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CONTRIBUTIONS TO REGULARIZED DETERMINANTS OF ELLIPTIC OPERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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1995

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To my mother and father
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CHAPTER I

Introduction

There are two extensions of the concept of determinant of an endomorphism $\alpha$ of a finite dimensional vector space $V$ to an endomorphism of an infinite dimensional vector space.

The first extension is to endomorphism $\alpha$ of a separable Hilbert space $H$ of the form $\alpha = Id + F$, where $F$ is a trace class operator, and is called the Fredholm determinant of $\alpha$.

The second is to elliptic pseudodifferential operator of positive order in a vector bundle over a smooth manifold having Agmon angle, and is called the regularized determinant. In particular, a self-adjoint elliptic $\Psi$DO of positive order has Agmon angles and therefore belongs to this class.

The regularized determinant of an elliptic $\Psi$DO has important applications in geometry, topology and mathematical physics and is the focus of this thesis.

Let us recall the definition of the Fredholm determinant. Let $H$ be a separable Hilbert space and let $F : H \rightarrow H$ be a linear map. Assume that $F$ is of trace class, i.e. $\sum_{\lambda \in \text{Spec}(F)} |\lambda| \leq tr((F^*F)^{1/2}) < \infty$. Then the Fredholm determinant $det_{F,}(I + F)$
is defined by

$$\det_{Fr}(I + F) = \prod_{\lambda \in \text{Spec}(F)} (1 + \lambda)$$

(1.1)

Since $1 + |\lambda| < e^{|\lambda|}$, (1.1) is well defined if $F$ is a trace class operator. It is well known (cf.[Si]) that for any two trace class operators $F_1, F_2$

$$\det_{Fr}((I + F_1)(I + F_2)) = \det_{Fr}(I + F_1) \cdot \det_{Fr}(I + F_2).$$

(1.2)

Let us describe the class of \(\Psi DO\)'s which have Agmon angles and recall the definition of the regularized determinant.

Let $M$ be a compact closed smooth manifold and let $E \rightarrow M$ be a smooth vector bundle of a finite rank. Let $A : \mathcal{C}^{\infty}(E) \rightarrow \mathcal{C}^{\infty}(E)$ be an invertible, elliptic $\Psi DO$ of order $k$, where $k$ is a positive real number. Assume that there is an angle $\theta'$ such that $\sigma_L(A)(x, \xi)$ does not have any eigenvalues in $\{re^{i\theta'} \in \mathbb{C} | r \geq 0\}$ for any $(x, \xi) \in T^*M - M$, where $\sigma_L(A)(x, \xi) : T^*M - M \rightarrow \text{End}(E)$ is the principal symbol of $A$. Such an angle $\theta'$ is called a principal angle.

Since $\sigma_L(A)(x, \xi)$ is a homogeneous polynomial of degree $k$ with respect to $\xi$, i.e. $\sigma_L(A)(x, t\xi) = t^k \sigma_L(A)(x, \xi)$ for $t > 0$, we can view $\sigma_L(A)(x, \xi)$ as a smooth section of the pull back of $\text{End}(E) \rightarrow M$ on $S^*M := \{(x, \xi) ||\xi|| = 1\}$, the cosphere bundle of $T^*M$. If $\sigma_L(A)(x, \xi)$ does not have any eigenvalues in $\{re^{i\theta'} \in \mathbb{C} | r \geq 0\}$ for all $(x, \xi) \in S^*M$, then from the compactness of $S^*M$, one can find a small $\delta' > 0$ such that for any $(x, \xi) \in S^*M$, $\sigma_L(A)(x, \xi)$ does not have any eigenvalues in $L_{[\theta' - \delta', \theta' + \delta']} := \{re^{i\phi} | \theta' - \delta' \leq \phi \leq \theta' + \delta', r \geq 0\}$. Hence for $(x, \xi) \in T^*M - M$, $\sigma_L(A)(x, \xi)$ does not have eigenvalues in $L_{[\theta' - \delta', \theta' + \delta']}$. 
Then it is well known (cf. [Sh]) that there exists $R > 0$ such that

$$\text{Spec}(A) \cap \{re^{i\varphi}|\theta' - \delta' \leq \varphi \leq \theta' + \delta', r \geq R\} = \emptyset.$$ 

Since $\text{Spec}(A)$ is discrete and $\{re^{i\varphi}|\theta' - \delta' \leq \varphi \leq \theta' + \delta', 0 \leq r \leq R\}$ is compact, for any $\epsilon > 0$ one can choose an angle $\theta$ and $\delta > 0$ such that $\text{Spec}(A) \cap L[\theta - \delta, \theta + \delta] = \emptyset$ and $L[\theta - \delta, \theta + \delta] \subset L[\theta - \epsilon, \theta + \epsilon]$. An angle $\theta$ which satisfies $\text{Spec}(A) \cap L[\theta - \delta, \theta + \delta] = \emptyset$ for small $\delta > 0$ is called an Agmon angle. Hence Agmon angles exist arbitrarily close to a given principal angle.

Let us choose a contour $\gamma$ enclosing all the eigenvalues of $A$, which lies in $\mathbb{C} - L[\theta - \delta, \theta + \delta]$,

$$\gamma = \{re^{i\theta}|\infty > \rho \geq \epsilon\} \cup \{ee^{i\varphi}|\theta \geq \theta \geq \theta - 2\pi\}$$

$$\cup \{pe^{i(\theta - 2\pi)}|\epsilon \leq \rho < \infty\}.$$ 

For any $z \in \mathbb{C}$ with $\text{Re}z < 0$,

$$A_z := \frac{1}{2\pi i} \int_{\gamma} \lambda^z (A - \lambda)^{-1}d\lambda$$

is well defined and satisfies

(i) $A_z A_w = A_{z+w}$ for $\text{Re}z < 0$, $\text{Re}w < 0$ and (ii) $A_{-k} = A^{-k}$ for $k \in \mathbb{Z}^+$.

Notice that

$$A^z := A^k \cdot A_{z-k} \text{ for } (\text{Re}z) - k < 0, k \in \mathbb{Z}^+$$

does not depend on the choice of $k$ and it satisfies (cf.[Se1],[Sh]):

(i) for $\text{Re}z < 0$, $A^z = A_z$

(ii) for any $z, w \in \mathbb{C}$, $A^{z+w} = A^z \cdot A^w$
(iii) $A^1 = A$, $A^0 = Id$ and $A^{-1}$ is the inverse of $A$.

Of course, $A^s$ depends on the choice of an Agmon angle $\theta$.

Define $\zeta_{A,\theta}(s) := tr A^{-s}$. Then for $Res > \frac{\text{dim}(M)}{\text{deg}(A)}$, $\zeta_{A,\theta}(s)$ is holomorphic in $s$ and it has a meromorphic continuation to the whole complex plane with at most simple poles and with 0 as a regular value. The simple poles occur at $(-s)\cdot (\text{ord}A) = -(\text{dim}M) + j$, $j = 0, 1, 2, \cdots$, i.e. $s = \frac{\text{dim}(M) - j}{\text{ord}A}$, $j = 0, 1, 2, \cdots$. The residues at these poles are computable in terms of symbols of $A$ (cf. [Sel],[Sh],[Ka]). If $A$ is a differential operator, then the residue of $\zeta_{A,\theta}(s)$ at $s = 0, -1, -2, \cdots$ is zero.

The regularized determinant is defined by

$$Det_{\theta}(A) := e^{-\zeta_{A,\theta}(0)}.$$ \hspace{1cm} (1.5)

$Det_{\theta}(A)$ does depend on the choice of $\theta$ but it is locally constant in $\theta$.

The regularized determinant was introduced by Ray and Singer in [RS] in order to define the analytic torsion. The regularized determinant does not satisfy the property (1.2) (cf. [KV]).

Let $Q : C^\infty(E) \rightarrow C^\infty(E)$ be a smoothing operator such that $I + Q$ is invertible. Since $Q$ is of trace class as a map from $L^2(E)$ to $L^2(E)$, $det_{Fr}(I + Q)$ is well defined. $I + Q$ can be also viewed as an elliptic $\Psi$DO of order 0 and therefore its composition with any other $\Psi$DO of positive order $k$ is again a $\Psi$DO of order $k$. The following is a well known theorem (cf. [KV]) which provides the exact relationship between the two extensions of determinants.

**Theorem 1.1** Let $A : C^\infty(E) \rightarrow C^\infty(E)$ be an invertible $\Psi$DO of order $k$ with $k$ positive real number and $Q : C^\infty(E) \rightarrow C^\infty(E)$ be a smoothing operator with $I + Q$
invertible. Then \( (I + Q)A \) is a \( \Psi DO \) of order \( k \). Assume that \( (I + Q)^s A \) for \( 0 \leq s \leq 1 \) have \( \theta \) as an Agmon angle. Then

\[
Det_\theta((I + Q)A) = Det_\theta(A) \cdot det_{Fr}(I + Q). \quad (1.6)
\]

The proof will be given in Appendix A.

The regularized determinant is a spectral invariant, i.e. two operators with the same spectrum have the same regularized determinants with respect to the same Agmon angle. Contrary to other spectral invariants (for instance, the residues of the zeta function \( \zeta_{A,\theta}(s) \) at the simple poles), the regularized determinant cannot be calculated as a "local quantity", i.e. the integral of a density on a manifold depending on the terms of the "asymptotic symbol". This can be actually seen from the above theorem since the asymptotic symbol of \( A \) and \( (I + Q)A \) are the same.

On the other hand, Burghelea, Friedlander and Kappeler proved in [BFK1] that if \( A(\lambda) \) is an invertible elliptic \( \Psi DO \) with parameter of weight \( \chi > 0 \) (Definition 2.1.1) and has the angle \( \theta \) as an Agmon angle for each \( \lambda \), then \( \log Det_\theta A(\lambda) \) has an asymptotic expansion for \( |\lambda| \to \infty \) of the form:

\[
\log Det_\theta A(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-\frac{j}{d}} + \sum_{j=0}^{d} q_j |\lambda|^{\frac{j}{d}} \log |\lambda|, \quad (1.7)
\]

where \( \pi_j \) and \( q_j \) are local quantities which can be computed in terms of the asymptotic symbol of \( A(\lambda) \) and \( \frac{\lambda}{|\lambda|} \).

In Chapter 2, we will include the proof of (1.7) which was originally proved in the Appendix of [BFK1]. In spite of explicit formulas for \( \pi_i \) and \( q_j \) which can be calculated in principle, the geometric interpretation of \( \pi_i \) and \( q_j \) is not well understood. The
results in Chapter 4 are contributions toward the understanding of the meaning of the coefficients $\pi_i$ and $q_j$.

We show (in Chapter 4) that for an operator elliptic with parameter $A(\lambda)$, one can associate an elliptic $\Psi DO P$, unique up to smoothing operators, and verify that $\pi_1$ is exactly the value at zero of the zeta function of $P$ multiplied by $\frac{m}{2}$, where $m$ is the order of the operator $A(\lambda)$ (Theorem 4.2.1).

The regularized determinant can be extended to be defined for an elliptic $\Psi DO$ on a compact manifold with boundary with an elliptic boundary value problem $B$ once the pair $(A, B)$ has $\theta$ as an Agmon angle. The determinant of an elliptic boundary value problem is not only a generalization but also a useful tool in calculating regularized determinant of elliptic operators on a closed manifold. We will be concerned in this work with Dirichlet and Neumann boundary conditions for self-adjoint positive definite operators of second order. For this type of elliptic boundary value conditions the pair $(A, B)$ has always Agmon angles. In fact, any angle $\neq 0$ is an Agmon angle.

Let $(M, g)$ be a closed oriented Riemannian manifold of dimension $d$ and $\Gamma$ be a submanifold of $M$ of codimension 1. Let $M_\Gamma$ be a compact manifold with boundary obtained by cutting $M$ along $\Gamma$ and denote by $p : M_\Gamma \to M$ the identification map. Let $E \to M$ be a smooth vector bundle and $E_\Gamma$ be a pull back of $E \to M$ to $M_\Gamma$ by $p$.

Given an elliptic differential operator $A$ on the bundle $E \to M$ and the complementary elliptic boundary value problems $B$ and $C$ along $\Gamma$, Burghelea, Friedlander and Kappeler have defined an elliptic $\Psi DO R$ on $\Gamma$ depending on $A, B, C$ and $\Gamma$. 
whose asymptotic symbol can be calculated in terms of the symbols of $A$, $B$, $C$ along $\Gamma$. They then proved the Mayer-Vietoris formula relating to the regularized determinants of $A$, $(A_\Gamma, B)$ and $R$.

This formula was applied to $A$ the Laplace type operator on forms and $B$, $C$ Dirichlet, Neumann boundary conditions respectively to get a new proof for the equality of analytic and Reidemeister torsion (cf. [BFK2]).

In Chapter 3, we will give a short-cut proof of this formula for an elliptic, essentially self-adjoint, positive definite differential operator $A$ of Laplace-Beltrami type, which states that

$$Det_\pi(A) = c \cdot Det_\pi(A_\Gamma, B) \cdot Det_\pi(R).$$

In this formula, $c$ can be explicitly computed in terms of the symbol of $A$ in a neighborhood of $\Gamma$ and $A_\Gamma$ is the extension of $A$ to $E_\Gamma \to M_\Gamma$.

In particular, if $(M, g)$ is a closed orientable Riemannian manifold of even dimension i.e. $d = 2n$, we will show that the constant $c$ in (1.8) is 1 (Theorem 3.3.3). Furthermore when $E = \bigwedge^q T^*M$ for $0 \leq q \leq 2n$ and $A = \Delta_q + \epsilon Id$ for $\epsilon > 0$, then we will also give a modified version of the formula (1.8) for the Laplace operator $\Delta_q$ which has zero modes by letting $\epsilon \to 0$ (Theorem 3.4.1). In Appendix B an alternative way of calculating the symbol of the operator $R$ is presented based on ideas of Gel'fand and Friedlander (unpublished).

In topology, the regularized determinants appear in the analytic definitions of various types of torsions. Here is an interesting situation.

Let $(M, g)$ be a closed oriented Riemannian manifold and let $\varphi : M \to M$ be an
orientation preserving diffeomorphism. The mapping torus $M_\varphi$ is defined by $M_\varphi = M \times I/(x, 1) \sim (\varphi(x), 0)$. Then $M_\varphi$ is a fiber bundle over $S^1$ with fiber $M$ denoted by $\pi : M_\varphi \to S^1$. Let $d\theta$ be a canonical 1-form on $S^1$. Now let $g_1$ be a Riemannian metric on $M_\varphi$ and define

$$d_q(t) : \Omega^q(M_\varphi) \to \Omega^{q+1}(M_\varphi) \text{ by } d_q(t) = d_q + t\pi^*d\theta \wedge$$

(1.9)

and

$$\Delta^t_{q}^{\pi^*d\theta} = \tilde{d}_q^*(t)\tilde{d}_q(t) - d_{q-1}(t)\tilde{d}_{q-1}^*(t), \quad (1.10)$$

where $d_q$ is an exterior derivative and $\tilde{d}_q^*$ is the adjoint of $\tilde{d}_q$. We define the torsion function $T_0(M, \varphi, g)(t)$ by

$$T_0(M, \varphi, g)(t) = \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} q \cdot \log \text{Det}_\pi(\Delta^t_{q}^{\pi^*d\theta}). \quad (1.11)$$

It is not hard to see that for $t$ away from the eigenvalues of $\varphi^* : H^*(M, \mathbb{R}) \to H^*(M, \mathbb{R})$ the homomorphism induced in cohomology by $\varphi$, the quantity $T_0(M, \varphi, g)(t)$ is independent of $g$. It is shown by J. Marcsik in [Ma] (also see [Mil]) that it can be expressed with the help of the Alexander polynomial, a rational function which is completely determined by the eigenvalues of $\varphi^*$ or equivalently by the “number” of periodic points of $\varphi$.

In particular, $T(M, \varphi)(t) := \frac{1}{2} (T(M, \varphi, g_1)(t) + T_0(M, \varphi^{-1}, g_2)(t))$ has an asymptotic expansion of the form $\frac{1}{2} \cdot \chi(M) \cdot t$ when $t \to \infty$, where $\chi(M)$ is the Euler characteristic of $M$ and $g_1, g_2$ are Riemannian metrics on $M_\varphi$ and $M_{\varphi^{-1}}$ respectively. To establish this result, at present one uses both analysis and topology and so far no entirely analytic proof has been found in literature.
In Chapter 5, we will give an analytic proof for the asymptotic formula \( T(M, \varphi)(t) \sim \frac{1}{2} \cdot \chi(M) \cdot t \) based on (1.7).
CHAPTER II

The Asymptotics of logDet of an Elliptic Pseudodifferential Operator with parameter

In this chapter, we will include the complete proof of the theorem of the asymptotic expansion of logDet of a classical elliptic PDO with parameter, which was proved in the appendix of [BFK1].

2.1 Statement of Theorem on the Asymptotics of logDet

Let V be an open angle in the complex λ-plane and P(λ), λ ∈ V, a family of PDO’s of order m, m a positive integer, acting on smooth sections of a vector bundle E → M of rank ν, where M denotes a closed smooth Riemannian manifold of dimension d.

**Definition 2.1.1** (cf. [Sh]) The family P(λ), λ ∈ V, is said to be a PDO with parameter of weight χ > 0 if in any coordinate neighborhood U of M, not necessarily connected, and for an arbitrarily fixed λ ∈ V, the complete symbol p(λ; x, ξ) of P is in $C^∞(U × \mathbb{R}^d, End(\mathbb{C}^ν))$ and, moreover, for any multiindices α and β, there exists a constant $C_{α,β}$ such that $|∂^α_ξ∂^β_x p(λ; x, ξ)| ≤ C_{α,β}(1 + |ξ| + |λ|^{1/χ})^{m-|α|}$.

**Definition 2.1.2** P(λ) is called classical if in any chart the complete symbol $p(λ; x, ξ)$
\( p(\lambda; x, \xi) \) admits an expansion of the form

\[
p(\lambda; x, \xi) \sim p_m(\lambda; x, \xi) + p_{m-1}(\lambda; x, \xi) + \cdots
\]  

(2.1)

where \( p_j(\tau^x\lambda; x, \tau\xi) = \tau^j p_j(\lambda; x, \xi) (\tau > 0, j \leq m) \). The family \( P(\lambda) \) is said to be elliptic with parameter if \( p_m(\lambda; x, \xi) \) is invertible for all \( x \in M, \xi \in T^*_x(M) \) and \( \lambda \in V \) satisfying \(|\xi| + |\lambda|^{\frac{1}{x}} \neq 0\).

**Definition 2.1.3** Let \( Q \) be an elliptic \( \Psi \text{DO} \). The angle \( \theta \) is called an Agmon angle for \( Q \) if for some \( \epsilon > 0 \), \( \text{spec}(Q) \cap \Lambda_{\theta, \epsilon} = \emptyset \), where \( \text{spec}(Q) \) denotes the spectrum of \( Q \) and \( \Lambda_{\theta, \epsilon} = \{ z \in \mathbb{C} \mid \theta - \epsilon < \arg(z) < \theta + \epsilon \} \) or \( |z| < \epsilon \).

**Remark** If a \( \Psi \text{DO} Q \) does not have any eigenvalues in \( \Lambda_{\theta, \epsilon} \), then it can be shown that \( \sigma_L, \) the principal symbol of \( Q \), does not have any eigenvalues in \( \Lambda_{\theta, \frac{\epsilon}{2}} \), and thus this definition is equivalent to the definition of an Agmon angle given in Introduction.

**Theorem 2.1.4** Let \( P(\lambda) \) be a classical \( \Psi \text{DO} \) of order \( m \in N \) with parameter \( \lambda \in V \) of weight \( \chi > 0 \) such that

(i) \( P(\lambda) \) is elliptic with parameter and

(ii) for each \( \lambda \in V, \ P(\lambda) \) has \( \theta \) as an Agmon angle.

Then \( \log \text{Det}_\theta P(\lambda) \) admits an asymptotic expansion for \( \lambda \in V, \ |\lambda| \to \infty, \) of the form

\[
\log \text{Det}_\theta P(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-\frac{j}{x}} + \sum_{j=0}^{d} q_j |\lambda|^{\frac{j}{x}} \log |\lambda|. \quad (2.2)
\]

The coefficients \( \pi_j \) and \( q_j \) can be evaluated in terms of the symbol of \( P(\lambda) \) and \( \frac{\lambda}{|\lambda|} \). In particular, \( \pi_0 \) is independent of perturbations by lower order operators, whose orders differ at least by \( d + 1 \) from the order of \( P(\lambda) \).
2.2 Proof of Theorem 2.1.4

We divide the proof into several steps. First to make writing easier, we assume that \( \theta = \pi \).

**Step 1** By a standard procedure we construct a parametrix for
\[
R(\mu, \lambda) = (\mu - P(\lambda))^{-1}(\mu \leq 0).
\]

**Step 2** Define \( R_N(\mu, \lambda) \) to be a conveniently chosen approximation of \( R(\mu, \lambda) \) and write \( P(\lambda)^{-s} = P_N(\lambda; s) + P_N(\lambda; s) \), where \( P_N(\lambda; s) = \frac{1}{2\pi i} \int_{\gamma} \mu^{-s} R_N(\mu, \lambda) d\mu \) and where \( \gamma \) denotes a contour around the negative axis, enclosing the origin in clockwise orientation. Then for \( s \in \mathbb{C} \) with \( \text{Res} \) sufficiently large, \( \zeta(s) = \zeta_N(s) + \tilde{\zeta}_N(s) \), where \( \zeta(s) = \text{tr} P(\lambda)^{-s}, \zeta_N(s) = \text{tr} P_N(\lambda, s), \) and \( \tilde{\zeta}_N(s) = \text{tr} \tilde{P}_N(\lambda, s) \).

**Step 3** Describe an asymptotic expansion of \( \frac{d}{ds} \mid_{s=0} \zeta_N(\lambda, s) \) as \( \lambda \to \infty \).

**Step 4** Provide an estimate for the remainder term \( \frac{d}{ds} \mid_{s=0} \tilde{\zeta}_N(\lambda, s) \) as \( \lambda \to \infty \).

**Step 5** Provide a formula for \( \pi_0 \).

**Step 1** We want to construct a parametrix for \( R(\mu, \lambda) = (\mu - P(\lambda))^{-1} \) \((\mu \leq 0)\). Consider the equation \( (\mu - p(\lambda; x, \xi)) \circ r(\mu, \lambda; x, \xi) = Id, \) where \( \circ \) denotes multiplication in the algebra of symbols.

Introduce, for \( \alpha = (\alpha_1, \ldots, \alpha_d) \), the standard notation \( \alpha! = \alpha_1! \cdots \alpha_d! \), \( \partial_\xi^\alpha = (\frac{\partial}{\partial \xi_1})^{\alpha_1} \cdots (\frac{\partial}{\partial \xi_d})^{\alpha_d} \) and \( D_\xi^\alpha = (\frac{1}{i})^\alpha \partial^\alpha \xi \).

Write \( r(\mu, \lambda; x, \xi) \sim r_{-m}(\mu, \lambda; x, \xi) + r_{-m-1}(\mu, \lambda; x, \xi) + \cdots \), where \( r_{-j}(\mu, \lambda; x, \xi) \) is positive homogeneous of degree \(-j\) in \((\xi, \mu^\frac{1}{m}, \lambda^\frac{1}{\lambda})\). Then we obtain the following formula:

\[
r_{-m}(\mu, \lambda; x, \xi) = (\mu - p_m(\lambda; x, \xi))^{-1}
\] \hspace{1cm} (2.3)
The functions $r_j(n, A; x, \xi)$ satisfy the following homogeneity condition

$$
(2.4) \quad r_{-m-j}(n, A; x, \xi) = -(\mu - p_m(\lambda; x, \xi))^{-1} \sum_{k=0}^{j-1} \sum_{|\alpha|+\ell+k=j} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-\ell}(\lambda; x, \xi) D_x^{\alpha} r_{-m-k}(\mu, \lambda; x, \xi).
$$

The functions $r_j(\mu, \lambda; x, \xi)$ satisfy the following homogeneity condition

$$
(2.5) \quad r_j(r^m \mu, r^x \lambda; x, r\xi) = r^j r_j(\mu, \lambda; x, \xi) \quad (r > 0).
$$

By a standard procedure, $\sum_{j \geq 0} r_{-m-j}(\mu, \lambda; x, \xi)$ gives rise to a $\Psi$DO with parameter, called a parametrix for $R(\mu, \lambda)$.

