Hybrid (MM–UTD) Analysis of EM Scattering by Finned Objects

A Dissertation
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the Degree Doctor of Philosophy in the
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by

Mimi Hsu

* * * * *

The Ohio State University
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Dissertation Committee:
Prof. Prabhakar Pathak
Prof. E.H. Newman
Prof. W.D. Burnside

Approved by:

Prabhakar Pathak
Advisor
Department of Electrical Engineering
Dedication

To my father, my mentor, and my best-friend,

Tse-Wen Hsu

whose spirit has, is, and always will be inside of me.

His cap and gown will be difficult to fill. Forever and a day daddy.
Acknowledgements

I would like to express my appreciation to my advisor, Dr. Prabhakar Pathak, for his guidance and most importantly his trust in allowing me the freedom to make my own mistakes and successes. Thanks also go to my committee members, Dr. Ed Newman and Dr. Dennie Burnside for their helpful comments and suggestions. My tenure here has taught me the meaning behind Nietzsche’s philosophy that, “Spirit is the life that cuts back into life; with it’s suffering it increases its knowledge.”[41]

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VITA

April 14, 1964 ......................... Born in Bryn Mawr, PA.
June 1982 ............................... Graduated Marple-Newtown H.S.,
                                          Newtown Square, PA
May 1985 ............................... B.S. in Engineering Math, University of
                                          California, Berkeley, CA.
June 1985 – September 1993 ............. Research Engineer, Lockheed Missiles and
                                          Space, Sunnyvale, CA.
December 1986 .......................... M.S.E.E., Purdue University, West
                                          Lafayette, IN.
September 1988 - Present ............. Graduate Research Associate, The
                                          ElectroScience Laboratory, The Ohio
                                          State University, Columbus, OH.

FIELDS OF STUDY

Major Field: Electrical Engineering

Electromagnetic Theory .................. Professor R.G. Kouyoumjian
Control Theory .......................... Professor S. Yurkovich
Mathematics ............................ Professor G.J. Majda
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CHAPTER I
Introduction

Hybrid methods, the combination of two or more analytical or numerical methods, are used for analyzing problems more accurately or efficiently than is generally possible using a single method alone. This dissertation presents a hybrid combination of the method of moments (MM) [1, 2] and the uniform geometrical theory of diffraction (UTD) [4] developed to solve the electromagnetic (EM) radiation or scattering from objects in free-space which have the following characteristics:

1. Main Body
   (a) Perfectly conducting (PEC) smooth convex surface
   (b) Large with respect to the wavelength

2. Appendages
   (a) Perfectly conducting
   (b) Possibly irregularly shaped
   (c) Small with respect to the wavelength

A geometry with these characteristics is the focus of the present development but not a limitation. This will be discussed further in Chapter IX. This development can be employed, for example, to analyze the EM scattering by or the radiation by antennas on a missile or other aerospace vehicles. A simple model is shown in Figure
where the large convex scatterer would model the fuselage and the appendages would represent the antennas or control surfaces. Inclusion of the effects of such things as jet engine inlets may be added in separately. In other words, the field due only to the inlets, for example, is calculated separately and then added to the field scattered by the rest of the body without the inlets. This is also the case for the radome/antenna structure.

Commonly used methods to handle the geometry shown in Figure 1 include the MM at low frequencies and the UTD at high frequencies. The MM is a very accurate mathematical technique which is commonly employed to solve the governing integral equation but usually involves the computation, storage, and inversion of an extremely large matrix when analyzing electrically large radiating objects. In the MM, the induced current on the radiating object, which is the unknown within the governing integral equation, is expanded in terms of a set of basis functions. These basis functions along with a set of testing functions are used to then convert the
integral equation into a system of linear equations which can be solved by matrix inversion. The number of basis functions is directly proportional to the physical extent of the unknown currents on the radiating/scattering object. Therefore, it is best used on objects which are small in terms of the wavelength. On the other hand, the UTD is an asymptotic high frequency method which describes through closed-form, physically insightful expressions the basic radiation mechanisms such as reflection from a surface, diffraction from discontinuities, and creeping waves, etc., provided the pertinent UTD solutions are available. However, there are only a limited number of ray mechanisms for which UTD expressions currently exist. Therefore, it would be advantageous to use the UTD whenever possible to account for the presence of large objects (i.e., the reflection, diffraction, etc. from them) and thus not require the unknowns in the integral equation to exist on those surfaces. This would dramatically reduce the number of unknowns which need to be solved for via the MM procedure.

Past work on a hybrid combination of the MM and the GTD (or UTD) techniques is divided into two categories:

1. an extension of the GTD (or UTD) using MM [5, 6, 7, 8, 9], and

2. an extension of MM using the GTD [10, 11, 12].

One major difference between the two techniques is that the basis functions cover the entire body in the former method but only part of the geometry in the latter.

In the former, the current on the object is approximated by

\[
\mathbf{J}_{\text{obj}} = \mathbf{J}_{\text{MM}} + \mathbf{J}_{\text{GTD}}
\]

\[
\approx \sum_{n=1}^{N} \alpha_n \mathbf{f}_n + \sum_{m=1}^{M} \beta_m \mathbf{g}_m
\]  

(1.1)
where the \( \tilde{f}_n \) are standard subsectional MM basis functions usually limited in use to the region on the object where the current is rapidly-varying or where the mathematical form of it is not known accurately. Elsewhere, the \( \tilde{g}_m \) are used; they are GTD/UTD-based entire domain physical basis functions which attempt to mimic the asymptotic physical behavior of the current. The \( \alpha_n \) and \( \beta_m \) are the unknown coefficients to be solved for via the MM procedure.

Burnside, Yu, and Marhefka [5] demonstrated this method on a two-dimensional (2-D) wedge (Figure 2a). Pulse basis functions were used around the corner of the wedge while, away from the corner, a GTD current form, \( e^{-jk\rho/\sqrt{\rho}} \), was used. Note that \( \rho \) is the distance along the wedge faces measured from the edge. The method worked well except for grazing incidence. Sahalos and Thiele [6] determined that the GTD current form used was not an adequate basis function to describe the behavior of the current for grazing incidence. This brings to light a limitation of this hybrid technique: a good a priori knowledge of the behavior of the current in the GTD region is required for an accurate determination of the \( \beta_m \). Medgyesi-Mitschang and Wang [7] looked at the scattering by a body of revolution with a discontinuity in curvature (Figure 2b). They used optics-derived physical basis functions deduced from Fock theory when applicable; however, in the region near the discontinuity where the optics-derived solutions are inapplicable or require correction, overlapping triangles were used. Use of the Fock currents allowed for accurate results even for grazing illumination. Also, their method yielded good results for objects near resonance. Chang and Mittra [9] analyzed flat plates using subsectional MM rooftop basis functions coupled with half-plane currents where the half-plane solution assumes that each edge is infinite. Half-plane currents alone will not represent the edge behavior very well. In the case of a PEC strip, MM basis functions were used at the edges to remedy this problem. In the case of a PEC triangular plate, MM
Figure 2: Applications of the hybrid method based on an extension of the GTD (or the UTD) using the MM. (a) Burnside, Yu, and Marhefka [5], (b) Medgyesi-Mitschang and Wang [7], (c) Chang and Mittra [9], (d) Srikanth, Pathak, and Chuang [8].
regions are as shown in Figure 2c. The regions were chosen to better account for the effects of tip and multiple diffraction. The results for a triangular plate were very good at broadside, but the current did not exhibit the correct edge behavior because of the approximate nature of the half-plane basis functions. Srikanth, Pathak, and Chuang [8] analyzed the scattering from a PEC, infinitely-long, semi-circular cylinder (Figure 2d). Subsectional pulse basis functions were used around the edges while the UTD-based basis functions were used to describe the behavior of the current on the remainder of the body. The results were very good even for the case of grazing incidence on the cylindrical portion or normal incidence on the flat portion for which there was no UTD solution available at the time. This example is interesting in that the phenomena of sharp edge diffraction and surface diffraction are coupled in this case. All four of these applications use conventional MM basis functions in a limited region and GTD/UTD-based physical basis functions to describe the current in the rest of the region. The four sets of results all indicate, that in order for this method to work, the physical basis function must be an accurate description of the basic behavior of the actual current in that region.

In the second hybrid approach, an extension of the MM using the GTD, currents on specific parts of the geometry are approximated by standard MM basis functions while the rest of the geometry is accounted for by a special Green's function. Thus, the MM matrix equation which needs to be solved is

\[ V = [Z_0 + \Delta Z] I \] (1.2)

where \(Z_0\) is the mutual impedance between two MM basis functions in free space while \(\Delta Z\) is the mutual impedance between the same two basis functions but in the presence of the rest of the object. The key thing to note here is that: (1) the size of the impedance matrix is dependent only on the physical extent of the appendages,
and (2) the calculation of the $\Delta Z$ matrix requires a calculation of a special Green's function. This hybrid technique was described in [10], [11], and [12] where it was used to solve problems in which there is an antenna near or on an electrically large conducting body. The MM accounts for the antenna while the GTD/UTD was used to characterize the effect of the conducting body on the antenna. Thiele and Newhouse [10] used pulse basis functions to represent the current on the dipole while Ekelman [11] and Henderson [12] used overlapping piecewise sinusoids. However, in all three references, the large PEC body was accounted for by a special Green's function which was approximated by the UTD.

The hybrid method developed here extends the latter approach first introduced by Thiele and Newhouse [10]. The difference between the conventional MM procedure and the hybrid MM-UTD scheme lies in the type of Green's function in the kernel of the integral equation for the currents induced on the scatterer by an external source. Calculation of this special Green's function is equivalent to determination of the electric field at an arbitrarily-located observation point due to an arbitrarily-located point source in the presence of a PEC smooth convex object. It was made possible by being able to extend the UTD to predict the scattering due to a near field source. Previous UTD solutions applicable to this problem existed only for the following limiting positions of a source and an observer:

1. Mutual coupling between source and observer on the surface of a PEC convex object [16, 20, 21, 22];

2. Radiation from a source on a PEC convex object [15];

3. Scattering from a PEC convex object due to a source in the far field [14].

However, from Figure 1, it is seen that if subsectional basis functions are used on the fins, there will be sources and observers on the appendages which may be close
to the convex scatterer. In anticipation of this situation, new UTD extensions to
the previous expressions have been developed to handle the following two cases:

4. Extended UTD coupling: source and observer close to but not necessarily on
the surface of a PEC circular cylinder (Chapter III);

5. Extended UTD radiation: source close to but not necessarily on the surface
while the observer is far from the PEC circular cylinder (Chapter IV).

These results for the canonical circular cylinder can be generalized to treat the case
of an arbitrary convex surface using the UTD concepts as long as the curvature of
the surface is slowly-varying. With this extended UTD, one can treat the com­
plex object described earlier. The hybrid MM–UTD scheme restricts the unknown
currents to the appendages by using this UTD–based Green’s function to account
for the presence of the rest of the body which is represented here by an electrically
large convex scatterer. Such a hybrid technique makes it possible to analyze the
high frequency radiation/scattering from complex objects in a far more efficient and
tractable fashion than is currently possible by either the MM or the UTD alone.

In Chapter II, the hybrid technique is detailed. An important property of this
hybrid method is the fact that it is a self-consistent technique. The only approxi­
mation lies in the calculation of the special Green’s function. The two chapters after
that detail the theoretical development of the extension of the UTD to handle near
field sources. Chapter III presents the development and verification of the extended
UTD coupling equations which describe the EM scattering involving a near field
source and a near field observer. Chapter IV details the development and verifica­
tion of the extended UTD radiation equations for a near field source and a far field
observer. Chapter V shows the advantages of using the piecewise sinusoid as a basis
function in this hybrid technique. The field from a monopole with a PWS current
distribution appears, mathematically, to be emanating from the endpoints. Thus, the monopole can be replaced by two point sources. This decomposition was found to be particularly well-suited for use with this hybrid method because the exact incident field from these point sources is ray optical which is required by the UTD. As a result, this decomposition provided the most accurate results when used with the UTD. Chapter VI shows the disadvantages of using the piecewise sinusoid with the UTD. Cancellation problems between these endpoint fields when the observer is on the monopole axis result in a singular total monopole field. This problem is resolved by using a version of flat plane image theory which includes the curvature of the scatterer; it is referred to as the "modified image technique". The details are included in Chapter VI. Chapter VII verifies the extended UTD results along with the modified image technique by showing how the fields calculated by the various procedures, each accurate in their regions of validity, blend into each other in the crossover regions. Finally, in Chapter VIII, the hybrid technique is applied to various finned cylinders to show its use, the necessity of it, and its accuracy. The hybrid method computer program, "HY-MOM", is too unwieldy to include in this dissertation. However, since the bulk of the HY-MOM algorithm lies in choosing which UTD equation to use, a flowchart detailing this process is included instead (see Appendix G). An $e^{+j\omega t}$ time convention is assumed and suppressed in the following development.
CHAPTER II
Hybrid Approach

In the following sections, a hybrid method which combines the MM and the UTD in a self-consistent manner is presented. The development is followed by a discussion on the specifics of its implementation such as choice of basis and test functions and possible difficulties.

2.1 Moment Method Matrix Equation

Given a complex object such as the one shown in Figure 1 consisting of an electrically large, PEC, smooth convex object (PSCO) with a set of plates on or near it, the electric field scattered by this object is [25]

\[ \mathbf{E}^s = -j\omega\mu \int_{PSCO+plates} \mathbf{J} \cdot \overline{G}_{\text{special}} \, dS' \]  

(2.1)

where \( j = \sqrt{-1}, \omega = 2\pi f \) where \( f \) is the frequency, and \( \mu \) is the permeability of the medium. In the conventional MM approach, \( \overline{G}_{\text{special}} \) is the free-space Green’s function; thus, the scattered field is due to the currents, \( \mathbf{J} \), on the surface of the entire complex object radiating in free-space. However, given a condition on the Green’s function such that

\[ \mathbf{n} \times \overline{G}_{\text{special}} = 0 \]

on the PSCO (2.2)

reduces the integration area in (2.1) to be only over the plates:

\[ \mathbf{E}^s = -j\omega\mu \int_{plates} \mathbf{J} \cdot \overline{G}_{\text{special}} \, dS' \]  

(2.3)
In this dissertation, the plates are assumed to be perfectly conducting. Enforcing the boundary condition on the plates yields

\[ \hat{n} \times \vec{E}^i = j \omega \mu \int_{\text{plates}} \hat{n} \times (\vec{J} \cdot \vec{G}_{\text{special}}) \, dS' \]  

(2.4)

Note that because of the special Green’s function being used here, the \( \vec{E}^i \) in (2.4) is the field at an observation point on the plates due to a plane wave radiating in the presence of the PSCO. This special Green’s function may be decomposed as

\[ \vec{G}_{\text{special}} = \vec{G}_o + \vec{G}_{\text{scat}} \sim \vec{G}_{\text{special}} \]  

(2.5)

where \( \vec{G}_o \) is the well-known free-space dyadic Green’s function and \( \vec{G}_{\text{scat}} \) represents the effect of the PSCO with the plates absent. Since the PSCO is assumed to be electrically large, \( \vec{G}_{\text{special}} \) may be adequately approximated by its UTD form denoted by \( \vec{G}_{\text{special}}^{UTD} \). Using (2.5) in (2.4) results in the following integral equation to be solved:

\[ \hat{n} \times \vec{E}^i = j \omega \mu \int_{\text{plates}} \hat{n} \times (\vec{J} \cdot \vec{G}_o) \, dS' + j \omega \mu \int_{\text{plates}} \hat{n} \times (\vec{J} \cdot \vec{G}_{\text{scat}}) \, dS' \]  

(2.6)

where it is noted that because the PSCO is electrically large \( \vec{G}_{\text{scat}} \) in (2.6) will be approximated by the UTD as \( \vec{G}_{\text{special}}^{UTD} \).

Next, the unknown current on the plates is represented in terms of a set of basis functions \( \vec{J}_n \).

\[ \vec{J} \approx \sum_{i=1}^{N} I_n \vec{J}_n \]  

(2.7)

Using the method of moments technique, equation (2.6) is tested with a set of weighting or test functions, \( w_m \) where \( m = 1, N \). The resulting matrix equation is

\[ [V_m] = \left[ Z_{\alpha mn} \right] [I_n] + \left[ \Delta Z_{\alpha mn} \right] [I_n] \]  

(2.8)

with

\[ V_m = \int_{m^{th \, \text{test}}} w_m \cdot (\hat{n} \times \vec{E}^i) \]  

(2.9a)
\[ Z_{mn} = j \omega \mu \int_{m^{th} \text{ test}} \int_{n^{th} \text{ basis}} \bar{w}_m \cdot \hat{\overline{\mathbf{E}}}^{i} \cdot \mathbf{G}_0 dS' \]  
\[ \Delta Z_{mn} = j \omega \mu \int_{m^{th} \text{ test}} \int_{n^{th} \text{ basis}} \bar{w}_m \cdot \hat{\overline{\mathbf{E}}}^{i} \cdot \mathbf{G}_{\text{scat}} dS' \]  
\[ I_n = \text{current coefficient [Amps]} \]

\( V_m \) is the tangential component of \( \hat{\overline{\mathbf{E}}}^{i} \), the field incident at the observation point in the presence of the convex body with the plates absent, integrated over the test function. It is commonly referred to as the "excitation vector". \( Z_{\text{omn}} \) is the mutual impedance between the basis and test functions with the main convex body absent and thus is dependent only on the geometry of the plates and not the main body. It describes the interaction between the current and fields of the \( N \) current basis elements making up the plates and is the conventional MM matrix if only the plates were present. On the other hand, \( \Delta Z_{mn} \) is the mutual impedance due to the presence of the main convex body and thus is dependent not only on the geometry of the plates but also of the convex scatterer. Due to the size of the PSCO, this latter contribution will be determined by the UTD. Finally, \( I_n \) is the coefficient of the current basis functions and from it the current on the plates is found to be

\[ \mathbf{J} = \sum_{n=1}^{N} I_n \overline{J}_n \]  

The choice of the basis and test functions used here is described in the next two sections.

### 2.2 Basis Function

In order to obtain quicker convergence to a more accurate answer, the basis function \( \overline{J}_n \) should be a good initial guess of the behavior of the actual current. Taking a hint from the free-space Green's function case, i.e. the plate in free-space, a piecewise sinusoid (PWS) is chosen. For a surface current this means that the basis
function is a piecewise sinusoid in the direction of the current and a unit pulse in the orthogonal direction (Figure 3). The PEC surface is approximated by quadrilateral patches; one of which is shaded in Figure 3. A single basis function extends over two such patches where the current goes from zero at the "endpoint" of the patch denoted by the $E$ in this figure to unity at the "terminal" point denoted by $T$. To insure continuity of current, the next basis function overlaps the previous one, i.e. the endpoint of $\vec{J}_{n+1}$ is the terminal point of $\vec{J}_n$. The 2-D surface current can be approximated by a weighted sum of one-dimensional (1-D) filamentary currents (Figure 4) such that

$$\vec{J}_n = \sum_{m=1}^{M} a_m \vec{f}_m$$

(2.11)

where $a_m$ is the weighting associated with the integration in the direction perpendicular to the current direction and the $m^{th}$ basis function representing the filamentary
Figure 4: Approximation of the surface current, $\vec{J}_n$, by a sum of filamentary currents, $\vec{f}_m$.

The current is

$$\vec{f}_m = \frac{\sin k(s - s_1)}{\sin k(s_2 - s_1)} \hat{s}_1 + \frac{\sin k(s_3 - s)}{\sin k(s_3 - s_2)} \hat{s}_2$$

(2.12)

which is non-zero only over the $m^{th}$ filament (see Figure 5). In (2.12), $k$ is the wavenumber, $s_1$ and $s_3$ are the endpoints of the filament, $s_2$ is the "terminal", and $\hat{s}_1$ and $\hat{s}_2$ are the unit vectors for the two "monopoles" comprising the "dipole filament." Note that the filament current has been normalized so that it is unity at the terminal. Thus, the magnitude of $\vec{J}_n$ at $s_2$ will be the value of current in amperes there. Because the current $\vec{f}_m$ is assumed to vary sinusoidally and to flow only along the axis of the filament, the two segments in Figure 5 are thought of as a dipole. The term "dipole filament" is to distinguish it from a "surface dipole" which describes the two patches shown in Figure 3 over which $\vec{J}_n$ flows.
Figure 5: Basis function for the filamentary current.

2.3 Test Function

With regards to the test function, many possibilities exist. The simplest to use would be the delta function which gives rise to what is referred to as point matching. However, it is not always the most accurate. Instead, Galerkin's method, which implies that the test function is the same as the basis function, will be used. Since the test and basis functions are identical, Galerkin's method results in a symmetric matrix which means that only half of the matrix needs to be calculated. It is assumed that the test functions lie in the same location as the basis functions so the test surface current can also be approximated by a sum of test filamentary current sources. Thus, the mutual impedance between a surface test function and a surface basis function is the double summation of the mutual impedances between filaments of current on a basis patch and filaments of current on a test patch (see Figure 6). As a consequence, the basic calculation in filling the impedance matrix is the determination of the mutual impedance between a test filament and a basis filament.

\[
\bar{f}_m = \frac{\sin k(s-s_1)}{\sin k(s_2-s_1)} \hat{s}_1 + \frac{\sin k(s_3-s)}{\sin k(s_3-s_2)} \hat{s}_2
\]
Figure 6: Approximation of surface currents by filamentary currents and the resulting approximation to the mutual impedance.
The main advantage of the PWS current distribution

\[ I(z) = \frac{I_1 \sin k(z - z_1) + I_2 \sin k(z_2 - z)}{\sin k(z_2 - z_1)} \]  \hspace{1cm} (2.13)

over any other non-delta type test function is that the field from a filamentary monopole, such as that shown in Figure 7, with this current distribution is available in closed-form [2]:

\[ E_\phi = 0 \]  \hspace{1cm} (2.14a)

\[ E_z = C \left[ \frac{(I_1 - I_2 \cos kd) e^{-jkR_2}}{R_2} + \frac{(I_2 - I_1 \cos kd) e^{-jkR_1}}{R_1} \right] \]  \hspace{1cm} (2.14b)

\[ E_\rho = \frac{C}{\rho} \left[ (I_1 e^{-jkR_1} - I_2 e^{-jkR_2}) j \sin kd \right] \]
\[ + (I_1 \cos kd - I_2) e^{-jkR_1 \cos \theta_1} + (I_2 \cos kd - I_1) e^{-jkR_2 \cos \theta_2} \] (2.14c)

where

\[ C = -\frac{jZ_0}{4\pi \sin kd} \] (2.15)

and \( Z_0 \) is the intrinsic impedance of free space. In addition, \( d = |z_2 - z_1| \) is the length of the monopole, \( z_i \) (\( i = 1 \) or \( 2 \)) denotes the endpoints of the monopole, \( I_i \) is the corresponding magnitude of the current (1 ampere or 0) at endpoint \( i \), \( R_i \) is the distance from endpoint \( i \) to the observer, and \( \theta_i \) is the angle made by the vector \( \hat{R}_i \) with the wire axis. These expressions do not include the field contributions from the point charges that would exist at the endpoints of the monopole where the current is discontinuous because these charges disappear when two monopoles are joined together to form a dipole. When used in the standard moment method which involves the free-space Green’s function, one integration (determining the field from the test dipole) is eliminated by using the piecewise sinusoid. This is true in the hybrid case also.

Looking at Figure 6, the mutual impedance between the two patches is the sum of the mutual impedance between the 5 basis filaments and the 5 test filaments (25 filamentary mutual impedances in all). In order to compute the self-impedance of a patch the 5 test filaments would exist at the same positions as the basis filaments. Thus, a problem may exist when computing the self-impedance between two filaments because they exist at the same location in space. This implies that the field is needed right at the source location. In dealing with this problem, let us look at how this singularity problem is dealt with when wires are involved. In thin wire theory, the current is assumed to flow on the surface of the wire in a direction parallel to the wire axis. The current is assumed to have no circumferential variation, only axial variation. When computing the mutual impedance between two wires that are
not close to each other, the approximation that the current is concentrated on the axis of the wire is used in order to simplify the calculations. When determining the self-impedance, because the integration over the tubular test dipole is symmetric about the axis, a filamentary test dipole placed at the axis of the expansion dipole can be used instead of a tubular test dipole. This is exact. Since a filamentary approximation is used to approximate the surface dipole, this same method will be used to calculate the self-impedance term here. The observer (basis filament) is placed on the wire surface, i.e. a “wire” radius away from the source (test filament). As was shown in Figure 6, an initial approximation to the surface current is as a sum of filamentary currents. This allows the use of the one-dimensional results. A better approximation to the surface current is a sum of strip currents, but then the wire results cannot be used. An alternative is to use a filament to mimic a strip dipole. This is done by finding the equivalent wire radius, \( a_{eq} \), such that the self-impedance of a cylindrical dipole of length \( 2l \) equals that of a slender strip dipole of the same length and width \( w \) where \( kw \ll 1 \) and \( w \ll l \). This is found to be [17]

\[
a_{eq} = 0.223w
\]  

(2.16)

(see Figure 8).

2.4 UTD-based Green's function

The difference between the conventional MM procedure and the hybrid MM-UTD scheme lies in the type of Green’s function used in the kernel of the integral equation for the currents induced on the scatterer by an external source. The former procedure uses a free-space Green’s function which results in the unknown currents lying on the entire complex structure. The hybrid scheme uses a special, UTD-based Green’s function which accounts for the presence of the electrically large convex
Figure 8: Equivalent wire radius used such that the self-impedance of the strip dipole is equivalent to that of the filament.

\[ a_{eq} = 0.223w \]
scatterer, and as a result, the unknown surface currents are restricted to the region of the appendages. Therefore, when the MM is used in the hybrid scheme to solve the integral equation for these currents, the resulting MM impedance matrix is significantly smaller and more manageable than in the conventional MM procedure.

The aforementioned special Green's function is the field due to a point source in the presence of the large convex scatterer. The location of source and observer are arbitrary. Previous UTD expressions applicable to this problem existed only for the following limiting positions of a source and an observer (illustrated in Figure 9a-c):

1. Mutual coupling between a source and an observer both on the surface of a PEC convex object [16]

2. Radiation from a source on a PEC convex object to an observer sufficiently far from the surface [15]

3. Scattering from a PEC convex object when the source and the observer are both sufficiently far from the surface [14]

However, in determining the Green's function, there will be sources and observers on the appendages which may be close to the convex scatterer. In anticipation of this, new extended UTD expressions have been developed to handle the following two cases (illustrated in Figures 9d and e):

4. Extended UTD coupling: source and observer close to the surface of a PEC convex object (Chapter III)

5. Extended UTD radiation: source close to but observer sufficiently far from the PEC convex object (Chapter IV)

Figure 10 illustrates the utility of these near-field solutions. Given a source near
Figure 9: The positions for source and observer in the presence of perfectly conducting convex body for which UTD and extended UTD solutions are available along with the names used to describe the situation.
Figure 10: Use of the UTD equations as the observer, $O$, is raised off the surface of the PEC cylinder given a source, $S$, close to the surface. (a) Geometry. (b) Expected blending of equations used to determine the electric field $E$ versus $d$, the normal distance of the observer off the surface.
the cylinder and an observer at a fixed $z$ but varying $\rho$, what UTD equation should be used to determine the field? When the observer in Figure 10a is between position 1 and 2, the extended UTD coupling solution will be used. As the observer moves farther from the PEC, from position 2 to 3, it would require that the extended UTD radiation formula be used. A similar situation is shown in Figure 11a where the UTD radiation solution is used at position 1, the extended UTD radiation solution is used from positions 1 to 2, and the UTD scattering solution is used thereafter. A significant question is where is point 2, the point of transition in both Figures 10a and 11a from one equation to another, located and can it be predicted a priori? Since none of the UTD or the extended UTD solutions have sharply defined regions of validity, it is expected that the answer to this question is based on empirical reasoning. It is expected that the curve produced using the fields calculated by one equation will "blend" into the curve produced by using the fields calculated by another equation as the source or observer remains fixed and the other moves continuously through various regions where different solutions are valid. The expected result is shown in Figure 10b and Figure 11b. This blending will be used to validate the new extended UTD equations.

The availability of this extended UTD is very important in solving the problem of interest here: a finned object. Figure 12 shows a typical situation encountered in the $Z$ matrix calculation. A single plate is shown for convenience but the observer could be on another plate. The free-space component shown is accounted for in the $Z_0$ matrix whereas the field scattered off the PSCO is added into the $\Delta Z$ matrix. In the calculation of the $\Delta Z$ matrix, the matrix elements involving the mutual impedance between test and basis functions which are close to the surface of the PSCO will be the largest in magnitude, i.e. a source and observer close to the PSCO. This is because the scattered field is more significant the closer the source
Figure 11: Use of the UTD equations as the observer, $O$, is raised off the surface of the PEC cylinder given a source, $S$, far from the surface. (a) Geometry. (b) Expected blending of equations used to determine the electric field $E$ versus $d$, the normal distance of the observer off the surface.
Figure 12: Example of source and observer on the fin and the ray paths by which the field travels to the observer.
and the observer are to the surface and to each other. Thus, being able to solve this near-field scattering problem is crucial because it affects the impedance matrix more significantly than the previously analyzed far-field case.
CHAPTER III

Extended UTD (EUTD) Green's Function for Source and Observer Close to the PEC Boundary

3.1 Introduction

The problem of interest was described earlier in Figure 9d – determine the field in the near zone of a perfectly conducting smooth convex object (PSCO) due to a near-zone source. The development of a solution to this problem begins with solving a related canonical problem in which the PSCO is replaced by an infinitely-long circular cylinder. The circular cylinder results can then be generalized to treat the case of an arbitrary convex surface using the localization property of high-frequency wave propagation.

The geometry of the canonical problem is shown in Figure 13. The source and observer current elements are close to the surface of the PEC, infinitely-long, circular cylinder. Pathak and Wang [16] analyzed this problem by extending their results for the mutual coupling between sources on the PEC surface. They generalized the \( u \) and \( v \) Fock functions in the on-surface coupling results to allow the source and/or observer to be slightly raised above the surface provided the observer is outside the paraxial region. In the paraxial region, the solution in [16] for the case of a raised source and observer is not valid; in fact, it is singular. However, the paraxial region is very important in this hybrid approach since one edge of the plate may lie along the cylinder axis, and thus, the source and observer will have the same \( \phi \) coordinate. Therefore, an alternative solution must be found to treat this case.
Figure 13: Canonical problem: Two current elements close to but not necessarily on a PEC, infinitely-long, circular cylinder.

The approach taken here begins with the exact radially propagating eigenfunction solution to the problem. Basically, the procedure involves using a Taylor series to expand terms involving $d'$ and $d$ since both are assumed to be small. Then asymptotics are used to simplify the resulting expression. Since a Taylor series is used, the result is expected to be valid as long as $d'$ and $d$ satisfy the initial assumption of being small. The result is referred to as the "extended UTD (EUTD) coupling" solution.

3.2 Methodology

An outline of the steps taken to derive the "extended UTD coupling" solution is given here.
1. Given the conventional, radially \((\rho)\) propagating eigenfunction representation for the field due to an electric current moment in the presence of a PEC, infinitely-long, circular cylinder.

2. Obtain an angularly \((\phi)\) propagating eigenfunction representation for the same canonical problem. This is done because the UTD solution is contained in this representation which more clearly exhibits surface rays or creeping wave effects.

3. Develop a Taylor series expansion about \(a\), the radius of the cylinder, for terms involving \(\rho\) and \(\rho'\), the radial coordinates of the observer and the source points, respectively.

4. Use the Debye approximation for the Hankel functions to generate a lit region result, valid in the paraxial region, which, for an observer in the deep lit region, goes, in the asymptotic high frequency limit, to the plane surface result.

5. Match this solution, valid in the paraxial region, to an expansion of the field in terms of Fock functions, valid elsewhere, to obtain the coefficients of the expansion.

3.3 Use of Debye-type Approximations in the Taylor Expansion

Of interest here are cylinders with fins whose one edge lies parallel to the cylinder axis which is herein assumed to be in the \(\hat{z}\) direction. Therefore, the source and observer current elements will only be in the \(\hat{\rho}\) and \(\hat{z}\) directions as shown in Figure 13. Although results will not be presented for a circumferential current moment, the following analysis is applicable to that case as well.

An electric current moment

\[
\vec{p}_e = p_e \hat{\rho}_e
\]  

(3.1)
with

\[
\rho_e' = \begin{cases} 
\hat{\rho}' \\
\hat{z}'
\end{cases}
\]  

(3.2)

is located at the point \(P'\) whose coordinates are

\[
\hat{r}' = (\rho', \phi', z') = (a + d', 0, z') \quad ; \quad d' << a
\]  

(3.3)

\(P'\) is assumed to be close to the PEC, infinitely-long, circular cylinder of radius \(a\) where \(a\) is assumed to be large with respect to the wavelength. Note that for simplicity \(\phi'\) is set to zero. The field is desired at the observation point \(P\) which is also assumed to be close to the cylinder. Its coordinates are

\[
\hat{r} = (\rho, \phi, z) = (a + d, \phi, z) \quad ; \quad d << a
\]  

(3.4)

The exact radially-propagating eigenfunction solution for this problem when \(\rho < \rho'\) is given in \([18]\) and derived in Appendix A. It is repeated below for both a radial and axial electric current moment. The \(\hat{\rho}\)- and \(\hat{z}\)-directed fields due to a radial current moment \((\rho_e = \rho')\) are

\[
E(P)_{\rho \rho'} = -\frac{k Z_0 \rho_e}{8\pi} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{-jk_z(z-z')} e^{jn\phi} 
\]

\[
\cdot \frac{n^2}{\rho \rho'} \frac{H_n^{(2)}(k_\ell \rho')}{k_\ell^2} \left[ J_n(k_\ell \rho) - \frac{J_n'(k_\ell a)}{H_n^{(2)}(k_\ell a)} H_n^{(2)'}(k_\ell \rho) \right] 
\]

\[
- \frac{Z_0 \rho_e}{8\pi k} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{-jk_z(z-z')} e^{jn\phi} 
\]

\[
\cdot k_z^2 H_n^{(2)'}(k_\ell \rho') \left[ J_n'(k_\ell \rho) - \frac{J_n(k_\ell a)}{H_n^{(2)}(k_\ell a)} H_n^{(2)'}(k_\ell \rho) \right] 
\]

and

\[
E(P)_{z \rho'} = -\frac{j Z_0 \rho_e}{8\pi k} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{-jk_z(z-z')} e^{jn\phi} 
\]

\[
\cdot k_z k_\ell H_n^{(2)'}(k_\ell \rho') \left[ J_n(k_\ell \rho) - \frac{J_n(k_\ell a)}{H_n^{(2)}(k_\ell a)} H_n^{(2)'}(k_\ell \rho) \right] 
\]  

(3.5)

(3.6)
respectively. The $\hat{\rho}$- and $\hat{z}$-directed fields due an axial current moment ($\hat{p}_e = \hat{z}'$) are

$$E(P)_{\rho z'} = \frac{j Z_0 p_e}{8\pi k} \int dk_z \sum_{n=-\infty}^{\infty} e^{-jk_z(z-z') + jn\phi}$$

and

$$E(P)_{zz'} = -\frac{Z_0 p_e}{8\pi k} \int dk_z \sum_{n=-\infty}^{\infty} e^{-jk_z(z-z') + jn\phi}$$

respectively. In equations (3.5) – (3.8), one should note that

$$k_t = \begin{cases} \sqrt{k^2 - k_z^2} & k > k_z \\ -j\sqrt{k_z^2 - k^2} & k < k_z \end{cases}$$

$Z_0$ is the intrinsic impedance of the medium surrounding the cylinder, and $k$ is the wavenumber of the medium. Henceforth, for brevity, only the radial component of the field due to a radially-directed dipole source, $E_{\rho\rho}$, will be analyzed. However, the following analysis has been applied to the other three components with the results being employed in the hybrid code and presented at the end of Section 3.4.

The eigenfunction representation in (3.5) is converted into an angularly-propagating representation via the following version of the Watson transformation [19]:

$$\sum_{n=-\infty}^{\infty} F_n(\xi) \approx \int_{-\infty}^{\infty} e^{-j\xi} d\nu \sum_{l=0}^{\infty} e^{-j2\pi\nu F(\xi) + F(-\nu(\xi))} e^{-j2\pi l\nu}$$

Using this on (3.5) and neglecting the contribution of the creeping waves which have travelled around the cylinder ($l > 0$), it is found that

$$E_{\rho\rho} = -\frac{k_p e Z_0 a^3}{8\pi \rho \rho'} \int dk_z dk_y e^{-jk_z(z-z') + k_z^2 H_n^{(2)}(k_t \rho')}$$

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Figure 14: Illustration of the two geodesic paths, $R_s$ and $R_L$ from $P'_N$ to $P_N$, the projections onto the cylinder surface of $P'$ and $P$, respectively.

\[
\begin{align*}
\cdot J_\nu(k\ell \rho) &- \frac{J'_\nu(k\ell a)}{H^{(2)}_\nu(k\ell a)} H^{(2)'}_\nu(k\ell \rho) \left(e^{-jk_ya\phi} + e^{-jk_ya(2\pi-\phi)}\right) \\
&- \frac{p_0 Z_0 a}{8\pi k} \int_{-\infty}^{\infty} dk_z \, dk_y \, e^{-jk_z(z-z')} k_z^2 H^{(2)'}_\nu(k\ell \rho') \\
&\cdot J'_\nu(k\ell \rho) - \frac{J_\nu(k\ell a)}{H^{(2)}_\nu(k\ell a)} H^{(2)'}_\nu(k\ell \rho) \left(e^{-jk_ya\phi} + e^{-jk_ya(2\pi-\phi)}\right)
\end{align*}
\]

where the change of variables $\nu = k_y a$ was used. The $l \neq 0$ terms in (3.10) correspond to multiple encirclements of the cylinder by the angularly propagating wave. Thus, if $ka$ is assumed to be large then these terms can be ignored because they are exponentially damped; i.e., they are proportional to $e^{-jk_ya\phi}$. As a result, only the $l = 0$ term is retained. The $e^{-jk_ya\phi}$ term in (3.11) involves the shortest geodesic ray path from $P'$ to $P$ denoted by $R_s$ in Figure 14. The $e^{-jk_ya(2\pi-\phi)}$ term represents the ray which travels from $P'$ to $P$ the opposite way around the cylinder (path $R_L$.

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in Figure 14) and, as a result, produces a field contribution which is usually small compared to the field associated with the shorter ray path.

