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ANISOTROPIC ELASTICITY ANALYSIS OF COMPOSITE PLATES
SUBJECTED TO NON-UNIFORM STRESSES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the
degree Doctor of Philosophy in the Graduate School of the
Ohio State University

BY

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* * * * * * * *

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JUNE, 1995

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Advisor
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To my parents

You have always reminded me of the value of education, this is for your endless efforts.

Amir
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VITA

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ANISOTROPIC ELASTICITY ANALYSIS OF LAMINATED COMPOSITE
PLATES SUBJECTED TO NON-UNIFORM STRESSES

ABSTRACT

Laminated composite plates are increasingly used in design of modern structures. Most of these structures are subjected to severe non-uniform in-plane axial and shear loading. An accurate knowledge of stresses throughout the plate for different types of non-uniform stress distribution, both axial and shear, is essential for a more efficient use of material in composite plates. Knowing the stress distribution throughout the plate resulting from non-uniform loading on the boundary of the plate will allow the study of Saint-Venant's end effects to be carried out for different types of laminates for both axial and shear loading.

In the present work a procedure for determining the anisotropic plane elasticity solution for the stresses throughout a thin, rectangular, laminated composite plate subjected to non-uniform loading is presented. Three types of non-uniform loading on the boundary of the plate are considered in detail: (1) uniaxial parabolic normal stress, (2) uniaxial cubic normal stress, and (3) self-equilibrated
cubically distributed shear stress. The plate is assumed to be a symmetric laminate, which eliminates bending-extension coupling. Also, no hygrothermal effects have been considered in the plate.

The Ritz method is used to solve for the displacements, which are expressed in the form of polynomial series. Then the strains and stresses are determined throughout the plate. Convergence of the displacements and stresses is studied in order to answer the question of how closely a relatively few terms of the series for assumed polynomials can approximate the exact solutions for the displacements and the stresses. The results are compared with the known results for certain special cases. The validity of Saint-Venant's principle is studied for the case of anisotropic rectangular plates. The characteristic decay length is estimated for different cases of isotropic, orthotropic, and anisotropic plates for the non-uniform axial and shear loadings, and the results are compared with those in the literature.
CHAPTER I

INTRODUCTION

1.1 Background Information

Laminated composite plates are increasingly used in design of modern structures such as space vehicles, airplane wings, automobiles, ships, submarines, and sporting goods. Due to the anisotropic material characteristics of laminated composite plates, they can be tailored for a specific point or component of the structure in order to carry the desired loading. Consequently, the structure as a whole becomes capable of performing under severe static or dynamic loading conditions. Economy, longer operating life, and high strength-to-weight ratio of the structure are other relevant factors. The analysis of laminated composite plates has attracted many researchers, who improve upon and add to the existing analytical results on the subject.

A laminated composite plate is made up of a multi-stack of lamina perfectly bonded together to form a
lamine. A lamina is a flat arrangement of unidirectional fibers suspended in a matrix material. Each lamina typically is considered to be homogeneous and orthotropic, with the fibers being equally spaced. A laminate, however, behaves as either anisotropic or orthotropic. Laminated composite plates of commercial importance include graphite, boron, glass, carbon, brass, or steel fibers imbedded in a matrix such as epoxy, ceramic, or metal.

Composite plates as components of modern designs, in most cases, will be under quite non-uniform inplane stresses. If one assumes a uniform loading, the problem becomes trivial and too far from being realistic in most practical cases. With the uniform or linearly varying normal stresses, the elasticity solutions have exact solutions; with uniform loading the stress resultants are uniform, and with linearly varying loading they are linearly varying throughout the plate. However, with loading other than the uniform or linearly varying loading, the plane elasticity problem typically has no exact solution and the primary objective, and often most difficult one, would be solving the plane problem of anisotropic elasticity. Furthermore, since fiber orientation through the lamina may be different, the stress distribution caused by the non-uniform stresses throughout
the laminate will be quite complex. One has to start with the basic equations of the theory of anisotropic elasticity to calculate the resulting stresses and deformation throughout the plate.

In the isotropic elasticity theory it is assumed that the plate is homogeneous and isotropic. That is, the number of independent elastic constants is equal to two and the properties of the plate remain the same in all directions. These assumptions result in relatively simplified calculations compared to the anisotropic plate theory. In anisotropic plate theory the material properties are different in all directions and the number of independent elastic constants can be considerably larger, as many as 21. The analysis of stress and strain is the same, however, whether the body is isotropic or anisotropic. Additionally, the fundamental equations such as equations of motion, strain-displacement relations, and compatibility conditions of isotropic elasticity remain valid in case of anisotropic elasticity. The sole difference is the complexity of the stress-strain equations in anisotropic elasticity compared to the isotropic case.

Saint-Venant in his classical paper (Saint-Venant, 1855) presented his famous principle: "If the forces acting on a small portion of an elastic body are replaced by a
statically equivalent system of forces acting on the same portion of the surface, this redistribution of forces produces substantial changes in the stresses locally but has a negligible effect on the stresses at distances which are larger than the dimension of the surface which the loading are changed." (Timoshenko and Goodier, 1970). By an appeal to this principle, many approximations in isotropic elasticity have been possible. Many theories in strength of materials, plates, and shells rely on this principle for a simplified development. Without this principle, a proper assessment of end effects in homogeneous isotropic elastic structures would not have been possible and consequently many structural designs would not have existed today. Saint-Venant's principle is also the basis for static mechanical tests of material properties for isotropic materials. For example, in measuring material properties of an isotropic material specimen, which could be in state of either plane stress or strain, local effects due to the clamping of the specimen are neglected due to this principle. The validity of this principle has been throughly investigated and proved to hold in the case of an isotropic material. In the case of laminated composite plates as anisotropic materials, the important question is "How drastically do the stresses diminish for different types of laminates as one moves away from the boundary of
the plate for different types of non-uniform, self-equilibrating loading (both axial and shear) at the boundary of the plate?”. Solving the plane elasticity problem throughout a laminated composite plate under different types of non-uniform inplane loading will allow us to study the Saint-Venant principle correctly and more accurately in different types of laminated composite plates. Consequently, having accurate decay lengths for different types of laminates will give the designers more information about composite plates in utilizing them in their designs.

1.2 Literature Review

One can find numerous publications that deal with the inplane loading of composite plates. However, many of these publications deal with free vibration and buckling of composite plates, and they have assumed a uniform inplane loading. The plane elasticity solution for such problems is very trivial; stress resultants will be uniform throughout the plate.

Inplane problems have been mentioned in the literature of the theory of anisotropic elasticity. However, no references have been found which present specific numerical results for stresses throughout a laminated composite plate
subjected to non-uniform axial and/or shear stresses. One excellent monograph in the theory of anisotropic elasticity is by Lekhnitskii (1963) who is one of the leading contributors to this field. This book was published in Russian in 1950 and translated into English three years later. It presents some of the significant results of his own investigations and some other researchers. He extended N.I. Muskhelishvili's work in the theory of plane isotropic elasticity to the anisotropic elasticity situation by essentially using the method of complex variables. In solving problems in anisotropic elasticity, Lekhnitskii's work remains one of the main references.

The case of a rectangular isotropic plate subjected to parabolically and cubic distributed tensile inplane stresses was studied (Timoshenko and Goodier, 1970). Using a stress function which satisfied all the boundary conditions, the strain energy of the plate was minimized and an approximate solution was presented for the stresses throughout the plate.

Significant work on the topic of Saint-Venant's principle in the case of anisotropic elasticity has been done by C.O. Horgan in the past two decades. His first paper on this subject (Horgan, 1972a) dealt with application
of methods involving energy-decay inequalities in investigating Saint-Venant's principle for the plane problem of linear elastostatics for wide class of anisotropic media. A lower bound in terms of elastic constants was obtained for the rate of exponential decay of stresses, and was compared with the known results for the isotropic case. In this work Horgan generalized earlier results of Knowles (Knowles, 1966) and Toupin (Toupin, 1965) for the isotropic case. In the same year (Horgan, 1972b), he investigated the behavior of a transversely isotropic medium, in the limit of extensibility and compressibility.

The validity of Saint-Venant's principle was studied by Folks and Arridge (1975). In this paper measurements of longitudinal shear modulus in single crystal samples were reported for a type of copolymer. As it was judged from the dependence of shear modulus on sample length/width ratio, the conditions of non-uniform stress at the sample ends arising from the clamping method decay away more slowly than expected on the basis of the Saint-Venant principle expressed in its usual form. It was also shown that, the characteristic decay length can be quite closely predicted from the experimental data and a simple block model.

The Saint-Venant principle was subsequently studied by Choi and Horgan (1978). The purpose of the paper was to draw
attention to the fact that the routine application of Saint-Venant's principle in the solution of elasticity problems involving highly anisotropic or composite material is not justified in general. This was illustrated in the context of the plane problem of elasticity for an anisotropic rectangular strip loaded with self-equilibrating stress only on the short ends.

The Saint-Venant end effects in composites were further examined by Horgan (1982). It was demonstrated that the neglect of end effects in the case of composite materials cannot be so easily justified by appealing to Saint-Venant's principle. In particular, for plain strain or generalized plane stress of an orthotropic rectangular plate the characteristic decay length was found to be in the order of \( b(E/G)^{1/2} \), where \( b \) is the maximum dimension perpendicular to the fibers and \( E, G \) are the longitudinal Young's modulus and shear modulus respectively. Thus when \( E/G \) is large, end effects propagate over a distance which is of the order of several specimen widths. The results of these investigations have been widely adopted in the literature on composite materials. For instance, the input of such results have been used in designing test specimen geometries for mechanical testing of fiber-reinforced composites in a monograph by Whitney et al (1982) and in the text by Carlson

A finite element analysis on the edge effects in graphite/epoxy was made by Carlson et al (1986), which showed very close agreement with the results obtained by Horgan. Also in finite element analyses there is the work of Arridge and Folks (1976) and of Holt and Hope (1979). The existence of these extended studies in Saint-Venant's principle has led to the modification of ASTM test D198 to measure the shear modulus of anisotropic beams (Gromala, 1985).

The most recent publications on Saint-Venant's principle are that of Crafter et al (1993) and Miller and Horgan (1994). Excellent results are obtained in the later work (Miller and Horgan, 1994) which is yet to be published in Mechanics of Composite Materials and Structures. In this work the linear theory of elasticity was used to study a homogeneous, anisotropic, semi-infinite strip. The semi-infinite strip is in the state of plane stress/strain and subject to a self-equilibrating end load. The problem considered is the anisotropic analog of the eigenvalue problem for the Fadle-Papkovich eigenfunctions arising in the isotropic case. For the strips made of specially
orthotropic materials, the stress decay was formulated in terms of a single dimensionless material parameter. For strips made of general anisotropic material, the problem was formulated using three dimensionless material properties. This scheme of reducing the material properties involved in the governing differential equation is termed "orthotropic rescaling" which simplifies the problem considerably for both orthotropic and anisotropic cases. The decay rate for Saint-Venant end effects was given by the eigenvalue with smallest positive real part which can be solved numerically. The tables and graphs were presented which could be used to determine the Saint-Venant decay length for the fully anisotropic strip.

1.3 Research Objectives

The primary objectives of this work are:

1. To derive the fundamental equations and energy functionals for the plane stress analysis of symmetrically laminated thin composite plates.

2. To develop a method of analysis for determining the displacements throughout a rectangular laminated composite plate subjected to various types of non-uniform axial and shear loading at the boundaries.
3. To determine strains and stresses throughout the plate after solving for the displacements.

4. To study the stress behavior throughout composite plates subjected to non-uniform inplane axial and shear stresses for different types of symmetric lamination.

5. To study Saint-Venant's principle in plane anisotropic elasticity and to find the characteristic decay lengths for normal stress loading in different types of laminates, comparing the results obtained with that of the known results in the literature for a semi-infinite strip loaded at one end.

6. To determine characteristic decay lengths of orthotropic and generally anisotropic laminated composite plates loaded by shear stresses.

1.4 Description of Organization

The previous section has summarized much of the research done on the end effects and Saint-Venant's principle in the case of anisotropic elasticity. The results for the characteristic decay length for the same material and the same loading will be compared here. Also, decay lengths for different types of anisotropic plates for the case of self-equilibrating shear stress will be found.
The fundamental equations and energy functionals are derived in Chapter 2. A general formulation for the determination of inplane stresses and the fundamentals of the Ritz method are presented in Chapter 3. In Chapter 4, convergence studies are carried out to determine the size of the series of assumed displacements required for the most efficient numerical results. Also in this chapter, numerical results obtained by the present analysis and those obtained by Timoshenko for an isotropic plate subjected to parabolic tensile loading are presented and compared both tabularly and graphically. In Chapter 5, results for stress distribution for different types of laminates with non-uniform inplane axial loading are presented. The results for shear stress resultants throughout rectangular laminates having different shear modulus, orthotropy ratio, and fiber angle are presented and discussed in Chapter 6.

In Chapter 7 Saint-Venant's effect is discussed and the characteristic decay length for laminates with different orthotropy ratios and fiber angles are obtained for plates loaded by normal stresses. Also in this Chapter the effect of self-equilibrating shear stress on the decay rate of stresses is examined. Finally, a summary of conclusions reached and suggestions for further research is presented in Chapter 8.
2.1 Introduction

Laminated composite plates are typically constructed of an arbitrary number of homogeneous orthotropic layers perfectly bonded together. Each layer has arbitrary elastic properties and orientation of orthotropic axes with respect to the plate axes. The complexity of the governing equations depends on the stacking sequence of the individual layers.

The layers of symmetric laminates studied in this work are arranged so that a midplane of symmetry exists; that is, for each layer above the midplane there is a corresponding layer identical in thickness, elastic properties, and orientation located at the same distance below the midplane. Also, there could be an even or odd number of layers. Since there is no bending-extension coupling in symmetric laminates, the midsurface strains, and therefore the stresses, in the plate can be determined
by using a simplified form of governing equations. Symmetric laminates are the most widely used and extensively studied type of laminates. This work entirely deals with symmetric laminated composite plates in the absence of hygrothermal effects.

2.2 Strain-Displacement Relations

Consider a thin symmetric laminated composite plate (Figure 2.1) referred to a right-handed x,y,z coordinate system. The strain-displacement relations of the plate are:

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]  \hspace{1cm} (2.1a)
\[ \varepsilon_y = \frac{\partial v}{\partial y} \]  \hspace{1cm} (2.1b)
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]  \hspace{1cm} (2.1c)

Where \( \varepsilon_x \), \( \varepsilon_y \), and \( \gamma_{xy} \) are midplane strains.

2.3 Stress-Strain Relations

The stress-strain relations for a two dimensional orthotropic \( k \)th layer of a laminate can be expressed as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}_k =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_k
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}_k
\]  \hspace{1cm} (2.2)
Figure 2.1 A thin rectangular laminated composite plate.
In which $[Q_y]$ is the stiffness matrix of the $k$th layer of the laminate and is defined in terms of material properties and fiber direction in each lamina. $\sigma_x$ and $\sigma_y$ are the normal stresses and $\tau_{xy}$ is the shear stress in each layer. As a consequence of stacking, these orthotropic layers of different fiber orientation could result in a generally anisotropic composite plate.

Based on the compatibility condition of the theory of elasticity,

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

(2.3)

the strain through the laminate must be continuous. This is not the case, however, for the stresses through the laminate. Since each lamina may have different fiber orientation, and therefore different stiffness properties, the stress distribution through the laminate need not be continuous.

2.4 Inplane Stress Resultants

In order to satisfy the equilibrium equations of the plate, the resultant of the laminate forces must be balanced with the integral of the internal stresses over the laminate thickness.
Figure 2.2 Positive convention for inplane stress resultants.
Figure 2.2 shows the positive sign convention for the laminate loads. Since a lamina generally consists separately, then the sums of the integrals of the stresses will equal the resultant forces. The inplane stress resultants, Figure 2.2, all per unit length of the plate are thus defined by

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz$$

(2.4)

where $n$ is the total number of laminae in the laminate. Substituting (2.2) into (2.4), the resulting integrals become simple integrals of $(1, z, z^2)$. After integration, the inplane stress resultants are expressed in matrix form as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

(2.5)

where the symmetric matrix $[A_y]$ is termed the extensional stiffness matrix. The components of this matrix are calculated using the laminate stacking sequence shown in Figure 2.3 from

$$[A_y] = \sum_{k=1}^{n} \left[ Q_{yk} \right] (z_k - z_{k-1})$$

(2.6)
Figure 2.3 Laminate stacking sequence.
By stacking the fibers at different angles in the laminate, one could have either quasi isotropic, orthotropic, or anisotropic laminate. These different cases will be apparent by examining the components of the extensional stiffness matrix \([A_{ij}]\) as follows:

- \(A_{11} = A_{22}\) and \(A_{16} = A_{26} = 0\) \(\Rightarrow\) Quasi isotropic
- \(A_{11} \neq A_{22}\) and \(A_{16} = A_{26} = 0\) \(\Rightarrow\) Orthotropic
- Fully populated \(\Rightarrow\) Anisotropic

2.5 Energy Functionals

The strain energy of a plate under a state of plane stress is defined as:

\[
U = \frac{1}{2} \iint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) dV
\]  

(2.7)

Where \(V\) denotes the volume of the plate. Substituting strains from equations (2.1), integrating over the thickness, and using equations (2.2) and (2.6), results in an area integral for the strain energy of the plate in the form
\[
U = \frac{1}{2} \iint_A \left[ \alpha_{11} \left( \frac{\partial u}{\partial x} \right)^2 + 2\alpha_{12} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \alpha_{22} \left( \frac{\partial v}{\partial y} \right)^2 \right. \\
+ 2\alpha_{16} \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + 2\alpha_{26} \frac{\partial v}{\partial y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
+ \left. \alpha_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] dxdy
\]

(2.8)

The work done by the applied inplane loads at the boundary of the plate is equal to:

\[ W_L = -\iint_A (N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy}) dxdy \]

(2.9)

and the total potential energy of the plate in equilibrium is

\[ V = U_s + W_L \]

(2.10)
CHAPTER III

DETERMINATION OF INPLANE STRESSES AND THE RITZ METHOD

3.1 Introduction

In today's modern designs which utilize composite plates as their components, in most cases the plate is subjected to quite severe non-uniform inplane loading. Based on the level of accuracy desired, one can model inplane loading as uniform, linear, quadratic, cubic, or even higher degree form. For inplane loads other than constant or linearly varying, solving the plane elasticity problem is more difficult, and only approximate solutions of it are generally possible.