**Step 2** Introduce a finite cover $(U_j)$ of $M$ by open charts and take a partition of unity $\varphi_j$, subordinate to $U_j$. Choose $\psi_j \in C^\infty_0(U_j)$ such that $\psi_j \equiv 1$ in some neighborhood of $\text{supp}\varphi_j$. Let us fix local coordinates in every $U_j$ and define the operators

$$(R_{(N)}^{(j)}(\mu, \lambda, f))(x) = \psi_j(x) \cdot \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dy \left( r^{(N)}(\mu, \lambda; x, \xi)e^{i(x-y)}\xi \varphi_j(y) f(y) \right), \quad (2.6)$$

where $r^{(N)}(\mu, \lambda; x, \xi) = \sum_{j=0}^{N-1} r_{-m-j}(\mu, \lambda; x, \xi)$ in the local coordinates of $U_j$. The approximation $R_N(\mu, \lambda)$ of the resolvent $R(\mu, \lambda)$ is defined by $R_N(\mu, \lambda) = \sum_j R_{(N)}^{(j)}(\mu, \lambda)$. We need an estimate of $R(\mu, \lambda) - R_N(\mu, \lambda)$ in trace norm. The latter is denoted by $\| \cdot \|$.

**Lemma 2.2.1** Choose $N > \frac{3d}{2} + m$. Then for $\lambda \in V_1$ and $\mu \in \mathbb{R}^-$ with $|\mu|$ sufficiently large

$$
||| R(\mu, \lambda) - R_N(\mu, \lambda) ||| < C_N (1 + |\lambda|)^{(N - \frac{3d}{2} - m)} (1 + |\lambda|)^{-2}, \quad (2.7)
$$

where $V_1$ is an angle whose closure is contained in $V$, $V_1 \ll V$. 

Proof Define $T_N(\mu, \lambda)$ by $(\mu - P(\lambda))R_N(\mu, \lambda) = Id - T_N(\mu, \lambda)$. From $(\mu - P(\lambda))R(\mu, \lambda) = Id$, we then conclude that $R(\mu, \lambda) - R_N(\mu, \lambda) = R(\mu, \lambda)T_N(\mu, \lambda)$. The claimed estimate of the lemma follows, once we have proved that for some $\tau > d$

$$\| R(\mu, \lambda) \|_{L^2 \to L^2} \leq C(1 + |\mu|)^{-1}.$$  \hfill (2.8)

$(\lambda \in \mathbb{V}_1 \ll V, \mu \in \mathbb{R}^-, |\mu| \text{ sufficiently large})$ and

$$\| T_N(\mu, \lambda) \|_{L^2 \to H^\tau} \leq C_\tau(1 + |\lambda|^{\frac{1}{2}} + |\mu|^{\frac{1}{8}})^{-N + \tau}$$  \hfill (2.9)

because from (2.9) we can conclude that $T_N(\mu, \lambda)$ is a $\Psi$DO of order $-\tau < -d$ and hence of trace class, when considered as an operator on $L^2$-sections of $E \to M$. The estimate (2.8) is standard (cf. e.g. [Sh]) and (2.9) follows from the fact that the symbol $t_N(\mu, \lambda; x, \xi)$ of $T_N(\mu, \lambda)$ satisfies

$$| D_\alpha^x D_\beta^\xi t_N(\mu, \lambda; x, \xi) | \leq C_{\alpha\beta}(1 + |\xi| + |\lambda|^{\frac{1}{2}} + |\mu|^{\frac{1}{8}})^{-N - |\beta|}.$$  \hfill (2.10)

Thus the norm of $T_N(\mu, \lambda)$ as an operator from $H^s$ to $H^{s+\tau}$ is

$$O(1 + |\lambda|^{\frac{1}{2}} + |\mu|^{\frac{1}{8}})^{-N + \tau}.$$  

Therefore

$$\| T_N \| = O(1 + |\lambda|^{\frac{1}{2}} + |\mu|^{\frac{1}{8}})^{-N + \frac{3d}{2}}.$$  \hfill (2.11)

□

Step 3 Next we study the asymptotic expansion of $\frac{\partial}{\partial s} |_{s=0} \zeta_N(\lambda, s)$ as $\lambda \to +\infty$. Recall that $P_N(\lambda, s) = \frac{1}{2\pi i} \int d\mu \mu^{-s} R_N(\mu, \lambda)$. Its Schwarz kernel is given by

$$P_N(\lambda, s; x, y) = \sum_j \psi_j(x) \phi_j(y) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi e^{i(x-y) \cdot \xi} \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s} \sum_{k=0}^{N-1} r_{-m-k}. $$  \hfill (2.12)
As a consequence

$$P_N(\lambda, s; x, x) = \sum_j \varphi_j(x) \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} I_k(s, \lambda, x) = \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} I_k(s, \lambda, x),$$  \hspace{1cm} (2.13)

where $I_j(s, \lambda, x) = \frac{1}{2\pi i} \int_{\mathbb{R}_d} d\xi \frac{1}{\lambda} f_j d\mu \mu^{-s} r_{m-j}(\mu, \lambda, \xi)$. By the change of variables $\xi = |\lambda|^\frac{1}{n} \xi', \mu = |\lambda|^\frac{n}{m} \mu'$ and by using the homogeneity of $r_{m-j}$, we obtain

$$I_j(s, \lambda, x) = \frac{1}{2\pi i} |\lambda|^{\frac{d-n-s-j}{n}} \int_{\mathbb{R}_d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{m-j}(\mu, |\lambda|^{-1}; x, \xi).$$  \hspace{1cm} (2.14)

We need to investigate $J_k(s, \omega, x) := \frac{1}{2\pi i} \int_{\mathbb{R}_d} d\xi \int_{\gamma} d\mu \mu^{-s} r_{m-k}(\mu, |\lambda|^{-1}; x, \xi)$.

**Lemma 2.2.2** Let $\omega \in V$ with $|\omega| = 1$. Then $J_k(s, \omega; x)$ is holomorphic in $s$ in the half plane $\text{Res} > \frac{d-k}{m}$ and it admits a meromorphic continuation in the complex $s$-plane. The point $s = 0$ is always regular and $J_k(0, \omega; x) = 0$ if $k > d$.

**Proof** (i) Integrating by parts with respect to $\mu$, one obtains

$$J_k(s, \omega; x) = \int_{\mathbb{R}_d} d\xi \frac{1}{(1-s)\cdots(l-s)} \int_{\gamma} (-1)^td\mu \mu^{-s+l} \frac{\partial^l}{\partial \mu^l} r_{m-k}(\mu, \omega; x, \xi).$$  \hspace{1cm} (2.15)

If $l > \text{Res} - 1$, the contour integral reduces to

$$-\frac{\sin \pi s}{\pi} \int_0^\infty d\mu \mu^{-s+l} \left( \frac{\partial^l}{\partial \mu^l} r_{m-k}\right)(-\mu, \omega; x, \xi).$$

Further, the matrix $\frac{\partial^l}{\partial \mu^l} r_{m-k}$ can be estimated

$$|\frac{\partial^l}{\partial \mu^l} r_{m-k}| \leq C(1+|\mu|^\frac{n}{m} + |\xi|)^{-m-k-ml}.$$  \hspace{1cm} (2.16)

Thus, for $\text{Res} > \frac{d-k}{m}$, the integral

$$\int_{\mathbb{R}_d} d\xi \int_0^\infty d\mu \mu^{-s+l} \left( \frac{\partial^l}{\partial \mu^l} r_{m-k}\right)(-\mu, \omega; x, \xi)$$
converges absolutely and therefore is a holomorphic function in $s$. Moreover, $\frac{\sin \pi s}{\pi (1-s) \cdots (l-s)}$ is entire. In all, we have proved that $J_k(s, \lambda; x)$ is holomorphic in $\text{Re}s > \frac{d-k}{m}$.

(ii) Next let us prove that $J_k(s, \lambda; x)$ can be meromorphically continued to the entire complex $s$-plane. To keep the exposition simple let us assume that $P(\lambda)$ is a scalar $\Psi DO$. The expressions $r_{-m-k}(\mu, \omega; x, \xi)$ have been defined in a recursive fashion and are sums of terms of the form $(\mu - p_m(\omega; x, \xi))^{-l}q_{l,k}(\omega; x, \xi)$ with $l \geq 1$, where $\text{ord}(q_{l,k}) = -m - k + ml$ and $q_{l,k}$ is an expression, independent of $\mu$, involving only the symbols $p_{m-j}(\omega; x, \xi)$ and their derivatives with $0 \leq j \leq k$.

It follows from the recursive definition of the $r_{-m-k}$ that $l$ has to satisfy $l \geq k + 1$ and thus, in the case $k \geq 1$, $J_k$ consists of a sum of terms of the form

$$
\int_{\mathbb{R}^d} d\xi q_{l,k}(\omega; x, \xi) \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s}(\mu - p_m(\omega; x, \xi))^{-l} \quad (2.17)
$$

where after integration by parts, we used Cauchy’s formula. As $|\omega| = 1$, it follows from Definition 2.1.1 that the integrand $q_{l,k}(\omega; x, \xi)p_m(\omega; x, \xi)^{-s-l+1}$ is absolutely integrable in $|\xi| \leq 1$. Thus one only needs to consider the integral over $|\xi| > 1$. For $\omega$ fixed, the symbols $q_{l,k}$ and $p_m$ are classical and admit an asymptotic expansion in $\xi$-homogeneous functions. Consider two cases:

Case 1: $k = 0$.

Using that $r_{-m}(\mu, \omega; x, \xi) = (\mu - p_m(\omega; x, \xi))^{-1}$ we conclude that

$$J_0(s, \omega; x) = \int_{\mathbb{R}^d} d\xi \frac{1}{2\pi i} \int_{\gamma} d\mu \mu^{-s}(\mu - p_m(\omega; x, \xi))^{-1} = \int_{\mathbb{R}^d} d\xi (p_m(\omega; x, \xi))^{-s}. \quad (2.18)$$
Recall that $\omega$ with $|\omega| = 1$ is fixed and thus $p_m(\omega; x, \xi)$ defines an elliptic $\Psi DO$ $P_m(\omega; x, D)$ and we can apply the standard theory of complex powers of elliptic operators (cf. e.g. [Se1]) to conclude that $\int_{\mathbb{R}^d} d\xi p_m(\omega; x, \xi)^{-s}$ has a meromorphic continuation in the whole complex $s$-plane, with at most simple poles and that $s = 0$ is a regular point. The poles are located at $s_j = \frac{d-j}{m}$ with $j \in \{0, 1, 2, \ldots\} \setminus \{d\}$.

Case 2: $k \geq 1$.

As it was observed by Guillemin [Gu] and Wodzicki [Wo2] in the context of non-commutative residues, $\int_{\mathbb{R}^d} q_{l,k}(\omega; x, \xi)(p_m(\omega; x, \xi))^{-s-l+1} d\xi$ admits a meromorphic continuation to the whole complex $s$-plane with at most simple poles. Thus $s \cdot \int_{\mathbb{R}^d} d\xi q_{l,k}(\omega; x, \xi)(p_m(\omega; x, \xi))^{-s-l+1}$ must be regular at $s = 0$. This shows that $J_k(s, \omega; x)(k \geq 1)$ is meromorphic and that $s = 0$ is a regular point.

(iii) Let $k > d + 1 - m$. Observe that

$$|r_{m-k}(\mu, \omega; x, \xi)| \leq C_k(1 + |\mu|^\frac{1}{m} + |\xi|)^{-m-k}. \quad (2.19)$$

As $m \geq 1$, the integral $J_k(s, \omega; x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_{\gamma} d\mu \mu^{-s-r_{m-k}(\mu, \omega; x, \xi)}$ converges absolutely at $s = 0$. Evaluating at $s = 0$, one obtains $\int_{\gamma} d\mu r_{m-k}(\mu, \omega; x, \xi) = 0$ and thus $J_k(0, \omega; x) = 0$ for $k > d$. □

By the above lemma, we see that

$$P_N(\lambda, s; x, x) = \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} |\lambda|^\frac{(d-2s-k)}{x} J_k(s, \frac{\lambda}{|\lambda|}; x). \quad (2.20)$$

Hence, with $N^* = \min(N - 1, d),

$$\frac{\partial}{\partial s} \bigg|_{s=0} P_N(\lambda, s; x, x) = \frac{1}{(2\pi)^d} \sum_{k=0}^{N-1} |\lambda|^\frac{d-k}{x} \frac{\partial}{\partial s} J_k(s, \frac{\lambda}{|\lambda|}; x) \bigg|_{s=0} -$$
Step 4 We have to estimate $\text{tr} \tilde{P}_N(\lambda, s)$, where

$$\tilde{P}_N(\lambda, s) = P(\lambda)^{-s} - P_N(\lambda, s) = \frac{1}{2\pi i} \int_\gamma d\mu \mu^{-s} (R(\mu, \lambda) - R_N(\mu, \lambda)).$$ (2.22)

The estimate of Lemma 2.2.1 implies that

$$| \text{tr} \frac{\partial}{\partial s} \tilde{P}_N(\lambda, s) |_{s=0} \leq C(1 + |\lambda|) \int_0^\infty d\mu \cdot \frac{|\log \mu|}{1 + |\mu|^2}$$ (2.23)

and thus the asymptotic expansion given in Theorem 2.1.4 is proved.

Step 5 In the notation introduced above, we obtain the following formula for $\pi_0$:

$$\pi_0 = \sum_j \frac{\partial}{\partial s} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dvol(x) J_d(s, \lambda, x; \varphi_j(x)) |_{s=0},$$ (2.24)

where $J_d(s, \lambda, x) = \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \int_\gamma d\mu \mu^{-s} r_{-m-d}^{(\mu, \lambda; x, \xi)}$.

As $r_{-m-d}^{(\mu, \lambda; x, \xi)}$ is defined recursively by ($j \geq 1$)

$$r_{-m-j}^{(\mu, \lambda; x, \xi)} =$$ (2.25)

$$-(\mu - p_{m-j}^{(\lambda; x, \xi)})^{-1} \sum_{k=0}^{j-1} \sum_{|\alpha|+k=j} \frac{1}{\alpha!} \partial^\alpha \varphi_j^{(p_{m-j}^{(\lambda; x, \xi)})} D_{x_{m-k}} r_{-m-k}^{(\mu, \lambda; x, \xi)},$$

we conclude that $\pi_0$ only depends on $p_{m-j}^{(\lambda; x, \xi)}$ for $0 \leq j \leq d$ and its derivatives up to order $d$. \(\square\)

The following result is due to Voros [Vo] and Friedlander [Fr], which will be used in Chapter 3 and Chapter 4. We include Voros' proof.

**Proposition 2.2.3** Let $\{\lambda_k\}_{k \geq 1}$ be a sequence in $V_{0, \frac{\pi}{2}} = \{z \in \mathbb{C} | -\frac{\pi}{2} + \epsilon < \arg(z) < \frac{\pi}{2} - \epsilon\}$, possibly with multiplicities, arranged in such a way that $0 < \text{Re}\lambda_1 \leq \text{Re}\lambda_2 \leq \ldots$. Then

$$\frac{1}{\chi(2\pi)^d} \sum_{k=0}^{N^*} |\lambda|^{\frac{d+k}{2}} \log |\lambda| \cdot J_k(0, \lambda; x).$$ (2.21)
Assume that the heat trace \( \theta(t) := \sum_{k=0}^{\infty} e^{-t\lambda_k} (t > 0) \) admits an asymptotic expansion for \( t \to 0 \) of the form
\[
\theta(t) \sim \sum_{n \geq 0} c_n t^{i_n},
\]
where \( i_0 < 0 \) and \( i_0 < i_1 < i_2 < \cdots \to +\infty \). For \( \lambda \) with \( \text{Re}\lambda > 0 \), let \( \zeta(s, \lambda) = \sum_{k=0}^{\infty} (\lambda_k + \lambda)^{-s} \). Then \( \pi_0 = 0 \), where \( \pi_0 \) is the constant term in the asymptotic expansion of \( -\frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \) for \( |\lambda| \to +\infty \).

**Proof** It is well known that for \( \text{Res} > -i_0, \zeta(s, \lambda) \) is well defined and \( \zeta(s, 0) = \frac{1}{\Gamma(s)} \int_0^\infty \theta(t) \cdot t^{s-1} dt \). Let
\[
\eta(s, \lambda) = \int_0^\infty \sum_{k=0}^{\infty} e^{-\lambda_k t} e^{-\lambda t^{s-1}} dt.
\]
Then \( \zeta(s, \lambda) = \frac{1}{\Gamma(s)} \eta(s, \lambda) \). For \( \text{Res} > -i_0, \eta(s, \lambda) \) can be expanded in \( \lambda \) for \( |\lambda| \to \infty \)
\[
\eta(s, \lambda) \sim \sum_{n \geq 0} c_n \lambda^{-s-i_n} \int_0^\infty t^{i_n+s-1} e^{-t} dt = \sum_{n \geq 0} c_n \Gamma(s+i_n) \lambda^{-s-i_n}.
\]
Thus
\[
\zeta(s, \lambda) = \frac{1}{\Gamma(s)} \eta(s, \lambda) \sim \frac{1}{\Gamma(s)} \lambda^{-s} \sum_{n \geq 0} c_n \Gamma(s+i_n) \lambda^{-i_n}.
\]
All functions involved are meromorphic functions of \( s \). Moreover \( s = 0 \) is a regular point of \( \zeta(s, \lambda) \) and thus \( \frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \) admits an asymptotic expansion in \( \lambda \) of the form
\[
\frac{d}{ds} \zeta(s, \lambda) \big|_{s=0} \sim \sum_{i_n \notin \mathbb{Z}^- \cup \{0\}} c_n \Gamma(i_n) \lambda^{-i_n} + \sum_{i_n \in \mathbb{Z}^-} \frac{c_n}{d \Gamma(s+i_n) \cdots (s-1)} \big|_{s=0} \lambda^{-i_n} - \sum_{i_n \in \mathbb{Z}^- \cup \{0\}} \frac{c_n}{(i_n) \cdots (-1)} \lambda^{-i_n} \log \lambda,
\]
where \( \mathbb{Z}^- \) is the set of all negative integers. This expansion shows that \( \pi_0 = 0 \). \( \square \)
CHAPTER III

Mayer-Vietoris Formula for Determinants of Laplace-Beltrami Type Operators

3.1 Statement of Mayer-Vietoris Formula for Determinants

Let \((M, g)\) be a closed oriented Riemannian manifold of dimension \(d\) and \(\Gamma\) be an oriented submanifold of codimension 1. We denote by \(\nu\) the unit normal vector field along \(\Gamma\). Let \(M_{\Gamma}\) be the compact manifold with boundary \(\Gamma^+ \sqcup \Gamma^-\) obtained by cutting \(M\) along \(\Gamma\), where \(\Gamma^+\) and \(\Gamma^-\) are copies of \(\Gamma\) and denote by \(p : M_{\Gamma} \to M\) the identification map. The vector field \(\nu\) has the lift on \(M_{\Gamma}\) which we denote by \(\nu\) again. Denote by \(\Gamma^+\) the component of the boundary where the lift of \(\nu\) points outward. Given a smooth vector bundle \(E \to M\), denote by \(E_{\Gamma}\) the pull back of \(E \to M\) to \(M_{\Gamma}\) by \(p\). Let \(A : C^\infty(E) \to C^\infty(E)\) be an elliptic, essentially self-adjoint (Definition 3.2.1), positive definite differential operator of Laplace-Beltrami type, where we say that \(A\) is of Laplace-Beltrami type if \(A\) is an operator of order 2 whose principal symbol is \(\sigma_L(x, \xi) = \|\xi\|^2 Id_x, \text{Id}_x \in \text{End}_x(E_x, E_x)\). We denote by \(A_{\Gamma} : C^\infty(E_{\Gamma}) \to C^\infty(E_{\Gamma})\) the extension of \(A\) to smooth sections of \(E_{\Gamma}\).

Consider Dirichlet and Neumann boundary conditions \(B, C\) on \(\Gamma^+ \sqcup \Gamma^-\) defined
as follows:

\[ B : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma \mid_{\Gamma^+ \cup \Gamma^-}), B(f) = f \mid_{\Gamma^+ \cup \Gamma^-} \quad (3.1) \]

\[ C : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma \mid_{\Gamma^+ \cup \Gamma^-}), C(f) = \nu(f) \mid_{\Gamma^+ \cup \Gamma^-} \quad (3.2) \]

Consider \( A_{\Gamma, B} = (A_\Gamma, B) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus C^\infty(E_\Gamma \mid_{\Gamma^+ \cup \Gamma^-}) \). From the properties of \( A \) it follows that \( A_{\Gamma, B} \) is invertible. Therefore we can define the corresponding Poisson operator \( P_B \) as the restriction of \( A_{\Gamma, B}^{-1} \) to \( 0 \oplus C^\infty(E_\Gamma \mid_{\Gamma^+ \cup \Gamma^-}) \). Denote by \( A_B \) the restriction of \( A_{\Gamma, B} \) on \( \{ u \in C^\infty(E_\Gamma) \mid B(u) = 0 \} \). Then \( A_B \) is also essentially self-adjoint and positive definite (cf. Lemma 3.2.2), and thus we can take \( \pi \) as an Agmon angle. In this chapter \( Det(A) \) means \( Det_\pi(A) \). This (using standard analytic continuation technique due to Seeley (cf. [Sel,2])) allows us to define

\[ \log Det(A) = -\frac{d}{ds} \mid_{s=0} tr \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - A)^{-1} d\lambda \quad (3.3) \]

\[ \log Det(A_\Gamma, B) = -\frac{d}{ds} \mid_{s=0} tr \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - A_B)^{-1} d\lambda, \quad (3.4) \]

where \( \gamma \) is a path around the negative real axis,

\[ \{ \rho e^{i\pi} \mid \infty > \rho \geq \epsilon \} \cup \{ \epsilon e^{i\theta} \mid \pi \geq \theta \geq -\pi \} \cup \{ \rho e^{-i\pi} \mid \epsilon \leq \rho < \infty \} \]

with \( \epsilon > 0 \) chosen sufficiently small to ensure that \( \Gamma \) does not separate the spectrum.

We define the Dirichlet to Neumann operator, associated to \( A, B \) and \( C \),

\[ R : C^\infty(E \mid_\Gamma) \to C^\infty(E \mid_\Gamma) \]

by the composition of the following maps

\[ C^\infty(E \mid_\Gamma) \xrightarrow{\Delta_{i\gamma}} C^\infty(E \mid_{\Gamma^+}) \oplus C^\infty(E \mid_{\Gamma^-}) \xrightarrow{P_B} C^\infty(E_\Gamma) \xrightarrow{\nu} C^\infty(E \mid_{\Gamma^+}) \oplus C^\infty(E \mid_{\Gamma^-}) \]
where $\Delta_{ia}(f) = (f, f)$ is the diagonal inclusion and $\Delta_{if}(f, g) = f - g$ is the difference operator. Then $R$ is an essentially self-adjoint, positive definite, elliptic operator of order 1 (cf. Corollary 3.2.10).

**Theorem 3.1.1 (Mayer-Vietoris Type Formula for Determinants [BFK])**

Let $(M, g)$ be a closed oriented Riemannian manifold of dimension $d$ and $A$ be an elliptic, essentially self-adjoint, positive definite differential operator of Laplace-Beltrami type acting on smooth sections of a vector bundle $E \to M$. Then $A_B$ and $R$ are essentially self-adjoint, positive definite elliptic operators and

$$\text{Det}(A) = c\text{Det}(A_{\Gamma}, B)\text{Det}(R),$$

(3.5)

where $c$ is a local quantity which can be computed in terms of the symbols of $A, B$ and $C$ along $\Gamma$.

**Remark** The above result can be extended to manifolds with boundary. E.g. consider an oriented, compact, smooth manifold $M$ whose boundary $\partial M$ is a disjoint union of two components $\partial_+ M$ and $\partial_- M$ with $\Gamma \cap \partial M = \emptyset$, an operator $A$ of Laplace-Beltrami type and differential elliptic boundary conditions $B_+$ respectively $B_-$ for $A$ on $\partial_+ M$ respectively $\partial_- M$. Denote by $A^{(0)}$ the operator $A$ with domain $\{ u \in C^\infty(E) \mid B_+ u = 0, B_- u = 0 \}$. Then Theorem 3.1.1 remains true with $A$ replaced by $A^{(0)}$.

