THERMAL COUNTERFLOW IN A DIVERGING RECTANGULAR CHANNEL: A STUDY OF SUPERFLUID TURBULENCE IN NONUNIFORM FLOW

DISSERTATION

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By

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In memory of my father

Donald Raymond Franklin
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The road has been long and often arduous, but the dream is finally at hand. Here will I give pause and humble thanks for all the support, guidance, assistance, friendship, and caring which has helped to move my feet and light my path.

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CHAPTER I

Introduction

1.1 The Two Fluid Model

At standard atmospheric pressure, helium liquefies at 4.2 K. As the liquid $^4$He is cooled below this temperature, nothing very remarkable happens until the special temperature $T_\lambda = 2.172$ K is reached, whereupon the helium enters a second fluid phase. Helium I, the first fluid phase, is an ordinary, viscous fluid, therefore Helium I is also called the normal fluid phase. Below $T_\lambda$, helium is a superfluid, called Helium II. This superfluid phase bears many extraordinary properties and is capable of complex flow states not possible for any normal, viscous fluid.

Our picture of $^4$He at absolute zero is that of a pure superfluid of density $\rho = \rho_s$, capable only of irrotational flow, and having no viscosity. Motion of this superfluid is described by a velocity field $v_s$. Viewed on a quantum level, the pure superfluid constitutes the ground state of the liquid. At finite temperatures, a gas of elementary thermal excitations is also present, which on the macroscopic level manifests itself as a normal, viscous fluid with density $\rho_n$. The average drift velocity of the excitations determines the velocity field $v_n$ of the normal fluid component. The macroscopic
density of the fluid $\rho$ is given by

$$\rho = \rho_s + \rho_n$$  \hspace{1cm} (1.1)

and the mass transport, or momentum density, of the total fluid can be described as the sum

$$\rho \mathbf{v}_m = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$  \hspace{1cm} (1.2)

of these two separate motions. The symbol $\mathbf{v}_m$ denotes the rate of mass flow, the macroscopic velocity of the fluid as whole.

This description of superfluid helium is known as the two fluid model, and was developed fully by Landau in the 1940s [Lan41, Lan47, Lan59] from the seminal ideas of Tisza [Tis38] and F. London [FLon38]. As the name suggests, this model presents the mental image of two fluids which coexist without any interaction. Although a very compelling idea, it must be remembered that two fluids exist only in the mind of the experimenter; Helium II is still a single fluid. The normal fluid and superfluid components of this superfluid phase are two aspects of a single fluid, not anything akin to the identifiably separate constituents of a mixture of two different, miscible fluids.

The densities $\rho_s$ and $\rho_n$ are functions of temperature, whereas the overall density $\rho$ is essentially constant, $\rho \simeq 0.145 \text{ g/cm}^3$, throughout the temperature range from absolute zero to $T_\lambda$. In the limit of absolute zero, the fluid is a pure superfluid, for which $\rho_s/\rho = 1$ and $\rho_n = 0$, whereas at the lambda transition the fluid becomes a pure normal fluid again, with $\rho_n/\rho = 1$ and $\rho_s = 0$. In the temperature range
explored in the present work, the superfluid component dominates, increasing from $\rho_s/\rho = 0.7685$ at 1.7 K to 0.9531 at 1.3 K.

Many types of experiments have been conducted since the discovery of this superfluid phase in 1938 [All38, Kap38] to unlock its mysteries and characterize its thermodynamic properties. Various types of experiments have explored the complicated flow states which can be produced. Studying the flow of helium in a channel of uniform cross section has been by far the most prevalent general type of experiment conducted. The channels used have ranged in length from 1 cm [Gri87] to many meters [Pes62, Mar85], and in their smallest cross sectional dimension from capillaries only a few tens of microns wide [Bre62, Lad78] to large tubes a centimeter wide [Aws84, Wan87]. In cross section, channels have ranged from circular [Chi76, dHaa76, Sle77, Bae85, Cou89] to square [Hen81] or rectangular [Lad79, Yar79], and even narrow annular gaps [Kel60, Kel61, Cra63], but the channels used over the past several decades have one important commonality: uniformity along their length. Heretofore, channel experiments have examined only one-dimensional flow through a tube of uniform cross section.

With a glance at equation 1.2, it should be obvious that a myriad of different flow states are possible, providing the motions of the normal and superfluid components can be experimentally controlled. The viscosity of the normal fluid component can serve to lock the normal fluid in place, either to create pure superflow where $v_n = 0$, or make a barrier permeable only to superfluid through which superfluid is forced at a controlled velocity $v_s$. The ability of the superfluid to creep as a thin film can
also be used to regulate $v_s$. In short, many techniques exist to achieve a desired flow state, with particular values of $v_n$ and $v_s$. The respective motions of the two fluid components are described by the coupled, two-fluid Navier-Stokes equations:

\[
\begin{align*}
\rho_n \frac{\partial v_n}{\partial t} + \rho_n(v_n \cdot \nabla)v_n &= \eta \nabla^2 v_n - \frac{\rho_n \nabla p}{\rho} - \rho_s S \nabla T \\
\rho_s \frac{\partial v_s}{\partial t} + \rho_s(v_s \cdot \nabla)v_s &= -\frac{\rho_s \nabla p}{\rho} + \rho_s S \nabla T
\end{align*}
\] (1.3)

(1.4)

where $\eta$ is the normal fluid viscosity and $p$ is the pressure. Note that in the special case of a steady, spatially uniform flow, the left-hand sides of these equations reduce to zero, and equation 1.4 yields the London equation

\[
\nabla p = \rho S \nabla T
\] (1.5)

relating the pressure and temperature gradients. This relation was first derived in 1939 by H. London on purely thermodynamic grounds [HLon39].

1.2 Thermal Counterflow

By far the easiest flow state to create experimentally, thermal counterflow of superfluid helium is the type of channel flow most frequently studied. One end of the channel is open to a large reservoir of helium maintained at a constant temperature, while the other end is connected to a small chamber containing a heater. Introducing a heat current $\dot{Q}$ in the sealed end causes the normal fluid to flow toward the reservoir end at an average velocity

\[
V_n = \langle v_n \rangle = \frac{\dot{Q}}{\rho S T A},
\] (1.6)
where $S$ is the entropy per unit mass of fluid and $\langle \rangle$ denotes temporal and spatial averaging of $v_n$ over the cross sectional area $A$ of the channel. While the normal fluid convects the heat away from the sealed end, the superfluid flows in the opposite direction with average velocity $V_s = \langle v_s \rangle$, in accordance with the constraint that no net mass be transferred

$$\rho_n v_n + \rho_s v_s = 0 \quad (1.7)$$

Thermal counterflow is therefore the special case of equation 1.2 where, at least on average, there is no momentum transfer of the fluid as a whole: $V_m = 0$. Nevertheless, the fluid is far from quiescent. A strong internal convection is taking place, transporting heat. It is this ability to convect heat internally rather than its low temperature alone which makes superfluid helium such a phenomenally good coolant. The rate of this internal convection between the two fluid components is given by the average relative velocity $V$. For thermal counterflow

$$V = V_n - V_s = \frac{\dot{Q}}{\rho_s S T A} \hat{n} \quad , \quad (1.8)$$

where $\hat{n}$ is a unit vector in the direction of the heat current, and equation 1.7 has been used to obtain $V$ in terms of $V_n$. The spatial averaging implicit in the counterflow condition is very important. It is not necessarily true at any point in the flow that the local velocity fields $v_n$ and $v_s$ would satisfy the much stronger criterion $\rho_n v_n = -\rho_s v_s$. Although the underlying vector fields $v_s$ and $v_n$ are unknown functions of position, the average relative velocity $V$ has a simple position dependence, dictated by energy conservation. For a uniform channel, meaning a channel with constant cross sectional area $A$, the relative velocity $V$ is constant everywhere in the channel, at least so long
as the overall temperature variation over the length of the channel is small enough that $\rho_s ST$ remains essentially constant throughout the channel. In a nonuniform channel, $A$ varies and therefore $V$ becomes a function of position along the channel length.

1.3 Superfluid Turbulence and the Vinen Model

At low relative velocities the flow is laminar and the viscous dissipation creates a laminar temperature gradient $\nabla T_L$ in the fluid, as predicted by equations 1.3 and 1.4. At a large enough relative velocity $V$ the fluid enters a new dynamical state characterized by a dense, random tangle of quantized vortex lines in the superfluid. The bulk of the fluid is still irrotational, but the circulation around any path enclosing a vortex filament becomes

$$\int (\nabla \times \mathbf{v}_s) \cdot d\mathbf{A} = \oint \mathbf{v}_s \cdot d\mathbf{l} = n\kappa,$$

where the quantum of circulation $\kappa = \hbar/m_4$ is the ratio of Planck’s constant to the mass of a $^4\text{He}$ atom, and $n$ is a positive integer.

With the formation of this vortex tangle in the superfluid, the normal fluid interacts locally with the superfluid. The normal fluid excitations scattering off the vortex cores create additional dissipation in the fluid manifested as a large, nonlinear temperature gradient $\nabla T'$. This state of superfluid turbulence was first observed in 1949 by Gorter and Mellink [Gor49], who proposed a mutual friction force $F_m$ acting between the normal fluid and superfluid to account for this new dissipation in excess
of the laminar dissipation,
\[ \nabla T' = \frac{F_m}{\rho_s S}. \]  
(1.10)

They found this excess dissipative force depended cubically on the relative velocity:
\[ F_m = A \rho_n \rho_s V^3, \]  
(1.11)

where the Gorter-Mellink coefficient \( A \) was fit to the experimental measurements.

The mutual friction \( F_n \) is a drag force exerted on the normal fluid by the superfluid, so is in the direction of motion of the superfluid. The mutual friction force density is added to the two fluid equations of motion:
\[
\rho_n \frac{\partial v_n}{\partial t} + \rho_n (v_n \cdot \nabla)v_n = \eta \nabla^2 v_n - \frac{\rho_n}{\rho} \nabla p - \rho_s S \nabla T + F_m  \tag{1.12}
\]

\[
\rho_s \frac{\partial v_s}{\partial t} + \rho_s (v_s \cdot \nabla)v_s = -\frac{\rho_s}{\rho} \nabla p + \rho_s S \nabla T - F_m  \tag{1.13}
\]

In general, it is far from clear that one should be able to add \( F_m \), which is some kind of spatially averaged field, to these dynamical equations describing local fields.

In adding the term \( F_m \), it is implied that the two-fluid equations are limited to spatially uniform flow. In the steady state, the left-hand sides of equations 1.12 and 1.13 then vanish, and the gradients on the right-hand sides yield constant terms. To treat flow in a uniform channel, it has been traditional to replace the local fields \( v_n \) and \( v_s \) with their cross-channel averages \( V_n \) and \( V_s \), and \( \nabla p \) and \( \nabla T \) with averages over the channel cross section, \( \langle \nabla p \rangle \) and \( \langle \nabla T \rangle \). This set of assumptions has come to be known as the mutual friction approximation. For a uniform channel, the average velocities \( V_n \) and \( V_s \) are constant everywhere in the channel, and the left-hand sides of the two-fluid equations still vanish. In this work, we adopt the same mutual friction
approximation, however the velocity derivatives do not vanish as for uniform flow, but instead reflect the downstream variations in $V_n$ and $V_r$.

In a set of landmark experiments in the mid-1950s, the properties of superfluid turbulence were investigated by Hall and Vinen using uniformly rotating Helium II [Hal56a, Hal56b], and by Vinen using thermal counterflow in uniform channels [Vin57a, Vin57b]. Their work was motivated in part by Feynman's exciting new proposal of a state of superfluid turbulence [Fey55]. Feynman suggested that a distribution of quantized vortex lines would form in the superfluid, and further that the normal fluid excitations would scatter off these vortices, exchanging energy. Independently and slightly earlier, Onsager had also proposed the idea of quantized vorticity [Ons49]. In Helium II undergoing uniform rotation, Feynman argued that the vortices would form a regular, rectilinear array, whereas in a uniform channel flow, he said, "a single initial vortex playing out from points in the wall upstream ... can soon fill the tube with a tangle of line." Hall and Vinen [Hal56b] and Vinen [Vin57c] showed that in either case the excess dissipation could be quantitatively accounted for by the microscopic process of elementary excitations scattering off the vortex lines. Vinen described the turbulent state in terms of the amount of vortex line length per unit volume, called the line density $L$. The Vinen equation [Vin57c] predicts the time evolution of the vortex line density to be

$$\frac{\partial L}{\partial t} = \frac{B \rho_n}{2 \rho} \chi_1 V L^{3/2} - \frac{\kappa}{2\pi} \chi_2 L^2,$$

(1.14)

where the Hall-Vinen friction coefficient $B$ and the ratio $\chi_1/\chi_2$ are experimentally determined parameters. Vinen used simple phenomenological arguments about the
growth and decay of vorticity to derive the form of this equation. The $L^{3/2}$ growth term is obtained from considering the growth of line length of a vortex ring being propelled by a normal fluid tailwind. The decay term scales as $L^2$, since two parallel rectilinear vortices with opposite circulation approaching each other will annihilate one another. In the steady state, $\partial L/\partial T$ is zero, meaning that the rates of production and destruction of vortex line length are equal, and the line density is given by

$$L = \gamma^2 V^2$$

(1.15)

where

$$\gamma = \pi B^{\rho_{n}\kappa \rho} \left( \frac{X_1}{X_2} \right)$$

(1.16)

Comparing equations 1.10, 1.11, and 1.15, we see that $\nabla T'$, and correspondingly $F_m$, is proportional to both $L$ and $V$. This basic relation is very compelling, showing that the mutual friction depends on the rate at which the elementary excitations are brought in contact with the vortex cores which scatter them, and the density of scatterers.

The basic functional dependences embodied in equations 1.10 and 1.15 have been well established by experimental measurements of uniform channel flows. The Vinen model has proven an adequate description of the turbulent states produced by thermal counterflow in uniform channels, but bears the limitation of being a phenomenological model.
1.4 Motivation

As mentioned above, extensive experiments have been conducted over the years in uniform channels of many cross sectional shapes and aspect ratios and a wide range of overall sizes. As well as thermal counterflow, more general flows with all possible combinations of $v_n$ and $v_x$ have been studied. For such flows in uniform channels, $V$ is still constant throughout the channel, but is no longer given by equation (1.8). Several distinct turbulent states are observed in uniform channels under these various conditions [Tou82, Don91].

Limiting our attention to thermal counterflow, a state of fully developed turbulence known as the T-II state is observed in uniform channels of either circular or rectangular cross section at high enough relative velocities. Ion trapping experiments have been successful in probing $L$ to a 1 mm accuracy in large rectangular channels with a width of 1 cm, confirming that the T-II state is spatially homogeneous to within 1 mm of the channel walls [Aws84]. The T-II state is modeled accurately by the theory of homogeneous superfluid turbulence developed by Schwarz [Sch85, Sch88]. Starting with an equation of motion describing the appropriate dynamics for a vortex line element, including mutual friction, Schwarz computes by direct numerical simulation the final steady state of an initially small, arbitrary arrangement of vortex lines evolving under the influence of a uniform driving velocity $V$. In either the Vinen or Schwarz formalism, this theory of homogeneous turbulence predicts the steady state line density $L$ as a function of $V$, and how the mutual friction yielding the temperature gradient $\nabla T'$ depends on $L$ and $V$. In contrast with the earlier Vinen model,
all parameters characterizing the predicted turbulent state are computed within the
Schwarz model; none are adjustable. The predictions of the Schwarz model are in
very good quantitative agreement with the observed T-II state in uniform channels.
This state of fully developed, homogeneous turbulence is thus fairly well understood.

In uniform channels of certain geometries at more moderate relative velocities, the
T-II state is preceded by another turbulent state, known as the T-I state, with a much
lower line density than would be characteristic of the T-II state at the same \(V\). This
lower level of turbulence has been observed only in channels of fairly small circular and
low-aspect-ratio rectangular (square) cross section, and is conspicuously absent from
high-aspect-ratio rectangular channels. Schwarz and Rozen recently conjectured that
the anomalous T-I state may be a state of inhomogeneous turbulence which represents
a “spatially patchy precursor” to the fully developed, homogeneous T-II state [Sch91].
Another long-standing speculation holds that spatial inhomogeneity in the turbulence
could also result from a nonuniform cross-channel profile in the normal fluid velocity.

Several difficulties arise in trying to determine whether the T-I state is actually
one of inhomogeneous turbulence. First, the cross-channel \(v_n\) profile cannot be mea-
sured directly in such small channels, nor has it been predicted theoretically in the
presence of the vortex tangle. In laminar flow, the \(v_n\) profile is computed easily
enough by treating the normal fluid as an incompressible, viscous fluid and applying
the appropriate boundary conditions; however, once vortex lines are present the \(v_n\)
profile is also influenced by their frictional interaction with the normal fluid, in a way
that is not well understood and cannot presently be modeled. Second, it is equally
difficult to observe any resulting spatial inhomogeneity in $L$ in the T-I state. Channels in which the T-I state has been observed range in diameter from only 100 $\mu$m up to about 1 mm, well under the limit of resolution for an ion trapping experiment.

Once $L$ becomes large enough, the presence of such a dense vortex tangle presumably influences the normal fluid velocity profile so significantly that any cross-channel spatial variation in $v_n$ is obliterated. If so, the T-II state would thus conform to the condition of uniform relative velocity assumed in the Schwarz model of homogeneous turbulence. This scenario suggests a reason behind the transition from the T-I to the T-II state, but does nothing to predict the value of the relative velocity at which this transition occurs. Why the T-I state is not observed in rectangular channels is also unclear. One possibility is that the T-I state does in fact exist in rectangular channels, but over such a narrow range of velocities that its presence before the onset of the T-II state is indiscernible. The lack of axial symmetry in the rectangular channel flow may also play a role, making $v_n$ renormalize to a flat profile more readily in this geometry.

The idea that inhomogeneous turbulence could be the result of a spatially nonuniform normal fluid velocity field provided one impetus for the present work. Our experiment was designed to study the effect on homogeneous superfluid turbulence of a small but well-controlled deviation from a spatially uniform driving velocity. Any cross-channel variations in the local relative velocity $v_n - v_s$ cannot be controlled, but by constructing a channel which opens up gradually along its length, the average relative velocity $V$ can be varied in a known way along the channel length. The
primary question explored in this research is whether the character of the T-II state would remain largely unaffected or be radically altered by this small perturbation imposed on a uniform flow.

For this experiment, I used a channel of high-aspect-ratio rectangular cross section to isolate the influence of a nonuniform velocity on homogeneous turbulence from any ambiguity introduced by the presence of the T-I state. Moreover, I wanted to measure the temperature of the helium at several positions along the length of the channel, not just at the channel ends, to map out the position dependence of the temperature gradient. For simplicity, I also wanted the cross sectional area to vary in a precisely controlled and well defined manner. A rectangular channel could be constructed by machining, permitting both exact control of the channel shape and easy incorporation of local temperature probes. An analogous set of experiments conducted by Murphy, Castiglione, and Tough used a flaring circular channel made from drawn glass tubing, in which the channel diameter could not be as precisely controlled, and no temperature probes could be inserted along the channel length [Mur93a]. The width of the rectangular channel increases linearly along its length, so the cross sectional area $A$ increases linearly with position $r$ down the channel. This geometry produces a small, known divergence in the downstream velocity: $V \propto \dot{Q}/r$. This experimental design emulates the radial heat transfer problem of a heated wire or cylinder immersed in a large bath of superfluid helium.
1.5 The Local Uniformity Approximation

We require a valid method by which to compare our experimental results with existing theory. It seems at least plausible to expect that the turbulent state forming in a flow which is only slightly nonuniform might not differ dramatically from homogeneous turbulence. The weakness of the divergence in our nonuniform flow suggests that our data may be adequately described by applying the Schwarz model of homogeneous turbulence locally. Indeed, this approach seems the only logical recourse available in lieu of a proper theory of inhomogeneous turbulence.

We make this "local uniformity approximation" as follows. First, we assume not only that the density of vortex lines $L(r)$ at any point in this weakly nonuniform flow is determined solely from the local value of the velocity $V(r)$, but further that $L(r)$ bears exactly the same functional dependence on $V(r)$ as it would have in a uniform flow at that velocity. Similarly, we assume that the turbulent temperature gradient $\nabla T'(r)$ at each point $r$ has the same functional dependence on $L(r)$ and $V(r)$ as it would in a uniform flow at equal velocity. The resulting local $\nabla T'(r)$ predicted within the local uniformity approximation can then be integrated to obtain the temperature difference between any two positions in the channel. The local uniformity approximation is analogous to the local equilibrium approximation of kinetic theory, in which the equilibrium distribution function appropriate to the local values of the system thermodynamic variables is used as a substitute at every point for the actual distribution function in a nonequilibrium state, which cannot be determined.
It is entirely possible that the local uniformity approximation will fail to describe even a weakly nonuniform turbulent flow, for any number of reasons. All the parameters predicted by the Schwarz model are computed using the homogeneous distribution of vortex lines resulting from a uniform velocity, therefore these parameters may well take on quite different values in the case of inhomogeneous turbulence, or even become spatial functions. The relation between $L$ and $V$ for a nonuniform flow might also differ in some fundamental way from that predicted for a uniform flow, or the dependence of $\nabla T'$ on $L$ and $V$ might be altered. Underlying all these possibilities is the fact that the dynamical scaling arguments at the heart of the Schwarz model are worked out under the assumption of a uniform driving velocity. The local uniformity approximation consists of thumb-tacking a prediction obtained under the guise of a uniform flow onto a situation for which the prediction might prove to be quite different. No such after-the-fact modification should necessarily be expected to adjust the Schwarz model of homogeneous turbulence properly to describe inhomogeneous turbulence. If the local uniformity approximation proves inadequate, perhaps altering the underlying scaling arguments in the Schwarz model to incorporate the symmetries of the nonuniform flow appropriately and conduct a proper simulation of the vortex dynamics is all that will be needed to extend the Schwarz description to inhomogeneous turbulence.

On the other hand, perhaps a more fundamental change in the underlying vortex dynamics will be required instead. New dynamic mechanisms which do not come into play or are not discernable in uniform flows might be needed. One recent suggestion
of Schwarz along these lines seems worth further investigation [Sch93]. In being convected from one region to another in a nonuniform flow, a vortex line, being an object which exists only as a pattern in the flow itself, is "stretched," becoming longer or shorter as the flow diverges or converges. Such a vortex stretching represents a mechanism of line length production different from that already present in a uniform flow. A new line production term might need to be incorporated in the dynamics to account for this additional vortex growth mechanism.

A suggestion of a quite different nature resides in a new hydrodynamic model of superfluid turbulence recently proposed by Geurst, which introduces the effects of a nonuniform line density explicitly by including a new term in the internal energy that depends on the gradient of \( L \) [Geu89, Geu92, Geu94]. Because the Geurst model purports to shed new light on the very situation created in this experiment, its predictions were compared to the experimental results. This analysis was by necessity limited to the version of the Geurst model describing a one-dimensional flow at constant velocity. Despite allowing for the possibility of inhomogeneous turbulence, this one-dimensional Geurst model could be applied to our nonuniform flow only in the local uniformity approximation, therefore many of the same caveats cited above concerning why the comparison may not be fair still apply.

1.6 Organization of this Thesis

The apparatus and the measurement technique used in this experiment are explained fully in Chapter II. A special thermocouple inductively coupled to an rf SQUID is
used to make a very sensitive differential measurement of the temperature within the channel. Two sections of this chapter are devoted to detailed discussions of the design and construction of the diverging channel and the thermocouple, including estimates of the expected measurement sensitivity. A review of the rest of the thermometry follows. Due to the unusually large temperature differences generated across the channel in this experiment, a special nonlinear fit had to be used in calibrating the resistance thermometers. The actual performance characteristics and operation of the SQUID/thermocouple system are discussed in detail. Problems with rf noise frequently impaired SQUID operation, impeding data collection. All steps taken to diagnose, and where possible, alleviate or remedy these noise problems are reviewed. Finally, with the wisdom of hindsight, several ideas for improvements to the experimental design are offered.

In Chapter III, the experimental findings and all analyses of the measurements are presented. In each of four data series, the temperature difference was measured over a different region of the channel. The chapter begins with a review of this sequence of data acquisition and the procedures followed in conducting a data run, and a description of the general features of typical data set. Before superfluid turbulence forms, the flow is laminar. Comparing the observed and predicted laminar dissipation provides an important check on the experimental operation. Considerable effort was therefore devoted to understanding the laminar data, which showed some unexpected results, so the analysis and findings are described in detail.
The analysis of the turbulent state data begins with comparing the measured turbulent dissipation to the Schwarz model, wherein the validity of the local uniformity approximation is tested. Likewise, the results of similarly applying the Geurst model are described in detail. The section discussing the Geurst model closes with a description of how, after much of this analysis had been completed, I identified a severe limitation in interpreting the Geurst model which renders the model, in my judgement, untenable. Finally, I describe a parametric fit which characterizes the observed inhomogeneous turbulent state by modifying the vortex line density from the functional dependence predicted in the Vinen or Schwarz model. Chapter III closes with an examination of the onset of turbulence. In this final section, evidence that a stationary turbulent-laminar front is present at moderate relative velocities is presented. This observation confirms that stationary fronts can and do form in a nonuniform flow.

The primary focus of this work is to test whether the description of homogeneous superfluid turbulence can be extended to describe inhomogeneous turbulence in a weakly nonuniform flow. Inasmuch as the Schwarz model is central to the data analysis, this document would be incomplete without a thorough overview of this model, which constitutes the final section of this introduction. The reader already familiar with the Schwarz model, or more concerned with either the design and execution of this experiment or specific aspects of the data analysis, may wish to proceed at this point to other sections of this document, returning to Section 1.7 only as needed.
1.7 The Schwarz Model

In this Section, a brief overview of the Schwarz model of homogeneous superfluid turbulence is presented. The intent herein is only to familiarize the reader with the essential features and predictions of this model, which applies to uniform flows. For further details, the reader is referred to [Sch85] and [Sch88]. Application of the Schwarz model to a nonuniform flow is discussed in Section 3.4 of Chapter III.

Using a direct numerical simulation, Schwarz computes the state of superfluid turbulence that evolves under the condition of a spatially uniform driving velocity. The simulation begins with an initial seeding of a few vortex rings in an arbitrary configuration, and follows their subsequent motion and growth into a dense tangle, as depicted in figure 1.1. Once a steady state is reached, the turbulence is found to be spatially homogeneous, although not isotropic. The macroscopic, statistically averaged properties of the homogeneous turbulent state are computed by integrating the corresponding local quantities over the tangle distribution. All such characteristics of homogeneous turbulence are therefore predicted within the model; none constitute adjustable parameters that must be fit to experimental data.

1.7.1 Motion of a Vortex Line

The motion, and subsequent growth or decay, of each element of vortex line within the tangle is determined by an equation of motion that describes the instantaneous velocity of a given point on the vortex line. This velocity is comprised of three basic
Figure 1.1: Results of a typical simulation: growth of a vortex tangle at $T_0 = 1.62K$ from an initial configuration of six rings [Sch88].
constituents. First, each vortex element experiences the local velocity field produced by all other vortex elements. In addition to this self-induced motion, being that the vortex filament is a feature in the superfluid, it is carried along by the bulk flow of the superfluid. Finally, the vortex experiences a frictional force due to the normal fluid streaming past the vortex core. This last contribution to the motion is described in Section 1.7.2.

The position of a point on a vortex filament is parametrized with the vector $s(\xi)$, where $\xi$ measures the distance along the vortex line from some reference point on the line. As shown in figure 1.2, a useful triad of mutually perpendicular vectors is formed by $s'$, $s''$, and $s' \times s''$, where the primes denote differentiation with respect to $\xi$. Note that $s'$ is a unit vector, locally tangent to the vortex filament and pointing in the direction of the circulation $\kappa$. The second derivative $s''$ lies along the principal normal, and has magnitude $1/R$, where $R$ is the local radius of curvature of the vortex filament. The cross product $s' \times s''$ therefore also has length $1/R$ and points along the binormal.

In direct analogy to the magnetic field produced by a current element, the velocity at position $r$ in the fluid produced by an element of vortex line $ds_1$ is given by the Biot-Savart expression

$$v(r,t) = \frac{\kappa}{4\pi} \int_{\mathcal{L}} \frac{(s_1 - r) \times ds_1}{|s_1 - r|^3}. \quad (1.17)$$

This expression is valid as long as $(s_1 - r) \gg a_0$, so the integration is to be carried out over the set $\mathcal{L}$ of all such points. The self-induced velocity $\dot{s}$ of a point on the line itself is computed by setting $r = s$. Obviously, if the vortex were treated as
an infinitely thin filament, the integrand would diverge as \( s_1 \) approaches \( s \). This divergence may be handled mathematically by introducing a cutoff length \( \delta \), so the integration is only carried out to within \( \delta \) of \( s \). For a vortex ring, this integration is then straightforward to perform. The resulting self-induced velocity of the ring is given by

\[
\dot{s} = -\frac{\kappa}{4\pi R} \ln \left( \tan \frac{\delta}{4R} \right) \hat{e}_\perp,
\]

where \( \hat{e}_\perp \) is a unit vector in the direction of \( s' \times s'' \). The cutoff length is on the order of the core radius \( a_0 \) and is given by

\[
\ln \frac{2\delta}{a_0} = c_s
\]

where \( c_s \) depends on the internal structure of the core. For a classical vortex with a hollow core, \( c_s = 1/2 \), and for a Rankine vortex, where the fluid in the core is in solid

![Figure 1.2: Parametrization of a vortex line \( s(\xi) \) showing the triad of mutually perpendicular vectors. The local self-induced velocity causes the vortex to move in the direction of \( s' \times s'' \).](image-url)
body rotation, \( c_s = 1/4 \). For quantum vortices, the internal structure of the core is unknown, and \( a_0 \) is considered an adjustable parameter determined from experiment. Choosing \( c_s = 1/4 \), the effective core radius is found experimentally [Ray64, Gla71] to be about \( 1.3 \times 10^{-8} \) cm. The self-induced velocity of a vortex ring therefore becomes

\[
\hat{s} = \frac{\kappa}{4\pi R} \ln \left( \frac{8R}{e^{1/4}a_0} \right) \hat{e}_\perp.
\] (1.20)

For a curved vortex filament, the largest contribution to the self-induced velocity comes from those points \( s_1 \) closest to \( s \). The integral may be formally broken into local and nonlocal contributions to \( \hat{s} \). For the local contribution \( \hat{s}_l \), the integration is carried out only over the region from \( \ell_- \) to \( \ell_+ \) around \( s \) (see figure 1.3). The result is

\[
\hat{s}_l = \frac{\kappa}{4\pi R} \ln \left( \frac{2\sqrt{\ell_+\ell_-}}{e^{1/4}a_0} \right) \hat{e}_\perp,
\] (1.21)

This "localized-induction approximation" was first developed for infinitesimally thin classical vortices by Arms and Hama [Ham63, Arms65] and is simply referred to as the "local approximation" by Schwarz. The size of the local region from \( \ell_- \) to \( \ell_+ \) must be kept small enough that use of the small angle approximation for \( \tan(\phi/4) \) remains valid. A practical limit is \( \ell_\pm \leq R \). Schwarz replaces \( (\ell_+\ell_-)^{1/2} \) by \( R \), and further by the characteristic or average radius of curvature of the tangle, \( \bar{R} = |\bar{s}''| \). Recalling that \( s' \times s'' \) has magnitude \( 1/R \), the local self-induced velocity may be written as

\[
\hat{s}_l = \frac{\beta_s}{R} \hat{e}_\perp = \beta_s s' \times s'' \hat{e}_\perp
\] (1.22)

where the Schwarz parameter \( \beta_s \) is defined to be

\[
\beta_s = \frac{\kappa}{4\pi} \ln \left( \frac{e\bar{R}}{a_0} \right)
\] (1.23)
Adding the local and nonlocal contributions of the self-induced velocity to the background superfluid field, the velocity of a point on a vortex line (neglecting frictional forces) is given by

$$\hat{s}_0 = v_s + \hat{s}_l + \hat{s}_{nl} \ .$$

(1.24)

The zero subscript indicates that $\hat{s}_0$ is the velocity of the vortex line in the absence of frictional forces, and the nonlocal self-induced velocity $\hat{s}_{nl}$ is obtained by integrating
over the remaining vortex distribution \( \mathcal{L}' \), excluding the local region from \( l_- \) to \( l_+ \):

\[
\dot{s}_{nl} = \frac{\kappa}{4\pi} \int_{\mathcal{L}'} \frac{(s_1 - s) \times ds_1}{|s_1 - s|^3},
\]

(1.25)

The nonlocal contribution \( \dot{s}_{nl} \) is typically about 10\% as large as the local self-induced velocity \( \dot{s}_l \).

If boundaries are present, an additional velocity \( \dot{s}_b \) must be added to equation 1.24. This term is chosen to satisfy the boundary condition that the superfluid velocity perpendicular to the wall must be zero. The boundary is handled using the concept of mirror vortices. The mirror image of a vortex has opposite circulation, and \( \dot{s}_b \) is computed from a Biot-Savart integral over all such mirror vortices. Any vortex terminating on a boundary is assumed to be oriented perpendicular to the boundary at the interface, and continue smoothly into the surface.

The nonlocal term becomes significant whenever two vortices approach each other closely, causing the vortices to break and reconnect. Likewise, the boundary term becomes large whenever a vortex approaches a boundary closely, in which case the vortex line undergoes a reconnection with its mirror image. Schwarz introduces appropriate reconnection rules in his simulation to handle these line-line crossings and line-boundary interactions.
1.7.2 Including Friction

The normal fluid streaming past the vortex core creates a frictional drag on the vortex line. This frictional interaction can be thought of as the net result of the elementary excitations scattering off the vortex core, and constitutes a momentum transfer between the normal and superfluid components. The frictional drag is proportional to the difference between the velocity of the normal fluid \(v_n\) and that of the vortex line \(s\). The normal fluid velocity \(v_n\) is the average drift velocity of the excitation gas, in other words the velocity of the normal fluid far from the vortex core. In the immediate vicinity of a vortex, the excitation gas may be entrained by the core, distorting \(v_n\) locally. It is assumed that the frictional force only acts in a direction locally perpendicular to the vortex. A general expression for the drag force exerted per unit length on the vortex can therefore be written as

\[
f_D = -\gamma_0 s' \times (v_n - \hat{s}) + \gamma_0' s' \times (v_n - \hat{s}) .
\]  

(1.26)

where \(\gamma_0\) and \(\gamma_0'\) are temperature-dependent parameters to be determined from experiment. The term in \(\gamma_0\) is in the direction of the component of \((v_n - \hat{s})\) perpendicular to the vorticity direction \(s'\), and the term in \(\gamma_0'\) lies in the other transverse direction, perpendicular to the first. The total velocity of the vortex line \(s\) now includes the motion \(\hat{s}_f\) induced by friction:

\[
\dot{s} = \dot{s}_0 + \dot{s}_f .
\]  

(1.27)

A second equation is needed in order to solve for \(\hat{s}\).

A cylinder rotating in a classical viscous fluid of density \(\rho\) will set up a flow around it, with circulation \(\Gamma\) in the axial direction. If the cylinder also moves with
a transverse velocity $v$ through the quiescent fluid, a lift force will be created by the Bernouilli effect, $f = \rho \Gamma \times v$. This phenomenon is known as the Magnus effect. For a quantized vortex with circulation $\kappa = \kappa s'$ moving at a velocity different from the local superfluid velocity $\hat{s}_0$, one would expect a Magnus force

$$f_M = \rho \kappa s' \times (\hat{s} - \hat{s}_0)$$  \hspace{1cm} (1.28)

to act on the vortex core. The vortex core is presumably hollow, therefore massless. In this context, the Magnus force is better understood as the reaction to the drag force, so we write $f_D + f_M = 0$.

The velocity due to friction will likewise be directed perpendicular to the vortex line and may be written in terms of frictional coefficients $\alpha$ and $\alpha'$:

$$\hat{s}_f = \alpha s' \times (v_n - \hat{s}_0) - \alpha' s' \times [s' \times (v_n - \hat{s}_0)] \hspace{1cm} (1.29)$$

The parameters $\alpha$ and $\alpha'$ are related to $\gamma_0$ and $\gamma'_0$. In either form, these parameters constitute properties of the quantized vortex line which have been measured in a variety of experiments and their temperature dependence is now well known [Tou82, Don91]. Hall and Vinen first determined their respective values as a function of temperature by measuring second sound attenuation in a rotating bucket apparatus, where the quantized vortices exist in a regular hexagonal array rather than a random tangle [Hal56a, Hal56b]. The friction coefficients are often expressed in terms of the the Hall-Vinen coefficients $B$ and $B'$: $\alpha = B \rho_n / (2 \rho)$ and $\alpha' = B' \rho_n / (2 \rho)$. Inserting equation 1.29 for $\hat{s}_f$ into equations 1.26 and 1.28 and setting $f_D + f_M = 0$ yields the relations
\[ \alpha = \frac{\gamma_0 \kappa \rho_s}{\gamma_0^2 + (\gamma_0' - \kappa \rho_s)^2}, \]
\[ \alpha' = \frac{\gamma_0^2 + \gamma_0' (\gamma_0' - \kappa \rho_s)}{\gamma_0^2 + (\gamma_0' - \kappa \rho_s)^2}. \]  \hspace{2cm} (1.30)

The frictional force may now be expressed in terms of \((v_n - \dot{s}_0)\) instead of \((v_n - \dot{s})\). Schwarz describes this quantity as the reaction force \(f\) per unit length exerted on the superfluid in the neighborhood of the core, rather than as a drag force exerted on the core. The result is

\[ \frac{f}{\rho_s \kappa} = -\alpha s' \times [s' \times (v_n - \dot{s}_0)] - \alpha' s' \times (v_n - \dot{s}_0). \]  \hspace{2cm} (1.31)

The motion of the vortex due to friction is correspondingly written as

\[ \dot{s}_f = \frac{s' \times f}{\rho_s \kappa}. \]  \hspace{2cm} (1.32)

Although Schwarz never refers to this relation as the Magnus effect, comparing equation 1.28 to the above expression shows that \(f = f_D = -f_M\) since \(\dot{s}_f = \dot{s} - \dot{s}_0\). Equation 1.29 is recouped upon inserting equation 1.31 for \(f\) into the above expression and employing the highly useful identity that for any vector \(A\) locally perpendicular to the vortex line, \(s' \times (s' \times A) = -A\).

Finally, we have a complete expression for the motion of a given point on a vortex filament. If nonlocal and boundary effects are ignored, \(\dot{s}_0\) simplifies to \(v_s + \dot{s}_t\) and \(v_n - \dot{s}_0 = v - \dot{s}_t\) where \(v\) is the relative velocity \(v_n - v_s\) between the normal and superfluids, so the velocity of the vortex line \(\dot{s}\) can be written as

\[ \dot{s} = v_s + \dot{s}_t + \alpha s' \times (v - \dot{s}_t) = \alpha' s' \times [s' \times (v - \dot{s}_t)]. \]  \hspace{2cm} (1.33)
Even though it does not contain the nonlocal and boundary terms, equation 1.33 provides a good approximation of the vortex motion, accurate to within 10% as long as the vortex-vortex or vortex-image spacing is greater than $\Delta \approx 2R/\ln(cR/a_0)$.

Let us now examine the effect of friction on the motion of a vortex line. Without friction, a vortex ring maintains a constant size as it moves perpendicular to its plane of orientation at velocity $\dot{s}_t = \beta_S/R$, while being carried downstream with the superfluid at $v_\ast$. Since the second friction parameter $\alpha'$ is an order of magnitude smaller than $\alpha$, it can be ignored for the purpose of this simple analysis. Adopting spherical-polar coordinates, for a ring initially in the x-y plane moving in the z-direction, $s' = -\sin \phi \dot{x} + \cos \phi \dot{y}$, where $\phi$ is the angle $s$ makes with the x-axis. If the relative velocity $v$ lies in the y-z plane, i.e. $v = v_\ast (\sin \theta \dot{y} + \cos \theta \dot{z})$, then equation 1.33 with $\alpha' = 0$ yields

$$\dot{s} = v_\ast + \frac{\beta_S}{R} \dot{z} + \alpha (v \cos \theta - \frac{\beta_S}{R}) \dot{r} + \alpha v \sin \theta \sin \phi \dot{z},$$

(1.34)

where $\dot{r} = \cos \phi \dot{x} + \sin \phi \dot{y}$ is a unit vector directed radially outward from the ring.

While moving in the $\dot{z}$-direction at velocity $\beta_S/R$, the ring expands outward at velocity

$$\frac{dR}{dt} = \alpha (v \cos \theta - \frac{\beta_S}{R})$$

(1.35)

and changes orientation at a rate

$$R \frac{d\theta}{dt} = -\alpha v \sin \theta.$$  

(1.36)

The frictional interaction causes a ring to shrink if $\beta_S/R > v \cos \theta$, and grow otherwise. Under a relative velocity "tailwind" ($\cos \theta$ positive), small rings shrink and
large rings grow, whereas rings travelling against the relative velocity always shrink. As a ring travels, its plane of orientation rotates so as to decrease $\theta$, preferentially becoming oriented perpendicular to $v$. Thus, under the right conditions, a ring initially travelling mostly upstream and shrinking can flip around to end up travelling downstream and growing.

**Preferentially, this frictional interaction annihilates highly curved vortex filaments and produces large vortices of low curvature. Driven solely by the relative velocity, these large vortices would continue to balloon outward until all were annihilated at the channel boundaries, and a self-sustaining tangle would never form. Through the process of lines crossing and reconnecting, small vortex loops of high curvature and random orientation are continuously being created. These line-line and line-boundary reconnections are essential to sustaining the vortex tangle.**

### 1.7.3 Predictions of the Schwarz Model

The evolution of the vortex distribution is computed numerically from the equation of motion of a vortex line (equation 1.27). The details of this simulation are described fully in [Sch85] and [Sch88]. Each macroscopic property of the vortex tangle state predicted within the Schwarz model is computed by integrating the corresponding local variable over the vortex distribution. Once the tangle reaches a steady state, these macroscopic properties of the tangle no longer evolve in time, but merely fluctuate about steady average values, following the fluctuations in the tangle itself. The simplest property to compute is the total length of vortex line per unit volume, called
the vortex line density

\[ L = \frac{1}{\Omega} \int d\xi \ , \quad (1.37) \]

where \( \Omega \) is the integration volume. To average out the intrinsic fluctuations, Schwarz considers the ensemble average of each property of interest by averaging the integral in question over all possible tangle configurations, weighted by their respective probability of occurrence. In the equations that follow, \( L \) will be understood to refer to the ensemble average of equation 1.37, and similarly for all other macroscopic properties defined by integrals over the vortex distribution. In the steady state, the ensemble average is the same as the temporal average, but the ensemble average can be used to average the fluctuations out of even time-dependent quantities.

Schwarz uses some simple dynamical scaling arguments to derive the form of the dependence on \( L \) and the relative velocity \( v \) that the various macroscopic properties will exhibit. The essential results obtained from these scaling arguments are quoted below. Most important of the scaling relations, the line density is found to depend quadratically on the relative velocity:

\[ L = \frac{c_L^2 v^2}{\beta s^2} \ . \quad (1.38) \]

Similarly, the line-length weighted curvature, defined by

\[ \bar{|s'|} = \frac{1}{\Omega L} \int |s'| d\xi \ , \quad (1.39) \]

is found to obey

\[ \bar{|s'|} = c_1 L^{1/2} \ . \quad (1.40) \]
Recalling that $|\bar{s}''| = 1/R$, this relation can be used to recast $\beta_s$ from equation 1.23 in terms of $L$, revealing the logarithmic dependence of $\beta_s$ on $L$:

$$\beta_s = \frac{\kappa}{4\pi} \ln \left( \frac{1}{c_1 L^{1/2} a_0} \right).$$  (1.41)

The constant $c \sim 1$ appearing previously in $\beta_s$ has been absorbed into $c_1$, which is also on the order of one. The variation of $\beta_s$ with the line density is not dramatic (as $L$ increases from 1 to $10^{10}$ cm$^{-2}$, $\beta_s$ decreases from about 18 to 7); nevertheless, it is an important and observable feature which distinguishes the Schwarz model from the earlier phenomenological model of Vinen.

Equation 1.38 alone demonstrates that the vortex tangle is homogeneous, since the temporal average of $L$ depends only on the relative velocity and not on position or any other geometric feature such as channel shape. The vortex tangle may be highly anisotropic due to the frictional force discussed above, with the orientation of lines polarized perpendicular to the flow direction; nevertheless, the steady-state vortex distribution produced by numerical simulation proves to be spatially homogeneous [Sch88]. Dynamical scaling reveals this homogeneity to be a direct consequence of having imposed a spatially uniform drive velocity. The dynamical scaling argument leading to equation 1.38 is simple and straightforward, making this technique all the more powerful since it renders such an important result so easily. For dynamical scaling to work, however, all dimensions in the problem must scale equally to one another, likewise all velocities and all times. A uniform flow is readily scaled in this way.
Dynamical scaling cannot be performed for a nonuniform flow, since like quantities no longer all scale in the same way. Anticipating the present work, it may seem that, to predict the (inhomogeneous) vortex line density resulting in a nonuniform flow, all one need do is replace the uniform velocity in equation 1.38 with the appropriate nonuniform velocity. Indeed, it is precisely this hypothesis that will be tested in analyzing the results of the present experiment. Such a replacement of the uniform velocity with its nonuniform counterpart presumes that the relation between $L$ and the velocity remains unchanged, even though the flow is no longer uniform. Lacking the ability to make dynamical scaling arguments for a nonuniform flow, the dependence of $L$ on $v$, or on any other variable in the problem, cannot be predicted a priori. Equation 1.38 may still be valid in the case of nonuniform flow; one is simply no longer able to assert its validity through a scaling argument.

A multitude of other properties of the vortex tangle are predicted in the Schwarz model by integration over the vortex distribution. The anisotropy is traditionally measured in terms of three parameters, $I_\parallel$, $I_\perp$, and $I_\ell$, defined by

\begin{align}
I_\parallel &= \frac{1}{\Omega L} \int \left[ 1 - (s' \cdot \hat{r}_\parallel)^2 \right] d\xi , \\
I_\perp &= \frac{1}{\Omega L} \int \left[ 1 - (s' \cdot \hat{r}_\perp)^2 \right] d\xi , \\
I_\ell \hat{r}_\parallel &= \frac{1}{\Omega L^{3/2}} \int s' \times s'' d\xi .
\end{align}

(1.42)

Here $\hat{r}_\parallel$ and $\hat{r}_\perp$ are unit vectors parallel and perpendicular to $v$, respectively. For an isotropic tangle, $I_\parallel = I_\perp = \frac{2}{3}$ and $I_\ell = 0$. If instead the tangle consisted solely of
vortices lying in planes perpendicular to \( \mathbf{v} \), then \( I_\parallel = 1 \) and \( I_\perp = \frac{1}{2} \), while \( I_t \) would depend on the exact distribution of the binormal.

Equation 1.31 describes the local frictional force which the normal fluid exerts on the superfluid. To obtain the average mutual friction force density \( F_m \) exerted by the normal fluid on the superfluid, this local force is integrated over the vortex distribution:

\[
F_m = -\rho_s \kappa \alpha \frac{1}{\Omega} \int \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v} - \beta_s \mathbf{s}' \times \mathbf{s}'')] \, d\xi
\]

Choosing a spherical-polar coordinate system and aligning \( \mathbf{v} \) in the azimuthal direction so \( \hat{r}_\parallel = \hat{z} \), and arbitrarily setting \( \hat{r}_\perp = \hat{\mathbf{z}} \), the unit vector \( \mathbf{s}' \) is given by

\[
\mathbf{s}' = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}
\]

and the terms in \( \mathbf{v} \) in the integrand become

\[
\begin{align*}
\mathbf{s}' \times \mathbf{v} &= v [\sin \theta \sin \phi \hat{x} - \sin \theta \cos \phi \hat{y}], \\
\mathbf{s}' \times (\mathbf{s}' \times \mathbf{v}) &= v [\sin \theta \cos \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} - \sin^2 \theta \hat{z}].
\end{align*}
\]

Over the tangle, the vectors \( \mathbf{s}' \) and \( \mathbf{s}'' \) will be distributed equally over all angles \( \phi \), therefore the \( x \)- and \( y \)-components of these terms vanish by symmetry upon integration, leaving only the \( z \)-component. Since \( \mathbf{s}' \) and \( \mathbf{s}'' \) are mutually perpendicular, \( \mathbf{s}' \times (\mathbf{s}' \times \mathbf{s}'') = -\mathbf{s}'' \), so upon integration the term in \( \mathbf{a}' \) in equation 1.43 vanishes entirely by symmetry. Recognizing that \( \mathbf{s}' \cdot \hat{r}_\parallel = \cos \theta \), the integral becomes

\[
F_m = \rho_s \kappa \alpha \left\{ \frac{v}{\Omega} \int [1 - (\mathbf{s}' \cdot \hat{r}_\parallel)^2] \, d\xi - \frac{\beta_s}{\Omega} \int \mathbf{s}' \times \mathbf{s}'' \, d\xi \right\}.
\]
Comparing the above expression to the anisotropy coefficients 1.30 given above, and using the relation between $L$ and $v$ (equation 1.38), the mutual friction force density is recognized to be

$$F_{sn} = \rho_s \kappa \alpha \left( I_\parallel - c_L I_\parallel L \right) L \cdot v \quad (1.47)$$

Finally, the excess dissipation predicted by the Schwarz model is

$$\nabla T' = -\frac{F_{sn}}{\rho_s S} = -\frac{\kappa \alpha}{S} \left( I_\parallel - c_L I_\parallel L \right) L \cdot v \quad (1.48)$$

Following Schwarz's notation, the direction of $F_{sn}$ is opposite to $\nabla T'$, since $F_{sn}$ is the force exerted by the normal fluid on the superfluid. Elsewhere in the literature, $F_{sn}$ is used to denote the force exerted by the superfluid on the normal fluid (see for example equations 1.12 and 1.13. Usually one is only concerned with the magnitude of $F_{sn}$ so the direction of this vector is unimportant, but this distinction between the convention adopted by Schwarz and that used by other authors can lead to significant confusion.

This turbulent temperature gradient is in excellent agreement with the T-II state observed in uniform flows. Furthermore, this model of homogeneous turbulence does not contain any adjustable parameters to describe the turbulent state: the anisotropy coefficients $I_\parallel$, $I_\perp$ and $I_\parallel$, and the coefficients $c_1$ and $c_L$ are all temperature-dependent parameters whose values are determined absolutely within the model. In the case of inhomogeneous turbulence, any or all of these parameters may take on quite different values, or even become spatial functions. Moreover, the underlying relations defining these parameters may even need to be altered, since these relations were determined by dynamical scaling arguments which hinged on the flow being uniform.
Two other relations derived within the Schwarz model are germane to future discussions, so warrant presentation here. The vortex line drift velocity taken in the superfluid rest frame is given by

$$v_t = \frac{1}{\Omega L} \int \dot{s} \, d\xi - v_s$$  \hfill (1.49)

where $s$ is given by equation 1.33. Employing the relations given in equations 1.42 and 1.45, upon integration the term in $\alpha$ vanishes entirely by symmetry, and the remainder reduces to

$$v_t = \alpha' I_{||} v + (1 - \alpha') I_t \beta_s L^{1/2} \dot{r}_{||}$$

$$= [\alpha' I_{||} + c_L (1 - \alpha') I_t] v . \hfill (1.50)$$

In considering the dynamical balance struck between vortex line growth and decay processes, Schwarz derives the time rate of change of the line density, starting with the instantaneous fractional rate of change of line length for a particular point on a vortex. He obtains the Vinen equation in terms of his more fundamental parameters:

$$\frac{\partial L}{\partial t} = \alpha I_t v L^{3/2} - \frac{\alpha I_t}{c_L} \beta_s L^2 . \hfill (1.51)$$

Comparing to equation 1.14, the identification can be made that $\chi_1 = I_t$ and $\chi_2 = 2\pi \alpha I_t \beta_s / (\kappa c_L)$.

Finally, it should be noted that in his thesis Aarts points out one nonlocal effect not included in the vortex line equation of motion, which would not be seen in a homogeneous tangle but which might well play a role in an inhomogeneous tangle [Aar93]: Being that the line density is independent of position in a homogeneous tangle, at any
point in the fluid the induced velocity averages to zero over the tangle distribution, even though the orientation of vortices in the tangle becomes polarized by friction. For a homogeneous tangle, therefore, the vortices do not contribute collectively to the superfluid velocity field. For an inhomogeneous tangle, the induced velocity can add a non-zero contribution to \( v_s \).
CHAPTER II

Experimental Apparatus

This experiment follows a standard thermal counterflow design. The same experimental probe, measurement technique, and control electronics were previously used in experiments by D. Griswold [Gri86] and C. P. Lorenson [Lor85]. A special diverging flow channel and a new thermocouple were constructed for this experiment. Following a brief overview of the apparatus, a detailed description of the flow channel design and assembly is presented in Section 2.2. Construction of the thermocouple, including a minor design modification adopted for this experiment, is described in detail in Section 2.3. The thermometry used to monitor the temperature of the helium and maintain temperature regulation is discussed in Section 2.4. Because of the large temperature differences generated in the flow channel, the thermometers must be calibrated using a nonlinear fit, described in Section 2.5. Normal operating behavior of the SQUID/thermocouple measurement system is outlined in Section 2.6. The behavior of the SQUID when beset with rf interference is described in Section 2.7, which also documents how specific problems with rf noise were remedied to the extent possible. Finally, in Section 2.8 several modifications are outlined which would
address specific design problems recognized in the course of this work and improve upon the design of this experiment.

2.1 Probe Overview

A schematic overview of the experimental apparatus is presented in figure 2.1. The flow channel connects a large (~ 400 ml) helium reservoir to a small chamber containing a resistance heater. A regulator circuit controlled by an ac resistance bridge and lock-in amplifier holds the reservoir at a constant temperature $T_0 \pm 20 \mu K$. Introducing power to the heater establishes a heat current $\dot{Q}$ in the channel, and the temperature at the heated end of the flow channel rises to some temperature $T_0 + \Delta T$, determined by the total thermal resistance of the helium in the flow channel.

A vacuum jacket, referred to in previous experiments as the vacuum can, thermally isolates the active experimental region, consisting of the reservoir, flow channel, and heater chamber, from the surrounding 4.2 K helium bath. All of the electrical leads entering the experimental region through the two feedthroughs are thermally anchored to the outside of the reservoir. The success of this experiment relies on knowing exactly how much heat is being input to the heated chamber at the narrow end of the flow channel, therefore a direct heat leak from the 4.2 K bath into the heated chamber, or elsewhere into the helium inside the flow channel, would be disastrous. Heat-sinking the wiring at the reservoir guarantees that all the heat travelling along these wires into the experimental region from the surrounding 4.2 K bath is deposited in the reservoir. This heat is added to the variable amount of heat input to the reservoir by
Figure 2.1: Overview of the experimental apparatus. The thermocouple is shown connected across the entire channel, between $r_A$ (heated end) and $r_H$ (reservoir end), but either side can be moved to be connected instead to any other intermediate location, labelled B–G.
the temperature regulation heater. Meanwhile, heat is continuously being removed from the reservoir by evaporative cooling (pumping on the reservoir). The regulation circuit dynamically adjusts its contribution to the total heat input to the reservoir to maintain the remaining helium in the reservoir at a constant temperature.

The remaining structural portion of the probe immersed in the 4.2 K bath is not shown in figure 2.1. A detailed schematic of this structural “skeleton” of support rods, pumping lines, and radiation baffles is presented as figure 7 in [Gri86] and as figure 5 in [Lor85]. The details of constructing the probe, including the vacuum jacket, helium reservoir, and all electrical feedthroughs are given in [Lor85].

Two nested glass dewars provide the thermal isolation between the 4.2 K bath and room temperature. The inner dewar containing the 4.2 K helium bath is immersed in a nitrogen bath at 77 K filling the outer dewar. The vacuum space in the outer dewar, separating the nitrogen bath from room temperature, is sealed and therefore not actively pumped. The inner dewar vacuum space separating the nitrogen and helium baths, and the space inside the vacuum jacket isolating the experiment at $T_0$ from the helium bath, are both actively pumped. (Figure 8 in [Gri86] schematically depicts this arrangement of liquid and vacuum spaces.) The helium reservoir and the main helium bath can each be cooled to superfluid temperatures by pumping on the liquid. During daily operation, only the helium reservoir is cooled in this manner. The main bath is cooled below 4.2 K only when a temperature calibration is performed. The helium reservoir is refilled from the main bath through a needle valve.
Mechanical vibrations of the experimental probe are highly undesirable for two basic reasons. First, vibrations can easily precipitate the transition to turbulence, precluding a proper study of the metastable laminar state. Second, motions of any electrical leads can create electromagnetic noise to which the SQUID/thermocouple circuit is very susceptible. The entire cryostat is therefore vibration isolated during a data run. The framework supporting the cryostat and probe is isolated from the floor by four large springs. (At all times except when vibration isolation is needed, this framework is otherwise supported on four large lab jacks.) The framework also contains two boxes, one on either side of the probe, each filled with 1000 lb. of sand to damp vibrations further. Any pumping line which must remain attached during data acquisition runs through an additional sandbox to damp vibrations. The helium recovery line is connected by a long segment of lightweight flexible tubing. All additional pumping and plumbing lines can be disconnected during data runs.

2.2 The Flow Channel

The flow channel is depicted schematically in figure 2.2. Stycast 1266 epoxy is machined and assembled to form a 10 cm long rectangular channel of constant height \( h = 0.025 \text{ cm} \), but varying width. The width increases linearly with distance along the channel length from 0.2 cm at the narrow end to 1.0 cm at the wide end. This expansion of the aspect ratio from 8:1 to 40:1 represents a constant opening angle of \( \theta = 0.08 \text{ rad} \). Expressing the location along the channel length as a radial position \( r \) in cylindrical coordinates, the width at \( r \) is given by \( w = \theta r \). Counterflow
Figure 2.2: Schematic detail of the flow channel. The expanded 3-D view of a probe and tab (drawn to approximate scale) shows how the copper wire has been cut away during the machining step that forms two adjacent walls of the flow channel, such that the exposed copper surface lies flush with these walls.
in this geometry therefore emulates the two-dimensional problem of radial heat flow in a cylindrical geometry. One may think of this radially diverging flow as originating from a line source at \( r = 0 \), entering the channel at \( r = 2.5 \text{ cm} \) and exiting at \( r = 12.5 \text{ cm} \).

The laminar or turbulent dissipation associated with this counterflow establishes a temperature gradient throughout the channel, so the temperature of the fluid at position \( r \) is raised above that of the regulated reservoir by an amount \( \Delta T(r) = T(r) - T_0 \). Although the present experiment studies diverging flow, the apparatus is designed so that the flow channel could be easily inverted to study converging flow. Another heater chamber is therefore located at the wide end of the flow channel, identical to the one at the narrow end but unused in the present experiment (see figure 2.1). Each heater chamber, made of Stycast, is connected to a copper thermometer block. Depending on whether it is to serve as the reservoir or heated end, respectively, the thermometer block is then either attached to the mating flange on the large helium reservoir, or blanked off with a copper flange, using an indium o-ring seal.

### 2.2.1 Mixing and Molding Stycast

All the pieces used in making the flow channel and heater chambers are machined from Stycast 1266 epoxy resin [EC]. Stycast 1266 was chosen for its low thermal conductivity, low thermal contraction, long curing time (8-12 hours), and durability under repetitive thermal cycling. The assembly process therefore begins with mixing the epoxy and molding pieces of appropriate size for machining. Several secrets to
mixing and molding a successful batch of Stycast were discovered with some experimentation.

To make each half of the flow channel, a rectangular block of Stycast approximately $4.5 \times 1.5 \times 0.75$ inches in size had to be molded. Reusable molds were made by cutting thin aluminum plate into a rectangle, then cutting out small squares from each corner and folding the edges up to form a rectangular box. The dimensions of the box did not have to be precise, but in order to provide enough material to work with, the resulting Stycast block should provide at least 0.25 inches of extra material for each machining cut. The aluminum molds were lined with aluminum foil to allow the Stycast to be released from the mold. The aluminum does not react with the Stycast, and can be peeled off easily afterwards (for visual inspection) if it is not too wrinkled. The heater cells were machined from cylindrical pieces. Ordinary plastic film canisters (the type used to package standard 35mm photographic film) proved to be of precisely the right diameter, about 1.25 inches, to serve as molds. The film canisters were easily cut and broken off the molded piece and did not need to be lined.

Although mixing epoxy does not seem a particularly challenging task, obtaining the best possible results required some care and attention, and not a little trial and error. Stycast 1266, unlike many other epoxies, is fairly fussy in the proportions of catalyst to resin used. Slight deviation from the recommended proportions results either in soft epoxy that is tacky to the touch, or in epoxy that sets up too fast. The temperature at which the epoxy is cured also affects the curing rate. Since the curing reaction is exothermic, too high a curing rate results in epoxy which burns.
slightly as it cures. The resulting epoxy is amber in color and slightly opaque. If the reaction proceeds too quickly or too large a volume of epoxy is mixed in one step, the epoxy will turn out riddled with trapped bubbles caused by the released gases which expand and form bubbles in the hot mixture faster than they can escape the quickly hardening material. Although this burning does not affect the thermal and other properties much, air bubbles were to be avoided since the machined surfaces ought to be smooth and free of defects. Since one cannot visually judge whether or not the discolored epoxy contains small bubbles, any burned pieces were discarded.

