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ANALYSIS OF SOIL-REINFORCEMENT INTERACTION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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* * * * *

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To My Mother

and

The Memory of My Father
Acknowledgment

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TABLE OF CONTENTS

Acknowledgments........................................................................................................ iii
Vita....................................................................................................................................... iv
List Of Tables..................................................................................................................... xi
List Of Figures.................................................................................................................. xii

CHAPTER

I. Introduction.................................................................................................................. 1

II. Literature Review....................................................................................................... 5

  2.1 Introduction ........................................................................................................... 5
  2.2 Mechanisms Of The Soil-Reinforcement Interaction ......................................... 5
      2.2.1 Load Transfer By Friction ........................................................................ 6
      2.2.2 Load Transfer By Passive Earth Pressure ............................................. 8
      2.2.3 Forces In The Reinforcement ............................................................... 9
  2.3 Analytical Procedures .......................................................................................... 11
      2.3.1 Analyses Based On The Limit Equilibrium Approach .................... 11
          2.3.1.1 Methods That Consider The Soil-Reinforcement Interaction .... 12
              a. Davis Method ................................................................................. 12
              b. German Method ............................................................................ 15
              c. French Method ............................................................................. 15
              d. Local Stability Method ................................................................. 21
              e. Kinematical Limit Equilibrium Method ...................................... 22
          2.3.1.2 Methods That Assume A Deformation Pattern For The Reinforcement ... 25
3.6.4 Comparison Between The Beam Model And Schlosser's Model ................................................................. 119
3.7 Distribution Of Stresses And Strains ................................................. 127
  3.7.1 Distribution Of Stresses And Strains In A Soil Reinforced By One Reinforcement ................................................................. 128
  3.7.2 Distribution Of Stresses And Strains In A Soil Reinforced By A Group Of Reinforcement .................................................. 135
3.8 Summary And Conclusions .............................................................. 140

IV. Simplified Finite Element Model For Soil-Reinforcement Interaction ........................................................................ 143

  4.1 Introduction ................................................................................. 143
  4.2 General Description Of The Model ................................................ 145
    4.2.1 Load-Displacement Relationship For The Reinforcement System ............................................................ 147
    4.2.2 The External Force Vector ..................................................... 151
    4.2.3 Governing Equation ............................................................. 154
  4.3 Stiffness Matrix Of The Reinforcement Element ......................... 156
    4.3.1 Basic Definitions ................................................................. 156
    4.3.2 Relationship Between The Force Vector \(\{\Delta s\}\) And The Deformation Vector \(\{\Delta v\}\) ................................................................. 158
      4.3.2.1 The Yield Surface ......................................................... 163
      4.3.2.2 Control Of The Force Drift From The Yield Surface .... 164
    4.3.3 Relationship Between The Local Force Vector \(\{\Delta \delta\}\) And The Local Displacement Vector \(\{\Delta \delta\}\) ................................................................. 167
    4.3.4 Relationship Between The Global Force Vector \(\{\Delta s\}\) And The Global Displacement Vector \(\{\Delta v\}\) ................................................................. 169
  4.4 Constitutive Relations For The Subgrade Springs ....................... 171
    4.4.1 Independent Constitutive Relations ........................................ 172
    4.4.2 Interacting Constitutive Relations ........................................ 175
    4.4.3 Representation Of The Constitutive Relations For The Subgrade Springs In The Computer Program .................. 179
  4.5 Solution Technique For The Governing Equation ......................... 180
4.6 Description Of The Computer Program ............................................................... 185
4.7 Verification Of The Computer Program .......................................................... 187
   4.7.1. Large Deflection Analysis Of A Cantilever Beam ................................ 187
   4.7.2. Elastic-Plastic Analysis Of A Beam Fixed at Both Ends ....................... 187
4.8. Applications ........................................................................................................... 190
   4.8.1. Analyses Of A Soil Reinforced By Straight Bars ............................ 190
      4.8.1.1. Comparison With Chapter (3) Models .................. 190
      4.8.1.2. Effect Of The Reinforcement Orientation .................. 193
      4.8.1.3 Contribution Of The reinforcement's Axial And Bending Stiffnesses To The Composite Strength .... 195
   4.8.2. Analysis Of A Soil Reinforced By Piles .................................. 200
   4.8.3. Analysis Of A soil Reinforced By Tree Roots ............................... 204
4.9 Summary And Conclusions ............................................................................... 209

V. Conclusions .................................................................................................... 210

List Of references ................................................................................................... 214
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Cases To Investigate The Effect Of Different Model Assumptions</td>
<td>67</td>
</tr>
<tr>
<td>3.2 Cases In The Parametric Study</td>
<td>96</td>
</tr>
<tr>
<td>3.3 Properties Of The Soil Material</td>
<td>96</td>
</tr>
<tr>
<td>3.4 Properties Of The Reinforcement Material</td>
<td>96</td>
</tr>
<tr>
<td>4.1 Suggested Values For The Initial Slope And The Ultimate Resistance Of The Soil Response Curves</td>
<td>174</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Frictional Transfer Between Soil And Reinforcement (From Mitchell and Villet 1987)</td>
<td>7</td>
</tr>
<tr>
<td>2.2 Restrained Dilatancy Effect On Soil-Reinforcement Friction (From Schlosser et al. 1983)</td>
<td>7</td>
</tr>
<tr>
<td>2.3 Effect Of The Reinforcement Rigidity On The Soil-Reinforcement Interaction (From Mitchell And Villet 1987)</td>
<td>10</td>
</tr>
<tr>
<td>2.4 Theoretical Relationships Between Bearing Stress And Frictional Angle (From Mitchell And Villet 1987)</td>
<td>10</td>
</tr>
<tr>
<td>2.5 Davis Method (From Elias And Juran 1990)</td>
<td>14</td>
</tr>
<tr>
<td>2.6 The German Method (From Elias And Juran 1990)</td>
<td>16</td>
</tr>
<tr>
<td>2.7 The French Method (From Mitchell And Villet 1987)</td>
<td>20</td>
</tr>
<tr>
<td>2.8 Local Stability Method (From Mitchell And Villet 1987)</td>
<td>23</td>
</tr>
<tr>
<td>2.9 Kinematical Limit Equilibrium Method (From Christopher et al. 1990)</td>
<td>24</td>
</tr>
<tr>
<td>2.10 Assumed Deformation Pattern In Wu's Model (1976) And Waldron's Model (1977)</td>
<td>28</td>
</tr>
<tr>
<td>2.11 Cylindrical Reinforcement Element At Equilibrium (Waldron 1977)</td>
<td>28</td>
</tr>
<tr>
<td>2.12 Distribution Of The Tensile Stress In The Reinforcement (Waldron 1977)</td>
<td>28</td>
</tr>
<tr>
<td>2.13 Assumed Deformation Pattern In Gray And Ohashi's Model (1983)</td>
<td>31</td>
</tr>
<tr>
<td>2.14 Assumed Deformation Pattern In Shewbridge And Sitar's Model (1985)</td>
<td>31</td>
</tr>
<tr>
<td>2.15 Empirical Earth Pressure Design Diagrams Used For Nailed Walls (From Christopher et al. 1990)</td>
<td>39</td>
</tr>
<tr>
<td>2.16 Variation Of The Coefficient Of Lateral Earth Pressure As A Function Of Depth And Reinforcement Type (From Christopher et al. 1990)</td>
<td>39</td>
</tr>
<tr>
<td>3.1 Elements Used To Represent Direct Shear Test</td>
<td>53</td>
</tr>
<tr>
<td>3.2 The Interface Between Soil And Reinforcement</td>
<td>54</td>
</tr>
</tbody>
</table>
3.3 Drucker-Prager Model: Yield Surface In The p-q Plane ........................................ 58
3.4 Cap Model: Yield Surface And Flow Potential In The p-q Plane ......................... 58
3.5 Finite Element Mesh ................................................................................................. 68
3.6 Load-Displacement Curves For Different Soil Models ............................................ 71
3.7 Forces In The Reinforcement For Different Soil Models ........................................ 72
3.8 Load-Displacement Curves For Different Reinforcement Models ......................... 73
3.9 Forces In The Reinforcement For Different Reinforcement Models ....................... 74
3.10 Load-Displacement Curves For Different Contact Between Soil And Reinforcement .......................................................... 75
3.11 Forces In The Reinforcement For Different Contact Between Soil And Reinforcement .......................................................... 76
3.12 Load-Displacement Curves For Different Kinematics ........................................... 77
3.13 Forces In The Reinforcement For Different Kinematics ........................................ 78
3.14 Finite Element Mesh (Simulating Gray And Ohashi's Experiments) ..................... 82
3.15 Comparison Between The Results Of The Finite Element Analyses And The Results Of Gray And Ohashi's Test On Unreinforced Soil ........................................ 83
3.16 Comparison Between The Results Of The Finite Element Analyses And The Results of Gray And Ohashi's test On Soil Reinforced By Six #2 Reeds ................. 83
3.17 Finite Element Mesh (Simulating Shewbridge And Sitar's Experiments) ............... 86
3.18 The Arrangement Of Shewbridge And Sitar's Shear Box ...................................... 87
3.19 Comparison Between The Results Of The Finite Element Analysis And The Results Of Shewbridge And Sitar's Test On Unreinforced Soil .............................. 91
3.20 Comparison Between The Results Of The Finite Element Analysis And The Results Of Shewbridge And Sitar's Test On Soil Reinforced By Fourteen Wood Dowels .......................................................... 91
3.21 Deformation Pattern Given By Equation (2.33) ..................................................... 92
3.22 Deformation Pattern Computed In The Finite Element Analysis ......................... 92
3.23 Soil Dilation Measured In The Experiments ............................................................ 93
3.24 Soil Dilation Computed In The Finite Element Analyses ....................................... 93
3.25 Effect Of Soil Relative Density On The Load-Displacement Behavior .................. 100
3.26 Effect Of Soil Relative Density On The Reinforcement Forces ............................. 101
3.27 Effect Of Reinforcement Type On The Load-Displacement Behavior ................. 102
3.28 Effect Of Reinforcement Type On The Reinforcement Forces ............................. 103
3.29 Effect Of Confining Pressure On The Load-Displacement Behavior ...................... 104
3.30 Effect Of Confining Pressure On The Reinforcement Forces ........................................ 105
3.31 Failure Envelope Of Unreinforced And Reinforced Dense Sand .................................. 106
3.32 Failure Envelope Of Unreinforced And Reinforced Medium Dense Sand ..................... 106
3.33 Failure Envelope Of Unreinforced And Reinforced Loose Sand ................................... 106
3.34 Distribution Of The Normal Pressure On The Reinforcement And The Developed Forces In The Reinforcement ..................................................... 122
3.35 Distribution Of The Net Normal Pressure On A Wood Dowel Imbedded In Dense And Loose Sands .................................................................................. 123
3.36 Distribution Of The Normal And Shear Stresses On The Periphery Of A Reinforcement Section ................................................................................................. 124
3.37 Comparison Between The Axial Forces Obtained From The Finite Element Model And From The Beam Model ................................................................. 125
3.38 Comparison Between The Shear Forces Obtained From The Finite Element Model And From The Beam Model ................................................................. 126
3.39 The Planes Along Which The Distribution Of Stresses And Strains Are Represented .......................................................... 130
3.40 Distribution Of The Mean Normal Stresses (p) In A Dense Sand Reinforced By One Reinforcement ......................................................................................... 131
3.41 Distribution Of The Mises Stresses (q) In A Dense Sand Reinforced By One Reinforcement ........................................................................................................ 132
3.42 Distribution Of The Volumetric Plastic Strain (\(e_p^v\)) In A Dense Sand Reinforced By One Reinforcement ......................................................................................... 133
3.43 Distribution Of The Plastic Strain Magnitude (\(e_p^p\)) In A Dense Sand Reinforced By One Reinforcement ......................................................................................... 134
3.44 Distribution Of The Mean Normal Stresses (p) In A Dense Sand Reinforced By Six Reinforcement .......................................................................................... 136
3.45 Distribution Of The Mises Stresses (q) In A Dense Sand Reinforced By Six Reinforcement ........................................................................................................ 137
3.46 Distribution Of The Volumetric Plastic Strain (\(e_p^v\)) In A Dense Sand Reinforced By Six Reinforcement ......................................................................................... 138
3.47 Distribution Of The Plastic Strain Magnitude (\(e_p^p\)) In A Dense Sand Reinforced By Six Reinforcement ......................................................................................... 139
4.1 The Simplified Finite Element Model ................................................................. 146
4.2 Control Of Force Drift ..................................................................................... 166
4.3 Ramberg-Osgood Model (Desai And Kuppusamy 1980) .............................. 174
4.4 Solution Technique For The Governing Equation .............................................. 184
4.5 Different Slip Surfaces Which Can Be Considered By The Computer Program ................................................................. 186
4.6 Large Deflection Analysis Of A Cantilever Beam ........................................... 188
4.7 Elastic-Plastic Analysis Of A Beam Fixed At Both Ends ................................. 189
4.8 Comparison Between The Results Of Models (1) and (3) ............................ 197
4.9 Comparison Between The Ultimate Axial Forces Obtained From Models (1), (2) And (3) ........................................................................................................ 198
4.10 Comparison Between The Ultimate Shear Forces Obtained From Models (1), (2) And (3) ........................................................................................................ 198
4.11 Effect Of The Orientation Angle (θ) On The Developed Forces In The Reinforcement ............................................................................................................ 199
4.12 Analysis Of Piles Used To Stabilize A Soil Undergoing Lateral Movement ...... 203
4.13 In Situ Direct Shear Test On A c-ϕ Soil Reinforced By Vegetation Roots (Wu And Watson 1991) ................................................................. 207
4.14 Plane View For The Geometry Of The Instrumented Root ............................. 208
4.15 Load-Displacement curve For The Instrumented Root .................................. 208
CHAPTER I

Introduction

Soil reinforcement is a method of soil improvement in which reinforcement elements are placed in soil to increase the soil shearing resistance. Although soil-reinforcement is an old technique, it has been used in modern practice since the early sixties (Schlosser 1983). Currently, the technique of soil-reinforcement is used in stabilizing highway slopes, in construction of inclined and vertical retaining structures, in tunneling, and in other civil and industrial projects. This rapid acceptance of soil-reinforcement technique can be attributed to a number of factors including low cost, aesthetics, ability to adapt different site conditions and ability to withstand substantial deformation without distress (Mitchell and Christopher 1990).

A wide variety of reinforced-soil systems are currently used in practice. These different systems can be classified by the reinforcement geometry into two categories (Christopher et al. 1990). These categories are the bi-directional reinforcement systems and the unidirectional reinforcement systems. In the bi-directional reinforcement systems, the reinforcements can be sheets (geotextiles), grids (geogrids), or wire meshes. In the
unidirectional reinforcement systems, the reinforcement can be piles, nails, or strips. The reinforcement types concerned in this research fall within the second category. These reinforcement types can be detailed as follows:

**Rigid Piles**: Piles can be used to stabilize landslides or to reduce the rate of creep in soils (Yamada et al. 1971; Poulus 1973; Ito and Mastui 1977; Sommer 1979). Usually, one or two rows of piles are placed at the toe of the slope to resist the downslope sliding of the soil.

**Soil Nails**: Rods or bars, called soil nails, can be used to stabilize slopes (Gassler and Gudehus 1981; Winter et al. 1983), in construction of retaining structures (Shen et al. 1981; Elias and Juran 1990), and in tunneling (Rabcewicz 1965). In this reinforcement system, relatively high density of nails are placed in rows almost perpendicular to the tending direction of soil movement.

**Micropiles (Root Piles)**: The micropile reinforcement system, developed by Lizzi (1977), consists of small diameter piles (5 to 30 cm) arranged in a pattern resembling the geometry of vegetation roots. This system differs from soil nails in that the piles are arranged in a pattern to form a network that encompasses the soil and in which the piles interact with each other. However, in a nailed system, the nails act as independent resisting elements (Mitchell and Christopher 1990). Micropiles can be used for underpinning structures, reinforcement of foundation soils, and slope stabilization (Mitchell and Villet 1987).

**Vegetation Roots**: Vegetation roots can be used to stabilize landslides and to prevent surficial erosion (Gray and Sotir 1992). Vegetation roots have been used for soil reinforcement since 1591 (Greenway 1987). However, the practice of using this reinforcing system in modern civil engineering project began in 1936 (Krabel 1936). The
role played by vegetation roots in specific geotechnical problems has been recognized in more recent times (Wu et al. 1979; Gray and Leiser 1982). Vegetation roots can be considered as an inexpensive natural alternative to soil nails or micropiles.

The mechanical concept of soil reinforcement can be explained as follows. The soil has good resistance to compression and significant resistance to shear while it has no tensile resistance. On the other hand, the reinforcements have high tensile and sometimes bending and shear strength. Through the stress transfer between the soil and the reinforcements (i.e., the soil-reinforcement interaction), the two materials produce a composite that combines their best characteristics. Hence, the soil-reinforcement composite acts as a coherent material which is more resistant and less deformable than the soil alone.

The transfer of stress between the soil and the reinforcement involves two basic mechanisms; namely, friction and passive soil pressure (Mitchell and Villet 1987). A successful analysis of a reinforced soil requires an accurate representation the above two mechanisms. Several analytical procedures have been developed for reinforced soils. Some of these procedures predict the behavior of reinforced soils only at failure condition while other procedures make the prediction at working as well as at failure conditions. A review for the mechanisms of soil-reinforcement interaction and a survey of the available analytical procedures are given in Chapter (2). This review shows that the mechanisms for the soil-reinforcement interaction have not yet been completely understood and the available analytical procedures still have several shortcomings. Therefore, the study performed in this research was undertaken.

The objectives of this research are to develop a better understanding of the soil reinforcement interaction mechanisms and to develop analytical procedures which can
accurately represent these mechanisms. The output of this research is the solution of the soil-reinforcement interaction problem using three models. These models can be detailed as follows:

**Model (1):** The finite element package “ABAQUS” is used to construct a three dimensional, nonlinear, finite element model for the problem. This model includes the effect of the parameters which were ignored or inaccurately considered in the previous works. The results of this model are used to develop better understanding of the soil-reinforcement interaction and to determine the important design parameters which should be included in the analysis.

**Model (2):** Based on the experience gained from the results of model (1), a model for computing the ultimate forces in the reinforcement is developed. This model will be referred as “the Beam Model” throughout this research. The beam model can be used to perform at failure analyses for reinforced soils.

**Model (3):** A simplified finite element model for the problem is developed. The difference between this model and model (1) is that the soil is represented in this model using the Winkler hypothesis while it is represented as a continuum in model (1). This model can be used to predict the forces in the reinforcement at working as well as at failure conditions.

Chapter (2) gives the review of the mechanisms of soil-reinforcement interaction and the survey of the available analytical procedures. Chapter (3) describes the analyses performed by model (1) and the development of model (2). Chapter (4) describes model (3). Chapter (5) gives a summary of the conclusions obtained from this research.
CHAPTER II

Literature Review

2.1 Introduction

The objective of this Chapter is to review the mechanisms of the soil-reinforcement interaction and to survey and evaluate the available design procedures for reinforced soils. This literature review helps to assess the limitations and the accuracy of available design procedures.

2.2 Mechanisms Of The Soil-Reinforcement Interaction

Stress transfer between soil and reinforcements (the soil-reinforcement interaction) involves two mechanisms: friction and passive soil pressure. In many reinforced-soil systems, both mechanisms are active and the relative contribution of each is indeterminate (Mitchell and Villet 1987). Together, they determine the behavior of the soil-reinforcement composite. The above two mechanisms are explained in the following two subsections.
2.2.1 Load Transfer By Friction

Frictional load transfer between soil and reinforcement is illustrated schematically in Figure (2.1). The frictional stress that can be transferred depends on the interface characteristics and the normal stress between the soil and the reinforcement. The limiting shear stress along the soil-reinforcement interface (τᵣ) can be written as,

\[ \tauᵣ = \mu \times \sigmaᵣ \]  

(2.1)

where:
- \( \sigmaᵣ \) = normal stress at the soil-reinforcement interface,
- \( \mu \) = the coefficient of friction between the soil and the reinforcement.

Normally, it is assumed that \( \sigmaᵣ \) equals the overburden pressure (\( \sigmaᵥ \)) on the reinforcement and \( \mu \) equals \( \tan(δ) \), where \( δ \) is the frictional angle between the soil and the reinforcement materials. Making these assumptions, it should be a simple matter to calculate the limiting frictional stress (\( \tauᵣ \)) or the pull-out resistance of the reinforcement. However, experimental results showed that the pull-out resistance of the reinforcement is always larger than that computed using equation (2.1), (Schlosser 1983). Schlosser et al. (1983) explained this phenomenon as follows. The pull-out of the reinforcement, Figure (2.2), induces shear displacement in a zone of surrounding soil. In compacted granular soils, the shear zone tends to dilate. However, the dilation of this zone is restrained by the surrounding soil and hence, there is an increase in the normal stresses (\( \Deltaσᵣd \)) on the reinforcement. This led to the definition of an apparent coefficient of friction (\( \mu^* \)) given by the equation,

\[ \mu^* = \mu \times \left( \frac{\sigmaᵥ + \Deltaσᵣd}{\sigmaᵥ} \right) \]  

(2.2)
Figure (2.1) Frictional Transfer Between Soil And Reinforcement
(From Mitchell And Villet 1987).

Figure (2.2) Restrained Dilatancy Effect On Soil-Reinforcement Friction
(From Schlosser et al. 1983).
Various values for $\mu^*$ ranging from 0.5 to considerably greater than 1.0 have been reported in the literature with the lower values corresponding to smooth reinforcements and higher overburden pressure (Mitchell and Villet 1987).

2.2.2 Load Transfer By Passive Earth Pressure

The load transfer between the soil and the reinforcement by passive earth pressure depends on the reinforcement rigidity and the relative displacement between the soil and the reinforcement. To illustrate the effect of the reinforcement rigidity on the soil-reinforcement interaction, the two limiting cases of flexible and completely rigid reinforcement are considered in Figure (2.3), (Mitchell and Villet 1987). A rigid reinforcement resists the soil deformation and develops passive pressure in the soil. However, the deformation of the flexible reinforcement is almost the same as the soil deformation and hence no passive earth pressure is developed. The development of passive earth pressure on the reinforcement is associated with the development of bending moments and shear forces in the reinforcement. These bending moments and shear forces along with the axial forces in the reinforcement constitute the reinforcement’s contribution to the composite strength.

It is convenient to express the ultimate value of the passive earth pressure (or the soil bearing capacity) as a function of the overburden pressure ($\sigma_v$) by the equation,

$$\sigma_b = N_q \sigma_v$$ \hspace{1cm} (2.3)

where:

- $\sigma_b =$ ultimate value of the passive soil pressure,
- $N_q =$ a bearing capacity factor.
The factor $N_q$ has been expressed as a function of the soil frictional angle ($\phi$), Figure (2.4). Rowe and Davis' (1982) curves in Figure (2.4) were obtained by finite element analyses of buried strips loaded horizontally. The curve labeled "constant volume" is for no dilatancy. Jewell et al. (1984) obtained the lower curve in Figure (2.4) using the slip line solution for the punching mode of failure. The curve marked "Prandtl" in Figure (2.4) is the classical Prandtl bearing capacity solution (Jewell et al. 1984).

The development of the passive earth pressure on the reinforcement can produce a significant change in the normal stress ($\Delta \sigma_{n}$) on the reinforcement. This change in the normal stress can be responsible for the increase in the apparent frictional coefficient ($\mu^*$) described in section (2.2.1). In fact, it is believed that the above change in the normal stress combined with the effect of the restrained soil dilatancy (Schlosser et al. 1983) produce the observed increase in the apparent frictional coefficient ($\mu^*$). It can further be argued that, since the passive pressure on the reinforcement ($\sigma_p$) is dependent on the soil dilation, the consideration of the above change in the normal stress can implicitly include the effect of the restrained soil dilatancy.

### 2.2.3 Forces In The Reinforcement

The friction on the reinforcement surface produces tensile axial forces in the reinforcement. On the other hand, the passive soil pressure on the reinforcement produces bending moments and shear forces in the reinforcement. In most reinforced-soil systems, both of the above mechanisms are active (Mitchell and Villet 1987). The relative contribution of the two mechanisms depends on the rigidity, the surface characteristics, and the geometry of the reinforcement, the soil properties, the in situ stress, and the relative soil to reinforcement deformations (Mitchell and Villet 1987). Ideally, none of the above mechanisms can be ignored (Schlosser and De Buhan 1990).
Figure (2.3) Effect Of The Reinforcement Rigidity On The Soil-Reinforcement Interaction (From Mitchell And Villet 1987).

Figure (2.4) Theoretical Relationships Between Bearing Stress And Frictional Angle (From Mitchell And Villet 1987).
2.3 Analytical Procedures

Several different procedures for analyzing reinforced soils have been developed. These analytical procedures can be classified as:

1- analyses based on the limit equilibrium approach.
2- analyses based on empirical methods for considering the working stress.
3- analyses performed using the Winkler hypothesis.
4- analyses performed using the finite element method.

Detailed descriptions of the above procedures are given in the following subsections.

2.3.1 Analyses Based On The Limit Equilibrium Approach

The limit equilibrium approach has been used extensively for the analysis of reinforced soils. In this approach, potential failure surfaces throughout the reinforced-soil mass are examined. Calculations are made to determine if the forces tending to cause movement along the failure surfaces can be resisted and held in equilibrium by the available shear strength of the soil and the ultimate forces which can be developed in the reinforcements. As the case in classical stability analyses, different shapes for the potential failure surface and different definitions for the factor of safety have been used. Also, different methods for computing the reinforcement forces have been used. The methods used for computing the reinforcement forces can be divided into two categories. These categories are:

1- methods that consider the soil-reinforcement interaction mechanisms (Stocker et al. 1979; Juran and Schlosser 1978; Shen et al. 1981; Schlosser 1983; Juran et al. 1990).
2- methods that assume a deformation pattern for the reinforcement (Wu 1976; Waldron 1977; Gray and Ohashi 1983; Shewbridge and Sitar 1985).

In addition to the above differences, a main difference concerning the reinforcement forces incorporated in the analysis can be detected. Some procedures (e.g., Shen et al. 1981), assume that the reinforcements are only capable of developing axial forces. On the other hand, other procedures (e.g., Schlosser 1983) consider both axial and bending stiffnesses of the reinforcements.

In the following subsections, a review of the different limit equilibrium analyses for reinforced soils is given. The presentation given herein is made using the classification by the two categories mentioned above.

### 2.3.1.1 Methods That Consider The Soil-Reinforcement Interaction

#### a. Davis Method

This method was developed by Shen et al. (1981) at the University of California, Davis, and is usually referred as the Davis method. In this method, the slip surface is assumed to be a parabola passing through the toe of the excavation, Figure (2.5). Shen et al. (1981) considered two conditions for the slip surface. In the first condition, the slip surface extends beyond the reinforcement, Figure (2.5.a). In the second condition, the slip surface lies entirely within the reinforced soil, Figure (2.5.b).

Shen et al. (1981) assumed that the reinforcements can carry only axial forces. The axial force is assumed to be the smaller of two: the tensile strength of the reinforcement material and the frictional resistance along the portion of the reinforcement behind the slip surface. Therefore, the axial force in the reinforcement is computed as the smaller of,
where:

- \( T_{\text{max}} \) = maximum axial force in the reinforcement,
- \( D \) = reinforcement diameter,
- \( L_a \) = length of the reinforcement behind the slip surface,
- \( \sigma_r \) = normal stress at the soil-reinforcement interface (assumed = \( \sigma_y \)),
- \( \delta \) = frictional angle between the soil and the reinforcement materials,
- \( c_a \) = adhesion between the soil and the reinforcement materials,
- \( \sigma_y \) = yield strength of the reinforcement material.