Both the source and observer are assumed to be close to the PEC surface such that \( d' \ll a \) and \( d \ll a \) ((3.3) and (3.4)). Consequently, all terms involving \( \rho \) and \( \rho' \) can be expanded in a Taylor series about the radius \( a \). For instance,

\[
Z_\nu(k t \rho) \approx Z_\nu(k t a) + k t (\rho - a) Z'_\nu(k t a) + \frac{k_t^2 (\rho - a)^2}{2} Z''_\nu(k t a) + \mathcal{O}(k_t^3 d^3)
\]

(3.12)

where \( d = \rho - a \ll a \) and \( Z_\nu(\cdot) = (J_\nu(\cdot), J'_\nu(\cdot), H^{(2)}_\nu(\cdot), \) or \( H^{(2)'}_\nu(\cdot) \)). A similar expression results for terms involving \( \rho' \) since \( d' = \rho' - a \ll a \). Expanding \( J_\nu(k t \rho) \) and \( H^{(2)}_\nu(k t \rho) \) in a Taylor series about \( k t a \) gives

\[
J_\nu(k t \rho) \approx \frac{J_\nu(k t a)}{H^{(2)}_\nu(k t a)} H^{(2)}_\nu(k t \rho) \approx \left( \frac{j^2 d}{\pi a H^{(2)}_\nu(k t a)} \right) \left[ 1 - \frac{d}{2a} - \frac{d^2}{6} \left( k^2_t - k^2_y \right) + \frac{d^2}{3a^2} \right]
\]

(3.13)

Similarly, expanding \( J'_\nu(k t \rho) \) and \( H^{(2)'}_\nu(k t \rho) \) results in

\[
J'_\nu(k t \rho) \approx \frac{J_\nu(k t a)}{H^{(2)}_\nu(k t a)} H^{(2)'}_\nu(k t \rho) \approx \left( \frac{j^2}{\pi k_t a H^{(2)}_\nu(k t a)} \right) \left\{ 1 - \frac{d}{a} - \frac{d^2}{2a^2} \left[ 1 - \frac{a^2 \left( k^2_t - k^2_y \right)}{2} \right] \right\}
\]

(3.14)

Multiplying (3.13) by the Taylor expansion of \( H^{(2)}_\nu(k t \rho') \) and (3.14) by the expansion for \( H^{(2)'}_\nu(k t \rho') \) and substituting into (3.11) results in

\[
E_{\rho \rho'} \approx \frac{j k_p e Z_0}{4\pi^2} \int_{-\infty}^{\infty} dk_z dk_y e^{-jk_z (z - z')} \frac{k^2_y}{k^3_t} \frac{H^{(2)}_\nu(k t a)}{H^{(2)'}_\nu(k t a)}
\]

\[
\left[ 1 - \left( \frac{d + d'}{a} \right) - \left( \frac{d^2 + d'^2}{2} \right) \left( k^2_t - k^2_y \right) + \mathcal{O} \left( \frac{d^2}{a^2}, \frac{d'^2}{a^2}, k^3_t d^3, k^3_t d'^3 \right) \right]
\]

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where only the shorter geodesic path is considered for now. In addition, the substitution \( y = a \phi \) has been made. Since a Taylor series about \( k_t a \) was used, the results are valid only if
\[
\frac{d'}{\lambda} \ll 1, \quad \frac{d}{\lambda} \ll 1
\] (3.16)
and in neglecting the higher order terms in (3.15), it is also assumed that
\[
\frac{d'}{a} \ll 1, \quad \frac{d}{a} \ll 1
\] (3.17)

Let \( E_{\rho \phi} \) be decomposed into
\[
E_{\rho \phi} = E_{\rho \phi}' + E_{\rho \phi}''
\] (3.18)
where \( E_{\rho \phi}' \) represents the first integral in (3.15) and \( E_{\rho \phi}'' \) represents the second integral. For brevity, only \( E_{\rho \phi}' \), will be analyzed here. Using the Debye approximation ([23], [26]) for \( H_{\nu}^{(2)}(z) \) reduces the ratio of Hankel functions to
\[
\frac{H_{\nu}^{(2)}(z)}{H_{\nu}^{(2)'}(z)} \sim \frac{z}{\sqrt{z^2 - \nu^2}} - \frac{z^3}{2(z^2 - \nu^2)^2} + \mathcal{O}\left(\frac{1}{\nu^2}\right), \quad z > \nu
\] (3.19)
The details are presented in Appendix B. A similar expression can be derived for \( H_{\nu}^{(2)'}(z)/H_{\nu}^{(2)}(z) \):
\[
\frac{H_{\nu}^{(2)'}(z)}{H_{\nu}^{(2)}(z)} = -\frac{\sqrt{z^2 - \nu^2}}{z} - \frac{z}{2(z^2 - \nu^2)}
\] (3.20)
These approximations reduce (3.15) to
\[
E_{\rho \phi}' \sim \frac{jkpe_z Z_0}{4\pi^2} \int_{-\infty}^{\infty} dk_z \; dk_y \; e^{-jk_z(z-z')-jk_yy}
\]
\[
\cdot \left\{ 1 - \left( \frac{d' + d}{a} \right) \right\} \left[ \frac{j}{\sqrt{k_1^2 - k_y^2}} - \frac{j\sqrt{k_1^2 - k_y^2}}{k_1^2} - \frac{k_y^2}{2a(k_1^2 - k_y^2)^2} \right] \\
- \left( \frac{d'^2 + d^2}{2} \right) \left[ \frac{jk_y^2\sqrt{k_1^2 - k_y^2}}{k_1^2} - \frac{k_y^2}{2a(k_1^2 - k_y^2)^2} \right] \right\} (3.21)
\]

An important step is to notice that the integrals in (3.21) can be written as derivatives with respect to \( z \) and \( y \) of the following integrals [20]:

\[
I_1(y, \tilde{z}) = \int_{-\infty}^{\infty} e^{-jkyy - jkz\tilde{z}} \frac{dk_y dk_z}{\sqrt{k^2 - k_y^2 - k_z^2}} = 2\pi j \frac{e^{-jks}}{s} (3.22a)
\]

\[
I_2(y, \tilde{z}) = \int_{-\infty}^{\infty} e^{-jkyy - jkz\tilde{z}} \frac{dk_y dk_z}{(k^2 - k_y^2 - k_z^2)^2} = -\frac{\pi^2 s}{2jk} H_1^{(2)}(ks) (3.22b)
\]

\[
I_3(y, \tilde{z}) = \int_{-\infty}^{\infty} dk_z dk_y e^{-jkyy - jkz\tilde{z}} \frac{\sqrt{k^2 - k_y^2 - k_z^2}}{k^2 - k_z^2} = \frac{-2\pi}{ks^2 \cos^2 \theta} e^{-jks} (3.22c)
\]

where

\[
\tilde{z} = z - z' = s \sin \theta \quad (3.23a)
\]

\[
y = a\phi = s \cos \theta \quad (3.23b)
\]

Thus,

\[
s = \sqrt{y^2 + \tilde{z}^2} \quad (3.24)
\]

is the distance along the surface of the body from the projection of \( P' \) onto the surface to the projection of \( P \) onto the surface (see Figure 15). Rewriting (3.21) in terms of \( I_1, I_2, \) and \( I_3 \) results in

\[
E_{\rho \rho'}_1 \approx \frac{jkpeZ_0}{4\pi^2} \left\{ \left( 1 - \frac{d' + d}{a} \right) \left( jI_1 - jI_3 + \frac{1}{2a} \frac{\partial^2 I_2}{\partial y^2} \right) \\
- \left( \frac{d'^2 + d^2}{2} \right) \left[ -j \frac{\partial^2 I_3}{\partial y^2} + \frac{1}{2a} \frac{\partial^2}{\partial y^2} \left( k^2 + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) I_2 \right] \right\} (3.25)
\]
Figure 15: Ray picture for EUTD coupling.
The algebra involved in taking the derivatives will be omitted here. The result is

\[
E_{\rho \rho_1} \simeq \frac{-jkpe^*}{2\pi} e^{-jks/s} \left\{ \left[ 1 - \left( \frac{d' + d}{a} \right) \right] \times \left[ 1 - \frac{j}{ks \cos^2 \theta} - \left( \frac{jpe^*jks}{8a} \right) \left( H_0^{(2)}(ks) - ks \cos^2 \theta H_1^{(2)}(ks) \right) \right] \right. \\

- \left[ \frac{d'^2 + d^2}{2} \right] \left[ \frac{1}{s^2} \left( jks + 1 + \frac{3}{\cos^2 \theta} - \frac{j6}{ks \cos^4 \theta} \right) \right. \\

- \left. \frac{j\pi ke^*jks}{4a} \left( \cos 2\theta H_1^{(2)}(ks) - ks \cos^2 \theta H_0^{(2)}(ks) \right) \right\} \right. \\
\] (3.26)

Since the Debye-type approximation for the Hankel functions was used, it is believed that these results will provide the proper limiting value for \( E_{\rho \rho_1} \) for large \( ka \) and small \( \xi \) where

\[
\xi = 2^{-1/3} \frac{ks}{(ka)^{2/3} \cos^{4/3} \theta} \] (3.27)

This implies that (3.26) is valid in the lit region.

### 3.4 Asymptotic Matching to an Expansion Using Fock Functions

In the previous section, an analysis was shown that provides a solution which should remain accurate in the limit of small \( \xi \) and even in the paraxial region. It is desirable to have an equation which is valid for any value of \( \xi \). To this end, the behavior of the field for large \( \xi \) is examined. For large values of \( \xi \), it is deduced from (3.27) that \( s \) is large and \( \theta \) cannot be 90°. Therefore, a large value of \( \xi \) implies that the field point is in the deep shadow region outside the paraxial region where the field is known to decay exponentially. This behavior is best reproduced by the \( u \) and \( v \) surface-type Fock functions since they have the correct limiting behavior [16]. The Fock integral representation is good for both large and small \( \xi \) for an observer in the shadow region outside the paraxial region. It would be of interest to extend
this Fock representation into the paraxial region. In order to accomplish this, the \( u \) and \( v \) Fock functions and their derivatives (which are independent) will be used as basis functions in an expansion of the field. The coefficients of this expansion may be found by matching it for small \( \xi \) to the solution based on the Debye approximation where both representations are valid. By looking at the Fock–type Airy function representation of \( H_{\nu}^{(2)}(\cdot) \) and \( H_{\nu}^{(2)'}(\cdot) \) [16, p. 56]

\[
H_{\nu}^{(2)}(k_4a) \sim \frac{j}{\sqrt{\pi}} \frac{w_2(\tau)}{m_4} \quad (3.28a)
\]

\[
H_{\nu}^{(2)'}(k_4a) \sim -\frac{j}{\sqrt{\pi}} \frac{w_2'(\tau)}{m_4^2} \quad (3.28b)
\]

where

\[
m_4 = \left( \frac{k_4a}{2} \right)^{1/3} \quad (3.29a)
\]

\[
\nu = k_4a + m_4 \tau \quad (3.29b)
\]

and the integral representation of the Fock–type \( u \) and \( v \) functions [16, p. 58,60]

\[
u(\xi) = \frac{e^{j3\pi/4}}{\sqrt{\pi}} \xi^{3/2} \int_{-\infty}^{\infty} d\tau \frac{w_2'(\tau)}{w_2(\tau)} e^{-j\xi \tau} \quad (3.30a)
\]

\[
u(\xi) = \frac{e^{j\pi/4}}{2\sqrt{\pi}} \xi^{1/2} \int_{-\infty}^{\infty} d\tau \frac{w_2'(\tau)}{w_2(\tau)} e^{-j\xi \tau} \quad (3.30b)
\]

where \( w_2(\tau) \) is the Airy function (see Appendix V of [16] for more information), it can be deduced that \( v(\xi) \) results from integrals involving \( H_{\nu}^{(2)}(k_4a)/H_{\nu}^{(2)'}(k_4a) \) while \( u(\xi) \) results from integrals involving \( H_{\nu}^{(2)'}(k_4a)/H_{\nu}^{(2)}(k_4a) \). Looking at (3.15) and using the above information as a guide to how the \( u \) and \( v \) functions can be used as basis functions to expand the field, the following form for \( E_{pp'}_1 \) in the shadow region is postulated.

\[
E_{pp'}_1 \sim \frac{k_2^2 Z_0}{2\pi j} \frac{e^{-jks}}{k_3}
\]
This representation is valid in the limiting case of the deep shadow region ($\xi$ large); whereas (3.26), derived using Debye-type approximations, is valid in the limiting case of the paraxial region ($\xi \to 0$). The coefficients in (3.31) are determined as mentioned earlier by asymptotically matching it to the solution employing the Debye approximation (equation (3.26)) for small $\xi$ where both representations are valid. This extends the validity of the Fock representation into the paraxial region.

This asymptotic matching technique is applied as follows:

**Step 1**: Looking at the Fock representation first, since we are interested in extending this solution into the paraxial region, $\xi$ small is of interest. Thus, the small argument form of $v$ and $v'$, [16, p. 92]

\[ v(\xi) \simeq 1 - \frac{\sqrt{\pi}}{4} e^{\frac{j\pi}{4}} \xi^{3/2} \]  
\[ v'(\xi) \simeq -\frac{3\sqrt{\pi}}{8} e^{\frac{j\pi}{4}} \xi^{1/2} \]  

is applicable. Using these in (3.31) gives

\[ E_{pp'1} = \frac{k^2 p_0 Z_0}{2\pi j} \frac{e^{-jks}}{ks} \left[ S_1 + \frac{(d^2 + d'^2)}{s^2} S_2 + \left( \frac{d + d'}{s} \right) S_3 \right] \]  

where

\[ S_i = A_i + \frac{e^{-j\frac{\pi}{4}}}{4ka} \sqrt{\frac{\pi ks}{2}} \left( \frac{3}{2} B_i - jksA_i \right) \cos^2 \theta \]  

where $i = 1, 2, \text{and} 3$.

**Step 2**: Next, consider the solution using the Debye approximation. Near the paraxial region, $\cos \theta$ is small so $ks$ need not be small. If this is true, then the large
argument form of the Hankel function [23, Equation 9.2.8]

\[ H^{(2)}_{\nu}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-jz+j\nu \frac{\pi}{2} + j\frac{\pi}{4}} \left[ 1 - \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128z^2} - j \frac{(4\nu^2 - 1)}{8z} \right] \] (3.35)

which is valid for \( \nu \) fixed and \(|z| \to \infty\) can be used to reduce (3.26) to

\[ E_{\rho \rho'} \sim \frac{k^2\rho e Z_0}{2\pi j} \frac{e^{-jk\rho}}{ks} \left\{ \left[ 1 - \left( \frac{d' + d}{a} \right) \right] \left[ 1 - \frac{j}{ks \cos^2 \theta} \right] \\
+ \frac{e^{-j\frac{\pi}{4}}}{4ka} \sqrt{\frac{\pi ks}{2}} \left( 1 - jks \cos^2 \theta - \frac{3}{8} \cos^2 \theta + \frac{j}{8ks} \left( 1 - \frac{15}{16} \cos^2 \theta \right) \right) \right\} \\
+ \frac{k}{2s} \left( d^2 + d'^2 \right) \left[ j3\tan^2 \theta \left( \frac{j}{ks} + \frac{2}{k^2 s^2} + \frac{2}{k^2 s^2 \cos^2 \theta} \right) - j - \frac{4}{ks} + j \frac{6}{k^2 s^2} \right] \\
+ \frac{k}{s} \left( d^2 + d'^2 \right) \\
\left[ \frac{e^{-j\frac{\pi}{4}}}{4ka} \sqrt{\frac{\pi ks}{2}} \left( ks \cos^2 \theta + j - \frac{15}{8} \cos^2 \theta + \frac{3}{8ks} - \frac{105 \cos^2 \theta}{128ks} \right) \right] \] (3.36)

Step 3: Equating the results for these two limiting cases ((3.33) and (3.36)) determines the coefficients \( A_i \) and \( B_i \) in (3.34):

\[ A_1 = 1 - \frac{j}{ks \cos^2 \theta} \] (3.37a)

\[ B_1 = \frac{4}{3 \cos^2 \theta} \left( 1 + \frac{j}{16ks} \right) - \frac{1}{4} \left( 1 + \frac{j5}{16ks} \right) \] (3.37b)

\[ A_2 = \frac{ks}{2} \left[ j3\tan^2 \theta \left( \frac{j}{ks} + \frac{2}{k^2 s^2} + \frac{2}{k^2 s^2 \cos^2 \theta} \right) - j - \frac{4}{ks} + j \frac{6}{k^2 s^2} \right] \] (3.37c)

\[ B_2 = \frac{2}{3} jks A_2 \]

\[ + \frac{2}{3 \cos^2 \theta} \left( ks \cos^2 \theta + j \left( 1 - \frac{15}{8} \cos^2 \theta \right) + \frac{3}{8ks} \left( 1 - \frac{35}{16} \cos^2 \theta \right) \right) \] (3.37d)
\[ A_3 = 0 \]  
\[ B_3 = -\frac{8}{3} \frac{e^{j\frac{\pi}{4}}}{\cos^2 \theta} \sqrt{\frac{2ks}{\pi}} \left(1 - \frac{j}{ks \cos^2 \theta}\right) \]

(3.37e)  
(3.37f)

An expression for \( E_{\rho p'2} \), similar to (3.26), can be determined, and an expansion in the form of (3.31) can be written except with \( v(\xi) \) and \( v'(\xi) \) replaced by \( u(\xi) \) and \( u'(\xi) \), respectively. Using the same asymptotic matching technique, a general expression for \( E_{\rho p'2} \) can be determined. Adding this to \( E_{\rho p'1} \) produces the total \( \rho' \) field due to a \( \rho' \) current element:

\[ E_{\rho p'} \sim \frac{k^2 p_e Z_0}{2\pi j} \frac{e^{-jk_s}}{ks} (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \]

(3.38)

where

\[ \varepsilon_1 = \frac{j}{ks} \tan^2 \theta (u - v) - \frac{u}{k^2 s^2} + \left(1 - \frac{j}{ks}\right) v \]

\[ + \frac{j\xi}{ks} \left[ v' \left( \frac{4\tan^2 \theta}{3} \left(1 + \frac{j}{16ks}\right) + \frac{13}{12} + \frac{j}{192ks} \right) + \frac{u'}{-\frac{j}{ks} \tan^2 \theta} \left(\frac{7}{12} - \frac{j19}{64ks} + \frac{1}{4k^2 s^2}\right) \right] \]

(3.39a)

\[ \varepsilon_2 = \frac{k^2 (d^2 + d'^2)}{m^2 \xi} \]

\[ \times \left\{ \left( v' - \frac{3}{2} v \right) \left[ -\frac{j\tan^2 \theta}{2} \left(\frac{j}{ks} + \frac{2}{k^2 s^2} \left(2 + \tan^2 \theta\right)\right) + \frac{j}{6} + \frac{2}{3ks} - \frac{j}{k^2 s^2} \right] \right. \]

\[ + \frac{j}{2} (u' - \frac{3}{2} u) \left[ -\frac{1}{k^2 s^2} \left(1 - \frac{3j}{ks}\right) + \tan^2 \theta \left(\frac{j}{ks} + \frac{2}{k^2 s^2} + \frac{2}{k^2 s^2 \cos^2 \theta}\right) \right] \]

\[ + \frac{j\xi v'}{3ks \cos^2 \theta} \left[ ks \cos^2 \theta + j \left(1 - \frac{15}{8} \cos^2 \theta\right) + \frac{3}{8ks} \left(1 - \frac{35}{16} \cos^2 \theta\right) \right] \}

(3.39b)

\[ \varepsilon_3 = -\frac{k(d + d')}{m \sqrt{\xi}} \left(\frac{j\xi 4e^{j\frac{\pi}{4}}}{ks 3\sqrt{\pi}}\right) \left[ \frac{j\tan^4 \theta}{ks} (u' - 2v') \right. \]

\[ + \frac{j\tan^2 \theta}{ks} \left(1 + \frac{j}{ks}\right) u' + 2 \tan^2 \theta \left(1 - \frac{j^2}{ks}\right) v' - \frac{u'}{k^2 s^2} + 2 \left(1 - \frac{j}{ks}\right) v' \]

42
The previous analysis has been applied to the \( \dot{z} \) field component and the result is shown below.

\[
E_{\dot{z}d} \sim \frac{kZ_0pe^{-jks}}{4\pi k^2s^2} d\sin\theta \left\{ \left( 2 - \frac{d}{a} - \frac{2d'}{a} \right) \left[ \left( jk^2 + \frac{3k}{s} - \frac{j3}{s^2} \right) u(\xi) \right. \right.
\]
\[
+ \frac{2\xi u'(\xi)}{3s^2\cos^2\theta} \left( ks \left( -1 + \frac{11}{8} \cos^2\theta \right) + \frac{j3}{8} \left( 5 - \frac{187}{16} \cos^2\theta \right) \right.
\]
\[
+ \frac{105}{128 ks} \left( 1 - 4 \cos^2\theta \right) \left. \right\} \right[ \left( 2 + 3 \cos^2\theta \right) + \left( -4 + 5 \cos^2\theta \right) \right] + \frac{4}{3} \left( d^2 + d'^2 \right)
\]

The field components due to a \( \dot{z}' \)-directed current element are

\[
E_{\dot{z}d} \sim \frac{kZ_0pe^{-jks}}{4\pi k^2s^2} d'\sin\theta \left\{ \left( 2 - \frac{d}{a} - \frac{2d'}{a} \right) \left[ \left( jk^2 + \frac{3k}{s} - \frac{j3}{s^2} \right) u(\xi) \right. \right.
\]
\[
+ \frac{2\xi u'(\xi)}{3s^2\cos^2\theta} \left( ks \left( -1 + \frac{11}{8} \cos^2\theta \right) + \frac{j3}{8} \left( 5 - \frac{187}{16} \cos^2\theta \right) \right.
\]
\[
+ \frac{105}{128 ks} \left( 1 - 4 \cos^2\theta \right) \left. \right\} \right[ \left( 2 + 3 \cos^2\theta \right) + \left( -4 + 5 \cos^2\theta \right) \right] + \frac{4}{3} \left( d^2 + d'^2 \right)
\]

and

\[
E_{zz} \sim -\frac{jkpeZ_0}{2\pi s^3} \left[ \left( 2 - \frac{1}{2a} \left( d + d' \right) \right) \right. \left[ \left( 2 - 3 \cos^2\theta \right) + jks \cos^2\theta - \frac{j3}{ks} \left( -4 + 5 \cos^2\theta \right) \right]
\]
\[-\frac{3}{k^2a^2} (-4 + 5 \cos^2 \theta) u(\xi) \]
\[-\frac{2\xi u'(\xi)}{3 \cos^2 \theta} \left[ \cos^2 \theta \left( 2 - \frac{15}{8} \cos^2 \theta \right) + \frac{j}{k \xi} \left( 3 - \frac{\cos^2 \theta}{4} \left( 57 - \frac{1545}{32} \cos^2 \theta \right) \right) \right] \]
\[-\frac{1}{k^2a^2} \left( -\frac{45}{8} + \cos^2 \theta \left( \frac{2067}{64} - 15 \cos^2 \theta \right) \right) + \frac{j315}{16k^5a^3} \left( -\frac{1}{8} + \cos^2 \theta \sin^2 \theta \right) \] 
\[+ \frac{k}{6a} (d' + d^2) \xi u'(\xi) \left[ \frac{4ka}{3ks \cos^2 \theta} \sqrt{\frac{2}{\pi k s}} e^{j\pi/4} \left[ j3ks \cos^2 \theta + 4(-7 + 10 \cos^2 \theta) \right] \right] \]
\[\frac{j9}{ks} (-10 + 13 \cos^2 \theta) - \frac{45}{k^2a^2} (-6 + 7 \cos^2 \theta) + \frac{j45}{k^3a^3} (-6 + 7 \cos^2 \theta) \] 

Reiterating, the circumferential current elements have not been considered here, but the analysis can be easily extended in an analogous fashion to include this case if need be. The results in (3.38)-(3.42) constitute the extended UTD coupling solution.

3.5 Numerical Results

As discussed in Chapter II, equations already exist for the limiting cases shown in Figure 9a-c. Equations (3.38)-(3.42) provide the missing link necessary to do the arbitrary source and observer problem. However, there should be a smooth transition as predicted in Figure 10 and 11 from the field determined by the EUTD coupling solution to the field derived from existing UTD equations for the mutual coupling, radiation, and scattering case. The existence of a smooth transition will be used as a test to verify the accuracy of the EUTD coupling results.

Tests for two \( \hat{p} \)-directed current elements as shown in Figure 16 are presented in the following pages. Blending for the three other combinations of source and observer orientations \( (\hat{p}\hat{z}', \hat{z}\hat{p}', \hat{z}\hat{z}') \) were done but, for brevity, are not shown here. The results and conclusions are the same however. The blending tests presented here focus on the the magnitude of the field; however, a similar type of investigation
Figure 16: Current element near the surface of the PEC, infinitely-long cylinder at \( \vec{r}' = (a + d', 0, 0) \) with observer at \( \vec{r} = (\rho, 0, 2\lambda) \) where \( \rho > a \).

regarding the phase of the field were investigated but, for brevity, are not presented here.

3.5.1 Transition from EUTD Coupling to UTD Mutual Coupling

First, consider the mutual coupling case: source and observer on the surface of the cylinder. This implies that \( d = d' = 0 \) in (3.38). Note that when these two variables are set to zero, (3.38) correctly reduces to the result given in [16]. Thus, the field predicted by the EUTD coupling solution goes exactly to that predicted by the mutual coupling solution as both source and field point go onto the surface of the cylinder.
3.5.2 Transition from EUTD Coupling to UTD Radiation

Next, the transition to the radiation case will be considered. Unlike in the mutual coupling case, the field predicted by the EUTD coupling solution does not directly transform mathematically into the UTD radiation equation [15] when \( d' = 0 \) and \( d \to \infty \). As a result, verification of the EUTD coupling solution in this situation is done by looking at the numerical results from test runs to see if the curve of the field predicted by one equation blends into the curve predicted by the other. Figure 16 depicts the test geometry: a PEC, infinitely-long circular cylinder of radius \( a \) excited by a \( \rho' \) current element on the cylinder surface, i.e., at \( (\rho' = a, \phi' = 0, z' = 0) \); the fields are observed at \( \phi = 0 \) but two wavelengths away along the \( z \)-axis, i.e., at \( (\rho, \phi = 0, z = 2\lambda) \). The verification procedure involves obtaining the field as the observer moves away from the cylinder in the \( \rho \) direction. Thus, \( \rho = a + d \) where \( d > 0 \). The UTD radiation solution is accurate for an observer not close to the cylinder surface. On the other hand, the EUTD coupling solutions blows up when the observer is far from the cylinder because \( d/a \) and/or \( d/\lambda \) are no longer small, and thus, the assumptions made in the EUTD coupling derivation are violated. Therefore, it is expected that the EUTD coupling solution should be used for small values of \( d/\lambda \) and the UTD radiation solution should be used when \( d/\lambda \) is large. This hypothesis is verified by the next two plots.

Consider a cylinder radius of \( 20\lambda \). The purpose of using such a large cylinder is to be able to check the results against flat plane image theory. The variation of the magnitude of \( E_{\rho\rho'} \) with \( d \), the normal distance of the observer from the cylinder surface, is shown in Figure 17. The radiated field is calculated using the EUTD coupling equations (dashed line) and the UTD radiation equation (dot dash line). Image theory (solid line) is included for reference purposes. The singular behavior of the radiation equation for \( d \) small is expected, as is the singular behavior of
Figure 17: \( |E_{\rho\phi}| \) as a function \( d \), the normal distance of the observer from the cylinder \((a = 20\lambda)\), with \( d' = 0 \). The field is calculated using the EUTD coupling and radiation equations with flat plane image results provided for reference.
the EUTD coupling equation for $d$ large. Visually, it is seen that there is a smooth transition from one curve to another when $d$ is approximately $0.3 \lambda$. An approximate transition point is indicated by the $TP$ on the graph. Therefore, for $0 < d < 0.3\lambda$ the EUTD coupling equations are used to determine $E_{p,p'}$ and for $d > 0.3\lambda$ the UTD radiation equations are used. Since the cylinder radius is so large, image theory should generate a reasonably accurate result for the field. Comparison with image theory verifies that indeed $TP$ is a good transition point.

Figure 18 shows a similar result for a smaller cylinder radius of $2\lambda$. The transition point labeled by a $TP$ is around $d = 0.5\lambda$. The gap between the two curves
around the transition point widens as the radius gets smaller so the transition from one curve to another is not as smooth and continuous as for the large cylinder. The results are still good; the gap between the two solutions at the point where they should merge is only about 1 dB. Image theory is shown for comparison purposes only since it is not expected to be as accurate as it was for the larger radius cylinder. As for the larger cylinder, UTD radiation is very close to image theory; however, the result generated by the EUTD coupling solution deviates as expected. For a smaller cylinder, a flat plane should not be a good approximation to the cylinder. Thus, we have confidence that the EUTD coupling results are correct.

3.5.3 Transition from EUTD Coupling to UTD Scattering

Now let $d' = 0.02\lambda$ in Figure 16 in order to verify blending from EUTD coupling to scattering. Note $d'$ cannot be too large; otherwise, the EUTD coupling solution will not be as accurate. Like radiation, scattering is most accurate for a far-away observer. Therefore, it is expected that the EUTD coupling solution should be used for small values of $d$ and UTD scattering should be used when $d$ is large. Again, a large, $20\lambda$-radius cylinder is considered first. Figure 19 shows the variation in the magnitude of $E_{\rho'd'}$ as a function of $d$, the normal distance of the observer from the cylinder surface, predicted by the EUTD coupling solution (dashed line) and UTD scattering equation (dot-dashed line). Here as well, image theory is included for reference and should be accurate since the radius of the cylinder is large. For such a large cylinder, the results are expected to be very good as they were for the source on the cylinder in Figure 17. The UTD scattering solution, like the UTD radiation solution, breaks down as $d \to 0$ while the EUTD coupling solution breaks down as $d \to \infty$. From Figure 19, it can be seen that the transition point should be somewhere around $d = 0.3\lambda$. Again, for the large cylinder, the transition from one
Figure 19: $|E_{p_{\rho\phi}}|$ as a function $d$, the normal distance of the observer from the cylinder ($a = 20\lambda$), with $d' = 0.02\lambda$. The field is calculated using the EUTD coupling and UTD scattering equations. Flat plane image theory is included for reference.
Figure 20: $|E_{pd}|$ as a function $d$, the normal distance of the observer from the cylinder ($a = 2\lambda$), with $d' = 0.02\lambda$. The field is calculated using the EUTD coupling and UTD scattering equations. Flat plane image theory is included for reference.

curve to the other is smooth and continuous. Figure 20 shows the results for the same situation but a smaller cylinder radius of $2\lambda$. Note how, like for the UTD radiation case, the gap between the curves widens as the cylinder gets smaller. Though again, the gap is not too large.

3.5.4 Summary

Numerical results were generated for a source location which was fixed and a field point which moved away from the cylinder in the $\hat{\rho}$ direction. In order to test the accuracy of the EUTD coupling solution, the electric field was determined using
the EUTD coupling solution and existing UTD solutions. The results were examined to see whether or not there exists a transition point where one goes from using the EUTD coupling solution to using the existing UTD solutions. First, the source was placed on the cylinder surface in order to test the blending between EUTD coupling and UTD radiation. Then the source was moved slightly off the cylinder surface in order to test the blending between EUTD coupling and UTD scattering. Both set of tests show that indeed there is a transition point where the field goes from being calculated by the EUTD coupling solution to the appropriate, existing UTD equation valid far from the surface. The larger the cylinder radius, the more smooth and continuous the transition. For the smaller radius cylinder, the curves generated by the EUTD and the existing UTD do not touch so there is a step in the field at the transition point. However, in the cases considered here, the step is not too large. This gives us confidence in the EUTD coupling solution. For a smaller radius cylinder or a source farther from the cylinder, the step may be larger. Thus, around the transition point, it seems that both the EUTD coupling solution and the usual UTD solutions are close to their limits of validity. This implies that an additional equation may be needed. The development of one, referred to as the "extended UTD (EUTD) radiation" solution, will be discussed in the next chapter.

3.6 Conclusion

An asymptotic solution to the problem of determining the field near a PEC circular cylinder due to a source located near the cylinder has been derived in this chapter. Debye–type approximations were used in a Taylor series about $k_4a$, where $a$ is the radius of the cylinder, for the angularly-propagating eigenfunction solution. The resulting equation is valid in the paraxial region. In order to obtain a solution
which is uniformly valid for all points about the cylinder (from the paraxial to the deep shadow region), the solution using the Debye approximations is asymptotically matched to the creeping wave representation utilizing Fock functions. This result, given in (3.38)–(3.42), constitute the EUTD coupling solution. Since the Taylor series was used to approximate the eigenfunction solution, the results are valid for sources and observers close to but not necessarily on the PEC surface.

Verification was made for all the components but, for brevity, only the $E_{p\rho}'$ results were presented here. It was shown that there is a point at which one switches from calculating the field by the EUTD coupling to calculating the field by an appropriate UTD solution valid far from the surface. The existence of this transition point validates the EUTD coupling solution.

Although not done here, these results for the canonical circular cylinder can be generalized via the GTD procedure to treat a PEC smooth convex object. This generalization procedure modifies the circular cylinder result to account for the curvature of the surface. Torsion terms are included as are surface divergence factors. Both of these represent the dependence of the behavior of the surface rays on the radius of curvature of the surface along the direction of the ray and normal to it. Consequently, these circular cylinder results can be generalized via the GTD concepts as long as the curvature is slowly-varying.
CHAPTER IV

Extended UTD (EUTD) Green's Function for Source On and Observer Far From the PEC Boundary

4.1 Impetus

As mentioned in Section 2.4, it may be necessary to analyze the situation shown in Figure 10: the field at an arbitrary location due to a source in the near field of the cylinder \(d' \ll a\). Referring to Figure 10, the field in the vicinity of the cylinder \(d < d_2\) can be determined via the EUTD coupling solution. However, \(d_2\), the transition point between EUTD coupling and EUTD radiation, is restricted to be much less than the radius of the cylinder, \(a\), since the error for EUTD coupling is on the order of \(\left(\frac{d}{a}\right)^2\). If \(d > d_2\), there is presently no UTD equation that can be used. UTD scattering is invalid because \(d'\) in Figure 10 is assumed to be small; in order for scattering to be used, \(d'\) along with \(d\) must be asymptotically large. UTD radiation cannot be used since \(d' \neq 0\). However, if \(d'\) is small, it seems reasonable to expect that the field at \(O\) produced by a source at \(S\) would differ little from that at \(O\) if the source were at \(S_N\), the normal projection of \(S\) onto the cylinder. Thus, a modification of the radiation result or the scattering result (combined with reciprocity) would be a natural solution to the problem. Since, in the hybrid technique, the source and observer can be anywhere on the appendages, it is clear that a new equation must be developed to fill this gap in the existing UTD.
4.2 Methodology

Since the reciprocal version of the geometry shown in Figure 10a has been solved in 2-D, it seems natural to use the reciprocal case as a starting point. The following methodology is used to analyze the problem of a source near the PEC circular cylinder while the observer is far from the cylinder (Figure 21b):

1. Extend the 2-D, near-field, UTD scattering results of Pathak [27] to 3-D to obtain the results for Figure 21a which is hereafter referred to as the "original problem". This is accomplished by

   (a) using a test source \( \tilde{p}_e \) not necessarily in the far field but which produces a local plane wave near the point \( P_N \), the normal projection of the observation point \( P \),

   (b) representing the \( \hat{z} \) components of the incident field as

\[
E_z^{3D} \approx e_z^{2D}(k_t) e^{-jk_{\hat{z}}z} \tag{4.1a}
\]

\[
H_z^{3D} \approx h_z^{2D}(k_t) e^{-jk_{\hat{z}}z} \tag{4.1b}
\]

which makes the 3-D problem an effective 2-D problem in the transverse-to-\( \hat{z} \) plane,

(c) projecting everything onto a given \( z \) plane since \( e_z^{2D} \) and \( h_z^{2D} \) are the incident fields propagating in the transverse-to-\( \hat{z} \) plane with wavenumber \( k_t \),

(d) using Pathak's 2-D results [27] to analyze the problem in (c) to obtain the total \( \hat{z} \)-directed fields, and

(e) using Maxwell's equations to determine the other field components.
Figure 21: Extended UTD radiation geometry.
2. Utilize reciprocity if the source is in the near field at $P$ and the observer is far from the cylinder at $Q$ (Figure 21b), hereafter referred to as the "reciprocal problem".

For ease of reference, the results from [27] for 2-D, plane-wave, near-field scattering from a PEC cylinder have been summarized in Appendix C.

4.3 Original Problem

4.3.1 Shadow Region

The geometry that will be analyzed is shown in Figure 22a. The field will be determined at an observer, $P_s$, which is in the shadow region of $p_e$ and in the vicinity of the cylinder. Although initially the test source is assumed to be in the far field of the observer and the cylinder, the final result will be modified to allow for a near-field test source. The electric test source has the form

$$\bar{p}_e = p_e^\theta \hat{\theta}' + p_e^\phi \hat{\phi}'$$

and radiates an incident field which can be approximated locally at a point far from $p_e$ as a plane-wave field. Consequently,

$$\bar{E}^i = \left[ f_{\theta'} \hat{\theta}' + f_{\phi'} \hat{\phi}' \right] e^{jk \cdot \hat{r}'}$$

$$\bar{H}^i = \left[ f_{\phi'} \hat{\phi}' - f_{\theta'} \hat{\theta}' \right] \frac{e^{jk \cdot \hat{r}'}}{Z_0}$$

where

$$f_{\theta'} = -\frac{jkZ_0}{4\pi} \frac{p_e^\theta e^{-jkr'}}{r'}$$

$$f_{\phi'} = -\frac{jkZ_0}{4\pi} \frac{p_e^\phi e^{-jkr'}}{r'}$$

$Z_0$ is the intrinsic impedance of free-space and $k$ is the associated wavenumber. Note that the far-field assumption has been used. The field can be decomposed into
Figure 22: A test source radiating a local plane wave. Observer is in the shadow region of the source and close to the infinitely-long circular cylinder. (a) Original 3-D problem and (b) equivalent 2-D problem.
its transverse-to-$\hat{z}$ and $\hat{z}$ components:

$$\bar{U} = U_{\hat{z}} \hat{z} + \bar{U}_t$$  \hspace{1cm} (4.5)$$

where $U$ may be either $E$ or $H$. Only the $\hat{z}$ component of the incident field

$$E_{\hat{z}}^i = -f_{\phi'} e^{jk\hat{r} \cdot \hat{r}'} \sin \theta_0$$ \hspace{1cm} (4.6a)$$

$$H_{\hat{z}}^i = -f_{\phi'} \frac{e^{jk\hat{r} \cdot \hat{r}'} \sin \theta_0}{Z_0}$$ \hspace{1cm} (4.6b)$$

is required in order to obtain the total $\hat{z}$-directed field from which the transverse fields can be determined via Maxwell's equations. Note that $\cos \theta_0 = -\hat{z} \cdot \hat{r}'$. The distance vectors are denoted by

$$\bar{r} = \hat{x} x + \hat{y} y + \hat{z} z$$ \hspace{1cm} (4.7)$$

and

$$\hat{r}' = \hat{x} \sin \theta_0 \cos \phi' + \hat{y} \sin \theta_0 \sin \phi' - \hat{z} \cos \theta_0$$ \hspace{1cm} (4.8)$$

For simplicity, $\phi' = 0$ will be assumed. This is equivalent to assuming that the $x$-$z$ plane is the plane of incidence. If this is not true, then the coordinate system is rotated appropriately. This assumption simplifies the exponential in (4.6) to

$$e^{jk\hat{r} \cdot \hat{r}'} = e^{jk (z \sin \theta_0 - z \cos \theta_0)}$$ \hspace{1cm} (4.9)$$

Defining the components of $\vec{k}$ as

$$k_z = k \cos \theta_0$$ \hspace{1cm} (4.10a)$$

$$k_t = k \sin \theta_0$$ \hspace{1cm} (4.10b)$$

allows (4.6) to be rewritten as

$$E_{\hat{z}}^i = -f_{\phi'} \sin \theta_0 e^{jk_t z} e^{-jk_z z}$$ \hspace{1cm} (4.11a)$$

$$H_{\hat{z}}^i = -f_{\phi'} \frac{\sin \theta_0}{Z_0} e^{jk_t z} e^{-jk_z z}$$ \hspace{1cm} (4.11b)$$
Since the z-dependence of the incident field in (4.11) is limited to the last exponential term in both (4.11a) and (4.11b), then for a given z, the problem is equivalently 2-D. This characteristic of the incident field allows the 3-D results to be obtained through use of the 2-D results in Appendix C and the method of separation of variables. Equation (4.11) can be written as

\[ U_z^i = u_z^i(k_t) \ e^{-jk_zz} \]  

where

\[ u_z^i(k_t) = u_0 \ e^{jk_tz} \]  

and

\[ e_0 = \frac{jk_z0}{4\pi} \ \sin \theta \ \rho \ e^{\theta_i} \ \frac{e^{-jkr'}}{r'} \]  

\[ h_0 = \frac{jk}{4\pi} \ \sin \theta \ \rho \ e^{\theta_i} \ \frac{e^{-jkr'}}{r'} \]  

In (4.12), \( U = E \) or \( H \) and \( u = e \) or \( h \). The small \( u \) distinguishes the field components with the z-dependence removed. Thus, \( u \) represents the field components for the 2-D problem; whereas, \( U \) represents those for the 3-D problem. \( u_z^{2D}(k_t) \) is the z-component of the incident field, which propagates with wavenumber \( k_t \), in the transverse-to-\( z \) plane and has the same spatial variation as \( u^i \) in (C.9) except that \( k \) in (C.9) is replaced with \( k_t \).