In this chapter, the plane elasticity problem for the case of a thin, rectangular, symmetrically laminated composite plate subjected to arbitrary types of non-uniform inplane (both normal and shear) loading is formulated. The Ritz method is used to solve the problem.
3.2 The Plane Elasticity Problem

The solution to the plane anisotropic elasticity problem must satisfy the boundary conditions as well as the basic equations of the plane anisotropic elasticity theory, summarized in tensorial notation as follows:

(i) Equations of equilibrium (no inertia or body forces):
\[ \sigma_{ij} = 0 \] (3.1)

(ii) Stress-strain relations
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \] (3.2)
where \( C_{ijkl} \) is the stiffness matrix.

(iii) Strain-displacement relations
\[ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \] (3.3)

(iv) Boundary conditions:
\[ \delta_{ij}^{(b)} = \sigma_{ij} n_i \] stresses prescribed at the boundary
\[ u_i = d_i \] displacements prescribed at the boundary

Equations (3.1)-(3.3) result in eight necessary equations in order to solve for eight unknowns which are:

- Three stress components \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \)
- Three strain components \( \varepsilon_x, \varepsilon_y, \) and \( \gamma_{xy} \)
- Two displacement components \( u \) and \( v \)
Since in the present plane problem the stresses will be prescribed at the boundary, one approach is to assume the solution in terms of stresses and after satisfying (3.1), strains can then be determined from (3.2), with the additional requirement of insuring the integrability of the strain field, by satisfying Saint-Venant's compatibility condition which is:

\[ \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \]  

(3.4)

The requirement of satisfying (3.4) will be eliminated with the alternative solution of assuming a solution to the problem in terms of displacements. If the assumed displacement functions have sufficient continuity throughout the plate, (3.4) is satisfied identically. Then the strains could be determined from (3.3) and stresses from (3.2). In general, an exact solution in the case of non-uniform stresses at the boundary is out of the question. Various approximate methods such as Galerkin, finite difference, finite element, and the Ritz method could be used to solve this plane elasticity problem. However, the Ritz method is one of the best established methods in formulating problems in the small displacement theory of elasticity (Washizu 1982).
In the present work, the Ritz method is used to solve for the approximate stresses throughout the plate.

### 3.3 Fundamentals of the Ritz Method

The Ritz method is based on the principle of minimum potential energy in obtaining approximate solutions of elasticity problems. The principle of minimum potential energy is established by the principle of virtual work, when the existence of a strain energy function and potential functions for the external forces is assured for the elastic body (Weinstock 1974). This method essentially bypasses the variational derivation of Euler equations (equilibrium equations) and directly minimizes the potential energy of the system by assuming suitable functions (admissible functions) for the displacements $u$ and $v$ in the form

\begin{align}
  u &= \sum_{i=1}^{L} \sum_{j=1}^{J} B_{ij} u_{ij}(x,y) \tag{3.5a} \\
  v &= \sum_{k=1}^{K} \sum_{l=1}^{L} C_{kl} v_{kl}(x,y) \tag{3.5b}
\end{align}

where the parameters $B_{ij}$ and $C_{kl}$ are the unknown coefficients to be determined. In order to ensure convergence of the Ritz approximation to the exact solution as the number of
terms taken for the displacements is increased, the known functions $u$ and $v$ are to be chosen such that

- At least the essential (geometric) boundary conditions of the problem are satisfied.
- The set of functions must be chosen from a complete set of functions.

The total potential energy of the elastic body is

$$ V = U + W_L $$

(3.6)

in which $U$ is the strain energy and $W_L$ is the potential energy of the external loads, both being functions of the displacements. In the present work they are given by eqs. (2.8) and (2.9). Substituting equations (3.5) into (3.6), the total potential energy becomes a function of the coefficients $B_y$ and $C_u$. These coefficients will be determined by minimizing the potential energy with respect to them, that is:

$$ \frac{\partial V}{\partial B_y} = 0 $$

(3.7a)

$$ \frac{\partial V}{\partial C_u} = 0 $$

(3.7b)

Equations (3.7) will result in the same number of simultaneous equations as the number of coefficients taken.
These equations may be solved for the unknown coefficients by a simultaneous equation technique.

3.4 General Description of the Problem

A typical plane elasticity problem which could be solved is depicted in Figure 3.1. A rectangular laminated composite plate with dimensions $a \times b$ is subjected to combination of cubic inplane normal stress in the $x$-direction and parabolically varying normal stress in the $y$-direction. In addition, boundary shear stresses may act. In the next section, the general formulation of the problem for different types of non-uniform loading using the Ritz method is presented.

3.5 Formulation of the Required Energy Functionals

The total potential energy of the plate with dimensions $a \times b$ is given by (2.10), where the strain energy and load potential are seen in (2.8) and (2.9). The load potential may be written directly in terms as the negative of the work done by boundary forces as:
Figure 3.1 Rectangular laminated composite plate subjected to non-uniform inplane loading.
\[ W_L = \int_{-b/2}^{b/2} \sigma_x u(-a/2,y)dy - \int_{-b/2}^{b/2} \sigma_x u(a/2,y)dy + \int_{-a/2}^{a/2} \sigma_y v(x,b/2)dx - \int_{-a/2}^{a/2} \sigma_y v(x,-b/2)dx - \int_{-a/2}^{a/2} \tau_{xy} u(x,b/2)dx + \int_{-a/2}^{a/2} \tau_{xy} u(x,-b/2)dx - \int_{-b/2}^{b/2} \tau_{xy} v(a/2,y)dy + \int_{-b/2}^{b/2} \tau_{xy} v(-a/2,y)dy \]

(3.8)

To simplify the evaluation of integrals, the \( x-y \) coordinate system is transformed into the non-dimensional coordinate system \( \xi - \eta \), where \( \xi = 2x/a \) and \( \eta = 2y/b \). The potential energy of the plate in terms of the non-dimensional coordinates becomes:

\[ V = \frac{1}{8} \int_{-1}^{1} \left[ \frac{4}{R} A_{11} \left( \frac{\partial u}{\partial \xi} \right)^2 + 8 A_{12} \frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \eta} + 8 A_{22} \left( \frac{\partial v}{\partial \eta} \right)^2 \right] d\xi d\eta + \int_{-1}^{1} \frac{b}{2} \sigma_\xi u(1,\eta) d\eta + \int_{-1}^{1} \frac{b}{2} \sigma_\xi u(-1,\eta) d\eta - \int_{-1}^{1} \frac{a}{2} \sigma_\eta v(\xi,1) d\xi + \int_{-1}^{1} \frac{a}{2} \sigma_\eta v(\xi,-1) d\xi - \int_{-1}^{1} \frac{a}{2} \tau_{\xi\eta} u(\xi,1) d\xi + \int_{-1}^{1} \frac{a}{2} \tau_{\xi\eta} u(\xi,-1) d\xi - \int_{-1}^{1} \frac{b}{2} \sigma_\eta v(1,\eta) d\eta + \int_{-1}^{1} \frac{b}{2} \tau_{\xi\eta} v(-1,\eta) d\eta \]

(3.9)
where $R=a/b$ is the aspect ratio of the plate. If the boundary loading is either symmetric or antisymmetric with respect to both coordinates, then the integrations need only be performed in one quadrant of the plate.

Polynomials would be suitable functions to assume for the displacements, since they are able to satisfy the geometric boundary conditions of the problem and they do, in fact, form a complete set of functions. Choosing the displacement polynomials from a set of ordinary polynomials as:

\[ u(\xi, \eta) = \sum_{i=0}^{I} \sum_{j=0}^{J} B_{ij} \xi^i \eta^j \quad (3.10) \]

\[ v(\xi, \eta) = \sum_{k=0}^{K} \sum_{l=0}^{L} C_{kl} \xi^k \eta^l \quad (3.11) \]

where the domain of indices $i,j,k,$ and $l$ in the series must be carefully chosen, depending on the type of laminate and the symmetry involved, in order to have a more efficient series for computation. As mentioned in Chapter II, stacking the fibers at different angles in the laminate will result in either quasi isotropic, orthotropic, or anisotropic laminates. The quasi isotropic and orthotropic laminates are doubly symmetric with respect to the $xy$-coordinates, and the anisotropic laminates will generally have no symmetry. In addition to the stacking sequence,
symmetry is determined by the shape of the plate, and the
symmetry of its boundary loading. Plates studied in this
work are rectangular, thus having double symmetry. Loadings
will be either symmetric or antisymmetric. Thus, if the
stacking sequence produces symmetry (quasi isotropic or
orthotropic laminates), then the symmetry or antisymmetry of
the problem is the same as that of the loading.

If the problem is doubly symmetric, then $i$ and $l$ are
odd only, and $j$ and $k$ are even only. For double
antisymmetry, these are reversed. The completely annotated
series are summarized as follows:

- Doubly symmetric:
  \[ u(\xi, \eta) = \sum_{i} \sum_{j} B_{ijkl} \xi^{i} \eta^{j} \]
  \[ i = 1, 3, 5, \ldots \]
  \[ j = 0, 2, 4, \ldots \]  \hspace{1cm} (3.12)
  \[ v(\xi, \eta) = \sum_{k} \sum_{l} C_{ijkl} \xi^{k} \eta^{l} \]
  \[ k = 0, 2, 4, \ldots \]
  \[ l = 1, 3, 5, \ldots \]

- Doubly antisymmetric:
  \[ u(\xi, \eta) = \sum_{i} \sum_{j} B_{ijkl} \xi^{i} \eta^{j} \]
  \[ i = 0, 2, 4, \ldots \]
  \[ j = 1, 3, 5, \ldots \]  \hspace{1cm} (3.13)
  \[ v(\xi, \eta) = \sum_{k} \sum_{l} C_{ijkl} \xi^{k} \eta^{l} \]
\[ k = 1, 3, 5, \ldots \]
\[ l = 0, 2, 4, \ldots \]

- No symmetry:

\[
u(\xi, \eta) = \sum_{i} \sum_{j} B_{ij} \xi^i \eta^j
\]
\[ i = 0, 1, 2, 3, \ldots \]
\[ j = 0, 1, 2, 3, \ldots \]
\[ (3.14)\]

\[
u(\xi, \eta) = \sum_{k} \sum_{l} C_{kl} \xi^k \eta^l
\]
\[ k = 0, 1, 2, 3, \ldots \]
\[ l = 0, 1, 2, 3, \ldots \]

It must be mentioned that the sets for doubly symmetric and doubly antisymmetric equations contain one-fourth of the terms as that of the no symmetry case, and the missing terms contribute nothing to the solution sought. Also, the terms with \( i = j = k = l = 0 \) in each of the series are the rigid body translation terms and are removed from the series. Thus, the center of the plate \((x = y = 0)\) undergoes displacement when the loads are applied.

Substituting the displacement polynomials (3.10) and (3.11) into the potential energy functional (3.9) yields
The total potential energy (3.15) is minimized with respect to the coefficients $B_w$ and $C_w$ by differentiating it with respect to each of them. It is computationally more efficient to carry out these differentiations before integration. Doing so results in

$$V = \frac{1}{2R \Delta n} \left[ \sum \left( B_{\Delta n} \sum (C_w) \right)^2 + \sum \left( B_{\Delta n} \sum (C_w) \right)^2 \right]$$
\[ \frac{\partial V}{\partial B_{pq}} = A_{11} \int_{-1}^{1} \left\{ \left( p \xi^{r-1} \eta^{q} \right) \left[ \sum_{i} \sum_{j} \left( B_{yi} \xi^{i-1} \eta^{j} \right) \right] \right\} d\xi d\eta \\
+ A_{12} \int_{-1}^{1} \left\{ \left( p \xi^{r-1} \eta^{q} \right) \left[ \sum_{k} \sum_{l} \left( C_{ik} \xi^{k} \eta^{l} \right) \right] \right\} d\xi d\eta \\
+ A_{16} \int_{-1}^{1} \left\{ \left( q \xi^{r-1} \eta^{q-1} \right) \left[ \sum_{i} \sum_{j} \left( B_{yi} \xi^{i} \eta^{j} \right) \right] \right\} d\xi d\eta \\
+ A_{16} \int_{-1}^{1} \left\{ \left( q \xi^{r-1} \eta^{q-1} \right) \left[ \sum_{k} \sum_{l} \left( C_{ik} \xi^{k} \eta^{l} \right) \right] \right\} d\xi d\eta \\
+ A_{16} \int_{-1}^{1} \left\{ \left( p \xi^{r-1} \eta^{q} \right) \left[ \sum_{k} \sum_{l} \left( C_{ik} \xi^{k} \eta^{l} \right) \right] \right\} d\xi d\eta \\
+ A_{26} \int_{-1}^{1} \left\{ \left( r \xi^{r-1} \eta^{r} \right) \left[ \sum_{i} \sum_{j} \left( B_{yi} \xi^{i} \eta^{j} \right) \right] \right\} d\xi d\eta \\
+ A_{26} \int_{-1}^{1} \left\{ \left( r \xi^{r-1} \eta^{r} \right) \left[ \sum_{k} \sum_{l} \left( C_{ik} \xi^{k} \eta^{l} \right) \right] \right\} d\xi d\eta \\
+ A_{66} \int_{-1}^{1} \left\{ \left( q \xi^{r-1} \eta^{q-1} \right) \left[ \sum_{i} \sum_{j} \left( B_{yi} \xi^{i} \eta^{j} \right) \right] \right\} d\xi d\eta \\
+ A_{66} \int_{-1}^{1} \left\{ \left( q \xi^{r-1} \eta^{q-1} \right) \left[ \sum_{k} \sum_{l} \left( C_{ik} \xi^{k} \eta^{l} \right) \right] \right\} d\xi d\eta + \frac{\partial W}{\partial B_{pq}} = 0 \ (3.16) \]
where the terms $\partial W_L/\partial B_p$ and $\partial W_L/\partial C_n$ are the partial derivatives of the load potential, and will differ depending on the type of loads applied at the boundary.

Since the series involved in the above equations is continuous, the order of summation and integration are interchangeable. This will simplify evaluation of integrals drastically. Integrations involved in above equations are typified by:

$$\sum_{i} \sum_{j} \frac{A_{ij}}{R} \int_{-1}^{1} \int_{-1}^{1} B_{ij} \eta^{p+2} \eta^{q+1} d\xi d\eta$$

which can be evaluated exactly without using a numerical procedure. The result of the integration will be zero if either $\xi$ or $\eta$ has an odd exponent. If the integral is formed by terms of $u$ and $v$ from the same class of symmetry, both exponents will be even. Otherwise, at least one is odd and the result of the integral will be zero.

