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A STUDY OF RADIATION CONDUCTION INTERACTION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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The Ohio State University

1995

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1995
ACKNOWLEDGEMENTS

My experience in the Department of Mechanical Engineering at the Ohio State University has been a pleasant one. I would like to thank all the faculty and students for making school life stimulating and enjoyable.

More specifically, I would like to thank my adviser, Professor Seppo A. Korpela for his generosity, humor, and vitality and for providing an insight into what a decent life should be. I would also like to thank the other members of my committee, Professor Bernard J. Hamrock, Prabhat K. Gupta, and Michael R. Foster for their time and generosity.

I would like to thank the Research Institute of Science and Technology in Korea for the financial support and encouragement in finishing the graduate studies.

Lastly, I am grateful to my family, who endured losing her husband and their daddy for four long years.
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CHAPTER I

INTRODUCTION

1.1 Background

The work described in this dissertation was motivated by the need to understand how radiative heat transfer interacts with conduction in semi-transparent solids. There are several practical situations where combined radiation-conduction heat transfer is important. One is in the glass industry. Glass can absorb and emit radiation energy in certain frequency bands. Thus, when a temperature distribution or the heat flux is desired, for example in the heat treatment of glass panels or in molten glass in a furnace, radiation needs to be accounted for together with conduction. A second application is in cooling of fibers and electro-optical crystals as they are drawn from melts. Large temperature gradients in the radial direction are detrimental because they may cause dislocations, decreasing the strength of the fibers or deteriorating the electro-optical properties of the crystals.

In a semi-transparent material the equation governing the radiative transport is called the equation of radiative transfer, or transfer equation for short. The mathematical complexity in solving the equation of radiative transfer stems from its integro-differential character, the integrals arising from directions and frequencies of radia-
tion. The thermal energy balance including both conduction and radiation is also nonlinear. Thus, hope for finding a closed form solution is nil and even a numerical treatment can be cumbersome.

When the equations that govern the temperature distribution and the radiation intensity field are put into a non-dimensional form, two non-dimensional parameters appear: the optical thickness and the conduction-radiation parameter. The former describes the absorbing capacity of the medium; the latter denotes the relative importance between conduction and radiation. Many of the available results are for moderate or large values of the conduction-radiation parameter and the optical thickness. In many situations of crystal growth, temperatures are very high and therefore the conduction-radiation parameter is quite small for nominal values of optical thickness. In such cases numerical solutions become increasingly more difficult to carry out as this parameter decreases. The reason for this is that thin layers develop in which temperature and to a lesser extent radiation intensity change rapidly. Indeed, it is well known that the solution of pure radiation leads to so-called radiation slip at the bounding surfaces. This slip is a sign that this limit is singular, and suggests that perturbation methods can be used to complement numerical solutions at this limit.

At the beginning of the present century Prandtl showed how it was possible to analyze viscous flows precisely. He divided the flow field into two regions: a very thin boundary layer in the neighborhood of the solid body, where friction plays an essential part, and the remaining region outside this layer, where friction may be neglected. This boundary-layer theory, after mathematical justification, has been developed in
the framework of a more general theory designed to calculate asymptotic expansions of the solutions to the complete equations. The method of matched asymptotic expansions is now used in applications in various fields in addition to fluid mechanics. Asymptotic expansions in certain radiative problems were used in the 1960's. Interest in this kind of analysis has subsided somewhat during the past couple of decades, apparently owing to the possibility of tackling the analysis by numerical methods. Nevertheless, finding a simple expression for both the heat flux and temperature field, when a radiation parameter is either small or large, is a worthy goal, for it would appear that greater physical understanding can be gained by contemplating such a result than by studying graphs generated from numerical results. As computers have made the work of numerical computations lighter, the theory of asymptotic expansions has also benefited from computer development. Symbolic analysis tools have become more powerful and easier to use, which in turn helps reduce the drudgery that is often necessary in constructing a perturbation series. This is likely to spur further development of analytic approximations. The efforts reported here are in this spirit.

1.2 Literature Survey

The early papers on radiative heat transfer were contributed by astrophysicists in the early part of this century (Schuster [45], Schwarzschild [46], Milne [35], and Eddington [17]). They were interested in how radiation is transported in stellar atmospheres, and they discussed it in one-dimensional media in which radiation dominated conduction. Owing to the integrals originating from directional and frequency dependencies of the radiation intensity, the equation for radiation intensity is an integro-differential
equation. It can be cast into an integral equation, but even for the simplest problem its solution is usually obtainable only through approximation techniques (for example, Chandrasekhar[12] and Kourganoff[27]). Although the subject of radiative transfer has been investigated principally by astrophysicists, it has attracted the attention of physicists also, for essentially the same problems arise in the theory of the diffusion of neutrons (for example, Davison[14], Case and Zweifel[9], and Duderstadt and Martin[16]). In the early 1960's radiative heat transfer in engineering problems was widely discussed in relation to the reentry process of satellites and spacecraft. A strong activity in the study of radiative heat transfer between gas and solid surface was the consequence. Since then radiative heat transfer has been treated by heat transfer engineers and a number of texts dealing with this topic have been written (for example, Sparrow and Cess[49], Vincenti and Kruger[52], Özişik[39], Siegel and Howell[47], and Modest[37]).

Since the early 1960's numerous articles on combined radiation and conduction problems have appeared. In 1962 Viskanta and Grosh [54] formulated and solved the problem of combined radiation and conduction in a one-dimensional plane parallel absorbing and emitting medium. They assumed the medium to be gray, meaning that its radiative properties are independent of frequencies of radiation. This assumption allows the integral over frequencies to be dropped. Two non-dimensional parameters, namely optical thickness, $\tau$, and conduction-radiation parameter, $N$,

$$\tau = \kappa L, \quad N = \frac{k\kappa}{4\sigma_0 T_r^3},$$

(1.1)

were employed to demonstrate the results. Here, $\kappa$ is the absorption coefficient, $L$
a characteristic thickness of the slab, $k$ the thermal conductivity, $\sigma_0$ the Stefan-Boltzmann constant, and $T_r$ a reference temperature. Numerical results for the temperature distribution and the heat flux were obtained by an iterative method. Although the method was not extended to a multi-dimensional situation, the solutions have been used as a benchmark for various approximation methods. In a review article [53], Viskanta and Anderson have dealt with this exact formulation thoroughly and provided an extensive list of references on this topic.

Asymptotic analyses (for example, Van Dyke [51] or Kevorkian and Cole [22]) have been developed by Lick [29], Cess [10] and by Wang and Tien [55, 56] for the same one-dimensional problem. In a paper that appeared in 1963 Lick [29] employed the exponential kernel approximation (Krook [28]) for gray media and the picket fence model (see, for example, Modest [37, Sec.17.4]) for a non-gray medium to transform the equations governing the heat flux from their integro-differential form into differential equations. He solved these differential equations by perturbation methods for the limiting solutions of optically thick, radiation-dominant and conduction-dominant limits and reported the leading order solution for heat flux and temperature field for each case.

In a review article [10] Cess described limiting solutions of optically thin and thick limits with the exact formulation of radiation and observed that both $N$ and $N/\tau^2$ are measures of the relative importance of conduction versus radiation; the former being more important in an optically thick medium, the latter in an optically thin medium. He also discussed the interaction of radiation and convection in a laminar
flow of an absorbing gas across a flat plate.

Motivated by the study of Lick, in 1966 Wang and Tien [55] employed the method of matched asymptotic expansions associated with the differential approximation of radiation to get limiting solutions of radiation- and conduction-dominant cases. Unfortunately, they neglected some of the gauge functions and thus were not able to obtain higher order terms in the situation dominated by radiation. They [56] also reported analyses of the opaque limit, the transparent limit, the radiation-dominant limit, and the limit dominated by conduction, using the exact formulation of radiation. Their analyses revealed that when radiation is combined with conduction, the radiation-dominant limit and the optically transparent limit are not treated satisfactorily in terms of the parameters, $\tau$ and $N$. However, these difficulties could have been overcome by considering the limits carefully. The incompleteness of these analyses is one reason for the work described in this dissertation.

Owing to the mathematical complexities, numerical methods are necessary to obtain accurate solutions for combined radiation and conduction problems for a broad range of parameters, as well as for most multi-dimensional problems. For the numerical solution of the exact integro-differential equation for combined radiation and conduction various numerical methods have been employed. The zonal method was used by Smith et al. [48] and Lin and Motakef [30]. Kholodov et al. [23], and Abed and Sacadura [1] have used the Monte Carlo method. Finite element methods have been applied by Wu et al. [57], Fernandes et al. [18], and Razzaque et al. [43, 44]. Finite difference methods with incorporation of a generalized exponential integral
function were used by Yuen and Wong [60] and Yuen and Takara [59]. Ducharme et al. [15] devised a finite difference method with a truncated expression for radiation intensity.

The more classical methods for solving the equation of radiative transfer include the discrete ordinates method and the differential approximation. Both strive to form equations independent of the directional dependence of radiation intensity; the former by approximating integrals over solid angles as numerical quadrature, the latter by transforming the integro-differential equation into a set of simultaneous partial differential equations.

The method of discrete ordinates has been developed on the idea that the directional variation of the radiation intensity can be represented as a sum of values in a finite number of directions, spanning the total solid angle range of $\pi$, and that a solution is found by solving the equation of transfer for a set of discrete ordinates. One of the earliest workers to set out this method was Chandrasekhar [12], and the method was extensively developed in the 1960's by Carlson and Lathrop [8] for application to neutron transport problems. The $S_4$ approximation, the simplest one, which includes three discrete ordinates per octant, has been shown to work well, not only for a non-scattering medium, but also for a medium that scatters isotropically. Kim and Lee [24] have used high-order approximations to investigate the effect of anisotropic scattering. Fiveland [19] has described how to select proper discrete ordinate quadrature sets. For combined radiation and conduction problems, Yücel and Williams [58] used this method for axisymmetric enclosures, Jones and Bayazitoğlu

The differential approximation transforms the integro-differential equation into a set of simultaneous partial differential equations. Approximate solutions to these partial differential equations can then be sought either by numerical or by analytic methods. The differential approximation has a long history in the analysis of radiative heat transfer. Astrophysicists such as Milne [35], Eddington [17], and Chandrasekhar [12] were early contributors to this method. They considered stellar atmospheres in which conduction can be neglected and radiation takes place in a medium without solid boundaries. The simplest set of equations goes by the name of Milne-Eddington approximation or $P_1$ approximation. The latter designation links this to a spherical harmonic expansion of the radiation intensity. The absence of integrals leads to the name of differential approximation for these methods. The same set can also be arrived at by taking moments of the equation of radiative transfer with respect to a direction vector (for example, Vincenti and Kruger [52]). The mathematical equations that need to be solved in the differential approximation are also equivalent to those from the $S_4$ approximation in the method of discrete ordinates, since both satisfy the moment equations up to second order (Krook [28]).

The application of the differential approximation to multiple dimensions and the corresponding boundary conditions to be used have been discussed by Mark [32, 33] and Marshak [34]. A detailed derivation by using the spherical harmonic expansions in Cartesian coordinates has been given by Cheng [13]. Traugott [50] extended its application to a non-gray medium in a manner where the approximate expression
of the equation of radiative transfer yields the correct asymptotic limits when the medium is either optically thin or thick. Arpaci and Gözüm [5] used it to solve the Bénard problem of non-gray radiating fluids.

In a situation of pure radiation, as is shown for example in Vincenti and Kruger [52], the differential approximation reduces to the optically thick limit when emission from the medium dominates emission from the bounding surfaces. This situation takes place physically when the temperatures of all bounding surfaces are much lower than the temperature of the medium or when the medium is optically thick. Sparrow and Cess [49] have pointed out that the differential approximation may be good for all values of optical thickness, because the divergence of radiative heat transfer from the differential approximation is identical to that from the exact formulation. Cess [11] has further shown that in an optically thin medium the differential approximation requires that the bounding surfaces must be isothermal and that the radiosity ought not depart greatly from uniformity over each surface. He illustrated that, if these requirements are satisfied, the differential approximation is fairly good for all values of optical thickness for a plane slab and a circular cylinder and that for two concentric cylinders the differential approximation does not yield the correct optically thin limit. That is, for certain geometries of multi-dimensional problems the differential approximation may be substantially in error in an optically thin medium. Various methods have been developed to overcome this drawback. Bayazitoglu and Higenyi [7] used the so-called $P_3$ approximation, which is a higher order spherical harmonic expansion. It improved the accuracy of the solution to some extent, but the higher
order methods become cumbersome in multi-dimensional geometries. Olfe [38] and Modest [36] devised modifications of the differential approximation that treat wall emission and emission from the medium separately. Despite the increased accuracy, the modifications introduce some integral expressions that remove the element of simplicity afforded by the differential approximation, and they are cumbersome in cases where radiation is coupled with other modes of heat transfer.

For combined radiation and conduction problems, Amlin and Korpela [3] used the differential approximation to solve the problem of heat transfer in a one-dimensional medium and in a two-dimensional rectangular enclosure. Ratzel and Howell [41, 42] discussed its application and that of the $P_3$ approximation to a one-dimensional medium and a two-dimensional rectangular enclosure. Their results are for moderate or large values of the conduction-radiation parameter and optical thickness.

In the next chapter the differential approximation is reviewed for phenomena associated with combined radiation and conduction. It is customary to define the optical thickness and the conduction-radiation parameter as they have been defined in equation (1.1). This leads to slight ambiguity in deciding what the proper thin and pure radiation limit should be. It is clearly better to define these parameters in such a way that the absorption coefficient appears only in one of them, for then it alone governs the opacity of the medium. This kind of reasoning is also necessary for treating the spectral properties effectively, as has been demonstrated by Anderson and Viskanta [4]. An alternative definition for the conduction-radiation parameter, named the Planck number, will be introduced and the spectral dependence is treated in the
manner suggested by Traugott [50] and used in Arpaci and Gözüm [5] and Amlin and Korpela [3]. For a one-dimensional medium, numerical solutions are sought with optical thickness and Planck number as parameters. By comparing the results with those of the exact solution, the accuracy of the differential approximation in one-dimensional combined radiation and conduction problems will be evaluated.

The issue of developing perturbation solutions for the various limiting situations is taken up in the following chapters, where the method of matched asymptotic expansions is described. Since the choice of suitable gauge function is essential to the success of this method, it is discussed in Chapter III for a situation involving small temperature variations. In the subsequent chapters the asymptotic expressions for the temperature and intensity fields and the heat flux are obtained for a plane slab and a circular cylinder of semi-infinite length.

The notations for various kinds of equality and ordering employed in this dissertation follow the standard nomenclature as presented, for example, in Kevorkian and Cole [22].
CHAPTER II

RADIATION COMBINED WITH
CONDUCTION

In heat transfer problems which take place in a radiatively participating medium, a photon energy balance known as the equation of radiative transfer needs to be considered, together with the thermal energy balance that accounts for conduction and convection. In the first section a balance of the radiation energy is described in terms of radiation intensity for an emitting, absorbing, and scattering medium. It is seen that the mathematical character of the equation of radiative transfer is integro-differential, the integrals arising from directions and frequencies of radiation. It is also noted that the thermal energy balance equation is nonlinear in temperature when radiation is coupled to conduction. In the following section the differential approximation of radiation is reviewed. The spectral dependence of the equation of radiative transfer and of the corresponding boundary conditions is treated in a manner similar to that suggested by Traugott [50].

For a simple one-dimensional problem the approximate solution obtained by using the differential approximation is compared to that obtained by solving the equation of radiative transfer in its exact form, and the validity of the differential approximation is examined in the situation of combined radiation and conduction. An alternative
definition for the conduction-radiation parameter is introduced as a measure of relative importance between conductive and radiative contributions.

2.1 Equation of Radiative Transfer

In a radiatively participating medium, a balance equation of the radiation energy that is carried by photons traveling in the direction $\hat{s}$ governs the change in radiation intensity in space. This equation is found by accounting for the contributions at position $\mathbf{r}$ from emission, absorption, and scattering out and into the direction $\hat{s}$. In the medium whose radiation properties are rotationally invariant (Case and Zweifel [9, p. 6]), the balance equation for radiation energy is given by (for example, Vincenti and Kruger [52, Ch. XI])

$$
\frac{dI_{\nu}(\mathbf{r}, \hat{s})}{ds} = \hat{s} \cdot \nabla I_{\nu}(\mathbf{r}, \hat{s}) = \kappa_{\nu} I_{b\nu}(\mathbf{r}) - (\kappa_{\nu} + \sigma_{\nu}) I_{\nu}(\mathbf{r}, \hat{s}) + \frac{\sigma_{\nu}}{4\pi} \int_{4\pi} I_{\nu}(\mathbf{r}, \hat{s}_i) \Phi_{\nu}(\hat{s}_i \cdot \hat{s}) d\Omega, \quad (2.1)
$$

in a situation in which local thermodynamic equilibrium prevails (for example, Chandrasekhar [12, p. 288], and Siegel and Howell [47, p. 446]). The spectral radiation intensity, $I_{\nu}$, is defined as the rate at which radiation energy flows per unit solid angle, unit frequency, and unit area normal to the traveling direction $\hat{s}$. The proportionality constants in the equation are the absorption coefficient $\kappa_{\nu}$ and the scattering coefficient $\sigma_{\nu}$. The parameter $\Omega$ is a solid angle and subscript $\nu$ denotes that the property is dependent on frequency. The function $\Phi_{\nu}$ is called the scattering phase function and it describes the probability that photons traveling in direction $\hat{s}_i$ are scattered into the direction $\hat{s}$ in such a way that scattering depends only on the angle between
these two unit vectors. Thus the scattering phase function can be written in terms of the inner product between the two unit vectors alone. The interpretation as a probability density leads then to the condition

$$\int_{4\pi} \Phi_\nu(\hat{s}_i \cdot \hat{s}) d\Omega_i = 4\pi. \quad (2.2)$$

The spectral blackbody intensity, $I_{\nu}$, or the Planck function, is given by

$$I_{\nu}(T) = \frac{2h\nu^3n^2}{c_0^2[e^{h\nu/kT} - 1]}, \quad (2.3)$$

where $T$ is absolute temperature, $h$ Planck's constant, $k$ Boltzmann's constant, $n$ refractive index, and $c_0$ speed of light in vacuum. Its integral over all frequencies leads to the well-known Stefan-Boltzmann radiation law

$$I_b(T) = \int_0^\infty I_{\nu}(T) d\nu = \frac{n^2\sigma_0 T^4}{\pi}, \quad (2.4)$$

where $\sigma_0$ is the Stefan-Boltzmann constant.

The spectral radiative heat flux $q_{R\nu}$ that crosses an imaginary surface element with the unit normal $\hat{n}$ is expressed in terms of intensity as

$$q_{R\nu}(r) \cdot \hat{n} = \int_{4\pi} I_\nu(r, \hat{s}) \hat{n} \cdot \hat{s} d\Omega,$$

and by removing the surface normal $\hat{n}$,

$$q_{R\nu}(r) = \int_{4\pi} I_\nu(r, \hat{s}) \hat{s} d\Omega. \quad (2.5)$$

The radiative heat flux $q_R$ is obtained by integrating over all frequencies,

$$q_R(r) = \int_0^\infty q_{R\nu}(r) d\nu = \int_0^\infty \int_{4\pi} I_\nu(r, \hat{s}) \hat{s} d\Omega d\nu. \quad (2.6)$$
Integrating the equation of radiative transfer (2.1) over all solid angles yields a volumetric balance of the radiation energy

\[ \int_{4\pi} \hat{s} \cdot \nabla I_\nu(r, \hat{s}) d\Omega = \int_{4\pi} \kappa_\nu I_{b\nu}(r) d\Omega - (\kappa_\nu + \sigma_\nu) \int_{4\pi} I_\nu(r, \hat{s}) d\Omega + \int_{4\pi} \sigma_\nu d\Omega \int_{4\pi} I_\nu(r, \hat{s}_i) \Phi_\nu(\hat{s}_i \cdot \hat{s}) d\Omega, \]

or by rearranging and employing equation (2.2)

\[ \nabla \cdot \int_{4\pi} \hat{s} I_\nu(r, \hat{s}) d\Omega = 4\pi \kappa_\nu I_{b\nu}(r) - (\kappa_\nu + \sigma_\nu) \int_{4\pi} I_\nu(r, \hat{s}) d\Omega + \frac{\sigma_\nu}{4\pi} \int_{4\pi} I_\nu(r, \hat{s}_i) d\Omega \int_{4\pi} \Phi_\nu(\hat{s}_i \cdot \hat{s}) d\Omega \]

\[ = 4\pi \kappa_\nu I_{b\nu}(r) - (\kappa_\nu + \sigma_\nu) \int_{4\pi} I_\nu(r, \hat{s}) d\Omega + \sigma_\nu \int_{4\pi} I_\nu(r, \hat{s}_i) d\Omega \]

\[ = \kappa_\nu \left[ 4\pi I_{b\nu}(r) - \int_{4\pi} I_\nu(r, \hat{s}) d\Omega \right]. \quad (2.7) \]

Integrating both sides of the above equation over all frequencies results in the divergence of the radiative heat flux as

\[ \nabla \cdot \mathbf{q}_R(r) = \int_0^\infty \kappa_\nu \left[ 4\pi I_{b\nu}(r) - \int_{4\pi} I_\nu(r, \hat{s}) d\Omega \right] d\nu. \quad (2.8) \]

The thermal energy balance equation for simultaneous conduction and radiation in a radiatively participating medium is

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = -\nabla \cdot \mathbf{q}_C - \nabla \cdot \mathbf{q}_R. \quad (2.9) \]

Parameter \( \rho \) denotes the density of the medium, \( c_p \) is its constant-pressure heat capacity per unit mass, and \( \mathbf{u} \) is the velocity of the medium. The conductive heat flux, \( \mathbf{q}_C \), is given by Fourier's law of conduction as

\[ \mathbf{q}_C = -k \nabla T. \quad (2.10) \]
Equations (2.1) and (2.9), together with (2.8) and (2.10), serve as the governing equations for combined radiation and conduction. Since equation (2.1) is a first-order partial differential equation in intensity, it requires that the intensity leaving the boundary into the medium be given everywhere on the bounding surfaces,

\[ I_\nu(r_w, \hat{s}) = I_{\nu u}(r_w, \hat{s}), \quad \hat{n} \cdot \hat{s} > 0, \quad (2.11) \]

where \( I_{\nu u}(r_w, \hat{s}) \) denotes the intensity emanating into the medium at the boundary surface and \( \hat{n} \) is the unit surface normal pointing into the medium. Actually, the boundary conditions are usually described in terms of surface properties of radiation, surface temperature, and the incoming intensity. Thus, the boundary intensity \( I_{\nu u}(r_w, \hat{s}) \) needs to be expressed in these variables. By definition, the intensity leaving a surface is described for an opaque surface as a sum of contributions from the surface emission and the reflected part of the incoming radiation from the medium. For a partially transparent surface it is a sum of the reflection of the incoming radiation from the medium and the penetration of the incoming radiation from the background. Both types of boundary can be accounted for by the general expression

\[
I_{\nu u}(r_w, \hat{s}) = \epsilon_{\nu}(r_w, \hat{s})I_{b\nu}(T_w) + \frac{\rho_{\nu}(r_w, \hat{s})}{\pi} \int_{\hat{n} \cdot \hat{s}_i < 0} I_{\nu u}(r_w, \hat{s}_i) \hat{n} \cdot \hat{s}_i \; d\Omega_i \\
+ \frac{1 - \rho_{\nu}(r_w, \hat{s})}{\pi} \int_{\hat{n} \cdot \hat{s}_i > 0} I_{\nu u}(r_w, \hat{s}_i) \hat{n} \cdot \hat{s}_i \; d\Omega_i, \quad (2.12)
\]

where parameters \( \epsilon_{\nu}(r_w, \hat{s}) \) and \( \rho_{\nu}(r_w, \hat{s}) \) denote \textit{directional, spectral emissivity} and \textit{directional, spectral reflectivity} of the surface, respectively. The intensity of background radiation is \( I_{b\nu}(r_w, \hat{s}) \) and \( T_w \) is the surface temperature. When the boundary is opaque, \( I_{b\nu}(r_w, \hat{s}) \) is zero and the identity, \( \rho_{\nu}(r_w, \hat{s}) = 1 - \epsilon_{\nu}(r_w, \hat{s}) \), is used to elim-
inate \( \rho_v(r_w, \mathbf{\hat{s}}) \). For a partially transparent boundary, the emissivity \( \epsilon_v(r_w, \mathbf{\hat{s}}) \) is zero in the above expression.

From the equations above, the mathematical complexity is clearly seen to stem from the integro-differential character of the equation of radiative transfer and the boundary condition. The intensity is also included in the thermal energy balance as the divergence of the radiative heat flux (2.8), which is an integral of the spectral intensity over all solid angles and frequencies. That is, the dependence of intensity on direction and frequency is the primary cause of the mathematical difficulties.

### 2.2 Differential Approximation

The differential approximation accounts for the directional dependence of intensity by expanding it in terms of its moments (for example, Vincenti and Kruger [52, p. 491-495], Duderstadt and Martin [16, p. 225-238]). The spectral mean intensity, \( J_v \), defined as

\[
J_v(r) = \int_{4\pi} I_v(r, \mathbf{\hat{s}}) d\Omega,
\]

and the spectral radiative heat flux, \( q_{R_v} \), defined by equation (2.5), work as the zeroth and the first moments of intensity, respectively. With these, the intensity expanded as

\[
I_v(r, \mathbf{\hat{s}}) \approx \frac{1}{4\pi} \left[ J_v(r) + 3q_{R_v}(r) \cdot \mathbf{\hat{s}} \right]
\]

is valid, as is easily checked out by taking the zeroth and the first moments. To carry out the integration, the integrals of the direction vector, \( \mathbf{\hat{s}} \), over all solid angles,

\[
\int_{4\pi} \mathbf{\hat{s}} d\Omega = 0, \quad \int_{4\pi} \mathbf{\hat{s}} \mathbf{\hat{s}} d\Omega = \frac{4\pi}{3} \delta, \quad \int_{4\pi} \mathbf{\hat{s}} \mathbf{\hat{s}} \mathbf{\hat{s}} d\Omega = 0,
\]

(2.15)
are needed, where $\delta$ is the second-order unit tensor.

Substituting the expanded intensity equation (2.14) into the equation of radiative transfer (2.1) and taking its zeroth moment result in

$$\nabla \cdot \mathbf{q}_{R\nu} = \kappa_{\nu}(4\pi I_{b\nu} - J_{\nu}),$$  \hspace{1cm} (2.16)

which is the same as the exact formula of the divergence of the radiative heat flux given by equation (2.7). The first moment gives

$$\nabla J_{\nu} = -3[\kappa_{\nu} + \sigma_{\nu}(1 - a_{1\nu})] \mathbf{q}_{R\nu},$$  \hspace{1cm} (2.17)

where the constant $a_{1\nu}$ is an expansion coefficient of the scattering phase function in a Legendre series

$$\Phi_{\nu}(\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}) = 1 + \sum_{m=1}^{\infty} a_{m\nu}(2m + 1) P_m(\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}).$$  \hspace{1cm} (2.18)

Here the function $P_m(\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}})$ denotes the Legendre polynomial, and the first term of unity is necessary in order to satisfy the condition given by equation (2.2). In taking moments of the equation of radiative transfer, the integral

$$\int_{4\pi} \hat{\mathbf{s}}_i \Phi_{\nu}(\hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}) d\Omega_i = 4\pi a_{1\nu} \hat{\mathbf{s}}$$  \hspace{1cm} (2.19)

is used together with equation (2.2). Eliminating the heat flux from equations (2.16) and (2.17) yields an equation for $J_{\nu}$

$$\nabla \cdot \left\{ \frac{1}{3[\kappa_{\nu} + \sigma_{\nu}(1 - a_{1\nu})]} \nabla J_{\nu} \right\} = \kappa_{\nu}(J_{\nu} - 4\pi I_{b\nu}).$$  \hspace{1cm} (2.20)

Consequently, once $J_{\nu}$ is determined by solving the differential equation (2.20), the radiative heat flux and its divergence are obtained from equations (2.17) and (2.16), respectively.
The exact boundary condition defined by equation (2.11) needs to be approximated, since it depends on direction. Two different approximations for boundary conditions have been developed in works related to neutron transport theory, named Mark's [32, 33] and Marshak's [34] boundary conditions. It has turned out that Marshak's boundary condition gives better results and thus is widely used. It is defined in the form of the integral of intensity at a boundary as

$$\int_{\hat{n} \cdot \hat{s} > 0} I_\nu(r, \hat{s}) \hat{n} \cdot \hat{s} d\Omega = \int_{\hat{n} \cdot \hat{s} > 0} I_{w\nu}(r_w, \hat{s}) \hat{n} \cdot \hat{s} d\Omega .$$

By considering the definition of the radiative heat flux, inspection shows that this boundary condition conserves heat flux. Upon substituting the intensity expanded by the differential approximation, integrating and rearranging lead to the boundary condition for the spectral mean intensity, $J_\nu(r_w)$, as

$$\frac{2}{3 [\kappa_\nu + \sigma_\nu (1 - a_{1\nu})]} \hat{n} \cdot \nabla J_\nu(r_w) = -\frac{\epsilon_\nu(r_w)}{1 + \rho_\nu(r_w)} 4\pi I_{b\nu}(T_w)$$

$$+ \frac{1 - \rho_\nu(r_w)}{1 + \rho_\nu(r_w)} \left[ J_\nu(r_w) - 4 \int_{\hat{n} \cdot \hat{s} > 0} I_{o\nu}(r_w, \hat{s}_i) \hat{n} \cdot \hat{s}_i d\Omega_i \right] , \quad (2.21)$$

where parameters $\epsilon_\nu(r_w)$ and $\rho_\nu(r_w)$ are hemispherical, spectral quantities defined as

$$\epsilon_\nu(r_w) = \frac{1}{\pi} \int_{\hat{n} \cdot \hat{s} > 0} \epsilon_\nu(r_w, \hat{s}) \hat{n} \cdot \hat{s} d\Omega , \quad \rho_\nu(r_w) = \frac{1}{\pi} \int_{\hat{n} \cdot \hat{s} > 0} \rho_\nu(r_w, \hat{s}) \hat{n} \cdot \hat{s} d\Omega . \quad (2.22)$$

In order to take care of the frequency dependence of intensity, a method in conjunction with the differential approximation was suggested by Traugott [50] by using appropriate mean values for absorption coefficient in a non-scattering medium. The basic idea of his method is that when the medium is either optically thin or thick, the approximate equation of radiative transfer should yield the correct asymptotic
limit. It is a small matter to extend this to an emitting, absorbing, and scattering
medium, in which the two mean coefficients, Planck's mean absorption coefficient \( \kappa_P \)
and Rosseland's mean extinction coefficient \( \beta_R \), are defined as [52, p. 465–469]

\[
\kappa_P = \frac{\int_0^\infty \kappa_\nu I_{b\nu} d\nu}{\int_0^\infty I_{b\nu} d\nu} = \frac{\pi}{n^2\sigma_0 T^4} \int_0^\infty \kappa_\nu I_{b\nu} d\nu,
\]

\[
\frac{1}{\beta_R} = \frac{\int_0^\infty 1/\left[\kappa_\nu + \sigma_\nu (1 - a_{1\nu})\right] dI_{b\nu}/dT d\nu}{\int_0^\infty dI_{b\nu}/dT d\nu} = \frac{\pi}{4n^2\sigma_0 T^3} \int_0^\infty \frac{1}{\kappa_\nu + \sigma_\nu (1 - a_{1\nu})} \frac{dI_{b\nu}}{dT} d\nu.
\]  

(2.23)

When the medium is non-scattering, that is \( \sigma_\nu = 0 \), Rosseland's mean extinction
coefficient reduces to Rosseland's mean absorption coefficient \( \kappa_R \). With these two
mean coefficients, integrating equations (2.16) and (2.17) over all frequencies gives
the expressions

\[
\nabla \cdot \mathbf{q}_R = \kappa_P (4\pi I_b - J), \quad \mathbf{q}_R = -\frac{1}{3\beta_R} \nabla J,
\]

(2.24)

which are now independent of frequency. Here \( J = \int_0^\infty J_\nu d\nu \).

When the boundary is gray, the boundary condition (2.21) is reduced into an
expression independent of frequency in accord with equations (2.24),

\[
\frac{2}{3\beta_R} \hat{n} \cdot \nabla J(r_w) = -\frac{\epsilon(r_w)}{1 + \rho(r_w)} 4\pi I_b(T_w)
+ \frac{1 - \rho(r_w)}{1 + \rho(r_w)} \left[ J(r_w) - 4 \int_0^\infty d\nu \int_{\hat{a}, s_\nu > 0} I_{\nu\nu}(r_w, s_i) \hat{n} \cdot \hat{s}_i d\Omega_i \right].
\]  

(2.25)

This expression is identical to those for special cases discussed by Arpaci and Gözüm
[5] and Amlin and Korpela [3]. For an opaque surface [5], that is, \( I_{\nu\nu} = 0 \) and \( \rho = 1 - \epsilon \),
the above expression yields
\[
\frac{2}{3\beta_R} \hat{n} \cdot \nabla J(r_w) = \frac{\epsilon(r_w)}{2 - \epsilon(r_w)} [J(r_w) - 4\pi I_b(T_w)].
\]

When the boundary is partially transparent, that is, \(\epsilon(r_w) = 0\), and the background intensity is direction and frequency independent [3], the expression (2.25) reduces to
\[
\frac{2}{3\beta_R} \hat{n} \cdot \nabla J(r_w) = \frac{1 - \rho(r_w)}{1 + \rho(r_w)} [J(r_w) - 4\pi I_b(r_w)].
\]

For non-gray boundaries 
\textit{hemispherical, spectral-mean} surface parameters can be assumed as
\[
\epsilon(r_w) = \frac{\int_0^\infty \epsilon_\nu(r_w) I_{b\nu}(T_w) d\nu}{\int_0^\infty I_{b\nu}(T_w) d\nu} = \frac{\pi \int_0^\infty \epsilon_\nu(r_w) I_{b\nu}(T_w) d\nu}{I_b(T_w)},
\]
\[
\rho(r_w) = \frac{\int_0^\infty \rho_\nu(r_w) I_{b\nu}(T_w) d\nu}{\int_0^\infty I_{b\nu}(T_w) d\nu} = \frac{\pi \int_0^\infty \rho_\nu(r_w) I_{b\nu}(T_w) d\nu}{I_b(T_w)},
\]

although these cannot rigorously be derived from the boundary condition.