The total field can also be split up as in (4.12). Consequently,

\[ U_z = u_z(k_t) \ e^{-jk_zz} \]  

As a result, the problem simplifies to determining the 2-D field \( u_z \) at a given \( z \) due to the incident field in (4.13). This is illustrated in Figure 22b. Since the objective is to determine the field at \( P_z \), then the \( z = z_0 \) plane is the transverse plane of interest.
where $z_0$ is the $z$-coordinate of the observer. $\hat{k}_t$ is the transverse-to-$\hat{z}$ component of $\hat{k}$ while $Q_{1t}$ is the projection of $P_1$, the 3-D tangent point, onto the transverse plane. Figure 22b resembles the 2-D scattering problem shown in Figure 82. Thus, the 2-D results in Appendix C can be used to determine the $\hat{z}$-directed total field due to the incident field in (4.13). Using (C.1) to determine $u_z$, the total field at $P_s$ due to $U_z^i$ is

$$U_z(P_s) \approx u_z^i(Q_{1t}) \hat{p}_g \ e^{-jk_{tt}t} \ e^{-jk_{zz}z_0} \ \Omega_{s,h} (Z_{st}, h_t) \quad (4.16)$$

where $\hat{p}_g$ is the bracketed term in (C.1) which is equal to unity for the circular cylinder and $\Omega_s$ and $\Omega_h$ are given in (C.3). The "t" is included in the subscript of $Z_s$ and $t$ as a reminder that they are measured in the transverse-to-$\hat{z}$ plane (the equivalent 2-D plane). $u_z^i(Q_{1t})$ is the incident field at the attachment point $Q_{1t}$ in the $z = z_0$ plane as shown in Figure 22b. From (4.13),

$$u_z^i(Q_{1t}) = u_0 \ e^{jk_{tt}z} \ |Q_{1t} = u_0(Q_{1t}) \quad (4.17)$$

since $\phi' = 0$, and thus, as shown in Figure 22b, $Q_{1t}$ is at $\phi = 90^\circ$. Given the $\hat{z}$-component of the field, the transverse components may be found from the relationships [13, p.79]

$$\tilde{E}_t = \frac{j\omega e}{k_t^2} \frac{\hat{t} \times \nabla t H_z - jk_z \frac{\nabla t E_z}{k_t^2}} {k_t^2} \quad (4.18a)$$

$$\tilde{H}_t = \frac{-j\omega e}{k_t^2} \frac{\hat{t} \times \nabla t E_z - jk_z \frac{\nabla t H_z}{k_t^2}} {k_t^2} \quad (4.18b)$$

where an $e^{-jk_{zz}z}$ dependence is assumed. Substituting (4.16) into (4.18) results in

$$\tilde{E}_t = \frac{e^{-jk_{zz}z_0}}{k_t^2} \left[ j\omega e \left( \frac{\partial h_z}{\partial \rho} - \frac{1}{\rho} \frac{\partial h_z}{\partial \phi} \right) - jk_z \left( \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} \right) \right] \quad (4.19)$$

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and
\[ h_t = \frac{e^{-jk_{2}z_0}}{k_t^2} \left[ -j\omega \epsilon \left( \frac{\partial e_z}{\partial \rho} - \frac{1}{\rho} \frac{\partial e_z}{\partial \phi} \right) - jk_z \left( \frac{\partial h_z}{\partial \rho} + \frac{1}{\rho} \frac{\partial h_z}{\partial \phi} \right) \right] \]  

(4.20)

Thus, given \( E_z \) and \( H_z \), the transverse components, \( E_\phi \), \( E_\rho \), \( H_\phi \), and \( H_\rho \) can be determined.

For brevity, only the \( \rho \)-component of the electric field will be derived. The other components in (4.19) and (4.20) are found similarly. From equation (4.19), \( E_\rho \) due to the test source \( \tilde{p}_e \) is

\[ E_\rho = - \frac{e^{-jk_{2}z_0}}{k_t^2} jk_z \frac{\partial e_z}{\partial \rho} \]  

(4.21)

It can be seen from (4.14) that \( e_z \) is produced by \( p_\rho^{\phi} \) and \( h_z \) is produced by \( p_\rho^{\rho} \); as a consequence, the problem can be decomposed into the following two independent problems:

\[ E^{\phi}_{\rho}^{\phi} = - \frac{e^{-jk_{2}z_0}}{k_t^2} jk_z \frac{\partial e_z}{\partial \rho} \]  

(4.22a)

\[ E^{\rho}_{\rho}^{\rho} = - \frac{j\omega \mu}{\rho} e^{-jk_{2}z_0} \frac{\partial h_z}{\partial \phi} \]  

(4.22b)

where the superscript on \( E_\rho \) indicates the orientation of the test source. From (4.16) and (4.17), \( e_z \) and \( h_z \) are

\[ e_z \sim e_0(Q_{1t}) e^{-jk_{1}t} \]

\[ \cdot \left[ h_t \tilde{g}(Z_{st}) + j \frac{h_t^3}{3!} \tilde{g}'(Z_{st}) - \frac{2h_t^4}{4!} \tilde{g}(Z_{st}) - \frac{h_t^5}{5!} \tilde{g}''(Z_{st}) + \mathcal{O}(h_t^6) \right] \]  

(4.23a)

\[ h_z \sim h_0(Q_{1t}) e^{-jk_{1}t} \]

\[ \cdot \left[ g(Z_{st}) + j \frac{h_t^2}{2!} g'(Z_{st}) - \frac{h_t^3}{3!} g(Z_{st}) - \frac{h_t^4}{4!} g''(Z_{st}) - j \frac{4h_t^5}{5!} g'(Z_{st}) + \mathcal{O}(h_t^6) \right] \]  

(4.23b)

where

\[ h_t = \frac{k_t}{m_t} (\rho - a) \]  

(4.24)
with

\[ m_t = m^3D \sin \theta_0 \quad (4.25a) \]

\[ m^3D = \left( \frac{ka}{2 \sin^2 \theta_0} \right)^{1/3} \quad (4.25b) \]

As a result,

\[ \frac{\partial h_t}{\partial \rho} = \frac{k_t}{m_t} = \frac{k}{m^3D} \quad (4.26) \]

Similarly, using the definition for \( Z_s \) in (C.4a), which is equivalent to \( Z_{st} \),

\[ Z_{st} = m_t \psi_t \quad (4.27) \]

where \( \psi_t \) is shown in Figure 22b and

\[ \psi_t = \left[ |\phi| - \frac{\pi}{2} \right] \text{ for } |\phi| < \pi \quad (4.28) \]

Thus

\[ \frac{\partial Z_{st}}{\partial \phi} = \begin{cases} m_t & \text{for } 0 < \phi < \pi \\ -m_t & \text{for } -\pi < \phi < 0 \end{cases} \quad (4.29) \]

From equation (C.8),

\[ t_t = a \psi_t \quad (4.30) \]

then

\[ \frac{\partial t_t}{\partial \phi} = \begin{cases} a & \text{for } 0 < \phi < \pi \\ -a & \text{for } -\pi < \phi < 0 \end{cases} \quad (4.31) \]

Using (4.26), (4.29), and (4.31) in (4.22a) and (4.22b) results in the \( \hat{\rho} \)-directed field at the near-field observer, \( P_s \), due to \( \hat{p}_e \) in the far field (Figure 22a) –

\[ E^x_{\rho} \sim -e_0(Q_{1t}) e^{-j(k_t t_t + k_z z_0)} \]

\[ \cdot \frac{jk_z}{k_t m_t} \left[ \tilde{g}(Z_{st}) + j \frac{h_t^2}{2!} \tilde{g}'(Z_{st}) - \frac{2 h_t^3}{3!} \tilde{g}(Z_{st}) - \frac{4 h_t^4}{4!} \tilde{g}''(Z_{st}) + O(h_t^5) \right] \quad (4.32) \]
Figure 23: Geometry of interest in order to move the origin to the attachment point $P_1$.

and

$$E_p^{(e)} \sim \mp \frac{j \omega \mu}{\rho} h_0(Q_{1t}) \frac{e^{-j(k_{1t}r_1 + k_2 z_0)}}{k^2_t}
\cdot \left[ -j k_{1t} a g(Z_{st}) + m_{t} g'(Z_{st}) + j \frac{k_t^2}{2!} \left[ -j k_{1t} a g'(Z_{st}) + m_{t} g''(Z_{st}) \right] + O(k^3_t) \right]$$

(4.33)

where the negative sign is used when $0 < \phi < \pi$ and the plus sign is used when $-\pi < \phi < 0$.

In order to obtain an expression that is more physically insightful, the origin is moved to the point $P_1$ in Figure 23 where the incident ray attaches to the cylinder. Since $\vec{p}_e$ is in the far field, then the norm of $\vec{r}' = -\vec{r}_{P_1} - \vec{r}'_1$ can be approximated
by

\[ r' \approx \begin{cases} \frac{r'_1 + (\bar{\tau}_{P_1} \cdot r')}{r'_1} & \text{for phase terms} \\ r'_1 & \text{for magnitude terms} \end{cases} \quad (4.34) \]

where \( r'_1 \) is the distance from \( P_1 \) to \( \bar{\rho}_e \) and \( \bar{\tau}_{P_1} \) is the vector to the point \( \bar{P}_1 \). So by shifting the origin to \( P_1 \), the distance dependence in (4.14) can be approximated by

\[ \frac{e^{-jkr'}}{r'} \approx \frac{e^{-jkr'_1}}{r'_1} e^{-jk\bar{\tau}_{P_1} \cdot r'} \quad (4.35) \]

In addition, if the origin is moved to \( P_1 \), then by looking at the geodesic in Figure 23 from the attachment point \( P_1 \) to \( P_N \), the projection of \( P_s \) onto the cylinder, it is found that

\[ t_t = s \cos \gamma \quad \text{and} \quad \Delta z = s \sin \gamma \quad (4.36) \]

where \( s \) is the path length of the geodesic, \( t_t \) is the arclength along the cylinder in the 2-D plane, \( \Delta z = z_o - z_{P_1} \), and \( \gamma \) is 90° minus the angle between the geodesic and the \( \hat{z} \)-axis. This allows for simplification of the exponential term in equation (4.32) and (4.33) through use of the identity

\[ k_t t_t + k_z z_o = k_s + k_z z_{P_1} \quad (4.37) \]

Since \( \bar{\tau}_{P_1} \) can be decomposed as

\[ \bar{\tau}_{P_1} = z_{P_1} \hat{z} + \bar{\tau}_{P_1t} \quad (4.38) \]

and

\[ k' = -k_z \hat{z} - \bar{k}_t \quad (4.39) \]

then

\[ k(\bar{\tau}_{P_1} \cdot r') = -k_z z_{P_1} - \bar{k}_t \cdot \bar{\tau}_{P_1t} \quad (4.40) \]
But from Figure 22b it can be seen that

\[ \mathbf{k}_\ell \cdot \hat{r}_{P_{1t}} = 0 \]  

(4.41)

So

\[ e^{-jk \rho P_{1} \cdot \hat{r}^f} e^{-j(k \ell_{t} + k_{z} z_{0})} = e^{-jk s} \]  

(4.42)

Using (4.42) along with (4.35), the \( \hat{r} \)-directed electric field at \( P_{1} \) due to \( p_{e}^{\rho} \) and \( p_{e}^{\phi} \) in the far field can be expressed as

\[ E_{\rho}^{\phi} \sim \frac{k Z_{0}}{4 \pi} \frac{p_{e}^{\rho}}{p_{e}^{\phi}} \frac{e^{-jk \rho_{1}^{'}}}{r_{1}^{'}} e^{-jk s} \cot \theta_{0} \frac{\partial h_{t}(Z_{st}, h_{t})}{\partial h_{t}} \]  

and

\[ E_{\rho}^{\phi} \sim \frac{jk Z_{0}}{4 \pi} \frac{p_{e}^{\rho}}{p_{e}^{\phi}} \frac{e^{-jk \rho_{1}^{'}}}{r_{1}^{'}} e^{-jk s} \left[ a \Omega_{h}(Z_{st}, h_{t}) + \frac{jm^{3D}}{k} \frac{\partial h_{t}(Z_{st}, h_{t})}{\partial g} \right] \]  

(4.44)

respectively.

The underscored term in (4.43) and (4.44), denoted by \( R \), is the standard far-zone distance dependence for a point source. However, this is an inaccurate representation in the near zone. Since \( R \) distinguishes the fields as far-zone fields, then it is \( R \) which must be modified if \( p_{e} \) is in the near zone. \( R \) can be heuristically generalized according to ray optics to account for near-field spreading by replacing the far-zone ray divergence factor of \( r_{1}^{-1} \) with its near-zone value, \( [r_{1}^{'}(s + r_{1}^{'})]^{-1/2} \).

Thus, for \( p_{e} \) in the near zone and the origin at the attachment point \( P_{1} \), the incident field in (4.14) is modified to be

\[ e_{0} \sim \frac{jk Z_{0}}{4 \pi} \sin \theta_{0} \frac{p_{e}^{\rho}}{p_{e}^{\phi}} \frac{e^{-jk \rho_{1}^{'}}}{\sqrt{r_{1}^{'}(s + r_{1}^{'})}} e^{-jk \rho P_{1} \cdot \hat{r}^f} \]  

(4.45a)

\[ h_{0} \sim \frac{jk}{4 \pi} \sin \theta_{0} \frac{p_{e}^{\phi}}{p_{e}^{\rho}} \frac{e^{-jk \rho_{1}^{'}}}{\sqrt{r_{1}^{'}(s + r_{1}^{'})}} e^{-jk \rho P_{1} \cdot \hat{r}^f} \]  

(4.45b)
As a result, the $\rho$-directed total electric field at $P_s$ due to $p_e^{\rho'}$ and $p_e^{\phi'}$ in the near zone of the cylinder is

$$E_{\rho}^{\rho'} \sim \frac{kZ_0}{4\pi} p_e^{\rho'} \frac{e^{-jk_{1}'r}}{\sqrt{r_{1}'(s + r_{1}')}} e^{-jks} \cos \theta_0 \frac{\partial \Omega_s(Z_{st}, h_t)}{\partial h_t}$$

(4.46)

and

$$E_{\phi}^{\phi'} \sim \pm \frac{jkZ_0}{4\pi} p_e^{\phi'} \frac{e^{-jk_{1}'r}}{\sqrt{r_{1}'(s + r_{1}')}} e^{-jks} \left[ a \Omega_h(Z_{st}, h_t) + \frac{jm^3D}{k} \frac{\partial \Omega_h(Z_{st}, h_t)}{\partial h} \right]$$

(4.47)

where $\rho$ is the cylindrical coordinate of the observer and

$$\Omega_s(D, h_t) = h_t g(D) + j \frac{h_t^2}{3!} g'(D) - \frac{2h_t^4}{4!} g''(D) - \frac{h_t^5}{5!} g'''(D)$$

+ $O(h_t^6)$

(4.48a)

$$\Omega_h(D, h_t) = g(D) + j \frac{h_t^2}{2!} g'(D) - \frac{h_t^3}{3!} g''(D) - \frac{h_t^4}{4!} g'''(D) - j \frac{h_t^5}{5!} g''(D)$$

+ $O(h_t^6)$

(4.48b)

The angle $\theta_0$ is defined as

$$\cos \theta_0 = - \hat{r}_1 \cdot \hat{z}$$

(4.49a)

$$\sin \theta_0 = \| \hat{r}_1 \times \hat{z} \|$$

(4.49b)

The $\hat{z}$-directed electric field, $E_z$, was given in (4.16). Similar modifications can be made to it to account for a near-zone source and to relocate the origin to $P_1$. The resulting $\hat{z}$-directed electric field due to $p_e^{\rho'}$ is

$$E_z^{\rho'} \sim \frac{jkZ_0}{4\pi} p_e^{\rho'} \sin \theta_0 \frac{e^{-jk_{1}'r}}{\sqrt{r_{1}'(s + r_{1}')}} e^{-jks} \Omega_s(Z_{st}, h_t)$$

(4.50)

and that due to $p_e^{\phi'}$ is

$$E_z^{\phi'} \sim 0$$

(4.51)
since $p^q$ produces only an $H_z$ component. The $\phi$-directed fields, $(E_\phi, H_\phi)$, are derived similarly to the $\rho$-directed electric field. However, since the fins of interest at the present time lie in the $\rho$-$z$ plane, the results for $E_\phi$ and $H_\phi$ are not presented here.

**4.3.2 Lit Region**

A derivation similar to that in Section 4.3.1 can be done for an observer in the lit region (Figure 24a). Again $\phi' = 0$ will be assumed. If everything is projected onto the transverse $z = z_o$ plane, then Figure 24b, the equivalent 2-D problem, results. This resembles the 2-D scattering problem shown in Figure 82, but now the incident field has a wavevector $\vec{k}_t$ instead of $\vec{k}$. From (4.13), the incident field at $P_N$, the projection of $P_l$ onto the cylinder, in the $z_o$ plane is

\[
\begin{align*}
\psi^i(P_N) &= u_0 e^{j k_t z} \big|_{P_N} \\
&= u_0 e^{j k_t a \cos \theta^i_t} 
\end{align*}
\]

(4.52)

where $\theta^i_t$, shown in Figure 24b, is the incident angle at $P_N$ measured in the transverse plane. Using equation (C.2) to determine the resulting field at $P_l$ due to the incident field in (4.12) gives

\[
\begin{align*}
u^i_z(P_l) \sim \psi^i_z(P_N) e^{-j k_t z_o} e^{-j Z_l^3/3} \Omega_{s,h} (Z_l, h_t)
\end{align*}
\]

(4.53)

As in the previous section, the transverse-to-$\hat{z}$ components of the field are obtained by taking the derivative of (4.53) with respect to $\rho$ and $\phi$ and substituting the results into (4.19) and (4.20).

Again, for brevity, only the $\rho$-component of the field is derived. The main difference between the form of the field in the lit and that in the shadow region is the argument of the Fock function, which in the lit zone case is given by (C.4b).
Figure 24: A test source radiating a local plane wave. Observer is in the lit region of the source and close to the infinitely-long circular cylinder. (a) Original 3-D problem and (b) equivalent 2-D problem.
Thus,

\[ Z_{lt} = -m_t \cos \phi = -m_t \cos \theta^i_t \]  

(4.54)

The derivative of \( Z_{lt} \) is

\[ \frac{\partial Z_{lt}}{\partial \phi} = m_t \sin \phi = m_t \sin \theta^i_t \]  

(4.55)

The rest of the derivation is similar to that in Section 4.3.1 and thus is omitted. The resulting \( \rho \)-directed field at an observer in the lit region of the source, \( p^\phi_e \), and in the near field of the cylinder is

\[
E^\phi_\rho (P_t) \sim \frac{p^\phi_e Z_0}{4\pi} \cot \theta_{PN} \frac{e^{-jkr^i_N}}{r^i_N} \frac{e^{-jkr^i_N \cdot r_{PN}}}{r^i_N} e^{jkr^i_N} e^{-jkz_0} \\
\times \left[ \frac{k}{m^3D} e^{-jZ^3_t/3} \frac{\partial \Omega_s (Z_{lt}, h_t)}{\partial h_t} \right] 
\]

(4.56)

where \( \bar{r}_N = \bar{r}_{PN} - \bar{r}' \). The angle \( \theta_{PN} \) is shown in Figure 25 and is defined as

\[ \cos \theta_{PN} = -\hat{z} \cdot \hat{r}_N \]  

(4.57a)

\[ \sin \theta_{PN} = \| \hat{z} \times \hat{r}_N \| \]  

(4.57b)

and the observer coordinates are \((\rho, \phi, z_0)\). The field generated with \( p^\phi_e \) as the source is

\[
E^\phi_\rho (P_t) \sim \frac{p^\phi_e Z_0}{4\pi \rho \sin \theta_{PN}} \sin \theta^i_t \frac{e^{-jkr^i_N}}{r^i_N} \frac{e^{-jkr^i_N \cdot r_{PN}}}{r^i_N} e^{jkr^i_N} e^{-jkz_0} \\
\times \left\{ e^{-jZ^3_t/3} \frac{\partial \Omega_h (Z_{lt}, h_t)}{\partial g} \right\} 
\]

(4.58)

As in Section 4.3.1, to facilitate physical insight, the origin has been moved from \( O \) to \( P_N \). Additionally, note that in the lit region, no additional changes, i.e. a spread factor, are required to account for a near-field test source.
Figure 25: Geometry of interest in order to move the origin to the point $P_N$. 
The $\hat{z}$-directed field was given in (4.53). After moving the origin to $P_N$, the field due to $p^\phi_e$ and $p^\phi_t$ is found to be

$$E^\phi_{z}(P_{t}) \sim p^\phi_e \frac{j k Z}{4 \pi} \sin \theta_{P_N} \frac{e^{-j k r_{N}^I}}{r_{N}^I} e^{-j k r_{t}^I} \cdot \hat{r}_{P_N} e^{j k \alpha \cos \theta_{t}^I} e^{-j k z_{0}}$$

$$e^{-j Z_{t}^2 / 3} \Omega_{s}(Z_{t}, h_{t})$$

and

$$E^\phi_{z}(P_{t}) \sim 0$$

respectively. Again, due to a lack of need at the present time for $E^\phi$, the results for the $\hat{\phi}$-directed field are not presented here.

4.4 Reciprocal Problem

The results in Section 4.3.1 and 4.3.2 analyzed the situation shown in Figure 21a: a test source radiating a local plane wave field and an observer in the near field of the cylinder. Reciprocity is used to obtain the results for the case shown in Figure 21b: an electric current element in the near field of the cylinder radiating a field which an observer detects in the far field of the cylinder. Consider an electric current source, $\bar{J}$, at the near-field point $P$ and an electric test source $\bar{p}_e$ at $Q$ in the presence of a PEC circular cylinder as shown in Figure 21b. The current on the test source is given by

$$\bar{J}_t = \bar{p}_e \delta(||\bar{r} - \bar{r}_t||)$$

(4.61)

where $\bar{p}_e$ was given in (4.2), and the electric current source is

$$\bar{J} = \bar{p}^I_e \delta(||\bar{r} - \bar{r}_J||)$$

(4.62)

Applying reciprocity between $\bar{p}^I_e$ and the test source $\bar{p}_e$ results in

$$\int_V \bar{J} \cdot \bar{E}^4 (\bar{r}_J) \ dV = \int_V \bar{J}_t \cdot \bar{E}^I (\bar{r}_t) \ dV$$

(4.63)
which shows that $\mathbf{E}^J(\mathbf{r}_t)$, the field radiated by $p_e^J$ at the test source, can be determined if given $\mathbf{E}^t(\mathbf{r}_J)$, the electric field radiated by the test source $p_e$ at $J$ near the cylinder. Given the representation for the currents in (4.61) and (4.62), equation (4.63) reduces to

$$p_e^J \cdot \mathbf{E}^t(\mathbf{r}_J) = p_e \cdot \mathbf{E}^J(\mathbf{r}_t)$$

$$= p_e^{\phi'} E_{\phi'}^J(\mathbf{r}_t) + p_e^{\rho'} E_{\rho'}^J(\mathbf{r}_t)$$

(4.64)

4.4.1 A $\rho$–directed current source $\mathbf{J}$

First, it is assumed that the electric current source is in the normal direction to the cylinder,

$$\mathbf{p}_e^J \triangleq \hat{\rho} p_e$$

(4.65)

then (4.64) further simplifies to

$$p_e^\rho E_{\rho}^t(\mathbf{r}_J) = p_e^{\rho'} E_{\rho'}^J(\mathbf{r}_t) + p_e^{\phi'} E_{\phi'}^J(\mathbf{r}_t)$$

(4.66)

where $E_{\rho}^t(\mathbf{r}_J)$, the $\rho$–directed field at $\mathbf{r}_J$ produced by the test source $p_e$, is known from the previous two sections. Because $p_e$ was decomposed as in (4.2), $E_{\rho}^t$ is determined by summing up the independently calculated $\rho$–directed field due to $p_e^{\rho'}$ and $p_e^{\phi'}$.

$$E_{\rho}^t = E_{\rho}^{\rho'} + E_{\rho}^{\phi'}$$

(4.67)

This reduces the problem of solving (4.66) to two independent problems:

$$p_e^\rho E_{\rho}^{\rho'}(\mathbf{r}_J) = p_e^{\rho'} E_{\rho'}^{\rho'}(\mathbf{r}_t)$$

(4.68a)

$$p_e^\rho E_{\rho}^{\phi'}(\mathbf{r}_J) = p_e^{\phi'} E_{\phi'}^{\rho'}(\mathbf{r}_t)$$

(4.68b)
Using (4.68), the problem shown in Figure 21b – observer in the far-zone due to a \( \hat{\rho} \)-directed electric current element near the cylinder – can be solved. These results follow.

4.4.1.1 Shadow Region

Using equation (4.46) in (4.68a) results in the \( \hat{\rho} \)-directed electric field at an observer at \( \vec{r}_t \) in the far field due to the \( \hat{\rho} \)-directed current source near the cylinder.

\[
E^J_{\hat{\rho}}(\vec{r}_t) \sim p e \rho \left( \frac{k Z_0}{4\pi} \right) \frac{e^{-jk r'_t}}{\sqrt{r'_t(s + r'_t)}} e^{-jks} \frac{\cot \theta_t}{m^3 D} \frac{\partial \Omega_s}{\partial h}(Z_{st}, h_t) \tag{4.69}
\]

Similarly, if (4.47) is used in (4.68b), the \( \hat{\rho} \)-directed field component is obtained.

\[
E^J_{\hat{\rho}}(\vec{r}_t) \sim \pm p e \rho \left( \frac{j k Z_0}{4\pi} \right) \frac{e^{-jk r'_t}}{\sqrt{r'_t(s + r'_t)}} e^{-jks} \rho J
\]

\[
\cdot \left[ \rho_0(Z_{st}, h_t) + j m^3 D \frac{\partial \Omega_s}{\partial g}(Z_{st}, h_t) \right] \tag{4.70}
\]

where \( \rho_J \) denotes the radial coordinate of the current source \( J \) in the near field of the cylinder.

4.4.1.2 Lit Region

The lit region results are obtained by using (4.56) and (4.58) in (4.68a) and (4.68b), respectively, to obtain the fields due to an electric current element \( J \) in the \( \hat{\rho} \)-direction at the observer at \( \vec{r}_t \).

\[
E^J_{\hat{\rho}}(\vec{r}_t) \sim p e \rho \frac{Z_0}{4\pi} \cot \theta P_N e^{-jkr'_N} e^{-jk \rho_t} \cdot \frac{\partial P_N}{\partial \rho} e^{j k a \cos \theta_t} e^{-j k z_{z0}}
\]

\[
\cdot \left[ \frac{k}{m^3 D} e^{-jZ_{t}^3/3} \frac{\partial \Omega_s}{\partial h}(Z_{st}, h_t) \right] \tag{4.71}
\]

and

\[
E^J_{\hat{\rho}}(\vec{r}_t) \sim \frac{p e \rho Z_0}{4\pi \rho \sin \theta P_N} \sin \theta_t e^{-jkr'_N} e^{-jk \rho_t} \cdot \frac{\partial P_N}{\partial \rho} e^{j k a \cos \theta_t} e^{-j k z_{z0}}
\]
\[ e^{-j\frac{Z_1}{3}} \left\{ -j \left( k_t a + m_t Z_{t_1}^2 \right) \Omega_h (Z_{t_1}, h_t) + m_t \frac{\partial \Omega_h (Z_{t_1}, h_t)}{\partial g} \right\} \] (4.72)

### 4.4.2 A \( \hat{z} \)-directed current source \( \hat{J} \)

If

\[ \hat{p}_e = \hat{z} p_{ez} \] (4.73)

the problem resembles those in (4.68); i.e.,

\[ p_{ez} E_{z}^{\theta'} (\bar{r}_f) = p_{e}^{\theta'} E_{\theta'}^{\hat{J}} (\bar{r}_t) \] (4.74a)

\[ p_{ez} E_{z}^{\phi'} (\bar{r}_f) = p_{e}^{\phi'} E_{\phi'}^{\hat{J}} (\bar{r}_t) \] (4.74b)

Using (4.74), the problem shown in 21b, for a \( \hat{z} \)-directed current element near the cylinder, can be solved. The results follow.

#### 4.4.2.1 Shadow Region

Using equation (4.50) and (4.51) in (4.74), results in the \( \hat{\theta}' \)-directed and \( \hat{\phi}' \)-directed electric fields at the observer at \( \bar{r}_t \) in the far field due to the \( \hat{z} \)-directed current source near the cylinder. These are

\[ E_{\theta'}^{\hat{J}} (\bar{r}_t) \sim p_{ez} \frac{j k Z_0}{4\pi} \sin \theta_0 \frac{e^{-jk r_1}}{\sqrt{r_1'(s + r_1')}} e^{-jks} \Omega_s (Z_{s_1}, h_t) \] (4.75)

and

\[ E_{\phi'}^{\hat{J}} (\bar{r}_t) \sim 0 \] (4.76)
4.4.2.2 Lit Region

The field for an observer in the lit region of the source are obtained similarly. These are

\[ E_z^\phi(r_k) \sim \rho r_z \sin \theta_P \cdot \frac{-jkr'_N}{r'_N} e^{-jkr'_N} \cdot e^{-jkr'_N} \cdot e^{jkt \cos \theta_i} e^{-jkrz_0} \]

\[ \cdot \left[ e^{-jZt'^i/3} \Omega_s(Zt^i, h_t) \right] \quad (4.77) \]

and

\[ E_z^\phi(r_k) \sim 0 \quad (4.78) \]

4.5 Conclusion

An asymptotic solution to the problem of determining the field at a point close to but not necessarily on the PEC boundary has been presented in this chapter. It is referred to as the extended UTD (EUTD) radiation solution. The 2-D results of Pathak [27] were generalized to 3-D. Since a Taylor series for \( h_t \) small was used to approximate the Airy functions, the EUTD radiation solution is most accurate if the observer is close to the PEC surface. Verification of this solution will be presented in Chapter VII. Similar to the process used to verify the EUTD coupling solution, the existence of a transition point from the EUTD radiation solution to both the EUTD coupling solution and to the existing UTD solutions will be shown. As for the EUTD coupling case, the EUTD radiation solution for the canonical circular cylinder can be generalized via GTD concepts to apply to the PEC smooth convex object.
CHAPTER V
Equivalent Point Source Representation for a Monopole with a
Piecewise Sinusoidal (PWS) Current Distribution

5.1 Introduction

In filling the MM impedance matrix, the most rudimentary calculation which
must be performed is to determine the field due to a test filamentary dipole. For
a test dipole with a general current distribution, the field is approximated as being
produced by a series of current elements, i.e. point sources, along the length of the
dipole weighted by the value of the current at that location, \( \bar{J}(t') \). In mathematical
terms, this amounts to determining the field of \( \bar{J} \) by

\[
\mathbf{E}^i(\mathbf{R}) = \int_0^L \bar{J}(t') \cdot \bar{G}(\mathbf{R}, t') \, dt'
\]

(5.1)

where \( L \) is the length of the dipole and \( \bar{G} \) is the field at \( \mathbf{R} \) from a point current source
at \( t' \). However, a test monopole, such as the one shown in Figure 7 of Section 2.3
with a piecewise sinusoidal current distribution (equation (2.13)) produces a field
which has a closed-form expression (equation (2.14)). This is the major reason for
using it in the conventional MM so that the integration in (5.1) is eliminated.

With regards to the MM-UTD combination, the PWS has the added feature
that the field from a monopole with this current distribution can be represented
as being produced by two equivalent point sources located at the endpoints of the
monopole. This is a very important feature because the fields of point sources are
spherical waves which can be described ray optically and a direct application of the
UTD requires that the incident field in the UTD be ray optical. This requirement
can also be satisfied by representing the incident field as in (5.1); however, as will be shown, this sometimes does not produce very good results especially in the very important self-impedance case.

5.2 Equivalent PWS Point Source Fields

This concept of thinking of the field from a monopole with a piecewise sinusoidal current distribution as emanating from two distinct point sources was previously used by Ekelman [11] for analyzing wire antennas in the presence of a circular cylinder. He showed the decomposition of the field from a straight dipole. The derivation for the V-dipole case is detailed here. The geometry is shown in Figure 26.

Using equation (2.14), the electric field from the bottom monopole ($i = 0$, $i_2 = 1$) of the V-dipole in Figure 26 is

$$E_z^{M1} = C_{M1} \left( \frac{e^{-jkR_1}}{R_1} - \cos kd \frac{e^{-jkR_2}}{R_2} \right)$$

(5.2a)

$$E_{\rho}^{M1} = \frac{C_{M1}}{\rho} \left[ e^{-jkR_2} \left( \cos kd \cos \theta_2 - j \sin kd \right) - e^{-jkR_1} \cos \theta_1 \right]$$

(5.2b)

where

$$C_{M1} = - \frac{jZ_0}{4\pi \sin kd}$$

(5.3)

and $\rho = R_1 \sin \theta_1 = R_2 \sin \theta_2$. The field for the top monopole ($i_1 = 1$, $i_2 = 0$) is

$$E_z^{M2} = C_{M2} \left( \frac{e^{-jkR_3}}{R_3} - \cos kd \frac{e^{-jkR_2}}{R_2} \right)$$

(5.4a)

$$E_{\rho}^{M2} = \frac{C_{M2}}{\rho'} \left[ e^{-jkR_2} \left( \cos kd \cos \theta'_2 + j \sin kd \right) - e^{-jkR_3} \cos \theta_3 \right]$$

(5.4b)

where

$$C_{M2} = - \frac{jZ_0}{4\pi \sin kd}$$

(5.5)
Figure 26: V-dipole with a piecewise sinusoidal current distribution.
and $\rho' = R_2 \sin \theta'_2 = R_3 \sin \theta_3$. The $\hat{z}'$ and $\hat{r}'$ axis are the standard axis in a cylindrical coordinate system where the $\hat{z}'$ axis coincides with the axis of the top monopole and the origin is at its bottom endpoint. By combining (5.2) and (5.4), the field from the V-dipole in Figure 26 is

$$\vec{E}^{\text{dip}} = E_z \hat{z} + E_\rho \hat{r}$$

(5.6)

where

$$E_z = \left( E_z^{M1} + E_z^{M2} \cos \theta - E_\rho^{M2} \sin \theta \right)$$

$$= C_{M1} \frac{e^{-jkR_1}}{R_1} + C_{M2} \frac{e^{-jkR_3}}{R_3} \left( \cos \theta + \cot \theta_3 \sin \theta \right)$$

$$- \frac{e^{-jkR_2}}{R_2} \left[ C_{M1} \cos k\hat{d} \right.$$

$$+ C_{M2} \left( \cos kd \cos \theta + \frac{j \sin kd \sin \theta}{\sin \theta_2} + \cos kd \cot \theta_2 \sin \theta \right) \left. \right]$$

(5.7)

and

$$E_\rho = \left( E_\rho^{M1} + E_\rho^{M2} \cos \theta + E_\rho^{M2} \sin \theta \right)$$

$$= -C_{M1} \frac{e^{-jkR_1}}{R_1} \cot \theta_1 - C_{M2} \frac{e^{-jkR_3}}{R_3} \cos \theta \cot \theta_3$$

$$+ \frac{e^{-jkR_2}}{R_2} \left\{ \frac{C_{M1}}{\sin \theta_2} \left( \cos k\hat{d} \cos \theta - j \sin kd \right) \right.$$ 

$$+ \frac{C_{M2}}{\sin \theta_2} \left( j \sin kd + \cos kd \cos \theta_2 \right) - \sin \theta \cos kd \right\}$$

(5.8)

This is not in a form conducive for use with the conventional UTD which requires the incident field to be ray optical and hence polarized transverse to the ray direction. To accomplish this, (5.6) is rewritten such that the field is decomposed into three distinct terms: two of which are associated with the two endpoints of the V-dipole
and one which is associated with the terminal point. Term 1 will be associated with the endpoint of the V-dipole where the current \( I \) starts from while term 3 is associated with the endpoint where the current ends. Term 2 will be associated with the terminal point. To this end, let

\[
E_{\text{dip}} = E_1 i_1 + E_2 i_2 + E_3 i_3
\]

(5.9)

where \( E_i \) is dependent only on variables having to do with the \( i^{th} \) point source. The unit vector \( i_i \) is in the direction transverse to \( \hat{R}_i \), the vector from the \( i^{th} \) endpoint to the observer; i.e.,

\[
i_i = \frac{\hat{R}_i \times \hat{z} \times \hat{R}_i}{|| \hat{R}_i \times \hat{z} \times \hat{R}_i ||}
\]

(5.10)

Since \( E_1 \) is independent of endpoints 2 and 3, it can be determined by equating only those terms in (5.6) and (5.9) which are dependent on \( R_1 \) (denoted below by \( E_z(R_1) \) and \( E_\rho(R_1) \)). The result is

\[
E_1 = E_z(R_1) \sin \theta_1 - E_\rho(R_1) \cos \theta_1
\]

(5.11)

A similar result is found for the other non-terminal endpoint of the dipole.

\[
E_3 = E_z(R_3) \sin(\theta_3 + \theta) - E_\rho(R_3) \cos(\theta_3 + \theta)
\]

(5.12)

By using (5.6), \( E_2 \) automatically includes the field from the terminal point of both monopoles.

\[
E_2 = E_z(R_2) \sin \theta_2 - E_\rho(R_2) \cos \theta_2
\]

(5.13)
where the relationship $\theta_2 = \theta_2' + \theta$ has been used to reduce (5.13). However, from (5.13), the contributions to $E_2$ from the top and bottom monopole can be isolated resulting in

$$E_{2}^{\text{bot}} = -C_{M1} \frac{e^{-jkR_2}}{R_2} \frac{(\cos k\tilde{d} - j \sin k\tilde{d} \cos \theta_2)}{\sin \theta_2}$$  \hspace{1cm} (5.14a)$$

$$E_{2}^{\text{top}} = -C_{M2} \frac{e^{-jkR_2}}{R_2} \frac{(\cos k\tilde{d} + j \sin k\tilde{d} \cos \theta'_2)}{\sin \theta'_2}$$  \hspace{1cm} (5.14b)$$

These results can be easily verified by using the following relationships:

$$\hat{i}_1 \cdot \hat{z} = \sin \theta_1$$  \hspace{1cm} (5.15a)$$

$$\hat{i}_2 \cdot \hat{z} = \sin \theta_2$$  \hspace{1cm} (5.15b)$$

$$\hat{i}_3 \cdot \hat{z} = \sin(\theta_3 + \theta)$$  \hspace{1cm} (5.15c)$$

and

$$\hat{i}_1 \cdot \hat{\rho} = -\cos \theta_1$$  \hspace{1cm} (5.16a)$$

$$\hat{i}_2 \cdot \hat{\rho} = -\cos \theta_2$$  \hspace{1cm} (5.16b)$$

$$\hat{i}_3 \cdot \hat{\rho} = -\cos(\theta_3 + \theta)$$  \hspace{1cm} (5.16c)$$

to prove that (5.11) - (5.13) do indeed recombine to give $E_z$ and $E_\rho$ as per

$$E_z = \left( E_1 \hat{i}_1 + E_2 \hat{i}_2 + E_3 \hat{i}_3 \right) \cdot \hat{z}$$  \hspace{1cm} (5.17)$$

and

$$E_\rho = \left( E_1 \hat{i}_1 + E_2 \hat{i}_2 + E_3 \hat{i}_3 \right) \cdot \hat{\rho}$$  \hspace{1cm} (5.18)$$

Thus the field can be represented in cylindrical (equation (5.6)) or ray-fixed coordinates (equation (5.9)); i.e.,

$$E^{\text{dip}} = E_\rho \hat{\rho} + E_z \hat{z} = E_1 \hat{i}_1 + E_2 \hat{i}_2 + E_3 \hat{i}_3$$  \hspace{1cm} (5.19)$$

This shows that the electric field from a V-dipole can be viewed as
1. being produced by three distinct point sources located at the endpoints and terminal point of the V-dipole;

2. emanating from each equivalent point source as a spherical wave, i.e., \( \frac{e^{-jkR}}{R} \) (pattern factor); and

3. radiating from each point source in a direction transverse to the ray from the source to the observer.