Storing Stycast in a refrigerator prolongs its shelf life, but it should be mixed only at room temperature. Sometimes small crystals form in the epoxy resin, especially under refrigeration. The resin is still perfectly good and usable, and needs only to be heated at 150°F for a couple of hours to dissolve the crystals. If all the crystals have not dissolved, a little stirring and heating for an additional hour should finish the job. This procedure can be repeated several times during the shelf life of the product without affecting it. Refrigeration is needed only for long term storage; when in regular use, the Stycast should be kept at room temperature. We avoided using any Stycast that had exceeded its shelf life.

To mix a batch of Stycast, up to 100 ml of the epoxy resin was weighed in a clean, preweighed polyethylene beaker. The appropriate amount of catalyst was then added with an eyedropper until total weight of the beaker with resin and catalyst equaled the amount calculated to ensure the proper weight ratio of 28:100 for catalyst:resin. In this manner, the amount of catalyst could be controlled very precisely. Batches
on the order of 65 g of resin to 18 g of catalyst were always mixed, even when the quantity needed was much smaller, since it was found that mixing smaller batches resulted in inadequate control of the catalyst:resin ratio.

Considerable care was taken to eliminate air bubbles from the Stycast mixture. After stirring the mixture slowly for 10 minutes, as much air as possible was removed by putting the mixture under vacuum. The beaker was placed in a bell jar which was evacuated by an old rotary pump, affectionately known in the lab as “glop-glop” for its sacrificial service ingesting Stycast-laden air and the peculiar noise it makes while running. The mixture should be watched carefully throughout the pumping process. For the first 10 minutes or so, the bubbles released are small, but after about 15–20 minutes the vapor pressure becomes low enough that any remaining air bubbles expand very rapidly and the mixture suddenly foams. Pinching off the hose to the vacuum pump slows this foaming action. At this point, the pump is shut off and the pressure bled slowly up to atmospheric pressure (quick release of pressure can result in a terrible mess). The beaker used for mixing the Stycast should always be at least twice as large as the volume of Stycast mixed, and leaving the Stycast unattended while pumping is not recommended unless one relishes cleaning up the sticky aftermath! The mixture was then poured into the mold or molds, taking care to pour slowly and smoothly to avoid introducing new air bubbles. No additional vacuum evacuation was needed after pouring.

The manufacturer states that the pot life will be 1/2 hour, but the mixture remains fluid enough to work with and pour for up to an hour. The manufacturer recommends
curing at temperatures of 125–150°F, but elevated curing temperatures caused the burning described above. Batches were allowed to cure either at room temperature or under mild cooling, which lowers the curing rate and ensures that the Stycast does not overheat and burn as it hardens. Both refrigeration and an ice bath were tested as cooling methods for the large rectangular pieces. Setting the aluminum molds on a metal plate in contact with an ice bath seemed to give the best results by keeping the Stycast cool during the beginning of the curing time, then allowing the Stycast to warm slowly to room temperature, once the ice had melted, for the duration of the curing time. Refrigeration kept the Stycast from setting up fully: after 12 hours it was still a viscous liquid. Once removed from the refrigerator, the Stycast did harden fully, but remained tacky to the touch and was deemed unusable. Room temperature curing worked well for small pieces, but larger pieces fared better when cooled with the ice bath.

2.2.2 Flow Channel Design and Assembly

To form the flow channel, two rectangular pieces are machined in such a way that when fit together, they create an open channel running lengthwise through the middle of an otherwise solid rectangle (see figure 2.3). The cross section of the channel is a rectangle of constant height but expanding width. To form this rectangular wedge-shaped channel, a lengthwise step is first milled out along one side of each of the two rectangular pieces. Topologically, each piece is now L-shaped in cross section, although the two are not identical. To distinguish the pieces, one will be referred to
hereafter as Part A and the other as Part B. The step in Part A spans about 1/3 of its width. The step in Part B is of equal depth to that in Part A, but spans about 2/3 of its width. At this point, the two pieces would fit together exactly to form a single solid rectangle. A second step is then milled along the middle of Part A, into the surface adjacent to the first cut. This second step is much shallower than the first step. The result is that the two pieces still fit together, but now a small gap has been opened up in the middle. This gap (or hole) forming the channel is the height and width of the second step, which is cut on a slight diagonal with respect to the lengthwise edge of the piece. The rectangular-shaped hole therefore increases linearly in width along the length of the assembled pieces.

Figure 2.3: Cross sectional views of the assembled flow channel, showing how the two halves, Parts A and B, fit together. The channel height is exaggerated and not to scale.
The mechanical drawing for Part A of the flow channel is shown in figure 2.4. Initial rough cuts square off the Stycast block to 1.0"×0.25"×10 cm. These dimensions do not need to be rendered to extremely tight tolerances. Once clamped in place for the subsequent finishing cuts, the squared-off piece should not be removed from or repositioned in the mill. In the first finishing cut, a step 0.303" wide and 0.125" deep is cut along the length of Part A. The depth of this cut is critical: it can exceed 0.125" slightly, but cannot be shallower than this dimension. Before the second finishing cut is begun, the piece is rotated with respect to the milling direction to achieve the angle of 0.08 radians. The second step is then cut to a depth of 8 mils, which is again a critical dimension requiring 0.5 mil tolerance, erring on the side of being too shallow if anything. This step starts out 1.0 cm (0.394") wide at one end, and ends up only 0.2 cm wide at the other end.

The second half of the flow channel, Part B, was made in two stages. In preparation for the first machining stage, the block is squared off, in this case to 1.0"×9/32"×10 cm. (The extra 1/32" thickness is subsequently removed by an initial finishing cut in the second machining stage.) After the rectangular block was squared off, six holes were drilled along one long edge to accommodate the temperature probes. The spacing of these holes is shown in figure 2.5, the mechanical drawing for the first machining stage of Part B. The temperature probes were spaced unevenly, being closest to one another where the temperature gradient was expected to be largest. The holes are 0.025" in diameter by 0.375" deep, and inset 0.125" from the outer surface and 5/32" from what is to be the inner surface of this half of the flow channel. The outer
Figure 2.4: Mechanical drawing of Part A of the flow channel.
Figure 2.5: Mechanical drawing for Part B of the flow channel, showing the first machining stage.
edges of the holes were countersunk (not shown) to about twice the diameter of the hole.

Each temperature probe is nothing more than a piece of 22 gauge copper wire, to which a small tab designed to hold one end of the thermocouple is attached. A detail of the finished copper probe and tab is shown in figure 2.2. For each probe, a small rectangle of thin gauge copper sheet (1/32" thick) is crimped around the wire, forming a flag or tab approximately 7/16" long by 3/8" wide. This tab is hard soldered to one end of the wire, and the wire trimmed to a total length of one inch. Two small holes are drilled or punched in the tab, large enough to accommodate two tiny brass bolts (size 2-56). A second smaller rectangle with matching holes is cut from the same copper sheet and sized to 1/4" by 7/16" to fit over the tab. The bolts are installed, holding the rectangle and tab together, and the nuts are soft soldered to the back side of the tab (not visible in the expanded view in figure 2.2). Care must be taken, and solder applied sparingly, so as not to solder the rectangle and tab pieces together, or wick solder up into the threads of the nuts and bolts.

The wire probe end of a finished tab assembly was then epoxied into each of the holes drilled into Part B. The tabs were first cleaned in an acetone bath in an ultrasonic cleaner for 20 minutes. The wires were dipped in Stycast and inserted in the holes, with Part B standing on the opposite edge so the holes were vertical. Since many small bubbles were introduced as the wire was inserted, the piece was vacuum evacuated for about 20 minutes, which got rid of most of the small bubbles. Additional Stycast was added as needed to fill in the countersunk area and build up
a small collar of Stycast around the wire to provide strain relief. The Stycast had to seal completely around the wire to prevent superleaks.

Once the Stycast had cured, Part B was ready for the second machining stage. The mechanical drawing for this stage is given in figure 2.6. An initial finishing cut leveled the inner surface, removing the extra 1/32" of material allowed for this purpose and reducing the thickness to 0.25". After this cut had been made, the copper probe wires were positioned exactly halfway between the inner and outer surfaces. Next, a step was cut 0.125" deep and 0.697" wide, starting from the side opposite that containing the probe wires. Since the wires extended into the material 0.375", and the remaining material not cut away by the step was only 0.303" wide, this step cut into the end of each wire, planing through approximately half its diameter and exposing a copper surface roughly 0.18 x 0.06 cm, or 0.01 cm², in size. The expanded view in figure 2.2 illustrates how the wire is exposed by this cut.

Parts A and B are now ready to be assembled to form the flow channel as shown in figure 2.3. The exposed copper surface of each temperature probe lies flush within two walls of the channel, therefore the copper comes in direct contact with the helium, but does not protrude into or otherwise obstruct the flow in any way. These temperature probes extend a constant distance, about 0.18 cm, across the channel, which constitutes about 90% of the total channel width for the probe at position B, but only about 20% of the total width for the probe at position G (see figure 2.2). Any cross-channel temperature variations are believed to be extremely small, therefore it is assumed that detecting the temperature over a variable fraction of the channel
Figure 2.6: Mechanical drawing for Part B of the flow channel, showing the second machining stage.
width does not present a problem. The copper probes are large enough that the
temperature of the helium will vary locally over the exposed surface, however small
those variations are, so it is true that the probe senses some kind of average of the
local temperature distribution over its surface. One can therefore entertain questions
as to whether this averaging, and also the uncertainty in the radial position of the
probes, lead to any significant systematic discrepancies in the measurements. Our
assessment is that these sources of error are negligible.

The astute reader will have noticed by now that the mechanical drawings specify a
channel height of 8 mils or 0.020 cm, whereas the beginning of this section states that
the channel height is 0.025 cm, or 10 mils. The channel was designed to be 0.020 cm
in height, but the actual channel height turned out to be 2 mils thicker, or 0.025 cm.
Accounting for the mysterious increase, and making an accurate determination of the
actual channel height, has proven a vexing problem throughout these investigations.
The process of dealing with this problem is described fully in Section 2.8.2 below.

To prepare them for epoxying, the two halves of the flow channel were cleaned
as follows. First, the pieces were cleaned in an ultrasonic cleaner for 5 minutes, in a
solution of 1 teaspoon of Alconox soap added to 1 liter of deionized, demineralized
(de-I) water. All subsequent handling of the cleaned pieces was done wearing surgical
latex gloves. The pieces were rinsed in clean de-I water and dried with a Kay-dry
towel (a low-lint tissue) and set on a clean aluminum foil surface. Any visible dust
was removed using an anti-static brush.
The Stycast used to glue the two halves together was applied to the cleaned surfaces with a clean, unused paint brush. A small, fairly stiff artist’s brush worked well. There are two mating surfaces, or seams, to fill with glue (see figure 2.3). The one on the side opposite the probes is flat, and parallel to the broad sides of the flow channel. The seam on the probe side is L-shaped, the leg of the L being where the two sides of the step cuts meet. In the first attempt at gluing, the Stycast was applied only along the outside portions of the two seams, leaving the last 3 mm nearest to the flow channel free of glue. No Stycast was applied around the bend in the L-shaped seam. The idea was that capillary action would carry the Stycast in towards the flow channel, completely filling the seams. It was feared that too much Stycast, or applying Stycast too close to the flow channel, would result in Stycast oozing into and obstructing the channel. The two halves were held together while drying by clamping the two broad sides of the channel between machinist’s parallels.

This first gluing attempt was far from successful. Along the flat seam, large gaps were visible where the glue did not migrate in far enough to reach the edge of the flow channel. The two pieces had also slipped sideways, since they were not clamped in that direction. As a result, a large gap had opened up along the inner leg of the L-shaped seam. The glue had not spread around the bend in the L-shaped seam at all, nor would there have been enough glue to fill this large gap if it had. The resulting channel was itself L-shaped instead of rectangular in cross section! The two halves of the flow channel were rebuilt, and the gluing attempted again.
One problem with the first gluing attempt was that the liquid Stycast did not readily wet the dry Stycast surfaces. Clamping forced the glue to propagate along unpredictable paths. In the second gluing attempt, the seams were first wet with a thin layer of glue. After applying this layer, most of the glue was wiped away again using the brush, leaving only a very thin layer wetting the surface. A thicker layer of glue was then applied as before to the outer portions of the wet surfaces. The thin wetting layer was supposed to guide the flow of glue throughout the seams, but ensure that it stopped short of oozing into the flow channel. Figure 2.7 illustrates the areas where the two layers of glue were applied. For the wetting layer on Part A, the entire surface of both the L-shaped and flat seams were wetted. On Part B, only the outer portion of the surface for the L-shaped seam and the first 1.5 mm continuing around the bend into the inner leg of that seam were wetted. The second, thicker layer of glue was applied only to the two parallel seam surfaces on Part A.

![Figure 2.7](image)

Figure 2.7: Placement of the epoxy used to glue Parts A and B together. A thin Stycast layer (dashed line) was applied first to wet the surface, followed by a thick Stycast layer (heavy solid line)
flat seam, the thick layer was applied to within about 2 mm of the flow channel edge. On the L-shaped seam, glue was applied only up to the bend in the L, and not along the inner leg.

The way the flow channel was clamped for drying was also changed to prevent the pieces from moving laterally. First, the ends were lined up with two parallels. Next, the long edges were clamped with three C-clamps. Parallels could not be used along these edges because of the copper probes and tabs along one side, but the C-clamps prevented any sideways slippage. Finally, the two broad sides were sandwiched between two parallels and held with two more C-clamps. All clamps were tightened only until barely finger-tight. The assembly was laid horizontally, with Part B on the bottom, and allowed to set overnight.

This time the glue job was successful. The epoxy spread well over the L-shaped seam. Some voids were visible in the middle of the flat seam, but the epoxy had sealed the inner edge of that seam along the length of the flow channel. Next, the heater chambers were made and attached to the two ends of the flow channel.

2.2.3 Heaters and Heater Chambers

At either end of the flow channel, a small cylindrical heater chamber made of Stycast is attached. Each heater chamber contains two noninductively wound ohmic heaters. Only one heater is needed to create the heat current $\dot{Q}$, but because once the heater chambers were installed there was no way to open them and replace a defective heater, each chamber held two heaters to provide redundancy. Only the heater chamber at the
narrow end of the flow channel was used in the present experiment to drive a diverging flow, but the flow channel was designed symmetrically, with heater chambers at both ends, in anticipation that converging flow could be studied in the same channel simply by inverting the channel and inputting heat at the wide end of the channel instead.

Each heater chamber is connected to a copper thermometer block via large diameter stainless steel tubing. This design, which separates the heat source from the point where the temperature is measured, circumvents the problem of Kapitza resistance. Because of this large thermal resistance present at the boundary between a heated object and superfluid helium, the heater is at a significantly higher temperature than the surrounding helium. If the heater were installed in the same copper block which holds the thermometers, the thermometers would record the temperature of the heater, not of the helium. By physically separating the heater from the thermometers, the thermometers measure what we want them to—the temperature of the helium.

The thermometer block is a cylindrical copper piece with an indium o-ring seat on the inside, and two small wells drilled into the outer surface to hold encapsulated resistance thermometers. The block is designed to attach to a matching flange on a large copper standoff pipe leading to the helium reservoir. The thermometer block at the end of the channel to be heated is blanked off with a matching cap, whereas the one at the other end (to be maintained at \( T_0 \)) is attached to the flange on the helium reservoir. To press-fit the indium o-ring seal, each thermometer block has six holes drilled in it, arranged symmetrically around the edge and threaded for 5-40
stainless steel cap screws. The mating flange and blank cap each have six unthreaded 1/8" holes in the same arrangement for the screws, plus two extra “pusher” holes on opposite sides for releasing the o-ring seal.

A mechanical drawing of the heater chambers is shown in figure 2.8. A cylindrical piece of Stycast is cut in the shape of a pillbox of inner diameter 5/8" and outer diameter 1" to form each chamber. The outside height of the piece is 9/16", but the interior space is cut to a depth of only 5/16", leaving a bottom on the pillbox 0.25" deep. A slot is cut across the bottom of the pillbox on the outside to accept the flow channel assembly. This slot is as wide as the flow channel assembly (0.5"), and 0.125" deep, so it cuts only halfway through the bottom. To connect the flow channel to the interior of the heater chamber, an elongated viaduct is milled through the remaining 0.125" of material in the center of the bottom. This viaduct is 0.125" wide and 0.5" long, to ensure that the flow channel (which is only 0.01" by 0.394" at its wide end) opens up into a space much larger than the channel itself.

The heater chamber is closed off with an endcap or lid, cut from another cylindrical piece of Stycast as shown in figure 2.9. A hole is drilled through the center of the cap to accept the 3/32" stainless steel tubing connecting the heater chamber to the thermometer block. Four 0.05" holes are drilled through one side of each heater chamber to serve as feedthroughs for the heater wires. A larger hole is drilled through the opposite side of each chamber. Six-inch long pieces of copper-nickel capillary tubing (0.047" o.d., 0.041" i.d.) were epoxied into these latter holes to allow the future possibility of connecting a pressure transducer to either end of the flow channel. A
Top view:

Cross section views:

Figure 2.8: Mechanical drawing of the heater chambers.
Figure 2.9: Mechanical drawing of the heater chamber caps.
pressure transducer had been used in the previous experiments conducted with the same probe as described in [Lor85] and [Gri86]. This capability was never used in the present work, and the tubing was sealed at the other end by inserting a thick copper wire (0.032" diameter), tinned with lead-tin (Pb-Sn) soft solder (60% Sn/40% Pb), most of the way into the tubing and soft-soldering at the end to plug the tube.

The heaters which delivered the heat current $\dot{Q}$ were made of Evanohm wire [EVAN] with a resistivity $\rho = 199.7 \, \Omega/\text{ft}$. Since 1000 $\Omega$ heaters were desired, the finished heater should be about 5 ft in length. For each heater, a 6 ft length of wire was cut and folded in half, then twisted, starting at the center, to create a noninductive winding. The twisted wire was then wound around a small diameter (3/32") wooden dowel, making a coil 1 cm long. About half of the 2.5 ft length of twisted wire was wound evenly in one direction along the dowel, and the remainder formed a second layer wound in the reverse direction back over the first layer, to further ensure that the heater be noninductive. Once wound, the coil was covered with a thin layer of GE 7031 varnish [GC], just enough to hold the wire in place. The last inch or so (of the 2.5 ft of twisted wire) was not coiled, but was left free to pass through the feedthroughs.

The portion of the heater wires passing through the feedthroughs had to be free of insulation because at low temperatures the insulation could crack, creating a super-leak through the feedthrough. The insulation was therefore stripped off the wires to within 0.25" of the coil. Rubbing with fine steel wool was the only method found to remove the insulation from the Evanohm. The wires were trimmed to two inches and
the ends tinned with silver solder, as ordinary Pb-Sn soft solder would not adhere to
the Evanohm. The silver solder was melted in a copper block by heating the block
with an acetylene torch. The copper keeps the solder molten for a few minutes. The
bare Evanohm wire is drawn quickly through the molten silver solder to tin it. Copper
leads (5 mil diameter) were then soldered to the heater wires using soft solder.
The resistance of each heater was measured, and the copper leads attached at the
appropriate points to make the heater resistance be as close to 1000 Ω as possible.
Any excess Evanohm wire was then trimmed off. Before each heater was used, it
was thermally cycled several times by dipping in liquid nitrogen. The resistance was
measured at room temperature and nitrogen temperature to make sure the heater
neither shorted out nor became an open circuit.

Small cylindrical rings, or collars, were machined from Stycast to hold the heaters
in place inside the heater chambers. These rings were cut to fit snugly inside the
heater chamber. Each ring was 0.125" high, with an inner and outer diameter of
0.5" and 0.625", respectively. Two rings were stacked atop one another, with the
heaters sandwiched between them, one heater on either side of the heater chamber.
Two hemispherical grooves cut in one surface of each ring accommodated the wooden
dowels on which the heaters were coiled. The grooves run parallel to each other, one
cutting a chord across one side of the ring, the other on the opposite side. To cut
these grooves, the two rings were stacked together and clamped, standing vertically,
in a drill press. Two parallel holes were drilled across the rings, one one either side,
centered on the interface between the two rings. An additional small hole was needed
on one side, arranged symmetrically between the two grooves, so the rings would not block the hole for the pressure transducer capillary tube. On the opposite side, four more vias were later reamed by hand, extending the feedthroughs for the four heater leads through the rings.

In the first step of assembling the heater chambers, the plugged Cu-Ni capillary tubing was installed in the heater chambers, and the 3/32" stainless steel tubing installed in the heater caps. Previously, the other end of the 3/32" tubing, cut to a length of 0.75", had been soldered into the copper thermometer block. The Stycast parts were cleaned in an Alconox bath and rinsed by the same method as described above for Parts A and B of the flow channel. After the metal tubes were cleaned with an Alconox solution (on the outside only) and rinsed in methanol, the end of each tube was painted with Stycast and pushed into its respective hole. Additional Stycast was applied around each joint on the outside to build up a small collar of Stycast and create a superleak-tight seal. The heater cap and thermometer block assemblies were dried with the cap on the bottom and the block on the top so gravity held the Stycast down around the joint, rather than allowing it to run away from the joint and along the tubing. Likewise, the heater chambers were suspended from the Cu-Ni tubing to dry, to make a nice collar around the juncture of the tubing with the body of the heater chamber. This first gluing step was actually done with the same batch of Stycast mixed for gluing the flow channel together.

In the second gluing step, the heaters were installed and the heater wire feedthroughs created. First, the positioning of the heater support rings was tested, and
the heater wire feedthrough holes reamed through the rings by hand. When stacked, the rings had to be slightly shy of 0.25" high to ensure that the heater cap would fit snugly. (Two small pieces of thread tied to the bottom ring allowed it to be easily and repeatedly lifted out of the heater chamber.) After pressing the bottom ring in place, pieces of small fiberglass "spaghetti" sheathing (Bentley-Harris S.T. #28) were threaded through the feedthrough holes. To accomplish this pesky task, a wire was inserted into a 3" length of fiberglass sheathing. One end of the sheathing was tacked to the wire with a little GE varnish, and this end pushed through the feedthrough hole. Once through the hole, the wire was removed and the sheathing trimmed to extend a bit (1-2mm) beyond the rings on the inside, and about 1 cm on the outside. The heaters were emplaced and the wires threaded through the fiberglass sheathing. Next, the sheathing was saturated with Stycast. A drop or two of Stycast was put on the outside portion of the sheathing, and the sheathing drawn inward to spread the Stycast along the hole. More Stycast was applied to the portion of sheathing now inside the heater chamber and the sheathing drawn back out to its original position. Additional Stycast was applied and carried into the feedthrough by capillary action. In the process, a few small bubbles formed in the feedthrough which could not be eliminated, but the bubbles did no harm as long as a solid, superleak-tight seal was created at one point along the feedthrough. With the Stycast applied to the sheathing on all feedthroughs, the top ring was emplaced and the heater cap put on temporarily to push the rings in place, then removed. After the Stycast had dried, a second coat was applied to the fiberglass sheathing extending outside the heater chamber,
to help ensure a good seal had been made. Finally, the point where the heater wire emerged from the now hardened end of the sheathing was coated with Liquid Tape to electrically insulate the soldered junction of the heater wire to its copper lead, and also provide strain relief to ensure the wire did not break off at that delicate point.

Next, the heater chambers were installed on either end of the flow channel. By completing this step before the heater chamber caps were put on, the ends of the flow channel could be visually checked to be certain no epoxy had migrated near or blocked the entrances to the channel. After all the parts in this gluing step had dried, the heater caps were glued onto the heater chambers in the final gluing step. The caps were oriented such that the two holes on the underside of each thermometer block (for the encapsulated thermometers) fell one on either side of the flow channel, allowing easy access for installing and removing the thermometers.

2.2.4 Flow Channel Design Considerations

Dimensions and Volume

The dimensions of the flow channel were chosen to maximize the anticipated temperature difference. The expected geometric dependence was estimated by assuming the dissipation would be predicted by the Schwarz model applied in the local uniformity approximation (see Section 3.4 of Chapter III). For a given heat current $\dot{Q}$, the flow velocity depends inversely on the cross sectional area of the channel, so the velocity was maximized by making the channel height as small as possible. The
turbulent temperature difference was expected to depend on the height as $h^{-3}$, so a small height dramatically increases the measured signal. The expected temperature difference $\Delta T'(r_i, r_j)$ to be measured between two positions $r_i$ and $r_j$ varies as $r_i^{-2} - r_j^{-2}$. This temperature difference is therefore increased by making the narrow end of the channel correspond to as small a position $r_i$ as is feasible, and making the overall channel length as long as possible. The experiment also had to fit inside the existing dewar system. Rather than use a full cylindrical geometry with inner radius $r_i$ and outer radius $r_j$, the choice was made to construct small wedge-shaped angular section of the full cylinder (like a piece of a pie). If a full cylinder were used, the outer radius would have been limited to 3.5 cm, the maximum size that would fit inside the vacuum jacket. With the wedge design, the length of the channel, given by $r_j - r_i$, could be as large as 10 cm. It was desired to keep the opening angle $\theta$ small so the divergence would represent a small perturbation of uniform flow. The combination of radial positions of the channel ends and opening angle $\theta$ were selected to maximize the temperature difference while meeting these other constraints of size and geometry.

Heater Surface Area

When a heated surface is brought in contact with superfluid helium, film boiling can occur at the interface if the heat flux (heat current per unit area) exceeds a nominal value. A heat flux of 0.5 W/cm² is considered a safe working limit to avoid film boiling. The heaters were designed to operate well below this limit. Each heater coil
was approximately 0.85 cm long and 0.24–0.25 cm in diameter. The surface area of each coil is estimated to be at least 0.64 cm². Even at the maximum heat current used, 10 mW, the heat flux at the heater surface is therefore only 0.016 W/cm².

This problem of film boiling exists only at the interface between the helium and the heated surface. The helium itself can support a much higher heat flux before reaching the limit where internal cavitation might occur. Since the cross sectional area of the channel entrance is only 0.005 cm², a heat current of 10 mW corresponds to a heat flux of 2 W/cm² at the channel entrance. If the heat current had been delivered by a heated surface at the end of the channel, the maximum usable heat current would have been about 2 mW. By delivering the heat through a large surface area into a large volume heater chamber, the heat flux is kept necessarily low at the heater. The heat flux is then concentrated or intensified as the same amount of heat energy passes through the constricted surface area at the channel entrance.

Having gone to significant effort to build heaters of an appropriate size and enclose them inside the heater chambers, in retrospect this design proved to be unnecessarily troublesome to build; moreover, once assembled the heater feedthroughs were fragile and not conducive to being repaired if damaged. An improved design which would have been preferable is described in Section 2.8.1 below.

**Thermal Conductivity**

The thermal resistance $R$ is the proportionality factor between the rate of heat flow along any thermal path (the "heat current" $\dot{q}$) and the temperature difference $\Delta T$. 
driving that heat flow: $\Delta T = \dot{q} R$, in analogy to Ohm's law $V = IR$ for an electrical circuit. Several thermal paths exist in parallel with the helium in the channel, each connecting the heated end of the channel to the reservoir. If the experiment is to work, the thermal resistance of each of these other paths must be orders of magnitude greater than that of the helium, ensuring that essentially all of the heat generated at the heated end is transported to the reservoir by the helium in the channel, not via one of these other thermal paths. The thermocouple and its electrical leads, the Stycast forming the channel itself, and the leads on various thermometers each provide thermal paths in parallel with the helium.

Of course, the actual thermal resistance $R_{He}$ of the helium in the flow channel cannot be known before the experiment is conducted and the turbulent temperature difference $\Delta T'$ generated for a given heat current $\dot{Q}$ has been measured. To estimate $R_{He}$, the expected turbulent temperature difference was computed from the Schwarz model applied in the local uniformity approximation. This estimate involved inserting the local relative velocity (equation 1.8) into the expression for the turbulent temperature gradient in the Schwarz model (equation 1.48) and integrating. For this estimate, the thermodynamic and Schwarz parameters were held fixed at the values appropriate to $T_0$; any variation (presumed small) of these parameters within the channel due to the increase in temperature along the channel length was neglected. The thermal resistance is thereby estimated to be

$$R_{He} \equiv \frac{\Delta T'}{\dot{Q}} = \frac{\kappa \alpha}{2S} (I_{||} - c_L I_L) \left( \frac{c_L}{\beta_S} \right)^2 \left( \frac{1}{\rho_* ST \theta h} \right)^3 \left[ \frac{1}{\tau_A^2} - \frac{1}{\tau_H^2} \right] \dot{Q}^2 \quad (2.1)$$
All the symbols have already been defined in Chapter I. The values of all the parameters in the Schwarz model are interpolated at the desired temperature as described later in Section 3.4.1 (also see the graphs of these parameters in Appendix B). The parameter $\beta_s$ varies only slightly with $\dot{Q}$ and $\tau$, and so is taken to be a constant, $\beta_s = 0.001 \text{ cm}^2/\text{s}$, for the purpose of this rough estimate. Note that the thermal resistance $R_{He}$ is a strong function of the heat current $\dot{Q}$.

The thermal resistance of the helium in the flow channel must be much larger than the thermal resistance of any other volume of helium in series with it, otherwise a measurable temperature drop would be created across these other volumes and the measured temperature difference would not be generated solely across the flow channel. The heater chambers and the stainless steel tube leading to the upper thermometer block are all large enough in their smallest dimensions to ensure that no appreciable amount of superfluid turbulence would form in these spaces, even at the highest heat current intended to be used, about 10 mW. The thermal resistance of these other volumes of helium therefore remains negligibly small, and only the helium in the channel has a thermal resistance of any significance.

One other important design consideration hinges on the magnitude of $R_{He}$: the thermal response time of the helium is given by $\tau = R_{He}C$, where $C$ is the total heat capacity of the volume of helium contained in the heater chambers, flow channel, and connecting vias. In analogy to an electrical RC circuit, the heat capacity acts as a thermal capacitor. For ease of data collection, $\tau$ needed to be small, preferably less than a second. To gauge the thermal response time of the helium, in addition
to estimating the thermal resistance, the total volume of helium in the flow channel assembly must be determined. The important criterion in deciding the dimensions the heater chambers and other helium-filled spaces is to strike a balance between making these spaces large enough to ensure a small thermal resistance while not making them so big that the total heat capacity results in too great a time constant for the experiment.

From the dimensions of the flow channel, its interior volume is computed to be 0.156 cm$^3$. With the heater support rings installed, the volume inside each heater chamber is 0.804 cm$^3$. Of this volume, the heaters take up about 0.04 cm$^3$ each. Each viaduct connecting the heater chamber to the flow channel adds a volume of 0.121 cm$^3$. The copper-nickel capillary tubing has an interior volume of 0.130 cm$^3$, but this volume is decreased by about 0.046 cm$^3$, due to the wire inserted in the tubing to plug it. The 3/32" stainless steel tubing connecting the heater chamber to the thermometer block has a wall thickness of 0.008", so the inner diameter is 0.197 cm, and the volume inside each 0.75" length of tubing is 0.058 cm$^3$. The total volume of each heater chamber, including its connecting vias to the flow channel and thermometer block, is therefore estimated to be 0.986 cm$^3$. Within the accuracy of all estimated dimensions, the small volumes of helium filling the heater chambers at either end of the flow channel may be taken to be about 1 cm$^3$, compared to the 0.156 cm$^3$ of helium within the channel. The total helium volume in the flow channel assembly is therefore no more than 2.16 cm$^3$. 
Table 2.1 lists the estimated thermal resistance at $\dot{Q} = 1 \text{ mW}$ computed from equation 2.1 for various reservoir temperatures. The thermal time constant $\tau$ is then estimated from the total heat capacity, computed by multiplying the specific heat per unit volume $c_v$ by the total volume of helium, 2.16 cm$^3$. The thermal resistance and time constant at 10 mW should be 100 times larger than the respective values at 1 mW. Because it is based on a theoretical thermal resistance rather than the actual measured value, the estimate of $\tau$ in Table 2.1 should be considered no more than a rough guide to the actual response time of the helium; nonetheless, this analysis shows that $\tau$ should be acceptably small. At 1 mW, $\tau$ should be on the order of 10–100 ms, and at the maximum heat current used, 10 mW, $\tau$ should be at most
a few seconds. Since the response time of the SQUID/thermocouple circuit is very short, the helium should provide the dominant time constant in the system, but this time constant is still acceptably short. These estimated values of $R_{He}$ and $\tau$ should prove to be at least of the same order of magnitude as the actual values.

Of all the thermal paths in parallel with the helium, the Stycast channel walls have the smallest total thermal resistance. The thermal conductivity of Stycast was measured by Armstrong et al. [Armst78], and found to obey a power law $k = (4.9 \pm 0.5) \times 10^{-4} T^{1.98}$ W/cm K over the range of temperatures measured, 0.05-0.45 K. Assuming this same $T^2$ behavior continues to higher temperatures, up to 2 K, the thermal resistance of the Stycast block forming the flow channel walls can be estimated. The Stycast block has a cross-sectional area of 2.42 cm$^2$ and is 10 cm long. That a small interior volume is missing from the block (0.16 cm$^3$, comprising the flow channel) can be ignored in this simple estimate, given the uncertainty in $k$. The thermal resistance of the Stycast channel walls is therefore estimated to be $5.0 \times 10^3$ K/W at 1.3 K and $3.0 \times 10^3$ K/W at 1.7 K. Only at the highest heat current of 10 mW does the thermal resistance of the helium become large enough for this second thermal path to be even remotely significant in comparison. The thermal resistance of the flow channel could have been increased by decreasing the cross-sectional area of the channel walls, but the Stycast was machined as thin as was feasible without running the danger of severely warping Parts A and B and distorting the flow channel shape. The finished Stycast channel was deemed to have an acceptably low thermal
resistance. Estimates of the thermal resistance of the other thermal paths, due to the thermocouple wire and various electrical leads, are provided below in Section 2.3.3.

2.3 The Thermocouple

The total temperature difference $\Delta T$ is measured with a gold-iron (Au-Fe) thermocouple. The thermocouple wire is made of nearly spectroscopically pure gold which has been doped with a small concentration of iron impurities, giving the material a high thermoelectric power, or thermopower, $S$. A temperature gradient $\nabla T$ applied to the wire generates an electrochemical field $E = SV T$ in it. Upon integration, this expression becomes $V = S\Delta T$, where $V$ is the voltage generated between the ends of the wire when a temperature difference $\Delta T$ is applied across it. When the thermocouple wire is connected in a closed circuit, this potential drives a current $I$ to flow through the thermocouple circuit.

The current produced in the thermocouple circuit is detected by an rf SQUID (Superconducting QUantum Interference Device) [SHE] used as an ammeter, so that the change in voltage $\Delta V$ output by the SQUID is linearly proportional to $\Delta T$. This technique has recently gained favor over standard resistance thermometry [Armb81, Mae83, Gri87] for situations requiring a very sensitive differential measurement of temperature. With this SQUID/thermocouple system, a minimum temperature difference $\Delta T$ of 1 $\mu$K can be resolved from background fluctuations, and voltage signals corresponding to temperature differences larger than 1 mK measured to within $\pm 0.1\%$ accuracy.
2.3.1 Thermocouple Assembly

The basic design and construction technique for building the thermocouple was adopted from [Gri86], with some minor modifications. The thermocouple is made of annealed Au-Fe wire supplied by Sigmund Cohn Corporation [COHN], which is 0.07 at. % Fe, 5 mils in diameter and coated with Teflon insulation. Throughout the ensuing description of the thermocouple assembly, the reader is referred to the schematic of the finished thermocouple in figure 2.10.

The thermocouple was made long enough to be connected between the two thermometer blocks (see figure 2.1). The finished length of the thermocouple had to be 5.625", so the Au-Fe wire was initially cut to 7.125", allowing 0.75" extra on each end to work with while attaching the leads. The Teflon coating was stripped from the extra 0.5" on either end by scraping gently with a clean razor blade to score it along one side, then peeling off the loosened coating.

Each end of the thermocouple is connected to the SQUID input by electrical leads. As reported in both [Lor85] and [Gri86], these electrical leads are Niobium-Titanium (Nb-Ti) superconducting wire throughout, except for the small portions going through the superconducting feedthrough at the top of the vacuum jacket, which are copper wire. Being a superconductor, Nb-Ti has zero thermopower at helium temperatures, but the material is being used here primarily for its low thermal conductivity, not for its other superconducting properties. (The total thermopower of the thermocouple circuit in view of the overall operation and sensitivity of the thermocouple is discussed below in Section 2.3.3). Niobium-Titanium has a much lower thermal conductivity
Figure 2.10: Thermocouple assembly, showing all layers (not drawn to scale).
than other metals such as copper; however, the Nb-Ti wire is clad with copper. To reduce the thermal conductivity along the thermocouple leads, the copper cladding was stripped from a small segment of each lead nearest the thermocouple, leaving bare Nb-Ti wire. As will be discussed in more detail later, these leads should have a very low thermal conductivity because at best they make imperfect thermal contact at the helium reservoir, therefore they could potentially create a significant heat leak from the 4.2 K helium bath into the point in the flow channel where the thermocouple is attached. The thermal conductivity of all thermal paths between the heated chamber and the reservoir or bath is discussed in more detail in Section 2.3.3.

The superconducting wire used consists of an inner core, 2 mil in diameter, of Nb-Ti, which is clad with copper to 3.2 mil and coated with Formvar insulation to 4.2 mil. Two equal lengths were cut, each about 7.125" long. The Formvar was first removed with Strip X [GC], and the wire rinsed with methanol. The copper cladding was then stripped from a 4.125" region in the center of the wire, using a 45% solution of nitric acid (HNO$_3$) applied with a cotton swab. If the copper cladding is not removed, the wire will not have the high thermal resistance desired, but leaving the copper on the ends permits easy soldering. The copper cladding was left on the last inch of wire on one end, and two inches on the other. The wires were dipped in a baking soda (sodium bicarbonate) solution to neutralize the acid, then rinsed in methanol.

Both ends of the Au-Fe wire, and the end with one inch of copper remaining on each Nb-Ti lead, were then tinned with indium solder using the following procedure:
a tiny piece of indium solder is cut and placed on a clean glass slide, then the wire to be coated is placed over the indium piece, a small drop of solder flux added, and the wire is heated from above and pressed into the indium with a small solder iron. Once the indium begins to melt, the wire is drawn through the molten indium. Very little solder flux is needed, and a clean coating of indium is applied in preparation for the later step when the Nb-Ti leads are attached to the Au-Fe wire. The other end of each Nb-Ti lead, with two inches of copper cladding, was tinned with Pb-Sn solder. For this and all subsequent soldering steps with Pb-Sn solder, solid solder (not flux-core solder) was used. When flux was needed, both Pb-Sn and indium solders were applied with Stay-clean [sc] solder flux. Afterwards, the flux was always neutralized with a baking soda solution and the part cleaned by rinsing with methanol.

The Au-Fe wire and Nb-Ti leads must be magnetically shielded. To form the shielding, copper-nickel (Cu-Ni) capillary tubing was tinned with Pb-Sn solder, which becomes a superconductor at helium temperatures. The Teflon coating on the Au-Fe wire electrically insulates it from the bare Nb-Ti leads, but these leads must in turn be insulated from the Cu-Ni shielding tubes which will house them. Teflon tubing supplied by Alpha Wire [ALPHA] provided this second insulating layer. The thermocouple was built in the shape of a T, with the Au-Fe wire running along the top of the T (see figure 2.10). Following an innovation reported by Yamaguchi et al. [Yam87], the nested tubes housing the thermocouple wire and leads were filled with Apiezon N grease [APIEZ] to prevent the wires from vibrating, since motion of the wires in any magnetic field present induces noise. The nested structure of Teflon
and Cu-Ni tubing will be described first, then the actual sequence of steps followed to achieve this arrangement of grease-filled tubes and wires is explained.

Along the top of the T, two equal lengths of #30 gauge Teflon tubing are threaded over the Au-Fe wire, each about half the length of the Teflon-coated portion of the Au-Fe wire. The two Nb-Ti leads run inside these Teflon tubes, from either end of the Au-Fe wire to the center juncture of the T. The remainder of the Nb-Ti leads are then routed along the stem of the T as follows. Sections of #19 and #30 gauge Teflon tubing are nested coaxially to insulate the Nb-Ti leads from each other and the surrounding shielding tube. One lead is threaded through the center of the #19 gauge tube, and the second lead is threaded through the space between the two coaxial tubes. For the two shielding tubes along the top of the T, a long enough section of Cu-Ni tubing (0.031\" o.d.) was first tinned with Pb-Sn solder, then cut to the appropriate lengths. For the stem of the T, a larger diameter shielding tube had to be used to accommodate the larger #19 gauge Teflon layer. Since large enough Cu-Ni tubing was not available, a piece of stainless steel tubing (0.062\" o.d.) was used instead. Tinning the stainless steel with Pb-Sn solder was a little more difficult, and required that the soldering to be done at a higher temperature than was used for the Cu-Ni tubing. Each successively larger tube in the nested structure was cut to be slightly shorter than the tube immediately inside it, to ensure that the exposed Nb-Ti wires could not touch either each other or the shielding tubes.

To get the grease inside the tubes, a small quantity of Apiezon N grease was melted in an aluminum dish over a hot plate, making sure the grease did not get too
hot and burn. The same grease could be repeatedly remelted and used. A vacuum system was improvised to suck the grease into the tubes. A small stopper, with a tiny hole through the center, was inserted in the end of a piece of tygon tubing. The other end of the tygon tubing was connected to a rotary pump (poor glop-glop again). One end of the Teflon tube to be filled with grease was inserted into the hole in the stopper and sealed in place with Apiezon Q grease. This admittedly poor vacuum system provided enough suction to draw the grease inside the Teflon tubing. The grease solidified quickly once it entered the tubing, so to get the grease to fill any appreciable length of tubing, the tubing was immersed in the molten grease over most of its length. An alternative might be to gently heat the tubing with a heat gun as the grease was being drawn in.

Two methods of getting the wires inside the grease-filled tubes were tried. In the first attempt, the Teflon tubing was filled with grease first, then the wires were pushed through the grease-filled tubes. This method only worked for the shorter lengths of nested #19 and #30 tubing housing the Nb-Ti leads. For the longer sections of #30 gauge tubing, the Au-Fe and Nb-Ti wires could not both be pushed all the way through the same section of grease-laden tubing as required. Instead, the wires had to be threaded through the tubes first, then the tubes filled with grease. The Au-Fe wire and one Nb-Ti wire were threaded through a Teflon tube until only the 1" ends, tinned with indium, emerged from the tube. The tube was then filled with grease, and the second tube and Nb-Ti lead likewise installed on the other end of the Au-Fe wire. This whole operation proved slightly tricky, because the wires had to remain
properly positioned inside their respective tubes while the grease was being drawn in with the vacuum apparatus; however, an easier method of getting the grease and wires inside the tubes was not found for these longer tubes.

For the stem of the T housing the two Nb-Ti leads, a piece of #30 gauge Teflon tubing was first filled with grease and inserted through a shorter segment of #19 gauge tubing. Grease was then drawn into the annular gap between the two tubes. The nested tubes were then trimmed to the appropriate length by cutting through both tubes at the appropriate point to achieve the desired final length of the outer tube (just over 1" long), then trimming only the inner tube (which protruded from the outer tube) at the other end, to a length about 1 mm longer than the outer tube. The rest of the Nb-Ti leads were threaded through these nested grease-filled tubes as described above and shown in figure 2.10.

Next, each shielding tube was cut to a slightly shorter length than the largest diameter Teflon tube it would encase. The Teflon tubes fit snugly enough inside the shielding tubes that no additional grease was needed. As small a gap as possible should exist where the three tube segments meet at the juncture of the final T-shaped structure. At this point, the thermocouple is ready for making the electrical connections which join the Au-Fe wire to the Nb-Ti leads.

Armbrüster and Kirk report making this electrical junction by soldering the Au-Fe wire to a Nb$_3$Sn foil substrate with indium solder [Armb81, Armb82], then spot-welding the Nb-Ti lead to another location on the substrate where the tin has been scraped off to expose the niobium. Indium solder is used instead of Pb-Sn because
gold migrates freely in lead and the gold wire will therefore eventually dissolve in lead solder. Maeno, Haucke, and Wheatley [Mae83] wrapped pure niobium wire around the Au-Fe wire and heliarc welded the two together under an argon atmosphere to form a junction with low electrical and thermal resistance, then soldered the junction with indium solder into a well drilled in a copper block, ostensibly to ensure good thermal contact by increasing the surface area at the junctions.

Some experimentation with resistive spot welding was tried, using spare Au-Fe and Nb-Ti wires, but this method proved too difficult. Usually it burned through the tiny wires. In those few cases when the wires were joined, they most often fell apart again when placed under slight mechanical stress. Instead, it was decided to form the junction by soldering with indium. The 1" lengths of Au-Fe and Nb-Ti wire emerging from each end of the thermocouple were first twisted together, then soldered with indium solder. The joined wires were trimmed to about 0.25".

To increase the surface area of the junction between the wires enough to make good thermal contact, the junction was sandwiched inside a folded piece of copper foil. Five mil copper foil was first rolled down to a thickness of 3 mils, then cleaned of oxide with fine steel wool and thoroughly rinsed with acetone followed by methanol. Grease residues were likewise cleaned from the indium-tinned wires with acetone first, then methanol. A small rectangle of foil was scored down the middle with a razor blade, and creased, folding the score mark inward. This piece was placed inside one of the copper tabs on the flow channel, crease-edge inward, and the location of the two bolt holes in the tab marked on the copper foil piece. Two slightly larger holes were
then punched in the copper foil piece, and the piece trimmed down to the dimensions of the tab. The final surface area of the copper foil piece was about 0.56 cm$^2$ on one side.

The copper foil piece was placed over the thermocouple junction with the wires running between the two holes and oriented perpendicular to the crease. The foil piece was soldered shut with indium solder, heating just long enough to ensure that the solder had flowed throughout the interior region and filled any voids. The total electrical resistance of the finished thermocouple was measured (between the ends of the Nb-Ti leads) to check continuity. Before assembly, the two bare sections of the Nb-Ti wires had measured 32.91 $\Omega$ and 33.21 $\Omega$ respectively, and the Au-Fe wire 0.392 $\Omega$, at room temperature. A four-terminal measurement of the finished assembly yielded 66.68 $\Omega$, in good agreement with the individual segments summed. This room temperature measurement does not assess the low-temperature performance of the soldered junctions between the Au-Fe and Nb-Ti wires. For the thermocouple to achieve maximum sensitivities, these junctions should have much lower electrical and thermal resistance at helium temperatures than the Au-Fe wire itself.

The juncture of the T-shaped thermocouple is held rigidly and shielded magnetically with lead foil. Unlike the thermocouple built for previous experiments [Gri86], no Stycast was applied at this juncture to create mechanical rigidity because it was found to be unnecessary and prone to cracking. The exposed Nb-Ti wires were insulated from the lead foil with Mylar. A square of 3 mil Mylar was folded diagonally over the juncture, forming a triangle, and glued in place with GE varnish. Before
and after applying the Mylar, the placement of the various tubes and the bare Nb-Ti wires at the juncture was checked, both visually and electrically, to ensure that the wires did not cross or touch each other or the Pb-Sn coated shielding tubes. A slightly larger square of lead foil was cut and likewise folded diagonally and placed over the Mylar sheet. The seams of the lead foil were soldered shut to form a continuous lead to lead-tin shield everywhere around the thermocouple. Only the very ends of the thermocouple wire, inside the copper foil pieces, were not shielded. A final electrical check verified that the thermocouple was not shorted to the shielding, nor were the leads shorted to one another.

For the first data series, the thermocouple was connected between the two copper thermometer blocks to measure the temperature difference across the entire channel. The copper foil pieces at the ends of the thermocouples had to be securely clamped to the thermometer blocks to ensure good thermal contact. Specially designed ring clamps (not shown in figure 2.1) were made for this purpose. These ring clamps worked like hose clamps. For each clamp, a 3/8" wide strip of copper, slightly shorter in length than the circumference of the thermometer block, was bent into a ring, and specially machined brass fittings were mounted on either end. A bolt ran through a hole in one fitting to a mating threaded hole in the other, allowing the ring to be cinched tight around the thermometer block. The upper thermometer block (reservoir end) is electrically connected to the rest of the metal body of the entire probe assembly, therefore the thermocouple must be electrically insulated from this thermometer block. This electrical insulation was achieved by attaching pieces of 3 mil Mylar sheet
with a thin coat of GE varnish to the portions of the thermometer block and copper ring clamp surfaces where the copper foil piece was held. No Mylar was used to electrically insulate the other end of the thermocouple where it was clamped to the lower thermometer block (heated end) because that thermometer block was already electrically isolated from the rest of the probe by the Stycast flow channel. Very thin layers of Apiezon N grease were spread on both surfaces meeting the copper foil piece, to ensure that good thermal contact was achieved over the entire surface area of the copper foil piece. The ring clamps were tightened over the copper foil pieces such that the wires themselves were not crimped by the clamps at the point where they emerged from the copper foil. Overtightening the ring clamps could also pinch the Au-Fe thermocouple wire enough to sever it, even through the copper foil, thereby reducing the surface area for thermal contact between the wire and the thermometer block, and possibly causing the wire to subsequently fall off altogether. The ring clamps were therefore tightened "finger-tight." The rings were made of copper, the same material as the thermometer blocks, to preclude loosening of the clamps due to differential thermal contraction.

For subsequent data series, one or both ends of the thermocouple were unclamped and moved to the desired intermediate position along the channel. Both halves of the copper tab (see figure 2.2) were greased with Apiezon N grease and the copper foil piece was clamped into the tab. This repositioning required that the tinned Cu-Ni tubing be bent appropriately to reroute the thermocouple to the new location. Mechanically, the thermocouple was fairly robust everywhere except at the point
where the Au-Fe and Nb-Ti wires emerged from the copper foil piece and entered the Teflon tubing. The copper foil piece weighted the ends of the wires, and they were subject to shearing off at this point. Care had to be taken as the screws were tightened not to torque the copper foil piece or move the tab, which could result in breaking the thermocouple wire. An improved design, which would have entirely eliminated the need to bend or move the thermocouple once installed, is discussed below in Section 2.8.3.

Two small Stycast blocks (not shown in figure 2.1) were glued to the outer broad side of the flow channel, adjacent to the heater chambers, to hold the thermocouple. A small groove was cut into each of these blocks to hold the Cu-Ni tubing containing the thermocouple wire. After the flow channel was installed below the reservoir, the body of the thermocouple was positioned in the grooves and secured with thread and GE varnish, and the copper foil pieces on the ends of the thermocouple clamped into position, initially between the upper and lower thermometer blocks. Once the thermocouple was in position, the electrical connection to the SQUID was completed by soldering the Nb-Ti leads on the thermocouple to the existing Nb-Ti leads on the probe, at the point where the thermocouple used in previous experiments was removed.

The two leads on the probe side are individually enclosed in Pb-Sn solder tubes (resin core solder from which the resin center has been removed to create a hollow tube) to shield them, and run as a twisted pair up to the “superconducting” feed-through (as the feedthrough for the SQUID/thermocouple leads was named). The
room-temperature resistance of these leads measures 2.2 Ω and 2.3 Ω, respectively, and they are each approximately 40 cm long. From the measured resistance of the ~4" segments of bare Nb-Ti wires on the thermocouple, about 33 Ω each, the resistance of the Nb-Ti core in these leads is estimated to be 128 Ω for a 40 cm length. Knowing the resistivity of copper at room temperature is $\rho = 1.72 \ \mu\Omega\text{-cm}$, the resistance of the copper cladding is estimated to be 2.18 Ω. The total room-temperature resistance of each copper-clad Nb-Ti lead is therefore estimated to be 2.14 Ω, in good agreement with the measured values.

The copper-clad portions of the leads on the thermocouple side were left a generous 2" long to facilitate joining them to the probe leads. Once joined, any extra wire emerging from the solder connection was trimmed. The two solder connections were insulated from each other and their surroundings with Mylar, then the entire joint was wrapped in lead foil to shield it magnetically. The seam in this foil shielding, and all gaps between it and the Pb-Sn coated tubing on one side and the two Pb-Sn solder tubes on the other, were sealed shut with solder.

The entire thermocouple circuit is thereby shielded as completely and continuously as possible. The shielding starts at the working end of the SQUID probe, which is encased in a niobium sleeve. Within the vacuum jacket, and outside in the 4.2 K bath, the Nb-Ti leads are shielded with Pb-Sn solder tubing. The electrical connection passes through the vacuum jacket at the "superconducting" feedthrough. The copper wires running through this feedthrough are individually shielded by Cu-Ni tubing
tinned with Pb-Sn solder (see [Lor85]). Breaks in the shielding therefore exist only above and below this feedthrough, and as the leads enter the SQUID probe.

Consider first the wires in the bath which connect the SQUID to the top of the feedthrough. At the SQUID end, the Pb-Sn solder tube enters the small opening in the endcap on the niobium sleeve, and a small wooden dowel forces the solder tube firmly against the niobium, making electrically good contact between the two materials. The solder tubes extend about 2 cm into the niobium shield to ensure continuous shielding. Outside the niobium shield, the solder tubes were encased in fiberglass spaghetti sheathing to insulate them electrically from the rest of the probe. At the feedthrough end, there is no way to establish any direct contact between the solder tube shielding and the tinned Cu-Ni shielding tube inside the feedthrough, because the latter is entirely covered by the Stycast 2850 sealing the feedthrough. After electrically insulating the soldered connections between the copper wires emerging from the feedthrough and the Nb-Ti leads outside, the entire top of the feedthrough, that part inside the bath space, is wrapped with lead foil to further shield it as much as possible.

The remaining section of Nb-Ti leads inside the vacuum jacket, establishing connection between the feedthrough and the thermocouple, are also encased in Pb-Sn solder tubes, but no attempt is made to connect this shielding to the tinned Cu-Ni tubing emerging at the bottom of the feedthrough, for one important reason explained below.
2.3.2 Shielding the Thermocouple Circuit

The "superconducting" feedthrough is so named only because it routes the Nb-Ti superconducting leads between the thermocouple and the SQUID, although the short segments of wire within the feedthrough itself are copper. The tinned Cu-Ni capillary tubing in this feedthrough is intended to shield the wires from external magnetic fields, which it does adequately, but this tubing is soft soldered to the brass cup at the top of the feedthrough. The brass cup was first hard soldered to a stainless steel standoff tube, which was in turn soft soldered into the flange to which the vacuum jacket attaches. The Cu-Ni capillary tubes shielding the SQUID/thermocouple leads within this feedthrough are therefore connected directly to all the metal of the entire probe assembly: the vacuum jacket, stainless steel support rods, and radiation baffles. The entire probe acts as one huge antenna electrically connected to the SQUID shielding. This arrangement constituted a serious design flaw in the probe apparatus inherited for use in this experiment, not for lack of ability to shield out magnetic fields, but for the undesired exposure of this shielding to rf noise. To quote from the operating manual for the SQUID: "...the input circuitry should be floating (to eliminate ground loop problems) and completely shielded against rf interference" [SHE]. For this probe design, the input circuitry could be disconnected from all ground references as required, but was hardly well shielded from rf interference. Having the shielding grounded to the probe was presumed to be the major source of the considerable, yet highly variable, rf noise problem which hampered data collection throughout the experiment. Nothing could be done to rectify the situation short of constructing
a new experimental probe. This substantial modification was implemented for the next stage in the ongoing investigations, after the experiment reported herein was concluded.

How this noise problem was addressed for the present work is discussed further in Section 2.7.2. Some systematic tests determined that the best SQUID operation would be achieved by securely grounding the remaining sections of the thermocouple shielding (those not already well grounded) to the metal of the probe, since one section was already so grounded. Inside the vacuum jacket, the Pb-Sn solder tubing was grounded to a copper support rod on the helium reservoir (not shown in figure 2.1) with conductive silver epoxy. In the bath, a tinned copper wire was soldered at one end to the lead foil covering the superconducting feedthrough and wrapped at the other end around one of stainless steel pumping lines, where it was secured with silver epoxy.

2.3.3 Thermocouple Operation and Design Considerations

If the electrical resistance and thermopower of the thermocouple wire at helium temperatures are known, the smallest temperature difference detectable by the thermocouple can be estimated by considering the minimum current $I$ observable above the Johnson noise threshold.
Electrical Resistivity

At room temperature, the resistance of the 5.625" length of Au-Fe wire measured 0.392 Ω. The corresponding room temperature resistivity is therefore $\rho = 3.48 \, \mu\Omega\cdot cm$, as compared to $\rho = 2.59 \, \mu\Omega\cdot cm$ measured for the 0.03 at.% Au-Fe wire used by Maeno et al. [Mae83]. Assuming the ratio of resistivities for these two wires remains the same at all temperatures, the low-temperature resistivity of the 0.07 at.% Fe wire is estimated to be 0.409 $\mu\Omega\cdot cm$, given that $\rho = 0.305 \, \mu\Omega\cdot cm$ for the 0.03 at.% wire at $T = 3.94$ K. Likewise, the low temperature resistance of the thermocouple wire is expected to be about 0.046 Ω.

Thermopower

The thermopower of the 0.07 at.% wire was estimated from a variety of previously reported measurements. The thermopower measured for 0.03 at.% wire at 1 K ranged from $S = -8.4 \, \mu V/K$ [Mae83] to close to $-10 \, \mu V/K$ [Ber64, Cha82b]. The data of Berman et al. [Ber64] for wires of various iron concentrations show that at low temperatures the thermopower decreases as the Fe concentration increases. Below 3 K, the thermopower for a 0.06 at.% wire is about 75% that of a 0.03 at.% wire. At 1 K, the thermopowers measured for 0.03 at.% and 0.07 at.% wires were about $-10 \, \mu V/K$ and $-7 \, \mu V/K$, respectively. Below 5 K, the measured thermopower increases monotonically with temperature, becoming less negative as $T$ increases. From these various data, the thermopower for our wire must lie somewhere in the range from $-5 \, \mu V/K$ to $-7 \, \mu V/K$ for temperatures between 1–2 K.
Thermal noise

The Johnson noise current generated in a wire of resistance $R$ is given by

$$\langle I_j^2 \rangle^{1/2} = \sqrt{\frac{4k_B T \Delta f}{R}},$$  \hspace{1cm} (2.2)

where $k_B$ is the Boltzmann constant and $\Delta f$ is the frequency bandwidth. A 1 Hz or 10 Hz low-pass filter is engaged in the SQUID control electronics while the dc current in the thermocouple is measured. Evaluating this expression at 1 K, and using the estimated resistivity of the wire, $\rho = 0.382 \, \mu\Omega\text{-cm}$, the Johnson noise current per unit bandwidth in a length of wire $\ell$ (in cm) is

$$\langle I_j^2 \rangle^{1/2} = 1.35 \times 10^{-10} \ell^{1/2} \, \text{A}/\sqrt{\text{Hz}}.$$  

With the inputs shorted by a low inductance superconductor, the device noise at the SQUID output is reported to be $2 \times 10^{-11} \, \text{A}/\sqrt{\text{Hz}}$, so thermal noise dominates over the device noise. In a 15 cm length of wire, the minimum detectable temperature difference at 1 K, assuming a bandwidth of 10 Hz, is therefore

$$\Delta T_{\text{min}} = \frac{\langle I_j^2 \rangle^{1/2} R}{S} \leq 1 \, \mu\text{K}.$$  \hspace{1cm} (2.3)

This threshold of 1 $\mu$K for detecting a temperature difference is acceptably low.

Thermal Conductivity

Maeno et al. measure a thermal conductivity of 0.102 W/cm K for wire that was 0.03 at.% Fe. Assuming the 0.07 at.% wire has a comparable thermal conductivity, the thermal resistance of our 15 cm long thermocouple wire is $1.2 \times 10^6 \, \text{K}/\text{W}$. This thermal resistance is many orders of magnitude greater than the thermal resistance...
estimated for the helium (see Table 2.1). The thermal resistance is proportional to the length of the wire, whereas the minimum detectable signal $\Delta T \propto l^{1/2}$. A large thermal resistance is desirable, but for maximum sensitivity the minimum detectable $\Delta T$ should be as small as possible. The thermal resistivity is so large that the latter consideration far outweighs the former, therefore the thermocouple wire should be made as short as possible for maximum sensitivity.

The Pb-Sn shielding tubes housing the Nb-Ti thermocouple leads are thermally anchored to the reservoir, held at $T_0$. Being a superconductor at helium temperatures, Pb-Sn solder has a very low thermal conductivity, so the shielding tubes do not thermally short the flow channel. On the other hand, the copper-clad Nb-Ti wires inside these tubes at best make poor thermal contact with the tubes and hence the reservoir, but they continue into the main bath where they are immersed in helium at 4.2 K. The concern with the leads is therefore not of thermally shorting the flow channel to the reservoir, but rather of introducing a heat leak into the flow channel from the 4.2 K bath. The copper cladding is stripped from a portion of each lead to break this thermal path. Even a small segment of bare Nb-Ti superconducting wire is enough to act as thermal "open circuit" here.