### 2.3 One-Dimensional Plane Parallel Medium

In order to examine the usefulness of the differential approximation, solutions by the exact formulation and by the differential approximation will be compared. For this purpose consider a conducting and radiatively absorbing-emitting slab of thickness \(L\). The absorption coefficient of the medium is \(\kappa\) and its thermal conductivity is \(k\). The boundary conditions are chosen such that both bounding surfaces are black and one surface is held at constant temperature \(T_0\) and the other at \(T_1\).

Since Viskanta and Grosh [54] solved this problem with the exact formulation in 1962, their solution has been used as a benchmark for various approximation
methods. It is so used here also. The conduction-radiation parameter, named the Planck number, is introduced in the non-dimensionalizing procedure, and it is used as a measure of the dominance of conduction versus radiation.

2.3.1 Exact Formulation

Since the radiative properties of the medium are assumed to be constant, the dependence of intensity on frequency disappears and consequently the operations on frequency can be dropped. In this situation the medium is called gray.

For a gray non-scattering medium, the equation of radiative transfer (2.1) reduces to

\[ \frac{dI}{dz} = \kappa (I_b - I), \quad (2.27) \]

where the \( z \) axis is chosen perpendicular to the bounding surfaces and \( \mu \) is the direction cosine with respect to the \( z \) axis. The divergence of the radiative heat flux according to equation (2.8) yields

\[ \frac{dq_R}{dz} = \kappa \left[ 4\pi I_b(z) - \int_{4\pi} I(z, \hat{s})d\Omega \right] = \kappa \left\{ 4\pi \sigma_0 T^4 - 2\pi \int_0^1 \left[ I^+(z, \mu) + I^-(z, -\mu) \right] d\mu \right\}, \quad (2.28) \]

where superscripts + and − denote the intensity of radiation energy emanating in the positive \( z \) direction and in the negative direction, respectively. The thermal energy balance equation (2.9) becomes

\[ k \frac{d^2T}{dz^2} - \frac{dq_R}{dz} = 0 \quad (2.29) \]

once Fourier's law of conduction is incorporated.
These equations are cast into the non-dimensional forms

\[
\begin{align*}
\frac{d\mu}{dx} &= \tau (\Theta^4 - i), \\
\frac{d^2\Theta}{dx^2} &= \frac{\tau}{\mathcal{P}} \left\{ \Theta^4 - \frac{1}{2} \int_0^1 \left[ i^+(x, \mu) + i^-(x, -\mu) \right] d\mu \right\},
\end{align*}
\]

with the non-dimensional parameters and variables defined as

\[
x = \frac{z}{L}, \quad \tau = \kappa L, \quad \mathcal{P} = \frac{k}{4\pi^2 \sigma_0 T_0^3 L}, \quad \Theta = \frac{T}{T_0}, \quad i = \frac{\pi I}{\nu^2 \sigma_0 T_0^4}.
\]

Here, \(T_0\) is a reference temperature and usually takes the value of the maximum possible temperature in the domain. The parameter \(\tau\), optical thickness, describes the optical opacity of the medium. In other words, optical thickness denotes the absorbing capacity of the medium. When the parameter \(\tau\) is small, the medium is classified as optically thin and is barely participating in radiative transfer. Physically, the medium absorbs a small amount of the radiative energy transferring through, and the radiation energy emanating from a surface is not attenuated much. When \(\tau\) is large, the medium is classified as optically thick and the situation is the opposite. Optical thickness is related to transmissivity, \(\tau_r\), by the equation (Modest [37] p. 29)

\[
\tau_r = e^{-\tau}.
\]

The parameter \(\mathcal{P}\), named the Planck number, is the ratio of maximum conductive heat flux to maximum radiative heat flux that can be emitted from a surface at temperature \(T_0\). It serves as a measure of the relative dominance of conductive heat flux versus radiative heat flux regardless of absorption coefficient, whereas the usual conduction-radiation parameter, \(N = k \kappa / (4\pi^2 \sigma_0 T_0^3)\), depends on absorption coefficient. The
basic idea in the definition of $P$ is that the absorption coefficient should appear only in one of the parameters, for then it alone governs the opacity of the medium [25].

The formal solution of equation (2.30) is

$$i(x, \mu) = e^{-\tau x / \mu} \left[ C + \tau \int_0^\infty \Theta^4(\zeta) e^{\tau \zeta / \mu} \frac{d\zeta}{\mu} \right],$$

where $C$ is an integrating constant. Now $i^+$ and $i^-$ can be expressed in terms of the boundary intensities, $i(0)$ and $i(1)$, as

$$i^+(x, \mu) = e^{-\tau x / \mu} \left[ i(0) + \tau \int_0^\infty \Theta^4(\zeta) e^{\tau \zeta / \mu} \frac{d\zeta}{\mu} \right],$$

$$i^-(x, -\mu) = e^{\tau x / \mu} \left[ i(1) e^{-\tau / \mu} - \tau \int_1^\infty \Theta^4(\zeta) e^{-\tau \zeta / \mu} \frac{d\zeta}{\mu} \right],$$

and the integral of intensity as

$$\int_0^1 [i^+(x, \mu) + i^-(x, -\mu)] d\mu = i(0) E_2(\tau x) + \tau \int_0^\infty \Theta^4(\zeta) E_1(\tau(x - \zeta)) d\zeta$$

$$+ i(1) E_2(\tau(1 - x)) + \tau \int_1^\infty \Theta^4(\zeta) E_1(\tau(\zeta - x)) d\zeta$$

$$= 2\Theta^4(x) + [i(0) - \Theta^4(0)] E_2(\tau x) + [i(1) - \Theta^4(1)] E_2(\tau(1 - x))$$

$$- \int_0^\infty \frac{d\Theta^4}{d\zeta}(\zeta) E_2(\tau(x - \zeta)) d\zeta + \int_0^1 \frac{d\Theta^4}{d\zeta}(\zeta) E_2(\tau(\zeta - x)) d\zeta,$$

in which

$$E_n(x) = \int_0^1 \mu^{n-2} e^{-\tau x / \mu} d\mu$$

is the exponential integral function [2]. Equation (2.31) now reduces to

$$\frac{d^2 \Theta}{dx^2} = \frac{\tau}{2P} \left\{ \left[ \Theta^4(0) - i(0) \right] E_2(\tau x) + \left[ \Theta^4(1) - i(1) \right] E_2(\tau(1 - x)) \right\}$$

$$+ \frac{\tau}{2P} \left\{ \int_0^x \frac{d\Theta^4}{d\zeta}(\zeta) E_2(\tau(x - \zeta)) d\zeta - \int_x^1 \frac{d\Theta^4}{d\zeta}(\zeta) E_2(\tau(\zeta - x)) d\zeta \right\}. \quad (2.34)$$
In principle, this nonlinear integro-differential equation gives the solution of temperature for given boundary conditions; however, a closed-form solution seems to be unobtainable. Once the temperature field is determined, the radiative heat flux, $\Psi_R$, can be evaluated as

$$\Psi_R(x) = 2 \int_0^1 [i^+(x, \mu) - i^-(x, -\mu)] \mu \, d\mu$$

$$= 2 \left[ i(0) - \Theta^4(0) \right] E_2(\tau x) - 2 \left[ i(1) - \Theta^4(1) \right] E_3(\tau(1 - x))$$

$$- 2 \int_0^x \frac{d\Theta^4}{d\zeta} E_2(\tau(x - \zeta)) \, d\zeta - 2 \int_x^1 \frac{d\Theta^4}{d\zeta} E_3(\tau(\zeta - x)) \, d\zeta,$$

(2.35)

where $\Psi_R$ is a non-dimensional quantity defined by

$$\Psi_R = \frac{q_R}{n^2 \sigma_0 T_0^4}.$$

(2.36)

When the bounding surfaces are black and their temperatures are held constant, the boundary conditions for temperature reduce to

$$\Theta(0) = 1, \quad \Theta(1) = \Theta_L = \frac{T_1}{T_0},$$

(2.37)

and for intensity they are from equation (2.12)

$$i(0) = \Theta^4(0) = 1, \quad i(1) = \Theta^4(1) = \Theta_L^4.$$

(2.38)

Equations (2.34) and (2.35) can then be reduced to

$$\frac{d^2 \Theta}{dx^2} = \frac{\tau}{2P} \left[ \int_0^x \frac{d\Theta^4}{d\zeta} E_2(\tau(x - \zeta)) \, d\zeta - \int_x^1 \frac{d\Theta^4}{d\zeta} E_2(\tau(\zeta - x)) \, d\zeta \right],$$

(2.39)

$$\Psi_R(x) = -2 \left[ \int_0^x \frac{d\Theta^4}{d\zeta} E_2(\tau(x - \zeta)) \, d\zeta + \int_x^1 \frac{d\Theta^4}{d\zeta} E_3(\tau(\zeta - x)) \, d\zeta \right].$$

(2.40)
Small and Large Planck Number Limits

The definition of Planck number allows it to be used as a measure of the relative importance of conduction versus radiation. Large $\mathcal{P}$ means similarly large conductivity such that conduction is dominant, and small $\mathcal{P}$ leads to negligible conduction, which consequently results in radiation dominance. The same conclusion can be drawn from the expression for the total heat flux, since it is expressed as a sum of a conductive part and a radiative part,

$$\Psi = \frac{q_c + q_R}{n^2 \sigma_0 T_0^4} = -4\mathcal{P} \frac{d\Theta}{dx} + \Psi_R. \quad (2.41)$$

That is, when $\mathcal{P}$ approaches infinity, this expression for heat flux reduces to

$$\Psi \sim -4\mathcal{P} \frac{d\Theta}{dx}, \quad (2.42)$$

and the heat transfer process is dominated by conduction. Equation (2.39), which governs the temperature field, then yields simply

$$\frac{d^2\Theta}{dx^2} = 0, \quad (2.43)$$

which is the equation for pure conduction with the solution

$$\Theta(x) = 1 - (1 - \theta_L)x \quad (2.44)$$

for the boundary conditions (2.37). In this situation the heat flux is

$$\Psi \sim 4\mathcal{P}(1 - \theta_L). \quad (2.45)$$

When $\mathcal{P}$ approaches zero, the situation is nearly the reverse. The heat flux is

$$\Psi \sim \Psi_R. \quad (2.46)$$
Equation (2.39) gives

\[ \int_0^x \frac{d\Theta^4}{d\zeta}(\zeta)E_2(\tau(x - \zeta))d\zeta - \int_x^1 \frac{d\Theta^4}{d\zeta}(\zeta)E_2(\tau(\zeta - x))d\zeta = 0, \]

or by integration by parts

\[ \Theta^4(x) = \frac{1}{2} \left[ E_2(\tau x) + \theta_1^4 E_2(\tau(1-x)) + \tau \int_0^1 \Theta^4(\zeta)E_1(\tau|\zeta - x|)d\zeta \right]. \] (2.47)

Since total heat flux must be independent of \( x \), it may be evaluated at any location, conveniently chosen as \( x = 0 \). Then, equation (2.40) yields

\[ \Psi \sim -2 \int_0^1 \frac{d\Theta^4}{d\zeta}(\zeta)E_2(\tau\zeta)d\zeta, \]

and upon integration by parts and substituting the boundary conditions of temperature, it reduces to

\[ \Psi \sim 1 - 2\theta_1^4 E_3(\tau) - 2\tau \int_0^1 \Theta^4(\zeta)E_2(\tau\zeta)d\zeta. \] (2.48)

These expressions are exactly the same as those for pure radiation discussed in [20]. Once the temperature field is determined by solving equation (2.47), heat flux is evaluated by using equation (2.48). Unfortunately, no closed-form solution is available.

### 2.3.2 Differential Approximation

Although the medium is assumed as gray in the previous section, in what follows the problem will be formulated for a general non-gray scattering medium. For purposes of comparison with the exact formulation, grayness and lack of scattering will be applied at the final step. The governing equations for combined radiation and conduction with
the differential approximation of radiation fall into the following non-dimensional forms

\[
\frac{d^2 \Theta}{dx^2} = \frac{\tau \eta}{\mathcal{P}} (\Theta^4 - H),
\]

\[
\frac{d^2 H}{dx^2} = -3\tau^2 (\Theta^4 - H),
\]

(2.49)

with the non-dimensional parameters and variables

\[
x = \frac{z}{L}, \quad \Theta = \frac{T}{T_0}, \quad H = \frac{J}{4\pi^2 \sigma_0 T_0^4},
\]

\[
\tau = \sqrt{\kappa_F \beta_R} L, \quad \eta = \sqrt{\kappa_F / \beta_R}, \quad \mathcal{P} = \frac{k}{4\pi^2 \sigma_0 T_0^8 L}.
\]

The parameter \(\sqrt{\kappa_F \beta_R}\) is called a spherical mean extinction coefficient. It and the non-grayness factor \(\eta\) describe the frequency dependence of the medium [5]. Non-dimensional radiative heat flux, \(\Psi_R\), is now

\[
\Psi_R = \frac{q_R}{n^2 \sigma_0 T_0^4} = -\frac{4\eta}{3\tau} \frac{dH}{dx},
\]

and thus total heat flux is

\[
\Psi = \frac{q_c + q_R}{n^2 \sigma_0 T_0^4} = -4\mathcal{P} \frac{d\Theta}{dx} - \frac{4\eta}{3\tau} \frac{dH}{dx}.
\]

(2.50)

The boundary conditions for temperature remain

\[
\Theta(0) = 1, \quad \Theta(1) = \theta_L,
\]

(2.51)

but the conditions for boundary intensities for black surfaces, from equation (2.25), take the forms

\[
\frac{dH}{dx}(0) = \frac{3\tau}{2\eta} [H(0) - 1], \quad \frac{dH}{dx}(1) = \frac{3\tau}{2\eta} [H(1) - \theta_L^4].
\]

(2.52)
Small and Large Planck Number Limits

In a manner similar to that used for the exact formulation, a limit analysis on Planck number associated with the differential approximation is carried out here. As $\mathcal{P}$ approaches infinity, the first of equations (2.49), which governs the temperature field, yields

$$\frac{d^2 \Theta}{dx^2} = 0,$$  \hspace{1cm} (2.53)

and it is again the equation for pure conduction, with the solution

$$\Theta(x) = 1 - (1 - \theta_L)x$$  \hspace{1cm} (2.54)

for boundary conditions (2.51). The heat flux is

$$\Psi \sim 4\mathcal{P}(1 - \theta_L).$$  \hspace{1cm} (2.55)

These results are exactly the same as equations (2.44) and (2.45) by exact formulation.

As $\mathcal{P}$ approaches zero, equations (2.49) return

$$\Theta^4 = H, \quad \frac{d^2 H}{dx^2} = 0.$$  \hspace{1cm} (2.56)

With boundary conditions (2.52), the solution for intensity is

$$H(x) = -\frac{3\tau}{4\eta + 3\tau}(1 - \theta_L^4)x + 1 - \frac{2\eta}{4\eta + 3\tau}(1 - \theta_L^4),$$  \hspace{1cm} (2.57)

and the heat flux is

$$\Psi \sim \frac{4\eta}{4\eta + 3\tau}(1 - \theta_L^4).$$  \hspace{1cm} (2.58)

The temperature field obtained by the first of equations (2.56) is inconsistent at the boundaries, which suggests that boundary layers may exist near both surfaces.
2.4 Numerical Solution

In the previous section combined radiation and conduction taking place in a slab has been formulated both by the exact formulation and by the differential approximation. Neither method has given an explicit solution for temperature and intensity. Thus numerical solutions are sought here as an alternative. Since the medium is assumed to be gray and non-scattering, all the absorption coefficients are equivalent,

\[ \kappa = \kappa_\nu = \kappa_P = \kappa_R, \tag{2.59} \]

and the non-grayness parameter \( \eta \) equals unity. A finite difference method with a relaxation scheme was used. The domain was discretized with equally spaced nodes, and central differences were employed to retain an overall truncation error of \( O(\Delta x^2) \), where \( \Delta x \) is the distance between two adjacent nodes.

2.4.1 Numerical Solution of Exact Formulation

To solve equation (2.39) together with its associated boundary conditions (2.37), the left-hand side of this equation is discretized as

\[
\left( \frac{d^2 \Theta}{dx^2} \right)_i = \frac{\Theta_{i+1} - 2\Theta_i + \Theta_{i-1}}{\Delta x^2} + O(\Delta x^2).
\]

With piecewise integration, the integral at the right-hand side can be expressed as

\[
\frac{\tau}{2P} \sum_{j=2}^{i} \frac{\Theta_j^4 - \Theta_{j-1}^4}{\Delta x} \int_{x_{j-1}}^{x_j} \frac{E_2(\tau(x_i - \zeta)) \tau d\zeta}{2P} \sum_{j=i+1}^{j} \frac{\Theta_j^4 - \Theta_{j-1}^4}{\Delta x} \int_{x_{j-1}}^{x_j} E_2(\tau(\zeta - x_i)) \tau d\zeta + O(\Delta x^2)
\]
Thus, the finite difference approximation of equation (2.39) is given by

\[ \Theta_{i-1} - 2\Theta_i + \Theta_{i+1} = \frac{\tau}{2P} \sum_{j=2}^{J} \Theta_j^4 - \Theta_{j-1}^4 \left[ E_3(\tau|x_i - x_j|) - E_3(\tau|x_{i-1} - x_j|) \right] + O(\Delta x^2) \]

for \( i = 2, 3, \ldots, (J - 1) \). From the boundary conditions, \( \Theta_1 = 1 \) and \( \Theta_J = \theta_L \). The corresponding numerical solution was first worked out by Viskanta and Grosh [54].

To solve this system of nonlinear equations, iterations were necessary with relaxation. Rewriting the system of equations for increments of temperature leads to

\[ M_{ij}(\Delta \Theta_j)_m = (F_i)_m - M_{ij}(\Theta_j)_m, \quad (2.61) \]

where \( M_{ij} \) is a constant matrix. The vector \((F_i)_m\) and the nodal temperatures \((\Theta_j)_m\) constitute the right-hand side of equation (2.60), and the subscript \( m \) denotes an iteration index. The solution procedure is as follows

1. Guess a temperature field for \((\Theta_j)_m\) when \( m = 1 \).
2. Evaluate the right-hand side of equation (2.60), \((F_i)_m\), based on \((\Theta_j)_m\).
3. Determine increment of temperature field \((\Delta \Theta_j)_m\) by solving equation (2.61).
4. Check convergence by monitoring the maximum norm of \((\Delta \Theta_i)_m\).
5. If convergence is not achieved, update the temperature field as

\[ (\Theta_j)_{m+1} = (\Theta_j)_m + s(\Delta \Theta_j)_m, \quad (2.62) \]
where \( s \) is a relaxation parameter. After this repeat steps 2 to 4.

Once the temperature field is determined, total heat flux, as a sum of conductive and radiative parts, is evaluated at every point using

\[
(\Psi)_i = (\Psi_C + \Psi_R)_i \\
\sim -\frac{2P}{\Delta x}(\Theta_{i+1} - \Theta_{i-1}) \\
-\frac{2}{\tau \Delta x} \sum_{j=2}^{i} (\Theta_j^4 - \Theta_{j-1}^4)[E_4(\tau(x_i - x_j)) - E_4(\tau(x_i - x_{j-1}))] \\
+\frac{2}{\tau \Delta x} \sum_{j=i+1}^{J} (\Theta_j^4 - \Theta_{j-1}^4)[E_4(\tau(x_j - x_i)) - E_4(\tau(x_{j-1} - x_i))],
\]

for \( i = 2, 3, \ldots, (J - 1) \).

**Relaxation Parameter \( s \)**

A relaxation parameter \( s \) in equation (2.62) is obtained dynamically by invoking a plan related to the conjugate gradient method (for example, [40] p. 77-79), and is chosen as a constant that satisfies the equation

\[
(\Delta \Theta_i)_m [(F_i)_m + 1 - M_{ij}(\Theta_j)_m + 1] = 0.
\]

Substituting \((\Theta_j)_m + 1\) into the above equation and solving for \( s \) yields

\[
s = \frac{(\Delta \Theta_i)_m M_{ij}(\Delta \Theta_j)_m}{(\Delta \Theta_i)_m M_{ij}(\Delta \Theta_j)_m - (\Delta \Theta_i)_m (\Delta F_i)_m + 1},
\]

where

\[
(\Delta F_i)_m + 1 = \frac{(F_i)_m + 1 - (F_i)_m}{s} \\
\approx \frac{2}{\rho \Delta x} \sum_{k=2}^{J} (\Delta \Theta_k^3 \Theta_k - \Delta \Theta_{k-1}^3 \Theta_{k-1})_m [E_3(\tau|x_i - x_k|) - E_3(\tau|x_i - x_{k-1}|)].
\]

Here, it is presumed that |\( \Delta \Theta_i | < |\Theta_i |\).
2.4.2 Numerical Solution of Differential Approximation

To obtain a solution to the differential approximation, equations (2.49) with \( \eta = 1 \) and the associated boundary conditions given by equations (2.51) and (2.52) must be solved. This numerical solution has been given in several articles [3, 41]. The first of equations (2.49) yields

\[- \Theta_{i-1} + 2\Theta_i - \Theta_{i+1} = -\frac{\tau \eta}{P} \Delta x^2 (\Theta_i^4 - H_i), \tag{2.65}\]

for \( i = 2, 3, \ldots, (J - 1) \), and the corresponding boundary conditions are

\[\Theta_1 = 1, \quad \Theta_J = \theta_L. \tag{2.66}\]

The second equation gives

\[-H_{i-1} + 2H_i - H_{i+1} = 3\tau^2 \Delta x^2 (\Theta_i^4 - H_i), \tag{2.67}\]

for \( i = 2, 3, \ldots, (J - 1) \), and the corresponding boundary conditions lead to

\[ (2 + 3\tau \Delta x + 3\tau^2 \Delta x^2)H_1 - 2H_2 = (3\tau \Delta x + 3\tau^2 \Delta x^2), \]

\[-2H_{J-1} + (2 + 3\tau \Delta x + 3\tau^2 \Delta x^2)H_J = (3\tau \Delta x + 3\tau^2 \Delta x^2)\theta_L^4. \tag{2.68}\]

Rewriting the discretized equations in incremental forms yields

\[M_{ij}^\Theta (\Delta \Theta_j)_m = -\frac{\tau \eta}{P} \Delta x^2 (\Theta_i^4 - H_i)_m - M_{ij}^\Theta (\Theta_j)_m,\]

\[M_{ij}^H (\Delta H_j)_m = 3\tau^2 \Delta x^2 (\Theta_i^4 - H_i)_m - M_{ij}^H (H_j)_m, \tag{2.69}\]

where \( M_{ij} \) is a constant matrix and the subscript \( m \) is again an iteration index. The solution procedure is similar to that discussed in the previous section. More
specifically it consists of the following steps:

1. Guess a temperature field \((\Theta_i)_m\) and an intensity field \((H_i)_m\) when \(m = 1\).
2. Evaluate the right-hand sides of equations (2.69) based on \((\Theta_i)_m\) and \((H_i)_m\).
3. Determine increments of solutions \((\Delta \Theta_i)_m\) and \((\Delta H_i)_m\) by solving equations (2.69) separately.
4. Check convergence of the solutions by monitoring the maximum norm of either \((\Delta \Theta_i)_m\) or \((\Delta H_i)_m\).
5. If convergence is not achieved, update solutions as

\[
(\Theta_i)_{m+1} = (\Theta_i)_m + s (\Delta \Theta_i)_m, \\
(H_i)_{m+1} = (H_i)_m + t (\Delta H_i)_m,
\]

with relaxation parameters \(s\) and \(t\), and repeat steps 2 to 5.

Once solutions are determined, the total heat flux is evaluated as

\[
\Psi_i = (\Psi_c + \Psi_R)_i;
\]

\[
\sim -\frac{2\eta}{\Delta x} (\Theta_{i+1} - \Theta_{i-1}) - \frac{2\eta}{3\tau \Delta x} (H_{i+1} - H_{i-1}),
\]

for \(i = 2, 3, \ldots, (J - 1)\).

Relaxation parameters \(s\) and \(t\) are obtained dynamically in the manner previously mentioned. They are chosen as constants that make the following norms zero:

\[
(\Delta \Theta_i)_m \left\{ -\frac{\eta}{\Delta x^2} (\Theta_i^4 - H_i)_{m+1} - M_{ij}^\Theta (\Theta_j)_{m+1} \right\} = 0,
\]

\[
(\Delta H_i)_m \left\{ 3\tau^2 \Delta x^2 (\Theta_i^4 - H_i)_{m+1} - M_{ij}^H (H_j)_{m+1} \right\} = 0,
\]
which, upon substituting \((\Theta_i)_{m+1}\) and \((H_i)_{m+1}\) and rearranging, are approximated to

\[
(\Delta \Theta_i)_m M^\Theta_{ij}(\Delta \Theta_j)_m \approx s\{(\Delta \Theta_i)_m M^\Theta_{ij}(\Delta \Theta_j)_m + 4\frac{\tau^2}{P} \Delta x^2(\Delta \Theta_i[\Delta \Theta \Theta^3]_i)_m
\]

\[
- t\frac{\tau^2}{P} \Delta x^2(\Delta \Theta_i\Delta H_i)_m,
\]

\[
(\Delta H_i)_m M^H_{ij}(\Delta H_j)_m \approx -s12\tau^2 \Delta x^2(\Delta H_i[\Delta \Theta \Theta^3]_i)_m
\]

\[
+ t\{(3\tau^2 \Delta x^2(\Delta H_i\Delta H_i)_m + (\Delta H_i)_m M^H_{ij}(\Delta H_j)_m\}.
\]  

2.5 Results and Discussion

To examine the usefulness of the method of differential approximation of radiative transfer in combined radiation-conduction problems, a specific situation has been worked out in detail. The medium is one-dimensional of constant properties, and has black bounding surfaces, which are held at constant temperatures. The results are limited to establishing the ranges of optical thickness, \(\tau\), and of Planck number, \(P\), for which the differential approximation is valid.

The results are summarized in Tables 1, 2, and 3 for various values of the boundary temperature ratio \(\theta_L\). The radiative and conductive heat fluxes are almost uniform in the entire domain, except near both surfaces, although their sum, the total heat flux, must remain constant. The values of heat flux used to construct these tables were picked from the center of the domain, corresponding to \(x = 0.5\). The errors in radiative heat flux, conductive heat flux, and temperature are defined as

\[
\Delta \Psi_R = \frac{[\Psi_{\text{differential}} - \Psi_{\text{exact}}]_R}{\Psi_{\text{exact}}},
\]

\[
\Delta \Psi_C = \frac{[\Psi_{\text{differential}} - \Psi_{\text{exact}}]_C}{\Psi_{\text{exact}}},
\]
Table 1: Non-dimensional heat flux at $x = 0.5$, its error, and the error in temperature field by the differential approximation for combined radiation and conduction across a gray slab bounded by black surfaces with a temperature ratio $\theta_L = 0.1$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$P$</th>
<th>Exact</th>
<th>Differential</th>
<th>Errors(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Psi_R$</td>
<td>$\Psi_C$</td>
<td>$\Psi_R$</td>
<td>$\Psi_C$</td>
</tr>
<tr>
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<td>0.</td>
<td>0.9902</td>
<td>0.</td>
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</tr>
<tr>
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<td>0.</td>
<td>0.9921</td>
<td>0.</td>
<td>0.9945</td>
</tr>
<tr>
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<td>0.</td>
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<td>0.0021</td>
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</tr>
<tr>
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</tr>
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Table 2: Non-dimensional heat flux at $x = 0.5$, its error, and the error in temperature field by the differential approximation for combined radiation and conduction across a gray slab bounded by black surfaces with a temperature ratio $\theta_L = 0.5$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$\mathcal{P}$</th>
<th>Exact $\Psi_R$</th>
<th>Exact $\Psi_C$</th>
<th>Differential $\Psi_R$</th>
<th>Differential $\Psi_C$</th>
<th>Errors(%) $\Delta \Psi_R$</th>
<th>Errors(%) $\Delta \Psi_C$</th>
<th>Errors(%) $\Delta \Theta$</th>
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</thead>
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Table 3: Non-dimensional heat flux at $x = 0.5$, its error, and the error in temperature field by the differential approximation for combined radiation and conduction across a gray slab bounded by black surfaces with a temperature ratio $\theta_L = 0.9$.

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respectively, and the error in total heat flux may be evaluated as the sum of $\Delta \Psi_R$ and $\Delta \Psi_C$. An optical thickness of 0.01 corresponds to 0.99 in transmissivity, 0.1 to 0.9, and an optical thickness of 2.3 leads to a transmissivity 0.1. It is easily seen that the optically thin or thick limits are achieved for the optical thicknesses used and that the radiation- or conduction-dominant limits are achieved for the Planck numbers considered.

The errors in heat flux and temperature produced by the differential approximation do not show a great dependence on the ratio of boundary temperatures, $\theta_L$. The ratio simply has an effect on the total heat flux; as the ratio decreases, the total heat flux increases, as expected. On the whole the method incorporated with the differential approximation overestimates the radiative heat flux and underestimates the conductive one. In sum, it overestimates the total heat flux. It is seen that the discrepancies in heat flux come mainly from the radiation part. It is also seen that the discrepancies in heat flux have their maxima around the midrange of optical thickness $\tau = 1$. The disparity seems to be related to the boundary conditions, since Marshak's approximation focuses on the radiative heat flux at the boundaries whereas the exact boundary conditions are imposed on intensity [31].

The error in the temperature field increases as the Planck number decreases for a fixed optical thickness. In the range of Planck number where the radiative heat flux is greater than the conductive one, the error in the temperature field is smallest for thin media and greatest in the midrange of optical thickness. For Planck numbers for
which $\Psi_{R}/\Psi_{C}$ is less than one, the error is within 1% regardless of optical thickness.

Error in total heat flux by the differential approximation is shown in Figure 1 for various Planck numbers. The curves beyond $\mathcal{P} = 1$ can be safely extrapolated and they merely indicate that as the Planck number approaches infinity the error must reduce to zero, as was seen in the limit analysis on Planck number discussed above. At the limit of zero Planck number the errors go to their pure radiation limits, which depend on optical thickness. Interestingly, the error has a maximum in the Planck number range $10^{-3} < \mathcal{P} < 10^{-1}$, the location depending on optical thickness. These have not been observed in investigations in which the usual conduction-radiation parameter, $N$, defined by equation (1.1) has been used.

For a fixed value of Planck number, the distributions of radiative heat flux and conductive heat flux are shown in Figures 2, 3, and 4 for three values of optical thickness. In an optically thin medium, shown in Figure 2, the discrepancy of heat fluxes from the differential approximation and the exact formulation is quite uniform in the whole domain. As optical thickness increases, the difference in $\Psi_{R}$ near the boundaries decreases as shown in Figures 3 and 4, but it increases in the bulk of the domain, where the difference is quite uniform. The discrepancy in $\Psi_{C}$ is very small and shows a slight increase near the boundaries.

For various values of $\mathcal{P}$, temperature fields are shown in Figure 5 for optical thickness $\tau = 0.01$, an optically thin medium, and in Figure 6 for $\tau = 1$, a moderately thick medium. For thin media the temperature field by the differential approximation shows excellent agreement with the exact solution as already seen in the previous
Figure 1: Error in total heat flux by the differential approximation for combined radiation and conduction across a gray slab bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
Figure 2: Distributions of radiative heat flux, $\Psi_R$, and conductive heat flux, $\Psi_C$, for combined radiation and conduction across a gray slab of optical thickness $\tau = 0.01$ and of Planck number $\mathcal{P} = 0.01$ bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
Figure 3: Distributions of radiative heat flux, $\Psi_R$, and conductive heat flux, $\Psi_C$, for combined radiation and conduction across a gray slab of optical thickness $\tau = 1$ and of Planck number $\mathcal{P} = 0.01$ bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
Figure 4: Distributions of radiative heat flux, $\Psi_R$, and conductive heat flux, $\Psi_C$, for combined radiation and conduction across a gray slab of optical thickness $\tau = 2.3$ and of Planck number $P = 0.01$ bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
tables. Even in the worst cases shown in Figure 6 the temperature fields are still adequately represented.
Figure 5: Non-dimensional temperature distributions for combined radiation and conduction across a gray slab of optical thickness $\tau = 0.01$ bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
Figure 6: Non-dimensional temperature distributions for combined radiation and conduction across a gray slab of optical thickness $\tau = 1$ bounded by black surfaces with a temperature ratio $\theta_L = 0.5$. 
2.6 Summary

In this chapter the method of differential approximation of radiative transfer associated with combined radiation and conduction has been examined. In doing that the spectral dependence of radiation intensity and of surface properties has been treated in a manner similar to that suggested by Traugott. For a one-dimensional plane parallel medium of constant properties confined between two parallel black surfaces, the solutions were obtained numerically and compared with those by the exact formulation. The numerical solutions were obtained with enough resolution that the errors can be attributed entirely to the differential approximation. The results were organized in terms of two non-dimensional parameters, optical thickness \( \tau \) and Planck number \( \mathcal{P} \), with which the optically thin or thick limits and the radiation- or conduction-dominant limits can be handled separately. On the whole, the errors of the solution by the differential approximation were below 5% for both heat flux and temperature.

The error in radiative heat flux was found to be mainly responsible for the overall error in the solutions obtained by the differential approximation. For a fixed optical thickness, the error in total heat flux converged to zero as the Planck number increased. It converged to the pure radiation limit as the Planck number decreased, and had its maximum at the midrange of the Planck number at a location that depends on optical thickness. For a fixed Planck number, the error in total heat flux had its maximum at the midrange of optical thickness. The reason seems to be related more to the inadequacy of the boundary conditions than to the differential
approximation itself.

The error in the temperature field generally increased as Planck number decreased for a fixed optical thickness. When radiative heat flux was less than conductive heat flux, the error was less than 1% regardless of optical thickness. When Planck number was fixed, the error was smaller in a thin medium than in a thick medium and its maximum was at the midrange of optical thickness.
CHAPTER III

LINEARIZED ONE-DIMENSIONAL PROBLEM

In the previous chapter, the validity of the differential approximation was examined for various values of optical thickness and Planck number. It was found that the differential approximation is accurate to within 5% in temperature and heat flux in a one-dimensional semi-transparent medium. These results were obtained by solving the governing equations numerically, since the nonlinearity of the differential equations generally makes it impossible to obtain solutions in closed form.

When temperature variations are small, it is possible to eliminate the nonlinearity arising from the fourth power of temperature by linearization. In this chapter the linearized problem will be discussed and its solution will be obtained explicitly for the various situations, such as radiation-dominant, optically thin, and optically thick limits. The results have some restrictions when applied to real situations, but they offer invaluable insights into the structure of the solution to the original nonlinear problem.
3.1 Formulation and Exact Solution

Consider a semi-transparent slab of thickness $L$ having two infinite parallel boundar­ies. The absorption coefficient of the medium is $\kappa$, its scattering coefficient $\sigma$, and its thermal conductivity $k$. The left boundary surface is opaque with surface emis­sivity $\varepsilon$ and it is held at temperature $T_0$. The right surface is partially transparent with surface reflectivity $\rho$. Radiation escaping through this surface is intercepted by a far away surface at temperature $T_s$ without any attenuation. In addition to ra­diative heat exchange, heat is also convected from the right boundary to an ambient at temperature $T_a$, at a rate that depends on the heat transfer coefficient $h$. That is, the ambient is regarded as being perfectly transparent to radiation transfer. The problem, first discussed by Amlin and Korpela [3], includes various realistic boundary conditions for a radiatively participating solid.