Statement (3) can be strengthened by showing that the field from each endpoint has no component along the ray paths. To prove this, let us postulate that there are field components along the ray paths. To the transverse-to-the-ray components in (5.9) we add

\[
\tilde{E}_{dip}^{R} = \tilde{E}_1 \hat{R}_1 + \tilde{E}_2 \hat{R}_2 + \tilde{E}_3 \hat{R}_3 \quad (5.20)
\]

Just as (5.9) was equated with (5.6) to determine the \( E_i \)'s, equations (5.6) and (5.20) are equated to find the \( \tilde{E}_i \)'s:

\[
\tilde{E}_1 = E_z(R_1) \cos \theta_1 + E_\rho(R_1) \sin \theta_1 = 0 \quad (5.21a)
\]

\[
\tilde{E}_2 = E_z(R_2) \cos \theta_2 + E_\rho(R_2) \sin \theta_2 = 0 \quad (5.21b)
\]

\[
\tilde{E}_3 = E_z(R_3) \cos \theta_3 + E_\rho(R_3) \sin \theta_3 = 0 \quad (5.21c)
\]

Thus for an arbitrary \( \theta \), there are no field contributions along any of the three ray paths. This is exactly in accordance with the assumptions of the UTD.
In summary, a monopole with a piecewise sinusoidal current distribution can be thought of as two distinct point sources radiating the field

\[ E_{\text{mon}} = E_{nt}(R_{nt}) \hat{i}_{nt} + E_{t}(R_{t}) \hat{i}_{t} \]  

(5.22)

where the subscript \( nt \) refers to the non-terminal endpoint of the monopole where the current is zero and the subscript \( t \) refers to the terminal point where the current is unity. The vector components are

\[ E_{nt} = C \frac{e^{-jkR_{nt}}}{R_{nt}} \left( \frac{1}{\sin \theta_{nt}} \right) \]  

(5.23a)

\[ E_{t} = C \frac{e^{-jkR_{t}}}{R_{t}} \left[ (-1)^{I_1} j \sin (k dL) \cot \theta_{t} - \frac{\cos kdL}{\sin \theta_{t}} \right] \]  

(5.23b)

where

\[ C = -\frac{jZ_0}{4\pi \sin k dL} \]  

(5.24)

and \( dL \) is the length of the monopole, \( R_{nt} \) and \( R_{t} \) are the distances between the observer and the non-terminal and terminal point, respectively, and \( \theta_{nt} \) and \( \theta_{t} \) are the angles between the axis of the monopole and \( R_{nt} \) and \( R_{t} \), respectively. So by using the PWS as a basis function for the current, an incident field is generated which satisfies, without necessitating approximations, the three major assumptions of the UTD (page 83). As a consequence, when using the UTD to determine the field from such a monopole in the presence of a PEC, the problem can be decomposed into finding the field from the endpoints of the monopole separately, and then summing the two contributions. Not only is this way numerically more efficient than using (5.1) to compute the incident field, but it will be shown to be a more accurate method also.
5.3 Reciprocity of Monopole–Monopole Impedance $\Delta z_{ij}$

Recall that the impedance matrix $Z$ is divided as:

$$Z_{mn} = Z_{omn} + \Delta Z_{mn}$$  \hspace{1cm} (5.25)

where $Z_{omn}$ is the mutual impedance between dipoles $m$ and $n$ in free-space while $\Delta Z_{mn}$ is the mutual impedance due to the presence of the PSCO. This latter contribution is determined by the UTD. The dipole impedance $Z$ is the sum of four monopole–to–monopole impedances denoted by $z_{ij}$.

An interesting point has been found regarding the reciprocity of the $\Delta z_{ij}$ between two monopoles. The mutual impedance between two monopoles in free-space, $z_{ij}$, calculated using the free-space Green’s function, is not reciprocal because the field contribution from the point charges has been omitted in (2.14). In contrast, $\Delta z_{ij}$, the mutual impedance due to the cylinder effect calculated via the UTD, is reciprocal.

This is easily seen by examining the field due to the point charges $-I_1/j\omega$ and $I_2/j\omega$ at the endpoints $z_1$ and $z_2$ respectively, of the line source in Figure 7. At endpoint $i$, the point charge generates the field [3]

$$\vec{E}_i = (-1)^{i+1} I_i \frac{Z_0}{j4\pi k} \nabla \left( \frac{e^{-jkR_i}}{R_i} \right)$$

$$= (-1)^{i+1} I_i \frac{Z_0}{j4\pi k} \left[ \frac{e^{-jkR_i}}{R_i} \left( -jk\hat{R}_i + \frac{\hat{R}_i}{R_i} \right) \right]$$  \hspace{1cm} (5.26)

where $i = 1$ or $2$, $I_i$ is the value of the current ($0$ or $1$) at $z_i$, and $R_i$ is the distance from $z_i$ to the observer. The field from each point charge is along the incident ray direction, $\hat{R}_i$, so it would not contribute to the UTD scattered field because, as was previously pointed out, the UTD only utilizes those incident field components transverse to the incident ray direction. So the exclusion of the point charge is
inconsequential to the calculation of the UTD-based field scattered by the PSCO and thus to the mutual impedance \( \Delta z_{ij} \). This is summarized in Figure 27.

This is important because in the free-space case, omission of the point charge is unimportant only because it is assumed that the monopole will be connected to form dipoles and thus the point charge contribution from the two monopoles at the terminal point will cancel each other. However, this last conclusion assumes that the two ends of the monopoles touch when they are connected which is not always guaranteed by computer codes. If the monopoles are not joined at the terminal point, a plus and a minus point charge physically exist which is not mathematically accounted for, but yet radiates a free-space field. This may cause numerical problems when determining \( Z_{omn} \). However, the fact that the point charges do not produce a UTD-based reflected field at the observer means that we do not have to worry about their omission.

5.4 Mutual Impedance Between Two Dipoles in the Presence of a PEC Circular Cylinder

5.4.1 Calculation of the Incident Field

There are two methods to determine the incident field from a monopole (or dipole) with a piecewise sinusoidal current distribution:

1. Sum up, i.e., integrate, the incident field from a series of infinitesimal current elements weighted by the value of the current at the integration point.

or

2. Add the field from the two endpoints of the monopole (see equation (5.22)).

The first method is more general and must be used when the current distribution is not a piecewise sinusoid. However, if the basis function is a piecewise sinusoid, the
Figure 27: Effect of the field from the point charge at the terminal point of the monopole. (a) Field at the observer. (b) Contribution to the monopole to monopole mutual impedance.
second method is preferable not only because there is no integration involved, but because, as will be shown, it is more accurate.

When using the UTD, the incident field is assumed to radiate from a point source, as \( e^{-jkR/R} \), in the direction transverse to the incident ray. This is important because only the field components which are transverse to the ray will be scattered from the cylinder. Therefore, in order to use the UTD with an incident field determined by method 1, the Green’s function for the current elements used in method 1 must radiate only the far-zone fields [25]:

\[
\vec{E}' \sim -\frac{jkZ_0}{4\pi} I \, dl \, e^{-jkR/R} \sin \theta \hat{\theta}
\]  

(5.27)

where \( R \) is the distance in Figure 28 from the current element \( I \, dl \) to the observer and \( \theta \) is the angle between the axis of \( I \, dl \) and \( \vec{R} \). As expected, this is only a good approximation when, in using the UTD, the reflection point on the PEC is
sufficiently far from the source which is often not the case. As shown in Section 5.2, the field used in method 2 automatically satisfies the requirements on the source necessitated by the UTD with no approximations required.

The two methods are compared using the geometry shown in Figure 29. This is a very simple example used to show, without involving the UTD yet, the error in ignoring the near-zone field contribution. The following four methods of calculating the incident field are used:

1. The exact closed-form expression in (5.6) denoted by $\vec{E}^{PWS}$. 

Figure 29: Dipole with a piecewise sinusoidal current distribution.
2. Integration over a series of current elements denoted by $\vec{E}_k^I$ —

$$
\vec{E}_k^I = \int \vec{I}_{PW} \cdot \vec{e}_k^I \, dl
$$

(5.28)

where $k$ indicates how many terms are included in $\vec{e}_k^I$. Thus,

(a) if $k = 1$, then only the far-zone term is included, and

$$
\vec{e}_1^I \triangleq \frac{j k Z_0}{4\pi} \frac{e^{-j k R}}{R} \sin \theta \hat{\theta}
$$

(5.29)

(b) if $k = 2$, then the induction field term is added in and defined as

$$
\vec{e}_2^I \triangleq \vec{e}_1^I + \left[ \frac{\hat{\theta}}{2} \sin \theta + \hat{R} \cos \theta \right] \frac{Z_0 e^{-j k R}}{2\pi R^2}
$$

(5.30)

(c) and if $k = 3$, then the near-zone term is added in and defined as

$$
\vec{e}_3^I \triangleq \vec{e}_2^I - \left[ \frac{\hat{\theta}}{2} \sin \theta + \hat{R} \cos \theta \right] \frac{j Z_0 e^{-j k R}}{2\pi k R^3}
$$

(5.31)

Note that $\vec{e}_1^I$ is the exact field from a current element. The results are shown in Table 1. The results for $\vec{E}_k^I$ remain the same even if the number of integration points per

---

Table 1: $\vec{E}_{PW}^I$ versus $\vec{E}_I$ with 8 integration points per monopole (geometry shown in Figure 29). $\vec{E}_{PW}^I$ = exact field from PWS. $\vec{E}_I^I$ = field from PWS determined by $\int \vec{I}_{PW} \cdot \vec{e}_k^I \, dl$ where $\vec{e}_k^I$ = field from a current element with $k$ terms.

<table>
<thead>
<tr>
<th>E-field</th>
<th>$E_y$ (V/m)</th>
<th>$E_z$ (V/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{E}_{PW}^I$</td>
<td>$-23.5 + j 26.7$</td>
<td>$-12.5 - j 3.0$</td>
</tr>
<tr>
<td>$\vec{E}_1^I$</td>
<td>$0.83 + j 1.7$</td>
<td>$-5.4 - j 9.2$</td>
</tr>
<tr>
<td>$\vec{E}_2^I$</td>
<td>$-27.5 + j 18.5$</td>
<td>$-13.4 - j 5.3$</td>
</tr>
<tr>
<td>$\vec{E}_3^I$</td>
<td>$-23.5 + j 26.7$</td>
<td>$-12.5 - j 3.0$</td>
</tr>
</tbody>
</table>
monopole is increased from 8 to 40 implying that the answers have converged. This example isolates the problem: $\vec{E}_I$ will always be deficient in the near field no matter the radius of the cylinder; i.e., regardless of the accuracy of the UTD. It is interesting that although the three equivalent PWS point sources radiate fields with only $1/R$ terms (equation (5.22)), these fields are exact. This is because the field in (5.22) was derived from a Green's function with $1/R^3$ terms so it is exact even in the near field.

The above example shows the limitation in using only the far-zone term of the current element. The inaccuracy in the incident field caused by the use of only the far-zone field of the current element will severely restrict our use of the UTD with method 1 on page 86.

5.4.1.1 $\Delta Z$

The mutual impedance between the two dipoles in Figure 30 is the sum of the mutual impedance between the two in free-space ($Z_o$) and the contribution to the mutual impedance due to the presence of the cylinder ($\Delta Z$). This latter contribution is calculated via the UTD. Concentrating on the $\Delta Z$ term since $Z_o$ can be calculated using $\vec{E}^{PWS}$, we find that the two different methods of calculating the incident field (page 86) results in two vastly different answers for $\Delta Z$. Image theory is given as a reference solution.

Near-zone Reflection Point If in Figure 30, $d_s = 0.2\lambda$, $s = 0.001\lambda$, and $d_f = 0.4\lambda$, then the reflection point for an integration point along the source dipole can be as close as $R = 0.2\lambda$ away from the source current element, i.e. the integration point. The results shown in Table 2 compare $\Delta Z$ found with the UTD-based scattered field versus image theory using both methods of calculating the incident
Figure 30: Two dipoles with piecewise sinusoidal current distributions in the presence of a $1\lambda$ radius cylinder.
field. Image theory is included to show that the UTD produces reasonable results (compared with image theory) regardless of the method used to determine the incident field. However, using the current element method to calculate the incident field generates erroneous results independent of whether the UTD or image theory is used to determine the scattered field. In this case, not only is $R$ not large and thus the observer is not in the far zone, but the radial field, which is excluded, is dominant because of the $\cos \theta$ term. Thus, the results generated using the current element method are not expected to be good unless one finds a way to predict the scattering of the $\hat{R}$ component of the incident field.

**Far-zone Reflection Point** If the reflection point is in the far zone, then using the series of current elements gives a reasonable answer. For example, as shown in Figure 30, let the distance between the dipoles, $s$, be $0.6\lambda$ and the source and observer dipole be $0.4\lambda$ and $0.2\lambda$ off the cylinder ($d_s = 0.4\lambda$ and $d_f = 0.2\lambda$), respectively.
Then the mutual impedance between the two effected by the cylinder calculated using the equivalent PWS points sources is almost the same as that determined using the series of current elements. The results are shown in Table 3. Again, image theory is included as the reference solution to show that the UTD-based results are reasonable. In this case, not only is the reflection point farther from the source dipole than in the previous example, but the field component along the ray which is excluded is not significant compared to the transverse-to-the-ray term. These factors result in better agreement among the results shown in Table 3 generated by the four methods.

Aside from the accuracy of the calculation of $\Delta Z$, one must consider its contribution to the total mutual impedance, $Z$, which is comprised of $Z_0$, the free-space contribution and $\Delta Z$, which includes the effect of the cylinder. If the dipoles are far from the cylinder, $\Delta Z$ is small compared to $Z_0$, and thus using either the PWS point sources method or the sum of current elements method will generate about the same answer for $Z$. However, as the dipoles get closer to the cylinder and to each other,
\( \Delta Z \) will contribute more to the total mutual impedance. If it is calculated using the current element approach, it may be orders of magnitude too small because the conventional UTD restricts the field to only include the far-zone, \( 1/R \), terms. Using the PWS point sources method is more accurate because the field from each point source is exactly proportional to \( 1/R \) and is in the transverse-to-the-ray direction, conditions of the UTD. From the examples given, the latter method gives at least a non-zero answer and is closer to image theory; whereas, the current element method gives basically a zero result for \( \Delta Z \). The choice of which method to use is an important decision because (1) Table 2 showed a very significant problem caused by using the current element method for a near-field reflection point and (2) the largest terms in the impedance matrix are the self-impedance terms for dipoles close to the PEC which involve a near-field reflection point and a significant \( \Delta Z \). Therefore, the PWS point source method is a better choice for representing the incident field from the monopole with a piecewise sinusoidal current distribution.

### 5.5 Equivalent Current Moment of the PWS Point Source

All the equations comprising the UTD along with its extended versions assume a point source with magnitude \( I \ dl \) which generates the electric field in (5.27). If the PWS point source method is used, each equivalent PWS point source generates the field in (5.23a) or (5.23b) depending on whether the point is a non-terminal or a terminal point. In order to use this method with the UTD, it is necessary to determine the current element with a magnitude \( I^{PWS} \ dl \) which generates the fields in (5.23). To this end, we equate (5.23) and (5.27) and solve for \( I^{PWS} \ dl \). The result is

\[
I^{PWS} \ dl = \frac{C_{idl}}{k \sin kd \sin^2 \theta}
\]

(5.32)
where

\[
C_{idl} = \begin{cases} 
1 & \text{if } I_1 = 0, \text{endpt} = 1 \\
-\cos kd + j \sin kd \cos \theta_2 & \text{if } I_1 = 0, \text{endpt} = 2 \\
-\cos kd - j \sin kd \cos \theta_2 & \text{if } I_1 = 1, \text{endpt} = 1 \\
1 & \text{if } I_1 = 1, \text{endpt} = 2
\end{cases}
\]  

(5.33)

and \(d\) is the length of the monopole. The fact that \(\hat{\mathbf{e}}\) in (5.22) is equivalent to \(-\hat{\theta}\) in (5.27) has been used in deriving (5.32). An equivalent PWS current element with this magnitude will generate a far field, given in (5.27), which is equivalent to the PWS fields given in (5.23).

In determining the free-space field due a current element with the magnitude given in (5.32), the variable \(\theta\) is the angle shown in Figure 28 between the axis of the current element and the vector \(\mathbf{R}\) from \(I \, dl\) to the observer. However, when using the UTD, the definition of \(\theta\) is not always quite so simple.

**UTD Scattering**  The UTD scattering equations are the simplest to use with the PWS point source method. In this case, \(\theta\) is simply the angle between the axis of the equivalent PWS point source and the vector from \(I \, dl\) to the observer, which is not always so simple.

**Extended UTD Radiation**  When using the EUTD radiation solution to calculate the scattered field from an equivalent PWS point source, the definition of \(\theta\) is not quite so obvious. The field determined by the EUTD radiation solution at the observer in Figure 31b is essentially a modification of the field at \(P_N\), the projection
Figure 31: $\theta$ used in $I_{PWS}^o \, dl$ when using (a) UTD scattering, (b) EUTD radiation with the observer in the near field of the surface, (c) EUTD radiation with the source in the near field of the surface, (d) EUTD coupling if the axis of the current element is non-coincident with the geodesic, or (e) EUTD coupling if the axis of the current element is coincident with the geodesic.
of the near-field observer onto the PEC. Thus, $\theta$ is the angle between the axis of $I\, dl$ and $\hat{R}_P$, the vector from $I\, dl$ to $P_N$. For the reciprocal case shown in Figure 31c, $\theta$ is defined the same except $\hat{R}_P$ is now the vector from $P_N$ to the far-field observer.

**Extended UTD Coupling**  Extended UTD coupling, like extended UTD radiation, involves modification of the surface field. So a similar approach to that taken to find $\theta$ when using the EUTD radiation solution is used here. The results are shown in Figure 31d–e. If the monopole axis and thus the current element axis is not in the direction of the geodesic as shown in Figure 31d, $\theta$ is the angle between the axis of $I\, dl$ and the vector from the projection of $I\, dl$ onto the PEC along the geodesic to the projection of the observer onto the surface. If however, the axis of $I\, dl$ is in the direction of the geodesic, then $\sin \theta$ computed using the angle shown in Figure 31d would be zero and thus $I^{PWS} \, dl$ would be infinite. Therefore, an alternate method to derive $I^{PWS} \, dl$ is required. In this case, we equate the total monopole field given in (5.2a) or (5.4a) to the field in (5.27) radiating from a current element at the center of the monopole. The result is

$$I^{PWS} \, dl = -\frac{j \, r^2 \, e^{jkr}}{2 \, \sin kd} \left[ C(1) \frac{e^{-jkR_1}}{R_1} + C(2) \frac{e^{-jkR_2}}{R_2} \right] \quad (5.34)$$

where $R_i$ is the distance from endpoint $i$ of the monopole to the observer, $r$ is the distance from the midpoint of the monopole to the observer,

$$C(1) = \begin{cases} 1 & \text{if } I_1 = 0 \\ -\cos kd - j \sin kd \cos \theta_2 & \text{if } I_1 = 1 \end{cases} \quad (5.35)$$

and

$$C(2) = \begin{cases} -\cos kd + j \sin kd \cos \theta_2 & \text{if } I_1 = 0 \\ 1 & \text{if } I_1 = 1 \end{cases} \quad (5.36)$$
The geometry is shown in Figure 31e.

An expression for $I^{PWS} \, dl$, the magnitude of the equivalent PWS point source, has been derived for use with the UTD. It was given in (5.32) where $\theta$ has been shown in Figure 31 when using various UTD equations. However, for the special case when using the EUTD coupling solution where $\theta$ is zero, (5.34) must be used. Its geometrical representation was shown in Figure 31e.
CHAPTER VI
Modified Image

6.1 Introduction

A cursory examination of equation (2.14c) for the \( \rho \)-directed field from a monopole with a PWS current distribution suggests that the field is infinite when \( \rho = 0 \). However, further probing reveals that cancellation between singular terms causes the field to remain well-behaved. In using the UTD with the PWS point source method, discussed in Chapter V, to determine the scattered field when the excitation is a monopole, the response due to each monopole endpoint is determined separately and then summed together. This isolation interferes with the cancellation necessary to keep the field due to the entire monopole excitation well-behaved because each endpoint excitation produces different scattered field contributions via the UTD. However it was shown in the previous chapter that the PWS point source method is best suited for use with the UTD. Therefore, a method to eliminate the singular terms in the UTD-based scattered field calculation needs to be found.

To accomplish this, first, a detailed analysis of the cancellation which allows the incident field to remain finite as \( \rho \to 0 \) is provided in order to show how this cancellation occurs. Then the UTD-based scattered field from the monopole is determined using the PWS point source method to determine when and if the cancellation occurs when the UTD is used. If it does not, a method to eliminate the singularity is found.
Figure 32: (a) Realistic wire with current flowing on the surface and (b) its thin-wire approximation.

6.2 Free-space Field on the Axis of a Monopole

Figure 32 shows a realistic wire and its thin-wire approximation. The notation used in this figure will be used in the next two sections.

6.2.1 Between the Wire Endpoints \((z_1 < z < z_2)\)

The field given in equation (2.14) is exact and valid everywhere except when the observer is on the axis of the monopole and either between or near the endpoints of the monopole. This is expected once we recall that the field from a simple current element, which the monopole can be thought of as being composed of, becomes
singular as the distance between the source and the observer goes to zero. This singularity is not a problem though because, for our purposes, the field will not be required at these locations. In reality, the geometry is a 3-D wire with current flowing on its surface (Figure 32a). Using the thin-wire approximation referred to in Section 2.3, the real surface current in Figure 32a is represented by the line current defined in equation (2.13) and shown in Figure 32b which has only an axial variation; i.e., no circumferential dependence, and it is limited to flow only on the axis of a 1-D filament. Thus, a point on the axis of the 1-D filament in Figure 32b is, in practice, a point inside the 3-D wire of Figure 32a. So in order to avoid the singularity and since this is not a realistic location at which to determine the field, \( \rho \) is restricted to be no smaller than the radius of the wire if \( z_1 < z < z_2 \). This equivalently limits the observer to be on the surface of the wire or farther.

6.2.2 Outside the Wire \((z < z_1 \text{ or } z > z_2)\)

If \( z < z_1 \) or \( z > z_2 \) and \( \rho = 0 \), then the observer is located on the axis of the 3-D wire but outside its physical confines. In this case, \( E_z \), the field along the axis of the filament, goes to a finite number and \( E_\rho \), the field normal to the axis of the filament, goes to zero. This can be predicted by using the current element analogy discussed in the previous section. However, from (2.14c), \( E_\rho \), which is proportional to \( 1/\rho \) seems like it should be singular for \( \rho = 0 \). The \( E_\rho \) component of the field will be shown to be bounded as a result of cancellation.

Considering the top monopole \((I_1 = 1, I_2 = 0)\) first, (2.14c) gives, for the \( \rho \) component of the field,

\[
E_\rho = \frac{-j Z_0}{4\pi \rho \sin k d} A(x, \rho, d) \quad (6.1)
\]
where

\[ A(x, \rho, d) = (\cos kd \cos \theta_1 + j \sin kd) e^{-jkR_1} - e^{-jkR_2} \cos \theta_2 \]  \hspace{1cm} (6.2) \]

If \( \rho \ll x \triangleq (z - z_2) \) in Figure 32b, then

\[ R_1 = (x + d) \left[ 1 + \frac{1}{2} \left( \frac{\rho}{x + d} \right)^2 \right] + O \left( \frac{\rho^4}{(x + d)^3} \right) \]  \hspace{1cm} (6.3a) \]

\[ R_2 = x \left[ 1 + \frac{1}{2} \left( \frac{\rho}{x} \right)^2 \right] + O \left( \frac{\rho^4}{x^5} \right) \]  \hspace{1cm} (6.3b) \]

where \( d = z_2 - z_1 \). Using (6.3), \( A \) can be approximated by

\[ A(x, \rho, d) \approx e^{-jkx} \left( e^{-\frac{j\rho^2}{2(x+d)}} - e^{-\frac{j\rho^2}{2x}} \right) \]  \hspace{1cm} (6.4a) \]

\[ \lim_{\rho \to 0} e^{-jkx} \left\{ \left[ 1 - \frac{j}{x + d} \left( \frac{k\rho^2}{2} \right) \right] - \left[ 1 - \frac{j}{x} \left( \frac{k\rho^2}{2} \right) \right] \right\} \]  \hspace{1cm} (6.4b) \]

\[ \lim_{\rho \to 0} e^{-jkx} \left( \frac{jk\rho^2}{2} \right) \left( \frac{d}{x(x + d)} \right) \]  \hspace{1cm} (6.4c) \]

where the approximation \( \theta_1 \approx \theta_2 \approx 0 \) has been used. Since the \( \rho^2 \) in (6.4c) cancels the \( \rho \) in (6.1),

\[ E_\rho \xrightarrow{\rho \to 0} C \cdot \rho \]  \hspace{1cm} (6.5) \]

where \( C \) is a constant. So indeed the \( \rho \)-directed field for the monopole with a PWS current distribution remains finite as \( \rho \to 0 \). However, this is true only because the singular \( 1/\rho \) terms for each endpoint, represented by the boldface ones in (6.4b), exactly cancel each other. If this were not true, then the field would be proportional to \( 1/\rho \); therefore, this cancellation is crucial. The same result is found for the bottom monopole.
6.3 Singularity in the Scattered Field due to Excitation from a Monopole Normal to PEC Boundary Using the PWS Point Source Method

The UTD-based scattered field response when the excitation is a dipole is determined by summing the individually-computed scattered fields from each of the three PWS point sources which are proportional to the incident field at the three associated reflection points. However, the incident field from each of the equivalent PWS point sources is singular as $\rho \to 0$. But as shown in the previous section, because the $1/\rho$ contributions from the monopole endpoints cancel, the total free-space monopole field is finite as $\rho \to 0$. The question is whether the total scattered field remains finite as $\rho \to 0$ as it should.

First, two examples will be given to answer this question and to show the severity of the problem. Since an unbounded field cannot physically exist, then it is deduced that the problem is theoretically based. The mathematical cause for it will be investigated in order to determine if a solution exists so that the UTD/PWS point source combination can still be used.

6.3.1 Examples
6.3.1.1 $E_{sc}$

The potential problem can be seen by analyzing the test case shown in Figure 33. The UTD-based scattered field, $E_{sc}$, due to the $0.4\lambda$ dipole as a function of $\hat{\rho}$, the radial distance of the observer from the axis of the dipole, is examined as $\hat{\rho} \to 0$. The results are shown in Figure 34. The scattered field determined by flat plane image theory ($a \to \infty$) for an observer at $\hat{\rho} = 0.002\lambda$, a point very close to the dipole axis, is given as a reference. Only the scattered field is plotted; the incident field is not included because it is not a problem. Figures 34a and 34b show the real and imaginary part of $E_{sc}^r$, respectively, while Figures 34c and 34d show the
Figure 33: Dipole radiating in the presence of a PEC cylinder.
Figure 34: Scattered field due to the 0.4λ-length dipole with a PWS current distribution illuminating the cylinder shown in Figure 33. Calculations made using UTD scattering and flat plane image theory for $\bar{\rho} = 0.002\lambda$. (a) Real part of $E^sc_y$, (b) imaginary part of $E^sc_y$, (c) real part of $E^sc_z$, and (d) imaginary part of $E^sc_z$. 

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real and imaginary part of $E_{2}^{sc}$, respectively. $E_{y}^{sc}$, which is not proportional to $1/\tilde{\rho}$, remains finite as the observer moves toward the dipole axis; however, $E_{x}^{sc}$, which is proportional to $1/\tilde{\rho}$, becomes singular. From the planar image theory results, it is gleaned that this latter result is incorrect. In fact, $E_{x}^{sc}$ should be small since, as shown in the previous section, the limit as $\tilde{\rho} \to 0$ is zero for $a \to \infty$. Whether the UTD or geometrical optics (GO), which is equivalent to setting $R_{H}$ equal to one in the UTD, is used, the general trend is still the same: $E_{x}^{sc} \to \infty$ as $\tilde{\rho} \to 0$.

### 6.3.1.2 Reciprocity of $\Delta Z$

The singularity found in the UTD–based scattered field from a dipole is important because it affects reciprocity which is very important in the calculation of the impedance matrix. To see this problem, the mutual impedance between the two dipoles in Figure 35 is analyzed. This particular orientation was chosen because it is one that is expected to be encountered in the future when analyzing a finned cylinder. Figure 36 shows $\Delta Z$, the part of the mutual impedance which is due to the scattered field, as a function of the distance, $\tilde{\rho}$, between the dipoles. If dipole 2 is the source in the $\Delta Z$ calculation, then the result is more stable; whereas, if dipole 1 is the source, the imaginary part of $\Delta Z$ starts to increase quickly without bound. This variance is due to the difference in $\rho_{Q_{R}}$, the distance from the source monopole axis to the reflection point as shown in 33 for example, for the two cases. If dipole 2 is the source, then for all values of $\tilde{\rho}$, the $\rho_{Q_{R}}$ for all reflection points is the distance to the cylinder, which in this case is $0.7\lambda$. However, if dipole 1 is the source, $\rho_{Q_{R}} \to 0$ as $\tilde{\rho} \to 0$, and as a result, $E_{\rho} \to \infty$. Therefore, in the latter case, using the UTD with the equivalent PWS point sources will not give acceptable results for $\Delta Z$ as the reflection point goes toward the axis of dipole 1. This large difference between $\Delta Z_{12}$ and $\Delta Z_{21}$ as $\tilde{\rho} \to 0$ violates reciprocity and indicates a
Figure 35: Two dipoles in the presence of a PEC circular cylinder.
Figure 36: Mutual impedance between the two dipoles in Figure 35 calculated using the scattered field only (i.e. $\Delta Z$). Comparison of $\Delta Z_{12}$ (dipole 2 is the source) and $\Delta Z_{21}$ (dipole 1 is the source) calculated using UTD scattering. Calculation using planar image theory included for reference purposes. Note $\rho = \rho'$. 

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potentially serious problem in the impedance matrix calculation; clearly, this is a problem which must be corrected.

6.3.2 Cause of the Singularity in the Scattered Field

6.3.2.1 Scattered Field Due to a Single PWS Point Source Illuminating the PEC Cylinder

In order to determine why, for an observer on the dipole axis, the cylinder scattered field is singular but the free-space field remains finite, the UTD-based scattered field response from a cylinder illuminated by a single PWS point source is examined first. Using the geometry shown in Figure 37, the incident field from each PWS point source on the top monopole ($I_1 = 1, I_2 = 0$) is, from (5.23),

$$
\vec{E}^i_k(Q_{R_k}) = C W_k \frac{e^{-jks_ik}}{\rho_{R_k}} \hat{i}_k
$$

(6.6)

where $C$ was defined in (5.24) and

$$
W_k = \begin{cases} 
-\cos kd - j \sin kd \cos \theta_2 & \text{if } k = 2 \\
1.0 & \text{if } k = 3
\end{cases}
$$

(6.7)

The reflection point associated with the $k^{th}$ PWS point source is denoted by $Q_{R_k}$, $s_{ik}$ is the distance from the $k^{th}$ PWS point source to its reflection point, and $\rho_{R_k}$ is the radial distance from the axis of the monopole to $Q_{R_k}$. The scattered field from the $k^{th}$ PWS point source is [14]

$$
\vec{E}^{sc}_k \sim (\vec{E}^i_k(Q_{R_k}) \cdot \vec{e}_{||}) \hat{\vec{e}}_{||k} R_{H_k} \sqrt{(\rho_{1_k} + \rho_{2_k})(\rho_{1_k} + s_{rk})} e^{-jks_{rk}}
$$

(6.8)

where the caustic distances are

$$
\rho_{1_k} = \frac{a s_{ik}}{a + 2 s_{ik} \cos \beta_k}
$$

(6.9)

and

$$
\rho_{2_k} = s_{ik}
$$

(6.10)
Figure 37: Monopole over a PEC cylinder.
The radius of the cylinder is $a$ and $\theta_k^i$ is the angle between the vector $\bar{s}_{ik}$ and the normal to the cylinder. The spread factor in (6.8) for the $k^{th}$ endpoint is defined to be

$$S_k \triangleq \sqrt{\frac{\rho_1 \rho_2}{(\rho_1 + s_{r_k})(\rho_2 + s_{r_k})}} = \frac{s_{ik}}{R_{Ik} \sqrt{1.0 + \frac{2s_{ik}s_{r_k}\cos\theta_k^i}{aR_{Ik}}}}$$

(6.11)

which along with (6.6) reduces (6.8) to

$$\bar{E}_{k}^{sc} \sim \hat{e}_{\parallel k} C \mathcal{W}_k \left( \frac{s_{ik}}{\rho R_k} \right) R_{H_k} e^{-jkR_{I_k}}$$

(6.12)

where the relationship

$$\hat{t}_3 = \hat{e}^{\parallel}_3$$

(6.13a)

$$\hat{t}_2 = \hat{e}^{\parallel}_2$$

(6.13b)

has been used. The variable $R_{Ik} \triangleq s_{ik} + s_{r_k}$ is the distance from the image of the $k^{th}$ PWS point source to the observer and $R_{k}^{curve}$ is the distance from the observer to an image location altered from the flat plane image location by the curvature of the surface. It is defined as

$$R_{Ik}^{curve} \triangleq R_{Ik} \sqrt{1 + \frac{2s_{ik}s_{r_k}\cos\theta_k^i}{aR_{Ik}}}$$

(6.14)

As expected, when the radius of the cylinder becomes infinite

$$\bar{E}_{k}^{sc} \xrightarrow{a \to \infty} \hat{e}_{\parallel k} C \mathcal{W}_k \left( \frac{s_{ik}}{\rho R_k} \right) e^{-jkR_{I_k}}$$

(6.15)

which is exactly the field predicted by flat plane image theory. This is seen by comparing (6.15) with (6.6).

6.3.2.2 Scattered Field From a Monopole and Development of Modified Image

It is evident that the $1/\rho$ singularity always exists for a single equivalent PWS point source whether dealing with the incident field or the scattered field. Only
because of exact cancellation does this singularity not exist for the incident field from a monopole or, equivalently, the scattered field from a flat plane illuminated by a monopole calculated by image theory. Since cancellation of the singular terms does occur when computing the scattered field using planar image theory (calculated via (6.15)) but does not occur when using the UTD (calculated via (6.12)), it is surmised that the square root term in $R^\text{curve}_k$ (equation (6.14)) is the culprit since if $a \rightarrow \infty$, it is eliminated and the resulting scattered field is bounded. This hypothesis will be checked in this section along with the determination of a solution to the cancellation problem. In order to use the UTD in conjunction with the PWS point source method near the monopole axis, some modification needs to be made in order to eliminate the singularity. To this end, let us first determine exactly why cancellation does not occur when calculating the UTD–based scattered field.

Using equation (6.12), the scattered field at the observer due to the non-terminal point of the monopole, i.e., where the current is zero, in Figure 37 is

$$\vec{E}^{\text{sc}}_3 \sim \epsilon_{\parallel 3} \frac{C \mathcal{W}_3}{\rho R_3} \frac{e^{-j k R_{13}}}{R_{I3}} \mathcal{R}_H \frac{s_{i3}}{\sqrt{1 + \frac{2s_{i3}z_3}{a R_{I3}}}} \tag{6.16}$$

A similar equation results for the scattered field caused by the terminal point of the monopole where the magnitude of the current is one:

$$\vec{E}^{\text{sc}}_2 \sim \epsilon_{\parallel 2} \frac{C \mathcal{W}_2}{\rho R_2} \frac{e^{-j k R_{12}}}{R_{I2}} \mathcal{R}_H \frac{s_{i2}}{\sqrt{1 + \frac{2s_{i2}z_2}{a R_{I2}}}} \tag{6.17}$$

The focus of this analysis will be on the $\hat{\rho}$ component of the field since it is the one which involves the $1/\hat{\rho}$ singularity. Using the following relationships,

$$\epsilon_{\parallel 2} \cdot \hat{\rho} = -\frac{z_2}{s_{i2}} \tag{6.18a}$$

$$\epsilon_{\parallel 3} \cdot \hat{\rho} = -\frac{z_2 + d}{s_{i3}} \tag{6.18b}$$
the \( \hat{\rho} \) component of (6.16) and (6.17) are

\[
E_{\rho_3}^{sc} \sim -\frac{C W_3}{\rho_{R_3}}\left(\frac{e^{-j k R_{I_3}}}{R_{I_3}}\right) (1.0) \frac{z_2 + d}{\sqrt{1 + \frac{2 s_{R_3} z_3}{a R_{I_3}}}} \tag{6.19a}
\]

\[
E_{\rho_2}^{sc} \sim -\frac{C W_2}{\rho_{R_2}}\left(\frac{e^{-j k R_{I_2}}}{R_{I_2}}\right) (1.0) \frac{z_2}{\sqrt{1 + \frac{2 s_{R_2} z_2}{a R_{I_2}}}} \tag{6.19b}
\]

The 1.0 in parentheses represents the UTD reflection coefficient. For simplicity, it was set to 1.0 and then left in the equation as a reminder of this assumption.