It is sufficient to estimate the thermal resistance of the bare Nb-Ti portion of the lead, since the smaller thermal resistance provided by the long copper-clad portion just adds in series to increase the total thermal resistance. An exact value for the thermal conductivity of Nb-Ti wire in zero field could not be found. A call to the manufacturer, who in turn called the National Bureau of Standards, located data
for 60:40 Nb:Ti at 4.2 K, which is reported to have a thermal conductivity \( k = 10^3 \, \text{W/cm K} \) at 4 Tesla, but only \( k = 5.0 \times 10^{-4} \, \text{W/cm K} \) at 2 Tesla. The wire we use is 48:52 Nb:Ti. Using the 2 Tesla value as a fair, albeit crude, guess of the thermal conductivity of our wire in zero field, the thermal resistance of the 10 cm of bare 2.4 mil wire would be \( R = 7 \times 10^8 \, \text{K/W} \). The temperature difference between the thermocouple end of the lead and the 4.2 K bath is at most \( \Delta T = 3 \, \text{K} \), creating a heat leak of \( \dot{q} = \Delta T/R = 4 \, \text{nW} \). This heat enters the helium at whatever point the thermocouple is attached, and is added to the heat current \( \dot{Q} \) generated by the heater. The heat current \( \dot{Q} \) is known and controllable only to the microwatt level. Even if the actual thermal resistance of the Nb-Ti lead is three orders of magnitude smaller than estimated here, the heat leak along this lead would be acceptably small in comparison to \( \dot{Q} \).

One resistance thermometer was installed in the thermometer block at the heated end of the flow channel, and several additional thermometers were installed at various probe locations along the flow channel. A four terminal measurement is made of each of these resistors, so each has four leads which are thermally anchored to the helium reservoir at the other end. The leads for these thermometers are 15 cm lengths of 32 gauge (0.2 mm diameter) phosphor bronze wires attached to 8 mil copper wire approximately 10 cm long. Because phosphor-bronze is an alloy, and due to the small diameter of the leads, the thermal conductivity of these leads is small enough not to thermally short the channel.
Good thermal contact must be made between the Au-Fe wire and the thermometer block. Any thermal resistance at these junctions acts in series with the thermal resistance of the Au-Fe thermoelement, creating the thermal equivalent of a voltage divider. If one of these contacts becomes highly resistive (make poor thermal contact), the temperature difference across the thermocouple wire becomes some fraction of the total temperature difference to be measured, since a portion of the temperature drop now occurs across the resistive contact. Because the electrochemical potential generated in the thermoelement is proportional only to the temperature difference applied across the thermoelement, the current input to the SQUID will be reduced. This thermal voltage divider effect is a linear phenomenon, therefore the current input to the SQUID will still be proportional to the total temperature difference to be measured, but a smaller current will be generated in the thermocouple circuit by a given $\Delta T$, decreasing the sensitivity of the SQUID/thermocouple measurement.

Of course, the thermal contact resistance would have to be poor indeed to become significant compared to the $1.2 \times 10^6 \, \text{K/W}$ resistance estimated for the Au-Fe wire. The Mylar and Apiezon N grease layers were made as thin as possible to keep the thermal resistance of these contacts low. From [Lou74], among other sources of information on low temperature properties of materials, the thermal conductivity at 1 K is $k = 10^{-6} \, \text{W/K cm}$ for Apiezon grease and $k = 3.3 \times 10^{-5} \, \text{W/K cm}$ for GE varnish. For the $0.56 \, \text{cm}^2$ area of the copper foil piece, the thermal resistances of a thickness $d$ cm of these materials is therefore $1.8 \times 10^6 d \, \text{K/W}$ and $5.4 \times 10^4 d \, \text{K/W}$, respectively. The total Kapitza, or thermal boundary layer, resistance of these sandwiches
of material must also be considered. Again from Lounasmaa, the thermal boundary resistivity of a copper-Mylar-copper sandwich held together with Epibond (comparable to our layers held together with GE varnish and Apiezon N grease) obeys the approximate relation $\mathcal{R}_K T^3 = 1.2 \times 10^{-3} \text{K}^4 \text{m}^2/\text{W}$. At 1 K, the Kapitza resistance of each of our junctions is therefore estimated to be about 0.2 K/W. As long as the Ge varnish and Apiezon N grease layers are thicker than a few microns each, the thermal conductivity through the layers will dominate over Kapitza resistance. On the other hand, the thickness of these layers would have to approach 0.1 cm before their thermal resistance would be even 1% as large as that of the thermocouple wire. The actual thickness of these layers is at most a few mils ($\sim 50 \mu\text{m}$).

Finally, the thermal anchoring of the thermocouple shielding to the reservoir raises the concern that a path of low thermal resistance could exist through the cylindrical "sandwich" of Apiezon N grease and Teflon tubing separating the Au-Fe and Nb-Ti wires from the shielding tube. A worst-case estimate of the thermal resistance of this sandwich of materials can be obtained by assuming that the surrounding Cu-Ni tubing is at $T_0$ along its entire length, whereas the Au-Fe wire is at $T_0$ at one end and $T_0 + \Delta T$ at the other end. The thermal conductivity of Apiezon N grease and Teflon are both very low, about $3 \times 10^{-6} \text{W/cm K}$ and $2 \times 10^{-4} \text{W/cm K}$, respectively, at 2 K. Using these values, and integrating over the cylindrical geometry, the effective thermal resistance between the thermocouple wire and the surrounding shielding is crudely estimated to be on the order of $10^3 \text{K/W}$. The actual resistance should be significantly larger, since the thermal conductivity of Teflon and Apiezon grease is
even smaller for \( T_0 < 2 \text{ K} \), and this estimate assumes the Cu-Ni tubing has infinite thermal conductivity.

In summary, careful analysis of all thermal paths in parallel with the helium in the flow channel shows that the helium has a thermal resistance several orders of magnitude lower than that of any other thermal path. None of the other materials spanning the heated chamber and reservoir comes close to thermally shorting the helium. Also, no appreciable amount of heat reaches the helium in the flow channel directly from the surrounding 4.2 K bath, since the Nb-Ti thermocouple leads have a high enough thermal resistance to prevent a heat leak from the bath of any significance.

2.4 Thermometry and Temperature Regulation

Germanium or carbon glass resistors are used as thermometers for regulation, for monitoring the temperature of the helium at various locations, and for calibrating the SQUID output. Two types of resistance thermometers are used: carbon glass (series CGR-1-500 and CGR-1-1000) and germanium (series GR-200A-500), both supplied by LakeShore Cryotronics [LAKE]. At cryogenic temperatures, the resistance of both these materials increases dramatically as the temperature decreases. To ensure optimal thermal contact with the helium, the two resistors installed inside the helium reservoir for temperature regulation, one germanium and one carbon glass, were unencapsulated. The resistors at all other locations were encapsulated. To make good thermal contact, each encapsulated resistor was greased with Apiezon N grease,
then either fit into a well drilled into the copper thermometer block or securely tied to one of the copper tabs emerging from the flow channel.

The germanium thermometers exhibit a very stable and reproducible resistance as a function of temperature and are therefore used to select the desired operating temperature $T_0$. Over a time span of months, the temperature corresponding to a given resistance reading fluctuates by ±0.5 mK under the daily thermal cycling of a few Kelvins. In preparation for a data run, the germanium resistor inside the helium reservoir is monitored as the reservoir is cooled below 4.2 K, until the resistance at $T_0$ is reached. The nominal resistance of the first germanium resistor used in the reservoir ranged from approximately $R = 3.02$ kΩ at $T_0 = 1.7$ K to $R = 5.90$ kΩ at $T_0 = 1.3$ K. The sensitivity, or slope of the resistance curve $R(T)$, was $dR/dT = -4$ Ω/mK and $dR/dT = -12$ Ω/mK at these two temperatures, respectively. This resistance was measured to ±0.1 Ω using a SHE Model 120 Picowatt Resistance Bridge [SHE].

Unlike the germanium resistors, the carbon glass thermometers do not repeatedly return to the same resistance value after experiencing large temperature excursions. For example, the resistance $R$ of the carbon glass resistor in the reservoir fluctuated from day to day by as much as ±10 Ω at 1.7 K, and ±60 Ω at 1.3 K. These daily variations, about 0.06% of $R$, are the result of the mild temperature cycling between $T_0$ and 4.2 K. Between cooldowns, the resistance shifted by much larger amounts, as much as 3%, due to the more extreme thermal cycling between 4.2 K and 300 K. Because the resistance at a given temperature was not highly reproducible from day to
day, the carbon glass resistors could not be used to locate the operating temperature \( T_0 \) with precision.

What the carbon glass thermometers lack in stability they make up for in sensitivity. These thermometers are therefore used to detect the changes in temperature \( \delta T \) within a small range about \( T_0 \), once the operating temperature \( T_0 \) has been located with the germanium thermometer. Two factors determine the overall sensitivity of a resistance thermometer: the sensitivity, or magnitude of the response to a change in temperature, as measured by the slope \( dR/dT \) of the resistance curve \( R(T) \), and the resolution, or accuracy to which the resistance \( R \) can be measured. The nominal resistance and sensitivity of the carbon glass resistor in the reservoir (a CGR-1-500) ranged from about \( R = 19 \, \text{k\Omega} \), \( dR/dT = -56 \, \text{\Omega/\text{mK}} \) at 1.7 K to \( R = 82 \, \text{k\Omega} \), \( dR/dT = -370 \, \text{\Omega/\text{mK}} \) at 1.3 K. The resistance of this particular thermometer could be measured to the nearest ohm, yielding a maximum sensitivity of \( \pm 3 \mu\text{K} \).

This carbon glass thermometer was used to regulate the reservoir once the operating temperature \( T_0 \) had been reached, as ascertained by the germanium thermometer. The carbon glass regulation thermometer was placed in an ac resistance bridge circuit and the bridge initially nulled by tuning a variable resistor box to balance the resistance \( R_0 \) of the carbon glass resistor at \( T_0 \). As the reservoir warmed or cooled slightly from \( T_0 \), the resistance of the carbon glass regulation thermometer would shift by an amount \( \delta R \), throwing the bridge circuit out of balance. This off-balance signal was detected by a lock-in amplifier. The output of the lock-in (proportional to \( \delta R \)) was input to a negative feedback circuit which drove a heater that warmed or cooled the
reservoir as needed to bring it back to $T_0$. The regulation circuit and its operation is outlined in Section 2.11 of [Gri86] and discussed fully Sections 3.2 and 3.3 of [Lor85] and therefore will not be described in further detail here.

During a data run, the helium reservoir is regulated to within $\pm 20 \mu K$ of the desired temperature $T_0$. These regulation fluctuations are larger than the smallest temperature differences $\Delta T(r)$ to be measured; however, because the regulation fluctuations cause the temperature at every point in the channel to change simultaneously by the same amount, a measurement of the temperature difference between two positions in the channel is impervious to this overall variation in the absolute temperature. Regulation fluctuations therefore necessitate making a differential rather than absolute measurement of temperature within the channel.

Although the regulation control electronics and the overall probe apparatus are exactly the same as were used previously by Griswold and Lorenson, the regulation achieved in this experiment is not quite as good as the $\pm 15 \mu K$ they achieved, for several reasons. First, the flow channel assembly is considerably larger; their flow channel was only 1 cm long and 130 $\mu m$ in diameter and their flow channel assembly held only 0.057 cm$^3$ of helium. The much larger volume of helium in the present flow channel assembly changes the time constants involved in regulation. Moreover, the maximum heat current used in this experiment was 10 mW, whereas these previous experiments never exceeded 200 $\mu W$. For heat currents above 1 mW, a much higher pumping rate must be used to cool the reservoir sufficiently than what is needed to handle heat currents in the microwatt range. For a data run spanning the full 10 mW
range in heat current, the pumping rate and corresponding dc component of the heat
delivered to the reservoir by the regulation heater would have to be readjusted several
times during the run. As the heat current $Q$ through the flow channel was increased,
the dc setting on the regulation heater would have to be decreased and/or the rate
of evaporative cooling increased to compensate. Such a comprehensive data run
typically lasted about 6-8 hours. At low temperatures, the maximum heat current
achievable was limited more by the ability of the pumping system to continue cooling
the reservoir to maintain regulation as $Q$ was further increased than by considerations
of creating too large a heat flux or of generating too large a temperature difference.
The regulation system in fact performed remarkably well, considering that it was
originally designed for more modest performance. Even over the longest and most
demanding data runs, the reservoir temperature did not drift by more than 0.3 mK,
as indicated by the germanium thermometer.

Encapsulated carbon glass resistors were installed on the copper tabs at various
positions, as well as in the two thermometer blocks. Because these resistors measured
the absolute temperature of the helium at their respective positions, they could not
detect small temperature signals above the regulation fluctuations. These resistors
were insufficiently sensitive to measure changes in temperature on the order of 20 $\mu$K
anyway, and became useful only once the temperature difference $\Delta T(r)$ to be mea-
sured was greater than about 1 mK. The most important of the carbon glass resistors
was the one installed in the lower thermometer block, a series CGR-1-1000. Not
only was this resistor used to calibrate the SQUID output, but since it measured the
temperature at the heated end of the channel, $T(t_a)$, it provided a crucial check on the operation of the experiment. By verifying that the same $\Delta T$ was always generated in the channel for a given heat current, this thermometer demonstrated that the same flow was always present under the same conditions for all cooldowns. Because the thermocouple measured a different portion of the channel for each cooldown, the SQUID/thermocouple could not be used to make this determination. Only the carbon glass resistor measuring $T(t_a)$ could test whether the channel was blocked by ice, or reveal any other changes in the turbulent flow from one cooldown to the next.

The nominal resistance and sensitivity of this thermometer ranged from $R = 78.2 \, \text{k}\Omega$, $dR/dT = -0.29 \, \text{k}\Omega/\text{mK}$ at 1.7 K to $R = 517 \, \text{k}\Omega$, $dR/dT = -3 \, \text{k}\Omega/\text{mK}$ at 1.3 K. The resolution to which the resistance could be measured also increased from $\pm 10 \, \Omega$ at 1.7 K to $\pm 1 \, \text{k}\Omega$ at 1.3 K. Although the nominal resistance was very large, and the slope $dR/dT$ correspondingly steep, the sensitivity of the temperature measurement made with this resistor was severely limited by the resolution to which the resistance could be determined with the Picowatt bridge. At the high end of this temperature range, $R$ could be read to four digit accuracy, and a change in $T(t_a)$ as small as $\pm 30 \, \mu\text{K}$ resolved. At 1.3 K, the low three-digit accuracy in reading $R$ meant that the smallest resolvable change in temperature was $\pm 300 \, \mu\text{K}$. At the intermediate temperatures of 1.4, 1.5, and 1.6 K, the resolution of this thermometer was determined to be $\pm 100 \, \mu\text{K}$. This resolution range of $\pm 30$–$300 \, \mu\text{K}$ constitutes the size of the error bars on measuring $T(t_a)$ with this carbon glass resistor. This uncertainty is comparable to the $\pm 20 \, \mu\text{K}$ regulation fluctuations. Both exceed the
±1 μK uncertainty in the differential SQUID/thermocouple measurement by more than an order of magnitude.

The CGR-1-1000 resistors have a larger response $dR/dT$ than the CGR-1-500, so one would think they would be more sensitive, however for some reason the resistance of the CGR-1-1000 cannot be resolved with nearly as much accuracy at low temperatures as that of the CGR-1-500. The last number in the series designation gives the nominal resistance at 4.2 K. The suggested useful temperature range for the #500 resistor is 1.0–77 K, whereas for the #1000 it is 1.4–100 K. It would have been better to use only #500 resistors, since the #1000 are not meant to extend to the lowest temperatures used in this experiment. Operationally, the resistance of the #1000 was so very large at 1.3 K that the Picowatt bridge could not measure it with much sensitivity, although it was not altogether clear whether this behavior represented more of a limit in the Picowatt bridge or in the nature of the resistor itself. The use of a #1000 instead of a #500 resistor to measure $T(r_A)$ was therefore somewhat unfortunate, although at the time of installation, the resistor in question had been misidentified as a #500. Other resistors later used at positions B, C, D, E, and F were all type CGR-1-1000, simply because all the ones remaining in the supply we already owned were that type.

In addition to the resistors already mentioned, encapsulated carbon glass and germanium resistors were also installed in the upper thermometer block. These latter two provided an important redundancy for the unencapsulated resistors inside the reservoir. In fact, at some point the germanium resistor inside the reservoir failed,
after which the germanium resistor in the upper thermometer block was used instead to select $T_0$. A last carbon glass resistor glued to the outside of the reservoir built in even more redundancy, probably more than was needed.

The final data series was not very successful as all the resistors used to measure the temperature at various locations in the channel (five carbon glass resistors in all, at $r_A$, $r_B$, $r_E$, $r_F$, and the one inside the reservoir, used to regulate) failed, all more or less at the same time. The culprit responsible for this mass annihilation of resistors turned out to be a modification intended to improve matters. Because resistors had been installed at a larger number of positions than in all previous cooldowns, a special breakout box had been constructed to allow each resistor in turn to be connected to the picowatt resistance bridge simply by turning a dial to select the desired resistor, rather than physically disconnecting and reconnecting the respective cables to the back of the bridge. Prior to building the breakout box, the procedure to switch between resistors had been to turn down the bridge excitation fully to remove power from the resistor before detaching the cable and switching to the new resistor. It was believed that the same procedure was being followed with the new breakout box: the bridge excitation was turned down to zero before the switch was thrown to select the new resistor. Somehow these two operations were not identical, however, perhaps because the electrical connection was broken and reinstated suddenly as the old resistor was switched out and the new one switched in. As a result, the resistors experienced sharp transient currents which eventually destroyed the delicate junctures between the resistive material and its leads. As the resistor started to fail, a variable contact
resistance would first develop in one or more of these junctures, causing the resistor to self-heat and the resistance to drift unpredictably. Eventually the resistor would become an open circuit. Fortunately, enough data were gathered at one temperature before the resistors at positions $r_B$, $r_E$ and $r_F$ failed. Since these positions were not otherwise observed in any other data series or with the SQUID/thermocouple, these measurements, scanty though they were, proved invaluable in building a more complete picture of the temperature profile in the channel. This last data series is further discussed in Section 3.2.6 of Chapter III.

2.5 Temperature Calibrations

Before data can be taken at each operating temperature $T_0$, all the resistors to be used must first be calibrated against the 1958 $^4$He temperature scale. The same basic calibration procedure used in previous experiments [Gri86, Lor85] was followed here. During the calibration, the helium vapor pressure is measured with an MKS Baratron pressure transducer [MKS] system consisting of a Type 270 controller and the transducer unit. Two different transducers were used: a Model 370H-10, which makes a differential measurement of pressure, and a Model 390HA-100, which measures absolute pressure. The vapor pressure readings are then converted to temperature using the 1958 $^4$He temperature scale.

The probe used in this experiment had unfortunately been designed with a single pumping line connected to the reservoir. Because this same line could not be used simultaneously to pump on the reservoir and to monitor the helium vapor pressure
in the reservoir, the entire bath had to be cooled during a calibration. The needle valve was opened to bring the bath and reservoir in thermal contact while the bath was pumped on by a Stokes pump, thereby also cooling the reservoir through the needle valve. The reservoir pressure was monitored through the reservoir pumping line. During a calibration, the same reservoir heater and feedback circuit normally used to regulate the reservoir alone were used to regulate the reservoir and bath together. The thermal contact between the bath and reservoir did not have to be perfect, just good enough to allow this feedback circuit to hold the reservoir and bath at a steady temperature for a short time, long enough to read the pressure at that temperature and the resistance of the five or six thermometers being calibrated. The temperature of the bath and reservoir did not even have to coincide, since none of the thermometers being calibrated was located in the bath space. In fact, the bath was probably much colder than the reservoir throughout this calibration procedure, which did not matter as long as the reservoir temperature could be held steady long enough to take the necessary calibration readings.

Having to pump the entire bath for calibrations was both wasteful of helium and made calibrations difficult, since the regulation feedback circuit was really only designed to regulate the small volume of the reservoir. The new experimental probe built for future investigations was designed with two pumping lines on the reservoir so one could be used to measure the vapor pressure while the reservoir was pumped through the other one, therefore only the reservoir will need to be cooled during a calibration.
For a calibration at each $T_0$, resistance readings are taken at a relatively large number of temperatures, at least 8–10 in all, spanning a range extending about $\pm 20$ mK around $T_0$. Typically, a calibration started at the high end of this temperature range and readings for all thermometers being calibrated were taken as the temperature was first successively decreased, then increased, to cover the full temperature range and check for hysteresis (none was ever observed).

Once the set of resistance readings $R(T)$ has been obtained for a particular thermometer, the usual procedure is to find the best linear fit, the slope $dR/dT$ serving as a simple multiplicative constant to convert the measured change in resistance into the corresponding change in temperature. This linear interpolation works as long as the thermometer will operate over a small enough range in temperature around $T_0$ that the nonlinearity in $R(T)$ can be ignored. Because of the very large temperature differences generated across the channel in this experiment, a nonlinear fit to $R(T)$ had to be used here instead. The curvature in $R(T)$ is not very pronounced when viewed only within the 20 mK range around each $T_0$ spanned by the calibration data, but temperature differences as large as 60 mK were generated across the flow channel at the highest heat currents used. A typical set of calibration data shown in figure 2.11 clearly illustrates the need to use a nonlinear fit. For $\Delta T = 50$ mK, the temperature at the heated end of the channel reaches the halfway point between two reservoir temperatures. To enhance the overall accuracy of the nonlinear fit, the calibration data sets for all the reservoir temperatures were combined and the best-fit polynomial was found for the entire temperature range from just under 1.3 K
to almost 1.8 K. Sometimes a single third- or fourth-order polynomial would fit the entire temperature range reasonably well. In other cases, the best-fit polynomial (up to fifth order) for the full temperature range did not accurately fit the data at one end of that range. The calibration data for the end temperature, $T_0 = 1.3$ K or 1.7 K as the case might be, along with the adjacent temperature, 1.4 or 1.6 K, respectively, were then separately fit to a second polynomial. In converting resistance readings to temperature, this second polynomial fit was used for the data taken at the end temperature, whereas the original polynomial fit was used for data gathered at all other reservoir temperatures.

Figure 2.11 displays the calibration data for the carbon glass thermometer in the lower thermometer block, measuring $T(\tau_A)$. Note that the calibration data are plotted "inverted" as $T$ versus $R$. By finding the best polynomial fit to $T(R)$ rather than to $R(T)$, the coefficients of this inverted fit can then be easily used to convert a measured change in resistance directly into the corresponding change in temperature. The entire set of calibration data was fit to a fifth-order polynomial in this case (solid line in figure 2.11), but even so the fit does not represent the data well below 1.4 K. As discussed above, a separate polynomial fit, fourth-order this time, was fit to the 1.3 K and 1.4 K data (dashed line in figure 2.11), and used for the data gathered at 1.3 K. New calibrations had to be performed for each cooldown.

As described in Section 2.4 above, the resistance of the carbon glass resistors drifted around slightly with the daily temperature cycling of the reservoir. This drift in effect shifts the curve $T(R)$ up or down a bit in $R$ each day, but does not
Figure 2.11: Example of a typical calibration, in this case for the carbon glass resistance thermometer in the lower thermometer block, which measures $T(t_A)$. Solid line: a fifth-order polynomial fit to the entire set of calibration data is used to interpolate resistor readings at $T_0 = 1.4-1.7$ K. Dashed line: a fourth-order polynomial fit to the calibration data at 1.3 K and 1.4 K improves the interpolation of thermometer data for $T_0 = 1.3$ K.
change the shape of the curve. The data gathered on any given day must first be normalized by adding or subtracting this shift in resistance before the polynomial fit can be used to convert the resistance readings to temperature. The original "baseline resistance" $R_0$ is defined as that resistance which yields the temperature $T_0$ when inserted into the $i$th-order polynomial fit to the calibration data: $T(R) = \sum_0^i a_n R^n$.

For any given data run, the resistor will have new baseline resistance $R_0'$ on that particular day, given by the resistance reading at $Q = 0$ when the entire channel is at $T_0$. This daily baseline resistance is measured at the beginning and end of the data run, and if there is any change, the average of the two readings is used for $R_0'$. During a data run, the resistance $R$ is measured as the heat current is increased. To convert these readings of $R$ into temperature readings $T$, the decrease in resistance below the daily baseline $\Delta R = R - R_0'$ is first computed ($\Delta R$ is negative since $R$ decreases as $T$ increases). The original baseline resistance $R_0$ is then added to $\Delta R$ to obtain the adjusted resistance readings $R_{adj} = R_0 + \Delta R$. Note that this procedure simply normalizes the raw resistance readings $R$ for a given data run from the daily baseline of $R_0'$ to the original baseline $R_0$, since $\Delta R = R_{adj} - R_0 = R - R_0'$. The adjusted resistance readings $R_{adj}$ are then converted to temperature readings using the appropriate polynomial fit to find $T(R_{adj})$ for the particular resistor and cooldown in question. The polynomial fit renders the absolute temperature in K, from which the temperature difference can be obtained by subtracting $T_0$. 
2.6 SQUID Operation

The SQUID electronics consist of the SQUID probe which attaches to a Model 330 rf head that is connected in turn by a cable to a Model 30 control box. The working end of the SQUID probe contains the superconducting current loop with a Josephson junction weak link and therefore must remain immersed in the 4.2 K helium bath to function properly. The thermocouple provides the input to the SQUID circuit. The voltage output by the SQUID control box changes by an amount $\Delta V$ in response to a temperature difference $\Delta T$ imposed across the thermocouple. This voltage response is linear, so converting the raw voltage signal to the measured temperature difference requires that the linear conversion factor $\Delta T / \Delta V$ first be determined. Further details of the SQUID operation are outlined in Section 2.6.2 after the method used to calibrate the SQUID output is described below.

2.6.1 Calibrating the SQUID Output

The resistors located at the two positions spanned by the thermocouple provide a direct measurement of $\Delta T$ corresponding to the voltage $\Delta V$ output by the SQUID. The SQUID output is thus calibrated by simultaneously measuring $\Delta V$ and $\Delta T$ for various values of $\dot{Q}$ in the course of one or more data runs. When the thermocouple spans the entire channel only one resistor, the one measuring $T(\tau_A)$, is needed to calibrate the SQUID output since the temperature at the other end is held fixed at $T_0$. Only values of $\dot{Q}$ which produce large temperature differences are useful for
this calibration, since the resistors cannot resolve small values of $\Delta T$ with sufficient accuracy.

Several approaches to finding the best average value of the voltage conversion factor were tried. Smoothing the curves of $\Delta T$ versus $\dot{Q}$ and $\Delta V$ versus $\dot{Q}$ by fitting each to a polynomial, then dividing the respective polynomials, produced too much numerical noise in the form of wiggles in the resulting ratio. In the end, each individual measurement of $\Delta T$ was divided by the corresponding $\Delta V$ and the resulting ratio plotted as function of $\dot{Q}$. Readings at lower heat currents exhibited considerable scatter, which gradually diminished at successively higher $\dot{Q}$. The best average value and uncertainty for the voltage conversion factor was determined from the high $\dot{Q}$ portion of the graph for each $T_0$. Each data series at a given reservoir temperature, for which the SQUID/thermocouple was positioned over a different range in $r$, was treated separately. Only the first two series, in which the thermocouple spanned large portions of the channel, were very useful in determining this ratio. The third series confirmed the conversion factors deduced for the first two, but did not enhance the accuracy of this determination.

**Absolute Error in $\Delta T$**

The voltage conversion factor decreases monotonically with temperature from $0.59 \pm 0.02$ mK/V at 1.3 K to $0.51 \pm 0.01$ mK/V at 1.7 K. At the intermediate temperatures 1.4, 1.5, and 1.6 K, this ratio was found to be $0.58 \pm 0.02$, $0.57 \pm 0.02$, and $0.54 \pm 0.02$ mK/V, respectively. The uncertainty has been estimated generously to
encompass slight discrepancies measured among the various cooldowns and the somewhat large scatter found about the average value. The uncertainty in this voltage conversion factor, which constitutes an overall systematic error of 2–3.5% depending on the reservoir temperature, provides by far the largest single source of error in $\Delta T$. This systematic error shall be referred to as the absolute error in the SQUID/thermocouple measurements of $\Delta T$.

One may question whether finding slightly different voltage conversion factors at a given temperature for different cooldowns is a real effect, revealing shifts of the SQUID operation due to temperature cycling between cooldowns, or merely the accumulation of various random errors due to calibrating the thermometers and any other systematic factors involved in measuring $\Delta T$ with the resistance thermometers. The SQUID transfer function $F$, the change in output voltage $\Delta V$ in response to a known input current $I$, was measured after two cooldowns and found to remain constant, so any scatter in conversion factors over the various data series is likely attributable to random error, not to systematic changes in the SQUID performance between cooldowns. The gradual decrease in measurement sensitivity (i.e., increase in the ratio $\Delta T/\Delta V$) as the temperature is lowered most likely is due to a decrease in the thermopower of the Au-Fe wire, since the thermopower is a rather strong function of temperature. It does not represent a temperature dependence in the SQUID circuit: the SQUID probe remains at 4.2 K; only the thermocouple input circuit is cooled to $T_0$. 
Measuring the SQUID transfer function amounts to measuring the ratio of thermopower to resistance of the thermocouple wire. The input current satisfies the relation \( IR = S\Delta T \) (see the discussion of thermopower in section 2.3.3). The transfer function is therefore given by \( F = \frac{\Delta V}{I} = \frac{\Delta V}{\Delta T}(R/S) \). On the most sensitive setting \((\times 100)\) the SQUID transfer function was measured to be \( F = (1.87 \pm 0.01) \times 10^7 \text{ V/A} \), in good agreement with the nominal value of \( 2 \times 10^7 \text{ V/A} \) quoted in the SQUID manual \([SHE]\). This transfer function was likewise measured to be \((1.86 \pm 0.01) \times 10^6 \text{ V/A} \) on the \( \times 10 \) setting, and \((1.88 \pm 0.01) \times 10^5 \text{ V/A} \) on the \( \times 1 \) setting. Having measured the voltage conversion ratio \( \Delta T/\Delta V \) (using the \( \times 100 \) setting for the SQUID), the ratio of electrical resistance to thermopower of the thermocouple is computed to be \( \frac{R}{S} = F(\Delta T/\Delta V) = (1.07 \pm 0.02) \times 10^4 \text{ K/A} \) at 1.5 K, in good agreement with the ratio of 9200 K/A obtained from the estimated values \( R = 0.046\Omega \) and \( S = -5 \times 10^{-6} \text{ V/K} \) (see section 2.3.3). This verification of the SQUID operation and the measured voltage conversion factor together provide an important check on the expected properties of the thermocouple.

### 2.6.2 Measurement Accuracy

**Relative Error in \( \Delta T \)**

In comparing the various data sets for a given reservoir temperature, the relative measurement uncertainty is much smaller than the absolute error, being limited by the SQUID sensitivity in measuring \( \Delta V \). Due to rf interference and thermal noise, the
SQUID voltage signal fluctuates by ±1–2 mV about a steady value (corresponding to underlying noise fluctuations in the thermocouple current of less than ±0.1 nA). Our SQUID/thermocouple system can therefore measure a small $\Delta T$ signal to a relative accuracy of ±1 $\mu$K.

A different criterion determines the uncertainty in the SQUID measurement for larger temperature differences ($\Delta T > 1$ mK). The SQUID voltage signal $\Delta V$ rides on top of a base voltage level. This baseline shifts up or down at random intervals by multiples of a characteristic step voltage ($2.124 \pm 0.001$ V and $2.119 \pm 0.001$ V, respectively, for the two different SQUID control boxes used). These discrete jumps occur because the SQUID temporarily "unlocks" and resets whenever it has exceeded its maximum output voltage of 10 V, and also when the level of ambient rf noise becomes too great. The measurement $\Delta V$ must be extracted from the actual voltage reading by adding or subtracting the appropriate number of voltage steps, producing an uncertainty in $\Delta V$ of ±0.04%. The actual relative uncertainty for a large $\Delta T$ signal is taken to be somewhat larger, ±0.1%, to account for a slight, slow drift in the baseline voltage which sometimes occurs, another subtle effect of either rf noise or imprecise tuning of the SQUID control circuit. To illustrate, a measurement of $\Delta T = 50$ mK at $T_0 = 1.5$ K corresponds to a total $\Delta V$ of about 88 V, or approximately 41 multiples of 2.124 V, and is uncertain to within ±50 $\mu$K. The effects of rf noise mentioned here and other more serious difficulties with rf interference experienced by the SQUID circuit are discussed further in Section 2.7 below.
An offset control allows the initial baseline voltage to be set to zero at \( \hat{Q} = 0 \), slightly simplifying the computation of \( \Delta V \) from the output voltage reading \( V_{\text{out}} \). All one has to keep track of thereafter is how many multiples of the step voltage to add or subtract from \( V_{\text{out}} \) at any given moment to get the actual measurement of \( \Delta V = V_{\hat{Q}} - 0 \). In other words, the actual reading at \( \hat{Q} \) is \( V_{\hat{Q}} \), but the output of the SQUID could read \( V_{\text{out}} = V_{\hat{Q}} \pm n(2.124) \), where \( n \) is any integer which does not make the output exceed \( \pm 10 \) V. In taking data, one must keep track of which one of the entire set of possible readings \( \{V_{\text{out}}\} \) is the actual datum \( V_{\hat{Q}} \). At any time, rf noise may cause the output voltage to jump, or reset, from one possible reading \( V_{\text{out}} \) to another. This jumping behavior is unpredictable, and the number of voltage steps which must to be added to \( V_{\text{out}} \) to extract \( V_{\hat{Q}} \) changes with each jump. The number of multiples of 2.124 V to be added also changes whenever the output exceeds \( \pm 10 \) V and resets to a value close to zero. A reset button likewise allows the operator to reset the voltage, returning \( V_{\text{out}} \) to whatever value in the set of possible values is closest to zero. If the output voltage is jumping frequently due to noise, it is often advantageous to use the reset button to bring the \( V_{\text{out}} \) back near zero.

Any form of automated data collection was entirely impractical, for at least two reasons. Due to the jumping and resetting behavior described above, the number of voltage steps to be added to \( V_{\text{out}} \) to extract \( \Delta V \) fluctuated constantly throughout the data run. It was far easier to take readings and convert them to \( \Delta V \) by hand than to devise an automated system which could do so reliably. Also, the length of time needed to obtain a reading depended on how frequently these resets were occurring.
At the noisiest limit, still within some tolerable level for data collection, such resets would occur every few seconds, whereas on quieter days there might be at most one or two resets due to noise during a data run lasting several hours. A very sophisticated time-averaging scheme would have been needed, one that could detect the difference between small fluctuations and jumps or resets in $V_{\text{out}}$, sampling and averaging over the former yet not over the latter. These are but the most prevalent reasons data acquisition was best done manually.

Two limits determined the maximum feasible heat current at each $T_0$. To ensure that the heat flux remained well below the value sufficient to initiate film boiling at the heater surface due to Kapitza resistance, data collection was restricted to heat currents below $Q = 10$ mW. Moreover, measurements were not extended beyond heat currents for which the temperature of the heated chamber exceeded that of the reservoir by more than about 50 mK, for three reasons. First, converting the resistance readings of the thermometers into temperature becomes increasingly uncertain as the temperature range widens. The calibrations for various temperatures $T_0$ were performed at significantly different times within a cooldown, so thermal drifts in the carbon glass resistors become a factor. The conversion of $\Delta R$ into $\Delta T$ is not expected to be as accurate for $\Delta T$ exceeding the limited 20 mK range about $T_0$ spanned by the calibration data taken at $T_0$. If the temperature difference generated becomes very large, a measurable thermal drift can occur in the resistors even within a single data run. Second, extracting $\Delta V$ from the voltage reading output by the SQUID becomes more painstaking due to the unlocking discussed above. Finally, any comparison
of the data to theory becomes much more complicated if the temperature from one location in the fluid to another varies enough that the thermodynamic parameters describing the helium can no longer be treated as having constant values throughout the flow. Within the temperature range studied, an increase in temperature of 10–20 mK is sufficient to alter the predicted behavior by about 10%. Beyond $\Delta T \sim 20$ mK, the discrepancy between incorporating the temperature dependence of parameters versus not doing so increases dramatically. Figure 3.8 in Chapter III illustrates this difference. Extending our measurements beyond $\Delta T = 50$ mK seemed pointless in light of the ensuing computational difficulty in performing an accurate theoretical calculation. Since no interesting new features were discovered in the data at the highest heat currents used, we were not compelled to relax these two self-imposed and somewhat arbitrary limits on the range of heat currents studied.

2.7 rf Noise

Ideally, one would like to perform this experiment in a screened room which completely filters out all ambient rf noise and has active filters to block any rf noise from entering along power lines. Lacking such a luxury, we conducted this experiment in an ordinary room and continually battled the many sources and effects of rf interference. The cryostat and the SQUID input circuitry were shielded as much as possible. The problems with rf noise generally fell in one of three categories: incorrect or inadequate grounds on the shielding forming antennas to pick up rf noise, ground loops, and operating the experiment in a “dirty” rf environment. Considerable effort went
into diagnosing and remediying, to the extent possible, the first two types of problems, but nothing could be done to change the fact that working in an unshielded environment was far from ideal for this measurement technique. The discussion in Sections 2.7.2 and 2.7.3 outlines these diagnostic efforts and the resolution to various noise problems. Finally, Section 2.7.4 describes the characteristics of the various types of noisy behavior, correlating their respective signatures and causes where possible.

Shielding helped to mollify the impact of rf noise, but only when a proper grounding configuration for all elements of the experiment was achieved. This discussion of rf noise therefore begins with a description of the shielding configuration and ground connections.

2.7.1 Ground Configuration of Shielding

The cryostat was shielded with 1/16" brass mesh screening wrapped around the outside of the nitrogen dewar. All seams in this screening were soldered shut and at the top the screening was clamped around large cylindrical metal housing from which the cryostat was suspended, in an attempt to build as leakproof a Faraday cage as possible. Still, holes had to be created in this shielding for the inlet and outlet tubes for the nitrogen bath and the helium dewar wall pumping line, compromising the shielding. This screening was at best only marginally effective at cutting down on rf noise inside the cryostat.

To facilitate the ensuing discussion of ground references and connections, figure 2.12 schematically illustrates the shielding on all the parts of the experiment and
how each piece of shielding is or can be electrically grounded. For the electronic components such as the oscilloscope, digital multimeter, SQUID control box, or rf head, the shielding consists of the metal case or chassis of the equipment. A niobium case creates a superconducting shield around the SQUID probe. The SQUID input circuit, consisting of the thermocouple and its leads, is shielded in three disconnected segments. The shielding on the input circuit has already been described in detail in Section 2.3.1. The three boxes in figure 2.12 represent the lead foil and Pb-Sn tubing on the thermocouple and leads inside the vacuum can, in the superconducting feedthrough, and in the 4.2 K bath. The connections drawn in figure 2.12 show how each piece of shielding is, or can be, referenced to ground. Only the connections to and among the shielding elements are shown; the routing of the various electrical circuits inside this shielding is not depicted in this figure. The switches S1–S6 are not actual switches, but rather points where the ground connection can be made or broken at will. Two different sources of Earth ground are used: the ground terminal on the power line, and a water plumbing line. The dashed line offsets the parts of the experiment maintained at cryogenic temperatures, and corresponds roughly to the wire mesh shielding around the cryostat, not otherwise shown.

The chassis of most of the electronic devices receive a ground reference through the ground terminal on the power line. Most of these devices are bolted into the same equipment rack, which further references them all to the same power line ground, and grounds any chassis that otherwise floats. The grounding of the SQUID control box warrants special consideration, and is discussed further below.
Figure 2.12: Schematic depiction of the ground connections to all shielding.
In terms of its ground configuration, the experimental probe is divisible into two basic functional elements. The first of these is the metal structure of the probe, consisting of the vacuum can, reservoir, pumping lines, and radiation baffles. Switch S6 represents a multistranded stainless steel braid, bolted to the metal flange atop the probe and securely clamped at the other end to a water pipe, making a grounding strap that could be easily connected and disconnected to ground the probe or leave it floating, respectively. The coaxial shielding on all the electrical leads constitutes the second major element of the probe to be grounded. The leads for all thermometers and heaters consists of shielded microcoax cable inside the cryostat and ordinary shielded coax cable outside the cryostat. The coax shields on all leads inside the cryostat are soldered together and grounded to the metal of the probe. Outside the cryostat, all the coax shielding is again electrically tied together and referenced to the ground pin on each amphenol connector. The room temperature portion of the coax shielding therefore becomes grounded to the experimental rack when at least one of these cables is connected to its respective electronic device. Outside the cryostat, the coax shields on all the leads—except for those on the SQUID circuit itself—therefore share a common ground with the electronic devices.

The SQUID circuit is isolated from the rest of the electronics. A continuous metal shield around the SQUID electronics is formed by the chassis of the SQUID control box, the cable connecting it to the rf head, the rf head, and the metal casing of the SQUID probe. This shield extends to part of the SQUID input circuit: the portion of the thermocouple leads in the 4.2 K bath. As mentioned earlier, it would have
been desirable to enclose the entire input circuit in this shielding, but that was not possible because the portion of the shielding running through the superconducting feedthrough was grounded (soldered) to the rest of the metal probe.

Grounding the chassis of the SQUID control box automatically grounds the shielding on the entire SQUID circuit, up to the superconducting feedthrough, so the grounding configuration of the SQUID control box is very important. In all, three SQUID control boxes were used interchangeably. Although all three were supposedly identical Model 30 control boxes, the chassis of one was not internally referenced to the power line ground, whereas for the other two, it was. For the latter two SQUID control boxes, switch S1 being open represents using a 3-to-2 prong plug converter on the power plug to float the chassis of the SQUID control box. In the case of the first control box, S1 was always open by default. All three SQUID control boxes could be referenced to ground by physically bolting them into the experimental rack, as represented by connection S2. If S2 was to remain open, care had to be taken that the chassis of the SQUID control box did not touch any of the other electronics. Switch S3 represents plugging the SYNC output of the control box into the time base of the oscilloscope, a step necessary when tuning the SQUID circuit. The time base input was a grounded bnc jack, and the oscilloscope chassis was grounded to the rack, so establishing this connection grounded the shielding on the SQUID circuit. According to the SQUID manual, the voltage output terminal on the SQUID control box is a floating bnc jack, and should only be connected to a measuring instrument with a differential or floating input. One of the digital multimeters used to read this output
may not have met this criterion, since it was observed that connecting the SQUID output to the multimeter in question sometimes seemed to cause noise problems. The exact grounding configuration of this connection is unclear, but switch S4 is drawn to show that an undesirable ground connection may have thereby been made, although whether or not this connection actually established contact with the SQUID control box chassis was never determined.

The experimental configuration, as described to this point, was unaltered from the previous experiments conducted with the same apparatus. These experiments also used the SQUID/thermocouple measurement system, although they did not rely on the SQUID as the sole means of data collection as in the present work. Many of the rf noise problems addressed in this experiment were therefore built into the existing setup. In the course of this investigation, many improvements to the grounding configuration of the shielding and various electronic configurations were implemented. Each of these improvements came at the end of a long, frustrating process of first being stymied by trouble with rf noise, then systematically searching for the source of the problem, and finally making the appropriate changes. The SQUID circuit exhibits complex and variable behavior in reaction to rf noise, and the rf environment in the room is constantly shifting from day to day, making the process of identifying cause and effect of the noisy behavior painstaking indeed.

Not all the problems were diagnosed and cured at once; in fact, many of the best data runs were obtained in quiescent lulls amidst periods of severe difficulties with noise. The nature of rf noise is such that the severity of the noisy behavior
varied extremely from one day to the next. This observation has to do with the fact that the SQUID circuit operates as a tuned, phase-locked electrical circuit. If the rf interference becomes too great, the SQUID circuit simply unlocks. This behavior is also by its nature discrete: a small change in the ambient rf noise level or type can make the difference between the SQUID functioning well enough to take data and not working at all. Noise so severe as to preclude data collection one day would not be manifest the next. In addition to making active attempts to cure noise problems, many days were spent simply monitoring the system while it was noisy, waiting for the noise to die down so a data run could be started. More often than not, data runs were begun after 4:00 pm and continued into the wee hours of the night. Typically, although far from reliably, the level of rf noise was lower in the middle of the night, rising again around 5:00 am, and lower on weekends than weekdays. Patience and persistence were as important as actively remedying noise problems to the ultimate success of this experiment.

2.7.2 Inadvertent rf Noise Antennae

The worst noise problem faced in the experimental design was having the shielding inside the superconducting feedthrough soldered to the rest of the metal probe. The entire probe acted as an rf antenna attached to this portion of the shielding on the SQUID input circuit, which ideally should be floating. Initially, nothing special was done either to securely ground or float the other two portions of the shielding on the SQUID input circuit. As a result, the problems with noise were initially very
severe. In trying to diagnose this noisy behavior, it was discovered quite by accident that systematic mechanical vibrations significantly reduced the noise on the SQUID. One evening was spent turning a household fan placed on the frame supporting the cryostat on and off. With the fan running, the SQUID worked; without it, noise overwhelmed the SQUID. This behavior suggested that something was loose inside the cryostat and making intermittent contact to ground. The experiment was warmed up to find and fix the problem.

Systematic tests were conducted with a mock-up of the SQUID input circuit being immersed in liquid helium inside a storage dewar to identify the exact source of the problem. To simulate the effect of having an rf antenna connected to the shielding, a wire was attached to the shielding on an old (broken) thermocouple input to the SQUID probe. These dip tests confirmed that by far the best (least noisy) behavior was achieved when no antenna was attached and the shielding on the entire input circuit was grounded to the SQUID shielding. With the antenna attached, the best possible configuration was to have the input circuit and SQUID shielding securely grounded to each other, but floating from Earth ground. Floating the SQUID shielding while securely tying the input circuit shielding to Earth ground via the antenna did not degrade the SQUID performance much, but increasing the resistance of this path to ground by a few ohms made the noisy behavior much worse. Since a touch contact between the shielding and the metal of the probe created exactly this kind of resistive path to ground, better grounds on the input circuit shielding were established at this point, achieving the grounding configuration on the input circuit.
described in Section 2.3.2 and depicted in figure 2.12. The improved grounding of the
input circuit shielding apparently remedied this particular noise problem, for no sens-
sitivity to mechanical vibrations was ever again detected, and fewer noise problems
altogether were encountered thereafter.

As shown in figure 2.12, the metal of the probe could either be referenced to
ground via the water pipe or left floating. Choosing one configuration over the other
(connection S6 open or closed) did not usually make much difference, because either
way—grounded or floating—the entire probe assembly acted as a huge antenna, in-
troducing rf noise to the shielding on the SQUID input circuit. This antenna could be
eliminated only by building a new probe with a different feedthrough design, a step
not taken during the present work. The remedy effected was simply to clarify the
ground configuration of this shielding, thus reducing the magnitude and variability
of this rf antenna effect.

2.7.3 Ground Loops

By far the most severe, but also the most controllable, noise problems resulted from
the creation of ground loops. The first axiom for success when using a SQUID,
especially in a poorly shielded environment, is that ground loops must be hunted
down and eliminated. The corollary to this rule is that when you think you have
found them all but are still having problems with noise, look again!

Some ground loops proved to be relatively easy to track down; others were either
more subtle, or simply went unnoticed for a long time. The first ground loop was found
when it was observed that establishing connection S3 (see figure 2.12) caused noisy behavior. The shielding on the SQUID circuit was receiving two ground references, one through the power plug and the other through the oscilloscope. With both S1 and S3 closed, a ground loop was created.

One major ground loop discovered is not shown in figure 2.12. Initially, only one power strip of three on the electronics rack was powered through the ultraisolation transformer. The other two were plugged directly into the power line, bypassing the ultraisolation transformer. This transformer supposedly blocked all significant rf noise on the power line which otherwise would have been carried inside the cryostat along the various leads. At first, only those electronic devices connected to leads running inside the cryostat received power through the ultraisolation transformer. Other devices, for example the oscilloscope and a chart recorder used only occasionally, received ordinary unfiltered power. Having two paths from the electronics rack to the power line ground created a huge ground loop. Since this ground loop had been present from the day this experimental apparatus was first set up, it was a wonder that the previous experiments which Griswold and Lorenson conducted with this same setup were not more hampered by noise problems than they were. This ground loop was not entirely eradicated but was made significantly smaller in size (and therefore in effect) by plugging the other two power strips in series into a second outlet available on the ultraisolation transformer, thereby routing all power through the transformer.

The most subtle ground loop lurking in the experiment proved to be the second connection between the metal probe and the water pipe, represented by switch S5.
At some point a new pumping station for the MKS Baratron was designed and built to replace the original, which had developed leaks, and allow both the new Model 390HA-100 Baratron pressure sensor as well as the old Model 370H-10 to be used. On the old station, the copper pumping lines on the cryostat and the station had been connected via a short segment of rubber hosing, whereas on the new station a brass connector coupled the two pipes. With this connection (S5) closed, the probe metal was linked by a continuous metal path to the water cooling line for the diffusion pump on the Baratron station. Once the effects of this ground loop became manifest, literally every other connection to ground had to be removed to track down this source, although it seemed quite obvious once it had been found.

2.7.4 Symptoms of rf Noise

Tuning Behavior

Severe interference from rf noise prevents proper tuning of the rf circuit in the SQUID. As part of the tuning procedure, a characteristic triangle waveform pattern is produced. The appearance of this triangle pattern provided a very good indication of the ambient rf noise level present. Being able to tune the SQUID well enough to produce a clear triangle pattern with an amplitude of 60–80 mV p-p indicated that no appreciable rf interference was impairing its performance. The SQUID manual says the triangle pattern will be reduced in amplitude and less distinct if the SQUID picks up an interfering signal outside the bandwidth of the rf amplifier. Such a reduction in the
amplitude of the triangle pattern was frequently observed. Even though the circuit was carefully tuned, the triangles produced were often only 30-50 mV p-p. As long as the noise was not so severe as to destroy this triangle pattern altogether, however, the SQUID circuit usually locked in and worked reasonably well. This degradation of the triangle pattern was typical evidence that a bad ground loop was introducing broad-band rf noise into the SQUID circuit.

Sometimes the triangle pattern would be evident, but could not be properly triggered or synchronized on the oscilloscope. This effect was never clearly understood, although it was obviously evidence of rf noise. The SQUID would usually operate normally despite this behavior, as long as the unsynchronized triangle pattern was still large enough in amplitude.

Another type of rf interference discussed in the SQUID manual is that of brief noise spikes visible in the triangle pattern, caused by interference spikes being picked up by the high Q tuned circuit (housed in the rf head) coupled to the SQUID. These short bursts of noise can cause the SQUID to unlock and reset to a new output voltage which differed from the prior output voltage by an integer multiple of the discrete voltage step characteristic to the SQUID output. This unlocking and resetting behavior has already been described in Section 2.6.2. As long as such intermittent jumps occurred only occasionally, they did not prevent data from being gathered. The triangle pattern is observed from the detector output jack (DET) with the SQUID control box operating in the SET1 mode. With either the SET2 or operational mode engaged instead, the triangle pattern disappears, but this type of rf noise spike would typically still show
up on the oscilloscope trace of the DET output jack as large bursts of intermittent noise disrupting the otherwise small-amplitude trace. With the SQUID in operating mode, these noise bursts were clearly correlated with the unlocking and jumps in the output voltage. This type of behavior was induced by the antenna effect described in Section 2.7.2 above, as well as by ground loops.

Another clear indication during the tuning procedure that rf noise would be too severe that day to allow data acquisition was that, with the RF LEVEL control set to produce a maximum output, the maximum meter deflection found by varying the TUNE control remained below about 25% of full-scale deflection, even with the GAIN control turned all the way up. The RF LEVEL adjusts the amplitude of the 19 MHz carrier signal applied to the tuned circuit, the TUNE control tunes the probe to resonance by adjusting a variable capacitor in this circuit, and the GAIN adjusts the open-loop gain of the system. If the maximum meter deflection remains too low even though these three controls are optimized, it means the rf noise is effectively blocking all resonant behavior of the tuned circuit. In the SET2 mode, a 50 kHz square-wave modulation is turned on. The MOD control is adjusted until a minimum meter deflection is found to achieve the maximum slew rate (needed to track fast changes in the input signal). On a good day, this minimum would be below 5% of the full scale meter deflection, but on a noisy day the minimum would be about 30–50%, again indicating that the SQUID was not well enough tuned to function.
Operating Mode

In operating mode, the feedback loop between the SQUID control box and the rf head is closed and the output voltage is proportional to the change in flux trapped in the SQUID loop. This loop is inductively coupled to the current input to the SQUID, hence the voltage responds linearly to changes in the thermocouple current. Three slew rates can be selected: slow, medium, and fast. The slew setting determines the maximum rate at which the output voltage can track a change in flux at the input. Typically, the lowest slew rate possible should be used, so during normal operation the SLOW setting was usually chosen. On somewhat noisy days the rf interference would often continuously unlock the SQUID on the SLOW setting, whereas the output would still be perfectly steady on the MED or FAST settings. One peculiar observation was that when the SQUID was completely overwhelmed by noise in both the SLOW and FAST settings, it would nevertheless often operate perfectly well in the MED setting.

For any given slew setting, three sensitivity settings were available: $\times 1$, $\times 10$, and $\times 100$. These settings change the output voltage by factors of ten by correspondingly changing the SQUID transfer function (ratio of output voltage to input current). For data collection, only the maximum sensitivity setting, $\times 100$, was useful.

In operating mode, the most common behavior of the SQUID when beset with rf noise would be for the output voltage to drift continuously in one direction, either increasing or decreasing. In general, the greater the drift rate, the worse the noise problem. This drift appeared quite different on the various sensitivity settings, depending on the severity of the drift. A gradual drift on the $\times 1$ setting would of
course be magnified by a factor of ten on the ×10 setting. Once the output voltage drifted to the maximum voltage of ±10 V, the output would reset to zero. On a chart recorder the steady drift therefore looked like a sawtooth wave, as the voltage climbed steadily in one direction, reset to zero, and started climbing again. On the ×100 setting, however, this drift looked like a random walk comprised of rectilinear steps, rather than a sawtooth wave. The voltage does not, in fact, change smoothly and continuously as it appears to on the ×1 and ×10 settings, but suddenly jumps up or down by the discrete step voltage characteristic of the SQUID, as the flux in the SQUID loop changes by one quantized unit after another under the influence of rf noise. On the ×100 setting, these jumps appear to be random, rather than always progressing in the same direction. They are random to some extent: the SQUID is temporarily unlocking and resetting, usually, but not always, to an adjacent voltage level. On average, the voltage progresses in one direction, either up or down. The voltage limit of ±10 V is exceeded after only a few steps in one direction, whereupon the output resets closer to zero, adding to the random appearance of the jumps on the ×100 setting.

The voltage drift described above was always present to some extent and in principle did not hamper data collection, as long as the drift rate was low enough. Calling this behavior a drift is somewhat misleading. A "discretized" drift might be more accurate, since the baseline voltage increases or decreases progressively through a sequence of discrete voltage steps in one direction. In some sense, no true drift is present at all. Like the progress of a winding number, the baseline voltage is not
really changing, just jumping to one of the other multiples of itself. Data collection is therefore possible as long as the drift rate (semantics aside) is low enough that the output remains constant for a few seconds at a time, so that a reading may be taken. Of course, the less frequent the jumps, the easier data collection becomes.

This discrete drift is caused either by rf interference or by a poorly tuned SQUID circuit; the two conditions cannot be distinguished by the appearance of the output. This type of drift was the typical signature of a ground loop problem. When a bad ground loop was present, the rate and direction of this voltage drift changed throughout the day, but often did not die down to the point where data could be collected. Sudden shifts or alterations in the drifting behavior often seemed to correlate with temporal events or the status of equipment in the building, any of which might produce changes in the rf noise in the room: the ringing of class bells, the proximity of the janitor's floor polisher, the time of day, whether the heating or air conditioning is running, and so forth. Just as often, no immediate causal event was evident. As well as shifting in degree of severity, the noise could just as suddenly die away altogether or appear, seeming to somehow be turned off or on. One was compelled to switch between trying to diagnose the problem whenever the SQUID was too noisy to take data and collecting data whenever the SQUID suddenly resumed quiescent behavior. This volatility in the SQUID behavior is implicit in the nature of ground loops. How well rf noise will be fed into the SQUID electronics by a ground loop will depend on the relative orientation of the loop to the electromagnetic fields comprising the rf signal, as well as the frequency spectrum of the rf noise.
If this discrete drift becomes so rapid that the resets on the \( \times 100 \) setting happen more frequently than the output can track, then the output simply jitters around zero instead of producing the random steplike pattern described above. The SQUID is unlocking so frequently that the output jitters in an analogy to Brownian motion, never having time to adjust after one jump before it is hit with the next. On a chart recorder, the output on the \( \times 100 \) setting will not look as noisy as when the drift rate is slightly lower, but the noise will actually be worse. This inverse relation between the severity of the noise and its appearance or manifestation on the SQUID output further confuses the diagnostic process, as one is easily fooled into thinking the noise has diminished upon effecting some change in the system, whereas in fact it has increased. The converse can also be true. Many times the SQUID output would suddenly change, apparently for the worse, either because the rf level in the environment shifted, or in response to a change in grounding or some other factor controllable by the experimenter. In truth, the change brought about a decrease in the noise, but because the SQUID circuit was so poorly tuned at the time, the improvement actually made the output look worse instead of better. All that was required at that point was to retune the SQUID circuit to restore the SQUID to quiescent behavior. When the SQUID was noisy, it was impossible to achieve accurate tuning, and anyway the optimal tuning depended on whatever specific rf noise was affecting the SQUID at the time. In summary, improvements could masquerade as degradations, and worsened noise as a decrease in noise. The same noisy behavior
could take on extremely different appearances when viewed on different sensitivity scales or slew rates.

As well as setting the slew rate and the sensitivity, one could select from several different low-pass filters on the SQUID output. Typically, a 1 Hz or 0.1 Hz filter was used. Since the steady turbulent state was being studied, fast tracking of transient behavior was not required. Use of these low frequency filters therefore did not introduce any limitations, but did filter out the small, fast random fluctuations of ±1–2 mV on the output, making the output easier to read. When rf noise was causing the SQUID to jump and reset every few seconds, a higher frequency filter, at least 10 Hz, had to be used instead. A 60 Hz notch filter was usually engaged, but whether or not this notch filter was used did not seem to make much difference.

**Anomalous Noisy Behavior**

In addition to the noisy behaviors described above, which were more or less typical, several other more anomalous noisy behaviors were also occasionally observed. These difficulties were usually not reproducible enough to determine the nature of what caused them, but frequent enough to be observed a number of times each.

Sometimes the SQUID output voltage would suddenly shift by a small amount that did not correspond to any integer multiple of the normal jump voltage. These shifts were typically much smaller than the step voltage, ranging from a few millivolts to a few tenths of volts, and never repeated. They sometimes occurred when the cryostat was mechanically disturbed, such as when a helium transfer was performed
or the needle valve was opened to refill the reservoir, and also when the reservoir pumping rate was periodically adjusted to maintain regulation. Changing the rate of cooling the reservoir should not have had any effect on the SQUID probe, which was immersed in the bath; however, turning the valve to adjust the pumping rate apparently caused enough mechanical vibrations to bring about a shift. Even a door slamming could cause such a shift.

This phenomenon could have been as much due to electrostatic effects introducing transient currents in the shielding as to mechanical vibrations. Alternatively, both types of disturbance could have caused these shifts equally. With either the metal of the probe or the SQUID circuit floating, isolated from ground, touch contacts to the probe, the SQUID control box, or even the thermometer cables also produced shifts in the SQUID output, especially in the winter when the air was dry and these touch contacts were accompanied by a small electrostatic shock. So could some of the abrupt changes in electrical circuits within the building mentioned above, such as the classroom bells ringing or a piece of equipment being turned on. It was obvious that such shifts in the SQUID output were not attributable to a newly discovered metastable turbulent state; these events were recognizable as singular electronic or mechanical disturbances which moved the SQUID output to a new operational set point, nothing more. The shifts were less likely to occur when the metal of the probe was grounded than when it was floating.

These anomalous shifts could also occur as a result of random bursts of noise interrupting otherwise quiescent behavior for no apparent reason. These short noise
bursts appeared as extremely violent disruptions in the SQUID output, which would quiet down as suddenly as it had become noisy, but at a new baseline voltage. The only electronic device ever identified which upset the SQUID in this manner was the microwave oven in the room, which when running caused the SQUID output on the \( \times 100 \) sensitivity to jitter as described above when the noise was at its worst. Whatever the cause, these irregular shifts were just subtracted from all voltage readings subsequently taken. All data massaged in this manner agreed perfectly well with data gathered in other runs when no such strange shifts had occurred.

Sometimes a true drift in the baseline voltage would be present. A drift in the baseline voltage (\( \dot{Q} = 0 \) value) of a few millivolts in several hours, over the course of a long data run, did not affect the measurement accuracy enough to matter. Occasionally a much larger drift would be seen. The baseline voltage would continually wander around, never settling down to a steady reading. This behavior occurred whenever the helium level in the bath dropped below the top of the SQUID probe so the probe started to warm slightly, and also if the SQUID circuit was slightly mis-tuned, but the influence of rf noise could also be to blame. For this type of drift, no clear cause and effect relationship was ever worked out when rf noise was the culprit, and all that could be done was to wait to take data until the drifting stopped.