The classical method of slices (Fellenius’ method) is used to evaluate the global safety factor for the reinforced-soil system. The assumption is made that the factor of safety with respect to the limit shear strength of the soil is equal to the factor of safety with respect to the pull-out resistance of the reinforcement. As with conventional slope stability analysis, direct computation of the safety factor is not possible because the equations for both the driving force and the resisting force contain the factor of safety as unknown term. Accordingly, solution uses an iterative method most readily done by the computer.
Figure (2.5.a) Slip Surface Extends Beyond The Reinforcements.

Figure (2.5.b) Slip Surface Lies Entirely Within The Reinforced Soil.

Figure (2.5) Davis Method (From Elias And Juran 1990).
b. German Method

Stocker et al. (1979) and Gassler and Gudehus (1981) proposed the force equilibrium method shown in Figure (2.6). In this method, a bilinear slip surface is assumed, Figure (2.6.a). The inclination angle \( \theta_B \) of the second portion of the slip surface is iteratively determined and the inclination angle \( \theta_A \) of the first portion of the slip surface is taken equal to \( \left( \frac{\pi}{2} - \frac{\phi}{2} \right) \). Stocker and his co-workers assumed that the soil shear strength is fully mobilized along the failure surface and only axial forces can be developed in the reinforcements. By considering the equilibrium of the forces acting on the rigid soil wedge limited by the potential failure surface, Figure (2.6.b), the required contribution of the reinforcement \( \Sigma T \) is obtained. Then, the global factor of safety \( F_g \) is defined as,

\[
F_g = \frac{\Sigma T_{\text{max}}}{\Sigma T}
\]

(2.6)

where:

\( \Sigma T_{\text{max}} \) = the sum of the maximum axial forces which can be developed in the reinforcements. The value of \( T_{\text{max}} \) for each reinforcement is considered as the smaller of equation (2.4) and (2.5).

c. French Method

Schlosser (1983) developed a limit equilibrium method for reinforced soils which is usually referred as the French method. The French geotechnical company "TERRASOL" developed the computer program "TALREN" based on the above method. In this method, circular or non-circular slip surface can be assumed. The choice of the shape of the slip surface is left to the designer. The overall stability is performed by the conventional method of slices (Fellenius' method), Figure (2.7.a). Both the axial and the bending
Figure (2.6.a) The Slip Surface In The German Method.

Figure (2.6.b) Force Polygon In The German Method.

Figure (2.6) The German Method (From Elias And Juran 1990).
stiffnesses of the reinforcement are considered. The reinforcement forces are computed using the following criteria.

**Frictional Resistance Along The Reinforcement**

The maximum tensile force in the reinforcement is computed by the equation,

\[ T_{\text{max}} = \pi D L_n f_{\text{max}} \]  \hspace{1cm} (2.7)

where:

- \( f_{\text{max}} \) = the limiting unit skin friction between soil and reinforcement;
- \( = \mu \sigma_v \).

Schlosser (1983) recommended to obtain the value of \( f_{\text{max}} \) from pull-out tests.

**Passive Earth pressure On The Reinforcement**

As explained in section (2.2.3), the passive earth pressure on the reinforcement produces bending moments and shear forces in the reinforcement. To compute these bending moments and shear forces, Schlosser (1983) used the closed-form solution developed by Hetenyi (1946) for an elastic, semi-infinite beam supported on elastic foundation and loaded by a concentrated load at its edge. This solution gives the maximum bending moment and the maximum shear force as,

\[ M_{\text{max}} = 0.16 \times \sigma_b D L_o^2 = 0.32 \times Q_{\text{max}} L_o \]  \hspace{1cm} (2.8)

\[ Q_{\text{max}} = \sigma_b \frac{D}{2} L_o \]  \hspace{1cm} (2.9)

where:

- \( \sigma_b \) = ultimate passive earth pressure of the soil,
- \( D \) = reinforcement diameter,
L₀ = a transfer length,
= \sqrt[4]{\frac{4E_f I_f}{C_y D}}

E_f = Young’s modulus of the reinforcement material,
I_f = moment of inertia of the reinforcement cross section,
C_y = coefficient of subgrade reaction of the elastic foundation.

Note that Qₘₐₓ develops at the point of intersection with the shear plane, point (o) in Figure (4.7.b), and Mₘₐₓ develops at a distance \( \frac{\pi}{4} L_0 \) from point (o).

**Yield Moment Of The Reinforcement**

The maximum bending moment in the reinforcement is restrained not to exceed the bending moment which produces a plastic hinge in the reinforcement cross section (Mₚ). Hence, the values of Mₘₐₓ and Qₘₐₓ given by equations (2.8) and (2.9) should not exceed,

\[
M_{\text{max}} \leq M_p
\]

\[
Q_{\text{max}} \leq \frac{M_p}{0.32 \times L_0}
\]

**The Reinforcement Yield Criterion Under Combined Loading**

Since the reinforcement has to withstand axial force, shear force, and bending moment, a yield criterion for the reinforcement material which considers such a combination of loading should be used. Schlosser (1983) used the following yield criterion,

\[
\frac{T_{\text{max}}^2}{T_p^2} + \frac{Q_{\text{max}}^2}{Q_p^2} \leq 1.0
\]
where:

\[ T_p = \text{tensile yield strength of the reinforcement} = \frac{\pi D^2}{4} \sigma_y, \]
\[ Q_p = \text{shear yield strength of the reinforcement} = \frac{T_p}{2}. \]

Using the above yield criterion and the principle of maximum plastic work, Figure (2.7.c), Schlosser (1983) obtained the following limiting values for the \( Q_{\text{max}} \) and \( T_{\text{max}} \),

\[ Q_{\text{max}} = \frac{Q_p}{\left[1 + 4 \times \tan^2\left(\frac{\pi}{2} - \alpha_r\right)\right]^{\frac{1}{2}}} \]  \hspace{1cm} (2.13)

\[ T_{\text{max}} = 4 \times Q_{\text{max}} \tan\left(\frac{\pi}{2} - \alpha_r\right) \]  \hspace{1cm} (2.14)

where:

\[ \alpha_r = \text{the angle between the reinforcement and the tangent to the slip surface.} \]

Note that equations (2.13) and (2.14) imply that, for \( \alpha_r = 0 \), only tensile forces are developed in the reinforcement and for \( \alpha_r = \frac{\pi}{2} \), only shear forces are developed in the reinforcement.

In the stability analysis, the axial force in the reinforcement is taken as the smaller of equations (2.7) and (2.14) and the shear force is taken as the smaller of equations (2.9), (2.11), and (2.13). The computer program TALREN allows for applying different values for the safety factors with respect to the different failure criteria. Schlosser (1983) suggested a safety factor of 1.5 for the soil shear strength, a safety factor of 1.5 for the pull-out resistance of the reinforcement, a safety factor of 2.0 for the ultimate soil bearing resistance \( (\sigma_b) \), and a safety factor of 1.0 for the tensile yield strength of the reinforcement material.
(2.7.a) The Slip Surface In The French Method.

Figure (2.7.b) Theoretical Solution For Semi-Infinite Long Bar Adopted For Design Purpose.

Figure (2.7.c) Determination Of The Maximum Forces In The Reinforcement.

Figure (2.7) The French Method (From Mitchell and Villet 1987).
d. Local Stability Method

The previous methods allow the computation of the global factor of safety along an assumed slip surface. However, they do not allow the computation of the distribution of forces between the different layers of the reinforcements. This distribution of forces can be computed by a method developed by Juran and Schlosser (1978) and it is usually called the Local Stability method. This method was originally developed for soil reinforced by smooth strips with vertical wall and horizontal ground surface. However, Juran et al. (1990) generalized this method as will be explained in method (e).

The slip surface in this method is assumed to be a logarithmic spiral perpendicular to the ground surface at the top, Figure (2.8). The normal stress distribution on the slip surface is computed by Kött's equation,

$$\frac{\partial \sigma}{\partial \ell} + 2 \sigma \tan \frac{\partial \xi}{\partial \ell} = \gamma \cos(\xi + \phi) \cos \phi$$  \hspace{1cm} (2.15)

where:

- $\sigma$ = the normal stress acting on the slip surface,
- $\xi$ = angle between the tangent to the spiral and the vertical,
- $\phi$ = angle of shearing resistance of the soil,
- $\ell$ = length along the spiral,
- $\gamma$ = unit weight of the soil.

The reinforcement is assumed to be able to carry only axial forces. The axial force ($T$) at any reinforcement level is calculated by considering the horizontal equilibrium of a horizontal slice of thickness $S_y$, bounded by the wall face and the slip surface, Figure (2.8). Consequently, $T$ can be expressed as,
The solution of equations (2.15) and (2.16) gives the values of the tensile force (T) at each reinforcement level and the distance (X) between the locus of the slip surface and the vertical wall.

**e. Kinematical Limit Equilibrium Method**

Juran et al. (1990) extended method (d) by considering the bending stiffness of the reinforcement and allowing for inclined wall and ground surface. Here again, the slip surface is a unique logarithmic spiral with normal stress follow Kötter's equation, Figure (2.9.a). The axial force (T) in the reinforcement is computed by considering the horizontal equilibrium of a slice with thickness $S_v$ and bounded by the wall face and the slip surface. Juran et al. (1990) assumed that the state of stress in the reinforcement at the point of intersection with the slip surface follows the stress condition shown by the Mohr circle in Figure (2.9.b). Hence, Juran et al. (1990) expressed the shear force (Q) as a function of the axial force (T) and the reinforcement inclination using the equation,

$$Q = \frac{1}{2} \cot [2(\xi - \beta_t + d\beta_t)] \times T$$  \hspace{1cm} (2.17)

where:

- $\beta_t = \text{the initial slope of the reinforcement with respect to the horizontal,}$
- $d\beta_t = \text{the rotation of the reinforcement at the point of intersection with the slip surface,}$
- $\sqrt{\frac{2Q}{C_y DL_o^2}}$
Figure (2.8) Local Stability Method (From Mitchell and Villet 1987).
Figure (2.9.a) Slip Surface In The Kinematical Limit Equilibrium Method.

Figure (2.9.b) State Of Stress In The Reinforcement At The Point Of Intersection With The Slip Surface.

Figure (2.9) Kinematical Limit Equilibrium Method (From Christopher et al. 1990).
Juran et al. (1990) suggested to check the values of $T$ and $Q$, equations (2.16) and (2.18), with respect to the failure criteria given by Schlosser (1983), equations (2.7) through (2.14).

### 2.3.1.2 Methods That Assume A Deformation Pattern For The Reinforcement

#### a. Wu's Method

Wu (1976) assumed the deformation pattern shown in Figure (2.10) for the soil-reinforcement composite. The soil is penetrated by a vertical reinforcement and it is sheared in a horizontal zone of thickness ($z$). The distortion angle of the reinforcement ($\theta$) can be calculated using the equation,

$$\theta = \tan^{-1}\left(\frac{\Delta}{z}\right)$$

(2.18)

where:

$\Delta$ = the shear displacement.

Wu (1976) assumed that the reinforcement is flexible and can carry only axial forces. Wu (1976) assumed also that the reinforcement is sufficiently long such that it will fail by breakage. Hence, the tensile force in the reinforcement can be expressed as,

$$T_{\text{max}} = \sigma_f A_r$$

(2.19)

where:

$A_r$ = cross sectional area of the reinforcement.

By considering the horizontal and the vertical components of $T_{\text{max}}$, the contribution of the reinforcement to the composite shear strength can be written as,
\[ \Delta s_r = \frac{T_{\text{max}}}{A_s} (\sin \theta + \cos \theta \tan \phi) \]  

(2.20)

where:

\( \Delta s_r \) = the contribution of the reinforcement to the composite shear strength,

\( A_s \) = cross sectional area of the soil at the shear plane.

**b. Waldron's Method**

Waldron (1977) assumed the same deformation pattern given by Wu (1976), Figure (2.10). However, Waldron considered the frictional resistance between the soil and the reinforcement in the computation of the axial force in the reinforcement as follows.

Assuming that there is a complete bond between the soil and the reinforcement in the shear zone \( z \), then, the extension of the reinforcement can be computed as,

\[ \Delta L = z \sec \theta - 1 \]  

(2.21)

Waldron (1977) assumed that only a length \( L_f \) of the reinforcement is subjected to tension (where \( L_f \) is the distance between points M and Q in Figure (2.10)). Hence, Waldron computed the average tensile stress in the reinforcement by the expression,

\[ \sigma_{av} = \frac{\Delta L}{L_f} E_r \]  

(2.22)

Waldron (1977) assumed that the frictional resistance between the soil and the reinforcement \( \tau_r \) will be fully mobilized along the length \( L_f \). Then, by considering the equilibrium of an element of the reinforcement along the length \( L_f \), Figure (2.11), the following expression can be obtained,

\[ \frac{d\sigma_n}{dL} = \frac{4\tau_r}{D} \]  

(2.23)
where:

\( \sigma_a \) = tensile stress in the reinforcement element,
\( \tau_r = \sigma_f \tan \delta \),
\( \sigma_f \) = normal stress on the reinforcement element,
\( \delta \) = frictional angle between the soil and the reinforcement.

Waldron (1977) assumed that the tensile stress along the reinforcement is linearly distributed, Figure (2.12), and has a maximum value at the slip surface (point N in Figure (2.10)). Hence, equation (2.23) can be integrated as,

\[
\sigma_{\text{max}} = \frac{4\tau_r \times L_f}{D \times 2}
\]  

(2.24)

where:

\( \sigma_{\text{max}} \) = the maximum tensile stress in the reinforcement which occur at the point of intersection with the slip surface.

Therefore, the average tensile stress in the reinforcement can be written as,

\[
\sigma_{\text{av}} = \frac{\tau_r L_f}{D}
\]  

(2.25)

By equating equations (2.22) and (2.25), then,

\[
L_f = \left( \frac{D \times zE_r}{\tau_r} \right)^{\frac{1}{2}} (\sec \theta - 1)^{\frac{1}{2}}
\]  

(2.26)

Substitute in equation (2.24), then,

\[
\sigma_{\text{max}} = \left( \frac{4\tau_r \times zE_r}{D} \right)^{\frac{1}{2}} (\sec \theta - 1)^{\frac{1}{2}}
\]  

(2.27)
Undeformed Reinforcement

Deformed Reinforcement

Figure (2.10) Assumed Deformation Pattern In Wu's Model (1976) And Waldron's Model (1977).

Figure (2.11) Cylindrical Reinforcement Element At Equilibrium (Waldron 1977).

Figure (2.12) Distribution Of The Tensile Stress In The Reinforcement (Waldron 1977).
The axial force in the reinforcement at the point of intersection with the slip surface can be written as,

$$T_{\text{max}} = \sigma_{\text{max}} A_r$$  \hspace{1cm} (2.28)

Substitute (2.27) and (2.28) in (2.20), then,

$$\Delta s_r = \frac{A_r}{A_s} \left( \frac{4 \tau_t \times z E_t}{D} \right)^{\frac{1}{2}} (\sec \theta - 1)^{\frac{1}{2}} (\sin \theta + \cos \theta \times \tan \phi)$$  \hspace{1cm} (2.29)

c. Gray And Ohashi's Method

Gray and Ohashi (1983) presented a model similar to Waldron model. However, Gray and Ohashi (1983) assumed that the tensile stress in the reinforcement is parabolically distributed, (instead of the linear distribution assumed by Waldron). Therefore the tensile stress in the reinforcement at the shear plane was expressed as,

$$\sigma_{\text{max}} = \left( \frac{8 \tau_t}{3D} \times z E_t \right)^{\frac{1}{2}} (\sec \theta - 1)^{\frac{1}{2}}$$  \hspace{1cm} (2.30)

Gray and Ohashi (1983) considered also the case when the reinforcements initially inclined by an angle (i) to the slip surface, Figure (2.13). In that case, the distortion angle of the reinforcement ($\zeta$) was expressed as,

$$\zeta = \tan^{-1} \left( \frac{1}{A/z + 1/\tan(i)} \right)$$  \hspace{1cm} (2.31)

And the contribution of the reinforcement to the composite shear strength is,

$$\Delta s_r = \sigma_{\text{max}} \left( \frac{A_r}{A_s} \right) \left( \sin (90 - \zeta) + \cos (90 - \zeta) \times \tan \phi \right)$$  \hspace{1cm} (2.32)
**d. Shewbridge And Sitar’s Method**

Shewbridge and Sitar (1985) developed a model in which they considered the tensile and the bending stiffness of the reinforcement. Shewbridge and Sitar (1985) assumed that the deformation pattern for the soil-reinforcement composite is a smooth asymptotic curve given by the equation, Figure (2.14),

\[ X_1 = \frac{\Delta}{2} - \frac{\Delta}{2} \times e^{-b|x_2|} \]  

(2.33)

where:

- \( X_1 \) = coordinate axis parallel to the shear direction,
- \( X_2 \) = coordinate axis perpendicular to the shear direction,
- \( \Delta \) = the displacement of the free field,
- \( b \) = parameter depends on the reinforcement stiffness, the reinforcement concentration, and the bond condition between the soil and the reinforcement.

Shewbridge and Sitar obtained the parameter (b) from the results of direct shear tests performed on different soil-reinforcement composites.

Shewbridge and Sitar used the above deformation pattern along with relations from continuum mechanics to derive the work required to shear an unreinforced soil (\( W_1 \)) and the work required to shear a reinforced soil (\( W_2 \)). The difference between \( W_2 \) and \( W_1 \) is the additional work due to the reinforcement existence. Then, by dividing this additional work by the displacement (\( \Delta \)), the average increase in the composite shear strength due to the reinforcement is obtained.
Figure (2.13) Assumed Deformation Pattern In Gray And Ohashi's Model (1983).

Figure (2.14) Assumed Deformation Pattern In Shewbridge And Sitar's Model (1985).
2.3.1.3 Summary And Discussion

Although the limit equilibrium approach was historically the first procedure used for designing reinforced soils, it is still the most common method in use today. In the limit equilibrium approach, an evaluation of the safety factor against an assumed failure mechanism is performed. However, this procedure does not provide any information concerning the working stresses or the deformations of the reinforced soil. Consequently, to obtain safe design using the limit equilibrium method, different safety factors are imposed to indirectly maintain the stresses everywhere within the allowable strength and to limit the soil movements. Unfortunately, the margin of safety against failure alone is not sufficient for judging the damages which can happen to the surrounding building or installations.

Numerous methods have been developed based on the limit equilibrium approach. A review of these methods shows the following. Concerning the methods described in Section (2.3.1.1), the ultimate forces in the reinforcement were computed by considering the soil-reinforcement interaction mechanisms. The Davis method, the German method, and the local stability method consider only the frictional resistance between the soil and the reinforcement. On the other hand, the French method and the Kinematical Limit Equilibrium method consider both the frictional resistance between the soil and the reinforcement and the passive soil pressure on the reinforcement. A generalized method for analyzing reinforced soils should consider both mechanisms (Mitchell and Villet 1987; Schlosser and De Buhan 1990).

The frictional coefficient (μ) between the soil and the reinforcement represents an important parameter in the design methods described in Section (2.3.1.1). The Davis method and the German method assume that μ=\tan\delta, where \delta is the frictional angle
between the soil and the reinforcement materials. However, as explained in Section (2.1), the results of full-scale and laboratory tests showed that the pull-out resistance of the reinforcement is larger than the value computed using $\mu = \tan \delta$. Schlosser et al. (1983) stated that this phenomenon occurs due the restrained dilatancy of the soil. Therefore, in the French method and the Kinematical Limit Equilibrium method, the frictional resistance along the reinforcement $f_{\text{max}}$ is used as input in the analysis. It was recommended that the value of $f_{\text{max}}$ should be obtained from pull-out tests. Also, Elias and Juran (1990) modified the Davis method such that the pull-out resistance of the reinforcement is computed using $f_{\text{max}}$ instead of $\mu \sigma$. This modified version of the Davis method is called "the Modified Davis method".

The Davis method, the German method, and the French method provide information about the global factor of safety along the potential failure surface. However, observations from full-scale structures and reduced-scale laboratory tests showed that pull-out failure is a progressive phenomenon which is generally started by the pull-out of the upper reinforcements (Elias and Juran 1990). Therefore, methods which consider the local stability at each reinforcement level (e.g., the Local Stability method and the Kinematical Limit Equilibrium method) have been recommended (Elias and Juran 1990).

Several criticisms have been made of the Kinematical Limit Equilibrium method. Leshchinsky (1991) showed that the Kinematical Limit Equilibrium method does not satisfy the stress boundary condition and the vertical equilibrium requirement. The method for computing the shear forces in the reinforcement, equation (2.16), is questionable (Jewell 1991). Also, Juran et al. (1990) claimed that the Kinematical Limit Equilibrium method can be used to predict the forces in the reinforcement at working condition.
However, according to Schlosser and De Buhan (1990), the use of Kötters's equation is mathematically equals to the fact that the soil is in at-failure condition along the slip surface.

In spite of the shortcomings in the above methods, they have been used as the available state-of-art to design reinforced soils. The margin factor of safety used in these methods can overcome (or hide) their shortcomings. The design by the above methods with using the recommended safety factors is generally conservative (Mitchell and Villet 1987). The Federal Highway Administration, Christopher et al. (1990), recommended that both local stability analysis by the Kinematical Limit Equilibrium method and global stability analysis by the Modified Davis method (Elias and Juran 1990) be done. The Modified Davis method was recommended because of its availability in the public domain. However, to be consistent with the assumptions of the Kinematical Limit Equilibrium method, it would be more appropriate to use the French method to perform the global stability analysis.

The methods described in Section (2.3.1.2) contain many over-simplified assumptions and hence, they did not find the wide acceptance given to the methods described in Section (2.3.1.1). Although the thickness of the shear zone (z) in Wu's method, Waldron's method, and Gray and Ohashi’s method has a significant effect on the analysis, none of the above methods gave any verified estimate for the value of (z). Wu (1976) considered only the tensile strength of the reinforcement material. Waldron (1977) and Gray and Ohashi (1983) attempted to consider also the frictional resistance along the reinforcement. However, the method of including the frictional resistance in these methods is questionable. The assumption that the frictional resistance will develop only along a length (L_f) around the shear zone is inconsistent with the soil-reinforcement interaction behavior. The development of the frictional resistance depends on the relative displacement between the soil and the reinforcement in the axial direction of the
reinforcement. If the relative displacement between the soil and the reinforcement is sufficient to develop the frictional resistance along the length \(L_f\) of the reinforcement, it will be sufficient to develop the frictional resistance along the reminder of the reinforcement. The above inconsistency in Waldron's method and Gray and Ohashi's method occurred because of attempting to compute the frictional resistance by considering the reinforcement deformation in the horizontal direction, equation (2.21), instead of considering the relative displacement between the soil and the reinforcement in the axial direction.

In Shewbridge and Sitar model, the results of the analysis are significantly affected by the parameter \((b)\). Although Shewbridge and Sitar obtained this parameter experimentally, the use of the values of \((b)\) suggested by Shewbridge and Sitar can not be generalized since Shewbridge and Sitar obtained the values of \((b)\) for limited number of soil-reinforcement composites using a special shear box. Since Shewbridge and Sitar assumed that the reinforcement will be elastic all times, the use of a deformation pattern such as that given by Hetenyi (1946) would be more general. The method of including the frictional resistance in Shewbridge and Sitar model suffers from the same shortcoming as in Waldron's model and Gray and Ohashi's model.

Wu (1976), Waldron (1977) and Gray and Ohashi (1983) neglected the bending stiffness of the reinforcements. As explained before, the bending stiffness can not be neglected in the analysis of many reinforced-soil systems (Schlosser and De Buhan 1990). Shewbridge and Sitar (1985) considered the bending stiffness of the reinforcement but they neglected the limiting value of this stiffness (i.e., they assumed that the reinforcement is elastic all times). A limit equilibrium analysis of reinforced-soil systems that neglect the limiting value of the bending moment of the reinforcement can significantly overestimate the contribution of the reinforcement shear forces to the composite strength.
In spite of the shortcomings of the methods described in Section (2.3.1.2), the attempt to include the deformation as a variable in the analysis is worth noticing. Results of full-scale and laboratory tests showed that the development of the soil-reinforcement interaction mechanisms are highly dependent on the deformations (Mitchell and Villet 1987). Therefore, for accurate analysis of a reinforced-soil system, the deformation should be included. However, deformation calculations are generally difficult to perform and need more rigorous procedures than those given in Section (2.3.1.2).

It is interesting to note also Waldron's attempt to consider the effect of the horizontal deformation in the computation of the axial forces in the reinforcement. From the mechanics point of view, the above attempt equals the consideration of the geometric nonlinearity of the problem. For flexible reinforcement subjected to large displacements, the effect of the geometric nonlinearity may be significant. Therefore, it may be necessary to consider the geometric nonlinearity if an appropriate deformation analysis is performed.

2.3.2 Analyses Based On Empirical Methods For Considering Working Stress

By definition, working stress analysis should be based on the stress-deformation behavior of the reinforced soil under externally applied loads (Mitchell and Villet 1987). However, since the calculations of the deformations of reinforced soils are difficult to perform, empirical methods were developed to estimate the working stress in the reinforcements. These empirical methods can be classified as,

1- methods used empirical earth pressure diagrams.
2- methods based on observations of full-scale tests.

The above methods are described in the following subsections.
2.3.2.1 Empirical Earth Pressure Diagrams

Juran and Elias (1987) assumed that the deformation of the reinforced soil structure (nailed wall) will be in the range of the deformation observed for braced excavation. Therefore, they used the empirical earth pressure diagrams suggested by Terzaghi and Peck (1967), Figure (2.15), to estimate the working forces in the reinforcements. Using these diagrams, the reinforcement working axial force can be estimated as,

\[ T = T_N (\gamma H) \times S_h S_v \]  

(2.34)

where:

\[ T_N = \left\{ \begin{array}{l}
0.65 \times k_a & \text{for cohesionless soils} \\
\kappa_a \times \left( 1 - \frac{4c}{\gamma H} \times \frac{1}{\sqrt{k_a}} \right) & \text{for cohesive soils}
\end{array} \right. \]

(2.35)

\( k_a = \) the Rankin active coefficient of earth pressure,
\( H = \) the total excavation height,
\( \gamma = \) the unit weight of the soil,
\( S_h \) and \( S_v = \) the horizontal and vertical spacing between the reinforcements.

2.3.2.2 Observations From Full Scale Tests

Observations made on many instrumented reinforced-soil structures showed that the coefficient of earth pressure (\( k \)) can be equal to or greater than the at rest earth pressure coefficient (\( k_0 \)) at the top of the wall. However, in the lower part of the wall, the earth pressure is smaller than that in the upper part of the wall and it is approximately equal to the active earth pressure. Christopher et al. (1990) reported that the coefficient of earth pressure on the wall (\( k \)) is dependent on the reinforcement extendibility and the depth. Figure (2.16) shows a conservative distribution for the coefficient of earth pressure (\( k \)) as a function of the reinforcement type and the depth (Christopher et al. 1990). This Figure
can be used to estimate the horizontal stresses at each reinforcement level ($\sigma_h$). Then, the axial force in the reinforcement at working condition can be written as (Christopher et al. 1990),

$$T = \sigma_h \times S_h \times S_v$$  \hspace{1cm} (2.36)

### 2.3.2.3 Summary And Discussion

Because of the difficulty to perform deformation calculations, empirical methods to estimate the working stress in the reinforcement have been developed. The use of these methods presents sever limitations. In particular, the earth pressure diagrams used in these methods correspond to special cases with simple geometry of a vertical wall and a horizontal ground surface. Therefore, these methods can not be used to assess the effect of design parameters such as inclination of the wall, inclination of ground surface, etc. Also, these methods do not provide any estimates of the shear forces and the bending moments that can develop in the reinforcement. Due to the above limitations, the Federal Highway Administration, Christopher et al. (1990), recommended to use the above methods only to perform preliminary design.

