S\)
Problem 8: Select only one category per student.

Which is larger, $2n$ or $n + 2$? Explain.

CODING:
(0) No response
(1) Student says $2n$ is larger because it multiplies the $n$; $n + 2$ is just adding.
(2) Makes only correct statements: Student (a) realizes DIFFERENT relationships hold between $2n$ and $n + 2$ depending on the value of $n$; (b) correctly describes the relationship between $2n$ and $n + 2$ for some combination of continuous intervals (at least one continuous interval is mentioned) and discrete values of $n$, but doesn't specify what occurs for all values of $n$.
(3) Correctly specifies the relationship between $n + 2$ and $2n$ for two or more discrete values of $n$ only by substituting and calculating; doesn't generalize to other values of $n$ and doesn't describe what occurs on any continuous interval.
(4) Student says, "Depends on what $n$ is."
(5) Student says: If $n < 2$, then $n + 2$ is larger. If $n = 2$, $2n = n + 2$. If $n > 2$, then $2n$ is larger.
(6) Correctly describes what happens for $n < 2$ and $n > 2$ and omits what occurs for $n = 2$.
(7) States only $2n > n + 2$ OR $n + 2 > 2n$ OR $2n = n + 2$.
(8) Decides which expression is larger by choosing one value for $n$ and calculating.
(9) Other response
(10) Student states $2n$ is larger unless $n$ is (one, two, or three discrete values of $n$ are listed).
(11) Student makes conditional statements; makes at least one incorrect statement; and makes at least one correct statement.

Problem 9: Code answer based on most abstract approach attempted.

What can $x$ equal in the following equation?

$$\frac{12x - 2x + 16}{2} = 2(x + 4) + 3x$$

Explain.

CODING:
(0) No response (0, 1)
(2) Used only a procedural approach (i.e., substituted values to find solutions) (0, 1)
(3) Used only a procedural approach and concluded that they found the solution set: solution set consists of (some or all) of the substituted numbers, or some proper subset of reals (0, 1)
(4) Used only a procedural approach and concluded $x$ can equal any number (0, 1)
(5) Used a structural approach (i.e., algebraic simplification and manipulation) to find solutions (0, 1)
(6) Algebraically simplified each step correctly (0, 1)
(7) Used a structural approach, simplified correctly, and concluded \( x \) can equal any number (0, 1)
(8) Other response (0, 1)

**Problem 10**

Which of the following expressions are always equivalent? sometimes equivalent?

A) \( b - a + c \)
B) \( c + a - b \)
C) \( c - b + a \)
D) \( c - a + b \)

Explain.

**CODING:**
(1) No response (0, 1)
(2) Student writes at least one of the following: A and D are (always) equivalent. B and C are (always) equivalent. (0, 1)
(3) Student writes: (Some combination(s) of A, B, C and D other than A and D; B and C) are sometimes equivalent if (mentions at least one of the following): \( a = b; a = b = c; a = b = 0; a = b = c = 0 \). (0, 1)
(4) Method: Used only substitution of numbers (0, 1)
(5) Other response (didn't do anything that is described in VAR1, VAR2, VAR3, VAR4, VAR6) (0, 1)
(6) Student writes at least one of the following: C and D are (always) equivalent. A and B are (always) equivalent. (0, 1)

**Problem 11: For A, C, and D, select one category per student.**

(A) I understand all the details of the proof and the proof seems correct to me.

Yes / No (Circle one)

(C) If you think the teacher has given a correct proof for the theorem "every whole number of the form \( n^3 - n \) is divisible by 6", then answer the following question:

Do you think that further checks (by substituting numbers) are necessary in order to verify the validity of the theorem?

Yes / No (Circle one)

Explain.

(D) Victor is a doubter. He thinks that we have to check at least a hundred numbers in order to be sure that the theorem is correct. What is your opinion? Explain your answer.
PART A:
(0) No response
(1) Answer to Part A: Yes
(2) Answer to Part A: No

PART C:
(0) No response
(1) Answer to Part C: Yes
(2) Answer to Part C: No

PART D:
(0) No response
(1) Answer to Part D: Agree with Victor
(2) Answer to Part D: Disagree because the algebraic proof is sufficient. Checks won't add anything.
(3) Answer to Part D: Disagree because while a finite number of numerical checks are needed to verify (be sure of) the correctness of the theorem (i.e., they are a necessary part of the proof), the number of checks needed is less than 100.
(4) Answer to Part D: Disagree; Other explanation.
(5) Answer to Part D: Disagree; Nothing can be proven like this. The theorem can be true for 100 numbers, and false for the 101st number.