As mentioned in Section 6.3.1.1, setting \( R_I \) to unity, which generates the GO-based scattered field, produces the same overall result as using the UTD reflection coefficient. Note that the only approximations made so far are due to the asymptotic approximations of the UTD. From Figure 37,

\[
\rho_{R_2} = \hat{\rho} \left(\frac{z_2}{2 z_2 + d + \tilde{z}}\right) \tag{6.20a}
\]

\[
\rho_{R_3} = \hat{\rho} \left(\frac{z_2 + d}{2 z_2 + 2 d + \tilde{z}}\right) \tag{6.20b}
\]

Using this in (6.19),

\[
E_{\rho_3}^{sc} \sim -C W_3 \frac{e^{-j k R_{I_3}}}{\hat{\rho}} \left(\frac{2 x_2 + 2 d + \tilde{z}}{R_{I_3}}\right) \tag{6.21a}
\]

\[
E_{\rho_2}^{sc} \sim -C W_2 \frac{e^{-j k R_{I_2}}}{\hat{\rho}} \left(\frac{2 x_2 + d + \tilde{z}}{R_{I_2}}\right) \tag{6.21b}
\]

If \( \hat{\rho} \) is small, approximations can be made for the distance terms. The distances from the two endpoints of the monopole to their respective reflection points are

\[
s_{I_2} = z_2 \left[1 + \frac{1}{2} \left(\frac{\rho_{R_2}}{z_2}\right)^2\right] + \mathcal{O} \left(\frac{\rho_{R_2}^4}{z_2^3}\right) \tag{6.22a}
\]

\[
s_{I_3} = (z_2 + d) \left[1 + \frac{1}{2} \left(\frac{\rho_{R_3}}{z_2 + d}\right)^2\right] + \mathcal{O} \left(\frac{\rho_{R_3}^4}{(z_2 + d)^3}\right) \tag{6.22b}
\]

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and the distances from those reflection points to the observer are

$$s_{r_2} = (z_2 + d + \hat{z}) \left[ 1 + \frac{1}{2} \left( \frac{\hat{\rho} - \rho_{R_2}}{z_2 + d + \hat{z}} \right)^2 \right] + \mathcal{O} \left( \frac{\left( \frac{\hat{\rho} - \rho_{R_2}}{z_2 + d + \hat{z}} \right)^4}{(z_2 + d + \hat{z})^3} \right) \quad (6.23a)$$

$$s_{r_3} = (z_2 + d + \hat{z}) \left[ 1 + \frac{1}{2} \left( \frac{\hat{\rho} - \rho_{R_3}}{z_2 + d + \hat{z}} \right)^2 \right] + \mathcal{O} \left( \frac{\left( \frac{\hat{\rho} - \rho_{R_3}}{z_2 + d + \hat{z}} \right)^4}{(z_2 + d + \hat{z})^3} \right) \quad (6.23b)$$

The distance from the endpoints of the image monopole to the observer are

$$R_{I_2} = (2z_2 + d + \hat{z}) \left[ 1 + \frac{1}{2} \left( \frac{\hat{\rho}^2}{(2z_2 + d + \hat{z})^2} \right) \right] + \mathcal{O} \left( \frac{\hat{\rho}^4}{(2z_2 + d + \hat{z})^3} \right) \quad (6.24a)$$

$$R_{I_3} = (2z_2 + 2d + \hat{z}) \left[ 1 + \frac{1}{2} \left( \frac{\hat{\rho}^2}{(2z_2 + 2d + \hat{z})^2} \right) \right] + \mathcal{O} \left( \frac{\hat{\rho}^4}{(2z_2 + 2d + \hat{z})^3} \right) \quad (6.24b)$$

Using these approximations, (6.21a) becomes

$$E_{p_3}^{sc} \approx - \frac{C \chi_3}{\hat{\rho}} (1.0) e^{-j k (2z_2 + 2d + \hat{z})} e^{-j \frac{k}{2} (2z_2 + 2d + \hat{z})} \cdot \left[ 1 + \frac{\hat{\rho}^2}{2(2z_2 + 2d + \hat{z})} \right]^{-1} \left[ 1 + \frac{2 (z_2 + d)(z_2 + d + \hat{z})}{2z_2 + 2d + \hat{z}} \right] - \frac{1}{2} \quad (6.25)$$

Multiplying the Taylor expansion of the second exponential with that for the first bracketed term in (6.25) results in the term

$$1 - \frac{j k}{2} \frac{\hat{\rho}^2}{(2z_2 + 2d + \hat{z})} \left[ 1 + \frac{2 (z_2 + d)(z_2 + d + \hat{z})}{2z_2 + 2d + \hat{z}} \right] - \frac{1}{2} \left( \frac{\hat{\rho}^2}{(2z_2 + 2d + \hat{z})^2} \right) + \mathcal{O} \left( \frac{\hat{\rho}^4}{(2z_2 + 2d + \hat{z})^3} \right) \quad (6.26)$$

which reduces (6.25) to

$$E_{p_3}^{sc} \approx - \frac{C \chi_3}{\hat{\rho}} (1.0) e^{-j k (2z_2 + 2d + \hat{z})} \left[ 1 + \frac{2 (z_2 + d)(z_2 + d + \hat{z})}{2z_2 + 2d + \hat{z}} \right] - \frac{1}{2} \left( \frac{\hat{\rho}^2}{(2z_2 + 2d + \hat{z})^2} \right) \quad (6.27)$$

A similar result occurs for the scattered field due to the terminal point radiating:

$$E_{p_1}^{sc} \approx - \frac{C \chi_2}{\hat{\rho}} (1.0) e^{-j k (2z_2 + 2d + \hat{z})} \left[ 1 + \frac{2 z_2 (z_2 + d + \hat{z})}{2z_2 + 2d + \hat{z}} \right] - \frac{1}{2} \left( \frac{\hat{\rho}^2}{(2z_2 + 2d + \hat{z})^2} \right) \quad (6.28)$$
These are the UTD scattered fields at an observer near the monopole axis ($\rho$ small) due to the PWS point sources at the ends of the top monopole which are radiating.

Now, combining (6.27) and (6.28) and replacing the $\cos \theta_2$ in $W_2$ with the approximation in (6.22a) for $s_{i_2}$ if $\rho$ is small, i.e.,

$$\cos \theta_2 = \frac{-z_2}{s_{i_2}} = \sqrt{1 - \frac{1}{2} \left( \frac{\rho}{2z_2 + d + \tilde{z}} \right)^2 + \mathcal{O} \left( \frac{\rho^4}{(2z_2 + d + \tilde{z})^4} \right)}$$  

results in the field for the top monopole near its axis:

$$E_\rho^{sc}(I_1 = 1, I_2 = 0) \approx -C(1.0) e^{-jkzI_3}$$

$$\cdot \left\{ \frac{(I_3 - I_2)}{\rho} - \frac{\rho}{2} \left[ \frac{1}{z_{I_3}} \left( jk + \frac{1}{z_{I_3}} \right) I_3 - \frac{1}{z_{I_2}} \left( jk + \frac{1}{z_{I_2}} \right) I_2 \right] \right\}$$  

where

$$z_{I_3} \triangleq 2z_2 + 2d + \tilde{z}$$  

$$z_{I_2} \triangleq 2z_2 + d + \tilde{z}$$  

and

$$I_2 = \left[ 1 + \frac{2}{a} \frac{z_2 (z_2 + d + \tilde{z})}{z_{I_2}} \right]^{-\frac{1}{2}}$$  

$$I_3 = \left[ 1 + \frac{2}{a} \frac{(z_2 + d) (z_2 + d + \tilde{z})}{z_{I_3}} \right]^{-\frac{1}{2}}$$

$I_k$, henceforth referred to as the "spread factor coefficient" for the $k^{th}$ PWS point source, can be written as

$$I_k = \left( 1 + \frac{2}{a} \frac{s_{r_k} z_k}{R_{I_k}} \right)^{-\frac{1}{2}}$$  

It is related to the spread factor by

$$S_k = \frac{s_{i_k}}{R_{I_k}} I_k$$  

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and modifies the spreading of the field by the curvature of the PEC surface. C, defined previously, is reiterated for completeness:

\[ C = \frac{-jZ_0}{4\pi\sin kd} \]  

(6.35)

Note that \( I_k \) is the square root term in (6.14) that was suspected earlier to be the cause of the cancellation problem. From (6.30), it is found that:

1. If \( a \to \infty \) then \( I_k \to 1 \) and the coefficient of the inverse \( \hat{\rho} \) term is zero; so indeed,

\[ E_{\rho}^{sc} \propto \hat{\rho} \]  

(6.36)

2. However, for \( a \neq \infty \),

\[ E_{\rho}^{sc} \propto \frac{1}{\hat{\rho}} [I_3 - I_2] \]  

(6.37)

which increases without bound as the observer moves closer to the axis of the monopole. The \( I_3 \) and \( I_2 \) terms ensue from the different spread factors for the two PWS point sources.

The second observation shows that in order to eliminate the singular term, the numerator in (6.37) must be zero. If this occurs, then the resulting scattered field is proportional to \( \hat{\rho} \) for all values of \( a \) and thus remains finite as \( \hat{\rho} \to 0 \). This is the desired behavior. This can be accomplished by using \( I_2 \) in place of \( I_3 \) or, in the case of the bottom monopole, by using \( I_1 \). If \( I_3 \) is replaced with \( I_2 \), the scattered field in (6.30) becomes

\[ E_{\rho}^{mi}(I_1 = 1, I_2 = 0) \sim -C (1.0) e^{-jkzI_3} \]

\[ \cdot \left\{ -\frac{\hat{\rho}}{2} I_2 \left[ \frac{1}{zI_3} \left( jk + \frac{1}{zI_3} \right) - \frac{1}{zI_3} \left( jk + \frac{1}{zI_2} \right) \right] \right\} \]  

(6.38)
The only difference between the field in (6.38) and the UTD scattered field for the flat plane case is that \( I_2 \) in the latter case is unity. The existence of \( I_2 \), which is curvature dependent, is the reason for referring to this version of UTD scattering as "modified image"; the superscript \( \text{mi} \) in (6.38) distinguishes the field as such.

6.3.2.3 Error Analysis for Modified Image

Before using \( I_2 \) in the spread factor for both endpoints of the monopole, an error bound is put on the approximation. Using Figure 37, \( I_k \) in (6.33) can be simplified by noting that

\[
\frac{s_{rk}}{R_{Ik}} = \frac{z_{obs}}{z_{obs} + z_k}
\]

(6.39)

where \( z_{obs} \) is the distance of the observer from the PEC in the direction along the axis of the monopole. Thus, \( I_k \) for the \( k^{th} \) endpoint is

\[
I_k = \left(1 + \frac{2}{a} \frac{z_k z_{obs}}{z_k + z_{obs}}\right)^{-\frac{1}{2}}
\]

(6.40)

Expanding \( I_k \) in a Taylor series about a point \( z_o \) results in

\[
I_k \approx I_o + (z_k - z_o) \frac{dI_k}{dk} \bigg|_{z_k = z_o} + \mathcal{O}\left((z_k - z_o)^2\right)
\]

(6.41)

where

\[
I_o = I_k \big|_{z_k = z_o} = \left(1 + \frac{z_o z_{obs}}{a z_o + z_{obs}}\right)^{-\frac{1}{2}}
\]

(6.42)

and

\[
\frac{dI_k}{dz_k} \bigg|_{z_k = z_o} = -\frac{I_o^3}{a} \frac{1}{\left(1 + \frac{z_o}{z_{obs}}\right)^2}
\]

(6.43)
Thus,

\[ I_k \approx I_0 \]  

if

1. \((z_k - z_0) \ll a\),

2. \((1 + \frac{z_0}{z_{obs}})^{-1} \ll 1\) which implies that \(z_0 \gg z_{obs}\), or

3. \(I_0 \ll 1\) which is satisfied as long as \(z_{obs} \gg a\).

Since modified image involves setting \(I_k = I_2\) for all values of \(k\), then let

\[ z_0 = z_2 \]  

(6.45)

where \(z_2\) is the terminal point of the monopole. Neither condition (2) nor (3) is always true. However, condition (1), the length of the monopole being much less than the radius of the cylinder, is usually satisfied because in order for the calculation of the free-space part of the impedance matrix to converge, the monopole length must be \(0.25\lambda\) or smaller. Additionally, the radius of the cylinder must be greater than \(0.25\lambda\) in order for the UTD to be accurate. So, a reasonable approximation to the spread factor for the \(k^{th}\) endpoint is

\[ S_k \approx \frac{s_{ik}}{R_{I_k}} I_2 \]

\[ \approx \frac{s_{ik}}{R_{I_k}} \left(1 + \frac{2}{a} \frac{z_2 z_{obs}}{z_2 + z_{obs}}\right)^{-\frac{1}{2}} \]  

(6.46a)

The error in using \(I_2\) is

\[ E = I_k - I_2 = (z_2 - z_k) \frac{I_2^3}{a} \frac{1}{(1 + \frac{z_2}{z_{obs}})^2} + \mathcal{O}((z_k - z_2)^2) \]  

(6.47)
which is small because $z_k - z_2 \ll a$ is always true. However, $E$ will be smallest if $z_{obs}$ satisfies conditions (2) and (3); i.e. observers closer to the cylinder than the terminal point of the monopole is but whose distance from the cylinder is greater than the radius of the cylinder. This implies that modified image becomes more accurate the higher the source and the observer are off the cylinder which is where flat plane image theory is most inaccurate. In fact, if $z_{obs} \approx z_2$ and $z_{obs} = 3a$, then

$$I_2 = (1 + \frac{2z_{obs}}{a})^{-\frac{1}{2}} = \frac{1}{2} = \frac{1}{2} I_{flat image} \quad (6.48)$$

Thus, for a source and an observer far from the cylinder, there is quite a difference between what modified image predicts for the scattered field and what flat plane image theory predicts. On the other hand, as $z_k \to 0$ (source on PEC), $I_k \to 1$, and modified image theory goes to flat plane image theory.

6.4 Scattered Field due to Excitation From a Monopole Parallel to PEC Using the PWS Point Source Method

As shown in Section 6.3.1.2, when the dipole whose axis is parallel to the cylinder is the source, $\Delta Z$ remains finite as the distance, $\rho$, between the two dipoles goes to zero because the scattered field remains finite. This is in contrast to the singular behavior of $\Delta Z$ if the axis of the dipole is normal to the cylinder. This difference warrants closer examination since there seems to be no need for any cancellation of singularities for a source monopole parallel to the PEC.

The geometry in Figure 38 will be used for the following analysis. The spread factor is the same as that for the dipole whose axis is normal to the PEC except for the definition of $\cos \theta_k^i$ which is now

$$\cos \theta_k^i = \frac{d_s}{s_{ik}} \quad (6.49)$$
Figure 38: Monopole with a PWS current distribution whose axis is parallel to the cylinder axis.
Using (6.49),

\[ S = \frac{s_i}{R_l \sqrt{1 + \frac{2s_d d_s}{a R_l}}} \]  

(6.50)

As \( d_s \to 0 \), i.e., as the source monopole gets closer to the PEC, this spread factor for the UTD scattering looks more like that for flat plane image theory. The scattered field from the cylinder illuminated by the PWS point sources that form the monopole in Figure 38 is found in a manner similar to that used to obtain (6.12). The result is

\[ \vec{E}_{k}^{sc} \sim \vec{\epsilon}_{ll} \frac{C W_k}{d_s R_{I_k}} e^{-jk R_{I_k}} \mathcal{R}_{h_k} \frac{s_{ik}}{\sqrt{1 + \frac{2s_d d_s}{a R_{I_k}}}} \]  

(6.51)

where \( k = 2 \) or \( 3 \) and \( W_k \) was defined in (6.7). The focus of this analysis will be the radial component, \( E_y \); it is the component that will be affected if a singularity exists. Using the following relationship,

\[ \vec{\epsilon}_{ll} \cdot \hat{y} = \frac{z Q_k}{s_{ik}} \]  

(6.52)

the \( y \)-component of the field for each endpoint is

\[ E_{y_3}^{sc} \sim C W_3 \frac{e^{-jk R_{I_3}}}{\rho + d_s} (1.0) \frac{\tilde{z}}{R_{I_3}} \frac{1}{\sqrt{1 + \frac{2s_d d_s}{a R_{I_3}}}} \mathcal{I}_3 \]  

(6.53a)

\[ E_{y_2}^{sc} \sim C W_2 \frac{e^{-jk R_{I_2}}}{\rho + d_s} (1.0) \frac{\tilde{z} + d}{R_{I_2}} \frac{1}{\sqrt{1 + \frac{2s_d d_s}{a R_{I_2}}}} \mathcal{I}_2 \]  

(6.53b)

where again, for simplicity, \( \mathcal{R}_H \) has been set to unity. The spread factor coefficients, \( \mathcal{I}_2 \) and \( \mathcal{I}_3 \), for the two PWS point sources have been noted in (6.53). Using equivalent triangles, these are found to be equal:

\[ \mathcal{I}_2 = \left[ 1 + \frac{2\rho d_s}{a (\rho + d_s)} \right]^{-\frac{1}{2}} = \mathcal{I}_3 \]  

(6.54)
This bodes well since using the same spread factor coefficients for both monopole endpoints is what makes modified image non-singular. Indeed, the total GO-based scattered field from the monopole

\[
E_{y}^{sc} \sim C(1.0) \left( 1 + \frac{2 d}{a(1 + \frac{d}{\rho})} \right)^{-\frac{1}{2}} \left( \mathcal{W}_3 \frac{e^{-jkR_I} \bar{z}}{\rho + d} \frac{\bar{z}}{R_I} + \mathcal{W}_2 \frac{e^{-jkR_{I_2}} \bar{z} + d}{\rho + d} \frac{\bar{z} + d}{R_{I_2}} \right)
\sim \mathcal{I}_2 \cdot E_{y}^{flat\ image}
\]

(6.55)

found by adding (6.53b) and (6.53a) is finite for all observer locations. In fact, (6.55) is what was referred to earlier as modified image: flat plane image theory modified to include the effect of curvature. Thus, for the \(\hat{z}\)-directed monopole there are no cancellation problems! This feature is demonstrated in Figure 40 where the scattered field is plotted as a function of \(\rho\), the distance of the observer from the PEC. The field remains finite even for an observer located at a point along the monopole axis, i.e., \(\rho = 0.1\lambda\).

6.5 Conclusion

The \(\hat{\rho}\)-directed field for a single PWS point source is singular for an observer on or near the axis of the relevant monopole where the \(\hat{\rho}\) direction is normal to the monopole axis. However, cancellation of the \(1/\rho\) terms in the field associated with each endpoint comprising a monopole allows the incident field from the monopole to remain finite. On the other hand, the UTD-based scattered field from the cylinder illuminated by a monopole whose axis is normal to the PEC, found by summing the individually-calculated scattered field of the equivalent PWS point source, is infinite along the monopole axis. This is remedied by using the same coefficient of the UTD spread factor for both monopole endpoints. The resulting method is referred to as modified image because the result is equivalent to flat plane image.
Figure 39: Test geometry – a 0.4λ length dipole whose axis is parallel to the cylinder axis – to show that the scattered field remains finite for all observer locations.

Figure 40: Scattered field from the cylinder in Figure 39 illuminated by 0.4λ-length dipole oriented parallel to the cylinder axis. The results calculated by flat plane image theory (labeled "image") are compared with those calculated using GO reflection.
theory modified by a multiplicative factor which accounts for the curvature of the PEC surface. No modifications are necessary for a monopole whose axis is parallel to the PEC because the spread factor coefficients for the monopole endpoints are equal. As a result, cancellation of the singular $1/\rho$ terms is guaranteed.
CHAPTER VII

Blending Between the Different UTD Green's Functions and Their Regions of Validity

7.1 Introduction

None of the equations that comprise the original UTD ([14], [15], [16]) nor the new modified versions that extend the UTD (extended UTD coupling, extended UTD radiation) have exact error bounds provided with them. Asymptotic approximations were used in all the derivations. As a result, the equations are valid only when the asymptotic parameter, usually $ka$, is large. This is equivalent to requiring that the radius of the cylinder be sufficiently large with respect to the wavelength. Additionally, in the case of the extended UTD coupling solution and the extended UTD radiation solution, a Taylor expansion was also used. Its validity depends on the step size, $h$, being small in the Taylor series. Also, modified image, derived from GO for use when the observer is near the monopole axis, can only be used if $\rho$, the distance from the monopole axis to the observer, is small. Clearly, the requirement that certain parameters be "large" or "small" as mentioned above needs to be quantified for applications. When using many different equations in a computer code which must decide which equation to use given only the source location, the observer location, and the cylinder size, it is necessary to provide the program with a well-defined set of a priori rules for choosing which equation to use and where.

The goal in this chapter is to derive equations which define the boundaries for the regions where the following four equations are used: modified image, UTD
scattering, extended UTD coupling, and extended UTD radiation. When the latter three equations predict infinite fields because of singularity cancellation problems, modified image is used. However, since these three equations may also incorrectly predict large, but not infinite fields, a measure to determine when the result starts to become singular needs to be found. With regards to blending among the last three types of equations, the boundaries on the regions of validity will be empirically determined by examining the results of numerous runs with different source and observer locations and different cylinder radii. The resulting boundaries which would delineate the regions of validity of the different solutions will be found to be functions of the radius of the cylinder, the length of the source dipole, and its location relative to the cylinder.

7.2 M-cone: Modified Image versus UTD Scattering

7.2.1 Need for a Delineation Curve

Modified image theory, a variation of the GO reflection solution, was developed because the field is singular when the PWS point source method is used in conjunction with the UTD scattering solution to determine the field near the monopole axis. Removal of this singular behavior distinguishes modified image theory from the GO solution. Basically, modified image theory was created as a non-singular alternative to the UTD scattering solution for use when the latter method results in an unbounded field. However, modified image theory involves an approximation because the same factor, \( I_2 \), is used in the spread factor for both endpoints of the monopole. As a consequence, it is preferable to use the GO/UTD scattering solution as long as the field remains bounded. Therefore, it seems reasonable to use the GO/UTD scattering solution as long as \( \rho \), the distance of the observer from the source monopole, is large and thus the field remains finite. However, when the value
of the resulting field incorrectly begins to approach an infinite value, the modified image solution is used. An important task is the determination of what the value of \( \rho \) is at this transition point and what measure should be used to decide this point.

First, in order to show the practical importance of this, a typical situation that the hybrid code will encounter will be examined. Figure 41 shows the geometry of interest: a 0.4\( \lambda \) length dipole is radiating in the presence of a cylinder of radius 2\( \lambda \). The scattered field is determined as a function of \( \rho \), the distance along the cylinder axis that the observer is from the dipole axis, while the observer's distance from the cylinder surface is fixed at 0.6\( \lambda \). The results are shown in Figure 42. Image theory is included to provide a guide as to what range the answer should be in. As expected, for \( \rho \) large, modified image theory and the GO reflection solution predict the same scattered field values. However, as \( \rho \) goes to zero, the field calculated using the GO reflection equation does indeed become unbounded. It deviates from the field calculated by modified image theory at about \( \rho = 0.3\lambda \). On the other hand, by comparing the behavior for \( \rho \) small of the scattered field computed using modified image theory with that using flat plane image theory, it is concluded that modified image theory generates reasonable results. Therefore, from Figure 42, it is surmised that modified image theory should be used for \( \rho < \rho_T \) while the GO/UTD scattering solution is reserved for \( \rho > \rho_T \) where \( \rho_T \) is the value of \( \rho \) at the transition point which is 0.3\( \lambda \) for this example.

The transition point, \( \rho = \rho_T \), was identified after the fields had already been calculated. It is not feasible for the hybrid code to perform such a search for each different source dipole location. For realistic geometries, there will be thousands of dipoles to check. Thus, it is essential that the value of \( \rho \) at the transition point be able to be predicted a priori. To this end, a curve will be derived which divides the
Figure 41: A $0.4\lambda$ length dipole radiating in the presence of a cylinder of radius $2\lambda$.

Figure 42: Scattered $E_z$ field is calculated by four different methods to examine the existence or non-existence of a singularity in the field as the observer moves near the dipole axis. The geometry is shown in Figure 41.
region about the source dipole into two: one in which modified image theory is used and the other in which the GO/UTD scattering solution will be used.

7.2.2 Derivation of the M-cone

Note first that the transition point is not a definitive point because the correct answer for the field is not known since a computer code for the 3-D eigenfunction solution is not available at the present time. As a consequence, it is not known, for a given value of $\rho$ in Figure 42, whether the result generated by the UTD scattering solution or that generated by the modified image solution is correct. Instead, it is hypothesized that, in general, the UTD scattering solution is more accurate than the modified image solution because the latter method is an approximate version of the former. However, because the UTD-based field is singular for $\rho$ small, which is physically incorrect, it is concluded that the UTD scattering result in this region is in error. At this point, the transition to modified image theory is made. Since the transition is made because the UTD-based scattered field becomes singular, a reasonable starting point for deriving an estimate of the transition point is to examine the UTD-based field about the point when it becomes unbounded. This would be where the $1/\rho$ term dominates the terms of order $\rho$. Since modified image theory is a variation of GO, it will be compared with the GO reflection solution ($R_H = 1$ in the UTD) rather than with the UTD scattering solution for the purpose of deriving a boundary. However, as seen in Figure 42, the singularity exists regardless of the value of $R_H$. The only difference between GO and modified image theory is the elimination of the singular $1/\rho$ term in modified image theory. Hence, any difference in the resulting field calculated by the two methods is due solely to the non-cancellation of the $1/\rho$ term when using GO. Thus, comparing GO and modified
image theory gives an indication of the behavior of the $1/\rho$ term versus the terms of order $\rho$.

Based on an examination of the plots from many different test cases, the measure used to decide the transition point $\rho_T$ is chosen as follows:

$$20 \log_{10} |E^{GO}(\rho_T)| - 20 \log_{10} |E^{MI}(\rho_T)| = 3.5 \text{ dB} \quad (7.1)$$

Thus, the transition between GO and modified image theory is made when the magnitude of the electric field determined by the two differ by 3.5 dB. The choice of 3.5 dB as the acceptable deviation was determined empirically. The value can be greater if the GO result is allowed to deviate from the non-singular modified image theory result by a larger amount. Using (7.1), a boundary, henceforth referred to as the "M-cone", between the region where modified image theory is valid and where the GO/UTD scattering solution is valid can be determined. The derivation is as follows.

The limit of the scattered electric field calculated by GO reflection generated by the top monopole at an observer near to the monopole axis was given in Section 6.3.2.2, specifically in equation (6.30). It is reiterated below.

$$E^{GO}_\rho (I_1 = 1, I_2 = 0) \approx - C(1.0) e^{-jkzI_3}$$

$$\left\{ \frac{I_3 - I_2}{\rho} - \frac{1}{2} \left[ \frac{1}{zI_3} \left( jk + \frac{1}{zI_3} \right) I_3 - \frac{1}{zI_2} \left( jk + \frac{1}{zI_2} \right) I_2 \right] \right\} \quad (7.2)$$

where $zI_2$ and $zI_3$ were defined in (6.31). The geometry of interest was depicted in Figure 37. For the same configuration, modified image theory predicts the field for small $\rho$ to be

$$E^{MI}_\rho (I_1 = 1, I_2 = 0) \approx - C(1.0) e^{-jkzI_3}$$
The zero in (7.3) is included to emphasize the absence of the singular $1/\rho$ term. The spread factors, $I_2$ and $I_3$, given in (6.32) are related by

$$I_3 = I_2 \left[ 1 + s \frac{2}{a} \frac{z_2 (z_2 + d + \tilde{z})}{z_{I_2}} \right]^{-\frac{1}{2}} \quad (7.4)$$

where

$$s \triangleq \frac{d (z_2 + d + \tilde{z})}{z_2 (2z_2 + 2d + \tilde{z})} \quad (7.5)$$

The variables in (7.4) have been defined in Figure 37. If $X \ll 1$ in (7.4), then $I_3$ can be approximated by

$$I_3 \approx I_2 \left[ 1 - \frac{2}{a} \frac{d}{z_{I_2}} (1 - \frac{z_{I_2}^2}{z_{I_1}^2})^2 \right] \quad (7.6)$$

It can be verified that this condition on $X$ is indeed satisfied. Using (7.6), equation (7.2) and (7.3) can be simplified as follows:

$$E^G_{\rho}(I_1 = 1, I_2 = 0) \approx -C (1.0) e^{-jkzI_3} (A + B) \quad (7.7)$$

and

$$E^M_{\rho}(I_1 = 1, I_2 = 0) \approx -C (1.0) e^{-jkzI_3} (0.0 + B_M) \quad (7.8)$$

where

$$A \triangleq \frac{(I_3 - I_2)}{\rho}$$

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\[
\approx - \frac{T_2^3 \, \frac{d}{\rho}}{a} \left( 1 - \frac{x_2}{z_{I_2}} \right)^2 \tag{7.9a}
\]

\[
B \triangleq -\frac{\rho}{2} \left[ jk \left( \frac{T_3}{z_{I_3}} - \frac{T_2}{z_{I_2}} \right) + \left( \frac{T_3}{(z_{I_3})^2} - \frac{T_2}{(z_{I_2})^2} \right) \right]
\]

\[
\approx - \frac{T_2 \rho}{2 \, z_{I_2}} \left\{ jk \left[ -1 + \frac{1}{1 + \frac{d}{z_{I_2}}} \right] + \frac{1}{z_{I_2}} \left[ -1 + \frac{1}{1 + \frac{d}{z_{I_2}}} \right] \right\} + \epsilon \tag{7.9b}
\]

\[
B_M \triangleq -\frac{\rho}{2} \left[ \frac{1}{z_{I_3}} \left( jk + \frac{1}{z_{I_3}} \right) - \frac{1}{z_{I_2}} \left( jk + \frac{1}{z_{I_2}} \right) \right] T_2
\]

\[
\approx - \frac{T_2 \rho}{2 \, z_{I_2}} \left\{ jk \left[ -1 + \frac{1}{1 + \frac{d}{z_{I_2}}} \right] + \frac{1}{z_{I_2}} \left[ -1 + \frac{1}{1 + \frac{d}{z_{I_2}}} \right] \right\} \tag{7.9c}
\]

where the \(\epsilon\) term in (7.9b) is small compared to the preceding term. It is proportional to \(T_2^2 d^2 \). The error measure in (7.1) is equivalent to the criterion

\[
\frac{\text{Field found by GO Reflection}}{\text{Field found by Mod Image}} = 1.5 \tag{7.10}
\]

which, by using (7.7) and (7.8), is equal to

\[
1.5 = \frac{A + B}{B_M} \approx 1 - \left( \frac{z_2 + d + \tilde{z}}{\rho} \right)^2 \frac{T_2^2}{a} \frac{d}{jk d + \frac{d}{z_{I_2}}} \tag{7.11}
\]

Looking at the form of this equation, it can be deduced that the values of \(\rho\) that would satisfy the equality form a cone where the axis of the source monopole is the axis of the cone and \(\rho\) is the radius of it for a given value of \(\tilde{z}\) (see Figure 43). This latter quantity will be denoted by \(\rho^C\). In order to find the \(\rho\) which satisfies (7.11) for a given \(\tilde{z}\), (7.11) is represented as

\[
\Upsilon = 1 + \frac{C}{\rho^2} \left( \frac{1}{\tilde{a} + \tilde{b} j} \right) \tag{7.12}
\]
Figure 43: M-cone for a monopole whose axis is normal to the cylinder axis.

where

\[ C = \frac{2d}{a} \left( z^2 + d + \bar{z} \right)^2 I_2^2 \]  \hspace{1cm} (7.13a)

\[ \bar{a} = -\frac{d}{2z_2 + d + \bar{z}} \]  \hspace{1cm} (7.13b)

\[ \bar{b} = -kd \]  \hspace{1cm} (7.13c)

and \( \Upsilon \) is the value of the ratio in (7.10) which was chosen to be 1.5 (equivalent to 3.5 dB). Note that \( d \), the length of the source monopole, and \( z_2 \), the distance of the terminal point from the PEC, are shown in Figure 43. The variable \( a \) is the radius of the cylinder and \( I_2^2 \) is the spread factor coefficient associated with the terminal point. The equation for \( I_2^2 \) was given in (6.32a). Equating the magnitude of both
sides of (7.12) and setting
\[ \tilde{c} = \frac{C}{\tilde{a}^2 + \tilde{b}^2} \]  
(7.14)
results in
\[ \Upsilon^2 = 1 + \frac{2\tilde{a}}{\rho} + \frac{\tilde{c}^2 (\tilde{a}^2 + \tilde{b}^2)}{\rho^2} \]  
(7.15)
which can be solved for \( \rho \). The result is denoted as \( \rho^c \) since it is the radius of the
M–cone at a particular value of \( \tilde{z} \):
\[ \rho^c (\tilde{z}) = \left[ -\frac{B \pm \sqrt{B^2 - 4C}}{2} \right]^{-\frac{1}{2}} \]  
(7.16)
where
\[ B = \frac{2\tilde{a}}{\tilde{C}} \]  
(7.17a)
\[ C = (1 - \Upsilon^2) \frac{(\tilde{a}^2 + \tilde{b}^2)}{C^2} \]  
(7.17b)
This conical boundary is an approximate delineation surface. For instance, \( \Upsilon \) can
be adjusted to give a tighter or looser criterion for the region in which the GO/UTD
scattering solution can be used. A larger value of \( \Upsilon \) produces a narrower cone and
thus a larger region in which the UTD can be used. In addition, if \( \varepsilon \) was included in
\( B \) when deriving (7.11), a smaller \( \rho^c \) would have resulted, and as a consequence, a
narrower cone would have divided the region. In general, (7.16) gives a conservative
estimate of when to use the UTD.

7.2.3 Summary

Modified image theory was developed in response to the singularity in the UTD–
based scattered field when used with the PWS point source method for an observer
near the monopole axis. Since modified image theory is an approximate version of the GO reflection solution, it is preferable to use the UTD scattering solution when possible. Thus, a measure was needed to distinguish between when to use modified image and when to use UTD scattering. This criterion was developed by determining when the singular $1/\rho$ term in UTD scattering dominated the result. The delineation surface was found to be a cone. Its radius at a given observer location was given in (7.16). If the monopole axis is oriented normal to the cylinder axis, the resulting cone was depicted in Figure 43 where the M-cone axis is coincident with the monopole axis. However, if the monopole axis is tilted at an angle $\theta$ relative to the cylinder axis, then the axis of the M-cone is tilted. The derivation and depiction of the associated M-cone is given in Appendix D. The variables to be used in (7.16) for such an arbitrarily-oriented monopole are also derived in Appendix D. If the monopole axis is tilted such that its axis is parallel to the cylinder axis, no cone is required because, as shown in Section 6.4, the field remains bounded regardless of the observer location.

7.3 I-cone: Modified Image Theory versus the EUTD Radiation Solution

Modified image theory was derived in response to a singularity exhibited when the UTD scattering solution is used with the PWS point source method and the observer lies along the monopole axis. It is of interest to examine if the same problem exists when the UTD scattering solution is replaced by the EUTD radiation solution.

7.3.1 Need for a Delineation Curve

The geometry of interest is shown in Figure 44. Using the results from Section 4.4.1.2, the far-zone, $\theta$-directed, electric field due to a $\rho'$ current element in the near
Figure 44: Geometry used to investigate the existence of a singularity when using the EUTD radiation solution with the PWS point source method: monopole whose axis is normal to the cylinder axis.
field of the cylinder is

$$E_{\theta \rho} \sim \left( \frac{p_{e \rho} Z_0 \cot \theta_{PN}}{4\pi} \right) \left( \frac{e^{-jk_{\rho} r_{\rho}^{\prime}}}{r_{N}^{\prime}} \right) \frac{e^{jk_{\lambda} a \cos \theta_{i}^{\lambda}}}{r_{N}^{\prime}} e^{-jk_{\zeta} z_{0}} \cdot \left[ \frac{k}{m^{3} D} e^{-jZ_{t}^{3}/3} \frac{\partial \Omega_{s}(Z_{t}, \lambda)}{\partial h_{t}} \right]$$

(7.18)

where $\Omega_{s}$ was defined in (4.48a). The angle $\theta_{i}^{\lambda}$ was shown in Figure 24b. For the geometry shown in Figure 44, $\theta_{i}^{\lambda} = 0$ for a PWS point source located at either monopole endpoint. This is not a limitation, rather this is the orientation that is of most interest when analyzing the finned cylinder. However, the results can be generalized if need be. Since $\theta_{i}^{\lambda} = 0$, the exponentials reduce as

$$e^{-jk_{\rho} r_{\rho}^{\prime}} e^{jk_{\lambda} a \cos \theta_{i}^{\lambda}} e^{-jk_{\zeta} z_{0}} = 1$$

(7.19)

From Figure 44,

$$\theta = \theta_{PN} - 90^\circ$$

(7.20)

where $\theta$ is the angle between the axis of the monopole and the vector from the point $P_{N}$, the projection of the monopole endpoints onto the cylinder, to the observer. Using (7.19) and (7.20) reduces (7.18) to

$$E_{\theta \rho} \sim \left( \frac{p_{e \rho} k Z_0 \sin \theta}{4\pi} \right) \frac{e^{-jkR}}{R} \left( \frac{ka \cos \theta}{2} \right)^{-1/3} \cdot \left[ \tilde{G}(Z_{t}) + j \frac{h_{t}^{2}}{2!} e^{-jZ_{t}^{3}/3} \tilde{g}^{\prime}(Z_{t}) - \frac{2h_{t}^{3}}{3!} \tilde{G}(Z_{t}) + O(h_{t}^{4}) \right]$$

(7.21)

where

$$\tilde{G}(D) = e^{-jD^{3}/3} \tilde{g}^{\prime}(D)$$

(7.22)

and $R$ is the distance from $P_{N}$ to the observer. Note that in this particular case, radiation by a $\rho$-directed monopole, the $\theta$-component of the field from both PWS
point sources is in the same direction because the location of $P_N$ is the same for both endpoints. This allows straightforward addition of the $E_{\theta_p}$ fields from the two PWS point sources. However, what distinguishes the two fields is the value of the normal distance of the observer from the cylinder and $p_{e_p}$, the equivalent current magnitude. For the terminal point,

$$p_{e_2} = \frac{-\cos kd - j \sin kd \cos \theta}{k \sin kd \sin^2 \theta}$$  (7.23)

and

$$h_{t_2} = \frac{k_t}{m_t} z_2$$  (7.24)

For the non-terminal point,

$$p_{e_3} = \frac{1}{k \sin kd \sin^2 \theta}$$  (7.25)

and

$$h_{t_3} = \frac{k_t}{m_t} (z_2 + d)$$  (7.26)

By adding the contribution from the two PWS endpoints, the $\hat{\theta}$ field due to the monopole is

$$E_{\theta_p}^{\text{mon}} \approx \frac{Z_0}{4\pi \rho \sin kd} \left( \frac{ka}{2} \cos \theta \right)^{-1/3} e^{-jkR}$$

$$\cdot \left\{ \tilde{G}(Z_{t_1}) \left[ 1 - (\cos kd + j \sin kd \cos \theta) \right] + \frac{j}{2} e^{-jZ_{t_1}^2/3} \tilde{g}'(Z_{t_1}) \left( \frac{k}{m^3D} \right)^2 \left[ (z_2 + d)^2 - (\cos kd + j \sin kd \cos \theta) z_2^2 \right] - \frac{\tilde{G}(Z_{t_1})}{3} \left( \frac{k}{m^3D} \right)^3 \left[ (z_2 + d)^3 - (\cos kd + j \sin kd \cos \theta) z_2^3 \right] \right\}$$  (7.27)
The concern is when $p$, in the coefficient in (7.27), goes to zero, i.e., for an on-axis observer. Of interest is the behavior of the field near the monopole axis. Thus, $\rho \ll Z \triangleq z_2 + d + \bar{z}$ in Figure 44 is assumed. Then

$$\frac{e^{-jkR}}{\rho} \approx e^{-jkZ} \left( \frac{1}{\rho} - \frac{jk}{2Z} \rho^2 - \frac{k^2}{8Z^2} \rho^3 \right)$$

(7.28)

and the equivalent $\hat{\theta}$-directed field for $\rho \ll Z$ is

$$E'_{\rho \theta} \ll Z \sim \frac{Z_0}{4\pi \sin kd} \left( \frac{2}{ka\cos \theta} \right)^{1/3} e^{-jkZ} \left( \frac{1}{\rho} - \frac{jk}{2Z} \rho^2 \right)$$

$$\cdot \left[ \hat{G}(Z_{1t}) \cdot A + \frac{j}{2!} e^{-j^2/4} \hat{g}'(Z_{1t}) \left( \frac{k}{m^3D} \right)^2 \cdot B - \frac{2}{3} \hat{G}(Z_{1t}) \left( \frac{k}{m^3D} \right)^3 \cdot E \right]$$

(7.29)

where

$$A = 1.0 - (\cos kd + j \sin kd \cos \theta)$$

(7.30a)

$$B = (z_2 + d)^2 - (\cos kd + j \sin kd \cos \theta) z_2^2$$

(7.30b)

$$E = (z_2 + d)^3 - (\cos kd + j \sin kd \cos \theta) z_2^3$$

(7.30c)

Equation (7.29) shows that as $\rho$ goes to zero, the field goes to infinity because of the presence of the $1/\rho$ term.