### 2.3.3 Analyses Based On The Winkler Hypothesis

This method has been used extensively in the design of pile foundations. For reinforced earth, this method was used to study the stability of slopes reinforced by nails or piles (Sommer 1979; Gudehus 1983; Carter and Gigan 1983). Two models based on this approach have been developed. Descriptions for these two models are given in the following subsections.
Notes:
- Vertical cut slope
- Horizontal upper surface

\[ T_N = 0.65 K_a \]

\[ K_a = \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \]

\( T_N \) = lateral earth pressure  
\( \gamma_h \) = overburden pressure

\[ C L A Y E Y \]
\[ T_N = 0.2 \gamma H \rightarrow 0.4 \gamma H \]

\[ C L A N D; \quad T_N = K_a \left( 1 - \frac{4c}{\gamma H \sqrt{K_a}} \right) \leq 0.65 K_a \]

\[ S A N D; \quad \frac{c}{\gamma H} \leq 0.05 \]

Figure (2.15) Empirical Earth Pressure Design Diagrams Used For Nailed Walls (From Christopher et al. 1990).

Figure (2.16) Variation Of The Coefficient Of Lateral Earth Pressure As A Function Of Depth And Reinforcement Type (From Christopher et al. 1990).
2.3.3.1 Juran et al.'s Model

Juran et al. (1981) assumed that the reinforcement can be treated as a laterally loaded pile. Therefore, they adopted the method normally used to analyze laterally loaded piles. In this method, the pile is considered as an elastic beam supported on nonlinear lateral springs (Winkler-type foundation). The governing equation for this problem is,

\[
\frac{d}{dx^2} \left( E_r I_r \frac{dv_y^2}{dx^2} \right) = p_y
\]  

(2.37)

where:

- \( E_r \) = Young's modulus of the reinforcement material,
- \( I_r \) = moment of inertia of the reinforcement cross section,
- \( v_y \) = lateral displacement of the reinforcement,
- \( p_y \) = lateral soil pressure on the reinforcement,
- \( = DC_y (w_y - v_y) \)
- \( C_y \) = coefficient of subgrade reaction in the lateral direction,
- \( w_y \) = the free field soil displacement in the lateral direction,
- \( D \) = reinforcement diameter.

Most commonly, equation (2.37) is solved numerically using the finite difference method (e.g., Reese 1977). The solution for this equation gives the distribution of the lateral displacements, shear forces, and bending moments in the reinforcement.
2.3.3.2 Wu et al.'s Model

In Juran et al.'s model, only the bending stiffness of the reinforcement was considered. However, for most reinforced-soil systems, the axial forces in the reinforcement constitute the dominant reinforcing effect and it cannot be ignored (Schlosser and De Buhan 1990). Therefore, Wu et al. (1988) developed a model in which they considered both the axial and bending stiffnesses of the reinforcement. In this model, the reinforcement was represented as a tie (or beam column) supported on nonlinear axial and lateral springs. This model was also used in the analysis of generally loaded piles (e.g., Awoshika and Reese 1971).

The governing equations in the above model can be written as,

\[ E_r A_r \frac{d^2 v_x}{dx^2} = p_x \quad (2.38) \]

\[ \frac{d}{dx^2} \left( E_r I_r \frac{dv_y}{dx^2} \right) - \frac{d}{dx} \left( T_x \frac{dv_y}{dx} \right) = p_y \quad (2.39) \]

where:

- \( A_r \) = cross sectional area of the reinforcement,
- \( v_x \) = the displacement of the reinforcement in the axial direction,
- \( p_x \) = frictional stress developed on the reinforcement circumference,
- \( = BC_x (w_x - v_x) \)
- \( B \) = reinforcement parameter,
- \( C_x \) = coefficient of subgrade reaction in the axial direction,
- \( w_x \) = the free field soil displacement in the axial direction,
- \( T_x \) = the axial force at the considered point along the reinforcement.
Wu et al. (1988) adopted a closed-form solution given by Scott (1981) to solve the above system of equations. However, this system of equations can more conveniently be solved numerically (Awoshika and Reese 1971). The solution of the above system of equations gives the reinforcement displacements in the axial and lateral directions and the distribution of the axial forces, shear forces, and bending moments in the reinforcement.

2.3.3.3 Summary And Discussion

Two models have been developed based on the procedure described in this Section. In Juran et al.'s model (1981), only the bending stiffness of the reinforcement is considered. However, for most reinforced-soil systems, the axial force in the reinforcement constitutes the dominating reinforcing effect and it can not be ignored, (Schlosser and De Buhan 1990). On the other hand, Wu et al.'s model (1988) considered both the axial and the bending stiffness of the reinforcement. However, this model has the disadvantage of assuming that the reinforcement material is linearly elastic at all times. This assumption can overestimate of the reinforcement's contribution to the composite shear strength.

As stated before, working stress analysis should be based on the stress-deformation behavior of the reinforced soil under an externally applied load (Mitchell and Villet 1987). The method described in this Section has the advantage of computing the reinforcement forces based on its deformations and taking account of the soil-reinforcement interaction mechanisms. Therefore, this method can be considered as a basis for working stress analysis. However, the application of this method encounters two difficulties. First, the definition of the externally applied loads can not be made easily. This difficulty can be overcome by making the assumption that the soil consists of two zones, a moving zone and a resisting zone, separated by a slip surface. Also, it can be assumed that the
soil in the moving zone will rotate (or slide) on the slip surface as a quasi-rigid body. By making the above assumptions, the reinforcement forces due to any imposed soil deformation can be computed and hence the requirement of the working stress analysis will completely be met. The second difficulty of using the above method lies in determining the subgrade coefficients $C_x$ and $C_y$. In pile analyses, the coefficients $C_x$ and $C_y$ are determined from nonlinear soil response curves which are known as the t-z and the p-y curves. A vast literature on the soil response curves for piles can be found (Matlock 1970; Reese et al. 1974; 1975; 1984; Kraft et al. 1981; among others). However, experience with reinforced soil has been limited and the applicability of the soil response curves for piles has not been yet proven.

2.3.4 Analyses Performed Using The Finite Element Method

The finite element method has been used by several investigators to analyze the behavior of reinforced soils (Romstad et al. 1976; Shen et al. 1981; Juran et al. 1985; Adib et al. 1990; Plumelle et al. 1990; Ho and Rowe 1994). A brief review of these studies is given herein.

Romstad et al. (1976) assumed that the reinforced-soil mass can be treated as a homogenous material with orthotropic properties. This modeling method can not represent any of the soil-reinforcement interaction mechanisms.

Shen et al. (1981) performed two dimensional, finite element analyses for reinforced walls. In these analyses, Shen et al. (1981) represented the soil, the reinforcement and the interface between the soil and the reinforcement separately. The soil was represented by plain strain elements and the hyperbolic stress-strain material model developed by Duncan and Chang (1970). The reinforcements were represented by bar elements and they were assumed to be elastic all times. The soil-reinforcement interface
was represented by an element developed by Goodman et al. (1968). The interface element was assumed to be elastic-perfectly plastic governed by the Coulomb's failure criterion. Shen et al. (1981) compared the results of the above model with the results of full-scale tests and they found good agreement.

Adib et al. (1990) performed finite element analyses similar to those performed by Shen et al. (1981). However, Adib et al. (1990) used a computer program developed by Seed (1983) which can represent the effect of soil compaction.

Ho and Rowe (1994) performed two dimensional finite element analyses similar to those carried out by Shen et al. (1981). However, Ho and Rowe (1994) represented the soil material by the Mohr-Coulomb model with nonassociated flow rule, instead of the hyperbolic model in Shen et al.’s analyses.

Other finite element analyses similar to the ones outlined above were performed by Herrmann and Al-Yassin (1978); Juran et al. (1985); Plumelle et al. (1990).

2.3.4.1 Summary And Discussion

In spite of the capabilities of the finite element method, the finite element analyses performed for reinforced soils suffered from all or some of the following shortcomings.

1- All analyses were performed assuming that plain strain condition prevails. The two dimensional analysis has the shortcoming that reinforcements such as nails, strips, etc., are modeled as sheets.

2- The choices of the material models for the soil, the reinforcement, or the interface were inappropriate in some analyses. For instance, the representation of the soil material by the hyperbolic model can not represent the behavior of real soil at or after failure (Duncan 1994). Also, the hyperbolic model can not represent the soil
dilation which is an important factor in the behavior of the soil-reinforcement interaction. On the other hand, the representation of the reinforcement material as linear elastic does not allow the calculation of the ultimate forces which can be carried by the reinforcement.

3- The bending stiffness of the reinforcement was neglected in most analyses. This implies the neglect of one of the fundamental soil-reinforcement interaction mechanisms. As stated before, this mechanism can not be ignored for most reinforced-soil systems (Schlosser and De Buhan 1990).

4- In most analyses, several unverified assumptions were made (e.g., the assumption that there is a complete bond between the soil and the reinforcement, the assumption that the geometric nonlinearity can be neglected, etc.). The results of the finite element analysis can be inaccurate if the problem is sensitive to the above assumptions.

2.4 Conclusions

1- The soil reinforcement interaction involves two mechanisms; namely, friction and passive soil pressure. An accurate analysis of reinforced soils should consider both mechanisms.

2- The design methods based on the limit equilibrium approach are the most common methods in use today. These methods provide only an evaluation of the stability against an assumed failure mechanism. However, these methods do not provide any information concerning the actual stresses or the deformations of the reinforced-soil system.
3- Empirical methods to compute the forces in the reinforcement at working conditions have been developed. However, these methods can only be used in aid of measurements made on actual structures.

4- The procedure based on the Winkler hypothesis can be considered as a rational method to compute the reinforcement forces at working condition. That is because of the fact that this procedure can compute a complete load-deformation behavior for the reinforcement taking account of the soil-reinforcement interaction mechanisms.

5- Finite element analyses for reinforced soils have been carried out. However, these analyses had several shortcomings. These shortcomings can be attributed to the complexity of modeling the problem and the high cost of the finite element analysis.

2.5 Scope And Objectives

Although several analytical procedures for reinforced soils have been developed, the behavior of the soil-reinforcement interaction has not been completely understood. The objectives of this research are to develop a better understanding of the soil-reinforcement interaction mechanisms and to develop design procedures which can accurately consider the above mechanisms. To achieve these objectives, the following steps were performed.

1- The finite element package "ABAQUS" is used to perform careful analyses for the problem. The results of these analyses help to develop better understanding of the soil-reinforcement interaction mechanisms and to determine the important design parameters which should be included in the analysis.

2- Based on the experience gained from the analyses performed in (1), a beam model for computing the ultimate forces in the reinforcement is developed. The results of
this model are represented in the form of design charts which can be used in practice. These results can be used to perform limit equilibrium analyses for reinforced soils.

3- A simplified finite element model is developed. The difference between this model and the model mentioned in (1) is that the soil is represented in this model using the Winkler hypothesis while it is represented in model (1) as a continuum. This model can be used to predict the behavior of reinforced soils at working as well as at failure conditions.

The above three steps are described in the following two chapters.
CHAPTER III

Finite Element Analysis Of Soil-Reinforcement Interaction

3.1 Introduction

In the previous chapter, a review of the available soil-reinforcement interaction models is presented. It is also explained that most of the models used in practice for designing reinforced soils are based on the limit equilibrium approach. This approach has the disadvantage that it does not provide any information concerning the actual stresses or deformations of the reinforced-soil system. On the other hand, models that consider the working condition have been developed. However, these models contain many oversimplified assumptions and hence, they might not be sufficiently accurate.

The finite element method provides a convenient tool to overcome the shortcomings mentioned above. It allows the consideration of many factors excluded in other analyses and it allows for a complete computation of the distribution of stresses and deformations in the components of the studied problem. Therefore, the finite element method is employed in this study to perform three dimensional, nonlinear analyses of the soil-reinforcement interaction problem. The scheme of these analyses can be summarized as follows:
1- The problem is analyzed using different assumptions concerning the soil, the reinforcement, the contact between the soil and the reinforcement, and the kinematics. For convenience, these assumptions will be called the model assumptions. Based on the results of the above analyses, an evaluation of the effect of the model assumptions on the predicted behavior of the soil-reinforcement composite is performed. The results are then used to choose the most appropriate assumptions and finite element representation for the problem.

2- The chosen finite element representation is used to simulate laboratory experiments on different soil-reinforcement composites. Comparisons between the results of the finite element analyses and the results of the experiments are made to check the validity of the finite element representation. The laboratory experiments simulated in this study are direct shear tests by Gray and Ohashi (1983) and by Shewbridge and Sitar (1985).

3- The chosen finite element representation is used to perform a parametric study of the problem. The parameters included in this study are the soil type, the reinforcement stiffness, and the confining pressure on the soil-reinforcement composite. The results of this study help to develop better understanding of the behavior of the soil-reinforcement composites and to determine the important parameters which should be included in the analysis.

4- Based on the experience gained from the analyses performed in (1) through (3), a model for computing the ultimate forces in the reinforcement is developed. This model will be referred as "The Beam Model". The results of this model are represented in the form of design charts which can be used in practice. These results can be used to perform limit equilibrium analyses for reinforced soil systems.

5- The results obtained in (2) and (3) are used to perform a detailed study of the distribution of stresses and deformations in different soil-reinforcement composites.
This chapter is divided into eight Sections. In Section (3.2), a description of the finite element model is given. In Section (3.3), the study of the effect of the model assumptions on the behavior of the composite is given. Section (3.4) gives the comparison between the results of the finite element analyses and the results of the laboratory experiments. In Section (3.5), the parametric study of the problem is given. Section (3.6) describes the beam model for computing the ultimate forces in the reinforcement. Section (3.7) gives the detailed study of the distribution of stresses and deformations in the different soil-reinforcement composites. Finally, a summary of the results and conclusions obtained from this study are given in Section (3.8).

3.2 Finite Element Model

The finite element method is used to analyze reinforced soils subjected to the loading condition imposed in the direct shear test. Figure (3.1.a) shows the arrangement of the direct shear test. In this test, the soil-reinforcement composite is placed inside a box which has two halves, a movable half and a fixed half. The load is applied in two steps. In the first step, a uniform pressure ($\sigma_0$) is applied on the top surface of the soil. In the second step, the soil-reinforcement composite is sheared by applying incremental displacements on the movable half of the box.

The finite element package ABAQUS, version 5.3 (Hibbitt, Karlson, and Sorensen, Inc. 1990) is used to perform three dimensional, nonlinear, finite element analyses simulating the above test. Detailed description of the finite element meshes, elements, material models, kinematics, loading sequences and boundary conditions which are used in this study are given in the following subsections.
3.2.1 Finite Element Meshes And Elements

Finite element meshes of different dimensions, number of reinforcement, and arrangements are used to perform the analyses of parts (1) through (3) described in Section (3.1). Detailed description of these meshes will be given in Sections (3.3), (3.4), and (3.5).

Figures (3.1.b) through (3.1.d) show the different elements and boundaries used in the finite element model. The soil is represented in this model by three dimensional, eight nodal, solid elements (C3D8 elements in ABAQUS element library). The reinforcement is represented by spatial, two nodal, beam elements (B31 elements in ABAQUS element library). The sides and the bottom of the box and the loading plate at the top of the soil, Figures (3.1.b), (3.1.c), and (3.1.d), are represented as rigid surfaces using the rigid surface option in ABAQUS element library.

The interface between the soil and the box surfaces and the interface between the soil and the loading plate are modeled by rigid-surface contact elements (IRS4 elements in ABAQUS element library), Figures (3.1.b), (3.1.c) and (3.1.d). The IRS4 element represents the interaction between a three dimensional deformable body (the soil) and a rigid surface (the shear box sides) and allows sliding or separation to occur between the deformable body and the rigid surface.

The soil-soil interface at the potential failure surface, the surface between the movable and the fixed halves of the box, is represented by the slide-line contact elements (ISL31 elements in ABAQUS element library), Figures (3.1.b) and (3.1.c). The ISL31 element represents the interaction between two, three-dimensional, deformable bodies and allows sliding or separation to occur between the two deformable bodies.
Two conditions are considered for the interface between the soil and the reinforcement: complete bond and slip. The complete bond condition is modeled by giving the soil and the reinforcement the same nodal numbering. In the slip condition, the interface between the soil and the reinforcement is modeled by the slide-line contact elements (ISL31) as shown in Figure (3.2).

Detailed description of the elements mentioned above and the formulation for these elements can be found in ABAQUS, Version 5.3 (Hibbitt, Karlson, and Sorensen, Inc. 1990).

3.2.2 Material Models and Properties

3.2.2.1 Soil

Two models are used to represent the soil: the Drucker-Prager model (Drucker et al. 1957) and the Cap model (DiMaggio and Sandler 1971). In the Drucker-Prager model, Figure (3.3), the yield surface is defined as,

$$ F_D = q - (\tan \beta) p - d = 0 $$

(3.1)

where:

$q$ = Mises equivalent stress

$$ q = \frac{1}{\sqrt{2}} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6\sigma_{xy}^2 + 6\sigma_{yz}^2 + 6\sigma_{zx}^2 \right]^{\frac{1}{2}} $$

(3.2)

$p$ = mean normal stress

$$ p = \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) $$

(3.3)

$\beta$ = slope of the yield surface in the p-q plane,

d = q intercept of the yield surface.
Figure (3.1.a) Isometric View

Figure (3.1.b) Sectional Elevation

Figure (3.1.c) Sectional Side View

Figure (3.1.d) Sectional Plane

Figure (3.1) Elements Used To Represent Direct Shear Test.
Node (1): A node in the reinforcement element, (beam element).

Nodes (2) Through (9): Nodes on the periphery of the reinforcement. These nodes are constrained by connecting them to node (1) by rigid beams.

Nodes (10) Through (17): Nodes on the soil elements. These nodes are in contact with nodes (2) through (9).

Figure (3.2) The Interface Between Soil And Reinforcement.
Inside the yield surface, the soil is assumed to be elastic. When the stress reaches the yield surface, plastic flow occurs. The direction of the incremental plastic strain is normal to the potential function. In ABAQUS, the potential function for the Drucker-Prager model is defined by the equation,

$$g_D = q - (\tan \eta)p = 0 \quad (3.4)$$

where:

$$\eta = \text{dilation angle in the p-q plane.}$$

Note that if $\eta$ is equal to $\beta$, the flow rule is associated, otherwise, the flow rule is non-associated.

The material properties in the Drucker-Prager model are the Young's modulus ($E_s$), the Poisson's ratio ($\nu_s$), $d$, $\beta$, and $\eta$. The material properties $d$, $\beta$, and $\eta$ can be related to the soil cohesion ($c$), the angle of internal friction in plane strain ($\phi_{ps}$) and the dilation angle ($\psi$) by the following equations (Walters and Thomas 1982; ABAQUS, version (5.3), Hibbitt, Karlson, and Sorensen, Inc. 1990),

$$\eta = \tan^{-1} \left( \frac{\sqrt{3} \sin \psi}{1 + \frac{1}{3} \sin^2 \psi} \right) \quad (3.5)$$

$$d = \frac{9c \cos \phi_{ps}}{\sqrt{3(9 - \tan^2 \eta)} + \sin \phi_{ps} \tan \eta} \quad (3.6)$$

$$\beta = \tan^{-1} \left( \frac{9 \sin \phi_{ps}}{\sqrt{3(9 - \tan^2 \eta)} + \sin \phi_{ps} \tan \eta} \right) \quad (3.7)$$

Also, the angles $\phi_{ps}$ and $\psi$ can be related to the angle of internal friction in direct shear ($\phi_{ds}$) and the critical state angle ($\phi_{cv}$) by the following equations (Jewell and Wroth 1987),
\[
\phi_{ps} = \phi_{cv} + 0.8\psi \quad (3.8)
\]

\[
\sin \phi_{ps} = \frac{\tan \phi_{es}}{\cos \psi (1 + \tan \psi \tan \phi_{es})} \quad (3.9)
\]

In the Cap model, the yield function consists of the Drucker-Prager surface and a cap, Figure (3.4.a). The Drucker-Prager surface is given by equation (3.1). The cap is,

\[
F_c = \sqrt{(p - p_a)^2 + (Rq)^2} - R(d + p_a \tan \beta) = 0 \quad (3.10)
\]

where:

\[R = \text{material property which control the shape of the cap},\]

\[p_a = \frac{p_a - Rd}{1 + R \tan \beta} \quad (3.11)\]

\[p_b = \text{the hydrostatic stress at the cap position, Figure (3.4.a)}.\]

The position of the cap \((p_a)\) is related to the plastic volumetric strain \((e_v^p)\) by the equation,

\[p_b = p_b(e_v^p) \quad (3.12)\]

The above equation is the hardening / softening law. In ABAQUS, equation (3.12) is provided by the user as a piecewise linear function.

Inside the yield surface given by equations (3.1) and (3.10), the soil is assumed to be elastic. When the stresses reach the yield surface, plastic flow occurs. The direction of the incremental plastic strain is normal to the potential function. ABAQUS uses associated flow rule on the cap and non-associated flow rule on the Drucker-Prager surface. The potential function on the cap is,

\[g_c = \sqrt{(p - p_a)^2 + (Rq)^2} \quad (3.13)\]

and the potential function on the Drucker-Prager surface is,
\[ g_D = \sqrt{\left[(p_a - p)\tan\beta\right]^2 + q^2} \]  

(3.14)

Note that the two elliptical portions given by equations (3.13) and (3.14) form a continuous and smooth potential surface, Figure (3.4.b). The material properties in the Cap model are \( E_s, v_s, d, \beta, R \), the initial position of the cap \( (p_a) \), and the hardening/softening law defined by equation (3.12).

Comparing the Drucker-Prager model and the Cap model, the following can be observed. In both models, dilation occurs when stresses reach the Drucker-Prager yield surface. In the classical Drucker-Prager model, the potential function given by equation (3.4) with \( \eta = \beta \) is used to compute the direction of the plastic strain increment. The horizontal component of this strain represents the soil dilation. The amount of dilation computed using the classical Drucker-Prager model is usually larger than the dilation observed in tests (Davis 1968). To reduce this dilation, two alternatives can be used. In the first alternative, the dilation is reduced by using non-associated flow rule with \( \eta < \beta \). The other alternative is to use the Cap model. The addition of the cap to the analysis helps to reduce the volumetric strain as follows. When the soil dilates on the Drucker-Prager yield surface, the cap softens following equation (3.12). This softening is associated with a modification of the potential function along the Drucker-Prager yield surface with a reduction of the horizontal component of the plastic strain.

In this study, analyses with using the Drucker-Prager model and the Cap model are compared. To study the effect of the soil dilation on the behavior of the composite, analyses with different values of the parameters which control the dilation are performed. The parameter that controls the dilation in the Drucker-Prager model is the angle \( \eta \). The parameters that control the dilation in the Cap model are the cap hardening/softening law,
Figure (3.3) Drucker-Prager Model: Yield Surface In The p-q Plane.

Drucker Prager Surface, $F_D$

$\beta$

$\eta$

$\gamma$

$p_a$

$P_a$

$P_b$

$R(d + p_a \tan \beta)$

Figure (3.4.a) Cap Model: Yield Surface In The p-q Plane.

Cap, $F_C$

$d + p_a \tan \beta$

$R(d + p_a \tan \beta)$

Figure (3.4.b) Cap Model: Flow Potential In The p-q Plane

$g_D$ (On Drucker-Prager Surface)

$g_C$ (On Cap)

$d + p_a \tan \beta$

$R(d + p_a \tan \beta)$

$p_a$

$P_a$

$P_b$
equation (3.12), and the initial position of the cap \( p_{\text{in}} \). The initial position of the cap affects the soil dilation because the cap is not allowed to soften behind this position.

### 3.2.2.2 Reinforcement

Two material models are used to represent the reinforcement: linear-elastic and ideally elastic-plastic. For the linear-elastic material, the material properties are the Young's modulus \( E_r \) and the Poisson's ratio \( \nu_r \). For the ideally elastic-plastic material, the material properties are \( E_r, \nu_r, \) and the yield strength \( \sigma_y \).

### 3.2.2.3 Interfaces

ABAQUS uses the following procedure for the ISL31 and the IRS4 elements. After an iteration, the normal and the shear stresses at any two points in contact with each other are computed. Then, the following criteria are applied.

1- If the computed normal stress between the two points is tensile, a gap opens between the two points. Then, the normal and the shear stresses are updated to zero and another iteration is performed.

2- If the computed normal stress between the two points is compressive, the shear stress is compared with Coulomb's Law,

\[
\tau_c = \mu_c \sigma_c \tag{3.15}
\]

where:

- \( \tau_c \) = the ultimate shear stress,
- \( \sigma_c \) = the normal stress between the two points,
- \( \mu_c \) = the coefficient of friction.
If the computed shear stress between the two points is smaller than \( \tau_c \), then it is a sticking condition and no other iteration is performed. For this condition, ABAQUS uses the penalty stiffness method in the analysis. This means that a small elastic slip between the two nodes occurs even when the shear stress is less than \( \tau_c \).

On the other hand, if the computed shear stress between the two points is larger then \( \tau_c \), then it is a sliding condition. At this condition, the shear stress is updated to equal \( \tau_c \), sliding between the two points is allowed and another iteration is performed.

The material properties required for the ISL31 and the IRS4 elements are the frictional coefficient \( (\mu_c) \) and the maximum allowable elastic slip distance \( (\Delta L_{\text{max}}) \). These material properties are taken as follows. The coefficient of friction for the soil-soil interface is taken equal to \( \tan \phi_{ds} \). It is assumed that the surfaces of the box are smooth and the coefficient of friction between the soil and the box surfaces is zero. The coefficient of friction for the soil-reinforcement interface is taken equal to \( \tan \delta \), where \( \delta \) is the friction angle between the soil and the reinforcement. The friction angle \( (\delta) \) is estimated by the relation (Potyondy 1961),

\[
\tan(\delta) = \phi \times \tan(\phi_{ds})
\]

(3.16)

where \( (\phi) \) is a constant which depends on the soil and the reinforcement materials. The value of the allowable elastic slip is taken equal to,

\[
\Delta L_{\text{max}} = \frac{\mu_c \sigma_i}{G_s} L_{sv}
\]

(3.17)

where:

- \( \mu_c = \) coefficient of friction as defined above,
- \( \sigma_i = \) initial mean normal stress on the soil-reinforcement composite,
\( G_s \) = shear modulus of the soil,
\( L_{av} \) = average length of the contact elements.

### 3.2.3 Kinematics

Two kinematics are considered in this study. One neglects the geometric nonlinearity and the other considers the geometric nonlinearity.

### 3.2.4 Loading Sequences

The stresses in a soil in the at rest condition can be written as (Duncan and Seed 1986; Hanna and Ghaly 1992),

\[
\begin{align*}
\sigma_{yy} &= \sigma_i \\
\sigma_{xx} &= \sigma_{zz} = k_o \sigma_{yy}
\end{align*}
\]

where:

- \( \sigma_{yy} \) = vertical stress,
- \( \sigma_{xx} \) and \( \sigma_{zz} \) = lateral stresses,
- \( \sigma_i \) = applied vertical stress at the surface,
- \( k_o \) = coefficient of earth pressure at rest.

For normally consolidated soil, the coefficient of earth pressure at rest \( (k_o) \) is,

\[
k_o = \frac{\nu_s}{1 - \nu_s}
\]

For over-consolidated or compacted soils, the coefficient of earth pressure at rest \( (k_o) \) is larger than that given by equation (3.19). A review of the values of \( k_o \) for over-consolidated or compacted soils was given by Hanna and Ghaly (1992).
For the direct shear test, the stress condition in the soil after the application of the vertical pressure ($\sigma_v$) should follow equation (3.18). In the finite element analysis, if the vertical pressure ($\sigma_v$) is applied as the first step of loading, the lateral stresses will be equal to ($k_0\sigma_v$) with the value of $k_0$ as given by equation (3.19). This represents the case for normally consolidated soils. To be able to represent the case for compacted soils (where $k_0$ is higher than that given by equation (3.19)), the loading is applied using the following two steps.