After evaluating the integrals and normalizing the stiffness coefficients with respect to $A_{11}$, (3.16) and (3.17) can be written in much more simple form as:
\[
\frac{\partial V}{\partial B_{pq}} = \sum_i \sum_j \left\{ \frac{ip}{R(p+i-1)(q+j+1)} \left[ 1 + (-1)^{p+i} \right] \left[ 1 + (-1)^{q+j} \right] \right. \\
+ \left( \frac{A_{16}}{A_{11}} \right) \frac{pj}{(p+i)(q+j)} \left[ 1 - (-1)^{p+i} \right] \left[ 1 - (-1)^{q+j} \right] \right. \\
+ \left( \frac{A_{16}}{A_{11}} \right) \frac{qi}{(p+i)(q+j)} \left[ 1 - (-1)^{p+i} \right] \left[ 1 - (-1)^{q+j} \right] \\
+ \left( \frac{A_{66}}{A_{11}} \right) \frac{Rqj}{(p+i+1)(q+j-1)} \left[ 1 + (-1)^{p+i} \right] \left[ 1 + (-1)^{q+j} \right] B_y \\
+ \sum_k \sum_l \left\{ \left( \frac{A_{12}}{A_{11}} \right) \frac{pl}{(p+k)(q+l)} \left[ 1 - (-1)^{p+k} \right] \left[ 1 - (-1)^{q+l} \right] \right. \\
+ \left( \frac{A_{16}}{A_{11}} \right) \frac{pk}{R(p+k-1)(q+l+1)} \left[ 1 + (-1)^{p+k} \right] \left[ 1 + (-1)^{q+l} \right] \\
+ \left( \frac{A_{28}}{A_{11}} \right) \frac{Rql}{(p+k+1)(q+l-1)} \left[ 1 + (-1)^{p+k} \right] \left[ 1 + (-1)^{q+l} \right] \\
+ \left( \frac{A_{66}}{A_{11}} \right) \frac{qk}{(p+k)(q+l)} \left[ 1 - (-1)^{p+k} \right] \left[ 1 - (-1)^{q+l} \right] \right. \} C_k \\
+ \frac{\partial W_L}{\partial B_{pq}} = 0 \\
(3.18)
\]

\[
\frac{\partial V}{\partial C_m} = \sum_i \sum_j \left\{ \left( \frac{A_{12}}{A_{11}} \right) \frac{is}{(r+i)(s+j)} \left[ 1 - (-1)^{r+i} \right] \left[ 1 - (-1)^{s+j} \right] \right. \\
+ \left( \frac{A_{16}}{A_{11}} \right) \frac{ir}{R(r+i-1)(s+j+1)} \left[ 1 + (-1)^{r+i} \right] \left[ 1 + (-1)^{s+j} \right] \right. \\
+ \left( \frac{A_{26}}{A_{11}} \right) \frac{Rjs}{(r+i+1)(s+j-1)} \left[ 1 + (-1)^{r+i} \right] \left[ 1 + (-1)^{s+j} \right] \\
+ \left( \frac{A_{66}}{A_{11}} \right) \frac{jr}{\left( r+i+1 \right) \left( s+j \right)} \left[ 1 + (-1)^{r+i} \right] \left[ 1 + (-1)^{s+j} \right] B_y \\
+ \sum_k \sum_l \left\{ \left( \frac{A_{22}}{A_{11}} \right) \frac{Rsl}{(r+k+1)(s+l-1)} \left[ 1 - (-1)^{r+k} \right] \left[ 1 - (-1)^{s+l} \right] \right. \\
+ \left( \frac{A_{36}}{A_{11}} \right) \frac{rl}{\left( r+k \right) \left( s+l \right)} \left[ 1 - (-1)^{r+k} \right] \left[ 1 - (-1)^{s+l} \right] \right. \} \\
\]
The terms in (3.18) and (3.19) contain expressions like 
\( r+k-1 \), which can become zero in their denominator, and are therefore troublesome for computer programming. Those terms must be skipped and their true value must be assigned by logical statements in the computer program. This is one disadvantage of polynomials compared to trigonometric series.

Equations (3.18) and (3.19) will result in \((i+1)\times(j+1)+(k+1)\times(l+1)\) linear, simultaneous equations in the unknowns \( B_{ij} \) and \( C_{kl} \). These equations are formulated into a computer program and the unknowns are determined by using the Gaussian Elimination Method, as well as by using an iterative refinement procedure to enhance the accuracy of the calculated results. Because the algebraic polynomials used in (3.10) and (3.11) form complete sets of functions, convergence to the exact solution is guaranteed.

Substituting \( B_{ij} \) and \( C_{kl} \) into (3.10) and (3.11) completely determines the inplane displacements. Substituting the displacements into strain-displacement relations (2.1) determines the strains. Finally,
substituting the strains into the constitutive equations (2.5) determines the stress resultants throughout the plate.

3.6 Formulation of Stresses at the Boundaries

In this section the plane elasticity problem presented in the previous section is formulated for certain specific boundary loadings. Three types of inplane, non-uniform (both normal and shear) loading on the boundaries of the plate are formulated. (1) biaxial, parabolically distributed normal stress; (2) biaxial self-equilibrating, parabolic normal stress; (3) biaxial cubic distributed stress; and (4) self-equilibrating, cubically varying shear stress.

The work done by the stresses at the boundary of the plate \( W_L \) is expressed in terms of the assumed displacements defined in (3.10) and (3.11). The potential energy of the applied stresses at the boundaries is defined as:

\[
W_L = \frac{b}{2} \int_{-1}^{1} \sigma_{\xi} u(-1, \eta) d\eta - \frac{b}{2} \int_{-1}^{1} \sigma_{\xi} u(1, \eta) d\eta \\
+ \frac{a}{2} \int_{-1}^{1} \sigma_{\eta} v(\xi, 1) d\xi - \frac{a}{2} \int_{-1}^{1} \sigma_{\eta} v(\xi, -1) d\xi \\
- \frac{a}{2} \int_{-1}^{1} \tau_{\eta \xi} u(\xi, 1) d\xi + \frac{a}{2} \int_{-1}^{1} \tau_{\eta \xi} u(\xi, -1) d\xi \\
- \frac{b}{2} \int_{-1}^{1} \tau_{\eta \eta} u(1, \eta) d\eta + \int_{-1}^{1} \tau_{\eta \eta} u(-1, \eta) d\eta
\]  
(3.20)
In the next subsections of this section, the load potentials of different types of applied stresses at the boundary of the plate are derived for the cases investigated. The cases of uniform and linearly varying normal stresses along the boundaries yield uniform and linearly varying elementary exact solutions, respectively, which have no end effects and so they will not be discussed here. The parabolic stress distribution which is the first even deviation from the uniform normal stress distribution is carried out first. Second, the self-equilibrating parabolic normal stress is formulated which is the superposition of parabolic and constant distributed stress. Third, the cubic form of normal stress is applied at the boundary which is the first odd deviation from the linearly varying stress. Finally, formulation of self-equilibrating, cubically varying shear stress applied at the two opposite edges of the plate is presented. This form of shear stress is carefully chosen. Compared to the linearly or parabolically varying shear stress forms, the cubic one is the only self-equilibrating one which produces zero shear at the corners of the plate. This zero shear at the corners of the plate is essential so that shear stresses are applied only at two opposite sides of the plate.
In order to get the formulation of any combination of the above loading on the boundary of the plate, the obtained formulation must be added to the potential energy of the applied loads.

3.6.1 Parabolically Distributed Normal Stress at the Boundaries

Applied parabolic normal stress at the boundaries is shown in Figure 3.2, where the applied stresses have equations of the form:

- Stress in the $\xi$-direction:
  \[ \sigma_\xi = T(1 - \eta^2) \]

- Stress in the $\eta$-direction:
  \[ \sigma_\eta = T(1 - \xi^2) \]

where $T$ and $W$ are the maximum stress per unit length of the plate in $\xi$ and $\eta$-directions, respectively. Substituting the above stresses in (3.20) gives:
\[ W_L = \frac{b}{2} \int_{-1}^{1} T(1 - \eta^2) \sum_{i}^{L} \sum_{j}^{J} B_{ij} \eta' d\eta - \frac{b}{2} \int_{-1}^{1} (T(1 - \eta^2)) \sum_{i}^{L} \sum_{j}^{J} B_{ij} (-1)^\prime \eta' d\eta \]
\[ + \frac{a}{2} \int W(1 - \xi^2) \sum_{k}^{K} \sum_{l}^{L} C_{kl} \xi^l d\xi - \frac{a}{2} \int (W(1 - \xi^2)) \sum_{k}^{K} \sum_{l}^{L} C_{kl} (-1)^\prime \xi^l d\xi \] (3.21)

and taking the partial derivatives of above with respect to the constants \( B_{pq} \) and \( C_{rs} \) and evaluating the integrals results in:

\[ \frac{\partial W_L}{\partial B_{pq}} = Tb \left[ \frac{1}{(q+1)(q+3)} \right] (1-(-1)^p)(1+(-1)^q) \] (3.22)

\[ \frac{\partial W_L}{\partial C_{rs}} = Wa \left[ \frac{1}{(r+1)(r+3)} \right] (1+(-1)^r)(1-(-1)^s) \] (3.23)
Figure 3.2 Rectangular laminated composite plate subjected to biaxial parabolic stresses.
3.6.2 Self-Equilibrating Parabolic Stress at the Boundaries

Applied self-equilibrating parabolic stress is shown in figure 3.3, where the applied stresses have equations of the form:

\[ \sigma_\xi = T - T_\eta \eta^2 \]  \hspace{1cm} (3.24)

\[ \sigma_\eta = W - W_\xi \xi^2 \]

\( T, T_\eta, W, \) and \( W_\xi \) are the stresses at the middle and the edges, all per unit length of the plate, as shown in figure 3.3.

For self-equilibrating stress, the conditions are:

\[ \int_{-1}^{1} \sigma_\xi d\eta = 0 \]  \hspace{1cm} (3.25a)

\[ \int_{-1}^{1} \sigma_\eta d\xi = 0 \]  \hspace{1cm} (3.25b)

Substituting (3.24) into equations (3.25) gives:

\[ \int_{-1}^{1} (T - T_\eta \eta^2) d\eta = (T - \frac{T_\eta}{3}) = 0 \Rightarrow T_\eta = 3T \]  \hspace{1cm} (3.26a)

\[ \int_{-1}^{1} (W - W_\xi \xi^2) d\xi = (W - \frac{W_\xi}{3}) = 0 \Rightarrow W_\xi = 3W \]  \hspace{1cm} (3.26b)

With the above conditions, the self-equilibrating stresses become:

\[ \sigma_\xi = T(1 - 3\eta^2) \]  \hspace{1cm} (3.27a)

\[ \sigma_\eta = W(1 - 3\xi^2) \]  \hspace{1cm} (3.27b)
Figure 3.3 Rectangular laminated composite plate subjected to self-equilibrating parabolic stresses.
Substituting (3.27) into (3.25) gives:

\[ W_L = \frac{b}{2} \int_{-1}^{1} T(1-3\eta^2) \sum_{i} \sum_{j} B_{ij} \eta' d\eta \]
\[ - \frac{b}{2} \int_{-1}^{1} (T(1-3\eta^2)) \sum_{i} \sum_{j} B_{ij} (-1)^i \eta' d\eta \]
\[ + \frac{a}{2} \int_{-1}^{1} W(1-3\xi^2) \sum_{k} \sum_{l} C_{kl} \xi^k d\xi \]
\[ - \frac{a}{2} \int_{-1}^{1} (W(1-3\xi^2)) \sum_{k} \sum_{l} C_{kl} (-1)^i \xi^k d\xi \]

(3.28)

and taking the partial derivatives of the above with respect to the coefficients \( B_{pq} \) and \( C_{rs} \) and evaluating the integrals results in:

\[ \frac{\partial W_L}{\partial B_{pq}} = Th \left[ \frac{q}{(q+1)(q+3)} \right] (1-(-1)^p)(1+(-1)^q) \]
\[ \frac{\partial W_L}{\partial C_{rs}} = Wa \left[ \frac{r}{(r+1)(r+3)} \right] (1+(-1)^r)(1-(-1)^s) \]

(3.29)
(3.30)

3.6.3 Cubically Distributed Normal Stress at the Boundaries

Applied normal stress varying cubically at the boundaries is shown in Figure 3.4, where the applied stresses have equations of the form:

- Stress in the \( \xi \)-direction:
  \[ \sigma_\xi = T\eta^3 \]
• Stress in the $\eta$-direction:

$$\sigma_\eta = W \xi^3$$

where $T$ and $W$ are the maximum stress per unit length of the plate in $\xi$ and $\eta$-directions, respectively. Substituting the above stresses in (3.20) gives:

$$W_L = \frac{b}{2} \int T \eta^3 \sum_i \sum_j B_i \eta' d\eta$$

$$- \frac{b}{2} \int (T \eta^3) \sum_i \sum_j B_i (-1)^i \eta' d\eta$$

$$+ \frac{a}{2} \int W \xi^3 \sum_k \sum_i C_{\xi^k} \xi d\xi$$

$$- \frac{a}{2} \int (W \xi^3) \sum_k \sum_i C_{\xi^k} (-1)^i \xi d\xi$$

(3.31)

and taking the partial derivative of the above with respect to the constants $B_{pq}$ and $C_{rs}$ and evaluating the integrals results in:

$$\frac{\partial W_L}{\partial B_{pq}} = Tb \left[ \frac{1}{2(q+4)} \right] \left( 1 - (-1)^q \right) \left( 1 + (-1)^q \right)$$

(3.32)

$$\frac{\partial W_L}{\partial C_{rs}} = Wd \left[ \frac{1}{2(r+4)} \right] \left( 1 + (-1)^r \right) \left( 1 - (-1)^r \right)$$

(3.33)
Figure 3.4 Rectangular laminated composite plate subjected to cubically distributed inplane shear stresses.
3.6.4 Self-Equilibrating Cubically Distributed Shear Stress at Two Opposite Edges

Self-equilibrating cubic shear stress at the two ends of a rectangular laminated composite plate is shown in Figure 3.5. This self-equilibrating stress vanishes to zero at the ends and the middle of two opposite sides of the plate. The algebraic form of this stress is given in the form:

\[ \tau_{\eta} = \frac{3\sqrt{3}}{2} S \eta (1 - \eta^2) \quad \text{along} \ \xi = \pm 1 \quad (3.34) \]

where \( S \) is the maximum intensity of shear stress which occurs at \( \eta = \pm \sqrt{3}/3 \) on both ends of the plate.

The work done by the applied shear stress at the two ends of the plate is given by:

\[ W_L = \frac{b}{2} \int_{-1}^{1} \tau_{\eta} \sum_{k} \sum_{l} C_{u} \eta' d\eta - \frac{b}{2} \int_{-1}^{1} \tau_{\eta} \sum_{k} \sum_{l} C_{u} (-1)^k \eta' d\xi \quad (3.35) \]

Sustituting (3.34) into (3.35) and taking the partial derivatives with respect to the constants \( C_{u} \) and evaluating the integrals results in:

\[ \frac{\partial W_L}{\partial C_{u}} = \frac{3\sqrt{3}}{2} S b \left[ \frac{1}{(s+2)(s+4)} \right] (1 - (-1)^s)(1 - (-1)^s) \quad (3.36) \]
Figure 3.5 Applied self-equilibrating shear stress at two ends of a laminated composite plate.
Having formulated these different types of boundary stresses, any combination of these loadings, either biaxially or uniaxially, can be applied on the boundaries of the plate. For the combination of loading presented in Figure 3.1, the stress variations throughout the plate will be very complex. In order to observe the stress behavior of different types of laminates under non-uniform loading conditions discussed in this section, one should apply these loadings individually. After analyzing the stress behavior of the plate under uniaxial cases of the loading conditions for single layered laminates, one would have a better understanding of stress behavior throughout the plate under the combination of the inplane loading conditions such as those in Figure 3.1.
CHAPTER IV

CONVERGENCE STUDY

4.1 Introduction

Minimization of the energy functional over the admissible functions by the Ritz Method may result in exact values for the displacements as sufficient terms are used. However, convergence depends mainly on the choice of the admissible functions associated with the method. As discussed in Chapter 3, ordinary polynomials are chosen for the displacements in this work. The approximated values will get closer to the exact values as the number of parameters taken in the assumed displacements, equations (3.1), are increased (i.e., \(I, J, K, L \to \infty\)). However, the only problem associated with using ordinary polynomials is the "ill-conditioning" phenomenon as the number of parameters is increased beyond a certain limit. To avoid this phenomenon, orthogonal polynomials may be used as an alternative. Nevertheless, orthogonal polynomials generally consume much more computation time than ordinary polynomials. One must
seriously consider savings in computation time as opposed to avoiding "ill-conditioning". Realistically speaking, in order to study the stress behavior of different types of laminates under different non-uniform loading three significant figures in converged stresses would be sufficient.

In this Chapter convergence of the results is studied to answer the question of: How closely can a relatively few terms of the series in equations (3.1) approximate the exact solutions for the displacements? In this study, it is expected to have unbounded results for displacements and less accurate results for the strains and the stresses. In the limit, however, as sufficient terms of the assumed displacements functions are taken, exact minimization of the energy is achieved, which also results in exact satisfaction of the differential equation and the boundary conditions.