Finally, it should be noted that the performance of the SQUID probe appeared to slowly degrade over time. Despite identifying and correcting many specific problems with the shielding and grounding configuration, on average the best performance of the SQUID got worse over time rather than better. Perhaps the best indicator of
this slow decline was the amplitude of the triangle pattern. At the beginning of these investigations, the best performance of the SQUID yielded a triangle pattern that was 80 mV p-p. A year and three cooldowns later, the best triangle pattern seen was never better than 60 mV p-p. By the end of these investigations, on the best days the triangle pattern was only 40 or 50 mV p-p.

The final data series was curtailed early due to a number of problems which prevented data collection. During this final series, the SQUID/thermocouple system never operated correctly. Once or twice, the SQUID appeared to tune and function normally, but the voltage output ΔV was far smaller than it should have been as indicated by the thermometers. The SQUID may well not have been at fault, but the problem was never diagnosed. After the experiment was warmed up for the final time, some simple bench tests were performed on the SQUID probe to assess its condition. In all previous iterations of these same tests, it had performed within specifications, but this time the SQUID failed to work at all.

One possible explanation is that the SQUID probe worked intermittently. If so, it might have been working intermittently for a long time. Perhaps a dysfunctional SQUID probe, rather than a highly variable rf noise environment, was partly responsible for the extremely erratic behavior of the SQUID. As well as being plausible, for SQUIDs can eventually degrade, this hypothesis contains the comfort of explaining why the many battles waged to gain control over the noise problems felt more like stalemates than victories. It is otherwise somewhat hard to understand why, in the investigations preceding these, rf noise was not more of a hindrance than it was.
There is no way to discern after the fact whether or not the SQUID was actually fail-
ing intermittently, but no correlation was ever found to time of day, day of the week, time of year, or any piece of equipment being operated in the building to explain what change in the rf environment caused the SQUID to change states so suddenly from quiescent and working to noisy and untunable. This sudden switching behavior, on time scales of minutes to hours, seemed most prevalent during the final year of operation.

2.8 Design Improvements

In retrospect, several aspects of the original experimental design and implementation could have been significantly improved. Each of the problems encountered and how it was addressed is described here, as well as the modification or improvement needed to circumvent the problem. This discussion is offered primarily in the interest of assisting any researcher who might wish to replicate any part of the present design, or use it as a template for a designing a similar experimental apparatus.

2.8.1 Heater Location

There was no reason to install the heaters inside the heater chambers at all. A much better design would have been to leave the heaters outside any sealed Stycast regions, and simply deliver the heat to the interior of the Stycast heater chamber. All that is needed inside the heater chamber is a piece of metal with a large enough surface area
in contact with the helium to dissipate the maximum heat current without inducing film boiling. A piece of copper foil would work perfectly well. The foil should be soldered to a thick copper wire which passes through the heater chamber wall, and the heater wrapped around a cylindrical copper support soldered to the wire outside the chamber. If the heater later failed, it could then be easily replaced.

The inability to service the heaters was one of two serious drawbacks to the present design. Being installed inside the Stycast chambers, the heaters could not be replaced in the (relatively unlikely) event that a heater shorted or became an open circuit. In fact, two heaters were installed in each chamber for exactly this reason: to build in redundancy in case one heater failed. In the end, the feedthroughs through which the heater wires exited the heater chambers proved to be a worse Achilles’ heel in the design. These feedthroughs, being built up of Stycast and fiberglass sheathing, were very rigid. Even though Liquid Tape was applied as stress relief where the wire emerged from the end of the feedthrough, this minor protection did not prevent the wires from eventually shearing off at that point during periods of subsequent handling over the experimental lifetime of the flow channel assembly.

Eventually, at least one wire broke off of each of the two heaters at the wide end of the channel, rendering both heaters unusable for the study of converging flow. Cutting through the Stycast feedthrough to expose more of the heater wire and reattach an electrical lead presented a host of difficult and delicate problems, and was not a feasible solution overall. Fortunately, the Cu-Ni capillary tubing running through the heater chamber wall, heretofore unused, provided exactly the elements needed
to effect an easy repair. A new heater was wrapped around the tubing, outside the heater chamber. The tubing had more than enough interior surface area in contact with the helium to dissipate the maximum heat current. The capillary tubing was never meant to serve as a heater surface, but this unintentional redundancy and flexibility in the design proved to be extremely fortuitous in the end. As a last comment, building the original heaters and Stycast rings to hold them in place, and in particular constructing the feedthroughs for the heater wires, can only be described as pesky work. The modified design described here would have been much simpler to implement, as well as more rugged and conducive to easy repairs, but such is usually the nature of hindsight.

2.8.2 Determining the Actual Channel Height

The height of the flow channel was designed to be 8 mils, or 0.020 cm. The two halves of the flow channel, Parts A and B, were machined to very tight tolerances to ensure that the finished channel had a height that was not only constant along the length of the channel, but as close to \( h = 0.020 \text{ cm} \) as possible. Unfortunately, no attempt was made immediately after epoxying the two Stycast pieces together to verify whether or not this crucial dimension was actually what it was supposed to be. It would have been relatively easy to measure \( h \) optically at the two ends of the channel before the heater cells were attached; however, the need to verify this dimension became apparent only much later, after laminar data had been taken.
Before attaching the heater chambers, the height could easily have been measured optically at both ends of the channel, using a microscope with a calibrated length scale. A cruder form of measurement would have been to insert shims of varying thicknesses into the channel from either end, to set upper and lower bounds on $h$ at the channel ends. Other more sophisticated techniques could also have been employed. One possibility would have been to fill the channel with a viscous and noncorrosive fluid such as glycerin, then compute the average height from the difference in weights of the empty and filled channel. The height was small enough that a technique which measures the gap between two closely spaced parallel surfaces by creating an interference pattern of ultrasound waves reflected off the two surfaces could have been attempted. This last method holds the potential advantage of determining, to some degree, the uniformity of the channel height along its length, not just the height at the ends or the average height. Once the flow channel was fully assembled, it was not possible to measure the channel height $h$ directly, either optically or by any other nondestructive means, except at the wide end of the channel.

The experimental evidence that the channel height is indeed larger than expected is found in the laminar data, which is discussed in detail in Section 3.3 of the following chapter. A straightforward calculation predicts the laminar temperature difference $\Delta T_L(r_i, r_j)$ between two radial positions $r_i$ and $r_j$ in this diverging geometry. Treating the channel height as a tunable parameter in this calculation, it was found that a height of $0.0246 \pm 0.001 \text{ cm}$ agreed best with the temperature difference measured across the entire channel, $\Delta T_L(r_A, r_H)$. For the second data series which measured
between positions \( r_A \) and \( r_D \), the laminar data \( \Delta T_L(r_A, r_D) \) suggested an even larger channel height, \( h = 0.0275 \pm 0.001 \text{ cm} \). The laminar data therefore present two mysteries. First, the overall channel height is apparently much larger than expected. The measurements further suggest that the channel height is not constant, but is larger toward the narrow end than at the wide end, perhaps with some anomalous feature in the vicinity of position \( r_D \).

A comparable anomaly in the temperature gradient was observed in a previous experiment by Henberger and Tough, which used a uniform rectangular channel of somewhat similar design, and was explained in terms of an aberration in the height within one section of the channel [Hen82a]. The channel used in that experiment was formed by gluing two polished Stycast slabs together with Mylar spacers to create a channel of height 0.008 cm and width 0.11 cm. Six evenly spaced temperature probes measured the temperature profile along the channel length. The temperature gradient was found to be anomalously small in one section of the channel. After the experiment was concluded, the channel was sectioned transversely, and the height in the vicinity of the anomaly was found to be about 25% greater than in the rest of the channel. A side gap between the epoxy slab and the Mylar spacer also gave the channel an irregular cross section within that same section.

The variation in the channel height in Henberger's experiment can possibly be attributed to polishing the Stycast slabs used to form the channel, which might have created a slight wavy distortion in the surface. At the time, the effects of surface
roughness were unknown and polishing was deemed necessary. Subsequent experiments comparing the use of smooth-walled glass tubes to ones artificially roughened with small spheres have shown that surface roughness has but a minor effect on the turbulent state, changing the measured dissipation by only about 5% [Cou89].

The laminar data in the present experiment certainly suggest a similar anomaly: the height of our channel might change locally, or the cross section be otherwise distorted, in the vicinity of position $r_p$. Given the care with which the Stycast Parts A and B were machined, it was difficult to understand what could have caused an overall increase, let alone a local distortion, in the channel height. The Stycast was not polished after it was machined, precisely to avoid introducing wavelike distortions in the surface as much as possible, and also because the small-scale roughness of the machined surface was deemed unlikely to affect the turbulent state in any significant way. Finding out whether or not an imperfection is present in the channel near $r_p$ will have to wait until the channel can be destructively sectioned. Two tests were performed to answer two basic questions: 1) could the epoxy layer gluing the two Stycast pieces together be thick enough to account for the unexpected 0.005 cm increase in height and 2) what is the actual channel height?

To test the hypothesis that the layer of epoxy used to join the two Stycast slabs could be thick enough to significantly increase $h$, two plain rectangular pieces were glued together following the exact procedure used for the flow channel. First, two identical rectangular pieces were machined, each measuring $10 \text{ cm} \times 1'' \times 0.25''$. When glued together, these two pieces formed a rectangular block with the same outer
dimensions as the flow channel assembly. The large face of one piece was marked off into 16 equal subdivisions in a 2×8 array and, holding the two pieces together firmly (by hand, not using any clamps), the combined thickness of the two pieces was measured with a micrometer. Three measurements in all were taken within each subdivision and the results averaged. The two pieces were then epoxied together, maintaining the same orientation, and the thickness measured again. The entire set of measurements, both before and after epoxying, was carried out independently by two individuals. Each set of 16 measurements before and after gluing was averaged to find the average thickness. These two independent determinations of the average thickness after gluing agreed very well, but the corresponding readings before gluing differed slightly. One found the overall thickness had increased by 0.0003 cm, while the other found the thickness had decreased by 0.0001 cm. This difference in pre-gluing values can probably be attributed to how tightly each individual held the two pieces together. In bonding the two surfaces together, the liquid Stycast likely melts a small layer of the solid Stycast and the total thickness does not change appreciably. In any case, the change in thickness caused by the gluing process is well within the machining tolerances of ±0.001 cm, and an order of magnitude too small to account for an increase in \( h \) from 0.020 cm to 0.025 cm.

The gluing may also have changed the degree to which the two slabs were canted lengthwise relative to one another. Both sets of measurements showed small systematic changes in the average thickness from one end of the slabs to the other. Whether the slabs were closer to being parallel to one another before or after gluing could
not be determined, but both individuals found the combined thickness had increased slightly at one end while decreasing at the other end. The two individual measurement sets differed most with regard to this latter observation, again probably because the two slabs were held together by hand before gluing. While discounting the idea that the glue added thickness between Parts A and B to increase \( h \), this test did reveal that two sides of the flow channel may end up not exactly parallel.

Although the thickness of the epoxy glue layer cannot account for the increase in overall height, the explanation might still lie in the gluing process. Because Stycast is such a soft material, it is impossible to avoid all warping and distortion of the material when it is clamped, either during machining or for gluing. The interior surfaces of Part A and B, milled last, turned out flat, but the outer sides were bowed slightly inward (becoming concave in the transverse direction). For Part A, this lateral curvature was enough that if the curved outer side were placed against a truly flat surface, the gap between the two surfaces would be as large as 1.5–2 mils at the center. It is possible that the pressure exerted in clamping Parts A and B together while gluing was enough to reverse this curvature, making the outer sides flat and parallel but causing the inner surfaces to become laterally bowed, therefore separating the two walls of the flow channel by about 2 mils.

Once all the investigations into diverging flow reported herein had been completed and the experiment warmed up for the last time, an attempt was made to measure \( h \) directly. This measurement was of necessity rather crude, but at least it served to establish upper and lower limits on \( h \) at the wide end of the channel. Because
the experimental apparatus was to be reused in a future study of converging flow in the same channel, this measurement had to be strictly nondestructive. A set of thin and narrow shims was constructed of brass in approximately 1 mil increments of thickness and calibrated. With the flow channel assembly detached from the helium reservoir, successively thicker shims were inserted into the wide end of the flow channel through the stainless steel tube connecting the copper thermometer block to the heater chamber. The Stycast is transparent enough that one can watch the shims and guide their progress into the channel. The 7, 7.8, and 9 mil shims fit inside the channel easily, whereas the 9.9 mil shim could not be inserted beyond the very entrance. From this test, the channel height, at the wide end at least, was determined to be less than 10 mils, or 0.0254 cm. (The height at the narrow end could not be similarly measured since the channel opening at that end is only 0.2 cm wide and not collinear with the stainless steel tube, preventing the shims from being inserted.) The height at $r_H$ is definitely larger than the design specification of 0.020 cm, otherwise the 9 mil shim would not have fit. Whatever the cause of the increase in $h$, the laminar data and this shim test consistently indicate that this crucial channel dimension turned out to be about 0.005 cm larger than planned. The value $h = 0.025 \text{ cm} \pm 0.001 \text{ cm}$ was consequently used for the channel height in all data analysis.

The procedure of using the laminar data to determine the smallest dimension of a channel is a standard technique. The problem of laminar Poiseuille flow in a uniform channel of circular, square or rectangular cross section is solved analytically, and the best fit value of the channel diameter or height is determined by comparison.
to the measured laminar dissipation. For very small channels, an accurate, direct measurement of the smallest dimension is simply not feasible by any other means. The materials most often used to make small channels are obtained from commercial sources, and although the nominal dimensions are specified by the manufacturer, most researchers instead use the laminar data as the most accurate indication of the exact dimensions of the channel. Moreover, only the average dimensions can be deduced by this method, except in those few experiments where the temperature is probed at a number of locations along the channel length, not just at the ends of the channel. Only when the temperature profile is measured, such as in Henberger's experiment, can any information about the degree of uniformity of the channel dimensions be gleaned. The boundary conditions in the diverging geometry make only an approximate analytic solution possible, but otherwise the method of finding the channel height from the laminar data is every bit as valid here as in all previous uniform channel experiments, and the shim test confirms the average height deduced from the data.

This technique of treating the small dimension of the channel as a tunable parameter to be fit by comparing the measured laminar behavior to the solution of the laminar flow problem rests on the assumption that the analytic solution, which treats the normal fluid as an ideal viscous fluid, truly represents the physics of two-fluid laminar flow. Nothing in the wealth of uniform channels has ever suggested this assumption is anything but justified; nevertheless, none of the uniform flow experiments which employed this assumption to determine the smallest channel dimension could
have revealed any evidence to the contrary, and precious few experiments performed to date actually test this assumption, even indirectly.

2.8.3 Thermocouple Connections

In retrospect, the thermocouple should not have been connected directly to the positions along the flow channel to be observed, necessitating that one or both ends of the thermocouple be disconnected, moved, and reconnected every time a new measurement was needed. Ideally, one would want to be able to switch the positions to be spanned by the thermocouple remotely while the experiment remains cold, but because solid mechanical contact must be made to provide good thermal contact, the experiment had to be warmed up to change the connection point. Less ideally, one might imagine having enough SQUID probes and thermocouples to connect one over each range of positions to be measured. This experiment used an existing experimental apparatus which accommodated only one SQUID probe, nevertheless there was still no reason to have to bend and move the thermocouple repeatedly to reach new connection points, then reestablish thermal contact at the ends of the thermocouple. As mentioned earlier, the fragile Au-Fe thermocouple wire was exposed and subject to breaking at the points where it entered the copper foil pieces attached to either end, given too much handling. Instead, the thermocouple should have been permanently mounted, and movable thermal links devised to connect the various positions in the channel with the ends of the thermocouple. These connections could have been as simple as two fairly large gauge copper wires. One end of each wire would be
permanently connected to the thermocouple, whereas the other end could be moved freely and repeatedly, without being damaged, between the various connection points on the flow channel and thermometer blocks.

These thermal links would have to meet only two criteria to function as well as connecting the thermocouple directly to points along the flow channel. First, this design would add two new segments to the SQUID circuit and therefore requires four instead of two junctions where good thermal contact must be made. As has already been explained at the close of Section 2.3.3, poor thermal contact reduces the sensitivity of the SQUID/thermocouple measurement. The junctions at the thermocouple end of the links could be permanent soldered connections, ensuring low thermal resistivity. The junctions at the movable ends, attaching to the flow channel, would be the same clamped connections as described above, except that heavy gauge copper wire would be much more durable than fine Au-Fe wire, and able to withstand the repeated handling required.

Second, whatever material is used to make the thermal links should have a low thermopower compared to that of the Au-Fe thermoelement, otherwise the thermopower of the connectors would add in series to that of the Au-Fe wire, creating a voltage divider by which the electrochemical potential generated in the thermocouple, and hence the input current detected by the SQUID, would be reduced. As before, such an effect does not invalidate the measurement of $\Delta T$ being made, since the SQUID input current is still proportional to $\Delta T$, but it reduces the measurement
sensitivity, because a smaller input current is produced for a given $\Delta T$ to be measured. Either extra thermal resistance or extra thermopower in the thermocouple circuit therefore produces the same result, nevertheless these two effects are distinguishable and should not be confused with one another. Regardless of what material is used to make the thermal links, it is unlikely that its thermopower would be at all significant in comparison with that of the iron-doped gold wire, which was chosen for the thermoelement precisely for its extremely large thermopower relative to other materials.
CHAPTER III

Results

3.1 Notation

This experiment involved measuring the absolute temperature at, and temperature differences between, several locations in our diverging channel. Because keeping track of the various temperature measurements and many positions involved can become somewhat confusing, a careful definition of the notation used to refer to our measurements is in order before beginning a discussion of our results. The thermocouple can be connected between any two of eight locations along the channel length. These radial positions are labelled alphabetically A–H in figures 2.1 and 2.2. The absolute temperature at a radial position \( r_i \) is denoted \( T(r_i) \), whereas the temperature difference between two positions \( r_i \) and \( r_j \) is denoted \( \Delta T(r_i, r_j) = T(r_i) - T(r_j) \), where \( i, j \in (A, H) \). The wide end of the channel, located at \( r_H \), is connected to the large reservoir held at temperature \( T_0 \), so \( T(r_H) = T_0 \). The increase in temperature at a particular position \( r_i \) over that of the reservoir is the quantity of primary interest, so we shorten the notation \( \Delta T(r_i, r_H) \) to simply \( \Delta T(r_i) \), with the caution that one must clearly distinguish between \( T(r_i) \) and \( \Delta T(r_i) = T(r_i) - T_0 \).
3.2 Data Collection and General Features

The local temperature gradient is a function of both the position in the channel and the heat current $\dot{Q}$. We sought to map out these two dependences and understand the interplay between them. We proceeded by measuring the temperature difference $\Delta T(r_i, r_j)$ over the fixed range in $r$ from $r_i$ to $r_j$ while varying $\dot{Q}$, then moving the thermocouple to span a new range in $r$ and repeating the measurements in $\dot{Q}$. Because the thermocouple connection can only be moved by warming up the apparatus, the data for this experiment were taken in four series, each series corresponding to a different thermocouple positioning.

3.2.1 First Data Series: $\Delta T(r_A)$

For the first data series, the thermocouple was connected between the heated chamber at the narrow end of the channel at $r_A = 2.5$ cm and the reservoir at $r_H = 12.5$ cm to measure the temperature difference $\Delta T(r_A)$ across the entire flow channel. This series of measurements was conducted at five reservoir temperatures: $T_0 = 1.3, 1.4, 1.5, 1.6,$ and $1.7$ K. The data for $T_0 = 1.5$ K are shown in figure 3.1 and are representative of data at other reservoir temperatures. Measuring $\Delta T(r_i, r_j)$ over other ranges in $r$ yielded data with qualitatively the same features as are shown here for $\Delta T(r_A)$. 
Figure 3.1: The total temperature difference across the entire channel, $\Delta T(\tau_A)$, as a function of heat current for a typical data set, $T_0 = 1.5$ K. Only data for $\dot{Q} < 2$ mW are shown here to emphasize features. The full data set extends to $\Delta T \approx 58$ mK at $\dot{Q} = 10$ mW. The laminar flow regime ($\Delta T$ linear in $\dot{Q}$) is clearly distinguishable from the turbulent regime (roughly cubic in $\dot{Q}$). For heat currents beyond $\dot{Q}_c$, the laminar state is metastable.
3.2.2 The $\dot{Q}$-dependence

All data series reveal a laminar flow regime for sufficiently small $\dot{Q}$, where the temperature difference

$$\Delta T = \Delta T_L$$  \hspace{1cm} (3.1)

depends linearly on $\dot{Q}$. Above a certain critical value of the heat current $\dot{Q_c}$ the flow undergoes a discontinuous transition to a turbulent state characterized by a much larger temperature difference with an essentially cubic dependence on $\dot{Q}$. This transition is hysteretic, as can be seen in figure 3.1. The turbulent temperature difference $\Delta T'$ is taken to be the excess dissipation beyond the laminar value, so the total temperature difference in the turbulent state is written as

$$\Delta T = \Delta T_L + \Delta T'$$  \hspace{1cm} (3.2)

That this division of the measured temperature difference into a laminar and a turbulent piece is appropriate once the flow is turbulent is an assumption, but one well grounded in convention and supported by experimental evidence [Tou82]. Adopting this assumption, we therefore subtract $\Delta T_L$ from our measured signal to obtain $\Delta T'$. For values of $\dot{Q}$ much greater than $\dot{Q_c}$, the linear laminar piece is so insignificantly small in comparison to $\Delta T'$ that whether or not $\Delta T_L$ has been subtracted becomes immaterial. The general appearance of the turbulent state data is qualitatively similar enough to that observed for a uniform high-aspect-ratio rectangular channel [Lad79] to give one the initial impression that the homogeneous T-II state observed in uniform flows remains largely unaltered by the weak divergence imposed in this flow.
The same data displayed in figure 3.1 are graphed in a more informative manner in figure 3.2. To reveal the basic cubic dependence of $\Delta T'$ on $\dot{Q}$, the data have here been plotted as $\Delta T'^{1/3}$ versus $\dot{Q}$. Linearizing the data in this manner immediately reveals two general features. First, the line formed does not extrapolate to intersect the origin, but rather is slightly offset, indicating that $\Delta T' \propto (\dot{Q} - \dot{Q}_o)^3$ would be a more appropriate functional form to describe the data. Second, rather than forming a perfectly straight line, the data exhibit a slight downward curvature at large $\dot{Q}$, appearing to indicate that the power-law dependence is not exactly cubic. This curvature can largely be attributed to the temperature dependence of the thermodynamic parameters characterizing the turbulent He-II, and would be present even if the power-law dependence on the heat current proved to be exactly cubic.

3.2.3 A Typical Data Set: Procedures and Observations

Several individual data runs are combined to form one data set such as that exemplified in figure 3.1. In a typical individual data run, the heat current was first increased in small steps, starting from zero, to some large heat current (up to 10 mW), then decreased, again in small steps, back to zero. A reading of $\Delta T$ was taken for each increment in $\dot{Q}$. A single data run from the set at $T_0 = 1.5$ K already shown in figure 3.1 is redisplayed in figure 3.3, in which the measurements made while increasing $\dot{Q}$ are distinguished from those for decreasing $\dot{Q}$.

As the heat current was increased from zero, the flow was initially in the laminar state. Eventually, at some increment of $\dot{Q}$, the helium would suddenly make the
Figure 3.2: Linearized plot of same data as shown in figure 3.1. The laminar contribution to $\Delta T$ has been subtracted to yield the turbulent temperature difference $\Delta T'$. The cube root of $\Delta T'$ is graphed here to reveal the basic cubic dependence of $\Delta T'$ on $\dot{Q}$. 

$T_0 = 1.5$K
Figure 3.3: A single data run from the data displayed in figure 3.1. The filled circles indicate data obtained while increasing \( \dot{Q} \), and the open circles indicate data gathered while decreasing \( \dot{Q} \). The laminar/turbulent transition occurred in this case at \( \dot{Q} = 1.23 \) mW, and the minimum heat current at which turbulence can be sustained is \( Q_c = 0.55 \) mW.
transition to the turbulent state, recognizable by a large shift in $\Delta T$. At this onset of turbulence, the SQUID voltage signal measuring $\Delta T$ would initially change very rapidly, then continue to drift upward more slowly until a steady state was established. It often took several seconds for the SQUID voltage to stabilize so that a reading of this first turbulent data point could be taken. Several time constants in the system contributed to this overall slow approach to a steady reading. First, the turbulence itself was forming. The SQUID signal adjusted rapidly enough to track this evolving signal without any appreciable lag. By far the longest time constant in the system was that of the temperature regulator. With the onset of turbulence, the thermal impedance of the channel would change abruptly. This sudden shift in thermal impedance created sharp transients in the heat current being delivered to the reservoir, which in turn would throw off the dynamic balance established by the temperature regulation system. The slow approach to a steady reading reflects the long time, on the order of a few seconds, it took this feedback circuit to reestablish the reservoir at $T_0$. This long time constant in the regulator electronics precluded any useful analysis of the dynamics involved in the initial formation and growth of a vortex tangle, which might otherwise have been resolved by the SQUID circuitry.

Once turbulence had been initiated, the flow was never observed to spontaneously reenter the laminar state, either upon increasing or decreasing the heat current slightly, or upon holding the heat current constant for a long time. In a typical data run, once the turbulent state was reached, the heat current continued to be increased incrementally to some large value of $\dot{Q}$, then successively decreased. No
hysteresis in the turbulent state was ever observed: as figures 3.2 and 3.3 show, the same curve of $\Delta T$ versus $\dot{Q}$ is observed for increasing and decreasing $\dot{Q}$. Eventually, the heat current at which the flow had undergone the laminar/turbulent transition was again reached on the way down in $\dot{Q}$. This time the flow would remain turbulent all the way down to $\dot{Q}_c$. Below $\dot{Q}_c$, the flow reentered the laminar state, and the same laminar curve seen for increasing $\dot{Q}$ was traced out as the heat current was decreased to zero.

Called the critical heat current, $\dot{Q}_c$ is the lowest heat current for which turbulence can be sustained within the region of the channel being observed. The laminar state is clearly observed for heat currents much larger than $\dot{Q}_c$, but is metastable beyond this critical heat current, whereas the turbulent state, once entered, is stable. The heat current at which the transition to turbulence occurred was often well beyond $\dot{Q}_c$ (see figure 3.3), but the laminar state never persisted at heat currents greater than about three or four times the value of $\dot{Q}_c$.

For each reservoir temperature, two to three data runs were completed as described above to establish repeatability in the measurements. Once the entire range in $\dot{Q}$ (up to 10 mW, or until $\Delta T(r_A)$ reaches about 50 mK) had been explored in this manner, a few additional data runs were often needed to better render certain features or fill in any ranges in $\dot{Q}$ with scanty data coverage. For example, it might take several additional tries to successfully map out the laminar state well beyond $\dot{Q}_c$. Once turbulence had formed on a given day, the flow would become turbulent immediately upon exceeding $\dot{Q}_c$ during further runs on the same day, even if the helium had been
left sitting quiescently at $Q = 0$ for several hours between runs. Reaching a high heat current in the laminar state could usually only be achieved during the first run of the day, but even during that initial run any small disturbance such as tapping on the cryostat would cause the flow to become turbulent. As often as not, the laminar/turbulent transition occurred spontaneously soon after $Q_c$ was exceeded, even in the absence of any apparent vibrations in the room or other disturbance. Another feature needing several runs to clearly establish was the position of $Q_c$, which required two or more runs with good laminar data (out to at least twice $Q_c$), and one or more runs in which, once turbulence had been established, the heat current was carefully and repeatedly decreased and increased around $Q_c$. The value of $Q_c$ was difficult to measure accurately since at that point the turbulent signal $\Delta T'$ is diminishing to zero, therefore becoming absorbed in the natural scatter of the laminar signal.

Since in a single data run reaching the highest allowable heat current via small increments could take as long as eight hours, sometimes the heat current could only be increased during one run, and data for decreasing $Q$ would have to be gathered in a subsequent run. To this end, once the transition to turbulence had occurred so no more laminar data could be obtained in that run, the heat current was immediately increased to a very large value in order to map out the turbulent state for decreasing $Q$. That the observed turbulent state was unique was verified in a similar manner, by establishing laminar flow at a low heat current, then quickly ramping up the heat current to a very high value to suddenly force the flow into a highly turbulent state. Alternatively, laminar flow was established at a relatively high heat current, then
the transition to turbulence was induced by tapping on the cryostat. Only a single robust turbulent state was ever observed; no evidence was found that more than one turbulent state exists at any heat current. This finding lies in contrast with that of a similar experiment in a diverging circular channel where, over a limited range of heat currents, multiple metastable turbulent states have been observed [Cas95b], but are not currently understood.

3.2.4 Second Data Series: $\Delta T(r_A, r_D)$

For the second data series, one end of the thermocouple was moved from the reservoir end, $r_H = 12.5$ cm, to an intermediate position in the channel, $r_D = 4.725$ cm. The other end was left attached to the heated end at $r_A = 2.5$ cm, so the SQUID/thermocouple system then measured the temperature difference $\Delta T(r_A, r_D)$. To extract $\Delta T(r_D)$ from the measured quantity, since

$$\Delta T(r_A) = \Delta T(r_A, r_D) + \Delta T(r_D), \quad (3.3)$$

a polynomial fit was first obtained to the corresponding data set of the previous series, $\Delta T(r_A)$ as a function of $\dot{Q}$, allowing interpolation of $\Delta T(r_A)$ at the values of $\dot{Q}$ for which $\Delta T(r_A, r_D)$ was measured, then the latter data set was subtracted from the fit to the former to yield $\Delta T(r_D)$. It would have been more straightforward to measure $\Delta T(r_D)$ directly, but without knowing a priori just how large $\Delta T(r_D)$ would prove to be, we opted for observing the complementary range in $r$ to ensure that we would have a detectable signal, even at small $\dot{Q}$ and high $T_0$. Under the
local uniformity approximation, $\Delta T(r_A, r_D)$ is predicted to be about 75% of the total temperature difference $\Delta T(r_A)$. Measurements of $\Delta T(r_A, r_D)$ were taken at the same five reservoir temperatures as before.

3.2.5 Third Data Series: $\Delta T(r_C, r_D)$

The third data series again involved moving only one end of the thermocouple, this time the one attached at $r_A$, to the new position $r_C = 3.47$ cm, so for this series $\Delta T(r_C, r_D)$ was measured, from which the temperature difference $\Delta T(r_C)$ was extracted. We were interested in examining a region of the flow far enough removed from the entrance and exit of the channel to avoid the influence of any possible end effects. In this data series, measurements were taken at only three reservoir temperatures: $T_0 = 1.4, 1.5$, and $1.6$ K.

3.2.6 Fourth Data Series: $\Delta T(r_A, r_B)$

For the last data series the thermocouple spanned the smallest region possible within the entrance of the channel, from $r_A$ to $r_B = 2.87$ cm. Unfortunately, the SQUID apparatus did not function correctly after this final cooldown, but direct measurements of the temperature at $r_B$ were still obtained for $T_0 = 1.5$ K from a carbon glass thermometer attached to this probe location. Temperature difference data were similarly gathered in these last two series by resistance thermometers placed at two additional locations in the channel, $r_E = 6.5$ cm and $r_F = 8.5$ cm. Measurements at $r_E$ were
taken at $T_0 = 1.3$ and 1.5 K and at $r_F$ only at $T_0 = 1.5$ K. Resistance thermometers lack the sensitivity of the SQUID/thermocouple system to discern a small $\Delta T$ signal from background noise, in part because they measure the absolute temperature rather than temperature differences, so the measurements at these last three positions are useful only at the larger heat currents; nevertheless, the added information was gathered simultaneously with the SQUID/thermocouple data and proved valuable in quickly building a more complete picture of the temperature profile throughout the channel.

3.2.7 Mapping the $r$-dependence

Once the temperature difference as a function of $\dot{Q}$ had been mapped out over several different ranges in $r$ for a given reservoir temperature, the data could be graphed in a manner that better reveals the dependence of $\Delta T'$ on position $r$. The temperature profile throughout the channel, $\Delta T'(r)$, is plotted as a function of $r$ for a fixed value of $\dot{Q}$. Figure 3.4 shows this temperature profile at $T_0 = 1.5$ K for a representative value of the heat current, $\dot{Q} = 5$ mW.

This profile graph is a composite of measurements from all data series. Each point on a graph of $\Delta T'$ versus $r$ is derived from one of the data series as follows. Each data set, $\Delta T'(r_i)$ versus $\dot{Q}$ for position $r_i$, $i \in (A, B, C, D, E, F)$, is fit to a third- or fourth-order polynomial, allowing interpolation of $\Delta T'(r_i)$ at a specific value of $\dot{Q}$. These fits were obtained with a polynomial regression routine in Sigmaplot [SIGMA]. In order to obtain a good enough fit to the data at all heat currents within a given
Figure 3.4: Profile in $r$ of the turbulent temperature difference $\Delta T'(r)$ at a typical heat current, $\dot{Q} = 5 \text{ mW}$, for $T_0 = 1.5 \text{ K}$
data set, it was usually necessary to break the full range in \( \dot{Q} \) up into sub-ranges, and fit each sub-range separately. The fit over the whole range in \( \dot{Q} \) is then formed by splicing together the polynomials over each sub-range. To interpolate \( \Delta T'(r_i) \) at a certain \( \dot{Q} \) desired, the fit corresponding to the appropriate sub-range in \( \dot{Q} \) must be selected.

A procedural rule of thumb in understanding the results of any study of superfluid turbulence in a channel flow has been to first check that the laminar observations make sense before proceeding to analyse the turbulent results. The next section of this chapter describes the analysis of the laminar data.

### 3.3 Laminar flow

One should be able to predict the laminar dissipation \( \Delta T_L(r) \) exactly by applying classical hydrodynamics in the two-fluid model to the specific geometry of the flow. Agreement of this theoretical prediction with the laminar data provides several essential checks on the experimental operation and design. First, if the functional forms of the predicted and measured laminar dissipation agree, then one may be confident that the flow channel has the shape expected and is not obstructed. Comparing resistance thermometry readings among the different data series independently verifies that the flow channel is not obstructed, at least by frozen air, since the apparatus is warmed up to room temperature between series. Second, finding the value of the channel height \( h \) which makes the predicted \( \Delta T_L \) best fit the observed value provides an experimental determination of this crucial channel dimension, which is difficult
to measure exactly by other means. Last, and most important to our purposes, the laminar prediction tests the validity of making the local uniformity approximation, in essence providing a measure of the weakness with which the flow diverges.

3.3.1 The Integrated Rectangle Solution

Although the geometry of the flow channel does not readily lend itself to an analytic solution, an approximate solution is easily derived from the result for a uniform rectangular channel. An exact solution for the laminar temperature gradient in a uniform rectangular channel is obtained from the two-fluid equations of motion [Cor28, Hen82b] and is given by

$$\nabla T_L = -\frac{12 \eta F}{\rho S h^2} V_n,$$  

(3.4)

where the negative sign indicates that the normal fluid flows through the cross sectional area $A = h w$ of height $h$ and width $w$ in the direction of decreasing temperature (see equation 1.6). The aspect ratio $w:h$ of the channel determines the value of the geometric factor $F$, which ranges from $F = 2.371$ for a square channel ($w:h = 1$) to $F = 1$ for flow between two flat parallel plates ($w:h = \infty$). An analytic expression for $F$ in terms of a series can be found in Appendix A of [Hen82b].

For this diverging channel, where the width of the channel varies with position $r$ as $w = \theta r$, we apply the local uniformity approximation by assuming that the local temperature gradient $\nabla T_L(r)$ at position $r$ in our diverging flow is given by equation (3.4) upon replacing the constant value of $V_n$ for uniform flow with the
local value at position \( r \)

\[
V_n(r) = \frac{\dot{Q}}{\rho S T \theta h r} \dot{r} = C_n \frac{\dot{Q}}{r} \dot{r}
\]  

(3.5)

where the coefficient \( C_n \) is defined for convenience:

\[
C_n \equiv \frac{1}{\rho S T \theta h}.
\]  

(3.6)

The local temperature gradient is then integrated over \( r \) to obtain the laminar temperature difference between any two positions in the channel. Other than \( V_n \), the only factor in \( \nabla T_L \) that varies with position is \( \mathcal{F} \), which quantifies the influence of the channel sidewalls. At the narrow end of our channel, where \( w : h = 8, \mathcal{F} = 1.086 \), and at the wide end, where \( w : h = 40, \mathcal{F} = 1.016 \). Because the variation in \( \mathcal{F} \) over the length of the channel is so small, \( \mathcal{F} \) is treated as a constant to simplify the integration. Since \( \nabla T_L \) is largest at small \( r \), we use \( \mathcal{F} = 1.086 \); therefore we may be slightly overestimating the actual predicted value of \( \Delta T_L \), but not by more than 7% at most. Integrating the uniform rectangle \( \nabla T_L(r) \) in the local uniformity approximation, we obtain the predicted laminar temperature difference between position \( r \) and the wide end of the channel at \( r_H = 12.5 \, \text{cm} \):

\[
\Delta T_L(r) = \frac{12 \eta \mathcal{F}}{\rho S h^2} C_n \dot{Q} \ln \left( \frac{r_H}{r} \right),
\]  

(3.7)

where \( C_n \) is given by equation 3.6. This "integrated rectangle" solution does not account for the divergent nature of the flow, since the temperature gradient appropriate to a one-dimensional flow in a uniform rectangular channel was used. Before comparing this solution with the measured laminar dissipation, we will first explore two other approximate laminar solutions.
To determine whether the radial divergence alters the above prediction for $\Delta T_L$ substantially, we compared the integrated rectangle solution to two other approximate solutions, both of which incorporate the divergence explicitly but neglect the presence of the sidewalls. These solutions are discussed below in Sections 3.3.2 and 3.3.3, respectively. We expect the effect of the diverging sidewalls to be negligible, because the normal fluid velocity profile should be dominated by the much closer spacing of the two parallel walls separated by $h$. To the extent that the sidewalls in our channel do not significantly influence the flow, these latter two solutions can be expected to accurately describe our laminar data.

3.3.2 The Elkouh Solution

A solution for the radial laminar flow of an ordinary viscous fluid between two parallel disks has been presented by Savage [Sav64] and expanded upon by Elkouh [Elk70]. Our radially diverging flow represents an angular slice $\theta = 0.08\, \text{rad}$ of the full $2\pi$ geometry treated by Elkouh. In cylindrical coordinates $(r, \phi, z)$ the parallel disks are located at $z = \pm a$, so for our channel $a = h/2$.

We generalized the Elkouh solution to the two-fluid model, making the minimal additional assumptions that the superfluid is incompressible and irrotational, with the result that the superfluid only has a component of velocity in the radial direction. The normal fluid velocity is assumed to have only radial and azimuthal components, and the solutions for $v_n$ and $\nabla T_L$ are obtained via power series expansion. The two-fluid equations are thereby reduced to an infinite series of ordinary differential
equations which, when solved for the boundary conditions involved, yield the following expression for the laminar temperature difference:

\[
\Delta T_L(r) = \frac{4 \eta}{\rho S h^2} \left\{ 3 \ln \left( \frac{r_H}{\tau} \right) C_n \hat{Q} + \frac{1}{2} \left[ \frac{54}{35} - \left( \frac{\rho_n}{\rho} \right)^2 \right] \frac{h^2}{4 \nu_n} \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right) C_n^2 \hat{Q}^2 + \frac{78}{2695} \left( \frac{h^2}{4 \nu_n} \right)^2 \left( \frac{1}{r_H^4} - \frac{1}{r^4} \right) C_n^3 \hat{Q}^3 + \cdots \right\}. \tag{3.8}
\]

For a complete description of how the Elkouh approach was generalized to the two-fluid model, see Appendix A.

The first term in this expansion is equal to the "integrated rectangle" solution with \( F = 1 \), the value for flow between infinite parallel plates (compare equation 3.7). This leading-order term therefore corresponds to mono-directional flow between two parallel plates. The additional terms may be thought of as corrections that describe the divergent nature of the actual radial flow. The lowest order correction, which is quadratic in \( \hat{Q} \), is positive for \( T_0 \geq 2.0 \text{ K} \) and becomes negative just below 2.0 K. The significance of this quadratic term and its change in sign with temperature is discussed further in Section A.2 of Appendix A.

### 3.3.3 The Parabolic \( v_n \) Solution

The other approximate solution for \( \Delta T_L(r) \), which we term the "parabolic \( v_n \)" solution, is obtained by invoking reasonable and minimal assumptions regarding the symmetries and properties of the flow in order to solve the two-fluid equations in closed form. To obtain an analytic solution for \( \nabla T_L \), the assumption that the cross-channel normal fluid velocity profile is parabolic is incorporated as an ansatz.
First, it is assumed that the normal and superfluid components are incompressible ($\nabla \cdot \mathbf{v}_n = \nabla \cdot \mathbf{v}_s = 0$) and that $\mathbf{v}_n$ and $\mathbf{v}_s$ have only radial components, which depend only on $r$ and $z$:

\[
\begin{align*}
\mathbf{v}_n(r) &= C_n \frac{\dot{Q}}{r} Z_n(z) \hat{r}, \\
\mathbf{v}_s(r) &= -\frac{\rho_n}{\rho_s} C_n \frac{\dot{Q}}{r} Z_s(z) \hat{r},
\end{align*}
\] (3.9)

where $C_n$ is given by equation 3.6. The cross-channel averages of $Z_n$ and $Z_s$ are normalized ($\langle Z_n \rangle = \langle Z_s \rangle = 1$) so as to give the correct value of $V_n$ (see equation 3.5) and satisfy the condition of thermal counterflow. Upon imposing the condition that $\mathbf{v}_s$ is irrotational ($\nabla \times \mathbf{v}_s = 0$), it immediately follows that $Z_s(z) = 1$. Inserting these expressions for $\mathbf{v}_n$ and $\mathbf{v}_s$ into the two-fluid equations yields a scalar equation for the laminar temperature gradient in terms of $Z_n(z)$:

\[
\rho \mathbf{s} \frac{dT_L}{dr} = \eta C_n \frac{\dot{Q}}{r} Z_n'' + \rho_n \left[ Z_n^2 - \left( \frac{\rho_n}{\rho_s} \right)^2 \right] C_n^2 \frac{\dot{Q}^2}{r^3}. \tag{3.10}
\]

To obtain an analytic solution to the above equation for $\nabla T_L$, the cross-channel normal fluid velocity profile is further assumed to be parabolic:

\[
Z_n(z) = \frac{3}{2} \left[ 1 - \left( \frac{2z}{h} \right)^2 \right]. \tag{3.11}
\]

Inserting this expression for $Z_n$ into equation 3.10 and integrating, we obtain the laminar temperature difference under the assumption of a parabolic normal fluid profile:

\[
\Delta T_L(r) = \frac{4 \eta}{\rho_s h^2} \left\{ 3 \ln \left( \frac{r_H}{r} \right) C_n \dot{Q} + \frac{1}{2} \left[ 6 - \left( \frac{\rho_n}{\rho_s} \right)^2 \right] \frac{h^2}{4 \nu_n} \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right) C_n^2 \dot{Q}^2 \right\}. \tag{3.12}
\]
To second order in $\dot{Q}$, the only difference between this solution and the Elkouh expression above (compare to equation 3.8) lies in the coefficient in square brackets in the term quadratic in $\dot{Q}$. Here the quadratic term is negative for $T_0 < 1.98$ K. All three approximate solutions are compared in Figure 3.5, in which the thermal resistance $\Delta T_L(\tau)/\dot{Q}$ is plotted as a function of $\tau$ for $\dot{Q} = 1$ mW at $T_0 = 1.5$ K. The integrated rectangle solution is depicted in Figure 3.5 for $F = 1.086$ and $F = 1$. At this reservoir temperature, the parabolic $v_n$ solution is slightly smaller than the integrated rectangle solution with $F = 1$ and nearly indistinguishable from the Elkouh solution. The data at this $T_0$ are included in this figure for comparison. How well these laminar solutions compare to the observed laminar behavior is discussed in detail in the following section.

3.3.4 The Measured Laminar Dissipation

Only the first two of the four data series, for which the thermocouple spanned the entire channel $(r_A, r_H)$ and the region $(r_A, r_D)$, respectively, measured $\Delta T$ over a large enough range in $\tau$ to obtain accurate laminar data. Direct measurements of $\Delta T(r_E)$ and $\Delta T(r_F)$ made with resistors, as well as the SQUID/thermocouple measurements at other positions, could at best be used to deduce an upper bound for the laminar thermal resistance $\Delta T_L/\dot{Q}$ at these other positions. None of the measurements of $\Delta T_L$ exhibited any clear, systematic or repeatable deviations from a linear dependence on $\dot{Q}$. Within the range of heat currents for which the laminar state was observed, the predicted quadratic deviation from a linear $\dot{Q}$-dependence remains too small an effect
Figure 3.5: Measured and predicted thermal resistance $\Delta T_L(r)/\dot{Q}$ as a function of $r$ at $\dot{Q} = 1$ mW for $T_0 = 1.5$ K. Solid lines: integrated rectangle solution with $F = 1.086$ (upper line) and $F = 1.000$ (lower line). Short dashed line: parabolic $v_{\text{a}}$ solution. Long dashed line: Elkouh solution.
Table 3.1: Measured laminar thermal resistances at each $T_0$. The thermal resistances $\Delta T(r_A)/\dot{Q}$ and $\Delta T(r_A,r_D)/\dot{Q}$ were measured directly, whereas $\Delta T(r_D)/\dot{Q}$ is obtained by subtracting these two measurements. The maximum heat current at which the laminar state was observed is given in the last column.

<table>
<thead>
<tr>
<th>$T_0$ (K)</th>
<th>$\Delta T_L(r_A)/\dot{Q}$ ($\mu$K/mW)</th>
<th>$\Delta T_L(r_A,r_D)/\dot{Q}$ ($\mu$K/mW)</th>
<th>$\Delta T_L(r_D)/\dot{Q}$ ($\mu$K/mW)</th>
<th>Laminar state maximum $\dot{Q}$ (mW)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>127. ± 4.</td>
<td>35.7 ± 1.3</td>
<td>91.7 ± 4.6</td>
<td>1.15</td>
</tr>
<tr>
<td>1.4</td>
<td>45.9 ± 2.</td>
<td>13.1 ± 0.6</td>
<td>32.8 ± 2.</td>
<td>1.5</td>
</tr>
<tr>
<td>1.5</td>
<td>18.5 ± 0.9</td>
<td>4.73 ± 0.17</td>
<td>13.8 ± 0.8</td>
<td>1.8</td>
</tr>
<tr>
<td>1.6</td>
<td>8.26 ± 0.3</td>
<td>1.62 ± 0.06</td>
<td>6.64 ± 0.4</td>
<td>1.95</td>
</tr>
<tr>
<td>1.7</td>
<td>3.79 ± 0.13</td>
<td>0.1 {$^+0.3$}</td>
<td>3.69 ± 0.13</td>
<td>2.1</td>
</tr>
</tbody>
</table>

to be detected in the data. We therefore compared our results to the integrated rectangle solution, which only has a linear $\dot{Q}$ dependence. Table 3.1 summarizes the laminar results, listing the measured thermal resistances (i.e. the best linear fit to the slope of $\Delta T_L$ versus $\dot{Q}$ for each range in $r$). The laminar state becomes increasingly less stable as $\dot{Q}$ is increased far beyond $\dot{Q}_c$. The maximum heat current reached in the laminar state at each reservoir temperature is listed in the last column of Table 3.1.

Figure 3.5, for $T_0 = 1.5K$, shows that $\Delta T_L(r_A)$ agrees very well with the prediction of the integrated solution with $F = 1.086$, but the measurement of $\Delta T_L(r_D)$ is significantly higher than the corresponding predicted value. The measurements at other temperatures all show this same behavior. This finding is illustrated much more clearly by examining the temperature dependence of the thermal resistance $\Delta T_L(r)/\dot{Q}$
for each position \( r \) rather than looking at the \( r \)-dependence of the thermal resistance at a single temperature. The integrated rectangle solution, with \( \mathcal{F} = 1.086 \), is compared to the data in this manner in Figures 3.6 and 3.7. At all reservoir temperatures, the predicted laminar temperature difference is in excellent agreement with measurements of \( \Delta T_L(r_A) \), as can be seen in Figure 3.6, but is consistently lower than the measurements at the intermediate position \( r_D = 4.725 \) cm, as shown in Figure 3.7. Since \( \mathcal{F} = 1.086 \) is the value appropriate for \( r_A \), if anything the predicted \( \Delta T_L(r_D) \) should be too large, but instead it is smaller than the measured result by about 20%.

It must be noted that the theoretical laminar curves in figures 3.5, 3.6 and 3.7 were all computed assuming a channel height of \( h = 0.025 \) cm. As has already been discussed in Chapter II, the height was designed to be 0.020 cm, but both the laminar data and direct measurement of the height at the wide end of the channel indicate the actual height turned out to be much larger than planned. By treating the height as a tunable parameter in equation 3.7 and taking \( \mathcal{F} = 1.086 \), the value of \( h \) best agreeing, on average, with the measured laminar slopes listed in Table 3.1 can be determined for each of the three ranges in \( r \). The first data series, measuring \( \Delta T_L(r_A) \), revealed that an average height of \( 0.0246 \pm 0.0002 \) cm provided the best fit to the data. In this sense, the excellent agreement depicted in figure 3.6 is engineered and not at all surprising.

After the experiment concluded, the wide end of the channel was probed with shims, confirming that the height was close to 9.9 mils, or 0.025 cm. Taking these two pieces of evidence together, the actual channel height was therefore determined to be
Figure 3.6: Temperature dependence of the laminar thermal resistance $\Delta T_L (r_A)/\dot{Q}$. Data are compared to predictions of the integrated rectangle solution with $\mathcal{F} = 1.086$. 
Figure 3.7: Temperature dependence of the laminar thermal resistance $\Delta T_L(r_D)/\dot{Q}$. Data are compared to predictions of the integrated rectangle solution with $F = 1.086$. 
0.025 ± 0.001 cm rather than 0.020 cm as designed. In Section 2.8.2 of Chapter II, we have already described the tests conducted to assess the gluing process and measure the height, and discussed how the assembly process might have brought about this increase in the height.

In defense of our analysis, the procedure of using the laminar data to determine the smallest dimension of a channel is a standard technique. The problem of laminar Poiseuille flow in a uniform channel of circular, square or rectangular cross section is solved analytically, and the best fit value of the channel diameter or height is determined by comparison to the measured laminar dissipation. For very small channels, an accurate, direct measurement of the smallest dimension is simply not feasible by any other means. The materials most often used to make small channels are obtained from commercial sources, and although the nominal dimensions are specified by the manufacturer, most researchers instead use the laminar data as the most accurate indication of the exact dimensions of the channel. Moreover, only the average dimensions can be deduced by this method, except in those few experiments where the temperature is probed at a number of locations along the channel length, not just at the ends of the channel. Only when the temperature profile is measured, such as in Henberger's experiment, can any information about the degree of uniformity of the channel dimensions be gleaned. The boundary conditions in the diverging geometry make only an approximate analytic solution possible, but otherwise the method of finding the channel height from the laminar data is every bit as valid here as in all
previous uniform channel experiments, and the shim test confirms the average height deduced from the data.

This technique of treating the small dimension of the channel as a tunable parameter to be fit by comparing the measured laminar behavior to the solution of the laminar flow problem rests on the assumption that the analytic solution, which treats the normal fluid as an ideal viscous fluid, truly represents the physics of two-fluid laminar flow. Nothing in the wealth of uniform channels has ever suggested this assumption is anything but justified; nevertheless, none of the uniform flow experiments which employed this assumption to determine the smallest channel dimension could have revealed any evidence to the contrary, and precious few experiments performed to date actually test this assumption, even indirectly.

Having resolved as far as possible this issue of the increase in the average channel height, only the question of the disagreement between the measured and predicted values of $\Delta T_L(r_D)$ remains. We considered the nature and possible causes of this discrepancy at some length. That the relative size of the discrepancy remains constant over the whole range of temperature points to some small but not insignificant systematic effect which the laminar solution fails to capture, rather than some more serious shortfalling of the laminar solution itself, or an inability to apply the uniform rectangular channel laminar solution in the local uniformity approximation. Having solved the laminar flow problem in three different approximations and arrived at essentially the same result each time, we are confident that these laminar solutions are free of any computational errors and that the true solution the actual boundary value
problem would not differ much from these three approximations to it. Consequently, we thoroughly examined several possible explanations for the observed discrepancy.

One factor not accounted for in the theoretical prediction of $\Delta T_L$ is that the normal fluid velocity profile must develop over some entrance length, which could alter $\Delta T_L(r)$ slightly; however, no clear evidence for a significant laminar entrance region has been observed in comparable experiments in uniform high-aspect-ratio rectangular channels [Lad79]. To estimate the length over which entrance effects might influence the laminar temperature gradient in our experiment, we estimated the distance (from the narrow end of the channel) over which $v_n$ develops into a parabolic profile by adapting a simple empirical formula applicable to classical fluids [Gol38]. This entrance length is given by

$$\ell_e = \frac{0.06 k^2}{\nu_n} V_n(r_A),$$

(3.13)

where $\nu_n$ is the kinematic viscosity, and the normal fluid velocity used is that at the narrow end of the channel. Note that $\ell_e$ depends on the heat current. In the temperature range 1.3–1.7 K, for $\dot{Q} = 1$ mW, this entrance length ranges between 2.4–2.7 cm. This estimate should amount to an upper bound for $\ell_e$ in the diverging channel geometry, because the velocity decreases as $1/r$, so the entrance length should be somewhat foreshortened by the decrease in $V_n(r)$ over the distance $\ell_e$. At best, this entrance region barely extends beyond $r_D = 4.725$ cm, and can hardly be expected to account for the observed 20% increase in $\Delta T_L(r_D)$ over the theoretical result. We therefore believe laminar entrance effects, if any, to be unimportant.
Since $\Delta T_L$ depends on the channel height as $h^{-3}$, it must be asked whether a slight variation in this dimension could account for the larger $\Delta T_L(r_D)$ observed. On average, the channel height between $r_D$ and $r_H$ would have to be $h = 0.022 \pm 0.001$ cm in order to increase $T_L(r_D)$ by 20%, considerably smaller than the overall best-fit value of $h = 0.025 \pm 0.001$. Furthermore, the channel height would have to be much larger than expected towards the narrow end of the channel if the total laminar temperature difference $\Delta T_L(r_A)$ is to match the observed value. Between $r_D$ and $r_A$, a better average height deduced from the measured $\Delta T_L(r_A, r_D)$ would be $h = 0.028 \pm 0.001$.

Assuming any variation in $h$ would have to be gradual and smooth, we modeled the idea that during assembly, the two Stycast slabs forming the flow channel became canted instead of remaining parallel, resulting in a smooth increase in the height $h$ from $r_H$ to $r_A$. Alternatively, the channel could perhaps be bowed inward or outward longitudinally, with $h$ smaller or larger in the middle than at the ends.

Our computations, which assumed several plausible distortions of $h$, fell far short of explaining the measured deviation in $\Delta T_L$ from the theoretical prediction. Any such smooth variation in $h$ would have to be more extreme than is credible to account for the anomalous 20% increase in $\Delta T_L(r_D)$.

Moreover, the shim test contradicts the notion that $h$ is 0.22 cm on average between $r_D$ and $r_H$. From this test, $h$ is decidedly greater than 0.023 cm at $r_H$, otherwise the 9 mil shim would not have fit; in fact, $h$ is close to 0.025 cm at $r_H$. Rather than a smooth variation in $h$ over the channel length, the data suggest a sharper aberrance in $h$ near $r_D$. One such possibility is that incomplete coverage of the epoxy used to glue
the two Stycast pieces together resulted in a gap or small crevice being created in the channel wall, forming a "side channel" with a height on the order of 1–2 mils or less. How such a gap might have formed is touched upon in Section 2.2.2, where the flow channel assembly is described in detail. If a gap exists, it must further be supposed that the normal fluid flows within the gap, contributing to $\Delta T_L$. One would expect the normal fluid to be immobilized in such a very narrow gap, so the reasonableness as well as the likelihood of this scenario explaining the unexpected $r$-dependence is debatable.

The systematic variation between the laminar data and the predicted $r$-dependence remains somewhat puzzling, but we believe it can be explained satisfactorily by a combination of slight variations in $h$ along the channel length, with perhaps an unexpected sharper feature around $r_D$, such as a local change in the height or a small gap opening up in the channel sidewall. Alternatively, the laminar flow may prove more complicated than envisioned, but to date we have found no plausible alternative or reason to suspect the integrated rectangle solution incorrectly represents the physics involved. Overall, locally applying $\nabla T_L$ for a uniform rectangular flow satisfactorily predicts the observed laminar temperature profile in this diverging flow. Our strongest evidence of this basic agreement lies in figure 3.7, where it can be seen that the measured and theoretical temperature dependences are in very close agreement functionally, despite the measured $\Delta T_L(r_D)$ being systematically larger than is predicted. The laminar data therefore provides convincing evidence that the divergence is indeed weak, in that the local uniformity approximation successfully adapts the
uniform laminar solution to our nonuniform flow. It therefore seems reasonable to employ the same local uniformity approximation as was used in modeling the laminar state to model the turbulent state.

3.4 Applying The Schwarz Model Locally

A brief overview of the Schwarz model of homogeneous superfluid turbulence is given in Section 1.7 of Chapter I. Following the same prescription as for laminar flow, the Schwarz model is applied to our diverging flow by invoking the local uniformity approximation. Section 3.4.1 discusses the exact procedures and assumptions used in implementing the calculation. The predictions of the Schwarz model applied locally are then compared to the measured excess dissipation in Section 3.4.2.

We begin by recalling from Section 1.7.3 the key predictions of the Schwarz model needed here. Recognizing that for a spatially uniform velocity there is no distinction between the local relative velocity \( \mathbf{v} = \mathbf{v}\hat{r} \) and its spatial average over a cross-section perpendicular to the flow direction \( \mathbf{V} = \mathbf{V}\hat{r} \), we substitute \( \mathbf{V} \) for \( \mathbf{v} \) to ensure consistent notation throughout this section. The vortex line density is given in the Schwarz model as

\[
L = \frac{c_L^2 V^2}{\beta_S^2}
\]  

(3.14)

where

\[
\beta_S = \frac{\kappa}{4\pi} \ln \left( \frac{1}{c_1 L^{1/2} a_0} \right).
\]  

(3.15)
Here $c_L$ and $c_1$ are temperature-dependent parameters predicted within the Schwarz model, $a_0 \simeq 1.3 \times 10^{-8}$ cm is the effective core radius of a vortex line, and the quantum of circulation $\kappa = \hbar/m_4$, where $\hbar$ is Planck's constant and $m_4$ is the mass of a helium atom. The mutual friction force density exerted on the superfluid by the normal fluid is mediated by the vortex tangle, and is given by

$$F_m = \rho_s \kappa \alpha \left( I_{\parallel} - c_L I_t \right) L V \hat{r} \quad (3.16)$$

The excess dissipation is therefore predicted in the Schwarz model to be

$$\nabla T' = -\frac{F_m}{\rho_s S} = -\frac{\kappa \alpha}{S} \left( I_{\parallel} - c_L I_t \right) L V \hat{r} \quad (3.17)$$

The friction coefficient $\alpha$ quantifies the mutual friction force exerted between the superfluid and normal fluid components, and $I_{\parallel}$ and $I_t$ characterize the anisotropy of the vortex tangle; all are temperature-dependent parameters.

In the case of homogeneous turbulence, the relative velocity $V$ appearing in each of the above equations is a constant, independent of position. The local uniformity approximation is invoked by inserting the local average relative velocity $V(\tau) = (\rho/\rho_s) V_n(\tau)$, where for our diverging flow $V_n(\tau)$ is given by equation 3.5, in place of the constant $V$. As prescribed by this spatial variation in the relative velocity, $L$, $F_m$ and $\nabla T'$ all become spatial functions in the local uniformity approximation. The dependence of each of these quantities on position $\tau$ and the drive parameter $\dot{Q}$ enters only through $V$, therefore $\dot{Q}$ and $\tau$ are coupled in a specific way in the local uniformity approximation, in this case appearing only in multiples of the combination $(\dot{Q}/\tau)$. 
3.4.1 Integrating the local temperature gradient

For a fixed value of the heat current \( \dot{Q} \), the excess temperature difference \( \Delta T'(r) \) is found by integrating equation 3.17 from \( r_H \) to \( r \). This integration must be performed numerically since the integrand contains \( \beta_S \), which depends logarithmically on \( r \) through \( L \). A FORTRAN routine was written to iteratively compute \( \beta_S \) at each \( r \) for a fixed value of \( \dot{Q} \), then integrate \( \nabla T'(r) \) over \( r \), and repeat the calculation for successive increments of \( \dot{Q} \). The code for program SCHWARZ is listed in Appendix B.