Problem 12: Select only one category per student.

Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If \( b \) is the number of blue pencils bought and if \( r \) is the number of red pencils bought, write an equation involving \( b \) and \( r \).

CODING:
(0) No response
(1) Student writes \( 5b + 6r = 90 \) (British) or \( 5m + 6n = 90 \) (Russian)
(2) Student writes incorrect equation
(3) Student writes any other response

Problem 14: Select only one category per student.

Mary's basic wage is 20 dollars per week. She is also paid another 2 dollars for each hour of overtime that she works. If \( h \) stands for the number of hours of overtime that she works, and if \( W \) stands for her total wage (in dollars), write down an equation connecting \( W \) and \( h \).

CODING:
(0) No response
(1) Student writes \( W = 20 + 2h \) (British) or \( N = 20 + 2M \) (Russian)
(2) Student writes any other response
Problem 17A: INDEPENDENT REASONING ONLY. Use written test (i.e., independent written work on the test prior to the interview) combined with interviewer's clarification of the original work only.

Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . . Now do it with another three consecutive numbers . . . . What happens? Can you prove it will always work?

CODING:
(1) No response (0, 1)
(2) Formulated correct algebraic proof (0, 1)
(3) Formulated algebraic proof with a small error (0, 1)
(4) Used numerical examples only (0, 1)
(5) Decides some generalization will always hold for an infinite set on basis of numerical examples only (0, 1)
(6) Used some algebraic work (0, 1)

Problem 17(B): PROMPTED REASONING. Code response after interviewer asks student to use algebra. Choose one code per student.

Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two . . . . Now do it with another three consecutive numbers . . . . What happens? Can you prove it will always work?

CODING:
(0) Could not formulate an algebraic proof
(1) Formulated a correct algebraic proof (without additional prompting)
(2) Formulated an algebraic proof with minor errors (without additional prompting)

Problem 18A: INDEPENDENT REASONING ONLY. Use written test (i.e., independent written work on the test prior to the interview) combined with interviewer's clarification of the original work only. Choose only one code per student; code for the "highest" level attained.

If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

CODING:
(0) No response
(1) Gave correct verbal rule (i.e., rhetorical solution)
(2) Correct Diophantine set-up (i.e., system of equations with use of letters as unknowns, but not as givens)
(3) Correct Diophantine solution
(4) Correct Vietan set-up (i.e., system of equations with use of letters as unknowns and as givens)
(5) Correct Vietan solution
(6) Used one or more numerical examples
(7) Gives correct solution using algebraic letters with no algebraic derivation (e.g., uses \(n\) as sum, \(p\) as difference, and/or \(a\) and \(b\) as original numbers, and generalizes numerical pattern with algebraic symbolism)

(8) Other response

**Problem 18(B): PROMPTED REASONING.** Code response after interviewer asks student to use algebra. Choose one code per student.

If you are given the sum and the difference of any two numbers, show that you can always find out what the numbers are.

**CODING:**
(0) Could not formulate an algebraic proof
(1) Formulated a correct Vieta proof (without additional prompting)
(2) Formulated a correct Diophantine proof (without additional prompting)

**Problem 19: Select only one category per student.**

For the following problem:

Assume the first two sentences (in bold) are true. Make a conclusion from the assumptions. (Choose a, b, c, or d.)

All fahmooth numbers can be divided evenly by 8.
26 is a fahmooth number.
Therefore . . .

a) 26 must not be a fahmooth number.
b) 26 is an exception to the rule.
c) It is probably true that fahmooth numbers cannot be divided evenly by 8.
d) 26 can be divided evenly by 8.

**CODING:**
(0) No response
(1) A
(2) B
(3) C
(4) D

**Problem 20:**

Which of the following expressions are equivalent?