### 3.2 Auxiliary Results for the Proof of Theorem 3.1.1

We begin from the definition of essential self-adjointness (cf. [ES] p.131).
**Definition 3.2.1** Let $H$ be an infinite dimensional separable Hilbert space. Let $A : D(A) \to H$ be an unbounded operator, where $D(A)$, the domain of $A$, is a dense subset of $H$. Denote by $\text{Graph}(A) := \{(x, Ax) | x \in D(A)\}$, $\text{Graph}(\bar{A}) := \{(x, \bar{A}x) | x \in D(A)\}$ the graphs of $A$ and $\bar{A}$ respectively, where $\bar{A}$ is the closure of $A$. Then $A$ is called essentially self-adjoint if (i) $\text{Graph}(A) = \text{Graph}(\bar{A})$ (ii) $\bar{A}$ is self-adjoint in the sense that for any $f, g \in D(\bar{A})$, $\langle \bar{A}(f), g \rangle = \langle f, \bar{A}(g) \rangle$.

Now we are going to collect a number of results about operators related to $A$ and $\Gamma$. Denote by $H^s(\Gamma)$ the Sobolev spaces of $\Gamma$-valued sections. Throughout section 3.2 and section 3.3 we assume that $A$ satisfies the hypothesis of Theorem 3.1.1. Fix $\epsilon > 0$, so that the spectrum of $A$ is bounded from below by $\epsilon$.

**Lemma 3.2.2** (i) The operator $A_B : \{u \in C^\infty(\Gamma) \mid B(u) = 0\} \to C^\infty(\Gamma)$ has a self-adjoint extension $\bar{A}_B : D(\bar{A}_B) \to L^2(\Gamma)$, where $D(\bar{A}_B) := \{u \in H^2(\Gamma) \mid B(u) = 0\}$.

(ii) The operator $\bar{A}_B$ is positive definite and its spectrum is bounded below by $\epsilon$.

(iii) The operator

$$(A_\Gamma, B) : C^\infty(\Gamma) \to C^\infty(\Gamma) \oplus C^\infty(\Gamma_{\Gamma+\Gamma-})$$

defined by $(A_\Gamma, B)(u) = (A_\Gamma(u), B(u))$ can be extended to an invertible operator $(A_\Gamma, B)^\dagger$.

$$(A_\Gamma, B)^\dagger : H^2(\Gamma) \to L^2(\Gamma) \oplus H^{2-\frac{1}{2}}(\Gamma_{\Gamma+\Gamma-}).$$

**Proof** (i) Let $\{U_\alpha\}$ be an open cover of $M_\Gamma$ and $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. For each $U_\alpha$, let $(x_1, x_2, \cdots, x_d)$ be a local coordinate system and $(e_1, e_2, \cdots, e_k)$ be an orthonormal frame of $\pi^{-1}(U_\alpha)$. Then in each $U_\alpha$, $A$ can be
expressed by $k \times k$ matrix of differential operators in $(x_1, x_2, \cdots, x_d)$, say, $A = (a_{ij}(x))$, where each $a_{ij}(x)$ is a differential operator of order 2.

Let $u, v \in C^\infty(E_T)$ with $B(u) = B(v) = 0$. For each $U_\alpha$, choose a smooth function $\tau_\alpha$ such that $\tau_\alpha = 1$ on $\text{supp} \rho_\alpha$ and $\text{supp} \tau_\alpha \subset U_\alpha$. Then

$$< A_B u, v > = \sum_\alpha < A_B \rho_\alpha u, v > = \sum_\alpha < A_B \rho_\alpha u, \tau_\alpha v >$$

$$= \sum_\alpha \int_{U_\alpha} (\sum_{i,j} a_{ij}(x) \rho_\alpha u_j e_i, \sum_l \tau_\alpha \eta_l e_l) d\text{vol}$$

$$= \sum_\alpha \int_{U_\alpha} \sum_{i,j} (a_{ij}(x) \rho_\alpha u_j) (\tau_\alpha \eta_l) d\text{vol}$$

$$= \sum_\alpha \int_{U_\alpha} \sum_{i,j} (\rho_\alpha u_j)(a_{ij}(x)^* \tau_\alpha \eta_l) d\text{vol}$$

$$= \sum_\alpha \int_{U_\alpha} (\rho_\alpha u, A^*(\tau_\alpha v)) d\text{vol}$$

$$= < u, A_B v >, \text{ since } A^* = A. \quad (3.6)$$

Here we used integration by parts and the fact that $\rho_\alpha u_j$ and $\tau_\alpha u_i$ are 0 on the boundary of $U_\alpha$ on the fourth equality.

Hence $A_B$ is self-adjoint. From the essential self-adjointness of $A$, the domain of $A$ can be extended to $H^2(E)$ with preserving the self-adjointness. Clearly $\tilde{A}_B$ is well defined and also self-adjoint from the same argument as above.

(ii) Claim that for any $u \in C^\infty(E_T)$ with $u \mid_{\Gamma^+ \cup \Gamma^-} = 0$, one can find a sequence $\{\phi_n\}$ in $C^\infty(E_T)$ such that $\text{supp} (\phi_n) \subset M - \Gamma$ and $\phi_n$ converges to $u \in H^1(E_T)$.

Indeed, for each sufficiently small $\epsilon > 0$, we define a function $f_\epsilon : \mathbb{R} \to \mathbb{R}$ as follows.
\[
\begin{align*}
    f_\epsilon(t) &= \begin{cases} 
    0 & \text{if } 0 \leq t \leq \epsilon \\
    \frac{1-2\epsilon}{\epsilon} + 5\epsilon - 2 & \text{if } 2\epsilon \leq t \leq 3\epsilon \\
    1 & \text{if } t \geq 4\epsilon \\
    f_\epsilon(-t) & \text{if } t < 0
    \end{cases} \\
    \text{smoothly increasing with } |f'_\epsilon(t)| \leq \frac{1-2\epsilon}{\epsilon} \\
    \text{if } \epsilon \leq t \leq 2\epsilon \text{ or } 3\epsilon \leq 4\epsilon
\end{align*}
\]  

(3.7)

Then \(f_\epsilon(t)\) converges to a constant function 1 in \(H^1(\mathbb{R})\). Let \(W\) be a tubular neighborhood of \(\Gamma\) which is diffeomorphic to \(\Gamma \times (-1,1)\) with a diffeomorphism \(\varphi : W \to \Gamma \times (-1,1)\). Define \(\tilde{f}_\epsilon : \Gamma \times (-1,1) \to \mathbb{R}\) by \(\tilde{f}_\epsilon(x,t) = f_\epsilon(t)\) for \((x,t) \in \Gamma \times (-1,1)\). Setting \(\phi_n = (\tilde{f}_n \circ \varphi)u\), then \(\phi_n\) converges to \(u\) in \(H^1(E_\Gamma)\). Using \(\langle A_B(\phi_n), \phi_n \rangle = \langle A\phi_n, \phi_n \rangle \geq \epsilon \|\phi_n\|^2\) and integrating by parts, one concludes that

\[
\langle Au, u \rangle = \lim_{n \to \infty} \langle A_B\phi_n, \phi_n \rangle = \lim_{n \to \infty} \langle A\phi_n, \phi_n \rangle \geq \epsilon \|u\|^2.
\]

(3.8)

Thus (ii) follows.

(iii) As \(A_B^*\) is injective (cf. [ES]), so is the extension \((A_\Gamma, B)\): To prove that this extension is onto, consider \(f \in L^2(E_\Gamma)\) and \(\varphi \in H^{2-\frac{1}{2}}(E_\Gamma |_{\Gamma^+ \cup \Gamma^-})\). Choose any section \(v \in H^2(E_\Gamma)\) so that \(Bv = \varphi\). As \(\tilde{A}_B\) is invertible, there exists \(w \in H^2(E_\Gamma)\) satisfying \(\tilde{A}_Bw = f - \tilde{A}_\Gamma v\) and the boundary conditions \(Bw = 0\). Therefore \(u := w + v\) is an element in \(H^2(E_\Gamma)\) with \((A_\Gamma, B)u = (f, \varphi)\). Altogether one concludes that \((A_\Gamma, B)\) is an isomorphism. □

Set \(\alpha_k = e^{i\pi k d/\pi}\) for \(0 \leq k \leq d - 1\), where \(d = \text{dim}(M)\).

Lemma 3.2.3 The following operators are invertible for \(0 \leq k \leq d - 1\) and \(t \geq 0\)

\[
(A_\Gamma - \alpha_k t, B) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus C^\infty(E_\Gamma |_{\Gamma^+ \cup \Gamma^-}).
\]

Proof As \(\alpha_k \in \mathbb{C} - \mathbb{R}^+\) and thus, for \(t \geq 0\), \(\alpha_k t \notin \text{Spec}(A_B)\), the operator
$(A_{\Gamma} - \alpha_k t, B)$ is injective. By Lemma 3.2.2,

$$(A_{\Gamma} - \alpha_k t, B) : H^2(E_{\Gamma}) \to L^2(E_{\Gamma}) \oplus H^{2-\frac{1}{2}}(E_{\Gamma} |_{\Gamma+\Gamma-})$$

is injective. Since for any $t \geq 0 \ Index(A_{\Gamma}, B) = Index(A_{\Gamma} - \alpha_k t, B) = 0,$ $(A_{\Gamma} - \alpha_k t, B)$ is surjective and thus is invertible. □

Since $(A_{\Gamma} - \alpha_k t, B)$ is invertible, we can define the Poisson operator $P(\alpha_k t)$ associated to $(A_{\Gamma} - \alpha_k t, B), P(\alpha_k t) : C^\infty(E_{\Gamma} |_{\Gamma+\Gamma-}) \to C^\infty(E_{\Gamma}),$ i.e. for $\varphi \in C^\infty(E_{\Gamma} |_{\Gamma+\Gamma-}), u = P(\alpha_k t)\varphi$ is the solution in $C^\infty(E_{\Gamma})$ of $(A_{\Gamma} - \alpha_k t)u = 0$ with boundary conditions $u |_{\Gamma+\Gamma-} = \varphi.$

Let $R(\alpha_k t) : C^\infty(E |_{\Gamma}) \to C^\infty(E |_{\Gamma})$ be the Dirichlet to Neumann operator corresponding to $A_{\Gamma} - \alpha_k t, B$ and $C.$ Then the following result holds:

**Lemma 3.2.4** For $0 \leq k \leq d - 1,$ and $t \geq 0, R(\alpha_k t)$ is an invertible \Psi DO of order 1, which is elliptic with parameter $t$ of weight 1.

**Proof** In a sufficiently small collar neighborhood $U$ of $\Gamma,$ choose coordinates $x = (x', s)$ such that $(x', 0) \in \Gamma$ and $\frac{\partial}{\partial s} |_{(x',0)} = \nu_{(x',0)}.$ Let $\xi = (\xi', \eta)$ be coordinates in the cotangent space corresponding to the coordinates $(x', s).$ Let $D_s = \frac{1}{i} \frac{\partial}{\partial s}$ and write

$$(A - \alpha_k t) = A_2 D_s^2 + A_1 D_s + A_0,$$

where the $A_j$'s are differential operators of order at most $2 - j.$ The $A_j$'s induce, when restricted to $\Gamma,$ differential operators, again denoted by $A_j,$ $A_j : C^\infty(E |_{\Gamma}) \to C^\infty(E |_{\Gamma}).$ Note that $\sigma_L(x, (\xi', \eta)) = \| (\xi', \eta) \|^2$ and $g$ looks like

$$
\begin{pmatrix}
g_{ij}(x', 0) & 0 \\
0 & 1
\end{pmatrix}
$$

(3.9)
on $\Gamma$ since $\nu_{(x',0)}$ is the unit normal to $\Gamma$ at $(x',0).$ Hence one has $A_2(x) = Id_x \in \text{End}_x(E_x, E_x)$ on $\Gamma.$
For any $\varphi \in C^\infty(E|_T)$ and $t \geq 0$, we can choose $u \in C^\infty(E_T) \cap C(E)$ such that $(A - \alpha_k t)u = 0$ on $M - \Gamma$ and $u|_{\Gamma^+} = \varphi = u|_{\Gamma^-}$. Then $\frac{\partial u}{\partial s}(x', s)$ has a jump across $\Gamma$, which is $-R(\alpha_k t)(\varphi)(x')$. Hence

$$\frac{\partial u}{\partial s}(x', s) = -R(\alpha_k t)(\varphi)(x')H(s) + v(x', s), \quad (3.10)$$

where $v(x', s) \in C^\infty(E_T|_U) \cap C(E|_U)$ and $H(s)$ is the Heavyside function. Therefore, on $U$

$$(A - \alpha_k t)u = A_2 R(\alpha_k t)(\varphi) \otimes \delta_T - A_2 \frac{\partial u}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u. \quad (3.11)$$

Since $(A - \alpha_k t)u = 0$ on $M - \Gamma$, we conclude that, on $U \cap (M - \Gamma)$,

$$-A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u = 0. \quad (3.12)$$

As $-A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u \in L^2(E|_U)$, it follows that

$$(A - \alpha_k t)u = A_2 \cdot (\cdot \otimes \delta_T) \cdot R(\alpha_k t)\varphi. \quad (3.13)$$

Using that $A_2 = Id$ on $\Gamma$, one therefore obtains $Id = J \cdot (A - \alpha_k t)^{-1} \cdot (\cdot \otimes \delta_T) \cdot R(\alpha_k t)$ where $J$ is the restriction operator to $\Gamma$. From this identity it follows that $R(\alpha_k t)$ is invertible. Moreover, setting $\phi = R(\alpha_k t)\varphi$,

$$R(\alpha_k t)^{-1} \phi = J \cdot (A - \alpha_k t)^{-1} \cdot (\phi \otimes \delta_T) \quad (3.14)$$

$$= \int_{\mathbb{R}^{d-1}} e^{ix' \cdot \xi'} \int_{\mathbb{R}} \sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta)\hat{\phi}(\xi') \cdot \frac{1}{\sqrt{2\pi}} \text{d}\eta \text{d}\xi'.$$

Hence $R(\alpha_k t)^{-1}$ is a classical $\Psi$DO of order -1 with symbol

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta) \text{d}\eta, \quad (3.15)$$
and therefore $R(\alpha_k t)$ is a $\Psi$DO of order 1 with parameter $t$ of weight 1. The ellipticity with parameter of $R(\alpha_k t)$ follows from the explicit formula of the symbol. □

**Corollary 3.2.5** If $\sigma((A - \alpha_k t)^{-1})(x', s, \xi', \eta) \sim \sum_{j=0}^{\infty} a_{-2-j}(x', s, \xi', \eta)$ is the complete asymptotic symbol of $(A - \alpha_k t)^{-1}$, then the complete asymptotic symbol of $R(\alpha_k t)^{-1}$ is

$$
\sigma((R(\alpha_k t))^{-1})(x', \xi') \sim \sum_{j=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_{-2-j}(x', 0, \xi', \eta) d\eta.
$$

(3.16)

Hence $(R(\alpha_k t)^{-1})$ is a classical $\Psi$DO of order -1.

**Remark** $d\eta$ in the above formula is a normalized Lebesgue measure by $\frac{1}{\sqrt{2\pi}}$, i.e., $d\eta = \frac{1}{\sqrt{2\pi}} d\tilde{\eta}$, where $d\tilde{\eta}$ is a Lebesgue measure. In Appendix A, we will give another way of computation of the symbol of $R$ when $A = \Delta_q + \lambda$, where $\Delta_q$ is the Laplacian acting on differential $q$-forms and $\lambda$ is a positive real number.

**Proof** From Lemma 3.2.4, we obtained

$$
\sigma(R(\alpha_k t)^{-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta) d\eta.
$$

(3.17)

Since $\sigma((A - \alpha_k t)^{-1}) - a_{-2}$ is a symbol of order -3, we get

$$
|\sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta) - a_{-2}(x', 0, \xi', \eta)| \leq C \left( 1 + \sqrt{|\xi'|^2 + |\eta|^2 + |t|^2} \right)^{-3}
$$

$$
\leq C(|\xi'|^2 + |\eta|^2 + |t|^2)^{-\frac{3}{2}} \text{ for } |\xi'| \neq 0.
$$

(3.18)

Thus

$$
\lim_{|\xi'| \to \infty} \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta) d\eta}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_{-2}(x', 0, \xi', \eta) d\eta} = \lim_{|\xi'| \to \infty} \frac{\int_{-\infty}^{\infty} a_{-2}(x', 0, \xi', \eta) d\eta + \int_{-\infty}^{\infty} (\sigma((A - \alpha_k t)^{-1}) - a_{-2})(x', 0, \xi', \eta) d\eta}{\int_{-\infty}^{\infty} a_{-2}(x', 0, \xi', \eta) d\eta}
$$

(3.19)
Thus

\[ \sigma(R(\alpha_k t)^{-1}) \sim \]

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_{-2}(x', 0, \xi', \eta) d\eta + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma((A - \alpha_k t)^{-1}) - a_{-2})(x', 0, \xi', \eta) d\eta. \]

Continuing this argument, the result follows. □

Now we are going to verify that \( R(\alpha_k t) \) has \( \pi \) as an Agmon angle. By the assumption, \( A : C^\infty(E) \to C^\infty(E) \) is essentially self-adjoint and positive definite. Let \( \{\phi_j\}_{j \geq 1} \) be a complete orthonormal system of eigensections of \( A \) with corresponding eigenvalues \( \{\lambda_j\}_{j \geq 1} \). Since the complex power of \( A - \alpha_k \) is well defined, \( (A - \alpha_k)^{-\frac{1}{2}} \) is also invertible, elliptic PDO of order -1 with adjoint \( (A - \alpha_k)^{-\frac{1}{2}} \). Note that for any \( \varphi \in C^\infty(E |_\Gamma) \), \( \varphi \otimes \delta_\Gamma \in H^{-1}(E) \) and hence \( (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma) \in L^2(E) \).

**Lemma 3.2.6** For \( \varphi \in C^\infty(E |_\Gamma) \),

\[ < (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma), \phi_j > = (\lambda_j - \alpha_k t)^{-\frac{1}{2}} \int_{\Gamma} (\varphi, \phi_j) d\mu_\Gamma, \]  

where \( d\mu_\Gamma \) is the volume form on \( \Gamma \) induced from the metric on \( M \) and \((\cdot, \cdot)\) is the Hermitian inner product on \( E \).

**Proof** Since \( (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma) \in L^2(E) \), there exists a sequence \( \{f_n\} \) in \( C^\infty(E) \) such that \( f_n \) converges to \( (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma) \) in \( L^2(E) \). Then

\[ < (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma), \phi_j > = (\lambda_j - \alpha_k t)^{-\frac{1}{2}} < (A - \alpha_k t)^{-\frac{1}{2}}(\varphi \otimes \delta_\Gamma), (A - \alpha_k t)^{\frac{1}{2}} \phi_j > \]
\begin{align*}
    \lim_{n \to \infty} (\lambda_j - \alpha_k t)^{-\frac{1}{2}} f_n, (A - \alpha_k t)^{\frac{3}{2}} \phi_j > \\
    \lim_{n \to \infty} (\lambda_j - \alpha_k t)^{-\frac{1}{2}} < (A - \alpha_k t)^{\frac{3}{2}} f_n, \phi_j >
\end{align*}

(3.22)

Since \((A - \alpha_k t)^{\frac{1}{2}} : L^2(E) \to H^{-1}(E)\) is a bounded map with respect to given Sobolev norms, \((A - \alpha_k t)^{\frac{1}{2}} f_n\) converges to \((A - \alpha_k t)^{\frac{1}{2}} (A - \alpha_k t)^{-\frac{1}{2}} (\varphi \otimes \delta_\Gamma) = \varphi \otimes \delta_\Gamma\) in \(H^{-1}(E)\).

Now consider the pairing \((,)_0 : H^{-1}(E) \times H^1(E) \to \mathbb{C}.\) Then

\begin{equation}
    ((A - \alpha_k t)^{\frac{1}{2}} f_n - \varphi \otimes \delta_\Gamma, \phi_j) \leq \|(A - \alpha_k t)^{\frac{1}{2}} f_n - \varphi \otimes \delta_\Gamma\|_{-1} \cdot \|\phi_j\|_1,
\end{equation}

which tends to 0 as \(n \to 0.\) Thus

\begin{equation}
    \lim_{n \to \infty} (\lambda_j - \alpha_k t)^{-\frac{1}{2}} < (A - \alpha_k t)^{\frac{3}{2}} f_n, \phi_j > = (\lambda_j - \alpha_k t)^{-\frac{1}{2}} (\varphi \otimes \delta_\Gamma, \phi_j) = (\lambda_j - \alpha_k t)^{-\frac{1}{2}} \int_\Gamma (\varphi, \phi_j) d\mu_\Gamma.
\end{equation}

□

Corollary 3.2.7

\begin{equation}
    (A - \alpha_k t)^{-\frac{1}{2}} (\varphi \otimes \delta_\Gamma) = \sum_{j=1}^{\infty} (\lambda_j - \alpha_k t)^{-\frac{1}{2}} \int_\Gamma (\varphi, \phi_j) d\mu_\Gamma \cdot \phi_j \text{ in } L^2(E).
\end{equation}

Lemma 3.2.8 For \(\varphi_1, \varphi_2 \in C^\infty(E \mid \Gamma)\)

\begin{equation}
    ((A - \alpha_k t)^{-1}(\varphi_1 \otimes \delta_\Gamma), \varphi_2 \otimes \delta_\Gamma) = < (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_1 \otimes \delta_\Gamma), (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_2 \otimes \delta_\Gamma) >.
\end{equation}

Proof Since \((A - \alpha_k t)^{-\frac{1}{2}} (\varphi_2 \otimes \delta_\Gamma) \in L^2(E),\) there exists a sequence \(\{f_n\} \in C^\infty(E)\) such that \(f_n \to (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_2 \otimes \delta_\Gamma)\) in \(L^2(E).\) Then

\begin{equation}
    < (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_1 \otimes \delta_\Gamma), (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_2 \otimes \delta_\Gamma) >
\end{equation}
\[
\lim_{n \to \infty} < (A - \alpha_k t)^{-\frac{1}{2}}(\varphi_1 \otimes \delta_T), f_n >
\]
\[
= \lim_{n \to \infty} < (A - \alpha_k t)^{-\frac{1}{2}}(\varphi_1 \otimes \delta_T), (A - \alpha_k t)^{-\frac{1}{2}}(A - \alpha_k t)^{\frac{1}{2}} f_n >
\]
\[
= \lim_{n \to \infty} < (A - \alpha_k t)^{-1}(\varphi_1 \otimes \delta_T), (A - \alpha_k t)^{\frac{1}{2}} f_n > .
\] (3.27)

Since \((A - \alpha_k t)^{\frac{1}{2}} : H^0(E) \to H^{-1}(E)\) is a bounded map, \((A - \alpha_k t)^{\frac{1}{2}} f_n\) converges to \(\varphi_2 \otimes \delta_T\) in \(H^{-1}(E)\). Then

\[
\left| < (A - \alpha_k t)^{-1} (\varphi_1 \otimes \delta_T), (A - \alpha_k t)^{\frac{1}{2}} f_n - \varphi_2 \otimes \delta_T > \right| \leq \| (A - \alpha_k t)^{-1} (\varphi_1 \otimes \delta_T) \|_1 \cdot \| (A - \alpha_k t)^{\frac{1}{2}} f_n - \varphi_2 \otimes \delta_T \|_{-1} \to 0 \text{ as } n \to \infty.
\] (3.28)

Hence

\[
\lim_{n \to \infty} < (A - \alpha_k t)^{-1} (\varphi_1 \otimes \delta_T), (A - \alpha_k t)^{\frac{1}{2}} f_n >
\]
\[
= < (A - \alpha_k t)^{-1} (\varphi_1 \otimes \delta_T), \varphi_2 \otimes \delta_T > .
\] (3.29)

□

**Lemma 3.2.9** For \(\varepsilon' < \frac{\pi}{d}\) sufficiently small and \(0 \leq k \leq d - 1\), the operator \(R(\alpha_k t)\) does not have any eigenvalues in \(\Lambda_{\varepsilon'}\), where \(\Lambda_{\varepsilon'} = \{ z \in \mathbb{C} \mid \pi - \varepsilon' < \arg(z) < \pi + \varepsilon' \text{ or } |z| < \varepsilon' \}\). Hence \(R(\alpha_k t)\) has \(\pi\) as an Agmon angle.