The temperature dependence of all parameters in equation 3.17 had to be self-consistently taken into account throughout the integration. First, the values of the Schwarz and thermodynamic parameters had to be interpolated for any temperature in the range of 1.3-1.8K. The Schwarz parameters were obtained from Table I of [Sch88], and are summarized here in Table 3.2. In addition, Table I of [Sch85] contains the values of \( \alpha \) and \( \alpha' \) at many other temperatures. For a given temperature, corresponding to a specific value of \( \alpha \) and \( \alpha' \), Schwarz ran his simulation at two different relative velocities and obtained slightly different values for \( c_L, c_1, I_||, \) and \( I \). The difference (at most a few percent) is attributed to the fluctuating nature of the tangle, and represents the uncertainty within which these parameters can be determined by the simulation. The values of \( c_L, c_1, I_||, \) and \( I \) listed in Table 3.2 are therefore the average of the pair of values given in [Sch88]. The best third- or fourth-order polynomial fit to each parameter as a function of temperature was identified and incorporated in a subroutine, allowing determination of the parameter value at the desired temperature. The thermodynamic parameters were obtained from [Don92]
Table 3.2: Schwarz parameters as a function of temperature

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.07K</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.010</td>
</tr>
<tr>
<td>$\alpha'$</td>
<td>0.005</td>
</tr>
<tr>
<td>$c_L$</td>
<td>0.0359</td>
</tr>
<tr>
<td>$c_L'$</td>
<td>2.86</td>
</tr>
<tr>
<td>$I_{</td>
<td></td>
</tr>
<tr>
<td>$I_{\perp}$</td>
<td>0.4325</td>
</tr>
</tbody>
</table>

and likewise interpolated with polynomial fits. Graphs showing the temperature dependence and interpolation of each of the parameters listed in Table 3.2, and also $\rho_s$ and $S$, are included in Appendix B.

The integrand was then determined over a small range of $r$ values, starting at $r_H$ where $T = T_0$, and the integration carried out to the endpoint $r < r_H$ to yield $\Delta T'(r)$. The values of all temperature-dependent parameters were updated to the new temperature at the endpoint, $T(r) = T_0 + \Delta T'(r)$, before the integrand was computed over the next small range in $r$ and the integration repeated from $r_H$ to the new endpoint $r$. Continuing in this way, the temperature difference profile $\Delta T'(r)$ predicted by the Schwarz model in the local uniformity approximation was computed,
as accurately as possible, at each \( \dot{Q} \). Note that the laminar contribution to the increase in temperature was neglected in updating the temperature. For reasonably large heat currents, the turbulent contribution to \( \Delta T \) completely overwhelms the laminar contribution, whereas at low enough heat currents that \( \Delta T_L \) and \( \Delta T' \) are comparable in size, the total increase in \( T \) over \( T_0 \) is so small that the parameters have not yet shifted significantly from their values at \( T_0 \). The laminar contribution to \( \Delta T \) can therefore be safely ignored without compromising the accuracy of the temperature profile calculation.

Program \textsc{Schwarz} allows the user to control the integration range, the integration step size, and the number of divisions into which the range is divided for the temperature update. Typically, the integration range from 2.5 cm to 12.5 cm was divided into 1000 integration steps (0.01 cm step size) and 100 divisions for the temperature correction (parameters updated every tenth step or 0.1 cm). The algorithm was found to be stable with respect to changes in the granularity of these spatial divisions. Testing finer and coarser spatial resolutions showed that a step size of 0.01 cm was small enough to ensure accuracy in the calculation, even at the maximum heat currents of interest. Updating the temperature at least every 0.1 cm was necessary, especially at small \( r \) where the temperature gradient becomes large, to maintain an accurate and self-consistent prediction of the increase in temperature.

Two different versions of program \textsc{Schwarz} were used. Originally, the program was written to determine the \( \dot{Q} \)-dependence, outputting \( \Delta T'(r_i) \) over a specified range in heat currents for a single position \( r_i \). Later, the program was modified to determine
the $r$-dependence, outputting $\Delta T'(r)$ over a specified range in $r$ at selected heat currents. The version listed in Appendix B outputs the $\dot{Q}$-dependence, and includes other user-selected options to be discussed in Sections 3.6 and 3.7 of this chapter. To solve for $\beta_S(\dot{Q}, r)$, the expression for $L$ is inserted into the expression for $\beta_S$ (see equations 3.14 and 3.15), resulting in an iterative equation for $\beta_S$ which is solved in a loop that repeats until the level of accuracy in determining $\beta_S$ specified by the user is reached. Since $\beta_S$ is on the order of $10^{-3}$, the default level of $\pm 10^{-9}$ ensures that $\beta_S$ is computed to at least five significant figures.

3.4.2 Comparison to the measured $\Delta T'$

No unusual variations or trends were manifest in the data as a function of the reservoir temperature, therefore to minimize the number of graphs shown in this section the results will be displayed for only one representative temperature. The observations at all other reservoir temperatures are qualitatively similar to those depicted here for $T_0 = 1.5$ K, the sample temperature selected. Graphs for the corresponding data sets at all other temperatures are contained in Appendix C.

Figure 3.8 compares the prediction of the Schwarz model applied in the local uniformity approximation to the temperature difference measured across the entire channel as a function of $\dot{Q}$ for $T_0 = 1.5$ K. This figure contains the same data at $T_0 = 1.5$ K as were previously shown in figure 3.2. That the predicted $\dot{Q}$-dependence (solid line) curves slightly downward at large $\dot{Q}$ is the result of the temperature dependence of the parameters and not a change in the functional dependence on $\dot{Q}$. The
Figure 3.8: $\dot{Q}$-dependence of the turbulent temperature difference measured across the entire channel for $T_0 = 1.5$ K. Solid line: prediction of the Schwarz model applied in the local uniformity approximation, including temperature correction of parameters. Dashed line: no temperature correction; parameters held constant at 1.5 K values.
same downward curvature is evident in the data, but the excess dissipation $\Delta T'(r_A)$ predicted by the theory is consistently larger than the data at all temperatures. Shown as a dashed line for comparison, integrating $\nabla T'(r)$ without self-consistently updating all temperature-dependent parameters in the integrand to the local temperature as described above (i.e. instead, leaving all temperature-dependent parameters "frozen" at the values appropriate to $T_0$) results in a slight upward curvature with increasing $\dot{Q}$. This upward curvature is produced by the logarithmic dependence of $\beta_s$ on $\dot{Q}$.

Figures 3.9 through 3.13 contain the linearized graphs of $\Delta T'(r_i)$ versus $\dot{Q}$ at other positions $r_i$ studied at $T_0 = 1.5K$. At each of these other positions, similar systematic discrepancies are found between the observed and predicted $\Delta T'(r_i)$. At $r_A$, $r_B$, and $r_C$, the measured temperature difference is larger than predicted over the entire range in $\dot{Q}$, and at $r_E$ and $r_F$, it is smaller than predicted, except at very small heat currents. The measured and predicted values of $\Delta T'(r_D)$ are about equal. Looking at this data set only, one might believe that the Schwarz model applied locally fits the observed behavior reasonably well, but upon viewing the entire series of $\dot{Q}$-dependence graphs it becomes clear that the observed $r$-dependence is quite different from what is predicted. The close agreement between data and theory around $r_D$ is therefore understood to be nothing more than serendipity. At any position, the data extrapolate to a positive intercept in $\dot{Q}$, suggesting a $\dot{Q}$-dependence of the form $(\dot{Q} - \dot{Q}_0)^3$. The Schwarz model applied locally cannot account for this non-zero $\dot{Q}$-intercept seen in the data.
Figure 3.9: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_B = 2.868$ cm and the reservoir end held at $T_0 = 1.5$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.10: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_C = 3.468$ cm and the reservoir end held at $T_0 = 1.5$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.11: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_D = 4.725$ cm and the reservoir end held at $T_0 = 1.5$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.12: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_E = 6.5$ cm and the reservoir end held at $T_0 = 1.5$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.13: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_p = 8.5$ cm and the reservoir end held at $T_0 = 1.5$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
The $r$-dependence is more clearly illustrated by graphing the temperature profile as a function of $r$ for a single value of the heat current. In figures 3.14 through 3.17, the predictions of the Schwarz model applied locally are compared to the measured temperature profile in $r$ of $\Delta T'(r)$ at several values of the heat current for $T_0 = 1.5$ K. Figure 3.16 for $\dot{Q} = 5$ mW contains the same data as appeared in figure 3.4.

Plotted in this way, the theoretical $r$-dependence might not appear to be too dramatically different from that seen in the data, but the discrepancy between the two is actually quite large. The predicted $r$-dependence in $\nabla T'$ is very close to $r^{-3}$, since $V \propto r^{-1}$ and $\nabla T' \propto V^3$. The only deviations from a purely $r^{-3}$ dependence come from the logarithmic $r$-dependence enfolded in $\beta_s$ and the implicit $r$-dependence introduced through the temperature variation of parameters. These latter two dependences add a gradual, smooth change in $\nabla T'$ with $r$ to the basic $r^{-3}$ dependence. Integrating $\nabla T''(r)$ therefore yields an $r$-dependence of $\Delta T'(r)$ close to $(r^{-2} - r_H^{-2})$. This factor is divided out from the same data and predictions already shown in figures 3.14 through 3.17, and the results displayed in figures 3.18 through 3.21. In this latter set of figures, $\nabla T' \propto r^{-3}$ would appear as a straight horizontal line. We can deduce from these figures that the turbulent temperature gradient does not depend on any single power of $r$ throughout the channel. The data suggest that $\nabla T' \propto r^{-x}$ where at small $r$, $x \geq 3$ and at large $r$, $x \approx 1$. The small deviation from a purely cubic $r$-dependence inherent in the theory is manifestly different from the substantial and more complicated bimodal deviation exhibited in the data.
Figure 3.14: $r$-dependence of the turbulent temperature difference at $\dot{Q} = 3 \text{ mW}$ for $T_0 = 1.5 \text{ K}$. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.15: \( r \)-dependence of the turbulent temperature difference at \( \dot{Q} = 4 \text{ mW} \) for \( T_0 = 1.5 \text{ K} \). The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.16: $r$-dependence of the turbulent temperature difference at $\dot{Q} = 5 \text{ mW}$ for $T_0 = 1.5 \text{ K}$. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.17: $r$-dependence of the turbulent temperature difference at $\dot{Q} = 6 \text{ mW}$ for $T_0 = 1.5 \text{ K}$. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure 3.18: $r$-dependence of $\Delta T'$ at $\dot{Q} = 3 \text{ mW}$ for $T_0 = 1.5 \text{ K}$. The factor $1/r^3 - 1/r_H^3$ has been divided out to emphasize deviations from cubic $r$-dependence: $\nabla T' \propto r^{-3}$ exactly would appear as a straight horizontal line. The data are compared to the Schwarz model applied locally.
Figure 3.19: \( r \)-dependence of \( \Delta T' \) at \( \dot{Q} = 4 \ mW \) for \( T_0 = 1.5 \ K \). The factor \( 1/r^2 - 1/r_H^2 \) has been divided out to emphasize deviations from a cubic \( r \)-dependence: \( \nabla T' \propto r^{-3} \) exactly would appear as a straight horizontal line. The data are compared to the Schwarz model applied locally.
Figure 3.20: $r$-dependence of $\Delta T'$ at $\dot{Q} = 5$ mW for $T_0 = 1.5$ K. The factor $1/r^2 - 1/r_H^2$ has been divided out to emphasize deviations from a cubic $r$-dependence: $\nabla T' \propto r^{-3}$ exactly would appear as a straight horizontal line. The data are compared to the Schwarz model applied locally.
Figure 3.21: $r$-dependence of $\Delta T'$ at $\dot{Q} = 6$ mW for $T_0 = 1.5$ K. The factor $1/r^2 - 1/r_H^2$ has been divided out to emphasize deviations from a cubic $r$-dependence: $\nabla T' \propto r^{-3}$ exactly would appear as a straight horizontal line. The data are compared to the Schwarz model applied locally.
Clearly, the Schwarz model of homogeneous turbulence applied in the local uniformity approximation does not produce the correct functional dependence on either position $r$ or heat current $\dot{Q}$ to agree with the data. We explored several ad hoc modifications to the Schwarz temperature gradient to see if some simple adjustment could better account for the observed temperature profile. One obvious possibility is that the Schwarz parameters might retain their respective roles, but simply have different values for inhomogeneous and homogeneous turbulence. The amount of vorticity resulting at a certain relative velocity is quantified in terms of $c_L$ (see equation 3.14). Arbitrarily increasing or decreasing $c_L$ scales the overall magnitude of $\nabla T'$, shifting the theoretical curve in figures 3.14 through 3.17 (also figures 3.18 through 3.21) so it intersects the measured profile at a new position $r$, but the functional dependence in $r$ and $\dot{Q}$ is unaffected and the overall shape of the curve remains the same. Scaling any other single parameter in the Schwarz model likewise cannot improve the fit to the data. In the Schwarz model the $\dot{Q}$- and $r$-dependences enter together through $V \propto \dot{Q}/r$. Scaling the Schwarz parameters does nothing to modify the $\dot{Q}$- or $r$-dependence inherent in the homogeneous model. A correct theory of inhomogeneous turbulence must either modify the functional form of $L$ itself, so that $L$ is no longer proportional to $V^2$, or the relation $\nabla T' \propto L V$, or both. Whatever this modification might prove to be, it must serve to substantially alter the coupling between the $\dot{Q}$-dependence and $r$-dependence from that built into the homogeneous model.

A plausible first-order correction to the vortex line density was tested by adding a term proportional to $\nabla L$ to the expression for $L$ given by equation 3.14 above. In
taking the gradient of the homogeneous line density, the $r$-dependence of $\beta S$ was neglected, so the effect of this modification was to multiply the original $L$ by a function of the form $(1 + B/r)$, therefore adjusting $\nabla T''$ by the same multiplicative factor. For negative $B$, the $r$-dependence of $\Delta T'$ reverses in curvature at small $r$, but no such behavior is evident in the data. For positive $B$, the curvature becomes steeper, in reasonable agreement with the data at small $r$, but widening the disagreement at larger $r$. Furthermore, this adjustment to $L$ alters the $r$-dependence, but leaves the $Q$-dependence of $\nabla T'$ unchanged. The $r$-dependence was further explored by arbitrarily multiplying the local temperature gradient (equation 3.17) by other simple polynomial functions of $r$. This exercise served only to confirm that a more appropriate $r$-dependence would result if $\nabla T'$ were multiplied by $(Ar^2 + B/r)$ or $(Ar^3 + B)$.

Although a numerical simulation of spatially varying flows may someday be performed, the possibility also remains that a new hydrodynamic model is needed to better describe inhomogeneous turbulence. At about the time this analysis applying the Schwarz model in the local uniformity approximation had been completed, the Geurst model emerged, promising to contain just the new ingredients needed. The analysis using this model is described in the following section.

### 3.5 The Geurst Three-Fluid Model

In a series of recent papers, Geurst has presented a new hydrodynamic model of turbulent superfluid helium [Geu89, Geu92, Geu94]. The Geurst model treats the vortex tangle as a third independent fluid, with velocity $v_\xi$ equal to the drift velocity
of the tangle and density $L$. This "flow" of vortex lines is on equal footing with
the mass flow of the total fluid, with velocity $v$ and density $\rho$, and of the normal
fluid component, with velocity $v_n$ and density $\rho_n$, respectively. Geurst uses the
entropy density $S$ rather than $\rho_n$ as the appropriate density to complete his set of six
independent variables. Both are proportional to the number density of elementary
excitations which comprise the normal fluid. This three-fluid hydrodynamic model is
of particular interest to the present work in that it purports to directly address the
case of an inhomogeneous vortex line density. Only those key features and predictions
needed to explain how the model was applied to our results will be summarized in this
section. The underlying derivation of the Geurst model is reviewed in much greater
detail in Appendix D. In Section 3.5.1, the methodology and necessary assumptions
used to apply the Geurst model to our results is described.

The Geurst model was initially presented for the simplest case of uniform flow
in one dimension [Geu89]. The 1D Geurst model contains six adjustable parameters
which depend only on temperature and must be determined either by careful compar­
ison to experiment or to a microscopic numerical simulation. Only after the relations
between five of the six parameters and the analogous set of Schwarz parameters ($c_L$,
$\alpha$, $\alpha'$, $I_\parallel$, and $I_\perp$) were presented in a subsequent paper [Geu92] could the 1D Geurst
model be applied in a predictive sense. Appendix E summarizes these relations and
lists the FORTRAN code used to compute the predicted behavior.

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1In keeping with Geurst's notation, throughout this section and Appendices D and E, $v$ will be
used to refer to the mass flow of the total fluid rather than the relative velocity. The latter will
be denoted as $v_n - v_s$. 

In this macroscopic hydrodynamic model, Geurst employs a generalized form of Lin's variational principle to derive a Lagrangian density, and from it the set of evolution equations which determine the physical variables $\rho, S, L, v, v_n,$ and $v_t$. This set consists of three conservation equations for $\rho, S,$ and $L,$ and three corresponding equations of motion for the respective generalized momenta. In particular, in the absence of dissipation, line length is conserved according to

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(Lv_t) = 0 .$$  \hspace{1cm} (3.18)

The equations of conservation of momentum and energy can then be derived, either using the Lagrangian density, or directly from the evolution equations. In contrast to other hydrodynamic models, such as proposed by Nemirovskii and Lebedev [Nem83] or van Beelen, von Joolingen, and Yamada [vBee88], this three-fluid model explicitly accounts for the energy density and impulse (i.e. pseudo-momentum) of the vortex tangle in the internal energy. The internal energy density includes the energy density of the vortex tangle associated with the "microscopic" circular motion of the superfluid around the vortex cores, and also the total impulse (or pseudo-momentum) of the vortex tangle.

By adding dissipative terms to account for the production of entropy and vortex line length, and appropriate internal and external forces, the kinematic equation the vortex line density becomes

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(Lv_t) = \frac{\beta}{2\gamma} |v_t - v|L^{3/2} - \frac{\beta_s}{\gamma} L^2 .$$  \hspace{1cm} (3.19)

Since this equation is for one-dimensional flow, vector notation is not used. The coefficients $\beta$ and $\gamma$ are two of the five adjustable parameters in the Geurst model, and
\( \beta_s \) is the Schwarz parameter already defined in equation 3.15. The Geurst parameters \( \beta \) and \( \gamma \) are related to the Schwarz parameters by

\[
\frac{\beta}{2} = \frac{c_L}{\alpha' I_{\|} - \frac{\rho_n}{\rho} + (1 - \alpha') I_L c_L}
\]

and

\[
\frac{\beta}{2\gamma} = \frac{\alpha I_L}{\left( \alpha' I_{\|} - \frac{\rho_n}{\rho} \right)}
\]

For the case of a homogeneous tangle, where \( \frac{\partial}{\partial x} (L v_t) = 0 \), Geurst derives that the vortex line drift velocity \( v_t \) satisfies

\[
v_t - v = \frac{1}{\Gamma} \left[ (\gamma_{11} + \gamma_{22}) (v_n - v) + \frac{\beta}{2\gamma} \beta_s L^{1/2} \hat{w}_t \right],
\]

where \( \hat{w}_t \) is a unit vector in the direction of \( (v_t - v) \) and \( \Gamma \) is defined by

\[
\Gamma = \frac{\beta^2}{4\gamma} + \gamma_{11} + 2\gamma_{12} + \gamma_{22}.
\]

The coefficients \( \gamma_{11}, \gamma_{12}, \) and \( \gamma_{22} \) are the other three adjustable parameters in the Geurst model. They also enter into the expression for the mutual friction force \( F_{nm} \), and hence the turbulent dissipation \( \nabla T' \). The interpretation of the Geurst parameters in terms of the Schwarz parameters, and several useful relations among them, are presented in Section E.1 of Appendix E. The vortex line drift velocity is given in terms of the Schwarz parameters by

\[
v_t - v = \left( \alpha' I_{\|} - \frac{\rho_n}{\rho} \right) (v_n - v_s) + (1 - \alpha') I_L \beta_s L^{1/2} \hat{w}_t,
\]

where \( (\rho/\rho_s)(v_n - v) = (v_n - v_s) \) has been used. In the 1D model, \( \hat{w}_t \) may either be parallel or antiparallel to \( \hat{w}_n \), the direction of \( (v_n - v) \). This expression for \( v_t \), and
equations 3.20 and 3.21 for $\beta$ and $\gamma$, respectively, are all given here in the limit where the second-order friction parameter $\alpha''$ is taken to be zero. This approximation is valid since $\alpha''$ is very small and can be neglected in comparison to $\alpha'$.

In the case of a large gradient in the vortex line density, such as might be present at a turbulent-to-laminar front, Geurst postulates adding a new term to the internal energy proportional to the gradient of the line density. Because this adjustment treats a spatial variation in the vortex line density, this new model held the promise of addressing our experimental situation. This added term strongly modifies the kinematic equation for the line density. The result is a second-order differential equation for the line density (compare to equation 3.19):

\[
\frac{\partial L}{\partial t} + \frac{\partial}{\partial x} (L v_t) + \kappa \frac{\gamma}{\gamma} \left( \frac{2}{L} \left( \frac{\partial L}{\partial x} \right)^2 - \frac{\partial^2 L}{\partial x^2} \right) - \frac{\kappa}{\rho_s \gamma} \frac{\partial}{\partial x} \left( \rho_s \gamma t \right) \frac{\partial L}{\partial x} = \frac{\beta}{2\gamma} |v_t - u| L^{3/2} - \frac{\beta s}{\gamma} L^2 .
\]

The sixth and last parameter in the 1D Geurst model, $\gamma_t$, determines the strength of the new gradient term added to the internal energy and therefore has no analogue in the Schwarz model, so $\gamma_t$ must be treated as a free and tunable parameter.²

²In his latest paper [Geu95], Geurst corrects his derivation of this equation to show that the factor of two should not appear in the $\left( \frac{\partial L}{\partial x} \right)^2$ term. This minor correction does not affect our subsequent analysis, except to modify the values of $\gamma_t$ obtained. See the discussion following equation D.69 in Appendix D.
3.5.1 Applying the Geurst Model Locally

Provided with this link to the Schwarz model, I applied the 1D Geurst model to our results. First, the 1D equations of motion were generalized to three dimensions (as much as possible) by substituting for all vector quantities the appropriate three-dimensional analogues in cylindrical coordinates. It must be emphasized that this step alone falls far short of properly generalizing the Geurst model to three dimensions, since all the parameters involved, as well as their interpretation in the context of the Schwarz model, were determined for a one-dimensional, uniform flow. In fact, Geurst and van Beelen have recently provided a full three-dimensional (3D) version of this three-fluid model [Geu94]. By contrast to the 1D version, the equations of motion in the full 3D model are much more complicated and contain no less than 23 independent parameters which have yet be evaluated by comparison to an experiment or simulation. For the special case of homogeneous turbulence, the 3D Geurst model reduces to a slightly more general form of the 1D model, with the kinematic equation emerging in the form given in equation 3.19 and the number of parameters collapsing to the above-mentioned five. The 3D version for the most general case of inhomogeneous turbulence in a nonuniform flow is daunting in its complexity, and not usable until the host of parameters it generates can be interpreted. The best I can do, therefore, is to employ the 1D Geurst model in the local uniformity approximation, cognizant of the underlying assumption, now embedded in many stages of the argument, that on a local level the turbulence bears the characteristics of a homogeneous tangle.
Converting the 1D kinematic equation for the vortex line density in the presence of a large gradient in $L$ to a three-dimensional vector equation yields (compare to equation 3.25)

\[
\frac{\partial L}{\partial t} + \nabla \cdot (L \mathbf{v}_L) + \frac{\gamma}{\kappa} \left[ \frac{2}{L} (\nabla L)^2 - \nabla^2 L \right] - \frac{\gamma}{\rho_s \gamma} \nabla (\rho_s \gamma) \cdot \nabla L = \frac{\beta}{2 \gamma} L^{3/2} |\mathbf{v}_L - \mathbf{v}| - \frac{\beta S}{\gamma} L^2 .
\] (3.26)

It should be noted that in the case of a spatially homogeneous $L$, all the gradient and divergence terms vanish and, upon inserting $\mathbf{v}_L$, the kinematic equation leads as expected to the original Vinen (or Schwarz) equation for homogeneous turbulence. See Section D.3 of Appendix D for further details.

I seek the steady-state solution of equation 3.26 for thermal counterflow or zero net mass flow ($\mathbf{v} = 0$). This highly nonlinear second-order differential equation can only be reasonably solved in some approximation which simplifies it. The simplification I explored was to neglect the second derivative term $\nabla^2 L$ in favor of the first derivative squared term, $L^2 / L$. In justifying this approximation, I first observe that if the line density varies as a simple power of $r$: $L = C r^n$, then $L^2 / L = \nabla^2 L = \nabla^2 L = n^2 C r^{n-2}$. For such a simple line density, since these two terms would have the same functional dependence, each would introduce a variation that decouples the $Q$- and $r$-dependences in $L$ in exactly the same way. The character of the line density which is an exact solution to equation 3.26, if it is to fit our data, must be such that these two terms still behave very analogously to one another and the above remarks would still largely hold true. Choosing which of these terms to retain and which to discard seems more a matter of computational convenience than a substantive issue, so it
seems reasonable to replace the difference of these two terms with a single term written either as $\nabla^2 L$ or $(\nabla L)^2/L$. Because the prefactor multiplying the expression in square brackets in equation 3.26 contains the adjustable parameter $\gamma_\ell$, the choice of which one of these two terms is used becomes absorbed, to some extent, in the magnitude and algebraic sign of $\gamma_\ell$, therefore the entire possible range of positive and negative $\gamma_\ell$ values was considered. The parameter $\gamma_\ell$ depends only on temperature, therefore as long as the temperature variation in the flow remains modest, it is reasonable to discard the cross term $\nabla (\rho_* \gamma_\ell) \cdot \nabla L$ also.

Choosing to drop the Laplacian term in favor of the $(\nabla L)^2$ term yields, in the steady state for $v = 0$,

$$\frac{2\kappa}{L} \gamma \frac{\gamma_\ell}{L} (\nabla L)^2 + \nabla \cdot (L v_\ell) + \frac{\beta S}{\gamma} L^2 - \frac{\beta}{2 \gamma} L^{3/2} |v_\ell| = 0 . \tag{3.27}$$

The second term in this equation can be expanded according to $\nabla \cdot (L v_\ell) = v_\ell \cdot \nabla L + L \nabla \cdot v_\ell$. In general, the contribution from the divergence $\nabla \cdot v_\ell$ is very small, but not necessarily zero. Presuming that $(v_\ell - v)$ is in the direction of $(v_n - v)$, hence $(v_n - v_*)$, the divergence term is computed from equation 3.24 for $v_\ell$, yielding

$$\nabla \cdot v_\ell = (1 - \alpha') I_t \left[ \frac{\beta_s}{\tau} L^{1/2} + \left( \frac{\beta_s}{2} - \frac{\kappa}{8 \pi} \right) \frac{\nabla L}{L^{1/2}} \right] . \tag{3.28}$$

Inserting this result into equation 3.27 for the line density produces a quadratic equation in $\nabla L$ of the form

$$A(\nabla L)^2 + B \nabla L + C = 0 , \tag{3.29}$$
where the coefficients $A$, $B$, and $C$ are functions of $L$ and the Geurst parameters:

$$A = \frac{2\kappa \gamma L}{\gamma},$$

$$B = |\mathbf{v}_t| + (1 - \alpha') I_t L^{1/2} \left( \frac{\beta_s}{2} - \frac{\kappa}{8\pi} \right),$$

$$C = \frac{\beta_s}{\gamma} L^2 - \left[ \frac{\beta}{2\gamma} |\mathbf{v}_t - \mathbf{v}| - (1 - \alpha') \frac{I_t \beta_s}{r} \right] L^{3/2}.\quad (3.30)$$

With these approximations, equation 3.26 is reduced to a first-order equation that is relatively easy to solve. Upon inserting equation 3.24 for $\mathbf{v}_t$ and simplifying, the expressions for $B$ and $C$ become

$$B = \left( \alpha' L - \frac{\beta_n}{\rho} \right) |\mathbf{v}_n - \mathbf{v}_s| + (1 - \alpha') I_t L^{1/2} \left( \frac{3}{2} \beta_s - \frac{\kappa}{8\pi} \right),$$

$$C = \frac{\alpha I_t \beta_s}{c_L} L^2 - \left[ \alpha |\mathbf{v}_n - \mathbf{v}_s| - (1 - \alpha') \frac{\beta_s}{r} \right] I_t L^{3/2}.\quad (3.31)$$

The question as to whether the vortex lines drift in the same direction as or opposite to the normal fluid bears some further examination. Geurst remarks that in characterizing homogeneous superfluid turbulence, the direction of $(\mathbf{v}_t - \mathbf{v})$ enters his 1D model as a free parameter. For a homogeneous tangle, choosing $(\mathbf{v}_t - \mathbf{v})$ parallel to $(\mathbf{v}_n - \mathbf{v})$ recoups the standard form of the Vinen equation and means that the $|\mathbf{v}_t - \mathbf{v}| L^{3/2}$ term in equation 3.25 behaves as a line production term. Geurst states that this choice is the appropriate one near equilibrium (steady state), and that the three-dimensional version of his theory generates an additional equation which determines the direction of $\mathbf{v}_t$. See the discussions of equations 4-14, 4-16, and also equation 3-15 ff. in §3 in [Geu92].
Experiments which have attempted to directly measure the magnitude and direction of the vortex line drift velocity have obtained somewhat ambiguous and conflicting results [Ash75, Wan87]. In the Schwarz model, it is assumed that \( \dot{w}_n \) is parallel to \( \dot{w}_n \). Because sufficient uncertainty remains regarding the actual direction in which the vortex lines drift, I was compelled to consider both possible directions in my analysis. Initially, I concluded that the effect of reversing the direction of \( (v_L - v) \) is limited to reversing the algebraic sign of \( B \nabla L \), which comes from the divergence term \( \nabla \cdot (L v_L) \). Since only the magnitude \( |v_L - v| \) enters in to the \( L^{5/2} \) term in \( C \), I concluded that \( C \) is unaffected by a reversal in the direction of \( (v_L - v) \). These conclusions proved to be incorrect, a fact not discovered until a thorough analysis using them had been completed. A simple correction is needed to properly account for reversing the direction of \( (v_L - v) \), as discussed in Section 3.5.3 below. Making this correction would be straightforward, and would require only that the analysis for the case of \( \dot{w}_L \) opposite to \( \dot{w}_n \) be redone, following the same procedure as before but using corrected expressions for \( B \) and \( C \). I was about to embark on this reanalysis when I discovered a serious inconsistency in relating the Geurst and Schwarz models: the interpretation of the Geurst parameters in terms of the Schwarz parameters is incompatible with several requisites of the Geurst model. The problem is severe enough to obviate the need for any further analysis using the Geurst model. Section 3.5.3 describes this incompatibility and how it was discovered. For the sake of completeness, in the rest of this section I first outline the methodology used in comparing the Geurst model to our results, and briefly summarize the outcome of that analysis in Section 3.5.2,
before addressing the inconsistency I found in being able to relate the Geurst and Schwarz models.

Because the divergence of \( \mathbf{v}_\ell \) is very small, it may be neglected without much loss of accuracy, resulting in the coefficients in equation 3.29 being simplified to

\[
\begin{align*}
A &= \frac{2 \kappa \gamma \ell}{L \gamma} \\
B &= |\mathbf{v}_\ell| \\
C &= \frac{\beta s}{\gamma} \frac{L^2}{2 \gamma} - \frac{\beta}{2} |\mathbf{v}_\ell| L^{3/2}.
\end{align*}
\]

This latter form of the coefficients \( A, B \) and \( C \) was reported in [Kaf94]; however, the previous expressions for these coefficients, given in equation 3.30, were actually used in the calculation which determined \( L(\tau) \). The divergence of \( \mathbf{v}_\ell \) is so small that more or less the same \( L(\tau) \) resulted in either case, regardless of which set of expressions was used.

I attempted to account for reversing the direction of \( \mathbf{v}_\ell \) by introducing an adjustable parameter \( \phi \) multiplying \( B \), with value constrained to the range \(-1 < \phi < 1\). Changing this "fudge factor" from \(+1\) to \(-1\) reverses the sign of the term linear in \( \nabla L \) in equation 3.29. Picking a value of \( \phi \) other than \( \pm 1 \) also adjusts the strength of the linear term.

Being first order, equation 3.29 requires only one boundary condition, the value of \( L \) at the entrance (or exit) of the channel, \( L(r_A) = L_e \). Equation 3.29 predicts the derivative of \( L \) at a given position \( r \), not \( L \) itself. The two roots to this equation are referred to here as the "plus" and "minus" roots, corresponding to choosing the plus or minus sign, respectively, before the square root of the discriminant \( (B^2 - 4AC) \). To
solve for $L(r)$, one root of the equation is selected, and values of the three adjustable parameters $L_e$, $\gamma$, and $\phi$ are chosen. Having thus fixed both $L$ and $\nabla L$ at $r = r_A$ (or $r_H$), an iterative procedure can then used to map out $L(r)$ throughout the channel for the particular combination of parameters and root specified.

I wrote a FORTRAN routine to compute $L(r)$, insert the solution into the expression for $\nabla T'$, and numerically integrate the result to obtain $\Delta T'(r)$ as was done in the program SCHWARZ. Mapping out $L(r)$ was accomplished by multiple calls to an external subroutine called DIVPRK, part of the IMSL math and statistics library of subroutines [IMSL]. DIVPRK is a double-precision version of IVPRK, which uses a Runge-Kutta-Verner method to solve the initial-value problem for ordinary differential equations of the form $y' = f(x, y)$ where $y(x_0) = y_0$. This subroutine is appropriate when moderate accuracy is required in the result, and the problem is not stiff. As the degree of stiffness of this differential equation could not be easily checked, the use of DIVPRK was deemed appropriate as long as this method yielded stable and well-behaved solutions for $L$ over a wide range of parameter space. Were this not the case, some other differential equation solver would have been required instead. Fortunately, the use of DIVPRK proved adequate.

Once $L(r)$ had been predicted, the next step was to insert $L(r)$ into the local temperature gradient and integrate to obtain the predicted temperature profile $\Delta T'(r)$. Writing the appropriate equation of motion (equation 5.5 in [Geu89]) in terms of the pseudo-momentum of elementary excitations $P_n = \rho_n(v_n - v_s)$, and assuming no
external forces act on the normal fluid, yields:

$$\frac{\partial P_n}{\partial t} + P_n(\nabla \cdot v_n) + \nabla (v_n \cdot P_n) + \rho S \nabla T' = F_{ln} \quad . \quad (3.33)$$

where the internal force density $F_{ln}$ exerted by the vortex tangle on the normal fluid is given by (see equation D.52 in Appendix D)

$$F_{ln} = -F_{nl} = \rho_s \kappa L [\gamma_{12} (v_t - v) + \gamma_{22} (v_n - v_t)]$$

$$= \rho_s \kappa L [(\gamma_{12} + \gamma_{22}) (v_t - v) - \gamma_{22} (v_n - v)] \quad . \quad (3.34)$$

In [Geu92], Geurst derives the relations between the parameters $\gamma_{12}$ and $\gamma_{22}$ and the corresponding Schwarz parameters for the case of homogeneous turbulence. In the limit that the second-order friction coefficient $\alpha'' = 0$, these relations are given by (for the full expressions, see equations E.7 and E.8 in Appendix E)

$$\gamma_{12} = \frac{\alpha}{1 - \alpha'} \left( \frac{\rho}{\rho_s} \right)^2 (1 - I_{||}) \quad , \quad (3.35)$$

$$\gamma_{22} = \frac{\alpha}{1 - \alpha'} \left( \frac{\rho}{\rho_s} \right) - \gamma_{12} \quad , \quad (3.36)$$

Upon replacing $(v_n - v)$ with $(\rho / \rho_s)(v_n - v_s)$, and inserting $(v_t - v)$ from equation 3.24, it can be shown that equation 3.34 becomes

$$F_{ln} = -\rho \kappa \alpha \left[ I_{\parallel} (v_n - v_s) - I_{\parallel} \beta S L^{1/2} \dot{\omega}_l \right] L = -\frac{\rho}{\rho_s} F_{ns} \quad . \quad (3.37)$$

In the Schwarz model $\beta S L^{1/2} \dot{\omega}_l = c_L (v_n - v_s)$, therefore the force $F_{nl}$ is recognized to be equal to $(\rho / \rho_s) F_{ns}$, where in keeping with Geurst's notation, $F_{ns}$ denotes the force exerted by the normal fluid on the superfluid. (Schwarz calls this force $F_m$; compare to equation 1.47.) In [Geu89], Geurst erroneously omits the factor $\rho / \rho_s$ in relating $F_n$ to $F_{ns}$, but this relation appears correctly in his subsequent paper, [Geu92].
In part, invoking the local uniformity approximation involves replacing the local fields $v_n$ and $v_s$ with the cross-channel average velocities $V_n(r)$ and $V_s(r)$ for our diverging flow (see equations 1.6 and 1.8). Writing $|V_n - V_s| = V_{ns}$ and computing terms in equation 3.33: $\nabla \cdot V_n = 0$, and

$$\nabla (V_n \cdot P_n) = \frac{\rho_n \rho_s}{\rho} \nabla (V_{ns}^2) = -2 \frac{\rho_n \rho_s}{\rho} \frac{V_{ns}^2}{r} \hat{w}_n .$$

Finally, in the steady state the turbulent temperature gradient predicted by the 1D Geurst model in the local uniformity approximation is given by

$$\nabla T' = -\frac{\kappa \alpha}{S} \left[ I_{\parallel} (V_n - V_s) - I_{\parallel} \beta S L^{1/2} \hat{\omega}_\parallel \right] L - 2 \left( \frac{\rho_n \rho_s}{\rho^2 S} \right) \frac{V_{ns}^2}{r} \hat{w}_n .$$

Assuming $\hat{\omega}_\parallel = \hat{w}_n$, comparison of this expression to the corresponding temperature gradient in the Schwarz model,

$$\nabla T' = -\frac{\kappa \alpha}{S} \left[ I_{\parallel} (V_n - V_s) - I_{\parallel} \beta S L^{1/2} \hat{w}_n \right] ,$$

reveals that the only difference between these two predictions is the additional inertial term, $\nabla (v_n \cdot P_n)$. Within the range in $r$ explored in our diverging flow, and assuming the cross-channel average velocities reasonably approximate the actual local velocity fields, this inertial term is extremely small and may be neglected. The 1D Geurst and Schwarz models therefore predict ostensibly the same dependences of the mutual friction and temperature gradient on $L$ and the drive velocity. The only significant distinction between these two models is found in the line density each predicts.
3.5.2 Searching Parameter Space

Due to the highly nonlinear nature of equation 3.29 and the resulting strong dependence of both the sign and magnitude of $\nabla L(r)$ on $L(r)$ and $\gamma_\ell$, I was compelled to explore the full range of parameter space for both roots. In all, eight possible classes of the solution to equation 3.29 had to be considered, corresponding to the eight combinations of choices of the root of the equation (plus/minus) and the algebraic signs of $\gamma_\ell(\pm)$ and $\phi(\pm)$.

For each of these eight regions in parameter space, only certain ranges of the values of $L_e$, $\gamma_\ell$, and $\phi$ lead to a viable solution for all $r$ within $2.5 \leq r \leq 12.5$ cm. For combinations of $(L_e, \gamma_\ell, \phi)$ outside the viable ranges, the solution $L(r)$ yields a negative discriminant $B^2 - 4AC$ at some position $r$ within 2.5–12.5 cm. Operationally, program GEURST crashes upon trying to compute the square root of a negative number. Since the subroutine which computes $\nabla L$ is called many thousands of times by DIVPRK, no IF statement was incorporated in it to check the discriminant each time and exit the program nicely upon detecting a negative discriminant.

Once the mathematically viable solutions have been identified, the majority of these must then be rejected on physical grounds. The fact that $\Delta T'$ is observed to decrease with increasing $r$ alone suggests that, on the whole, $L$ must decrease more or less smoothly with increasing $r$. A crude estimate of $L$ can be obtained by estimating $\nabla T'(r)$ from the data. For heat currents in the range of 1–10 mW, one expects $L_e$ to be large, on the order of at least $10^5$ to $10^8$, near the channel entrance. At the channel entrance (depending on the values of $L_e$, $\gamma_\ell$, and $\phi$ chosen) the respective algebraic
signs of $\nabla L$ could be positive for both roots, negative for both, or positive for one and negative for the other. One is tempted to conclude that those cases yielding a positive gradient can be summarily disregarded because $L(\tau)$ should decrease with $\tau$. This simple reasoning makes the tacit assumption that the algebraic sign of $\nabla L$ is fixed once the choice of root has been made, and $L(\tau)$ merely increases or decreases monotonically from $L_e$. Both $B$ and $C$ are highly nonlinear functions of $L$ and $\tau$, and are subject to changing sign at some position $\tau$ as $L(\tau)$ varies, in some cases causing the algebraic sign of $\nabla L$ to reverse. Nor can the data be used to rule out the possibility that $L$ might do something other than decrease monotonically. For example, the line density might be very small at the channel entrance, but grow to a large value within a short distance of $r_A$, then decrease monotonically throughout the rest of the channel. The data lack the spatial resolution needed to resolve or preclude any such small-scale behavior in $L(\tau)$ which might be present. In no case can either root be summarily excluded from consideration simply because it yields a positive gradient for $L(\tau)$ at $r_A$.

Exploring the behavior of both roots in the limit that $\gamma_\ell$ becomes small allows one some further insight into the nature of the various solutions possible. The parameter $\gamma_\ell$ appears only in the quadratic coefficient $A$ (see equation 3.30), therefore using a small $\gamma_\ell$ amounts to taking the limit as $A$ tends to zero. In this limit, one root of the equation reduces to the solution of the linear equation $B\nabla L + C = 0$:

$$\lim_{\gamma_\ell \to 0} \frac{dL}{d\tau} = -\frac{C}{B}, \quad (3.41)$$
whereas the other root becomes

$$\lim_{\gamma \to 0} \frac{dL}{dr} = \lim_{A \to 0} \frac{B}{A} - \frac{C}{B} = \frac{-B}{A}.$$  \hspace{1cm} (3.42)

If \(B\) is positive, then the "plus" root approaches \(-C/B\) and the "minus" root \(-B/A\); however, if \(B\) is negative then the roles of the two roots reverse so the minus root reduces to \(-C/B\) and the plus root to \(-B/A\). Multiplying \(B\) by a negative \(\phi\) likewise reverses the sign of the linear term in the quadratic, therefore switching the respective limiting behavior of the two roots for small \(\gamma\). Whichever root approaches \(-C/B\), this limiting behavior is independent of \(\gamma\), therefore the cases of positive and negative \(\gamma\) will approach the same limiting behavior as \(|\gamma|\) tend to zero. The other root will diverge to \(+\infty\) for one sign of \(\gamma\), and to \(-\infty\) for the other sign of \(\gamma\), as \(\gamma\) becomes small. \textsc{Maple} [\textsc{maple}] was used to further explore the nonlinear behavior of the coefficients as a function of position, temperature, heat current, and choice of parameters.

The coefficient \(A\), which scales as \(\gamma/\ell\), contains no explicit dependence on either \(r\) or \(\dot{Q}\). Notice that if \(L\) were constant throughout the channel, \(A\) would be also. The strength of the \((\nabla L)^2\) term would then be independent of position, but this term would also reduce to zero since \(\nabla L = 0\) if \(L\) is constant. This behavior is to be expected, since the role the \(A(\nabla L)^2\) term in equation 3.29 is to describe the presence of a sharp gradient in \(L\). Similarly, postulating too large a value for \(\gamma\) allows the term in \(A\) to dominate over \(B\) and \(C\), effectively reducing equation 3.29 to \(A(\nabla L)^2 = 0\) and forcing the solution \(L(r) = \text{constant}\). The criterion that \(L\) is expected to be inhomogeneous therefore sets a practical upper bound for the size of \(\gamma\).
Of the eight classes of solution, four lead to unphysical solutions where $L$ either decays much too rapidly to zero, or diverges equally rapidly, from the starting $L_e$. The first two of these classes arise from considering the cases of $\gamma_\ell$ positive and negative, respectively, while choosing the plus root with $\phi > 0$; the other two are obtained by choosing the minus root with $\phi < 0$. For some of these classes, within particular ranges of allowed $L_e$ and $\gamma_\ell$ values, solutions are also obtained where the line density bears a nearly identical $r$-dependence to that predicted by the Schwarz model. In at least one of these cases, the line density initially decreases more steeply than is predicted by the Schwarz model, but at some position $r$ the combination of $L(r)$ and $\dot{Q}$ is such that the roles of $A$ and $B$ are diminished in comparison to $C$, enough that equation 3.29 approaches $C = 0$. A Schwarz-like solution for $L(r)$, and hence $\nabla T'$, is recovered at all subsequent positions $r$.

Two more classes of the solution, obtained by considering both signs of $\gamma_\ell$ for the plus root with $\phi < 0$, and starting from a small initial value $L_e$, yield a solution where $L$ increases very gradually over some length $\delta$ within the channel entrance, then over a very small range in $r$ about $r = r_A + \delta$, $L$ grows explosively to a value $L \gg L_e$. With this increase in $L$, the relative importance of the terms in $A$ and $B$ in equation 3.29 diminishes in comparison to the role of $C$ in determining the steady-state line density, so for $r > r + \delta$, $L(r)$ is nearly the same as is predicted in the Vinen or Schwarz model. This solution reproduces a result first obtained by Murphy, Tough, and Fiszdon, who added a term of the form $\nabla \cdot (L v_\ell)$, with $v_\ell < 0$, to the Vinen equation [Mur92]. This solution is nothing more than what the Geurst model
gives in the absence of a large gradient in $L$ (see equation 3.19), or correspondingly, equation 3.29 in the limit that $A$ can be neglected in comparison to $B$ and $C$. The line density obtained in this solution matched that given by the Schwarz (or Vinen) model of homogeneous turbulence in the local uniformity approximation everywhere in the channel except within a distance $\delta$ of the entrance, where $L$ decreased rapidly until it had "accommodated" to the much lower value outside the channel. Upon completion of only the first two series of measurements, I had only the two data points at $r_A = 2.5$ cm and $r_D = 4.725$ cm in figure 3.16, from which it seemed that $\Delta T'(r_D)$ was correctly predicted by the Schwarz model, but $\Delta T'(r_A)$ was much smaller than expected. Our initial findings could be explained by supposing that a fairly large entrance region existed, on the order of $\delta = 0.5$ cm in size, over which the vortex line density was suppressed much below the value expected from the Schwarz model. After obtaining a more complete picture of the temperature profile $\Delta T'(r)$, however, it became obvious that no such pronounced entrance effect exists, and the apparent agreement between the Schwarz model and the data in the vicinity of $r_D$ is serendipitous.

The last two classes of the Geurst solution, corresponding to choosing the minus root to the quadratic and $\phi > 0$, produce line density profiles where $L(r)$ decreases smoothly with increasing $r$ over a wide range of parameter space. These two solutions converge in the limit that $\gamma_L \to 0$. For larger $\gamma_L$, only one of these two solutions, the case of $\gamma_L > 0$, has both a suitable functional form and large enough overall magnitude to potentially describe our data.
For the sake of comparison, the resulting Geurst $L(r)$ was inserted into both the Geurst and the slightly simpler Schwarz expressions for $\nabla T'$ given in equations 3.39 and 3.40, respectively. The respective temperature gradients were then integrated to obtain $\Delta T'(r)$. The only difference between the Geurst and Schwarz expressions for $\nabla T'$, as expressed above, is the additional inertial term in the Geurst expression. In this case, the Geurst and Schwarz expressions for $\nabla T'$ yield nearly the same result for the range of parameter space in question, so using the simpler Schwarz expression for $\nabla T'$ does not introduce any inconsistency. In cases where the resulting line density differs little from that predicted within the Schwarz model, the $\beta_s L^{1/2} \dot{\psi}_t$ term in either temperature gradient could be replaced with its equivalent in the Schwarz model, $c_L(v_n - v_s)$. This last substitution would be inappropriate in the Geurst $\nabla T'$ whenever the line density predicted by the Geurst model differed from that predicted by the Schwarz model, for then the Schwarz relation that $\beta_s L^{1/2} = c_L|v_n - v_s|$ would no longer be satisfied by the Geurst $L(r)$.

In seeking a fit to the data, the line density $L_e$ at the channel entrance will of course increase with $\dot{Q}$, but $\gamma_\ell$ should depend on temperature only, so the same value of $\gamma_\ell$ should provide a reasonable fit to the temperature profile $\Delta T'(r)$ at different $\dot{Q}$ values for a given $T_0$. No such overall fit to the data was found. At best, for a given $\dot{Q}$, a combination of $L_e$ and $\gamma_\ell$ could be identified that yielded a reasonable fit to the temperature profile data at small $r$ ($r < 3.5$ cm). Figure 3.22 depicts the result of this fit for $T_0 = 1.5$ K and $\dot{Q} = 5$ mW. Upon increasing $\dot{Q}$ to 6 mW, but holding the value of $\gamma_\ell$ fixed at $(2.84 \pm 0.02) \times 10^8$, no new value of $L_e$ could be found to similarly
fit our results over any range in $r$. To fit the data at 6 mW, $\gamma_L$ had to be increased by about 50% to $(4.37 \pm 0.02) \times 10^6$. In Figure 3.23, the fits for these two values of $\gamma_L$ at 6 mW are compared, where the respective values of $L_e$ paired with each $\gamma_L$ were chosen to fit the data at $r_A$. Using $\gamma_L = 2.84 \times 10^6$ obviously no longer yields an adequate fit to the data at any other position. Increasing $\gamma_L$ recoups a reasonably close fit for small $r$, but $\gamma_L$ should remain constant. It is also interesting to note that a comparable fit at small $r$ to the “closest fit” achieved with the Geurst model (e.g. that shown in figure 3.22) can be obtained from the Schwarz model simply by decreasing $c_L$ by 16%.

The integration routine used in applying the Geurst model was thoroughly checked for accuracy and sensibility. Inputting the line density at the channel exit as the boundary condition to iteratively map out $L(r)$ while decreasing $r$ yielded the same result as did starting at the entrance and increasing $r$. By varying the integration step size, it was found that a step size of 0.01 cm was sufficient at large $r$, but for $r < 6$ cm the step size was decreased by an order of magnitude to 0.001 cm to ensure that the temperature correction of helium parameters, performed every 10 steps, was carried out in small enough increments in $r$. As for parameters internal to DIVPRK, setting the maximum number of steps to PARAM(4) = 5000 and the tolerance to TOL = .0001 was found to work better than using the default values of 500 and 0.001, respectively. As a final check on my assumptions, I estimated the size of the neglected $\nabla^2 L$ term in comparison to the $(\nabla L)^2/L$ term kept. For the “closest fit” obtained at a given
Figure 3.22: Fit to temperature profile data at $T_0 = 1.5K$ and $\dot{Q} = 5 \, \text{mW}$. Parameters used are $L_e = 9.8 \times 10^6 \, \text{cm}^{-2}$ and $\gamma_e = 2.84 \times 10^9$. 
Figure 3.23: Fits to temperature profile data at $T_0 = 1.5K$ and $\dot{Q} = 6$ mW. Dashed line: $\gamma_t = 2.84 \times 10^6$ and $L_e = 1.7 \times 10^7$ cm$^{-2}$. Solid line: $\gamma_t = 4.37 \times 10^6$ and $L_e = 1.43 \times 10^7$ cm$^{-2}$. 
\( \dot{Q} \), \( \nabla^2 L \) was of the same order of magnitude, but somewhat smaller than, \( 2(\nabla L)^2 / L \).

More important, these two terms exhibited nearly the same \( r \)-dependence.

### 3.5.3 A Problem with the Geurst Model

As stated above, simply multiplying \( B \) by \( \phi = -1 \) does not correctly reverse the direction of \( v_\ell \) into equation 3.29. The vector equation for \( (v_\ell - v) \), equation 3.24, may be written as

\[
|v_\ell - v| \dot{\hat{w}}_\ell = \left( \alpha' I_\| - \frac{\rho n}{\rho} \right) |v_n - v_s| \hat{w}_n + (1 - \alpha') I_\ell \beta S L^{1/2} \dot{\hat{w}}_\ell ,
\]

(3.43)

where \( \hat{w}_n \) is a unit vector in the direction of \( (v_n - v_s) \). Following Geurst, the parameter \( \varepsilon \equiv \hat{w}_\ell \cdot \hat{w}_n \) is introduced, so \( \varepsilon = +1 \) corresponds to \( (v_\ell - v) \) being in the direction of \( \hat{w}_n \), and \( \varepsilon = -1 \) to \( (v_\ell - v) \) antiparallel to \( \hat{w}_n \). The magnitude of \( (v_\ell - v) \) may then be expressed as

\[
|v_\ell - v| = \varepsilon \left( \alpha' I_\| - \frac{\rho n}{\rho} \right) |v_n - v_s| + (1 - \alpha') I_\ell \beta S L^{1/2} .
\]

(3.44)

Reversing the direction of \( v_\ell \) is effected by changing \( \varepsilon \) from \( +1 \) to \( -1 \). The magnitude of \( v_\ell \) is changed by this reversal in the direction of \( v_\ell \). In other words, it is impossible to simply reverse the direction of the vortex line drift velocity without affecting its magnitude as well. To represent \( v_\ell \) flowing opposite to \( v_n \), it is not sufficient to simply multiply \( B \) by \( \phi = -1 \). The magnitudes of both coefficients \( B \) and \( C \) are altered by reversing the direction of \( v_\ell \). Inserting the above expression for \( |v_\ell| \) into \( B \) and \( C \), and properly accounting for vector directions in the linear term \( \nabla \cdot (Lv_\ell) \),
these coefficients become (compare equations 3.30 and 3.31)

\[
B = \epsilon \left[ |v_t| + (1 - \alpha') I_t L^{1/2} \left( \frac{\beta_S}{2} - \frac{\kappa}{8\pi} \right) \right]
\]

\[
= \epsilon^2 \left( \alpha' I_{||} - \frac{p_n}{\rho} \right) |v_n - v_s| + \epsilon (1 - \alpha') I_t L^{1/2} \left( \frac{3\beta_S}{2} - \frac{\kappa}{4\pi} \right) \quad (3.45)
\]

and

\[
C = \frac{\beta_S}{\gamma} L^2 - \left[ \frac{\beta}{2\gamma} |v_t - v| - \epsilon (1 - \alpha') \frac{I_t \beta_S}{\tau} \right] L^{3/2}
\]

\[
= \frac{\alpha I_t \beta_S}{c_L} \frac{L^3}{3} - \epsilon \left[ \frac{\alpha |v_n - v_s| - (1 - \alpha') \beta_S}{\tau} \right] I_t L^{3/2} \quad . \quad (3.46)
\]

The relations between Schwarz and Geurst parameters given in equations 3.20 and 3.21 have been used to derive the final expression for \( C \). An overall factor of \( \epsilon \) multiplies \( B \) to account for the relative direction between \( v_t \) and \( \nabla L \), just as \( B \) was earlier multiplied by \( \phi \); however, the term in \( B \) involving \( |v_n - v_s| \) contains another factor of \( \epsilon \), so this term does not in fact reverse in sign since \( \epsilon^2 = +1 \). Thus changing the direction of \( v_t \) affects the magnitude of \( B \), not just its algebraic sign. Likewise, the magnitude of \( C \) changes when \( v_t \) is reversed. For \( \epsilon = -1 \), \( B \) becomes always negative, and \( C \) always positive, over the ranges of absolute temperature, heat current, and position of interest.

In addition to correcting \( B \) and \( C \), the expressions for \( F_{ns} \) and \( \nabla T' \) in program GEURST must be corrected to read (compare to equations 3.37 and 3.39)

\[
F_{ns} = \rho_s \kappa \alpha [I_{||} |v_n - v_s| - \epsilon I_t \beta_S L^{1/2}] L \tilde{w}_n \quad (3.47)
\]

and

\[
\nabla T' = -\frac{\kappa \alpha}{S} [I_{||} (v_n - v_s) - \epsilon I_t \beta_S L^{1/2}] - 2 \left( \frac{\rho_n \rho_s}{\rho^2 S} \right) \frac{V_{ns}^2}{\tau} \tilde{w}_n \quad . \quad (3.48)
\]
The case of $\varepsilon = +1$ corresponds exactly to the analysis performed for $\phi = +1$, so no correction to the previous analysis is needed for the four classes of solution with $\phi = +1$. The other four classes, with $\phi = -1$, warrant reexamination in light of the corrected expressions for $B$, $C$, and $\nabla T'$ above for $\varepsilon = -1$.

At the point where the reanalysis for the case of $\varepsilon = -1$ was about to be performed, I discovered a very serious logical inconsistency in the ability to interpret the Geurst model in terms of the Schwarz model. The difficulty first came to light upon closely examining equation 3.44 for the magnitude of $(v_\varepsilon - v)$. The left-hand side of the equation, which is just $|v_\varepsilon - v|$, is by definition positive definite. Geurst states that a definite flow state for $(v_\varepsilon - v)$ will only be found when the right-hand side is also positive definite, and when a series of inequalities involving the parameters $\gamma$ and $\gamma_{ij}$ are also met. The inequalities in question must be satisfied in order to ensure that entropy production is non-negative. These inequalities which must be upheld are given in equation 2.24 in [Geu92], which says:

$$
\gamma \geq 0 , \quad \gamma_{11} \geq 0 , \quad \gamma_{22} \geq 0 ,
$$

$$
\gamma_{11} \gamma_{22} - \gamma_{12}^2 \geq 0 ,
$$

$$
\gamma_{11} + 2\gamma_{12} + \gamma_{22} \geq (\gamma_{11}^{1/2} + \gamma_{22}^{1/2})^2 .
$$

There is nothing obviously wrong with requiring that any of these relations hold or with the Geurst model per se. The problem arises only when the Geurst parameters are related to the corresponding set of Schwarz parameters and evaluated at a given temperature.
For the case of \( \varepsilon = +1 \), and provided that \( L \) is not very large (below about \( 10^8 \text{ cm}^{-2} \)), the right-hand side of equation 3.44 turns out to be negative over most of the superfluid temperature range, yet it should always be positive definite. This paradox could be avoided simply by allowing only those solutions \( L(r) \) which render equation 3.44 sensible, thereby ensuring that \( v_t \) assumes a definite flow state. The parameters in the inequalities of equation 3.49 depend only on temperature, therefore each inequality either is or is not satisfied at a given temperature. No ancillary restriction imposed on the line density can restore one of these inequalities, should it be violated at some temperature.

Turning our attention to the first of the inequalities, \( \gamma_{11} \) is found to be negative for temperatures above \( T_0 = 1.26 \text{ K} \). The full expression for \( \gamma_{11} \) in terms of the Schwarz parameters is (see equation E.6 in Appendix E)

\[
\gamma_{11} = \frac{\alpha}{1 - \alpha'} \left[ \left( \frac{\rho}{\rho_*} \right)^2 (1 - \alpha'')(I_{||} - 1) - \frac{\rho}{\rho_*} + \frac{\beta}{2c_L} \right] \tag{3.50}
\]

where

\[
\frac{\beta}{2c_L} = \left[ (\alpha' - \alpha'')I_{||} - \frac{\rho_n}{\rho} + \alpha'' + (1 - \alpha')I_{cL} \right]^{-1}. \tag{3.51}
\]

The second-order friction coefficient \( \alpha'' \) is indeed very small, and may be neglected. In the limit that \( T \to 0 \), \( \alpha' \) also approaches zero, and \( \rho_n/\rho \to 0 \). It is this low temperature limit that Geurst professes to consider when he evaluates these coefficients in the final section of [Geu92]. In the limit \( T \to 0 \), the Geurst parameters do satisfy all of the inequalities listed in 3.49. It is my opinion that Geurst takes this low temperature limit incorrectly, however, because he sets the modified friction coefficients\(^3\) \( \alpha'_* = \)

\(^3\text{rendered in the mass flow frame rather than the superfluid rest frame—see equation E.1 in Appendix E}\)
\( \alpha''_v = 0 \) while inserting the values of the Schwarz coefficients at a finite temperature. Geurst gives one example, using the values of the Schwarz parameters at 1.26 K: \( c_L = 0.0729, I_{\parallel} = 0.749, \) and \( I_\ell = 0.442, \) as given in [Sch88] for the higher flow velocity, from which he obtains \( \gamma_{11} = 0.97. \) He takes this result to be a reasonable estimate of the true value of \( \gamma_{11} \) at 1.26 K. In fact, setting \( \alpha'' = 0 \) but retaining the value of \( \alpha' = 0.0125 \) used by Schwarz at 1.26 K yields \( \gamma_{11} = 10.3 \) instead, quite a different result. Only in the limit \( T \to 0 \) is it true that \( \alpha'_v = \alpha' = 0. \) It is inappropriate to set this friction coefficient to zero at higher temperatures!

The value of \( \gamma_{11} \) depends largely on the sign and value of \( \beta/2c_L. \) Above 1.3 K, the subtraction of \( \rho_n/\rho \) drives \( \beta/2c_L \) negative. As the temperature decreases, at some point the denominator in \( \beta/2c_L \) passes through zero, and for lower temperatures \( (T \leq 1.26 \text{ K}) \) \( \beta/2c_L \) is positive. As a function of decreasing temperature, \( \gamma_{11} \) is at first negative, diverging to \(-\infty\), then abruptly switches to being positive and decreasing from \(+\infty\). The temperature at which \( \gamma_{11} \) changes sign is just above \( T = 1.26 \text{ K}. \) At 1.3 K, using \( \alpha' = 0.014 \) given in Table I of [Sch85], I obtain \( \beta/2c_L = -1550 \) and \( \gamma_{11} = -57. \) The slightly lower temperature of 1.26 K yields \( \beta/2c_L = +341 \) and \( \gamma_{11} = +10.3. \)

Similarly, the value of \( \gamma \) can turn out to be negative, but only at intermediate temperatures. Setting \( \alpha'' = 0, \) the expression for \( \gamma \) reduces to

\[
\gamma = \frac{c_L \left( \alpha' I_{\parallel} - \frac{\rho_n}{\rho} \right)}{\alpha I_\ell \left[ \alpha' I_{\parallel} - \frac{\rho_n}{\rho} + (1 - \alpha') I_\ell c_L \right]}, \tag{3.52}
\]
At higher temperatures \((T \geq 1.3 \text{ K})\), both the numerator and denominator are negative, so \(\gamma\) is positive. In the \(T = 0\) limit, \(\gamma = 0\). At some intermediate range in temperature, at least between \(1-1.26\) K, \(\gamma\) is negative. Of the five inequalities in equation 3.49, only \(\gamma_{12} \geq 0\) and \(\gamma_{22} \geq 0\) are upheld over the entire range of superfluid temperatures.