A) \(685 - 492 + 947\)
B) \(947 + 492 - 685\)
C) \(947 - 685 + 492\)
D) \(947 - 492 + 685\)

Explain how you figured this out.
**CODING:**

(1) No response (0, 1)
(2) Responded A and D are equivalent; and B and C are equivalent (0, 1)
(3) Responded with one of these variants: \( x - y \) is the same as \( y - x \); \( x - y + z = z + y - x \); "reversible" reading of operations; or no attention to operations. For example: "A and B are equivalent because they both show that you add 947 and 492, then subtract 685." "A and B are the same because it is the same sum turned around." "A and B. They both have 492 and 947, then -685. C and D. They both have 492 and 685, then -947." "C and D are equivalent because the sets of numbers are the other way around. When adding, it doesn't matter." (0, 1)
(4) Method: Only computed (0, 1)
(5) Method: Only correctly applied principles (0, 1)
(6) Method: Computed and correctly applied principles (0, 1)
(7) Reasoning or method not clear (0, 1)

**Problem 22: Select only one category per student.**

Part of this figure is not drawn. There are \( n \) sides altogether. All sides have length 2. What is the perimeter of the figure? ____________

![Figure](image)

**CODING:**

(0) No response
(1) \( 2n \)
(2) Gives specific number (e.g., 16, 32, 40)
(3) Gives incorrect response involving a variable (e.g., \( 4n, 2^n, 2a, 36 + n, n^2, 40n, \ 32n, n^2 \))
(4) Gives any other response

**Problem 23A: INDEPENDENT REASONING ONLY.** Use written test (i.e., independent written work on the test prior to the interview) combined with interviewer's clarification of the original work only.

Choose any number between 1 and 10. Write your number in this box:

![Box](image)

Add the number in the box to 10 and write down the answer here: _________.
Now take the number in the box, subtract it from 10, and write down the answer here: _________.
Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.
**CODING:**
(1) No response (0, 1)
(2) Valid verbal argument with no use of algebra (0, 1)
(3) Used numerical examples only (0, 1)
(4) Used numerical examples only; and decided some generalization will hold for $R$ or for some infinite set because it worked for a few examples. (0, 1)
(5) Used numerical examples only; and decided some generalization will always hold for numbers 1–10 and checked every whole number from 1–10. (0, 1)
(6) Used numerical examples only and decided there is no pattern from numerical examples (0, 1)
(7) Used some algebraic work (0, 1)
(8) Reasoning: Used some algebra, but final judgment rested on numerical examples. (Note: If their argument seems to be primarily inductive and numerically based, then give them a 1; if their argument seems to be primarily deductive and algebraically based, then give them a 0. If they use a procedural interpretation only to decide, and never refer to cancellation, give them a 1.) (0, 1)
(9) Interpreted algebraic equation(s) ONLY procedurally. (Note: Applies if person uses algebra as a formula; i.e., substitutes numbers for letters, or uses letters to show how to calculate.) (0, 1)
(10) Reasoning: Independently completed correct algebraic proof. (Note: They make a conclusion and justify the conclusion by referring to cancellation.) (0, 1)
(11) Formulated algebraic equation $10 + x + 10 - x = 20$ or algebraic expression $10 + x + 10 - x$ (0, 1)
(12) Correctly simplified $10 + x + 10 - x$ (0, 1)
(13) Correctly simplified $10 + x + 10 - x$, and accept the equation/expression and the simplification as evidence that the generalization will always hold. (0, 1)
(14) Expressed the belief that numerical substitution is a necessary part of the algebraic proof (0, 1)

**Problem 23B:** Code response after interviewer asks student to use algebra.

Choose any number between 1 and 10. Write your number in this box:

Add the number in the box to 10 and write down the answer here: ________.
Now take the number in the box, subtract it from 10, and write down the answer here: ________.
Add your two answers. What results do you get? Will the result be the same no matter what number you choose to put in the box? Prove that your answer is right.