**Proof** Let \(\varphi_1, \varphi_2 \in C^\infty(E |_\Gamma)\). Since \(R(\alpha_k t)^{-1} = J \cdot (A - \alpha_k t)^{-1} \cdot (\cdot \otimes \delta_T)\),

\[
< R(\alpha_k t)^{-1} \varphi_1, \varphi_2 >
\]
\[
= < J \cdot (A - \alpha_k)^{-1} (\varphi_1 \otimes \delta_T), \varphi_2 >_{\Gamma}
\]
\[
= < (A - \alpha_k t)^{-1} (\varphi_1 \otimes \delta_T), \varphi_2 \otimes \delta_T >_{M}
\]
\[
= < (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_1 \otimes \delta_T), (A - \alpha_k t)^{-\frac{1}{2}} (\varphi_2 \otimes \delta_T) >
\]
by Lemma 3.2.8 and Corollary 3.2.7. Together with Lemma 3.2.4 this implies that $\Lambda^\nu$ has an empty intersection with $\text{Spec}R(\alpha_k t)$. □

Using the above formula, one obtains as an immediate consequence the following

**Corollary 3.2.10** The operator $R = R(0)$ is essentially self-adjoint and positive definite.

Next we are collecting a number of results about operators involving the $d$–th power of $A$ and the submanifold $\Gamma$.

Consider the families of operators $A^d + t^d$ and $A^d_t + t^d$ for nonnegative real numbers $t$. Then $A^d + t^d$ and $A^d_t + t^d$ are elliptic differential operators with parameter, where the weight of $t$ is 2. Note that

$$A^d_t + t^d = (A^d - te^{i\pi d})(A^d - te^{i\pi d}) \cdots (A^d - te^{i\pi d}).$$  \hspace{1cm} (3.31)

Let us introduce the boundary conditions $B_d(t), C_d(t)$ by setting

$$B_d(t) = (B, B(A^d - \alpha_0 t), B(A^d - \alpha_t t)(A^d - \alpha_{0} t), \cdots, B(A^d - \alpha_{d-2} t) \cdots (A^d - \alpha_{0} t)), \hspace{1cm} (3.32)$$

and

$$C_d(t) = (C, C(A^d - \alpha_0 t), C(A^d - \alpha_t t)(A^d - \alpha_{0} t), \cdots, C(A^d - \alpha_{d-2} t) \cdots (A^d - \alpha_{0} t)). \hspace{1cm} (3.33)$$

It follows from Lemma 3.2.3 that the following operator is invertible

$$\left(A^d_t + t^d, B_d(t)\right) : C^\infty(E_\Gamma) \to C^\infty(E_\Gamma) \oplus (\oplus_d C^\infty(E_\Gamma|_{\Gamma+\cup\Gamma-})). \hspace{1cm} (3.34)$$
Therefore the corresponding Poisson operator $\tilde{P}_d(t) : \oplus_d C^\infty(E_{t+\mathrm{L}_t^*}) \to C^\infty(E_t)$ is well defined.

**Lemma 3.2.11** The Poisson operator $\tilde{P}_d(t)$ associated to $(A^d_t + t^d, B_d(t))$ is given by

$$\tilde{P}_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = P(\alpha_0 t)\varphi_0 + (A_G - \alpha_0 t)^{-1}P(\alpha_1 t)\varphi_1 + \cdots + (A_G - \alpha_d t)^{-1}P(\alpha_{d-1} t)\varphi_{d-1},$$

where $(A_G - \alpha_k t)_B$ is the restriction of $A_G - \alpha_k t$ to \{u \in $C^\infty(E_t) | Bu = 0\}.

**Proof** Denoting the right hand side of the claimed identity by $Q_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ one obtains

$$(A^d_t + t^d) \cdot Q_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = 0. \quad (3.36)$$

Moreover, for $0 \leq k \leq d - 1$, $Q_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ satisfies the boundary conditions

$$(B(A_G - \alpha_{k-1} t)(A_G - \alpha_{k-2} t)\cdots(A_G - \alpha_0 t))Q_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = (3.37)$$

$$B(A_G - \alpha_{k-1} t)\cdots(A_G - \alpha_0 t)P(\alpha_0 t)\varphi_0 + \cdots + B(A_G - \alpha_{k-1} t)\cdots(A_G - \alpha_0 t)(A_G - \alpha_0 t)^{-1}B_1P(\alpha_k t)\varphi_k + \cdots + B(A_G - \alpha_{k-1} t)\cdots(A_G - \alpha_0 t)(A_G - \alpha_0 t)^{-1}B_1P(\alpha_{d-1} t)\varphi_{d-1} = \varphi_k,$$

since $(A_G - \alpha_j t)P(\alpha_j t) = 0$ and $B(A_G - \alpha_j t)_B^{-1} = 0$. These two properties of $Q_d(t)$ establish the claimed identity. □

Further let us consider the boundary conditions $B_d(t)$ and $C_d(t)$ for $t = 0$. Note that

$$B_d(0) = (B, BA_G, \cdots, BA^d_G); C_d(0) = (C, CA_G, \cdots, CA^d_G). \quad (3.38)$$
Let $\Omega(t)$ be the following lower triangular $d \times d$ matrix

$$
\Omega(t) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
\alpha_0 t & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_0^{d-1} t^{d-1} & t^{d-2} & \cdots & 1 \\
\end{pmatrix}.
$$

(3.39)

Then $B_d(0) = \Omega(t)B_d(t)$ as well as $C_d(0) = \Omega(t)C_d(t)$. Let $P_d(t) := \tilde{P}_d(t)\Omega(t)^{-1}$ and notice that $P_d(t)$ is the Poisson operator corresponding to $(A_1^d + t^d, B_d(0))$.

Consider the Dirichlet to Neumann operator $\tilde{R}_d(t) = \Delta_{if} \cdot C_d(t) \cdot \tilde{P}_d(t) \cdot \Delta_{ia}$ corresponding to $A_1^d + t^d, B_d(t)$ and $C_d(t)$. Then

$$
\tilde{R}_d(t)(\varphi_0, \ldots, \varphi_{d-1}) = 
$$

$$
\Delta_{if} \cdot (C, C(A_\Gamma - \alpha_0 t), \ldots, C(A_\Gamma - \alpha_{d-2} t) \cdots (A_\Gamma - \alpha_0 t)).
$$

(3.40)

$$
(P(\alpha_0 t) \Delta_{ia} \varphi_0 + (A_\Gamma - \alpha_0 t)_B^{-1} P(\alpha_1 t) \Delta_{ia} \varphi_1 + \cdots + 
$$

$$(A_\Gamma - \alpha_0 t)_B^{-1} (A_\Gamma - \alpha_1 t)_B^{-1} \cdots (A_\Gamma - \alpha_{d-2} t)_B^{-1} P(\alpha_{d-1} t) \Delta_{ia} \varphi_{d-1}).
$$

(3.41)

Thus $\tilde{R}_d(t) : \oplus_d C^\infty(E |_{\Gamma}) \to \oplus_d C^\infty(E |_{\Gamma})$ can be represented by a $d \times d$ matrix of upper triangular form,
where $R(\alpha_k t)$ is the Dirichlet to Neumann operator corresponding to $A_{\Gamma} - \alpha_k t$, $B$ and $C$ defined earlier. In particular, we conclude that $\tilde{R}_d(t)$ is invertible and has $\pi$ as an Agmon angle.

Finally introduce the Dirichlet to Neumann operator $R_d(t)$ associated to $A_{\Gamma}^d + t^d, B_d(0)$ and $C_d(0)$. Then

$$
\tilde{R}_d(t) = \Delta_{ij} \cdot C_d(t) \cdot \tilde{P}_d(t) \cdot \Delta_{ia} = \Delta_{ij} \cdot \Omega(t)^{-1} \cdot C_d(0) \cdot P_d(t) \cdot \Omega(t) \cdot \Delta_{ia}
$$

$$
= \Omega(t)^{-1} \cdot \Delta_{ij} \cdot C_d(0) \cdot P_d(t) \cdot \Delta_{ia} \cdot \Omega(t) = \Omega(t)^{-1} \cdot R_d(t) \cdot \Omega(t).
$$

(3.42)

As a consequence, $R_d(t)$ has the same spectrum as $\tilde{R}_d(t)$ and therefore, $R_d(t)$ is invertible, has $\pi$ as an Agmon angle and satisfies $\log \text{Det}(R_d(t)) = \log \text{Det}(\tilde{R}_d(t))$. In view of the fact that $\tilde{R}_d(t)$ is of upper triangular form one has

$$
\log \text{Det}(\tilde{R}_d(t)) = \sum_{k=0}^{d-1} \log \text{Det}(R(\alpha_k t)).
$$

(3.43)

As $A$ is positive and essentially selfadjoint, the operator $A_{\Gamma}^d + t^d : C^\infty(E) \to C^\infty(E)$ is invertible for $t \geq 0$. Using the kernel $k_d(x, y)$ of $(A_{\Gamma}^d + t^d)^{-1}$ this operator can be extended to $C^\infty(E_{\Gamma})$ by setting ($u \in C^\infty(E_{\Gamma})$)

$$
((A_{\Gamma}^d + t^d)^{-1})_\Gamma u(x) = \int_{M_{\Gamma}} k_d(x, y) u(y) dy.
$$

(3.44)

It follows from Lemma 3.2.3 that

$$(A_{\Gamma}^d + t^d, B_d(t)) : C^\infty(E_{\Gamma}) \to C^\infty(E_{\Gamma}) \oplus (\oplus_d C^\infty(E_{\Gamma} \mid_{\Gamma + \Gamma}))$$

is invertible. Thus, since $B_d(0) = \Omega(t)B_d(t)$, we conclude that $(A_{\Gamma}^d + t^d, B_d(0))$ is invertible as well. Denote by $(A_{\Gamma}^d + t^d)_{B_d(0)}$ the restriction of $A_{\Gamma}^d + t^d$ to

$\{u \in C^\infty(E_{\Gamma}) \mid B_d(0)u = 0\}$

and let $(A_{\Gamma}^d + t^d)_{B_d(0)}^{-1}$ be its inverse.
Lemma 3.2.12

\[(A^d_t + t^d)^{-1} = ((A^d + t^d)^{-1})_\Gamma - P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma. \tag{3.45}\]

**Proof** Denote by \(Q(t)\) the right hand side of the claimed identity. One verifies that for \(u \in C^\infty(E_\Gamma)\)

\[(A^d_t + t^d)Q(t)u = u \tag{3.46}\]

and

\[B_d(0)Q(t)u = B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma u - B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma u = 0. \tag{3.47}\]

These two identities imply that \(Q(t) = (A^d_t + t^d)^{-1}B_d(0)\). \(\square\)

Lemma 3.2.13

(i) \(\frac{d}{dt} P_d(t) = -dt^{d-1}(A^d_t + t^d)^{-1}B_d(0) \cdot P_d(t). \tag{3.48}\)

(ii) \(R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = -dt^{d-1}R_d(t)^{-1} \cdot \Delta_{\text{ia}} \cdot C_d(0) \cdot (A^d_t + t^d)^{-1}B_d(0) \cdot P_d(t) \cdot \Delta_{\text{ia}}. \tag{3.49}\)

In particular, \(d\) being the dimension of \(M\), \(R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t)\) is of trace class.

**Proof** (i) Derive \((A^d_t + t^d) \cdot P_d(t) = 0\) with respect to \(t\) to obtain

\[(A^d_t + t^d) \cdot \frac{d}{dt} P_d(t) = -\frac{d}{dt}(A^d_t + t^d) \cdot P_d(t) = -dt^{d-1}P_d(t). \tag{3.50}\]

Similarly, deriving \(B_d(0) \cdot P_d(t) = Id\) with respect to \(t\) yields \(B_d(0) \frac{d}{dt} P_d(t) = 0\). Hence

\[(A^d_t + t^d)_{B_d(0)} \cdot \frac{d}{dt} P_d(t) = -dt^{d-1}P_d(t) \tag{3.51}\]

and therefore

\[\frac{d}{dt} P_d(t) = -dt^{d-1}(A^d_t + t^d)^{-1}B_d(0) \cdot P_d(t). \tag{3.52}\]
(ii) follows from the definition of \( R_d(t) \) and (i). □

Taking into account that \( \triangle_{ia}(\oplus_d C^\infty(E \, | \, \Gamma) = \{(\varphi, \varphi) \mid \varphi \in \oplus_d C^\infty(E \, | \, \Gamma)\} \) we may define \( P_{\Gamma} : \triangle_{ia}(\oplus_d C^\infty(E \, | \, \Gamma) \to \oplus_d C^\infty(E \, | \, \Gamma) \) by \( P_{\Gamma}(\varphi, \varphi) = \varphi \).

**Corollary 3.2.14**

\[
R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = dt^{d-1} P_{\Gamma} \cdot B_{d}(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} \cdot P_{d}(t) \cdot \triangle_{ia} \tag{3.53}
\]

**Proof** By Lemma 3.2.12 and 3.2.13

\[
R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = -dt^{d-1}(\triangle_{if} \cdot C_a(0) \cdot P_d(t) \cdot \triangle_{ia})^{-1}. \tag{3.54}
\]

Now consider \( \triangle_{if} \cdot C_a(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} \cdot P_d(t) \cdot \triangle_{ia} : \oplus_d C^\infty(E \, | \, \Gamma) \to \oplus_d C^\infty(E \, | \, \Gamma) \).

For any \( \varphi \in \oplus_d C^\infty(E \, | \, \Gamma) \), \( P_d(t) \cdot \triangle_{ia}(\varphi) \in (C^\infty(E_{\Gamma}) \cap C^0(E)) \subset L^2(E) \). Hence

\[
(\triangle_{if} \cdot C_a(0) \cdot ((A^d + t^d)^{-1})_{\Gamma}) \mid_{H^0(E)} \text{ is the composition of the following bounded maps}
\]

\[
H^0(E) \xrightarrow{(A^d + t^d)^{-1}} H^{2d}(E) \xrightarrow{C_a(0)} H^{2d-1-\frac{1}{2}}(E \, | \, \Gamma) \oplus H^{2d-3-\frac{1}{2}}(E \, | \, \Gamma) \oplus \cdots \oplus H^\frac{1}{2}(E \, | \, \Gamma)
\]

Clearly \( \triangle_{if} \cdot C_a(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} u = 0 \) for \( u \in C^\infty(E) \). Since \( C^\infty(E) \) is dense in \( H^0(E) \), \( \triangle_{if} \cdot C_a(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} = 0 \) on \( H^0(E) \) and thus \( \triangle_{if} \cdot C_a(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} \cdot P_d(t) \cdot \triangle_{ia} = 0 \). Therefore

\[
R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = dt^{d-1}(\triangle_{if} \cdot C_a(0) \cdot P_d(t) \cdot \triangle_{ia})^{-1} \cdot \triangle_{if} \cdot C_a(0) \cdot P_d(t) \cdot B_{d}(0) \cdot ((A^d + t^d)^{-1})_{\Gamma} \cdot P_d(t) \cdot \triangle_{ia}.
\]

Note that for any \( u \in C^\infty(E_{\Gamma}) \), the boundary values of \((A^d + t^d)^{-1})_{\Gamma} u \) on \( \Gamma^+ \) and \( \Gamma^- \) are the same, i.e.

\[
B_{d}(0)((A^d + t^d)^{-1})_{\Gamma} u \mid_{\Gamma^+} = B_{d}(0)((A^d + t^d)^{-1})_{\Gamma} u \mid_{\Gamma^-}. \tag{3.55}
\]
Hence

\[ B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia} = \Delta_{ia} \cdot P_r_\Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia}. \]  
(3.56)

\[ \square \]

As \((A^d + t^d)^{-1}, (A^d + t^d)^{-1}_B(0)\) and \(R_d(t)^{-1} \frac{d}{dt} R_d(t)\) are of trace class, we can apply the well known variational formula for regularized determinants:

**Lemma 3.2.15** Let \(Q(t)\) denote any of the operators \(A^d + t^d, (A^d + t^d)_B(0)\) or \(R_d(t)\). Then, for any \(t \geq 0\),

\[ \frac{d}{dt} \log \text{Det} Q(t) = tr(Q(t)^{-1} \frac{d}{dt} Q(t)). \]  
(3.57)

**Proof**

\[
\frac{d}{dt} \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - Q(t))^{-1} d\lambda \right)
= tr \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s} \frac{d}{dt} (\lambda - Q(t))^{-1} d\lambda \right)
= tr \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s} (\lambda - Q(t))^{-1} (-\frac{d}{dt} Q(t)) (\lambda - Q(t))^{-1} d\lambda \right)
\]
(3.58)

Since \((\lambda - Q(t))^{-1}\) is bounded along \(\gamma\), \((\lambda - Q(t))^{-1} (-\frac{d}{dt} Q(t)) (\lambda - Q(t))^{-1}\) is of trace class. Thus

\[
tr \left( (\lambda - Q(t))^{-1} (-\frac{d}{dt} Q(t)) (\lambda - Q(t))^{-1} \right) = tr \left( -\frac{d}{dt} Q(t) (\lambda - Q(t))^{-2} \right)
\]
(3.59)

and

\[
(3.58) = tr \left( -\frac{d}{dt} Q(t) \cdot \frac{1}{2\pi i} \int_\gamma \lambda^{-s} (\lambda - Q(t))^{-2} d\lambda \right)
= tr \left( -\frac{d}{dt} Q(t) \cdot \frac{1}{2\pi i} \int_\gamma \lambda^{-s} \frac{d}{d\lambda} (- (\lambda - Q(t))^{-1}) d\lambda \right)
\]
\[ = \text{tr} \left( \frac{d}{dt} Q(t) \cdot \frac{1}{2\pi i} \int_{\gamma} (-s)^{-s-1}(\lambda - Q(t))^{-1} d\lambda \right) \text{ by integration by parts} \]
\[ = -s \cdot \text{tr} \left( \frac{d}{dt} Q(t) \cdot Q(t)^{-s-1} \right). \]  

(3.60)

Since for \( s \geq 0, \frac{d}{dt} Q(t) \cdot Q(t)^{-s-1} \) is of a trace class,

\[ \frac{d}{dt} \log \text{Det} Q(t) = -\frac{d}{dt} \left|_{s=0} \frac{1}{2\pi i} \text{tr} \int_{\gamma} \lambda^{-s}(\lambda - Q(t))^{-1} d\lambda \right| \]
\[ = -\frac{d}{ds} \left|_{s=0} \frac{1}{2\pi i} \text{tr} \int_{\gamma} \lambda^{-s}(\lambda - Q(t))^{-1} d\lambda \right| \]
\[ = \frac{d}{ds} \left|_{s=0} s \cdot \text{tr} \left( \frac{d}{dt} Q(t) \cdot Q(t)^{-s-1} \right) \right| \]
\[ = \text{tr} \left( \frac{d}{dt} Q(t) \cdot Q(t)^{-1} \right). \]  

(3.61)

\[ \square \]

### 3.3 Proof of Theorem 3.1.1

In the case where the operator \( A^{-1} \) is of trace class, the proof of Theorem 3.1.1 is considerably simpler. Unfortunately, this is the case only if the dimension \( d \) of \( M \) is equal to 1. Our strategy is to first prove a version of Theorem 3.1.1 for \( A^d \) (Lemma 3.3.1), using the fact that \( (A^d)^{-1} \) is of trace class. Together with the auxiliary results of Section 3.2 and the asymptotic expansion derived in Chapter 2, the proof of Theorem 3.1.1 is then completed.

**Lemma 3.3.1** Let \( A^d + t^d \) and \( (A^d_t + t^d, B_d(0)) \) be as above. Then, for \( t \geq 0, \)

\[ \frac{d}{dt} \left( \log \text{Det}(A^d + t^d) - \log \text{Det}(A^d_t + t^d, B_d(0)) \right) = \frac{d}{dt} \log \text{Det}(A^d(t)). \]  

(3.62)
**Proof** Define \( w(t) := \frac{d}{dt}(\log \text{Det}(A^d + t^d) - \log \text{Det}((A^d_t + t^d), B_d(0))) \). By Lemma 3.2.15 and Lemma 3.2.12

\[
\begin{align*}
w(t) &= \text{tr}(\frac{d}{dt}(A^d + t^d) \cdot (A^d + t^d)^{-1} - \frac{d}{dt}(A^d_t + t^d)_{B_d(0)} \cdot (A^d_t + t^d)_{B_d(0)}^{-1}) \\
&= d \cdot t^{d-1} \text{tr}((A^d + t^d)^{-1} - (A^d_t + t^d)_{B_d(0)}^{-1}) \\
&= d \cdot t^{d-1} \text{tr}(((A^d + t^d)^{-1})_\Gamma - (A^d_t + t^d)_{B_d(0)}^{-1}) \\
&= d \cdot t^{d-1} \text{tr}(P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma). \tag{3.63}
\end{align*}
\]

On the other hand, by Lemma 3.2.15, Corollary 3.2.14 and the commutativity of the trace,

\[
\begin{align*}
\frac{d}{dt} \log \text{Det}R_d(t) &= \text{tr}(\frac{d}{dt}R_d(t) \cdot R_d(t)^{-1}) \\
&= d \cdot t^{d-1} \text{tr}(P \Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma \cdot P_d(t) \cdot \Delta_{ia}) \\
&= d \cdot t^{d-1} \text{tr}(P_d(t) \cdot \Delta_{ia} \cdot P \Gamma \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma) \\
&= d \cdot t^{d-1} \text{tr}(P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})_\Gamma). \tag{3.64}
\end{align*}
\]

Combining the above two identities shows that

\[
w(t) = \frac{d}{dt} \log \text{Det}R_d(t). \tag{3.65}
\]

\(\square\)

Since \( \log \text{Det}R_d(t) = \sum_{k=0}^{d-1} \log \text{Det}(\alpha_k t) \), we conclude from Lemma 3.3.1 that

\[
\log \text{Det}(A^d + t^d) - \log \text{Det}((A^d_t + t^d), B_d(0)) = \tilde{c} + \sum_{k=0}^{d-1} \log \text{Det}(\alpha_k t), \tag{3.66}
\]

where \( \tilde{c} \) is independent of \( t \).
Note that $\log\text{Det}(A^d + t^d)$, $\log\text{Det}(A^d + t^d, B_d(0))$ and $\log\text{Det}R(\alpha_k t)(0 \leq k \leq d - 1)$ have asymptotic expansions as $t \to +\infty$. Since the eigenvalues of $A^d + t^d$ and $(A^d + t^d)_B(0)$ satisfy the condition in Proposition 2.2.3, the constant terms in the asymptotic expansions of $\log\text{Det}(A^d + t^d)$ and $\log\text{Det}((A^d + t^d), B_d(0))$ are zero. Let $\pi_0(R(\alpha_k t))$ be the constant term in the asymptotic expansion of $\log\text{Det}R(\alpha_k t))$. Then $\bar{c} = -\sum_{k=0}^{d-1} \pi_0(R(\alpha_k t))$, which is computable in terms of the symbol of $R(\alpha_k t)$ by Theorem 2.1.4.

**Lemma 3.3.2** (i) $\text{Det}(A^d, B_d(0)) = (\text{Det}(A^d, B))^d$; (ii) $\text{Det}(A^d) = (\text{Det}A)^d$.

**Proof** Statement (i) follows from the fact that $\lambda$ is an eigenvalue of $A_B$ if and only if $\lambda^d$ is an eigenvalue of $(A^d)_B(0)$ and (ii) is proved in the same way. □

**Proof of Theorem 3.1.1** Setting $t = 0$ in (3.66), one obtains

$$
\log\text{Det}A^d - \log\text{Det}(A^d, B_d(0)) = \bar{c} + \log\text{Det}R_d(0). \quad (3.67)
$$

By Lemma 3.3.2, $\log(\text{Det}A)^d - \log(\text{Det}(A^d, B))^d = \bar{c} + \log(\text{Det}R)^d$. Hence

$$
\log\text{Det}A = \log(c) + \log\text{Det}(A^d, B) + \log\text{Det}R, \text{ where } \log(c) = -\frac{1}{d} \sum_{k=0}^{d-1} \pi_0(R(\alpha_k t)).
$$

Using the result of Theorem 2.1.4, Theorem 3.1.1 follows.