Under these settings the thermal energy balance together with the differential approximation of radiation can be put into the following non-dimensional forms:

$$
\frac{d^2 \Theta}{dx^2} = \frac{\tau \eta}{\mathcal{P}} (\Theta^4 - H), \quad \frac{d^2 H}{dx^2} = -3\tau^2 (\Theta^4 - H), \quad (3.1)
$$

where the non-dimensional variables and parameters

$$
x = \frac{z}{L}, \quad \Theta = \frac{T}{T_0}, \quad H = \frac{J}{4\pi^2\sigma T_0^4},
\tau = \sqrt{\kappa \beta R} L, \quad \eta = \sqrt{\kappa / \beta R}, \quad \mathcal{P} = \frac{k}{4\pi^2\sigma T_0^3 L},
$$

have been used. Here, the frequency dependence of the equations is treated in the same way as was introduced in the previous chapter. The parameter $\kappa$ is again
the Planck mean absorption coefficient and \( \beta_R \) is the Rosseland mean extinction coefficient. The boundary conditions, from equation (2.25), are

\[
\begin{align*}
\Theta(0) &= 1, \\
\frac{dH}{dx}(0) &= \frac{\tau \gamma_0}{\eta} [H(0) - 1], \\
- \frac{d\Theta}{dx}(1) &= B[\Theta(1) - \theta_s], \\
\frac{dH}{dx}(1) &= -\frac{\tau \gamma_1}{\eta} [H(1) - \theta_s^4],
\end{align*}
\]

(3.2)

where

\[
B = \frac{hL}{k}, \quad \theta_a = \frac{T_1}{T_0}, \quad \theta_s = \frac{T_s}{T_0}, \quad \gamma_0 = \frac{3}{2} \frac{\epsilon}{(2 - \epsilon)}, \quad \gamma_1 = \frac{3}{2} \frac{1 - \rho}{(1 + \rho)}.
\]

The \( z \)-axis is chosen perpendicular to the bounding surfaces, and \( T_0 \) is used as a reference temperature. The parameters \( \epsilon \) and \( \rho \) are the hemispherical spectral-mean emissivity and reflectivity, respectively, as defined in equations (2.26), and \( B \) is the Biot number. In this formulation there are five radiative parameters, two of which, \( \gamma_0 \) and \( \gamma_1 \), describe the surface properties and other two, \( \tau \) and \( \eta \), the properties of the medium. The last one, \( \mathcal{P} \), is a measure of the relative dominance of heat transfer modes between radiation and conduction.

When the temperature variation is small, \( |\Theta(x) - \theta_s| \ll 1 \), and say, \( \theta_s = \theta_a \), the approximation

\[
\Theta(x)^4 \sim \theta_s^4 + 4 \theta_s^3 (\Theta(x) - \theta_s) = 4 \theta_s^3 \Theta(x) - 3 \theta_s^4
\]

(3.3)

can be invoked. With this approximation the nonlinear equations (3.1) become linear,

\[
\begin{align*}
\frac{d^2 \Theta}{dx^2} &= \frac{\tau \eta}{\mathcal{P}} (4 \theta_s^3 \Theta - 3 \theta_s^4 - H), \\
\frac{d^2 H}{dx^2} &= 3 \tau^2 (H - 4 \theta_s^3 \Theta + 3 \theta_s^4).
\end{align*}
\]

(3.4)

Elimination of \( H \) between these gives a fourth-order equation for \( \Theta \),

\[
\frac{d^2 \Theta}{dx^2} \left( \frac{d^2 \Theta}{dx^2} - \frac{(4 \tau \eta \theta_s^3 + 3 \tau^2 \mathcal{P})}{\mathcal{P}} \Theta + \frac{3 \tau \eta \theta_s^4}{\mathcal{P}} \right) = 0.
\]

(3.5)
A second independent equation is

\[ \frac{d^2 H}{dx^2} = -\frac{3 \tau \mathcal{P}}{\eta} \frac{d^2 \Theta}{dx^2}. \]  

(3.6)

The general solution of equation (3.5) is

\[ \Theta(x) = A \exp \left(-\frac{m(1-x)}{\sqrt{\mathcal{P}}} \right) + B \exp \left(-\frac{mx}{\sqrt{\mathcal{P}}} \right) + Cx + D, \]  

(3.7)

where

\[ m^2 = 4 \tau \eta \theta_s^2 + 3 \tau^2 \mathcal{P}, \]  

(3.8)

and from equation (3.6) and the first of equations (3.4), \( H(x) \) is obtained as

\[ H(x) = -\frac{3 \tau \mathcal{P}}{\eta} \Theta(x) + \frac{m^2}{\tau \eta} Cx + \frac{m^2}{\tau \eta} D - 3 \theta_s^4. \]  

(3.9)

Non-dimensional heat flux \( \Psi \), composed of radiative and conductive parts, is

\[ \Psi = \frac{q_c + q_r}{n^2 \sigma T_0^4} = -4 \mathcal{P} \frac{d \Theta}{dx} - \frac{4 \eta}{3 \tau} \frac{dH}{dx} = -\frac{4m^2}{3 \tau^2} C; \]  

(3.10)

The constants \( A, B, C, \) and \( D \) can be determined by applying boundary conditions.

The temperature condition at the left boundary gives

\[ \Theta(0) = A \exp \left(-\frac{m}{\sqrt{\mathcal{P}}} \right) + B + D = 1, \]  

(3.11)

so that

\[ D = 1 - A \exp \left(-\frac{m}{\sqrt{\mathcal{P}}} \right) - B, \]  

(3.12)

and the solutions can be written as

\[ \Theta(x) = A \exp \left(-\frac{m(1-x)}{\sqrt{\mathcal{P}}} \right) + B \exp \left(-\frac{mx}{\sqrt{\mathcal{P}}} \right) + Cx + 1 - A \exp \left(-\frac{m}{\sqrt{\mathcal{P}}} \right) - B, \]

\[ H(x) = -\frac{3 \tau \mathcal{P}}{\eta} \Theta(x) + \frac{m^2}{\tau \eta} Cx + \left\{ \frac{m^2}{\tau \eta} \left[ 1 - A \exp \left(-\frac{m}{\sqrt{\mathcal{P}}} \right) - B - 3 \theta_s^4 \right] \right\}. \]  

(3.13)
The other boundary conditions are used to evaluate the other three constants. The intensity condition on the left boundary,

$$ \frac{dH}{dx}(0) = \frac{\tau \gamma_0}{\eta} [H(0) - 1], \quad (3.14) $$

gives

$$ -3\tau P \left\{ \frac{m}{\sqrt{\rho}} [A \exp \left( -\frac{m}{\sqrt{\rho}} \right) - B] + C \right\} + \frac{m^2}{\tau} C = \tau \gamma_0 \left\{ -\frac{3\tau P}{\eta} + \frac{m^2}{\tau\eta} [1 - A \exp \left( -\frac{m}{\sqrt{\rho}} \right) - B] - 3\theta_s^4 - 1 \right\}. \quad (3.15) $$

The convective boundary condition on the right boundary,

$$ -\frac{d\Theta}{dx}(1) = B[\Theta(1) - \theta_s], $$

leads to

$$ -\frac{m}{\sqrt{\rho}} \left[ A - B \exp \left( -\frac{m}{\sqrt{\rho}} \right) \right] - C = B \left[ A + B \exp \left( -\frac{m}{\sqrt{\rho}} \right) + C + 1 - A \exp \left( -\frac{m}{\sqrt{\rho}} \right) - B - \theta_s \right]. \quad (3.16) $$

And the intensity condition at the partially transparent boundary on the right,

$$ \frac{dH}{dx}(1) = -\frac{\tau \gamma_1}{\eta} [H(1) - \theta_s^4], $$

gives

$$ -3\tau P \left\{ \frac{m}{\sqrt{\rho}} [A - B \exp \left( -\frac{m}{\sqrt{\rho}} \right)] + C \right\} + \frac{m^2}{\tau} C = -\tau \gamma_1 \left\{ -\frac{3\tau P}{\eta} [A + B \exp \left( -\frac{m}{\sqrt{\rho}} \right) + C + 1 - A \exp \left( -\frac{m}{\sqrt{\rho}} \right) - B] \right. $$

$$ + \frac{m^2}{\tau\eta} [1 - A \exp \left( -\frac{m}{\sqrt{\rho}} \right) - B] - 4\theta_s^4 \right\}. \quad (3.17) $$
These equations can be put into matrix form

\[
\begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \begin{bmatrix}
A \\
B \\
C
\end{bmatrix} = \begin{bmatrix}
-\tau \gamma_0(1 - 4\theta_2^3 + 3\theta_4^4) \\
-B(1 - \theta_s) \\
-4\tau \gamma_1 \theta_2(1 - \theta_s)
\end{bmatrix},
\]

(3.18)

where

\[
\begin{align*}
M_{11} &= \left(\frac{\gamma_0}{\eta} m^2 - 3\tau m \sqrt{\mathcal{P}}\right) \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right), \\
M_{12} &= \frac{\gamma_0}{\eta} m^2 + 3\tau m \sqrt{\mathcal{P}}, \\
M_{13} &= 4\eta \theta_2^3, \\
M_{21} &= B[1 - \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right)] + \frac{m}{\sqrt{\mathcal{P}}}, \\
M_{22} &= -B[1 - \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right)] - \frac{m}{\sqrt{\mathcal{P}}} \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right), \\
M_{23} &= 1 + B, \\
M_{31} &= -3\tau(m \sqrt{\mathcal{P}} + \frac{\tau \gamma_1 \mathcal{P}}{\eta}) - 4\tau \gamma_1 \theta_2^3 \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right), \\
M_{32} &= 3\tau(m \sqrt{\mathcal{P}} - \frac{\tau \gamma_1 \mathcal{P}}{\eta}) \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right) - 4\tau \gamma_1 \theta_2^3, \\
M_{33} &= 4(\eta + \tau \gamma_1) \theta_2^3.
\end{align*}
\]

By inspection and with the aid of the symbolic mathematical program Maple V, the solutions of equations (3.18) can be cast into the following forms:

\[
\begin{align*}
A &= \frac{A_0 + A_1 \exp(-m/\sqrt{\mathcal{P}})}{M_0 + M_1 \exp(-m/\sqrt{\mathcal{P}}) + M_2 \exp(-2m/\sqrt{\mathcal{P}})}, \\
B &= \frac{B_0 + B_1 \exp(-m/\sqrt{\mathcal{P}})}{M_0 + M_1 \exp(-m/\sqrt{\mathcal{P}}) + M_2 \exp(-2m/\sqrt{\mathcal{P}})}, \\
C &= \frac{C_0 + C_1 \exp(-m/\sqrt{\mathcal{P}}) + C_2 \exp(-2m/\sqrt{\mathcal{P})}}{M_0 + M_1 \exp(-m/\sqrt{\mathcal{P}}) + M_2 \exp(-2m/\sqrt{\mathcal{P}})}.
\end{align*}
\]

(3.19)
The expressions for the embedded coefficients are very lengthy. Thus, they will not be displayed explicitly. For certain limiting situations, they can be expressed as asymptotic series and so can the solutions. These representations will be taken up in the subsequent sections for the situations in which radiation dominates conduction, and when the medium is either optically thin or optically thick.

### 3.2 Radiation-Dominant Situation

When the radiant heat flux is much greater than the conductive one, the situation is categorized as radiation dominant. In the non-dimensionalized equations, it can be described by making \( \mathcal{P} \) small. In this situation the constants given by equations (3.19) can be written as the following asymptotic series in \( \mathcal{P} \):

\[
A \sim \sqrt{\mathcal{P}} \frac{1}{4\theta_s^3(4\tau\eta\theta_s^3)^{1/2}} \frac{\gamma_0(\tau\gamma_1 - \eta\dot{B})(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} + O(\mathcal{P}),
\]

\[
B \sim \frac{1}{4\theta_s^3} \left[ \frac{\eta\gamma_1(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} - (1 - 4\theta_s^3 + 3\theta_s^4) \right]
- \sqrt{\mathcal{P}} \frac{3\tau\eta(\eta + \gamma_1)}{4\theta_s^3(4\tau\eta\theta_s^3)^{1/2}} \left[ \frac{\eta\gamma_1(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)^2} - \frac{(1 - 4\theta_s^3 + 3\theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} \right] + O(\mathcal{P}),
\]

\[
C \sim -\frac{\tau\gamma_1\gamma_0}{4\theta_s^3(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} (1 - \theta_s^4)
- \sqrt{\mathcal{P}} \frac{3\tau^2\eta\gamma_1}{4\theta_s^3(4\tau\eta\theta_s^3)^{1/2}} \left[ \frac{\eta\gamma_1(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)^2} - \frac{(1 - 4\theta_s^3 + 3\theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} \right] + O(\mathcal{P}),
\]

\[
D = 1 - A \exp\left(-\frac{m}{\sqrt{\mathcal{P}}}\right) - B \sim 1 - B + O(\mathcal{P}).
\]

The temperature field is now given by the series

\[
\Theta(x) \sim \frac{1}{4\theta_s^3} \left\{ 3\theta_s^4 + \frac{(\eta\gamma_1\theta_s^4 + \eta\gamma_0 + \tau\gamma_1\gamma_0)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} - \frac{\tau\gamma_1\gamma_0(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} x
+ \left[ \frac{\eta\gamma_1(1 - \theta_s^4)}{(\eta\gamma_1 + \eta\gamma_0 + \tau\gamma_1\gamma_0)} - (1 - 4\theta_s^3 + 3\theta_s^4) \right] \exp\left(-\frac{m\gamma_1}{\sqrt{\mathcal{P}}}\right) \right\}
\]
and the intensity field is

\[ H(x) \sim \frac{\eta \gamma_1 \theta_s^4 + \eta \gamma_0 + \tau \gamma_1 \gamma_0}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)} - \frac{\tau \gamma_1 \gamma_0 (1 - \theta_s^4)}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)} x \]

\[ + \sqrt{\mathcal{P}} \left\{ \frac{3 \tau \eta}{(4 \pi \theta_s^3)^{1/2}} \frac{(1 - \theta_s^4)}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)^2} \frac{(1 - 4 \theta_s^3 + 3 \theta_s^4)}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)} \right\} + O(\mathcal{P}). \]  

(3.21)

Non-dimensional heat flux \( \Psi \) becomes

\[ \Psi \sim \frac{4}{3} \frac{\eta \gamma_1 \gamma_0}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)} (1 - \theta_s^4) \]

\[ + \sqrt{\mathcal{P}} \frac{4 \pi \eta^2 \gamma_1}{(4 \pi \theta_s^3)^{1/2}} \left\{ \frac{\eta \gamma_1 (1 - \theta_s^4)}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)^2} - \frac{(1 - 4 \theta_s^3 + 3 \theta_s^4)}{(\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0)} \right\} + O(\mathcal{P}). \]  

(3.22)

These results show that the asymptotic expansions of the solutions require powers of \( \sqrt{\mathcal{P}} \) as gauge functions. The terms containing an exponential function in the expression of temperature are transcendentally small as \( \mathcal{P} \) goes to zero. This is commonly a sign for the existence of a boundary layer, since these exponential functions are effective at each specific region and are asymptotically zero away from that region.

Thus, it can be concluded that there exist thermal boundary layers at both boundaries. The one near the left boundary is strong, since it appears at the leading order; the other near the right boundary is relatively weak. The intensity field does not
show a boundary layer character. However, an intensity boundary layer is expected to appear at higher orders since the equations are coupled. The representation of heat flux is independent of conduction up to this order. This is reasonable for small values of \( P \), because radiation dominates and a conduction contribution is expected to be seen only at higher orders.

### 3.3 Optically Thin Medium

When the medium is almost transparent, in other words, barely participating in radiative transfer, it is called optically thin. This condition prevails if either the absorption coefficient \( \kappa \) or the thickness of the medium \( L \) is small. That is, the optical thickness is small. At this situation the four constants in the solution can be expressed as the following asymptotic series in \( \tau \),

\[
A \sim \frac{1}{\sqrt{\tau}} \frac{P}{4(\eta \theta_0^2)^{1/2}} \frac{B}{1 + B} (1 - \theta_s) + \left[ \frac{1 - \theta_s}{2(1 + B)} - \frac{\gamma_0}{\gamma_0 + \gamma_1} \frac{1 - \theta_s^4}{8 \theta_0^2} \right] + O(\sqrt{\tau}),
\]

\[
B \sim \frac{1}{\sqrt{\tau}} \frac{P}{4(\eta \theta_0^2)^{1/2}} \frac{B}{1 + B} (1 - \theta_s) + \left[ \frac{1 - \theta_s}{2} - \frac{\gamma_0}{\gamma_0 + \gamma_1} \frac{1 - \theta_s^4}{8 \theta_0^2} \right] + O(\sqrt{\tau}),
\]

\[
C \sim -\tau \frac{1}{4 \eta \theta_0^3} \left[ \frac{P B}{1 + B} (1 - \theta_s) + \frac{\gamma_0 \gamma_1}{\gamma_0 + \gamma_1} (1 - \theta_s^4) \right] + O(\tau^2),
\]

\[
D \sim \theta_s + \frac{\gamma_0}{\gamma_0 + \gamma_1} \frac{1 - \theta_s^4}{4 \theta_0^2} + O(\tau).
\]  

(3.24)

The temperature field becomes

\[
\Theta(x) \sim 1 - \frac{B}{1 + B} (1 - \theta_s) x + O(\sqrt{\tau}),
\]  

(3.25)

where the series representation of the exponential function for small \( \tau \),

\[
\exp\left(-\frac{\eta \theta_0^2}{\sqrt{P}}\right) = \exp\left(-\sqrt{\frac{4 \eta \theta_0^2 + 3 \eta^2 P}{P}} x\right)
\]
\[ \sim 1 - \sqrt{\frac{4 \eta \theta^3_s}{P}} x + \tau \frac{2 \eta \theta^3_s}{P} x^2 + O(\tau^{3/2}), \]

has been used. The intensity field is

\[ H(x) \sim \frac{\gamma_0 + \gamma_1 \theta^4_s}{\gamma_0 + \gamma_1} + O(\tau), \quad (3.26) \]

and the non-dimensional heat flux becomes

\[ \Psi \sim \left[ 4 \frac{PB}{1 + B} (1 - \theta_s) + \frac{4}{3} \frac{\gamma_0 \gamma_1}{\gamma_0 + \gamma_1} (1 - \theta^4_s) \right] + O(\tau). \quad (3.27) \]

These results suggest that the asymptotic expansions of the solutions may only require powers of \( \tau \) as gauge function, although the expansions for the constants need powers of \( \sqrt{\tau} \). Actually, terms of order \( \sqrt{\tau} \) in \( A \) and \( B \) cancel each other out when they are substituted into the solutions. The intensity field obtained above satisfies both of the intensity conditions at the boundaries, since for small \( \tau \) they reduce to

\[ \frac{dH}{dx}(0) = 0, \quad \frac{dH}{dx}(1) = 0. \]

The temperature field also satisfies its boundary conditions at both ends. Thus, a boundary layer character is not seen at these leading order solutions.

The optically thin medium is barely participating in radiative transfer, so that the radiative properties of the medium are less decisive than those of the bounding surfaces. Consequently, the leading order solution of intensity is mainly governed by the properties of the boundary surfaces and by the background intensity, and does not include the radiative properties of the medium, such as the non-grayness factor \( \eta \). It also results in no interrelation between the temperature and intensity fields at the leading order. In addition, the heat flux obtained above is merely the sum of separate and independent parts of conduction and radiation.
3.4 Optically Thick Medium

When the medium is actively participating in radiative transfer, it is called optically thick. This situation is seen when either the absorption coefficient $\kappa$ or the thickness of the medium $L$ is large so that their product, the optical thickness, is large. In this situation, the parameter $1/\tau$ is small and the four constants in the solution can be written as the following asymptotic series in $1/\tau$:

$$
A \sim \frac{1}{\tau} \frac{8(1 - \theta_s)\theta_s^3 \gamma_1}{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\sqrt{3\theta_s^3 \eta \gamma_1}} + O\left(\frac{1}{\tau^2}\right),
$$

$$
B \sim -\frac{1}{\tau} \frac{\gamma_0}{3P(\sqrt{3\eta} + \gamma_0)}(1 - 4\theta_s^3 + 3\theta_s^4) + O\left(\frac{1}{\tau^2}\right),
$$

$$
C \sim -\frac{\sqrt{3PB}(\sqrt{3\eta} + \gamma_1) + 4\theta_s^3 \gamma_1}{\sqrt{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\theta_s^2 \eta \gamma_1}}(1 - \theta_s) + O\left(\frac{1}{\tau}\right),
$$

$$
D \sim 1 + \frac{1}{\tau} \frac{\gamma_0}{3P(\sqrt{3\eta} + \gamma_0)}(1 - 4\theta_s^3 + 3\theta_s^4) + O\left(\frac{1}{\tau^2}\right), \quad (3.28)
$$

From these the temperature field can be cast as

$$
\Theta(x) \sim 1 - \frac{\sqrt{3PB}(\sqrt{3\eta} + \gamma_1) + 4\theta_s^3 \gamma_1}{\sqrt{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\theta_s^2 \eta \gamma_1}}(1 - \theta_s) x + O\left(\frac{1}{\tau}\right)
$$

$$
\sim 1 - \frac{B}{1 + B}(1 - \theta_s) \frac{\sqrt{3PB(\sqrt{3\eta} + \gamma_1) + 4\theta_s^3 \gamma_1}}{1 + B \sqrt{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\theta_s^2 \eta \gamma_1}} x + O\left(\frac{1}{\tau}\right), \quad (3.29)
$$

and the intensity field is

$$
H(x) \sim 4\theta_s^3 \left[1 - \frac{\sqrt{3PB}(\sqrt{3\eta} + \gamma_1) + 4\theta_s^3 \gamma_1}{\sqrt{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\theta_s^2 \eta \gamma_1}}(1 - \theta_s) x - 3\theta_s^4\right]
$$

$$
-\frac{3P}{\eta} \left[\frac{8(1 - \theta_s)\theta_s^3 \gamma_1}{3P(1 + B)(\sqrt{3\eta} + \gamma_1) + 4\sqrt{3\theta_s^3 \eta \gamma_1}} \exp\left(-\frac{m(1 - x)}{\sqrt{P}}\right) \right]
$$

$$
-\frac{\gamma_0}{3P(\sqrt{3\eta} + \gamma_0)}(1 - 4\theta_s^3 + 3\theta_s^4) \exp\left(-\frac{mx}{\sqrt{P}}\right) + O\left(\frac{1}{\tau}\right). \quad (3.30)
$$
Non-dimensional heat flux is obtained as

\[ \Psi \sim \frac{4\mathcal{P}[\sqrt{3}\mathcal{P}B(\sqrt{3}\eta + \eta_1) + 4\theta_2^2\eta_1]}{\sqrt{3}\mathcal{P}(1 + B)(\sqrt{3}\eta + \eta_1) + 4\theta_2^2\eta_1}(1 - \theta_s) + O\left(\frac{1}{\tau}\right) \]

or

\[ \sim 4\frac{\mathcal{P}B}{1 + B}(1 - \theta_s) + \frac{\mathcal{P}}{1 + B}\frac{16\theta_2^2(1 - \theta_s)\eta_1}{\sqrt{3}\mathcal{P}(1 + B)(\sqrt{3}\eta + \eta_1) + 4\theta_2^2\eta_1} + O\left(\frac{1}{\tau}\right). \quad (3.31) \]

These equations show that the asymptotic expressions for the solutions require powers of $1/\tau$ as gauge functions. The exponential functions in the expression for intensity are transcendentally small as optical thickness $\tau$ tends to infinity. This suggests that there exist intensity boundary layers near both boundaries.

The leading order expression for heat flux does not include the contribution from the intensity field, although it includes radiative parameters. The reason for this is that the optically thick medium participates so actively in radiation transfer that the interaction of radiation and conduction results in a correction in the temperature field. The last term in the expression for the temperature field supports this fact. Consequently, radiation makes a contribution to heat flux through the temperature field and not through the intensity field.

### 3.5 Summary

The interaction of radiation and conduction in a one-dimensional semi-transparent slab has been examined when temperature variations are small. The linearization has made it possible to obtain a closed-form solution, although the constants embedded in the expression are too lengthy to be displayed explicitly. For certain limiting situations the solution has been expanded as asymptotic series. The expanded solutions have shown the structure of the solution at each limiting situation.
radiation dominates conduction and $P$ was small, the solution required powers of $\sqrt{P}$ as gauge functions and thermal boundary layers appear near both boundaries. For the optically thin and thick media, the asymptotic expansions suggest that gauge functions contain powers of $\tau$ and $1/\tau$, respectively. The leading order solution for a thin medium did not show a boundary layer character. In the case of a thick medium, intensity boundary layers existed near both boundaries.

The results obtained here have some limitations when used in the original non-linear problem because perturbation expansions were taken after linearization of the fourth power of temperature. The linearization gave

$$\Theta^4 \sim 4\theta_s^2 \Theta - 3\theta_s^4 + O(|1 - \theta_s|^2).$$

Substituting the perturbation expansions of $\Theta$ and collecting like powers of $\varepsilon$ yielded

$$\Theta^4 \sim 4\theta_s^3[\Theta_0 + \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \cdots] - 3\theta_s^4 + O(|1 - \theta_s|^2)$$

$$\sim (4\theta_s^3\Theta_0 - 3\theta_s^4) + \varepsilon[4\theta_s^3\Theta_1] + \varepsilon^2[4\theta_s^3\Theta_2] + O(\varepsilon^3) + O(|1 - \theta_s|^2). \quad (3.32)$$

Thus, the discussion in this chapter is valid under the assumption that

$$O(|1 - \theta_s|^2) = O(\varepsilon^n), \quad (3.33)$$

where $n$ is the order of the truncated part of the solution. Physically, it means that the temperature difference $(1 - \theta_s)$ is small for linearization to be valid, but is larger than any perturbation parameter.
In the previous chapter, the interaction of radiation and conduction has been discussed for a semi-transparent slab under the assumption of small temperature variations. It was found that in the situation dominated by radiation the gauge functions for the asymptotic expansions of the temperature and intensity fields need to be powers of $\sqrt{\tau}$ and that in the limits of opacity they should be powers of $\tau$. Each limiting solution has been obtained by expanding the exact solution asymptotically for various limiting situations. In this chapter the same one-dimensional problem is discussed again, and limiting solutions are sought by using the method of matched asymptotic expansions. Unlike in the previous chapter, the assumption of small temperature variations is not used. Thus, the governing equations and the boundary conditions are

$$\frac{d^2 \Theta}{dx^2} = \frac{\tau \eta}{\rho} (\Theta^4 - H),$$
$$\frac{d^2 H}{dx^2} = 3 \tau^2 (H - \Theta^4),$$

$$\Theta(0) = 1,$$
$$-\frac{d \Theta}{dx}(1) = \mathcal{B}[\Theta(1) - \theta_e],$$
$$\frac{d H}{dx}(0) = \frac{\tau \gamma_0}{\eta} [H(0) - 1],$$
$$\frac{d H}{dx}(1) = -\frac{\tau \gamma_1}{\eta} [H(1) - \theta_s^4].$$

(4.1)
where the non-dimensional variables and parameters are

\[
x = \frac{z}{L}, \quad \Theta = \frac{T}{T_0}, \quad H = \frac{J}{4\pi^2\sigma_0 T_0^4},
\]
\[
\tau = \sqrt{\kappa P \beta R L}, \quad \eta = \sqrt{\kappa P / \beta R}, \quad \mathcal{P} = \frac{k}{4\pi^2\sigma_0 T_0^3 L},
\]
\[
B = \frac{hL}{k}, \quad \theta_a = \frac{T_a}{T_0}, \quad \theta_s = \frac{T_s}{T_0}, \quad \gamma_0 = \frac{3}{2}(2 - \epsilon), \quad \gamma_1 = \frac{3}{2}(1 - \rho).
\]

4.1 Radiation-Dominant Situation

In the situation dominated by radiation, the parameter \( \mathcal{P} \) is small. Thus, an asymptotic expansion can be constructed as a series in powers of \( \epsilon \), where \( \epsilon = \sqrt{\mathcal{P}} \). The perturbation expansions of temperature and intensity therefore take the forms

\[
\Theta(x) = \Theta_0(x) + \epsilon \Theta_1(x) + \epsilon^2 \Theta_2(x) + \cdots,
\]
\[
H(x) = H_0(x) + \epsilon H_1(x) + \epsilon^2 H_2(x) + \cdots. \tag{4.3}
\]

Substituting these into the governing equations (4.1) and the boundary conditions (4.2) and equating like powers of \( \epsilon \) yield

\[
\frac{d^2 \Theta_0}{dx^2} = \frac{d^2 H_0}{dx^2} = 0, \tag{4.4}
\]
\[
\frac{d^2 \Theta_1}{dx^2} = \frac{d^2 H_1}{dx^2} = 0, \tag{4.5}
\]
\[
\frac{d^2 \Theta_2}{dx^2} = r \eta (4\Theta_0^2 \Theta_2 + 6\Theta_0^3 \Theta_1^2 - H_2), \quad \frac{d^2 H_2}{dx^2} = -\frac{3\tau d^2 \Theta_0}{\eta dx^2}, \tag{4.6}
\]

and the corresponding boundary conditions are

\[
\Theta_0(0) = 1, \quad \Theta_n(0) = 0, \quad (n = 1, 2, \cdots),
\]
\[ -\frac{d\Theta_0}{dx}(1) = B(\Theta_0(1) - \theta_a), \quad -\frac{d\Theta_n}{dx}(1) = B\Theta_n(1), \quad (n = 1, 2, \ldots), \]
\[ \frac{dH_n}{dx}(0) = \frac{\tau \gamma_0}{\eta} H_n(0), \quad (n = 1, 2, \ldots), \]
\[ \frac{dH_0}{dx}(1) = -\frac{\tau \gamma_1}{\eta} (H_0(1) - \theta_s^1), \quad \frac{dH_n}{dx}(1) = -\frac{\tau \gamma_1}{\eta} H_n(1), \quad (n = 1, 2, \ldots). \quad (4.7) \]

### 4.1.1 Outer Expansion

The leading order solutions can be written as

\[ H_0(x) = a_0^4 + (b_0^4 - a_0^4)x, \quad \Theta_0(x) = [a_0^4 + (b_0^4 - a_0^4)x]^{1/4}. \quad (4.8) \]

The boundary conditions for the intensity give for the constants

\[ a_0^4 = \frac{\eta \gamma_1 \theta_s^4 + \eta \gamma_0 + \tau \gamma_1 \gamma_0}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0}, \quad b_0^4 = \frac{\eta \gamma_1 \theta_s^4 + \eta \gamma_0 + \tau \gamma_1 \gamma_0 \theta_s^4}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0}. \quad (4.9) \]

Since the leading order of the intensity field satisfies the boundary conditions, it is completely determined by the outer expansion. The leading order solution for the temperature, given by the first of equations (4.4), does not satisfy either of the temperature conditions at the boundaries. Thus, thermal boundary layers must exist near both the left and the right boundaries and, through these, temperature adjusts to its appropriate values. That is, boundary layers exist for the sake of the temperature and provide a correction for the higher order terms of the intensity field.

The higher order solutions can be similarly cast into the forms

\[ H_1(x) = a_1 + (b_1 - a_1)x, \]

\[ \Theta_1(x) = \frac{H_1}{4\Theta_0^3}, \quad (4.10) \]

\[ H_2(x) = -\frac{3\tau}{\eta} \Theta_0 + a_2 + \frac{3\tau}{\eta} a_0 + [b_2 - a_2 + \frac{3\tau}{\eta} (b_0 - a_0)]x, \]
\[
\Theta_2(x) = \frac{1}{4\Theta_3^3} \left( \frac{1}{\tau \eta} \frac{d^2 \Theta_0}{dx^2} - 6\Theta_0^2 \Theta_1^2 + H_2 \right), \quad (4.11)
\]
\[
H_3(x) = -\frac{3\tau}{\eta} \Theta_1 + a_3 + \frac{3\tau a_1}{4\eta a_3^3} + \left[ b_3 - a_3 + \frac{3\tau}{4\eta} \left( \frac{b_3}{b_0} - \frac{a_1}{a_3^3} \right) \right] x,
\]
where constants \(a_n\) and \(b_n\) will be determined from matching to the inner expansions.