1- Vertical pressure ($\sigma_v$) is applied on the loading plate at the top surface of the soil. During the same step, initial lateral stresses ($\sigma_{xx}$ and $\sigma_{zz}$) equal to $k_0\sigma_v$ are applied to represent the effect of soil compaction. The stress condition in the soil at the end of this step is given by equation (3.18).

2- The box is sheared by applying incremental displacements on the rigid surfaces which represent the walls of the movable half of the box.

3.2.5 Boundary Conditions

During the first step of loading, the boundary conditions are as follows. The rigid surfaces which represent the box walls are not allowed to displace or rotate. The rigid surface which represents the loading plate is allowed to displace only in the vertical direction. During the second step of loading, the boundary conditions are changed by applying incremental horizontal displacement on the walls of the movable half of the box.
3.3 Effect Of The Model Assumptions

3.3.1 Studied Cases

Finite element analyses are carried out for a dense sand reinforced by one wood dowel with diameter of 0.32 cm. The model assumptions are changed one at a time and the effect on the behavior of the composite is studied. Table (3.1) gives a list of the cases analyzed in this Section.

Figure (3.5) shows the finite element mesh used in this Section. The dimensions of this mesh are as follows. The cross sectional area of the box is taken equal to the cross sectional area of the conventional direct shear box. The height of the box is taken equal to 20 cm. This height of the box is sufficiently long to ensure that the reinforcement will behave as a long reinforcement (Scott 1981).

The soil is a dense sand with the properties (Jewell and Milligan 1989; Lade 1988),

\[ E_s = \text{Young's modulus of the soil} = 20,000 \, \text{kPa}, \]
\[ \nu_s = \text{Poisson's ratio of the soil} = 0.17, \]
\[ c = \text{soil cohesion} = 0.0 \, \text{kPa}, \]
\[ \phi_{ps} = \text{angle of internal friction in plane strain} = 50^\circ, \]
\[ \phi_{ds} = \text{angle of internal friction in direct shear} = 44^\circ, \]
\[ \psi = \text{soil dilation angle} = 20^\circ. \]

Both the Drucker-Prager model and the Cap model are used to represent the soil material. The properties of the Drucker-Prager model are computed using equations (3.5) through (3.9) as:
\[ E_s = 20,000 \text{ kPa}, \]
\[ \nu_s = 0.17, \]
\[ d = 0.0 \text{ kPa}, \]
\[ \beta = 51.2^\circ, \]
\[ \eta = 30.2^\circ, \]

For the Cap model, \( E_s, \nu_s, d, \) and \( \beta \) are same as given above. The parameters \( R, P_{o_a}, \) and the cap hardening / softening law are estimated as follows. The hardening / softening law, equation (3.12), is assumed to have the form (DiMaggio and Sandler 1971),

\[ \varepsilon_p^e = \kappa_1 \left( 1 - e^{-3\kappa_2 p} \right) \quad (3.20) \]

where:
- \( \varepsilon_p^e \) = the plastic volumetric strain,
- \( p \) = the mean normal stress,
- \( \kappa_1 \) and \( \kappa_2 \) = additional material constants.

The parameters \( \kappa_1 \) and \( \kappa_2 \) are estimated from published data (e.g., Wu and Khosla 1976; Baladi 1989; Saxena et al. 1989) as 0.015 and \( 1 \times 10^{-5} \text{ kPa}^{-1} \), respectively. Also, the parameter \( R \) is estimated from the above data as 0.6.

The initial position of the cap (\( P_{o_a} \)) can be defined as the maximum past mean pressure applied to the soil and depends on the over-consolidation ratio of the soil (OCR). The over-consolidation ratio of the soil is estimated using the following equation (Duncan and Seed 1986; Hanna and Ghaly 1992),

\[ \text{OCR} = \left( \frac{k_o (1 - \nu_s)}{\nu_s} \right)^\frac{1}{2} \quad (3.21) \]

where:
\( v_s \) = the Poisson's ratio of the soil,
\( k_o \) = the coefficient of earth pressure at rest,
\( \chi \) = a factor that depends on the soil friction angle. For dense sand, this factor is approximately equal to 0.75 (Duncan and Seed 1986).

Then, the value of \( p_{b_e} \) can be estimated as,

\[
p_{b_e} = \frac{1}{3} \sigma_t \times OCR \times \left( 1 + \frac{2v_s}{1 - v_s} \right)
\]

(3.22)

where \( \sigma_t \) is the applied pressure on the top surface of the soil.

Two cases are analyzed using the above material properties, cases (1) and (2). Four additional analyses are performed to study the effect of the soil dilation, cases (3) through (6). In case (3), the Drucker-Prager model is employed with the value of \( \eta \) equal to \( \beta \), (i.e., an associated flow rule is assumed). In case (4), the analysis is performed using the Drucker-Prager model with the value of \( \eta \) equal to zero (i.e., the soil is assumed to be non-dilational). In cases (5) and (6), the cap model is employed with varying the soil dilation by changing the value of \( p_{b_e} \). In case (5), \( p_{b_e} \) is taken equal to two times the value computed by equation (3.22). In case (6), \( p_{b_e} \) is taken equal to one half the value computed by relation (3.22).

In all cases shown Table (3.1), except case (7), the reinforcement material is assumed to be ideally elastic-plastic with the properties (Wood Handbook 1955),

\( E_r = 1.19 \times 10^7 \) kPa,

\( v_r = 0.3 \),

\( \sigma_y = 25,000 \) kPa.
To study the effect of neglecting the nonlinear behavior of the reinforcement, the analysis of case (7) is performed for a linear-elastic reinforcement with the values of $E_r$ and $v_r$ given above.

The friction angle for the soil-soil interface at the potential failure surface is taken equal to $\phi_{ds}$. The friction angle between the soil and the box walls is taken equal to zero. To study the effect of the contact condition between the soil and the reinforcement, two cases are considered for the interface between the soil and the reinforcement. In case (8), complete bond between the soil and the reinforcement is assumed. In all other cases, the reinforcement is allowed to slip inside the soil. For these cases, the friction angle $\delta$ is estimated using equation (3.16) with the parameter $\vartheta$ equal to 0.7 (Potyondy 1961).

To study the effect of neglecting the geometric nonlinearity, two kinematics are used in this study. In case (9), the geometric nonlinearity is neglected. In all other cases, the geometric nonlinearity is considered.

The loading in these analyses is applied in two steps as mentioned in Section (3.2). The value of the coefficient of earth pressure at rest (used to compute the initial stresses $\sigma_{xx}$ and $\sigma_{zz}$, equation (3.18)) is taken equal to 1.0. In all studied cases, the applied vertical pressure ($\sigma_t$) is 23.6 kPa.
Table (3.1)

Studied Cases To Investigate The Effect Of Different Model Assumptions.

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Studied Parameter</th>
<th>Soil Model</th>
<th>Reinforcement Model</th>
<th>Contact Between Soil And Reinforcement</th>
<th>Kinematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>Drucker-Prager, ( \eta = 30.2 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>Cap, ( P_b = 100 \text{KPa} )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>3</td>
<td>Soil Properties</td>
<td>Drucker-Prager, ( \eta = 50.5 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>Drucker-Prager, ( \eta = 0.0 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>Cap, ( P_b = 200 \text{KPa} )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>Cap, ( P_b = 50 \text{KPa} )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>7</td>
<td>Nonlinear Behavior Of Reinforcement</td>
<td>Drucker-Prager, ( \eta = 30.2 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>8</td>
<td>Contact Condition Between Soil And Reinforcement</td>
<td>Drucker-Prager, ( \eta = 30.2 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Nonlinear</td>
</tr>
<tr>
<td>9</td>
<td>Geometric Nonlinearity</td>
<td>Drucker-Prager, ( \eta = 30.2 )</td>
<td>Elastic-Plastic</td>
<td>Slip</td>
<td>Geometrically Linear</td>
</tr>
</tbody>
</table>
Figure (3.5) Finite Element Mesh
3.3.2 Results

3.3.2.1 Effect Of Soil Properties

Figure (3.6) shows the load-displacement curves obtained from the analyses of cases (1) through (6). Figure (3.7) shows the computed axial and shear forces in the reinforcement at its point of intersection with the shear plain versus the applied displacement on the movable half of the box. From these figures, it can be observed that the soil dilation has a significant effect on the behavior of the soil-reinforcement composite. The ultimate strength of the soil-reinforcement composite and the axial force in the reinforcement increase significantly with increasing angle $\eta$ of the Drucker-Prager model or with increasing the value of $p_{\text{se}}$ in the Cap model.

The results of cases (1) and (2) are close to each other. This indicates that, with careful choice of the parameters that control the soil dilation, both the Drucker-Prager and the Cap models can be used. Therefore, in the rest of this chapter, the soil will be represented by the Drucker-Prager model with non-associated flow rule. This model is chosen rather than the Cap model because its parameters have physical meaning and they can more easily be related to the well-known soil properties ($C$, $\phi_{\text{ds}}$, and $\psi$).

3.3.2.2 Effect Of Nonlinear Behavior Of Reinforcement

Figure (3.8) shows the load-displacement curves obtained from the analyses of cases (1) and (7). Figure (3.9) shows the computed axial and shear forces in the reinforcement obtained from the analysis of the above two cases. From these figures, it can be noted that the neglect of the material nonlinearity of the reinforcement causes a significant overestimation of the ultimate strength of the composite and of the forces in the reinforcement. In conclusion, the assumption that the reinforcement material is linear
elastic produces inaccurate results and hence, this assumption should not be made in the analysis of the soil-reinforcement interaction problem.

### 3.3.2.3 Effect Of Soil-Reinforcement Bond

Figure (3.10) shows the load-displacement curves obtained from the analyses of cases (1) and (8). Figure (3.11) shows the computed axial and shear forces in the reinforcement obtained from the analysis of the above two cases. From these figures, it can be noted that the assumption of a complete bond between the soil and the reinforcement causes a significant overestimation of the ultimate strength of the composite and of the axial force in the reinforcement. In conclusion, the assumption of complete bond between the soil and the reinforcement produces inaccurate results and hence, this assumption should not be made in the analysis of the soil-reinforcement interaction problem.

### 3.3.2.4 Effect Of Geometric Nonlinearity

Figure (3.12) shows the load-displacement curves obtained from the analyses of cases (1) and (9). Figure (3.13) shows the computed axial and shear forces in the reinforcement obtained from the analysis of the above two cases. From these figures, it can be noted that the neglect of the geometric nonlinearity causes underestimation of the composite strength and of the axial force in the reinforcement. Figure (3.13.a) shows that, at an applied displacement of 1.5 cm, the axial force for case (9) is about one third the axial force for case (1). As a result, the increase in the strength due to the reinforcement ($\Delta s_r$ = the difference between the peak strength of the reinforced and the unreinforced soils) for case (9) is about one half that of case (1). In conclusion, the neglect of the geometric nonlinearity can produce inaccurate results and hence it should be considered in the analysis of the problem.
Figure (3.6) Load - Displacement Curves For Different Soil Models.
Figure (3.7) Forces In The Reinforcement For Different Soil Models.

Figure (3.7.a) Axial Forces

Figure (3.7.b) Shear Forces
Figure (3.8) Load-Displacement Curves For Different Reinforcement Models.
Figure (3.9) Forces In The Reinforcement For Different Reinforcement Models.

Figure (3.9.a) Axial Forces.

Figure (3.9.b) Shear Forces.
Figure (3.10) Load-Displacement Curves For Different Contact Between Soil And Reinforcement.
Slipage Between Soil And Reinforcement Is Allowed
Complete Bond Between Soil And Reinforcement

Figure (3.11.a) Axial Forces.

Slipage Between Soil And Reinforcement Is Allowed
Complete Bond Between Soil And Reinforcement

Figure (3.11.b) Shear Forces.

Figure (3.11) Forces In The Reinforcement For Different Contact Between Soil And Reinforcement.
Figure (3.12) Load-Displacement Curves For Different Kinematics.
Geometric Nonlinearity Is Considered

Geometric Nonlinearity Is Neglected

Figure (3.13.a) Axial Forces

Figure (3.13.b) Shear Forces

Figure (3.13) Forces In The Reinforcement For Different Kinematics.
3.3.3. Summary and Conclusions

The results given in the previous Section can be summarized as follows.

1- The behavior of the soil-reinforcement composite is significantly affected by the dilation characteristics of the soil. Therefore, the soil dilation should be carefully modeled in the analysis of the problem.

2- With careful choice of the parameters that control the soil dilation, either the Drucker-Prager model with non-associated flow rule or the Cap model can be used to represent the soil material.

3- The nonlinear behavior of the reinforcement significantly affects the behavior of the soil-reinforcement composite and it should be considered in the analysis of the problem.

4- The assumption that there is a complete bond between the soil and the reinforcement significantly affects the accuracy of the analysis and hence, this assumption should not be made.

5- The effect of the geometric nonlinearity on the accuracy of the analysis appears to be less significant than that of the factors mentioned in (1), (3) and (4). However, at large displacements, the neglect of the geometric nonlinearity gives errors in computed axial force in the reinforcement and in the computed amount of increase in the composite strength (Δs_f) which exceeded 100%.

In the analyses given in the rest of this chapter, the soil material will be represented by the Drucker-Prager model with non-associated flow rule. Also, the nonlinear behavior of the reinforcement, the slippage between the soil and the reinforcement, and the geometric nonlinearity of the problem will be considered.
3.4 Simulation Of Laboratory Experiments

3.4.1 Simulation Of Gray And Ohashi's Tests

3.4.1.1 Description Of The Analyzed Cases

The finite element model described in Section (3.2) is used to simulate two of Gray and Ohashi's (1983) direct shear tests. The simulated tests were performed on an unreinforced dense sand and on a dense sand reinforced by six #2 reeds with a diameter of 0.18 cm. Figure (3.14) shows the finite element mesh used to simulate these tests. The shear box in this mesh had the same height as that of the box used in the experiments. However, to simplify the mesh, a box with a square cross sectional area is used in the finite element analyses instead of the circular box used in the experiments.

The soil used in the above experiments was a clean, quartz sand from Muskegan Michigan. This sand had a mean diameter of 0.23 cm, a coefficient of uniformity of 1.5, a maximum density of 1.73 gm/cm³ and a minimum density of 1.50 gm/cm³. For the two simulated tests, the sand was at a relative density of 100%. At this relative density, the values of E_s and \( \phi_{ds} \) were given by Gray and Ohashi (1983) as 21,000 kPa and 39°, respectively. The Drucker-Prager model is used to represent the soil. The Young's Modulus for the soil is taken as given above. The Poisson's ratio is assumed to be equal to 0.2. This value is a reasonable estimate for the Poisson's ratio for most sands (Lade 1988). The dilation angle is estimated as 15° (Jewell and Milligan 1989). The properties \( d \), \( \beta \), and \( \eta \) are computed using relations (3.5) through (3.9) and were 0.0 kPa, 49.5°, and 24°, respectively.

The reinforcement is assumed to be ideally elastic-plastic with the properties \( E_r = 1.52 \times 10^6 \) kPa, \( \nu_r = 0.3 \), \( \sigma_y = 33,500 \) kPa (Gray and Ohashi 1983).
The friction angle for the soil-soil interface at the potential failure surface is taken equal to $\phi_{ds}$. The friction angle between the soil and the reinforcement ($\delta$) is taken equal to $30^\circ$ (Gray and Ohashi 1983). The friction angle between the soil and the box walls is taken equal to zero.

As explained in Section (3.2), the loading in the finite element analyses is applied in two steps. The value of the coefficient of earth pressure at rest (used to compute the initial stresses $\sigma_{xx}$ and $\sigma_{zz}$, equation (3.18)) is taken equal to 1.0. The value of vertical pressure ($\sigma_v$) used in the simulated tests is 75.5 kPa.

3.4.1.2 Comparison Between The Results Of Experiments And The Results Of Finite Element Analyses

Figures (3.15) and (3.16) show the load-displacement curves obtained from the experiments and those obtained from the finite element analyses for the two simulated tests. From these figures, it can be observed that the results of the experiments and the results of the finite element model are in reasonable agreement. The finite element analyses and the experiments gave almost the same peak strength of the soil-reinforcement composites and the same amount of increase in the composite strength ($\Delta_s_r$) due to the reinforcement. However, the finite element model cannot simulate the soil softening after reaching the peak strength.
Figure (3.14) Finite Element Mesh (Simulating Gray And Ohashi's Experiments).
Figure (3.15) Comparison Between The Results Of The Finite Element Analysis And The Results Of Gray And Ohashi's Test On Unreinforced Soil.

Figure (3.16) Comparison Between The Results Of The Finite Element Analysis And The Results Of Gray And Ohashi's Test On Soil Reinforced By Six #2 Reeds.
3.4.2 Simulation Of Shewbridge And Sitar's Tests

3.4.2.1 Description Of The Analyzed Cases

The finite element model described in Section (3.2) is used to simulate two of Shewbridge and Sitar's (1985) direct shear tests. The simulated tests are a test performed on an unreinforced dense sand and a test performed on a dense sand reinforced by fourteen wood dowels with a diameter of 0.32 cm. Figure (3.17) shows the finite element mesh used to simulate these tests. The dimensions and the arrangement of this mesh are the same as those used in the experiments.

The soil used in the above experiments was a dry, cohesionless, uniformly-graded, Monterey #0 sand. This sand had a mean diameter of 0.5 mm, a maximum density of 1.68 gm/cm$^3$, and a minimum density of 1.36 gm/cm$^3$. The direct shear tests were performed on sand with a relative density of 0.71. The Drucker-Prager model is used to represent the soil in the finite element model. Since the information given by Shewbridge and Sitar is not sufficient to estimate the material properties required for the Drucker-Prager model, the results of laboratory experiments performed on the same sand by Lade and Duncan (1973) and by Durgunoglu and Mitchell (1973) are used to estimate these material properties. From these tests, the values of the material properties are estimated as, $E_s=30,000$ kPa, $v_s=0.17$, $d=0.0$ kPa, $\beta=51.5^\circ$, and $\eta=35^\circ$.

The reinforcement is assumed to be ideally elastic-plastic. The Young's modulus of the reinforcement ($E_r$) was given by Shewbridge and Sitar (1985) as $1.19\times10^7$ kPa. The Poisson's ratio ($v_r$) and the yield strength ($\sigma_y$) are taken equal to 0.3 and 25,000 kPa, respectively (Wood Handbook 1955).
The friction angle for the soil-soil interface at the potential failure surface is taken equal to \( \phi_{ds} \). The value of \( \phi_{ds} \) is assumed to be equal to 45°. The friction angle between the soil and the reinforcement (\( \delta \)) was given by Shewbridge and Sitar (1985) as 35°. The friction angle between the soil and the box walls is assumed to be equal to zero.

Figure (3.18) shows a sketch of the box used in Shewbridge and Sitar tests. In these tests, vertical pressure (\( \sigma_t = 10.25 \) kPa) was applied on the top surface of the soil (surface (a-e-k-l-g-f-b) in Figure (3.18)). Then, the box was sheared along the vertical plane (e-f-i-j) in Figure (3.18). It should be noted that, for this box arrangement, the normal stress on the shear plane was unknown and it was not constant during the test. The initial value of this normal stress equaled to \( (k_0 \sigma_t) \). Therefore, the strength of the composite was highly dependent on the value of \( k_0 \). The value of \( k_0 \) is estimated as follows. The ultimate shear strength obtained from the test performed on the unreinforced soil was 23.0 kPa. Assuming that the soil friction angle in direct shear was 45°, the normal stresses on the shear plane can be computed as 23.0 kPa. Then, the value of \( k_0 \) can be obtained by dividing the value of this normal stress by the value of the applied vertical pressure \( \sigma_t \). This gives a value for \( k_0 \) equals to 2.25.

As explained in Section (3.2), the loading is applied in two steps. The value of the coefficient of earth pressure at rest (used to compute the initial stresses \( \sigma_{xx} \) and \( \sigma_{zz} \), equation (3.18)), is taken equal to 2.25. The applied vertical pressure (\( \sigma_t \)) is 10.25 kPa.
Figure (3.17) Finite Element Mesh (Simulating Shewbridge and Sitar's Experiments).
Figure (3.18) The Arrangement Of Shewbridge And Sitar's Shear Box.
3.4.2.2 Comparison Between The Results Of Experiments And The Results Of Finite Element Analyses

In this Section, comparisons between the results of the finite element analyses and the results reported by Shewbridge and Sitar (1985) are given. The results reported by Shewbridge and Sitar were:

1. The load-displacement curves for the different soil-reinforcement composites,
2. The deformation mode for the composites,
3. The profile of the increase in the sample height for some of the tests.

Figures (3.19) and (3.20) show the load-displacement curves obtained from the experiments and those obtained from the finite element analyses for the two simulated tests. From these figures, it can be noted that the results of the experiments and the results of the finite element model are in reasonable agreement. The finite element analyses and the experiments gave almost the same peak strength of the soil-reinforcement composites and the same amount of increase in the composite strength \( \Delta s_r \) due to the reinforcement. However, the finite element model cannot simulate the softening behavior which happened for the unreinforced soil after reaching the peak strength. Also, the initial slopes of the load-displacement curves obtained using the finite element analyses are larger than those obtained in the experiments.

Shewbridge and Sitar (1985) mapped the deformation pattern of the composite as follows. They fixed seven, 0.32 cm diameter Bungy cords on the glass bottom of the box. During the tests, they traced the deformation shape of these cords. Then, they obtained the following regression equation for the deformation pattern of the middle of these cord, Figure (2.14):
\[ X_1 = \frac{\Delta}{2} - \frac{\Delta}{2} \times e^{-bX_2} \]  

(2.33)

where:

\( \Delta \) = the boundary displacement,

\( b \) = parameter with units one over length. This parameter was obtained using statistical regression,

\( X_2 \) = distance along the reinforcement length,

\( X_1 \) = lateral displacement.

Shewbridge and Sitar assumed that the above equation represented the deformation mode for the composite and they developed a soil-reinforcement interaction model based on this equation.

Figure (3.21) shows the regression equations obtained by Shewbridge and Sitar for the two studied composites. Figure (3.22) shows the displacement shapes along the line where Shewbridge and Sitar placed the middle Bungy cord obtained from the finite element analyses of the two studied composites. These figures show that the displacement shapes computed in the finite element analyses are generally in agreement with the displacement shapes given by equation (2.33). In both the experiments and the finite element analyses, the deformation spread over a larger shear zone for the reinforced soil than for the unreinforced soil. However, It should be mentioned here that one should not make much emphasis on equation (2.33) as was carried out in Shewbridge and Sitar model. That is because the computed forces in the reinforcement are very sensitive to the parameter \( b \) such that a variation in this parameter within the experimental error, say 10\%, causes a significant difference in the computed forces in the reinforcement.

Both the experiments and the finite element analyses showed the occurrence of soil dilation during shearing of the soil-reinforcement composites. Figure (3.23) shows the
profiles of the increase in the sample height measured in the tests for an unreinforced soil and a soil reinforced by twelve, 0.78 cm diameter wood dowels. Shewbridge and Sitar did not report the profile of the increase in the sample height for the soil reinforced by fourteen, 0.32 cm diameter wood dowels (the case which is simulated in the finite element analysis). Figure (3.24) shows the profiles of the increase in the sample height computed in the finite element analyses for the unreinforced soil and the soil reinforced by fourteen, 0.32 cm diameter wood dowel. These figures show that the computed soil dilation in the finite element analysis of the unreinforced soil is in reasonable agreement with that measured in the experiments. In both the finite element analyses and the experiments, a bulge about the shear plane occurred. The profiles of the bulges obtained from the analytical model and the experiments are generally the same and the maximum value of increase in the sample height (which occurs at the shear plane) are approximately equal. Quantitative comparison between the computed and measured soil dilation for the reinforced soil could not be made because the shown results are for different composites. Nevertheless, the general trend obtained from the finite element analyses and from the tests is the same. Both the finite element analyses and the experiments showed that the soil dilation increases due to the reinforcement existence and the zone around the shear plane which undergoes volumetric changes is larger for the reinforced soil than for the unreinforced soil.

3.4.3 Summary

In this Section, comparisons between the results of the finite element model and the results of laboratory experiments were performed. These comparisons showed that the finite element model can successfully predict the behavior of the soil-reinforcement composite. The peak strength of the composite, the amount of increase in the composite strength, the deformations, and the dilation behavior of the soil computed in the finite
Figure (3.19) Comparison Between The Results Of The Finite Element Analysis And The Results Of Shewbridge And Sitar's Test On Unreinforced Soil.

Figure (3.20) Comparison Between The Results Of The Finite Element Analysis And The Results Of Shewbridge And Sitar's Test On Soil Reinforced By Fourteen Wood Dowels.
Figure (3.21) Deformation Pattern Given By Equation (2-33).

Figure (3.22) Deformation Pattern Computed In The Finite Element Analyses.
Figure (3.23) Soil Dilation Measured In The Experiments.

Figure (3.24) Soil Dilation Computed In The Finite Element Analyses.
element analyses are in reasonable agreement with those measured in the experiments. However, the finite element analyses fail to simulate the soil softening after reaching the peak strength of the composite.

3.5 Parametric Study Of The Problem

3.5.1 Studied Cases

The parametric study of the soil-reinforcement interaction problem is given in this Section. The parameters included in this study are the soil type, the reinforcement stiffness, and the confining pressure on the soil-reinforcement composite. To perform this study, the finite element model described in Section (3.2) is used to perform thirty-six analyses on different soil-reinforcement composites. Table (3.2) gives a list of these analysis cases. The finite element mesh used to perform these analyses is the mesh described in Section (3.3.1) and shown in Figure (3.5).

Three different soil types are used in this study. These soils are a dense sand, a medium dense sand, and a loose sand. The Drucker-Prager model is used to represent the soil material in the finite element analyses. The material properties used for the different soils are given in Table (3.3). These material properties are determined as follows.

The Young’s modulus of the soil ($E_s$) depends on the soil density and the confining pressure (Duncan and Chang 1970). Therefore, the values of $E_s$ for the different soil types are estimated using the relation (Duncan and Chang 1970),

$$E_s = A\sigma_{\text{ult}} \left( \frac{\sigma}{\sigma_{\text{ult}}} \right)^m$$  \hspace{2cm} (3.23)

where:

$A$ and $m$ = material constants,
\[ \sigma_{atm} = \text{atmospheric pressure expressed in the same pressure units as } E_s, \]
\[ \sigma_1 = \text{the initial value of the mean confining stress in the composite.} \]

The used values of \( A \) and \( m \) for the different soil types are given in Table (3.3). In all analysis cases, the Poisson’s ratio of the soil \( (\nu_s) \) is assumed to be equal to 0.17. This value is a reasonable estimate of \( \nu_s \) for most sands (Lade 1988). The values of the soil cohesion \( (c) \), the friction angle in direct shear \( (\phi_{ds}) \), and the dilation angle \( (\psi) \) for the different soils are assumed as given in Table (3.3), (Jewell and Milligan 1989). Then, the material properties for the Drucker-Prager yield surface \( (\phi, \beta, \eta) \) are computed using equations (3.5) to (3.9).

One reinforcement with a diameter of 0.32 cm is used in the different analysis cases. Two different material types are used for the reinforcement. These materials are wood and steel. The reinforcement materials are assumed to be ideally elastic-plastic with the properties given in Table (3.4).

The friction angle for the soil-soil interface at the potential failure surface is taken equal to \( \phi_{ds} \). The friction angle between the soil and the box walls is assumed to be equal to zero. The friction angle between the soil and the reinforcement \( (\delta) \), is estimated using equation (3.16). The values of the parameter \( (\delta) \) for sand-wood and for sand-steel interaction are estimated as 0.7 and 0.455, respectively (Potyondy 1961).

As explained in Section (3.2), the loading is applied in two steps. The value of the coefficient of earth pressure at rest \( (\sigma_{xx} \text{ and } \sigma_{zz}, \text{ equation (3.18)}) \) is taken equal to 1.0. The values of vertical pressure \( (\sigma_1) \) used in the different analysis cases are given in Table (3.2).
### Table (3.2)
Cases In The Parametric Study.