**Theorem 3.3.3** If $M$ is an even dimensional manifold, i.e. $d$ is even, then the constant $c$ in Theorem 3.1.1 is 1 and therefore

$$
\text{Det}(A) = \text{Det}(A^d, B) \cdot \text{Det}(R). \quad (3.68)
$$

**Proof** From (3.67), we obtained that

$$
\log c = -\frac{1}{d} \sum_{k=0}^{d-1} \pi_0(R(\alpha_k)). \quad (3.69)
$$
We are going to prove that for each \( k \), \( \pi_0(R(\alpha_k)) = 0 \). Let \( \sigma(R(\alpha_k t)) \sim p_1 + p_0 + p_{-1} + \cdots \) be the asymptotic symbol of \( R(\alpha_k t) \) in some local coordinate neighborhood.

From step 5 in Chapter 2, we get a formula for \( \pi_0 \) as follows:

\[
\pi_0(R(\alpha_k t)) = \frac{\partial}{\partial s} \bigg|_{s=0} \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{1}{2\pi i} \int_{\gamma} \mu^{-s} tr r_{-1-(d-1)}(\mu, \alpha_k; x', \xi') d\mu d\xi' d\text{vol}(\Gamma)
\]

Here \( r_{-1-j} \) is the homogeneous part of the asymptotic symbol of the resolvent \((\mu - R(\alpha_k t))^{-1}\) i.e.

\[
r_{-1} = (\mu - p_1(\alpha_k t; x', \xi'))^{-1},
\]

\[
r_{-1-j} = -(\mu - p_1(\alpha_k t; x', \xi'))^{-1} \sum_{k=0}^{j-1} \sum_{|\omega| + l = j-k} \frac{1}{\omega!} \partial_{\xi_l}^\omega p_{1-l}(\alpha_k t; x', \xi') D_{\xi_l} D_{\xi_l}^{-1} r_{-1-k}(\mu, \alpha_k t; x', \xi').
\]

The proof of Theorem 3.3.3 reduces to the verification of the following equation:

\[
p_{-1-j}(\alpha_k t; x', -\xi') = (-1)^j p_{-1-j}(\alpha_k t; x', \xi').
\]

Then

\[
r_{-1-j}(\mu, \alpha_k; x', -\xi') = (-1)^j r_{-1-j}(\mu, \alpha_k; x', \xi'),
\]

hence when \( d \) is even, \( r_{-1-(d-1)}(\mu, \alpha_k; x', \xi') \) is odd with respect to \( \xi' \). Thus the integrand of \( \mathbb{R}^{d-1} \) is zero and \( \pi_0(R(\alpha_k t)) = 0 \).

Let \( \sigma(R(\alpha_k t)^{-1}) \sim \sum_{j=0}^{\infty} q_{-j}(\alpha_k t; x', \xi') \). From Corollary 3.2.5, we get a formula

\[
q_{-j} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \beta_{2-j}(\alpha_k t; x', 0, \xi', \eta) d\eta,
\]

where \( \sigma((A - \alpha_k t)^{-1}) \sim \sum_{j=0}^{\infty} \beta_{2-j}(\alpha_k t; x', s, \xi', \eta) \). Set \( \sigma(A - \alpha_k t) = w_2 + w_1 + w_0 \),

where \( w_2(x', s, \xi', \eta) = (\|\xi', \eta\|^2 - \alpha_k t)\text{Id} \).
Then \( w_{2-j}(x', s, -\xi', -\eta) = (-1)^j w_{2-j}(x', s, \xi', \eta) \) and hence
\[
\beta_{2-j}(\alpha_k t; x', s, -\xi', -\eta) = (-1)^j \beta_{2-j}(\alpha_k t; x', s, \xi', \eta).
\]

From these facts, we can conclude that \( q_{-j}(\alpha_k t; x', -\xi') = (-1)^j q_{-j}(\alpha_k t; x', \xi') \) and hence \( p_{1-j}(\alpha_k t; x', -\xi') = (-1)^j p_{1-j}(\alpha_k t; x', \xi') \). □

3.4 Mayer-Vietoris Formula for the Laplace Operator on an Even Dimensional Manifolds

In this section, we will consider the special case of the Mayer-Vietoris formula proved in the previous section. Throughout this section \((M, g)\) is an even dimensional closed Riemannian manifold. Let \( \Lambda^q T^*M \) be the \( q \)-th exterior product of the cotangent bundle. Set \( A = \Delta_q + \lambda \), where \( \Delta_q \) is the Laplacian operator acting on differential \( q \)-forms and \( \lambda \) is a positive real number. Then from Theorem 3.3.3, we get

\[
\det(A_q + \lambda) = \det(\Delta_q; \Gamma + \lambda, B) \cdot \det(R(\lambda)) \quad \text{for any } \lambda > 0,
\]

where \( B \) is the Dirichlet boundary condition and \( R(\lambda) \) is the Dirichlet to Neumann operator.

Now we are going to consider the case of \( \lambda = 0 \). By Corollary 3.2.10, \( R(\lambda) \) is a positive definite, self-adjoint, elliptic \( \Psi \text{DO} \). When \( \lambda = 0 \), both the Laplacian \( \Delta_q \) and \( R := R(0) \) have zero eigenvalues and so \( \det(\Delta_q) = \det(R) = 0 \). In this case we can define the modified determinants \( \det^* \Delta_q \) and \( \det^* R \) to be the determinants of \( \Delta_q \) and \( R \) respectively, when restricted to the orthogonal complement of the null space i.e. for \( P = \Delta_q \) or \( R \), \( \det^* P = \exp(-\langle \zeta^*_P \rangle(0)) \), where \( \zeta^*_P(s) = \sum_{\substack{\lambda_j \in \text{Spec}(P) \\lambda_j \neq 0 \\lambda_j \neq 0}} \lambda_j^{-s} \).
Let $H_q$ be the space of harmonic $q$-forms equipped with the natural inner product 

\[ \langle \varphi, \psi \rangle = \int_M \varphi \wedge \ast \psi = \int_M (\varphi, \psi) d\text{vol}(M), \]

where $(,)$ is a metric in $E = \wedge^q T^* M$ induced by the Riemannian metric $g$ on $M$. Let $H_q |_\Gamma$ be the restriction of harmonic forms to $\Gamma$. Define an inner product on $H_q |_\Gamma$ by \( \langle \alpha, \beta \rangle_\Gamma = \int_\Gamma (\alpha, \beta) d\mu_\Gamma \), where $d\mu_\Gamma$ is a volume element of $\Gamma$ coming from $g$ restricted to $\Gamma$.

Suppose $k = \dim H_q$, and let $\psi_1, \ldots, \psi_k$ be an orthonormal basis of $H_q$ and $\phi_1, \ldots, \phi_k$ be an orthonormal basis of $H_q |_\Gamma$. Let $J : H_q \to H_q |_\Gamma$ denote the restriction map. Set $J(\psi_i) = a_{ij} \phi_j$ and $A = (a_{ij})_{1 \leq i, j \leq k}$.

**Theorem 3.4.1** For an even dimensional manifold $M$,

\[ \text{Det}^* \Delta_q = \frac{1}{(\text{det} A)^2} \text{Det}(\Delta_q, B) \cdot \text{Det}^* R. \]  

**Remark** If $q = 0$, then $E = M \times \mathbb{R}$ and $A$ is a $1 \times 1$ matrix. If $\omega$ and $\phi$ are orthonormal harmonic 0-forms on $M$ and $\Gamma$ respectively, then $\omega = \frac{1}{\sqrt{V}}$ and $\phi = \frac{1}{\sqrt{l}}$, where $V$ and $l$ are the volumes of $M$ and $\Gamma$ respectively. Consider $J : H_0 \to H_0 |_\Gamma$. Then $J(\omega) = \frac{1}{\sqrt{V}} 1_\Gamma = \frac{1}{\sqrt{V}} \cdot \phi$. Hence $A = \left(\frac{1}{\sqrt{V}}\right)$ and we obtain

\[ \text{Det}^* \Delta_0 = \frac{V}{l} \text{Det}(\Delta_0, B) \cdot \text{Det}^* R, \]

which is established in [BFK1].

Now we are going to prove Theorem 3.4.1.

Let $\{\psi_j\}_{j \geq 0}$ be the complete orthonormal system of eigenforms of $\Delta_q$ with eigenvalue $\lambda_j$ in $L^2(E)$. Let $k = \dim H_q$. Then $\psi_1, \ldots, \psi_k$ are harmonic forms and $\lambda_1 = \cdots = \lambda_k = 0.$
Lemma 3.4.2

\[ \log \text{Det}(\Delta_q + \epsilon) = k \log \epsilon + \log \text{Det}^*(\Delta_q) + o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \quad (3.78) \]

**Proof** For \( Res \gg 0 \), \( \zeta_{\Delta_q+\epsilon}(s) = k \epsilon^{-s} + \sum_{\lambda_i > 0} (\lambda_i + \epsilon)^{-s} \). Then

\[ \zeta'_{\Delta_q+\epsilon}(s) = -k \log \epsilon \cdot \epsilon^{-s} - \sum_{\lambda_i > 0} \log(\lambda_i + \epsilon)(\lambda_i + \epsilon)^{-s}. \quad (3.79) \]

Thus

\[ -\zeta'_{\Delta_q+\epsilon}(s) - (k \log \epsilon + \sum_{\lambda_i > 0} \log \lambda_i \cdot \lambda_i^{-s}) = k \log \epsilon (\epsilon^{-s} - 1) + \sum_{\lambda_i > 0} \left( \log(\lambda_i + \epsilon) \cdot (\lambda_i + \epsilon)^{-s} - \log \lambda_i \cdot \lambda_i^{-s} \right). \quad (3.80) \]

Since \( \sum_{\lambda_i > 0} \log \lambda_i \cdot \lambda_i^{-s} \) converges uniformly for \( Res \gg 0 \) and has a meromorphic continuation with 0 as a regular value,

\[ \lim_{\epsilon \to 0} \sum_{\lambda_i > 0} \left( \log(\lambda_i + \epsilon) \cdot (\lambda_i + \epsilon)^{-s} - \log \lambda_i \cdot \lambda_i^{-s} \right) = 0 \quad \text{at} \quad s = 0. \quad (3.81) \]

Thus we get

\[ \log \text{Det}(\Delta_q + \epsilon) - (k \log \epsilon + \log \text{Det}^*\Delta_q) = o(\epsilon). \quad (3.82) \]

\[ \square \]

**Proof of Theorem 3.4.1** Let \( \mu_j(\epsilon)(j \geq 1) \) be the eigenvalues of \( R(\epsilon) \) with \( 0 < \mu_1(\epsilon) \leq \cdots \leq \mu_k(\epsilon) < \mu_{k+1}(\epsilon) \leq \cdots \). It is clear that \( \lim_{\epsilon \to 0} \mu_j(\epsilon) = 0 \) for \( 1 \leq j \leq k \). Then by the same argument as Lemma 3.4.3,

\[ \log \text{Det} R(\epsilon) = \log(\mu_1(\epsilon) \cdot \mu_k(\epsilon)) + \log \text{Det}^* R + o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \quad (3.83) \]

Now we want calculate \( \mu_1(\epsilon) \cdots \mu_k(\epsilon) \). Let \( \phi_1(\epsilon), \cdots, \phi_k(\epsilon) \) be orthonormal eigenforms of \( R(\epsilon) \) corresponding to eigenvalues \( \mu_1(\epsilon), \cdots, \mu_k(\epsilon) \). Then \( \phi_j(\epsilon) \to \phi_j \) as \( \epsilon \to 0 \),
where $\phi_j$ is the restriction of a harmonic form to $\Gamma$ with $\langle \phi_j, \phi_j \rangle_{\Gamma} = 1$. Let $a_{ij}(e) = \langle \psi_i, \phi_j(e) \rangle_{\Gamma} \; (1 \leq i, j \leq k)$ and $A(e) = (a_{ij}(e))$. Note that $\psi_i |_{\Gamma} = a_{ij}(e)\phi_j(e) + \psi_i(e) |_{\Gamma}$ for some $\psi_i(e) |_{\Gamma} \in \langle \text{span}\{\phi_1(e), \ldots, \phi_k(e)\} \rangle$. Define

$$I : C^\infty(E |_{\Gamma}) \to C^\infty(E |_{\Gamma}) \text{ by}$$

$$\varphi \mapsto \sum_{j=1}^{k} \int_{\Gamma} (\varphi, \psi_j)d\mu_{\Gamma} \cdot \psi_j |_{\Gamma} = \sum_{j=1}^{k} \langle \varphi, \psi_j \rangle_{\Gamma} \cdot \psi_j |_{\Gamma} . \quad (3.84)$$

Then

$$\langle I(\phi_i(e)), \phi_j(e) \rangle_{\Gamma} = \sum_{i=1}^{k} a_{ii}(e)a_{ij}(e) = (t^t A(e)A(e))_{ij}. \quad (3.85)$$

Define

$$G_{\epsilon} : C^\infty(E |_{\Gamma}) \to C^\infty(E |_{\Gamma}) \text{ by}$$

$$\varphi \mapsto R(\epsilon)^{-1}(\varphi) - \frac{1}{\epsilon}I(\varphi). \quad (3.86)$$

Then $G_{\epsilon}$ is a bounded operator with $\| G_{\epsilon} \| = \frac{1}{\mu_{k+1}(e)}$, which converges to $\frac{1}{\mu_{k+1}(0)}$ as $\epsilon \to 0$. Clearly

$$\langle R(\epsilon)^{-1}\phi_i(e), \phi_j(e) \rangle = \frac{1}{\epsilon} \langle I(\phi_i(e), \phi_j(e)) + \langle G_{\epsilon}(\phi_i(e)), \phi_j(e) \rangle. \quad (3.87)$$

For $1 \leq j \leq k$,

$$\frac{1}{\mu_j(e)} = \langle R(\epsilon)^{-1}\phi_j(e), \phi_j(e) \rangle$$

$$= \frac{1}{\epsilon} \langle I(\phi_j(e), \phi_j(e)) + \langle G_{\epsilon}(\phi_j(e)), \phi_j(e) \rangle$$

$$= \frac{1}{\epsilon} (t^t A(e)A(e))_{jj} + N_j(e), \quad (3.88)$$

where $N_j(e) := \langle G_{\epsilon}(\phi_i(e)), \phi_j(e) \rangle$ is bounded as $\epsilon \to 0$. For $i \neq j, 1 \leq i, j \leq k$,

$$0 = \langle R(\epsilon)^{-1}(\phi_i(e)), \phi_j(e) \rangle$$
\[
\begin{align*}
\frac{1}{\epsilon} (\langle I_1(\phi_1(\epsilon)), \phi_2(\epsilon) \rangle + \langle G_1(\phi_1(\epsilon)), \phi_2(\epsilon) \rangle) \\
= \frac{1}{\epsilon} (\langle t^A(\epsilon)A(\epsilon) \rangle_{ij} + \langle G_1(\phi_1(\epsilon)), \phi_2(\epsilon) \rangle).
\end{align*}
\]

(3.89)

Since \((t^A(\epsilon)A(\epsilon))_{ij}\) and \(\langle G_1(\phi_1(\epsilon)), \phi_2(\epsilon) \rangle\) are bounded, \((t^A(\epsilon)A(\epsilon))_{ij} \to 0\). Thus

\[
\frac{1}{\mu_1(\epsilon) \cdots \mu_k(\epsilon)} = \left( \frac{1}{\epsilon} (t^A(\epsilon)A(\epsilon))_{11} + N_1(\epsilon) \right) \cdots \left( \frac{1}{\epsilon} (t^A(\epsilon)A(\epsilon))_{kk} + N_k(\epsilon) \right)
= \frac{1}{\epsilon^k (\det A(\epsilon))^2} \left( \frac{(t^A(\epsilon)A(\epsilon))_{11}(t^A(\epsilon)A(\epsilon))_{22} \cdots (t^A(\epsilon)A(\epsilon))_{kk}}{(\det A(\epsilon))^2} + \epsilon \cdot \frac{\tilde{N}(\epsilon)}{(\det A(\epsilon))^2} \right),
\]

(3.90)

where \(\tilde{N}(\epsilon)\) is bounded as \(\epsilon \to 0\). Hence

\[
\log \det R(\epsilon) = k \log \epsilon - \log(\det A(\epsilon))^2 + \log \det^* R + o(\epsilon) \quad \text{as} \quad \epsilon \to 0.
\]

(3.91)

From the Lemma 3.4.3 and Theorem 3.4.1, we get

\[
\log \det (\Delta q + \epsilon) = k \log \epsilon + \log \det^* (\Delta q) + o(\epsilon)
\]

(3.92)

\[
\log \det (\Delta q + \epsilon) = \log \det (\Delta q, B) + \log \det (R(\epsilon)).
\]

(3.93)

From these three equations, we get

\[
\log \det^* \Delta q = -\log(\det A)^2 + \log \det^* R + \log \det (\Delta q, B).
\]

(3.94)
CHAPTER IV

The Coefficient $\pi_1$ on the Asymptotic Expansion of $\log \text{Det}$ of Elliptic Operators

4.1 Statement of Theorem

Let $M$ be a closed oriented Riemannian manifold of dimension $d$ and let $E \xrightarrow{\pi} M$ be a vector bundle of rank $k$. Let $A(\lambda) : C^\infty(E) \to C^\infty(E)$ be a classical elliptic pseudodifferential operator of order $m > 0$ with parameter $\lambda$ of weight $\chi > 0$ (see Definition 2.1.1), where $\lambda$ is a nonnegative real number. That is, the symbol of $A(\lambda)$ has an asymptotic expansion as in (4.2).

We assume that there is an angle $\theta$ such that the principal symbol $a_m(x, \xi, \lambda)$ does not have any eigenvalues on the ray $\{z \in \mathbb{C} | z = \rho e^{i\theta}, \rho \geq 0\}$ for $|\xi| + |\lambda|^\frac{1}{\chi} \neq 0$ and that $A(\lambda)$ does not have any eigenvalues in a sector $L_{[\theta - \epsilon, \theta + \epsilon]} = \{z \in \mathbb{C} | \theta - \epsilon \leq \arg z \leq \theta + \epsilon\}$ for some small $\epsilon > 0$. We call this $\theta$ an Agmon angle (cf. Definition 2.1.3). In fact from the compactness of the set $\{(x, \xi, \lambda) | x \in M, |\xi|^2 + |\lambda|^\frac{2}{\chi} = 1\}$, we know that $a_m(x, \xi, \lambda)$ does not have any eigenvalues in a sector $L_{[\theta - \delta, \theta + \delta]}$ for sufficiently small $\delta > 0$.

It is shown in Chapter 2 that as $\lambda \to +\infty$, $\log \text{Det}_\theta A(\lambda)$ admits an asymptotic
expansion of the following form:

\[ \log \text{Det}_\theta A(\lambda) \sim \pi_{-d} \lambda^{d} + \pi_{-d+1} \lambda^{d-1} x + \cdots + \pi_0 + \pi_1 \lambda^{-\frac{1}{2}} x + \cdots + \sum_{j=0}^{d} q_j \lambda^j \log \lambda \] (4.1)

as \( \lambda \to +\infty \). Here each coefficient \( \pi_i \) and \( q_j \) is computable in terms of the asymptotic symbol of \( A(\lambda) \).

Let \( \sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda) \) be an asymptotic symbol of \( A(\lambda) \) for some local coordinate \( U \). We also assume that for each \( j \), the function \( \tilde{a}_{m-j} : U \times \mathbb{R}^d \times \mathbb{R} \to \{k \times k \text{ matrices}\} \) defined by \( \tilde{a}_{m-j}(x, \xi, \lambda) = a_{m-j}(x, \xi, \min(\lambda, |\xi|)) \) is smooth. Then from \( A(\lambda) \) we can construct a classical elliptic pseudodifferential operator \( P \) with the same Agmon angle \( \theta \) on \( M \times S^1 \), where \( P \) is uniquely determined up to smoothing operators. Let \( \zeta_P(s) \) be the zeta function constructed from the eigenvalues of \( P \).

In this chapter, we will construct the elliptic operators \( P \) on \( M \times S^1 \) and will show that \( \pi_1 = \frac{m}{2} \zeta_P(0) \).

### 4.2 Construction of \( P \)

Suppose that in a local coordinate system \( U \), the asymptotic symbol of \( A(\lambda) \) is

\[ \sigma(A(\lambda)) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda). \]

Then

\[ a_{m-j}(x, \xi, \lambda) : U \times \mathbb{R}^d \times [0, \infty) \to \{k \times k \text{ matrices}\} \] (4.2)

with

\[ a_{m-j}(x, t\xi, t^\lambda) = t^{m-j} a_{m-j}(x, \xi, \lambda) \quad \text{for} \quad t > 0. \]
Now for each \( j \), we extend \( a_{m-j} \) to

\[
\tilde{a}_{m-j} : U \times S^1 \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \{k \times k \text{ matrices}\} \quad \text{by (4.3)}
\]

\[
(x, t, \xi, \lambda) \mapsto a_{m-j}(x, \xi, |\lambda|^\chi).
\]

Then \( \tilde{a}_{m-j} \) is smooth by the assumption of the smoothness of \( a_{m-j}(x, \xi, |\lambda|^\chi) \). Note that \( \tilde{a}_{m-j}(x, t, \xi, \lambda) \) is a homogeneous function of degree \( m - j \) with respect to \( \xi, \lambda \).

Consider a diagram

\[
p^* E \longrightarrow E \quad \downarrow \quad \downarrow \pi
\]

\[
M \times S^1 \quad \longrightarrow \quad M
\]

where \( p \) is the natural projection.

Choose an elliptic \( \psi DO \bar{A} : C^\infty(p^*E) \longrightarrow C^\infty(p^*E) \) whose asymptotic symbol is \( \sigma(\bar{A}) \sim \sum_{j=0}^{\infty} \tilde{a}_{m-j}(x, t, \xi, \lambda) \) in a local coordinate chart \( U \times S^1 \). Since the principal symbol \( a_m(x, \xi, |\lambda|^\chi) \) of \( \bar{A} \) does not have any eigenvalues in the sector \( L_{\theta-\delta, \theta+\delta} \) for \( |\xi| + |\lambda|^\chi \neq 0 \), there exists \( R > 0 \) such that \( \text{Spectrum}(\bar{A}) \cap \{z||z| \geq R, \theta - \delta \leq \arg z \leq \theta + \delta\} \) is empty (see [Sh] for details). Since the spectrum of \( \bar{A} \) is discrete, there are only finitely many eigenvalues of \( \bar{A} \) in \( \{z||z| < R, \theta - \delta \leq \arg z \leq \theta + \delta\} \). Note that each eigenspace corresponding to an eigenvalue is a finite-dimensional vector space by the ellipticity of \( \bar{A} \) of order \( m > 0 \). Let \( Q \) be the span of \{eigenvectors of \( \bar{A} \) whose eigenvalues are \( \rho e^{i\theta} \) for some \( \rho, 0 \leq \rho \leq R \}. Then \( Q \) is a finite-dimensional vector space. Define \( \phi : C^\infty(p^*E) \rightarrow C^\infty(p^*E) \) as the natural projection onto \( Q \).

Define \( P = \bar{A} - \text{Re}^{i\theta} \phi \). Then \( P \) is injective with an Agmon angle \( \theta \) and the asymptotic symbol of \( P \) is exactly the same as the asymptotic symbol of \( \bar{A} \). Define \( \zeta_P(s) = \sum_{i} \lambda_i^{-s} \), where \( \lambda_i \)'s are the eigenvalues of \( P \). Then by [Se1](also see [Wo1]),
\( \zeta_P(s) \) is regular at 0 with

\[
\zeta_P(0) = -\frac{e^{i\theta}}{m(2\pi)^{d+1}} \int_{M \times S^1} d\text{vol}(x, t) \int_{|\xi|^2 + \lambda^2 = 1} d(\xi, \lambda) \int_0^\infty \text{tr} \tilde{r}_{-m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) d\mu,
\]

where \( \tilde{r}_{-m-d-1} \) is the homogeneous part of degree \(-m-d-1\) in the asymptotic symbol of the resolvent \((P - \mu I)^{-1}\). Note that \( \tilde{r}_{-m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) \) does not depend on \( t \) in this case.

Recall that \( \log \text{Det}_g A(\lambda) \sim \pi_{-d} \lambda^d + \cdots + \pi_0 + \pi_1 \lambda^{-\frac{1}{2}} + \cdots + \sum_{j=0}^d q_j \lambda^j \log \lambda \) as \( \lambda \to +\infty \). Then our goal in this chapter is to prove the following theorem.