It seems that Geurst was a little overzealous in taking the low temperature limit and setting \(\alpha'\) to zero instead of using the experimentally determined value. He therefore does not recognize the ensuing difficulty with evaluating his parameters at higher temperatures. In fact, in a recent preprint \([\text{Geu95}]\) Geurst argues that the values of \(\alpha\) and \(\alpha'\) are not well known, and the values of \(c_L\), \(I_L\), and \(I_{||}\) have been computed by Schwarz, and are therefore known, at only very few discrete temperatures. He implies that the degree of uncertainty in these quantities justifies his peculiar method of taking the low temperature limit to evaluate his parameters. In the preprint in question, he uses the same 1D three-fluid model of inhomogeneous turbulence under discussion here to predict the formation of steady propagating turbulent-laminar fronts. His derivations are limited by the assumption that \(\gamma = 0\) (the low temperature limit), yet he computes front velocities for \(T = 1.26\) K and \(T = 1.62\) K, and compare these predictions to data taken at \(1.4\) K. He consistently uses this incompatible mixture of Schwarz parameters at finite temperatures with the \(T = 0\) limit of the expressions for his parameters, creating in the process a mathematical "mixed metaphor." Let us examine more closely the issue of how well each of these parameters is known as a function of temperature.
Although the Schwarz parameters $c_L$, $I_{||}$ and $I_\perp$ have only been determined by simulations at a handful of temperatures within 1.07–2.15 K, each proves to be a smooth, gradual function of temperature, at least below 2 K. It seems eminently reasonable to interpolate the Schwarz parameters at intervening temperatures in this range. The interpolation procedure used is discussed at length in Section 3.4.1, and graphs of the resulting interpolations are included in Appendix B.

Granted, $\alpha'$ is an experimentally measured quantity, and its precise value at any temperature is subject to some uncertainty. It is usually small in comparison to $\alpha$, and in [Sch85] Schwarz states that "it is generally an excellent approximation to set $\alpha' = 0." He made this statement in the context of considering the respective weighting of the terms in $\alpha$ and $\alpha'$ in the equation of motion of a point on a vortex line (see equation 1.33). In simulations performed for diagnostic and intercomparison purposes, Schwarz often sets $\alpha' = 0$, but for the final simulations of vortex motion, he retains $\alpha'$. It does not follow that $\alpha'$ may be neglected in an expression in which $\alpha$ does not even enter. The expressions for $F_n$, and $\nabla T'$ do not depend on $\alpha'$. On the other hand, $\gamma_{||}$ and $\gamma$, and hence the coefficients $A$, $B$ and $C$ which determine the line density in the Geurst model, depend critically on $\alpha'$ and temperature, as well as on $\alpha$.

The friction coefficients $\alpha = \rho_n B/2\rho$ and $\alpha' = \rho_n B'/2\rho$, where $B$ and $B'$ are the Hall-Vinen coefficients, have been measured in a number of experiments. The values of these parameters are known to within at least 10–20% for any given temperature above 1 K. The review article by Barenghi et al. [Bar83] summarizes many of these
results. The important point to be made is that any uncertainty in $\alpha'$ simply contributes to the uncertainty in knowing the exact temperature at which $\beta$, and hence $\gamma_{11}$, changes sign. At sufficiently low temperatures, $\gamma_{11}$ is assuredly positive, but above some finite temperature within the superfluid range, $\gamma_{11}$ will just as assuredly be negative. Likewise, $\gamma$ will be negative over some intermediate temperature range.

The exact temperature at which $\gamma_{11}$ and $\gamma$ change sign is not the issue; it is the fact that these parameters are negative at any temperature—indeed, over a significant range of temperatures—which creates the problem. Being that the inequalities in equation 3.49 are necessary to ensure that entropy production is non-negative, a fundamental requirement of the Geurst model is thereby violated. To continue trying to apply the Geurst model in light of this discovery seemed senseless, so the reanalysis for the case of $\varepsilon = -1$ was not completed.

A private communication from this author regarding this matter prompted Geurst to write a new appendix to his preprint [Geu95], in which he demonstrates that all the inequalities in equation 3.49 will be simultaneously satisfied providing the following inequality involving $\alpha''$ is satisfied:

$$\alpha'' \geq \frac{\rho - \alpha' I_{\parallel}}{1 - I_{\parallel}}.$$  \hspace{1cm} (3.53)

In this appendix, Geurst states that this inequality "implies that taking ... $\alpha'' = 0$ at temperatures above 1.3 K is at variance with the condition of non-negative entropy production."

Unfortunately, this restriction on the value of $\alpha''$ cannot be justified in light of the best experimental evidence, which indicates that $\alpha''$ is most likely identically zero.
over the full superfluid temperature range. Several experiments have attempted to
measure \( \alpha'' = \rho_n B''/2\rho \), where \( B'' \) is the axial mutual friction coefficient, by measuring
the attenuation of a second sound wave propagating parallel to the vortex lines in
a rotating annulus of helium, with various and conflicting results. One group found
\( B'' = 0 \) [Sny66], whereas the others measured different non-zero values of \( B'' \) for
the same temperature. This body of experimental evidence is reviewed in a recent
paper by Swanson and Donnelly [Swa93], who conclude that \( B'' \) is in fact zero, and
the experiments finding some finite value for \( B'' \) were instead observing spurious
attenuation from residual vorticity transverse to the axial direction. Unless another
explanation can be found to resolve these experimental measurements, one which
allows \( \alpha'' \) to satisfy equation 3.53 at finite temperatures, the Geurst model is rendered
invalid. The fact that at least one experimental group found \( \alpha'' = 0 \) seems damning
enough, since it is difficult to imagine how they could have managed to measure zero
axial attenuation if the mutual friction in the axial direction is large.

I note in closing that the criterion whereby the Geurst parameters must satisfy the
set of inequalities listed in equation 3.49 in order for the model to hold meaning must
be a subtle point. Although the Geurst model never produced behavior in agreement
with our data, in all the cases tried and over the entire range of parameters tested,
spanning many orders of magnitude, no behavior was generated that stood out as
anomalous or indicative that something was amiss in the underlying equations or
values of parameters. Moreover, for those cases where the Geurst model predicted
a line density which decreased gradually with \( r \), nothing particularly alarming or
nonsensical was discovered in the mutual friction or temperature gradient. True, it is possible to generate a line density which decreases smoothly through zero to negative values, but this was judged to be a mathematical solution which bore no physical meaning. Operationally, the line density was set to zero for all \( r \) beyond the point where \( L \) became negative. Likewise, if the values of parameters chosen made \( L(r) \) very small overall, the inertial term in \( \nabla T' \) would dominate and \( \Delta T' \) would turn out negative, but this behavior only occurred when the solution \( L(r) \) was much too small to even come close to agreeing with the data.

In the end, the kinematic equation for the vortex line density which incorporates the possibility of a large gradient in \( L \) is found to be unable to produce any agreement with the data better than that already achieved with the Schwarz model, at least when applied with the simplifying assumptions made (see equation 3.27). One robust limiting behavior of this equation in fact reproduces the functional form of the line density generated by the Schwarz model of homogeneous turbulence. The Geurst model may one day prove useful in describing the situation of an extremely large gradient in the line density, such as would be realized if a sharp turbulent front were present, for example in a pulsed heat experiment or at the channel entrance or exit in a steady flow. For the experimental situation realized within our diverging channel, where the line density presumably varies by several orders of magnitude over the channel length, but the variation is smooth and gradual in comparison to the formation of a front, the Geurst model does not offer any additional insight. Because fundamental criteria are violated, the interpretation of the Geurst model in
terms of the Schwarz microscopic simulation is at best suspect at all but very low temperatures. Unless a satisfactory response to this last concern exists, the Geurst model is effectively rendered useless to our purposes, above and beyond being found to yield predictions not in accord with our data.

3.6 A Modified Line Density

Since none of the existing theoretical models adequately describe our results, we present an empirical fit that succinctly summarizes our data. The following modification to the vortex line density captures the general features of our observations and constitutes a three-parameter fit to the data. The modified line density is comprised of two pieces. The first part consists of the line density predicted by the Schwarz model of homogeneous turbulence, scaled by a multiplicative factor $g_0$. The Schwarz line density is used in the local uniformity approximation, as given by equation 3.14, and will here be denoted as $L = L_h$, the "local homogeneous" line density. The descriptor "homogeneous" merely labels $L_h$ as being generated by the homogeneous model$^4$ and is not meant to imply that $L_h$ is spatially homogeneous. Quite the contrary, $L_h$ depends on position $r$, hence it retains a "local" character. A polynomial in $\dot{Q}$ is added to the local homogeneous piece to form the total modified line density:

$$L = g_0 L_h + g_1 \dot{Q} + g_2 \dot{Q}^2 .$$

$^4$"Homogeneous model" refers to the Schwarz model of homogeneous turbulence. Throughout this section, the Schwarz model applied in the local uniformity approximation, as described in Section 3.4, will be referred to as the "local homogeneous model."
The three adjustable parameters $g_0, g_1, \text{and} g_2$ which have been introduced depend on temperature only, so this second contribution to $L$ is spatially homogeneous. The modified line density is inserted in the Schwarz expression for $\nabla T'(r)$ given in equation (3.17).

For each reservoir temperature, a unique set of parameters $g_0, g_1 \text{and} g_2$ which provide excellent agreement with our data was identified. Although these parameters depend on temperature only, it is possible that enfolded in their values is some "hidden" dependence on the specific geometry of our experiment. What is meant by "hidden" is some factor which remains fixed throughout the experiment, such as the opening angle $\theta$ or the channel height $h$. Therefore it should not be construed that these parameters are universal functions of temperature such that the same parameters could be used to describe the results for a different experimental geometry. Likewise, the exact form of the modified line density above should not necessarily be expected to describe inhomogeneous flows in other geometries or flow states other than thermal counterflow. Nevertheless, any forthcoming model of inhomogeneous turbulence, or any generalization or alteration of existing theory purporting to address nonuniform flows, ought to yield a line density in general agreement with the above form when applied to counterflow in this diverging rectangular geometry.

The addition of the polynomial function in $\dot{Q}$ to the homogeneous line density adjusts both the $\dot{Q}$- and $r$-dependence of $\nabla T'(r)$ appropriately. Since $L$ is multiplied by $V$ in forming $\nabla T'$, and $V = C\dot{Q}/r$, the functional form of $\nabla T'$ becomes

$$\nabla T' = -F(T) \left[ g_0 C^2 \left( \frac{c_L}{g} \right)^2 \frac{\dot{Q}^3}{r^3} + g_1 \frac{\dot{Q}^2}{r} + g_2 \frac{\dot{Q}^3}{r} \right] \hat{r},$$

(3.55)
where all other temperature and geometric dependence are contained in $C$ and $F(T)$. Specifically, $C$ and $F(T)$ are given by

$$ C = \frac{\rho C_n}{\rho_*} = \frac{1}{\rho_* ST \theta h} $$

(3.56)

and

$$ F(T) = \frac{\kappa \alpha}{S} \left( I_{||} - c_L I_z \right) C. $$

(3.57)

At sufficiently small $r$, this modified temperature gradient is dominated by the local homogeneous contribution, which varies as $r^{-3}$. For $r$ less than about 4 cm, the $r$-dependence of $L_h$ follows that of the data, but the local homogeneous model results in a temperature difference that is always much too large in magnitude at small $r$ (see figures 3.14–3.17, or equivalently figures 3.18–3.21). To scale back the contribution of the local homogeneous piece, $L_h$ is multiplied by $g_0$, where $0 < g_0 < 1$. Adding the polynomial in $\dot{Q}$ to $L_h$ introduces an $r^{-1}$ dependence in $\nabla T'$ which allows the bimodal dependence on $r$ observed in the temperature profile to be fit reasonably well throughout the channel. Upon integrating $\nabla T'$ from $r_H$ to $r$, the resulting $\Delta T'(r)$ becomes a polynomial in $\dot{Q}$ with a quadratic as well as cubic term. Together these two terms act as the two highest order terms in the expansion of $(\dot{Q} - \dot{Q}_0)^3$. At all reservoir temperatures and for any position $r_i$ measured, the linearized graph of the measured temperature difference $\Delta T'(r)$ as a function of $\dot{Q}$ extrapolates to a positive $\dot{Q}$-intercept. Figures 3.8–3.13 for $T_0 = 1.5$ K, and the corresponding data displayed in Appendix C for the other reservoir temperatures, all clearly exhibit this behavior.

The added quadratic piece, $\Delta T' \propto g_1 \ln(\tau_H/\tau) \dot{Q}^2$, accounts for the non-zero $\dot{Q}$-intercept, provided that $g_1 < 0$. The new cubic piece $\Delta T' \propto g_2 \ln(\tau_H/\tau) \dot{Q}^3$ combines with
the local homogeneous piece, also cubic in $\dot{Q}^3$, therefore increasing $g_2$ increases the slope of $\Delta T'(r_4)^{1/3}$ versus $\dot{Q}$.

### 3.6.1 Determining the Fit Parameters

Program SCHWARZ integrates the modified line density version of the temperature gradient (equation 3.55) by asking the user to input values of $g_0$, $g_1$, and $g_2$. The integration routine incorporates the temperature variation of all other parameters, as already described in Section 3.4.1. The following procedure was used to obtain the values of $g_0$, $g_1$, and $g_2$ which best fit the data.

It is easiest to begin the fitting procedure using the temperature difference measured across the entire channel. The first step is to find combinations of values for the parameters $g_0$, $g_1$, and $g_2$ which match the slope and intercept in the linearized plot for this first data series, $\Delta T'(r_4)^{1/3}$ versus $\dot{Q}$. Picking an initial value of $g_0$ in the range 0.25–0.75, and temporarily fixing $g_1$ to some arbitrary value, program SCHWARZ is run for various $g_2$ values and the results compared to the data. The value of $g_2$ providing the best match to the slope of the data is selected. Next, an appropriate value of $g_1$ must be found for this preliminary combination of $g_0$ and $g_2$. Since $g_1$ controls the $\dot{Q}$-intercept of the curve, $g_1$ is varied until a reasonable fit to the intercept is found. Varying $g_1$ does affect the slope slightly, so the values of $g_2$ and $g_1$ are iteratively fine-tuned until a satisfactory overall fit is achieved for the particular $g_0$ value in question. The whole procedure is then repeated, starting with
a new value of $g_0$. The new combination of $g_2$ and $g_1$ yielding the best fit for the new $g_0$ is identified.

From the first data series, the "candidate pairs" of $(g_0, g_2)$ values are thereby matched to their $g_1$ "running mates" to form candidate trios of parameters. Using only the information contained in the first data series, a unique trio cannot be identified. Measurements involving at least one other position in the channel must be used to sort through all possible trios and identify the "winning ticket" which best fits all the data at a given reservoir temperature. Said another way, the first data series maps out a linear region in $(g_0, g_1, g_2)$ parameter space. Another data series maps out a different line in parameter space, and the trio of parameters being sought is found at the intersection of the two lines. The second data series, measuring $\Delta T''(r_D)$, was used to identify the best-fit trio at each temperature.

The results of this fitting procedure are illustrated here for $T_0 = 1.5$ K. The temperature differences $\Delta T''(r_A)$ and $\Delta T''(r_D)$ predicted by four candidate trios are compared to the data in figures 3.24-3.27. As can be clearly seen in part (a) of these figures, all four trios yield equally good agreement with the first data series, measuring $\Delta T''(r_A)$. Of the four trios, only the one shown in figure 3.26 also yields good agreement with the second data series, measuring $\Delta T''(r_D)$. This combination of $g_0 = 0.51$, $g_1 = -7 \times 10^4$ mW$^{-1}$cm$^{-2}$, and $g_2 = 4.8 \times 10^4$ (mW cm)$^{-2}$ provides the best overall fit to all the data at $T_0 = 1.5$ K.

The excellent agreement in the $\dot{Q}$-dependence of this fit with the data from the first and second data series, for $r_A$ and $r_D$, has already been illustrated in figure 3.26.
Figure 3.24: First candidate trio at $T_0 = 1.5$ K: $g_0 = 0.30$, $g_1 = -8 \times 10^4$ mW cm$^{-2}$, $g_2 = 8.5 \times 10^4$ (mW cm)$^{-2}$. The $Q$-dependence of this fit is compared to measurements of (a) $\Delta T'(r_A)$ and (b) $\Delta T'(r_D)$. 
Figure 3.25: Second candidate trio at $T_0 = 1.5$ K: $g_0 = 0.40$, $g_1 = -8 \times 10^4$ mW$^{-1}$cm$^{-2}$, $g_2 = 7.0 \times 10^4$ (mW cm)$^{-1}$. The $\dot{Q}$-dependence of this fit is compared to measurements of (a) $\Delta T'(r_A)$ and (b) $\Delta T'(r_D)$. 
Figure 3.26: Third candidate trio at $T_0 = 1.5$ K: $g_0 = .51$, $g_1 = -7 \times 10^4$ mW$^{-1}$cm$^{-2}$, $g_2 = 4.8 \times 10^4$ (mW cm)$^{-2}$. The $\dot{Q}$-dependence of this fit is compared to measurements of (a) $\Delta T''(r_A)$ and (b) $\Delta T''(r_D)$. 
Figure 3.27: Fourth candidate trio at $T_0 = 1.5$ K: $g_0 = 0.60$, $g_1 = -6 \times 10^4$ mW$^{-1}$cm$^{-2}$, $g_2 = 3.0 \times 10^4$ (mW cm)$^{-2}$. The $\dot{Q}$-dependence of this fit is compared to measurements of (a) $\Delta T'(r_A)$ and (b) $\Delta T'(r_D)$. 
To complete the picture, this fit is also compared to the measurements for the remaining positions \( r_B, r_C, r_E \) and \( r_F \) in figures 3.28–3.31, respectively. For comparison, the modified line density fit (solid line) is contrasted with the prediction of the local homogeneous model (dashed line). The modified line density fit is obviously not perfect at these other positions, as it only approximately reproduces the measurements. Nevertheless, at each of these other positions, the deviation from the data is much smaller for the modified line density than for the local homogeneous model. The modified line density provides a significant improvement over the local homogeneous model, in that it captures the essential functional dependence on \( \dot{Q} \) exhibited by the data, and closely reproduces the temperatures \( T'(r) \) measured throughout the channel.

Examining the temperature profile to focus on the \( r \)-dependence at a single heat current is also instructive. Figure 3.32 compares the same four candidate trios to the temperature profile measured at \( \dot{Q} = 5 \) mW. For comparison, the result of using the local homogeneous line density alone, scaled to fit \( \Delta T'(r_A) \) for \( \dot{Q} = 5 \) mW, is displayed in curve (5). Although scaling \( L_h \) by \( g_0 = 0.70 \) (setting \( g_1 = g_2 = 0 \)) appears to fit the temperature profile data fairly well at small \( r \), this fit bears the wrong \( \dot{Q} \)-dependence, so it does not agree with the data at other \( \dot{Q} \). Figure 3.33(a) compares the \( \dot{Q} \)-dependence of \( \Delta T'(r_A) \) for this scaled local homogeneous line density to the data. The agreement at \( r_A \) might seem reasonably good, although this scaled-\( L_h \) fit does not match the slope of the data quite as well as does the modified line density fit shown in figure 3.26(a). This scaled local homogeneous model does not agree at all with the data at position \( r_D \), however, as is clear from figure 3.33(b). As has already
Figure 3.28: $\dot{Q}$-dependence of $\Delta T'(r_B)$ for $T_0 = 1.5$ K. Solid line: modified line density with the best-fit trio of parameters $g_0 = 0.51$, $g_1 = -7 \times 10^4$ mW$^{-1}$ cm$^{-2}$, $g_2 = 4.8 \times 10^4$ (mW cm)$^{-2}$. Dashed line: local homogeneous model.
Figure 3.29: $\dot{Q}$-dependence of $\Delta T'(r_C)$ for $T_0 = 1.5$ K. Solid line: modified line density with the best-fit trio of parameters $g_0 = 0.51$, $g_1 = -7 \times 10^4$ mW$^{-1}$cm$^{-2}$, $g_2 = 4.8 \times 10^4$ (mW cm)$^{-2}$. Dashed line: local homogeneous model.
Figure 3.30: $\dot{Q}$-dependence of $\Delta T'(r_E)$ at $T_0 = 1.5$ K. Solid line: modified line density with the best-fit trio of parameters $g_0 = .51$, $g_1 = -7 \times 10^4$ mW$^{-1}$cm$^{-2}$, $g_2 = 4.8 \times 10^4$ (mW cm)$^{-2}$. Dashed line: local homogeneous model.
Figure 3.31: \( \dot{Q} \)-dependence of \( \Delta T'(r_F) \) at \( T_0 = 1.5 \) K. Solid line: modified line density with the best-fit trio of parameters \( g_0 = .51, g_1 = -7 \times 10^4 \) mW\(^{-1}\)cm\(^{-2}\), \( g_2 = 4.8 \times 10^4 \) (mW cm\(^{-2}\)). Dashed line: local homogeneous model.
Figure 3.32: \( r \)-dependence of \( \Delta T'(r) \) at \( \dot{Q} = 5 \, \text{mW} \) for \( T_0 = 1.5 \, \text{K} \). Line densities used are: (1)–(4) modified line density with four candidate trios of parameters (solid lines) and (5) scaled local homogeneous line density (dashed line).
Figure 3.33: $\dot{Q}$-dependence predicted by the scaled local homogeneous line density ($g_0 = 0.70$, $g_1 = g_2 = 0$) for $T_0 = 1.5$ K, shown for positions: (a) $\Delta T'(r_A)$ and (b) $\Delta T'(r_D)$
been discussed at length in Section 3.4.2, simply scaling the local homogeneous line density by a constant does not alter its $\dot{Q}$- or $r$-dependence and therefore cannot fit the data.

Returning to the $r$-dependence of the modified line density fit, all four trios provide a good fit to $\Delta T'(r_A)$, so curves (1) through (4) in figure 3.32 converge at $r_A$. The effect of adding the polynomial in $\dot{Q}$ to $L_h$ is to increase $\Delta T'(r)$ at large $r$. This increase in $\Delta T'(r)$ is primarily controlled by the value of $g_2$. As $g_2$ is increased, $g_0$ must be correspondingly decreased to maintain the fit at $r_A$. The role of $g_1$ diminishes in importance as $\dot{Q}$ increases.

The temperature profile data with the dominant $r$-dependence divided out are displayed in figures 3.34–3.37 for heat currents of 3 mW through 6 mW, respectively. The modified line density fit is shown with a solid line; for comparison the local homogeneous model (as previously displayed in figures 3.18–3.21) is shown with a dashed line. Again, the very good agreement of the modified line density fit with the data at $r_A$ and $r_D$ is evident. Perhaps more clearly than in figures 3.28–3.31 or figure 3.32, one can see here that the modified line density fit tracks the overall $r$-dependence much better than does the local homogeneous model over the entire range of heat currents measured.

The best-fit trio of parameters for each reservoir temperature is listed in Table 3.3. This parametric fit works as well at the other reservoir temperatures as has been illustrated here for $T_0 = 1.5$ K. Our measurements at any reservoir temperature can therefore be quickly and easily reconstructed with reasonable accuracy by using the
Figure 3.34: Temperature profile at $\dot{Q} = 3$ mW for $T_0 = 1.5$ K. Solid line: modified line density using best-fit trio of parameters. Dashed line: line density from the local homogeneous model.
Figure 3.35: Temperature profile at $\dot{Q} = 4$ mW for $T_0 = 1.5$ K. Solid line: modified line density using best-fit trio of parameters. Dashed line: line density from the local homogeneous model.
Figure 3.36: Temperature profile at $\dot{Q} = 5 \text{ mW}$ for $T_0 = 1.5 \text{ K}$. Solid line: modified line density using best-fit trio of parameters. Dashed line: line density from the local homogeneous model.
Figure 3.37: Temperature profile at $\dot{Q} = 6$ mW for $T_0 = 1.5$ K. Solid line: modified line density using best-fit trio of parameters. Dashed line: line density from the local homogeneous model.
Table 3.3: Best-fit values of parameters $g_0$, $g_1$, and $g_2$ at each reservoir temperature

<table>
<thead>
<tr>
<th>$T_0$ (K)</th>
<th>$g_0$ (mW$^{-1}$cm$^{-2}$)</th>
<th>$g_1$ (mW$^{-2}$cm$^{-2}$)</th>
<th>$g_2$ (mW$^{-2}$cm$^{-2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>0.50 ± 0.02</td>
<td>($-8.0 \pm 2.0$) $\times 10^4$</td>
<td>($1.4 \pm 0.1$) $\times 10^5$</td>
</tr>
<tr>
<td>1.4</td>
<td>0.49 ± 0.01</td>
<td>($-7.0 \pm 1.0$) $\times 10^4$</td>
<td>($8.6 \pm 0.2$) $\times 10^4$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.51 ± 0.01</td>
<td>($-7.0 \pm 0.5$) $\times 10^4$</td>
<td>($4.8 \pm 0.2$) $\times 10^4$</td>
</tr>
<tr>
<td>1.6</td>
<td>0.47 ± 0.01</td>
<td>($-6.0 \pm 0.5$) $\times 10^4$</td>
<td>($2.6 \pm 0.1$) $\times 10^4$</td>
</tr>
<tr>
<td>1.7</td>
<td>0.39 ± 0.01</td>
<td>($-4.0 \pm 0.5$) $\times 10^4$</td>
<td>($1.5 \pm 0.1$) $\times 10^4$</td>
</tr>
</tbody>
</table>

appropriate trio of parameters from Table 3.3 in the modified line density formula (equation 3.54). The measurements of $\Delta T''(r_A)$ and $\Delta T''(r_D)$ can be rendered with great precision, and the temperature differences at other positions $r$ approximated with only small systematic variations from the actual measured values, provided that the same procedures as were used here are followed in the computations. Several subtleties of the modified line density calculation are crucial to exactly duplicating the fits attained herein. The reader interested in using the contents of Table 3.3 to reconstruct our measurements is directed to the discussion of these details in Section 3.6.2 below. In addition, this section describes the effects of possible variations in the computational procedure, how the computation was checked for errors, and the
method used to assign the uncertainties listed in Table 3.3. The reader uninterested in these fine points may wish to skip Section 3.6.2.

Although we hesitate to assign physical significance to the form of the modified line density, it is interesting to note that since the polynomial $g_1 \dot{Q} + g_2 \dot{Q}^2$ contains no $r$-dependence, this piece of the modified line density is a constant everywhere in the channel for any given heat current $\dot{Q}$. It is as if the local homogeneous line density, which is determined by the local velocity, is superimposed on a constant or average background line density. In fact, spatially averaging the local homogeneous part of the line density $g_0 L_h$ over the channel volume yields a result about equal, at each $\dot{Q}$, to this added constant piece. It is worth pointing out that in the case of a uniform flow, the local homogeneous line density is itself constant throughout the channel, and therefore would be indistinguishable from a constant background line density. In the modified line density formula, $L_h$ is scaled by $g_0$. It would be entirely consistent with the data for uniform flows to similarly rescale the local homogeneous line density, then add to it a spatially averaged contribution to make the total line density once again equal to the original $L_h$ predicted by the Schwarz model. Although it would be presumptuous to push this speculation too far, the fact that in a uniform flow one cannot distinguish between a vortex line density governed solely by the local velocity field and a line density representing some sort of spatially averaged background level bears some consideration.
3.6.2 Details of the Modified Line Density Calculation

The fit parameters listed in Table 3.3 may be used to reconstruct our measurements over any range in \( r \), providing some care is taken to reproduce a calculation identical to that executed by program \textsc{SCHWARZ}. The code of program \textsc{SCHWARZ} is given in Appendix B, but the reader wishing to replicate the computation performed by this program should be aware of the following features and assumptions built into the calculation.

First, equation 3.54 yields a negative \( L \) over all or part of the range in \( r \) whenever the negative \( g_i \) term dominates, namely when \( \dot{Q} \) is small enough. As such a result is clearly nonphysical, whenever the calculation yields a negative line density at a particular \( \dot{Q} \), the line density is set equal to zero for that position and all subsequent larger \( r \).

Second, the temperature dependence contained in the turbulent temperature gradient (equation 3.55) is self-consistently incorporated into the calculation. The values of the temperature-dependent parameters contained in \( F(T) \) and \( C \) are iteratively adjusted to the local temperature \( T'(r) \) as the integration routine maps out \( \Delta T'(r) \), starting from the wide (cold) end of the channel. Specifics of this integration procedure have already been described in detail in Section 3.4.1. Without making this adjustment for the temperature variation in the channel, a reasonable fit to the data can still be achieved with the modified line density expression (equation 3.54), but with values of \( g_0, g_1, \) and \( g_2 \) significantly different from those listed in Table 3.3; however, the fit ceases to track the data at sufficiently large \( \dot{Q} \).
Although the fit parameters $g_0$, $g_1$, and $g_2$ also vary weakly with temperature, as can be seen in Table 3.3, this latter dependence is by necessity neglected in the temperature adjustment of the integrand. The fit parameters $g_0$, $g_1$, and $g_2$ had to be held constant during the computation, since their temperature dependence was not known a priori: the whole purpose of the fitting procedure was to identify the appropriate values of these parameters at each temperature.

Last, in iteratively calculating the value of $\beta_s$ at each $r$ and $\dot{Q}$, a choice must be made as to whether the total modified line density $L$, or only the local homogeneous piece $L_h$, is used to determine $\beta_s$. The resulting value of $\beta_s$ in turn determines the value of $L_h$. Assuming that the modified line density equation in fact predicts the actual line density, and does not simply represent an "effective" line density which describes a new dissipation mechanism in familiar language, it seems most reasonable to determine $\beta_s$ from the modified line density $L$, rather than from $L_h$ alone. This choice is therefore the one adopted in program SCHWARZ. This choice results in somewhat different values of $\beta_s$, and hence $L_h$, than were previously obtained in the local homogeneous model. To the extent that this procedure alters the magnitude of $L_h$ from the local homogeneous case, it is somewhat simplistic and therefore misleading to claim that all that has been done here is to multiply $L_h$ by a simple constant and add to it a polynomial in $\dot{Q}$. In other words, the actual rescaling of the local homogeneous line density $L_h$ is the combined effect of multiplying by $g_0$ and adjusting $\beta_s$. 
The modification to the local homogeneous model has been introduced through
the line density; the dependence of the local temperature gradient on the line density
remained unaltered. As a check on this methodology, the order of events was reversed,
and the local homogeneous line density $L = L_h$ was used to calculate $\beta_S$ at each $\dot{Q}$ and
$r$. In this latter approach, the line density is therefore not altered at all from the result
predicted by the local homogeneous model. The modification was instead introduced
directly in $\nabla T'$, by scaling the homogeneous temperature gradient (resulting from
$L_h$) by $g_0$ and adding to it the same polynomial in $\dot{Q}$ as before, to once again obtain
equation 3.55 for $\nabla T'$. This latter approach therefore modifies the expression for $\nabla T'$
in the same manner as before, but does not introduce the modification through $L$.
The only distinction in practice between these two approaches lies in the values of
$\beta_S$ computed and inserted in equation 3.55. An equally good fit to the data can still
be achieved, but with different values of $g_0$, $g_1$ and $g_2$ than are listed in Table 3.3.
Whether the modification to the local homogeneous model is introduced directly in
the line density or only in the temperature gradient is immaterial inasmuch as either
way this method is no more than a parametric fit to the data, but it seems more
sensible to modify the line density from the local homogeneous expression than to
alter the basic relation $\nabla T' \propto L V$.

To ascertain the uncertainties listed in Table 3.3 for the parameters $g_0$, $g_1$ and
g_2, two of the parameters were held constant while the third was varied. The range
of uncertainty was determined by eyeballing the respective fits to the data produced
on the linearized graphs of $\Delta T'(r_A)$ and $\Delta T'(r_D)$. The upper and lower bounds of
uncertainty for the parameter in question were drawn at the point where the fit first deviates visibly and noticeably from the data, lying outside of the error bars assigned to the measured temperature difference. Care was taken to simultaneously match the slope and magnitude of the data as accurately as possible, and over the widest possible range in $\dot{Q}$, at both positions $r_A$ and $r_D$. Consideration of any of the factors described below would significantly increase the uncertainties determined solely on this basis.

First, if all three parameters were varied simultaneously instead of each being varied individually to determine the bounds of uncertainty, those bounds would have to be drawn more generously. Roughly speaking, if one allowed the other two parameters to take on their upper or lower limiting values listed in Table 3.3 instead of being held fixed on the centers of their respective ranges, the uncertainty in the third parameter would have to be doubled in size. The pair $(g_0, g_2)$ provides one exception to this observation, since it is the combination of $g_0$ and $g_2$ which determines the slope. For the value of $g_1$ listed in Table 3.3, increasing either $g_0$ or $g_2$ to its upper limit while decreasing the other to its lower limit will produce a fit as good as the best-fit trio, so allowing that combination of variations of $g_0$ and $g_2$ would not necessitate increasing the uncertainty listed for $g_1$.

Second, the measurements at the various reservoir temperatures extend over different ranges in $\dot{Q}$, and to different maximum $\Delta T'(r_A)$. As $\dot{Q}$ increases and with it $\Delta T'(r_A)$, the deviation from cubic dependence on $\dot{Q}$ due to the temperature variation of the thermodynamic parameters grows more noticeable. Although the calculation
performed by program SCHWARZ also incorporates this temperature variation, it is
certainly true that if, for example, the data sets at 1.5 K were truncated to \( \dot{Q} \leq 5 \text{ mW} \)
instead of using the entire data set, which extends to \( \dot{Q} = 10 \text{ mW} \), slightly different
values of the fit parameters than are listed in Table 3.3 would have been obtained
from the truncated data. Although any such variation of the fit parameters prob­
ably lies well within the uncertainties listed in the table, it should be realized that
no clear criteria of equivalence between data sets at different temperatures can be
drawn. Should data sets extending to the same temperature difference across the
entire channel be considered comparable? Or to the same heat current? The highly
nonlinear nature of the turbulent temperature gradient and the strong temperature
dependence of \( F(T) \) in equation 3.55 make assigning such equivalences between data
sets at different reservoir temperatures difficult.

Last, if it were required that the uncertainties be increased enough to encompass
the measurements at \( r_B, r_C, r_E, \) and \( r_F \), the uncertainties would have to be consider­
ably more generous. No numerical estimate was made of just how much larger these
uncertainties would need to be, nor is there any sense in providing such a determi­
nation. The modified line density fit accomplishes two objectives. First, it reproduces
the gross, overall behavior seen in the data at all temperatures, as a function of both
the position \( r \) and the drive parameter \( \dot{Q} \). The reproduction of the data sets at
these other positions is admittedly crude. What is really needed is a proper model of
inhomogeneous turbulence that would better predict the actual inhomogeneous line
density and temperature gradient, not just quantify their gross behavior. Relaxing
the bounds of uncertainty on the modified line density fit would accomplish nothing toward this worthy goal; however, such action would severly dilute the second achievement of this parametric fit. It must be emphasized that the modified line density fit accurately reproduces the two data series at $r_A$ and $r_D$. Without question, all turbulent measurements at these positions are recreated by this fit, and the uncertainty bounds are tightly drawn to ensure that these data could be reconstituted with as little uncertainty as possible, should any future researcher wish to compare our results to a new theory or new measurements.

The modified line density fit constitutes our final effort to understand the inhomogeneous turbulent state. One last piece of information about the turbulent state remains to be gleaned from the data. Because measurements were made at several positions in the channel, it is possible to examine the transition to turbulence in this nonuniform flow. The final section of Chapter III describes our observations of a stationary turbulent-laminar front.

### 3.7 A Stationary Turbulent Front

The measurements of the temperature profile along the channel length provide insight into one final question: how does the onset regime for superfluid turbulence in a nonuniform flow compare to that in a uniform flow? At heat currents just above the minimum needed to sustain turbulence, we observe the formation of a stable and stationary laminar/turbulent front within our diverging channel. At such moderate values of $\dot{Q}$, a total temperature difference in excess of the laminar value is measured
at small r, whereas at larger r, $\Delta T(r) = \Delta T_L(r)$, indicating that at some intermediate position in the channel the flow undergoes a transition from a turbulent to the laminar state. Not unexpectedly, at the minimum $\dot{Q}$ for which the turbulent state persists, only a small region of the flow near the narrow end of the channel—where the local velocity is largest—is turbulent. As the heat current is increased, the laminar/turbulent front moves out to larger r, until eventually the front reaches the wide end and the entire channel is filled with turbulence.

The critical value of the heat current at which the front reaches each probe location $r_i$ is identified by carefully examining the graph of $\Delta T(r_i)$ versus $\dot{Q}$ for each position probed. Viewing the data for a particular position $r_i$, the critical heat current is, roughly speaking, the heat current at which the temperature signal becomes indistinguishable from that of the laminar state. Determining $\dot{Q}_c$ in our nonuniform channel geometry is somewhat more difficult than in a uniform channel experiment for the following obvious reason. Just above $\dot{Q}_c$ in a uniform channel, the entire channel is filled with a low level of turbulence, and the measured signal $\Delta T$ is considerably larger than it would be were only laminar flow present. Right at $\dot{Q}_c$ this turbulence vanishes throughout the channel, and a discrete and highly reproducible step can often be discerned between the turbulent and laminar states. By contrast, for any position $r$ in our diverging channel, as $Q_c(r)$ is approached from above the laminar/turbulent front is moving towards $r$, so very near $Q_c$ most of the flow beyond $r$ is laminar, and only a diminishingly thin slice beyond $r$ is turbulent. As $\dot{Q}$
is decreased, the measured signal $\Delta T$ smoothly approaches $\Delta T_L$, as the contribution $\Delta T'$ from this shrinking turbulent slice becomes vanishingly small. The greater difficulty in determining $\dot{Q}_c$ is reflected in the size of the error bars assigned.

The data set measuring $\Delta T(r_{rA})$ at $T_0 = 1.5$ K, which has already been displayed in figures 3.1 and 3.3, is replotted here as figure 3.38 to illustrate the procedure used to identify $\dot{Q}_c$. First, every data point in each data run is classified as belonging either to the turbulent or laminar state. As can be clearly seen in the single data run displayed in figure 3.3, the hysteretic nature of the transition to turbulence usually makes this task easy for the data gathered while $\dot{Q}$ is first increased from zero, since the flow usually remains in the laminar state well beyond $\dot{Q}_c$. Occasionally, the transition to turbulent flow will occur as soon as the heat current exceeds $\dot{Q}_c$. An example of this behavior can be seen in the close grouping of three data points (solid circles) which lie just above the laminar state data in the vicinity of $\dot{Q} = 0.65$ mW. In such a case, one must decide at what point the flow within the portion of the channel under observation has entered the turbulent state.

For data gathered while decreasing $\dot{Q}$, the main difficulty rests in deciding when the flow has reentered the laminar state. In the vicinity very near $\dot{Q}_c$, the turbulent state exhibits slightly irreproducible behavior, tracing out a marginally different path during each successive data run as it rejoins the laminar state. These small variations are sometimes more obvious than in figure 3.38, but at best are barely discernable above the natural scatter in the data. Nevertheless, they are sufficient to blur the location of $\dot{Q}_c$ slightly. This observed variability in the transition can be attributed
Figure 3.38: A sample data set showing how $\dot{Q}_c$ is determined. Open symbols designate laminar data and solid symbols turbulent data. Circles: data obtained while increasing $\dot{Q}$. Triangles: data taken while decreasing $\dot{Q}$. 

$T_0 = 1.5K$

- ○ Laminar
- • Turbulent
to the fluctuating nature of the turbulent state, which makes the exact location and degree of sharpness of the front hard to assess. To be counted as turbulent, a data point must lie well enough above $\Delta T_L$ that even within a generous assignment of uncertainty, the datum clearly does not belong to the laminar regime. To quantify this criterion, the uncertainty assigned to the best-fit laminar slope in Table 3.1 is doubled. If the data point is far enough above the best-fit laminar line to be outside this doubled uncertainty or beyond the experimental scatter of $\pm 0.001 \text{ mK}$, whichever is larger, it is classified as turbulent. When in doubt, the data point is classified as laminar.

Using only those data points identified under this criterion as turbulent, a smooth curve through the turbulent state data is extrapolated until it intersects the best-fit laminar line. The intersection marks the critical heat current $\dot{Q}_c$. The upper uncertainty limit on $\dot{Q}_c$ is set at a heat current for which at least one of the turbulent data points lies so far above the laminar data that the gap between the turbulent data point in question and the upper uncertainty bound for the laminar state exceeds the experimental scatter of $\pm 1 \mu\text{K}$. The lower uncertainty limit on $\dot{Q}_c$ is chosen to be the highest heat current for which all measurements unquestionably lie within the experimental scatter of the laminar regime. In figure 3.38, the laminar slope from Table 3.1 and a smooth curve through the turbulent data are shown with solid lines. To guide the eye, the generous uncertainty bounds for the laminar state are added with dotted lines. The critical heat current and its error bars, found to be $\dot{Q}_c = 0.55 \pm 0.05 \text{ mW}$, are marked with dashed vertical lines.
Table 3.4 lists the critical heat current $Q_c$ determined at each position $r_i$ for each reservoir temperature. Measurements were not made at all positions for all temperatures. Only those measurements made with the SQUID/thermocouple apparatus had sufficient resolution to accurately determine the critical heat current as described above. The second data series, which measured $\Delta T(r_A, r_D)$, provides a second direct measurement of the critical heat current at $r_A$. In order to deduce the critical heat current at $r_D$, the measurements of $\Delta T(r_A, r_D)$ were subtracted from fits to $\Delta T(r_A)$, yielding a derived measurement of $\Delta T(r_D)$. This subtraction procedure has been described fully in Section 3.2.4. The uncertainty reported for $Q_c$ at $r_D$ encompasses the additional uncertainty introduced by the subtraction. The critical heat current at $r_C$ is obtained directly from the third data series, measuring $\Delta T(r_C, r_D)$. Measurements
were made at this position at $T_0 = 1.4, 1.5, \text{ and } 1.6 \text{ K only. At } 1.4 \text{ K, the turbulent state measurements of } \Delta T(r_C, r_D) \text{ unfortunately did not extend to low enough } \dot{Q} \text{ to determine } \dot{Q}_c(r_C). \text{ The data gathered at } 1.6 \text{ K were sufficient to localize } Q_c(r_C) \text{ within reasonable bounds, but not to pinpoint the critical heat current more exactly within that range. The measurements at } r_E \text{ and } r_F \text{ cm were made with resistance thermometers, which are far less sensitive than the SQUID/thermocouple apparatus; therefore, only an upper bound for } \dot{Q}_c \text{ could be determined from these data sets. As a logical lower bound, } \dot{Q}_c \text{ must be larger at these locations than the value observed at smaller } r. \text{ Since the SQUID was not working correctly for the final data series, the only determination of } \dot{Q}_c \text{ at } r_B \text{ comes from the resistance thermometer at that position. This latter measurement proved too inaccurate to be of interest, given the much more precise SQUID measurements for the surrounding positions } r_A \text{ and } r_C. \text{ Rather than being directly measured as for the other positions, the critical heat current at the wide end of the channel, } r_H = 12.5 \text{ cm, is inferred from the linearized graph of } \Delta T'(r_D) \text{ versus } \dot{Q}, \text{ which exhibits a subtle but distinct change in slope in the region around } \dot{Q} = 1.4 \text{ mW, as shown in figure 3.39. For } \dot{Q} < 1.4 \text{ mW, the slope of the data is steeper than at large } \dot{Q}, \text{ and } \Delta T' \text{ is smaller, than would be estimated from an extrapolation of the high-} \dot{Q} \text{ behavior. This change in slope presumably occurs at the heat current for which the laminar/turbulent front is located at the wide end of the channel. Below } \dot{Q} = 1.4 \text{ mW, part of the channel is still in the laminar state and } \Delta T' \text{ is correspondingly smaller than if the entire flow were turbulent. Above } \dot{Q} = 1.4 \text{ mW, the entire channel is in the turbulent state.}
Figure 3.39: $\dot{Q}$-dependence of $\Delta T'(r_D)^{1/3}$ illustrating the change in slope at the heat current where the laminar/turbulent front exits the channel, at approximately $\dot{Q} = 1.4$ mW.
To test this interpretation of the change in slope, the presence of a laminar/turbulent front was simulated using the modified line density model described in the previous section. Although the exact critical condition causing the turbulent state to collapse is not known, a simple assumption is that turbulence cannot be maintained once the average relative velocity $V$ falls below some minimum critical value $V_e$. Adopting this criterion, the modified line density model was adjusted to arbitrarily set $L = 0$ for all positions $r$ for which $V$ was less than a chosen value $V_e$. Program SCHWARZ allows the user to input the desired value for the critical velocity. The modified line density model, supplemented in this manner by a front, yields a cusp in the slope of $\Delta T'^{1/3}$ versus $\dot{Q}$ at precisely the value of $\dot{Q}$ corresponding to the velocity at wide end of the channel being at $V_e$. Without a front, the model predicts a uniform slope$^5$ over the whole range in $\dot{Q}$.

Rather than a critical velocity, the condition determining where the laminar/turbulent transition occurs could be a minimum line density below which turbulence cannot be maintained. This criterion was also easily modeled with the modified line density formula, again resulting in a distinct change in slope in the linearized $\Delta T'$ graph similar to that generated by assuming a critical velocity. The code for program SCHWARZ listed in Appendix B allows the user either to input the critical velocity or elect to perform the integration without a critical velocity (i.e. without generating a front). Another version of the program instead allowed for a critical line density as the cutoff criterion. The result of choosing a minimum line density of

$^5$ with the exception of a gradual change in curvature at very small $\dot{Q}$ where the negative $g_{1}$ term gains in relative importance
$L = 25,600 \text{ cm}^{-2}$ is depicted in figure 3.40 with a solid line. This minimum condition is more appropriately reported in terms of the dimensionless quantity $L^{1/2}h = 4$, where $h = 0.025 \text{ cm}$ is the channel height. For comparison, the modified line density without a front is shown as a dashed line in figure 3.40. Setting a minimum line density produces a change in slope qualitatively similar to that produced by a critical velocity. The question of whether a minimum $L$ or $V$ provides the more appropriate criterion for sustaining turbulence is addressed below.

For a reasonable value of $V_e$ or minimum $L^{1/2}h$, this shift in slope is discernable in the modeled behavior of $\Delta T'(r)^{1/3}$ versus $\dot{Q}$ for any position $r$, but for small $r$ becomes a very subtle effect, too small to resolve in the data. That we do not observe a change in slope in the linearized graphs of $\Delta T'(r_A)$ or $\Delta T'(r_A, r_D)$ versus $\dot{Q}$ is explained by the highly nonlinear nature of the turbulent temperature gradient. The overwhelming contribution to the temperature difference across the entire channel is coming from the large $\nabla T'$ at the narrow end. Whether the flow farther down the channel is laminar or turbulent makes little difference since $\Delta T'(r_A)$ is so dominated by the action at the narrow end. For a range in position near the wide end of the channel, the laminar region beyond the front plays a larger relative role in reducing $\Delta T'$ and a more noticeable change in slope results. The cusp could only be distinguished in graphs of $\Delta T'(r_D)$. Since this data set is obtained by subtracting measurements of $\Delta T(r_A, r_D)$ from a fit to $\Delta T(r_A)$, generous error bars are assigned to the exact location of the cusp, and hence to $\dot{Q}_e$ at $r_H = 12.5 \text{ cm}$. Careful error analysis confirms that the change in slope is a real, albeit subtle, feature of the data and not some numerical
Figure 3.40: Effect of adding a laminar/turbulent front to the modified line density fit. Dashed line: no front. Solid line: front created where the line density reaches a minimum of $L^{1/2}h = 4$. 
artifact of the process used to extract \( \Delta T'(r_D) \) from the directly measured quantities. This painstaking analysis was only carried out fully for \( T_0 = 1.5 \text{ K} \), the temperature at which the most complete set of data had been gathered, since this data set presented the clearest opportunity to deduce \( \hat{Q}_c(r_H) \) without ambiguity.

Thermal counterflow in uniform channels exhibits a similar critical heat current corresponding to the laminar/turbulent transition. Whether the transition is from the T-II or the T-I state to laminar flow depends on the size and geometry of the flow channel, but in either case the turbulent state abruptly ceases at some well-defined \( Q_c \), below which only the laminar state is observed. In very long channels, the turbulence has been observed to originate in one location, but such a laminar/turbulent front is not stable, and quickly propagates through the channel [Men59]. Once the flow reaches a steady state, the entire channel is filled with turbulence. When possible to measure, the temperature gradient in such long channels has proven to be constant along the channel length, indicating that the turbulent state entered is homogeneous and no stationary laminar/turbulent fronts are present within the channel.

In only one experiment has a stable and stationary T-II/laminar front been directly observed in a uniform channel, in a flow regime removed from thermal counterflow. For a very narrow range of relative velocities in a regime where \( V_n \) and \( V_s \) are in the same direction, Slegtenhorst, Marees, and van Beelen observed such a stationary T-II/laminar front in one circular glass channel, but not in other channels of comparable geometry [Sle82a].
Restricting our attention to thermal counterflow, the following simple picture of the criterion determining the laminar/turbulent transition is consistent with all uniform channel data. The transition occurs at constant \( V_c d \), where \( d \) is the narrowest channel dimension. This observation implies that the line density reaches a critical minimum value below which the vortex tangle is not self-sustaining. Using the appropriate model for the T-I or T-II state, as applicable, the line density at \( V_c \) can be estimated. For a uniform flow, one cannot truly distinguish which is the more fundamental condition, that \( V \) or \( L \) reaches a critical minimum value, since these two criteria are essentially synonymous. A wide range of experiments in uniform channels are consistent with a critical condition of \( L^{1/2}d \simeq 2-3 \), independent of temperature, and regardless of whether the transition to laminar flow proceeds from a T-I or a T-II turbulent state [Tou82]. Since \( L^{-1/2} \) represents the average local radius of curvature of a vortex segment, thinking in terms of the simple geometry of a vortex ring yields the appealing picture that at \( \dot{Q}_c \), a vortex of average size would just "fit" within the channel in any orientation.

This experiment in a nonuniform channel offers the possibility of discerning which of the two quantities, \( V \) or \( L \), provide the more correct minimum condition. For our diverging flow, \( V_c h \) being constant does not correspond to constant \( L^{1/2}h \), since the local uniformity approximation does not hold and the local line density, as given by the modified line density formula, is not proportional to \( V^2 \). By using the modified line density formula, "isodensity" curves of constant \( L^{1/2}h \) may be predicted by finding, for a given heat current \( \dot{Q} \), the location \( r \) at which \( L \) (equation 3.54) reaches a
value corresponding to the chosen cutoff value of \( L^{1/2}h \). This procedure is at best questionable, since it presumes that the modified line density formula accurately characterizes the true local average line density at very low heat currents, just above the onset of turbulence. In defense of proceeding, estimating \( L \) at \( V_c \) from uniform channel data involves a comparable set of assumptions.

The critical heat current for \( T_0 = 1.5 \) K as a function of position is compared in figure 3.41 to \( \dot{Q}_c(r) \) predicted by assuming a critical velocity. The value \( V_c h = 0.072 \) cm\(^2\)/s is obtained from \( \dot{Q}_c = 0.55 \) mW measured at the narrow end of the channel. In figure 3.42 these same data are compared instead to the prediction of a critical line density. The isodensity curve of \( L^{1/2}h = 4 \) was selected as having the best overall agreement with the critical heat current data. Taking all the evidence together, including the inferred \( \dot{Q}_c \) at the wide end of the channel, our results clearly favor the interpretation that a minimum \( L^{1/2}h \) is the more fundamental criterion. Given the substantial differences apparent in the turbulent states involved, we consider our value of \( L^{1/2}h = 4 \) to be in excellent accord with the uniform channel result of \( L^{1/2}h \approx 2.5 \).

Last, the measurements of \( \dot{Q}_c \) at \( r_A \) are compared to the results of two experiments in uniform high-aspect-ratio rectangular channels, the closest uniform analogue to our diverging channel [Lad79, Yar79]. The temperature dependence of \( \dot{Q}_c \) for the diverging and uniform channels is displayed together in figure 3.43. The diverging channel data exhibit a monotonic temperature variation in good agreement with the uniform channel data. As for the way \( \dot{Q}_c \) increases with channel height \( h \), the diverging channel measurements fall in line amidst the sequence of uniform channel
Figure 3.41: The measured $r$-dependence of the critical heat current compared to $Q_c(r)$ predicted by assuming a critical velocity. The data for $T_0 = 1.5$ K are contrasted with a critical velocity of $V_c h = 0.072$ cm$^2$/s.
Figure 3.42: The measured $r$-dependence of the critical heat current compared to $\dot{Q}_c$ predicted by a minimum line density. The measurements at $T_0 = 1.5$ K agree well with $\dot{Q}_c(r)$ derived from a cutoff line density of $L^{1/2} \hbar = 4$. 
Figure 3.43: Temperature dependence of the critical heat current in uniform and diverging rectangular channels of aspect ratio $a =$width:height. Filled circles: diverging channel (present work). Open symbols: uniform channel data. Diamonds are data of Yarmchuck and Glaberson [Yar79] ($h = 0.092$ cm); all others are from Ladner and Tough [Lad79, Lad83].
results. This neat ordering of $Q_c$ by channel height is upset when a critical velocity is derived from the measured $Q_c$, whereupon $V_c h$ determined for the entrance of the diverging channel proves to be significantly larger than the respective quantity for any uniform rectangular channel of comparable aspect ratio, by as much as a factor of two. Taken together, the larger critical velocity and the agreement with $L^{1/2} h = 4$ instead of $L^{1/2} h = 2-3$ suggest that a larger minimum line density is needed to sustain turbulence in a diverging flow than in a comparable uniform flow. Nevertheless, the temperature dependence of $Q_c$ and its place in the ordering by channel height, together with the facts that the single turbulent state is akin to uniform T-II state and the transition to turbulence is hysteretic, all demonstrate that the inhomogeneous turbulent state in this diverging flow bears a striking similarity to the turbulent state observed in uniform rectangular channels.

The critical heat current data do locate the position of the front, but shed little light on the nature of the laminar/turbulent transition. Two major outstanding questions remain to be answered. First, how spatially well defined is the laminar/turbulent transition? It is very possible that instead of a sharp front, the line density decreases gradually to zero and the transition to laminar flow is spatially very broad. Even below $Q_c$, turbulence may still be present at some low level in the region of the channel under observation, but if the line density is too low the resulting dissipation would be immeasurably small. Second, is the turbulence in the vicinity of the transition in any way different from the bulk turbulent state? This question aims at discovering whether the laminar/turbulent transition in rectangular channels is the
same phenomenon as the laminar/T-I transition seen in low-aspect-ratio channels, or a transition directly between the T-II state and laminar flow. This latter question is presently being addressed by ongoing investigations, as described below.

Counterflow experiments in a nonuniform channel of circular cross section have uncovered surprising differences in the laminar/turbulent transition when the flow direction is reversed. Diverging as well as converging flow has been studied in this conical geometry by inverting the channel and therefore reversing the direction of the heat current [Cas95a, Cas95b, Cas96]. One important difference between the flows in this nonuniform circular channel geometry and the nonuniform high-aspect-ratio rectangular channel is immediately obvious: for either flow direction, two turbulent states, not one, are produced in the conical channel. These two states are the nonuniform extensions of the T-I and T-II states seen in uniform circular channels [Mur93a]. In fact, multiple metastable T-I states are seen in the conical channel, but only for the diverging flow direction [Cas95b]. For the converging direction, as in a uniform circular channel, a single T-I state is seen. By contrast, only the inhomogeneous analogue of the T-II state is manifest in the diverging rectangular channel, just as only the T-II state is observed in uniform high-aspect-ratio rectangular channels. A second remarkable clue to the nature of the laminar/T-I transition has now been added by studying both flow directions in the conical geometry [Cas95a]: the critical heat currents measured for converging and diverging flow are different!

The laminar/T-I transition exhibits the characteristics of a subcritical bifurcation, and therefore is described by a solution to the Ginzberg-Landau equation. This
application of stability analysis introduces a key new idea: the laminar/turbulent front propagates at an intrinsic velocity \( v_i \). This intrinsic velocity is a function of the magnitude of the relative velocity, and describes the rate at which the turbulent state either expands into, or contracts away from, the laminar state. The front is also subject to a second velocity: the vortex tangle moves at the vortex line drift velocity \( v_\ell \), which depends on the direction and magnitude of the relative velocity. The front velocity is the sum of the intrinsic and drift velocities. In a nonuniform channel, \( v_i \) and \( v_\ell \) are both strong functions of position, through their respective dependences on the relative velocity which drives the flow. A stationary front is therefore created at the position where these two velocities cancel each other. The observed asymmetry in \( \dot{Q}_c \) between converging and diverging flow is understood in terms of this simple picture, since the direction of the relative velocity controls the direction of \( v_\ell \). The position of the stationary front depends on the flow direction, therefore the front will reach the end of the channel at a higher heat current in one case than in the other.

Using previous observations of propagating fronts in uniform channels [Bha64] to fit the model parameters, this subcritical bifurcation model, developed by Castiglione et al. [Cas95a], provides a means to quantitatively analyze the difference in \( \dot{Q}_c \) measured for the cases of converging and diverging flow in the nonuniform circular channel.

As is true of the laminar/turbulent transition in uniform circular channels, the stationary front in the nonuniform circular channel marks a laminar/T-I transition. The stationary front in nonuniform rectangular channel appears to be a laminar/T-II
transition, the same type of laminar/turbulent transition observed in uniform rectangular channels. No evidence has ever been found of a T-I precursor to the T-II state in channels of rectangular cross section, either uniform or nonuniform. In uniform channels, the T-I state is believed to be an inhomogeneous state of low-level turbulence, a precursor to the T-II homogeneous state. The T-I state, which forms at low drive velocities, is commonly interpreted as being produced by a nonuniform cross-channel normal fluid velocity profile. Once the vortex line density grows large enough, it is believed that the strong coupling between the vortex tangle and the underlying velocity field forces the cross-channel velocity into a flat profile, making the flow in a uniform channel truly homogeneous. The T-I/T-II transition in circular channels is attributed to this annealing of the normal fluid velocity profile. The absence of the T-I state in the rectangular geometry has therefore never been understood. So strong is the belief that the T-I state must precede the T-II state that it is often proposed that the laminar/turbulent transition always involves the T-I state, but for some reason the T-I state is not stable in the rectangular geometry and immediately transforms into or precipitates the T-II state.

The interesting question to consider for the nonuniform rectangular channel geometry is whether or not a thin slice of the T-I state forms at the stationary front, acting as a kind of buffer separating the T-II state on one side of the front from laminar flow on the other. In other words, can the transition occur directly between the T-II state and laminar flow, or must the transition be precipitated by the T-I state? As
in uniform rectangular channels, the transition to turbulence in the diverging rectangular channel is hysteretic, with a metastable laminar state found well above $Q_c$. This hysteresis is the signature of a subcritical bifurcation, the same type of behavior seen in the laminar/T-I transition studied in circular channels. Does this similarity suggest that the T-I state is present even in rectangular channels, or only that the laminar/T-II transition is analogous to a first-order phase transition, just as is true for the laminar/T-I transition? These questions may soon be answered by experiments presently underway [Kli96] in which converging flow is being studied in the same nonuniform rectangular channel used in the present work to examine diverging flow. Preliminary results seem to indicate that, at least at high heat currents, the turbulent state is the same for converging and diverging flow, a finding which agrees with observations of the similar T-II state analogue in the nonuniform circular channel experiments. The crucial outstanding question is whether the critical heat currents for converging and diverging flow will differ or be the same, and what these results will indicate about the nature of the laminar/turbulent transition.

A model now exists which quantitatively predicts the critical heat current measurements in nonuniform circular channels as well as providing a reasonable and simple interpretation for observations of propagating fronts in uniform channels, which heretofore had defied explanation [Cas95a]. Our results for diverging flow in the nonuniform rectangular channel suggest that a minimum line density is necessary to sustain turbulence, but this observation alone offers no explanation as to why and how the laminar/turbulent transition occurs, or why a minimum threshold for the line
density should exist. Our results do favor a minimum line density over a critical velocity, but perhaps the true criterion which must be satisfied to sustain the turbulent state will be shown to have little to do with either of these concepts. The model of the laminar/turbulent transition now available provides testable predictions of where this transition occurs in a nonuniform channel. Perhaps the ongoing research following the present work will shed new light on these two persistent and related mysteries which have been the subject of so much speculation: the absence of a T-I state in high-aspect-ratio geometries and the nature of the laminar/turbulent transition.
CHAPTER IV

Conclusions

These investigations have established that inhomogeneous turbulence in a weakly diverging rectangular channel differs markedly from the homogeneous turbulent state (T-II) observed in uniform channels. The Schwarz model of homogeneous turbulence, so successful in describing the T-II state in uniform flows, when applied in the local uniformity approximation to this nonuniform flow predicts the wrong functional dependence to describe the observed dissipation in the inhomogeneous analogue of the T-II state.