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Description</th>
<th>Soil Type</th>
<th>Reinforcement Type</th>
<th>Confining Pressure, ( \sigma_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 4</td>
<td>Unreinforced</td>
<td>Dense Sand</td>
<td>-</td>
<td>10, 23.6, 50, 100 kPa</td>
</tr>
<tr>
<td>5 - 8</td>
<td>Unreinforced (Medium Dense Sand)</td>
<td>10, 23.6, 50, 100 kPa</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>9 - 12</td>
<td>Soil</td>
<td>Medium Dense Sand</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>13 _ 16</td>
<td>Soil Reinforced By Medium Dense Sand</td>
<td>Wood</td>
<td>10, 23.6, 50, 100 kPa</td>
<td></td>
</tr>
<tr>
<td>17 _ 20</td>
<td>One, 0.32 cm Diameter, Medium Dense Sand</td>
<td>Wood</td>
<td>10, 23.6, 50, 100 kPa</td>
<td></td>
</tr>
<tr>
<td>21 _ 24</td>
<td>Wood Dowel</td>
<td>Loose Sand</td>
<td>Wood</td>
<td>10, 23.6, 50, 100 kPa</td>
</tr>
<tr>
<td>25 _ 28</td>
<td>Soil Reinforced By Medium Dense Sand</td>
<td>Steel</td>
<td>10, 23.6, 50, 100 kPa</td>
<td></td>
</tr>
<tr>
<td>29 _ 32</td>
<td>One, 0.32 cm Diameter, Steel</td>
<td>Steel</td>
<td>10, 23.6, 50, 100 kPa</td>
<td></td>
</tr>
<tr>
<td>33 _ 36</td>
<td>Steel Rod</td>
<td>Loose Sand</td>
<td>Steel</td>
<td>10, 23.6, 50, 100 kPa</td>
</tr>
</tbody>
</table>

### Table (3.3)
Properties Of The Soil Material

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Properties for Dense Sand</th>
<th>Properties for Medium Dense Sand</th>
<th>Properties for Loose Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Properties</td>
<td>( E_i ) = A(\sigma_i)^m \</td>
<td>( E_i = A(\sigma_i)^m )</td>
<td>( E_i = A(\sigma_i)^m )</td>
</tr>
<tr>
<td></td>
<td>( A = 325 ) m=0.33</td>
<td>( A = 250 ) m=0.33</td>
<td>( A = 200 ) m=0.33</td>
</tr>
<tr>
<td></td>
<td>0.17</td>
<td>0.17</td>
<td>0.17</td>
</tr>
</tbody>
</table>
| Failure           | \( c \)                   | 0.0 kPa                         | 0.0 kPa                  | 0.0 kPa
|                   | \( \phi_\alpha \)         | 45                              | 37.5                     | 30
|                   | \( \phi_\beta \)          | 50                              | 42.5                     | 35
|                   | \( \psi \)                | 20                              | 10                       | 0
| Failure in The \( p-q \) Plane | \( \beta \) | 51.2                          | 48.5                     | 44.7
|                   | \( \eta \)                | 30.2                            | 17.5                     | 0.00

### Table (3.4)
Properties Of The Reinforcement Material

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Wood</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Properties</td>
<td>( E_i ) = 1.19 \times 10^7 \text{ kPa}</td>
<td>( 2.04 \times 10^8 \text{ kPa} )</td>
</tr>
<tr>
<td></td>
<td>( v_i ) = 0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Yield Strength</td>
<td>( \sigma_f ) = 25,000 \text{ kPa}</td>
<td>240,000 \text{ kPa}</td>
</tr>
</tbody>
</table>
3.5.2 Results

3.5.2.1 Effect Of Relative Density Of Soil

Figure (3.25) shows the load-displacement curves of unreinforced dense, medium dense, and loose sands and the load-displacement curves of the same sands reinforced by one wood dowel with a diameter of 0.32 cm. Figure (3.26) shows the computed axial and shear forces in the reinforcements versus the applied displacement on the movable half of the box. From these figures, the following can be observed.

1- The axial and shear forces in the reinforcement increase with increasing the soil relative density. Hence, the improvement in strength ($\Delta s_r$) due to the reinforcement increases with increasing the soil relative density.

2- The initial slopes of the load-displacement curves of unreinforced soils are larger than those of reinforced soils.

3- For reinforced loose sand, a large shear displacement is required before the improvement in shear resistance ($\Delta s_r$) is mobilized.

The last two results can be explained as follows. Upon the application of the vertical stresses on the top surface of the soil, the soil compresses and the reinforcement goes into compression. This compressional prestress in the reinforcement makes the initial strength of reinforced soils less than those of unreinforced soils. Therefore, the initial slopes of the load-displacement curves of reinforced soils appear to be smaller than those of unreinforced soils. However, with increasing shear displacement, tensile and shear forces develop in the reinforcement. The increase in shear strength starts to appear when shear displacement is sufficient to develop forces in the reinforcement which overcome the effect of the precompresional forces in the reinforcement. The amount of this shear
displacement depends on the soil type. Since dense sand has a large dilation angle, tensile elongation in the reinforcement occurs at smaller shear displacements and the increase in shear strength appears at smaller shear displacements. On the other hand, since loose sand has no dilation, tensile elongation in the reinforcement occurs at larger shear displacements and the increase in shear strength appears at larger shear displacements.

3.5.2.2 Effect Of Reinforcement’s Stiffness And Yield Strength

Figure (3.27) shows the load-displacement curve of an unreinforced dense sand and the load-displacement curves of the same sand reinforced by different types of reinforcement. Figure (3.28) shows the computed axial and shear forces in the reinforcements versus the applied displacement on the moveable half of the box. From these figures, the following can be observed.

1- The axial and shear forces in the reinforcement increase with increasing the reinforcement’s stiffness and yield strength. Hence, the improvement in strength ($\Delta s_r$) due to the reinforcement increases with increasing the reinforcement’s stiffness and yield strength.

2- The shear force in the reinforcement significantly increases with increasing the yield strength of the reinforcement. However, the increase of the axial force in the reinforcement is not proportional to the increase in the reinforcement’s stiffness or yield strength. The relations between the forces in the reinforcement and the reinforcement’s stiffness and yield strength will be given in Section (3.6).

3.5.2.3 Effect Of Confining Pressure

Figure (3.29) shows the load-displacement curves of an unreinforced dense sand subjected to different confining pressures and the load-displacement curves of the same
sands reinforced by one wood dowel with a diameter of 0.32 cm. Figure (3.30) shows the computed axial and shear forces in the reinforcement versus the applied displacement on the movable half of the box. From these figures, the following can be observed.

1- The axial and shear forces in the reinforcement increase with increasing the confining pressure on the composite. Hence, the improvement in strength ($\Delta s_f$) due to the reinforcement increases with increasing the confining pressure on the composite.

2- The axial force in the reinforcement significantly increases with increasing the confining pressure. However, the shear force in the reinforcement increases slightly with increasing the confining pressure. The relations between the forces in the reinforcement and the confining pressure will be given in Section (3.6).

3.5.2.4 Failure Envelope

Figures (3.31) through (3.34) show the failure envelopes for unreinforced and reinforced sands of different relative densities. From these figures, the following can be observed:

1- The reinforcement increases the shear strength of the soil. The magnitude of this increase depends on a number of sand-reinforcement parameters. Generally, the improvement in the shear strength due to the reinforcement increase with increasing soil relative density, reinforcement’s stiffness and yield strength, and confining pressure.

2- The apparent cohesion and friction angle for the failure envelopes of reinforced sands are larger than those of unreinforced sands.
Figure (3.25) Effect Of Soil Relative Density On The Load-Displacement Behavior.
Figure (3.26.a) Axial Forces

Figure (3.26.b) Shear Forces

Figure (3.26) Effect Of Soil Relative Density On The Reinforcement Forces.
Figure (3.27) Effect Of Reinforcement Type On The Load-Displacement Behavior.

- Unreinforced Dense Sand
- Dense Sand Reinforced By One Wood Dowel
- Dense Sand Reinforced By One Steel Rod
Figure (3.28.a) Axial Forces

Figure (3.28.b) Shear Forces

Figure (3.28) Effect Of Reinforcement Type On The Reinforcement Forces.
Figure (3.29) Effect Of Confining Pressure On The Load-Displacement Behavior.
Figure (3.30) Effect Of Confining Pressure On The Reinforcement Forces.

Figure (3.30.a) Axial Forces

Figure (3.30.b) Shear Forces
Figure (3.31) Failure Envelope Of Unreinforced And Reinforced Dense Sand.

Figure (3.32) Failure Envelope Of Unreinforced And Reinforced Medium-Dense Sand.

Figure (3.33) Failure Envelope Of Unreinforced And Reinforced Loose Sand.
3.6 Beam Model For Computing The Ultimate Forces In The Reinforcement

This Section describes a model for computing the ultimate forces in the reinforcement and compares the results of this method with the results of the finite element analyses. Schlosser (1983) developed a model similar to the model described in this Section. Schlosser's model was also used by Mitchell and Villet (1987) and by Juran et al. (1992) to perform limit equilibrium analysis for reinforced soils. A comparison between the model developed in this study and Schlosser's model will be given in Section (3.6.4).

3.6.1 Description Of The Model

As explained in Section (2.2), the forces in the reinforcement depend on two mechanisms: the friction along the reinforcement length and the passive soil pressure on the reinforcement. Both of the above mechanisms can be related to the normal pressure ($\sigma_r$) at the soil-reinforcement interface. Therefore, an accurate representation of the above mechanisms requires careful consideration on the magnitude and distribution of $\sigma_r$. This is performed in the beam model as will be explained in the following subsections.

3.6.1.1 Computation Of The Bending Moment And Shear Force

Initially, the normal pressure ($\sigma_r$) at the soil-reinforcement interface is uniformly distributed and equal to the initial mean normal pressure ($\sigma_i$), Figure (3.34.a). After shearing the soil on the shear plane, passive earth pressure is developed on the reinforcement. The development of this passive pressure produces a change in the distribution of $\sigma_r$ as shown in Figure (3.34.b). Note that the pressure distribution in Figure (3.34.b) has the following features. Within a distance ($X$) from the shear plane,
the pressure distribution is non-uniform and acts only on one side of the reinforcement. However, at large $x$, the normal pressure on both sides of the reinforcement, defined as $\sigma_u$ and $\sigma_d$ in Figure (3.34.b), are uniform and equal to $\sigma_i$. The shear force and bending moment in the reinforcement depend on the net normal pressure ($\sigma_n = \sigma_u - \sigma_d$) shown in Figure (3.34.e). Assuming that both soil and reinforcement behave as elastic materials, the distributions of the net normal pressure ($\sigma_n$), the shear forces ($Q$) and the bending moment ($M$) are respectively (Hetenyi 1946),

\[
\sigma_n = \frac{2\lambda Q_0 e^{-\lambda x} \cos(\lambda x)}{D} \quad (3.24)
\]

\[
M = \frac{Q_0}{\lambda} e^{-\lambda x} \sin(\lambda x) \quad (3.25)
\]

\[
Q = Q_0 e^{-\lambda x} \left[ \cos(\lambda x) - \sin(\lambda x) \right] \quad (3.26)
\]

where:

- $D$ = the reinforcement diameter,
- $x$ = distance along the reinforcement measured from the point of intersection of the reinforcement with the shear plane,
- $Q_0$ = the shear force at the point of intersection of the reinforcement with the shear plane,
- $\lambda$ = a parameter with dimension ($L^{-1}$) and the reciprocal of it is called the transfer length ($L_0$) of the soil-reinforcement system. The value of $\lambda$ is given by the relation,

\[
\lambda = \frac{C_y D}{4E_I I_t} \quad (3.27)
\]

$C_y$ = the coefficient of subgrade reaction in the lateral direction. This coefficient can be computed using the relation (Vesic 1961),
\[ C_y = \frac{0.65E_s}{D(1-\nu_s^2)\sqrt{\frac{E_sD^4}{E_rI_r}}} \tag{3.28} \]

\( E_s \) = Young's modulus of the soil,

\( \nu_s \) = Poisson's ratio of the soil,

\( E_r \) = Young's modulus of the reinforcement material,

\( I_r \) = moment of inertia of the reinforcement cross section.

Figures (3.34.c) through (3.34.e) show the distribution of \( \sigma_n \), \( Q \), and \( M \) given by the above equations. Note that the maximum net normal pressure \( (\sigma_{n_{\text{max}}}) \) and the maximum shear force \( (Q_{\text{max}}) \) occur at the shear plane. The maximum bending moment and its location are respectively,

\[ M_{\text{max}} = 0.3224 \frac{Q_n}{\lambda} \tag{3.29} \]

\[ L_b = \frac{\pi}{4\lambda} \tag{3.30} \]

Now considering the nonlinear behavior of the soil and the reinforcement materials, the following limitations should be included in the analysis.

1- The maximum bending moment in the reinforcement should not exceed the bending moment \( (M_p) \) which produces a plastic hinge in the reinforcement cross section. For a reinforcement with a circular cross section, \( M_p \) is given by the relation,

\[ M_p = \frac{\sigma_y D^3}{6} \tag{3.31} \]

where:

\( \sigma_y \) = yield strength of the reinforcement material.
Substituting (3.27), (3.28) and (3.31) in (3.29) and (3.24), the following equations can be obtained,

\[
Q_{\text{max}} = 3.1\lambda M_{\text{max}} = 0.517\lambda \sigma_y D^3 = 0.75\left(\frac{E_s}{E_r}\right)^{0.271} \sigma_y D^2
\]

(3.32)

\[
\sigma_{n_{\text{max}}} = \frac{6.2\lambda^2 M_{\text{max}}}{D} = 1.034\lambda^2 \sigma_y D^2 = 2.13\left(\frac{E_s}{E_r}\right)^{0.542} \sigma_y
\]

(3.33)

where:

- \( Q_{\text{max}} \) = the maximum shear force in the reinforcement,
- \( \sigma_{n_{\text{max}}} \) = the maximum net normal pressure on the reinforcement.

2- The maximum net normal pressure on the reinforcement (\( \sigma_{n_{\text{max}}} \)) should not exceed the ultimate bearing resistance of the soil (\( \sigma_b \)).

3- From (1) and (2), the value of \( \sigma_{n_{\text{max}}} \) should be the smaller of \( \sigma_b \) and the value given by equation (3.33). This finding is also observed in the results of the finite element analyses. For instance, Figures (3.35.a) and (3.35.b) show the distribution of \( \sigma_n \) computed in the finite element analyses for a wood dowel exist in dense and loose sands respectively. For the reinforcement in loose sand (where the soil bearing resistance controls), the distribution of \( \sigma_n \) does not follow equation (3.24) and Figure (3.34.c). That is because after \( \sigma_{n_{\text{max}}} \) reached the value of \( \sigma_b \) a modification in the distribution of \( \sigma_n \) as shown in Figure (3.35.b) occurred. Since the distribution of \( \sigma_n \) in this case does not follow equation (3.24), then, equations (3.30) through (3.33) can no longer be used and instead, the magnitude of the maximum bending moment (\( M_{\text{max}} \)), the position of the maximum bending moment (\( L_b \)) and the magnitude of the maximum shear force (\( Q_{\text{max}} \)) can approximately be computed using the equations (Jewell and Pedley 1992),
The distribution of the normal pressure ($\sigma_n$), Figure (3.34.b), can be assumed to be the sum of $\sigma_n$ and a uniform pressure $\sigma_l$. Making this assumption, the integration in relation (3.37) gives,
\[ T_{\text{max}} = T_i + T_n \]
\[ = \mu \pi DL_a \sigma_i + 1.4 \mu Q_{\text{max}} \]  
(3.38)

where:

- \( T_i \) = the portion of the frictional resistance which depends on \( \sigma_i = \mu \pi DL_a \sigma_i \)
- \( T_n \) = the portion of the frictional resistance which depends on \( \sigma_n = 1.4 \mu Q_{\text{max}} \)

Figure (3.36) shows the distribution of the stresses in polar coordinates on the periphery of a reinforcement section. In this Figure, \( \sigma_{rr} \) represents the normal stress, \( \tau_{rx} \) represents the axial shear stress, and \( \tau_{r\theta} \) represents the radial shear stress. In the finite element analyses, the following frictional criterion is used (ABAQUS, version (5.3), Hibbitt, Karlson, and Sorensen, Inc. 1990),

\[ \sqrt{\tau_{r\theta}^2 + \tau_{rx}^2} \leq \mu \sigma_{rr} \]  
(3.39)

Equation (3.39) gives a smaller value of \( \tau_{rx} \) than \( \tau_{r\theta} = \mu \sigma_{rr} \) in equation (3.37). This is most serious near the shear plane where \( \tau_{r\theta} \) is largest because the reinforcement moves through the soil. To consider the above effect, a reduction factor is applied to the second term (\( T_n \)) of equation (3.38). A value of 0.65 for the reduction factor is found to give the best fit to the results of the finite element analyses. After applying this reduction factor, equation (3.38) can be rewritten as,

\[ T_{\text{max}} = T_i + T_{nc} \]
\[ = \mu \pi DL_a \sigma_i + 0.91 \mu Q_{\text{max}} \]  
(3.40)

where:

- \( T_{nc} \) = the portion of the frictional resistance which depends on \( \sigma_n \) after applying the reduction factor.

Now, substitute (3.32) and (3.36) in (3.40), the following equations can be obtained,
for the cases where the bending resistance of the reinforcement controls:

\[ T_{\text{max}} = \mu \pi D L_a \sigma_i + 0.68\mu \left( \frac{E_b}{E_r} \right)^{271} \frac{\sigma_y}{\sigma_i} D^2 \]  

(3.41)

or:

\[ \frac{T_{\text{max}}}{\mu \pi D L_a \sigma_i} = 1 + 0.217 \left( \frac{D}{L_a} \right) \left( \frac{E_b}{E_r} \right)^{271} \left( \frac{\sigma_y}{\sigma_i} \right) \]  

(3.42)

and for the cases where the soil bearing resistance controls:

\[ T_{\text{max}} = \mu \pi D L_a \sigma_i + 0.527 \times \mu \sqrt{\sigma_y \sigma_b} D^2 \]  

(3.43)

or:

\[ \frac{T_{\text{max}}}{\mu \pi D L_a \sigma_i} = 1 + 0.169 \left( \frac{D}{L_a} \right) \frac{\sigma_y \sigma_b}{\sigma_i^2} \]  

(3.44)

Note that the value of \( T_{\text{max}} \) obtained from equations (3.41) or (3.43) should not exceed the tensile resistance of the reinforcement material \( (T_p) \). For a reinforcement with a circular cross section, \( T_p \) can be expressed as,

\[ T_p = \frac{\pi D^2}{4} \sigma_y \]  

(3.45)

### 3.6.1.3 Yield Criterion Of The Reinforcement

Since the reinforcement has to withstand axial force, shear force, and bending moment, a yield criterion for the reinforcement material which considers such a combination of loading should be used. Several yield criteria have been suggested for beams under combined loading. A discussion for these yield criteria will be given in Chapter (4). Here, we quote the yield criteria suggested by Argyris (1982) which is written as,
\[
\frac{T^2}{T_p^2} + \frac{Q^2}{Q_p^2} + \frac{M^2}{M_p^2} \leq 1.0
\]  
(3.46)

where:

- \(T\), \(Q\) and \(M\) = the axial force, shear force, and bending moment at any section along the reinforcement.
- \(T_p\), \(Q_p\) and \(M_p\) = the limiting values of the axial force, shear force and bending moment at the reinforcement section.

Ideally, equation (3.46) should be checked at each section along the reinforcement and iterations should be performed if equation (3.46) is violated (this procedure will be described in detail in Chapter (4)). For simplicity here, the following procedure can be followed.

The critical sections along the reinforcement are the section of maximum bending moment which exists at a distance \(L_b\) from the shearing plane and the section of maximum axial and shear forces which exist at the shear plane. At the section of maximum bending moment, the values of \(M\), \(Q\), and \(T\) are respectively,

\[
\begin{align*}
M &= M_{\text{max}} \\
Q &= 0 \\
T &= T_{c1}
\end{align*}
\]  
(3.47)

The value of \(T_{c1}\) can be computed using the procedure described in previous subsection and it is the smaller of,

\[
T_{c1} = \mu \pi D (L_a - L_b) \sigma_i + 0.65 \int_0^{L_a} \int_0^{2\pi} \frac{\mu \sigma_a}{2} D d\theta dL
\]

\[
= \mu \pi D (L_a - L_b) \sigma_i + 0.26 \mu Q_{\text{max}}
\]  
(3.48)

and

\[
T_{c1} = T_p
\]  
(3.49)
Substitute (3.47) in (3.46), then,

$$\frac{T_{el}^2}{T_p^2} + \frac{M_{max}^2}{M_p^2} \leq 1.0$$  \hspace{1cm} (3.50)

Making the assumption that the section of maximum bending moment will yield in bending, then, the value of $M_{max}$ is,

$$M_{max} = M_p \sqrt{1 - \left(\frac{T_{el}}{T_p}\right)^2}$$  \hspace{1cm} (3.51)

Considering the relations between $Q_{max}$ and $M_{max}$ (equations (3.32) and (3.36)), the maximum shear force in the reinforcement is,

for the cases where the bending resistance of the reinforcement controls, equation (3.32),

$$Q_{max} = 0.75 \left(\frac{E_s}{E_r}\right)^{0.271} \left(1 - \left(\frac{T_{el}}{T_p}\right)^2\right)^{0.5} \sigma_y D^2$$  \hspace{1cm} (3.52)

for the cases where the bearing resistance of the soil controls, equation (3.36),

$$Q_{max} = \sqrt{\frac{\sigma_b}{3\sigma_y}} \times \sqrt{1 - \left(\frac{T_{el}}{T_p}\right)^2} \sigma_y D^2$$  \hspace{1cm} (3.53)

At the section of maximum axial and shear forces, the values of $M$, $Q$, and $T$ are respectively,

\begin{align*}
M &= 0 \\
Q &= Q_{max} \\
T &= T_{max}
\end{align*}  \hspace{1cm} (3.54)

where $T_{max}$ is given by equations (3.41), (3.43) or (3.45) and $Q_{max}$ is given by equations (3.51) or (3.52). Substitute (3.54) in (3.46), then,
\[
\frac{T_{\text{max}}^2}{T_p^2} + \frac{Q_{\text{max}}^2}{Q_p^2} \leq 1.0 \tag{3.55}
\]

The value of \(Q_{\text{max}}\) depends on \(M_{\text{max}}\) which has been ensured not to violate the yield criteria at the section of maximum bending moment. Hence, the assumption can be made that if equation (3.55) is violated, yield will occur in tension and therefore, \(T_{\text{max}}\) should be reduced as,

\[
T_{\text{max}} = T_p \sqrt{1 - \left(\frac{Q_{\text{max}}}{Q_p}\right)^2} \tag{3.56}
\]

### 3.6.2 Summary Of The Steps Of Solution

The above approach for computing the ultimate forces in the reinforcement can be summarized in the following steps.

1- Determine the ultimate bearing resistance of the soil \((\sigma_b)\).
2- Compute the value of \(\sigma_{n_{\text{max}}}\) using relation (3.33) and compare it with \(\sigma_b\).
3- If \(\sigma_{n_{\text{max}}}\) is smaller than \(\sigma_b\), the distribution of the normal pressure on the reinforcement is controlled by the reinforcement bending resistance. Then, use equations (3.41) and (3.32) to compute the maximum axial and shear forces in the reinforcement.
4- If \(\sigma_b\) is smaller than \(\sigma_{n_{\text{max}}}\), the distribution of the normal pressure on the reinforcement is controlled by the bearing resistance of the soil. Then, use equations (3.43) and (3.36) to compute the maximum axial and shear forces in the reinforcement.
5- To satisfy the yield criterion at the section of maximum bending moment, reduce \(Q_{\text{max}}\) using equation (3.52) or (3.53).
6- Check the yield criterion at the section of maximum axial and shear forces, equation (3.55). If the yield criterion is violated, reduce $T_{\text{max}}$ using equation (3.56).

### 3.6.3 Comparison Between The Results of The Beam Model And The Results of The Finite element Analyses

The ultimate axial and shear forces in the reinforcement obtained by the beam model described above are compared with those obtained from the finite element analyses. Figure (3.37) shows the relation between \( \frac{T_{\text{max}}}{\mu \pi \delta L_s \sigma_i} \) and the dimensionless factor $\Phi_T$, where, for the cases where the bending resistance of the reinforcement controls, equation (3.42):

\[
\Phi_T = 0.217 \left( \frac{D}{L_s} \right) \left( \frac{E_s}{E_r} \right)^{0.271} \left( \frac{\sigma_t}{\sigma_i} \right) \quad (3.57)
\]

for the cases where the bearing resistance of the soil controls, equation (3.44):

\[
\Phi_T = 0.169 \left( \frac{D}{L_s} \right) \sqrt{\frac{\sigma_b \sigma_t}{\sigma_i^2}} \quad (3.58)
\]

Figure (3.38) shows the relation between \( \frac{Q_{\text{max}}}{\sigma_y D^2} \) and the dimensionless factor $\Phi_q$, where, for the cases where the bending resistance of the reinforcement controls, equation (3.52):

\[
\Phi_q = 0.75 \left( \frac{E_s}{E_r} \right)^{0.271} \left( 1 - \left( \frac{T_{\text{el}}}{T_p} \right) \right)^{0.5} \quad (3.59)
\]

for the cases where the bearing resistance of the soil controls, equation (3.53):

\[
\Phi_q = \sqrt{\frac{\sigma_b}{3\sigma_y}} \times \sqrt{1 - \left( \frac{T_{\text{el}}}{T_p} \right)^2} \quad (3.60)
\]
The solid line in Figure (3.37) represents relations (3.42) and (3.44) and the solid line in Figure (3.38) represents relations (3.52) and (3.53). The results of the finite element analyses are shown as points in Figures (3.37) and (3.38). The shown points represent the computed forces in the reinforcement at an applied displacement on the movable half of the box equals 1.5 cm. From the shown plots, it can be noted that the results of the beam model are in good agreement with the results of the finite element analyses.

The axial and shear forces in the reinforcement at different factors of safety are obtained from the results of the finite element analyses as follows. The factor of safety is defined as the global factor of safety for the soil-reinforcement composite. The force ($F_u$) required to shear the composite 1.5 cm is assumed to be the ultimate force for the soil-reinforcement composite. Then, the force applied on the composite at any factor of safety is obtained by dividing $F_u$ by the factor of safety. The axial and the shear forces in the reinforcement which correspond to the above force are obtained. The average relations for the developed axial and shear forces in the reinforcement at different factors of safety are plotted in Figure (3.37) and (3.38) as dotted lines. These relations can be used by the design engineers to obtain estimates of the reinforcement forces at different factors of safety.

Up to this point, two models have been described for the soil-reinforcement interaction problem. These models are the three dimensional, nonlinear, finite element model, which is computationally very expansive, and the beam model described in this Section, which needs only a pocket calculator. A comparison between the results of the beam model and the results of the finite element analyses showed that the beam model can successfully predict the ultimate forces in the reinforcement. The results of the beam model can be used to perform limit equilibrium analyses for reinforced soils. As explained before, the limit equilibrium approach has many limitations. Therefore, a third model
which can be used to predict the behavior of the soil-reinforcement composites at working condition will be developed in Chapter (4).

3.6.4 Comparison Between The Beam Model And Schlosser's Model

Schlosser (1983) developed a model (see Section 2.31.1.c) similar to the beam model described in this Section. The differences between the beam model and Schlosser's model are the following.

1- In the beam model, the change in the magnitude and distribution of the normal pressure on the reinforcement due to the development of the passive earth pressure is considered during the computation of the axial force in the reinforcement. Schlosser (1983) neglected the effect of the above change. The results of the finite element analyses showed that the above change significantly affect the frictional resistance along the reinforcement length and it should be considered.