**Theorem 4.2.1** \( \pi_1 = \frac{m}{2} \zeta_P(0) \).

### 4.3 Proof of Theorem 4.2.1

From Chapter 2, we get

\[
\pi_1 = \frac{\partial}{\partial s} \bigg|_{s=0} \frac{1}{(2\pi)^d} \int_M \int_{\mathbb{R}^d} \frac{1}{2\pi i} \int_{\gamma} \mu^{-s} \text{tr} r_{-m-d-1}(x, \xi, 1, \mu) d\mu d\xi d\text{vol}(x),
\]

where \( \gamma \) is the contour \( \{re^{i\theta} \mid \infty > r \geq \epsilon\} \cup \{re^{i\varphi} \mid \theta \geq \varphi \geq \theta - 2\pi\} \cup \{re^{i(\theta-2\pi)} \mid \epsilon \leq r < \infty\} \) for some small \( \epsilon > 0 \). Consider

\[
J := \frac{1}{2\pi i} \int_{\gamma} \mu^{-s} \text{tr} r_{-m-d-1}(x, \xi, 1, \mu) d\mu.
\]

Then

\[
J = \frac{1}{2\pi i} \int_{\infty}^{\epsilon} (\mu e^{i\theta})^{-s} \text{tr} r_{-m-d-1}(x, \xi, 1, e^{i\theta} \mu) e^{i\theta} d\mu + \frac{1}{2\pi i} \int_{\epsilon}^{\infty} (\mu e^{i(\theta-2\pi)})^{-s} \text{tr} r_{-m-d-1}(x, \xi, 1, e^{i(\theta-2\pi)} \mu) e^{i(\theta-2\pi)} d\mu.
\]
\[
\frac{1}{2\pi i} \int_{|z|=\epsilon} z^{-s} \text{trr}_{-m-d-1}(x, \xi, 1, z)dz = -e^{i\theta} \cdot \frac{\sin \pi s}{\pi} \cdot e^{-is(\theta-\pi)} \int_{\epsilon}^{\infty} \text{trr}_{-m-d-1}(x, \xi, 1, e^{i\theta} \mu)d\mu + \frac{1}{2\pi i} \int_{|z|=\epsilon} z^{-s} \text{trr}_{-m-d-1}(x, \xi, 1, z)dz.
\]

(4.8)

Since \(\frac{1}{2\pi i} \int_{|z|=\epsilon} z^{-s} \text{trr}_{-m-d-1}(x, \xi, 1, z)dz\) converges to 0 as \(\epsilon \to 0\),

\[
J = -e^{i\theta} \cdot \sin \pi s \cdot e^{-is(\theta-\pi)} \int_{0}^{\infty} \mu^{-s} \text{trr}_{-m-d-1}(x, \xi, 1, e^{i\theta} \mu)d\mu.
\]

(4.9)

Hence we get

\[
\pi_1 = \frac{-e^{i\theta}}{(2\pi)^d} \int_{M} \int_{\mathbb{R}^d} \int_{0}^{\infty} \text{trr}_{-m-d-1}(x, \xi, 1, e^{i\theta} \mu)d\mu d\xi d\text{vol}(x),
\]

(4.10)

where \(r_{-m-d-1}\) is the homogeneous part of degree \(-m - d - 1\) in the asymptotic symbol of the resolvent \((A(\lambda) - \mu I)^{-1}\).

Note the relation \(\tilde{r}_{-m-d-1}(x, t, \xi, \lambda, e^{i\theta} \mu) = r_{-m-d-1}(x, \xi, |\lambda|^x, e^{i\theta} \mu)\). Then

\[
\zeta_\mu(0) = -\frac{2e^{i\theta} \cdot 2\pi}{m(2\pi)^{d+1}} \int_{M} d\text{vol}(x) \int_{|\xi|^2 + \lambda^2 = 1} \int_{\lambda > 0} d(\xi, \lambda)' \int_{0}^{\infty} \text{trr}_{-m-d-1}(x, \xi, |\lambda|^x, e^{i\theta} \mu)d\mu
\]

(4.11)

since the integrand is even in \(\lambda\). Here \(d(\xi, \lambda)'\) means the volume element on the sphere \(|\xi|^2 + \lambda^2 = 1\).

Set (I) = \(\int_{|\xi|^2 + \lambda^2 = 1} d(\xi, \lambda)' \int_{0}^{\infty} \text{trr}_{-m-d-1}(x, \xi, |\lambda|^x, e^{i\theta} \mu)d\mu\). Then using the projection from the upper hemisphere to \(\{\xi \in \mathbb{R}^d | \xi| < 1\}\), we obtain

\[
(I) = \int_{|\xi| < 1} \int_{0}^{\infty} \text{trr}_{-m-d-1}(x, \xi, (\sqrt{1 - |\xi|^2})^x, e^{i\theta} \mu)(1 - |\xi|^2)^{-\frac{d}{2}} d\mu d\xi
\]

(4.12)

\[
= \int_{|\xi| < 1} \int_{0}^{\infty} (1 - |\xi|^2)^{-\frac{m+d+2}{2}} \times
\]
since the weight of \( \mu \) is \( m \).

Consider a map \( \Phi : \mathbb{R}^d \times (0, \infty) \rightarrow \{ \xi \in \mathbb{R}^d | |\xi| < 1\} \times (0, \infty) \) defined by

\[
(\eta_1, \ldots, \eta_d, \nu) \mapsto \left( \frac{\eta_1}{\sqrt{1 + |\eta|^2}}, \ldots, \frac{\eta_d}{\sqrt{1 + |\eta|^2}}, \frac{\nu}{\sqrt{1 + |\eta|^2}} \right) = (\xi_1, \ldots, \xi_d, \mu)
\]

Then \( \Phi \) is a diffeomorphism with

\[
\det(J(\Phi)) = (1 + |\eta|^2)^{-\frac{m + d + 2}{2}} = (1 - |\xi|^2)^{-\frac{m + d + 2}{2}}.
\]

Hence we can get \( (I) = \int_{\mathbb{R}^d} \int_0^\infty tr r_{-m-d-1}(x, \eta, 1, e^{i\theta} \nu) d\nu d\eta \). \( \Box \)
Chapter V

Mapping Torus and the Asymptotic Expansion of $T(M, \varphi)(t)$

5.1 Statement of Theorem 5.1.1

Let $(M, g)$ be a closed oriented Riemannian manifold of dimension $n$. Given an orientation preserving diffeomorphism $\varphi : M \to M$, we define a mapping torus $M_\varphi$ by $M_\varphi = M \times I/(x, 1) \sim (\varphi(x), 0)$, where $I = [0, 1]$. Then $M_\varphi$ is a fiber bundle over $S^1$ with the fiber $M$. Let $\pi : M_\varphi \to S^1$ be the natural projection. Denote by $d\theta$ the canonical 1-form on $S^1$ and let $g_1$ be a Riemannian metric on $M_\varphi$. Define

$$\tilde{d}_q(t) : \Omega^q(M_\varphi) \to \Omega^{q+1}(M_\varphi) \text{ by } \tilde{d}_q(t) = d_q + t \pi^* d\theta \wedge,$$

(5.1)

where $\Omega^q(M_\varphi)$ is the set of smooth $q$-forms on $M_\varphi$ and $d_q$ is the exterior differential operator. Since $\tilde{d}_{q+1}(t)\tilde{d}_q(t) = 0$, we can define the cohomology associated to $\tilde{d}_q(t)$ as

$$H^q(M_\varphi, \tilde{d}_q(t), \mathbb{R}) = \ker\tilde{d}_q(t)/\text{Im}\tilde{d}_{q-1}(t).$$

(5.2)

Denote by $\varphi^* : H^*(M, \mathbb{R}) \to H^*(M, \mathbb{R})$ the endomorphism of the cohomology group induced by $\varphi$. Then it is well known that for $t$ away from eigenvalues of $\varphi^*$, $H^*(M_\varphi, \tilde{d}_q(t), \mathbb{R}) = 0$. 

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We denote the Laplacian associated to \( d_q(t) \) by \( \Delta_{q}^{tr* d\theta} \), i.e. \( \Delta_{q}^{tr* d\theta} = \tilde{d}_q^*(t)d_q(t) + \tilde{d}_{q-1}(t)d_{q-1}(t) \), where \( \tilde{d}_q^*(t) \) is the adjoint of \( \tilde{d}_q(t) \).

Since \( H^q(M, \tilde{d}_q(t), \mathbb{R}) \) is isomorphic to \( \text{Ker}\Delta_{q}^{tr* d\theta} \), for \( t > 0 \) large enough \( \Delta_{q}^{tr* d\theta} \) is an invertible differential operator and thus we can define the regularized determinant \( \text{Det}(\Delta_{q}^{tr* d\theta}) := \text{Det}_\pi(\Delta_{q}^{tr* d\theta}) \) with \( \pi \) as an Agmon angle.

Then we define the torsion \( T_0(M, \varphi, g_1)(t) \) for \( t \gg 0 \) by

\[
T_0(M, \varphi, g_1)(t) = \frac{1}{2} \sum_{q=0}^{n+1} (-1)^q \cdot q \cdot \log \text{Det}(\Delta_{q}^{tr* d\theta}).
\] (5.3)

Since \( H^*(M, \tilde{d}_q(t), \mathbb{R}) = 0 \) for \( t \gg 0 \), \( T_0(M, \varphi, g_1)(t) \) does not depend on the choice of a Riemannian metric \( g_1 \) on \( M, \varphi \) (cf. [RS]).

We also define

\[
T(M, \varphi)(t) = \frac{1}{2} \left( T_0(M, \varphi, g_1)(t) + T_0(M, \varphi^{-1}, g_2)(t) \right),
\] (5.4)

where \( g_1, g_2 \) are Riemannian metrics on \( M, \varphi \) and \( M^{-1} \) respectively. In this chapter, we will prove the following theorem.

**Theorem 5.1.1** For \( t \to \infty \),

\[
T(M, \varphi)(t) \sim \frac{1}{2} \cdot \chi(M) \cdot t \text{ asymptotically }.
\] (5.5)

5.2 Proof of Theorem 5.1.1 in the case of \( M \times S^1 \)

Let \((M, g)\) be an oriented closed Riemannian manifold of dimension \( n \) and let \( \varphi : M \to M \) be an orientation preserving diffeomorphism. If \( \varphi \) is the identity on \((M, g)\),
then $M_p = M \times S^1 \rightarrow S^1$ is a trivial fiber bundle over $S^1$. Let us give a product metric $g \oplus d\theta^2$ on $M \times S^1$, where $d\theta^2$ is the canonical metric on $S^1$.

In this section, we will prove that

$$T(M, Id)(t) = T_0(M, Id, g \oplus d\theta^2)(t) \sim \frac{1}{2} \cdot \chi(M) \cdot t \text{ asymptotically.} \quad (5.6)$$

First of all, we will prove that $\Delta^q_{t\pi^*d\theta} = \Delta^q_{M \times S^1} + t^2 Id$, where $\Delta^q_{M \times S^1}$ is the usual Laplacian acting on $q$-forms on $M \times S^1$. Recall that for $0 \leq q \leq n + 1$,

$$\Omega^{q-1}(M \times S^1) \xrightarrow{\tilde{d}_q(t)} \Omega^q(M \times S^1) \xrightarrow{d_q(t)} \Omega^{q+1}(M \times S^1) \rightarrow. \quad (5.7)$$

We define $\tilde{d}_q(t) := d_q + t\pi^*d\theta \wedge$, $\tilde{d}_q^*(t) := d_q^* + (t\pi^*d\theta \wedge)^*$, the adjoint of $\tilde{d}_q(t)$ and we also define $\Delta^q_{t\pi^*d\theta} = \tilde{d}_q^*(t)\tilde{d}_q(t) + \tilde{d}_{q-1}(t)\tilde{d}_{q-1}^*(t)$. Here $d_q$ is the usual exterior derivative acting from $\Omega^q(M \times S^1)$ to $\Omega^{q+1}(M \times S^1)$ and $d_q^*$ is the adjoint of $d_q$. In fact, $d_q^*$ is defined by using the Hodge operator $*$. Let $m = n + 1$ be the dimension of $M \times S^1$. Then $* : \Omega^q(M \times S^1) \rightarrow \Omega^{n+1-q}(M \times S^1)$ is defined by the following relation.

For any $\omega \in \Omega^q(M \times S^1)$,

$$\omega \wedge *\omega = \langle \omega, \omega \rangle dvol(M \times S^1), \quad (5.8)$$

where $(\ ,\ )$ is the metric on $\wedge^q T^*(M \times S^1)$ induced by the Riemannian metric $g \oplus d\theta^2$ on $M \times S^1$. Then $*$ satisfies the following properties.

$$** \omega = (-1)^{q(m-q)} \omega \text{ and } d_{q-1}^* \omega = (-1)^{mq+m+1} * d * \omega, \quad (5.9)$$

where $\omega \in \Omega^q(M \times S^1)$.

Now we will start from calculating $\tilde{d}^*_q$. Let $I = \{i_1, \ldots, i_q\}$. Then by $dx_I$ we mean $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$.
Lemma 5.2.1 \( \tilde{d}_q^* (t) = d_q^* + t \mu_{\theta^q} \), where \( \mu_{\theta^q} \) is the interior product defined by

\[
\mu_{\theta^q} (d \theta \wedge dx_j) = dx_j \quad \text{and} \quad \mu_{\theta^q} (dx_j) = 0. 
\]  

(5.10)

Proof Since \( \tilde{d}_q^* (t) = (d_q + t \pi^* d \theta \wedge)^* = d_q^* + t (\pi^* d \theta \wedge)^* \), it is enough to show that \( (\pi^* d \theta \wedge)^* = \mu_{\theta^q} \).

Let \( \omega \in \Omega^q (M \times S^1) \) and \( \eta \in \Omega^{q+1} (M \times S^1) \). In some local coordinate neighborhood \( U \) in \( M \times S^1 \), let \( \{ \alpha_1, \cdots, \alpha_n, d \theta \} \) be an orthonormal basis of \( T^* (M \times S^1) |_U \). Without loss of generality, we can assume that

\[
\omega = f(x, \theta) \alpha_I + g(x, \theta) \alpha_{I'} \wedge d \theta 
\]  

(5.11)

\[
\eta = h(x, \theta) \alpha_J + k(x, \theta) \alpha_{J'} \wedge d \theta, 
\]  

(5.12)

where \( |I| = q = |J'|, |I'| = q - 1, |J| = q + 1 \) and if \( I = (i_1, \cdots, i_q) \), then \( \alpha_I \) means \( \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_q} \).

Then \( \langle \pi^* d \theta \wedge \omega, \eta \rangle = \int_{M \times S^1} (\pi^* d \theta \wedge \omega) \wedge * \eta. \)

Note that

\[
\pi^* d \theta \wedge \omega * \eta = d \theta \wedge \omega * \eta 
\]

\[
= d \theta \wedge f(x, \theta) \alpha_I \wedge *(k(x, \theta) \alpha_{J'} \wedge d \theta)
\]

\[
= (-1)^q f(x, \theta) k(x, \theta) \alpha_I \wedge d \theta \wedge * (\alpha_{J'} \wedge d \theta). 
\]  

(5.13)

Assume that \( d \text{vol}(M \times S^1) = \alpha_l \wedge \alpha_{J'} \wedge d \theta \), where \( |l| = n - q \).

Then

\[
(\alpha_{J'} \wedge d \theta) \wedge * (\alpha_{J'} \wedge d \theta) = \alpha_l \wedge \alpha_{J'} \wedge d \theta
\]

\[
= (-1)^{q(n-q)} \alpha_{J'} \wedge \alpha_l \wedge d \theta. 
\]  

(5.14)
Thus
\[ d\theta \wedge *(\alpha_J \wedge d\theta) = (-1)^q(n-q)\alpha_i \wedge d\theta. \] (5.15)

Similarly
\[ \alpha_J \wedge *(\alpha_J \wedge d\theta) = \alpha_i \wedge \alpha_J \wedge d\theta = (-1)^q(n-q)\alpha_i \wedge d\theta. \] (5.16)

Thus
\[ *(\alpha_J, \alpha_J) = (-1)^q(n-q)\alpha_i \wedge d\theta. \] (5.17)

From (5.15) (5.17), \( d\theta \wedge *(\alpha_J \wedge d\theta) = *\alpha_J, \) and
\[ \langle \pi^*d\theta \wedge \omega, \eta \rangle = \int_{M \times S^1} (-1)^q f(x, \theta)k(x, \theta)\alpha_I \wedge *\alpha_J. \] (5.18)

On the other hand
\[ \iota_{\partial_\theta}(\eta) = \iota_{\partial_\theta}(h(x, \theta)\alpha_J + k(x, \theta)\alpha_J \wedge d\theta) = (-1)^q k(x, \theta)\alpha_J. \] (5.19)

Thus
\[ \langle \omega, \iota_{\partial_\theta}\eta \rangle = \int_{M \times S^1} f(x, \theta)\alpha_I \wedge *((-1)^q k(x, \theta)\alpha_J) \]
\[ = \int_{M \times S^1} (-1)^q f(x, \theta)k(x, \theta)\alpha_I \wedge *\alpha_J, \] (5.20)

and hence from (5.18) and (5.20),
\[ \langle \pi^*d\theta \wedge \omega, \eta \rangle = \langle \omega, \iota_{\partial_\theta}\eta \rangle. \] (5.21)

\[ \square \]

**Lemma 5.2.2**
\[ \Delta^{\pi^*d\theta}_q = \Delta_q + t(L_{\partial_\theta} + L^*_{\partial_\theta}) + t^2Id, \] (5.22)
where $\Delta_q$ is the usual Laplacian acting on differential $q$-forms on $M \times S^1$, $L_{\partial \theta}$ is the Lie derivative and $L_{\partial \theta}^*$ is the adjoint of $L_{\partial \theta}$.

Proof

\[
\Delta_q^{\pi^*d\theta} = d_q^* (t) d_q (t) + d_{q-1}(t)^* d_{q-1}(t)
\]

\[
= (d_q^* + t \iota_{\partial \theta})(d_q + t \pi^* d\theta \wedge) + (d_{q-1} + t \pi^* d\theta \wedge)(d_{q-1}^* + t \iota_{\partial \theta})
\]

\[
= d_q^* d_q + t d_q^* (\pi^* d\theta \wedge) + t \iota_{\partial \theta} d_q + t^2 \iota_{\partial \theta} (\pi^* d\theta \wedge)
\]

\[
+ d_{q-1}^* d_{q-1} - t d_{q-1}^* (\pi^* d\theta \wedge) + t (\pi^* d\theta \wedge) d_{q-1}^* + t^2 \pi^* d\theta \wedge (\iota_{\partial \theta})
\]

\[
= \Delta_q + t(\iota_{\partial \theta} d_q + d_{q-1} \iota_{\partial \theta})
\]

\[
+ t(d_q^* (\pi^* d\theta \wedge) + (\pi^* d\theta \wedge) d_q^*) + t^2 (\iota_{\partial \theta} (\pi^* d\theta \wedge) + (\pi^* d\theta \wedge) (\iota_{\partial \theta}))
\]

\[
= \Delta_q + t(L_{\partial \theta} + L_{\partial \theta}^*) + t^2 \text{Id.} \quad (5.23)
\]

Remark When $q = 0$, then one can see easily that

\[
L_{\partial \theta} + L_{\partial \theta}^* = 0.
\]

Indeed, for $f \in C^\infty(M \times S^1)$,

\[
L_{\partial \theta}^* f = (d_{-1} \iota_{\partial \theta} + \iota_{\partial \theta} d_0)^* f = d_0^*(\pi d\theta \wedge f)
\]

\[
= d_0^*(f d\theta) = - \ast d \ast (f d\theta)
\]

\[
= - \ast d(f \ast d\theta) = - \ast (\frac{\partial f}{\partial \theta} d\theta \wedge \ast d\theta)
\]

\[
= - \ast (\frac{\partial f}{\partial \theta} d\text{vol}(M \times S^1))
\]

\[
= - \frac{\partial f}{\partial \theta} = - L_{\partial \theta} f. \quad (5.24)
\]

Hence, $(L_{\partial \theta} + L_{\partial \theta}^*) = 0.$
Now we are going to verify that \( L_\varphi + L^*_\varphi = 0 \) for differential forms of any degree so that we can conclude that \( \Delta_q^\varphi d\theta = \Delta_q + t^2 I d. \)

**Lemma 5.2.3** Let \(|I| = q\). Then

\[
\begin{align*}
(i) & \quad * (d\theta \wedge *(dx_I \wedge d\theta)) = (-1)^{n-q} dx_I \\
(ii) & \quad *(dx_i \wedge *(dx_I \wedge d\theta)) = (-1)^{n+q} d\theta \wedge *(dx_i \wedge dx_I).
\end{align*}
\]

**Proof** Since each term does not contain a differential operator, without loss of generality we can assume that \( \{dx_1, \ldots, dx_n, d\theta\} \) is an orthonormal basis of \( T^* (M \times S^1) |_U \). Let

\[
dvol(M \times S^1) = dx_I \wedge dx_I \wedge d\theta, \text{ where } |I| = n - q.
\]

(i) Note that

\[
(dx_I \wedge d\theta) \wedge *(dx_I \wedge d\theta) = dx_I \wedge dx_I \wedge d\theta = \begin{array}{c}
(-1)^{(n-q)(q+1)} dx_I \wedge d\theta \wedge dx_I.
\end{array}
\]

Thus

\[
*(dx_I \wedge d\theta) = (-1)^{(n-q)(q+1)} dx_I
\]

and therefore

\[
\text{LHS} = *(d\theta \wedge *(dx_I \wedge d\theta)) = (-1)^{(n-q)(q+1)} * (d\theta \wedge dx_I).
\]

\[
(dx_I \wedge dx_I) \wedge *(d\theta \wedge dx_I) = dx_I \wedge dx_I \wedge d\theta = (-1)^n d\theta \wedge dx_I \wedge dx_I
\]
\[ *(d\theta \wedge dx_l) = (-1)^n dx_l \]  

Hence

\[ LHS = (-1)^{(n-q)(q+1)} (-1)^n dx_I = (-1)^n q dx_I = RHS. \]  

(ii) If \( i \notin I \), then \( LHS = RHS = 0 \). Now we assume that \( i \in I \). From (5.29)

\[ LHS = (-1)^{(n-q)(q+1)} *(dx_i \wedge dx_l). \]  

From \((dx_i \wedge dx_l) \wedge *(dx_i \wedge dx_l) = dx_I \wedge dx_I \wedge d\theta,

\[ dx_i \wedge *(dx_i \wedge dx_l) = (-1)^{n-q} dx_I \wedge d\theta, \]  

and therefore

\[ *(dx_i \wedge dx_l) = (-1)^{n-q} \frac{\partial}{\partial x_i} dx_I \wedge d\theta. \]  

Thus

\[ LHS = (-1)^{q(n-1)} \frac{\partial}{\partial x_i} dx_I \wedge d\theta. \]  

From \( dx_I \wedge *dx_I = dx_I \wedge dx_I \wedge d\theta,

\[ *dx_I = (-1)^{q(n-q)} dx_I \wedge d\theta. \]  

Hence

\[ d\theta \wedge *(dx_i \wedge *dx_I) = (-1)^{q(n-q)} d\theta \wedge *(dx_i \wedge dx_I \wedge d\theta). \]  

\[ (dx_i \wedge dx_I \wedge d\theta) \wedge *(dx_i \wedge dx_I \wedge d\theta) = dx_I \wedge dx_I \wedge d\theta = (-1)^q dx_I \wedge d\theta \wedge dx_I. \]
\[
d x_i \wedge *(dx_i \wedge dx_I \wedge d\theta) = (-1)^{q+n-q+1} dx_I
\]

Thus

\[
*(dx_i \wedge dx_I \wedge d\theta) = (-1)^{n+1} \frac{\partial g}{\partial x_i} dx_I,
\]

and by (5.39) (5.42)

\[
RHS = (-1)^{n+q} d\theta \wedge *(dx_i \wedge *dx_I)
\]

\[
= (-1)^{n+q}(-1)^{q(q-n)}(-1)^{n+1} d\theta \wedge \frac{\partial g}{\partial x_i} dx_I
\]

\[
= (-1)^{n+q}(-1)^{q(q-n)}(-1)^{n+1}(-1)^{q-1} \frac{\partial g}{\partial x_i} dx_I \wedge d\theta
\]

\[
= (-1)^{q(n+1)} \frac{\partial g}{\partial x_i} dx_I \wedge d\theta = LHS
\]

\[
\square
\]