**4.1.2 Inner Expansion Near \(x = 1\)**

Near the right edge, the following variables are introduced:

\[
x = 1 - \varepsilon \tilde{x}, \quad \Theta(x) = \tilde{\Theta}(\tilde{x}), \quad H(x) = \tilde{H}(\tilde{x}). \quad (4.12)
\]

The solutions expanded as perturbation series

\[
\tilde{\Theta}(\tilde{x}) = \tilde{\Theta}_0(\tilde{x}) + \varepsilon \tilde{\Theta}_1(\tilde{x}) + \varepsilon^2 \tilde{\Theta}_2(\tilde{x}) + \cdots,
\]
\[
\tilde{H}(\tilde{x}) = \tilde{H}_0(\tilde{x}) + \varepsilon \tilde{H}_1(\tilde{x}) + \varepsilon^2 \tilde{H}_2(\tilde{x}) + \cdots, \quad (4.13)
\]
yield the equations

\[
\frac{d^2 \tilde{\Theta}_0}{d\tilde{x}^2} = \tau \eta (\tilde{\Theta}_0' - \tilde{H}_0), \quad \quad \frac{d^2 \tilde{H}_0}{d\tilde{x}^2} = 0, \quad (4.14)
\]
\[
\frac{d^2 \tilde{\Theta}_1}{d\tilde{x}^2} = \tau \eta (4\tilde{\Theta}_0^2 \tilde{\Theta}_1 - \tilde{H}_1), \quad \quad \frac{d^2 \tilde{H}_1}{d\tilde{x}^2} = 0, \quad (4.15)
\]
\[
\frac{d^2 \tilde{\Theta}_2}{d\tilde{x}^2} = \tau \eta (6\tilde{\Theta}_0^2 \tilde{\Theta}_1^2 + 4\tilde{\Theta}_0^2 \tilde{\Theta}_2 - \tilde{H}_2), \quad \quad \frac{d^2 \tilde{H}_2}{d\tilde{x}^2} = -\frac{3\tau}{\eta} \frac{d^2 \tilde{\Theta}_0}{d\tilde{x}^2}, \quad (4.16)
\]
\[
\vdots
\]

and the boundary conditions at \(\tilde{x} = 0\) are

\[
\frac{d\tilde{\Theta}_0}{d\tilde{x}}(0) = 0, \quad \quad \frac{d\tilde{H}_0}{d\tilde{x}}(0) = 0,
\]
\[
\frac{d\tilde{\Theta}_1}{d\tilde{x}}(0) = B[\tilde{\Theta}_0(0) - \Theta_2], \quad \quad \frac{d\tilde{H}_1}{d\tilde{x}}(0) = \frac{\tau \gamma_1}{\eta} [\tilde{H}_0(0) - \Theta_2'],
\]
\[
\frac{d\tilde{\Theta}_n}{d\tilde{x}}(0) = B\tilde{\Theta}_{n-1}(0), \quad \quad \frac{d\tilde{H}_n}{d\tilde{x}}(0) = \frac{\tau \gamma_1}{\eta} \tilde{H}_{n-1}(0) \quad (n = 2, 3, \cdots). \quad (4.17)
\]
Another set of conditions comes from matching the inner expansion to the outer expansion as \( \bar{x} \) tends to infinity. The matching conditions for the inner expansion can be obtained by expanding the outer expansion in terms of the inner variable and collecting like powers of \( \varepsilon \). When carried out, they are

\[
\begin{align*}
\lim_{\bar{x} \to \infty} \tilde{\Theta}_0(\bar{x}) &= b_0, \\
\lim_{\bar{x} \to \infty} \tilde{\Theta}_1(\bar{x}) &= \frac{1}{4b_0^3}[b_1 - (b_0^4 - a_0^4)\bar{x}], \\
\lim_{\bar{x} \to \infty} \tilde{\Theta}_2(\bar{x}) &= \frac{1}{4b_0^3} \left\{ b_2 - (b_1 - a_1)\bar{x} - \frac{3}{8b_0^4}[b_1 - (b_0^4 - a_0^4)\bar{x}]^2 - \frac{3}{16} \left( \frac{b_0^4 - a_0^4}{\eta b_0^2} \right)^2 \right\}, \\
\lim_{\bar{x} \to \infty} \tilde{H}_0(\bar{x}) &= b_0^4, \\
\lim_{\bar{x} \to \infty} \tilde{H}_1(\bar{x}) &= b_1 - (b_0^4 - a_0^4)\bar{x}, \\
\lim_{\bar{x} \to \infty} \tilde{H}_2(\bar{x}) &= b_2 - (b_1 - a_1)\bar{x}, \\
\lim_{\bar{x} \to \infty} \tilde{H}_3(\bar{x}) &= b_3 - [b_2 - a_2 + \frac{3\tau}{\eta}(b_0 - a_0) - \frac{3\tau}{4\eta} \frac{(b_0^4 - a_0^4)}{b_0^3}]\bar{x}. \tag{4.18}
\end{align*}
\]

Although the matching conditions are given for temperature and intensity separately, it was found in this case that when temperature solutions are matched the corresponding intensity solutions are automatically matched and vice versa.

After the matching conditions and the boundary conditions for the temperature field have been applied, the solutions can be written as

\[
\begin{align*}
\tilde{H}_0(\bar{x}) &= b_0^4, \\
\tilde{\Theta}_0(\bar{x}) &= b_0, \\
\tilde{H}_1(\bar{x}) &= b_1 - (b_0^4 - a_0^4)\bar{x}, \\
\tilde{\Theta}_1(\bar{x}) &= \frac{1}{4b_0^3}[b_1 - (b_0^4 - a_0^4)\bar{x}] + \tilde{f}_1 \exp\left(-[4\tau\eta b_0^3]^{1/2} \bar{x} \right), \tag{4.19}
\end{align*}
\]
\[ \tilde{H}_2(\tilde{x}) = b_2 - (b_1 - a_1)\tilde{x}, \]
\[ \tilde{\Theta}_2(\tilde{x}) = \frac{1}{4b_0^3} \left\{ b_2 - (b_1 - a_1)\tilde{x} - \frac{3}{8b_0^3} [b_1 - (b_0^4 - a_0^4)]^2 - \frac{3}{16} \frac{(b_0^4 - a_0^4)^2}{\tau b_0^2} \right\} \]
\[ + \tilde{f}_2(\tilde{x}) \exp(-[4\tau \eta b_0^3]^{1/2}\tilde{x}), \]  
(4.21)
\[ \tilde{H}_3(\tilde{x}) = -\frac{3\tau}{\eta} \tilde{\Theta}_1 + b_3 + \frac{3\tau b_1}{4\eta b_0^3} - [b_2 - a_2 + \frac{3\tau}{\eta} (b_0 - a_0)]\tilde{x}, \]

where,
\[ \tilde{f}_1 = -\frac{1}{[4\tau \eta b_0^3]^{1/2}} \left[ \frac{b_0^4 - a_0^4}{4b_0^3} + B(b_0 - \theta_a) \right], \]
\[ \tilde{f}_2(\tilde{x}) = \frac{1}{64b_0^3} \left\{ \frac{16}{[4\tau \eta b_0^3]^{1/2}} [a_1 - b_1 \left( \frac{a_0^4}{b_0^4} + B \frac{\theta_a}{b_0} \right)] + [4B(1 - \frac{\theta_a}{b_0}) + 1 - \frac{a_0^4}{b_0^4}] \right\} \]
\[ + \frac{2b_1}{[4\tau \eta b_0^3]^{1/2}} + \frac{4B \theta_a}{\tau \eta} + 6b_1 \tilde{x} - (b_0^4 - a_0^4)(3\tilde{x}^2 + \frac{3}{[4\tau \eta b_0^3]^{1/2}} \tilde{x} + \frac{7}{4\tau \eta b_0^2}) \]
\[ + \frac{1}{128\tau \eta b_0^2} [4B(1 - \frac{\theta_a}{b_0}) + 1 - \frac{a_0^4}{b_0^4}] \exp(-[4\tau \eta b_0^3]^{1/2}\tilde{x}). \]

The boundary conditions for the intensity lead to a set of relations between \( a_n \) and \( b_n \). For \( n = 0, 1 \) and \( 2 \), they are
\[ -b_0^4 + a_0^4 = \frac{\tau \gamma_1}{\eta} (b_0^4 - \theta_a^4), \]
\[ -b_1 + a_1 = \frac{\tau \gamma_1}{\eta} b_1, \]
\[ -\frac{3\tau}{\eta} B(b_0 - \theta_a) - b_2 + a_2 - \frac{3\tau}{\eta} (b_0 - a_0) = \frac{\tau \gamma_1}{\eta} b_2. \]  
(4.22)

A similar set will be drawn from the inner expansion at the other side.

### 4.1.3 Inner Expansion Near \( x = 0 \)

To develop the temperature and intensity fields near the left edge, the following variables are introduced:
\[ x = \varepsilon \tilde{x}, \quad \Theta(x) = \tilde{\Theta}(\tilde{x}), \quad H(x) = \tilde{H}(\tilde{x}), \]  
(4.23)
and the solutions expanded as perturbation series
\[ \tilde{\Theta}(\bar{x}) = \tilde{\Theta}_0(\bar{x}) + \epsilon \tilde{\Theta}_1(\bar{x}) + \epsilon^2 \tilde{\Theta}_2(\bar{x}) + \cdots, \]
\[ \tilde{H}(\bar{x}) = \tilde{H}_0(\bar{x}) + \epsilon \tilde{H}_1(\bar{x}) + \epsilon^2 \tilde{H}_2(\bar{x}) + \cdots. \] (4.24)

Substituting these into the governing equations and boundary conditions and equating like powers of \( \epsilon \) yield
\[ \frac{d^2 \tilde{\Theta}_0}{d\bar{x}^2} = \tau \eta (\tilde{\Theta}_0^3 - \tilde{H}_0), \]
\[ \frac{d^2 \tilde{H}_0}{d\bar{x}^2} = 0, \] (4.25)
\[ \frac{d^2 \tilde{\Theta}_1}{d\bar{x}^2} = \tau \eta (4 \tilde{\Theta}_0^3 \tilde{\Theta}_1 - \tilde{H}_1), \]
\[ \frac{d^2 \tilde{H}_1}{d\bar{x}^2} = 0, \] (4.26)
\[ \frac{d^2 \tilde{\Theta}_2}{d\bar{x}^2} = \tau \eta (6 \tilde{\Theta}_0^3 \tilde{\Theta}_2 + 4 \tilde{\Theta}_0^3 \tilde{\Theta}_0 - \tilde{H}_2), \]
\[ \frac{d^2 \tilde{H}_2}{d\bar{x}^2} = -\frac{3 \tau \eta}{\tilde{H}_0^2} \frac{d^2 \tilde{\Theta}_0}{d\bar{x}^2}, \] (4.27)
\[ \vdots \]
\[ \frac{d^2 \tilde{\Theta}_n}{d\bar{x}^2} = \frac{3 \tau \eta}{\tilde{H}_{n-1}^2} \frac{d^2 \tilde{\Theta}_{n-1}}{d\bar{x}^2}, \]
\[ \frac{d^2 \tilde{H}_n}{d\bar{x}^2} = 0, \]
\[ (n = 2, 3, \cdots). \] (4.28)

The boundary conditions at \( \bar{x}=0 \) are
\[ \tilde{\Theta}_0(0) = 1, \]
\[ \frac{d \tilde{H}_0}{d\bar{x}}(0) = 0, \]
\[ \tilde{\Theta}_1(0) = 0, \]
\[ \frac{d \tilde{H}_1}{d\bar{x}}(0) = \frac{\tau \gamma_0}{\eta} [\tilde{H}_0(0) - 1], \]
\[ \tilde{\Theta}_n(0) = 0, \]
\[ \frac{d \tilde{H}_n}{d\bar{x}}(0) = \frac{\tau \gamma_0}{\eta} \tilde{H}_{n-1}(0), \] (n = 2, 3, \( \cdots \)). (4.28)

The matching conditions require that
\[ \lim_{\bar{x} \to \infty} \tilde{\Theta}_0(\bar{x}) = a_0, \]
\[ \lim_{\bar{x} \to \infty} \tilde{\Theta}_1(\bar{x}) = \frac{1}{4a_0^3} [a_1 + (b_0^4 - a_0^4)] \bar{x}, \]
\[ \lim_{\bar{x} \to \infty} \tilde{\Theta}_2(\bar{x}) = \frac{1}{4a_0^3} \left\{ a_2 + (b_1 - a_1) \bar{x} - \frac{3}{8a_0^4} [a_1 + (b_0^4 - a_0^4)]^2 \bar{x}^2 - \frac{3}{16} \frac{(b_0^4 - a_0^4)^2}{\tau \eta a_0^4} \right\}, \]
\[ \lim_{\bar{x} \to \infty} \tilde{H}_0(\bar{x}) = a_0^4, \]
\[
\lim_{\xi \to \infty} \bar{H}_1(\bar{x}) = a_1 + (b_0^4 - a_0^4)\bar{x}, \\
\lim_{\xi \to \infty} \bar{H}_2(\bar{x}) = a_2 + (b_1 - a_1)\bar{x}, \\
\lim_{\xi \to \infty} \bar{H}_3(\bar{x}) = a_3 + [b_2 - a_2 + \frac{3\tau}{\eta} (b_0 - a_0) - \frac{3\tau}{4\eta a_0^4} (b_0^4 - a_0^4)]\bar{x}.
\]

(4.29)

The solutions for the intensity field \( \tilde{H}_n(\bar{x}) \) that satisfy the matching conditions can be written as

\[
\begin{align*}
\tilde{H}_0(\bar{x}) &= a_0^4, \\
\tilde{H}_1(\bar{x}) &= a_1 + (b_0^4 - a_0^4)\bar{x}, \\
\tilde{H}_2(\bar{x}) &= \frac{-3\tau}{\eta} \tilde{\Theta}_0(\bar{x}) + \frac{3\tau}{\eta} a_0 + [a_2 + (b_1 - a_1)]\bar{x}, \\
\tilde{H}_3(\bar{x}) &= \frac{-3\tau}{\eta} \tilde{\Theta}_1(\bar{x}) + a_3 + \frac{3\tau a_1}{4\eta a_0^4} + [b_2 - a_2 + \frac{3\tau}{\eta} (b_0 - a_0)]\bar{x}.
\end{align*}
\]

(4.30)

However, owing to the nonlinearity of the equations, obtaining a solution of the temperature field in closed form seems hopeless.

The boundary conditions for the intensity field give another set of relations between constants \( a_n \) and \( b_n \). For \( n = 0, 1, \) and \( 2 \) they are

\[
\begin{align*}
 b_0^4 - a_0^4 &= \frac{\tau \gamma_0}{\eta} (a_0^4 - 1), \\
 \frac{3\tau}{\eta} \frac{d \tilde{\Theta}_0}{d\bar{x}}(0) + b_1 - a_1 &= \frac{\tau \gamma_0}{\eta} a_1, \\
 \frac{-3\tau}{\eta} \frac{d \tilde{\Theta}_1}{d\bar{x}}(0) + b_2 - a_2 + \frac{3\tau}{\eta} (b_0 - a_0) &= \frac{\tau \gamma_0}{\eta} \left(\frac{3\tau}{\eta} \frac{a_0}{a_0 + a_2}\right).
\end{align*}
\]

(4.31)

The first equation and its counterpart in the set (4.22) yield \( a_0^4 \) and \( b_0^4 \) identical to those given in equations (4.9), which were obtained from the outer expansion. To solve for \( a_1 \) and \( b_1 \), it is necessary to determine \( d\tilde{\Theta}_0/d\bar{x}(0) \) in a closed-form. This is possible because the equation for \( \tilde{\Theta}_0 \) can be reduced in order and its matching
condition satisfied. To carry this out, both sides of the temperature equation in (4.25) are multiplied by \( d\theta_0 / d\bar{x} \), are integrated once, and have the matching condition as \( \bar{x} \) tends to infinity applied. These operations give

\[
\frac{d\theta_0}{d\bar{x}} = - \left[ \frac{2\tau \eta}{5} (\theta_0^5 - 5\alpha_0^4 \theta_0 + 4\alpha_0^5) \right]^{1/2}, \quad \theta_0(0) = 1. \quad (4.32)
\]

From this the temperature gradient at the wall, \( d\theta_0 / d\bar{x}(0) \), is obtained in terms of a known constant \( \alpha_0 \). Then solving the second equation in the set (4.31) and its counterpart in the set (4.22) together yields the expressions for \( a_1 \) and \( b_1 \) as

\[
a_1 = \frac{3(\eta + \tau \gamma_1)}{\eta(\gamma_1 + \gamma_0) + \tau \gamma_1 \gamma_0} \left[ \frac{2\tau \eta}{5} (1 - 5\alpha_0^4 + 4\alpha_0^5) \right]^{1/2},
\]

\[
b_1 = \frac{3\eta}{\eta(\gamma_1 + \gamma_0) + \tau \gamma_1 \gamma_0} \left[ \frac{2\tau \eta}{5} (1 - 5\alpha_0^4 + 4\alpha_0^5) \right]^{1/2}. \quad (4.33)
\]

The coefficients \( a_2 \) and \( b_2 \) involve the derivative of \( \bar{\theta}_1(\bar{x}) \). The equation for \( \bar{\theta}_1(\bar{x}) \) is a linear equation with a variable coefficient, the coefficient being a function of \( \bar{\theta}_0(\bar{x}) \). Thus, it cannot be determined analytically without an explicit expression for \( \bar{\theta}_0(\bar{x}) \).

**Approximation of \( \bar{\theta}_n(\bar{x}) \)**

By determining the first integral of the equation for \( \bar{\theta}_0(\bar{x}) \), it was possible to evaluate constants \( a_1 \) and \( b_1 \). However, the temperature field is still unknown. Numerical solutions can obviously be easily obtained. Here, an approximate expression for the analytic solution is derived as an alternative to a numerical solution. It is ad hoc in nature and thus stands apart from the method of matched asymptotic expansions used to develop the solution. To determine an approximation far away from \( \bar{x} = 0 \), one can start with the matching condition. It requires that the solution far away from
\(x = 0\) is in the form

\[
\tilde{\Theta}_0(\bar{x}) = a_0 + \tilde{\theta}_0(\bar{x}), \quad |\tilde{\theta}_0(\bar{x})| \ll a_0.
\]

Thus, substituting this into the equation and solving for \(\tilde{\theta}_0(\bar{x})\) yields

\[
\tilde{\Theta}_0(\bar{x}) \approx a_0 + \tilde{\Theta}_0(\bar{x}) \exp(-[4\tau \eta a_0^2]^{1/2}\bar{x}), \quad (4.34)
\]

where \(\tilde{\theta}_0(\bar{x})\) is undetermined. In a similar way, \(\tilde{\Theta}_1(\bar{x})\) can be written as

\[
\tilde{\Theta}_1(\bar{x}) \approx \frac{1}{4a_0}[a_1 + (b_0^4 - a_0^4)\bar{x}] + \tilde{\Theta}_1(\bar{x}) \exp(-[4\tau \eta a_0^2]^{1/2}\bar{x}), \quad (4.35)
\]

where \(\tilde{\Theta}_1(\bar{x})\) is also undetermined.

To obtain an approximation valid near \(x = 0\), \(\tilde{\Theta}_0(\bar{x})\) can be written as a power series

\[
\tilde{\Theta}_0(\bar{x}) = \sum_{n=0}^{\infty} \tilde{a}_n \bar{x}^n, \quad (4.36)
\]

with

\[
\tilde{a}_0 = 1, \quad \tilde{a}_1 = -\left[\frac{2\tau \eta}{5}(1 - 5a_0^4 + 4a_0^5)\right]^{1/2}, \quad \ldots. \quad (4.37)
\]

A series solution for \(\tilde{\Theta}_1(\bar{x})\) is

\[
\tilde{\Theta}_1(\bar{x}) = \sum_{n=1}^{\infty} \tilde{b}_n \bar{x}^n, \quad (4.38)
\]

with

\[
\tilde{b}_2 = -\frac{\tau \eta}{2} \tilde{a}_1, \quad \tilde{b}_3 = \frac{\tau \eta}{6}(4\tilde{b}_1 - b_0^4 + a_0^4), \quad \ldots,
\]

but \(\tilde{b}_1\) remains undetermined.

For a unique expression in the entire region of \(\bar{x}\), the functional forms in the far-away region of equations (4.34) and (4.35) are kept and made to satisfy the boundary
conditions and derivatives at $\bar{x} = 0$ as best as possible. Then the following forms become useful approximations for the first two orders of the temperature field:

$$
\bar{\Theta}_0(\bar{x}) \approx a_0 + (1 - a_0)\exp(-[4\tau\eta a_0^3]^{1/2}\lambda_0 \bar{x}),
$$

$$
\bar{\Theta}_1(\bar{x}) \approx \frac{1}{4a_0^2}[a_1 + (b_0^4 - a_0^4)\bar{x}] - \frac{a_1}{4a_0^3}\exp(-[4\tau\eta a_0^3]^{1/2}\bar{x}), \quad (4.39)
$$

where

$$
\lambda_0^2 = \frac{1 - 5a_0^4 + 4a_0^5}{10a_0^3(1 - a_0)^2}.
$$

An approximate expression for the temperature field can now be obtained in the spirit of composite expansions of the asymptotic solution as the sum of the inner and the outer expansions minus their common parts.

$$
\Theta(x) \approx \left[ a_0^4 + (b_0^4 - a_0^4)x \right]^{1/4} + (1 - a_0)\exp(-[4\tau\eta a_0^3]^{1/2}\lambda_0 \frac{x}{\sqrt{P}}) \\
+ \sqrt{P} \left\{ \frac{a_1 + (b_1 - a_1)x}{4[a_0^4 + (b_0^4 - a_0^4)x]^{3/4}} - \frac{a_1}{4a_0^3}\exp(-[4\tau\eta a_0^3]^{1/2}\frac{x}{\sqrt{P}}) \\
- \frac{1}{[4\tau\eta b_0^3]^{1/2}}\frac{b_0^4 - a_0^4}{4b_0^4} + B(b_0 - \theta_a) \right\}\exp(-[4\tau\eta b_0^3]^{1/2}\frac{1 - x}{\sqrt{P}}), \quad (4.40)
$$

This representation carries the character of the solution, such as a strong boundary layer near $x = 0$ and a relative weak one near $x = 1$, as was found in the linearized problem.

### 4.1.4 Composite Expansion and Heat Flux

Owing to the difficulties in getting an explicit solution of the temperature field near $x = 0$, only a two term expansion has been worked out. The intensity field is

$$
H(x) \sim [a_0^4 + (b_0^4 - a_0^4)x] + \sqrt{P}[a_1 + (b_1 - a_1)x] + O(P)
$$
\[
\begin{align*}
\sim & \left[ \frac{\eta \gamma \theta^4_s + \eta \gamma_0 + \tau \gamma_1 \gamma_0}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} - \frac{\tau \gamma_1 \gamma_0 (1 - \theta^4_s)}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} \right] x \\
& + \sqrt{\mathcal{P}} \left[ \frac{3(\eta + \tau \gamma_1)(\tau \eta)^{1/2}}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} - \frac{3 \tau \gamma_1 (\tau \eta)^{1/2}}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} \right] x \\
& + \left[ \frac{2 \eta \gamma_1 (1 - \theta^4_s)}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} - \frac{8}{5} \left[ \frac{\eta \gamma_1 \theta^4_s + \eta \gamma_0 + \tau \gamma_1 \gamma_0}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} \right]^{1/2} \right] \\
& + O(\mathcal{P}), \quad (4.41)
\end{align*}
\]

and the non-dimensional heat flux $\Psi$ is

\[
\begin{align*}
\Psi & = -4 \mathcal{P} \frac{d\Theta}{dx} - \frac{4 \eta}{3\tau} \frac{dH}{dx} \\
\sim & -4 \eta \frac{dH_0}{dx} - \sqrt{\mathcal{P}} \frac{dH_1}{dx} + \sum_{n=2}^{M-1} \mathcal{P}^{n/2} \left[ \frac{4}{3\tau} \frac{d\Theta_{n-2}}{dx} + \frac{4 \eta}{3\tau} \frac{dH_n}{dx} \right] + O(\mathcal{P}^M) \\
\sim & -4 \frac{\eta \gamma_1 \gamma_0}{3 \eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} (1 - \theta^4_s) \\
& - \sqrt{\mathcal{P}} \frac{4(\tau \eta)^{1/2} \eta \gamma_1}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} \left[ \frac{2 \eta \gamma_1 (1 - \theta^4_s)}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} - \frac{8}{5} \left[ \frac{\eta \gamma_1 \theta^4_s + \eta \gamma_0 + \tau \gamma_1 \gamma_0}{\eta \gamma_1 + \eta \gamma_0 + \tau \gamma_1 \gamma_0} \right]^{1/2} \right] \\
& + O(\mathcal{P}). \quad (4.42)
\end{align*}
\]

Since the two leading terms in both of the inner expansions near $x = 0$ and $x = 1$ are equal to the matching conditions in each region, the intensity field does not show any boundary layer character at the first two orders. The asymptotic expression for the heat flux reveals that the effect of conduction does not appear up to the first two orders. All these facts were already noted in the solution of the linearized problem.
4.1.5 Results and Discussion

From the explicit expressions for temperature and intensity fields and for heat flux, some information can be drawn in regard the influence of parameters on the solutions. As a special situation consider the radiation-dominant limit, that is, pure radiation between two black surfaces. Under this situation the expressions for the temperature and intensity fields reduce to

$$
\Theta(x; \mathcal{P} = 0, \gamma_0 = \gamma_1 = \frac{3}{2}) = \left[ \frac{2\eta(1 + \theta_4^4) + 3\tau}{4\eta + 3\tau} - \frac{3\tau(1 - \theta_4^4)}{4\eta + 3\tau} \right]^{1/4},
$$

and non-dimensional heat flux is

$$
\Psi(\mathcal{P} = 0, \gamma_0 = \gamma_1 = \frac{3}{2}) = \frac{4\eta}{4\eta + 3\tau}(1 - \theta_4^4).
$$

As expected, the so-called radiation slip is recovered at $x = 0$, since these expressions carry only the leading order of the outer expansions.

When the bounding surface at the right is a perfectly reflecting wall, that is, $\gamma_1 = 0$, which physically implies that the outgoing radiative heat flux through the boundary at $x = 1$ vanishes, then the radiation-collecting surface far away is ineffective and its temperature $\theta_s$ should disappear from the solutions. In this situation the temperature and intensity fields reduce to the expressions

$$
\Theta(x; \gamma_1 = 0) \sim 1 - \sqrt{\mathcal{P}} \frac{B(1 - \theta_s)}{2(\tau \eta)^{1/2}} \exp(-2[\tau \eta]^{1/2} \frac{1 - x}{\sqrt{\mathcal{P}}}) + O(\mathcal{P}),
$$

$$
H(x; \gamma_1 = 0) \sim 1 + O(\mathcal{P}).
$$

They show that at a short distance away from the right wall, temperature is almost
uniform. The heat flux is small, of order $O(\mathcal{P})$. The fact that there is no boundary layer near the left wall has been recovered in the above expressions, which could not be seen in the linearized problem.

Two term expansions of the temperature and intensity fields and of the heat flux for various values of Planck number have been evaluated and are compared with those obtained from the numerical solutions. In the comparison, the other parameters are held at $\tau = 1, \eta = 1, \epsilon = 0.5, \rho = 0.5, B = 1, \theta_s = 0.5$, and $\theta_a = 0.5$. In Figure 7, the magnitude of the non-dimensional heat fluxes given by equation (4.42) are shown together with those from the numerical solutions. The deviation is noticeable when the Planck number is greater than 0.01, and there two term expansion underestimates the heat flux. The temperature fields by the asymptotic expansions are shown with those from the numerical solutions in Figure 8. The asymptotic expansions near $x = 0$ should be obtained by solving the temperature equations for $\tilde{\Theta}_0$ and $\tilde{\Theta}_1$ in (4.25) and (4.26) numerically. Their graphical representations were almost the same as those from their approximate expression (4.40), and thus in the figure the results by using the approximate expression are shown. The numerical solutions were obtained by solving the governing equations (4.1) directly in the same way as was discussed in the previous chapter. The figure shows that the asymptotic expansions agree qualitatively with the numerical solutions, although the expression is an approximate one, and that for relatively large Planck numbers the temperature fields by the two term expansion are higher than those obtained numerically. Numerical values of heat flux and the errors in heat flux and temperatures are tabulated in Table 4 for various values of
Figure 7: Non-dimensional heat flux for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when \( \tau = 1, \eta = 1, \epsilon = 0.5, \rho = 0.5, \theta_\alpha = 0.5, B = 1, \) and \( \theta_\alpha = 0.5. \)
Figure 8: Non-dimensional temperature distributions for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when $\tau = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $B = 1$, and $\theta_a = 0.5$. 
Planck number. Each error is defined as

$$\Delta \Psi = \frac{|\Psi_{\text{asymptotic}} - \Psi_{\text{numerical}}|}{\Psi_{\text{numerical}}}, \quad \Delta \Theta = \frac{[\Theta_{\text{asymptotic}} - \Theta_{\text{numerical}}]_{\text{max}}}{[\Theta(0) - \Theta(1)]_{\text{numerical}}}. \quad (4.43)$$

At $P = 0.01$, where deviations in heat flux and in the temperature field between asymptotic and numerical solutions start to be noticeable, the error in heat flux is estimated as $0.9\%$ and for the temperature field it is $3.8\%$.

Table 4: Non-dimensional heat flux, its error and the error in temperature field by the method of matched asymptotic expansions for combined radiation and conduction across a slab, bounded by an opaque surface and a partially transparent surface, when $\tau = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $B = 1$, and $\theta_a = 0.5$.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\Psi_{\text{numerical}}$</th>
<th>$\Psi_{\text{asymptotic}}$</th>
<th>$\Delta \Psi$</th>
<th>$\Delta \Theta$</th>
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<td>3.8%</td>
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<td>0.3048</td>
<td>3.4%</td>
<td>9.4%</td>
</tr>
<tr>
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<td>0.3208</td>
<td>6.0%</td>
<td>14%</td>
</tr>
<tr>
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<td>0.3646</td>
<td>0.3337</td>
<td>8.5%</td>
<td>17%</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3979</td>
<td>0.3501</td>
<td>12%</td>
<td>22%</td>
</tr>
</tbody>
</table>
4.2 Optically Thin Medium

When the medium is optically thin, the parameter $\tau$ is small and an asymptotic expansion can be constructed in its powers. With $\varepsilon = \tau$, the perturbation expansions of temperature and intensity therefore take the forms

$$\Theta(x) = \Theta_0(x) + \varepsilon \Theta_1(x) + \varepsilon^2 \Theta_2(x) + \cdots,$$

$$H(x) = H_0(x) + \varepsilon H_1(x) + \varepsilon^2 H_2(x) + \cdots.$$  \hfill (4.44)

Substituting these into the governing equations (4.1) and equating like powers of $\varepsilon$ yield

$$\begin{align*}
\frac{d^2 \Theta_0}{dx^2} &= 0, \\
\frac{d^2 \Theta_1}{dx^2} &= \frac{\eta}{\mathcal{P}} (\Theta_0^2 - H_0) - \frac{3\mathcal{P}}{\eta} \frac{d^2 \Theta_1}{dx^2}, \\
\frac{d^2 \Theta_2}{dx^2} &= \frac{\eta}{\mathcal{P}} (4\Theta_0^2 \Theta_1 - H_1) - \frac{3\mathcal{P}}{\eta} \frac{d^2 \Theta_2}{dx^2}, \\
\frac{d^2 \Theta_3}{dx^2} &= \frac{\eta}{\mathcal{P}} (4\Theta_0^2 \Theta_2 + 6\Theta_0^2 \Theta_1^2 - H_2) - \frac{3\mathcal{P}}{\eta} \frac{d^2 \Theta_3}{dx^2},
\end{align*}$$

and similarly the boundary conditions become

$$\begin{align*}
\Theta_0(0) &= 1, \\
\Theta_n(0) &= 0, \quad (n = 1, 2, \cdots), \\
\frac{d\Theta_0}{dx}(1) &= B[\Theta_0(1) - \Theta_0], \\
\frac{d\Theta_n}{dx}(1) &= B\Theta_n(1), \quad (n = 1, 2, \cdots), \\
\frac{dH_0}{dx}(0) &= 0, \\
\frac{dH_1}{dx}(0) &= \frac{\gamma_0}{\eta} [H_0(0) - 1], \\
\frac{dH_n}{dx}(0) &= \frac{\gamma_0}{\eta} H_{n-1}(0), \quad (n = 2, 3, \cdots), \\
\frac{dH_0}{dx}(1) &= 0, \\
\frac{dH_1}{dx}(1) &= -\frac{\gamma_1}{\eta} [H_0(1) - \Theta_0^2], \\
\frac{dH_n}{dx}(1) &= -\frac{\gamma_n}{\eta} H_{n-1}(1), \quad (n = 2, 3, \cdots). \hfill (4.49)
\end{align*}$$
4.2.1 Perturbation Expansion

Since the equations are linear, solutions are easily obtained. For the leading order they are

\[ \Theta_0(x) = a_0 + b_0 x, \quad H_0(x) = c_0 + d_0 x. \]  

(4.50)

The boundary conditions for temperature and intensity give for constants

\[ d_0 = 0, \quad a_0 = 1, \quad b_0 = - \frac{B}{1 + B}(1 - \theta_c), \]  

(4.51)

and \( c_0 \) is undetermined. The next order solutions are

\[ \Theta_1(x) = a_1 + b_1 x + \frac{\eta}{\mathcal{P}} \left[ \frac{1}{30} \frac{(1 + b_0 x)^6}{b_0^2} - \frac{1}{2} c_0 x^2 \right], \]

\[ H_1(x) = c_1 + d_1 x, \]  

(4.52)

and the boundary conditions yield

\[ c_0 = \frac{\gamma_0 + \gamma_1 \theta_s^4}{\gamma_0 + \gamma_1}, \quad d_1 = - \frac{\gamma_1 \gamma_0}{\eta (\gamma_0 + \gamma_1)} (1 - \theta_s^4), \quad a_1 = - \frac{1}{30} \frac{\eta}{\mathcal{P} b_0^2}, \]

\[ b_1 = - \frac{\eta}{\mathcal{P}(1 + B)} \left\{ \left(1 + b_0^5 \right) \left(6 + B \right) + \frac{B}{30b_0^2} \left[ (1 + b_0^5) - 1 \right] - \left(1 + \frac{B}{2} \right) c_0 \right\}. \]  

(4.53)

The solutions of order \( \varepsilon^2 \) are

\[ \Theta_2(x) = a_2 + b_2 x + \frac{\eta}{\mathcal{P}} \left[ \frac{2}{15} b_0^3 b_1 x^6 + \left( \frac{1}{5} b_0^3 a_1 + 3 b_0^2 b_1 \right) x^5 \right. \]

\[ + \left( b_0 b_1 + b_0^2 a_1 \right) x^4 + \left( \frac{2}{3} b_1 + 2 b_0 a_1 - \frac{d_1}{6} \right) x^3 + \left( 2 a_1 - \frac{c_1}{2} \right) x^2 \]

\[ + \left( \frac{\eta}{\mathcal{P}} \right)^2 \left[ \frac{b_0^7}{825} x^{11} + \frac{b_0^6}{75} x^{10} + \frac{b_0^5}{15} x^9 + \frac{b_0^4}{5} x^8 \right. \]

\[ + \left( \frac{2}{5} - \frac{c_0}{21} \right) b_0^3 x^7 + \left( \frac{14}{25} - \frac{c_0}{5} \right) b_0^2 x^6 + \left( \frac{26}{25} - \frac{3}{10} c_0 \right) b_0 x^5 \]

\[ + \left( \frac{2}{5} - \frac{c_0}{6} \right) x^4 + \frac{1}{5b_0} x^3 + \frac{1}{15b_0^2} x^2 \left. \right], \]

\[ H_2(x) = c_2 + d_2 x - \frac{3\mathcal{P}}{\eta} \Theta_1(x), \]  

(4.54)
and their boundary conditions give

\[
c_1 = \frac{\gamma_0 + \gamma_1}{\gamma_0 + \gamma_1} \left\{ \frac{3}{5b_0} (1 + b_0)^5 - 1 - \frac{\gamma_1}{\gamma_0} \right\},
\]

\[
d_2 = \frac{3P}{\eta} b_1 + \frac{\gamma_0 + \gamma_1}{\gamma_0 + \gamma_1} \left\{ \frac{3}{5b_0} (1 + b_0)^5 + \frac{\gamma_1}{\gamma_0} \right\} - \frac{\gamma_1}{\eta} d_1 - 3c_0,
\]

\[
a_2 = 0,
\]

\[
b_2 = -\frac{\eta}{\mathcal{P}(1 + B)} \left[ \frac{2}{15} b_0^3 b_1 (6 + B) + \left( \frac{1}{5} b_0^3 a_1 + 3b_0^2 b_1 \right) (5 + B) \right.
\]

\[
+ (b_0 b_1 + b_0^2 a_1) (4 + B) + \left( \frac{2}{3} b_1 + 2b_0 a_1 - \frac{d_1}{6} \right) (3 + B)
\]

\[
+ \left( 2a_1 - \frac{c_1}{2} \right) (2 + B) \right]
\]

\[
- \frac{\eta^2}{\mathcal{P}^2(1 + B)} \left[ \frac{b_0^7}{825} (11 + B) + \frac{b_0^5}{75} (10 + B) + \frac{b_0^5}{15} (9 + B) + \frac{b_0^3}{5} (8 + B) \right.
\]

\[
+ \left( \frac{2}{5} - \frac{c_0}{21} \right) b_0^3 (7 + B) + \left( \frac{14}{25} - \frac{c_0}{5} \right) b_0^3 (6 + B)
\]

\[
+ \left( \frac{26}{25} - \frac{3}{10} c_0 \right) b_0 (5 + B) + \left( \frac{2}{5} - \frac{c_0}{6} \right) (4 + B)
\]

\[
+ \frac{1}{5b_0} (3 + B) + \frac{1}{5b_0} (2 + B) \right]. \tag{4.55}
\]

The solutions obtained above do not show any boundary layer character, and thus the problem is categorized as a regular perturbation problem. Higher order solution for intensity can be cast into

\[
H_n(x) = c_n + d_n x - \frac{3P}{\eta} \Theta_{n-1}(x), \tag{4.56}
\]

for \( n \geq 2 \). Thus, constants \( c_2 \) and \( d_3 \) can be obtained from \( H_5(x) \) as

\[
c_2 = \frac{1}{\gamma_0 + \gamma_1} \left\{ 3P \left[ \frac{d \Theta_2}{dx}(1) - b_2 \right] + \frac{3P \gamma_1}{\eta} \Theta_1(1) - \gamma_1 d_2 \right\},
\]

\[
d_3 = \frac{\gamma_0}{\eta} c_2 + \frac{3P}{\eta} b_2. \tag{4.57}
\]
From the above results non-dimensional heat flux $\Psi$ is derived as

$$\Psi = -4\mathcal{P} \frac{d\Theta}{dx} - \frac{4\eta}{3\tau} \frac{dH}{dx}$$

$$\sim -\frac{4\eta}{3\tau} \frac{dH_0}{dx} - (4\mathcal{P} \frac{d\Theta_0}{dx} + \frac{4\eta}{3} \frac{dH_1}{dx}) - \sum_{n=1}^{M} \tau^n \left(4\mathcal{P} \frac{d\Theta_n}{dx} + \frac{4\eta}{3} \frac{dH_{n+1}}{dx}\right) + O(\tau^{M+1})$$

$$\sim -(4\mathcal{P} b_0 + \frac{4\eta}{3} d_1) - \frac{4\eta}{3} \sum_{n=1}^{M-1} \tau^n d_{n+1} + O(\tau^M). \quad (4.58)$$

The quantity $d_n$ is an integration constant of the first integration for the intensity equation, given in equation (4.56). Owing to the fact that $dH_0/dx = 0$, the asymptotic expression for the heat flux shows that total heat flux at each order is an algebraic sum of conductive and radiative parts.