2- In the beam model, the effect of the relative stiffness between the soil and the reinforcement on the distribution of the net normal pressure ($\sigma_n$) is considered. Schlosser (1983) neglected the above effect and he assumed that the distribution of $\sigma_n$ will always follow the relation given by Hetenyi (1946) for elastic beams on elastic foundations. The results of the finite element analyses showed that the above effect can significantly affect the results and it should be considered.

3- In the beam model, the yield criterion for the reinforcement material is checked at two sections: the section of maximum bending moment and the section of maximum axial and shear forces. However, Schlosser (1983) checked the yield criterion only at the section of maximum axial and shear forces. In all finite element
analyses performed in this study, the section of maximum bending moment is the critical section and hence the yield criterion at this section governs the ultimate forces in the reinforcement. In most practical situations, the section of maximum bending moment is also expected to be the critical section (Jewell and Pedley 1992).

4- The procedure described by Schlosser (1983) to relate the axial and shear forces in the reinforcement, equations (2.13) and (2.14), is questionable. For instance, the above procedure gives that the axial force in a reinforcement perpendicular on the shearing plane equals zero. The finite element analyses performed in this study are for reinforcements perpendicular on the shearing plane and in these analyses, the maximum axial forces (or the maximum frictional resistance along the reinforcements) are completely mobilized. The beam model does not have the above shortcoming.
Figure (3.34.a) Initial Normal Pressure On The Reinforcement ($= \sigma_i$).

Figure (3.34.b) Normal Pressure On The Reinforcement After Shearing.

Figure (3.34.c) Net Normal Pressure On The Reinforcement ($\sigma_n$).
Figure (3.34.d) Distribution Of The Bending Moment.

Figure (3.34.e) Distribution Of The Shearing Force.

Figure (3.34) Distribution Of The Normal Pressure On The Reinforcement And The Forces In The Reinforcement.
Figure (3.35.a) Distribution Of The Net Normal Pressure ($\sigma_n$) On A Wood Dowel Imbeded In Dense Sand, Applied Displacement On The Movable Half Of The Box=0.5 cm.

Figure (3.35.b) Distribution Of The Net Normal Pressure ($\sigma_n$) On A Wood Dowel Imbeded In Loose Sand, Applied Displacement On The Movable Half Of The Box=0.5 cm.

Figure (3.35) Distribution Of The Net Normal Pressure ($\sigma_n$) On A Wood Dowel Imbeded In Dense And Loose Sands.
Figure (3.36.a) Distribution Of The Normal Stress $\sigma_{rr}$.

Figure (3.36.b) Distribution Of The Shear Stress $\tau_{rx}$.

Figure (3.36.c) Distribution Of The Shear Stress $\tau_{r\theta}$.

Figure (3.36) Distribution Of The Normal And Shear Stresses On The Periphery Of A Reinforcement Section.
Relations (3-42) and (3-44)

\[ \frac{T_{\text{max}}}{\mu \pi D L_a \sigma_i} \]

- FEM, Cases Controlled By The Fiber Bending Stiffness
- FEM, Cases Controlled By The Soil Bearing Resistance

Average FEM, Factor of Safety = 1.1
Average FEM, Factor of Safety = 1.2
Average FEM, Factor of Safety = 1.3
Average FEM, Factor of Safety = 1.4

\( \Phi_T = 0.217 \left( \frac{D}{L_a} \right) \left( \frac{E_k}{E_i} \right)^{0.271} \left( \frac{\sigma_f}{\sigma_i} \right) \) if \( \sigma_b > \sigma_{\text{a\_max}} \)

\( \Phi_T = 0.169 \left( \frac{D}{L_a} \right) \sqrt{\frac{\sigma_f \sigma_b}{\sigma_i^2}} \) if \( \sigma_b < \sigma_{\text{a\_max}} \)

Figure (3.37) Comparison Between The Axial Forces Obtained From The Finite Element Analyses And From The Beam Model.
Relations (3-52) and (3-53)

- FEM, Cases Controlled By The Fiber Bending Stiffness
- FEM, Cases Controlled By The Soil bearing Resistance

---

Average FEM, Factor of Safety=1.1
Average FEM, Factor of Safety=1.2
Average FEM, Factor of Safety=1.3
Average FEM, Factor of Safety=1.4

---

\[ \Phi_Q = 0.75 \left( \frac{E_s}{E_r} \right)^{0.271} \left( 1 - \left( \frac{T_{el}}{T_p} \right)^2 \right)^{0.5} \] if \( \sigma_b > \sigma_{n_{max}} \)

\[ \Phi_Q = \sqrt{\frac{\sigma_b}{3\sigma_y}} \times \sqrt{1 - \left( \frac{T_{el}}{T_p} \right)^2} \] if \( \sigma_b < \sigma_{n_{max}} \)

Figure (3.38) Comparison between The Shear Forces Obtained From The Finite Element Analyses And From The Beam Model.
3.7 Distribution Of Stresses And Strains

In this Section, the distributions of stresses and strains within different soil-reinforcement composites are studied. The objectives of this Section can be summarized as follows.

1- To investigate the distribution of stresses and strains in a soil reinforced by one reinforcement.

2- To investigate the distribution of stresses and strains in a soil reinforced by a group of reinforcement and to examine the group effect of the reinforcement.

To achieve the above objectives, the distributions of the stresses and strains within two of the already analyzed soil-reinforcement composites are examined. These composites are a dense sand reinforced by one wood dowel with a diameter of 0.32 cm (case (14) of Table (3.2)) and a dense sand reinforced by six reed reinforcements with a diameter of 0.18 cm (the reinforced soil of Gray and Ohashi’s test).

Throughout this Section, the stresses will be represented with the stress invariants \((p)\) and \((q)\) given by equations (3.2) and (3.3). Also, the plastic strains will be represented in terms of the strain invariants \((\varepsilon_p^p)\) and \((\varepsilon_p^s)\), where,

\[
\varepsilon_p^p = \text{the volumetric plastic strain} = \varepsilon_{xx}^p + \varepsilon_{yy}^p + \varepsilon_{zz}^p
\]

\[
(3.61)
\]

\[
\varepsilon_p^s = \text{the plastic strain magnitude} = \sqrt{\frac{2}{3} \left[ \varepsilon_{xx}^p \varepsilon_{yy}^p \varepsilon_{zz}^p + 2 \varepsilon_{xx}^p \varepsilon_{xy}^p \varepsilon_{yx}^p + 2 \varepsilon_{xx}^p \varepsilon_{xz}^p \varepsilon_{zx}^p + 2 \varepsilon_{yy}^p \varepsilon_{yz}^p \varepsilon_{zy}^p \right]^1}
\]

\[
(3.62)
\]

The use of the stress and strain invariants is preferred for presentation because being invariants, their magnitudes do not change as the reference axes are changed.
Figure (3.39) shows a sketch for a direct shear test performed on a reinforced soil. For convenience, the horizontal plane that exists at the potential failure surface on the bottom half of the direct shear box will be called “the shearing plane”. The vertical plane that passes through the reinforcement will be called “the longitudinal plane”. Throughout this Section, the stress and strain distributions will be shown for these two planes.

3.7.1 Distribution Of Stresses And Strains In A Soil Reinforced By One Reinforcement

Figures (3.40) through (3.43) show the computed distribution of \( p \), \( q \), \( \varepsilon_p^r \), and \( \varepsilon_p^t \) for a dense sand reinforced by one, 0.32 cm diameter, wood dowel. The shown figures are at an applied displacement on the moveable half of the box equals 1.5 cm. From these figures, the following can be observed.

Zones of highly concentrated normal and shear stresses are developed around the reinforcement. The concentrated normal stresses around the reinforcement shown in Figure (3.40.a) represent the normal soil pressure on the reinforcement. It can be seen that the distribution of this stresses is in agreement with the distribution given by relation (3.24) and Figure (3.34.b). As explained in Section (3.6), the development of this normal pressure contributes to the composite strength in two ways. First, it develops a shear force in the reinforcement which is added to the composite strength. Second, it increases the normal stresses on the reinforcement periphery and this increases the frictional resistance at the soil-reinforcement interface. This increase of the frictional resistance is the reason for the development of the highly concentrated shear stress around the reinforcement shown in Figure (3.41.a).

The axial force in the reinforcement produces the increase of the normal stresses on the shearing plane shown in Figure (3.40.b). This increase in the normal stresses is
associated with an increase of the strength of the soil surrounding the reinforcements. This is the reason for the development of the high shear stresses shown in Figure (3.41.b).

The distributions of strains shown in Figures (3.42) and (3.43) are compared with those of the same soil without reinforcement. This comparison shows that the strains for the reinforced soil are much higher than those of the unreinforced soil. For instance, the magnitude of the plastic strain at the maximum strained point in the unreinforced soil is less than 10%. However, the magnitude of the plastic strain at the maximum strained point in the reinforced soil is as much as 60%. Also, it is observed that the soil zone which undergoes deformation (the shear zone) is much wider for the reinforced soil than for the unreinforced soil. The above results can be explained as follows. For the unreinforced soil, shear and volumetric strains continue to develop until the maximum shear strength along the shear plane is developed. After that, the movable half of the box slides on the fixed half of the box as a rigid body with no further increase in the volumetric or shear strains. However, for the reinforced soil, the reinforcement act as a restraint to the soil displacement and it prevent the soil in its vicinity to slide on the shear plane. This results in continuous increase of the volumetric and the shear strains around the reinforcement and a widening of the soil zone which undergoes deformations.
Figure (3.39) The Planes Along Which The Distribution Of Stresses And Strains Are Presented.
Figure (3.40) Distribution Of The Mean Normal Stresses (p) In A Dense Sand Reinforced By One Reinforcement.
Figure (3.41.a) Distribution Of \( (q) \) On The Longitudinal Plane

Figure (3.41.b) Distribution Of \( (q) \) On The Shearing Plane

Figure (3.41) Distribution Of The Mises Stresses \( (q) \) In A Dense Sand Reinforced By One Reinforcement.
Figure (3.42-a) Distribution Of \( \epsilon_{p}^{p} \) On The Longitudinal Plane

Figure (3.42-b) Distribution Of \( \epsilon_{v}^{p} \) On The Shearing Plane

Figure (3.42) Distribution Of The Volumetric Plastic Strain \( \epsilon_{p}^{p} \) In A Dense Sand Reinforced By One Reinforcement.
Figure (3.43.a) Distribution Of $\varepsilon_p^p$ On The Longitudinal Plane

Figure (3.43.b) Distribution Of $\varepsilon_p^p$ On The Shearing Plane

Figure (3.43) Distribution Of The Plastic Strain Magnitude $\varepsilon_p^p$ In A Dense Sand Reinforced By One Reinforcement.
3.7.2 Distribution Of Stresses And Strains In A Soil Reinforced By A Group Of Reinforcement

Figures (3.44) through (3.47) show the computed distribution of \( p \), \( q \), \( \epsilon^e \), and \( \epsilon^h \) for the reinforced soil in Gray and Ohashi's test. The shown figures are at an applied displacement on the moveable half of the box equals 0.5 cm. From these figures, it can be noted that the behaviors explained in the previous subsection generally occur for the soil reinforced by group of reinforcements. However, it can also be observed that the developed normal and shear stresses on the perimeter of each reinforcement are affected by the developed stresses on the perimeters of the adjacent reinforcements. This interaction, or group effect is a common problem in soil mechanics which has not yet been sufficiently investigated. The finite element method has the advantage of including this effect in the analyses.

It has been stated in the literature for pile foundations (e.g., Scott 1981) that the group effect depends on several parameters including the reinforcement spacing, the number and arrangement of the reinforcements, and the relative stiffness between the soil and the reinforcements. Generally, the group effect can be neglected if the reinforcement spacing is larger than 25 times the reinforcement diameter \( D \), (Scott 1981). In the practice of soil nailing, the used reinforcement spacing is usually in the range of 15\( \times D \) to 30\( \times D \) (Elias and Juran 1990). For such composites, the group effect can be neglected (Mitchell and Christopher 1990).
Figure (3.44.a) Distribution Of (p) On The Longitudinal Plane

Figure (3.44.b) Distribution Of (p) On The Shearing Plane

Figure (3.44) Distribution Of The Mean Normal Stresses (p) In A Dense Sand Reinforced By Six Reinforcement.
Figure (3.45-a) Distribution Of (q) On The Longitudinal Plane

Figure (3.45-b) Distribution Of (q) On The Shearing Plane

Figure (3.45) Distribution Of The Mises Stresses (q) In A Dense Sand Reinforced By Six Reinforcement.
Figure (3.46.a) Distribution Of \( (\varepsilon^p_v) \) On The Longitudinal Plane

Figure (3.46.b) Distribution Of \( (\varepsilon^p_v) \) On The Shearing Plane

Figure (3.46) Distribution Of The Volumetric Plastic Strain \( (\varepsilon^p_v) \) In A Dense Sand Reinforced By Six Reinforcement.
Figure (3.47-a) Distribution Of $\varepsilon_p$ On The Longitudinal Plane

Figure (3.47-b) Distribution Of $\varepsilon_p$ On The Shearing Plane

Figure (3.47) Distribution Of The Plastic Strain Magnitude $\varepsilon_p$ In A Dense Sand Reinforced By Six Reinforcement.
3.8 Summary And Conclusions

The following conclusions can be obtained from the finite element analyses given in this chapter.

1- For an accurate analysis of the soil-reinforcement interaction problem, the nonlinear behavior of the soil, the dilation behavior of the soil, the nonlinear behavior of the reinforcement, the correct contact condition between the soil and the reinforcement and the geometric nonlinearity of the problem should be considered in the analysis.

2- The finite element analysis can successfully predict the behavior of the soil-reinforcement composites. In this study, the finite element method was used to simulate different laboratory experiments performed on reinforced soils. Reasonable agreement between the results of the experiments and the results of the finite element analyses was detected.

3- The parametric study of the problem showed that the behavior of the soil-reinforcement composite depends on a number of parameters including the soil relative density, the reinforcement stiffness, and the confining pressure. Generally, the improvement in the composite strength ($\Delta s_r$) due to the reinforcement increases with increasing the soil relative density, the reinforcement stiffness and the confining pressure on the composite.

4- The stress transfer between soil and reinforcement involves two mechanisms: the friction between the soil and the reinforcement and the passive soil pressure on the reinforcement. The above two mechanisms interact as follows: the passive soil pressure on the reinforcement controls the magnitude and the distribution of the
normal stress at the soil-reinforcement interface which, in turn, controls the frictional resistance along the reinforcement length.

5- The magnitude and the distribution of the normal pressure at the soil-reinforcement interface depends on the relative stiffness between the soil and the reinforcement, the ultimate bearing resistance of the soil, the yield strength of the reinforcement and the initial confining pressure on the soil-reinforcement composite. Therefore, the axial and shear forces in the reinforcement are function of the above parameters.

6- Although the finite element analyses performed in this chapter successfully predicted the behavior of the soil-reinforcement composites, such analyses are not feasible in most practical situations. These analyses require computation capabilities which are not available in practical situations and they are very expansive. For instance, the analysis of some cases (e.g., the reinforced soil of Shewbridge and Sitar's test) may need as much as 40 CPU hours on a Cray-YMP8 machine to be computed. The above reasons motivate the necessity for developing simplified models for the problem.

7- A simplified beam model using a modification of the closed-form solution for beams on elastic foundation was described in Section (3-6). In this model, the effect of results (4) and (5) were included in the analyses. The results of this model were compared with the results of the finite element analyses and good agreements were found.

8- The results of the above model were represented in the form of design charts which can be used to perform limit equilibrium analyses for reinforced soils. However, due to the limitations of the limit equilibrium approach, there is still a need for a
simplified model which can be used to study the behavior of reinforced soils under working loads. This model will be developed in chapter (4).

9- A study of the distributions of stresses and strains in reinforced soils was performed. This study showed that concentration of stresses and strains occurs in the soil around the reinforcement. The reasons for the development of these concentrated stresses and strains were explained in Section (3.8).

10- The finite element analysis has the advantage of being able to represent the group effect of the reinforcements. In practical situations where the distance between the reinforcements is sufficiently large, the group effect of the reinforcements can be ignored.
CHAPTER IV

Simplified Finite Element Model For Soil-Reinforcement Interaction

4.1 Introduction

A three dimensional, nonlinear, finite element model for the soil-reinforcement interaction problem is presented in the previous Chapter. This model gives accurate results which helped to understand the behavior of soil-reinforcement composites and to identify the important design parameters which should be included in the analysis. However, this model has the disadvantage of being computationally very expensive and hence, it might not be feasible in most practical situations. This disadvantage motivates the necessity of developing simplified models for the problem. A simplified model for computing the ultimate forces in the reinforcement is also described in the previous Chapter. However, this simplified model can not be used to study the behavior of the soil-reinforcement composites under working loads because it can not compute the soil or the reinforcement deformations.
To overcome the shortcomings of the above two models, a simplified finite element model is developed in this Chapter. In this model, an attempt is made to reduce the computation expanses without sacrificing the accuracy of the analysis. This accuracy is maintained by including the effect of the important design parameters which affect the results. The analyses performed in Chapter (3) showed that the important design parameters are:

1- the nonlinear and the dilative behavior of the soil,
2- the nonlinear behavior of the reinforcement,
3- the slip between the soil and the reinforcement,
4- the geometric nonlinearity of the problem.

To include the effect of the second and fourth of the above parameters, the reinforcement is represented in the new model by a beam element with material and geometric nonlinearity. However, to reduce the computation expanses, the soil reaction on the reinforcement is represented by the theory of subgrade reaction (the Winkler Hypothesis). In this theory, the assumption is made that the soil displacement at a given point is related to the forces applied only at that point. Hence, the subgrade around the reinforcement can be represented by sets of springs at discrete points along the reinforcement. By making this simplification, we can eliminate the soil elements from the analysis and hence the computation expanses can be reduced drastically. Also, by careful choice of the constitutive relations for the subgrade springs, the effect of the first and third of the above design parameters can be included in the analysis.

This chapter is divided into nine sections. Section (4.2) gives a general description of the proposed model. In Section (4.3), the formulation of the beam element used to represent the reinforcement is given. Section (4.4) gives a description of the constitutive relations used for the subgrade springs. The technique used to solve the system of
nonlinear equations of this model is given in Section (4.5). Section (4.6) gives a brief
description of the computer program developed in this study. In section (4.7), the
developed computer program is used to solve problems which have well-known theoretical
solutions. Comparisons between the results of the program and the theoretical solutions
are made to confirm the validity of the program. After confirming the validity of the
program, it is used to analyze different reinforced-soil systems. The results of these
analyses are give in Section (4.8). Finally, Section (4.9) gives a summary of the
conclusions drawn from the analyses performed in this Chapter.

4.2 General Description Of The Model

Figure (4.1.a) shows a reinforced soil with an assumed slip surface. Two types of
coordinate systems are shown in Figure (4.1.a), a global fixed coordinate system (X, Y,
Z) and a local convective coordinate system (x, y, z). In the global coordinate system, the
(X-Y) plane is the vertical plane and the (X-Z) plane is the horizontal plane. In the local
coordinate system, the (x) axis is coincident with the longitudinal axis of the reinforcement
and the (y) and (z) axes are the principle axes of the reinforcement.

As stated before, the subgrade is represented in this model by the theory of
subgrade reaction. Therefore, the subgrade is represented by sets of springs at discrete
points along the reinforcement, Figure (4.1.b). Each set of springs consists of four
springs. Two of these springs are in the local (y) and (z) directions and they represent the
lateral soil pressure on the reinforcement. The third spring is in the local axial direction and
it represents the friction between the soil and the reinforcement in the axial direction. The
fourth spring is a rotational spring around the local (x) axis and it represents the friction
between the soil and the reinforcement in the circumferential direction.
Figure (4.1.a) Definition Of The Coordinate Systems.

Figure (4.1.b) Modeling The Subgrade By Springs.

Figure (4.1) The Simplified Finite Element Model.
Now, the problem is reduced to that shown in Figure (4.1.b). In this Figure, the reinforcement is divided into (n) beam elements which are loaded (or supported) at their nodal points by the subgrade springs. To obtain the governing equation for this problem, the following steps can be performed.

1- Determine the load-displacement relationship for the reinforcement system.
2- Determine the external forces applied on the reinforcement by the subgrade springs.
3- Apply the equilibrium condition to determine the governing equation for the problem.

A general description of the above three steps are given in the following three subsections.

4.2.1 Load-Displacement Relationship For The Reinforcement System

In this study, the reinforcement is represented by two-nodal, spatial, beam elements with material and geometric nonlinearity. Since each node of this element has six degrees of freedom (three translations and three rotations), the displacement vector for any of the reinforcement elements (e.g., element (i-j) in Figure (4.1.b)) can be described in the local coordinate system as:

$$
\{\mathbf{v}\} = \begin{pmatrix}
\mathbf{v}_i^l & \mathbf{v}_j^l & \bar{\mathbf{v}}_i^l & \bar{\mathbf{v}}_j^l & \bar{\mathbf{v}}_i^l & \bar{\mathbf{v}}_j^l & \bar{\mathbf{v}}_i^l & \bar{\mathbf{v}}_j^l & \bar{\mathbf{v}}_i^l & \bar{\mathbf{v}}_j^l
\end{pmatrix}^T
$$

(4.1)

where:
- $\mathbf{v}_i^l$, $\mathbf{v}_j^l$, $\bar{\mathbf{v}}_i^l$ = the displacement of node (i) in the x, y, and z directions, respectively,
- $\mathbf{v}_i^l$, $\mathbf{v}_j^l$, $\bar{\mathbf{v}}_i^l$ = the displacement of node (j) in the x, y, and z directions, respectively,
- $\bar{\theta}_x^l$, $\bar{\theta}_y^l$, $\bar{\theta}_z^l$ = the rotation of element (i-j) at node (i) around the x, y, and z axes, respectively,
\( \theta_x^i, \theta_y^i, \theta_z^i = \) the rotation of element (i-j) at node (j) around the x, y, and z axes, respectively.

Also, the displacement vector of element (i-j) can be described with respect to the global coordinate system as,

\[
\{ \delta \} = \begin{pmatrix} \delta_x^i & \delta_y^i & \delta_x^i \end{pmatrix}^T
\]

where the components of the above vector have the same definition as given above but written with respect to the X, Y, and Z axes.

The relationship between the displacement vector \( \{ \delta \} \) and the displacement vector \( \{ \psi \} \) can be written as,

\[
\{ \psi \} = [t] \{ \delta \}
\]

where:

\([t] = \) a transformation matrix which transforms the local coordinate system to the global coordinate system.

The method of determining this matrix will be given in section (4.3).

Similar to the above representation, the internal forces developed in element (i-j) can be described with respect to the local coordinate system as,

\[
\{ \mathbf{s} \} = \begin{pmatrix} F_x^i & F_y^i & F_z^i \\ M_x^i & M_y^i & M_z^i \\ \end{pmatrix} \{ M_x^i \}
\]

where:
\( F_x^i, F_y^i, F_z^i = \) the forces at node (i) in the x, y, and z directions, respectively,
\( F_x^j, F_y^j, F_z^j = \) the forces at node (j) in the x, y, and z directions, respectively,
\( M_x^i, M_y^i, M_z^i = \) the moments at node (i) in the x, y, and z directions, respectively,
\( M_x^j, M_y^j, M_z^j = \) the moments at node (j) in the x, y, and z directions, respectively.
Also, the internal forces developed in element (i-j) can be described with respect to the global coordinate system as,

\[ \{ \bar{s} \} = \left( \bar{F}_x^i \bar{F}_y^i \bar{F}_z^i \bar{M}_x^i \bar{M}_y^i \bar{M}_z^i \right)^T \]

where the components of the above vector have the same definitions as given above but written with respect to the X, Y, and Z axes.

The relationship between the force vector \( \{ \bar{s} \} \) and the force vector \( \{ \hat{s} \} \) can be written as,

\[ \{ \bar{s} \} = [t]\{ \hat{s} \} \]  \hspace{1cm} (4.6)

Now, assume that the relation between the force vector \( \{ \bar{s} \} \) and the displacement vector \( \{ \bar{v} \} \) is,

\[ \{ \bar{s} \} = [\bar{k}_r]\{ \bar{v} \} \]  \hspace{1cm} (4.7)

where:

\[ [\bar{k}_r] = \text{the element stiffness matrix written in the local coordinate system.} \]

The method of determining the matrix \( [\bar{k}_r] \) will be given in section (4.3). By substituting relations (4.3) and (4.7) in relation (4.6), we obtain,

\[ \{ \bar{s} \} = [t]^T[\bar{k}_r][t]\{ \bar{v} \} = [\hat{k}_r]\{ \bar{v} \} \]  \hspace{1cm} (4.8)

where:

\[ [\hat{k}_r] = \text{the element stiffness matrix written in the global coordinate system.} \]

The vectors and matrices given above are for one reinforcement element (element (i-j)). Now we need to assemble the vectors and matrices for all reinforcement elements to obtain the load-displacement relationship for the reinforcement system. This can be carried out as follows.
The displacement in the global coordinate system for all nodes (node (1) through node (n+1) in Figure (4.1.b)) can be written in one vector \( \{ V \} \) as,

\[
\{ V \} = \left\{ \begin{array}{c}
\hat{v}_x^1 \\
\hat{v}_y^1 \\
\hat{v}_z^1 \\
\hat{\theta}_x^1 \\
\hat{\theta}_y^1 \\
\hat{\theta}_z^1 \\
\hat{v}_x^2 \\
\hat{v}_y^2 \\
\ddots \\
\hat{\theta}_{n+1}^2
\end{array} \right\}^T
\]  

(4.9)

Also, the internal forces at each node obtained from different elements sharing that node can be assembled in one vector \( \{ S \} \) as,

\[
\{ S \} = \left\{ \begin{array}{c}
\sum_{e=1}^{n} \hat{F}_x^1 \\
\sum_{e=1}^{n} \hat{F}_y^1 \\
\sum_{e=1}^{n} \hat{F}_z^1 \\
\sum_{e=1}^{n} \hat{M}_x^1 \\
\sum_{e=1}^{n} \hat{M}_y^1 \\
\sum_{e=1}^{n} \hat{M}_z^1 \\
\sum_{e=1}^{n} \hat{\phi}_x^2 \\
\sum_{e=1}^{n} \hat{\phi}_y^2 \\
\ddots \\
\sum_{e=1}^{n} \hat{\phi}_{n+1}^2
\end{array} \right\}^T
\]  

(4.10)

Finally, the stiffness matrices \( [\hat{k}_e] \) of the individual elements can be assembled in one matrix \( [K_r] \) by adding the coefficients of each matrix \( [\hat{k}_e] \) in the corresponding locations in the matrix \( [K_r] \). Now, the load-displacement relationship for the reinforcement system can be written as,

\[
\{ S \} = [K_r] \{ V \}
\]  

(4.11)

where:

\( \{ S \} \) = a vector contains the summation of the global internal forces at different nodes.

Note that for equilibrium, this vector should be equal to the vector of applied external forces on the reinforcement system \( \{ R \} \). The method for determining the vector \( \{ R \} \) will be given in the following subsection,

\( [K_r] \) = the global stiffness matrix for the reinforcement system,

\( \{ V \} \) = the vector of generalized global displacements of the nodes of the reinforcement system.
4.2.2 The External Force Vector

Consider again Figure (4.1.b) and let us define the soil displacement far away from the reinforcement at any depth \( d \) as "the free-field soil displacement at depth \( d \)". The free-field soil displacement at the depths correspond to points (i) and (j) of the reinforcement can be described with respect to the local coordinate system \((x, y, z)\) as,

\[
\{w\} = \begin{pmatrix}
  w_i \\
  w_j \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
\end{pmatrix}
\]

where:
- \( w_i \), \( w_j \), and \( w_k \) are the components of the free-field soil displacement at a depth corresponds to point (i) in the \( x \), \( y \) and \( z \) directions, respectively,
- \( w_i \), \( w_j \), and \( w_k \) are the components of the free-field soil displacement at a depth corresponds to point (j) in the \( x \), \( y \) and \( z \) directions, respectively.