4.2 Convergence Study

As discussed in Chapter 3, depending on the type of symmetry involved, the domain of the indices in equations (3.10)-(3.11) must be chosen in the forms listed in equations (3.12) through (3.14) for more efficient computation. Moreover, preliminary studies done on the convergence of stresses revealed that the rate of
convergence is affected by the orthotropy ratio \( (E_1/E_2) \) and fiber angle \( \theta \) of the laminate. Referring back to section 2.5, it was discussed that changing the fiber angle in a single layered laminate will cause the plate to be an orthotropic or anisotropic laminate. Additionally, increasing orthotropy ratio \( (E_1/E_2) \) will result in a more strongly orthotropic or anisotropic plate. Table 4.1 (Vinson, 1986) and (Miller, 1995) represents the material properties studied in this work. In this table, materials are given abbreviations in parentheses for repeated reference. To study all such effects on convergence of the results, three types of material with properties shown in Table 4.1 are chosen which are listed below:

1. E-glass/epoxy (LS) with \( E_1/E_2=2.45 \) (weak orthotropy)
2. Graphite/epoxy (GE) with \( E_1/E_2=15.4 \) (strong orthotropy)
3. Aluminum alloy 1100-H14 (AA) (isotropy)

A single-layer, square plate made of the above materials is subjected to a self-equilibrating parabolic normal stress \( (\sigma_t) \) at the boundaries \( \xi=\pm1 \). For the first two materials (LS and GE), convergences of the results are studied for fiber angles of 0, 30, and 90 degrees. The laminate with fiber angle of 30 degrees and \( E_1/E_2 \) ratio of 15.4 results in a stronger anisotropic case. Tabular results for the
convergence of displacements and stress resultants of this case only are presented in this Chapter, and remaining tables are shown in Appendix A. The input and output data
Table 4.1  Layer material properties for composite plates studied.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$G_{12}$</th>
<th>$v_{12}$</th>
<th>$E_1/E_2$</th>
<th>$G_{12}/E_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Strength E-glass/epoxy (LS)</td>
<td>60.7</td>
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<td>12.0</td>
<td>0.23</td>
<td>2.45</td>
<td>0.48</td>
</tr>
<tr>
<td>Graphite/epoxy (GE)</td>
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<td>8.96</td>
<td>7.1</td>
<td>0.30</td>
<td>15.40</td>
<td>0.80</td>
</tr>
<tr>
<td>Ultra-high Modulus/Graphite/epoxy (UM)</td>
<td>310.2</td>
<td>6.2</td>
<td>4.1</td>
<td>0.26</td>
<td>50</td>
<td>0.66</td>
</tr>
<tr>
<td>High Strength Graphite/epoxy 1 (HS1)</td>
<td>137.9</td>
<td>6.9</td>
<td>4.1</td>
<td>0.25</td>
<td>20</td>
<td>0.6</td>
</tr>
<tr>
<td>High Strength Graphite/epoxy 2 (HS2)</td>
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<td>11.0</td>
<td>5.5</td>
<td>0.35</td>
<td>11.6</td>
<td>0.5</td>
</tr>
<tr>
<td>Aluminum Alloy 1100-H14 (AA)</td>
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<td>70</td>
<td>26</td>
<td>0.3</td>
<td>1.0</td>
<td>0.37</td>
</tr>
</tbody>
</table>
and output data for the material to be presented in this Chapter are:

Input data:
GRAPHITE/EPOXY (G/E), $E_1/E_2=15.4$
$E_1=138$ GPA, $E_2=8.96$ GPA, $G_{12}=7.1$ GPA, $v_{12}=0.3$
SINGLE LAYER, FIBER ANGLE=30 DEG

Output data (stiffness coefficients $10^{-9}$, N/m):
$A_{11}=4.25$, $A_{12}=1.20$, $A_{16}=2.02$
$A_{22}=1.00$, $A_{26}=7.88$, $A_{66}=1.42$

All results are obtained for a quarter of the plate shown in Figure 4.2 with 0.25 increments in both $\xi$ and $\eta$ directions. The size of the characteristic determinant is obtained from $(i+1)\times(j+1)+(k+1)\times(l+1)-2$. For example, a determinant size of 96x96 is a result of taking $I=6$, $J=6$, $K=6$, and $L=6$ terms excluding rigid body translation terms (i.e. $B_{00}$ and $C_{00}$) which reduces the determinant size from 98x98 to 96x96. This particular problem has no symmetry, which requires one to retain all the displacement polynomial terms in the analysis. In the isotropic and orthotropic cases equations (3.10)-(3.11) would take the form of equations (3.12) to enforce the symmetry that would be present. In those cases the size of the characteristic determinant is considerably smaller than the anisotropic case and there is a increase in the rate of convergence of the results.
Figure 4.1 composite plate with single layer of 30 degree fiber orientation.
Figure 4.2 Quarter of the plate selected for convergence study.
The results in Table 4.2 and 4.3 show the convergence of the nondimensional displacements \((u/a)\) and \((v/a)\) respectively for the plate described above. In addition to the upper limits \((I,J,K,L)\) of terms retained in Equations (3.10) and (3.11), these Tables show the number of simultaneous equations to be solved in each case. The averages of the displacements are shown in the last rows of these tables. The displacements have converged without bound to at least three significant figures for most points in both the longitudinal and transverse directions of the applied load. As seen, the displacements as well as their average have not converged monotonically. However, the average displacement in both longitudinal and transverse directions have converged to at least three significant figures up to determinant size of 338X338. It was observed that for this particular plate (i.e. \(E_1/E_2=15.4\) and fiber angle \(\theta=30\) degree) after taking 14 terms for all the indices, which results in a 450X450 determinant size, ill-conditioning begins to occur.

Convergence of the nondimensional stresses is shown in Tables 4.4 and 4.5. The results in both tables are shown up to four decimal places. As mentioned earlier, three significant figures in converged stresses would be sufficient for stress analysis. Exact values of the desired parabolic stress resultant variation \((N_x/2T)\) along the
boundary $\xi=1.0$ at the points $\eta=0.00, 0.25, 0.50, 0.75$ and 1.00 are 0.500, 0.4062, 0.1250, -0.3438 and -1.0000, respectively.
Table 4.2 Nondimensional displacement \((u/a) \times 10^9\) convergence for a square \((a/b=1)\) plate (Graphite/epoxy \(E_1/E_2=15.4\)) subjected to uniaxial self-equilibrating parabolic stress at the boundaries; \(\theta = 30^\circ\).

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<th>(I=10, J=10, K=10, L=10)</th>
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Average 0.239038 0.240379 0.239232 0.239380 0.238996
Table 4.3 Nondimensional displacement \((v/a) \times 10^9\) convergence for a square \((a/b=1)\) plate (Graphite/epoxy \(E_1/E_2=15.4\)) subjected to uniaxial self-equilibrating parabolic stress at the boundaries; \(\theta = 30^\circ\).

<table>
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Table 4.4 Nondimensional stress resultant \( \left( \frac{N_{i}}{2T} \right) \) convergence for a square \( (a/b=1) \) plate (graphite/epoxy (GE) \( E_{1}/E_{2}=15.4 \)) subjected to uniaxial self-equilibrating parabolic stress at the boundaries; \( \theta = 30^\circ \).

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<th>( \eta )</th>
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Table 4.5 Nondimensional stress resultant \( (N_y/2T) \) convergence for a square \((a/b=1)\) plate (graphite/epoxy \(E_1/E_2=15.4\)) subjected to uniaxial self-equilibrating parabolic stress at the boundaries; \( \theta = 30^\circ \).

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \eta )</th>
<th>( K=6, L=6 )</th>
<th>( K=8, L=8 )</th>
<th>( K=10, L=10 )</th>
<th>( K=12, L=12 )</th>
<th>( K=14, L=14 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-0.1576</td>
<td>-0.1626</td>
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<tr>
<td>0.25</td>
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<td>-0.1434</td>
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<tr>
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<td>0.75</td>
<td>-0.0177</td>
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<td>-0.0224</td>
<td>-0.0225</td>
</tr>
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<td>1.00</td>
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<td>-0.0018</td>
<td>0.0030</td>
<td>-0.0010</td>
<td>-0.0001</td>
</tr>
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<td>-0.1283</td>
<td>-0.1313</td>
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<td>-0.1304</td>
<td>-0.1304</td>
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<td>0.25</td>
<td>-0.1414</td>
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<tr>
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<td>0.75</td>
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</tr>
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</tr>
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<td>-0.0396</td>
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<td>0.25</td>
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</tr>
<tr>
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<td>0.75</td>
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<td>-0.0514</td>
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<td>1.00</td>
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<td>0.0229</td>
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<td>-0.0002</td>
<td>0.0014</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.1094</td>
<td>0.1122</td>
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<td>0.1103</td>
<td>0.1106</td>
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<td>0.75</td>
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<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.3164</td>
<td>0.3188</td>
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<td>0.3198</td>
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</tr>
<tr>
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<td>0.25</td>
<td>0.3510</td>
<td>0.3542</td>
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<td>0.3525</td>
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<td>0.75</td>
<td>0.3267</td>
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<td>1.00</td>
<td>1.00</td>
<td>-0.0092</td>
<td>-0.0120</td>
<td>-0.0079</td>
<td>-0.0057</td>
<td>-0.0042</td>
</tr>
</tbody>
</table>
It is seen that the most accurate \((I=J=K=L=14)\) value in Table 4.4 for these points \((0.4996, 0.4069, 0.1239, -0.3429, -1.0077)\) has almost converged to three significant figures to the desired values. In Table 4.4 the nondimensional stress resultant \((N_x/2T)\) has converged up to three significant figures for most points and at least up to two throughout the plate. The "ill-conditioning" phenomenon occurs after taking 14 terms in equations (3.10) and (3.11) for this particular case of strongly anisotropic case. In fact the greater the anisotropy of the plate, that is when \([A_y]\) matrix is fully populated, the more number of terms is needed for convergence in equations (3.10) and (3.11), and also the "ill-conditioning" effect is increased.

To study the effect of orthotropy ratio on the convergence of stress resultants, Tables A.1 through A.5 in Appendix A show additional data for the convergence of nondimensional normal stress resultants \(N_x/2T\) for plates subjected to the same loading, made of materials with orthotropy ratios 2.54(LS) and 15.4(GE), and fiber angles of \(\theta=0, \theta=30, \text{and } \theta=90\) degrees. Also in Appendix A, Table A.6 contains the results for the stress resultant \((N_x/2T)\) for an isotropic plate subjected to the same loading. In Table A.1 which is for \(\theta=0\), stresses have converged up to four significant figures in more points throughout the plate than for the plate with \(\theta=30\) degrees (Table A.2). In Table A.1,
the solution size of 450x450 has resulted in strange results for the stress resultants which indicates ill-conditioning has occurred. In Table A.2 the stresses have converged up to at least three significant figures throughout the plate. This change in the number of converged significant figures with respect to change in the fiber angle is due to having an anisotropic plate instead of an orthotropic one. In Tables A.4 and A.5 the results of normal stress resultants for the plate with E1/E2=15.4 subjected to the same loading with fiber angles $\theta=0$ and 90 degrees are presented.

The results in Table 4.6 show the nondimensional normal stress resultant ($N_x/2T$) along the entire boundary ($\xi=1.0$) of the plate considered ($\theta=30^\circ$). The exact value of the parabolically applied load is shown in the last column of this Table. The percent errors of the calculated stress resultants compared to the exact ones at discrete points on the boundary of the plate are presented. In this Table, average difference between the approximate and exact boundary stresses are shown. These values decrease from 0.0085 which is for solution size of 98X98 to 0.0007 for solution size of 450X450.

Also in Table 4.6, the calculated force integrals along the boundary, which is zero for the exact value, are shown in the last row for different solution sizes.
Table 4.6 Nondimensional stress resultant ($N_x/2T$) at the boundary ($\xi=1.0$) of a square ($a/b=1$) plate (graphite/epoxy $E_1/E_2=15.4$) subjected to uniaxial self-equilibrating parabolic stress at the boundaries; $\theta=30^\circ$. 

<table>
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<tr>
<th>$\eta$</th>
<th>$98\times98$</th>
<th>$162\times162$</th>
<th>$242\times242$</th>
<th>$338\times338$</th>
<th>$450\times450$</th>
<th>Exact</th>
</tr>
</thead>
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<td>-0.9950</td>
<td>-0.9975</td>
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<tr>
<td></td>
<td>1.8%</td>
<td>0.9%</td>
<td>0.5%</td>
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<td></td>
</tr>
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<tr>
<td></td>
<td>2.1%</td>
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</tr>
<tr>
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<td>0.1250</td>
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<tr>
<td></td>
<td>2.3%</td>
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<td>0.5%</td>
<td>0.8%</td>
<td>0.4%</td>
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</tr>
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<td>0.27%</td>
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<td>0.78%</td>
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<tr>
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<td>0.4063</td>
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<tr>
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<td>2.6%</td>
<td>1.0%</td>
<td>0.5%</td>
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<td>0.0%</td>
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<tr>
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<td>0.43%</td>
<td>3.4%</td>
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<tr>
<td></td>
<td>7.2%</td>
<td>0.0%</td>
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</tr>
<tr>
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<td>-1.0386</td>
<td>-1.0251</td>
<td>-1.0154</td>
<td>-1.0096</td>
<td>-1.0077</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>3.9%</td>
<td>2.5%</td>
<td>1.5%</td>
<td>1.0%</td>
<td>0.8%</td>
<td></td>
</tr>
<tr>
<td>Average difference</td>
<td>0.0085</td>
<td>0.0036</td>
<td>0.0019</td>
<td>0.00010</td>
<td>0.0007</td>
<td>0.0000</td>
</tr>
<tr>
<td>Force</td>
<td>0.0182</td>
<td>0.0138</td>
<td>0.0132</td>
<td>0.0131</td>
<td>0.0131</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

(%) show the percent error compared to the exact stress applied at the boundary.
This integral is calculated at the boundary using 200 points by a numerical method imbedded in the main program and is equal to

\[ \text{Force} = \int_{-1}^{1} \left( \frac{N_s}{2T} \right) d\eta \]

As seen from Table 4.6, the force integral has decreased to a value of 0.0131 compared to the exact value of zero as the number of terms in the assumed displacements is increased. The percent errors for the points along the boundary of the plate show that the results have converged to the exact values with an error of at most 0.8 percent for the determinant size of 450X450. The percent errors for the determinant size of 338X338 is at most 1.0 percent of the exact values. Also in this Table, average differences between approximate and exact normal stresses along the boundary for the plate is calculated. These values are calculated using 2000 points in the boundary and they are the absolute values of the differences between the exact and the calculated stresses.

Table 4.7 represent the average difference between the approximate and the exact boundary stresses (\( \xi = 1.0 \)) for the cases considered in Tables 4.4, and A.1 through A.6. It is seen from this Table that ill-conditioning occurs for the Aluminum alloy and E-glass/epoxy at solution sizes of
Also the orthotropic cases resulting from zero degree fiber angle have closer values to the exact boundary values compared to the orthotropic cases arising from fiber angle of 90 degrees for both E-glass/epoxy and Graphite/epoxy materials. Comparing the average differences for $\theta = 30^\circ$ of E-glass/epoxy and Graphite/epoxy reveals that average boundary stresses are closer to the exact ones for the E-glass/epoxy material with orthotropy ratio of 2.45. Finally, Table 4.7 reveals that Graphite/epoxy with $E_1/E_2 = 15.4$ and $\theta = 0^\circ$ results in closer values to the exact ones at the boundary for the different solution sizes compared to E-glass/epoxy material.