### 4.2.2 Results and Discussion

Two term expansions for the temperature and intensity fields and heat flux are

$$\Theta(x) \sim 1 - \frac{B}{1 + B}(1 - \theta_s)x$$

$$+ \tau \left\{ b_1 x + \frac{\eta}{\mathcal{P}} \left[ \frac{(1 + b_0 x)^6 - 1}{30 b_0^2} \frac{\gamma_0 + \gamma_1 \theta_s^4}{2(\gamma_0 + \gamma_1)^2} \right] \right\} + O(\tau^2), \quad (4.59)$$

$$H(x) \sim \frac{\gamma_0 + \gamma_1 \theta_s^4}{\gamma_0 + \gamma_1}$$

$$+ \tau \left\{ \frac{\eta}{\gamma_0 + \gamma_1} \left[ \frac{3(1 + b_0 x)^5 - 1}{5 b_0} - \frac{3}{5 b_0} + \frac{\gamma_0 \gamma_1^2}{\eta^2(\gamma_0 + \gamma_1)} (1 - \theta_s^4) - \frac{3 \gamma_0 + \gamma_1 \theta_s^4}{\gamma_0 + \gamma_1} \right] \right\} - \frac{\gamma_0 \gamma_1}{\eta(\gamma_0 + \gamma_1)} (1 - \theta_s^4)x + O(\tau^2), \quad (4.60)$$

$$\Psi \sim 4\mathcal{P} b_0 \frac{B}{1 + B}(1 - \theta_s) + \frac{4}{3} \frac{\gamma_0 \gamma_1}{\gamma_0 + \gamma_1} (1 - \theta_s^4)$$

$$+ \tau \left\{ \frac{4\eta}{1 + B} \left[ \frac{B}{30 b_0^2} (1 + b_0 x)^6 - 1 + \frac{6 + B}{30 b_0} (1 + b_0)^5 - \frac{(1 + B)}{2} \frac{\gamma_0 + \gamma_1 \theta_s^4}{\gamma_0 + \gamma_1} \right] \right\}$$

$$- \frac{4\eta \gamma_0}{\gamma_0 + \gamma_1} \left[ \frac{(1 + b_0)^5 - \gamma_1}{5 b_0} + \frac{1}{5 b_0 \gamma_0} + \frac{\gamma_0 \gamma_1^2}{3 \eta^2(\gamma_0 + \gamma_1)} (1 - \theta_s^4) - \gamma_0 + \gamma_1 \theta_s^4 \right] \right\}$$

$$+ O(\tau^2). \quad (4.61)$$
For a perfectly transparent medium, in the limit of optically thin media between two black surfaces, the above expressions reduce to the leading order terms, which are

\[
\Theta(x; \tau = 0, \gamma_0 = \gamma_1 = \frac{3}{2}) = 1 - \frac{B}{1 + B}(1 - \theta_a)x,
\]
\[
H(x; \tau = 0, \gamma_0 = \gamma_1 = \frac{3}{2}) = \frac{1 + \theta_a^4}{2},
\]
\[
\Psi(\tau = 0, \gamma_0 = \gamma_1 = \frac{3}{2}) = \frac{PB}{1 + B}(1 - \theta_a) + (1 - \theta_a^4).
\]

They show that there is no interaction between conduction and radiation and that the temperature field is entirely governed by conduction and the intensity field takes a constant value which is the algebraic mean of the intensities of the backgrounds. Furthermore, heat flux is the algebraic sum of the separate and independent contributions of conduction and radiation. As expected, the intensity field and radiative heat flux do not depend on the radiative properties of the medium. All these observations are precisely the same as in the discussion of the optically thin limit by Sparrow and Cess [49, p. 206–207, 253], where they drew these results from the exact formulation.

Results were obtained by evaluating two term expansions of temperature, intensity, and heat flux and compared with numerical solutions. They correspond to relatively small values of optical thickness, and the other parameters were held at \( P = 1, \eta = 1, \epsilon = 0.5, \rho = 0.5, B = 1, \theta_s = 0.5, \) and \( \theta_a = 0.5 \). The temperature fields are shown in Figure 9. When optical thickness is less than 1, the temperature field from equation (4.59) is almost equal to that from the numerical solution. The deviation is noticeable for \( \tau = 1 \) and the asymptotic solution underestimates the temperature field. Numerical values of heat flux and the errors in heat flux and in
Figure 9: Non-dimensional temperature distributions for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when $\mathcal{P} = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_{a} = 0.5$, $\mathcal{B} = 1$, and $\theta_{a} = 0.5$. 
temperature, defined as in equations (4.43), are tabulated in Table 5. A graphical comparison of heat flux will be seen in the next section together with the results for optically thick medium.

Table 5: Non-dimensional heat flux, its error, and the error in temperature field by the method of matched asymptotic expansions for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when $P = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $B = 1$, and $\theta_a = 0.5$.

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<th>$\tau$</th>
<th>$\Psi_{\text{numerical}}$</th>
<th>$\Psi_{\text{asymptotic}}$</th>
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<th>$\Delta \Theta$</th>
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<td>-</td>
<td>-</td>
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<td>1.4263</td>
<td>9.1%</td>
<td>-3.7%</td>
</tr>
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</table>
4.3 Optically Thick Medium

When the medium is optically thick, a quantity $1/\tau$ is small and an asymptotic expansion can be constructed as a series of inverse powers of the optical thickness. With $\varepsilon = 1/\tau$, the perturbation expansions of temperature and intensity take the forms

$$
\Theta(x) = \Theta_0(x) + \varepsilon\Theta_1(x) + \varepsilon^2\Theta_2(x) + \cdots,
$$

$$
H(x) = H_0(x) + \varepsilon H_1(x) + \varepsilon^2 H_2(x) + \cdots.
$$

(4.62)

Substituting these into the governing equations (4.1) and the boundary conditions (4.2) and equating like powers of $\varepsilon$ yield the equations

$$
\Theta_0' - H_0 = 0, \quad \Theta_0'' = 0, \quad 4\Theta_0^3\Theta_1 - H_1 = 0, \quad \frac{d^2 H_0}{dx^2} = -3(4\Theta_0^3\Theta_2 + 6\Theta_0^2\Theta_1^2 - H_2),
$$

(4.63) (4.64) (4.65)

and the boundary conditions

$$
\Theta_0(0) = 1, \quad \Theta_n(0) = 0, \quad (n = 1, 2, \cdots),
$$

$$
\frac{d\Theta_0}{dx}(1) = B[\Theta_0(1) - \theta_s], \quad \frac{d\Theta_n}{dx}(1) = B\Theta_n(1), \quad (n = 1, 2, \cdots),
$$

$$
H_0(0) = 1, \quad H_n(0) = \frac{\eta}{\gamma_0} \frac{dH_{n-1}}{dx}(0), \quad (n = 1, 2, \cdots),
$$

$$
H_0(1) = \theta_s^4, \quad H_n(1) = -\frac{\eta}{\gamma_1} \frac{dH_{n-1}}{dx}(1), \quad (n = 1, 2, \cdots). \quad (4.66)
$$

(4.67)
4.3.1 Outer Expansion

The solutions for the leading order can be written as

\[ \Theta_0(x) = a_0 - (1 - b_0)x, \quad H_0(x) = [a_0 - (1 - b_0)x]^4. \]  (4.68)

The boundary conditions for temperature give for the constants

\[ a_0 = 1, \quad b_0 = \frac{1 + B\theta_a}{1 + B}. \]  (4.69)

With these constants the leading order solution for intensity does not satisfy its boundary condition at \( x = 1 \). This suggests that the intensity field may need a boundary layer near \( x = 1 \) to adjust it to its proper value at that boundary. The solutions at next order can be cast into the following forms

\[ \Theta_1(x) = a_1 - (1 - b_1)x - \frac{\eta}{3P}[a_0 - (1 - b_0)x]^4, \]
\[ H_1(x) = 4 [a_0 - (1 - b_0)x]^3 \left\{ a_1 - (1 - b_1)x - \frac{\eta}{3P}[a_0 - (1 - b_0)x]^4 \right\}. \]  (4.70)

The boundary condition for temperature at \( x = 0 \) gives

\[ a_1 = \frac{\eta}{3P}. \]  (4.71)

Intensity solution \( H_1(x) \) does not satisfy its boundary condition at \( x = 0 \). Thus, the intensity field needs a boundary layer near \( x = 0 \) as well. Consequently, the intensity field requires a boundary layer near both boundaries. Since temperature and intensity are coupled, these boundary layers are also expected to provide a correction to the temperature field. The solutions valid in these boundary layers will be sought in the following sections. In dealing with these inner expansions, the constant \( b_0 \) obtained
in equation (4.69) will be shown to be invalid, and the outer expansion has to be reworked.

The solutions at order $\epsilon^2$ are written as

$$
\Theta_2(x) = a_2 - (1 - b_2)x - \frac{\eta}{3\mathcal{P}}H_1(x),
$$

$$
H_2(x) = 4\Theta_0^3\Theta_2 + 6\Theta_0^2\Theta_1 + \frac{1}{3} \frac{d^2 H_0}{dx^2}. \quad (4.72)
$$

### 4.3.2 Inner Expansion Near $x = 0$

For a solution valid near $x = 0$, the following variables are introduced:

$$
x = \epsilon \bar{x}, \quad \Theta(x) = \bar{\Theta}(\bar{x}), \quad H(x) = \bar{H}(\bar{x}). \quad (4.73)
$$

Solutions expanded as the perturbation series

$$
\bar{\Theta}(\bar{x}) = \bar{\Theta}_0(\bar{x}) + \epsilon \bar{\Theta}_1(\bar{x}) + \epsilon^2 \bar{\Theta}_2(\bar{x}) + \cdots,
$$

$$
\bar{H}(\bar{x}) = \bar{H}_0(\bar{x}) + \epsilon \bar{H}_1(\bar{x}) + \epsilon^2 \bar{H}_2(\bar{x}) + \cdots. \quad (4.74)
$$

give the equations

$$
\frac{d^2 \bar{\Theta}_0}{d\bar{x}^2} = 0, \quad \frac{d^2 \bar{H}_0}{d\bar{x}^2} = -3(\bar{\Theta}_0^4 - \bar{H}_0), \quad (4.75)
$$

$$
\frac{d^2 \bar{\Theta}_1}{d\bar{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \bar{H}_0}{d\bar{x}^2}, \quad \frac{d^2 \bar{H}_1}{d\bar{x}^2} = -3(4\bar{\Theta}_0^3\bar{\Theta}_1 - \bar{H}_1), \quad (4.76)
$$

$$
\frac{d^2 \bar{\Theta}_2}{d\bar{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \bar{H}_1}{d\bar{x}^2}, \quad \frac{d^2 \bar{H}_2}{d\bar{x}^2} = -3(4\bar{\Theta}_0^3\bar{\Theta}_2 + 6\bar{\Theta}_0^2\bar{\Theta}_1^2 - \bar{H}_2), \quad (4.77)
$$

$$
\frac{d^2 \bar{\Theta}_n}{d\bar{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \bar{H}_n}{d\bar{x}^2}, \quad \vdots \quad (4.78)
$$

and the boundary conditions at $\bar{x} = 0$ are

$$
\bar{\Theta}_0(0) = 1, \quad \bar{\Theta}_n(0) = 0, \quad (n = 1, 2, \cdots),
$$

$$
\frac{d\bar{H}_0}{d\bar{x}}(0) = \frac{\gamma_0}{\eta} [\bar{H}_0(0) - 1], \quad \frac{d\bar{H}_n}{d\bar{x}}(0) = \frac{\gamma_0}{\eta} \bar{H}_n(0), \quad (n = 1, 2, \cdots). \quad (4.79)
$$
The matching conditions as \( \bar{x} \) tends to infinity are

\[
\begin{align*}
\lim_{\bar{x} \to \infty} \Theta_0(\bar{x}) &= a_0, \\
\lim_{\bar{x} \to \infty} \Theta_1(\bar{x}) &= a_1 - \frac{\eta}{3\mathcal{P}}a_0^4 - (1 - b_0)\bar{x}, \\
\lim_{\bar{x} \to \infty} \Theta_2(\bar{x}) &= a_2 - \frac{4\eta}{3\mathcal{P}}(a_1 - \frac{\eta}{3\mathcal{P}}a_0^4)a_0^3 - [1 - b_1 - \frac{4\eta}{3\mathcal{P}}(1 - b_0)a_0^3]\bar{x}, \\
\lim_{\bar{x} \to \infty} \bar{H}_0(\bar{x}) &= a_0^4, \\
\lim_{\bar{x} \to \infty} \bar{H}_1(\bar{x}) &= 4a_0^3(a_1 - \frac{\eta}{3\mathcal{P}}a_0^4) - 4a_0^3(1 - b_0)\bar{x}.
\end{align*}
\tag{4.80}
\]

The solutions of these equations and boundary conditions are developed as

\[
\begin{align*}
\Theta_0(\bar{x}) &= a_0, \\
\bar{H}_0(\bar{x}) &= a_0^4, \\
\Theta_1(\bar{x}) &= a_1 - \frac{\eta}{3\mathcal{P}}a_0^4 - (1 - b_0)\bar{x}, \\
\bar{H}_1(\bar{x}) &= 4a_0^3(a_1 - \frac{\eta}{3\mathcal{P}}a_0^4) - 4a_0^3(1 - b_0)\bar{x} - \frac{4\eta(1 - b_0)}{\sqrt{3}\eta + \gamma_0}\exp(-\sqrt{3}\bar{x}), \\
\Theta_2(\bar{x}) &= a_2 - \frac{4\eta}{3\mathcal{P}}(a_1 - \frac{\eta}{3\mathcal{P}}a_0^4)a_0^3 - [1 - b_1 - \frac{4\eta}{3\mathcal{P}}(1 - b_0)a_0^3]\bar{x} - a_2\exp(-\sqrt{3}\bar{x}),
\end{align*}
\tag{4.82}
\]

where the constant \( a_n \) is obtained from the boundary conditions for temperature as

\[
a_0 = 1, \quad a_1 = \frac{\eta}{3\mathcal{P}}, \quad a_2 = -\frac{4\eta^2(1 - b_0)}{3\mathcal{P}(\sqrt{3}\eta + \gamma_0)}. \tag{4.83}
\]

The constants \( b_n \) are still undetermined. As expected from the outer expansions, boundary layer makes its contribution to the temperature field at third order and to the intensity field at second order.
4.3.3 Inner Expansion Near $x = 1$

To develop solutions for temperature and intensity valid near $x = 1$, the following variables are introduced

$$x = 1 - \varepsilon \tilde{x}, \quad \Theta(x) = \tilde{\Theta}(\tilde{x}), \quad H(x) = \tilde{H}(\tilde{x}). \quad (4.84)$$

For the following perturbation expansions of the solutions

$$\tilde{\Theta}(\tilde{x}) = \tilde{\Theta}_0(\tilde{x}) + \varepsilon \tilde{\Theta}_1(\tilde{x}) + \varepsilon^2 \tilde{\Theta}_2(\tilde{x}) + \cdots, \quad (4.85)$$

$$\tilde{H}(\tilde{x}) = \tilde{H}_0(\tilde{x}) + \varepsilon \tilde{H}_1(\tilde{x}) + \varepsilon^2 \tilde{H}_2(\tilde{x}) + \cdots,$$

the equations are given in the exactly same form as those at near $x = 0$

$$\frac{d^2 \tilde{\Theta}_0}{d\tilde{x}^2} = 0, \quad \frac{d^2 \tilde{H}_0}{d\tilde{x}^2} = -3(\tilde{\Theta}_0^4 - \tilde{H}_0), \quad (4.86)$$

$$\frac{d^2 \tilde{\Theta}_1}{d\tilde{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \tilde{H}_0}{d\tilde{x}^2}, \quad \frac{d^2 \tilde{H}_1}{d\tilde{x}^2} = -3(4\tilde{\Theta}_0^3 \tilde{\Theta}_1 - \tilde{H}_1), \quad (4.87)$$

$$\frac{d^2 \tilde{\Theta}_2}{d\tilde{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \tilde{H}_1}{d\tilde{x}^2}, \quad \frac{d^2 \tilde{H}_2}{d\tilde{x}^2} = -3(4\tilde{\Theta}_0^3 \tilde{\Theta}_2 + 6\tilde{\Theta}_0^2 \tilde{\Theta}_1^2 - \tilde{H}_2), \quad (4.88)$$

$$\frac{d^2 \tilde{\Theta}_3}{d\tilde{x}^2} = -\frac{\eta}{3\mathcal{P}} \frac{d^2 \tilde{H}_2}{d\tilde{x}^2}, \quad \vdots \quad (4.89)$$

The boundary conditions at $\tilde{x} = 0$ are

$$\frac{d\tilde{\Theta}_0}{d\tilde{x}}(0) = 0, \quad \frac{d\tilde{\Theta}_1}{d\tilde{x}}(0) = B[\tilde{\Theta}_0(0) - \theta_s^4],$$

$$\frac{d\tilde{\Theta}_n}{d\tilde{x}}(0) = B\tilde{\Theta}_n(0), \quad (n = 2, 3 \cdots),$$

$$\frac{d\tilde{H}_0}{d\tilde{x}}(0) = \frac{\gamma}{\eta}[\tilde{H}_0(0) - \theta_s^4], \quad \frac{d\tilde{H}_n}{d\tilde{x}}(0) = \frac{\gamma}{\eta}\tilde{H}_n(0), \quad (n = 1, 2 \cdots). \quad (4.90)$$

The other set of conditions comes from the matching conditions as $\tilde{x}$ tends to infinity

$$\lim_{\tilde{x} \to \infty} \tilde{\Theta}_0(\tilde{x}) = \theta_0,$$
The solutions for temperature and intensity are obtained as

\[ \bar{\theta}_0(x) = b_0, \]
\[ \bar{H}_0(x) = b_0^4 - \frac{\sqrt{3}P}{\eta} [1 - b_0 + B(\theta_a - b_0)] \exp(-\sqrt{3}x), \quad (4.92) \]
\[ \bar{\theta}_1(x) = b_1 - 1 + \frac{\eta}{3P} (1 - b_0^4) - (1 - b_0)x \]
\[ + \frac{1}{\sqrt{3}} [1 - b_0 + B(\theta_a - b_0)] \exp(-\sqrt{3}x), \]
\[ \bar{H}_1(x) = 4b_0^3 \left[ \frac{\eta}{3P} (1 - b_0^4) - (1 - b_1) \right] + 4b_0^2 (1 - b_0)x \]
\[ + \frac{4b_0^3}{\sqrt{3}P^3} \left[ \eta (1 - b_0) + \gamma_1 (1 - b_1) - \frac{\eta \gamma_1}{3P} (1 - b_0^4) \right] \exp(-\sqrt{3}x), \quad (4.93) \]
\[ \bar{\theta}_2(x) = - \frac{4\eta^2 (1 - b_0)}{3P (\sqrt{3}P + \gamma_0)} - 1 + b_2 + \frac{4\eta}{3P} \left[ 1 - b_1 - \frac{\eta}{3P} (1 - b_0^4) \right] b_0^2 \]
\[ + \left[ 1 - b_1 - \frac{4\eta}{3P} (1 - b_0) b_0^2 \right] x \]
\[ - \frac{4\eta^2 b_0^2}{3P (\sqrt{3}P + \gamma_1)} \left[ \eta (1 - b_0) + \gamma_1 (1 - b_1) - \frac{\eta \gamma_1}{3P} (1 - b_0^4) \right] \exp(-\sqrt{3}x). \]

The boundary conditions for temperature at \( x = 0 \) yield expressions for the constants \( b_n \). For \( b_0 \) and \( b_1 \), they are

\[ \eta \gamma_1 b_0^4 + \sqrt{3}P (\sqrt{3}P + \gamma_1)(1 + B)b_0 = \sqrt{3}P (\sqrt{3}P + \gamma_1)(1 + B\theta_a) + \eta \gamma_1 \theta_a^4, \]
\[ b_1 = 1 - \frac{B \eta [\sqrt{3}P (1 - b_0^4) + \gamma_1 (1 - \theta_a^4)]}{4\sqrt{3}\eta \gamma_1 b_0^2 + 3P (\sqrt{3}P + \gamma_1)(1 + B)}. \]
The constant $b_0$ cannot be obtained explicitly. However, examination of its equation shows that among its four roots there is a real positive root, which is the proper one for the problem. It also illustrates that the constant $b_0$ obtained from the outer region as equation (4.69) is not valid.

### 4.3.4 Composite Expansion and Heat Flux

A composite expansion, which is the sum of the inner and the outer expansions minus their common parts, can now be developed. For the temperature field, it is

$$
\Theta(x) \sim 1 - (1 - b_0)x + \frac{1}{\tau} \left\{ \frac{\eta}{3\mathcal{P}} (1 - b_0)x - \frac{\eta}{3\mathcal{P}} [1 - (1 - b_0)x]^4 \right.
$$

$$
+ \frac{1}{\sqrt{3}} [1 - b_0 + B(\theta_a - b_0)] \exp(-\sqrt{3} \tau (1 - x)) \right\} + O(\frac{1}{\tau^2}), \quad (4.95)
$$

and for the intensity field it is given by

$$
H(x) \sim [1 - (1 - b_0)x]^4 - \frac{\sqrt{3} \mathcal{P}}{\eta} [1 - b_0 + B(\theta_a - b_0)] \exp(-\sqrt{3} \tau (1 - x))$

$$
+ \frac{1}{\tau} \left\{ 4[1 - (1 - b_0)x]^3 \left\{ \frac{\eta}{3\mathcal{P}} (1 - b_0)x - \frac{\eta}{3\mathcal{P}} [1 - (1 - b_0)x]^4 \right\} 
$$

$$
+ \sqrt{3}\eta \left[ \eta (1 - b_0) + \gamma_1 (1 - b_1) - \frac{\eta \gamma_1}{3\mathcal{P}} (1 - b_0^4) \right] \exp(-\sqrt{3} \tau (1 - x)) 
$$

$$
- \frac{4\eta(1 - b_0)}{\sqrt{3}\eta + \gamma_0} \exp(-\sqrt{3} \tau x) \right\} + O(\frac{1}{\tau^2}). \quad (4.96)
$$

Non-dimensional heat flux $\Psi$ is obtained as

$$
\Psi = -4\mathcal{P} \frac{d\Theta}{dx} - \frac{4\eta \frac{dH}{dx}}{3\tau \frac{dx}{dx}}$

$$
\sim -4\mathcal{P} \frac{d\Theta_0}{dx} - \sum_{n=1}^{M} \frac{1}{\tau^n} (4\mathcal{P} \frac{d\Theta_n}{dx} + \frac{4\eta}{3} \frac{dH_{n-1}}{dx}) + O(\frac{1}{\tau^{M+1}})$$
\[ \sim 4\mathcal{P} \sum_{n=0}^{M} \frac{1}{\tau^n} (1 - b_n) + O\left(\frac{1}{\tau^{M+1}}\right). \]  

(4.97)

The quantity \((1 - b_n)\) is introduced in order to write the outer expansion of temperature in the following form

\[ \Theta_n(x) = a_n - (1 - b_n)x - \frac{\eta}{3\mathcal{P}} H_{n-1} \quad (n = 0, 1, \ldots), \]

where \(H_{-1} = 0\). The heat flux expression shows that its leading order does not include the effect of the intensity field. All these findings confirm the results from the linearized problem discussed in the previous chapter.

### 4.3.5 Results and Discussion

Examination of the equation for \(b_0\) reveals that it can be simplified for three special cases. A first situation is that both boundary surfaces are opaque and they are held at constant temperatures. Let the non-dimensional temperature at the right by \(\theta_L\).

The constants \(b_0\) and \(b_1\) are obtained as

\[ b_0 = \theta_L, \quad b_1 = 1 - \frac{\eta}{3\mathcal{P}} (1 - \theta_L^4), \]  

(4.98)

and the expressions for the temperature and intensity fields and for the heat flux are reduced to

\[ \Theta(x) \sim 1 - (1 - \theta_L) x \]
\[ + \frac{1}{\tau} \frac{\eta}{3\mathcal{P}} \left\{ 1 - (1 - \theta_L^4) x - [1 - (1 - \theta_L) x]^4 \right\} + O\left(\frac{1}{\tau^2}\right), \]

\[ H(x) \sim [1 - (1 - \theta_L) x]^4 \]
\[ + \frac{1}{\tau} \left\{ \frac{4\eta}{3\mathcal{P}} [1 - (1 - \theta_L) x]^3 \left\{ 1 - (1 - \theta_L^4) x - [1 - (1 - \theta_L) x]^4 \right\} \right\} \]
The expression for the heat flux is identical to that of the optically thick limit given in the text by Sparrow and Cess [49, p. 254]), where they derived it from the exact formulation. Furthermore, the temperature field obtained is also independent of the surface parameters of radiation at this first two orders.

As a second case, when the relation
\[(1 + B)\theta_s = (1 + B\theta_s)\] (4.100)
is satisfied, a specially arranged condition of convection, then \(b_0\) and \(b_1\) are obtained as
\[b_0 = \theta_s, \quad b_1 = 1 - \frac{\eta}{3\mathcal{P}}(1 - \theta_s^4 - \hat{b}_1),\] (4.101)
where
\[\hat{b}_1 = \frac{3\mathcal{P}[\sqrt{3}\eta(1 - \theta_s^4) + \gamma_1(1 - 4\theta_s^3 + 3\theta_s^4)]}{4\sqrt{3}\eta\gamma_1\theta_s^3 + 3\mathcal{P}(\sqrt{3}\eta + \gamma_1)(1 + B)},\]
and the expressions for the temperature and intensity fields reduce to
\[
\Theta(x) \sim 1 - (1 - \theta_s) x \\
+ \frac{1}{\tau} \frac{\eta}{3\mathcal{P}} \left\{ 1 - (1 - \theta_s^4 - \hat{b}_1) x - [1 - (1 - \theta_s) x]^4 \right\} + O\left(\frac{1}{\tau^2}\right),
\]
\[
H(x) \sim [1 - (1 - \theta_s) x]^4 \\
+ \frac{1}{\tau} \left\{ \frac{4\eta}{3\mathcal{P}} [1 - (1 - \theta_s) x]^3 \{ 1 - (1 - \theta_s^4 - \hat{b}_1) x - [1 - (1 - \theta_s) x]^4 \} \right\}.
\]
Since under the condition (4.100) all boundary conditions are satisfied at the leading order, the boundary layers at the leading order have disappeared in the above expressions. The corresponding heat flux is

\[
\Psi \sim 4\mathcal{P}(1 - \theta_s) + \frac{4\eta}{3(1 - \theta_s^4 - \hat{b}_1)} + O\left(\frac{1}{\tau^2}\right). \tag{4.103}
\]

A third one is that for which \(\gamma_1 = 0\), a perfectly reflecting wall at the right. Then constants \(b_0\) and \(b_1\) are

\[
b_0 = 1 - \frac{B}{1 + B}(1 - \theta_a), \quad b_1 = 1 - \frac{\eta}{3\mathcal{P}} \frac{B}{1 + B}(1 - \theta_a^4). \tag{4.104}
\]

The temperature at far-away radiation collector, \(\theta_s\) has disappeared since it is not effective any more. The expressions for the temperature and intensity fields and for heat flux become:

\[
\Theta(x) \sim 1 - \frac{B}{1 + B}(1 - \theta_a)x
\]
\[
\quad + \frac{1}{\tau} \left\{ \frac{4\eta}{3\mathcal{P}} \left[ 1 - \frac{B}{1 + B}(1 - \theta_a)x \right]^4 \left[ 1 - \frac{B}{1 + B}(1 - \theta_a^4) \right] x^4 \right\} + O\left(\frac{1}{\tau^2}\right),
\]

\[
H(x) \sim [1 - \frac{B}{1 + B}(1 - \theta_a)x]^4
\]
\[
\quad + \frac{1}{\tau} \left\{ \frac{4\eta}{3\mathcal{P}} \left[ 1 - \frac{B}{1 + B}(1 - \theta_a)x \right]^3 \left[ 1 - \frac{B}{1 + B}(1 - \theta_a^4) \right] x^3 \right\} - \frac{4\eta}{\sqrt{3}\eta + \gamma_0} \frac{B}{1 + B}(1 - \theta_a) \exp(-\sqrt{3}\tau x)
\]
\[
\quad + \frac{4}{\sqrt{3}} \frac{B(1 + B\theta_a)^3}{(1 + B)^4} (1 - \theta_a) \exp(-\sqrt{3}\tau(1 - x)) \right\} + O\left(\frac{1}{\tau^2}\right),
\]

\[
\Psi \sim 4\mathcal{P}B(1 - \theta_a) + \frac{4\eta}{\tau} \frac{B}{1 + B} \left[ 1 - \left( \frac{1 + B\theta_a}{1 + B} \right)^4 \right] + O\left(\frac{1}{\tau^2}\right). \tag{4.105}
\]
By considering the boundary conditions (4.2), it is seen that this situation leads to no radiative heat flux through the right wall,

\[
\frac{dH}{dx}(1) = 0. 
\] (4.106)

The leading order of the outer expansions is seen to satisfy its conditions at both boundaries. Thus, the intensity boundary layer at the leading order has disappeared, and the thermal boundary layer at the second order has been eliminated. The leading order solutions are exactly equal to those of the second case, and the difference between these two cases is seen in the next order.

Figure 10 shows the temperature field for various values of the optical thickness. All other parameters are held at the same values as those used with the optically thin medium. The asymptotic results evaluated from the two term expansions are compared with numerical solutions. They show excellent agreement for \( \tau \geq 10 \), and

<table>
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<th>( \tau )</th>
<th>( \Psi_{\text{numerical}} )</th>
<th>( \Psi_{\text{asymptotic}} )</th>
<th>( \Delta \Psi )</th>
<th>( \Delta \Theta )</th>
</tr>
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<td>-</td>
<td>-</td>
</tr>
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<td>( \sim 0 )</td>
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</tr>
<tr>
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<td>1.6634</td>
<td>27 %</td>
<td>26%</td>
</tr>
</tbody>
</table>
Figure 10: Non-dimensional temperature distributions for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when $\mathcal{P} = 1$, $\eta = 1$, $\varepsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $B = 1$, and $\theta_a = 0.5$. 
the deviation starts to be noticeable at $\tau = 5$, with the asymptotic expansion giving larger temperature than the numerical solution. Numerical values of heat flux and the errors in heat flux and in temperature, defined as in equation (4.43), are tabulated in Table 6.

Two term expansion of non-dimensional heat flux, given by equation (4.61) for small optical thickness and by equation (4.97) for large optical thickness, are shown in Figure 11 together with numerical solution. It shows that the asymptotic results serve as useful analytical approximations in each limit.
Figure 11: Non-dimensional heat flux for combined radiation and conduction across a slab bounded by an opaque surface and a partially transparent surface, when $\mathcal{P} = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $\mathcal{B} = 1$, and $\theta_a = 0.5$. 
4.4 Summary

The interaction of radiation and conduction in a one-dimensional semi-transparent slab has been examined for certain limiting situations by using the method of matched asymptotic expansions. In the situation dominated by radiation, asymptotic expansions were constructed as a series in powers of $\sqrt{\mathcal{P}}$. Thermal boundary layers existed near both boundaries. The boundary layers also provided a correction for the higher order terms of the intensity field. Owing to the nonlinearity in the leading order equation of temperature, the first two terms were only developed in closed form. The asymptotic expression for temperature was obtained except in the region near $x = 0$. The heat flux did not give a contribution from conduction in the first two orders. Explicit expressions for the solution have been given for two special cases: for a pure radiation and for a perfectly reflective wall at the right boundary. Finally, comparison with numerical solutions showed that the discrepancies in the temperature field are noticeable at $\mathcal{P} = 0.01$, where the deviation in heat flux is 0.9% and in temperature field is 3.8%, and that as $\mathcal{P}$ increases the deviations increase, as expected.