An even more complicated hydrodynamic model of Geurst, purporting to treat the case of a nonuniform vortex line density existing in a uniform flow, yields predictions that differ only slightly from those of the Schwarz model, and likewise fails to characterize the experimental results. Furthermore, I discovered that evaluating the parameters in the 1-D Geurst model in terms of the Schwarz parameters and mutual friction coefficients leads to violation of several conditions requisite to upholding the Second Law of Thermodynamics. This finding is quite surprising, because the Geurst model is derived from such fundamental thermodynamic principles and conservation laws that it is not at all obvious how such an unphysical result is produced. The
Geurst model can be brought into accord with the Second Law providing the axial friction coefficient $\alpha''$ exceeds a minimum positive value which depends on temperature, yet the experimental evidence strongly suggests that $\alpha''$ is identically zero. I therefore conclude that this constraint renders the Geurst model unusable, at least in terms of obtaining definitive numerical predictions. Only some correction to or reinterpretation of either the Geurst model, the experimental record, or possibly the Schwarz model, could salvage the Geurst model.

A remarkably good parametric fit to the data is achieved by severely altering the line density from its expression in the case of uniform flow. Most compelling and most puzzling is the revelation that the modified line density fitting the data consists of two parts, a local piece (the homogeneous line density rescaled) and a nonlocal piece that does not have any spatial dependence, being a polynomial function only of the drive parameter $\hat{Q}$. The nonlocal piece may therefore be thought of as an average or background line density present throughout the channel. One might argue that this separation of $L$ into two pieces is artificial, just as is any expansion of one function into a sum of other functions. One might therefore caution against assigning too much meaning to the division. Until further experimental evidence is gathered, this modified line density must be considered an “effective” line density — it yields the correct behavior for $T(r)$ upon integration, but $L$ has not been measured directly. It must be emphasized that in the case of homogeneous turbulence in a uniform flow, there is no way to distinguish whether the line density has a solely local character or is comprised of two pieces, one depending on the local velocity and the other constant.
throughout the channel at any given heat current, since in a uniform flow these two pieces are indistinguishable.

My results underscore the need to simulate the vortex line distribution generated in a nonuniform velocity field. A comparison of the inhomogeneous superfluid turbulent state predicted by direct numerical simulation of this nonuniform flow to the experimental results is needed to determine how and why applying the homogeneous model locally is inadequate to the task of describing the observed dissipation. Because the functional dependences of the measured dissipation differ so markedly from the predictions of the homogeneous model used in the local uniformity approximation, I suspect that some new contribution to the underlying vortex dynamics must be at work in the inhomogeneous T-II state.

The vortex stretching which necessarily takes place as a vortex filament is transported from one region to another in a nonuniform flow is one mechanism not incorporated in the homogeneous model. The form of the new production term in the line density equation of state resulting from such vortex stretching can be predicted by simple arguments. Unfortunately, the resulting line density does not have the correct functional dependence to agree with the data, but this simple argument does not self-consistently couple the new stretching term to the vortex production present even in a uniform flow, namely the growth of line length due to the self-induced motion of a vortex filament. While a more careful treatment of vortex stretching might shed some light on the character of inhomogeneous turbulence, I do not believe that vortex
stretching contributes significantly to altering the T-II state so dramatically from the homogeneous case.

Also unlikely to account for the difference is any simple adjustment or extension of the parameters in the Schwarz model needed to extend it to nonuniform flow. Although the anisotropy of the tangle may be quite different from that seen in a uniform flow, and the Schwarz parameters might even become more complicated spatial functions, no longer depending solely on the local temperature, it is difficult to see how any extension of the homogeneous turbulence model to a nonuniform flow would alter the dependence of the line density on the local velocity drastically enough to agree with the observed inhomogeneous state.

Recent vortex simulations by Aarts and de Waele exhibit some interesting new features [Aar93, Aar94]. When they simulate the steady state line density in a uniform circular channel while constraining the normal fluid cross-channel profile to be parabolic, they find that the vortex line density is largest close to the walls, where the normal fluid velocity (and hence the relative velocity) is low. The spatial variation in the predicted line density exactly compensates for the variation in velocity so that the local mutual friction force, which depends on the product of the local line density \( L \) and the local relative velocity \( v \), remains essentially constant everywhere except very near the channel walls, where it decreases to zero. For uniform channel flows, the homogeneous model only asserts that \( \Delta T' \propto LV \), where \( V \) is the cross-channel average of \( v \), and \( L \) is constant on statistical average. The Aarts and de Waele simulations seem to suggest that this basic relation is even more robust, holding locally even in
situations where $L$ is not constant and does not depend on $v^2$. The situation they impose is somewhat artificial, in that the normal fluid velocity profile is not allowed to be influenced by the vortex tangle. A proper, self-consistent simulation of flow through a uniform channel, one that would solve for the normal fluid profile rather than imposing a fixed profile, has yet to be done. Nevertheless, the results obtained by Aarts and de Waele bear some similarity to my findings, a point which may require closer examination in the future.

According to the modified line density formula, $L$ is much larger than would be predicted by the homogeneous model ($L_h \propto V^2$) at the wide end of the channel, where $V$ is lowest, and correspondingly smaller than $L_h$ at the narrow end of the channel, where $V$ is highest. This situation bears at least a superficial resemblance to the simulation results of Aarts and de Waele, but the similarity may be deceptive. Since the relative size of the local homogeneous piece to the total modified $L$ is a strong function of both position and the heat current, whether the local value of $L$ is dominated by the local homogeneous piece or the nonlocal piece depends strongly on both $\tau$ and $\dot{\mathcal{Q}}$. Thus within some region in the channel, $L \propto V^2$, while elsewhere this relation no longer holds. This situation is quite different from that treated in the Aarts and de Waele simulations, which still pertain to flows that are uniform in the downstream direction.

Last, my data suggest that the laminar to turbulent transition is precipitated by the local velocity reaching a value sufficient to sustain a minimum line density. Unlike in a uniform flow, the velocity at which this critical line density is achieved
is not itself a constant, because of the complicated dependence of the line density on the local velocity and the underlying drive parameter $\hat{Q}$. A stationary turbulent-laminar front is clearly produced, but many outstanding questions remain about the character of this front. A higher line density is needed to sustain turbulence than is observed in uniform flows. I observe the critical condition to be $L^{1/2} h \approx 4$, as compared to $L^{1/2} h \approx 2.5$ for most uniform flows. This observation that the inhomogeneous turbulent state becomes unstable at a higher line density provides another hint towards understanding the underlying vortex dynamics.

One particularly interesting question is whether or not the transition to the T-II state proceeds directly from the laminar state, or is precipitated by a laminar/T-I transition. Since in low-aspect-ratio uniform channels, the laminar/T-I transition is hysteretic, behaving like a first-order phase transition, but the T-I/T-II transition is continuous, it has always been puzzling why the laminar/T-II transition seen in high aspect ratio channels is hysteretic. The suggestion has been made that the T-I state actually forms first, but for whatever reasons is not stable and gives way immediately to the homogeneous T-II state. One would like to probe the characteristics of the front region itself in a nonuniform flow, since a stationary front can be created and moved at will back and forth within a narrow spatial region. Even if one could achieve extremely good spatial resolution with, for example, two very closely spaced temperature probes, the turbulent temperature signal in the vicinity of $\hat{Q}_c$ is on the same order in size as the measurement fluctuations in the laminar signal. Furthermore, since it is not known how abrupt the T-II/laminar transition ought to be, any information gleaned
from even the most careful experimental observations of the front itself would be subject to interpretation as to whether it showed evidence for a small precursor T-I region, or simply a spatially broad transition directly to the T-II state. Much more promising is to experimentally determine only the gross behavior of the transition, such as the $\dot{Q}$-dependence of the front location, and deduce what these more easily measured features indicate about the transition to turbulence, based on a theoretical analysis of such transitions.

Future investigations using the same nonuniform rectangular channel will examine the case of converging rather than diverging flow. Although we expect the fully developed turbulent state will be largely unchanged, how the transition to turbulence will be affected by reversing the flow velocity is of great interest. Some significant differences have already been observed in the nature and stability of the front between the cases of diverging and converging flow in a nonuniform circular channel. Interpreting the behavior of these laminar/turbulent fronts is an ongoing focus of present and future investigations. I believe the question of understanding the transition to turbulence is closely linked to that of understanding the observed states of inhomogeneous turbulence, and how and why the inhomogeneous T-II state seen in this nonuniform rectangular channel, and in similar nonuniform circular channels at sufficiently high heat currents, is so different from the state of homogeneous superfluid turbulence.
Appendix A

The Elkouh Solution for the Laminar Temperature Difference

A.1 Adapting The Elkouh Solution to the Two-Fluid Model

Elkouh solves the Navier-Stokes equation for laminar source flow of a single viscous fluid with kinematic viscosity \( \nu = \eta / \rho \)

\[
(v \cdot \nabla) v = \frac{1}{\rho} \nabla p + \nu \nabla^2 v \quad \text{(A.1)}
\]

in the cylindrical geometry described in Section 3.3.2. In order to apply the Elkouh methodology to laminar flow in our channel, we must first reduce the two-fluid equations to a similar expression for the normal fluid velocity alone. For steady laminar flow, the two-fluid equations for the superfluid and normal fluid velocities become (see equations 1.12 and 1.13 in Chapter I, no mutual friction)

\[
(v_n \cdot \nabla) v_n = \nu_n \nabla^2 v_n - \frac{1}{\rho} \nabla p - \frac{\rho_s}{\rho_n} S \nabla T \quad \text{(A.2)}
\]

\[
(v_s \cdot \nabla) v_s = -\frac{1}{\rho} \nabla p + S \nabla T \quad \text{(A.3)}
\]

where the normal fluid kinematic viscosity is \( \nu_n = \eta / \rho_n \). Eliminating the temperature gradient term from equations A.2 and A.3, we obtain an expression involving only
the pressure gradient,

\[
(v_n \cdot \nabla) v_n + \frac{\rho_s}{\rho_n} (v_s \cdot \nabla) v_s = \nu_n \nabla^2 v_n - \frac{1}{\rho_n} \nabla p . \tag{A.4}
\]

In order to proceed further, one must obtain an expression for \(v_s\), which is possible if it can be assumed that \(v_s\) conforms to several reasonable conditions. From symmetry, no flow in the \(\phi\)-direction is expected (at least in the full \(2\pi\) geometry treated by Elkouh). Because the superfluid should not be affected by the presence of any walls, there should likewise be no flow of the superfluid in the azimuthal direction, therefore the superfluid velocity is assumed to have only a radial component: \(v_s = v_{sr} \hat{r}\). Expressing in cylindrical coordinates the condition that the superfluid is irrotational,

\[
\nabla \times v_s = 0 = \frac{\partial v_{sr}}{\partial z} \hat{\phi} - \frac{1}{r} \frac{\partial v_{sr}}{\partial \phi} \hat{\phi} , \tag{A.5}
\]

leads to the conclusion that \(v_{sr}\) depends only on \(r\). Finally, the condition that the superfluid is incompressible,

\[
\nabla \cdot v_s = 0 = \frac{1}{r} \frac{\partial}{\partial r} (rv_{sr}) , \tag{A.6}
\]

constitutes a simple differential equation for \(v_s\), which has the solution

\[
v_s = \frac{C_s}{r} \hat{r} . \tag{A.7}
\]

We may now use the above expression for \(v_s\) to compute the term involving \(v_s\) in equation A.4,

\[
(v_s \cdot \nabla) v_s = - \frac{C_s^2}{r^3} \hat{r} . \tag{A.8}
\]
Following Elkouh, the normal fluid velocity is assumed to be of the form

\[ v_n = u(r, z) \hat{r} + v(r, z) \hat{z} \]  
(A.9)

where again, by symmetry, no \( \hat{\phi} \)-component of \( v_n \) is expected, and the radial and azimuthal components \( u \) and \( v \) do not depend on \( \phi \). Equation A.4 now has the form of the Navier-Stokes equation presented in Elkouh. Writing out equation A.4 in the radial and azimuthal directions, respectively, yields

\[ \left( \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{z}} \right) - \frac{\rho_s \bar{C}_s^2}{\bar{\rho}_n \bar{r}^3} = - \frac{1}{\bar{\rho}_n} \frac{\partial \bar{p}}{\partial \bar{r}} + \nu_n \left[ \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \right], \]  
(A.10)

\[ \left( \frac{\partial \bar{v}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{z}} \right) = - \frac{1}{\bar{\rho}_n} \frac{\partial \bar{p}}{\partial \bar{r}} + \nu_n \left[ \frac{\partial^2 \bar{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{v}}{\partial \bar{r}} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} \right], \]  
(A.11)

where, following the notation used in Elkouh, an overbar has been added to indicate the dimensioned versions of quantities (dimensionless versions of these same quantities will soon be defined). The two-fluid nature of the flow is reflected only in the \( \left( \frac{\bar{C}_s^2}{\bar{r}^3} \right) \) term present in equation A.10. With the exception of this term, the equations above are identical to the Navier-Stokes equations for a single fluid solved by Elkouh (compare to equations 1 and 2 in [Elk70]). The equation of continuity becomes

\[ \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{r}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{z}} = 0. \]  
(A.12)

The boundary conditions are

\[ \bar{u}(\bar{r}, \pm a) = 0, \]  
(A.13)

\[ \bar{v}(\bar{r}, \pm a) = 0, \]
where the average radial normal fluid velocity is given by

\[
\langle u \rangle = \frac{\int_{-a}^{a} \bar{u} \, d\bar{z}}{\int_{-a}^{a} \, d\bar{z}} = V_n = C_n \frac{\dot{Q}}{\bar{r}} \quad (A.14)
\]

(For the definition of \( C_n \), see equation 3.6). We note that Elkouh allows for the possibility of porous walls, so that the boundary condition for \( \bar{v} \) used in [Elk70] is \( \bar{v}(\bar{r}, \pm a) = \pm V \), where \( V \) is the magnitude of the injection velocity through the walls.

We are concerned here only with the case of nonporous walls \( (V = 0) \), a fact which considerably simplifies not only the boundary conditions but much of the following derivation from the more general case treated by Elkouh. Defining a Reynolds number

\[
Re \equiv \frac{\bar{r} V_n}{\nu_n} = \frac{C_n \dot{Q}}{\nu_n} \quad (A.15)
\]

we can write the superfluid velocity in terms of \( Re \). By comparing the solution (equation A.7) for \( v_s \),

\[
V_s = \langle v_s \rangle = \frac{\bar{C}_s}{\bar{r}} \hat{r} \quad (A.16)
\]

(again, overbars have been added to indicate dimensioned quantities) to the condition of thermal counterflow (equation 1.7),

\[
V_s = - \left( \frac{\rho_n}{\rho_s} \right) V_n = - \left( \frac{\rho_n}{\rho_s} \right) \frac{\nu_n Re}{\bar{r}} \hat{r} \quad (A.17)
\]

we can now identify the constant \( \bar{C}_s \):

\[
\bar{C}_s = - \left( \frac{\rho_n}{\rho_s} \right) \nu_n Re = - \left( \frac{\rho_n}{\rho_s} \right) C_n \dot{Q} \quad (A.18)
\]

The radial and azimuthal equations can now be recast in terms of the dimensionless quantities.
We may now rewrite equations A.10 and A.11 in terms of these dimensionless variables:

\[
\begin{align*}
\left( u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} \right) &= -\frac{\partial p}{\partial r} + \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right] + \frac{C_s^2}{r^2}, \\
\left( u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} \right].
\end{align*}
\]

(A.20)

(A.21)

The equation of continuity in dimensionless variables is

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0
\]

(A.22)

with the boundary conditions now given as

\[
\begin{align*}
u(r, \pm 1) &= 0, \\
\psi(r, \pm 1) &= 0.
\end{align*}
\]

(A.23)

and

\[
\int_{-1}^{1} u \, dz = \frac{2 \text{Re}}{r}.
\]

(A.24)

Following Elkouh, we then introduce a dimensionless stream function \( \psi \) which satisfies the equation of continuity (equation A.22), defined by

\[
\begin{align*}
u &= \frac{1}{r} \frac{\partial \psi}{\partial z}, \\
\psi &= -\frac{1}{r} \frac{\partial \psi}{\partial r}.
\end{align*}
\]

(A.25)

The following power series expansions for \( \psi \) and the pressure are then assumed:

\[
\psi(r,z) = \text{Re} \left[ f_0(z) + \left( \frac{\text{Re}}{r^2} \right) f_1(z) + \left( \frac{\text{Re}}{r^2} \right)^2 f_2(z) + \cdots \right],
\]

(A.26)
\[ p(r, z) = h(z) + \Re \left[ h_0(z) \ln r + \left( \frac{\Re}{r^2} \right) h_1(z) + \left( \frac{\Re}{r^2} \right)^2 h_2(z) + \ldots \right] . \]

(A.27)

By substituting for \( \psi \) in equation A.25, the corresponding expansions for \( u \) and \( v \) are obtained (primes denote differentiation with respect to \( z \)):

\[
\begin{align*}
    u & = \frac{\Re}{r} \sum_{n=0}^{\infty} f_n' \left( \frac{\Re}{r^2} \right)^n , \\
    v & = \sum_{n=1}^{\infty} 2n f_n \left( \frac{\Re}{r^2} \right)^{n+1} .
\end{align*}
\]

(A.28)  (A.29)

Comparing these expansions for \( u \) and \( v \) to equations A.23 and A.24, respectively, determines the boundary conditions for the functions \( f_n \):

\[
\begin{align*}
    f_n'(\pm 1) & = 0 \quad n = 0, 1, 2, \ldots , \\
    f_n(\pm 1) & = 0 \quad n = 1, 2, 3, \ldots ,
\end{align*}
\]

(A.30)

and

\[ f_0(1) - f_0(-1) = 2 , \]

(A.31)

which upon choosing \( f_0(-1) = -1 \) gives \( f_0(1) = 1 \). By substituting the respective expansions for \( p, u, \) and \( v \) (equations A.27, A.28 and A.29) into equations A.20 and A.21 and equating coefficients of equal powers in \( r \), the Navier-Stokes equations are reduced to an infinite set of systems of simultaneous ordinary differential equations. The first four of these systems are given below:

**System I**

\[ h' = 0 \quad \text{or} \quad h = \text{constant} \]

(A.32)
System II

\[ h_0' = 0 \quad \text{or} \quad h_0 = \text{constant} \quad (A.33) \]
\[ f_0''' = h_0 \quad (A.34) \]

System III

\[ h_1' = 0 \quad \text{or} \quad h_1 = \text{constant} \quad (A.35) \]
\[ f_1''' = -2h_1 - (f_0')^2 - \left( \frac{C_s}{Re} \right)^2 \quad (A.36) \]

System IV

\[ f_1'' = \frac{Re}{2} h_2' \quad (A.37) \]
\[ f_2''' = 2f_0''f_1 - 4f_0'f_1' - \frac{8}{Re} f_1 - 4h_2 \quad (A.38) \]

Recall that the two-fluid nature of the He-II flow is manifested only by the \( \left( \frac{C_s^2}{r^3} \right) \) term appearing in equation A.20. Since this term consists of a single power in \( r \), this modification to the results obtained by Elkouh for an ordinary viscous fluid enters in to only one equation out of the above systems, namely equation A.36, which contains the additional term \( \left( \frac{C_s}{Re} \right)^2 \).

Integrating equations A.34, A.36, and A.38, respectively, and upon applying the boundary conditions expressed in equations A.30 and A.31 to evaluate the coefficients and constants of integration, we obtain the following solutions for \( f_0, f_1, \) and \( f_2 \):

\[ f_0(z) = \frac{1}{2} \left( 3z - z^3 \right), \quad (A.39) \]
\[ f_1(z) = \frac{3}{280} \left[ 5z - 11z^3 + 7z^5 - z^7 \right], \quad (A.40) \]
\[ f_2(z) = \frac{1}{431200} \left[ 5301z - 14780z^3 + 15246z^5 - 7524z^7 + 1925z^9 - 168z^{11} \right] \\
+ \frac{1}{2100} \left( \frac{1}{Re} \right) \left[ 93z - 236z^3 + 198z^5 - 60z^7 + 5z^9 \right]. \]  
(A.41)

In the process, we find the values of \( h_0, h_1, \) and \( h_2 \) to be:

\[
\begin{align*}
h_0 &= f_0''' = -3, \\
h_1 &= -\frac{27}{35} - \frac{1}{2} \left( \frac{C_s}{Re} \right)^2, \\
h_2 &= -\frac{78}{2695} + \frac{1}{700} \left( \frac{1}{Re} \right) \left[ 43 - 495z^2 + 525z^4 - 105z^6 \right]. 
\end{align*}
\]

(A.42)

We may now evaluate the dimensionless pressure (equation A.27). In doing so, we drop the \((1/Re)\) term in \( h_2 \). This simplification, which eliminates any \( z \)-dependence in the pressure to fourth order in \( 1/r \), is justified since \( z \in [-1,1] \) and at even the smallest \( \dot{Q} \) for which \( \Delta T_L \) becomes large enough to measure, \( Re \sim 100 \). The final remaining constant, \( h \), is evaluated by knowing the pressure at some position \( r \). Upon setting the pressure equal to \( p_0 \) at the wide end of the channel at \( r = r_H \), the pressure at other radial locations is

\[
p(r) = p_0 + 3Re \ln \left( \frac{r_H}{r} \right) + \left[ \frac{27}{35} + \frac{1}{2} \left( \frac{C_s}{Re} \right)^2 \right] Re^2 \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right) \\
+ \frac{78}{2695} Re^3 \left( \frac{1}{r_H^4} - \frac{1}{r^4} \right) + \cdots . \]  
(A.43)

Converting back to dimensioned variables using the relations A.19, the pressure is

\[
\bar{p}(r) = \bar{p}_0 + \frac{\rho_L \nu_L}{a^2} \left\{ 3 Re \ln \left( \frac{r_H}{r} \right) + \left[ \frac{27}{35} + \frac{1}{2} \frac{\rho_n}{\rho_s} \right] a^2 Re^2 \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right) \\
+ \frac{78}{2695} a^4 Re^3 \left( \frac{1}{r_H^4} - \frac{1}{r^4} \right) + \cdots \right\}. \]  
(A.44)

Finally, having determined the pressure, we may obtain the laminar temperature difference. Equation A.3 for the superfluid relates the temperature gradient to the
pressure gradient. Using equation A.8 to evaluate the velocity term, and allowing that to a very good approximation the pressure and temperature only depend on the radial coordinate \( \bar{r} \), the temperature gradient is given by

\[
\frac{dT}{d\bar{r}} = \frac{1}{\rho S} \frac{d\bar{p}}{d\bar{r}} - \frac{1}{S} \frac{\bar{C}^2}{\bar{r}^3},
\]  

(A.45)

which upon integrating yields the laminar temperature difference

\[
\Delta T_L(\bar{r}) = T(\bar{r}) - T_0 = \frac{1}{\rho S} \left( \bar{p}(\bar{r}) - \bar{p}_0 \right) - \frac{\bar{C}^2}{2S} \left( \frac{1}{\bar{r}_H^2} - \frac{1}{\bar{r}^2} \right).
\]  

(A.46)

Inserting the pressure difference from equation A.44 into the above expression, the laminar temperature difference becomes

\[
\Delta T_L(\bar{r}) = \frac{\rho_n \nu_n^2}{\rho S a^2} \left\{ 3 \frac{\bar{r}}{\bar{r}_H} \ln \left( \frac{\bar{r}_H}{\bar{r}} \right) + \left[ \frac{27}{35} - \frac{1}{2} \left( \frac{\rho_n}{\rho_s} \right)^2 \right] a^2 \bar{C}^2 \left( \frac{1}{\bar{r}_H^2} - \frac{1}{\bar{r}^2} \right) + \frac{78}{2695} a^4 \bar{C}^2 \left( \frac{1}{\bar{r}_H^4} - \frac{1}{\bar{r}^4} \right) + \cdots \right\},
\]  

(A.47)

where we have used the relation A.18 to evaluate \( \bar{C} \). Finally, substituting \( a = h/2 \) and using equation A.15 to evaluate \( \bar{C} \), we arrive at

\[
\Delta T_L(\bar{r}) = \frac{4\eta}{\rho S h^2} \left\{ 3 \frac{\bar{r}}{\bar{r}_H} \ln \left( \frac{\bar{r}_H}{\bar{r}} \right) C_n \hat{Q} + \frac{1}{2} \left[ \frac{54}{35} - \left( \frac{\rho_n}{\rho_s} \right)^2 \right] \frac{h^2}{4\nu_n} \left( \frac{1}{\bar{r}_H^2} - \frac{1}{\bar{r}^2} \right) C_n^2 \hat{Q}^2 \right. \\
+ \left. \frac{78}{2695} \left( \frac{h^2}{4\nu_n} \right)^2 \left( \frac{1}{\bar{r}_H^4} - \frac{1}{\bar{r}^4} \right) C_n^2 \hat{Q}^3 + \cdots \right\}.
\]  

(A.48)

In this final result, we have ceased using overbars to indicate dimensioned quantities in order for the notation in equation A.48 to agree with that used throughout Section 3.3 of Chapter III. The reader must bear in mind that the symbols \( r \) and \( r_H \) in equation A.48 refer to the dimensioned positions, heretofore denoted \( \bar{r} \) and \( \bar{r}_H \), not to their dimensionless counterparts.
The leading-order term in the above solution for $\Delta T_L$ arises from the viscous dissipation $\nu_n \nabla^2 v_n$ in the equation for the normal fluid (equation A.2). This term, logarithmic in $r$ and linear in $\dot{Q}$, is no more than the solution obtained for monodirectional flow between two infinite parallel plates. The major contribution to the laminar temperature difference therefore results from a parabolic cross-channel profile in $v_n$ forming between the closely spaced parallel walls. Indeed, the radial velocity obtained in the Elkouh solution is, to leading order (in dimensioned quantities)

$$u = \frac{3}{2} \left( 1 - \frac{z^2}{a^2} \right) \frac{C_v \dot{Q}}{r} \quad (A.49)$$

The origin of next higher-order term in $\Delta T_L$ is also readily identified, but the physics underlying this term, which is quadratic in $\dot{Q}$, is somewhat subtle and puzzling at first glance, so we explore the situation in some detail in the following section.

A.2 Radial Source Flow in an Infinite Cylindrical Geometry

To understand the nature of the term quadratic in $\dot{Q}$ in equation A.48, we consider the solution for the radial flow of heat from an infinitely long cylinder immersed in an infinite bath of He-II. In this simple cylindrical geometry with no boundaries (we consider only the external problem of $r$ greater than the cylinder radius) by symmetry the normal fluid and superfluid velocities must be purely radial, and cannot depend on $\phi$ or $z$. Applying the condition of incompressibility, $\nabla \cdot v_n = \nabla \cdot v_s = 0$, yields the following solutions for $v_n$ and $v_r$:

$$v_n = \frac{A}{r} \hat{r} \quad (A.50)$$
and

\[ v_\phi = \frac{B}{r} \hat{r}, \quad \text{(A.51)} \]

where \( A \) and \( B \) are related by the counterflow condition, \( \rho_\phi v_n = -\rho_\phi v_\phi \). Here the local velocity is everywhere the same at a given \( r \), so no distinction need be made between the local and average value of \( v_n(r) \). As before, the normal fluid velocity is determined by energy conservation, so we immediately identify the constants \( A \) and \( B \) to be

\[ A = \frac{\dot{Q}}{\rho ST 2\pi h} = C_n \dot{Q} \quad \text{(A.52)} \]

and

\[ B = -\frac{\rho_n}{\rho_\phi} A, \quad \text{(A.53)} \]

where \( \dot{Q}/h \) is the heat current per unit length driving the flow, and we have used the definition of \( C_n \) (equation 3.6) with the understanding that here \( \theta = 2\pi \). We may now compute the terms in equation A.4 from the above solutions for \( v_n \) and \( v_\phi \):

\[ (v_n \cdot \nabla) v_n = -\frac{A^2}{r^3} \hat{r}, \quad \text{(A.54)} \]

\[ (v_\phi \cdot \nabla) v_\phi = -\frac{B^2}{r^3} \hat{r}, \quad \text{(A.55)} \]

\[ \nabla^2 v_n = 0. \quad \text{(A.56)} \]

Since there are no boundaries to affect the normal fluid velocity field, the viscous term \( \nabla^2 v_n \) vanishes in this special geometry, and the pressure gradient is created solely by the inertial terms \( (v \cdot \nabla)v \). This situation is remarkably different from any flow in a channel or other bounded flow, where the contribution to the pressure gradient from these inertial terms is dwarfed by the viscous dissipation, or from a mono-directional
uniform flow, where all derivatives of the velocity vanish trivially. Here, the pressure gradient is

$$\frac{1}{\rho_n} \nabla p = \left( A^2 + \frac{\rho_s B^2}{\rho_n} \right) \frac{1}{r^3} \hat{r} .$$  \hspace{1cm} (A.57)

Although the normal and superfluid velocities are opposite in direction, both inertial terms, being quadratic in $v$, produce a positive pressure gradient. The superfluid equation relates this pressure gradient to the temperature gradient (see equation A.3):

$$\nabla T_L = \frac{1}{\rho S} \nabla p + \frac{1}{S} (v_s \cdot \nabla) v_s .$$  \hspace{1cm} (A.58)

Inserting $\nabla p$ into the above expression and evaluating $A$ and $B$, the temperature gradient becomes

$$\nabla T_L = \frac{1}{\rho S} \left( \rho_n A^2 + \rho_s B^2 - \rho B^2 \right) \frac{1}{r^3} \hat{r} .$$  \hspace{1cm} (A.59)

$$= \frac{\rho_n}{\rho S} \left( 1 - \rho_n^2 \right) \frac{C_n^2 \dot{Q}^2}{r^3} \hat{r} .$$  \hspace{1cm} (A.60)

Integrating, we obtain the temperature difference

$$\Delta T_L(r) = \int_{r_H}^r \nabla T_L(r') \, dr' = \frac{1}{2} \left[ 1 - \frac{\rho_n^2}{\rho_s^2} \frac{\rho_n}{\rho S} \left( \frac{1}{r_H^2} - \frac{1}{r^2} \right) C_n^2 \dot{Q}^2 .$$  \hspace{1cm} (A.61)

This temperature difference is identical to the terms quadratic in $\dot{Q}$ in the Elkouh and parabolic $v_n$ laminar solutions in all respects but one (compare to equation A.48, and equation 3.12 in Section 3.3.3). The only difference among these three second-order terms is the slight variation in the first numerical factor in square brackets, which has the value one here, and $54/35$ and $6/5$ in the Elkouh and parabolic solutions, respectively.
The inertial $v$, term in equation A.58 is negative, therefore the temperature gradient can be either positive or negative, depending on whether the positive pressure gradient or this negative inertial term dominates. Below 1.96 K, $T_L(r) < T_0$, and we see that with the outward radial flow of heat from a long cylinder, the temperature actually increases in the direction of heat flow. We have arrived at the paradox that heat is flowing from a cold to a hot region.

This counterintuitive result appears to violate the second law of thermodynamics. At first, the puzzle may seem to lie in having a positive pressure gradient in the absence of any viscous dissipation. That the pressure increases with $r$ can be understood as due simply to the Bernoulli principle, since the velocity decreases with increasing $r$. In fact, even if the direction of heat flow were reversed, exactly the same pressure and temperature gradients would result. The directions of the normal and superfluid velocities would reverse, but the inertial terms $(v \cdot \nabla)v = \nabla(\frac{1}{2}v^2)$ are quadratic in these velocities. The sign of $\nabla T$ depends only on temperature and not on the direction of heat flow!

This paradox is resolved by closely examining the fluxes of entropy and energy entering and leaving the fluid through the boundaries at $r_A$ and $r_H$. Requiring that both energy and entropy are conserved across the boundary leads to the conclusion that a temperature discontinuity is created at each boundary. We will briefly summarize the arguments presented by Putterman in §6 of [Put74] that lead to this conclusion. From the conservation of energy in the case of thermal counterflow, the energy entering the
fluid from the boundary is given as

$$\frac{\dot{Q}}{\text{Area}} \equiv \dot{W} = \rho S [T + (v_n \cdot \mathcal{A}) v_{n\perp}] ,$$

(A.62)

where $v_{n\perp}$ is the component of the normal fluid velocity perpendicular to the boundary and

$$\mathcal{A} = \frac{\rho_n}{\rho S} (v_n - v_s) .$$

(A.63)

The term involving $v_n \cdot \mathcal{A}$ in equation A.62 is so small in comparison to $T$ (e.g. at $\dot{Q} = 1 \text{ mW}$, ranging from less than $10^{-6}$ K at 1.7 K to $10^{-5}$ K at 1.3 K) that it can usually be neglected. The expression we normally use for $v_n$ ignores this term (compare equation 1.6). The new quantity $\mathcal{A}$, introduced by Putterman, is identified by Geurst as the pseudo-momentum per unit excitation. Geurst discusses the role of this pseudo-momentum at length in §5 of [Geu88]. The derivation of $\mathcal{A}$ in the two-fluid model is also discussed in Section D.1 of Appendix D.

Letting $T_{B1}$ be the temperature of the boundary at $r_A$, we must conserve entropy across the boundary, so the entropy flux into the fluid becomes

$$\frac{\dot{W}}{T_{B1}} = \rho S v_{n\perp} .$$

(A.64)

Reconciling equations A.62 and A.64, we see that a thermal discontinuity must exist at the boundary where the heat enters the He II, so that

$$T_{B1} = T + (v_n \cdot \mathcal{A}) .$$

(A.65)

This temperature discontinuity, it must be noted, has absolutely nothing to do with the much larger temperature discontinuity present at the heater surface as a result
of Kapitza resistance, so \( T_{B1} \) is the low temperature reached once the Kapitza step in temperature has been traversed. If a physical boundary, the heater surface, were present at the narrow end of our channel, then both these temperature discontinuities would occur within a narrow region at \( r_A \), the Kapitza jump from the internal temperature of the heater to \( T_{B1} \) as the heat leaves the heated surface, followed hard upon by this latter discontinuity, as the heat enters the fluid itself. In our channel geometry, the temperature discontinuity \( T_{B1} - T(r_A) \) occurring at \( r_A \) is physically far removed from the heater surface, and the boundary in question is a stationary surface at \( r_A \), a feature within the fluid identified by the temperature discontinuity demarcating it. Putterman calls this discontinuity “half a shock wave,” an apt phrase, recognizing a shock wave as any discontinuity in an intrinsic variable describing the fluid, and that this shock wave is at rest with respect to the boundary. This temperature discontinuity originates because the heat flow in He II involves a convection of the normal and superfluid components. This internal mass flow has inertia, so in establishing a heat flow, the normal and superfluid components must acquire kinetic energy. The work done on the fluid to establish the kinetic energy of this internal convection comes from the transfer of heat from the boundary at \( T_{B1} \) to the lower temperature of the fluid. The term \( \rho S(v_n \cdot A) \) in equation A.62 is the kinetic energy flux density of the heat flow. Thus this one-sided shock wave is located in the infinitesimal region over which the fluid components are accelerated from being at rest on one side of the boundary to being in motion at a large velocity on the other.
A similar temperature discontinuity occurs at \( r_H \), as the heat leaves the fluid through a second boundary. The temperature increases from \( T(r_H) \) to \( T_{b2} \) across this second boundary. The temperature increase from \( r_A \) to \( r_H \) in the bulk of the fluid, computed from equation A.61, is very small—ranging from 0.27 \( \mu K \) at 1.3 K to a mere 0.19 nK at 1.7 K for \( \dot{Q} = 1 \) mW—and is on the same order of magnitude as the temperature discontinuities at the boundaries. The temperature increase (from fluid to boundary) at the second boundary is smaller than the initial temperature drop (from boundary to fluid) at the first boundary. Neglecting all dissipative effects, we would find that \( T_{b2} = T_{b1} \) exactly, so the difference in the temperature discontinuities at the respective boundaries would be just equal to the temperature increase in the fluid, \( T(r_H) - T(r_A) \). The net effect, in this ideal fluid limit, is that the heat flows reversibly from \( r_A \) to \( r_H \), since the two boundaries are at the same temperature.

We have cheated somewhat in the preceding analysis, in that we have assumed the temperature gradient in the bulk of the fluid is due to the viscous dissipation in laminar flow, but at the boundaries we have assumed the conservation laws appropriate to dissipationless flow. As Putterman discusses, when discontinuities are present one cannot use the law of conservation of entropy, since dissipative effects within the thin shock layer become important. The derivation of the entropy conservation law made use of the first and second laws of thermodynamics for reversible processes and is only valid when variations from point to point in the thermodynamic variables can
be taken to be infinitesimally small. There must be a discontinuity at the boundary, however, since if none existed (assuming $T_{B1} = T(r_A)$) then entropy would be conserved, and the above equations would lead to the contradiction that $T_{B1} \neq T(r_A)$.

Including dissipative effects, the analysis becomes more complicated than in the preceding discussion in that entropy is not conserved. Nevertheless, the superfluid equation of motion, together with the laws of conservation of energy, mass, and momentum still result in sharp changes in the fluid temperature at the two boundaries. Putterman derives the magnitude of this temperature jump in §28 of [Put74], finding that it is now second order in the heat flow, rather than first order as in the above analysis. The heat flow is now irreversible, so the two boundaries will no longer be at the same temperature, but heat will always flow from the hotter to the colder boundary in agreement with the second law of thermodynamics. The temperature within the fluid itself can still rise in the direction of heat flow, without violating the second law.
Appendix B

Program SCHWARZ

This program integrates the local temperature gradient to obtain the turbulent temperature difference as a function of heat current for a specific location \( r \) in the channel. The program allows the user to select the temperature gradient appropriate to one of two models described in Chapter III: i) the Schwarz model applied locally, discussed in Section 3.4, or ii) the modified line density fit, described in Section 3.6. The formation of a laminar/turbulent front, as discussed in Section 3.7, can also be modeled by selecting a critical velocity, the minimum velocity for which turbulence is sustained.

For the integration of \( \nabla T'(r) \), a parabolic interpolation is performed as prescribed in Bevington [Bev69] once the integrand \( y(r_i) = \nabla T'(r_i) \) has been determined at points \( r_i = r_H - i \Delta r \), where \( i = (1, \ldots, N) \). This interpolation fits three adjacent points to a parabola which defines \( y(r) \) over the range \( \Delta r \) centered on the middle point. It can be shown that the total area under the resulting smooth curve of \( y(r) \) versus \( r \) is equal to the sum

\[
\text{Area} = \frac{3}{8}(y_1 + y_N) + \frac{7}{8}(y_2 + y_{N-1}) + \frac{23}{24}(y_3 + y_{N-2}) + \sum_{i=4}^{N-3} y_i. \tag{B.1}
\]
The uneven contributions of the six end points gives them a collective weight of only five points: \(2\left(\frac{3}{8} + \frac{7}{9} + \frac{23}{24}\right) = 5\), which follows from the fact that only \(N - 1\) divisions can be made between \(N\) evenly spaced points.

The Schwarz parameters \(c_L, c_1, I_z\), and \(I_{||}\), the friction coefficients \(\alpha\) and \(\alpha'\), and the entropy \(S\) and superfluid density \(\rho\), are all interpolated at the desired temperature by the subroutine TEMP. Third-order polynomial fits were used for the interpolations except where the third-order fit proved too inaccurate, in which case the fit was increased to forth order. Graphs of these interpolations are displayed following the code of program SCHWARZ.

program SCHWARZ

Integrates the local temperature gradient to compute the turbulent temperature difference \(\delta T'(r)\) as a function of \(Q\) at a user-specified output position \(r = r_{\text{min}}\)

Coded by J. Kafkalidis

First version created - 8/23/90
This version created - 10/7/93
Most recent update - 2/14/95

Features of calculation:

1) Adjusts temperature-dependent parameters as \(T\) increases

2) Calculates \(\delta T'\) as a function of \(Q\) for a single range in \(r\), from \(r_{\text{min}}\) to \(r_{\text{max}}\). Outputs \(\delta T'(r_{\text{min}}, Q)\).

3) Includes polynomial \(g(Q)\) in \(L\) if nonzero values of \(g_1\) and \(g_2\) are selected:

\[L_{\text{total}} = g_0 \cdot L_{\text{schwarz}} + g_1 \cdot Q + g_2 \cdot (Q^2)\]

For Schwarz calculation, select:

\(g_0 = 1.0\)
319

c

g1 = 0
g2 = 0

Note: setting g0 to some positive value other than 1.0 scales the Schwarz grad T by this multiplicative factor.

For modified line density calculation:
Select 0 < g0 < 1
   g1 < 0 (order of magnitude: -6.0E4)
   g2 > 0 (order of magnitude: 1.0E4 - 1.0E5)

4) Prevents L from becoming negative (unphysical result)
   L (= L_total) is reset to zero wherever L becomes negative.

5) Added 12/8/93:
   Queries user to input critical velocity V_c. Disallows turbulence in parts of channel where local relative velocity V(r,Q) < V_c. Determines cutoff r by:
   rcutoff = Q / (rho_s S T theta h V_c)

   For each Q, sets L (= L_total) = 0 and gradT(Q,r) = 0 for all r > rcutoff.

   Option for cutoff in r can be turned off by setting V_c small enough that at the minimum Q (= qmin), the cutoff r has a value greater than maximum r. (To prevent a divide-by-zero error, program will not allow user to set V_c = 0).

6) Outputs Q and delta_T(r) = T(r) - T_0 at endpoint r for each Q

User-customized ranges and outputs:

1) Allows integration range in r to be subdivided into up to 10,000 increments (default is 1000)

2) Allows user to elect to output L(r) and beta(r) for chosen Q values

Variables with fixed values:
real*8 a0  ! effective core radius in cm
real*8 kappa  ! quantum of circulation in cm^{-2}/sec
real*8 rkappa  ! kappa/(4*pi)
real*8 h  ! channel height in cm

Temperature variables:
real*8 Tstart  ! reservoir temperature T_0 in K
real*8 T  ! temperature at position r in K
real*8 gradT(101,10001)  ! grad T'(Q,r) in mK/cm
real*8 deltaT(101,1001)  ! delta T'(Q,r) in mK

Temperature-dependent parameters:
real*8 c1, cL, IL, ILL, alpha  ! Schwarz parameters
real*8 rhos  ! superfluid density
real*8 S  ! entropy
real*8 c, c2, c3, c4, c5  ! combinations of the above
real*8 tcoeff  ! temp. dependence of integrand

Heat current (Q) variables:
real*8 qmin  ! Minimum heat current Q in mW
real*8 qmax  ! Maximum Q in mW
real*8 qinc  ! Increment in Q in mW
real*8 qdiv  ! number of Q values
real*8 q(101)  ! Q array: stores qmin to qmax
real*8 qout

Position (r) variables:
real*8 r  ! position (running value) in cm
real*8 rmin  ! Limit of integration - minimum r
real*8 rmax  ! Limit of integration - maximum r
real*8 rinc  ! step size for integration in r
integer incr  ! number of subdivisions in r

Variables for beta_Schwarz:
real*8 bprev  ! previous value of beta in cm^{-2}/sec
real*8 bnew  ! new value of beta in cm^{-2}/sec
real*8 blimit  ! error limit for beta (cm^-2/sec)

Variables for polynomial g(Q) that modifies the line density:

real*8 g0  ! g0 (unitless)
real*8 g1  ! g1 in units of 1/(mW*cm^-2)
g0 (unitless)
real*8 g2  ! g2 in units of 1/(mW*cm)^-2
real*8 gtot  ! polynomial g(Q) in 1/cm^-2

Line density variables:

real*8 L_schwarz  ! Schwarz line density in 1/cm^-2
real*8 L_total  ! Total line density in 1/cm^-2
real*8 L_root  ! Square root of L_total
real*8 vcrit  ! Critical velocity in cm/sec
real*8 rcutoff  ! V(r) < vcrit for r > rcutoff

Integration and do-loop variables:

real*8 area  ! running sum of grad T(r)
real*8 sum  ! final sum of grad T(r)
integer m  ! index for increments in Q
g0 (unitless)
integer qlim  ! maximum m, = # increments in Q
integer k  ! index for increments in r
integer ndiv  ! maximum k, = # divisions in r
integer n  ! dummy index for r in sum loop
g0 (unitless)
integer ksum  ! maximum n
integer kbeg, kend, k1, k2  ! limiting k values in T update, sum
real*8 add  ! increment for kbeg, kend, k1, k2
integer nadd

Variables for outputting L(Q,r) and beta(Q,r):

real*8 r_out(201)  ! output @ these r
integer reave  ! number of r values output
real*8 save
integer nsave
real*8 r_output
integer l ! number of divisions in r output
real*8 qsave(10) ! output @ these Q
integer mqout(10) ! index m for q(m) = qsave
integer icount
integer ncount
integer number
integer mindex
real*8 beta(11,201) ! beta(Q,r) output
real*8 L_out(11,201) ! L_total(Q,r) output
real*8 L_schwarzout(11,201) ! L_schwarz(Q,r) output

character*1 qchange, rchange, tchange, bchange, L_output
character*12 fname, fname2

common T, c1, cL, Il, I11, alpha, rhos, S

== Initializing variables ==

a0 = 1.3E-8
kappa = 9.966305E-4
rkappa = 9.966305E-4/( 8*asin(1.0) )

h = 0.025
qmin = 0.1
qmax = 10.0
qinc = 0.1
blimit = 1.0E-9
rmax = 12.5
incr = 1000
ndiv = 100

write(5,*)' Enter reservoir temperature T_0 in K'
read(6,*') Tstart

10 bprev = 0.001
nsave = 10002
icount = 1

write(5,*)' Enter factor g0 to multiply Schwarz line density'
write(5,*)' For Schwarz calculation, set g0 = 1'
read(6,*) g0

write(5,*)' Enter value of g1 (should be negative)'
write(5,*)' For Schwarz calculation, set g1 = 0'
read(6,*) g1

write(5,*)' Enter value of g2 (should be positive)'
write(5,*)' For Schwarz calculation, set g2 = 0'
read(6,*) g2

------ Setting range in Q --------------------------------------
write(5,15) qmin, qmax, qinc
15 format(5x,' Heat flux range is set at qmin = ',F6.3,' mW, qmax = ',F6.3,' mW, qinc = ',F6.3,' mW.')
write(5,*)' Do you want to change?'
read(6,20) qchange
20 format(a1)
if( qchange.EQ.'Y' .OR. qchange.EQ.'y' ) then
write(5,*)' Input new heat flux range and increment'
read(6,*) qmin, qmax, qinc
end if

------ Setting range in r --------------------------------------
write(5,*)' Enter position r (in cm) for outputting Delta T(q,r)'
read(6,+) rmin
25 rinc = (rmax - rmin)/incr
write(5,30) incr
30 format(5x,' Integration range is divided into incr = ',15,F7.4, ' divisions')
write(5,31) rinc
31 format(1x,' of size rinc = ',F7.4,' cm.')
write(5,*)' Do you want to change the integration step size?'
read(6,20) rchange
if( rchange.EQ.'Y' .OR. rchange.EQ.'y' ) then
write(5,*)' Input new number of divisions in r (max=10000)'
read(6,*) incr
go to 25
end if

35 add = incr/ndiv
nadd = nint(add)

write(5,40) ndiv
format(1x,' Integration range is subdivided into ndiv = ',I5,
1 ' divisions for updating T. ')
write(5,45) nadd
format(1x,' T will be updated every ',I5,' integration steps. ')

write(5,*),' Do you want to change ndiv?'
read(6,20) tchange
if(tchange.EQ.'Y' .OR. tchange.EQ.'y') then
  write(5,*),' Input new number of divisions in r (max=1000)'
  read(6,*) ndiv
go to 35
end if

c
-------- Setting accuracy of determining beta  -------------------------
c
Note: preset value of blimit = 1.0E-9 is almost always suitable
c
write(5,50) blimit
format(1x,' Limit on accuracy of beta set at +/- ',E10.3)
write(5,*),' Do you want to change?'
read(6,20) bchange
if(bchange.EQ.'Y' .OR. bchange.EQ.'y') then
  write(5,*),' Input new limit on accuracy of beta'
  read(6,*) blimit
end if

c
-------- Setting the critical velocity  -------------------------
c
T = Tstart
call TEMP
c3 = 12.5 / (rhos*S*T*h)
c2 = 10000.0*c3*qmin
vmin = c2/rmax
write(5,*),' Enter critical velocity in cm/s'
write(5,56) vmin
format(1X,' For no critical velocity, enter vcrit < ',E11.4)
read(6,*) vcrit
if(vcrit .LE. 0 ) then
  write(5,*)' Critical velocity must be greater than zero'

go to 55
end if

c
write(5,*)' Do you want to save beta(r) and L(r) to a file?'
read(6,20) L_output

c
if ( L_output.EQ.'Y'.OR.L_output.EQ.'y' ) then
write(5,*)' Enter number of divisions in r for recording L(r) and beta(r) (maximum = 200)'
read(6,*) rsave
save = incr/rsave
nsave = nint(save)
write(5,*)' For how many Q values do you wish to output L and beta? (maximum number = 10)'
read(6,*) number
if ( number .GT. 10 ) then
write(5,*)' You cannot output L for that many Q values'
write(5,*)' Re-enter the number of Q values (maximum = 10)'
go to 60
end if
write(5,*)' Enter values of Q for which L and beta are to be output'
write(5,*)' These need not be listed in any numerical order,'
write(5,*)' but do not repeat the same value.'

c
do 100 i = 1, number
read(6,*) qout
qdiv = (qout-qmin)/qinc + .00001
mqout(i) = qdiv + 1
100 continue

c
end if

c-----------------------------------
c
NOTE: .00001 is added to qdiv to ensure that when qdiv is truncated to its integer part in the next step, the integer value does not turn out too small by one. It is assumed that qinc will always be much larger than .00001 (typically at least .001)
c
-----------------------------------
c
qdiv = (qmax - qmin)/qinc + .00001
qlim = qdiv + 1
mindex = 1

c
====== Entering do loop in Q ================

c
m counts the increment of Q

c
------------------------------------------------------------------------

do 500 m = 1, qlim
q(m) = qmin + (m-1)*qinc
gtot = (g1 + (g2*q(m)))*q(m) ! units: 1/(cm^-2)

c
------------------------------------------------------------------------

c Reset temperature-dependent parameters to initial values at
reservoir temperature (T = Tstart) for each iteration through
Q do-loop

c
------------------------------------------------------------------------

c T = Tstart
c

   call TEMP
c

c3 = 12.5 / (rhos*S*T*h) ! units: (sec^-2)/gm

c5 = (I11 - cL*I1) ! no units

c4 = (cL**2) * c5 ! no units

c2 = 10000.0*c3*q(m) ! units: (cm^-2)/sec

   Note: c2/r = local velocity v in cm/s

c
rcutoff = c2/vcrit
tcoeff = kappa*alpha*c5/S ! units: sec*K

c
kbegin = 1
kend = nadd
k1 = kend - 1
k2 = kend - 2
ksum = kend - 3
ncount = nsave
For a given Q (m value):

1) Calculate beta(m,k) and integrand gradT(m,k)

2) Integrate along the way to obtain delta T(m,1) and
the updated value of T:

T = Tstart + delta T(m,1)

-------------------------------------------------------------
   do 300 l=1,ndiv
   
   do 200 k = kbegin, kend
      r = rmax - (k*rinc)
      L_schwarz = ( cL*c2/(bprev*r) )**2
      L_total = (gO*L_schwarz) + gtot

      if ( L_total .GT. 0 ) then
         L_root = sqrt(L_total)
         c = 1/(c1*a0*L_root)
         bnew = rkappa * log(c)
      else
         L_total = 0.0
         bnew = bprev
         gradT(m,k) = 0.0
      go to 150
   end if

   if( abs(bnew - bprev) .GT. blimit ) then
      bprev = bnew
      go to 120
   else

j = 1
L_schwarz = ( cL*c2/(bnew*r) )**2
L_total = (gO*L_schwarz) + gtot
gradT(m,k) = tcoeff*L_total*c2/r

if ( r.GT.rcutoff ) then
    L_total = 0.0
    gradT(m,k) = 0.0
end if

c
ncount = ncount - 1
if ( ncount.LT.1 ) then
    r_out(j) = r
    L_schwarzout(mindex,j) = L_schwarz
    L_out(mindex,j) = L_total
    beta(mindex,j) = bnew
    ncount = nsave
    j = j + 1
end if

c
end if

c
continue

c
rectangle = 0.0
sum = 0.0

do 250 n = 4,ksum
    area = gradT(m,n) + area
250 continue

c
sum = 0.375 * (gradT(m,l) + gradT(m,kend))
sum = sum + ( (7.0/6.0) * (gradT(m,2) + gradT(m,k1)) )
sum = sum + ( (23.0/24.0) * (gradT(m,3) + gradT(m,k2)) )
deltaT(m,1) = (area + sum)*rinc
kbegin = kbegin + nadd
kend = kend + nadd
k1 = k1 + nadd
k2 = k2 + nadd
ksum = ksum + nadd

Update parameters to temperature at integration endpoint r:
T(r) = T_0 + delta T(Q,r)
T = Tstart + deltaT(m,1)
call TEMP

c3 = 12.5 / (rhos*S*T*h)
c5 = (Il1 - cL*Il)
c4 = (cL**2) * c5
c2 = 10000.0*c3*q(m)
rcutoff = c2/vcrit
tcoeff = kappa*alpha*c5/S

continue

if (mindex.GT.number) go to 500
do 400 i = 1, number
   if (m.EQ.mqout(i)) then
      qsave(icount) = q(m)
icount = icount + 1
      mindex = mindex + 1
   end if
400 continue
500 continue

outputting delta T(r) ==============

write(5,*)' Enter name of file for outputting delta T'
read(6,600) fname
600 format(a)
   open(unit=9,iostat=ios,status='new',file=fname,
      1 carriagecontrol='list')
   write(9,610) fname
610 format(1x,'FILE ',a)
write(9,*) ' Running program SCHWARZ'
write(9,*) ' Polynomial g(Q) added to Schwarz line density'
write(9,*) ' Temperature-dependent parameters adjusted as T
1 increases'
write(9,620) Tstart
620 format(1x, ' T = ',F4.2, ' K')
write(9,*) ' Parameter values:
write(9,621) g0
621 format(5x, 'g0 = ',F6.3)
write(9,622) g1
622 format(5x, 'g1 = ',E11.4, ' 1/mW*(cm^-2)')
write(9,623) g2
623 format(5x, 'g2 = ',E11.4, ' 1/(cm*mW^-2)')
write(9,624) rmin
624 format(5x, 'Rmin = ',F6.3, ' cm')
write(9,625) rmax
625 format(5x, 'Rmax = ',F6.3, ' cm')
write(9,626) h
626 format(5x, 'h = ',F7.4, ' cm')
write(9,627) rinc, incr
627 format(5x, 'Rinc = ',F8.5, ' cm (',I5, ' increments)')
write(9,628) qmin
628 format(5x, 'Qmin = ',F6.3, ' mW')
write(9,629) qmax
629 format(5x, 'Qmax = ',F6.3, ' mW')
write(9,630) qinc
630 format(5x, 'Qinc = ',F6.3, ' mW')
write(9,631) blimit
631 format(5x, 'Beta determined to +/- ',E10.3)
write(9,632) nadd
632 format(5x, 'T updated every ',I5, ' integration steps')
write(9,633) vcrit
633 format(5x, 'Critical velocity = ',F7.3, ' cm/s')
write(9,*)

A note on units:
deltaT(q,r) has been computed in K, whereas delT_out
is in mK:

delT_out = 1000*deltaT(q,r)
r_output = rmax - (incr*rinc)
write(9,640) r_output

format(' Delta T(Q) output for r = ',F6.3, cm)
write(9,*), ' Q  Delta T(Q)
write(9,*), ('mW) (mK)

do 700 i=1,qlim
delT_out = deltaT(i,ndiv)*1000
write(9,650) q(i), delT_out

700 continue

close(unit=9)

c ======== Outputting L(r) and beta(r) =============================
c
if( L.output.EQ.'Y' .OR. L.output.EQ.'y' ) then
  write(5,*), ' Enter name of file for outputting L and beta'
  read(6,600) fname2
  open(unit=10,iostat=ios,status='new',file=fname2,
1 carriagecontrol='list')
  write(10,610) fname2
  write(10,*), ' Running program SCHWARZ'
  write(10,710) fname
710 format(1x,F6.3,2x,E11.4)
write(10,*) ' Polynomial g(Q) added to Schwarz line density'
write(10,620) Tstart
write(10,*) ' Parameter values:'
write(10,621) g0
write(10,622) g1
write(10,623) g2
write(10,624) rmin
write(10,625) rmax
write(10,626) h
write(10,627) rinc, incr
write(10,631) blimit
write(10,632) nadd
write(10,633) vcrit
    do 800 i = 1, number
        write(10,*), '  '
write(10,750) qsave(i)
format(1x,'Q = ','F6.3,' mW')
write(10,*) ' r L_schwarz L_total'
do 770 l = 1, rsave
    write(10,760) r_out(l), L_schwarzout(i,l), L_out(i,l)
760    format(2x,F6.3,2(2x,E11.4))
770 continue
800 continue

c close(unit=10)
c
end if
c
write(5,850) Tstart, fname
850 format(1x,'Results for T = ',F4.2,'K output to file ',a)
c
write(5,*)' Next temperature? (0 = end)'
read(6,*) Tstart
if( Tstart .GT. 0 ) go to 10
999 end
c
============== SUBROUTINE TEMP ================
c
subroutine TEMP
c
Last updated on 2/14/95
c
Computes all temperature dependent parameters via
c polynomial fits to T
c
returns values at new T of:
c
superfluid density: rhos = gm/cm^3
c
entropy: S = erg/(gm K) = cm^2/(K s^2)
c
Schwarz temperature-dependent parameters:
alpha, c1, cL, II, III

c
real*8 T, c1, cL, II, III, alpha, rhos, S
common $T$, $c_1$, $c_L$, $II$, $III$, $\alpha$, $\rho_s$, $S$

cL = 0.624815 + (-2.203492 + (2.666013 + (-1.294606 + 
1 0.230740*T)*T)*T)*T

c1 = 35.800389 + (-69.925711 + (52.662187 + (-17.144197 + 
1 1.975564*T)*T)*T)*T

II = 0.118102 + (0.530756 +(-0.266411 + 0.042042*T)*T)*T

III = -0.306166 + (2.099670 +(-1.422467 + 0.333133*T)*T)*T

$\alpha$ = -0.493601 + (1.157230 +(-0.940296 + 0.279100*T)*T)*T

$\rho_s$ = .145050 + (.001855 + (-.011099 + (.021406 - 
1 .013133*T)*T)*T)*T

$S$ = -1.146272D7 + (2.896596D7 + (-2.573258D7 + 
1 8.26254D6*T)*T)*T

c

end
Figure B.1: Schwarz parameter $c_L$ as a function of temperature. The two values listed in [Sch88] for each $T$ are averaged, and fit to a forth-order polynomial.

$$c_L(T) = 0.624815 - 2.203492 T + 2.666013 T^2 - 1.294606 T^3 + 0.230740 T^4$$
Figure B.2: Schwarz parameter $c_1$ as a function of temperature. The two values listed in [Sch88] for each $T$ are averaged and fit to a forth-order polynomial.

$$c_1(T) = 35.800389 - 69.925711 T + 52.66219 T^2 - 17.144197 T^3 + 1.975584 T^4$$
Figure B.3: Schwarz parameter $I_{\parallel}$ as a function of temperature. The two values listed in [Sch88] for each $T$ (excluding 2.15 K) are averaged and fit to a third-order polynomial.
Figure B.4: Schwarz parameter $I_1$ as a function of temperature. The two values listed in [Sch88] for each $T$ (excluding 2.15 K) are averaged and fit to a third-order polynomial.

\[ I_1(T) = 0.118102 + 0.530756 T - 0.266411 T^2 + 0.042042 T^3 \]
Figure B.5: Friction coefficient $\alpha$ as a function of temperature. Values shown are from [Sch85] and [Sch88]. The third-order polynomial fit is generated from the values in [Sch88], excluding $\alpha = 1$ at 2.15 K.

$$\alpha(T) = -0.493601 + 1.157230 T - 0.940296 T^2 + 0.279100 T^3$$
Figure B.6: Friction coefficient $\alpha'$ as a function of temperature. Values are taken from [Sch85] and [Sch88]. As $\alpha'$ is not used in program SCHWARZ, no interpolation was necessary.
Figure B.7: Superfluid density $\rho_s$ as a function of temperature. Values are obtained from [Don92]. A forth-order polynomial fit to $\rho_s$ for $T \leq 1.85$ K is used in program SCHWARZ.
Figure B.8: Entropy per unit mass $S$ as a function of temperature. Values are obtained from [Don92]. A third-order polynomial fit to $S$ for $T \leq 2.0$ K is used in program SCHWARZ.

$$S(T) = -1.146272 + 2.896596 T - 2.573258 T^2 + 0.826254 T^3$$
Appendix C

Data Compared to the Schwarz Model Applied Locally

In the figures in this Section, measurements of $\Delta T(r_i)$ as a function of $\dot{Q}$ are compared to the predictions of the Schwarz model applied in the local uniformity approximation. Graphs for the reservoir temperatures $T_0 = 1.3, 1.4, 1.6, \text{ and } 1.7 \text{ K}$ are displayed here. The corresponding graphs for $T_0 = 1.5 \text{ K}$ appear as figures:

1) $\Delta T'(r_A)$: Figure 3.8
2) $\Delta T'(r_B)$: Figure 3.9
3) $\Delta T'(r_C)$: Figure 3.10
4) $\Delta T'(r_D)$: Figure 3.11
5) $\Delta T'(r_E)$: Figure 3.12
6) $\Delta T'(r_F)$: Figure 3.13

in Section 3.4.2 of Chapter III. Measurements were made at $r_A$ and $r_D$ for all five reservoir temperatures. Measurements at other positions were taken at selected reservoir...
temperatures only. For positions $r_B$, $r_E$, and $r_F$, data were gathered at $T_0 = 1.5$ K only. At $r_C$, data were collected at $T_0 = 1.4$, 1.5, and 1.6 K. The graphs in this appendix are arranged first by temperature, then ordered by position.
Figure C.1: $\dot{Q}$-dependence of the turbulent temperature difference measured across the entire channel for $T_0 = 1.3$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.2: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_D = 4.725$ cm and the reservoir end held at $T_0 = 1.3$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.3: $\dot{Q}$-dependence of the turbulent temperature difference measured across the entire channel for $T_0 = 1.4$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.4: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_C = 3.468$ cm and the reservoir end held at $T_0 = 1.4$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.5: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_D = 4.725$ cm and the reservoir end held at $T_0 = 1.4$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.6: $\dot{Q}$-dependence of the turbulent temperature difference measured across the entire channel for $T_0 = 1.6 \text{ K}$. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.7: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_c = 3.468$ cm and the reservoir end held at $T_0 = 1.6$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.8: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_D = 4.725$ cm and the reservoir end held at $T_0 = 1.6$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.9: $\dot{Q}$-dependence of the turbulent temperature difference measured across the entire channel for $T_0 = 1.7$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Figure C.10: $\dot{Q}$-dependence of the turbulent temperature difference measured between $r_D = 4.725$ cm and the reservoir end held at $T_0 = 1.7$ K. The data are compared to the Schwarz model applied in the local uniformity approximation.
Appendix D

The Geurst Three-Fluid Hydrodynamic Model

In this appendix, the derivation and predictions of the Geurst model [Geu89] are summarized for the case of one-dimensional flow. The vortex tangle constitutes a third independent fluid in this three-fluid hydrodynamic model. The simpler version of a two-fluid model, in which no vortex tangle is present, is outlined in Section D.1. The three-fluid model in the absence of dissipation is reviewed in Section D.2. Dissipative effects are incorporated by adding source terms to the conservation equations for line length and entropy, and internal and external forces to the equations of motion, respectively, as described in Section D.3. Finally, Geurst considers the circumstance that the line density possesses a large gradient. The resulting modifications to the model and the dynamical balance equation for the vortex tangle due to a large gradient in \( L \) are summarized in Section D.4. The intent in this appendix is not to repeat Geurst's derivations in full, but rather to present enough detail to elucidate the unique features of this three-fluid model and provide the necessary background for Section 3.5 of Chapter III, which describes how this model was applied to the present work.
D.1 The Two-Fluid Model and Lin’s Variational Principle

We begin by summarizing the results derived in §2 of [Geu89], in which Lin’s Hamilton principle for the two-fluid model is reviewed. This variational principle adopts the virtual-mass model for the elementary excitations by choosing \( v, \ v_n, \ \rho, \) and \( S \) as independent variables for the two fluids. Note that, throughout this appendix, \( v \) refers to the mass flow of the total fluid, not the relative velocity between the normal and superfluid components. A Lagrangian density \( \Lambda_0 = K_0 - U_0 \) is defined in which the kinetic and internal energy densities \( K_0 \) and \( U_0 \) are given by

\[
K_0 = \frac{1}{2} \rho v^2 \tag{D.1}
\]

\[
U_0 = \mu \rho + TS - \frac{1}{2} m (v_n - v)^2 \tag{D.2}
\]

where \( m \) is the “virtual-mass” density of the elementary excitations. The zero subscripts are used in this section to indicate that no vortex tangle is present; only two fluids are involved. In the virtual-mass model, the entropy density \( S \) is considered to be the density of elementary excitations per unit volume. Geurst uses the entropy per unit volume, \( S \). Elsewhere in this document \( S \) is used to refer to the entropy per unit mass, so the reader is cautioned that \( S = \rho S \). The thermodynamic potential \( \mu \), the absolute temperature \( T \), and the virtual-mass density \( m \) of the elementary excitations are functions of the independent variables \( \rho, S, \) and \( (v_n - v) \). The conditions that mass and entropy be conserved provide two constraints:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \tag{D.3}
\]

\[
\frac{\partial S}{\partial t} + \frac{\partial (S v_n)}{\partial x} = 0 \tag{D.4}
\]
By performing variations on the lagrangian density subject to these constraints, the corresponding generalized momentum variables are identified as

\[
\begin{align*}
    u_s &\equiv v - \frac{m}{\rho}(v_n - v), \\
    \mathcal{A} &\equiv \frac{m}{S}(v_n - v).
\end{align*}
\]  

The velocity \( u_s \) constitutes the generalized momentum of He-II per unit mass, and \( \mathcal{A} \) is the pseudo-momentum of the elementary excitations per unit excitation. The pseudo-momentum of the elementary excitations per unit volume is designated as \( P_n \):

\[
    P_n \equiv S\mathcal{A} = m(v_n - v).
\]  

It can be shown that the virtual-mass density is given by

\[
    m = \rho \frac{\rho_n}{\rho_s}.
\]

The equations of motion for \( u_s \) and \( \mathcal{A} \) are also derived:

\[
\begin{align*}
    \frac{\partial u_s}{\partial t} + \frac{\partial}{\partial x}(vu_s + \mu - \frac{1}{2}v^2) &= 0, \\
    \frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial x}(v_n\mathcal{A} + T) &= 0.
\end{align*}
\]

The differential form of the internal energy \( U_0 \) is

\[
\begin{align*}
    dU_0 &= \mu d\rho + TdS - m d\left(\frac{1}{2}(v_n - v)^2\right) \\
    &= \mu d\rho + TdS - P_n d(v_n - v). \quad (D.11)
\end{align*}
\]

The conservation of energy and momentum can be expressed, respectively, in terms of the energy density \( H_0 \) and energy flux \( Q_0 \), and momentum density \( P_0 \) and
momentum flux $\Pi_0$. Each of these quantities is determined from the lagrangian density. The resulting expressions for $H_0$, $Q_0$, $P_0$ and $\Pi_0$ are:

\[ H_0 = \frac{1}{2} \rho v^2 + U_0 + P_n(v_n - v), \quad \text{(D.12)} \]
\[ Q_0 = \rho v (v u_\ast + \mu - \frac{1}{2} v^2) + S v_n (v_n A + T), \quad \text{(D.13)} \]
\[ P_0 = \rho v = \rho u_\ast + P_n, \quad \text{(D.14)} \]
\[ \Pi_0 = \rho v v + P_n(v_n - v) + p, \quad \text{(D.15)} \]

where the pressure $p$ is defined by

\[ p = -U_0 + \rho \mu + ST. \quad \text{(D.16)} \]

Note that the energy density $H_0$ is not equal to $K_0 + U_0$ in the virtual-mass model. By transforming to the "drift-mass" model, where the independent variables are $\rho_n$, $\rho_\ast$, $v_n$ and $v_\ast$, new kinetic and internal energy densities $K_0^*$ and $U_0^*$ can be defined which do satisfy the relations $A_0 = K_0^* - U_0^*$ and $H_0 = K_0^* + U_0^*$:

\[ K_0^* = K_0 + \frac{1}{2} m (v_n - v)^2 = \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_\ast v_\ast^2, \quad \text{(D.17)} \]
\[ U_0^* = U_0 + \frac{1}{2} m (v_n - v)^2 = \mu \rho + TS. \]

In the drift-mass model, $\rho_n$ is the drift-mass density of the elementary excitations. Using the relation D.8 between $m$ and $\rho_n$, and the fact that $\rho v = \rho_\ast v_\ast + \rho_n v_n$ in this two-fluid model, the generalized momentum of He-II per unit mass (equation D.5) is recognized to be nothing more than the superfluid velocity:

\[ u_\ast = v - \frac{P_n}{\rho} = v_\ast. \quad \text{(D.18)} \]
D.2 The Three-Fluid Model: A Vortex Tangle Without Dissipation

In §3 and §4 of [Geu89], Lin's variational principle is generalized to a three-fluid model in which the vortex tangle is considered as a third independent fluid, with velocity $v_t$ and density $L$. The kinetic energy density and pseudo-momentum, or impulse, of the vortex tangle are taken into account by adding appropriate terms to the lagrangian and repeating the variation of $\Lambda$. The variation is performed in the absence of dissipation, so $\rho$, $S$, and $L$ are all conserved quantities. Dissipative effects can then be incorporated by adding the appropriate source terms to the conservation equations for $S$ and $L$, and external forces to the corresponding equations of motion.