From the results obtained for the stresses throughout the plate (Tables 4.4, 4.5 and A.1-A5) together with the results presented in Table 4.6, reveal that determinant size of $338 \times 338$ or 12 terms in each of the indices in equations (3.11) and (3.12) is a sufficient number of terms for desired accuracy in stresses. Taking advantage of the double symmetry and the double antisymmetry involved in isotropic and orthotropic cases, a determinant size of $338 \times 338$ is equivalent to taking 23 terms for each of the indices in equations (3.11) and (3.12) which is quite more terms than needed for the desired accuracy in the calculated results.
Table 4.7 Average difference between approximate and exact normal stresses along the boundary $\xi=1$.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\theta^*$</th>
<th>$i=6, J=6, K=6, L=6$</th>
<th>$i=8, J=8, K=8, L=8$</th>
<th>$i=10, J=10, K=10, L=10$</th>
<th>$i=12, J=12, K=12, L=12$</th>
<th>$i=14, J=14, K=14, L=14$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum alloy</td>
<td>$98 \times 98$</td>
<td>0.0052</td>
<td>0.0008</td>
<td>0.0004</td>
<td>0.0002</td>
<td>0.0005</td>
</tr>
<tr>
<td>E-glass/epoxy</td>
<td>$90 \times 90$</td>
<td>0.0005</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0003</td>
</tr>
<tr>
<td>Graphite/epoxy</td>
<td>$90 \times 90$</td>
<td>0.0006</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
4.3 Comparison of Results for an Isotropic Square Plate Subjected to Parabolically Distributed Load with that of Timoshenko

Timoshenko (1924) (see also Timoshenko and Goodier, 1951) presented the solution for a square isotropic plate subjected to tensile stresses parabolically distributed along two opposite edges. The energy method used by him to solve the problem assumes a stress function, which after non-dimensionalization, is

\[ \phi = \frac{1}{2} T \eta^2 \left(1 - \frac{1}{6} \eta^3 \right) + (\xi^2 - 1)^2 (\eta^2 - 1)^2 (\alpha_1 + \alpha_2 \xi^2 + \alpha_3 \eta^2 + \ldots) \]

where \( T \) is the maximum intensity of parabolic stress applied at the boundary, and \( \alpha_1, \alpha_2, \alpha_3, \ldots \) are constants to be determined by minimizing the strain energy of the plate with respect to them. This stress function satisfies all the boundary conditions including the zero shear exactly on all four sides of the plate and yields an approximate solution for the interior stresses. For a three term solution and a square plate \((a=b)\), Timoshenko obtained the three constants by minimizing the strain energy of the plate to be

\[ \alpha_1 = 0.04040 \frac{T}{a^6} \quad \alpha_2 = \alpha_3 = 0.01174 \frac{T}{a^3} \]

Then nondimensional stresses are found from
Table 4.8 presents values of nondimensional stress $N_x/T$ from Timoshenko's solution, and the values obtained by the present analysis for an isotropic square plate subjected to parabolically distributed stress at the boundaries. The data from the three term solution of Timoshenko's are compared to a solution from the present analysis which has 98 unknown coefficients. The present analysis for the isotropic material (which is a symmetric case) is obtained by taking $I=J=K=L=12$ in equations 3.11 and 3.12 since due to symmetry, half of the terms are not used in the assumed displacements. The data obtained from the present analysis uses the isotropic plate made of Aluminum Alloy 1100-H14. However, stresses are independent of the material properties of the plate in the isotropic case, which is as expected. It may be noticed from the results shown that the two approximate solutions are quite close and the error between the solutions does not exceed 3% which occurs mainly near the lateral edges of the plate. The present approach does not exactly satisfy the stress boundary conditions. However, from the computer results it was observed that the boundary conditions are satisfied very closely for the isotropic case; i.e., the maximum nondimensional shear stress resultant $N_y/T$ at the boundary was calculated to be 0.0010 at two points. Figure 4.3 is a plot that shows the
nondimensional stress $N_x/T$ along the three lines at the center ($\xi=0$), the quarter-point ($\xi=0.5$) and the edge ($\xi=1.0$). It must be mentioned that the stress resultants are nondimensionalized with respect to $T$ for the parabolic loading in this work.
Table 4.8 Comparison of $N_x/T$ for an isotropic plate subjected to parabolically distributed stress obtained by Timoshenko and by the present analysis.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\xi=0.0$</th>
<th>$\xi=0.5$</th>
<th>$\xi=1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>presentwork</td>
<td>Timoshenko</td>
<td>presentwork</td>
</tr>
<tr>
<td></td>
<td>$I=J=K=L=1$</td>
<td>3 term</td>
<td>$I=J=K=L=1$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>338X338</td>
<td>2</td>
</tr>
<tr>
<td>0.00</td>
<td>0.8591</td>
<td>0.8619</td>
<td>0.9180</td>
</tr>
<tr>
<td>0.20</td>
<td>0.8291</td>
<td>0.8306</td>
<td>0.8830</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7448</td>
<td>0.7434</td>
<td>0.7818</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6237</td>
<td>0.6206</td>
<td>0.6267</td>
</tr>
<tr>
<td>0.80</td>
<td>0.4969</td>
<td>0.4961</td>
<td>0.4400</td>
</tr>
<tr>
<td>1.00</td>
<td>0.4107</td>
<td>0.4171</td>
<td>0.2590</td>
</tr>
</tbody>
</table>
Figure 4.3 Stress resultants for an isotropic square plate subjected to parabolic end stresses compared to the known results of Timoshenko's.
CHAPTER V

RESULTS FOR NORMAL STRESS DISTRIBUTION THROUGHOUT THE PLATE

5.1 Introduction

In order to utilize laminated composite plates in modern design, knowledge of stress distribution throughout the laminate under different types of non-uniform normal inplane loading is essential. In the literature, inplane problems of composite plates have referred only to the uniform stress case and there have been almost no published specific numerical results for stresses throughout a rectangular composite plate under non-uniform inplane loading. Horgan (1982) studied Saint-Venant's principle in anisotropic elasticity by applying a self-equilibrating parabolic inplane stress on a semi-infinite strip. But, results for stress distribution throughout the strip was not presented. Decaying stresses from the boundary of the plate for orthotropic case were presented, and the decay rate was calculated from the obtained stresses for the orthotropic case. Miller and Horgan in a recent paper (1994) subjected
a semi-infinite strip to self-equilibrating parabolic inplane stress to study the decay rate of stresses for different cases of single layer composites with different orthotropy ratios. In this paper decay rates were calculated for a semi-infinite strip; however, no numerical results for stresses distribution throughout the strip were published. It is the intent of the present study to give specific and accurate numerical results for stresses in composite plates with various orthotropy ratios and fiber angles, for various types of non-uniform inplane loading at the two ends.

In order to study the stress distribution, a single layered rectangular composite plate with aspect ratio of $a/b$ has been chosen. The computer program developed to compute stresses can analyze inplane non-uniform loading in both longitudinal and transverse directions. However, results are presented only for the case of non-uniform stresses in the longitudinal direction and stress free on the remaining two sides of the plate. The case of non-uniform loading on the boundary of the plate are shown in Figure 5.1 and 5.2 and are as follows:

i) Parabolically distributed normal stress which is the first even deviation from uniform stress (Figure 5.1).
ii) Cubically distributed normal stress which is the first odd deviation from linear stress (Figure 5.2).
Figure 5.1 Applied parabolic normal stress on a rectangular composite plate.

Figure 5.2 Applied cubically distributed normal stress on the boundaries of a rectangular composite plate.
To study the effect of orthotropy on stress distribution throughout the plate, three types of materials with different orthotropy ratios have been considered. These three are as follows:

i) Aluminum Alloy 1100-H14 (AA) (isotropy)

ii) Low Strength E-glass/epoxy (LS) (weak orthotropy)

iii) High Strength Graphite/epoxy (HS1) (strong orthotropy)

The layer material properties for the above materials are shown in Table 4.1, which are repeated in Table 5.1 for convenience. As seen from this table, the first material is an isotropic one and the other two have a longitudinal to transverse moduli ratio ($E_1/E_2$) which ranges from 2.45 for E-glass/epoxy (LS) to 20 for the high strength graphite/epoxy (HS1) case. The purpose of this selection is to investigate the effect of weak and strong orthotropy ratios on the distribution of normal stresses. It should also be mentioned that the high strength graphite/epoxy chosen for the present study is the one for which Miller and Horgan (1994) presented stress decay rates for a case of semi-infinite strip loaded at one end.

The effect of fiber angle on stress distribution of the laminate has been studied for all of the above materials by varying it from 0 to 90 degrees in 15 degree increments. As
discussed in Chapter 3, by varying the fiber angle different cases of orthotropic and anisotropic plates can be obtained. In the case of small \((E_1/E_2)\) ratio (i.e. 2.45), the resulting plate is considered to be a weakly anisotropic plate and in the case of strong \(E_1/E_2\) (i.e. 20) it is strongly anisotropic. Those cases will also be apparent by examining the extensional stiffness matrix of the plate.

For all of the cases, the results presented in this Chapter are obtained for an aspect ratio \((a/b)\) of one, which corresponds to a square plate, and aspect ratio equal to twice the characteristic decay length (i.e. \(a/b=2\lambda\)) of the plate as estimated in this work, where the characteristic decay length \(\lambda\) of the plate is defined to be the length over which the applied stress at the boundary decays to 1% of its value at the middle \((\xi=0)\) of the plate. Table 5.2 contains the characteristic aspect ratios \((2\lambda)\) for different fiber angles for both e-glass/epoxy and high strength graphite epoxy. These values are calculated in the present work using a computer program which finds the aspect ratio of the plate for which the average self-equilibrating stress applied at the two ends of the plate decay to 1% of their original values at the middle \((\xi=0)\) of the plate. The advantage of having the results of stress distribution for plates with aspect ratio equal to twice their characteristic decay length is to study the decay pattern of
stresses going from the boundary to the middle of the plate. For both of the non-uniform loadings considered, we will see how the stresses decay to their average value across the plate for different types of laminates.

In this Chapter the study of above mentioned cases is organized so that a clear analysis of stress distribution for rectangular plates having different material, different fiber angle, different non-uniform loading, and different aspect ratios may be done. First, the results of stress distribution for an isotropic plate made of aluminum alloy is presented. Second, the results for the plate made of E-glass/epoxy (LS) with $E_1/E_2$ ratio of 2.45 is shown. Third, the plate made of graphite epoxy (HS1) with $E_1/E_2$ ratio of 20 subjected to the above mentioned non-uniform normal stresses is presented.

5.2 Isotropic Case

The case of an isotropic square plate subjected to parabolically distributed end stresses was studied by Timoshenko (1924) which was discussed in Chapter 4 of this work. Also, The case of an isotropic plate subjected to cubic loading at the two ends was studied by Goodier which is presented with parabolic case in (Timoshenko, 1970).
Table 5.1 Layer material properties for composite plates studied in this Chapter.

<table>
<thead>
<tr>
<th>Material</th>
<th>( E_1 ) (GPA)</th>
<th>( E_2 ) (GPA)</th>
<th>( G_{12} ) (GPA)</th>
<th>( \nu_{12} )</th>
<th>( E_1/E_2 )</th>
<th>( G_{12}/E_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Strength E-glass/Epoxy (LS)</td>
<td>60.7</td>
<td>24.8</td>
<td>12.0</td>
<td>0.23</td>
<td>2.45</td>
<td>0.48</td>
</tr>
<tr>
<td>High Strength Graphite/Epoxy 1  (HS1)</td>
<td>137.9</td>
<td>6.9</td>
<td>4.1</td>
<td>0.25</td>
<td>20</td>
<td>0.59</td>
</tr>
<tr>
<td>Aluminum Alloy 1100-H14 (AA)</td>
<td>70</td>
<td>70</td>
<td>26</td>
<td>0.3</td>
<td>1.0</td>
<td>0.37</td>
</tr>
</tbody>
</table>

Table 5.2 Characteristic aspect ratio corresponding to single layer fiber angle.

<table>
<thead>
<tr>
<th>Fiber Angle, ( \theta ) (Degrees)</th>
<th>E-glass/epoxy (LS) ( E_1/E_2 = 2.45 )</th>
<th>High strength Graphite/epoxy (HS1) ( E_1/E_2 = 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.60</td>
<td>10.42</td>
</tr>
<tr>
<td>10</td>
<td>3.19</td>
<td>7.12</td>
</tr>
<tr>
<td>15</td>
<td>2.84</td>
<td>6.00</td>
</tr>
<tr>
<td>20</td>
<td>2.48</td>
<td>5.75</td>
</tr>
<tr>
<td>30</td>
<td>2.06</td>
<td>4.58</td>
</tr>
<tr>
<td>40</td>
<td>2.02</td>
<td>3.63</td>
</tr>
<tr>
<td>45</td>
<td>1.94</td>
<td>3.00</td>
</tr>
<tr>
<td>50</td>
<td>1.91</td>
<td>2.63</td>
</tr>
<tr>
<td>60</td>
<td>1.99</td>
<td>2.00</td>
</tr>
<tr>
<td>70</td>
<td>1.99</td>
<td>1.88</td>
</tr>
<tr>
<td>75</td>
<td>2.15</td>
<td>2.16</td>
</tr>
<tr>
<td>80</td>
<td>2.02</td>
<td>2.18</td>
</tr>
<tr>
<td>90</td>
<td>2.30</td>
<td>2.33</td>
</tr>
</tbody>
</table>
The analysis of an isotropic plate being under the same loading conditions as an anisotropic plate is a much simpler case to deal with. Numerical calculation using the computer program developed in this work is a lot less time-consuming than the anisotropic case. Also, convergence of the results for the isotropic case is much faster than the anisotropic case and it takes lesser number of terms in the assumed polynomials to reach the same convergence. Having the stress distribution for the isotropic case under the same loading conditions will reveal how the anisotropy of the plate considered affects the stress distribution throughout the plate. In this section an isotropic plate made of Aluminum Alloy 1100-H14 with material properties given in Table 5.1, is subjected to the two types of non-uniform inplane loading mentioned above.

Figures 5.3 and 5.4 represent the nondimensional stress ($N_s/T$, where $T$ is maximum stress applied) distribution of an isotropic plate subjected to inplane parabolic normal stresses at the two ends. It must be mentioned that the nondimensional stress resultant in this case (i.e. parabolic loading), must not be mistaken for $N_s/2T$ for the self-equilibrating parabolic loading. The latter has been nondimensionalized by $2T$ instead of only $T$. Clearly, the resulting stress distribution will be even with respect to both axes. Therefore, the stress distribution is presented
only for one half of the plate. The loading is applied along the longitudinal direction of the plate. The stress resultants have been shown across five different cross sections of the plate as seen in the Figures. $\xi = 0$ represents the middle of the plate and $\xi = 1$ represents the boundary of the plate where the load is applied. Figure 5.4 represents the same plate under the same loading condition with aspect ratio of 2.18 which corresponds to twice the characteristic decay length $\lambda$ of this plate. As seen from this figure, the stress distribution has become more and more uniform as we move away from the boundary of the plate and finally has reached the average stress ($2/3T$) at $\xi = 0$ which is the middle of the plate. It should be mentioned that in order to obtain the stress distribution corresponding to a self-equilibrating parabolic load, one would have to subtract a constant stress equal to $2/3T$ from the results shown for the parabolic normal stress.

Figure 5.5 and 5.6 represent the same isotropic plate with two different aspect ratios subjected to a cubically varying inplane loading applied in the longitudinal direction of the plate as seen in Figure 5.2. Figure 5.5 is the plate with aspect ratio of one and Figure 5.6 is with aspect ratio equal to twice the characteristic decay length of the plate. Clearly, in both cases the resulting stress distribution will be odd with respect to the x-axis and even
with respect to the y-axis. As seen in Figure 5.6, the stress distribution has become more and more linear and finally at $\xi = 0$ it has reached the average linear stress with equation $N_z/T = 3/5\xi$. 
Figure 5.3  Nondimensional stress resultants for an isotropic square (a/b=1) plate subjected to parabolically end loading.
Isotropic Material, $a/b=2.18$
Aluminum Alloy

Figure 5.4 Nondimensional stress resultants for an isotropic rectangular ($a/b=2.18$) plate subjected to parabolic end loading.
Figure 5.5 Nondimensional stress resultants for an isotropic square ($a/b=1$) plate subjected to cubic end loading.
Figure 5.6 Nondimensional stress resultants for an isotropic rectangular (a/b=2.18) plate subjected to cubic end loading.
5.3 Weak Orthotropy Ratio

In this section, results for nondimensional normal stress resultants ($N_z/T$) for a single layered rectangular plate made of E-glass/epoxy (LS) with $E_1/E_2$ ratio of 2.45 are presented. The results are obtained for parabolic and cubic end loading. First the results of stress distribution for the parabolic loading are presented. Second, in this section, the results for cubic loading are shown. For both end loadings, the results are shown for fiber angles varying from 0 to 90 degrees in 15 degree increments for an aspect ratio of one, and an aspect ratio equal to twice the characteristic decay length ($a/b=2\lambda$) corresponding to the fiber angle of the laminate. Table 5.2 contains those values for this material.

Figures 5.7 through 5.13 represent normal stress resultant distribution throughout a single layered, square ($a/b=1$), composite plate subjected to parabolic end loading. These Figures also show the result for fiber angles $\theta$ from 0 to 90 degrees in 15 degree increments. The stress resultants are shown for five different cross sections of the plate, namely, $\xi=0, 0.25, 0.5, 0.75$, and 1.0. Figures 5.7 and 5.13 represent fiber angles of 0 and 90 degrees respectively. Both of these fiber angles result in an orthotropic case. As seen in Figures 5.7 and 5.13, the
stress distributions are even functions with respect to $\xi$ and $\eta$ axes at all cross sections of the plate, as is the applied parabolic load at $\xi = \pm 1$. However, this symmetry with respect to both axes is completely destroyed for the fiber angles of 15,30,45,60, and 75 degrees, as seen from Figures 5.8 through 5.12. Observing the stress distribution in Figures 5.7 through 5.13, the value of the stresses along the $\eta=0$ line, going from the boundary $\xi=1.0$ of the plate to $\xi=0$ which is the middle of the plate have decayed to 0.935 for $\theta=0$ to a minimum of 0.807 for $\theta=90$ degrees. However, the rate of decay of stresses is more for angles 0 to 45 degrees than for 60 to 90 degrees; the change in stress distribution for all cross sections between 60 and 90 degrees is very little as seen from Figures 5.11 through 5.13.

Figure 5.14 through 5.20 represent stress distribution for rectangular plates of characteristic aspect ratio made of E-glass/epoxy subjected to parabolic loading at the ends $\xi = \pm 1$ with fiber angles $\theta=0,15,30,45,60,75,$ and 90 degrees, respectively. For each fiber angle (Figures 5.14 through 5.20), results for the plate with aspect ratio equal to twice the characteristic decay length for that angle are shown. Table 5.2 shows the characteristic aspect ratios for this material. It is seen from Figures 5.14 through 5.20 that the stresses have essentially decayed to the average
stress of parabolic load equal to 2/3 for all fiber angles considered. Comparing the aspect ratios (a/b) associated with fiber angles \( \theta = 0 \) through 45 which range from 3.60 to 1.94 to the ones for fiber angles \( \theta = 60 \) through 90 degrees which range from 1.99 to 2.30 reveals that rate of decay of stresses is higher for fiber angles between 0 through 45 degrees.