For optically thin media, asymptotic expansions for the temperature and intensity fields were constructed with the optical thickness as a small parameter. The corresponding equations resulted in a regular perturbation problem and simple forms of the solutions were easily obtained. The expression for the heat flux at each order was an algebraic sum of conductive and radiative parts. The leading order solution for temperature corresponded to pure conduction. The leading order of the intensity field has been found to be the algebraic mean of the boundary intensity and the back-
ground intensity. Comparison with numerical solutions showed that the deviations in the temperature field are observable at $\tau = 1$, where the difference in heat flux is 9.1% and in temperature is 3.7%. As $\tau$ increased the discrepancies increased, as expected.

For optically thick media, asymptotic expansions were developed in inverse powers of the optical thickness. Intensity boundary layers existed near both boundaries. A two term expansion for the temperature and intensity fields was obtained in a simple form although its constants were not obtained explicitly. The leading order heat flux expression was determined from the temperature field alone. Explicit expressions of the solution have been obtained in three special cases: for an opaque surface held at constant temperature at the right, for a specially arranged condition of convection at the right boundary, and for a perfectly reflecting wall at the right. Comparison with numerical solutions showed that the deviations in the temperature field are noticeable at $\tau = 5$, at which the difference in heat flux is 1.6% and in temperature is 2.2%. As $\tau$ increased the asymptotic expansion approached the numerical solution.

Three limiting situations of combined radiation and conduction have been discussed. The situation dominated by conduction does not yield much new information and thus is not considered.
CHAPTER V

RADIATION-CONDUCTION INTERACTION IN SEMI-INFINITE CIRCULAR CYLINDER

In this chapter the interaction of radiation and conduction taking place in an axially-symmetric medium will be discussed by using the method of matched asymptotic expansions. In particular, radiation-conduction interaction is considered in a circular cylinder of radius $R$ moving along its axial direction at constant speed $U$. The absorption coefficient of the cylinder is $\kappa$, its scattering coefficient is $\sigma$, its thermal conductivity is $k$, and its thermal diffusivity is $\alpha$. For a stationary coordinate origin, the $z^*$-axis is taken to coincide with the axis of the cylinder. The base of the cylinder at $z^* = 0$ is assumed to be held at constant temperature $T_0$. It is also assumed to be opaque with surface emissivity $\epsilon$. The peripheral surface at $r^* = R$ is partially transparent with surface reflectivity $\rho$. Part of the thermal energy transmitted through this peripheral surface radiates without attenuation to a far-away black surface at temperature $T_s$, where it is absorbed. In addition to radiative heat exchange, heat is also convected from the peripheral surface to a radiatively non-participating ambient at temperature $T_s$ with heat transfer coefficient $h$. This model represents cooling of fibers or crystals drawn from melts with the pulling speed $U$, when the surface at $z^* = 0$ is regarded as a solidification front and the body in the negative $z^*$-region as
an opaque melt.

Under these settings the differential approximations of the thermal energy balance and the radiation energy balance can be put into the following non-dimensional forms

\[ \nabla^2 \Theta - P e \frac{\partial \Theta}{\partial z} = \frac{\tau \eta}{\mathcal{P}} (\Theta^4 - H), \quad \nabla^2 H = -3\tau^2 (\Theta^4 - H), \]  \tag{5.1}

and the boundary conditions for temperature and intensity become

\[ \begin{align*}
\Theta(r,0) &= 1, & \frac{\partial \Theta}{\partial z}(r,\infty) &= 0, \\
\frac{\partial \Theta}{\partial r}(0,z) &= 0, & \frac{\partial \Theta}{\partial r}(1,z) &= -\mathcal{B}[\Theta(1,z) - \Theta_s], \\
\frac{\partial H}{\partial z}(r,0) &= \frac{\tau \gamma_0}{\eta} [H(r,0) - 1], & \frac{\partial H}{\partial z}(r,\infty) &= 0, \\
\frac{\partial H}{\partial r}(0,z) &= 0, & \frac{\partial H}{\partial r}(1,z) &= -\frac{\tau \gamma_1}{\eta} [H(1,z) - \Theta_s^1].
\end{align*} \]  \tag{5.2}

The non-dimensional variables and parameters in these equations are

\[ \begin{align*}
r &= \frac{r^*}{R}, & z &= \frac{z^*}{R}, & \Theta &= \frac{T}{T_0}, & H &= \frac{J}{4\sigma_0 R^3}, \\
\mathcal{P} &= \frac{k}{4\sigma_0 T_0^3 R}, & \tau &= \sqrt{\frac{\kappa_F R}{\beta_R}}, & \eta &= \sqrt{\frac{\kappa_P}{\beta_R}}, \\
\mathcal{B} &= \frac{h R}{k}, & \theta_s &= \frac{T_s}{T_0}, & \gamma_0 &= \frac{3 - \epsilon}{2}, & \gamma_1 &= \frac{3}{2} \frac{1 - \rho}{1 + \rho}.
\end{align*} \]  \tag{5.3}

The thermal Péclet number \( P e \) is now present and it represents the convective heat transfer by the moving cylinder. The frequency dependence of the equations is handled as discussed in the previous chapters. Thus, parameters \( \kappa_P \) and \( \beta_R \) denote again Planck’s mean absorption coefficient and Rosseland’s mean extinction coefficient, respectively, and \( \epsilon \) and \( \rho \) are hemispherical spectral-mean quantities defined in equation (2.26).
5.1 Radiation-Dominant Situation

When radiation dominates conduction, the Planck number \( \mathcal{P} \) is small and thus asymptotic expansions can be constructed as a series in powers \( \sqrt{\mathcal{P}} \). With \( \epsilon = \sqrt{\mathcal{P}} \), the governing equations (5.1) can be written as

\[
\epsilon^2(\nabla^2 \Theta - Pe \frac{\partial \Theta}{\partial z}) = \tau \eta (\Theta^4 - H), \quad \nabla^2 H = -3\tau^2 (\Theta^4 - H),
\]

and the boundary conditions remains unchanged. With the perturbation expansions

\[
\Theta(r, z) = \Theta_0(r, z) + \epsilon \Theta_1(r, z) + \epsilon^2 \Theta_2(r, z) + \cdots,
\]

\[
H(r, z) = H_0(r, z) + \epsilon H_1(r, z) + \epsilon^2 H_2(r, z) + \cdots,
\]

for \( \Theta(r, z) \) and \( H(r, z) \), the non-dimensional heat flux \( \Psi(r, z) \) is developed as

\[
\Psi(r, z) = \frac{q_C + q_R}{n^2 \sigma_0 T_0^4} = -4\mathcal{P} \nabla \Theta(r, z) - \frac{4\eta}{3\tau} \nabla H(r, z)
\]

\[
\sim -\frac{4\eta}{3\tau} \nabla H_0 - \sqrt{\mathcal{P}} \frac{4\eta}{3\tau} \nabla H_1
\]

\[
- \sum_{n=2}^{M-1} \mathcal{P}^{n/2} \left( \frac{4\eta}{3\tau} \nabla H_n + 4 \nabla \Theta_{n-2} \right) + O(\mathcal{P}^{M/2}).
\]

This shows that two more terms need to be obtained for the intensity field than for the temperature field in order to evaluate the heat flux at a certain order.

5.1.1 Outer Expansion

Substituting the perturbation expansions of \( \Theta(r, z) \) and \( H(r, z) \) into equations (5.4) and collecting like powers of \( \epsilon \) lead to the equations

\[
\nabla^2 H_0 = 0, \quad \Theta_0 = H_0^{1/4},
\]

(5.7)
\[ \nabla^2 H_1 = 0, \quad \Theta_1 = \frac{H_1}{4\Theta_0^3}, \] (5.8)

\[ \nabla^2 H_2 = -\frac{3\tau}{\eta}(\nabla^2 \Theta_0 - Pe\frac{\partial \Theta_0}{\partial z}). \]

The corresponding boundary conditions are

\[
\begin{align*}
\Theta_0(r, 0) &= 1, & \Theta_n(r, 0) &= 0, \quad (n = 1, 2, \cdots), \\
\frac{\partial \Theta_0}{\partial z}(r, \infty) &= 0, & \frac{\partial \Theta_n}{\partial z}(0, z) &= 0, \quad (n = 0, 1, \cdots), \\
\frac{\partial \Theta_0}{\partial r}(1, z) &= -B[\Theta_0(1, z) - \theta_s], & \frac{\partial \Theta_n}{\partial r}(1, z) &= -B\Theta_{n-1}(1, z), \quad (n = 1, 2, \cdots), \\
\frac{\partial H_0}{\partial z}(r, 0) &= \frac{\tau \gamma_0}{\eta}[H_0(r, 0) - 1], & \frac{\partial H_n}{\partial z}(r, 0) &= \frac{\tau \gamma_0}{\eta}H_{n-1}(r, 0), \quad (n = 1, 2, \cdots), \\
\frac{\partial H_0}{\partial z}(r, \infty) &= 0, & \frac{\partial H_n}{\partial z}(0, z) &= 0, \quad (n = 0, 1, \cdots), \\
\frac{\partial H_0}{\partial r}(1, z) &= -\frac{\tau \gamma_1}{\eta}[H_0(1, z) - \theta_s^4], & \frac{\partial H_n}{\partial r}(1, z) &= -\frac{\tau \gamma_1}{\eta}H_{n-1}(1, z), \quad (n = 1, 2, \cdots). \\
\end{align*}
\]

The leading order solutions are obtained as an infinite series of Bessel functions

\[
H_0(r, z) = \theta_s^4 + (1 - \theta_s^4) \sum_{n=1}^{\infty} a_n \exp(-\mu_n z)J_0(\mu_n r),
\]

\[
\Theta_0(r, z) = [H_0(r, z)]^{1/4}. \tag{5.9}
\]

The boundary conditions for intensity give for the eigenvalue \( \mu_n \) the transcendental equation

\[
\mu_nJ_1(\mu_n) = \frac{\tau \gamma_1}{\eta}J_0(\mu_n) \tag{5.10}
\]

and for the expansion coefficient \( a_n \) the expression

\[
a_n = \frac{\tau \gamma_0}{(\gamma \mu_n + \tau \gamma_0)} \int_0^1 J_0(\mu_n r)rdr = \frac{2\tau^2 \eta \gamma_1 \gamma_0}{(\gamma \mu_n + \tau \gamma_0)(\gamma^2 \mu_n^2 + \tau^2 \gamma_0^2)}J_0(\mu_n). \tag{5.11}
\]

Thus, the intensity field is completely determined. The temperature field by the second of equations (5.9) satisfies neither its condition at \( r = 1 \) nor at \( z = 0 \). This suggests that thermal boundary layers must exist near both \( r = 1 \) and \( z = 0 \) boundaries.
to adjust the temperature field to its proper values at both boundaries. Consequently, boundary layers exist for the sake of the temperature and may provide a correction for the higher order terms in the representation of the intensity field.

The solutions at the next order can be cast into the forms

\[
H_1(r, z) = \sum_{n=1}^{\infty} b_n \exp(-\nu_n z) J_0(\nu_n r),
\]

\[
\Theta_1(r, z) = \frac{H_1}{4\Theta_0^3}, \quad (5.12)
\]

where \((b_n, \nu_n)\) are to be determined from matching to the inner expansion.

5.1.2 Inner Expansion Near \( r = 1 \)

To determine the temperature and intensity fields near \( r = 1 \), the following variables are introduced:

\[
1 - r = \varepsilon \bar{r}, \quad \Theta(r, z) = \bar{\Theta}(\bar{r}, z), \quad H(r, z) = \bar{H}(\bar{r}, z). \quad (5.13)
\]

In terms of these, the governing equations (5.4) lead to

\[
\begin{align*}
\left(\frac{\partial^2}{\partial \bar{r}^2} - \varepsilon \frac{\partial}{\partial \bar{r}} - \varepsilon^2 \frac{\partial^2}{\partial \bar{r}^2} + \varepsilon^2 \frac{\partial^2}{\partial z^2}\right) \bar{H} + O(\varepsilon^3) &= -3\varepsilon^2 \tau^2 (\bar{\Theta}^4 - \bar{H}), \\
\left(\frac{\partial^2}{\partial \bar{r}^2} - \varepsilon \frac{\partial}{\partial \bar{r}} - \varepsilon^2 \frac{\partial^2}{\partial \bar{r}^2} + \varepsilon^2 \frac{\partial^2}{\partial z^2}\right) \bar{\Theta} - \varepsilon^2 P e \frac{\partial \bar{\Theta}}{\partial z} + O(\varepsilon^3) &= \tau \eta (\bar{\Theta}^4 - \bar{H}). \quad (5.14)
\end{align*}
\]

The solutions expanded in the perturbation series

\[
\begin{align*}
\bar{\Theta}(\bar{r}, z) &= \bar{\Theta}_0(\bar{r}, z) + \varepsilon \bar{\Theta}_1(\bar{r}, z) + \varepsilon^2 \bar{\Theta}_2(\bar{r}, z) + \cdots, \\
\bar{H}(\bar{r}, z) &= \bar{H}_0(\bar{r}, z) + \varepsilon \bar{H}_1(\bar{r}, z) + \varepsilon^2 \bar{H}_2(\bar{r}, z) + \cdots. \quad (5.15)
\end{align*}
\]
when substituted into equation (5.14) and like powers of \( \varepsilon \) equated, give

\[
\begin{align*}
\frac{\partial^2 \bar{H}_0}{\partial \bar{r}^2} &= 0, \quad \frac{\partial^2 \bar{\Theta}_0}{\partial \bar{r}^2} = \tau \eta (\bar{\Theta}_0^4 - \bar{H}_0), \\
\frac{\partial^2 \bar{H}_1}{\partial \bar{r}^2} - \frac{\partial \bar{H}_0}{\partial \bar{r}} &= 0, \quad \frac{\partial^2 \bar{\Theta}_1}{\partial \bar{r}^2} - \frac{\partial \bar{\Theta}_0}{\partial \bar{r}} = \tau \eta (4 \bar{\Theta}_0^3 \bar{\Theta}_1 - \bar{H}_1), \\
\frac{\partial^2 \bar{H}_2}{\partial \bar{r}^2} - \frac{\partial \bar{H}_1}{\partial \bar{r}} - \bar{r} \frac{\partial \bar{H}_0}{\partial \bar{r}} + \frac{\partial^2 \bar{H}_0}{\partial z^2} &= -3\tau^2 (\bar{\Theta}_0^4 - \bar{H}_0),
\end{align*}
\]

(5.16)

(5.17)

(5.18)

The corresponding boundary conditions at \( \bar{r} = 0 \) are

\[
\begin{align*}
\frac{\partial \bar{\Theta}_0}{\partial \bar{r}} (0, z) &= 0, \quad \frac{\partial \bar{\Theta}_1}{\partial \bar{r}} (0, z) = B [\bar{\Theta}_0(0, z) - \theta_s], \\
\frac{\partial \bar{\Theta}_n}{\partial \bar{r}} (0, z) &= B \bar{\Theta}_n-1(0, z), \quad (n = 2, 3, \ldots), \\
\frac{\partial \bar{H}_0}{\partial \bar{r}} (0, z) &= 0, \quad \frac{\partial \bar{H}_1}{\partial \bar{r}} (0, z) = \frac{\tau \gamma_1}{\eta} [\bar{H}_0(0, z) - \theta_s^4], \\
\frac{\partial \bar{H}_n}{\partial \bar{r}} (0, z) &= \frac{\tau \gamma_n}{\eta} \bar{H}_{n-1}(0, z), \quad (n = 2, 3, \ldots).
\end{align*}
\]

(5.19)

(5.20)

Expanding the outer expansion in terms of the inner variable \( \bar{r} \) and collecting like powers of \( \varepsilon \) yield the matching conditions. The first three for intensity are

\[
\begin{align*}
\lim_{\bar{r} \to \infty} \bar{H}_0(\bar{r}, z) &= Z_{00}(z), \\
\lim_{\bar{r} \to \infty} \bar{H}_1(\bar{r}, z) &= \bar{r} Z_{01}(z) + Z_{10}(z), \\
\lim_{\bar{r} \to \infty} \bar{H}_2(\bar{r}, z) &= \frac{\bar{r}^2}{2} Z_{02}(z) + \bar{r} Z_{11}(z) + Z_{20}(z) - \frac{3\tau}{\eta} Z_{00}(z),
\end{align*}
\]

(5.21)

and for the first two orders for temperature they are

\[
\begin{align*}
\lim_{\bar{r} \to \infty} \bar{\Theta}_0(\bar{r}, z) &= Z_{00}(z), \\
\lim_{\bar{r} \to \infty} \bar{\Theta}_1(\bar{r}, z) &= \bar{r} \frac{Z_{01}(z)}{4 Z_{00}(z)} + \frac{Z_{10}(z)}{4 Z_{00}(z)}.
\end{align*}
\]

(5.22)

where

\[
Z_{00}(z) = \theta_s^4 + (1 - \theta_s^4) \sum_{n=1}^{\infty} a_n \exp(-\mu_n z) J_0(\mu_n),
\]
\[ Z_{01}(z) = (1 - \theta_z^4) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\mu_n z) \mu_n J_1(\mu_n), \]
\[ Z_{02}(z) = (1 - \theta_z^4) \sum_{n=1}^{\infty} a_n \exp(-\mu_n z) [\mu_n J_1(\mu_n) - \mu_n^2 J_0(\mu_n)], \]
\[ Z_{10}(z) = \sum_{n=1}^{\infty} b_n \exp(-\nu_n z) J_0(\nu_n), \]
\[ Z_{11}(z) = \sum_{n=1}^{\infty} b_n \exp(-\nu_n z) \nu_n J_1(\nu_n). \] (5.23)

The function \( Z_{20}(z) \) is undefined but would come from the outer expansion \( H_2(r, z) \).

By applying the matching conditions for temperature and intensity and the boundary conditions for temperature, the solutions can be written as

\[ \tilde{H}_0(\bar{r}, z) = Z_{00}^4(z), \]
\[ \tilde{\Theta}_0(\bar{r}, z) = Z_{00}(z), \] (5.24)
\[ \tilde{H}_1(\bar{r}, z) = Z_{01}(z) \bar{r} + Z_{10}(z), \]
\[ \tilde{\Theta}_1(\bar{r}, z) = \bar{r} Z_{01}(z) + \frac{Z_{10}(z)}{4 Z_{00}^3(z)} + \frac{1}{[4 \tau \eta Z_{00}^3(z)]^{1/2}} \left[ \frac{Z_{01}(z)}{4 Z_{00}^3(z)} - B(Z_{00}(z) - \theta_z) \right] \exp(-[4 \tau \eta Z_{00}^3(z)]^{1/2} \bar{r}), \] (5.25)
\[ \tilde{H}_2(\bar{r}, z) = \frac{\bar{r}^2}{2} Z_{02}(z) + \bar{r} Z_{11}(z) + Z_{20}(z) - \frac{3 \tau}{\eta} \tilde{\Theta}_0(\bar{r}, z). \]

The boundary condition for \( \tilde{H}_1(\bar{r}, z) \) gives the relation

\[ Z_{01}(z) = \frac{\tau \gamma_1}{\eta} [Z_{00}^4(z) - \theta_z^4], \]

which, with equations (5.23), turns into an equation for \( \mu_n \)

\[ \mu_n J_1(\mu_n) = \frac{\tau \gamma_1}{\eta} J_0(\mu_n). \]
It confirms that the eigenvalue obtained from the outer expansion is valid. The boundary condition for $\tilde{H}_2(\tilde{r}, \tilde{z})$ gives a relation for $Z_{1n}(z)$ as

$$Z_{11}(z) = \frac{\tau_{11}}{\eta} Z_{10}(z).$$

The orthogonality of Bessel functions then leads to the condition

$$\nu_n = \mu_n. \quad (5.26)$$

**5.1.3 Inner Expansion Near $z = 0$**

For the temperature and intensity fields valid near $z = 0$, the following variables are introduced,

$$z = \varepsilon \tilde{z}, \quad \Theta(r, \tilde{z}) = \tilde{\Theta}(r, \tilde{z}), \quad H(r, \tilde{z}) = \tilde{H}(r, \tilde{z}), \quad (5.27)$$

and the solutions are expanded in the perturbation series

$$\tilde{\Theta}(r, \tilde{z}) = \tilde{\Theta}_0(r, \tilde{z}) + \varepsilon \tilde{\Theta}_1(r, \tilde{z}) + \varepsilon^2 \tilde{\Theta}_2(r, \tilde{z}) + \cdots,$$

$$\tilde{H}(r, \tilde{z}) = \tilde{H}_0(r, \tilde{z}) + \varepsilon \tilde{H}_1(r, \tilde{z}) + \varepsilon^2 \tilde{H}_2(r, \tilde{z}) + \cdots. \quad (5.28)$$

Substituting these into the governing equations (5.4) and equating like powers of $\varepsilon$ yield equations at each order as follows:

$$\frac{\partial^2 \tilde{H}_0}{\partial \tilde{z}^2} = 0, \quad \frac{\partial^2 \tilde{\Theta}_0}{\partial \tilde{z}^2} = \tau \eta (\tilde{\Theta}_0^4 - \tilde{H}_0), \quad (5.29)$$

$$\frac{\partial^2 \tilde{H}_1}{\partial \tilde{z}^2} = 0, \quad \frac{\partial^2 \tilde{\Theta}_1}{\partial \tilde{z}^2} - P_e \frac{\partial \tilde{\Theta}_0}{\partial \tilde{z}} = \tau \eta (4 \tilde{\Theta}_0^3 \tilde{\Theta}_1 - \tilde{H}_1), \quad (5.30)$$

$$\frac{\partial^2 \tilde{H}_2}{\partial \tilde{z}^2} + \frac{\partial^2 \tilde{H}_0}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{H}_0}{\partial r} = -\frac{3 \tau}{\eta} \frac{\partial^2 \tilde{\Theta}_0}{\partial \tilde{z}^2} :$$
The corresponding boundary conditions at \( \tilde{z} = 0 \) are

\[
\tilde{\theta}_0(r, 0) = 1, \quad \tilde{\theta}_n(r, 0) = 0, \quad (n = 1, 2, \cdots), \tag{5.31}
\]

\[
\frac{\partial \tilde{H}_0}{\partial \tilde{z}}(r, 0) = 0, \quad \frac{\partial \tilde{H}_1}{\partial \tilde{z}}(r, 0) = \frac{\tau \gamma_0}{\eta} [\tilde{H}_0(r, 0) - 1],
\]

\[
\frac{\partial \tilde{H}_n}{\partial \tilde{z}}(r, 0) = \frac{\tau \gamma_0}{\eta} \tilde{H}_{n-1}(r, 0), \quad (n = 2, 3, \cdots). \tag{5.32}
\]

The matching conditions are, upon expanding the outer expansion for intensity in terms of the inner variable \( \tilde{z} \) and collecting like powers of \( \varepsilon \),

\[
\lim_{\tilde{z} \to \infty} \tilde{H}_0(r, \tilde{z}) = R_{00}(r),
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{H}_1(r, \tilde{z}) = \tilde{z} R_{01}(r) + R_{10}(r),
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{H}_2(r, \tilde{z}) = \frac{\varepsilon^2}{2} R_{02}(r) + \tilde{z} R_{11}(r) + R_{20}(r) - \frac{3\tau}{\eta} R_{00}(r), \tag{5.33}
\]

and for temperature they are

\[
\lim_{\tilde{z} \to \infty} \tilde{\theta}_0(r, \tilde{z}) = R_{00}(r),
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{\theta}_1(r, \tilde{z}) = \tilde{z} \frac{R_{01}(r)}{4R_{00}^3(r)} + \frac{R_{10}(r)}{4R_{00}^3(r)}, \tag{5.34}
\]

where

\[
R_{00}(r) = \theta_s^4 + (1 - \theta_s^4) \sum_{n=1}^{\infty} a_n J_0(\mu_n r),
\]

\[
R_{01}(r) = -(1 - \theta_s^4) \sum_{n=1}^{\infty} a_n \mu_n J_0(\mu_n r),
\]

\[
R_{02}(r) = (1 - \theta_s^4) \sum_{n=1}^{\infty} a_n \mu_n^2 J_0(\mu_n r),
\]

\[
R_{10}(r) = \sum_{n=1}^{\infty} b_n J_0(\mu_n r),
\]

\[
R_{11}(r) = - \sum_{n=1}^{\infty} b_n \mu_n J_0(\mu_n r). \tag{5.35}
\]
The function $R_{20}(r)$ is undetermined and related to the outer expansion $H_{2}(r,z)$. The equations and their matching conditions for intensity give for the first three orders of the solution

$$
\tilde{H}_0(r,\tilde{z}) = R_{00}^4(r),
$$
$$
\tilde{H}_1(r,\tilde{z}) = \tilde{z} R_{01}(r) + R_{10}(r),
$$
$$
\tilde{H}_2(r,\tilde{z}) = \frac{\tilde{z}^2}{2} R_{02}(r) + \tilde{z} R_{11}(r) + R_{20}(r) - \frac{3\tau}{\eta} \tilde{\Theta}_0(r,\tilde{z}). \quad (5.36)
$$

Owing to the nonlinearity of the equation for $\tilde{\Theta}_0$, attempts to obtain an explicit solution for the temperature field appear to be hopeless.

The boundary condition for $\tilde{H}_1(r,\tilde{z})$ gives a relation, from which an expression for the expansion constants $a_n$ is obtained, that is identical to that developed from the outer expansion. The boundary condition for $\tilde{H}_2(r,\tilde{z})$ leads to

$$
R_{11}(r) - \frac{3\tau}{\eta} \frac{\partial \tilde{\Theta}_0}{\partial \tilde{z}}(r,0) = \frac{\tau \gamma_0}{\eta} R_{10}(r), \quad (5.37)
$$

which is expected to give an expression for $b_n$. Although $\tilde{\Theta}_0(r,\tilde{z})$ cannot be obtained explicitly, the quantity $\partial \tilde{\Theta}_0/\partial \tilde{z}(r,0)$, necessary for the above equation, can be obtained, since the equation for $\tilde{\Theta}_0(r,\tilde{z})$ can be reduced in order and its matching condition satisfied. To carry this out, both sides of the second equation in (5.29) are multiplied by $\partial \tilde{\Theta}_0/\partial \tilde{z}$ and integrated once with respect to $\tilde{z}$. Applying the matching condition then leads to

$$
\frac{\partial \tilde{\Theta}_0}{\partial \tilde{z}} = -\frac{2\gamma_0}{5} \{ \tilde{\Theta}_0^5 - 5R_{00}^8(r)\tilde{\Theta}_0 + 4R_{00}^6(r) \}^{1/2}, \quad \tilde{\Theta}_0(r,0) = 1. \quad (5.38)
$$

This gives the temperature gradient at $\tilde{z} = 0$ in terms of an already developed function $R_{00}(r)$. Substituting the obtained gradient into equation (5.37) yields an equation for
constant \( b_n \) as

\[
- \sum_{n=1}^{\infty} b_n \mu_n J_0(\mu_n r) + \frac{3\tau}{\eta} \left[ \frac{2\tau \eta}{5} (1 - 5R_{00}^4(r) + 4R_{00}^5(r)) \right]^{1/2} = \frac{\tau \gamma_0}{\eta} \sum_{n=1}^{\infty} b_n J_0(\mu_n r).
\]

By invoking the orthogonality of Bessel functions, the constant \( b_n \) is evaluated as

\[
b_n = \frac{3\tau(2\tau \eta/5)^{1/2} \int_{0}^{1} J_0(\mu_n r)[1 - 5R_{00}^4(r) + 4R_{00}^5(r)]^{1/2} r dr}{(\eta \mu_n + \tau \gamma_0) \int_{0}^{1} J_0^2(\mu_n r) r dr}.
\]

\[
b_n = \frac{6\tau \eta^2 \mu_n^2 (2\tau \eta/5)^{1/2} \int_{0}^{1} J_0(\mu_n r)[1 - 5R_{00}^4(r) + 4R_{00}^5(r)]^{1/2} r dr}{(\eta \mu_n + \tau \gamma_0)(\eta^2 \mu_n^2 + \tau^2 \gamma_0^2) J_0^2(\mu_n)}.
\] (5.39)

**Approximation of \( \tilde{\Theta}_n(r, \tilde{z}) \)**

Owing to nonlinearity in the leading order equation, analytic solution for the temperature field could not be found. It can obviously be determined numerically. Rather than proceeding this way an approximate expression for the analytical solutions is sought as an alternative. The approach to obtain it is in the same ad hoc spirit as was done for the one-dimensional slab earlier and again stands apart from the method of matched asymptotic expansions. Near \( \tilde{z} = 0 \), a series solution gives a clue to a useful approximate form. Writing \( \tilde{\Theta}_n(r, \tilde{z}) \) as a power series in \( \tilde{z} \),

\[
\tilde{\Theta}_0(r, \tilde{z}) = \sum_{n=0}^{\infty} \tilde{a}_n(r) \tilde{z}^n,
\] (5.40)

the boundary condition gives for the first two coefficients the expressions

\[
\tilde{a}_0(r) = 1, \quad \tilde{a}_1(r) = -\left[ \frac{2\tau \eta}{5} (1 - 5R_{00}^4(r) + 4R_{00}^5(r)) \right]^{1/2}, \quad \ldots.
\]

In a similar way, \( \tilde{\Theta}_1(r, \tilde{z}) \) can be written as

\[
\tilde{\Theta}_1(r, \tilde{z}) = \sum_{n=1}^{\infty} \tilde{b}_n(r) \tilde{z}^n,
\] (5.41)
with

$$b_2(r) = \frac{1}{2}[Pe - \tau \eta R_{10}(r)], \quad b_3(r) = \frac{1}{6}\left\{Pe \bar{a}_1(1) + \tau \eta[4\bar{b}_1(r) - R_{01}(r)]\right\}, \quad \ldots,$$

but $\bar{b}_1(r)$ remains undetermined.

In the region far away from $\tilde{z} = 0$, by considering the matching conditions, the approximations at each order of the temperature field can be cast into the following forms

$$\tilde{\theta}_0(r, \tilde{z}) \approx R_{00}(r) + \tilde{\theta}_0(r) \exp(-[4\tau \eta R_{00}^3(r)]^{1/2} \tilde{z}),$$

$$\tilde{\theta}_1(r, \tilde{z}) \approx \tilde{z} \frac{R_{01}(r)}{4R_{00}^3(r)} + \frac{R_{10}(r)}{4R_{00}^3(r)} + \tilde{\theta}_1(r) \exp(-[4\tau \eta R_{00}^3(r)]^{1/2} \tilde{z}),$$

(5.42)

where both $\tilde{\theta}_0(r)$ and $\tilde{\theta}_1(r)$ are again undetermined.

From these two forms of solutions that are valid near $\tilde{z} = 0$ and in the region far away from $\tilde{z} = 0$, a unique expression can be constructed that is valid for all $\tilde{z}$ in such a way that the functional forms of equations (5.42) are kept and they are made to satisfy the boundary conditions and the first order derivative at $\tilde{z} = 0$. The following expressions become in this way useful approximations for first two orders of the temperature field:

$$\tilde{\theta}_0(r, \tilde{z}) \approx R_{00}(r) + [1 - R_{00}(r)] \exp(-[4\tau \eta R_{00}^3(r)]^{1/2} \lambda_{\tilde{z}}(r)), $$

$$\tilde{\theta}_1(r, \tilde{z}) \approx \tilde{z} \frac{R_{01}(r)}{4R_{00}^3(r)} + \frac{R_{10}(r)}{4R_{00}^3(r)} - \frac{R_{10}(r)}{4R_{00}^3(r)} \exp(-[4\tau \eta R_{00}^3(r)]^{1/2} \tilde{z}),$$

(5.43)

where

$$\lambda_{\tilde{z}}^2(r) = \frac{1 - 5R_{00}^4(r) + 4R_{00}^2(r)}{10R_{00}^3(r)(1 - R_{00}(r))^2}.$$
The parameter $Pe$ is not included because it appears first in the second order derivative in the expression for $\hat{\Theta}_1(r, \hat{z})$ near $\hat{z} = 0$.

Now, an approximate expression for the temperature field is obtained in the spirit of composite expansion of the asymptotic solution as the sum of the inner and the outer expansions minus their common parts,

$$
\Theta(r, z) \approx \Theta_0(r, z) + \left[ 1 - R_{00}(r) \right] \exp\left( -[4\tau\eta R_{00}^3(r)]^{1/2} \lambda_0(r) \frac{z}{\sqrt{P}} \right)
$$

$$
+ \sqrt{P} \left\{ \Theta_1(r, z) - \frac{R_{10}(r)}{4R_{00}^3(r)} \exp\left( -[4\tau\eta R_{00}^3(r)]^{1/2} \frac{z}{\sqrt{P}} \right) \right. 
+ \left. \frac{1}{[4\tau\eta Z_{00}^3(z)]^{1/2}} \left[ \frac{Z_{01}(z)}{Z_{00}^3(z)} - B(Z_{00}(z) - \theta_0) \right] \exp\left( -[4\tau\eta Z_{00}^3(z)]^{1/2} \frac{1 - r}{\sqrt{P}} \right) \right\}. 
$$

(5.44)

It carries the character of the solution in that the temperature field has a strong boundary layer near $z = 0$ and a relatively weak one near the peripheral surface.