To obtain further insight, the limit of (7.29) as $a \to \infty$ (cylinder approaching a flat plane) is examined. This allows for the replacement of the Fock functions with algebraic quantities. The argument of the Fock functions from (4.54) is

$$Z_{1t} \xrightarrow{a \to \infty} - \left( \frac{ka \cos \theta}{2} \right)^{1/3}$$

(7.31)

where (7.20) has been used along with the fact that $\theta_{1t} = 0$. From [15, p. 616], it is known that in the lit region

$$S \triangleq \frac{-j}{m^3D} \hat{g}(D) e^{-jD^3/3} \overset{D \leq 0}{\to} 2 \cos \theta$$

(7.32)
If \( a \to \infty \), then \( Z_{l_t} \), the argument of \( \tilde{G} \), is much less than zero and the above limit is applicable. Using (7.32),

\[
\tilde{G}(Z_{l_t}) \xrightarrow{a \to \infty} - j 2 Z_{l_t}
\]

(7.33a)

\[
\tilde{G}'(Z_{l_t}) \xrightarrow{a \to \infty} - j 2
\]

(7.33b)

Thus, the limit of (7.29) as the cylinder approximates a flat ground plane is

\[
E_{\theta \rho}^{\text{mon}} \xrightarrow{\rho \ll Z} \frac{j Z_0}{4\pi \sin kd} e^{-jkz} \left( \frac{1}{\rho} - \frac{jk\rho}{2Z} - \frac{k^2\rho^3}{8Z^2} \right) (2.0 A - k^2 \cos^2 \theta B)
\]

(7.34)

So increasing the radius of the cylinder does not alleviate the problem as it did in the UTD scattering case. This indicates that, regardless of the radius of the cylinder, the field near the monopole axis determined via the EUTD radiation solution is unbounded. The results are the same for the bottom monopole.

It has been shown that when the extended radiation solution is used in conjunction with the PWS point source method to analyze the field radiated from a monopole along its axis in the presence of a PEC cylinder, a singularity is present. Consequently, the EUTD radiation solution cannot be used in this circumstance. Since the EUTD coupling solution is not valid for this case because the observer is far from the cylinder and the UTD scattering solution contains the same singularity under these circumstances, modified image theory is used. The choice of where to make the transition between calculation of the field by modified image theory and by the EUTD radiation solution is the subject of the next section.

7.3.2 Derivation of the I–cone

Since modified image theory is an approximate form of GO reflection, it is preferable to use the EUTD radiation solution whenever possible. It will be used when \( \rho \) is large where the field predicted by the EUTD radiation solution is well–behaved.
However, as $\rho$ becomes small, the field predicted by the EUTD radiation solution becomes dominated by the $1/\rho$ term. When this singularity dictates the behavior of the field, modified image theory is used. This criterion is used to determine the transition point. Given a source monopole, the set of transition points determined for various observer locations forms a boundary between the region where the EUTD radiation solution is used to calculate the field and where modified image theory is used.

The boundary of interest isolates the region in which $\rho$ is small enough such that the $1/\rho$ term dominates the field behavior. Therefore, its derivation begins by looking at (7.34) as $\rho \to 0$. The approximation

$$\cos \theta \approx 1 - \frac{1}{2} \frac{\rho^2}{R^2}$$  \hspace{1cm} (7.35)

can be made if $\rho \ll R$ in Figure 44. Thus, the quantity $A$ in equation (7.34) can be further simplified for small $\rho$:

$$A \approx \begin{cases} 1 - e^{jkd} & \text{if } \rho \ll R \\ a_0 + \rho^2 a_1 & \text{if } \rho > R \end{cases}$$  \hspace{1cm} (7.36)

Furthermore, if $B$ is represented by $B = b_0 + b_1 \cos \theta$, then

$$B \cos \theta \approx \begin{cases} b_0 \left(1 - \frac{\rho^2}{R^2}\right) + \frac{3}{2} b_1 \left(1 - \frac{\rho^2}{R^2}\right) & \text{if } \rho \ll R \\ b_0 + b_1 & \text{if } \rho > R \end{cases}$$  \hspace{1cm} (7.37)

and the second term in parentheses in (7.34) can be approximated as

$$2A - k^2 \cos^2 \theta B \approx \begin{cases} \left(2a_0 - k^2 b_0 - k^2 b_1\right) + \rho^2 \left[2a_1 + \frac{k^2}{R^2} (b_0 + 1.5 b_1)\right] & \text{if } \rho \ll R \\ 2A_0 - k^2 b_0 - k^2 b_1 & \text{if } \rho > R \end{cases}$$  \hspace{1cm} (7.38)

As a result, for an observer near the monopole axis, (7.34) can be rewritten as

$$E_{p_d} \xrightarrow{\rho \to \infty, \rho, \frac{Z}{2}} \frac{j Z_0}{4\pi \sin kd} \left(1.0\right) e^{-jkZ} \left[ \frac{C}{\rho} + \rho \left( D - \frac{jkC}{2\pi}\right) \right]$$  \hspace{1cm} (7.39)
Using the definition of $C$ and $D$, the coefficient of the $\rho$ term is

$$D - \frac{jkC}{2zz} = \left( \mathcal{K}_1 + \frac{\mathcal{K}_2}{R^2} \right) + j \left( \mathcal{K}_3 + \frac{\mathcal{K}_4}{R^2} \right)$$ \hspace{1cm} (7.40)

where

$$\mathcal{K}_1 = \frac{k \sin kd}{2zz}(-2 + k^2 z_2^2)$$ \hspace{1cm} (7.41a)

$$\mathcal{K}_2 = k^2 \left[(z_2 + d)^2 - \cos kd z_2^2\right]$$ \hspace{1cm} (7.41b)

$$\mathcal{K}_3 = -\frac{k}{2zz} \left[2(1 - \cos kd) - k^2 \left((z_2 + d)^2 - \cos kd z_2^2\right)\right]$$ \hspace{1cm} (7.41c)

$$\mathcal{K}_4 = \sin kd + 1.5 k^2 \sin kd z_2^2$$ \hspace{1cm} (7.41d)

The coefficient of the $1/\rho$ term is

$$C = M_1 + jM_2$$ \hspace{1cm} (7.42)

where

$$M_1 = 2(1 - \cos kd) - k^2 \left[(z_2 + d)^2 - \cos kd z_2^2\right]$$ \hspace{1cm} (7.43a)

$$M_2 = \sin kd \left(-2 + k^2 z_2^2\right)$$ \hspace{1cm} (7.43b)

In order to determine when the behavior of (7.39) becomes dominated by the $1/\rho$ term, the value of $\rho$ at which the $1/\rho$ term is some fraction of the $\rho$ term is determined. This is done by setting

$$\left|\frac{C}{\rho}\right| = \alpha \left|D - \frac{jkC}{2zz}\right| \rho$$ \hspace{1cm} (7.44)

Using (7.40) and (7.42), this is equivalent to

$$\frac{1}{\rho^2} = \alpha \left[\left(\mathcal{K}_1 + \frac{\mathcal{K}_2}{R^2}\right)^2 + \left(\mathcal{K}_3 + \frac{\mathcal{K}_4}{R^2}\right)^2\right]^{\frac{1}{2}} \frac{1}{\left[M_1^2 + M_2^2\right]^{\frac{1}{2}}}$$ \hspace{1cm} (7.45)
From Figure 44, it is seen that \( R^2 = \rho^2 + Z^2 \). Making this substitution in the numerator of (7.45) results in the following fourth degree polynomial in \( x = \rho^2 \):

\[
L_1 x^4 + (2L_1 Z^2 + 2L_2) x^3 \\
+ (L_1 Z^4 + 2L_2 Z^2 + L_3 - \bar{M}) x^2 - 2\bar{M} Z^2 x - \bar{M} Z^4 = 0 \tag{7.46}
\]

where

\[
L_1 = K_1^2 + K_2^2 \tag{7.47a}
\]

\[
L_2 = K_1 K_2 + K_3 K_4 \tag{7.47b}
\]

\[
L_3 = K_2^2 + K_4^2 \tag{7.47c}
\]

\[
\bar{M} = \frac{M_1^2 + M_2^2}{\alpha^2} \tag{7.47d}
\]

This fourth degree equation can be solved exactly for \( \rho \) given \( \alpha \) and the observer location; the solution is denoted by \( \rho^c \). Given a source monopole, the set of \( \rho^c \)'s for various observer locations forms a conical boundary about the monopole referred to as the "I-cone". Its axis, like the M-cone, is coincident with the source monopole axis. In the hybrid code, \( \alpha \) is chosen to have the value 0.41. This results in the value of \( \rho \) such that the magnitude of \( \frac{C}{\rho} + \rho \left( D - \frac{jkC}{2zz} \right) \) differs from that of \( \rho \left( D - \frac{jkC}{2zz} \right) \) by 3 dB. Again, this value was chosen empirically. It was deemed an acceptable deviation of the field with the \( 1/\rho \) term and without.

### 7.3.3 Summary

Just as a residual singularity occurs when the UTD scattering solution is used in conjunction with the PWS point source method to determine the field near the monopole axis, a singularity likewise occurs in the field when the EUTD radiation solution is used in place of the UTD scattering solution. Since the EUTD coupling solution is not valid in this situation (an observer far from the cylinder) and the
UTD scattering solution exhibits the same singularity, modified image theory is used. Once again, because modified image theory is based on an approximation, it is preferable to use the EUTD radiation solution when possible. To this end, a boundary was developed to isolate the region in which modified image theory is used to determine the field. Its determination was based on estimating when the \(1/\rho\) term in the electric field was a fraction of the \(\rho\) term. The delineation surface was found to be a cone. Its radius is the \(\rho\) which is a solution to (7.46). For a source monopole whose axis is normal to the cylinder axis, the I-cone axis is coincident with the monopole axis. However, like the M-cone, if the monopole is tilted at an angle with respect to the cylinder axis, then I-cone axis is tilted also.

Appendix E shows that the field determined by the extended UTD radiation/PWS combination is not always singular for an on-axis observer. Interestingly enough, if the observer is closer to the cylinder than the source monopole is, the field remains finite. The difference between this orientation and the one shown in Figure 44 is that in the former the observer location is projected onto the cylinder; whereas, in the latter, the locations of the PWS point sources are projected onto the cylinder. Projecting the observer location onto the cylinder causes the geometry of the source monopole to be undisturbed. As a consequence, as when finding the free-space field, cancellation occurs when finding the scattered field via the EUTD radiation solution.

Another situation in which no delineation surface is required is detailed in Appendix F. It is shown that the field calculated by the EUTD radiation solution remains bounded if the monopole axis is parallel to the cylinder axis because again if the monopole is projected onto the cylinder it still maintains the same orientation.
7.4 S-can: Modified Image Theory versus the EUTD Coupling Solution

7.4.1 Introduction

Looking at the results in Chapter III, even for a simple current element, problems are expected when determining the field along the axis of the current element via the EUTD coupling solution. This problem is inevitable because all the field components are proportional to \(1/s\) where \(s\) is the length of the geodesic connecting the projection of the source onto the cylinder with the projection of the observer onto the cylinder. For an observer located along the current element axis, \(s = 0\); consequently, the field is infinite. This singularity found in the total field from a single point source is not alleviated through cancellation of the singular terms by adding together the fields of the two PWS point sources which comprise a monopole. Basically, the EUTD coupling equations themselves do not allow for the case of an observer located along the current element axis; therefore, the singularity is not a result of using the PWS point source method.

7.4.2 Need for a Delineation Curve

Referring to Figure 45, the \(\hat{z}\)-directed field at the observer generated by a \(\hat{\rho}\)-directed monopole in the presence of the cylinder will be derived. From equation (3.40), the \(E_{z\rho'}\) field generated by a current element located at one of the monopole endpoint locations is

\[
E_{z\rho'} \sim \frac{p_0 k Z_0}{4\pi} \frac{d_f e^{-jks}}{s^2} \left( 2 - \frac{d_f}{a} - \frac{2d_s}{a} \right) \left( j + \frac{3}{ks} - \frac{j3}{k^2s^2} \right) \tag{7.48}
\]

where \(d_f = Z \hat{\Delta} z_2 + d + \hat{z}\) is the distance of the observer from the PEC. The distance of the source from the PEC is \(d_s\) and is either \(z_2\) or \(z_2 + d\) depending on which monopole endpoint is being considered. The variable \(s\) is the length of the geodesic between \(P^s_N\), the projection of the source onto the cylinder, and \(P^o_N\), the
Figure 45: Geometry to test the accuracy of the EUTD coupling solution when determining the scattered field at an observer near the axis of a monopole which illuminates a PEC cylinder.
projection of the observer. The radius of the cylinder is \( a \) and \( k \) is the wavenumber of the medium. For the orientation shown in Figure 45, \( \theta \), the angle between the geodesic and the \( z \)-axis, is 90°. Using (5.32), the current magnitude of the two equivalent PWS point sources at \( \theta = 90° \) is

\[
\rho_{e_2} = -\frac{\cot kd}{k}
\]  

(7.49)

for the terminal endpoint of the top monopole and

\[
\rho_{e_3} = \frac{1}{k \sin kd}
\]  

(7.50)

for the non-terminal endpoint. By adding the field contributions from the two equivalent PWS point sources, the monopole field is found to be

\[
E_{zp}^{\text{mon}} \sim -\frac{Z_0}{4\pi \sin kd} \frac{e^{-jk\rho}}{\rho^2} \left( j + \frac{3k}{k_0} - \frac{j3}{k^2 \rho^2} \right)
\]

\[
\cdot \left[ \left( 2 - \frac{z_2 + d}{a} - \frac{2Z}{a} \right) \left( z_2 + d \right) - z_2 \cos kd \left( 2 - \frac{z_2}{a} - \frac{2Z}{a} \right) \right]
\]  

(7.51)

where the substitution \( \rho = s \) has been made to facilitate comparison with the previous sections dealing with the singularity problem when using the UTD scattering solution or the EUTD radiation solution. By expanding the exponential term in (7.51)

\[
E_{zp}^{\text{mon}} \sim \frac{Z_0}{4\pi \sin kd} \frac{A B}{k^2 \rho^2}
\]  

(7.52)

where only \( A \) is a function of \( \rho \):

\[
A = -\frac{j3}{\rho^2} - \frac{jk^2}{2} + O(\rho)
\]  

(7.53a)

\[
B = z_2 \left( 1 - \cos kd \right) \left( 2 - \frac{z_2}{a} - \frac{2Z}{a} \right) - \frac{z_2 d}{a} + d \left( 2 - \frac{z_2}{a} - \frac{2Z}{a} - \frac{d}{a} \right)
\]  

(7.53b)

Thus, the behavior as \( \rho \rightarrow 0 \) (observer located along the monopole axis) is

\[
E_{zp}^{\text{mon}} \sim O\left( \frac{j}{\rho^4} \right)
\]  

(7.54)
This result is reasonable if the coupling between two current elements is considered. In the EUTD coupling solution, the dominant term is the “mutual coupling” term \((d_f = d_s = 0\) in the EUTD coupling solution) which is the coupling between two current elements if they are located directly on the PEC. Considering this situation, even the current element field equation [25, eq. 2-113] predicts a singularity:

\[
\vec{E}^I = \hat{\theta} E_\theta \\
\propto \hat{\theta} \sin \theta \mathcal{F}\left(\frac{1}{\rho}, \frac{1}{\rho^2}, \frac{1}{\rho^3}\right) \bigg|_{\theta=90^\circ} \\
\propto \hat{\theta} 1.0 \left(\frac{1}{\rho}, \frac{1}{\rho^2}, \frac{1}{\rho^3}\right) \\
\rho \to 0 \Rightarrow \hat{\theta} \mathcal{O}\left(\frac{1}{\rho^3}\right) 
\]  
(7.55)

where \(\mathcal{F}\) denotes a complex function. Thus, even the exact current element field predicts a singular field as \(\rho \to 0\). As a result, it is concluded that there are no cancellation problems due to the PWS/EUTD coupling combination; rather, the singularity is purely caused by the EUTD coupling equations.

An analysis similar to that which resulted in equation (7.54) shows that the \(\hat{\rho} \hat{\rho}'\) electric field from a monopole calculated using the EUTD coupling solution behaves as

\[
E^\text{mon}_{\rho \rho'} \sim \mathcal{O}\left(\frac{j}{\rho^5}\right) 
\]  
(7.56)

This is expected; indeed, by using the current element analogy, it is found that the \(\hat{\rho} \hat{\rho}'\) field goes to infinity faster than the \(\hat{\rho} \hat{\rho}'\)–component. For completeness, the other components of the field behave as

\[
E^\text{mon}_{\rho z'} \sim \mathcal{O}\left(\frac{j}{\rho^4}\right) 
\]  
(7.57a)

\[
E^\text{mon}_{zz'} \sim \mathcal{O}\left(\frac{j}{\rho^5}\right) 
\]  
(7.57b)
Unlike when the UTD scattering solution or the EUTD radiation solution is used in conjunction with the PWS point source method, the field at a point along the monopole axis derived through use of the EUTD coupling solution exhibits a singularity which comes exclusively from the asymptotic approximations involved in developing this solution and not from the combination of the asymptotic approximations with the PWS point source method. As in the previous cases where singularities existed, modified image theory will be used in the region where the EUTD coupling solution is not valid. Therefore, a boundary between the domain of validity of modified image theory and that of the EUTD coupling solution needs to be determined.

7.4.3 Derivation of the S-cone

As in the previous two cases (UTD scattering case and EUTD radiation case), it is preferable to use the EUTD coupling solution whenever possible since modified image theory is an approximation. Thus, the boundary dividing the region where the EUTD coupling solution is used and where modified image theory is used is determined by the location where the EUTD coupling solution begins to be dominated by the $1/\rho^4$ or $1/\rho^5$ behavior, depending on the field component of interest. Unlike in the EUTD radiation or the UTD scattering solution where the singularity only occurred when determining the monopole field, the singularity associated with the EUTD coupling solution is due its own nature and has nothing to do with the monopole. Consequently, the analysis of the singularity can be limited to examining the EUTD coupling-based field for a single current element. The second bracketed term in (7.48) controls the singular behavior of $E_{z,f}$ as $s \to 0$. A similar type of measure to that used to obtain the I-cone is utilized: equation (7.48) is examined to find the value of $s$ at which the singular terms in (7.48) begin to dominate. The
following measure is used:

\[
20 \log_{10} \left| \frac{3}{k s_T} + j \left(1 - \frac{3}{k^2 s_T^2}\right) \right| - 20 \log_{10} |j| = 3 \text{ dB} \quad (7.58)
\]

where \( s_T \) is the value of \( s \) at the transition point. Again, the value 3 dB was empirically chosen. Note that \( s_T \) is not the point where the EUTD coupling solution differs by 3 dB from the field determined by modified image theory; rather it is the point where the terms involving \( 1/s \) dominate the constant term in (7.48) by 3 dB. This is in contrast to \( \rho_c \), the radius of the M–cone, where the UTD scattering solution differs from modified image theory by 3 dB. Solving for \( s_T \) in (7.58) results in a quadratic equation in \( x = s_T^2 \):

\[
x^2 - \frac{3k^2}{(\gamma^2 - 1) k^4} x - \frac{9}{(\gamma^2 - 1) k^4} = 0 \quad (7.59)
\]

where \( \gamma = 10^{3/20} \). Note that \( s_T \) as determined by (7.59) is independent of \( d_f \) and \( d_s \), the distance of the observer and the source from the PEC, respectively. The resulting surface is a cylinder; henceforth referred to as the “S–can”. Its radius is \( s_T \) and its axis is coincident with the current element axis. The boundary for \( E_{zz'} \) and \( E_{pz'} \) is the same since the term which controls the field behavior as a function of \( s \) is the same as the bracketed term in (7.48).

However, for \( E_{ppl} \) the field behaves as

\[
E_{ppl} \sim C \frac{e^{-jk s}}{s} \left[ \frac{1 - j k s}{k^2 s^2} + \frac{C_1}{s} \left( \frac{j}{4} - \frac{1}{k s} + \frac{j 9}{4 k^2 s^2} + \frac{9}{4 k^3 s^3} \right) \right] \quad (7.60)
\]

where

\[
C = \frac{p_c k Z_0}{2\pi j} \quad (7.61)
\]

and

\[
C_1 = 2k (d_o^2 + d_f^2) \quad (7.62)
\]
Using only the bracketed terms denoted by $A$ to compute the value of $s$ where the singular terms begin to dominate, the following condition on the transition point is obtained:

\[
20\log_{10}\left|\left(1 - \frac{1}{k^2 s_T^2} - \frac{C_1}{ks_T^2} + \frac{9C_1}{4k^3 s_T^4}\right) + j\left(-\frac{1}{ks_T^2} - \frac{C_1}{4s_T^2} + \frac{9C_1}{4k^2 s_T^4}\right)\right|
- 20\log_{10}|1.0| = 3\text{dB} \quad (7.63)
\]

Solving this equality results in the following quartic in $x = s_T^2$:

\[
x^4 + \frac{m_1^2 - 2m_2}{(1 - \gamma^2)} x^3 + \frac{m_2^2 + 2m_4 + 2m_1m_3}{(1 - \gamma^2)} x^2
+ \frac{m_3^2 - 2m_2m_4}{(1 - \gamma^2)} x + \frac{m_4^2}{(1 - \gamma^2)} = 0 \quad (7.64)
\]

where

\[
m_1 = -\frac{1}{k} - \frac{C_1}{4} \quad (7.65a)
\]
\[
m_2 = \frac{1}{k^2} + \frac{C_1}{k} \quad (7.65b)
\]
\[
m_3 = \frac{9C_1}{4k^2} \quad (7.65c)
\]
\[
m_4 = \frac{9C_1}{4k^3} \quad (7.65d)
\]

Upon solving the quartic in (7.64) by conventional mathematics for $d_s = d_f = 0.2\lambda$, it is found that an error results. $s_T$, for this case, is found to be 0.335. However, when this value is substituted back into (7.64), a zero does not result. This discrepancy results because the quartic is very sensitive to changes in the variable $s$ around the transition point. This is seen by looking at the $A$ term given in (7.60) about the actual transition point. This is found by calculating $|A|$ and determining the value of $s$ when $|A|$ is equal to $10^{3/20}$. The value of $s$, for $d_s = d_f = 0.2\lambda$, is found to be

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0.27. The variation of $|A|$ about this point is found by differentiating it with respect to $s$ and evaluating the result at 0.27. The answer is found to be

$$\frac{\partial |A|}{\partial s} \bigg|_{s=0.27} \approx -27.5$$

(7.66)

As seen, a slight error in $s$ produces a large error in $|A|$; therefore, this method to obtain $s_T$ cannot be used. Since the quartic obtained from the condition in (7.63) is too unstable to be solved by a computer accurately, a simpler equation, a singular term plus the constant term in (7.63), will be used. Thus, the equality

$$20 \log_{10} \left| 1 + \frac{9C_1}{4k^3 s_T^4} \right| - 20 \log_{10} |1.0| = 3\text{dB}$$

(7.67)

will be enforced instead. Since

$$\frac{9C_1}{4k^3} > 0$$

(7.68)

then the absolute values can be removed which results in

$$s_T = \left( \frac{9C_1}{4k^3(\gamma - 1)} \right)^{\frac{1}{4}}$$

(7.69)

For the previous case, $d_s = d_f = 0.2\lambda$, (7.67) predicts that $s_T = 0.38$. This gives a larger region than required where modified image theory is used. It gives a conservative estimate of when to make the transition from the EUTD coupling to modified image theory.

### 7.5 Empirically-derived Crossover Points

The previous three sections focused on mathematically deriving rules or boundaries to aid in determining when to use the UTD-based equations and when to use modified image theory. Modified image theory was only used if the UTD-based field was singular; in which case, it was obvious that the UTD and EUTD equations were
not valid. This singularity provided the key to determining the limit of the region of validity of the UTD equations. It is equally important to know which UTD solution to employ if the observer is in a region where multiple UTD equations seem valid, i.e., the field is bounded. In this regard, it is noted that since no eigenfunction computer code was available at that time to test which UTD equation was the most accurate, test runs were made and visual inspection was used to determine which equation generated a field closest to what was believed to be correct. Thus, the crossover point was determined empirically. The results follow.

7.5.1 EUTD Coupling Solution versus EUTD Radiation Solution

As mentioned in Section 2.4 and shown in Figure 10, it is expected that for a source and observer located close to the cylinder the EUTD coupling solution should be used to calculate the field. Not only is it best-suited for use under these conditions, but it is the only UTD-based equation that is valid. However, as either the source or the observer moves away from the cylinder surface, the EUTD coupling solution begins to break down because the assumptions used in deriving the Taylor series for the EUTD coupling solution are violated. Fortunately, the EUTD radiation solution is expected to be more accurate as either the source or the observer but not both moves into the far from the cylinder. It is hoped that there is some crossover point where both the EUTD coupling and the EUTD radiation solution are accurate and thus a switch can be made from calculating the field using one equation to using the other. If there exists such a point then it is hoped that it can be predicted a priori.

As in the previous three sections, a search is made to identify the term or terms which cause either the EUTD coupling or the EUTD radiation solution to break down. A basic assumption in the derivation of the EUTD coupling result is that
Figure 46: Geometry used to test the EUTD coupling versus the EUTD radiation solution: a current element at \((\rho' = a, \phi' = 0^\circ, z = 0)\) and oriented normal to the PEC circular cylinder of radius \(a\) with an observer at \((\rho = a + d, \phi = 0^\circ, z = s)\). The variable \(d\) is varied and the \(\rho\)-directed field is determined.

\[
\frac{d}{a} \ll 1 \quad \text{and} \quad \frac{d'}{a} \ll 1
\]

Thus, this is a requirement. No other assumptions were made in the derivation of the EUTD coupling. However, satisfying this assumption was found to be not enough. Figure 46 shows the geometry used to see if there is a smooth blending from the curve of the field determined using the EUTD coupling solution to that determined via the EUTD radiation solution. To this end, the source location is fixed at \(\rho' = a\), on the cylinder surface, while the observer is moved in the radial direction away from the cylinder surface. The resulting field is shown in Figure 47 for a large, \(20\lambda\) radius cylinder and in Figure 48 for a smaller, \(2\lambda\) radius cylinder.

These two plots show that the break-down of the EUTD coupling solution also
Figure 47: Use of the EUTD coupling and EUTD radiation solutions to determine the field for a 20 $\lambda$ radius cylinder.
Figure 48: Use of the EUTD coupling and EUTD radiation solutions to determine the field for a $2\lambda$ radius cylinder.
depends on \( s \), the length of the geodesic between the projection of the source and of the observer onto the cylinder. These two plots verify the behavior predicted in Figure 10: there exists a value of \( d \) at which a switch can be made from using one extended UTD equation to determine the field to another. In fact, the transition is shown to occur very smoothly, i.e., over a wide range of \( d \) instead of at a single point. From these two plots it can be gleaned that:

1. As \( a \to 0 \), the crossover occurs at larger values of \( d \) and

2. as \( s \to 0 \), the crossover occurs at smaller values of \( d \).

For this particular orientation, \( \theta = 90^\circ \) which implies that \( k_\parallel = k \) and thus \( k_\parallel d \), the Taylor series coefficient in the EUTD coupling derivation, is independent of \( s \). So using the Taylor series coefficient as a guide to the validity of the EUTD coupling is not enough, although it must still be checked to make sure that it is not greater than one. Looking at the two points surmised from the plots and enumerated above may aid in choosing a variable to indicate when to use the EUTD coupling solution and when to use the EUTD radiation solution. Point 1 indicates that the variable should be proportional to \( da \). Point 2 indicates that it should be proportional to \( d/s \). Using these two criteria, the variable \( \zeta^R \) will be used where

\[
\zeta^R = -\left( \frac{ka \sin \theta}{2} \right)^{\frac{1}{3}}
\]  

(7.70)

where \( \theta \) is the angle between the \( \hat{z} \)-axis and the vector from the projection of the source onto the cylinder \( (P^R_s) \) to the observer. Using the values of \( a \) and \( s \) corresponding to Figures 47 and 48, various \( d \) were chosen and \( \zeta^R \) calculated. Then the curves in Figure 47 and 48 were examined to visually choose the crossover point and the value of \( \zeta^R \) calculated there. This test along with many others indicated that a
value of $\xi^R = -0.95$ provided, overall, the best a priori guess to the location of the crossover point. The criterion is

\begin{verbatim}
If ($\xi^R < -0.95$) then
    use EUTD RADIATION solution
else
    use EUTD COUPLING solution
end if
\end{verbatim}

The following table illustrates the use of $\xi^R$ to decide whether to use the EUTD coupling solution or the EUTD radiation solution. The last column in Table 4 shows which equation would be used given that $\xi^R = -0.95$ is the crossover value. This is an approximate value; however, by comparing the guess derived by using the above criterion (as in Table 4) with the visual guess derived from examining the plots in Figures 47 and 48, it is seen that this criterion gives reasonable results.

7.5.2 EUTD Radiation Solution versus UTD Scattering Solution

In the previous section, the criterion used to decide whether the field is calculated by the EUTD coupling solution or by the EUTD radiation solution was derived. It was found empirically by varying the observer or the source while the other remained fixed at a position close to the PEC cylinder. The condition that either the source or observer be in the vicinity of the cylinder was required so that both the EUTD coupling solution and the EUTD radiation solution remained valid. This section deals with the case if neither the source nor the observer are in the near to the cylinder. In which case, the EUTD radiation solution is inaccurate. Fortunately, the UTD scattering solution is most accurate when both source and observer are far from the cylinder. A criterion is needed to decide which equation to use to calculate the field. As in the previous section this will be determined empirically.
Table 4: Using $\xi^R$ to find the transition point between the EUTD coupling solution (MC) and the EUTD radiation solution (MR) (Refer to Figure 47 and 48 where $d' = 0$).

<table>
<thead>
<tr>
<th>a (λ)</th>
<th>s (λ)</th>
<th>d (λ)</th>
<th>$\xi^R = -\left(\frac{ka\sin\theta}{2}\right)^{1/3}$</th>
<th>Equation to use</th>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td></td>
<td></td>
<td>-2.87</td>
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</tr>
<tr>
<td>0.3</td>
<td></td>
<td></td>
<td>-2.62</td>
<td>MR</td>
</tr>
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<td></td>
<td></td>
<td>-1.84</td>
<td>MR</td>
</tr>
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<td></td>
<td></td>
<td>-1.46</td>
<td>MR</td>
</tr>
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<td></td>
<td></td>
<td>-0.86</td>
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</tr>
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<td></td>
<td></td>
<td></td>
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</tr>
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<td></td>
<td></td>
<td>-2.31</td>
<td>MR</td>
</tr>
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<td></td>
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<td></td>
</tr>
<tr>
<td>0.6</td>
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</tr>
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<td>-0.977</td>
<td>MR</td>
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<tr>
<td>0.2</td>
<td></td>
<td></td>
<td>-0.855</td>
<td>MC</td>
</tr>
</tbody>
</table>
Figure 49: (a) Field at the observer at $O$ due to a current element at $P'$ can be approximated by (b) the field at $O$ due to the same current element at $P'_N$, the normal projection of $P'$ onto the PEC.

The EUTD radiation solution assumes that either the source or the observer is close to the PEC cylinder. Whichever point is closer to the PEC cylinder is projected onto it. The field predicted by the EUTD radiation solution at an observer in far from the cylinder due to a current element near to the PEC cylinder (see Figure 49a) is equal to the field due to a current element at $P'_N$, the projection of $P'$ onto the cylinder, (see Figure 49b) plus higher order terms which are proportional to $d'$, the radial distance of the source from the cylinder. The smaller the value of $d'$, the more accurate the approximation. As $P'$ moves farther from the PEC, $d'$ increases and the higher order terms begin to dominate the field which is unacceptable. These higher order terms result from the Taylor series upon which the EUTD radiation solution is based. In this case, the Taylor series variable is $h'_t = \frac{k}{m^2 \rho} d'$. If the observer is closer to the cylinder then, using reciprocity, the Taylor series variable is
Figure 50: Geometry used to determine $\tilde{h}_t$ at the transition point between calculating the field via the EUTD radiation solution and by the EUTD scattering solution. The $\hat{\rho}$-directed current element is radiating in the presence of a cylinder of radius $a$ and $E_{\rho f}$ is determined. The distances $d$ and $s$ are fixed while $d'$ is varied.

$$h_t = \frac{k}{m^{3D}}d.$$ For simplicity, $\tilde{h}_t$ will be used to encompass both cases. As $\tilde{h}_t$ increases, the EUTD radiation solution becomes more inaccurate. Thus, this is a reasonable criterion to decide when to calculate the field by the EUTD radiation solution and when to use the UTD scattering solution.

Figure 50 shows the geometry used to determine $\tilde{h}_t$ at this transition point. The observer is fixed at $(\rho = a + d, \phi = 0, z = 0)$ while the $\hat{\rho'}$-directed current element is located at $(\rho = a + d', \phi = 0, z = s)$ with $d'$ varying. The $E_{\rho f}$ field component is calculated by multiple methods. Figure 51 shows the resulting electric...
field component normal to the cylinder as a function of \( d' \), the distance of the source off the cylinder. The result found by using flat plane image theory is provided for reference. The EUTD coupling solution, as expected, is not valid for large values of \( d' \). However, even when \( d' \) is small, the EUTD coupling solution is being used at the limits of its validity because \( d \) is not small. Thus, one must be careful in relying on the EUTD coupling results for the small values of \( d' \). The UTD scattering solution is expected to provide the most accurate results when \( d' \) is large; whereas, the EUTD radiation solution is expected to generate a more accurate result than the UTD scattering solution for smaller values of \( d' \). Since \( d = 0.4\lambda \) is not too small, the field determined using the EUTD radiation solution is expected to be accurate as long as \( d' < d \) and thus \( P' \) is projected onto the cylinder rather than \( P \). From Figure 51, this is seen to be true. There does seem to be a value of \( d' (\approx 0.3\lambda) \) where a smooth transition can be made from calculating the field via the EUTD radiation solution to calculating it using the UTD scattering solution. However, the EUTD radiation–based results seem suspect because of the dip. In order to determine the cause of this dip, each of the terms in the Taylor series for the EUTD radiation solution will be examined. To facilitate this, let

\[
E_p = C \left( T_1 + T_2 + T_3 + T_4 \right) \tag{7.71}
\]

where

\[
T_1 = \tilde{G} (Z_{lt}) \tag{7.72a}
\]

\[
T_2 = \frac{j h_{lt}^2}{2} e^{-j z_{lt}^3/3} \tilde{g}' (Z_{lt}) \tag{7.72b}
\]

\[
T_3 = \frac{2 h_{lt}^3}{3!} \tilde{G} (Z_{lt}) \tag{7.72c}
\]

\[
T_4 = -\frac{h_{lt}^4}{4!} e^{-j z_{lt}^3/3} \tilde{g}'' (Z_{lt}) \tag{7.72d}
\]

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Figure 51: The $E_{\rho\phi}$ field component calculated via four different methods. The geometry is shown in Figure 50 with $a = 2\lambda$, $s = 1\lambda$, and $d = 0.4\lambda$ while $d'$ varies.
The quantity $S_i$, where

$$S_i = C \sum_{j=1}^{i} T_j$$  \hspace{1cm} (7.73)

will be examined. Figure 52 shows plots of $|T_i|$ and plots of $S_i$. Ideally, if the Taylor approximation is valid, the dominant term is the lowest order term and subsequent terms should not affect the field greatly by their addition. Figure 52 shows that to be false for this case. Figure 52a shows the magnitude of each of the terms that comprise $E_p$, i.e., the $A_i$. For $d' \gg 0.7\lambda$, the $h_t^2$ term is the dominant term in the series. Figure 52b shows the effect that adding higher order terms has on $E_p$. The UTD scattering solution is provided as a reference only. It is seen that the addition of higher order terms causes the dip. These two plots indicate that the problem is that $h_t$ is too large for $d'$ large. In fact, around the dip, $h_t$ is greater than one. Thus, $h_t^n$ is greater than $h_t^{n-1}$. This would render the Taylor approximation invalid. Indeed, $h_t$ must be small in order for modified radiation to be valid. This gives us confidence that for $d' < 0.3\lambda$ the EUTD radiation results are correct. An examination of various test cases indicates that $h_t = 0.55$ provides a good crossover point; i.e.,

If $(h_t < 0.55)$ then

use EUTD RADIATION solution

double else

use UTD SCATTERING solution
end if

This point is indicated on Figure 51 as “TP”. It does indeed give a reasonable transition point. It is shown to also be a good criterion for the crossover points for the two cases in Figure 53 and also for Figures 54 and 55.
Figure 52: Effect of higher order terms in the EUTD radiation result for the geometry in Figure 50 with $a = 2\lambda$, $s = 1\lambda$, and $d = 0.4\lambda$ while $d'$ varies. (a) $|T_i|$, $i = 1, \ldots, 4$, where $T_i$ is the term in the EUTD radiation equation involving $h_i$. (b) $E_\rho$ calculated by the EUTD radiation solution (MR) of various orders. The UTD scattering solution included for reference.