Figures 5.21 through 5.27 represent normal stress resultant distribution throughout a single layered, square, composite plate subjected to cubically varying normal stress at two opposite ends. Looking at Figures 5.21 through 5.23, it is seen that stresses have decayed to get closer and closer to their average value of \( (N_x/T = 3/5) \) in going from fiber angle \( \theta = 0 \) to \( \theta = 45 \) degrees. However, in going from fiber angle \( \theta = 45 \) to \( \theta = 90 \) degree there seems to be almost no change in the stress distribution throughout cross sections of the plate shown. This means that rate of decay of stresses for this material is greater between fiber angles of 0 through 45 degrees for this material.

Figures 5.29 through 5.35 represent results of normal stress distribution for the plates with characteristic aspect ratios shown in Table 5.2 for this material.
Figure 5.7 Nondimensional normal stress resultants for a square, single layered plate with \( \theta = 0 \) degrees made of E-glass/epoxy subjected to parabolic end loading.
Fiber Angle=15 Deg, a/b=1, E1/E2=2.45

Figure 5.8 Nondimensional normal stress resultants for a square single layered plate with θ=15 degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.9 Nondimensional normal stress resultants for a square, single layered plate with $\theta=30$ degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.10 Nondimensional normal stress resultants for a square, single layered plate $\theta=45$ degrees made of E-glass/epoxy subjected to parabolic end loading.
Fiber Angle=60 Deg, a/b=1 , E1/E2=2.45

Figure 5.11 Nondimensional normal stress resultants for a square, single layered plate θ=60 degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.12 Nondimensional normal stress resultants for a square, single layered $\theta=75$ degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.13 Nondimensional normal stress resultants for a square, single layered plate with $\theta=90$ degrees made of E-glass/epoxy subjected to parabolic end loading.
Fiber Angle=0 Deg, a/b=3.6 , E1/E2=2.45

Figure 5.14 Nondimensional normal stress resultants for a rectangular (a/b=3.60), single layered plate with \( \theta = 0 \) degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.15 Nondimensional normal stress resultants for a rectangular \((a/b=2.84)\), single layered plate with \(\theta=15\) degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.16 Nondimensional normal stress resultants for a rectangular (a/b=2.06), single layered plate with \( \theta = 30 \) degrees made of E-glass/epoxy subjected to parabolic end inplane loading.
Figure 5.17 Nondimensional normal stress resultants for a rectangular (a/b=1.94), single layered plate with $\theta=45$ degrees made of E-glass/epoxy subjected to parabolic loading.
Figure 5.18 Nondimensional normal stress resultants for a rectangular \((a/b=1.99)\), single layered plate with \(\theta=60\) degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.19 Nondimensional normal stress resultants for a rectangular (a/b=2.15), single layered plate with θ=75 degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.20 Nondimensional normal stress resultants for a rectangular (a/b=2.3), single layered plate with θ=90 degrees made of E-glass/epoxy subjected to parabolic end loading.
Figure 5.21 Nondimensional normal stress resultants for a square \((a/b=1.0)\), single layered plate with \(\theta=0\) degrees made of E-glass/epoxy subjected to cubic end loading.
Figure 5.22 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with $\theta=15$ degrees made of E-glass/epoxy subjected to cubic loading.
Figure 5.23 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with θ=30 degrees made of E-glass/epoxy subjected to cubic loading.
Figure 5.24 Nondimensional normal stress resultants for a square \((a/b=1.0)\), single layered plate with \(\theta=45\) degrees made of E-glass/epoxy subjected to cubic end loading.
Figure 5.25 Nondimensional normal stress resultants for a square \((a/b=1.0)\), single layered plate with \(0=60\) degrees made of E-glass/epoxy subjected to cubic loading.
Fiber Angle = 75 Deg, a/b = 1.0, E1/E2 = 2.45

Figure 5.26 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with θ=75 degrees made of E-glass/epoxy subjected to cubic loading.
Fiber Angle=90 Deg, a/b=1, E1/E2=2.45

Figure 5.27 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with θ=90 degree made of E-glass/epoxy subjected to cubic loading.
Figure 5.28 Nondimensional normal stress resultants for a rectangular (a/b=3.6), single layered plate with \( \theta = 0 \) degrees made of E-glass/epoxy subjected to cubic end loading.
Figure 5.29 Nondimensional normal stress resultants for a rectangular (a/b=2.84), single layered plate with θ=15 degrees made of E-glass/epoxy subjected to cubic end loading.
Figure 5.30 Nondimensional normal stress resultants for a rectangular (a/b=2.06), single layered plate with \( \theta=30 \) degrees made of E-glass/epoxy subjected to cubic loading.
Figure 5.31 Nondimensional normal stress resultants for a rectangular (a/b=1.94), single layered plate with θ=45 degrees made of E-glass/epoxy subjected to cubic end loading.
Fiber Angle=60 Deg, a/b=1.99, E1/E2=2.45

Figure 5.32 Nondimensional normal stress resultants for a rectangular (a/b=1.99), single layered plate with θ=60 degrees made of E-glass/epoxy subjected to cubic end loading.
Figure 5.33 Nondimensional normal stress resultants for a rectangular \((a/b=2.15)\), single layered laminated plate with \(\theta=75\) degree made of E-glass/epoxy subjected to cubic end loading.
Figure 5.34 Nondimensional normal stress resultants for a rectangular \( a/b=2.3 \), single layered laminated plate with \( \theta=90 \) degrees made of E-glass/epoxy subjected to cubic loading.
5.4 Strong Orthotropy Ratio Case

In order to study the effect of strong orthotropy ratio on stress distribution, in this section results for nondimensional normal stress resultants \( N_x/T \) for a single layered rectangular plate made of Graphite/epoxy (HS1) with \( E_1/E_2 \) ratio of 20 are presented. Again, the results are obtained for parabolic and cubic end loading. First the results of stress distribution for the parabolic loading is presented. Second, in this section, the results for cubic loading is shown. For both end loadings, the results are shown for fiber angles varying from 0 to 90 degrees in 15 degree increments for both aspect ratio of one and aspect ratio equal to twice the characteristic decay length (2\( \lambda \)), corresponding to the fiber angle of the laminate presented in Table 5.2 for this material (HS1).

Figures 5.35 through 5.41 represents normal stress resultant distribution throughout a single layered, square composite plate made of graphite/epoxy (HS1) subjected to parabolic end loading. These Figures again show the results for fiber angle \( \theta \) from 0 to 90 degrees in 15 degree increments. Observing the stress distribution in Figures 5.35 through 5.41, the value of the stresses along \( \eta=0 \) line, going from the boundary \( \xi=1.0 \) of the plate to \( \xi=0 \) which is
the middle of the plate have decayed from 0.994 for $\theta=0$ to a minimum of 0.751 for $\theta=90$ degrees. Figure 5.35 represents stress distribution for a fiber angle $\theta=0$ degrees. As seen from this figure, going from the boundary ($\xi=1$) to the middle ($\xi=0$) of the plate stresses have decayed very little. This is due to the effect of strong orthotropy ratio $E_1/E_2=20$ for this material compared to the weak one shown in Figure 5.7. Looking at the stress distribution for the $\xi=0.75$ cross section in Figures 5.36 through 5.39, which are for fiber angles of 15 through 60 degree in 15 degree increments, the stress near the upper boundary of the plate has become almost constant, which was not the case for the material with weak orthotropy ratio under the same loading (Figures 5.8-5.11).

Figure 5.42 through 5.48 represent stress distribution for rectangular plates made of Graphite/epoxy (HS1) subjected to parabolic loading, with plate aspect ratios for fiber angles of 0 through 90 degrees in 15 degrees increments which are the characteristic aspect ratios corresponding to the fiber angles given for this material in Table 5.2. As seen from Figures 5.42 through 5.48 that the stresses have decayed to the average stress of parabolic load equal to approximately $2/3$ for every fiber angle with different aspect ratios. As seen from Figures 5.42 trough
5.45, going from a fiber angle of 0 to 45 degrees, the stress distribution has decayed more rapidly compared to those shown in Figures 5.45 through 5.48, which are for fiber angles of 45 through 90 degrees.

Figures 5.49 through 5.55 represent normal stress resultant distributions throughout square Graphite/epoxy plates subjected to cubically varying normal stress at two opposite ends. As seen from these figures, the stress distribution remains odd with respect to the η=0 axis for fiber angles of 0 and 90 degrees only. For fiber angles other than 0 and 90 degrees, the stress distribution is no longer odd.

Figure 5.56 represents normal stress distribution for a plate made of the same material with aspect ratio of 3.0 and fiber angle of 45 degree subjected to cubically distributed end loading. As seen from this Figure that stresses have decayed to their average value at the middle of the plate.

Figures 5.57 represent the normal stress distribution at the middle (ξ=0.0) of a rectangular plate varying fiber angle from 0 to 90 degree in 15 degrees increments with aspect ratio of a/b=2.0 made of Graphite/epoxy (HS1) subjected to parabolic end loading. As seen from this Figure, stresses have decayed to the average 2/3T stress at the middle (ξ=0.0) of the plate for fiber angles θ=60 and 75
degrees. Figure 5.58 represents the difference of nondimensional normal stress resultants \( \frac{N_x}{T} \) and average stress variation \( \frac{3}{5} \eta T \) at the middle (\( \xi = 0.0 \)) for a single layered rectangular (\( a/b = 2.0 \)) plate made of Graphite/epoxy (HS1) for fiber angles from 0 to 90 degrees in 15 degrees increments subjected to cubic end loading. The purpose of having the difference of stress and the average stress variation is to be able to clearly show the stress distribution at the middle of the plate. Again, as seen from this Figure, stresses have decayed to the average variation (\( \frac{3}{5} \eta T \)) for angles 60 through 90 degrees indicating a shorter decay length compared to fiber angles of 0 to 30 degrees.
Figure 5.35 Nondimensional normal stress resultants for a square, single layered $\theta=0$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.36 Nondimensional normal stress resultants for a square single layered plate with $\theta=15$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.37 Nondimensional Normal stress resultants for a square, single layered plate with $\theta=30$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.38 Nondimensional normal stress resultants for a square, single layered plate with $\theta=45$ degree made of Graphite/epoxy (HS1) subjected to parabolic loading.
Figure 5.39 Nondimensional normal stress resultants for a square, single layered plate with $\theta=60$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.40 Nondimensional normal stress resultants for a square, single layered plate with θ=75 degrees made of Graphite/epoxy (HS1) subjected to parabolic loading.
Figure 5.41 Nondimensional normal stress resultants for a square (a/b=1), single layer plate with θ=90 degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.42 Nondimensional normal stress resultants for a rectangular (a/b=10.42), single layered plate with θ=0 degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.43 Nondimensional normal stress resultants for a rectangular \((a/b=6.00)\), single layered plate with \(\theta=15\) degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.44 Nondimensional normal stress resultants for a rectangular \((a/b=4.58)\), single layered plate with \(\theta=30\) degrees made of Graphite/epoxy (HS1) subjected to parabolic loading.
Figure 5.45 Nondimensional normal stress resultants for a rectangular \((a/b=3.0)\), single layered plate with \(\theta=45\) degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.46 Nondimensional normal stress resultants for a rectangular (a/b=2.0), single layered plate with $\theta=60$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end inplane loading.
Figure 5.47 Nondimensional normal stress resultants for a rectangular (a/b=2.16), single layered plate with $\theta=75$ degrees made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.48 Nondimensional normal stress resultants for a rectangular \((a/b=2.33)\), single layered plate with \(\theta=90\) degrees made of Graphite/epoxy (HS1) subjected to parabolic loading.
Figure 5.49 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with θ=0 degrees made of Graphite/epoxy (HS1) subjected to cubic end loading.
Figure 5.50 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with \( \theta=15 \) degrees made of Graphite/epoxy (HS1) subjected to cubic end loading.
Figure 5.51 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with $\theta=30$ degrees made of Graphite/epoxy (HS1) subjected to cubic end loading.
Figure 5.52 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with $\theta=45$ degrees made of Graphite/epoxy (HS1) subjected to cubic end loading.
Figure 5.53 Nondimensional normal stress resultants for a square (a/b=1.0), single layer plate with $\theta=60$ degrees made of Graphite/epoxy (HS1) subjected to cubic end loading.
Figure 5.54 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with $\theta=75$ degrees made of Graphite/epoxy (HS1) subjected to cubic loading.
Figure 5.55 Nondimensional normal stress resultants for a square (a/b=1.0), single layered plate with $\theta=90$ degrees made of Graphite/epoxy (HS1) subjected to cubic loading.
Figure 5.56 Nondimensional normal stress resultants for a rectangular \((a/b=3.0)\), single layered plate with \(\theta=45\) degrees made of Graphite/epoxy (HS1) subjected to cubic loading.
Figure 5.57 Nondimensional normal stress resultants at the middle \( \xi = 0.0 \) of a single layered rectangular \((a/b=2.0)\) plate with different fiber angles made of Graphite/epoxy (HS1) subjected to parabolic end loading.
Figure 5.58 The difference of nondimensional normal stress resultant and the average stress variation at the middle $\xi=0.0$ of a single layered rectangular $a/b=2.0$ plate made of Graphite/epoxy (HS1) for different fiber angles subjected to cubic end loading.
CHAPTER VI

RESULTS FOR SHEAR STRESS DISTRIBUTION THROUGHOUT THE PLATE SUBJECTED TO NON-UNIFORM SHEAR STRESSES AT THE TWO ENDS

6.1 Introduction

Laminated composite plates as components of a structure can be under states of severe non-uniform shear stresses. For example, laminated plates used in construction of an airplane wing can be under shear stresses during certain maneuvers of the flight. These shear stresses can be responsible for failure of the plate and consequently the structure as a whole. As revealed by a survey of the field of laminated composite plates, there are no known results available in the literature for stresses throughout the plate under non-uniform shear stresses. It is the intent of this Chapter to study the distribution of shear stresses throughout rectangular laminated plates subjected to non-uniform shear stresses.

In order to study shear stress distribution throughout the plate, a single layered rectangular composite plate is
subjected to self-equilibrating cubically distributed shear stress at the two ends $\xi = \pm 1$ of the plate as shown in Figure 6.1. This non-uniform shear is carefully chosen so that the stress resultants are zero at the corners of the plate and reach a maximum/minimum ($\pm S$) at $\eta = \pm 3\sqrt{3}/2$. The maximum/minimum values ($\pm S$) are normalized to become $\pm 1$. Thus, as described in Chapter 3 (section 3.6.4) the shear stress applied at the edges $\xi = \pm 1$ is

$$\tau_{\xi \eta} = \frac{3\sqrt{3}}{2} S \eta(1-\eta^2) \quad (6.1)$$

This shear stress resultant is positive between $\eta = 0$ and $\eta = 1$ and negative between $\eta = 0$ and $\eta = -1$ at the two ends. To study nondimensional shear stress distribution throughout the plate, the same three materials used in Chapter 5 are chosen:

i) Aluminum Alloy 1100-H14 (AA)

ii) Low Strength E-glass/epoxy (LS)

iii) High Strength Graphite/epoxy 1 (HS1)

The material properties for the above materials are given in Table 5.1.

In this Chapter, the results for the isotropic case are presented first. Then, the nondimensional shear stress resultants throughout the plate for the low strength
Graphite/epoxy (LS) with low orthotropy ratio ($E_1/E_2=2.45$) and higher modulus ($G_{12}/E_2=0.48$) are shown next. Finally, shear stress resultants for high strength Graphite/epoxy (HS1) with a higher orthotropy ratio ($E_1/E_2=20.0$) and shear modulus $G_{12}/E_2=0.59$ are given.
Figure 6.1 Self-equilibrating cubically distributed shear stress applied at the two ends of a composite plate.
6.2 Isotropic Case

Figure 6.2 represents the nondimensional shear stress resultants \( \left( N_{y}/S \right) \) for an isotropic square plate subjected to self-equilibrating cubically distributed shear stresses at the ends \( \xi = \pm 1 \). The shear stresses are shown for six cross sections of the plate namely, \( \xi = 0, 0.2, 0.4, 0.6, 0.8 \) and 1.0. The applied stress vanishes at the corners of the plate and reaches its maximum/minimum value \( \pm S \) at the points \( \eta = \pm 3\sqrt{3}/2 \) at the two ends of the plate. As seen from Figure 6.2 that nondimensional shear stress resultants for different cross sections of the plate are distributed cubically as the applied load is at the boundary. Also from Figure 6.2, it is seen that the shear stress resultants change sign in the vicinity of \( \xi = 0.6 \) as \( \xi \) is varied. To investigate this behavior in more detail, Figure 6.3 shows the distribution of nondimensional shear stresses between \( \xi = 0.6 \) and 0.7 in 0.01 increments. In this Figure, it is seen that the nondimensional shear stress has decayed to 1\% \( \left( N_{y}/S = \pm 0.0106 \text{ at } \eta = \pm 0.925 \right) \) of its maximum value (i.e. 1) applied at the ends for the cross section at \( \xi = 0.63 \). The stress resultants for this cross section is shown as a dashed line in the figure. Therefore, shear stress resultants in a square isotropic plate have decayed to 1\% of their original value at less than one strip width from the
end for an isotropic plate subjected to self-equilibrating end loads. But for smaller $\xi$, their values become quite large, including at the center ($\xi=0$) of the plate.
Figure 6.2  Non-dimensional shear stress resultants $N_{xy}/S$ for a square isotropic plate subjected to self-equilibrating cubically distributed shear stress at the two ends.
Figure 6.3 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of an isotropic, square plate subjected to self-equilibrating cubically distributed shear stress at the two ends.
6.3 Low Orthotropy Ratio Case

In this section, results for nondimensional shear stress resultants \( \left( \frac{N_y}{S} \right) \) for a single layered rectangular plate made of E-glass/epoxy (LS) with \( E_1/E_2 \) ratio of 2.45 are presented. In order to study the shear stress distribution for different fiber angles for this material, shear stress resultants have been obtained for a square plate varying the fiber angle from 0 to 90 degrees in 15 degree increments. Figures 6.4 through 6.10 represent such cases.