**5.1.4 Inner Expansion at Corner**

To obtain the inner expansion at the corner region, the following variables are introduced,

$$
1 - r = \varepsilon \tilde{r}, \quad z = \varepsilon \tilde{z}, \quad \Theta(r, z) = \hat{\Theta}(\tilde{r}, \tilde{z}), \quad H(r, z) = \hat{H}(\tilde{r}, \tilde{z}),
$$

(5.45)

which turn the governing equations (5.4) into the forms

$$
\left( \frac{\partial^2}{\partial \tilde{r}^2} - \varepsilon \frac{\partial}{\partial \tilde{r}} - \varepsilon^2 \tilde{r} \frac{\partial}{\partial \tilde{r}} + \frac{\partial^2}{\partial \tilde{z}^2} \right) \hat{H} + O(\varepsilon^3) \sim -3\varepsilon^2 \tau^2 (\hat{\Theta}^4 - \hat{H}),
$$

$$
\left( \frac{\partial^2}{\partial \tilde{r}^2} - \varepsilon \frac{\partial}{\partial \tilde{r}} - \varepsilon^2 \tilde{r} \frac{\partial}{\partial \tilde{r}} + \frac{\partial^2}{\partial \tilde{z}^2} \right) \hat{\Theta} - \varepsilon P \varepsilon \frac{\partial \hat{\Theta}}{\partial \tilde{z}} + O(\varepsilon^3) \sim \tau \eta (\hat{\Theta}^4 - \hat{H}).
$$
Solutions, expanded as the perturbation series

\[
\hat{\Theta}(\bar{r}, \bar{z}) = \hat{\Theta}_0(\bar{r}, \bar{z}) + \epsilon \hat{\Theta}_1(\bar{r}, \bar{z}) + \epsilon^2 \hat{\Theta}_2(\bar{r}, \bar{z}) + \cdots,
\]

\[
\hat{H}(\bar{r}, \bar{z}) = \hat{H}_0(\bar{r}, \bar{z}) + \epsilon \hat{H}_1(\bar{r}, \bar{z}) + \epsilon^2 \hat{H}_2(\bar{r}, \bar{z}) + \cdots, \tag{5.46}
\]

and substituted into the governing equations, give for equations at each order

\[
\left( \frac{\partial^2}{\partial \bar{r}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) \hat{H}_0 = 0,
\]

\[
\left( \frac{\partial^2}{\partial \bar{r}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) \hat{\Theta}_0 - \frac{\partial \hat{H}_0}{\partial \bar{r}} = 0, \tag{5.47}
\]

\[
\left( \frac{\partial^2}{\partial \bar{r}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) \hat{\Theta}_1 - \frac{\partial \hat{H}_0}{\partial \bar{r}} - \frac{\partial \hat{H}_1}{\partial \bar{r}} = \tau \eta \left( 4 \hat{\Theta}_0^2 \hat{\Theta}_1 - \hat{\Theta}_1 \right), \tag{5.48}
\]

\[
\left( \frac{\partial^2}{\partial \bar{r}^2} + \frac{\partial^2}{\partial \bar{z}^2} \right) \hat{H}_2 - \frac{\partial \hat{H}_1}{\partial \bar{r}} - \hat{r} \frac{\partial \hat{H}_0}{\partial \bar{r}} = -3 \tau^2 \left( \hat{\Theta}_0^4 - \hat{\Theta}_0 \right),
\]

The corresponding boundary conditions at \( \bar{z} = 0 \) are

\[
\hat{\Theta}_0(\bar{r}, 0) = 1, \quad \hat{\Theta}_n(\bar{r}, 0) = 0, \quad (n = 1, 2, \cdots),
\]

\[
\frac{\partial \hat{H}_0}{\partial \bar{z}}(\bar{r}, 0) = 0, \quad \frac{\partial \hat{H}_1}{\partial \bar{z}}(\bar{r}, 0) = \frac{\tau \gamma_0}{\eta} \left[ \hat{H}_0(\bar{r}, 0) - 1 \right],
\]

\[
\frac{\partial \hat{H}_n}{\partial \bar{z}}(\bar{r}, 0) = \frac{\tau \gamma_0}{\eta} \left[ \hat{H}_{n-1}(\bar{r}, 0) - 1 \right], \quad (n = 2, 3, \cdots), \tag{5.49}
\]

and at \( \bar{r} = 0 \) they are

\[
\frac{\partial \hat{\Theta}_0}{\partial \bar{r}}(0, \bar{z}) = 0, \quad \frac{\partial \hat{\Theta}_1}{\partial \bar{r}}(0, \bar{z}) = B \left[ \hat{\Theta}_0(0, \bar{z}) - \theta_s \right],
\]

\[
\frac{\partial \hat{\Theta}_n}{\partial \bar{r}}(0, \bar{z}) = B \hat{\Theta}_{n-1}(0, \bar{z}), \quad (n = 2, 3, \cdots),
\]
The matching conditions as $\bar{z}$ tends to infinity are derived through expanding the inner expansions $\bar{H}(\bar{r}, \bar{z})$ and $\bar{\Theta}(\bar{r}, \bar{z})$ near $r = 1$ in terms of $\bar{z}$ and equating like powers of $\varepsilon$. Those for $\bar{r}$ come in a similar way from the inner expansions $\bar{H}(r, \bar{z})$ and $\bar{\Theta}(r, \bar{z})$ near $z = 0$. Carrying this out gives for the $\bar{z}$-direction

$$\begin{align*}
\lim_{\bar{z} \to \infty} \bar{H}_0(\bar{r}, \bar{z}) &= \theta_s^4 + (1 - \theta_s^4) \sum_{n=1} a_n J_0(\mu_n), \\
\lim_{\bar{z} \to \infty} \bar{H}_1(\bar{r}, \bar{z}) &= \bar{r}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_1(\mu_n) - \bar{z}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_0(\mu_n) \\
&\quad + \sum_{n=1} b_n J_0(\mu_n), \\
\lim_{\bar{z} \to \infty} \bar{\Theta}_0(\bar{r}, \bar{z}) &= [\theta_s^4 + (1 - \theta_s^4) \sum_{n=1} a_n J_0(\mu_n)]^{1/4},
\end{align*}$$

(5.51)

and for the $\bar{r}$-direction

$$\begin{align*}
\lim_{\bar{r} \to \infty} \bar{H}_0(\bar{r}, \bar{z}) &= \theta_s^4 + (1 - \theta_s^4) \sum_{n=1} a_n J_0(\mu_n), \\
\lim_{\bar{r} \to \infty} \bar{H}_1(\bar{r}, \bar{z}) &= \bar{r}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_1(\mu_n) - \bar{z}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_0(\mu_n) \\
&\quad + \sum_{n=1} b_n J_0(\mu_n), \\
\lim_{\bar{r} \to \infty} \bar{\Theta}_0(\bar{r}, \bar{z}) &= \lim_{\varepsilon \to 0} \bar{\Theta}(\bar{r}, \bar{z}).
\end{align*}$$

(5.52)

Then, two leading terms for the intensity field are obtained as

$$\begin{align*}
\bar{H}_0(\bar{r}, \bar{z}) &= \theta_s^4 + (1 - \theta_s^4) \sum_{n=1} a_n J_0(\mu_n), \\
\bar{H}_1(\bar{r}, \bar{z}) &= \bar{r}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_1(\mu_n) - \bar{z}(1 - \theta_s^4) \sum_{n=1} a_n \mu_n J_0(\mu_n) + \sum_{n=1} b_n J_0(\mu_n)
\end{align*}$$
\[
\begin{align*}
&= \bar{r}(1 - \theta_4^4) \frac{T \gamma_1}{\eta} \sum_{n=1}^{\infty} a_n J_0(\mu_n) - \bar{z}(1 - \theta_4^4) \frac{T \gamma_0}{\eta} \left[ 1 - \sum_{n=1}^{\infty} a_n J_0(\mu_n) \right] \\
&+ \sum_{n=1}^{\infty} b_n J_0(\mu_n). \quad (5.53)
\end{align*}
\]

But, owing to the nonlinearity, an explicit solution for the temperature field is again unobtainable.

### 5.1.5 Composite Expansion and Heat Flux

A composite expansion of the intensity field is determined as

\[
H(r, z) \sim H_0(r, z) + \sqrt{\mathcal{P}} H_1(r, z) + O(\mathcal{P})
\]

\[
\sim \theta_4^4 + (1 - \theta_4^4) \sum_{n=1}^{\infty} \frac{2\tau^2 \gamma_1 \gamma_0}{(\eta \mu_n + \tau \gamma_0)(\eta^2 \mu_n^2 + \tau^2 \gamma_1^2)} \exp(-\mu_n z) J_0(\mu_n r)
\]

\[
+ \sqrt{\mathcal{P}} \sum_{n=1}^{\infty} \frac{6\tau \eta^2 \mu_n^2 (2\tau \eta/5)^{1/2}}{(\eta \mu_n + \tau \gamma_0)(\eta^2 \mu_n^2 + \tau^2 \gamma_1^2)} J_0(\mu_n r)
\]

\[
\int_0^1 J_0(\mu_n r) \left[ 1 - 5 R_{00}^4(r) + 4 R_{00}^5(r) \right]^{1/2} dr \exp(-\mu_n z) J_0(\mu_n r)
\]

\[
+ O(\mathcal{P}), \quad (5.54)
\]

because the two leading terms of the inner expansions near \( r = 1 \), near \( z = 0 \), and at the corner are equal to the matching conditions. It shows that the intensity field has no boundary layer at the first two orders.

The non-dimensional heat flux \( \Psi(r, z) \), defined by equation (5.6), can be expressed in terms of two components in the radial and axial directions. From the temperature and intensity fields obtained above, the radial component is

\[
[\hat{e}_r \cdot \Psi] = -4\mathcal{P} \frac{\partial \Theta}{\partial r} - \frac{4\eta}{3r} \frac{\partial H}{\partial r}
\]

\[
\sim \frac{4\eta}{3r} (1 - \theta_4^4) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\mu_n z) J_1(\mu_n r)
\]
and for the axial component it is

\[
\begin{align*}
[\hat{e}_z \cdot \Psi] &= -4P \frac{\partial \Theta}{\partial z} - \frac{4\eta \partial H}{3\tau \partial z} \\
&\sim \frac{4\eta}{3\tau} (1 - \theta^4) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\mu_n z) J_0(\mu_n r) \\
&\quad + \sqrt{P} \frac{4\eta}{3\tau} \sum_{n=1}^{\infty} b_n \mu_n \exp(-\mu_n z) J_0(\mu_n r) + O(\mathcal{P}).
\end{align*}
\] (5.56)

When a two term expansion is considered, the effect of conduction is not included in the expression for the heat flux, as is expected from its definition.

### 5.1.6 Results and Discussion

The heat flux at each bounding surface is evaluated by taking the inner product of heat flux vector and a unit outward normal to the surface. It gives at \( r = 1 \)

\[
\Psi(1, z) = [\hat{e}_r \cdot \Psi](1, z)
\]

\[
\sim \frac{4\eta}{3\tau} \sum_{n=1}^{\infty} [(1 - \theta^4) a_n + \sqrt{P} b_n] \mu_n J_1(\mu_n) \exp(-\mu_n z) + O(\mathcal{P}),
\] (5.57)

and its integral over \( z \) yields the total amount of heat transferred through the peripheral surface as

\[
Q(r = 1) = 2\pi \int_0^\infty \Psi(1, z) dz
\]

\[
\sim \frac{8\eta}{3\tau} \pi \sum_{n=1}^{\infty} [(1 - \theta^4) a_n + \sqrt{P} b_n] J_1(\mu_n) + O(\mathcal{P}).
\] (5.58)

Similarly, at \( z = 0 \) the heat flux is given as

\[
\Psi(r, 0) = [\hat{e}_z \cdot \Psi](r, 0)
\]
This can be written into a different expression by considering the definition of heat flux and the boundary conditions for intensity together,

\[
\Psi(r,0) \sim -\frac{4\eta}{3r} \left[ 3\gamma_0 [\bar{\theta}_0(r,0) - 1] - \sqrt{\mathcal{P}} \left\{ \frac{4}{3} \gamma_0 \bar{H}_1(r,0) + 4 \frac{\partial \bar{\theta}_0}{\partial \bar{R}}(r,0) \right\} + O(\mathcal{P}) \right] \\
\sim -\frac{4\eta}{3r} \left[ 3\gamma_0 [\bar{R}_{00}(r) - 1] - \sqrt{\mathcal{P}} \left\{ \frac{4}{3} \gamma_0 \bar{R}_{10}(r) \\
-4 \left[ \frac{2\pi\eta}{5} (1 - 5\bar{R}_{00}(r) + 4\bar{R}_{00}^5(r))^1/2 \right] \right\} + O(\mathcal{P}) \right. 
\]

The total amount of heat \(Q(z=0)\) through the surface at \(z=0\) is evaluated as

\[
Q(z=0) = 2\pi \int_0^1 \Psi(r,0) r dr \\
\sim \frac{8\eta}{3r} * \sum_{n=1}^{\infty} [(1 - \theta_n^4) a_n + \sqrt{\mathcal{P}} b_n] J_1(\mu_n) + O(\mathcal{P}),
\]

which is equal to that through the peripheral surface, as it should be. The contribution of the enthalpy influx \(\Delta H_i\) through the surface at \(z=0\), caused by the convective motion of the medium, is not seen here because it is of order \(O(\mathcal{P})\).

\[
\frac{\Delta H_i}{n^2 \sigma_0 T_0^4 R^2} = 4\pi \mathcal{P} Pe(1 - \theta_s). 
\]

Since the expression for temperature, intensity, and heat flux is an infinite series of Bessel functions, the effects of parameters are not easily deciphered. However, numerical results can be obtained without difficulties by using the numeric computational packages Matlab or Maple V. The temperature distributions along \(z\)-direction at radial positions of \(r=0\) and \(r=1\) are shown in Figure 12 for \(\mathcal{P} = 0.01, 0.001\), and
0. Other parameters are held at $\tau = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta = 0.5$, $Pe = 0$, and $B = 1$. These are obtained from equation (5.44) by summing up six terms in the series. In principle, the temperature fields near $z = 0$ should be obtained numerically. However, as seen with the plane slab problem the graphical representation of the approximate expression is expected to be almost the same as that obtained numerically. Inside the thermal boundary layer near $z = 0$ the temperature field shows a steep gradient and outside this layer temperature decreases relatively slowly as the exponential function $\exp(-\mu_n z)$. The case $P = 0$ denotes the radiation limit, in the absence of conduction. The so-called temperature jump is seen at $z = 0$. Figure 13 shows the two different representations of heat flux variation along the radial direction at $z = 0$ for $P = 0.01$. The heat flux by equation (5.59) was evaluated by summing 19 terms in the infinite series, whereas that by equation (5.60) is from only 6 terms. The graphs suggest that the expression of equation (5.60) is better for evaluating numerical values of the heat flux at $z = 0$, although both are mathematically equivalent to each other.
Figure 12: Non-dimensional temperature distributions along the axial direction for combined radiation and conduction in a semi-infinitely long circular cylinder bounded by an opaque surface and a partially transparent surface, when $\tau = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.5$, $Pe = 0$, and $B = 1$. 
Figure 13: Non-dimensional heat flux distributions for combined radiation and conduction in a semi-infinitely long circular cylinder bounded by an opaque surface and a partially transparent surface, when $\mathcal{P} = 0.01, \tau = 1, \eta = 1, \epsilon = 0.5, \rho = 0.5, \theta_s = 0.5, Pe = 0$, and $B = 1$. 
5.2 Optically Thin Medium

As has been discussed when radiation-conduction interaction was considered in a slab in an optically thin medium, $\varepsilon = \sqrt{\tau}$ can be used as a small perturbation parameter and the solution can be constructed as a series in powers of $\varepsilon$. The governing equations (5.1) can then be written as

$$\nabla^2 \Theta - P e \frac{\partial \Theta}{\partial z} = \varepsilon^2 \frac{\eta}{P} (\Theta^4 - H), \quad \nabla^2 H = -3\varepsilon^4 (\Theta^4 - H),$$

(5.63)

and the boundary conditions are

$$\Theta(r, 0) = 1, \quad \frac{\partial \Theta}{\partial z}(r, 0) = 0,$$

$$\frac{\partial \Theta}{\partial r}(0, z) = 0, \quad \frac{\partial \Theta(z, 0)}{\partial r} = -B[\Theta(1, z) - \Theta^4],$$

$$\frac{\partial H}{\partial r}(0, z) = 0, \quad \frac{\partial H(r, 0)}{\partial z} = 0,$$

$$\frac{\partial H}{\partial z}(1, z) = -\frac{\varepsilon^2 \gamma_1}{\eta} [H(1, z) - \Theta^4].$$

(5.64)

With the asymptotic expansions of $\Theta(r, z)$ and $H(r, z)$

$$\Theta(r, z) = \Theta_0(r, z) + \varepsilon \Theta_1(r, z) + \varepsilon^2 \Theta_2(r, z) + \cdots,$$

$$H(r, z) = H_0(r, z) + \varepsilon H_1(r, z) + \varepsilon^2 H_2(r, z) + \cdots,$$

(5.65)

the non-dimensional heat flux $\Psi(r, z)$ is expressed as

$$\Psi(r, z) = \frac{q_C + q_R}{n^2 \sigma_0 T_0^4} = -4\mathcal{P} \nabla \Theta(r, z) - \frac{4\eta}{3\tau} \nabla H(r, z)$$

$$\sim -\frac{4\eta}{\tau} \nabla H_0 - \frac{4\eta}{\sqrt{\tau}} \nabla H_1$$

$$- \sum_{n=0}^{M-1} T^{n/2} [4\mathcal{P} \nabla \Theta_n + \frac{4\eta}{3} \nabla H_{n+2}] + O(\tau^{M/2}).$$

(5.66)
Again, it turns out that two more terms are needed for the intensity field than the temperature field in order to evaluate heat flux at a certain order.

The expansions given by equations (5.65) serve as outer expansions, since the solutions of the first three orders for the intensity are

\[
H_0(r, z) = \theta^4_s, \quad H_1(r, z) = 0, \quad H_2(r, z) = c_0 - \frac{7b}{\eta}(1 - \theta^4_s)z
\]

and \(H_2(r, z)\) breaks down in a region far away from \(z = 0\). This requires the rescaling [26]

\[
z = \frac{\tilde{z}}{\epsilon}, \quad \Theta(r, z) = \tilde{\Theta}(r, \tilde{z}), \quad H(r, z) = \tilde{H}(r, \tilde{z}), \quad (5.67)
\]

for the inner expansion, valid in the far-away region. For the inner expansion the perturbation expansions

\[
\tilde{\Theta}(r, \tilde{z}) = \tilde{\Theta}_0(r, \tilde{z}) + \epsilon\tilde{\Theta}_1(r, \tilde{z}) + \epsilon^2\tilde{\Theta}_2(r, \tilde{z}) + \cdots, \\
\tilde{H}(r, \tilde{z}) = \tilde{H}_0(r, \tilde{z}) + \epsilon\tilde{H}_1(r, \tilde{z}) + \epsilon^2\tilde{H}_2(r, \tilde{z}) + \cdots, \quad (5.68)
\]

and the governing equations are

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \epsilon^2 \frac{\partial^2}{\partial \tilde{z}^2} - \epsilon Pe \frac{\partial}{\partial \tilde{z}} \right] \tilde{\Theta} = \epsilon^2 \frac{\eta}{P} (\tilde{\Theta}^4 - \tilde{H}), \\
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \epsilon^2 \frac{\partial^2}{\partial \tilde{z}^2} \right] \tilde{H} = -3\epsilon^4(\tilde{\Theta}^4 - \tilde{H}). \quad (5.69)
\]

There is a slight ambiguity in the use of the terms outer and inner here. The solution in the original physical coordinates is usually called the outer one, and thus this convention is adopted. In the following, the outer and the inner expansions are discussed together.
5.2.1 Perturbation Expansion

For the leading order solution of the temperature field, one can start with the inner expansion. The equation and the boundary conditions

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Theta_0}{\partial r} \right) = 0, \]
\[ \frac{\partial \Theta_0}{\partial r}(0, \bar{z}) = 0, \quad \frac{\partial \Theta_0}{\partial r}(1, \bar{z}) = -B[\Theta_0(1, \bar{z}) - \theta_s], \quad \frac{\partial \Theta_0}{\partial \bar{z}}(r, \infty) = 0, \]

give the leading order solution as

\[ \tilde{\Theta}_0(r, \bar{z}) = \theta_s. \] (5.70)

For the outer expansion the governing equation and the boundary conditions are

\[ \nabla^2 \Theta_0 - Pe \frac{\partial \Theta_0}{\partial \bar{z}} = 0, \]
\[ \frac{\partial \Theta_0}{\partial r}(0, z) = 0, \quad \frac{\partial \Theta_0}{\partial r}(1, z) = -B[\Theta_0(1, z) - \theta_s], \]
\[ \Theta_0(r, 0) = 1, \quad \Theta_0(r, \infty) = \theta_s. \]

The last one matches the outer expansion to the inner one. The solution to these is obtained as an infinite series of Bessel functions,

\[ \Theta_0(r, z) = \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n \exp(-\zeta_n z) J_0(\mu_n r), \] (5.71)

where the eigenvalue \( \mu_n \) is given from the boundary condition at \( r = 1 \) as

\[ \mu_n J_1(\mu_n) = BJ_0(\mu_n), \] (5.72)

and the parameter \( \zeta_n \) is

\[ \zeta_n = \sqrt{\left( \frac{Pe}{2} \right)^2 + \mu_n^2 - \frac{Pe}{2}}. \] (5.73)
The expansion coefficient $a_n$ is developed as

$$a_n = \frac{\int_0^1 r J_0(\mu_n r) dr}{\int_0^1 r J_0^2(\mu_n r) dr} = \frac{2J_1(\mu_n)}{\mu_n[J_1^2(\mu_n) + J_0^2(\mu_n)]} = \frac{2B}{[B^2 + \mu_n^2]J_0(\mu_n)}, \quad (5.74)$$

by applying the boundary condition at $z = 0$ and using the orthogonality of Bessel functions. Thus, the leading order solution is completely determined.

For the intensity field the governing equation and the boundary conditions valid at the far-away region are given as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{H}_0}{\partial r} \right) = 0, \quad \frac{\partial \tilde{H}_0}{\partial r}(0, \tilde{z}) = 0, \quad \frac{\partial \tilde{H}_0}{\partial r}(1, \tilde{z}) = 0, \quad \frac{\partial \tilde{H}_0}{\partial z}(r, \infty) = 0.$$

These show the leading order intensity field to be a function of $\tilde{z}$ only. Let this function $g_0(\tilde{z})$ and then the last condition requires that its derivative vanishes as $\tilde{z}$ tends to infinity. That is,

$$\tilde{H}_0(r, \tilde{z}) = g_0(\tilde{z}), \quad \frac{d g_0}{d \tilde{z}}(\infty) = 0. \quad (5.75)$$

For the outer expansion the governing equation and the boundary conditions

$$\nabla^2 H_0 = 0, \quad \frac{\partial H_0}{\partial r}(0, z) = 0, \quad \frac{\partial H_0}{\partial r}(1, z) = 0, \quad \frac{\partial H_0}{\partial z}(r, 0) = 0, \quad H_0(r, \infty) = g_0(0),$$

lead to

$$H_0(r, z) = g_0(0). \quad (5.76)$$

Again, the last condition for $H_0(r, z)$ matches the outer expansion to the inner one. Thus, the leading order solution is obtained as a function $z$ and is not completely
developed yet. It is expected to be determined when the higher order terms are analyzed.

At the next order the governing equation and the boundary conditions for \( \dot{\Theta}_1(r, \bar{z}) \) are

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \dot{\Theta}_1}{\partial r} \right) - P e \frac{\partial \dot{\Theta}_0}{\partial \bar{z}} = 0,
\]

\[
\frac{\partial \dot{\Theta}_1}{\partial r}(0, \bar{z}) = 0, \quad \frac{\partial \dot{\Theta}_1}{\partial r}(1, \bar{z}) = -B \dot{\Theta}_1(1, \bar{z}), \quad \frac{\partial \dot{\Theta}_1}{\partial \bar{z}}(r, \infty) = 0.
\]

Since \( \dot{\Theta}_0(r, \bar{z}) = \theta_s \), a homogeneous problem is obtained. Its solution is simply

\[
\dot{\Theta}_1(r, \bar{z}) = 0. \quad (5.77)
\]

The governing equation and boundary conditions for the second term of the outer expansion

\[
\nabla^2 \Theta_1 - P e \frac{\partial \Theta_1}{\partial \bar{z}} = 0,
\]

\[
\frac{\partial \Theta_1}{\partial r}(0, z) = 0, \quad \frac{\partial \Theta_1}{\partial r}(1, z) = -B \Theta_1(1, z), \quad \Theta_1(r, 0) = 0, \quad \Theta_1(r, \infty) = 0
\]

lead also to a trivial solution

\[
\Theta_1(r, z) = 0. \quad (5.78)
\]

For the intensity \( \dot{H}_1(r, \bar{z}) \) the governing equation and boundary conditions are

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \dot{H}_1}{\partial r} \right) = 0,
\]

\[
\frac{\partial \dot{H}_1}{\partial r}(0, \bar{z}) = 0, \quad \frac{\partial \dot{H}_1}{\partial r}(1, \bar{z}) = 0, \quad \frac{\partial \dot{H}_1}{\partial \bar{z}}(r, \infty) = 0.
\]

Similarly to \( \dot{H}_0(r, \bar{z}) \), these give

\[
\dot{H}_1(r, \bar{z}) = g_1(\bar{z}), \quad \frac{dg_1}{d\bar{z}}(\infty) = 0, \quad (5.79)
\]
where \( g_1(\tilde{z}) \) is again an undetermined function of \( \tilde{z} \). For the outer expansion the second term in the intensity field is governed by

\[
\nabla^2 H_1 = 0,
\]

\[
\frac{\partial H_1}{\partial r}(0, z) = 0, \quad \frac{\partial H_1}{\partial r}(1, z) = 0, \quad \frac{\partial H_1}{\partial z}(r, 0) = 0, \quad H_1(r, \infty) = g_1(0) + \frac{dg_0}{d\tilde{z}}(0)z.
\]

Its solution is seen to be

\[
H_1(r, z) = g_1(0)
\]

(5.80)

and the matching condition is used to impose a condition

\[
\frac{dg_0}{d\tilde{z}}(0) = 0,
\]

(5.81)

on the function \( g_0(\tilde{z}) \).

The equation for \( \Theta_2(r, z) \) is given as the non-homogeneous equation

\[
\nabla^2 \Theta_2 - Pe \frac{\partial \Theta_2}{\partial z} = \frac{\eta}{P}(\Theta_0^4 - H_0).
\]

In principle, the solution could be represented with the aid of a two-dimensional Green function. The solution would be cumbersome because the right-hand side includes a fourth power of an infinite series. Numerical solution could be obtained, but it was decided not to pursue the higher order solutions for temperature. However, the equations for intensity at higher orders are still manageable analytically, and they are expected to give \( g_0(\tilde{z}) \) and \( g_1(\tilde{z}) \) in closed form. They are discussed in the following:

The governing equation and boundary conditions for \( \tilde{H}_2(r, \tilde{z}) \) are

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \tilde{H}_2}{\partial r} \right) + \frac{\partial \tilde{H}_0}{\partial \tilde{z}} = 0,
\]

\[
\frac{\partial \tilde{H}_2}{\partial r}(0, \tilde{z}) = 0, \quad \frac{\partial \tilde{H}_2}{\partial r}(1, \tilde{z}) = -\frac{\gamma_1}{\eta} [\tilde{H}_0(1, \tilde{z}) - \theta_s^4], \quad \frac{\partial \tilde{H}_2}{\partial \tilde{z}}(r, \infty) = 0.
\]
Upon substituting $\tilde{H}_0(r, \hat{z}) = g_0(\hat{z})$, the governing equation leads to
\[
\tilde{H}_2(r, \hat{z}) = -\frac{d^2 g_0}{d\hat{z}^2} + g_2(\hat{z}),
\]
when the boundary condition at $r = 0$ has been applied and $g_2(\hat{z})$ is an undetermined function. The boundary condition at $r = 1$ gives a equation for $g_0(\hat{z})$ as
\[
-\frac{1}{2} \frac{d^2 g_0}{d\hat{z}^2} = -\frac{\gamma_1}{\eta} [g_0 - \theta_s^4],
\]
the solution of which when the conditions in (5.75) and (5.81) are applied, is
\[
g_0(\hat{z}) = \theta_s^4.
\] (5.82)

As a result the second order intensity field $\tilde{H}_2(r, \hat{z})$ is independent of $r$ and has a vanishing derivative as $\hat{z}$ tends to infinity. That is,
\[
\tilde{H}_2(r, \hat{z}) = g_2(\hat{z}), \quad \frac{d g_2}{d\hat{z}}(\infty) = 0.
\] (5.83)

The equation and the boundary conditions that the outer expansion must satisfy are, upon substituting $H_0(r, z) = g_0(0) = \theta_s^4$,

\[
\nabla^2 H_2 = 0, \quad \frac{\partial H_2}{\partial r}(0, z) = 0, \quad \frac{\partial H_2}{\partial r}(1, z) = 0, \quad \frac{\partial H_2}{\partial z}(r, 0) = -\frac{\gamma_0}{\eta} (1 - \theta_s^4), \quad H_2(r, \infty) = g_2(0) + \frac{d g_1}{d\hat{z}}(0) \hat{z}.
\]

The solution
\[
H_2(r, z) = g_2(0) - \frac{\gamma_0}{\eta} (1 - \theta_s^4)z
\] (5.84)
is consistent once the matching condition is used to give the condition

\[ \frac{dg_1}{dz}(0) = -\frac{\gamma_0}{\eta}(1 - \theta_s^4) \]  \hspace{1cm} (5.85)

for \( g_1(r, \tilde{z}) \).

Without going through the details, the equations and conditions at order \( \epsilon^3 \) yield similarly for \( g_1(\tilde{z}), \tilde{H}_3(r, \tilde{z}) \) and \( H_3(r, z) \) the expressions

\[ g_1(\tilde{z}) = \frac{\gamma_0}{\sqrt{2\eta \gamma_1}}(1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1}{\eta}} \tilde{z} \right), \]

\[ \tilde{H}_3(r, \tilde{z}) = -\frac{\gamma_0 \gamma_1}{\eta \sqrt{2\eta \gamma_1}}(1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1}{\eta}} \tilde{z} \right) \frac{r^2}{2} + g_3(\tilde{z}), \]

\[ H_3(r, z) = -\frac{\gamma_0 \gamma_1}{\eta \sqrt{2\eta \gamma_1}}(1 - \theta_s^4) \left( \frac{r^2}{2} - z^2 \right) + \frac{\gamma_0^2(1 - \theta_s^4)}{\eta \sqrt{2\eta \gamma_1}} z + g_3(0). \] \hspace{1cm} (5.86)

Similarly at order \( \epsilon^4 \), \( g_2(\tilde{z}) \), and \( \tilde{H}_4(r, \tilde{z}) \) are found to be

\[ g_2(\tilde{z}) = -\frac{\gamma_0^2}{2\eta \gamma_1}(1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1}{\eta}} \tilde{z} \right), \]

\[ \tilde{H}_4(r, \tilde{z}) = \frac{\gamma_0^2}{\eta^2}(1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1}{\eta}} \tilde{z} \right) \frac{r^2}{4} + g_4(\tilde{z}). \] \hspace{1cm} (5.87)

Here \( g_3(\tilde{z}) \) and \( g_4(\tilde{z}) \) denote undetermined functions of \( \tilde{z} \).

### 5.2.2 Composite Expansion and Heat Flux

A composite expansion, the sum of the inner and the outer expansions minus their common parts, can now be developed. For the temperature field it is

\[ \Theta(r, z) \sim \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n \exp(-\zeta_n z) J_0(\mu_n r) + O(\tau), \] \hspace{1cm} (5.88)

and for the intensity field it is

\[ H(r, z) \sim \theta_s^4 + r^{1/2} \frac{\gamma_0}{\sqrt{2\eta \gamma_1}}(1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1}{\eta}} \frac{r}{z} \right) \]
\[ -\tau \gamma_2^2 (1 - \theta_s^4) \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) \]

\[ + \tau^{3/2} \left[ \frac{\gamma_0 \gamma_1 (1 - \theta_s^4)}{2\eta \sqrt{2\eta \gamma_1}} \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) + g_3(\sqrt{\tau}z) \right] \]

\[ + \tau^2 \left[ \frac{\gamma_0^2 (1 - \theta_s^4)}{4\eta^2} \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) + H_4(r, z) \right] + O(\tau^{5/2}), \quad (5.89) \]

where \( H_4(r, z) \) denotes the undeveloped contribution from the outer expansion at order \( \epsilon^4 \).