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Figure 53: Example to illustrate blending between the EUTD radiation result and the UTD scattering result. $E_\rho$ at an observer at $(\rho = a + d, \phi = 0, z = 0)$ in the presence of a cylinder of radius $a = 2\lambda$. Source is a current element at $(\rho' = a + d', \phi' = 0, z' = s = 1\lambda)$ and in the $\hat{\rho}$ direction.
Figure 54: Example to illustrate blending between the EUTD radiation result and the UTD scattering result. Source is a current element at \((\rho' = a + d', \phi' = 0, z' = s = 1\lambda)\) and in the \(\hat{\rho}\) direction. Calculation of the \(E_{\rho}\) component of the field at an observer at \((\rho = a + d, \phi = 0, z = 0)\) in the presence of a cylinder of radius \(a = 2\lambda\) where \(d = 0.6\lambda\).
Figure 55: Example to illustrate blending between the EUTD radiation result and the UTD scattering result. Source is a current element at \((\rho' = a + d', \phi' = 0, z' = s = 1\lambda)\) and in the \(\rho\) direction. Calculation of the \(E_\rho\) component of the field at an observer at \((\rho = a + d, \phi = 0, z = 0)\) in the presence of a cylinder of radius \(a = 2\lambda\) where \(d = 0.80\lambda\).
7.6 Conclusion

A computer program needs to decide which UTD or EUTD equation to use for a mutual impedance calculation. Boundaries have been determined for selecting the most accurate equation to use given the source and observer location, the length of the source monopole, and the radius of the cylinder. The boundary between modified image theory and the UTD scattering solution, the EUTD radiation solution, and the EUTD coupling solution was determined by investigating when the latter three equations become singular. This investigation produced three boundaries: the M-cone, a conical surface, which separates the region where modified image theory is used and where the UTD scattering solution is used; the I-cone, a conical surface, which separates the region where modified image theory is used and where the EUTD radiation solution is used; and the S-can, a cylindrical surface, which separates the region where modified image theory is used and where the EUTD coupling solution is used. When multiple UTD or EUTD equations are valid, a judgment must be made to determine which equation is most accurate. Numerous test cases were run, and two empirically-derived boundaries were found: one delineating the region where the EUTD radiation solution is used and where the EUTD coupling solution is used, and another delineating the region where the EUTD radiation solution is used and where the UTD scattering solution is used. In both cases, the boundary was determined based on when the EUTD radiation result becomes inaccurate. In the former case, the condition that $-0.95 < \xi^R = -\left(\frac{ka \sin \theta}{2}\right)^{1/3} < 0$ indicates that the EUTD coupling result would be more accurate; otherwise, the EUTD radiation result is used to predict the field. For the second empirically-derived boundary, $\hat{h}_t > 0.55$ indicates that the Taylor series variable is too large and thus the UTD
scattering result would be better; otherwise, the EUTD radiation result can be used. These results are combined to form the flowcharts given in Appendix G.
CHAPTER VIII
Numerical Results

8.1 Introduction

This chapter contains various test cases to show both the usefulness and accuracy of the hybrid method. The geometries considered here are finned cylinders; however, the hybrid method is applicable to more complex geometries. The finned cylinder was chosen for simplicity. If the hybrid code for the simpler finned cylinder geometries generates accurate results in, most importantly, an efficient manner, then (1) more complicated and larger appendages can be analyzed by adding more memory and perhaps vectorizing the code and (2) more complicated main bodies can be analyzed by generalizing the modified UTD solutions for the circular cylinder to the perfectly conducting, smooth convex object (PSCO). Thus, although it is a fundamental geometry, the finned cylinder possesses the characteristics needed to test the efficiency and accuracy of the hybrid method.

Previous work on a related geometry was done by Ruckgaber and Schultz [29]. They obtained the eigenfunction solution to the two-dimensional problem of a single fin attached to a circular cylinder. Pathak et al. analyzed this three-dimensional geometry using high frequency techniques [28]. They presented a UTD analysis for calculating the backscatter from a PEC circular cylinder with a fin illuminated by a plane wave. An approximate solution to the problem was synthesized from the UTD solutions to two related problems. The total backscatter was the result of the UTD field backscattered by the cylinder alone plus a contribution from each of the visible
fins. This latter contribution was obtained from the 2-D UTD solution modified by a factor that accounts for the 3-D nature of the fin. This accounts for the interaction between the fins and the cylinder but not for interactions between the fins.

The cylinders treatable at the present time by this hybrid code are infinite in length. The extension to a finite cylinder requires that the excitation vector, $V$, be supplemented by the component of the incident ray which hits the cylinder rim and diffracts to an observer located on the fin. Although this is an important exclusion that limits our ability to treat more realistic geometries, it does not detract from the usefulness of these infinitely-long, finned cylinders to show the accuracy of the hybrid method since the most significant problems and dominant field contributions occur when calculating the impedance matrix. When determining $\Delta Z$, the observer is sometimes in a region which strains the validity of any of the presently available UTD solutions. However, the calculations involved in filling the $\Delta V$ vector involve far-zone fields. This is a condition which is already assumed in the relevant UTD and extended UTD solutions. As a consequence, it is expected that the accuracy of the hybrid method depends most critically on the accuracy of the $\Delta Z$ matrix calculation since when the UTD/PWS combination fails, it often does not fail gracefully. Instead, if the UTD or the aforementioned combination is used in a region where it is not valid, it may produce a singular result. Thus, if the hybrid method works on an infinitely-long cylinder, it is expected to work on a finite length cylinder.

8.2 CPU Time for the Hybrid Method

A common belief regarding methods that use the UTD is that although the UTD involves no time-consuming integrations as the MM does, it does involve the calculation of special functions and the tracking of rays, which are deemed to be even
more time-consuming. Indeed, in some cases, a single UTD calculation is as time-
consuming as numerous integrations. However, as a whole, the central processing
unit (CPU) time required to solve the finned cylinder scattering problem by the
hybrid method is found to be of the same order as the time required to solve the
equivalent flat plane image theory problem, i.e., a fin over a ground plane. The
following discussion concentrates on why this might be true.

Consider the finned cylinder shown in Figure 56a. The fin has been divided
into four patches which generates six basis functions. Calculating the elements of
the $6 \times 6$ matrix used in the hybrid method usually requires calculating, in addition
to the normal free-space terms, the elements of the $6 \times 6 \Delta Z$ matrix using the
UTD. For example, the UTD scattering solution and modified image theory generate
only the matrix element $\Delta Z_{mn}$ (equation (2.9c)), which involves only the scattered
field, and thus $Z_{0mn}$ (equation (2.9b)), which involves the incident field, must be
determined separately. However, $Z_{mn}$, the sum of $Z_{0mn}$ and $\Delta Z_{mn}$, is determined
at once by the extended versions of the UTD because the EUTD radiation and
EUTD coupling solutions generate the total field rather than the scattered field.
The patching associated with the equivalent flat plane image theory problem is
shown in Figure 56b. The patching in the original fin above the PEC is the same as
in Figure 56a, but now a mirror image is generated through image theory. This new
problem requires filling a $10 \times 10$ matrix whose elements are the mutual impedance
between dipoles that exist in free-space and, as a result, involve only the incident
electric field. This matrix is shown in Figure 57 for comparison purposes with
the $6 \times 6$ matrix required for the original problem solved by the hybrid method.
The lightly shaded elements in Figure 57 are those which are common to both
problems. In other words, although the method of calculating $\Delta Z_{mn}$ is different
when using the UTD to solve for the scattered field instead of image theory, these
(a) Original problem

(b) Equivalent image theory problem

Figure 56: Single-finned cylinder patched for the purpose of comparing with the equivalent image problem and in order to compare the $V$ vector calculated using the hybrid method, using a flat plane approximation, and using an eigenfunction solution.
Figure 57: View of the $Z$ matrix associated with the equivalent image theory problem shown in Figure 56b. The shaded areas are those elements which are also required in the matrix associated with the original problem. The darkly shaded elements with a dot in the center are the overlaps modes whose calculation is similar but not exactly the same as that required for the overlap modes of the original problem. The squares within the bold lines which have asterisks at the center are those extra elements that need to be calculated for the equivalent image problem. The rest of the matrix need not be calculated because a symmetric matrix is assumed.
elements must still be calculated by both methods of solution. The original problem, if solved using the hybrid method, requires approximately ten elements which involve calculation of the UTD-based scattered field instead of the incident field from the monopole (the lightly shaded squares in the upper right-hand corner of Figure 57a). The darker shaded squares with dots in the center are those terms which require almost the same calculations regardless of which method is used. The difference here is that these mutual impedances involve the overlap modes 5 and 6 in Figure 56a. These are monopoles. Their counterparts in Figure 56b are dipoles. Thus, the calculations involving them are slightly different. The squares with asterisks in the center bounded by the bold line sets off those matrix elements which are calculated only in the equivalent image theory problem. There are twenty-four of these additional terms to calculate, four of which are self-impedance terms which are always more time-consuming to calculate than off-diagonal mutual impedance terms. Thus, both methods of solution, the hybrid method and image theory, have their pros and cons. The hybrid method involves the calculation of ten extra terms which require the determination of special functions and the tracing of rays; whereas, using image theory to solve the problem requires calculating twenty-four extra terms, four of which are self-impedance calculations. As a result, the CPU time used to solve the two problems depicted in Figure 56 is comparable.

8.3 Measurement System Description

The following section gives a brief discussion on the compact range at the Ohio State University ElectroScience Laboratory (OSU/ESL) and the finned cylinder measurements made there. For a more detailed discussion of the development and performance of the OSU/ESL range, the reader is referred to [30] where Walton and
Young also included a discussion on the digital data processing algorithms used. For more details regarding the reflector used and its performance, the reader is referred to [31] where Burnside et al. detailed the benefits of using a reflector with a blended rolled edge versus one with a serrated edge.

8.3.1 Compact Range Discussion

The compact range is an electromagnetic system which involves the use of a large collimating device (a reflector) to produce a uniform plane wave across the aperture of an antenna or a scattering body without requiring the normal far-field separation [32]. Accurate measurement of antenna radiation or target scattering patterns requires that the object being tested be illuminated by a uniform plane wave. Conventional methods of satisfying this requirement involve locating the test object sufficiently far from the feed antenna such that the spherical wavefront from the transmitting antenna closely approximates a uniform plane wave incident upon the test object. This can be satisfied by using an outdoor range; however, an outdoor range is subject to adverse weather conditions. An indoor alternative involves the use of a parabolic reflector to collimate the spherical wavefront produced by the feed antenna and thus produce a plane wave across its aperture. This collimation is achieved by placing the feed at the focus of the reflector. Collectively, this system is referred to as a compact range. A typical configuration is shown in Figure 58. The reflector in the OSU/ESL compact range has a focal length of 12 feet. There are two separate horns used to generate the vertical and horizontal polarizations separately. They are separated by approximately 1.8° in the azimuth direction and are placed so that the combined phase center is at the focus of the reflector. The angular difference between the horns should be taken into account when determining
the location of $0^\circ$ in the azimuth direction since it will be slightly different for each polarization.

Because of such factors as taper in the feed horn illumination, the plane wave produced is not necessarily uniform across the aperture. However, it is only required to be uniform over a specified area where the target or antenna is located. This area is typically referred to as the target zone. The size of this area is very important because it limits the size of the target that can be measured. Unfortunately, not all of the energy radiated by the feed horn is directed by the reflector in the form of a plane wave into the target zone. For example, there may be diffraction from the reflector edges that contributes to the incident field impinging on the target or antenna. This affects the size of the target zone and, as a result, the size of the test objects which can be measured. Previously, only small structures could be measured due to the difference between the reflector and target zone sizes. Research in the area of reflector design [31] has resulted in the use of a blended rolled edge reflector at the
OSU/ESL range. This rolled edge on the top and sides of the reflector reduces the strength of the edge diffracted field contaminating the target zone without creating any new mechanisms which could perturb the plane wave illumination. This edge treatment resulted in the ability to handle much larger targets with an order of magnitude improvement in accuracy. The OSU/ESL compact range and anechoic chamber is shown in Figure 59. Note the rolled edge reflector termination.

The finned cylinders measured for this dissertation were small enough such that they indeed resided in the target zone, and thus, accurate backscatter measurements are expected. Additionally, the room is lined with microwave absorber in order to minimize reflection from the floors, walls, and ceiling. Such an absorber lined room
is referred to as an anechoic chamber. The anechoic chamber at ESL utilizes wedge shaped absorber in the area of the target zone in order to reduce the strength of the absorber diffracted fields that directly backscatter into the target zone time gate. Elsewhere, pyramidal absorber is used (see Figure 59).

Another important point to consider in the design of a compact range is the method used to support the target or antenna. Ideally, the backscatter of the support structure must not affect the return of the test target. To this end, a tilted metal tower whose cross-section is an ogive is used in the OSU/ESL compact range. The radar cross section (RCS) of this pedestal is below 
60 dbsm according to measurements [30]. In addition, a styrofoam mount (depicted as the cylinder between the target and the pedestal in Figure 58) is placed on top of the pedestal to support the test object.

All of these considerations have been incorporated into the anechoic chamber and compact range here at The Ohio State University in order to mimic a free-space environment in which to conduct accurate far-zone measurements. The measured data included in this dissertation was generated in this compact range.

8.3.2 Measurement Procedure

The finned cylinder was mounted as shown in Figure 60. The 3.5 inch diameter styrofoam mount has a saddle cut out at the top such that the 4 inch radius cylinder rests comfortably in it. The larger 8 inch cylinder required a larger mount. The styrofoam was then mounted on top of a smaller radius styrofoam cylinder with a metal plate on the bottom which could then be attached to the metal pedestal. Internal to the low RCS pedestal are mechanical devices that enable the target to be rotated in azimuth. The cylinder is physically rotated about the cylinder axis to generate the roll angle. The azimuth angle reference was determined by making
Figure 60: Finned cylinder setup for measurement in the OSU/ESL compact range.
an angle sweep measurement on the cylinder and identifying the angle at which the
data peaks. This is the angle at which the wave hits the cylinder broadside. This
angle is set to be 90°, and all azimuth angles, denoted as θ in the discussions that
follow, are referenced accordingly. The roll angle, denoted as φ in the discussions
that follow, was measured with a protractor as shown in Figure 61. This was a
difficult measurement to make since it required keeping the protractor level on the
fin. However, not only was the fin quite a bit smaller than the protractor, but it
was also often attached to the cylinder by copper tape which made keeping the
fin still a difficult task. This is an important point which is commented on later
in this chapter. Measurements were made by physically fixing the roll angle and
then rotating the target in azimuth. If only 1-D time domain data is required for
processing, then the data is taken only at select aspect angles. However, if an inverse
synthetic aperture radar (ISAR) [38] image is desired then the azimuthal increments
must be small enough so that the entire target is properly scanned. In our case,
the spatial sampling increments were Δθ = 0.25°. Typically, when obtaining data
to construct an ISAR image, the target is rotated through several different aspect
angles as shown in Figure 62, and at each angle, the radar is swept through a band
of frequencies to find the frequency response of the target. In our case, the frequency
range over which the target response was measured was 2 to 18 GHz, incremented
in 40 MHz steps.

The measured data must be calibrated to remove background clutter. The cali-
bration process involved measuring a reference target with a known RCS, in this case
a 6 inch metallic sphere. Two background measurements were also needed, one taken
immediately after the sphere response was obtained and the sphere removed. The
second background measurement was taken after the target response was obtained
and the target removed. The calibrated target response (Tc) was then computed by
Figure 61: Setup for measurement of roll angle.
the following equation (see [34, pp. 27–28] for reference)

\[ T_c = \left( \frac{T - TB}{S - SB} \right) S_E \]  

(8.1)

where \( T \) is the measured target response, \( TB \) is the target background (target is removed), \( S \) is the measured sphere response, \( SB \) is the sphere background (sphere is removed), and \( S_E \) is the exact solution for the RCS response of a sphere.

The hybrid calculations are for an infinitely-long cylinder; whereas, the measured data corresponds to a finite length cylinder. In this dissertation, this difference is eliminated by two methods. The measured data for the single finned cylinder in Section 8.4 was generated by Huan–Wan Tseng. He used subtraction to take out the cylinder effect. This is discussed further in Section 8.4. In the case of the 3-finned cylinders, measured by the author with help from various lab personnel, the cylinder effect was taken out by time-gating. This is discussed further in Section 8.5.

8.4 Infinitely-long Cylinder with 1 Fin

8.4.1 Measurement Setup

The analysis of the finned cylinder shown in Figure 63 was prompted by a
Figure 63: Actual cylinder measured for the Swedish Defense Material Administration. (a) Front view of the finite length cylinder with a single blade at its center with two bolts on either side in the ±z direction. (b) Top view of ogival-shaped blade with a 1 cm. × 6 cm. slot parallel to it on either side and two cylindrical bolts whose diameter is 0.9 cm. This all sits on a elliptical base. (c) Side view.
contract with the Swedish Defense Material Administration. The measurements were performed by Huan-Wan Tseng in the OSU/ESL compact range. The finned cylinder consists of a single polygonal blade aligned along the axis of a hollow cylinder with two cylindrical bolts on either side of the fin at approximately ±5.40 cm in the \( z \) direction. Although not obvious from Figure 63, the cylinder is much larger than the fin. As seen from the top and side view, the blade has some thickness to it. Its cross-section is ogive-shaped and becomes progressively thicker toward the base. As seen in the top view, there are slots in the antenna base on either side of the fin which are parallel to the fin; however, they were covered with copper tape because their contribution to the scattered field is not able to be included in the calculated results. Unfortunately, the contribution of the two bolts was included in the measurements but not in the calculated results. The blade sits on top of an elliptical base which is flush-mounted into an ogive-shaped cover plate with an elliptical hole cut out in the center (refer to Figure 64). The cover plate lays on top of the cylinder such that it is conformal to the cylinder. Copper tape is placed over the edges of the elliptical hole in order to minimize the diffraction caused by the transition from the antenna base to the cover plate. Copper tape is also used around the edge of the cover plate. Since the cover plate has some thickness to it, the transition to the cylinder is not smooth; thus, there may be some scattering from the edge. For further discussion on the blade-cylinder geometry and its construction, the reader is referred to [35].

The basic setup and measurement procedure was described in Section 8.3. Of interest was the fin scattering from 2-18 GHz. The scattering from the cylinder was removed via subtraction. First the cylinder with the blade/bolt structure was measured. Then the blade/bolt structure and the cover plate was removed. An ogive-shaped cover plate, without the elliptical hole, was then attached to the cylinder by copper tape in the position of the original cover plate. This geometry was
Figure 64: Blade on top of cylinder with end–cap treatment [35].
then measured, and the measurement was then subtracted from the first in order to obtain the blade and bolt scattering. The accuracy of the second measurement is crucial in order to obtain good subtraction. ISAR images generated from the measured data showed that the cylinder end effects were effectively subtracted out. Even if they were not, because the cylinder is so long, any contribution could be easily gated out. At least the subtraction helps minimize contamination of the peak due to the fin scattering by the sidelobes associated with the scattered field from the cylinder. However, the ISAR images do show that there is still some residual contribution from the edge of the cover plate which was not subtracted out. This will be illustrated in the next section. For more details on the measurement setup and the data obtained, the reader is referred to [35].

8.4.2 Theoretical Analysis

This is a perfect test case for the hybrid code developed here because the surface area of the cylinder is much larger than the surface area of the fin. This coincides with the assumptions of this dissertation that the objects of interest are those with an electrically large main body and electrically small appendages. In addition, because the cylinder is so large and the blade is so small no other program is presently available which can analyze the combination. The moment method program developed here at OSU, ESP4 [39], cannot handle the problem because even at the lowest frequency of interest, the method requires at least 6000 unknowns. The size of the cylinder limits the use of the conventional MM. On the other hand, the blade is so small, especially at the lower frequencies, that a high frequency scattering program like NEC-BSC [40], also developed at OSU, cannot analyze the combination either. The corners of the blade are too close for the program to distinguish them as separate scattering centers. However, this hybrid method is especially suited to handle such a
Table 5: Comparison of the number of unknowns required for the MM (ESP4 [39]) and for the hybrid method to analyze the single finned cylinder shown in Figure 63 over the frequency range of interest.

<table>
<thead>
<tr>
<th>freq (GHz)</th>
<th>λ (m)</th>
<th>modes required by ESP (approximate)</th>
<th>modes required by hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.15</td>
<td>6000</td>
<td>6</td>
</tr>
<tr>
<td>18</td>
<td>0.0167</td>
<td>400,000</td>
<td>135</td>
</tr>
</tbody>
</table>

case: an electrically large cylinder with an electrically small appendage. Accounting for the presence of the cylinder by a special Green’s function which is approximated by the UTD and its extensions allows for the removal of the basis functions on the entire cylinder surface which accounts for over 99% of the unknowns in the conventional MM. The unknowns on the fin are left intact. A comparison of the number of unknowns required for ESP4 versus the number needed by the hybrid method is shown in Table 5. In this case, where the surface area of the cylinder is much larger than that of the fin, the savings incurred by using the hybrid method is substantial.

Savings alone would not justify use of the hybrid method. A correct answer that takes days to compute is still more desirable than a quickly generated incorrect result. Thus, the accuracy of the hybrid method will be shown. The following plots show the radar cross section (RCS) of the fin determined by measurements and by the hybrid method for various incident angles over a frequency range of 2–8 GHz. The measured geometry was the one shown in Figure 63. Since only the blade scattering was desired, the cylinder end cap effect was also time-gated out, in addition to the subtraction that was previously described, to insure that the end cap effect was removed. The geometry that was analyzed is shown in Figure 65.
Figure 65: Infinitely-long finned cylinder used for hybrid and pseudo-hybrid calculations.

Note that, for simplicity, the blade was approximated by a square blade and the bolts were not included. Results from computer runs made using a square blade did not differ very much from those made using a polygonal blade. This was the approximation used in the original attempts to analyze the geometry [37] and are continued here. The exclusion of the bolts will be addressed later. As mentioned before, the large size of the cylinder restricts the use of the MM for the entire body. However, an initial attempt to analyze this geometry was performed by H. Tseng using the available tools at the time. He approximated the cylinder by an infinite ground plane and then used ESP4 to solve the resulting image theory problem — the polygon-shaped fin along with its flat plane image radiating in free-space. When available, this result is labeled “ESP/image”. Figure 66 shows the RCS for the fin if the incident angle is $\theta_i = 70^\circ$ and $\phi_i = 80^\circ$ where $\theta$ and $\phi$ are the usual spherical coordinates measured from the $z$ and $x$ axis, respectively, which are shown in Figure 65. This implies that the plane wave is incident almost at broadside to the fin.
Comparing the measured result with the ESP result shows that, in general, this flat plane approximation generates a result which gives the correct trend of the electric field as a function of frequency. However, it is shifted in magnitude by as much as 10 dB from the measurements. An alternate method is thus required.

A hybrid procedure appears to be a useful method of attack in this case due to the large size of the cylinder. A preliminary attempt at solving the problem using the hybrid method was performed by including the presence of the cylinder curvature only in the excitation vector, \( V \), as described in (2.9b). Since the hybrid code utilizes the reciprocal relationship between the excitation vector and the far-zone scattered field vector, the cylinder curvature is also accounted for in the calculation of the far-zone scattered fields. The curve generated using this method is labeled as "pseudo-hybrid". Comparing the pseudo-hybrid result with the ESP result in Figure 66 shows the large change in the RCS caused by accounting for the curvature of the cylinder in the analysis even if it was only included in the \( V \) vector calculation. This marked improvement can be seen by examining the change in the \( V \) vector caused by the inclusion of the curvature effect. The excitation vector, \( V \), involves the field due to a source in the far-field. Thus, the source will see the curvature of the cylinder no matter how large the cylinder may be. As a result, the excitation vector is expected to be significantly affected by the inclusion of information regarding the cylinder radius. Table 6 shows a comparison of the elements of the excitation vector calculated by an eigenfunction solution, image theory, and the hybrid method. Note that the results generated by the hybrid method is equivalent to those generated by the pseudo-hybrid method since the treatment of the excitation vector is the same in both. Note that \( V_2 \) and \( V_6 \) have been omitted because they are the same as \( V_1 \) and \( V_5 \), respectively. By examining Table 6, it can be seen that including the curvature of the cylinder results in much more accurate answers than using
Figure 66: RCS of the single fin shown in Figure 63 versus frequency ($\theta_i = 70^\circ$, $\phi_i = 80^\circ$).
Table 6: Comparison of the elements of the $V$ vector using field values calculated by an eigenfunction solution (referred to as the "exact" solution), image theory, and the hybrid method for Figure 63 at 2 GHz. The patching is the same as in Figure 56.

| $| V_i |$ |
|---|---|---|---|---|
| Using | 1 | 3 | 4 | 5 |
| exact  | 0.00201 | 0.00942 | 0.00322 | 0.00104 |
| image  | 0.00116 | 0.00539 | 0.00181 | 0.000586 |
| hybrid | 0.00203 | 0.00978 | 0.00323 | 0.00105 |

flat plane image theory. The results generated by including the cylinder curvature (labeled "hybrid") compare much better with the exact values of $|V_i|$. This is the source of the improvement in the pseudo-hybrid results over the ESP results in Figure 66.

The hybrid method which accounts for cylinder curvature in both the $Z$ and $V$ calculation generates a curve, labeled "hybrid" in Figure 66, which is only slightly different from the curve generated by the pseudo-hybrid method which accounts for cylinder curvature only in the $V$ vector. Inclusion of curvature into the UTD calculations in the impedance matrix is not expected to significantly affect the results because the fin is small. Since, this cylinder is very large with respect to the fin, a source and observer on the fin do indeed see the cylinder as a ground plane. Thus, replacing the cylinder with an infinite ground plane is a good approximation for this particular geometry.

This particular finned cylinder structure is difficult to analyze because modeling the geometry accurately is difficult. Inclusion of the bolts, as mentioned earlier, was
not done. Because they are small, it is extremely hard to mesh them up accurately. Another option was to account for the effect of the bolts separately. One could determine the RCS from the two bolts alone over a ground plane and then add it into the fin scattering. However, this does not include any interaction between the blade and the bolts. An alternative to including the bolts in the theoretical calculation is to remove their effect from the measured data. Because their location on the cylinder is quite close to the blade, the scattering from them cannot be isolated in the one-dimensional time domain data which only gives down range resolution, i.e., along the direction of the incident wave. However, a 2-D ISAR image provides both cross range and down range resolution because it uses not only frequency data but also angular data. An ISAR image of the data for $\phi_i = 80^\circ$ using all the measured data ($-5^\circ$ to $95^\circ$) was created. Using data from all the aspect angles gives an image which depicts the average of the target response over all the angles. The grey scale plot for the $\hat{\theta}\hat{\theta}$ (HH) polarization is shown in Figure 67. Recall from Figure 62 that the cross range and down range directions are defined to be perpendicular and parallel, respectively, to the direction of propagation of the wave. However, in Figure 67 the axis labels are defined with respect to the target. Thus, if the top view of the fin structure was overlayed on top of the image, the blade would lie parallel to the cross range axis. The data for $\theta = 0^\circ$ is the target response for a plane wave incident along the minus cross range direction while $90^\circ$ corresponds to a plane wave traveling in the minus down range direction. As a consequence, the dark circle centered at approximately ($-2.4 \text{ in.}$, $0.5 \text{ in.}$) is the scattered field associated with the front edge of the blade, and the dark circle centered at approximately ($-2.3 \text{ in.}$, $-0.75 \text{ in.}$) is the scattered field associated with the back edge of the blade. This hypothesis is confirmed by noting that the cross range separation between these two scattering centers is about $1.25 \text{ in.} \approx 3.175 \text{ cm}$. This is approximately the distance between
Figure 67: Single finned cylinder image ($\phi_i = 80^\circ$, $\theta_i = 45^\circ \pm 50^\circ$). $\hat{\theta}$ (HH) polarization.
the edges of the blade. The lighter circle centered at about \((-0.75 \text{ in.}, -1.8 \text{ in.})\) is further in down range than the other scattering centers; therefore, it is associated with a higher order scattering mechanism, e.g., creeping waves. The important point to glean from this image is that the bolts do not contribute significantly to the total field for this polarization. Using this dynamic range (30 dB), there is no visible scattering center that could be associated with either bolt. Thus, any difference between the measured and the calculated data for this polarization is probably not due to the exclusion of the bolt contribution in the calculated results. However, the scattering from the two bolts can be seen in Figure 68 which shows the ISAR image for \(\hat{\phi}\hat{\phi}\) (VV) polarization. The peak value at \((-2.4 \text{ in.}, 0.5 \text{ in.})\) is the scattered field associated with the front edge of the blade. For this polarization the scattering from the back edge is much weaker, almost 30 dB down. The response at \((-2.4 \text{ in.}, 2.0 \text{ in.})\) is from the front bolt, while the response 4 inches farther in cross range and at almost the same down range position is due to the rear bolt. This is consistent with the physical location of the bolts with respect to each other and to the blade. Again, the response behind the blade in the down range is related to higher order mechanisms. The two responses farther from the blade, at a cross range of \(\pm 6 \text{ inch}\) is scattering from the edge of the ogive-shaped cover plate. Thus, the subtraction was not able to completely eliminate this edge diffracted field. It can be concluded from these two plots that the HH result is basically the fin response; whereas, the VV result is the fin response coupled with the scattering from the bolt and the ogive. In order to do an accurate comparison of the measured response to the calculated response, these extra mechanisms must be removed. In order to do this, another image for VV is created; however, only angular data from \(\theta = 70^\circ \pm 25^\circ\) is used, since \(70^\circ\) is the angle of interest. The result is shown in Figure 69. The scattering centers are distinguishable enough such that the bolt and ogive response can be
Figure 68: Single finned cylinder image ($\phi_i = 80^\circ$, $\theta_i = 45^\circ \pm 50^\circ$).  $\hat{\phi}\hat{\phi} (VV)$ polarization.
Tomographic Processing, HAMMING down range, HAMMING cross range
Original Freq. (GHz): Fmin = 2.002  Fmax = 18.002  DeltaF = 0.020
Used Freq. (GHz): Flo = 2.002  Fhl = 18.002
Original Asp. (deg.): Tmin = -45.00  Tmax = 5.00  DeltaT = 0.25
Used Asp. (deg.): Tlo = -45.00  Thi = 5.00
Far Field, Tbisstatic = 0 deg.

Figure 69: Single finned cylinder image ($\phi_i = 80^\circ$, $\theta_i = 70^\circ \pm 25^\circ$).  $\phi\phi$ (VV) polarization.
accurately gated out. The gated result is shown in Figure 70. This gated result was transformed back to the frequency domain and is the curve labeled "measured" in Figure 66b. It should be noted that the result at 2 GHz, the end frequency, is not accurate due to the processing limitations which require the use of data consisting of zeros outside the measured frequency range of 2-18 GHz. This effect is often referred to as "Gibb's phenomena."

Another difference between the actual structure that was measured and the model of the finned cylinder used to generate the hybrid results was the thickness of the blade. As seen in Figure 63b and 63c, the actual blade has not only thickness but is also curved. However, the theoretical model uses a flat plate. At 12 GHz, the bottom of the blade is already a quarter of a wavelength thick, not a thin plate. A cursory study of the effect of modeling errors shows that at the lower frequencies, the thick blade can be modeled by a thin flat plate. Since the goal of this dissertation is to show the accuracy of the hybrid method, it suffices to compare the hybrid result with the measured result only at the lower frequencies. Any discrepancies at the higher frequencies are due mainly to blade modeling errors in either thickness or curvature or both. This is most important for HH polarization because, as seen in the ISAR image (Figure 67), the dominant scattering centers for this polarization are the blade edges. If their position relative to each other is not modeled correctly, the high frequency results will be greatly affected. Therefore, although measurements were taken from 2-18 GHz, the result from 8-18 GHz has been neglected for now. Future studies should account for the thickness and curvature of the blade.

Figure 71 shows the RCS of the fin at \( \theta_i = 30^\circ \) and \( \phi_i = 80^\circ \). ESP results for the equivalent flat plane problem were not available. The pseudo-hybrid results are not as good as the hybrid ones for the \( \hat{\theta}\hat{\theta} \) (HH) polarization in this case. However, it is noted that the RCS for the VV polarization, for which pseudo-hybrid is as
Figure 70: Single finned cylinder image \((\phi_i = 80^\circ, \theta_i = 70^\circ \pm 25^\circ)\) with response due to scattering from the bolts and the ogive gated out. \((\mathbf{\hat{\phi} \hat{\phi}})\) (VV) polarization.
Figure 71: RCS of the single fin shown in Figure 63 versus frequency ($\theta_i = 30^\circ$, $\phi_i = 80^\circ$).
accurate as the hybrid method, is dominant at the frequencies (2–8 GHz) where pseudo-hybrid has been calculated. The \( \phi \) angle is the same as in the previous case; however, \( \theta \) is smaller. Thus, the plane wave is incident closer to nose on. From the ISAR images (Figure 67 and 68), it was concluded that the bolts only affected the data for VV and not HH. As a result, the scattering associated with the bolts was time-gated out from the image data. As in the previous case, their contribution produced a distinct peak; therefore, they could be easily gated out. The gated time domain data was then transformed back to the frequency domain. The result is the curved labeled “measured” shown in Figure 71b. Again, because of the finite bandwidth effect, the measured result at 2 GHz and 18 GHz is inaccurate. This is confirmed by looking at the original raw data, i.e., before any processing was done. For VV, at 2 GHz, the raw data predicts a RCS value of approximately -18 dbsm, which matches with the hybrid result. For HH, at 18 GHz, the raw data is, again, closer to the hybrid result than the post-processed data. The difference in the location of the null in the measured data for HH and that predicted by the hybrid method is attributed to differences in the model of the blade that the hybrid method analyzes and the actual blade that was measured. Overall, the agreement between the measured RCS and that predicted by the hybrid method is very good.

Figure 72 shows the RCS of the fin for an incident wave impinging more at broadside to the fin; i.e., \( \theta_i = 80^\circ \) and \( \phi_i = 40^\circ \). Again the hybrid result follows the general trend of the measurements. The fast ripples in the measured data is caused by the bolts which have been excluded in the geometry analyzed by the hybrid method. As before, comparison between the raw data and the post-processed measured data show that there is a significant error in the post-processed measured data at 18 GHz for HH and at 2 GHz for VV. This is unavoidable at the present time due to the finite bandwidth problem. The difference in the location of the peak at 4
Figure 72: RCS of the single fin shown in Figure 63 versus frequency ($\theta_i = 80^\circ, \phi_i = 40^\circ$).
GHz in the $\hat{\theta}\hat{\theta}$ (HH) polarization predicted by hybrid and by the measured data is
due to the approximation of a planar fin in the calculated results. Preliminary test
runs show that accounting for the blade thickness would shift the peak appropriately.
However, overall, the agreement is very good.

Figure 73 shows the RCS of the fin at an incident angle closer to nose on,
$\theta_i = 10^\circ$ and $\phi_i = 60^\circ$. The hybrid method is able to reproduce the general trend
of the measurements. As for the other nose on case, pseudo-hybrid is as accurate
as hybrid only for the more dominant $\hat{\phi}\hat{\phi}$ (VV) polarization. Again, there seems to
be more ripples in the measured data than in the curve generated via the hybrid
method. Looking at the frequency of the ripples, this is again attributed to the
presence of the bolts. Note that the bolts are visible for both polarizations. Again,
an examination of the raw data indicates that the post-processed measured data,
shown in Figure 73, is in great error at 2 GHz for both polarizations. The raw data
predicts, as hybrid does, that the RCS starts out high at 2 GHz and becomes smaller
for higher frequencies. Unfortunately, due to time constraints, the bolts were not
gated out of the ISAR image for this look angle. However, based on past data, it is
expected that the measured data with the bolt effect removed would have less ripple
from 7–18 GHz, as the hybrid method predicts.

8.5 Infinitely-long Cylinder with 3 Fins

The single-finned cylinder of Section 8.4 shows the usefulness of the hybrid
(MM-UTD) method not only because of the large reduction of unknowns over the
conventional MM but also in its accuracy over flat plane image theory, the only
other analysis tool available to the author at present to handle the problem. For the
single-finned cylinder, the major gain over flat plane image theory was made when
Figure 73: RCS of the single fin shown in Figure 63 versus frequency ($\theta_i = 10^\circ, \phi_i = 60^\circ$).
the cylinder curvature was accounted for in the excitation vector. The following
3-finned cylinders are better examples of the necessity of the full hybrid method
rather than the pseudo-hybrid method. Unlike the previous single-finned cylinder,
flat plane image theory cannot be used in this case in the $Z$ matrix calculation.
Curvature must be included in order to obtain the mutual impedance between the
fins. Thus, the full hybrid method is required under these circumstances.

The measurement setup used to determine the RCS of the following two 3-
finned cylinders was shown and described in Section 8.3. In the first case (an 8 inch
diameter cylinder), the frequency sweep was done only at a few select aspect angles.
For the second cylinder, more aspect angles were sampled in order to produce an
ISAR image. In both cases, each fin was a piece of flat metal that was attached to
the cylinder by copper tape. After the measured data was obtained, it was processed
using the Radar Data Processing Program (RDP) developed at the ElectroScience
Lab. Since only the RCS of the fins were of interest, the cylinder end effect was
removed with RDP. RDP was used to transform the frequency domain data into the
time domain in order to more easily discern certain scattering mechanisms. Since
the cylinder is quite long, the effect of diffraction off the cylinder end caps was easily
identified and time gated out. The peaks at time zero were expected to be the fin
scattering and were thus gated and transformed via the fast fourier transform (FFT)
back to the frequency domain. In all the plots that follow, the end frequencies, 2
GHz and 18 GHz, are not accurate because of the processing limitations caused by
finite bandwidth (Gibb's phenomena).

8.5.1 3-Finned Test Cylinder

Figure 74 shows the geometry of interest: a cylinder with three fins. The
cylinder radius and length are 4 inches and 40 inches, respectively. Although not
Figure 74: 3-finned cylinder used to evaluate the hybrid method. A large, hollow cylinder with three small fins at $\phi = 0^\circ$, $90^\circ$, and $180^\circ$ which are all centered at $z = 0$. 
easily seen from the figure, the surface area of the fins is much smaller than that of the cylinder. Since this was meant as a test case, the fins are relatively small in order to keep the number of unknowns relatively low and thus not make the problem too computer intensive since a frequency sweep is involved. All three fins are centered at the middle of the cylinder axis. This was done to facilitate checking of the calculated results. The symmetry of the three fins to each other made checking the impedance and excitation matrix much easier.