As seen from Figures 6.4 through 6.10 there is not a drastic change in the shear stress resultant at the given cross sections as the fiber angles are changed. However, as seen from Figures 6.4 and 6.10, shear stress resultants remain odd with respect to the center of the cross section and the \( \xi \) axis for fiber angles of 0 and 90 degrees. This is not the case for fiber angles other than 0 and 90 degrees. As seen from Figures 6.5 through 6.9, for fiber angles other than 0 and 90 degrees, distribution of shear stress is distorted from being odd functions, and there is nonzero shear stress resultant at the center of the plate. For all the fiber angles, the shear stress resultants change from positive to negative between \( \xi = 0.6 \) and \( \xi = 0.7 \) cross sections. Figures 6.11 through 6.15 represent
nondimensional shear stress resultant distribution between $\xi=0.6$ and $\xi=0.7$ for fiber angles $\theta=0, 15, 30, 45,$ and $90$ degree respectively. The cross sections with nondimensional shear stress closest to zero are shown as dashed lines. As seen from Figures 6.11 through 6.15, the closest curve to zero is at the $\xi=0.63$ cross section for all the fiber angles except for $\theta=90$ degrees which is at $\xi=0.68$ cross section.
Figure 6.4 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=0$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.5 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=15$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.6 Non-dimensional shear stress resultants $N_{xy}/S$ for a square single layered plate with $\theta=30$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Fiber Angle=45 (Deg), a/h=1, E1/E2=2.45
LS

Figure 6.7 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=45$ degree made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Fiber Angle=60 (Deg), a/b=1, E1/E2=2.45

**Figure 6.8** Non-dimensional shear stress resultants for a square, single layered plate with \( \theta=60 \) degree made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.9 Non-dimensional shear stress resultants \( N_{xy} / S \) for a square, single layered plate with \( \theta = 75 \) degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.10 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=90$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.11 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of a single layered, square plate with $\theta=0$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.12 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of a single layered, square plate with $\theta=15$ degrees made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.12 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of a single layered, square plate with $\theta=30$ degree made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.13 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of a single layered, square plate with $\theta=45$ degree made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.14 Non-dimensional shear stress resultants $N_{xy}/S$ around the center of a single layered, square plate with $\theta=90$ degree made of E-glass/epoxy (LS) subjected to self-equilibrating cubic shear stress at the two ends.
6.4 Strong Orthotropy Ratio Case

In order to study the effect of material properties \((E_1/E_2\) and \(G_{12}/E_2\)) on shear stress distribution, in this section, results for nondimensional shear stress resultants \((N_n/T)\) for a single layered square plate made of Graphite/epoxy (HS1) are obtained. This material has an \(E_1/E_2\) ratio of 20 and a \(G_{12}/E_2\) ratio of 0.59. Figures 6.15 through 6.21 represent shear stress resultants for a square plate subjected to cubically distributed shear stress at the two ends.

Very similar stress patterns are obtained for fiber angles ranging from 15 to 45 degrees, as seen from Figures 6.16 to 6.18. However, for fiber angles 60, 75, and 90 degrees stress distributions for cross sections less than \(\xi=0.6\) (i.e. \(\xi=0, 0.2, 0.4\)) get closer together and approach zero while \(\theta\) increases to 90 degrees. This shows that shear stress decay happens between 0 and 45 degrees.
Figure 6.15 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=0$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.16 Non-dimensional shear stress resultants \( N_{xy} / S \) for a square, single layered plate with \( \theta = 15 \) degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.17 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=30$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.18 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=45$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.19 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=60$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.20 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=75$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
Figure 6.21 Non-dimensional shear stress resultants $N_{xy}/S$ for a square, single layered plate with $\theta=90$ degree made of Graphite/epoxy (HS1) subjected to self-equilibrating cubic shear stress at the two ends.
CHAPTER VII

SAINT VENANT'S PRINCIPLE

7.1 Introduction

After Saint-Venant stated his famous principle in his classical paper in 1855, many approximations in linear theory of elasticity relied on the validity of this principle for simplified analysis. Without an appeal to this principle, the neglect of end effects when determining stresses in rods, beams, plates, and shells would not have been justified. The validity of this principle has been investigated and proved to hold in the case of an isotropic material. However, a great deal needs to be said about this principle in the case of an anisotropic material.

One of the major investigators of Saint-Venant's principle in the last two decades is C.O. Horgan. In his first paper concerning the Saint-Venant principle (Horgan, 1972a), he used methods involving energy-decay inequalities of a type used earlier by Toupin (1965) and Knowles (1966)
to solve for the decay of stresses with distance from the boundary of an elastic orthotropic solid subjected to self-equilibrating end loads. Horgan's latest work (Miller and Horgan, 1994) on this topic, which is to be published in *Journal of Mechanics of Composite Materials and Structures*, provided a further analysis of the effects of material anisotropy on the exponential decay rate for stresses in a semi-infinite elastic strip. In this work a linearly elastic semi-infinite strip in a state of plane stress/strain subjected to a self-equilibrated end load was considered first for a specially orthotropic material, and then for generally anisotropic one.

In the present work, a rectangular, generally anisotropic laminated composite plate which is subjected to self-equilibrating inplane normal stresses at two ends is considered in order to study the exponential decay rate of stresses. It is also the purpose of this work to obtain the stress distribution, both for normal and shear, along the length of the plates made of different materials to study the effect of orthotropy ratio and fiber angle on the decay of stresses along the length. First, in this Chapter, a detailed analysis of the method used by Miller and Horgan in their latest work is given. Next, characteristic decay length for plates made of four different materials and fiber angles subjected to non-uniform, self-equilibrating normal
stress are presented and the results are compared to those found by Horgan for the semi-infinite strip loaded at two ends for the orthotropic case. Finally in this Chapter, characteristic decay length for the case of applied self-equilibrating shear stress at the two ends is presented and discussed.

7.2 Formulation of the Problem by Miller and Horgan

In this section a brief review of the solution procedure used by Horgan is given. Consider a homogeneous, anisotropic, linearly elastic semi-infinite strip as shown in Figure 7.1. There 2H is the width of the strip and 2T is its thickness, which is taken to be infinite (plane strain) or infinitesimal (plane stress). All surfaces of the body are traction free except for the surface \( x_2 = 0 \), where a prescribed self-equilibrated traction is applied.

Miller and Horgan defined an Airy stress function \( \phi(x_1, x_2) \) in terms of which the Cartesian components of the stress tensor \( \tau \) are given by

\[
\tau_{11} = \phi_{,x_1}, \quad \tau_{22} = \phi_{,x_2}, \quad \tau_{12} = \phi_{,x_1} \quad \tau_{12} = \phi_{,x_2} \quad (7.1)
\]

where a comma denotes partial differentiation. With the Airy stress function defined, the equations of equilibrium
are identically satisfied. Then, the compatibility equations yield the following differential equation,

$$c_{22}\phi_{,1111} - 2c_{26}\phi_{,1112} + (2c_{12} + c_{66})\phi_{,1122} - 2c_{16}\phi_{,1222} + c_{11}\phi_{,2222} = 0 \quad (7.2)$$

which is a fourth-order elliptic partial differential equation involving six elastic compliances of the material as its coefficients. After nondimensionalizing equation (7.2), nondimensional governing equation becomes

$$\frac{\phi_{,xxxx}}{\varepsilon_2} - \frac{2}{\varepsilon_2}\phi_{,xxy}, \frac{1}{\varepsilon_1}\phi_{,yyx} - \frac{2}{\varepsilon_3}\phi_{,yxy}, \phi_{,yxx} = 0 \quad (7.3)$$

Now the governing differential equations (7.2) is expressed in terms of three nondimensional parameters $\varepsilon_1, \varepsilon_2,$ and $\varepsilon_3$, which are defined in terms of the $c_{ij}$ of the material. This method of reducing the number of material constants involved in the governing differential equation from six to three is called "anisotropic rescaling". Therefore, for strips made of isotropic, orthotropic, or anisotropic material the problem is formulated in terms of one ($\varepsilon_1$), two ($\varepsilon_2$ and $\varepsilon_3$), and three dimensionless material parameters, respectively. This parameter reduction scheme simplifies the problem in both orthotropic and anisotropic cases considerably. Seeking solutions of (7.3) that decay in the axial direction in the form;
Figure 7.1 Semi-infinite elastic strip considered by Horgan.
\[ \phi = e^{\gamma \tau} F(\eta), \quad \gamma = \text{constant} \quad (7.4) \]

leads to an eigenvalue problem for fourth-order ordinary differential equation with generally complex eigenvalues. These can be solved by a numerical technique (see Crafter et al, 1993 and Choi et al, 1977). The stress decay rate is given by the real part of the eigenvalue of smallest real part. In the case of anisotropic material, Miller and Horgan introduced another dimensionless parameter called "reduced eigencondition", which is also determined from the elastic constants of the material. By interpolating from tables given in (Crafter et al, 1993) and scaling by another factor, Miller and Horgan obtained the real and imaginary parts of the eigenvalue \( \gamma \) of smallest real part. Miller and Horgan have obtained the characteristic decay length for several types of materials including materials abbreviated in the present analysis as HS1, HS2, UM (see Table 7.1) for the semi-infinite strip made of a single layer with fiber angles ranging from 0 to 90 degrees.

7.3 Decay of Normal Stresses Along the Length of the plate

In order to study the effect of orthotropy ratio on decay of normal stresses along the length of an anisotropic plate, four types of orthotropic materials with material
properties shown in Table 7.1 are considered. The orthotropy ratios $E_1/E_2$ of these materials range from 2.45 for that of E-glass/epoxy (LS) to 50 which is for ultra modulus graphite/epoxy case. Materials with abbreviations HS1, HS2, and UM are the ones that Horgan and Miller used to obtain results for an semi-infinite strip loaded at one end.

To investigate decay of stresses along the length, a rectangular plate (Figure 7.2) made of these materials is subjected to parabolically self-equilibrating normal stress at the two ends. It is the purpose of this work to study the effect of elastic constants on the normal stress decay. A computer program was written to calculate the characteristic decay length for different types of anisotropic plates, where the characteristic decay length is the length over which the stresses applied at the ends decay to 1% of their original value at the middle of the plate. A computer program based on the Ritz method has been developed in this work to calculate the characteristic decay length for different types of materials. This program first finds the aspect ratio of the plate for which the average stresses applied at the boundary decay to almost zero value (i.e. for example, 0.0001) at the middle ($\xi=0.0$) of the plate. This is to assure that the effect of stress applied at the other end of the plate has almost vanished. Then, the characteristic decay length of the plate is found by finding
the aspect ratio for which the stresses decay to 1% of their original value applied at the boundary. However, all calculations in finding the characteristic decay length are done using an iterative procedure and therefore, the effect of stresses due to the applied load at the other end of the plate is not vanished completely in calculating the characteristic decay length of the plate. Thus, small inaccuracies in the values of the calculated characteristic decay length is expected.

Table 7.2 represents the characteristic decay length $\lambda$ for four types of materials mentioned above with fiber angles ranging from 0 to 90 degrees. These values have been obtained by applying a pair of self-equilibrating parabolically distributed normal stress (Figure 7.2) at the two ends of a plate made of the chosen materials. The data obtained in Table 7.2 are shown graphically in Figure 7.3. Table 7.2 and Figure 7.3 show the effect of orthotropy ratio on characteristic decay length for all the materials. It is seen from Figure 7.3 that the characteristic decay length increases with increasing orthotropy ratio. Also the greatest decay length for all of the materials is at a fiber angle of 0 degree which represents orthotropic cases. Ultra modulus graphite epoxy (UM) has the greatest decay length at zero degree fiber angle, which is equal to 7.93 times the dimension (b) of the side of the plate where the loading is
applied. This means that the self-equilibrating stress applied at the two ends decay to 1% of their original value, at 7.93 times the width of the plate for the UM case. E-glass/epoxy, which has the lowest orthotropy ratio of $E_1/E_2=2.45$, has the least decay length for its orthotropic case (i.e fiber angle $\theta=0$).

As seen from Figure 7.3, the decay length for all the materials decreases with increasing fiber angle up to fiber angle of 60 degrees, where the decay length for all the materials comes very close to that of the isotropic case, which has been found in this work to be 1.09 times the width of the plate. Miller and Horgan (1994) have obtained results similar to the ones shown in Figure 7.3 for characteristic decay length of materials named HS1, HS2, and UM for a elastic semi-infinite strip loaded at one end. For the orthotropic cases (fiber angle $\theta=0$ degree), Miller and Horgan (1994) found the characteristic decay length $\lambda$ for materials abbreviated as UM, HS1, HS2, and the isotropic case to be 6.24, 4.06, 3.25, and 1.09 respectively, which are for the case of elastic strip subjected to self-equilibrating load at one end. However, the problem considered by Miller and Horgan is for a semi-infinite strip loaded at one end where in the present analysis, a finite plate is loaded at two ends. The two problems are not directly comparable to each other. The calculated
characteristic decay lengths in the present problem are expected to be longer than the semi-infinite strip considered by Miller and Horgan because of the effect of applied load at the other end of the plate.

Figures 7.4 through 7.12 are plotted to show how the normal stresses decay along the length of rectangular plates made of Aluminum alloy, E-glass/epoxy (LS), and high strength Graphite/epoxy (HS1). For all these graphs, stresses have been shown along five lines going from the middle to the boundary of the plate, namely η=0.2,0.4,0.6, and 0.8. The lengths of the plates used are the characteristic decay lengths for each material and fiber angle shown in Table 7.2. Figure 7.4 is for the isotropic case. As seen from this Figure, the plate with aspect ratio of 2.18 has normal stresses which have decayed to an average of 1% at ξ=0, which is the middle of the plate. So, it is justified that Saint-Venant's end effects are "negligible" at a distance of approximately one strip width from the end for isotropic materials. Figures 7.5 through 7.8 represent normal stress decay from the boundary to the middle of the plate for E-glass/epoxy for fiber angles of 0,30,45, and 90 degrees. As seen from these Figures, stresses for different cross sections across the length of the plate have decayed to 1% of their original values at the middle of the plate.
Figures 7.9 through 7.12 represent normal stress decay along the length for plates made of Graphite/epoxy (HS1) for fiber angles of 0, 30, 45, and 90 degrees. For this material as the fiber angle changes from 0 degree to 30 or 45 degree, the stress decay pattern changes. As seen from Figures 7.10 and 7.11 that for some cross sections through the width of the plate, the stresses have decayed to zero far before the characteristic decay length of the plate.
Table 7.1 Properties for the materials considered in this Chapter.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$G_{12}$</th>
<th>$v_{12}$</th>
<th>$E_1/E_2$</th>
<th>$G_{12}/E_2$</th>
</tr>
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<tr>
<td>Low Strength E-glass/epoxy (LS)</td>
<td>60.7 (GPA)</td>
<td>24.8 (GPA)</td>
<td>12.0 (GPA)</td>
<td>0.23</td>
<td>2.45</td>
<td>0.48</td>
</tr>
<tr>
<td>Ultra-high Modulus/Graphite Epoxy (UM)</td>
<td>310.2 (GPA)</td>
<td>6.2 (GPA)</td>
<td>4.1 (GPA)</td>
<td>0.26</td>
<td>50</td>
<td>0.66</td>
</tr>
<tr>
<td>High Strength Graphite/Epoxy 1 (HS1)</td>
<td>137.9 (GPA)</td>
<td>6.9 (GPA)</td>
<td>4.1 (GPA)</td>
<td>0.25</td>
<td>20</td>
<td>0.59</td>
</tr>
<tr>
<td>High Strength Graphite/Epoxy 2 (HS2)</td>
<td>127.5 (GPA)</td>
<td>11.0 (GPA)</td>
<td>5.5 (GPA)</td>
<td>0.35</td>
<td>11.6</td>
<td>0.50</td>
</tr>
<tr>
<td>Aluminum Alloy 1100-H14 (AA)</td>
<td>70 (GPA)</td>
<td>70 (GPA)</td>
<td>26 (GPA)</td>
<td>0.3</td>
<td>1.0</td>
<td>0.37</td>
</tr>
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</table>
Figure 7.2 Applied self-equilibrating parabolically distributed normal stress at the two ends of a laminated composite plate.
Table 7.2 Characteristic decay length $\lambda$ of rectangular plates subjected to a pair of self-equilibrating load at the two ends for materials considered