**Lemma 5.2.4**

\[
(L_{\frac{\partial}{\partial \theta}} + L^*_{\frac{\partial}{\partial \theta}}) \omega = 0 \text{ for any } q\text{-form } \omega, 0 \leq q \leq n + 1.
\]

**Proof** Let us set

\[
Q = L_{\frac{\partial}{\partial \theta}} + L^*_{\frac{\partial}{\partial \theta}}
\]

\[
= d_{q-1} \frac{\partial g}{\partial \theta} + \frac{\partial g}{\partial \theta} d_q + d_q^*(d\theta \wedge) + (d\theta \wedge) d_q^*.
\]

**Case 1** Let \( \omega = f(x, \theta) dx_I \wedge d\theta \), where \(|I| = q - 1\). Then

\[
Q(\omega) = (-1)^{q-1} d_{q-1}(f(x, \theta) dx_I) + \frac{\partial g}{\partial \theta} d_q(f(x, \theta) dx_I \wedge d\theta)
\]

\[
+ d\theta \wedge d_q^* f(x, \theta) dx_I \wedge d\theta
\]

\[
= (-1)^{q-1} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I + (-1)^{q-1} \frac{\partial f}{\partial \theta} d\theta \wedge dx_I
\]
\[ + \frac{\partial f}{\partial \theta} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I \wedge d\theta + (-1)^{(n+1)q+n+1+1} d\theta \wedge d*(f(x, \theta)dx_I \wedge d\theta) \]

\[ = (-1)^q \frac{\partial f}{\partial \theta} d\theta \wedge dx_I + (-1)^{(n+1)q+n} d*(f(x, \theta) * (dx_I \wedge d\theta)) \]

\[ = (-1)^q \frac{\partial f}{\partial \theta} d\theta \wedge dx_I + (-1)^{(n+1)q+n} d\theta \wedge * \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(dx_I \wedge d\theta) \right) \]

\[ + (-1)^{(n+1)q+n} d\theta \wedge * \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(dx_I \wedge (dx_I \wedge d\theta)) \right) \]

\[ = (-1)^q \frac{\partial f}{\partial \theta} d\theta \wedge dx_I + (-1)^{(n+1)q+n} \frac{\partial f}{\partial \theta} d\theta \wedge * \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(dx_I \wedge d\theta) \right) \]

\[ \text{Case 2} \]

Let \( \omega = f(x, \theta)dx_I \), where \(|I| = q\). Then

\[ Q(\omega) = \int_{\mathbb{R}^n} d_q(f(x, \theta)dx_I) + d^* \left( f(x, \theta) d\theta \wedge dx_I \right) + d\theta \wedge d^* \left( f(x, \theta)dx_I \right) \]

\[ = \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_I + \frac{\partial f}{\partial \theta} d\theta \wedge dx_I \right) \]

\[ + (-1)^{(n+1)q+n+1+1} d* \left( f(x, \theta) d\theta \wedge dx_I \right) \]

\[ + d\theta \wedge (-1)^{(n+1)q+n+1+1} * d * \left( f(x, \theta)dx_I \right) \]

\[ = \frac{\partial f}{\partial \theta} dx_I + (-1)^{(n+1)+1} * d \left( f(x, \theta) * (d\theta \wedge dx_I) \right) \]

\[ + (-1)^{(n+1)q+n} \frac{\partial f}{\partial \theta} d\theta \wedge * \left( f(x, \theta) * (dx_I) \right) \]

\[ = \frac{\partial f}{\partial \theta} dx_I + (-1)^{(n+1)+1} \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(d\theta \wedge dx_I) \right) \]

\[ + (-1)^{q(n+1)+1} \left( \frac{\partial f}{\partial \theta} d\theta \wedge *(d\theta \wedge dx_I) \right) \]

\[ + (-1)^{(n+1)q+n} d\theta \wedge * \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(dx_I) \right) \]

\[ + (-1)^{(n+1)q+n} d\theta \wedge * \left( \frac{\partial f}{\partial \theta} d\theta \wedge *(dx_I) \right) \]

\[ + (-1)^{(n+1)q+n} d\theta \wedge * \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge *(dx_I) \right) \]

\[ = \frac{\partial f}{\partial \theta} dx_I + (-1)^{(n+1)+1} \frac{\partial f}{\partial \theta} \left( d\theta \wedge *(d\theta \wedge dx_I) \right) \]
\[ (+1)^{q(n+1)+1} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \ast (dx_i \wedge \ast (d\theta \wedge dx_I)) \]
\[ + (+1)^{q(n+1)+n} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \ast (d\theta \wedge * (dx_i \wedge * (dx_I))). \]

(5.47)

From Lemma 5.2.3 (i),

\[
\frac{\partial f}{\partial \theta} dx_I + (-1)^{q(n+1)+1} \frac{\partial f}{\partial \theta} \ast (d\theta \wedge \ast (d\theta \wedge dx_I)) = \frac{\partial f}{\partial \theta} dx_I + (-1)^{q(n+1)+1} (-1)^q \frac{\partial f}{\partial \theta} dx_I \\
= \frac{\partial f}{\partial \theta} dx_I + (-1) \frac{\partial f}{\partial \theta} dx_I = 0.
\]

(5.48)

From Lemma 5.2.3 (ii)

\[
(-1)^{q(n+1)+1} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \ast (dx_i \wedge \ast (d\theta \wedge dx_I)) \\
+ (-1)^{q(n+1)+n} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \ast (d\theta \wedge * (dx_i \wedge * dx_I)) \\
= (-1)^{q(n+1)+1} (-1)^q \sum_{i=1}^{n} (-1)^{n+q} \frac{\partial f}{\partial x_i} \wedge \ast (dx_i \wedge * dx_I) \\
+ (-1)^{q(n+1)+n} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \wedge \ast (dx_i \wedge * dx_I) \\
= (-1)^{q(n+1)} \left((-1)^{n+1} + (-1)^n\right) \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \wedge \ast (dx_i \wedge * dx_I). \\
= 0.
\]

(5.49)

\[\square\]

So far, we have shown that $\Delta_q^{t^{*}d\theta} = \Delta^q_{M \times S^1} + t^2 Id$ when $M_\varphi = M \times S^1$.

Note that

\[
\Omega^q(M \times S^1) =
C^\infty(M \times S^1) \Omega^q(M) \otimes \Omega^0(S^1) \oplus C^\infty(M \times S^1) \Omega^{q-1}(M) \otimes \Omega^1(S^1).
\]

(5.50)
Hence

\[ \Delta^q_{M \times S^1} = \Delta^q_{M 	imes S^1} + t^2 \text{Id} \quad (5.51) \]

\[ = \begin{pmatrix} \Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^0_{S^1} + t^2 \text{Id} & 0 \\ 0 & \Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^1_{S^1} + t^2 \text{Id} \end{pmatrix}. \]

Now for each 0 \leq q \leq n + 1, consider

\[ \Delta^q_{M \times S^1} = \begin{pmatrix} \Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^0_{S^1} + t^2 \text{Id} & 0 \\ 0 & \Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^1_{S^1} \end{pmatrix}. \quad (5.52) \]

First of all, to apply the formula (2.30) in Proposition 2.2.3 we need to find an asymptotic expansion of \( \text{tr} e^{-t \Delta^q_{M \times S^1}} \) for \( t \to 0^+ \). Now

\[ \text{tr} e^{-t \Delta^q_{M \times S^1}} = \text{tr} \{ e^{-t(\Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^0_{S^1})} + e^{-t(\Delta^q_M \otimes \text{Id}_{S^1} + \text{Id}_M \otimes \Delta^1_{S^1})} \} \]

\[ = \text{tr} \left( e^{-t \Delta^q_M} \otimes e^{-t \Delta^0_{S^1}} \right) + \text{tr} \left( e^{-t \Delta^q_M} \otimes e^{-t \Delta^1_{S^1}} \right) \]

\[ = \text{tr} e^{-t \Delta^q_M} \cdot \text{tr} e^{-t \Delta^0_{S^1}} + \text{tr} e^{-t \Delta^q_M} \cdot \text{tr} e^{-t \Delta^1_{S^1}}. \quad (5.53) \]

Since \( \Delta^0_{S^1} = * \Delta^1_{S^1} \), where * is the Hodge operator on \( S^1 \), the eigenvalues of \( \Delta^0_{S^1} \) are exactly same as eigenvalues of \( \Delta^1_{S^1} \). Hence

\[ \text{tr} e^{-t \Delta^q_{M \times S^1}} = \left( \text{tr} e^{-t \Delta^q_M} + \text{tr} e^{-t \Delta^q_M} \right) \cdot \text{tr} e^{-t \Delta^0_{S^1}}. \quad (5.54) \]

Now

\[ \frac{1}{2} \sum_{q=0}^{n+1} (-1)^q + 1 \cdot q \cdot \text{tr} e^{-t \Delta^q_{M \times S^1}} \]
\[
\begin{align*}
&= \frac{1}{2} \text{tre}^{-t\Delta^0_{s1}} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot (\text{tre}^{-t\Delta^q_M} + \text{tre}^{-t\Delta^q_{M'}}) \\
&= \frac{1}{2} \text{tre}^{-t\Delta^0_{s1}} \left( \sum_{q=0}^{n} (-1)^{q+1} \cdot q \cdot \text{tre}^{-t\Delta^q_M} + \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \text{tre}^{-t\Delta^q_{M'}} \right) \\
&= \frac{1}{2} \text{tre}^{-t\Delta^0_{s1}} \left( \sum_{q=0}^{n} (-1)^{q+1} \cdot q \cdot \text{tre}^{-t\Delta^q_M} + \sum_{q=0}^{n} (-1)^{q} (q + 1) \text{tre}^{-t\Delta^q_M} \right) \\
&= \frac{1}{2} \text{tre}^{-t\Delta^0_{s1}} \sum_{q=0}^{n} (-1)^{q} \text{tre}^{-t\Delta^q_M}. \quad (5.55)
\end{align*}
\]

Since \(\sum_{q=0}^{n} (-1)^{q} \text{tre}^{-t\Delta^q_M}\) is equal to \(\text{Index}(\Delta_M)\) which is same as \(\chi(M)\), the Euler characteristic of \(M\) (see [Gi] for details), we get

\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \text{tre}^{-t\Delta^q_M \times s^1} = \frac{1}{2} \chi(M) \text{tre}^{-t\Delta^0_{s1}}. \quad (5.56)
\]

Note that \(\text{tre}^{-t\Delta^0_{s1}}\) has an asymptotic expansion as follows.

\[
\text{tre}^{-t\Delta^0_{s1}} \sim a_0 t^{-\frac{1}{2}} + a_1 + a_2 t^{\frac{1}{2}} + \cdots \text{ as } t \rightarrow 0^+. \quad (5.57)
\]

Here setting \(\sigma((\lambda - \Delta^0_{s1})^{-1}) \sim \sum_{j=0}^{\infty} p_{-2-j}(\theta, \xi, \lambda)\),

\[
a_j = \int_{S^1} \int_{\mathbb{R}} \frac{1}{2\pi} \int_\gamma e^{-\lambda} p_{-2-j} d\lambda \frac{d\xi}{\sqrt{2\pi}} \frac{d\theta}{\sqrt{2\pi}}, \quad (5.58)
\]

where \(\gamma\) is a contour enclosing nonnegative real axis counterclockwisely in the complex plane (see [Gi] for details).

Since \(p_{-2-j} = 0\) for \(j \geq 1\), \(a_1 = a_2 = \cdots = 0\).

From the fact that \(p_{-2} = \frac{1}{\lambda - \xi^2}\),

\[
a_0 = \int_{S^1} \int_{\mathbb{R}} \frac{1}{2\pi i} \int_\gamma e^{-\lambda} \frac{d\lambda}{\lambda - \xi^2} \frac{d\xi}{\sqrt{2\pi}} \frac{d\theta}{\sqrt{2\pi}} = \int_{S^1} \int_{\mathbb{R}} e^{-\xi^2} \frac{d\xi}{\sqrt{2\pi}} \frac{d\theta}{\sqrt{2\pi}} = \frac{1}{2\sqrt{\pi}}. \quad (5.59)
\]
Hence
\[ \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \text{tr} e^{-t \Delta_{M \times S^1}} = \frac{1}{4 \sqrt{\pi}} \chi(M) t^{-\frac{1}{2}} \text{ as } t \to 0^+ \] (5.60)

**Remark** In this chapter we normalized the volume of $S^1$ as $\text{vol}(S^1) = \text{vol}([0, 1]) = 1$.

To apply the formula (2.30) in Proposition 2.2.3, we need to distinguish zero eigenvalues and nonzero ones. For each $i \in \mathbb{Z}^+$, let $\lambda_{i,q}$ be a positive eigenvalue of $\Delta_{M \times S^1}^q$ and suppose that $\sum_{i=1}^{\infty} e^{-t \lambda_{i,q}} \sim \sum_{k=0}^{\infty} C_{k,q} t^{\frac{k-(n+1)}{2}}$ for some constants $C_{k,q}$. Let $\zeta_{M \times S^1,q}(s)$ be the zeta function consisting of eigenvalues of $\Delta_{M \times S^1}^q + \lambda I_d$ for a positive real number $\lambda$. Then
\[ \zeta_{M \times S^1,q}(s) = \beta_q \lambda^{-s} + \sum_{\lambda_{i,q} > 0} (\lambda_{i,q} + \lambda)^{-s}, \] (5.61)

where $\beta_q$ is the $q$-th Betti number of $M \times S^1$.

Now
\[ \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \text{tr} e^{-t \Delta_{M \times S^1}^q} \]
\[ \sim \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \beta_q + \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \sum_{k=0}^{\infty} C_{k,q} t^{\frac{k-(n+1)}{2}} \]
\[ = \frac{1}{2} \chi(M) + \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \sum_{k=0}^{\infty} C_{k,q} t^{\frac{k-(n+1)}{2}} \]
\[ = \frac{1}{4 \sqrt{\pi}} \chi(M) t^{-\frac{1}{2}} \text{ as } t \to 0^+ \text{ by (5.60)} \] (5.62)

Hence
\[ \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \sum_{k=0}^{\infty} C_{k,q} t^{\frac{k-(n+1)}{2}} = \frac{1}{4 \sqrt{\pi}} \chi(M) t^{-\frac{1}{2}} - \frac{1}{2} \chi(M). \] (5.63)

By the formula (2.30) in Proposition 2.2.3,
\[ \frac{d}{ds} \big|_{s=0} \left\{ \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \sum_{\lambda_{i,q} > 0} (\lambda_{i,q} + \lambda)^{-s} \right\} \sim \]
\[
\frac{1}{4\sqrt{\pi}}\chi(M)\Gamma(-\frac{1}{2})\lambda^{\frac{1}{2}} - \frac{1}{2}\chi(M)\log\lambda \text{ as } \lambda \to +\infty. \tag{5.64}
\]

Note that
\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \zeta_{M \times S^1, q}(s) = \frac{1}{2}\chi(M)\lambda^{-s} + \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \sum_{\lambda_{i,q} > 0} (\lambda_{i,q} + \lambda)^{-s}. \tag{5.65}
\]

Then by (5.64)
\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \zeta'_{M \times S^1, q}(0)
\sim -\frac{1}{2}\chi(M)\log\lambda + \frac{1}{4\sqrt{\pi}}\chi(M)\Gamma(-\frac{1}{2})\lambda^{\frac{1}{2}} + \frac{1}{2}\chi(M)\log\lambda
= \frac{1}{4\sqrt{\pi}}\chi(M)(-\frac{1}{2})\Gamma(-\frac{1}{2})\lambda^{\frac{1}{2}} = -\frac{1}{2\sqrt{\pi}}\chi(M)(\frac{1}{2})\lambda^{\frac{1}{2}}
= -\frac{1}{2}\chi(M)\lambda^{\frac{1}{2}} \text{ as } \lambda \to +\infty. \tag{5.66}
\]

Now replacing \( \lambda \) by \( t^2 \),
\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \log\text{Det}(\Delta_{M \times S^1}^{q} + t^2 Id)
= -\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \zeta'_{M \times S^1, q}(0)
\sim \frac{1}{2}\chi(M)t \text{ as } t \to +\infty. \tag{5.67}
\]

### 5.3 Proof of Theorem 5.1.1 for a General Mapping Torus \( M_\varphi \)

In Chapter 2, we proved that \( \log\text{Det}(\Delta_{q}^{tx^{*}d^q}) \) has an asymptotic expansion of the form
\[
\log\text{Det}(\Delta_{q}^{tx^{*}d^q}) \sim \sum_{j=0}^{\infty} \pi_j t^{n+1-j} + \sum_{j=0}^{n+1} q_j t^{n+1-j}\log t \text{ as } t \to +\infty. \tag{5.68}
\]
Here \( \pi_j \) and \( q_k \) are computed as follows. Let \( \sum_{k=0}^{\infty} r_{-2-j}(\mu, t, x, \xi) \) be the asymptotic expansion of the symbol of the resolvent \( (\mu - \Delta^*_{q} d\theta) \) and let \( \{\rho_j\} \) be a partition of unity subordinate to some local coordinate system. Then \( J_j(s, 1, x) := \frac{1}{2\pi i} \int_{\mathbb{R}^{n+1}} d\xi \int_{\gamma} \mu^{-s} r_{-2-j}(\mu, 1, x, \xi) d\mu \) has an analytic continuation with respect to \( s \in \mathbb{C} \) and 0 as a regular value. Then

\[
\pi_j = \frac{1}{2\pi^{n+1}} \frac{d}{ds} \bigg|_{s=0} \sum_k \int_{M^o} J_j(s, 1, x) \rho_k(x) d\text{vol}(x) \tag{5.69}
\]

and

\[
q_j = \frac{1}{2\pi^{n+1}} \sum_k \int_{M^o} J_j(0, 1, x) \rho_k(x) d\text{vol}(x). \tag{5.70}
\]

Note that in case of a mapping torus, \( \pi_j \) and \( q_k \) does not depend on the choice of Riemannian metrics for \( t \gg 0 \).

Now let us consider \( M \times S^1 \) with the product metric \( g \oplus d\theta^2 \), where \( g \) is a Riemannian metric on \( M \) and \( d\theta^2 \) is a canonical metric on \( S^1 \). Let \( \{(U_k, \phi_k)\} \) be a local coordinate system of \( M \) and \( \{\rho_k\} \) be a partition of unity subordinate to \( (U_k, \phi_k) \). Then \( \Delta^*_{q} d\theta^2 = \Delta^*_{M \times S^1} + \ell^2 \text{Id} \) and its symbol does not depend on \( S^1 \)-variable. Let \( (x, \theta) \) be a variable of \( M \times S^1 \). If we denote

\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \log \text{Det}(\Delta^*_{q} d\theta) \sim
\]

\[
\sum_{j=0}^{\infty} c_j \ell^{n+1-j} + \sum_{j=0}^{n+1} d_j \ell^{n+1-j} \log t \text{ as } t \to +\infty, \tag{5.71}
\]

then

\[
c_j = \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \frac{1}{(2\pi)^{n+1}} \frac{d}{ds} \bigg|_{s=0} \sum_k \int_{M \times S^1} J_j(s, 1, x) \rho_k(x) d\text{vol}(x, \theta) \]

by Section 5.2, where \( \delta_{nj} = 1 \) if \( j = n \) and 0 if \( j \neq n \).

Now let us denote \( S^1 = [0, 1]/0 \sim 1 \) and let \( V_1 = (\frac{1}{5}, \frac{2}{5}), \ V_2 = (\frac{3}{5}, \frac{4}{5}), \ V_3 = [0, \frac{1}{5} + \epsilon) \cup (\frac{4}{5} - \epsilon, 1) \) and \( V_4 = (\frac{2}{5} - \epsilon, \frac{3}{5} + \epsilon) \) for a sufficiently small \( \epsilon > 0 \). Let \( \{\eta_k\}_{1 \leq k \leq 4} \) be a partition of unity subordinate to \( \{V_k\}_{1 \leq k \leq 4} \).

Let \( g_1, g_2 \) be Riemannian metrics on \( M \). Choose a nondecreasing smooth function \( \omega(r) \) on \( \mathbb{R} \) such that \( \omega(r) = 0 \) if \( r \leq 0 \), 1 for \( r \geq 1 \) and \( \omega(r) \) is symmetric to the line \( r = \frac{1}{2} \). Set

\[
\omega_1(r) = \omega(5r - 1), \quad \omega_2(r) = \omega(5r - 3).
\]

Then we define a new metric \( G(r, \theta) \) on \( M \times S^1 \) as follows.

\[
G(x, \theta) = \begin{cases} 
  g \oplus d\theta^2 & 0 < \theta < 1 \\
  (1 - \omega_1(\theta))g + \omega_1(\theta)g' \oplus d\theta^2 & \frac{1}{5} \leq \theta \leq \frac{4}{5} \\
  g' \oplus d\theta^2 & \frac{3}{5} \leq \theta \leq \frac{4}{5} \\
  (1 - \omega_2(\theta))g' + \omega_2(\theta)g \oplus d\theta^2 & \frac{4}{5} \leq \theta \leq \frac{3}{5} \\
  g \oplus d\theta^2 & 1 \leq \theta \leq 2
\end{cases}
\]  

(5.74)

Then

\[
c_j = \sum_{q=0}^{n+1} (-1)^q \cdot |q| \sum_{l, k} \frac{d}{ds} \left|_{s=0} \frac{1}{(2\pi)^{n+1}} \int_{S^n} \rho_l(x) \eta_k(\theta) J_j(s, x, \theta, 1) d\omega(x, \theta) \right|
\]

(5.75)

Note that \( J_j(s, x, \theta, 1) \) coming from the product metric of the form \( g \oplus d\theta^2 \) does not depend on the \( S^1 \)-variable \( \theta \). Hence
\[ c_j = \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \sum_{k \neq 1,2} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_k(\theta) J_j(s, x, \theta, 1) \, dvol(x, \theta) \]

\[ + \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \times \left( \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_1(\theta) J_j dvol(x, \theta) + \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_2(\theta) J_j dvol(x, \theta) \right) \]

\[ = \left( \sum_{k \neq 1,2} \int_{\mathbb{S}^1} \eta_k(\theta) d\theta \right) \cdot \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \int_{M} \rho_l(x) J_j(s, x, 1) dvol(x) \]

\[ + \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \times \left( \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_1(\theta) J_j dvol(x, \theta) + \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_2(\theta) J_j dvol(x, \theta) \right) \]

\[ = \frac{1}{2} \chi(M) \delta_{nj} \left( \sum_{k \neq 1,2} \int_{\mathbb{S}^1} \eta_k(\theta) d\theta \right) + C(g, g') + C(g', g), \quad (5.76) \]

where

\[ C(g, g') = \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_1(\theta) J_j dvol(x, \theta) \quad (5.77) \]

and

\[ C(g, g') = \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \sum_{l} \frac{d}{ds} \left|_{s=0} \right. \frac{1}{(2\pi)^{n+1}} \int_{M \times \mathbb{S}^1} \rho_l(x) \eta_2(\theta) J_j dvol(x, \theta). \quad (5.78) \]

Hence

\[ \frac{1}{2} \chi(M) \delta_{nj} = \frac{1}{2} \chi(M) \delta_{nj} \left( \sum_{k \neq 1,2} \int_{\mathbb{S}^1} \eta_k(\theta) d\theta \right) + C(g, g') + C(g', g), \quad (5.79) \]

and since \( \sum_{k=1}^{4} \int_{\mathbb{S}^1} \eta_k(\theta) d\theta = 1 \),

\[ C(g, g') + C(g', g) = \frac{1}{2} \chi(M) \delta_{nj} \left( \int_{\mathbb{S}^1} \eta_1(\theta) d\theta + \int_{\mathbb{S}^1} \eta_2(\theta) d\theta \right). \quad (5.80) \]
Now let us consider a general mapping torus. Let \((M, g)\) be an oriented Riemannian manifold and let \(\varphi : M \to M\) be an orientation preserving diffeomorphism. Then \(\varphi\) is an isometry from \((M, \varphi^* g)\) to \((M, g)\). Note that

\[
M_{\varphi^{-1}} = M \times I/(x, 1) \sim (\varphi^{-1}(x), 0) = M \times I/(x, 0) \sim (\varphi(x), 1).
\] (5.81)

Then the map \(\Phi : M_{\varphi} \to M_{\varphi^{-1}}\) defined by \([x, t] \mapsto [x, 1 - t]\) is a diffeomorphism from \(M_{\varphi}\) to \(M_{\varphi^{-1}}\).