The above vector can also be written with respect to the global coordinate system \((X, Y, Z)\) as,

\[
\{\hat{w}\} = \begin{pmatrix}
  \hat{w}_x \\
  \hat{w}_y \\
  \hat{w}_z \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
\end{pmatrix}
\]

where the components of the above vector have the same definitions as given above but written with respect to the global coordinate system \((X, Y, Z)\).

The relation between the displacement vector \(\{w\}\) and the displacement vector \(\{\hat{w}\}\) can be written as,

\[
\{w\} = [t]\{\hat{w}\}
\]

where:
- \([t]\) is a transformation matrix which transforms the local coordinate system to the global coordinate system.
The displacement vectors \( \{ \vec{w} \} \) and \( \{ \vec{\hat{w}} \} \) represent the soil displacement far away from the reinforcement. However, around the reinforcement, the soil displacement is resisted by the reinforcement. Due to this resistance, the soil develops lateral pressure and frictional stresses on the reinforcement surface. Using the theory of subgrade reaction (Hetenyi 1946), the lateral pressure or the frictional stress at any point along the reinforcement \( a \) is function of the relative displacement between the free-field soil displacement and the reinforcement displacement at that point. Then, the lateral pressure and frictional stress developed at point \( (i) \) can be expressed as:

\[
\begin{bmatrix}
    p_x^l \\
    p_y^l \\
    p_z^l \\
    p_\theta^l
\end{bmatrix} =
\begin{bmatrix}
    C_x & 0 & 0 & 0 \\
    0 & C_y & 0 & 0 \\
    0 & 0 & C_z & 0 \\
    0 & 0 & 0 & C_\theta
\end{bmatrix}
\begin{bmatrix}
    \vec{w}_x^i - \vec{v}_x^i \\
    \vec{w}_y^i - \vec{v}_y^i \\
    \vec{w}_z^i - \vec{v}_z^i \\
    0 - \vec{\theta}_x^i
\end{bmatrix}
\]

where:

- \( p_x^l \) = the frictional stress in the axial direction (x),
- \( p_y^l \) = the soil pressure in the lateral direction (y),
- \( p_z^l \) = the soil pressure in the lateral direction (z),
- \( p_\theta^l \) = the frictional stress in the circumferential direction,
- \( C_x, C_y, C_z, \) and \( C_\theta \) = the coefficient of subgrade reaction in the axial direction (x), the lateral direction (y), the lateral direction (z), and the circumferential direction, respectively.

The coefficients \( C_x, C_y, C_z, \) and \( C_\theta \) are normally determined from nonlinear curves which are known as the soil response curves. A discussion about these curves will be given in section (4.4). Using relation (4.15), the external forces at node \( (i) \) due to the soil pressure developed on element \( (i-j) \) of the reinforcement can be expressed as,
where:

\( B \) = the average perimeter of the reinforcement at element (i-j),

\( D \) = the average diameter of the reinforcement at element (i-j),

\( L \) = the length of element (i-j),

\( f \) and \( m \) = external force and moment respectively.

A similar relation can be written to express the external forces at node (j) due to the soil pressure developed on element (i-j). Now, the external forces on nodes (i) and (j) due to the soil pressure developed on element (i-j) can be written in one vector as,

\[
\{ \bar{f} \} = \begin{bmatrix} BLC_x & 0 & 0 & 0 \\ 0 & \frac{DL}{2}C_y & 0 & 0 \\ 0 & 0 & \frac{DL}{2}C_z & 0 \\ 0 & 0 & 0 & \frac{BDL}{4}C_b \end{bmatrix} \begin{bmatrix} \bar{w}_x - \bar{v}_x \\ \bar{w}_y - \bar{v}_y \\ \bar{w}_z - \bar{v}_z \\ 0 - \bar{G}_x \end{bmatrix}
\]  

(4.16)

where:

\( \bar{w}_x \) and \( \bar{v}_x \) are the average perimeter of the reinforcement at element (i-j),

\( \bar{D}_x \) = the average diameter of the reinforcement at element (i-j),

\( \bar{L}_x \) = the length of element (i-j),

\( \bar{f}_x \) and \( \bar{m}_x \) = external force and moment respectively.

The vector \( \{ \bar{f} \} \) can be expressed with respect to the global coordinate system (X, Y, Z) using the transformation relation,

\[
\{ \bar{f}_x \} = [I]^{T}\{ \bar{f} \}
\]  

(4.19)

where:

\( \{ \bar{f}_x \} \) = the vector of external forces at nodes (i) and (j) written with respect to the global coordinate system.
Substituting (4.3), (4.14) and (4.17) in (4.19), we obtain,

\[
\{r\} = [l]^T [k_s] [l] \{w\} - \{v\}
\]
\[
= [k_s] \{w\} - \{v\}
\]  \hspace{1cm} (4.20)

Vector \{r\} gives the external forces at nodes (i) and (j) due to the soil pressure developed on element (i-j). Such a vector can be written for the different reinforcement elements. Then, these vectors can be assembled to the global force vector \{R\} which can be expressed as,

\[
\{R\} = [K_s] \{W\} - \{V\}
\]  \hspace{1cm} (4.21)

where:

\[K_s\] = the soil stiffness matrix for the reinforcement system. These matrix can be obtained by assembling the coefficients of each \[k_s\] matrix in the corresponding location in the \[K_g\] matrix,

\[W\] = the free-field soil displacement vector for all nodes of the reinforcement system. This vector can be expressed as,

\[
\{W\} = \begin{pmatrix} \hat{w}_x^1 & \hat{w}_y^1 & \hat{w}_z^1 & 0 & 0 & \hat{w}_x^2 & \cdots & \hat{w}_x^{n+1} & 0 & 0 \end{pmatrix}^T
\]  \hspace{1cm} (4.22)

\[V\] = the displacement vector for all nodes in the reinforcement system. This vector is given by relation (4.9).

### 4.2.3 Governing Equation

For equilibrium, the external load vector \{R\}, given by relation (4.21), should be equal to the internal force vector \{S\}, given by relation (4.11). Then, by equating relations (4.11) and (4.21), we obtain,

\[
\{S\} = \{R\}
\]  \hspace{1cm} (4.23.a)
or,
\[
[K_r]\{V\} = [K_r]\{(W) - \{V\}\}
\]  \hspace{1cm} (4.23.b)

Since the material and geometric nonlinearities of the reinforcement element are considered, the matrix \([K_r]\) is function of the displacement vector \(\{V\}\). Also, since the constitutive relations for the subgrade springs are represented by nonlinear curves, the matrix \([K_s]\) is function of the displacement vector \(\{V\}\). Therefore, equation (4.23.b) can be rewritten as,
\[
[K_r(\{V\})]\{V\} = [K_s(\{V\})]\{(W) - \{V\}\}
\]  \hspace{1cm} (4.23.c)

The above equation is the governing equation for our problem. Since the above system of equations is nonlinear, it should be solved using an incremental method. In this study, equation (4.23.c) is solved using the Newton-Raphson method (Householder 1953) which involves a series of solutions to linear incremental equations. A description of this method is given in Section (4.5).

The above formulation gives an outline of the proposed model. However, the following questions remain to be addressed,

1- How to determine the stiffness matrix \([\hat{K}_r]\) for the reinforcement element?

2- How to obtain appropriate constitutive relations for the subgrade springs (these constitutive relations give the values of the coefficients \(C_x, C_y, C_z\) and \(C_0\) used to construct the matrix \([K_s]\))?

3- How to effectively solve the system of nonlinear equations given by relation (4.23.c)?

The answers of these three questions are given in the following three sections.
4.3 Stiffness Matrix Of The Reinforcement Element

As stated before, the reinforcement is represented by two-nodal, spatial beam elements with geometric and material nonlinearity. The derivation of the stiffness matrix of a typical element is given in this section. In this derivation, an updated Lagrangian formulation (Bathe and Bolourchi 1979; Meek and Tan 1984) is adopted to account for the geometric nonlinearity. The material nonlinearity is taken into account by using the plastic hinge method (Ueda 1964; Argyris et al. 1982).

4.3.1 Basic Definitions

The definitions of the coordinate systems (the global and local coordinate systems) were given in section (4.2.1). The displacement and force vectors of any element were described in the global coordinate system by \( \{v\} \) and \( \{s\} \), and in the local coordinate system by \( \{\bar{v}\} \) and \( \{\bar{s}\} \). Here, we introduce a third definition, the element deformation vector \( \{\gamma\} \) and the element force vector \( \{\bar{s}\} \). The element deformation vector \( \{\gamma\} \) can be obtained by excluding the rigid body movements from the vector \( \{\bar{v}\} \) and it can be written as,

\[
\{\gamma\} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}^T
\]

where:

\[
v_1 = \bar{v}_k^i - \bar{v}_k^j = \text{axial extension},
\]

\[
v_2 = \bar{\theta}_k^i - \bar{\theta}_k^j = \text{axial twist},
\]

\[
v_3 = \text{rotation at end (i) about the y axis},
\]

\[
v_4 = \text{rotation at end (j) about the y axis},
\]

\[
v_5 = \text{rotation at end (i) about the z axis},
\]

\[
v_6 = \text{rotation at end (j) about the z axis}.
\]
The element force vector \( \{s\} \) is the force vector associated with the element deformation vector \( \{v\} \) and it can be written as,

\[
\{s\} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{pmatrix}^T \tag{4.25}
\]

where:

- \( s_1 = F_{x}^1 - F_{x}^i = \text{axial force} \),
- \( s_2 = M_{x}^1 - M_{x}^i = \text{torsional moment} \),
- \( s_3 = \text{moment of end } i \text{ about the } y \text{ axis} \),
- \( s_4 = \text{moment of end } j \text{ about the } y \text{ axis} \),
- \( s_5 = \text{moment of end } i \text{ about the } z \text{ axis} \),
- \( s_6 = \text{moment of end } j \text{ about the } z \text{ axis} \).

Now, the following steps can be used to drive the stiffness matrix \( [k] \).

1- Determine the stiffness matrix \( [k] \) which relates the force vector \( \{s\} \) to the deformation vector \( \{v\} \).
2- Perform the appropriate transformation to determine the stiffness matrix \( [\tilde{k}] \) which relates the force vector \( \{\tilde{s}\} \) to the displacement vector \( \{\tilde{v}\} \), relation (4.7).
3- Perform the appropriate transformation to determine the stiffness matrix \( [\hat{k}] \) which relates the force vector \( \{\hat{s}\} \) to the displacement vector \( \{\hat{v}\} \), relation (4.8).

However, since the material and the geometric nonlinearities are considered in this study, an incremental approach is implemented in our analysis. Therefore, in the analysis to follow, the stiffness matrices \( [k] \), \( [\tilde{k}] \), and \( [\hat{k}] \) which relate the increments of the force vectors to the increments in the displacement vectors are derived. The symbol \( (\Delta) \) will be used to indicate increments.
4.3.2. Relationship Between The Force Vector \( \{\Delta s\} \) And The Deformation Vector \( \{\Delta v\} \)

The relationship between the force vector \( \{\Delta s\} \) and the deformation vector \( \{\Delta v\} \) can be written as,

\[
\{\Delta s\} = [k_r] \{\Delta v\}
\]

(4.26)

where:

\([k_r]\) = the tangent stiffness matrix of the element.

For the beam element with geometric and material nonlinearity, the tangent stiffness matrix \([k_r]\) can be expressed as (Argyris et al. 1982; Orbison et al. 1982),

\[
[k_r] = [k_e] + [k_i] - [k_p]
\]

(4.27)

where:

\([k_e]\) = the linear elastic stiffness matrix,

\[
\begin{bmatrix}
\frac{E_r A_r}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{g_r I_x}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4E_r I_y}{L} & \frac{2E_r I_y}{L} & 0 & 0 \\
0 & 0 & \frac{2E_r I_y}{L} & \frac{4E_r I_y}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4E_r I_z}{L} & \frac{2E_r I_z}{L} \\
0 & 0 & 0 & 0 & \frac{2E_r I_z}{L} & \frac{4E_r I_z}{L}
\end{bmatrix}
\]

(4.28)

in which:

\(E_r\) and \(g_r\) = the Young's and shear modulus of the reinforcement material, respectively,

\(A_r\) = the cross sectional area of the reinforcement element,
\( I_x, I_y, I_z \) = the moment of inertia of the element section about the (x), (y) and the (z) axes, respectively,

\( L \) = the element length.

\([k_i]\) = the initial stress stiffness matrix,

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{4s_1L}{30} & -\frac{s_1L}{30} & 0 & 0 \\
0 & 0 & -\frac{s_1L}{30} & \frac{4s_1L}{30} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4s_1L}{30} & -\frac{s_1L}{30} \\
0 & 0 & 0 & 0 & -\frac{s_1L}{30} & \frac{4s_1L}{30}
\end{bmatrix}
\]  (4.29)

where:

\( s_1 \) = the axial force in the element.

\([k_p]\) = the plastic reduction matrix. This matrix ensures that, for any element end at which the element forces lie on a predefined yield surface, the changes in the forces during the next load increment will lie in a plane tangent to the yield surface and the resultant plastic deformations will be normal to the yield surface. The derivation of this matrix was presented by Argyris et al. (1982). A summary of their derivation is presented here to establish the notation used subsequently.

Assume that the incremental deformation vector \( \{\Delta v\} \) can be expressed as the sum of the elastic and plastic components, then,

\[
\{\Delta v\} = \{\Delta v_e\} + \{\Delta v_p\}
\]  (4.30)

where:
\{ \Delta v_e \} = \text{the elastic part of the incremental deformation vector},
\{ \Delta v_p \} = \text{the plastic part of the incremental deformation vector.}

Further, assume that the plastic part of the incremental deformation vector can be decomposed as,

\begin{equation}
\{ \Delta v_p \} = \{ \Delta v^1_p \} + \{ \Delta v^2_p \}
\end{equation}

where:

\{ \Delta v^1_p \}, \{ \Delta v^2_p \} = \text{the incremental plastic deformation due to yielding of the first and second ends of the element, respectively.}

Substitute relation (4.31) in relation (4.30), then,

\begin{equation}
\{ \Delta v \} = \{ \Delta v_e \} + \{ \Delta v^1_p \} + \{ \Delta v^2_p \}
\end{equation}

Next, assume the existence of a yield surface (\Gamma) at the cross section of each end of the element. These yield surfaces can be expressed by the functions,

\begin{align*}
\Gamma^1 = \Gamma^1(\{s\}) \\
\Gamma^2 = \Gamma^2(\{s\})
\end{align*}

where:

\Gamma^1, \Gamma^2 = \text{the yield functions at the first and second ends of the element, respectively.}

Also, assume that the yield surface can be treated as a plastic potential (i.e., associated flow rule), then, the plastic deformation increments can be written as,

\begin{align*}
\{ \Delta v^1_p \} &= \lambda_1 \left[ \frac{\partial \Gamma^1}{\partial s} \right] = \lambda_1 \{ G^1 \} \\
\{ \Delta v^2_p \} &= \lambda_2 \left[ \frac{\partial \Gamma^2}{\partial s} \right] = \lambda_2 \{ G^2 \}
\end{align*}

where:
\( \lambda_1, \lambda_2 \) = non-negative proportionality constants,

\( \{G^1\}, \{G^2\} \) = the gradient of the yield functions \((\Gamma^1)\) and \((\Gamma^2)\). These vectors can be expressed as,

\[
\{G^n\} = \left( \frac{\partial \Gamma^n}{\partial s_1} \quad \frac{\partial \Gamma^n}{\partial s_2} \quad \frac{\partial \Gamma^n}{\partial s_3} \quad \frac{\partial \Gamma^n}{\partial s_4} \quad \frac{\partial \Gamma^n}{\partial s_5} \quad \frac{\partial \Gamma^n}{\partial s_6} \right)^T, \eta = 1, 2 \quad (4.35)
\]

Apply the normality condition at both ends of the element, then,

\[
\lambda_1\{G^1\}^T\{\Delta s\} = 0
\]

\[
\lambda_2\{G^2\}^T\{\Delta s\} = 0 \quad (4.36)
\]

The incremental force vector \(\{\Delta s\}\) is related to the elastic part of the deformation by the relation,

\[
\{\Delta s\} = [k_i][\Delta v_e] \quad (4.37)
\]

where:

\[
[k_i] = [k_e] + [k_l]
\]

Substitute (4.32) and (4.34) in (4.37), then,

\[
\{\Delta s\} = [k_i] \{\Delta v - \lambda_1\{G^1\} - \lambda_2\{G^2\}\} \quad (4.38)
\]

By substituting (4.38) in (4.36), we obtain,

\[
\lambda_1\{G^1\}^T[k_i]\{\Delta v - \lambda_1\{G^1\} - \lambda_2\{G^2\}\} = 0
\]

\[
\lambda_2\{G^2\}^T[k_i]\{\Delta v - \lambda_1\{G^1\} - \lambda_2\{G^2\}\} = 0 \quad (4.39)
\]

The above two equations can be solved for the parameters \( \lambda_1 \) and \( \lambda_2 \) as,
\[ \lambda_1 = \left\{ b_1 \{G^1\}^T + b_2 \{G^2\}^T \right\} [k_i] [\Delta v] \]
\[ \lambda_2 = \left\{ b_3 \{G^1\}^T + b_4 \{G^2\}^T \right\} [k_i] [\Delta v] \]  \hspace{1cm} (4.40)

where:
\[ b_1 = \frac{c_2}{c_1 c_2 - c_{12}^2}, \quad b_2 = \frac{-c_{12}}{c_1 c_2 - c_{12}^2} \]
\[ b_3 = \frac{-c_{12}}{c_1 c_2 - c_{12}^2}, \quad b_4 = \frac{c_1}{c_1 c_2 - c_{12}^2} \]  \hspace{1cm} (4.41)

in which,
\[ c_1 = \{G^1\}^T [k_i] \{G^1\} \]
\[ c_2 = \{G^2\}^T [k_i] \{G^2\} \]
\[ c_{12} = \{G^1\}^T [k_i] \{G^2\} \]  \hspace{1cm} (4.42)

Substitute (4.40) in (4.38), then,
\[ \{\Delta s\} = \left[ [k_i] - [k_p] \right] [\Delta v] \]
\[ = [k_i] [\Delta v] \]  \hspace{1cm} (4.26)

where:
\[ [k_p] = \text{the plastic reduction matrix}, \]
\[ = [k_i] \left[b_1 \{G^1\} \{G^1\}^T + b_2 \{G^1\} \{G^2\}^T + b_3 \{G^2\} \{G^1\}^T + b_4 \{G^2\} \{G^2\}^T \right] [k_i] \]  \hspace{1cm} (4.43)

Note that the above expression for \([k_p]\) represents the plastic reduction matrix for the case when yielding has been reached at both ends of the element. For the case when yielding has been reached at the first end of the element while the second end is still elastic, \([k_p]\) reduces to,
\[ [k_p] = \frac{1}{c_1} [k_i] \{G^1\} \{G^1\}^T [k_i] \]  \hspace{1cm} (4.44)
For the case when yielding has been reached at the second end of the element while the first end is still elastic, \([k_p]\) reduces to,

\[
[k_p] = \frac{1}{c_2}[k_t][G^2][G^2]^T[k_t]
\]

(4.45)

For the case when yielding has not been reached at both ends of the element, \([k_p]\) reduces to a null matrix.

### 4.3.2.1. The Yield Surface

Generally, the yield surface for a beam element depends on the geometric shape of the cross section of the beam and the loading condition applied on it. There exits a vast literature on the subject of determining the yield surface for different types of beams under different loading conditions (e.g., Chen and Atsuta 1977). The mathematical expressions for the yield surfaces given in the literature have different degrees of complexity. Some of the developed yield surfaces are expressed using single equation while others are expressed as a composite of several surfaces with different equations. The use of a yield surface which can be expressed as a single function has a numerical advantage, (see the discussion by Orbison et al. 1982). Therefore, a single function yield surface is used in this study.

This yield function has the form (Wen and Farhoomand 1971),

\[
\Gamma = \left(\frac{F_x}{F_{xu}}\right)^2 + \left(\frac{M_x}{M_{xu}}\right)^2 + \left(\frac{M_y}{M_{yu}}\right)^2 + \left(\frac{M_z}{M_{zu}}\right)^2 - 1 = 0
\]

(4.46)

where:

- \(F_x, M_x, M_y, M_z\) = the axial force, twist moment, bending moment in the (y) direction, and bending moment in the (z) direction, respectively,
- \(F_{xu}, M_{xu}, M_{yu}, M_{zu}\) = the limiting values of the axial force, twist moment, bending moment in the (y) direction, and bending moment in the (z) direction, respectively.
4.3.2.2. Control Of The Force Drift From The Yield Surface

The plastic reduction matrix \([k_p]\) constrains the element forces at a plastic hinge to move tangent to the yield surface. Figure (4.2.a) shows a possible motion of the force vector at a plastic hinge. The element stiffness is formulated at point (a) with gradient \([G^a]\). The load increment is then applied and it results in the motion of the force vector to point (b). If the procedure is repeated at point (b), the force vector will move to point (c). This results in continuous drift of the force vector from the yield surface which may lead to unacceptable errors.

Two procedures can be used to prevent the above problem. The first procedure is an iterative procedure suggested by Wen and Farhoomand (1971). This procedure is shown in Figure (4.2.b) and it can be explained as follows. In the first iteration, the element stiffness is formulated at point (a) with gradient \([G^a]\) and it results in point (b). Then, the second iteration is performed with a plastic reduction matrix being formed with a gradient vector which is an average of the gradients at points (a) and (b). This can be expressed as,

\[
\{G\} = \frac{1}{2}\{\{G^a\} + \{G^b\}\}
\]

This produces the force at state (b'). The procedure is then repeated until the agreement between the forces and the yield surface converges to an acceptable tolerance. The above explained procedure may requires considerable computational effort to bring the force point onto the yield surface.

The second procedure to prevent the force drift is a one step procedure and it was suggested by Row et al. (1979). In this procedure, Figure (4.2.c), the force required to regain the yield surface is computed and applied in the succeeding increment. The force
required to regain the yield surface may be computed in several ways. For instance, The force may be computed through a path passing through the origin, point (b) in Figure (4.2.c), a path normal to the yield surface, point (c) in Figure (4.2.c), or a pass along a constant force, point (d) in Figure (4.2.c). The use of any of these return paths might include an error.

In this study, the second procedure is used to prevent the force drift from the yield surface. The return path passing through the origin is used. To reduce the error involved in using this procedure, the following limitations are applied to the increment size in any step.

1- The computed values of the incremental axial force, torsional moment, and bending moments in the (y) and (z) directions obtained in any loading step should not exceed a prescribed ratio of the limiting values $F_{xu}$, $M_{xu}$, $M_{yu}$, and $M_{zu}$, respectively.

2- The drift from the yield surface in any increment should not exceed a user specified limit.

In the analyses performed in this chapter, the ratio in (1) is taken equal to 10% and the drift limit in (2) is taken equal to 1%. If any of the above two limitations is violated, the increment size is automatically reduced.
Figure (4.2.a) Force Point Deviation From Yield Surface.

Figure (4.2.b) Control Of Force Drift Using The Iterative Method.

Figure (4.2.c) Control Of Force Drift Using The One Step Method.

Figure (4.2) Control Of The Force Drift.
4.3.3 Relationship Between The Local Force Vector \{\Delta \bar{s}\} And The Local Displacement Vector \{\Delta \bar{v}\}

The relationship between the incremental force vector \(\Delta s\) and the incremental deformation vectors \(\Delta v\) was obtained in the previous section as,

\[
\{\Delta s\} = [k_r]\{\Delta v\} \tag{4.26}
\]

The incremental displacement vector \(\Delta \bar{v}\) is related to the element deformation vector \(\Delta v\) by the relation,

\[
\{\Delta v\} = [\bar{a}]\{\Delta \bar{v}\} \tag{4.48}
\]

where:

\[ [\bar{a}] = \text{the displacement transformation matrix}, \]

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{L} & 0 & 1 & 0 & 0 & \frac{1}{L} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-1}{L} & 0 & 0 & 0 & 0 & \frac{1}{L} & 0 & 1 & 0 \\
0 & \frac{1}{L} & 0 & 0 & 0 & 1 & 0 & \frac{-1}{L} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{L} & 0 & 0 & 0 & 0 & \frac{-1}{L} & 0 & 0 & 0 & 1 \\
0 & \frac{1}{L} & 0 & 0 & 0 & 0 & \frac{-1}{L} & 0 & 0 & 0 & 1 \\
\end{bmatrix} \tag{4.49}
\]

Also, through equilibrium considerations, the local force vector \(\bar{s}\) is related to the element force vector \(s\) by the relation,

\[
\{\bar{s}\} = [\bar{a}]^T \{s\} \tag{4.50}
\]

By differentiating equation (4.50) and substituting from equation (4.26) and (4.48), we get,
\[
\begin{align*}
\{\Delta \bar{s}\} &= [\bar{a}]^T\{\Delta s\} + [\Delta \bar{a}]^T\{s\} \\
&= [\bar{a}]^T[k_r]\{\Delta v\} + [\Delta \bar{a}]^T\{s\} \\
&= [\bar{a}]^T[k_r][\bar{a}]\{\Delta v\} + [\Delta \bar{a}]^T\{s\}
\end{align*}
\]  

(4.51)

The second term in the above equation gives the change in the stiffness relation which arises from the changed position of the element.

The matrix \([\Delta \bar{a}]\) can be expressed as (Oran 1973; Meek and Tan 1984),

\[
[\Delta \bar{a}] =
\begin{bmatrix}
0 & -\eta_2 & -\eta_3 & 0 & 0 & 0 & 0 & \eta_2 & \eta_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\eta_3 & 0 & \eta_1 & 0 & 0 & -\eta_3 & 0 & -\eta_1 & 0 & 0 & 0 & 0 \\
\eta_3 & 0 & \eta_1 & 0 & 0 & -\eta_3 & 0 & -\eta_1 & 0 & 0 & 0 & 0 \\
-\eta_2 & -\eta_1 & 0 & 0 & 0 & \eta_2 & \eta_1 & 0 & 0 & 0 & 0 & 0 \\
-\eta_2 & -\eta_1 & 0 & 0 & 0 & \eta_2 & \eta_1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(4.52)

where:

\[
\eta_1 = \frac{\Delta \bar{v}_i^j - \Delta \bar{v}_i^j}{L^2}
\]

\[
\eta_2 = \frac{\Delta \bar{v}_y - \Delta \bar{v}_y}{L}
\]

\[
\eta_3 = \frac{\Delta \bar{v}_z^j - \Delta \bar{v}_z^j}{L}
\]

Substituting equation (4.52) in the second term of the right hand side of equation (4.51), then after rearrangement of terms, equation (4.51) can be rewritten as,

\[
\{\Delta \bar{s}\} = [\bar{a}]^T[k_r][\bar{a}]\{\Delta v\} + [\Delta \bar{a}]^T\{s\}
\]

(4.53)

where \([k_G]\) is the geometric stiffness matrix given by,
4.3.4 Relationship Between The Global Force Vector \(\{\Delta \mathbf{s}\}\) And The Global Displacement Vector \(\{\Delta \mathbf{v}\}\)

Equation (4.53) gives the equilibrium equation written with respect to the local coordinate system. To write this equation with respect to the global coordinate system, transformations similar to those given by relations (4.3) and (4.6) can be performed. These transformations can be written as,

\[
\{\Delta \mathbf{v}\} = [t]\{\Delta \mathbf{v}\} \quad (4.55)
\]

\[
\{\Delta \mathbf{s}\} = [t]^T\{\Delta \mathbf{s}\} \quad (4.56)
\]

where \([t]\) is the coordinate transformation matrix.