<table>
<thead>
<tr>
<th>Fiber Angle $\theta$ (degrees)</th>
<th>Characteristic Decay Length $\lambda = \lambda' / b$</th>
<th>LS $(E_1/E_2=2.45)$</th>
<th>HS2 $(E_1/E_2=11.6)$</th>
<th>HS1 $(E_1/E_2=20)$</th>
<th>UM $(E_1/E_2=50)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1.80</td>
<td>4.30</td>
<td>5.21</td>
<td>7.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.25)</td>
<td>(4.06)</td>
<td>(6.24)</td>
<td></td>
</tr>
<tr>
<td>10</td>
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(LS) e-glass/epoxy
(HS2) High strength graphite/epoxy 2
(HS1) High strength graphite/epoxy 1
(UM) Ultra modulus graphite/epoxy

* For Aluminum alloy (Isotropic material), $\lambda=1.09$

() represent results for orthotropic cases found by Miller and Horgan for semi-infite strip loaded at one end.
Figure 7.3 Characteristic decay length $\lambda$ Vs. fiber angle $\theta$ along the length of laminated composite plates.
Figure 7.4 Normal stress decay along the length of an isotropic rectangular ($a/b=2.18$) plate.
Figure 7.5 Normal stress decay along the length of a single layered orthotropic (θ=0 deg) rectangular plate (a/b=3.6) made of E-glass/epoxy (LS1).
Figure 7.6 Normal stress decay along the length of a single layered anisotropic ($\theta=30$ deg) rectangular plate ($a/b=2.06$) made of E-glass/epoxy (LS1).
Figure 7.7 Normal stress decay along the length of a single layered anisotropic ($\theta$=45 deg) rectangular plate ($a/b=1.94$) made of E-glass/epoxy (LS1).
Orthotropic, $\lambda = 1.15$, $a/b = 2.30$, $\theta = 90$ (Deg) 

Figure 7.8 Normal stress decay along the length of a single layered orthotropic ($\theta = 90$ deg) rectangular plate ($a/b = 2.30$) made of E-glass/epoxy (LS1).
Orthotropic, $\lambda=5.21$, $a/b=10.42, \theta=0$ (Deg) 

HS1

Figure 7.9 Normal stress decay along the length of a single layered orthotropic ($\theta=0$ deg) rectangular plate ($a/b=10.42$) made of Graphite/epoxy (HS1).
Anisotropic, $\lambda = 2.29$, $a/b = 4.58, \theta = 30$ (Deg)

**Figure 7.10** Normal stress decay along the length of a single layered anisotropic ($\theta = 30$ deg) rectangular plate ($a/b = 4.58$) made of Graphite/epoxy (HS1).
Anisotropic, $\lambda = 1.50$, $a/b = 3.0$, $\theta = 45$ (Deg)  
HS1

Figure 7.11 Normal stress decay along the length of a single layered anisotropic ($\theta = 45$ deg) rectangular plate ($a/b = 4.58$) made of Graphite/epoxy (HS1).
Orthotropic, $\lambda=1.17$, $a/b=2.33$, $\theta=90$ (Deg)

Figure 7.12 Normal stress decay along the length of a single layered orthotropic ($\theta=90$ deg) rectangular plate ($a/b=2.33$) made of Graphite/epoxy (HS1).
7.4 Decay of Shear Stress Resultants Through the Length

In order to observe the behavior of the shear stress resultant decay along the length of the plate for different types of materials considered, Figures 7.16 through 7.24 are presented for rectangular plate made of Aluminum alloy, E-glass/epoxy (LS), and high strength Graphite/epoxy (HS1) with properties shown in Table 7.1. For all these graphs, shear stresses are shown along four lines going from the boundary to the middle of the plate namely, $\eta = 0.2, 0.4, 0.6,$ and 0.8 (the shear stresses are zero everywhere along $\eta = 0$ and 1.0). The length of the plates considered are the characteristic normal stress decay length for each material and fiber angle shown in Table 7.2. A general observation made from Figures 7.16 through 7.24 is that shear stresses decay rapidly from the boundary, cross zero in the vicinity of $\xi = 0.8$, which is about 20 percent of their characteristic decay length. After changing sign, the $N_y/S$ for all materials and fiber angles considered reach a minimum value of as much as -0.2. Figure 7.16 is for the isotropic case. It is seen from this Figure that with an aspect ratio of 2.18 for the plate, shear stress resultants have not decayed to an average of 1% at $\xi = 0$ at the middle of the plate. Further investigation in this work has found that for the isotropic case, shear stress resultants decay to 1% of their
original value at an aspect ratio of 2.6 which would correspond to a characteristic decay length of 1.3 which is different than the one for the normal stress case. Figures 7.17 through 7.20 represent decay of nondimensional shear stress resultant along the length for plates made of E-glass/epoxy (LS) for fiber angles of 0, 30, 45, and 90 degrees, respectively. Figure 7.17 is for fiber angle of 0 degree. As seen from this Figure, the shear stress resultants have decayed to somewhat more than 1% of their original values (on the average) at the middle of the plate. However, for fiber angles of 30 and 45 degree for this material, Figures 7.18 and 7.19 show that the shear stress resultants do not decay, on the average, to 1% of their original values for the aspect ratios corresponding to 2.06 and 1.94 which are for the normal stress cases. Those aspect ratios have been calculated to be at least 2.1 for both 30 and 45 degree fiber angles for the stresses to decay to 1% of their value at the middle of the plate. However, these fiber angles are highly anisotropic cases and the results obtained in Figures 7.18 and 7.19 could be due to numerical inaccuracies of the solutions.

Figures 7.21 through 7.24 represent nondimensional shear stress decay along the length of plates made of Graphite/epoxy (HS1) for fiber angles of 0, 30, 45, and 90
degrees, respectively. As seen from these Figures that shear stress resultants have decayed to 1% of their original values at the middle of the plate. However, the values of shear stress resultant at the boundaries of the plate have not converged to the exact values of shear stress applied at the boundary. An attempt was made to improve these results by increasing the number of terms in the assumed displacements to 13 (i.e. \( I=J=K=L=13 \)), however the results did not improve considerably. For example, with solution size of 392X392 (i.e. \( I=J=K=L=13 \)) the value of the shear stress resultant for \( \eta=0.6 \) in Figure 7.21 increased from its present value of 0.830 to 0.890 compared to the exact value of 0.998. Comparing shear stress decay for Graphite/epoxy (HS1) material (Figures 7.21 through 7.24) with that of E-glass/epoxy (LS1) case (Figures 7.17 through 7.21) reveals that the rate of decay is faster for Graphite/epoxy (HS1) since stress curves decrease with a greater slope compared to e-glass/epoxy (LS1) case.
Isotropic, $\lambda = 1.08$, a/b = 2.18

Figure 7.13 Shear stress decay along the length of an isotropic rectangular (a/b=2.18) plate.
Orthotropic, $\lambda = 1.8$, $a/b = 3.60, \theta = 0$ (Deg)

$\begin{array}{c}
\text{LS} \\
\eta = 0.2 \\
\eta = 0.4 \\
\eta = 0.6 \\
\eta = 0.8
\end{array}$

$N_x$ vs. $\xi$

Figure 7.14 Shear stress decay along the length of a single layered orthotropic ($\theta = 0$ deg) rectangular plate ($a/b = 3.6$) made of E-glass/epoxy (LS1).
Anisotropic, $\lambda = a/b = 2.06, \theta = 30^\circ$ (Deg)

Figure 7.15 Shear stress decay along the length of a single layered orthotropic ($\theta = 30^\circ$ deg) rectangular plate ($a/b = 2.06$) made of E-glass/epoxy (LS1).
Figure 7.16 Shear stress decay along the length of a single layered orthotropic ($\theta=45^\circ$ deg) rectangular plate ($a/b=1.94$) made of E-glass/epoxy (LS1).
Figure 7.17 Shear stress decay along the length of a single layered orthotropic ($\theta=90$ deg) rectangular plate ($a/b=2.3$) made of E-glass/epoxy (LS1).
Orthotropic, $\lambda=5.21$, $a/b=10.42$, $\theta=0$ (Deg)

Figure 7.18 Shear stress decay along the length of a single layered orthotropic ($\theta=0$ deg) rectangular plate ($a/b=10.42$) made of Graphite/epoxy (HS1).
Figure 7.19 Shear stress decay along the length of a single layered orthotropic ($\theta=30$ deg) rectangular plate ($a/b=4.58$) made of Graphite/epoxy (HS1).
Figure 7.20 Shear stress decay along the length of a single layered anisotropic (θ=45 deg) rectangular plate (a/b=3.0) made of Graphite/epoxy (HS1).
Figure 7.21 Shear stress decay along the length of a single layered anisotropic ($\theta=90$ deg) rectangular plate ($a/b=2.33$) made of Graphite/epoxy (HS1).
CHAPTER VIII
SUMMARY AND CONCLUSIONS

In this work, anisotropic elasticity analysis of laminated composite plates subjected to non-uniform stresses has been presented. In Chapter 2, the basic governing equations of thin, symmetrically laminated plates and the energy functionals are shown. In Chapter 3, a general procedure is developed to obtain the anisotropic elasticity solution for symmetrically laminated composite plates subjected to inplane, non-uniform, normal and/or shear stresses. This procedure uses the Ritz method with ordinary polynomial series as assumed displacements and is capable of dealing with many forms of non-uniform loading, both normal and shear, applied at the boundaries of the plate. The applied loads at the boundaries are formulated in series form using the assumed displacements. Unsatisfactory results are obtained using using this method for the cases of point load or partially uniform loading applied at the boundaries because thousands of terms in the assumed polynomial series would be needed to represent such loading conditions accurately. The method is flexible and could be
adapted to unsymmetric laminates if needed. In Chapter 4, a convergence study presented reveals that 12 terms for upper limits of the polynomial series (i.e. I=J=K=L=12) in the assumed displacements results in reasonably accurate results for displacements and stresses for most of the materials considered. It was also seen in this study that for strongly anisotropic plates, ill conditioning begins to develop after taking 14 terms for the upper limits of the polynomial series. For the isotropic and orthotropic cases, one fourth of the assumed terms in the polynomial series is being used because of the symmetry involved. The anisotropic cases arising from fiber angles between 30 and 45 degrees result in slower convergence for the results compared to the ones closer to 0 or 90 degrees. Better accuracy was obtained for the displacements compared to the stresses, which is justified because the stresses are obtained from the strains, which are derivatives of the displacements themselves. Unbounded results are obtained for the displacements, including the their average values throughout the plate, which is one of the characteristics of the Ritz method. Nondimensional normal stresses at the boundary (ξ=1.0) of a rectangular, single layer composite plate were compared to the exact applied load, and the maximum percent difference for the 338X338 solution size (i.e. I=J=K=L=12) was calculated to be 1%. Average
differences between the approximate and exact normal stresses along the boundary (\( \xi=1.0 \)) for three types of materials with different orthotropy ratio and fiber angle revealed that a lesser number of terms is needed in the assumed displacements to achieve a desirable accuracy for materials of higher orthotropy ratio and fiber angle of 0 degrees. Also, at the boundary of the plate, stresses are closer to the exact ones for the materials with lower orthotropy ratio for the same solution size. Finally, in Chapter 4 the results for stress distribution for a 6 term solution of the present analysis for an isotropic square plate subjected to parabolic loading at the two ends was compared with that from a 3 term energy solution of Timoshenko's. Both tabular and graphical results for the comparison revealed a close agreement between the results for stresses throughout the plate.

The effect of orthotropy ratio, fiber angle, and aspect ratio on the normal stress distribution throughout plates made of three types of materials subjected to parabolic and cubic stresses at the ends was presented in Chapter 5. It was found that drastic changes in the orthotropy ratio of the material causes significant changes in the stress distribution throughout the plate. This change in the stress distribution pattern is more pronounced for the anisotropic cases than for the orthotropic cases (\( \theta = 0^\circ \) or
Also, stress distribution becomes more non-uniform for the anisotropic cases compared to the orthotropic ones. Also in Chapter 5, it was shown that stress distributions decay to the average stresses for plates with aspect ratios equal to twice their characteristic decay length.

The characteristic decay lengths for normal stress loading for materials with different types of orthotropy ratio was found. The method developed in this work was successfully used to find these lengths with reasonable accuracy. The results were compared to those found in the literature for a semi-infinite plate loaded at one end. It was found, as expected, that the characteristic decay lengths in this analysis are longer than the ones for the semi-infinite strip case. The normal stress distributions presented for different types of plates with aspect ratios equal to twice their characteristic decay length showed that normal stresses decay to 1% of their original values at the middle of the plate. Also, it was found that orthotropic plates with fiber angle of 0 degrees have the longest characteristic decay length, and this decay length is increased with increasing orthotropy ratio. For single layer composites it is found that increasing fiber angle decreases the characteristic decay length of the plate, which means that stresses decay in shorter distances away from the boundary of the plate. For self-equilibrating
normal or shear stresses applied at the two ends of a composite plate, it was found that the form of decay for normal stresses and shear stresses are not the same, and that shear stresses change sign and reach a minimum value along the length of the plate before they decay to zero.

Following is a list of the specific items which the author believes he has contributed to technical knowledge and understanding, which he believes were not previously known:

1. The method of analysis developed in this work has demonstrated that problems in anisotropic elasticity can be solved by a simpler method compared with finite elements, finite differences or the ones that were mentioned in the literature as being used in the past.

2. The method has paved the way for future researchers to utilize it for more complicated problems in anisotropic elasticity, with confidence that the results obtained will be accurate ones to rely on.

3. The method can be addressed in composite material texts and classroom lectures without a knowledge in applied mathematics numerical methods such as finite elements or finite differences being required, which
themselves would have to be learned before attempting to solve problems in anisotropic elasticity.

4. The results obtained in this work for stresses and displacements, which the author has not been able to find in the past literature, are significant in understanding of stress behavior of composite plates under non-uniform loading, and they could be used as a reference for future research.

5. The graphical results obtained in this work for normal stresses of composite plates for different materials and various fiber angles under two different non-uniform loading conditions helps the researchers and designers of composite materials to better understand the behavior of composite plates and take these variables into consideration in their research or design.

6. No results for the decay of stresses due to self-equilibrating boundary shear stress have previously appeared in the published literature. The results presented here alert designers about the different behavior of shear stresses in composite plates compared to normal stress distribution.

7. The characteristic decay lengths obtained in this work for different composite plates under uniaxial normal and shear stresses adds to the list of
information the researchers and designers have so far in the field of composite plates.

8. All of the results in this work are organized and obtained in enough detail so that they could be used in graduate level composite material or anisotropic elasticity classrooms for students to achieve better understanding of composite and anisotropic materials.

For future work, orthogonal polynomials could be used for the assumed displacements instead of ordinary ones. This will eliminate the ill-conditioning phenomenon which occurred in some cases in the present work. The results presented in this analysis are only for single layered composites, one could use the formulation presented in this work for laminates which have more than single layer. One could also generalize the analysis to be applicable to non-symmetric laminates.
REFERENCES


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APPENDIX A

A.1 Convergence Study

Table A.1 Nondimensional stress resultant \( (N_r/2T) \) convergence for a square \((a/b=1)\) single layered laminated composite plate \((E_{glass}/epoxy(LS))\ \(E_1/E_2=2.45\) with 0 degree fiber angle subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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Table A.2  Nondimensional stress resultant \( (N_x/2T) \) convergence for a square \((a/b=1)\) single layered laminated composite plate \((E\text{-}glass/epoxy(LS) \ E_1/E_2=2.45)\) with 30 degree fiber angle subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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Table A.3  Nondimensional stress resultant \( (N_t/2T) \) convergence for a square \((a/b=1)\) single layered laminated composite plate \((E\text{-glass}/\text{epoxy}(LS) \ E_1/E_2=2.45)\) with 90 degree fiber angle subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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Table A.4 Nondimensional stress resultant \( (N_x/2T) \) convergence for a square \((a/b=1)\) single layered laminated composite plate \((\text{graphite/epoxy (GE)} \ E_1/E_2=15.4)\) with 0 degree fiber angle subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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Table A.5 Nondimensional stress resultant \((N_i/2T)\) convergence for a square \((a/b=1)\) single layered laminated composite plate (graphite/epoxy (GE) \(E_1/E_2=15.4\)) with 90 degree fiber angle subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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Table A.6 Nondimensional stress resultant \( \left( \frac{N_x}{2T} \right) \) convergence for an isotropic square \( (a/b=1) \) plate (Aluminum Alloy, AA) subjected to uniaxial self-equilibrating parabolic stress at the boundaries.

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