Figure 75 shows the RCS of the 3 fins versus frequency given a plane wave incident at \( \theta_i = 60^\circ \) and \( \phi_i = 170^\circ \). The agreement is better for the \( \hat{\phi}\hat{\phi} \) (VV) polarization. For the \( \hat{\theta}\hat{\theta} \) (HH) polarization, the results generated by the hybrid method seem to be shifted in frequency compared to the measured data. This seems to indicate a possible alignment problem with the fins since the peaks and nulls are caused by interference between the fins. The fins are attached to the cylinder by copper tape, and as a result, they may not be exactly perpendicular to the surface of the cylinder as assumed in the model being analyzed. In addition, \( \phi_i \) was measured with an analog protractor referenced to the flat surface of the fin (see Figure 61). Not only was it difficult keeping the fin still, but the protractor was not as accurate as the digital one used in the next set of measurements. Therefore, the value of \( \phi_i \) may have a slight error in it. These are relatively small problems associated with the difficulty of setting up the measurement. They are somewhat expected and inherent in the process. However, because the RCS levels that are being looked at are quite low (in the \(-25\) to \(-30\) dB range) and the structure of the plot indicates an interference mechanism, even small setup errors can translate into significant errors in the resulting scattered field. These slight perturbations could contribute to the inability of the hybrid result to replicate \textit{exactly} the measured data.
Figure 75: RCS of the 3 fins shown in Figure 74 versus frequency ($\theta_i = 60^\circ, \phi_i = 170^\circ$).
With regards to the measured data, the result achieved after processing the data for the HH polarization was found to vary significantly with the time gating used. The size of the time gate affects whether some higher order terms are or are not included in the results. The source of these terms was difficult to identify; thus, it was difficult to determine the exact size of the gate to use. An ISAR image may have helped but was not available at the time. This indicates a possible source of error in the HH measured results. Again, the measured data at the ends of the frequency band is not accurate because of the finite bandwidth problem.

Overall, for both co-polarizations, the agreement between the measured data and the hybrid result is good. The hybrid method correctly predicts the trend. Knowing these possible sources of error, gives us confidence in the results generated via the hybrid method.

Figure 76 shows the results for an incident wave with $\theta_i = 60^\circ$ and $\phi_i = 140^\circ$. This wave impinges on the cylinder at an angle farther from broadside to the fin than the previous result. Again, the results seem better for the $\phi\phi$ polarization. However, overall the hybrid results, though not as good as the previous near-broadside results, do seem to predict the trend of the measured data. Also, the same finite bandwidth effect is seen in the measured data as seen before. Although this cylinder has three fins attached to it, the results are not those of three fins interacting. The cylinder radius is large enough that the fins at $\phi = 0^\circ$ and $\phi = 180^\circ$ do not see each other. Thus, the results in Figures 75 and 76 represent the interaction between two widely-separated fins. This is represented by the wide lobes in the RCS plot.

### 8.5.2 3–Finned Cylinder #2

The finned cylinder in Section 8.5.1 had relatively small fins compared to the size of the cylinder. In order to both test the code further and to generate a more
Figure 76: RCS of the 3 fins shown in Figure 74 versus frequency ($\theta_i = 60^\circ, \phi_i = 140^\circ$).
Figure 77: Second 3-finned cylinder measured and analyzed. A 4 inch, hollow cylinder with 3 fins at $\phi = 0^\circ$, $90^\circ$, and $180^\circ$. The two side fins are centered at $z = 0$ while the top fin is 4 inches from the endcap.

realistic looking geometry, the fin size is increased, the cylinder radius is decreased, and the top fin at $\phi = 90^\circ$ is moved toward the endcap. The geometry is shown in Figure 77. The first change, increasing the fin size, has the effect of increasing the size of the impedance matrix. For this number of unknowns the CPU time is still dominated by the fill time. Creation of results for a frequency sweep from 2–18 GHz required a large amount of CPU time for the matrix fill especially at the higher frequencies. The second change, decreasing the radius of the cylinder, allows for more stringent testing of the UTD. The larger the cylinder, the more likely it is that image theory would suffice. The third change, moving the top fin, was merely to make the geometry have a more aircraft like shape.
The second 3-finned cylinder that was analyzed had a length of 20 inches. This length was chosen in an attempt to mimic a more realistic geometry at the frequencies of interest (2-18 GHz). The RCS for the three fins if the plane wave is incident at an angle of $\theta_i = 20^\circ$ and $\phi_i = 45^\circ$ is shown in Figure 78. At the lower frequencies, the agreement between measurements and the hybrid calculation is very good. However, at the higher frequencies there seems to be a shifting of the peaks and nulls rather than a disagreement. This could be due to many things. The most likely cause is that the fins are not exactly at the location and orientation that we expect. As before, the fins are attached to the cylinder by copper tape. Each fin is assumed to be normal to the cylinder along one of its edges and parallel to the cylinder axis along the other edge. Although care was taken to insure that the fins were such, small errors in the placement of a single fin are possible. This is very important because the lobing structure, caused by interference between the plates, is very sensitive to the orientation of the fins. Even small errors in the placement of a single fin alone could account for significant errors in the interference pattern. Since the radius of the cylinder is smaller and the fins are larger for this 3-finned cylinder than that for the geometry in Section 8.5.1, there is more interaction between these fins than in the previous case. This accounts for the finer lobing structure exhibited in these results compared to those of Section 8.5.1. As a consequence, exact placement of the fins is more crucial in this case.

In addition, the problem with obtaining $\phi_i$ accurately was minimized but not eliminated by using a digital protractor. The protractor base rested on the flat side of the plate (refer to Figure 61). Then the angle was read. The reading fluctuated quite a bit, and it took a great deal of care to obtain the angle. Although the fluctuations were no greater than two to three degrees, such a deviation may contribute to the offsetting of the peaks and nulls.
Figure 78: RCS of the 3 fins shown in Figure 77 versus frequency for a cylinder of length $L = 20$ in. ($\theta_i = 20^\circ, \phi = 45^\circ$).
Another possible problem with the measurement data is that, at this incident angle, there may be possible target-mount interaction problems. Since the feed horn position is fixed and the finned cylinder rests on top of a foam mount on top of a pedestal, $\theta_i$ is varied by rotating the pedestal, but $\phi_i$ is varied by physically rotating the cylinder about the $\hat{z}$-axis. Refer to Figure 60. As it turns out, for $\phi_i = 45^\circ$, the side fin almost touches the mount so there may be interactions between the mount and the fin which cannot be taken out by the time gating nor has it been included in the hybrid calculation. In order to check this, more measurements were performed using the same fins but with a different length cylinder. The main reason to lengthen the cylinder was to be able to put the foam mount such that it was not directly under the fins, thus eliminating any significant mount interaction. This enables the mount to be more easily identified in the time domain measured data.

For the second set of measurements, the overall geometry remains the same as in Figure 77 except that the cylinder length is $L = 30$ inches. The placement of the fins relative to each other remains the same as does the distance from the top fin to the edge. The size of the fins and the cylinder radius remain the same. Thus, the hybrid calculations remain the same. The comparison with measurements is shown in Figure 79. The agreement between the RCS predicted by hybrid and that measured is even better with this new set of measurements. In this case, the peaks and nulls of the two curves coincide almost exactly. In this second set of measurements, greater care was taken with the alignment of the fins, their orientation, and the measurement of the angles of incidence because the results were believed to be very sensitive to these parameters. However, perfection is very difficult to accomplish as mentioned earlier. Care in the measurement setup is crucial. Seeing how sensitive the measurements are to fin orientation and thus how difficult it is to reproduce even measurement
Figure 79: RCS of the 3 fins shown in Figure 77 versus frequency using the longer 30 inch cylinder ($\theta_s = 20^\circ, \phi = 45^\circ$).
data, we are pleased with the agreement shown between the hybrid calculations and the measured results.

8.6 Conclusion

This chapter demonstrated the usefulness and accuracy of the hybrid method on various finned cylinders – a single-finned cylinder, a large radius 3 finned cylinder, and a small radius 3 finned cylinder. In all three cases, because the surface area of the fin was quite small in comparison to the surface area of the cylinder, using the hybrid method resulted in a significant reduction in the number of unknowns required compared to the number needed if the conventional MM was used. This was a consequence of being able, in the hybrid method, to remove the basis functions on the cylinder surface which can account for as much as 99% of the total number.

In the single-finned cylinder case, a flat plane approximation was able to be used. The resulting fin and image in free-space was analyzed using ESP4. Although the results correctly predicted the trend of the RCS from 2-18 GHz, the magnitude was incorrect. This case showed the significant improvement in accuracy achieved by accounting for the cylinder curvature in the excitation vector calculation. Although inclusion of this information in the impedance matrix calculation does not produce any noticeable improvement for the single finned cylinder case, the full hybrid method is required when analyzing the 3 finned cylinder structure in order to obtain the mutual impedance between the fins.

In all the structures examined, the hybrid method often did not analyze exactly the same geometry as was measured. In the single-finned cylinder case, bolts were included in the structure that was measured, but not in the geometry that was analyzed. In addition, the blade had some thickness to it and its face was curved.
The calculated results all assumed a thin flat plate. In the 3-finned cylinder cases, perfect fin orientation with respect to the cylinder and to each other was difficult to achieve. This caused a shifting in the peaks and nulls of the RCS versus frequency plot. Mount interaction with a fin was a problem; the remedy was a longer cylinder with the fins moved closer to the endcap and, as a result, away from the mount. Small inaccuracies in the incident angle measurement may also have accounted for discrepancies between the calculations and the measurements. When using the measured data to judge the hybrid calculations, the existence of these sources of errors should be taken into account. Doing this, we conclude that the hybrid method is a useful and accurate tool.
CHAPTER IX
Conclusion

This dissertation has presented a hybrid method (MM-UTD) to treat the EM scattering from complex structures. The prime motivation for this work was the desire to be able to analyze an object comprised of an electrically large PEC convex body with electrically small PEC appendages. However, this hybrid method can also be applied to deal with electrically large appendages with reduced efficiency, but it would still be more efficient than just a conventional MM treatment of the latter case. In addition, the present development can be extended to treat non-conducting appendages.

In the MM-UTD method, the MM was used to reduce the governing integral equation into a system of linear equations. Use of a special Green's function accounted for the large convex scatterer. As a result, the MM basis functions resided only on the appendages. Thus, the size of the impedance matrix is reduced by the number of MM basis functions that would have resided on the large scatterer. This is an extremely large savings in our case because the scatterer was assumed to be electrically large. This special Green's function represents the electric field at an arbitrary location due to an arbitrarily-located source in the presence of the perfectly conducting convex body without the appendages. Since the scatterer is electrically large, the special Green's function is approximated using the UTD. However, previous UTD solutions pertinent to this case existed only for the case of the source and the observer both on the surface of the body or the source and/or the observer
in the far field of the body. Lacking, but necessary, was the ability to determine the near field of a source close to but not necessarily on the body. Two solutions to this problem— the EUTD coupling solution and the EUTD radiation solution—were developed in this dissertation. Since there are no definite error bounds on the UTD equations and no exact eigenfunction solution based code was readily available, blending was used to verify the new equations. Numerous test runs showed such blending did exist between various equations. Determining the empirically-derived bounds on the various UTD equations encompassed a large part of this research effort. Chapter VII presented the details, and Appendix G summarizes the conclusions in a flowchart. This flowchart was included because it is a large part of the hybrid code which was too unwieldy to include here.

In the implementation of this hybrid scheme, the basis and test functions used were the same—piecewise sinusoids. Not only are they efficient when used in the conventional MM free-space computations, but they are extremely well-suited for use in this particular hybrid scheme. Chapter V showed that using them eliminates an integration in the calculation of the $\Delta Z$ matrix, and they provide the most accurate results when used with the UTD.

Numerical results were compared with measurements for various finned cylinders. The necessity for such a technique is illustrated by the first example: a cylinder with a single fin representing a blade antenna. No existing codes were able to analyze this geometry in an efficient manner. The conventional MM would have required, at minimum, 6000 unknowns over the frequency range of interest. This would have required a much larger and faster computer than the author has access to at the present time. Prior to the development of the hybrid code, the single fin geometry could only be analyzed by approximating the cylinder by a flat ground plane. This method generates a result which gives the general trend of the RCS of the fin versus
frequency but not necessarily the correct magnitude. Accounting for the cylinder curvature only in the excitation vector (pseudo-hybrid) translated into improved accuracy. The full hybrid method, i.e., including cylinder curvature in both the impedance matrix and the excitation vector calculations, did not improve the accuracy by too much in this case because the fin is electrically small. However, the full hybrid method is required in the 3 finned cylinder cases in order to obtain the mutual impedance between the fins. The accuracy of this technique was shown in the analysis of the 3 finned cylinders, especially the smaller radius one. Since the cylinder radius was small, there was a great deal of interference between the fins. This interference was exhibited in the rapid lobing structure of the RCS versus frequency plot. The ability of the hybrid code to replicate this lobing structure demonstrates its accuracy.

The advantage of this hybrid code is its computational speed. Use of a special Green’s function allowed a significant reduction in the size of the impedance matrix. Although calculation of the special Green’s function involves tracing rays and calculating special functions, it was found that this calculation is not exceptionally time-consuming. Thus, using the UTD is not a disadvantage of the method. Instead, what limits the MM is what limits this method; namely, the matrix fill time. The conventional MM could not even be used in most of the cases presented here because of the size of the associated impedance matrix over the frequency range of interest. For the finned cylinders analyzed here, the large surface area of the cylinder limited the use of the conventional MM and necessitated use of the hybrid method. However, as the physical extent of the appendages becomes larger with respect to the wavelength, the impedance matrix associated with the hybrid method may become unwieldy. A large impedance matrix is not a limitation of the hybrid method developed here. As in the conventional MM, if time is not of
the essence and computer resources are available, even large appendages can be analyzed. Theoretically, size is not a limitation. However, practically, a very large impedance matrix has a chance of being ill-conditioned. As a consequence, if the appendages are electrically large, then an extension of this method is desired. An even more general hybrid procedure would use, in addition to the MM subsectional basis functions, UTD-based basis functions over the portions of the appendages not close to discontinuities. This would further reduce the number of unknowns in the MM solution. Such a "super-hybrid" technique would encompass both of the hybrid methods discussed in Chapter I.
Derivation of the Radially Propagating Eigenfunction Solution for the Fields of a Source Radiating in the Presence of a Circular Cylinder

The exact radially propagating eigenfunction solution for the field from a current element in the presence of a circular cylinder of radius $a$ is derived here. These results are used in Chapter III to derive the EUTD coupling solution (refer to equations (3.5) - (3.8)).

A.1 $\hat{z}$-directed Current Element

![Diagram of a line source in the presence of a circular cylinder]

Figure 80: $\hat{z}$-directed line source in the presence of a circular cylinder.
The electric field due to the electric line source in Figure 80 radiating in the presence of a cylinder of radius $a$ is \( [25, 5-122] \),

\[
E_{z}^{2D} = \frac{-k^2 I}{4\omega \varepsilon} \sum_{n=-\infty}^{\infty} H_n^{(2)}(kr) \left[ J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right] e^{j\omega t - j\phi'}
\]

(A.1)

where

\[
\rho = \begin{cases} 
\rho & \text{if } \rho > \rho' \\
\rho' & \text{if } \rho' > \rho
\end{cases}
\]

(A.2)

and similarly

\[
\rho' = \begin{cases} 
\rho & \text{if } \rho < \rho' \\
\rho' & \text{if } \rho' < \rho
\end{cases}
\]

(A.3)

In addition, $\omega$ is the angular frequency and $\varepsilon$ and $k$ are the permittivity and wavenumber of the medium, respectively. This is the result for the two-dimensional case. To obtain the results for the three-dimensional case, the following transformations are used:

1. Let $k^2 \rightarrow k_t^2 = k^2 - k_{z}^2$,

2. \( E_{z}^{3D} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_{z} e^{j\omega (z-z')} E_{z}^{2D} \).

As a result, for a dipole source excitation of magnitude $I$, the $z'$-directed field is

\[
E_{z}^{3D} = -\frac{IZ_0}{8\pi k} \int_{-\infty}^{\infty} dk_{z} \sum_{n=-\infty}^{\infty} e^{j\omega (z-z')} e^{jn(\phi'-\phi')} k_t^2 H_n^{(2)}(k_t \rho)
\]

\[
\times \left[ J_n(k_t \rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k_t \rho) \right]
\]

(A.4)

where the relationship

\[
\frac{Z_0}{k} = \frac{1}{\omega \varepsilon}
\]

(A.5)
was used. The other field components are found by decomposing the problem into a transverse electric (TE) to \( z \) and a transverse magnetic (TM) to \( z \) case. Since \( H_z = 0 \), the TM to \( z \) field is desired. The magnetic vector potential is represented as \( \vec{A} = \hat{z} \psi \) and the electric vector potential is \( \vec{E} = 0 \). The resulting electric field is

\[ E_z = \frac{1}{j\omega e} \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \psi \quad (A.6) \]

Assuming that the wave function \( \psi \) has the same \( z \)-dependence as in \((A.4)\), \((A.6)\) reduces to

\[ E_z = \frac{k^2}{j\omega e} \psi \quad (A.7) \]

Comparing \((A.6)\) with \((A.4)\), it is found that

\[ \frac{\psi}{j\omega e} = -\frac{IZ_0}{8\pi k} \int dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z-z')} e^{-j\phi + jn(\phi - \phi')} \frac{J_n(k \rho \rho)}{H_n^2(k \rho \rho)} \left[ J_n(k_t \rho \rho) - \frac{J_n(k \rho)}{H_n^2(k \rho)} \frac{H_n^2(k_t \rho \rho)}{H_n^2(k \rho \rho)} \right] \quad (A.8) \]

All the other components of the electric field can now be found. Specifically, the \( \hat{\rho} \)-component is [25, 5-18]

\[ E_\rho = \frac{1}{j\omega e} \frac{\partial^2 \psi}{\partial \rho \partial z} \quad (A.9) \]

Substitution of \((A.8)\) gives

\[ E_{\rho z'} = -\frac{jIZ_0}{8\pi k} \int dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z-z')} e^{-j\phi + jn(\phi - \phi')} k_z k_t Q_1 \frac{H_n^2(k \rho \rho)}{H_n^2(k_t \rho \rho)} \left[ J_n(k \rho \rho) - \frac{J_n(k \rho)}{H_n^2(k \rho \rho)} \frac{H_n^2(k_t \rho \rho)}{H_n^2(k \rho \rho)} \right] \quad (A.10) \]

where

\[ Q_1 = \begin{cases} 1 & \rho < \rho' \\ \frac{\partial}{\partial (k \rho \rho)} & \rho > \rho' \end{cases} \quad (A.11) \]

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and

\[ Q_2 = \begin{cases} 
1 & \rho > \rho' \\
\frac{\partial}{\partial (k_\| \rho)} & \rho < \rho' 
\end{cases} \]  
(A.12)

The \( E_\phi \), \( H_\rho \), and the \( H_\phi \) components can be derived similarly from (A.8).

### A.2 \( \dot{\rho} \)-directed Current Element

Reciprocity is used to obtain the electric field due to a \( \dot{\rho} \)-directed current element in the presence of the circular cylinder in Figure 81 from the results for the \( \dot{z} \) current element excitation as shown in Figure 81a. Applying reciprocity on the geometry shown in Figure 81, it is found that

\[
\int \vec{E}^b \cdot \vec{J}^a dV = \int \vec{J}^b \cdot \vec{E}^a dV \tag{A.13}
\]

If \( \vec{J}^a = \dot{z} I^a \delta(\vec{R} - \vec{R}^a) \) and \( \vec{J}^b = \dot{\rho} I^b \delta(\vec{R} - \vec{R}^b) \), where \( \vec{R}^a = (z_a, \rho_a, \phi_a) \) and \( \vec{R}^b = (z_b, \rho_b, \phi_b) \), then (A.13) reduces to

\[
I^a E^b(z_a, \rho_a, \phi_a) = I^b E^a(z_b, \rho_b, \phi_b) \tag{A.14}
\]

where \( E^a_\rho \), the \( \dot{\rho} \) field due to \( \vec{J}^a \) is given in (A.10). Substituting the result in (A.10) into (A.14) yields

\[
E^b(z_a, \rho_a, \phi_a) = -\frac{j I^b Z_0}{8\pi k} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z_b-z_a)} + jn(\phi_b-\phi_a) k_z k_\| Q_2 \left[ J_n(k_\| \rho <) - \frac{J_n(k_\| \rho >)}{H_n^{(2)}(k_\| \rho >)} \right] \tag{A.15}
\]

where \( \rho < \) and \( \rho > \) are defined for the case where \( \vec{J}^a \) is the source. If \( \vec{J}^a \) is located as shown in Figure 81a, then \( \rho < \rho' \) where \( \rho = \rho_b \) and \( \rho' = \rho_a \). However, if \( \vec{J}^a \) is located as shown in Figure 81c, then \( \rho > \rho' \) where \( \rho = \rho_b \) and \( \rho' = \rho_a \). If \( \rho < \) and \( \rho > \)
(a) Given: $\rho < \rho'$

(b) Desired ($\rho' < \rho$)

(c) Given: $\rho > \rho'$

(b) Desired ($\rho' > \rho$)

Figure 81: Reciprocal problem used to obtain the solution for the $\hat{\rho}$-directed excitation from the $\hat{z}$-directed excitation.
are defined for \( J^b \) as the source, then (A.15) is transformed to

\[
E_{z,\rho'} = -\frac{j I^b Z_0}{8\pi k} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} k_z k \kappa \left\{ \begin{array}{l}
e^{jk_z(z'-z)+jn(\phi'-\phi)} H_n^{(2)} (k_l \rho') \left[ J_n^{\prime}(k_l \rho') - \frac{J_n(k_l a)}{H_n^{(2)}(k_l a)} H_n^{(2)} (k_l \rho') \right] \\
\text{for } \rho' < \rho \\
e^{jk_z(z'-z)+jn(\phi'-\phi)} H_n^{(2)}' (k_l \rho') \left[ J_n(k_l \rho) - \frac{J_n(k_l a)}{H_n^{(2)}(k_l a)} H_n^{(2)} (k_l \rho) \right] \\
\text{for } \rho < \rho' 
\end{array} \right.
\]  

(A.16)

This is the \( \hat{z} \)-directed field due to a \( \hat{\rho} \)-directed current element in the presence of a cylinder of radius \( a \).

In order to obtain the \( \hat{\rho} \)-directed field from \( J^b \), the problem is decomposed into the TE to \( z \) and the TM to \( z \) case; i.e., the \( \hat{\rho} \) field is related to the \( \hat{z} \) fields from both an electric and a magnetic source. Equation (A.16) is the result for the \( \hat{z} \) field due to a \( \hat{\rho} \) source; therefore, it is the result for the TM to \( z \) case. As in (A.7),

\[
E_{z,\rho}^{TM} = k_t^2 \frac{\psi_{TM}}{j\omega \epsilon}
\]  

(A.17)

Comparing this to (A.16) to obtain \( \psi_{TM} \) and then using the result in (A.9) results in the \( \hat{\rho} \) field for the TM to \( z \) case:

\[
E_{\rho,\rho}^{TM} = -\frac{j I^b Z_0}{8\pi k} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z'-z)+jn(\phi'-\phi)} k_t^2 H_n^{(2)}' (k_l \rho) \left[ J_n^{\prime}(k_l \rho) - \frac{J_n(k_l a)}{H_n^{(2)}(k_l a)} H_n^{(2)} (k_l \rho) \right]
\]  

(A.18)

To find the contribution to the \( \hat{\rho} \) field from the TE to \( z \) case, a magnetic dipole is used. Using duality and a slightly different reflection coefficient, the \( \hat{z} \)-directed...
magnetic field due to a \(z'\)-directed magnetic line source of magnitude \(K\) in the presence of a circular cylinder of radius \(a\) is [25, 5-12]

\[
H_{2D}^2 = - \frac{k^2 K}{4\omega \mu} \sum_{n=-\infty}^{\infty} H_n^{(2)}(k\rho) \left[ J_n(k\rho_0) + b_n H_n^{(2)}(k\rho_0) \right] e^{in(\phi - \phi')} (A.19)
\]

where

\[
b_n = - \frac{J_n'(ka)}{H_n^{(2)'}(ka)} (A.20)
\]

and \(\mu\) is the permeability of the medium. The corresponding three-dimensional result is

\[
H_{3D}^2 = - \frac{K}{8\pi \omega \mu} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z-z')} + jn(\phi - \phi') k_z^2 H_n^{(2)}(k_\rho) \\
\cdot \left[ J_n(k_\rho_0) + b_n H_n^{(2)}(k_\rho_0) \right] (A.21)
\]

For the TE to \(z\) case, \(A = 0\) and the electric vector potential is defined as \(\vec{E} = \hat{z}\psi\).

An equation similar to (A.7) exists:

\[
H_z = \frac{k_z^2 \psi}{j\omega \mu} (A.22)
\]

Again, \(\psi\) is assumed to be proportional to \(e^{jkzs}\). By comparing (A.21) and (A.22), it is deduced that the \(\psi\) that generates \(H_z\) is

\[
\psi = - \frac{jK}{8\pi} \int_{-\infty}^{\infty} dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z-z')} + jn(\phi - \phi') H_n^{(2)}(k_\rho) \\
\cdot \left[ J_n(k_\rho_0) + b_n H_n^{(2)}(k_\rho_0) \right] (A.23)
\]

As in the TM to \(z\) case, the field components can be derived from \(\psi\). The \(\hat{\rho}\)-directed component is [25, 5-19]

\[
E_\rho = - \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} (A.24)
\]
Using (A.23), the $\hat{z}$-directed field due to a $\hat{z}'$-directed magnetic source is

$$E_{\rho z'} = -\frac{\mathcal{K}}{8\pi\rho} \int dk_z \sum_{n=-\infty}^{\infty} e^{ik_z(z-z')} \left(\begin{array}{c} n \\ \phi-\phi' \end{array}\right) H_n^{(2)}(k_l\rho') \left[ J_n(k_l\rho) + b_n H_n^{(2)}(k_l\rho) \right]$$

\hspace{1cm} (A.25)

Next, if the electric source, $\tilde{J}^a$, in Figure 81a is replaced with a magnetic source, $\tilde{M}^a$, then using the reciprocity theorem,

$$-\int \tilde{M}^a \cdot \tilde{B}^b dV = \int \tilde{J}^b \cdot \tilde{E}^a dV \hspace{1cm} (A.26)$$

which if $\tilde{M}^a = \hat{z}\mathcal{K}^a \delta(\tilde{R} - \tilde{R}^a)$ reduces to

$$-\mathcal{K}^a H^b_z(z_a, \rho_a, \phi_a) = \tilde{I}_b E^a_{\rho}(z_b, \rho_b, \phi_b) \hspace{1cm} (A.27)$$

where $E^a_{\rho}$ is generated by $\tilde{M}^a$ and is known ((A.25)). Therefore, $H^b_z$, the field from $\tilde{J}^b$ which is in the $\hat{\rho}$ direction, is

$$H^b_z(z_a, \rho_a, \phi_a) = \frac{\tilde{I}_b}{8\pi} \int dk_z \sum_{n=-\infty}^{\infty} n \left\{ \begin{array}{ll}
 e^{ik_z(z_2-z_1)+jn(\phi_2-\phi_1)} \frac{H_n^{(2)}(k_l\rho_2)}{\rho_2} \left[ J_n(k_l\rho_2) + b_n H_n^{(2)}(k_l\rho_2) \right] \\
 \text{for } \rho' < \rho \\
 e^{ik_z(z_1-z_2)+jn(\phi_1-\phi_2)} \frac{H_n^{(2)}(k_l\rho_1)}{\rho_1} \left[ J_n(k_l\rho_2) + b_n H_n^{(2)}(k_l\rho_2) \right] \\
 \text{for } \rho' > \rho 
\end{array} \right. \hspace{1cm} (A.28)$$

where $\rho$ and $\rho'$ are referenced to $\tilde{J}^b$ as the source. This is the TE to $z$ contribution to the $\hat{\rho}$ field from $\tilde{J}^b$. Therefore, using (A.22), the vector potential $\psi$ that generates $H^TE_z$ and the other field components is found to be

$$\psi_{TE} = \frac{j\omega\mu I^b}{8\pi} \int dk_z \sum_{n=-\infty}^{\infty} \frac{n}{k_l^2\rho'} e^{ik_z(z'-z)+jn(\phi'-\phi)} H_n^{(2)}(k_l\rho') \left[ J_n(k_l\rho) + b_n H_n^{(2)}(k_l\rho) \right]$$

\hspace{1cm} (A.29)
Substituting this into (A.24) gives

\[ E_{\rho \rho}^{\text{TE}} = -\frac{\omega \mu I b}{8\pi \rho c} \int dk_z \sum_{n=-\infty}^{\infty} \frac{n^2}{k_z^2} e^{jk_z(z'-z)+jn(\phi'-\phi)} H_n^{(2)}(k_t \rho >) \]

\[ \cdot \left[ J_n(k_t \rho <) + b_n H_n^{(2)}(k_t \rho <) \right] \quad (A.30) \]

Adding the fields due to \( \psi_{TE} \) and \( \psi_{TM} \) ((A.18) and (A.30)) gives

\[ E_{\rho \rho} = -\frac{k Z_0 I}{8\pi \rho c'} \int dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z'-z)+jn(\phi'-\phi)} \frac{n^2}{k_z^2} H_n^{(2)}(k_t \rho >) \]

\[ \cdot \left[ J_n(k_t \rho <) - \frac{J_n'(k_t a)}{H_n^{(2)'}}(k_t \rho <) \right] \]

\[ -\frac{IZ_0}{8\pi k} \int dk_z \sum_{n=-\infty}^{\infty} e^{jk_z(z'-z)+jn(\phi'-\phi)} k_z^2 H_n^{(2)'}(k_t \rho >) \]

\[ \cdot \left[ J_n'(k_t \rho <) - \frac{J_n(k_t a)}{H_n^{(2)}}(k_t \rho <) \right] \quad (A.31) \]
From Watson [26], Section 8.4, the Debye asymptotic expansion for the Bessel and Neumann functions where $\nu > \nu \text{sech}\alpha$, both $\nu$ and $\nu \text{sech}\alpha$ are large and positive, and $\alpha$ is positive is

$$J_\nu(\nu \text{sech}\alpha) \sim \frac{e^{-\nu(\alpha-\tanh\alpha)}}{\sqrt{2\pi\nu \tanh\alpha}} \left\{ 1 + \frac{3 - 5 \coth^2\alpha}{24 \nu \tanh\alpha} + O\left(\frac{1}{\nu^2}\right) \right\} \quad (B.1a)$$

$$Y_\nu(\nu \text{sech}\alpha) \sim -\sqrt{\frac{2}{\pi}} \frac{e^{\nu(\alpha-\tanh\alpha)}}{\sqrt{\nu \tanh\alpha}} \left\{ 1 - \frac{3 - 5 \coth^2\alpha}{24 \nu \tanh\alpha} + O\left(\frac{1}{\nu^2}\right) \right\} \quad (B.1b)$$

Since $\alpha$ is positive then

$$0 \leq \tanh\alpha < 1 \quad (B.2)$$

and in this region

$$\alpha > \tanh\alpha \quad (B.3)$$

Since $\nu$ is assumed to be large and positive, the exponential term in the Bessel function will be small; whereas, the one in the Neumann function will be large. Thus, the Neumann function is a good approximation to the Hankel function.

$$H^{(2)}_\nu(\nu \text{sech}\alpha) \approx -jY_\nu(\nu \text{sech}\alpha) \quad (B.4)$$

Letting $z = \nu \text{sech}\alpha$ implies that

$$\cosh\alpha = \frac{\nu}{z} \quad (B.5a)$$

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\[
\frac{\sinh \alpha}{z} = \frac{\sqrt{\nu^2 + z^2}}{z} \quad \text{(B.5b)}
\]

\[
\tanh \alpha = \frac{\sqrt{\nu^2 + z^2}}{\nu} \quad \text{(B.5c)}
\]

Using (B.5) to rewrite (B.4) in terms of \( z \) results in

\[
H^{(2)}_{\nu}(z) \approx j \sqrt{\frac{2}{\pi}} \frac{e^{\nu \cosh^{-1} \frac{\nu}{z} - \sqrt{\nu^2 - z^2}}}{(\nu^2 - z^2)^{1/4}} \left[ 1 + \frac{2\nu^2 + 3z^2}{24 (\nu^2 - z^2)^{3/2}} + \cdots \right] \quad \text{(B.6)}
\]

Using term-by-term differentiation on (B.6), the derivative of the Hankel function is

\[
H^{(2)'}_{\nu}(z) \approx j \sqrt{\frac{2}{\pi}} (A_0 + A_1 + \cdots) \quad \text{(B.7)}
\]

where

\[
A_0 = \frac{\partial}{\partial z} \left[ \frac{e^{\chi}}{(\nu^2 - z^2)^{1/4}} \right] = e^{\chi} \left[ \frac{z}{2 (\nu^2 - z^2)^{5/4}} - \frac{(\nu^2 - z^2)}{z (\nu^2 - z^2)^{3/4}} \right] \quad \text{(B.8)}
\]

and \( \chi = \nu \cosh^{-1} \frac{\nu}{z} - \sqrt{\nu^2 - z^2} \). Using only the leading terms in (B.6) and (B.7),

\[
\frac{H^{(2)}_{\nu}(z)}{H^{(2)'}_{\nu}(z)} \approx \frac{1}{\frac{z^2}{2(\nu^2 - z^2)^{1/2}} - (\nu^2 - z^2)^{1/2}} \approx - \frac{z}{(\nu^2 - z^2)^{1/2}} - \frac{z^3}{2 (\nu^2 - z^2)^2} + O \left( \frac{z^4}{(\nu^2 - z^2)^{7/2}} \right) \quad \text{(B.9)}
\]

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APPENDIX G

Maps and Flowchart That the Hybrid Code Uses to Decide Which Equation to Use

In Chapter VII, methods were devised to aid in deciding which equation – EUTD coupling, EUTD radiation, UTD scattering, or modified image – to use. Three surfaces were derived which delineate the region between modified image and the other three equations. These are referred to as the M-cone, the I-cone, and S-can and were derived in Sections 7.2, 7.3, and 7.4, respectively. The boundary between the region where the EUTD radiation solution is used and the region where the EUTD coupling solution is used was derived empirically and uses \( \xi = -\left(\frac{k \sin \theta}{2}\right)^{1/3} \) as the variable to aid deciding which equation to use. The boundary between the region where the EUTD radiation solution is used and the one where the UTD scattering solution is used was derived empirically also but uses the EUTD radiation variable \( h_t \) as a guide. These conclusions are put together in the flowchart shown in Figure 93. It is the major flowchart that calls all the other flowcharts. It details exactly the procedure that the hybrid code follows to decide, given a source and observer location, the length of the source dipole, and the radius of the cylinder, the equation to use to determine the electric field. In addition, to help visualize how the decision making process is done, “maps” are provided which show very crudely how the lit region around the cylinder and the source is divided such that one equation is used in one area and another is used in another area. There are three basic maps – map A, map B–1, and B–2. Map A (Figure 88) shows how the region is divided
around a source which is on the PEC surface; whereas, map B-1 (Figure 89) and B-2 (Figure 90) account for the case when the source is not on the PEC. Map B-1 shows the divided region when the source is close to but not on the PEC, and map B2 shows the divisions when the source is far from the PEC. The three delineation surfaces are referred to in Maps A, B-1, and B-2 and are shown individually in Figures 91 and 92. All six of the aforementioned maps have been represented in the form of flowcharts which indicate in detail what decisions the hybrid code performs (see Figure 94 – 99). The flowcharts represent more exactly what the maps show.

Note the addition of a sphere around the source where modified image is used. This is the first decision in the flowchart for map A, B-1, and B-2. All three UTD equations assume that the observer is not close to the source otherwise a singularity occurs. Based on numerous test cases, a 0.2λ radius sphere was found to suffice in preventing the observer from being too close to the source. Inside the sphere, the UTD-based equations are inaccurate so modified image is used.

All the aforementioned maps refer to an observer in the lit zone of the source. For brevity, the shadow region mappings have not been included here. They are merely simplified versions of the lit region results because many of the boundaries are required because of singularities caused by the observer being too close to the source or to its axis. This never occurs in the shadow region so neither the cones nor the S-can is required. It should be noted that in the maps and flowcharts to follow EUTD radiation is referred to as modified radiation and EUTD coupling is referred to as modified coupling.
Figure 88: Map A: Source on the PEC. Delineation of the region about the source dipole. Note: EUTD radiation/coupling is referred to here as modified radiation/coupling.
Figure 89: Map B-1: Source off but close to the PEC ($h_i^l < 0.55$). Delineation of the region about the source dipole. Note: EUTD radiation/coupling is referred to here as modified radiation/coupling.
Figure 90: Map B-2: Source off and far from the PEC ($h_t > 0.55$). Delineation of the region about the source dipole. Note: EUTD radiation is referred to here as modified radiation.
Figure 91: M- and I-Cone: Boundaries for scattering/modified image and EUTD radiation/modified image, respectively. Note: EUTD radiation is referred to here as modified radiation.
Figure 92: S-Can: Boundary between modified coupling and modified image. Note: EUTD coupling is referred to here as modified coupling.
Figure 93: Flowchart that HY-MOM uses to determine which of the four following equations—modified image, UTD scattering, EUTD radiation, and EUTD coupling—to use.
Figure 93 (continued).

Continued from previous page

Given

\[ r(i) = \| \overline{ob} - \text{endpt}(ix, i) \| , \quad i = 1, 2 \]

\[ \text{Is } r(1) \leq r(2) \text{?} \]
\[ \text{Yes} \quad iclosest = 1 \]
\[ \text{No} \quad iclosest = 2 \]

\[ \text{Is } r(iclosest) \leq 0.2A \text{?} \]
\[ \text{Yes} \]
\[ \text{Is } d'(1) > d'(2) \text{?} \]
\[ \text{Yes} \quad isource = 1 \]
\[ \text{No} \]
\[ \text{Is } d'(1) < d'(2) \text{?} \]
\[ \text{Yes} \quad isource = 2 \]
\[ \text{No} \]

\[ isource = iclosest \]

\[ source(ix) = \text{endpt}(ix, isource) \]

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Figure 93 (continued).

Continued from previous page

B

Is \( d' = 0 \)?

Yes

PEC

\( \theta_n \)

\( \hat{n} \)

No

Goto Map B-1

Find \( \mathbf{E}_{SCAT}^{MOM} \)

Call UTD.EQN

End

Goto Map A

Is \( \theta_n = 90^\circ \)?

Yes

\( E_{TOT}^{PEC} = 0 \)

\( E_{UTD}^{TOT} = -E_{MOM} \)

PEC

No

End
Figure 94: Flowchart that details the decision making process that the hybrid code uses to decide which equation to use if the source is on the PEC. It is associated with map A shown in Figure 88.
Figure 95: Flowchart that details the decision making process that the hybrid code uses to decide which equation to use if the source is off the PEC but close to it. This flowchart is associated with map B-1 shown in Figure 89.
Figure 96: Flowchart that details the decision making process that the hybrid code uses to decide which equation to use if the source is off the PEC and far from it. This flowchart is associated with map B-2 shown in Figure 90.
Figure 97: Flowchart to depict the decision making process that HY-MOM uses to decide whether to use UTD scattering or modified image theory. It details the use of the M-cone.
Figure 98: Flowchart to depict the decision making process that HY-MOM uses to decide whether to use modified radiation or modified image theory. It details the use of the I-cone.
Figure 99: Flowchart to depict the decision making process that HY-MOM uses to decide whether to use modified coupling or modified image theory. It details the use of the S-can.
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