The non-dimensional heat flux \( \Psi(r, z) \), defined by equation (5.66), can be expressed in terms of its components. The component in the radial direction is

\[ [\mathbf{e}_r \cdot \Psi] = -4\mathcal{P} \frac{\partial \Theta}{\partial r} - \frac{4\eta}{3\tau} \frac{\partial H}{\partial r} \]

\[ \sim 4\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\zeta_n z)J_1(\mu_n r) \]

\[ + \sqrt{\tau} \frac{4\gamma_0 \gamma_1 (1 - \theta_s^4)}{3\sqrt{2\eta \gamma_1}} r \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) \]

\[ + \tau \left[ \frac{2\gamma_0^2 (1 - \theta_s^4)}{3\eta} \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) + \frac{\partial H_4}{\partial r} + \frac{\partial \Theta_2}{\partial r} \right] + O(\tau^{3/2}), \quad (5.90) \]

and in the axial direction it is

\[ [\mathbf{e}_z \cdot \Psi] = -4\mathcal{P} \frac{\partial \Theta}{\partial z} - \frac{4\eta}{3\tau} \frac{\partial H}{\partial z} \]

\[ \sim 4\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \zeta_n \exp(-\zeta_n z)J_0(\mu_n r) + \frac{4\gamma_0 (1 - \theta_s^4)}{3} \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) \]

\[ - \sqrt{\tau} \frac{2\gamma_0^2 (1 - \theta_s^4)}{3\sqrt{2\eta \gamma_1}} \exp \left( -\sqrt{\frac{2\gamma_1 \tau}{\eta}} \right) + O(\tau), \quad (5.91) \]

where \( \partial \Theta_2/\partial r \) is the contribution from the outer expansion of temperature at order \( \epsilon^2 \).
5.2.3 Results and Discussion

The components of the heat flux given by equations (5.90) and (5.91) show that
the radiation contribution appears from the leading order of the $z$-component and
from the second order of the $r$-component. Physically, it is reasonable because all
surroundings except the surface at $z = 0$ are isothermal and the intensities both at
the base and at the backgrounds are most effective in optically thin media. Thus,
a major part of radiation transfer takes place in the interaction with the surface at
$z = 0$, that is, in the axial direction. Although the radiation transfer in the radial
direction is minor, it occurs along an infinite length, and thus the total amount of
heat crossing the periphery, should be the same as that passing through the surface
at $z = 0$. The heat flux at $r = 1$ is evaluated as

$$
\Psi(1, z) = [\mathbf{e}_r \cdot \mathbf{\Psi}](1, z)
$$

\[\sim 4\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\zeta_n z) J_1(\mu_n)
\]

\[+ \sqrt{\tau} \frac{4\gamma_0 \gamma_1 (1 - \theta_s^2)}{3\sqrt{2\eta \gamma_1}} \exp\left(-\sqrt{\frac{2\eta \gamma_1}{\eta}} z\right)
\]

\[+ \tau \left[ \frac{2\gamma_s^2 (1 - \theta_s^4)}{3\eta} \exp\left(-\sqrt{\frac{2\gamma_s^2}{\eta}} z\right) + \frac{\partial H_1}{\partial r}(1, z) + \frac{\partial \Theta_2}{\partial r}(1, z) \right] + O(\tau^{3/2}),
\]

and its integration gives for the total amount of heat $Q(r=1)$ transferred through the
peripheral surface

\[Q(r = 1) = 2\pi \int_0^\infty \Psi(1, z) dz
\]

\[\sim 8\pi \mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n J_1(\mu_n) + \frac{4\pi}{3} \gamma_0 (1 - \theta_s^4)
\]

\[+ \sqrt{\tau} \left[ - \frac{4\pi \gamma_0^2 (1 - \Theta_s^4)}{3\sqrt{2\eta \gamma_1}} \right] + O(\tau).
\]

(5.92)
At \( z = 0 \), the corresponding expressions are

\[
\Psi(r, 0) = [\hat{e}_z \cdot \Psi](r, 0) \\
\sim 4\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \zeta_n J_0(\mu_n r) + \frac{4\gamma_0(1 - \theta_s^4)}{3} \\
\quad - \sqrt{\tau} \frac{2\gamma_0^2(1 - \theta_s^4)}{3\sqrt{2\eta \gamma_1}} + O(\tau), \\
Q(z = 0) = 2\pi \int_0^1 r\Psi(r, 0) dr \\
\sim 8\pi\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \frac{\zeta_n}{\mu_n} J_1(\mu_n) + \frac{4\pi}{3} \gamma_0(1 - \theta_s^4) \\
\quad + \sqrt{\tau} \left[ - \frac{4\pi}{3} \frac{\gamma_0^2(1 - \theta_s^4)}{\sqrt{2\eta \gamma_1}} \right] + O(\tau). \quad (5.93)
\]

The difference between the two total amounts of heat, coming from the conduction term, is shown to be the enthalpy influx through the surface at \( z = 0 \) caused by the convective motion of the medium,

\[
Q(r = 1) - Q(z = 0) \sim 8\pi\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \left( \frac{\mu_n}{\zeta_n} - \frac{\zeta_n}{\mu_n} \right) J_1(\mu_n) + O(\tau) \\
\sim 8\pi\mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \frac{Pe}{\mu_n} J_1(\mu_n) + O(\tau) \\
\sim 4\pi\mathcal{P} Pe(1 - \theta_s) + O(\tau). \quad (5.94)
\]

Here, the summation

\[
\sum_{n=1}^{\infty} a_n \frac{J_1(\mu_n)}{\mu_n} = \frac{1}{2}
\]

is used, which is easily established from the relation

\[
\sum_{n=1}^{\infty} a_n J_0(\mu_n r) = 1,
\]

by multiplying both sides by \( r \) and integrating with respect to \( r \) over the interval \([0, 1]\).
The temperature field given in equation (5.88) and the intensity field by equation (5.89) show that their interaction is not seen at the first two orders in temperature and for the first three terms in intensity. In particular, the intensity field shows that its leading order solution is not an algebraic mean of the intensities at the bounding surfaces, but the intensity of the far-away surface. This supports the concept that in the optically thin limit the intensity field takes the value of the backgrounds [25, p. 144].

The analytic approximations of the temperature and intensity fields, and heat flux obtained give a consistent picture for the optically thin limit. This problem is not one of the set discussed by Cess [11] for which he shows the differential approximation to be valid. Thus, it is another example of an axisymmetric thin medium where the differential approximation is useful.
5.3 Optically Thick Medium

When the medium is optically thick, a perturbation expansion is again constructed as a series in inverse powers of the optical thickness. With $\varepsilon = 1/\tau$, the governing equations (5.1) become

$$
\varepsilon(\nabla^2 \Theta - P e \frac{\partial \Theta}{\partial z}) = \frac{\eta}{\mathcal{P}}(\Theta^4 - H), \\
\varepsilon^2 \nabla^2 H = -3(\Theta^4 - H),
$$

and the boundary conditions are

$$
\Theta(r, 0) = 1, \quad \frac{\partial \Theta}{\partial z}(r, \infty) = 0, \\
\frac{\partial \Theta}{\partial r}(0, z) = 0, \quad \frac{\partial \Theta}{\partial r}(1, z) = -B[\Theta(1, z) - \theta_s], \\
\varepsilon \frac{\partial H}{\partial z}(r, 0) = \frac{\gamma}{\eta}[H(r, 0) - 1], \quad \frac{\partial H}{\partial z}(r, \infty) = 0, \\
\frac{\partial H}{\partial r}(0, z) = 0, \quad \varepsilon \frac{\partial H}{\partial r}(1, z) = -\frac{\gamma}{\eta}[H(1, z) - \theta_s^4].
$$

With the perturbation expansions

$$
\Theta(r, z) = \Theta_0(r, z) + \varepsilon \Theta_1(r, z) + \varepsilon^2 \Theta_2(r, z) + \cdots, \\
H(r, z) = H_0(r, z) + \varepsilon H_1(r, z) + \varepsilon^2 H_2(r, z) + \cdots
$$

for $\Theta(r, z)$ and $H(r, z)$ the non-dimensional heat flux $\Psi$ can be cast into the form

$$
\Psi(r, z) = \frac{q_C + q_R}{n^2 \sigma_0 T_0^4} = -4\mathcal{P} \nabla \Theta(r, z) - \frac{4\eta}{3\tau} \nabla H(r, z)
$$

$$
\sim -4\mathcal{P} \nabla \Theta_0 - \sum_{n=1}^{M-1} \frac{1}{\tau^n} [4\mathcal{P} \nabla \Theta_n + \frac{4\eta}{3} \nabla H_{n-1}] + O(\tau^{M/2}).
$$

It is seen that one more term should be obtained for the temperature field than for the intensity field in order to evaluate the heat flux at a certain order.
The problem posed by equations (5.95) does not yield an explicit solution owing to the nonlinearities. In a way to bypass this difficulty, in what follows the linearized problem will be discussed. Thus, the results are valid for the limited situation of small temperature variations. With the assumption $|1 - \Theta_s| \ll 1$, the linearization

$$\Theta^4 = [(\Theta - \Theta_s) + \Theta_s]^4 \sim 4\Theta_s^3\Theta - 3\Theta_s^4 + O(|1 - \Theta_s|^2)$$  \hspace{1cm} (5.99)

can be invoked. Then, the governing equations (5.95), upon substituting and rearranging, lead to

$$\nabla^2 \Theta - Pe \frac{\partial \Theta}{\partial z} = -\varepsilon \frac{\eta}{3P} \nabla^2 H,$$

$$\varepsilon^2 \nabla^2 H \sim -3(4\Theta_s^3\Theta - 3\Theta_s^4 - H) + O(|1 - \Theta_s|^2).$$  \hspace{1cm} (5.100)

### 5.3.1 Outer Expansion

For the naive expansions given by equation (5.97), the equations at each order are

$$\nabla^2 \Theta_0 - Pe \frac{\partial \Theta_0}{\partial z} = 0, \quad H_0 = 4\Theta_s^3\Theta_0 - 3\Theta_s^4,$$

$$\nabla^2 \Theta_1 - Pe \frac{\partial \Theta_1}{\partial z} = -\frac{\eta}{3P} \nabla^2 H_0, \quad H_1 = 4\Theta_s^3\Theta_1,$$

$$\nabla^2 \Theta_n - Pe \frac{\partial \Theta_n}{\partial z} = -\frac{\eta}{3P} \nabla^2 H_{n-1}, \quad H_n = 4\Theta_s^3\Theta_n + \frac{1}{3} \nabla^2 H_{n-2}, \quad (n = 2, 3, \cdots),$$

and the boundary conditions become

$$\Theta_0(r, 0) = 1, \quad (n = 1, 2, \cdots),$$
$$\Theta_n(r, 0) = 0, \quad (n = 0, 1, \cdots),$$
$$\frac{\partial \Theta_n}{\partial z}(r, \infty) = 0, \quad \frac{\partial \Theta_n}{\partial r}(0, z) = 0, \quad (n = 0, 1, \cdots),$$
$$\frac{\partial \Theta_0}{\partial r}(1, z) = -B[\Theta_0(1, z) - \Theta_s], \quad \frac{\partial \Theta_n}{\partial r}(1, z) = -B\Theta_n(1, z), \quad (n = 1, 2, \cdots),$$
$$H_0(r, 0) = 1, \quad H_n(r, 0) = \frac{\eta}{\gamma_0} \frac{\partial H_{n-1}}{\partial z}(r, 0), \quad (n = 1, 2, \cdots),$$
\[ \frac{\partial H_n}{\partial z}(r, \infty) = 0, \quad \frac{\partial H_n}{\partial r}(0, z) = 0, \quad (n = 0, 1, \ldots), \]

\[ H_0(1, z) = \theta_s^4, \quad H_n(1, z) = -\frac{\eta}{\gamma_1} \frac{\partial H_{n-1}}{\partial r}(1, z), \quad (n = 1, 2, \ldots), \]

The leading order expansion for the temperature field can be written as an infinite series of Bessel functions

\[ \Theta_0(r, z) = \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n \exp(-\zeta_n z) J_0(\mu_n r), \quad (5.101) \]

where the parameter \( \zeta_n \) is defined as

\[ \zeta_n = \sqrt{\left( \frac{P_e}{2} \right)^2 + \mu_n^2} - \frac{P_e}{2}. \quad (5.102) \]

The boundary condition at \( r = 1 \) leads to the expression

\[ \mu_n J_1(\mu_n) = B J_0(\mu_n), \quad (5.103) \]

for the eigenvalues \( \mu_n \), and the condition at \( z = 0 \) gives for the expansion coefficient

\[ a_n = \frac{\int_0^1 r J_0(\mu_n r) dr}{\int_0^1 r J_1^2(\mu_n r) dr} = \frac{2J_1(\mu_n)}{\mu_n [J_1^2(\mu_n) + J_0^2(\mu_n)]} \quad (5.104) \]

The leading order of intensity is simply obtained as

\[ H_0(r, z) = 4\theta_s^2 \Theta_0(r, z) - 3\theta_s^4. \quad (5.105) \]

This satisfies the boundary conditions at \( r = 0 \) and at \( z = \infty \) but does not satisfy the conditions at \( r = 1 \) and \( z = 0 \). This means that the intensity field may require boundary layers near both boundaries at \( r = 1 \) and at \( z = 0 \).
5.3.2 Inner Expansion Near \( r = 1 \)

To develop the inner expansion near \( r = 1 \), the following variables are introduced:

\[
1 - r = \varepsilon \bar{r}, \quad \Theta(r, z) = \bar{\Theta}(\bar{r}, z), \quad H(r, z) = \bar{H}(\bar{r}, z). \tag{5.106}
\]

In these variables the governing equations (5.100) become

\[
\frac{\partial^2}{\partial \bar{r}^2} \Theta - \varepsilon \frac{\partial}{\partial \bar{r}} - \varepsilon^2 \bar{r} \frac{\partial}{\partial \bar{r}} + \varepsilon^2 \frac{\partial^2}{\partial z^2} - \varepsilon^2 Pe \frac{\partial}{\partial z} \bar{\Theta} + O(\varepsilon^3)
\]

\[
\sim \frac{\bar{r}}{3 P} \left[ \frac{\partial^2}{\partial \bar{r}^2} - \varepsilon \frac{\partial}{\partial \bar{r}} - \varepsilon^2 \bar{r} \frac{\partial}{\partial \bar{r}} + \varepsilon^2 \frac{\partial^2}{\partial z^2} \right] \bar{H},
\]

\[
\left[ \frac{\partial^2}{\partial \bar{r}^2} - \varepsilon \frac{\partial}{\partial \bar{r}} - \varepsilon^2 \bar{r} \frac{\partial}{\partial \bar{r}} + \varepsilon^2 \frac{\partial^2}{\partial z^2} \right] \bar{H} + O(\varepsilon^3)
\]

\[
\sim -3(4\theta_s^3 \bar{\Theta} - 3\theta_s^4 - \bar{H}) + O(|1 - \theta_s|^2). \tag{5.107}
\]

By expanding the variables as

\[
\Theta(\bar{r}, z) = \Theta_0(\bar{r}, z) + \varepsilon \Theta_1(\bar{r}, z) + \varepsilon^2 \Theta_2(\bar{r}, z) + \cdots,
\]

\[
\bar{H}(\bar{r}, z) = \bar{H}_0(\bar{r}, z) + \varepsilon \bar{H}_1(\bar{r}, z) + \varepsilon^2 \bar{H}_2(\bar{r}, z) + \cdots, \tag{5.108}
\]

the governing equations at various orders are readily determined. By expanding the outer expansions in terms of inner variables and collecting like powers of \( \varepsilon \), the matching conditions to the outer expansions are given as

\[
\lim_{\bar{r} \to 0} \Theta_0(\bar{r}, z) = Z_{00}(z),
\]

\[
\lim_{\bar{r} \to 0} \Theta_1(\bar{r}, z) = \Theta_1(1, z) + \bar{r} Z_{01}(z), \tag{5.109}
\]

for the temperature field, and for the intensity field they are

\[
\lim_{\bar{r} \to 0} \bar{H}_0(\bar{r}, z) = 4\theta_s^3 Z_{00}(z) - 3\theta_s^4,
\]

\[
\lim_{\bar{r} \to 0} \bar{H}_1(\bar{r}, z) = 4\theta_s^3 [\Theta_1(1, z) + \bar{r} Z_{01}(z)], \tag{5.110}
\]
where $\Theta_1(1, z)$ denotes the contribution of the outer expansion of order $\varepsilon$ and

$$Z_{00}(z) = \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n J_0(\mu_n) \exp(-\zeta_n z),$$
$$Z_{01}(z) = (1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n J_1(\mu_n) \exp(-\zeta_n z).$$

For the leading order solution the governing equation and the boundary and matching conditions for temperature are

$$\frac{\partial^2 \bar{\Theta}_0}{\partial \bar{r}^2} = 0,$$
$$\frac{\partial \bar{\Theta}_0}{\partial \bar{r}}(0, z) = 0, \quad \bar{\Theta}_0(\infty, z) = Z_{00}(z), \quad \frac{\partial \bar{\Theta}_0}{\partial z}(r, \infty) = 0.$$

These show that the solution takes its matching function

$$\bar{\Theta}_0(\bar{r}, z) = Z_{00}(z). \quad (5.111)$$

For the intensity field the equation and boundary conditions are

$$\frac{\partial^2 \bar{H}_0}{\partial \bar{r}^2} = -3(4\theta_3^3 \bar{\Theta}_0 - 3\theta_4^4 - \bar{H}_0),$$
$$\frac{\partial \bar{H}_0}{\partial \bar{r}}(0, z) = \frac{\gamma_1}{\eta} [\bar{H}_0(0, z) - \theta_3^4], \quad \bar{H}_0(\infty, z) = 4\theta_3^3 Z_{00}(z) - 3\theta_4^4, \quad \frac{\partial \bar{H}_0}{\partial z}(r, \infty) = 0.$$

Upon substituting $\bar{\Theta}_0(\bar{r}, z) = Z_{00}(z)$, the solution is obtained as

$$\bar{H}_0(\bar{r}, z) = 4\theta_3^3 Z_{00}(z) - 3\theta_4^4 - \frac{4\theta_3^3 \gamma_1}{\sqrt{3}\eta + \gamma_1} [Z_{00}(z) - \theta_3] \exp(-\sqrt{3}\bar{r}). \quad (5.112)$$

The equation and the conditions for $\bar{\Theta}_1$ are

$$\frac{\partial^2 \bar{\Theta}_1}{\partial \bar{r}^2} - \frac{\partial \bar{\Theta}_0}{\partial \bar{r}} = \frac{\eta}{P} (4\theta_3^3 \bar{\Theta}_0 - 3\theta_4^4 - \bar{H}_0),$$
$$\frac{\partial \bar{\Theta}_1}{\partial \bar{r}}(0, z) = B[\bar{\Theta}_0(0, z) - \theta_3], \quad \bar{\Theta}_1(\infty, z) = \Theta_1(1, z) + \bar{r} Z_{01}(z), \quad \frac{\partial \bar{\Theta}_1}{\partial z}(\bar{r}, \infty) = 0.$$
Upon substituting \( \Theta_0(\bar{r}, z) \) and \( \bar{H}_0(\bar{r}, z) \), the solution can be cast into the form

\[
\Theta_1(\bar{r}, z) = \Theta_1(1, z) + \bar{r} Z_{01}(z) + \frac{4 \theta_0^2 \eta \gamma_1}{3 \mathcal{P}(\sqrt{3} \eta + \gamma_1)} [Z_{00}(z) - \theta_s] \exp(-\sqrt{3} \bar{r}),
\]

(5.113)

and the boundary condition at \( \bar{r} = 0 \) leads to

\[
-\frac{4 \theta_0^2 \eta \gamma_1}{\sqrt{3} \mathcal{P}(\sqrt{3} \eta + \gamma_1)} [Z_{00}(z) - \theta_s] + Z_{01}(z) = B[Z_{00}(z) - \theta_s].
\]

Upon substituting \( Z_{00}(z) \) and \( Z_{01}(z) \), this gives an equation for \( \mu_n \) as

\[
\left[ \frac{4 \theta_0^4 \eta \gamma_1}{\sqrt{3} \mathcal{P}(\sqrt{3} \eta + \gamma_1)} + B \right] J_0(\mu_n) = \mu_n J_1(\mu_n).
\]

(5.114)

It shows that the eigenvalue given by equation (5.103), which was determined from the outer expansion, is not valid. In the former equation the influence of radiation was absent. Here, the first term on the left-hand side gives the radiation correction. Its contribution through the eigenvalues is an interesting aspect of the analysis.

### 5.3.3 Inner Expansion Near \( z = 0 \)

For the inner expansion valid near \( z = 0 \), the following variables are introduced:

\[
z = \varepsilon \bar{z}, \quad \Theta(r, z) = \bar{\Theta}(r, \bar{z}), \quad H(r, z) = \bar{H}(r, \bar{z}),
\]

(5.115)

and their perturbation expansions are

\[
\bar{\Theta}(r, \bar{z}) = \bar{\Theta}_0(r, \bar{z}) + \varepsilon \bar{\Theta}_1(r, \bar{z}) + \varepsilon^2 \bar{\Theta}_2(r, \bar{z}) + \cdots,
\]

\[
\bar{H}(r, \bar{z}) = \bar{H}_0(r, \bar{z}) + \varepsilon \bar{H}_1(r, \bar{z}) + \varepsilon^2 \bar{H}_2(r, \bar{z}) + \cdots.
\]

(5.116)

The governing equations (5.100) lead to

\[
\begin{align*}
\left[ e^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \bar{z}^2} - \varepsilon \mathcal{P} e^{-1} \frac{\partial}{\partial \bar{z}} \right] \bar{\Theta} &= -\varepsilon \frac{\eta}{3 \mathcal{P}} \left[ e^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \bar{z}^2} \right] \bar{H} \\
\left[ e^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \bar{z}^2} \right] \bar{H} &\sim -3(4 \theta_0^2 \bar{\Theta} - 3 \theta_0^4 - \bar{H}) + \mathcal{O}(|1 - \theta_s|^2).
\end{align*}
\]

(5.117)
In order to match to the outer expansion, the solution must satisfy the following conditions at each order:

\[
\lim_{\tilde{z} \to \infty} \tilde{\Theta}_0(r, \tilde{z}) = R_{00}(r),
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{H}_0(r, \tilde{z}) = 4\theta_s^3 R_{00}(r) - 3\theta_s^4,
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{\Theta}_1(r, \tilde{z}) = \Theta_1(r,0) + \tilde{z} R_{01}(r),
\]

\[
\lim_{\tilde{z} \to \infty} \tilde{H}_1(r, \tilde{z}) = 4\theta_s^3[\Theta_1(r,0) + \tilde{z} R_{01}(r)]. \tag{5.118}
\]

Here, $\Theta_1(r,0)$ denotes again the contribution from the outer expansion of order $\varepsilon$ and

\[
R_{00}(r) = \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n J_0(\mu_n r),
\]

\[
R_{01}(r) = -(1 - \theta_s) \sum_{n=1}^{\infty} a_n \zeta_n J_0(\mu_n r).
\]

The governing equation and the boundary and matching conditions for the leading order temperature are

\[
\frac{\partial^2 \tilde{\Theta}_0}{\partial \tilde{z}^2} = 0,
\]

\[
\tilde{\Theta}_0(r, 0) = 1, \quad \tilde{\Theta}_0(r, \infty) = R_{00}(r), \quad \frac{\partial \tilde{\Theta}_0}{\partial r}(0, \tilde{z}) = 0.
\]

The equation and the matching condition require that

\[
\tilde{\Theta}_0(r, \tilde{z}) = R_{00}(r). \tag{5.119}
\]

The boundary condition at $\tilde{z} = 0$ leads to an equation for $a_n$ as

\[
R_{00}(r) = \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n J_0(\mu_n r) = 1,
\]
which gives, upon applying the orthogonality of Bessel functions, an expression for $a_n$ identical to equation (5.104), determined from the outer expansion. For the intensity the governing equation and the boundary and matching conditions are

$$\frac{\partial^2 \tilde{H}_0}{\partial \tilde{z}^2} = -3(4\theta_s^2 \tilde{\Theta}_0 - 3\theta_s^4 - \tilde{H}_0),$$

$$\frac{\partial \tilde{H}_0}{\partial \tilde{z}}(r,0) = \frac{\gamma_0}{\eta}[\tilde{H}_0(r,0) - 1], \quad \tilde{H}_0(r,\infty) = 4\theta_s^3 R_{oo}(r) - 3\theta_s^4, \quad \frac{\partial \tilde{H}_0}{\partial r}(0,\tilde{z}) = 0.$$

Upon substituting $\tilde{\Theta}_0(r,\tilde{z}) = R_{oo}(r) = 1$, the solution is

$$\tilde{H}_0(r,\tilde{z}) = 4\theta_s^3 - 3\theta_s^4 + \frac{\gamma_0}{\sqrt{3\eta + \gamma_0}}(1 - 4\theta_s^3 + 3\theta_s^4)\exp(-\sqrt{3}\tilde{z}). \quad (5.120)$$

The governing equation and boundary and matching conditions for the next order are

$$\frac{\partial^2 \tilde{\Theta}_1}{\partial \tilde{z}^2} - Pe \frac{\partial \tilde{\Theta}_0}{\partial \tilde{z}} = -\frac{\eta}{3P} \frac{\partial^2 \tilde{H}}{\partial \tilde{z}^2}$$

$$\tilde{\Theta}_1(r,0) = 0, \quad \tilde{\Theta}_1(r,\infty) = \Theta_1(r,0) + \tilde{z}R_{01}(r), \quad \frac{\partial \tilde{\Theta}_1}{\partial r}(0,\tilde{z}) = 0.$$

Substituting $\tilde{\Theta}_0$ and $\tilde{H}_0$, and applying the matching condition yields the solution

$$\tilde{\Theta}_1(r,\tilde{z}) = \Theta_1(r,0) + \tilde{z}R_{01}(r) - \frac{\eta\gamma_0}{3P(\sqrt{3\eta + \gamma_0})}(1 - 4\theta_s^3 + 3\theta_s^4)\exp(-\sqrt{3}\tilde{z}), \quad (5.121)$$

and its boundary condition at $\tilde{z} = 0$ leads to

$$\Theta_1(r,0) = \frac{\eta\gamma_0}{3P(\sqrt{3\eta + \gamma_0})}(1 - 4\theta_s^3 + 3\theta_s^4).$$

### 5.3.4 Composite Expansion and Heat Flux

A composite expansion, the sum of the inner and the outer expansions minus their common parts, can now be developed. For the temperature field it is

$$\Theta(r,z) \sim \theta_s + (1 - \theta_s) \sum_{n=1}^{\infty} a_n \exp(-\zeta_n z)J_0(\mu_n r)$$
\[ + \frac{1}{\tau} \left\{ \Theta_1(r, z) + \frac{4\theta_s^3 \gamma_1}{3\rho(\sqrt{3}\eta + \gamma_1)} [Z_{00}(z) - \theta_s] \exp[-\sqrt{3}\tau(1 - r)] \right. \\
\left. - \frac{\eta\gamma_0}{3\rho(\sqrt{3}\eta + \gamma_0)} (1 - 4\theta_s^3 + 3\theta_s^4) \exp(-\sqrt{3} \tau z) \right\} \\
+ O\left(\frac{1}{\tau^2}\right), \quad (5.122) \]

and for the intensity field it takes the form

\[ H(r, z) \sim \Theta_1^4 + 4\theta_s^3(1 - \theta_s) \sum_{n=1}^{\infty} a_n \exp(-\zeta_n z) J_0(\mu_n r) \]

\[ - \frac{4\theta_s^3 \gamma_1}{\sqrt{3}\eta + \gamma_1} [Z_{00}(z) - \theta_s] \exp[-\sqrt{3}\tau(1 - r)] \\
+ \frac{\gamma_0}{\sqrt{3}\eta + \gamma_0} [1 - 4\theta_s^3 + 3\theta_s^4] \exp[-\sqrt{3}\tau z] + O\left(\frac{1}{\tau}\right), \quad (5.123) \]

where \( \Theta_1(r, z) \) denotes the undeveloped outer expansion of temperature of order \( \varepsilon \).

The eigenvalue \( \mu_n \) is determined from equation (5.114) and not from (5.103).

The radial and axial components of the non-dimensional heat flux \( \Psi(r, z) \), defined by equation (5.98), are evaluated as

\[ [\hat{e}_r \cdot \Psi] = -4\rho \frac{\partial \Theta}{\partial r} - \frac{4\eta}{3\tau} \frac{\partial H}{\partial r} \]

\[ \sim 4\rho(1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n \exp(-\zeta_n z) J_1(\mu_n r) + O\left(\frac{1}{\tau}\right), \quad (5.124) \]

\[ [\hat{e}_z \cdot \Psi] = -4\rho \frac{\partial \Theta}{\partial z} - \frac{4\eta}{3\tau} \frac{\partial H}{\partial z} \]

\[ \sim 4\rho(1 - \theta_s) \sum_{n=1}^{\infty} a_n \zeta_n \exp(-\zeta_n z) J_0(\mu_n r) + O\left(\frac{1}{\tau}\right). \quad (5.125) \]

A contribution of the intensity field to the leading order, coming from its inner expansion, cancels out a contribution at order \( \varepsilon \) from the inner expansion for the temperature field. Thus, the leading order term for the heat flux is determined from the
conduction part alone. However, it is not equal to that for the case of pure conduction
because the radiation field has already changed the temperature field, as seen in the
equation for the eigenvalue.

5.3.5 Results and Discussion

The total amount of heat $Q(r = 1)$ transferred through the peripheral surface is
evaluated as

$$Q(r = 1) = 2\pi \int_0^\infty [\hat{e}_r \cdot \vec{\Psi}](1, z)dz$$

$$\sim 8\pi \mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \mu_n \frac{\hat{\zeta}_n}{\zeta_n} J_1(\mu_n) + O\left(\frac{1}{r}\right), \quad (5.126)$$

and for $Q(z = 0)$ through the surface at $z = 0$, it is

$$Q(z = 0) = 2\pi \int_0^\infty [\hat{e}_z \cdot \vec{\Psi}](r, 0) rdr$$

$$\sim 8\pi \mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \frac{\hat{\zeta}_n}{\mu_n} J_1(\mu_n) + O\left(\frac{1}{r}\right). \quad (5.127)$$

The difference between the two should be the enthalpy influx through the surface at
$z = 0$ related to the convective motion of the medium,

$$Q(r = 1) - Q(z = 0) \sim 8\pi \mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \left(\frac{\mu_n}{\zeta_n} - \frac{\hat{\zeta}_n}{\mu_n}\right) J_1(\mu_n) + O\left(\frac{1}{r}\right)$$

$$\sim 8\pi \mathcal{P}(1 - \theta_s) \sum_{n=1}^{\infty} a_n \frac{Pe}{\mu_n} J_1(\mu_n) + O\left(\frac{1}{r}\right)$$

$$\sim 4\pi \mathcal{P}(1 - \theta_s) Pe + O\left(\frac{1}{r}\right), \quad (5.128)$$

where the identity

$$\sum_{n=1}^{\infty} a_n \frac{J_1(\mu_n)}{\mu_n} = \frac{1}{2},$$
has been used since the relation $\sum_{n=1}^{\infty} a_n J_0(\mu_n r) = 1$ holds.

Figure 14 shows the temperature distributions along the axial direction at radial positions of $r = 1$ and $r = 0$ for the optically thin and for the optically thick limits. To demonstrate both results, the background temperature $\theta_s$ is chosen as 0.9. Since the only difference between the two solutions is the eigenvalue, given in equation (5.72) for thin media and by (5.114) for thick media, the temperature fields show the same trend. Inspection of both expressions reveals that the difference between the two results arises from the contribution of the radiation transfer, since the temperature field for the optically thin limit is determined by conduction alone. Physically, this must be so because the radiation transfer from the medium to its surroundings is more active in thick media than in thin media, and thus the temperature for the thick limit should be lower than that for the thin limit.
Figure 14: Non-dimensional temperature distributions along the axial direction for combined radiation and conduction in a semi-infinitely long circular cylinder bounded by an opaque surface and a partially transparent surface, when $\mathcal{P} = 1$, $\eta = 1$, $\epsilon = 0.5$, $\rho = 0.5$, $\theta_s = 0.9$, $P\epsilon = 0$, and $B = 1$. 
5.4 Summary

In this chapter the interaction of radiation and conduction that takes place in a circular semi-transparent rod moving at a constant speed in the axial direction has been discussed for certain limiting situations. The method of matched asymptotic expansions has been used in order to obtain analytic approximations of the solutions in a simple form for the radiation-dominant situation, the optically thin medium, and the optically thick medium. In the situation dominated by radiation, asymptotic expansions of the solutions were constructed as series in powers of $\sqrt{y}$. Thermal boundary layers were observed near $z = 0$ and near $r = 1$; the former at the leading order, the latter at the second order. The temperature field valid near $z = 0$ could not be developed explicitly owing to the nonlinearity in the equation at the leading order. Except for that region, the first two terms for the temperature field were obtained. The intensity field did not show any boundary layer character up to the first two terms, and thus the two term expansion for the intensity field was determined from the outer expansion alone. A two term expansion of the heat flux did not show any contribution of conduction, and its sum over each bounding surface, the total amount of heat transferred, was conserved.

For optically thin media, perturbation expansions for the temperature and intensity fields were constructed as power series in $\sqrt{r}$. The inner expansions now occupied the region far away from $z = 0$. The leading order of the temperature field was identical to the solution for pure conduction, and that for the intensity field took the value of the background intensity. The heat flux at the surface $z = 0$ and along the surface
at $r = 1$ were not conserved at each order, but its integral, the total amount of heat
at each surface, was conserved when the enthalpy influx of the convective motion was
considered.

For optically thick media, asymptotic expansions of the solutions were constructed
as a series in inverse of the optical thickness. To bypass the difficulties arising from
nonlinearity, the solutions are sought for a restricted situation of small temperature
variations. Intrinsically, the intensity field had boundary layers near $z = 0$ at the
second order and near $r = 1$ at the leading order. The linearization introduced
an intensity boundary layer near $z = 0$ at the leading order as a side effect. The
leading order of the temperature field included the contribution of radiation via the
eigenvalue, and with other parameters held fixed, its temperature at any axial location
was lower than that for the thin limit. Again, the total amount of heat through each
bounding surface was conserved by including the enthalpy influx of the convective
motion.

The three limiting situations mentioned above have been discussed for an axisym-
metric medium. Basically, the solutions carry the characteristics of the corresponding
limiting solutions observed in the plane slab problem. The variations are caused by
the infinite length of the medium. The problems discussed here model the cooling
process of fibers or crystals drawn from melts, when materials are opaque to thermal
radiation as liquid but semi-transparent as solid and it is further assumed that a
constant diameter is reached at the solidification front. The heat flux developed can
be used in estimating the pulling speed for a stationary solidification front as

\[ U = \frac{Q(z = 0)}{\pi R^2 L}. \]  \hspace{1cm} (5.129)

Here, \( Q(z = 0) \) denotes the total amount of heat evaluated at \( z = 0 \), and the parameter \( L \) is the latent heat of solidification.
CHAPTER VI

CONCLUSION

The aim of this work was to enlarge understanding in regard to how radiative heat transfer interacts with conduction in semi-transparent materials. The differential approximation in Traugott's formulation was invoked in order to bypass the mathematical complexities in the equation of radiative transfer. For general absorbing, emitting, and scattering media the differential approximation has been reworked to take care of the directional dependence of radiation. Furthermore the frequency dependence has been handled by following Traugott's formulation. When the equations for the temperature and intensity fields and the corresponding boundary conditions were put into a non-dimensional form, there were five non-dimensional radiative parameters: two, the optical thickness and the non-grayness factor, that characterize the medium; two surface parameters; and a conduction-radiation parameter called the Planck number. The main feature in the choice of these non-dimensional parameters is that the optical thickness alone governs the opacity of the medium. The solutions for a one-dimensional slab having two parallel, infinite, black surfaces have clearly demonstrated the merit of this formulation: radiation or conduction dominance through the Planck number, and opacity via the optical thickness.

Simple expressions for the temperature and intensity fields and for the heat flux
have been developed when a radiative parameter is either small or large by using the method of matched asymptotic expansions. Two problems were discussed. The first was a combined radiation and conduction heat transfer taking place in a one-dimensional slab. The two boundary surfaces were assumed to be parallel and infinite; one is opaque and the other partially transparent. Explicit expressions for the solutions were obtained for the three limiting situations: the situation dominated by radiation, the optically thin medium, and the optically thick medium. It has also been demonstrated how the various parameters influence the solutions. Comparing with numerical solutions gives additional information in regard to the valid ranges of these asymptotic expansions. A circular cylindrical medium moving axially was a second example considered. For the same three limiting situations as discussed for the plane slab, explicit expressions for the temperature and intensity fields and for the heat flux have been obtained for the moving cylinder. Owing to the lack of previous studies, the validity of the solutions has been shown in a sense that the total amount of heat transferred through the medium is conserved.

Use of the method of matched asymptotic expansions has been demonstrated in connection with the combined radiation and conduction problems. The choice of proper gauge functions is crucial to the success of this method. The six examples discussed above are good illustrations of this. In spite of the restricted usage of the asymptotic analysis, the fact that the solutions are obtained in explicit forms is a big advantage. Specifically, they give insights into the exact solutions in the proper limits. The current developments in computer like symbolic calculation tools were a
great aid in the asymptotic analysis that was carried out.
BIBLIOGRAPHY