The effects of dissipation are outlined in the following section.

The lagrangian density must include the additional kinetic and internal energies due to the presence of the vortex tangle

$$\Lambda = \Lambda_0 + \tilde{K}_t - \tilde{U}_t . \tag{D.19}$$

The added kinetic energy density accounts for the pseudo-momentum, or impulse, of the vortex tangle

$$\tilde{K}_t = \frac{1}{2} \tilde{m}_t (v_t - v)^2 , \tag{D.20}$$

where $\tilde{m}_t$ is in general a function of the independent variables $\rho$, $S$, $L$, $v_n - v$ and $v_t - v$. The additional internal energy density $\tilde{U}_t$ due to the presence of the vortex tangle is given by

$$\tilde{U}_t = \rho s \frac{\kappa^2}{4\pi} \ln \left( \frac{c}{a_0 L^{1/2}} \right) L . \tag{D.21}$$
This added internal energy actually accounts for the energy density of the vortex tangle due to the "microscopic" circular motion of the superfluid around the vortex cores. A derivation of this energy density is given in Khalatnikov for the simple geometry of a rectilinear vortex line in a rotating bucket experiment [Kha65]. Since \( \int v_s \cdot dl = \kappa \), the local superfluid velocity in the vicinity of a vortex core is given by \( v_s = \kappa/(2\pi r) \). Integrating the kinetic energy per unit volume \( \frac{1}{2} \rho_s v_s^2 \) over the cross-sectional area in cylindrical coordinates \( dA = 2\pi r \, dr \) yields the kinetic energy per unit line length

\[
E_K = \int_{a_0}^{b} \frac{1}{2} \rho_s v_s^2 \, 2\pi r \, dr = \rho_s \frac{\kappa^2}{4\pi} \ln \left( \frac{b}{a_0} \right),
\]

where \( b \) is taken to be some reasonable external dimension of the vortex. If \( N \) is the number of vortices per unit surface area, then \( N = \frac{1}{\pi b^2} \). For a vortex of length \( \ell \), the vortex line length per unit volume is given by \( L = \frac{\ell}{\pi b^2} = N \), therefore \( b = \frac{1}{\sqrt{\pi L}} \).

A similar argument could be made for any arbitrary geometry, including a random tangle, simply by choosing a suitable cutoff distance \( b \) for the integration. Generally, the cutoff distance may be written as \( b = \frac{c}{L^{1/2}} \), where \( c \) is a constant of order one.

Finally, multiplying the kinetic energy per unit line length by the line density yields the kinetic energy per unit volume: \( \tilde{U}_\ell = E_K L \). Although this term is derived by integrating the local kinetic energy density of the superfluid, this microscopic motion does not produce any macroscopic motion of the superfluid; therefore, \( \tilde{U}_\ell \) contributes to the internal energy density rather than to the kinetic energy.

By incorporating the impulse of the tangle \( \tilde{K}_\ell \) into the internal energy rather than the kinetic energy, Geurst writes the lagrangian as \( \lambda = \Lambda_0 - U_\ell \) where \( \Lambda_0 = K_0 - U_0 \).
and
\[ U_t = \tilde{U}_t - \frac{1}{2} \tilde{m}_t (v_t - v)^2. \] (D.23)

The total internal energy density \( U = U_0 + U_t \) is now a function of \( \rho, S, L, (v_n - v) \), and \( (v_t - v) \) which can be put in the form
\[ dU = \mu d\rho + T dS + \mu_t dL - P_n d(v_n - v) - P_t d(v_t - v). \] (D.24)

The new variable \( P_t \) is introduced to represent the impulse density of the vortex tangle. In analogy to equation D.7 defining \( P_n \), the pseudo-momentum density of the elementary excitations, the impulse density \( P_t \) is defined by
\[ P_t \equiv m_t (v_t - v), \] (D.25)

where \( m_t \) is the "virtual-mass" density of the vortex tangle. The "line-length potential" \( \mu_t \) is defined as
\[ \mu_t \equiv \frac{\partial U}{\partial L} = \frac{\partial \tilde{U}_t}{\partial L} - \frac{1}{2} \frac{\partial \tilde{m}_t}{\partial L} (v_t - v)^2, \] (D.26)

and \( m_t \) is given by
\[ m_t = \tilde{m}_t + \frac{1}{2} (v_t - v)^2 \frac{\partial \tilde{m}_t}{\partial \left(\frac{1}{2} (v_t - v)^2\right)}. \] (D.27)

Geurst argues that \( \tilde{m}_t = \rho_* \tilde{m}_t \) where the dimensionless quantity \( \tilde{m}_t \) should depend only on the physical parameters of the vortex tangle, \( \kappa, L, \) and \( (v_t - v) \), and possibly also on temperature through the ratio \( \rho_* / \rho \). By introducing a dimensionless combination of the vortex parameters, called the "Vinen number" \( \text{Vi} \)
\[ \text{Vi} = \frac{\kappa L^{1/2}}{|v_t - v|}, \] (D.28)
$\dot{m}_t$ is assumed to be a function only of $V_i$ and $\rho_s/\rho$. The equations D.26 and D.27 therefore become

$$\mu_t = \frac{\partial \tilde{U}_t}{\partial L} - \rho_s \frac{\kappa^2}{4} \frac{1}{V_i} \left( \frac{\partial \tilde{m}_t}{\partial V_i} \right),$$

$$m_t = \tilde{m}_t - \frac{1}{2} V_i \left( \frac{\partial \tilde{m}_t}{\partial V_i} \right).$$

(D.29) (D.30)

The exact relationship between $m_t$ and $\tilde{m}_t$ is deduced below.

In the absence of dissipation, conservation of vortex line length adds a third constraint equation,

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x} (Lv_t) = 0.$$  

(D.31)

Performing independent variations on the quantities $\rho, S, L, v, v_n,$ and $v_t$ yields three generalized momenta (compare equations D.5 and D.6 in the absence of the vortex tangle):

$$u_s \equiv v - \frac{P_n}{\rho} - \frac{P_t}{\rho},$$

$$A \equiv \frac{P_n}{S},$$

$$B \equiv \frac{P_t}{L}.$$  

(D.32)

The new quantity $B$, the impulse of the vortex tangle per unit line length, joins the set of generalized momenta. Note that the generalized velocity $u_s$ is no longer equal to the superfluid velocity $v_s$. The equations of motion for $u_s$ and $A$ remain unchanged (see equations D.9 and D.10). The third equation of motion, for $B$, is found to be:

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x} (v_t B + \mu_t) = 0.$$  

(D.33)
As before, expressions for the energy density $H$, the energy flux $Q$, the momentum density $P$ and momentum flux $\Pi$ can be obtained:

\begin{align*}
H &= \frac{1}{2} \rho v^2 + P_n(v_n - v) + P_{\ell}(v_{\ell} - v) + U, \quad \text{(D.34)} \\
Q &= \rho v(v_u + \mu - \frac{1}{2} v^2) + S v_n(v_n A + T) + L v_{\ell}(v_{\ell} B + \mu_{\ell}), \quad \text{(D.35)} \\
P &= \rho v = \rho u_s + P_n + P_{\ell}, \quad \text{(D.36)} \\
\Pi &= \rho v v + P_n(v_n - v) + P_{\ell}(v_{\ell} - v) + p, \quad \text{(D.37)}
\end{align*}

where the pressure $p$ is now given by

\begin{equation}
p = -U + \rho \mu + ST + L \mu_{\ell}. \quad \text{(D.38)}
\end{equation}

The total energy density $H$ may be written as the sum of two parts: the original energy density in the absence of the tangle, $H_0$, plus an additional term $H_{\ell}$, which represents the total energy density of the vortex tangle:

\begin{equation}
H = H_0 + H_{\ell}. \quad \text{(D.39)}
\end{equation}

Comparing equations D.12 and D.34, it is obvious that $H_{\ell}$ is given by

\begin{align*}
H_{\ell} &= P_{\ell}(v_{\ell} - v) + U_{\ell} \\
&= m_{\ell}(v_{\ell} - v)^2 - \frac{1}{2} \tilde{m}_{\ell}(v_{\ell} - v)^2 + \tilde{U}_{\ell}. \quad \text{(D.40)}
\end{align*}

Geurst argues that the total energy density of the vortex tangle ought to be attributed solely to the energy density $\tilde{U}_{\ell}$ associated with the “microscopic” motion of the superfluid component around the vortex cores. Requiring that

\begin{equation}
H_{\ell} = \tilde{U}_{\ell} \quad \text{(D.41)}
\end{equation}
leads to the conclusion that

\[ \bar{m}_t = 2m_t \quad (D.42) \]

Inserting this relation into equation D.30, it is obvious that \( \bar{m}_t \) and \( m_t \) are linear functions of the Vinen number \( V_i \). Geurst therefore writes

\[ m_t = \rho_s \beta V_i \quad (D.43) \]

The proportionality constant \( \beta \) depends only on temperature through the mass ratio \( \rho_s/\rho \), and is one of the adjustable parameters in the Geurst model. The functional dependence of the impulse density of the tangle \( P_t \) is now clearly identified:

\[ P_t = \rho_s \kappa \beta L^{1/2} \hat{w}_t \quad (D.44) \]

where \( \hat{w}_t \) is a unit vector in the direction of \( (v_t - v) \). Evaluating terms in equation D.26, the line-length potential is given by

\[ \mu_t = \rho_s \frac{\kappa^2}{4\pi} \ln \left( \frac{c'}{a_0 L^{1/2}} \right) - \rho_s \kappa \frac{\beta}{2L^{1/2}} |v_t - v| \quad (D.45) \]

where \( \ln c' = \ln c - \frac{1}{2} \). Comparison of the above expression to equation 3.15 in Section 3.4 reveals that the first term includes the Schwarz parameter

\[ \beta_S = \frac{\kappa}{4\pi} \ln \left( \frac{c'}{a_0 L^{1/2}} \right) \quad (D.46) \]

(Note that in [Geu92], Geurst states explicitly that \( c' = c/c_1 \), where in our notation in Section 3.4, \( c \) has been absorbed into the value of \( c_1 \) quoted in equation 3.15). From equation D.45, the "equilibrium condition" \( \mu_t = \frac{\partial U_t}{\partial L} = 0 \) is achieved only when

\[ \frac{\beta}{2} |v_t - v| = \beta_S L^{1/2} \quad (D.47) \]
D.3 Adding Dissipation

To account for dissipation, appropriate source terms are added to the right-hand side of the conservation equations for $S$ and $L$ (equations D.4 and D.31), and external force terms are added to the equations of motion for $u_s$, $A$, and $B$ (equations D.9, D.10, and D.33). Rather than repeating the full derivation presented in §5 of [Geu89], only the essential results will be summarized here. The rate of production $r_\ell$ of vortex line density is proportional to the line-length potential $\mu_\ell$

$$- \mu_\ell = C r_\ell ,$$  \hspace{1cm} (D.48)

where the material coefficient $C$ can only depend on the physical parameters describing the vortex tangle. Dimensional analysis shows that

$$C = \rho_s^\kappa \frac{\kappa}{L^2} \gamma$$  \hspace{1cm} (D.49)

where the Geurst parameter $\gamma$ is a dimensionless quantity that can depend only on temperature through $\rho_s/\rho$. The kinematic equation for the vortex tangle,

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(L u_\ell) = r_\ell = - \frac{L^2}{\rho_s \kappa \gamma} \mu_\ell ,$$  \hspace{1cm} (D.50)

becomes, upon inserting $\mu_\ell$ from equation D.45:

$$\frac{\partial L}{\partial t} + \frac{\partial}{\partial x}(L u_\ell) = \frac{\beta}{2\gamma} |v_\ell - v| L^{3/2} - \frac{\beta_s L^2}{\gamma} .$$  \hspace{1cm} (D.51)

This equation appears to be an extension of the Vinen equation, except that it involves the drift velocity $(v_\ell - v)$ of the vortex tangle, rather than the relative drift velocity $(v_n - v)$ of the elementary excitations. The Vinen equation is traditionally interpreted
as being comprised of terms describing the production ($\sim L^{3/2}$) and annihilation ($\sim L^2$) of quantized vorticity. Geurst points out that $\mu_\ell$ in equation D.50 plays the role of a "generalized force" which is derived from a potential energy $U_\ell$. In equilibrium, for a uniform and steady vortex tangle, $\mu_\ell$ vanishes. When $\mu_\ell$ is negative, the vortex tangle is not in equilibrium and the line length density grows, and likewise when $\mu_\ell$ is positive, $L$ decays. Equation D.50 therefore acts as a balance equation for the line density $L$.

Four other similar material coefficients $C_{ij}$, where $i,j \in (1,2)$, are introduced by considering the relations between the internal force densities $F_{st}$ and $F_{nt}$ exerted on the vortex tangle by the superfluid and normal fluid components, respectively:

$$\begin{pmatrix} -F_{st} \\ -F_{nt} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} v_t - v \\ v_t - v_n \end{pmatrix}.$$

(D.52)

These forces are incorporated in the equations of motion for $u_\ast$, $A$ and $B$, although the specific relations will not be quoted here. It can be shown by dimensional analysis that the coefficients $C_{ij}$ are given by

$$C_{ij} = \rho_\ast \kappa L \gamma_{ij}$$

(D.53)

where $\gamma_{ij}$ are temperature-dependent parameters, and further that $C_{12} = C_{21}$; therefore only three of the coefficients $\gamma_{ij}$ are independent. In all, five independent parameters have been introduced in the Geurst model so far: $\beta$, $\gamma$, $\gamma_{11}$, $\gamma_{12}$, and $\gamma_{22}$. When the flow is steady and uniform, it can be shown that the mutual friction force $F_{ns}$ exerted by the normal fluid on the superfluid is simply given by

$$F_{ns} = \frac{\rho_\ast}{\rho} F_{nt} = -\frac{\rho_\ast}{\rho} F_{st}.$$

(D.54)
The factor $\rho_s/\rho$ is missing from both equation 5.29 and the right-hand side of equation 5.30 in [Geu89]. Geurst corrected this small error in his subsequent paper (see the discussion following equation 3.9 in [Geu92]). Also note that Geurst writes the force on the superfluid as $F_n$, whereas Schwarz denotes this force as $F_m$.

Recouping the Vinen equation for the case of homogeneous turbulence requires combining equation D.51 with the dynamic equation for the vortex tangle. Adding in the dissipative forces, this equation of motion for $B$ becomes

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial x}(v_t B + \mu_t) = \frac{1}{L}(F_{nt} + F_{st}) - \frac{r_t}{L} B + \frac{1}{L} F_t . \quad \text{(D.55)}$$

The force density $F_t$ represents external reactive and dissipative forces on the vortex tangle, such as the drag produced by vortices pinning to the channel walls, and will be neglected. For the case of homogeneous turbulence, the spatial derivatives vanish. Eliminating the time derivative of $L$ between the above expression and the conservation equation for the vortex tangle, equation D.50 leads to an expression for the vortex line drift velocity for the case of homogeneous turbulence. Using equations D.32, D.44, D.45, D.48, D.49, D.52, and D.53, it can be shown that the vortex line drift velocity is given by

$$\left(\frac{\beta^2}{4\gamma} + \gamma_{11} + 2\gamma_{12} + \gamma_{22}\right)(v_t - v) = \left(\gamma_{12} + \gamma_{22}\right)(v_n - v) + \frac{\beta}{2\gamma} \beta_s L^{1/2} \tilde{\omega}_t \quad \text{(D.56)}$$

when the turbulence is homogeneous. From the relations between the Geurst and Schwarz parameters presented in Appendix E, this expression may be written as

$$v_t - v = \left[(\alpha' - \alpha'')I_{\|} - \frac{\rho_n}{\rho} + \alpha''\right](v_n - v_s) + (1 - \alpha')I_t \beta_s L^{1/2} \tilde{\omega}_t . \quad \text{(D.57)}$$
In comparing this expression to what Schwarz derives for \( v_t \) (see equation 1.50), it must be remembered that Schwarz gives \( v_t \) with respect to the superfluid rest frame, whereas Geurst measures \( v_t \) in the mass flow rest frame. Transforming the Geurst \( v_t \) to the superfluid rest frame by subtracting \( v_\star \) from both sides of the above expression yields \( v_t \) in the superfluid frame:

\[
v_t = \left[ (\alpha' - \alpha'') I_{||} + \alpha'' \right] (v_n - v_\star) + (1 - \alpha') I_t \beta_S L^{1/2} \dot{\omega}_t .
\]  

(D.58)

which agrees with Schwarz upon taking \( \alpha'' = 0 \) and \( \dot{\omega}_t = \dot{\omega}_n \).

Finally, inserting this expression for \( (v_t - v) \) into equation D.51 yields the Vinen equation in terms of the Schwarz coefficients (compare to equation 1.51):

\[
\frac{\partial L}{\partial t} = \alpha I_t \epsilon |v_n - v_\star| L^{3/2} - \frac{\alpha I_t}{c_L} \beta_S L^2 .
\]  

(D.59)

where \( \epsilon \equiv \hat{\omega}_t \hat{\omega}_n = \pm 1 \) specifies the relative direction (parallel or antiparallel) between \( (v_t - v) \) and \( (v_n - v) \). The reader is reminded that the flow must be homogeneous to arrive at this result: equation D.56 and the subsequent expressions for \( (v_t - v) \) only pertain when the vortex tangle is homogeneous, and the divergence term in equation D.51 only vanishes if \( L \) is spatially invariant. In [Geu95], Geurst points out that the Vinen equation is rederived in his model for the case of homogeneous flow only by combining the kinematic and dynamic equations for \( L \) (equations D.50 and D.55). He concludes, “It is therefore not clear how the single Vinen equation might be extended, in an appropriate way, to inhomogeneous flow phenomena” and cautions that the kinematic equation for the line density, equation D.50, is not in itself the same thing as Vinen equation. For a homogeneous flow, when \( \partial L/\partial t = 0 \), equation D.51
reduces to equation D.47 and the Schwarz result that $\beta_3 L^{1/2} = \epsilon c_L |v_n - v_n|$ emerges. Schwarz assumes that $\epsilon = +1$. Geurst states that $\epsilon = +1$ is the appropriate choice "when the vortex tangle is in a state near to equilibrium." I interpret this statement to mean that only $\epsilon = +1$ works if $\partial L/\partial t = 0$ is close to being satisfied, otherwise $L^{1/2}$ would not be positive. Far from equilibrium, line length is being produced or destroyed, and either sign of $\epsilon$ is possible.

D.4 Including a Large Gradient in $L$

In a turbulent flow that is not in equilibrium, it is possible for the vortex line density to be nonuniform. As one example of such behavior, a sharp transition from laminar to turbulent flow can form and propagate through the fluid. The laminar-turbulent transition has been observed to occur as a sharp, propagating front in a number of experiments [Mar87, Sle82b, Dzi62, Bha64, Pes62, Men59]. When such a large gradient in the vortex line density is present, Geurst asserts in §6 of [Geu89] that "the vortex tangle is not sufficiently characterized by the local values of $L$ and $v_I - v$. The internal energy density $U$ should include the gradient $\partial L/\partial x$ as an additional macroscopic variable." He defines a new thermodynamic parameter $g_t$ for the new contribution to the internal energy. In differential form, the energy becomes (compare equation D.24)

$$dU = \mu d\rho + T dS + \mu dL + g_t d\left(\frac{\partial L}{\partial x}\right) - P_n d(v_n - v) - P_d d(v_I - v) . \quad (D.60)$$
By regrouping the differential terms in $L$, the internal energy may be written in the alternative form

$$dU = \mu d\rho + TdS + \mu_\text{t} dL - P_n d(v_n - v) - P_d (v_t - v) + \frac{\partial}{\partial x}(g_t dL) . \quad (D.61)$$

where

$$\dot{\mu}_t = \mu_t - \frac{\partial g_t}{\partial x} . \quad (D.62)$$

When this new expression for $U$ is inserted into the Lagrangian density $\Lambda$, the resulting equations of motion and of conservation of energy and momentum can be obtained by performing the following substitutions in the equations in Section D.3:

$$\mu_t \rightarrow \dot{\mu}_t , \quad (D.63)$$

$$Q \rightarrow Q - g_t \frac{\partial L}{\partial t} , \quad (D.64)$$

$$\Pi \rightarrow \Pi + g_t \frac{\partial L}{\partial x} . \quad (D.65)$$

The quantity $g_t$ is taken to be

$$g_t = \frac{\partial L}{\partial x} \quad (D.66)$$

where $\dot{g}_t$ depends only on the variables describing the vortex tangle, namely $\rho, \rho_s, L,$ and $\kappa$. By dimensional analysis,

$$\dot{g}_t = \rho_s \frac{\kappa^2}{L^2} \gamma_t , \quad (D.67)$$

where $\gamma_t$ is a dimensionless coefficient that only depends on temperature through $\rho_s/\rho$. The coefficient $\gamma_t$ is the sixth and last parameter defined in the 1-D Geurst model. Finally, the dissipative relation is modified according to

$$-\dot{\mu}_t = C r_t . \quad (D.68)$$
This relation yields an extended version of the balance equation for the vortex tangle:

\[
\frac{\partial L}{\partial t} + \frac{\partial}{\partial x} (Lv_t) + \kappa \frac{\gamma L}{2} \left[ \frac{2}{L} \left( \frac{\partial L}{\partial x} \right)^2 - \frac{\partial^2 L}{\partial x^2} \right] - \frac{\kappa}{\rho_s \gamma} \frac{\partial L}{\partial x} (\rho_s \gamma L) = \frac{\beta}{2\gamma} \left| v_t - v \right| |L|^{3/2} - \frac{\beta_s}{\gamma} L^2 .
\]  
(D.69)

This expression predicts the evolution of the vortex line density in situations where a large gradient in \( L \) exists. A turbulent-laminar front need not be present to create a sufficiently large gradient in \( L \) for this expression to be valid. Because large gradients in \( L \) are produced in our diverging flow, this prediction was tested against our experimental results as described in Section 3.5 of Chapter III.

In a recent preprint [Geu95], Geurst makes a small correction to his derivation of equation D.69. This correction comes from noting that the line-length potential \( \mu_t \) appearing in D.62 is no longer given by equation D.45, the appropriate expression in the absence of a large gradient in \( L \). In the presence of a large gradient in \( L \), the line-length potential contains an additional contribution, becoming

\[
\mu_t = \rho_s \kappa \beta_s - (v_t - v) \frac{\partial P}{\partial L} - \rho_s \frac{\kappa^2}{L^3} \gamma L \left( \frac{\partial L}{\partial x} \right)^2 .
\]  
(D.70)

The last term in equation D.70 originates from the additional line-length energy \( g \frac{d}{dx} \left( \frac{\partial L}{\partial x} \right) \) which appears in equation D.60. Considring this new contribution to \( dU \) alone,

\[
dU_{\text{new}} = g \frac{d}{dx} \left( \frac{\partial L}{\partial x} \right) = \rho_s \frac{\kappa^2}{L^3} \gamma L \frac{\partial L}{\partial x} \frac{d}{dx} \left( \frac{\partial L}{\partial x} \right) = \rho_s \frac{\kappa^2}{L^3} \gamma L \left[ \frac{1}{2} \left( \frac{\partial L}{\partial x} \right)^2 \right] ,
\]  
(D.71)

the new contribution to the line-length energy is obviously

\[
U_{\text{new}} = \frac{1}{2} \rho_s \frac{\kappa^2}{L^3} \gamma L \left( \frac{\partial L}{\partial x} \right)^2 .
\]  
(D.72)
As expressed in equation 6.5 of [Geu89], the generalized line-length potential is equal to the functional derivative of the total internal energy density

\[
\hat{\mu}_t = \frac{\delta U}{\delta L} = \frac{\partial U}{\partial L} - \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial (\partial L/\partial x)} \right).
\]

Upon adding \(U_{\text{new}}\) to the previous result for \(U\) (see equations D.26 and D.60), differentiation yields

\[
\hat{\mu}_t = \mu_t - \frac{\rho_s \kappa^2}{L^3} \gamma_t \left( \frac{\partial L}{\partial x} \right) - \frac{\partial}{\partial x} \left[ \frac{\rho_s \kappa^2}{L^2} \gamma_t \frac{\partial L}{\partial x} \right],
\]

where the \(\mu_t\) in this expression is that of equation D.45. The term in square brackets is nothing more than \(g_t\) (see equations D.66 and D.67). The only way to reconcile the above expression with equation D.62 for \(\hat{\mu}_t\) is to realize that \(\mu_t\) in the presence of a large gradient in \(L\) is given by equation D.70. In other words, the modification to the line-length potential is actually split between this additional term hidden in \(\mu_t\) and the added derivative \(\partial g_t/\partial x\). When \(\hat{\mu}_t\) is inserted into equation D.68, the heretofore neglected contribution to \(\mu_t\) adds the following term in \(r_t\):

\[
r_t = -\frac{1}{C} \left[ -\frac{\rho_s \kappa^2}{L^3} \gamma_t \left( \frac{\partial L}{\partial x} \right)^2 \right] = \frac{\kappa}{L} \frac{\gamma_t}{\gamma} \left( \frac{\partial L}{\partial x} \right)^2,
\]

which must be added to the right-hand side of equation D.69. The correct balance equation for the vortex tangle is therefore exactly as given in equation D.69, except that the \((\partial L/\partial x)^2\) term is not multiplied by two. Because the second derivative term \((\partial^2 L/\partial x^2)\) is dropped in the simplification of equation D.69 employed in the present work, implementing this correction to the \((\partial L/\partial x)^2\) term merely changes the values obtained for the fit parameter \(\gamma_t\) by a factor of two, and bears no other significant consequence.
Appendix E

The Geurst Parameters and Program GEURST

E.1 Interpreting the Geurst Parameters

In [Geu92], Geurst relates his coefficients $\beta$, $\gamma$, $\gamma_{11}$, $\gamma_{12}$, and $\gamma_{22}$ to the Schwarz coefficients $c_L$, $I_L$, $I_{||}$, $\alpha$, $\alpha'$, and $\alpha''$. The last of these Schwarz parameters is a second-order friction coefficient that is sometimes used, but usually neglected, in the equation describing the local motion of a vortex line. The relations quoted here include $\alpha''$, but in the end the limit $\alpha'' = 0$ will be adopted for the present work.

Geurst expresses quantities and relations in terms of the mass velocity $v$ instead of the superfluid velocity $v_s$ used by Schwarz. The Schwarz coefficients in the mass flow frame are denoted by the subscript $v$, and the corresponding Geurst coefficients are indicated with a hat. The relations between the Schwarz coefficients in the two frames are

\[
\begin{align*}
\beta_v &= \frac{\rho_s}{\rho} \beta_S , \\
\alpha_v &= \frac{\rho}{\rho_s} \alpha , \\
1 - \alpha_v' &= \frac{\rho}{\rho_s} (1 - \alpha') , \\
1 - \alpha_v'' &= \frac{\rho}{\rho_s} (1 - \alpha'') .
\end{align*}
\] (E.1)

Likewise, the Geurst coefficients in the two frames are related by
\[ \dot{\beta} = \frac{P_s}{P} \beta, \quad \dot{\gamma} = \frac{P_s}{P} \gamma, \]
\[ \dot{\gamma}_{ij} = \frac{P_s}{P} \gamma_{ij} \quad \text{where} \quad i, j \in (1, 2), \quad (E.2) \]
\[ \dot{\Gamma} = \frac{\dot{\beta}^2}{4\dot{\gamma}} + \dot{\gamma}_{11} + 2\dot{\gamma}_{12} + \dot{\gamma}_{22}. \]

For the case of homogeneous turbulence, the Geurst model derives expressions for \( F_{n_s}, \)
\((v_t - v),\) and the generalized Vinen equation in terms of the parameters \( \beta, \gamma, \gamma_{11}, \gamma_{12}, \) and \( \gamma_{22}.\) By comparing these expressions to the analogous equations for the same quantities derived in the Schwarz model, Geurst arrives at five independent equations which define the relation between his parameters and the Schwarz parameters. Solving these five equations for the modified coefficients \( \dot{\beta}, \dot{\gamma}, \dot{\gamma}_{11}, \dot{\gamma}_{12}, \) and \( \dot{\gamma}_{22} \) yields

\[ \dot{\beta} = \frac{c_L}{(\alpha' v - \alpha'' v)I_\| + \alpha'' v + (1 - \alpha'' v)I_e c_L}, \]
\[ \dot{\gamma} = \frac{\alpha v}{\alpha v I_t} \left( \frac{\dot{\beta}}{2} \right), \]
\[ \dot{\gamma}_{11} = \frac{\alpha v}{1 - \alpha' v} \left[ (1 - \alpha'' v)(I_\| - 1) - 1 + \frac{\dot{\beta}}{2 c_L} \right], \quad (E.3) \]
\[ \dot{\gamma}_{12} = \frac{\alpha v}{1 - \alpha' v} (1 - \alpha'' v)(1 - I_\|), \]
\[ \dot{\gamma}_{22} = \frac{\alpha v}{1 - \alpha' v} \left[ (1 - \alpha'' v)(I_\| - 1) + 1 \right]. \]

By using the relations given in E.1 and E.2 above, the Geurst parameters in the superfluid frame are found to be given in terms of the Schwarz parameters by

\[ \frac{\beta}{2 c_L} = \left[ (\alpha' - \alpha'')I_\| - \frac{P_n}{\rho} + \alpha'' + (1 - \alpha')I_e c_L \right]^{-1}, \quad (E.4) \]
\[ \frac{\beta}{2\gamma} = \frac{\alpha I_t}{(\alpha' - \alpha'')I_\| - \frac{P_n}{\rho} + \alpha''}, \quad (E.5) \]
\begin{align*}
\gamma_{11} &= \frac{\alpha}{1 - \alpha'} \left[ \left( \frac{\rho}{\rho_*} \right)^2 (1 - \alpha'')(I_{||} - 1) - \frac{\rho}{\rho_*} + \frac{\beta}{2c_L} \right], \\
\gamma_{12} &= \frac{\alpha}{1 - \alpha'} \left( \frac{\rho}{\rho_*} \right)^2 (1 - \alpha'')(1 - I_{||}), \\
\gamma_{22} &= \frac{\alpha}{1 - \alpha'} \left[ \left( \frac{\rho}{\rho_*} \right)^2 (1 - \alpha'')(I_{||} - 1) + \frac{\rho}{\rho_*} \right],
\end{align*}

(E.6)\tag{E.6}

(E.7)\tag{E.7}

(E.8)\tag{E.8}

Some useful relations involving combinations of the above parameters can be derived:

\begin{align*}
\gamma_{12} + \gamma_{22} &= \frac{\alpha}{1 - \alpha'} \left( \frac{\rho}{\rho_*} \right), \\
\gamma_{11} + 2\gamma_{12} + \gamma_{22} &= \frac{\alpha}{1 - \alpha'} \left( \frac{\beta}{2c_L} \right), \\
\Gamma &\equiv \frac{\beta^2}{4\gamma} + \gamma_{11} + 2\gamma_{12} + \gamma_{22} = \frac{1}{(1 - \alpha')I_L} \left( \frac{\beta}{2\gamma} \right).
\end{align*}

(E.9)\tag{E.9}

(E.10)\tag{E.10}

(E.11)\tag{E.11}

\section*{E.2 Program GEURST}

This program computes the line density $L(r)$ in the Geurst model, then integrates the local temperature gradient to obtain the turbulent temperature difference to obtain $\Delta T'(r)$ a function of $r$. The program asks the user to specify $T_0$, $\dot{Q}$, $L$, and $\gamma_\ell$, as well as the integration range in $r$ and step size $\Delta r$. The integration is performed in the same manner as in program SCHWARZ (see Appendix B). This version of the program outputs $\Delta T'(r)$ at a user-specified number of positions $r$ to one file, and $L(r)$ and $\beta_\ell(r)$ to another file. Another version of the program produced $\Delta T'$ as a function of $\dot{Q}$ for a single position $r$. 
program GEURST

C Coded by: J. Kafkalidis

C First version written: May 1992
C This version last updated: Dec 1994

C Features of this version:

C --- FORM OF CALCULATION -----------------------------------------------

C Active code:
C Value of epsilon selects direction of vortex line drift velocity:
C epsilon = +1 : v_l parallel to v_n
C epsilon = -1 : v_l antiparallel to v_n

C Suppressed code (commented out):
C Can switch to earlier version of code wherein coefficient B
C (linear dL/dr piece) is multiplied by ad hoc fudge factor phi
C to change its relative magnitude and/or algebraic sign.

C Active code:
C Uses negative root in the quadratic equation as solution for dL/dr

C Suppressed code (commented out):
C Can switch to using positive root in quadratic equation as solution
C for dL/dr

C --- CONTROL OF NUMERICAL ACCURACY -------------------------------------

C Integration step size rdiv is decreased by a factor of ten at
C r = 6.0 cm, from delta_r = .01 cm to delta_r = .001 cm.
C Can vary step size by changing values of ndiv1 and ndiv2.

C Adopts double precision for all real variables

C Overrides default maximum number of iterations (500) set by DIVPRK by
C asking user to input value of MXSTEP, i.e. PARAM(4). This allows the
C user to subdivide range in r into any number of divisions by inputting
C ndiv (so long as MXSTEP is set sufficiently large as rdiv gets small).

C If L(r) becomes less than 1.0E-37 at some position r, sets L = 0 and
grad_T(r) = 0 at all r beyond that point, and continues to compute
delta T. Program therefore won't crash because L becomes too small -
will only crash if (B^2 - 4AC) becomes negative.

--- FORM OF OUTPUT --------------------------------------------------

Output L(r) and gradT(r) to a file designated by the user.

Integrates gradT(r) to get delta T(r) for specified number of
subdivisions in r, and outputs the result to a second file.

For each value of r between rmin and rmax at which delta T(r) is
computed, the range of integration is from the endpoint rmax to r.

Includes rmax and excludes rmin in integration sum (a similar version
of this program includes rmin and excludes rmax - makes at most a small
difference in the exact delta T(r) computed).

---------------------------------------------------------------------

real*8 a0, kappa, rkappa, T(5), c1(5), cL(5), Ill(5)
real*8 alpha(5), aprime(5), rhos(5), rhon(5), rho(5), S(5)
real*8 rmin, rmax, rdiv, h, q, param(50), rinc1, rinc2, tol
real*8 rhosratio, rhonratio, eta1, eta2, c3, beta2, gamma
real*8 coeff2, eta4, aratio, gamma12, gamma22, Y(1), Linit
real*8 Lroot, bschwarz, eta3, velocityL, B, Cpart, C, gamlimit
real*8 gammal, coeff1, r, rend, end1, end2, Yprime(1), Lout(5001)
real*8 rout(5001), vns, gradT1, gradT2, gradT(5001)
real*8 area, sum, part1, div2, deltaT(501)
real*8 area1, sum1, area2, sum2, delTprev, div1

real*8 epsilon

Can switch between versions using epsilon and phi by switching roles
of active and commented statements involving epsilon and phi
throughout program

real*8 phi

character*12 fname, fname2
integer ndiv1, ndiv2, neq, ido, nendl, nend2, nstart2, nout
integer npartl, npart2, idummy, idiv2, kstart, kend, klim
integer kbegin, k2, k4, k1, k3, idiv1, istart

common j, rkappa, cl, a0, gamma, eta2, eta3, coeff1, coeff2, epsilon
common j, rkappa, c1, a0, gamma, eta2, eta3, coeff1, coeff2, phi

external gradL

c a0 = 1.3E-8
kappa = 9.966305E-4
rkappa = 9.966305E-4 / (8*asin(1.0))

c open(unit=8,iostat=iostat,status='old',file='geurstin.;')
do 10 i=1,5
   read(8,*), T(i)
10 continue
do 11 i=1,5
   read(8,*), c1(i)
11 continue
do 12 i=1,5
   read(8,*), cL(i)
12 continue
do 13 i=1,5
   read(8,*), Il(i)
13 continue
do 14 i=1,5
   read(8,*), Ill(i)
14 continue
do 15 i=1,5
   read(8,*), alpha(i)
15 continue
do 16 i=1,5
   read(8,*), aprime(i)
16 continue
do 17 i=1,5
   read(8,*), rhos(i)
17 continue
do 18 i=1,5
   read(8,*), rhon(i)
continue
do 19 i=1,5
read(8,*) rho(i)
continue
do 20 i=1,5
read(8,*) S(i)
continue
close(unit=8)
c
rmin = 2.5
rmax = 12.5
rdiv = 6.0
h = 0.025
c
cwrite(5,21) rdiv
format(1x,'r is divided into two regions around r = ',F6.3)
cwrite(5,*) 'Enter temperature (3 = 1.3K, 4 = 1.4K, 5 = 1.5K, 6 = 1.6K, 7 = 1.7K)'
cread(6,*) j
c
write(5,*) 'Enter heat flux in mW'
cread(6,*) q
ndiv1 = 10000
ndiv2 = 1000
cwrite(5,*) 'Enter maximum allowed number of steps for DIVPRK'
cwrite(5,*) '0 sets value to default of 500'
c
read(6,*) param(4)
rinc1 = (rmax - rmin)/ndiv1
rinc2 = (rmax - rmin)/ndiv2
cwrite(5,*) 'Enter tolerance level'
cwrite(5,*) '(maximum suggested value = 0.001)'
cread(6,*) tol
c
read(6,*) epsilon
c
write(5,*) 'Enter epsilon: +1 sets v_L parallel to v_nB'
cwrite(5,*) '-1 sets v_L antiparallel to v_nB'
c-----------------------------------------------------------
c write(5,*) 'Enter fudge factor for multiplying B (=vL)'
c read(6,*) phi

c----------------------------------------------------------
rhosratio = rhos(j)/rho(j)
rhonratio = rhon(j)/rho(j)
eta1 = (aprime(j)*Ill(j)) - rhonratio
eta2 = (1.0-aprime(j))*Il(j)
c3 = 12.5 / (rhos(j)*S(j)*T(j)*h)
beta2 = 1.0/( ( etal/cL(j)) + eta2 )
gamma = ( etal*beta2 )/( alpha(j)*Il(j) )

rhosratio = rhos(j)/rho(j)
rhonratio = rhon(j)/rho(j)
eta1 = (aprime(j)*Ill(j)) - rhonratio
eta2 = (1.0-aprime(j))*Il(j)
c3 = 12.5 / (rhos(j)*S(j)*T(j)*h)
beta2 = 1.0/( ( etal/cL(j)) + eta2 )
gamma = ( etal*beta2 )/( alpha(j)*Il(j) )

-------------------
c c3 in units of (sec^-2)/gm, c4 unitless
 c beta2 = beta/2
 c beta2, gamma, eta1, eta2, eta4 unitless

-------------------
coeff2 = alpha(j)*Il(j)/etal
eta4 = { aprime(j)*Ill(j) ) - 1.0
aratio = alpha(j)/( 1.0 - aprime(j) )
gamma22 = (aratio/rhosratio) - gammal2
write(5,23) rmin
23 format(lx,'Enter initial L value at r = ',F6.3)
read(6,*) Y(l)
Linit = Y(l)
r = rmin
Lroot = sqrt(Y(1))
bschwarz = rkappa*log( 1.0/(c1(j)*a0*Lroot) )
eta3 = etal*10000.0*c3*q
velocityL = epsilon*(eta3/r) + (eta2*Lroot*bschwarz)
B = epsilon*( velocityL + ( eta2*Lroot*(bschwarz - rkappa)/2 ) )
Cpart = ( (Y(l)/gamma) + (epsilon*eta2*Lroot/r) )*bschwarz

--------------------
C = ( Cpart - (coeff2*velocityL*Lroot) )*Y(l)
gamlimit = (B**2)*gamma*Y(l)/( 8.0*kappa*C )
write(5,25) gamlimit
format(ix,'Enter gamma_l parameter: limiting value is ',E11.4)
read(6,*) gammal
coeffl = 4.0*kappa*gammal/gamma

vns in units of cm/s
L = Y(l) in units of 1/cm^2
gradT(r) in units of mK/cm

write(5,*),'Enter name of output file for L(r) & grad_T(r).'
read(6,30) fname

open(unit=9,iostat=ios,status='new',file=fname,
1 carriagecontrol='list')
write(9,31) fname
write(9,32) T(j)
write(9,33) cL(j)
write(9,34) Linit
write(9,35) gammal
write(9,36) epsilon
write(9,37) tol
write(9,38) rmin
write(9,39) rmax
write(9,40) h
write(9,41) q
write(9,42) Q
write(9,42)
format(4x,'No.',4x,'r',8x,'L(r)',7x,'gradT(r)')
c
neq = 1
ido = 1
rend = rmin
end1 = (rdiv - rmin)/rinc1
nend1 = nint(end1)
end2 = (rmax - rdiv)/rinc2
nend2 = nend1 + nint(end2)
nstart2 = nendl - t - 1

do 50 i = 1,nendl
rend = rend + rincl
Lout(i) = Y(l)
rout(i) = r
Lroot = sqrt(Y(l))
bschwaxz = rkappa*log( 1.0/(c1(j)*a0*Lroot) )

vns = c3*q*10000.0/r
gradT1 = kappa*Y(l)*alpha(j)*((epsilon*Il(j)*Lroot*bschwarz)
  1 - (Ill(j)*vns))/S(j)

gradT2 = 2*rhonratio*rhosratio*(vns**2)/( S(j)*r )
write(9,46) i, rout(i), Lout(i), gradT(i)
46 format(1x,I5,2x,F6.3,2(2x,E11.4))
if ( Y(l) .LT. 1.0E-37 ) go to 51

50 continue
go to 54

c
51 idummy = i + 1

do 52 i = idummy,nendl
r = r + rincl
Lout(i) = 0.0
rout(i) = r
gradT(i) = 0.0
write(9,46) i, rout(i), Lout(i), gradT(i)
52 continue
r = r + rinc1
do 53 i = nstart2, nend2
Lout(i) = 0.0
rout(i) = r
gradT(i) = 0.0
write(9,46) i, rout(i), Lout(i), gradT(i)
r = r + rinc2
continue
ido = 3
i = nend2 + 1
rout(i) = r
Lout(i) = 0.0
gradT(i) = 0.0
write(9,46) i, rout(i), Lout(i), gradT(i)
call DIVPRK(ido,neq,gradL,r,rend,tol,param,Y)
close(unit=9)
go to 59

c
54 do 55 i = nstart2, nend2
rend = rend + rinc2
Lout(i) = Y(i)
rout(i) = r
Lroot = sqrt(Y(i))
bschwarz = rkappa*log(1.0/(c1(j)*a0*Lroot))
vns = c3*q*10000.0/r
gradT1 = kappa*Y(i)*alpha(j)*(epsilon*Il(j)*Lroot*bschwarz)
1 -(111(j)*vns)/S(j)
-------------------------------------------------------------------------
gradT2 = 2*rhonratio*rhosratio*(vns**2)/(S(j)*r)
gradT(i) = 0.0 - (1000.0*(gradT1 + gradT2))
write(9,46) i, rout(i), Lout(i), gradT(i)
if ( Y(1) .LT. 1.0E-37 ) go to 56
call DIVPRK(ido,neq,gradL,r,rend,tol,param,Y)
55 continue
go to 58

c
56 idummy = i + 1
do 57 i = idummy,nend2
r = r + rinc2
rout(i) = r
Lout(i) = 0.0
gradT(i) = 0.0
write(9,46) i, rout(i), Lout(i), gradT(i)

57 continue
ido = 3
i = nend2 + 1
r = r + rinc2
rout(i) = r
Lout(i) = 0.0
gradT(i) = 0.0
write(9,46) i, rout(i), Lout(i), gradT(i)
call DIVPRK(ido,neq,gradL,r,rend,tol,param,Y)
close(unit=9)
go to 59

c
58 ido = 3
i = nend2 + 1
rout(i) = r
Lout(i) = Y(1)
Lroot = sqrt(Y(1))
bschwarz = rkappa*log(1.0/(c1(j)*a0*Lroot))
vns = c3*q*10000.0/r
gradT1 = kappa*Y(1)*alpha(j)*((epsilon*Il(j)*Lroot*bschwarz)
1 - (Il(j)*vns))/S(j)

c gradT1 = kappa*Y(1)*alpha(j)*((Il(j)*Lroot*bschwarz)
c 1 - (Il(j)*vns))/S(j)
c-----------------------------------------------
c gradT2 = 2*rhonratio*rhosratio*(vns**2)/ ( S(j)*r )
gradT(i) = 0.0 - (1000.0*(gradT1 + gradT2))
write(9,46) i, rout(i), Lout(i), gradT(i)
call DIVPRK(ido,neq,gradL,r,rend,tol,param,Y)
c
close(unit=9)
c
59 write(5,60) T(j), fname
60 format(1x,'L(r), gradT(r) for T = ',F4.2,'K output to file ',a)
c
write(5,*) 'Enter number of divisions in r for outputting delta T(r). Maximum = 500.'
write(5,*) 'Both ',ndiv1,' and ',ndiv2,' must be integer
1 multiples of this value.
read(6,*), nout
write(6,*), 'Enter name of file for saving delta T(r)'
read(6,30), fname2
open(unit=10,iostat=ios,status='new',file=fname2,
carriagecontrol='list')
write(10,31), fname2
write(10,32), T(j)
write(10,*), 'Parameter values:
write(10,33), cL(j)
write(10,34), Linit
write(10,35), gammal
c--------------------------
c write(10,36), phi
c--------------------------
write(10,36), epsilon
write(10,37), tol
write(10,38), rmin
write(10,39), rmax
write(10,40), h
write(10,41), q
write(10,*), 'r delta T(r)' 
c area = 0.0
sum = 0.0
part1 = nout*(rdiv - rmin)/(rmax - rmin)
npart1 = nint(part1)
npart2 = nout - npart1
div2 = ndiv2/nout
idiv2 = nint(div2)
kstart = nend2 - idiv2 + 2
kend = nend2 + 1
k2 = kend - 1
k4 = kend - 2
klim = kend - 3
kbegin = kstart + 3
do 100 i = 1, npart2
   do 70 k = kbegin, klim
      area = gradT(k) + area
      continue
   70 sum = 0.375*(gradT(kbegin) + gradT(kend))
k1 = kstart + 1
385

\[ \text{Bum} = \text{Bum} + \left( \frac{7.0}{6.0} \right) \left( \text{gradT}(k_1) + \text{gradT}(k_2) \right) \]

\[ k_3 = k_{\text{start}} + 2 \]

\[ \text{sum} = \text{sum} + \left( \frac{23.0}{24.0} \right) \left( \text{gradT}(k_3) + \text{gradT}(k_4) \right) \]

\[ \text{deltaT}(i) = \left( (\text{area} + \text{sum}) \times rinc_2 \right) \]

\[ k_{\text{out}} = k_{\text{start}} - 1 \]

\[ \text{write}(10,75) \text{ rout}(k_{\text{out}}), \text{deltaT}(i) \]

\[ \text{format}(1x,F6.3,2x,E11.4) \]

\[ k_{\text{start}} = k_{\text{start}} - idiv_2 \]

\[ k_{\text{begin}} = k_{\text{start}} + 3 \]

\[ \text{area} = 0.0 \]

\[ \text{sum} = 0.0 \]

\[ 100 \quad \text{continue} \]

\[ \text{c} \]

\[ k_{\text{start}} = k_{\text{start}} + idiv_2 \]

\[ \text{area}_1 = 0.0 \]

\[ \text{sum}_1 = 0.0 \]

\[ \text{do 80 } k = k_{\text{start}}, k_{\text{lim}} \]

\[ \text{area}_1 = \text{gradT}(k) + \text{area}_1 \]

\[ 80 \quad \text{continue} \]

\[ \text{sum}_1 = 0.375 \times \text{gradT}(k_{\text{end}}) \]

\[ \text{sum}_1 = \text{sum}_1 + \left( \frac{7.0}{6.0} \right) \times \text{gradT}(k_2) \]

\[ \text{sum}_1 = \text{sum}_1 + \left( \frac{23.0}{24.0} \right) \times \text{gradT}(k_4) \]

\[ \text{deltaT}_{\text{prev}} = \left( \text{sum}_1 + \text{area}_1 \right) \times rinc_2 \]

\[ \text{c} \]

\[ \text{area}_2 = 0.0 \]

\[ \text{sum}_2 = 0.0 \]

\[ \text{div}_1 = \text{n}\text{div}/\text{nout} \]

\[ \text{idiv}_1 = \text{uint}(\text{div}_1) \]

\[ k_{\text{start}} = k_{\text{start}} - \text{idiv}_1 \]

\[ k_{\text{begin}} = k_{\text{start}} + 3 \]

\[ k_{\text{end}} = n_{\text{end}} + 1 \]

\[ i_{\text{start}} = n_{\text{part}}^2 + 1 \]

\[ \text{do 150 } i = i_{\text{start}}, \text{nout} \]

\[ \text{do 90 } k = k_{\text{begin}}, k_{\text{end}} \]

\[ \text{area}_2 = \text{gradT}(k) + \text{area}_2 \]

\[ 90 \quad \text{continue} \]

\[ \text{sum}_2 = 0.375 \times \text{gradT}(k_{\text{start}}) \]

\[ k_1 = k_{\text{start}} + 1 \]

\[ \text{sum}_2 = \text{sum}_2 + \left( \frac{7.0}{6.0} \right) \times \text{gradT}(k_1) \]

\[ k_3 = k_{\text{start}} + 2 \]

\[ \text{sum}_2 = \text{sum}_2 + \left( \frac{23.0}{24.0} \right) \times \text{gradT}(k_3) \]

\[ \text{deltaT}(i) = \left( (\text{area}_2 + \text{sum}_2) \times rinc_1 \right) + \text{deltaT}_{\text{prev}} \]
kout = kstart - 1
write(10,75) rout(kout), deltaT(i)
kstart = kstart - idiv1
kbegin = kstart + 3
area2 = 0.0
sum2 = 0.0
150 continue

c
close(unit=10)
c
write(5,*) 'Next temperature? (3 = 1.3K, 4 = 1.4K, 5 = 1.5K,
1 6 = 1.6K, 7 = 1.7K, 0 = end)'
read(6,*) j
if(j.GT.0) goto 22
end

c
subroutine gradL(neq,r,Y,Yprime)
c
real*8 a0, rkappa, c1(5), gamma, eta2, eta3, coeff1, coeff2
real*8 Y(l), Yprime(l), Lroot, bschwarz, Cpart
real*8 velocityL, rad, radical, A, B, C, Lminus
c
real*8 epsilon
c-----------------------------------
c real*8 phi
c-----------------------------------
c
integer j, neq

c
common j, rkappa, c1, a0, gamma, eta2, eta3, coeff1, coeff2,
1 epsilon
c------------------------
c common j, rkappa, c1, a0, gamma, eta2, eta3, coeff1, coeff2, phi
c------------------------
c
Lroot = sqrt(Y(1))
bschwarz = rkappa*log( 1.0/(c1(j)*a0*Lroot) )
A = coeff1/Y(1)
c Note: This 'A' equals 2A in the standard quotation of the roots to
c the quadratic equation

```
velocityL = epsilon*(eta3/r) + (eta2*Lroot*bschwarz)
B = epsilon*( velocityL + ( eta2*Lroot*(bschwarz-rkappa)/2 ) )
Cpart = ( (Y(1)/gamma) + (epsilon*eta2*Lroot/r) )*bschwarz
```

c velocityL = (eta3/r) + (eta2*Lroot*bschwarz)
```
B = phi*(velocityL + (eta2*Lroot*(bschwarz-rkappa)/2) )
Cpart = ( (Y(1)/gamma) + (eta2*Lroot/r) )*bschwarz
```

c C = ( Cpart - (coeff2*velocityL*Lroot) )*Y(1)
```
rad = (B**2) - (2.0*A*C)
```
rational = sqrt(rad)
```
Lminus = -(B + rational)/A
```
Yprime(l) = Lminus
```

c The negative root to the quadratic has been chosen. To select the
positive root, comment out the above two lines and substitute:
```
Lplus = -(B - radical)/A
Yprime(l) = Lplus
```
```
end
```

Input file GEURSTIN for program Geurst:

```
1.3
1.4
1.5
1.6
1.7
1.872670
1.667933
1.541372
1.458880
1.391074
0.080602
0.099320
0.116928
0.133688
```
0.150415
0.450216
0.454358
0.456703
0.457503
0.457011
0.751715
0.759453
0.767111
0.776302
0.789024
0.036
0.052
0.073
0.098
0.127
0.014
0.017
0.018
0.016
0.012
0.1383
0.1342
0.1285
0.1211
0.1116
0.006812
0.010950
0.016620
0.024050
0.033630
0.145103
0.145121
0.145151
0.145195
0.145258
0.08534E7
0.1329E7
0.1966E7
0.2853E7
0.4024E7
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