Now we give metrics \(G\) and \(G'\) on \(M_{\varphi}, M_{\varphi^{-1}}\) as follows.

\[
G(x, \theta) = \begin{cases} 
\varphi^* g \oplus d\theta^2 & 0 \leq \theta \leq \frac{1}{2} \\
((1 - \omega_1(\theta))\varphi^* g + \omega_1(\theta)g) \oplus d\theta^2 & \frac{1}{2} \leq \theta \leq \frac{3}{4} \\
g \oplus d\theta^2 & \frac{3}{4} \leq \theta \leq 1 
\end{cases}
\] (5.82)

\[
G'(x, \theta) = \begin{cases} 
g \oplus d\theta^2 & 0 \leq \theta \leq \frac{3}{4} \\
((1 - \omega_2(\theta))g \oplus \omega_2(\theta)\varphi^* g) \oplus d\theta^2 & \frac{3}{4} \leq \theta \leq 1 \\
\varphi^* g \oplus d\theta^2 & \frac{1}{2} \leq \theta \leq \frac{3}{4} 
\end{cases}
\] (5.83)

Then \(\Phi\) is an (orientation reversing) isometry from \((M_{\varphi}, G)\) to \((M_{\varphi^{-1}}, G')\). Denote by \(\Delta_q^{t* d\theta}, \tilde{\Delta}_q^{t* d\theta}\) the Laplacians on \((M_{\varphi}, G)\) and \((M_{\varphi^{-1}}, G')\) respectively and suppose that

\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \log \text{Det}(\Delta_q^{t* d\theta}) \sim \sum_{j=0}^{\infty} c_j t^{n+1-j} + \sum_{j=0}^{n+1} d_j t^{n+1-j} \log t \quad \text{as } t \to \infty \quad (5.84)
\]

\[
\frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \log \text{Det}(\tilde{\Delta}_q^{t* d\theta}) \sim \sum_{j=0}^{\infty} \tilde{c}_j t^{n+1-j} + \sum_{j=0}^{n+1} \tilde{d}_j t^{n+1-j} \log t \quad \text{as } t \to \infty \quad (5.85)
\]

Now

\[
c_j = \frac{1}{2} \sum_{q=0}^{n+1} (-1)^{q+1} \cdot q \cdot \left. \frac{d}{ds} \right|_{s=0} \sum_k \int_{M_{\varphi}} \rho_i(x) \eta_k(\theta) J_j(s, x, \theta, 1) d\text{vol}(x, \theta)
\]
If \( k \neq 1 \), on \( V_k \times M \), \( J_j(s, x, \theta, 1) \) comes from the product metric, so it does not depend on \( \theta \). Hence by (5.72) and (5.77)

\[
c_j = \frac{1}{2} \chi(M) \delta_{n_j} \left( \int_{S^1} \sum_{k, k \neq 1} \eta_k(\theta) d\theta \right) + C(\varphi^* g, g). \tag{5.87}
\]

By the same argument on \( M_{\varphi^{-1}} \) with \( \varphi^{-1} : (M, g) \to (M, \varphi^* g) \), we get

\[
\hat{c}_j = \frac{1}{2} \chi(M) \delta_{n_j} \left( \int_{S^1} \sum_{k \neq 2} \eta_k(\theta) d\theta \right) + C(g, \varphi^* g). \tag{5.88}
\]

By adding (5.87) and (5.88),

\[
(c_j + \hat{c}_j)
\]

\[
= \frac{1}{2} \chi(M) \delta_{n_j} \left( \int_{S^1} \sum_{k \neq 1} \eta_k(\theta) d\theta + \int_{S^1} \sum_{k \neq 2} \eta_k(\theta) d\theta \right) + C(\varphi^* g, g) + C(g, \varphi^* g)
\]

\[
= \frac{1}{2} \chi(M) \delta_{n_j} \left( \int_{S^1} \sum_{k \neq 1} \eta_k(\theta) d\theta + \int_{S^1} \sum_{k \neq 2} \eta_k(\theta) d\theta \right)
\]

\[
+ \frac{1}{2} \chi(M) \delta_{n_j} \left( \int_{S^1} \eta_1(\theta) d\theta + \int_{S^1} \eta_2(\theta) d\theta \right) \text{ by (5.80)}
\]

\[
= \frac{1}{2} \chi(M) \delta_{n_j} (1 + 1) = \chi(M) \delta_{n_j}. \tag{5.89}
\]

Hence

\[
\frac{1}{2} (c_j + \hat{c}_j) = \frac{1}{2} \chi(M) \delta_{n_j}. \tag{5.90}
\]

We can use the same argument to show that \( \frac{1}{2} (d_j + \tilde{d}_j) = 0 \). □
Appendix A

Proof of Theorem 1.1

In this part, we are going to prove Theorem 1.1. First, let us recall the statement of the theorem. The following proof can be found in [KV].

**Theorem 1.1** Let \( A : C^\infty(E) \to C^\infty(E) \) be an invertible \( \Psi \)DO of order \( k \) with \( k \) a positive real number and \( Q : C^\infty(E) \to C^\infty(E) \) be a smoothing operator with \( I + Q \) invertible. Then \( (I + Q)A \) is a \( \Psi \)DO of order \( k \). Assume that \( (I + Q)^s A \) for \( 0 < s < 1 \) have \( \theta \) as an Agmon angle. Then

\[
\text{det}_\theta((I + Q)A) = \text{det}_\theta(A) \cdot \text{det}_F(I + Q). \tag{A.1}
\]

**Proof** Since \( Q \) is a compact operator, its spectrum is discrete except possibly at \( 0 \). Thus \( \text{Spec}(I + Q) \) is discrete except possibly at \( 1 \) and there is a small neighborhood \( U \) of \( 0 \) such that \( U \cap \text{Spec}(I + Q) = \emptyset \), since \( I + Q \) is invertible. Since \( \text{Spec}(I + Q) \) is a bounded set in \( \mathbb{C} \), there exists a ray \( \{re^{i\theta} \in \mathbb{C} | r \geq 0 \} \) such that for some neighborhood \( V \) of \( \{re^{i\theta} | r \geq 0 \} \), \( V \cap \text{Spec}(I + Q) = \emptyset \).

Now choose \( R > 0, \epsilon > 0 \) so that for any eigenvalue \( 1 + \lambda \) of \( I + Q \),

\[
0 < \epsilon < |1 + \lambda| \leq 1 + |\lambda| < R. \tag{A.2}
\]
Let $\Gamma$ be a contour enclosing all the eigenvalues of $I + Q$ in $\mathbb{C} - \{re^{i\theta}|r \geq 0\}$, i.e.

$$\Gamma = \{xe^{i\theta}|R \geq x \geq \epsilon\} \cup \{ee^{i\varphi}|\theta \geq \varphi \geq \theta - 2\pi\}$$

$$\cup \{xe^{i(\theta - 2\pi)}|\epsilon \leq x \leq R\} \cup \{Re^{i\varphi}|\varphi - 2\pi \leq \theta \leq \varphi\}.$$ 

(A.3)

Now we define

$$C := \log_\theta(I + Q) = \frac{1}{2\pi i} \int_\Gamma \log_\theta \lambda \cdot (\lambda - (I + Q))^{-1} d\lambda.$$  

(A.4)

Since $\Gamma$ is compact and $(\lambda - (I + Q))^{-1}$ is a bounded map, this definition is well defined. Moreover, $C$ is a smoothing operator, hence of trace class.

Set $A(t) = \exp(tC)A$ for $0 \leq t \leq 1$. Then $\exp(tC) = (I + Q)^t$ for $0 \leq t \leq 1$, where

$$\lambda^t = \frac{1}{2\pi i} \int_\Gamma \lambda^t \lambda - (I + Q))^{-1} d\lambda.$$  

(A.5)

Hence $A(t)$ is an invertible PDO and $A(t)$ has $\theta$ as an Agmon angle by the assumption.

Thus for $\text{Res} \gg 0$, we have

$$\zeta_{A(t),\vartheta}(s) := \text{tr} \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - A(t))^{-1} d\lambda \right),$$

(A.6)

where $\gamma = \{xe^{i\theta}|\infty \geq x \geq \epsilon\} \cup \{ee^{i\varphi}|\theta \geq \varphi \geq \theta - 2\pi\} \cup \{xe^{i(\theta - 2\pi)}|\epsilon \leq x < \infty\}$.

Then for $\text{Res} \gg 0$,

$$\frac{d}{dt} \zeta_{A(t),\vartheta}(s) = \text{tr} \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(\lambda - A(t))^{-1}(CA(t))(\lambda - A(t))^{-1} d\lambda \right)$$

$$= \text{tr} \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(CA(t))(\lambda - A(t))^{-2} d\lambda \right)$$

$$= \text{tr} \left( \frac{1}{2\pi i} \int_\gamma \lambda^{-s}(-\frac{d}{d\lambda}CA(t)(\lambda - A(t)^{-1}) d\lambda \right)$$

$$= \text{tr} \left( \frac{1}{2\pi i} \int_\gamma -s\lambda^{-s-1}CA(t)(\lambda - A(t))^{-1} d\lambda \right)$$
\[ = -s \cdot \text{tr} \left( CA(t) \frac{1}{2\pi i} \int_{\gamma} \lambda^{-s-1} (\lambda - A(t))^{-1} d\lambda \right) \]
\[ = -s \cdot \text{tr} \left( CA(t) \cdot A(t)^{-s-1} \right) = -s \cdot \text{tr}(CA(t)^{-s}). \quad (A.7) \]

Hence for \( \text{Res} \gg 0 \), we get
\[ \frac{d}{dt} \zeta_{A(t),0}(s) = -s \cdot \text{tr}(CA(t)^{-s}). \quad (A.8) \]

Since \( C \) is a smoothing operator, \( \text{tr}(CA(t)^{-s}) \) is well defined for any complex number \( s \). Thus
\[ \frac{d}{dt} \frac{d}{ds} \mid_{s=0} \zeta_{A(t),0}(s) = -\text{tr}(C), \quad (A.9) \]
and \( \frac{d}{dt} \log\text{Det}_\theta(A(t)) = \text{tr}(C) \). Therefore,
\[ \frac{\text{Det}_\theta((I + Q)A)}{\text{Det}_\theta A} = \exp \left( \int_0^1 \frac{d}{dt} \log\text{Det}_\theta(A(t)) dt \right) \]
\[ = \exp \left( \int_0^1 \text{tr}(C) dt \right) = \exp(\text{tr}(C)) = \text{det}_F\exp(C) \]
\[ = \text{det}_F\exp(\log(I + Q)) = \text{det}_F(I + Q). \quad (A.10) \]

\( \square \)
Appendix B

Neumann Operator and Symbol of $R(\lambda)$

Let $(M, g)$ be a compact oriented Riemannian manifold of dimension $d$ and let $\Gamma$ be a submanifold of $M$ with dimension $d - 1$ such that $\Gamma$ has a collared neighborhood $U$ diffeomorphic to $\Gamma \times (-1, 1)$. Let $M_\Gamma$ be the compact manifold with boundary $\Gamma \sqcup \Gamma$ obtained by cutting $M$ along $\Gamma$. Let $E = \wedge^q T^* M$ be a $q$-th exterior product of the cotangent bundle $T^* M$ for $0 \leq q \leq d$, $p : M_\Gamma \to M$ be the identification map and $E_\Gamma := p^* E$.

We can lift $g$ to $M_\Gamma$, denote again by $g$ and choose the unit normal vector field $\nu$ along $\Gamma \sqcup \Gamma$ on $\partial M_\Gamma$. Denote by $\Delta$ the Laplacian acting on differential $q$-forms on $M$. Then we can define the Dirichlet to Neumann operator $R(\lambda)$ associated to $(\Delta + \lambda, B)$ and $(\Delta + \lambda, C)$, where $\lambda$ is a positive real number and $B$, $C$ are Dirichlet and Neumann boundary conditions respectively as in Section 3.1 and 3.4.

In this part, we are going to compute the asymptotic symbol of $R(\lambda)$ by using Neumann operators and a first order differential equation.

Let $U$ be a collared neighborhood of $\Gamma$ diffeomorphic to $\Gamma \times (-1, 1)$ with diffeomorphism $\eta : U \to \Gamma \times (-1, 1)$ and let $\Gamma_\varepsilon := \eta^{-1}(\Gamma \times \{t\})$. We extend $\nu$ to $\Gamma \times (-1, 1)$ and denote again by $\nu$. Let $\nu_\varepsilon$ be the restriction of $\nu$ to $\Gamma_\varepsilon$. Consider $M_{\Gamma_\varepsilon}$ and denote
by $\Gamma^+_t$ and $\Gamma^-_t$ the components of the boundary of $M_{\Gamma_t}$ where the lift of $\nu_t$ points outward and inward respectively. Clearly $\Gamma^+_0 = \Gamma^-$.  

We define Neumann operators $N^\pm_t$ on $\Gamma^\pm_t$ as follows. For $\varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t})$, choose $u \in C^\infty(E_{\Gamma_t})$ such that

$$ (\Delta + \lambda)u = 0 \text{ in } M - \Gamma_t \text{ and } u|_{\Gamma_t} = \varphi. \quad (B.1) $$

Then

$$ N^+_t(\varphi) := (-\nu_t)(u)|_{\Gamma^+_t} \text{ and } N^-_t(\varphi) := \nu_t(u)|_{\Gamma^-_t}, \quad (B.2) $$

and

$$ R(\lambda) = N^+_0 + N^-_0. \quad (B.3) $$

Now we can choose a local coordinate system such that the first fundamental form looks like

$$ \begin{pmatrix} g_{ij}(x,0) & 0 \\ 0 & 1 \end{pmatrix} \quad (B.4) $$

on $\Gamma \times \{0\}$ so that the Laplacian $q$-forms on $U$ is given by

$$ \Delta = -A(x,t) \frac{d^2}{dt^2} + \sum_{j=1}^d H_j(x,t) \frac{\partial}{\partial x_j} \frac{d}{dt} + F(x,t) \frac{d}{dt} + \Delta_t, \quad (B.5) $$

where $\Delta_t$ is the Laplacian on $\Gamma_t$ and $A(x,t)$, $H_j(x,t)$ and $F(x,t)$ are $C^\infty$-function valued $\left( \begin{smallmatrix} d \\ p \end{smallmatrix} \right) \times \left( \begin{smallmatrix} d \\ p \end{smallmatrix} \right)$ matrices. Note that from the choice of the coordinate system with (B4), i.e. $A(x,0) = Id$ and $H_j(x,0) = 0(0 \leq d)$.

**Lemma 1**

(i) \[ \frac{dN^-_t}{dt} = -(N^-_t)^2 + A(x,t)^{-1} \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N^-_t + F(x,t)N^-_t + (\Delta_t + \lambda) \right) \quad (B.6) \]
\begin{align*}
(\text{ii}) \quad \frac{dN_t^+}{dt} &= (N_t^+)^2 + A(x,t)^{-1} \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^+ + F(x,t)N_t^+ - (\Delta t + \lambda) \right). \quad (B.7)

\textbf{Proof.} (i) Let } \varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t}). \text{ Choose } u(x,t) \in C^\infty(M_{\Gamma_t}, E_{\Gamma_t}) \text{ such that } (\Delta + \lambda)u(x,t) = 0 \text{ on } M - \Gamma_t \text{ and } u(x,t)|_{\Gamma_t} = \varphi. \text{ Then }
\frac{d}{dt} u(x,t) &= N_t^- (u(x,t)). \quad (B.8)
\frac{d^2}{dt^2} u(x,t) &= \frac{d}{dt} (N_t^-(u(x,t))) = \frac{dN_t^-}{dt} (u(x,t)) + N_t^- \left( \frac{du}{dt} \right)
&= \left( \frac{dN_t^-}{dt} + (N_t^-)^2 \right) u(x,t). \quad (B.9)

\text{Thus }
A(x,t) \frac{d^2}{dt^2} u(x,t) &= \sum_j H_j(x,t) \frac{\partial}{\partial x_j} \frac{du}{dt} + F(x,t) \frac{du}{dt} + (\Delta_t + \lambda)u(x,t)
&= \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^- + F(x,t)N_t^- + (\Delta_t + \lambda) \right) u(x,t). \quad (B.10)

\text{Hence } \frac{dN_t^-}{dt} + (N_t^-)^2 = A(x,t)^{-1} \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^- + F(x,t)N_t^- + (\Delta_t + \lambda) \right) \text{ and }
\frac{dN_t^-}{dt} = -(N_t^-)^2 + A(x,t)^{-1} \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^- + F(x,t)N_t^- + (\Delta_t + \lambda) \right). \quad (B.11)

(ii) \text{ We can use the same argument.}
\begin{align*}
(-\nu_t)u(x,t) &= -\frac{d}{dt} u(x,t) = N_t^+(u(x,t)). \quad (B.12)
\frac{d^2}{dt^2} u(x,t) &= -\frac{d}{dt} N_t^+(u(x,t)) = -\frac{dN_t^+}{dt}(u(x,t)) + N_t^+ \left( -\frac{d}{dt} u(x,t) \right)
&= -\frac{dN_t^+}{dt}(u(x,t)) + (N_t^+)^2 u(x,t). \quad (B.13)
\end{align*}

\text{Since } (\Delta + \lambda)u = -A(x,t) \frac{d^2}{dt^2} u + \sum_j H_j(x,t) \frac{\partial}{\partial x_j} \frac{du}{dt} + F(x,t) \frac{du}{dt} u + (\Delta_t + \lambda)u = 0,
A(x,t)^{-1} \left( -\sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^+ - F(x,t)N_t^+ + (\Delta_t + \lambda) \right) = -\frac{dN_t^+}{dt} + (N_t^+)^2. \quad (B.14)
Hence

\[
\frac{dN_t^+}{dt} = (N_t^+)^2 + A(x,t)^{-1} \left( \sum_j H_j(x,t) \frac{\partial}{\partial x_j} N_t^+ + F(x,t) N_t^+ - (\Delta_t + \lambda) \right). \tag{B.15}
\]

\[\square\]

**Corollary 2**

(i) \[\frac{dN_t^-}{dt} \big|_{t=0} = -(N_t^-)^2 \big|_{t=0} + F(x,0) N_t^- \big|_{t=0} + (\Delta_0 + \lambda) \] \tag{B.16}

(ii) \[\frac{dN_t^+}{dt} \big|_{t=0} = -(N_t^+)^2 \big|_{t=0} + F(x,0) N_t^+ \big|_{t=0} - (\Delta_0 + \lambda), \tag{B.17}\]

where \(\Delta_0\) is the Laplacian on the submanifold \(\Gamma\).

Now let us consider the operators at \(t = 0\). Introduce the symbol expansion

\[
\sigma(N_t^+) \sim \alpha_1 + \alpha_0 + \cdots + \alpha_{1-i} + \cdots \tag{B.18}
\]

\[
\sigma(N_t^-) \sim \beta_1 + \beta_0 + \cdots + \beta_{1-i} + \cdots \tag{B.19}
\]

\[
\sigma(\Delta_t + \lambda) \sim (\sigma_2 + \lambda) + \sigma_1 + \sigma_0. \tag{B.20}
\]

Since \(R(\lambda) = N_0^+ + N_0^-\),

\[
\sigma(R(\lambda)) \sim (\alpha_1 + \beta_1)_{|t=0} + (\alpha_0 + \beta_0)_{|t=0} + \cdots. \tag{B.21}
\]

Note that

\[
\sigma_2 + \lambda = \left( \sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda \right) Id. \tag{B.22}
\]

\[
\sigma((N_t^+)^2) \sim \sum_{k=0}^{\infty} \sum_{\omega\chi + \iota\iota = k \atop i,j \geq 0} \frac{1}{\omega!} d_{\xi}^\omega \alpha_{1-i} D_x^\omega \alpha_{1-j} \tag{B.23}
\]

\[
\sigma((N_t^-)^2) \sim \sum_{k=0}^{\infty} \sum_{\omega\chi + \iota\iota = k \atop i,j \geq 0} \frac{1}{\omega!} d_{\xi}^\omega \beta_{1-i} D_x^\omega \beta_{1-j}, \tag{B.24}
\]
where $\omega$ is a multi-index and $D_x = \frac{1}{i} \frac{d}{dx}$.

Since $\frac{dN^+}{dt}$, $\frac{dN^-}{dt}$ are first order operators, $\alpha_i^2 - (\sigma_2 + \lambda) = 0$ and $-\beta_i^2 + (\sigma_2 + \lambda) = 0$.

Thus $\alpha_1 = \beta_1 = \sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda \text{Id}}$ and

$$\alpha_1 + \beta_1 = 2 \sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda \text{Id}.} \quad \text{ (B.25)}$$

Note that $\frac{d\alpha_0}{dt} = (2\alpha_0 \alpha_1 + \frac{d\xi}{dt} \alpha_1 \cdot D_x \alpha_1) + F\alpha_1 - \sigma_1$ and

$$\frac{d\beta_0}{dt} = -(2\beta_0 \beta_1 + \frac{d\xi}{dt} \beta_1 \cdot D_x \beta_1) + F\beta_1 + \sigma_1. \quad \text{Hence}$$

$$\alpha_0 = \frac{1}{2} \alpha_1^{-1} \left( \frac{d\alpha_1}{dt} - \frac{d\xi}{dt} \alpha_1 \cdot D_x \alpha_1 - F\alpha_1 + \sigma_1 \right) \quad \text{ (B.26)}$$

$$\beta_0 = \frac{1}{2} \beta_1^{-1} \left( -\frac{d\beta_1}{dt} - \frac{d\xi}{dt} \beta_1 \cdot D_x \beta_1 + F\beta_1 + \sigma_1 \right) \quad \text{ (B.27)}$$

Since $\alpha_1 = \beta_1$, we get

$$\alpha_0 + \beta_0 = \alpha_1^{-1} (-\frac{d\xi}{dt} \alpha_1 \cdot D_x \alpha_1 + \sigma_1). \quad \text{ (B.28)}$$

Recursively we can calculate

$$\alpha_{1-k} = \frac{1}{2} \alpha_1^{-1} \left( \frac{d\alpha_{1-(k-1)}}{dt} - \sum_{i+j+|\omega|=k \atop 0 \leq i,j \leq k-1} \frac{1}{\omega!} d^\omega \alpha_{1-i} D_x^\omega \alpha_{1-j} - F(x,t)\alpha_{1-(k-1)} \right) \quad \text{ (B.29)}$$

$$\beta_{1-k} = \frac{1}{2} \beta_1^{-1} \left( -\frac{d\beta_{1-(k-1)}}{dt} - \sum_{i+j+|\omega|=k \atop 0 \leq i,j \leq k-1} \frac{1}{\omega!} d^\omega \beta_{1-i} D_x^\omega \beta_{1-j} + F(x,t)\beta_{1-(k-1)} \right). \quad \text{ (B.30)}$$

From (B.25), we get the principal symbol of $R(\lambda)$ as follows

$$\sigma_L(R(\lambda)) = 2 \sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda \text{Id}.} \quad \text{ (B.31)}$$
Now let us compare this with the result of Corollary 3.2.5. From Corollary 3.2.5, we get

$$\sigma_L(R(\lambda)^{-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_{-2}(x', 0, \xi', \eta) \frac{d\eta}{\sqrt{2\pi}}, \quad (B.32)$$

where \(d\eta\) is Lebesgue measure and

$$a_{-2} = \sigma_L((\Delta + \lambda)^{-1}) = \frac{1}{\left(\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda\right) + \eta^2} Id. \quad (B.33)$$

Thus

$$\sigma_L(R(\lambda)^{-1}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{\left(\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda\right) + \eta^2} Id \right) \, d\eta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda}} \cdot \pi Id = \frac{1}{2\sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda}} Id. \quad (B.34)$$

As a consequence

$$\sigma_L(R(\lambda)) = 2 \sqrt{\sum_{i,j=1}^{d-1} g^{ij} \xi_i \xi_j + \lambda} Id \quad (B.35)$$

and therefore we get the same result.

**Remark** The idea to consider the Neumann operator as a solution of operator valued differential equations is due to I.M.Gel’fand.
BIBLIOGRAPHY