**CODING:**
(1) Used some algebraic work (0, 1)
(2) Reasoning: Used some algebra, but final judgment rested on numerical examples. (Note: If their argument seems to be primarily inductive and numerically based, then give them a 1; if their argument seems to be primarily deductive and algebraically based, then give them a 0. If they use a procedural interpretation only to decide, and never refer to cancellation, give them a 1 here.)
(3) At some point in the solution, interpreted algebraic equation(s) procedurally. (Note: Only applies if person uses algebra as a formula; i.e., substitutes numbers for letters, or uses letters to show how to calculate.) (0, 1)
(4) Expressed the belief that numerical substitution is a necessary part of the algebraic proof (0, 1)
(5) Reasoning: Independently completed correct algebraic proof and interpreted it structurally. (Note: They make a conclusion and justify the conclusion by referring to cancellation.) (0, 1)
(6) Used letters as generalized numbers (0, 1)
(7) Formulated algebraic equation 10 + x + 10 - x = 20 or algebraic expression 10 + x + 10 - x without prompting (0, 1)
(8) Formulated algebraic equation 10 + x + 10 - x = 20 or algebraic expression 10 + x + 10 - x with prompting or correction (0, 1)
(9) Formulated multiple algebraic equations/expressions (0, 1)
(10) Interpreted algebraic equation(s) or expression(s) structurally with prompting. (Note: They accept the equation/expression and the simplification as evidence that the generalization will always hold.)
(11) Interpreted algebraic equation(s) or expression(s) structurally without prompting (0, 1) (Note: They accept the equation/expression and the simplification as evidence that the generalization will always hold.)
(12) Simplified \((10 + x) + (10 - x) = 20\) with prompting (0, 1)
(13) Simplified \((10 + x) + (10 - x) = 20\) without prompting (0, 1)

**Problem 24:**

In this question you just have to tell whether or not the sentences show correct reasoning. All of the sentences are really nonsense, but you are to think only about the reasoning.
Circling **Yes** means the reasoning is good; circling **No** means the reasoning is not good.
Please choose **Yes** or **No** for each sentence:

**CODING:**

(1) If all birds have purple tails and all cats are birds, then all cats have purple tails.

**P24VAR1**

(0) No response
(1) Yes
(2) No
(3) Both "Yes" and "No" circled

(2) If all cars have sails and some swimming pools are cars, then some swimming pools have sails.

**P24VAR2**

(0) No response
(1) Yes
(2) No
(3) Both "Yes" and "No" circled

(3) If no skunks have green toes and all skunks are pigs, then no pig has green toes.

**P24VAR3**

(0) No response
(1) Yes
(2) No
(3) Both "Yes" and "No" circled

(4) If all horses have wings and no turtle has wings, then no turtle is a horse.

**P24VAR4**

(0) No response
(1) Yes
(2) No
(3) Both "Yes" and "No" circled

(5) If some men are purple and everything which is purple is a horse, then some horses are men.

P24VAR5

(0) No response
(1) Yes
(2) No
(3) Both "Yes" and "No" circled
APPENDIX F
PERCENTAGES FOR PROBLEM SOLUTIONS
### Table 15. Percentages for problem solutions

<table>
<thead>
<tr>
<th></th>
<th>NMP 10-14 years</th>
<th>NMP 14-16 years</th>
<th>E-NEX 10-14 years</th>
<th>E-NEX 14-16 years</th>
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<th>R-NEX 10-14 years</th>
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<td>19</td>
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<td>Use of correct algebraic proof (Problem 17)</td>
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<td>Expressed belief: Algebraic proof requires empirical support (Problem 11D)</td>
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<td>Expressed belief: Algebraic proof establishes universal validity (Problem 11D)</td>
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Table 15 (continued). Percentages for problem solutions

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<th>DV 14-16 years</th>
<th>R-NEX 10-14 years</th>
<th>R-NEX 14-16 years</th>
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<tr>
<td>Formulated correct algebraic equation: $S=6T$ (Problem 7)</td>
<td>63 60 11 28 60 83 20 53</td>
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<td>Formulated correct algebraic equation: $5b + 6r = 90$ (Problem 12)</td>
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<td>Used only principles and correctly selected all equivalent numerical expressions (Problem 3)</td>
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<td>Calculated all expressions to determine equivalence (Problem 3)</td>
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<td>Used a correct structural approach to find solutions, correct conclusion (Problem 9)</td>
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<td>Used only a procedural approach to find solutions (Problem 9)</td>
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<tr>
<td>Used only a procedural approach to find solutions and concluded found all solutions (Problem 9)</td>
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Table 15 (continued). Percentages for problem solutions

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<th>R-NEX 10-14 years</th>
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<td>Generalized number: correct response (Problem 1)</td>
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<td>Variable: Correct conditional statements (Problem 8)</td>
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LIST OF REFERENCES