Substitute (4.53) and (4.55) in (4.56), then,
\{Δς\} = [t]^T(\bar{[k]} + [k_a])\{Δv\}
\[= [t]^T(\bar{[k]} + [k_a][t]\{Δv\} = [k_t]\{Δv\}\] (4.57)

The coordinate transformation matrix \([t]\) can be written as, (Bathe and Bolourchi 1979),

\[[t] = \begin{bmatrix}
[p] & [0] & [0] & [0] \\
[0] & [p] & [0] & [0] \\
[0] & [0] & [p] & [0] \\
[0] & [0] & [0] & [p]
\end{bmatrix}\] (4.58)

where:

\[[p] = \begin{bmatrix}
1 & 0 & 0 & \cos(β)\cos(α) & \sin(β) & \cos(β)\sin(α) \\
0 & \cos(γ) & \sin(γ) & \sin(β)\cos(α) & \cos(β) & \sin(β)\sin(α) \\
0 & -\sin(γ) & \cos(γ) & -\sin(α) & 0 & \cos(α)
\end{bmatrix}\] (4.59)
in which:

\[
\cos(α) = \frac{X_j - X_i}{L_i} \quad \text{sin}(α) = \frac{Z_j - Z_i}{L_i}
\]

\[
\cos(β) = \frac{L_i}{L} \quad \text{sin}(β) = \frac{Y_j - Y_i}{L}
\]

\[
L_i = \sqrt{\left(\left(X_j - X_i\right)^2 + \left(Z_j - Z_i\right)^2\right)}
\]

\[
L = \sqrt{\left(\left(X_j - X_i\right)^2 + \left(Y_j - Y_i\right)^2 + \left(Z_j - Z_i\right)^2\right)}
\]

\(X_i, Y_i, Z_i = \) the coordinates of node (i) at the beginning of the increment, \(X_j, Y_j, Z_j = \) the coordinates of node (j) at the beginning of the increment, \(γ = \) the average twist angle at the beginning of the increment

\[= \frac{θ_x + \bar{θ}_x}{2}\]
\( \theta_{xo} \) = the initial twist angle of the element, \\
\( \overline{\theta}_x^i \) and \( \overline{\theta}_y^i \) = the twist of the element ends about its chord at the beginning of the increment. These twist angles can be calculated using the relations,

\[
\overline{\theta}_x^i = \left( \frac{X_j - X_i}{L} \right) \hat{\theta}_x^i + \left( \frac{Y_j - Y_i}{L} \right) \hat{\theta}_y^i + \left( \frac{Z_j - Z_i}{L} \right) \hat{\theta}_z^i
\]

\[
\overline{\theta}_y^i = \left( \frac{X_j - X_i}{L} \right) \hat{\theta}_x^i + \left( \frac{Y_j - Y_i}{L} \right) \hat{\theta}_y^i + \left( \frac{Z_j - Z_i}{L} \right) \hat{\theta}_z^i
\]

\( \hat{\theta}_x^i, \hat{\theta}_y^i, \) and \( \hat{\theta}_z^i \) = the rotations of node (i) about the global axes X, Y and Z at the beginning of the increment,

\( \hat{\theta}_x^j, \hat{\theta}_y^j, \) and \( \hat{\theta}_z^j \) = the rotations of node (j) about the global axes X, Y and Z at the beginning of the increment.

### 4.4 Constitutive Relations For The Subgrade Springs

As explained in Section (4.2), the subgrade is represented in this study by sets of springs at the reinforcement nodal points. Each set of springs consists of four springs. Two of these springs represent the lateral soil pressure on the reinforcement, the third spring represents the friction along the axial direction of the reinforcement, and the fourth spring represents the friction in the circumferential direction of the reinforcement. In the model developed in this study, the constitutive relations for these springs are represented using two procedures. Descriptions of these two procedures are given in the following two subsections.
4.4.1 Independent Constitutive Relations

In this procedure, the constitutive relations of the lateral, axial, and torsional springs are represented by three, independent, nonlinear, load-displacement curves (soil response curves) which are provided by the user. This procedure is most commonly used in the analysis of pile foundations (Reese et al. 1974, 1977, 1984; Nogmi et al. 1983, 1985; among others) and the soil response curves in the lateral, axial, and torsional directions are usually called the p-y, t-z, and M-θ curves, respectively.

Several studies have been carried out to establish the soil response curves for different soil types (Reese et al. 1970, 1974, 1975, 1984; Kraft et al. 1981; O’Neill 1964; among others). Most of these studies are based on limited number of field or laboratory tests and consequently, their range of application may be limited. Therefore, in order to follow the actual soil response behavior for a specific problem, it might be necessary to conduct extensive experimental studies which give the soil response curves taking into account the soil type, the reinforcement properties, and the loading condition. If such experimental data are not available, empirical relations such as the hyperbolic relationship (Kraft et al. 1981) or the Ramberg-Osgood relationship (Desai and Kuppusamy 1980) can be used to represent the soil response curves. For instance, Desai and Kuppusamy (1980) expressed the soil response in the lateral, axial and circumferential directions using the Ramberg-Osgood’s relationship, Figure (4.3),

\[
p_J = \frac{(C_{jj} - C_{HF})\delta_j}{1 + \left(\frac{(C_{jj} - C_{HF})\delta_j}{P_{J(UT)}}\right)^{\theta_j}} + C_{HF}\delta_j \quad ; J = x, y, z \text{ or } \theta
\]  

(4.60)

where:
\[ p_j = \text{the soil lateral pressure for the p-y curve}, \]
\[ = \text{the frictional stress in the axial direction for the t-z curve}, \]
\[ = \text{the frictional stress in the circumferential direction for the M-0 curve}, \]
\[ C_{ji} = \text{the initial slope of the p-y, t-z, or M-0 curves}, \]
\[ C_{jf} = \text{the final slope of the p-y, t-z, or M-0 curves}, \]
\[ \delta_j = \bar{w}_j - \bar{v}_j, J = x, y, z \text{ or } \theta \]
\[ = \text{the relative displacement between the free field soil displacement and the} \]
\[ \text{reinforcement displacement in the lateral, axial, or circumferential direction}, \]
\[ p_{j(\text{ult})} = \text{the ultimate resistance of the p-y, t-z, or M-0 curves}, \]
\[ \omega_j = \text{the order of the curve}. \]

Note that for \( \omega_j = 1 \) and \( C_{jf} = 0 \), equation (4.60) reduces to a hyperbola.

The use of an empirical relation such as the one given by relation (4.60) requires the
definition of the parameters \( C_{ji}, C_{jf}, p_{j(\text{ult})}, \) and \( \omega_j \). If sufficient information is not
available, the values of \( C_{jf} \) can be taken equal to zero and the values of \( \omega_j \) can be taken
equal to one. Suggested values for \( C_{ji} \) and \( p_{j(\text{ult})} \) for the lateral, axial and torsional
response curves are given in Table (4.1).

The procedure explained above has the advantage of being able to model the
increase in the soil strength with depth or layered soils. This can simply be done by
defining different sets of soil response curves at different depths. However, this
procedure assumes that the soil response in the lateral, axial and torsional directions are
independent. This assumption is not accurate because the developed lateral pressure on the
reinforcement causes an increase of the confining pressure on the soil and hence an increase
in the frictional resistance between the soil and the reinforcement.
Figure (4.3) Ramberg-Osgood Model (Desai And Kuppusamy 1980).

Table (4.1)

Suggested Values For The Initial Slope And The Ultimate Resistance Of The Soil Response Curves.

<table>
<thead>
<tr>
<th>Soil Response Curve</th>
<th>$C_h$</th>
<th>$P_{h(ult)}$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial Response Curve (T-z curve)</td>
<td>$C_h = C_{si} = \frac{G_s}{2D}$</td>
<td>$P_{h(ult)} = P_{x(ult)} = \sigma_j \tan \phi' + c_a$</td>
<td>Scott (1981), Kraft (1981)</td>
</tr>
<tr>
<td>Lateral Response Curve (p-y curve)</td>
<td>$C_h = C_{pi} = C_{pi} = \frac{0.65E_s}{(1 - V_s^2)D} \sqrt{\frac{E_sD^3}{E_i'}}$</td>
<td>$P_{h(ult)} = P_{y(ult)} = P_{h(ult)} = N_c c + N_q\sigma_i$</td>
<td>Vesic (1961), Hansen (1961), Broms (1964-a), Broms (1964-b), Jewell (1983), Among Others.</td>
</tr>
<tr>
<td>Torsional Response Curve (M- $\theta$ curve)</td>
<td>$C_h = C_{ti} = 2G_s$</td>
<td>$P_{h(ult)} = P_{t(ult)} = \sigma_i \tan \phi' + c_a$</td>
<td>Scott (1981)</td>
</tr>
</tbody>
</table>

$E_s$ = the Young's modulus of the soil,
$V_s$ = the Poisson's ratio of the soil,
$G_s$ = the shear modulus of the soil,
$N_c$ and $N_q$ = bearing capacity factors. Values for these factors can be found in the mentioned references,
$c$ = cohesion,
$c_a$ = adhesion between soil and reinforcement,
$\phi'$ = friction angle between soil and reinforcement,
$\sigma_i$ = the initial confining pressure.
4.4.2 Interacting Constitutive Relations

In this procedure, the above explained shortcoming is overcome as follows. The constitutive relations for the lateral springs are represented by the p-y curve as explained in the previous subsection. However, the constitutive relations for the axial and torsional springs are assumed to be elastic-perfectly plastic with given initial slopes ($C_{xi}$ and $C_{\theta i}$) and ultimate resistance that follow Coulomb's frictional law. In computing the ultimate resistance of the axial and torsional response curves, the change in the confining pressure due to the forces developed in the lateral springs is considered. The above procedure can be explained as follows.

Consider any reinforcement point, e.g., point (i) in Figure (4.1.b). Initially, the normal pressure around the reinforcement at point (i) is uniform and equals to the initial confining pressure ($\sigma_i$). However, after shearing the soil-reinforcement composite, the normal pressure increases in front of the reinforcement while it decreases in back of the reinforcement. Let us define the average normal pressure in front of the reinforcement as "$\sigma_u$" and the average normal pressure in back of the reinforcement as "$\sigma_d$". Further, assume that the change in the average normal pressure in front of the reinforcement is equal in magnitude to that in back of the reinforcement. Therefore, the values of $\sigma_u$ and $\sigma_d$ can be written as,

\begin{align*}
\sigma_u &= \sigma_i + \Delta \sigma \quad (4.61) \\
\sigma_d &= \sigma_i - \Delta \sigma \quad \geq 0.0 \quad (4.62)
\end{align*}

where:

$\Delta \sigma = \text{the change in the confining pressure on both sides of the reinforcement due to the shearing of the composite.}$
Note that, for $\Delta \sigma > \sigma_1$, the value of $\sigma_d$ is taken equal to zero. That is because, for $\Delta \sigma > \sigma_1$, separation between the soil and the reinforcement occurs. The average net normal pressure on the reinforcement at point (i) can be written as,

$$\sigma_n = \sigma_u - \sigma_d$$

(4.63)

substituting (4.61) and (4.62) in (4.63) with considering the constrain that $\sigma_d \geq 0.0$, Then,

$$\sigma_n = 2\Delta \sigma \quad \text{if} \quad \Delta \sigma < \sigma_1$$

$$\sigma_n = \sigma_u \quad \text{if} \quad \Delta \sigma \geq \sigma_1$$

(4.64)

The average net normal pressure ($\sigma_n$) is related to the resultant of the forces developed in the two lateral springs at point (i). This resultant force can be computed as follows. Considering relations (4.15) and (4.16), The forces in the two lateral springs at point (i) due to the pressure developed on element (i-j) can be computed as,

$$f_y^i = \frac{DL}{2} p_y^i$$

(4.65)

$$f_z^i = \frac{DL}{2} p_z^i$$

(4.66)

The resultant of the above lateral forces is,

$$f_n^i = \frac{DL}{2} \sqrt{(p_y^i)^2 + (p_z^i)^2}$$

(4.67)

Then, the average net lateral pressure ($\sigma_n$) can be computed as,

$$\sigma_n = \frac{f_n^i}{\pi D} \times \frac{L}{2}$$

$$= \frac{2}{\pi} \sqrt{(p_y^i)^2 + (p_z^i)^2}$$

(4.68)
Substitute (4.64) and (4.68) in (4.61) and (4.62), then,

\[
\sigma_u = \sigma_i + \frac{1}{\pi} \sqrt{(p_y^1)^2 + (p_z^1)^2} \quad \text{if} \quad \sigma_i > \frac{\sigma_n}{2}
\]

\[
\sigma_d = \sigma_i - \frac{1}{\pi} \sqrt{(p_y^1)^2 + (p_z^1)^2} \quad \text{if} \quad \sigma_i \leq \frac{\sigma_n}{2}
\]

and,

\[
\sigma_u = \frac{2}{\pi} \sqrt{(p_y^1)^2 + (p_z^1)^2} \quad \text{if} \quad \sigma_i > \frac{\sigma_n}{2}
\]

\[
\sigma_d = 0 \quad \text{if} \quad \sigma_i \leq \frac{\sigma_n}{2}
\]

The average confining pressure on all the reinforcement circumference can now be calculated as,

\[
\sigma_{av} = \frac{\sigma_u + \sigma_d}{2}
\]

Substitute (4.69) and (4.70) in (4.71), then,

\[
\sigma_{av} = \sigma_i \quad \text{if} \quad \sigma_i > \frac{\sigma_n}{2}
\]

\[
\sigma_{av} = \frac{1}{\pi} \sqrt{(p_y^1)^2 + (p_z^1)^2} \quad \text{if} \quad \sigma_i \leq \frac{\sigma_n}{2}
\]

Now, recall the discussion of the distribution of the normal pressure on the reinforcement given in Section (3.6). In this discussion, it was stated that, at large distance from the shearing plane, the normal pressure on both sides of the reinforcement is approximately uniform and equal to the initial confining pressure \((\sigma_i)\). However, close to the shearing plane, the normal pressure on the reinforcement is non uniform and it acts only on one side of the reinforcement. Note that the pressure distribution given by relations (4.72) and (4.73) matches the above described distribution. Also, by constraining the value of \(\sigma_d\) to be \(\geq 0.0\), the effect of the one-sided pressure distribution close to the shearing plane was considered. However, the approximation in relations
(4.72) and (4.73) lays in averaging the normal pressure on the total circumferential area of
the reinforcement.

Similar to the above representation, the axial and tangential forces on an arc \((da)\) on
the circumference of the reinforcement at point \((i)\) can be expressed as,

\[
\begin{align*}
    f_x^i &= (da) \frac{L}{2} p_x^i \\
    f_t^i &= (da) \frac{L}{2} p_t^i
\end{align*}
\]  

(4.74)  
(4.75)

The resultant of the above two frictional forces is,

\[
    f_r^i = (da) \frac{L}{2} \sqrt{(p_x^i)^2 + (p_t^i)^2}
\]  

(4.76)

Then, the average shear stress on the reinforcement circumference at point \((i)\) can
approximately be expressed as,

\[
    \tau_{av} = \frac{f_r^i}{(da) \frac{L}{2}} = \sqrt{(p_x^i)^2 + (p_t^i)^2}
\]  

(4.77)

After each iteration in the analysis, the values of \(\sigma_{av}\), relations (4.72) and (4.73),
and \(\tau_{av}\), relation (4.77), are computed. Then, \(\tau_{av}\) is compared with the ultimate
frictional resistance given by the equation,

\[
    \tau_r = c_a + \sigma_{av} \tan \phi_a
\]  

(4.78)

where:

\(\tau_r\) = the ultimate frictional resistance at the soil-reinforcement interface,

\(c_a\) = the adhesion between the soil and the reinforcement material,
\( \phi_a = \) the frictional angle between the soil and the reinforcement materials.

As long as the value of \( \tau_{av} \) does not exceed the value of \( \tau_r \), the axial and torsional springs continue to have constant stiffnesses \( C_{xi} \) and \( C_{\theta i} \), respectively. When the frictional stress \( (\tau_{av}) \) exceed the frictional limit \( (\tau_r) \) the ultimate frictional resistance in the axial and circumferential directions are calculated as,

\[
\begin{align*}
P_x(\text{ult}) &= \frac{P_x}{\tau_{av}} \times \tau_r \\
P_{\theta}(\text{ult}) &= \frac{P_{\theta}}{\tau_{av}} \times \tau_r
\end{align*}
\]

(4.79)

where:

\( P_x(\text{ult}) = \) the ultimate frictional resistance in the axial direction,

\( P_{\theta(\text{ult})} = \) the ultimate frictional resistance in the circumferential direction.

Then, the stiffnesses of the axial and torsional springs are updated such that the developed frictional stresses equal to the frictional limits given by equation (4.79), and another iteration is performed.

4.4.3 Representation Of The Constitutive Relations For The Subgrade Springs In The Computer Program

In the computer program written in this study, the option is given to the user to choose either of the above two procedures for his analysis. If the first procedure is chosen, the entries for the program are sets of soil response curves at different depths. If the second procedure is chosen, the entries for the program are sets of lateral response curves, the initial slopes of the axial and torsional response curves, and the parameters of Coulomb's frictional law at different depths. In both procedures, the user has the option to divided the soil up to twenty layers with different soil properties.
4.5. Solution Technique For The Governing Equation

Recall the governing equation,

\[ \{S\} = \{R\} \]  \hspace{1cm} (4.23.a)

or,

\[ [K_r(\{V\})]\{V\} = [K_r(\{V\})][\{W\} - \{V\}] \]  \hspace{1cm} (4.23.c)

Since the above system of equations is nonlinear, relation (4.23) will not be satisfied at any stage of the computation. In other words, there will always be a load imbalance which can be written as,

\[ \{U\} = \{R\} - \{S\} \neq \{0\} \]  \hspace{1cm} (4.80)

However, relation (4.23) can be satisfied within a given tolerance by solving it using an appropriate technique. In this study, equation (4.23) is solved by the Newton-Raphson method (Householder 1953). This method is chosen because it gives accurate results and it converges in few iterations (Bathe 1980). In the Newton-Raphson method, the load \{R\} is applied incrementally and within each increment, iterations are performed till prescribed convergence criteria are satisfied. This incremental-iterative approach can be detailed as follows.

Assume that the free-field soil displacement is applied in increments \{ΔW\}_1, \{ΔW\}_2, \ldots,\{ΔW\}_n. Assume also that the solution for the reinforcement displacement vector \{V\} and the reinforcement force vector \{S\} up to the end of increment (i-1) are known and it is required to obtain the solution due to an applied increment in the free-field soil displacement \{ΔW\}_i. Iterations within increment (i) are performed to obtain the value of \{V\}_i which satisfy the convergence criteria. Assume that (j-1) iterations have been performed within increment (i) and estimate for the displacement vector \{V\}_i was
obtained as \( \{V\}_i^{j-1} \). If the convergence criteria are not satisfied at the end of iteration \((j-1)\), another iteration \(j\) is performed. The linearized form of the governing equation which is used during iteration \(j\) can be obtained by considering the first order Taylor’s series expansion of equation (4.80) about \( \{V\}_i^{j-1} \) (Householder 1953). This linearized form of the governing equation can be written as,

\[
\left[ [K_r]_i^{j-1} + [K_s]_i^{j-1} \right] \{\Delta V\}_i^j = \left[ \tilde{K}_s \right]_i^{j-1} \{W\}_i^{j-1} - \{V\}_i^{j-1} \} - \{S\}_i^{j-1} + [K_s]_i^{j-1} \{\Delta W\}_i^j \tag{4.81}
\]

where:

\( \{V\}_i^{j-1} \) = the reinforcement displacement vector at the end of iteration \((j-1)\),

\( \{\Delta V\}_i^j \) = the increment in the reinforcement displacement vector computed in iteration \(j\),

\( \{W\}_i^{j-1} \) = the free-field soil displacement vector at the end of iteration \((j-1)\),

\( \{\Delta W\}_i^j \) = the increment in the free-field soil displacement vector applied in iteration \(j\),

\( = \{\Delta W\}_i^j; \text{ if } j = 1 \)

\( = \{0\}; \text{ if } j > 1 \)

\( \{S\}_i^{j-1} \) = the reinforcement force vector at the end of iteration \((j-1)\),

\( [K_r]_i^{j-1} \) = the tangent stiffness matrix for the reinforcement computed based on the latest known displacement vector \( \{V\}_i^{j-1} \),

\( [K_s]_i^{j-1} \) = the tangent stiffness matrix for the soil computed based on the latest known relative displacement vector \( \{W\}_i^{j-1} - \{V\}_i^{j-1} \),

\( [\tilde{K}_s]_i^{j-1} \) = the secant stiffness matrix for the soil computed based on the latest known relative displacement vector \( \{W\}_i^{j-1} - \{V\}_i^{j-1} \),

the subscript \((i)\) = increment number,

the superscripts \((j-1)\) and \((j)\) = iteration number.

The solution of equation (4.81) gives the incremental displacement vector \( \{\Delta V\}_i^j \). Then, the total displacement vector \( \{V\}_i^j \), the internal force vector \( \{S\}_i^j \) and the external force vector \( \{R\}_i^j \) at the end of iteration \(j\) can be obtained the equations,
\[
\{V\}_i^j = \{V\}_i^{j-1} + \{\Delta V\}_i^j \tag{4.82}
\]
\[
\{S\}_i^j = \{S\}_i^{j-1} + \{\Delta S\}_i^j \tag{4.83}
\]
\[
\{R\}_i^j = [K_s]^j_i \{(W)\}_i - \{V\}_i^j \} \tag{4.84}
\]
\[
\{W\}_i^j = \{W\}_i^{j-1} + \{\Delta W\}_i^j \tag{4.85}
\]

The convergence criteria are checked again using the values of the vectors given above. If the convergence criteria are not satisfied, another iteration (j+1) is performed. Otherwise, the iteration process is terminated and another increment (i+1) is considered.

The above solution technique can be illustrated graphically as follows. Figure (4.4.a) shows a schematic diagram which represents the subgrade response. Figure (4.4.b) shows a schematic diagram represents the behavior of the reinforcement. Assume that the reinforcement displacement vector and the reinforcement force vector at the end of increment (i-1) are \(\{V\}_{i-1}^i\) and \(\{S\}_{i-1}^i\), respectively. Then, the steps to obtain the solution for increment (i) are:

1- apply an increment in the free-field soil displacement \(\{\Delta W\}_i^i\).
2- consider \(\{W\}_i^0 = \{W\}_{i-1}^i\), \(\{V\}_i^0 = \{V\}_{i-1}^i\), and \(\{S\}_i^0 = \{S\}_{i-1}^i\).
3- using the value of \(\{(W)\}_i^0 - \{V\}_i^0\), compute the tangent stiffness matrix \([K_s]^0_i\) and the secant stiffness matrix \([K_s]^0_i\), Figure (4.4.a).
4- using the value of \(\{V\}_i^0\), compute the tangent stiffness matrix \([K_s]^0_i\), Figure (4.4.b)
5- substitute in relation (4.81) and solve for \(\{\Delta V\}_i^1\).
6- determine the internal forces in the reinforcement elements and then determine the force vector \(\{\Delta S\}_i^1\), Figure (4.4.b).
7- determine the reinforcement displacement vector \( \{V\}_i \), the internal forces vector \( \{S\}_i \), and the free-field soil displacement vector \( \{W\}_i \) using equations (4.82), (4.83) and (4.85).

8- compute the updated value of the relative displacement between the free-field soil displacement and the reinforcement displacement as \( \{W\}_i - \{V\}_i \).

9- using the value of \( \{W\}_i - \{V\}_i \), compute the secant stiffness matrix \( [\bar{K}_s]_i \), Figure (4.4.a).

10- compute the external force vector \( \{R\}_i \) using equation (4.84), Figure (4.4.a).

11- compute the unbalance force vector \( \{U\} \) using relation (4.80), Figure (4.4.b).

12- check for the convergence criteria. If the convergence criteria are satisfied, go to step (16).

13- using the value \( \{W\}_i - \{V\}_i \), compute the tangent stiffness matrix \( [K_s]_i \), Figure (4.4.a).

14- using the latest known reinforcement displacement \( \{V\}_i \), compute the tangent stiffness matrix \( [K_r]_i \), Figure (4.4.b).

15- repeat steps (5) through (14) until the convergence criteria are satisfied.

16- consider a new increment, (i+1).

The specified convergence criteria used herein is based on the convergence of both the force and the displacement vectors. Hence, convergence is assumed to occur when the following two inequalities are satisfied,

\[
\left[ \sum L (AV_L)^2 \right]^{\frac{1}{2}} \leq e_1
\]

\[
\left[ \sum L (U_L)^2 \right]^{\frac{1}{2}} \leq e_2
\] (4.86)
Figure (4.4.a) Schematic Diagram Represents The Subgrade Response.

Figure (4.4.b) Schematic Diagram Represents The Behavior Of The Reinforcement.

Figure (4.4) Solution Technique For The Governing Equation.
where \( e_1 \) and \( e_2 \) are prescribed tolerances for the displacement and force convergence criteria respectively.

4.6. Description Of The Computer Program

A computer program was written based on the model described in this chapter. This program was written in a general format such that it can be used in the analysis of different soil-structure interaction problems (such as reinforced-soil slopes, pile foundations, pipelines, etc.) or frame structures. This flexibility of the program is made by adding the following options.

1- The applied loads in the inclusion can be due to the soil response (as in the case of reinforced-soil slopes), or due to concentrated forces and moments at different points along the inclusion (as in the case of frame structures), or due to the above two loading conditions (as in the case of piles).

2- For the analysis of reinforced-soil slopes, the user has the option to choose between the shape of infinite slip surface (Figure (4.5.a)), planer slip surface (Figure (4.5.b)), or circular slip surface (Figure (4.5.c)).

3- For the analysis of reinforced-soil slopes or piles, the user can model the increase in soil strength with depth or layered soils. Up to twenty different soil layers with different properties can be included in the analysis.

4- Any of the two procedures described in section (4.4) can be used to represent the constitutive relations of the subgrade springs.

5- The analysis can be performed using only material nonlinearity, only geometric nonlinearity, or both nonlinearities.

6- The program is aided by an automatic incremental procedure. In this procedure, the increment size is automatically reduced if slow or no convergence occurs. Also,
Figure (4.5.a) Infinite Slip Surface

Figure (4.5.b) Planer Slip Surface

Figure (4.5.c) Circular Slip Surface

Figure (4.5) Different Slip Surfaces Which Can Be Considered By The Computer Program.
the increment size is automatically increased if convergence in each of the previous
two load increments occurred in number of iterations less than a prescribed value.

4.7. Verification Of The Computer Program

The aforementioned computer program is used to solve two problems which have
well-known theoretical solutions. Comparisons between the results of the program and the
theoretical solutions are made to provide verification of the program. The results of the
above two problems are given in the following subsections.

4.7.1. Large Deflection Analysis Of A Cantilever Beam

Figure (4.6.a) shows the geometry and the material properties of the problem. The
exact solution for this problem was obtained by Bisshopp and Drucker (1945), where both
the exact curvature expression and the shortening in the moment arm were considered. The
same problem is solved using the computer program developed in this chapter. Figures
(4.6.b) and (4.6.c) show the results obtained from the computer program as well as the
exact solution given by Bisshopp and Drucker. As can be seen, the results of the program
developed in this chapter are in excellent agreement with the exact solution.

4.7.2. Elastic-Plastic Analysis Of A Beam Fixed At Both Ends

Figure (4.7.a) shows the geometry and the material properties of the problem. Hodge
(1966) solved this problem using the plastic design method. The same problem is
solved using the computer program developed in this chapter. Figure (4.7.b) shows the
results obtained from the computer program as well as the results given by Hodge (1966).
Excellent agreement between the results of the program and the results of the theoretical
solution can be observed.
Figure (4.6.a) Dimensions And Material Properties Of The Problem

- $E_x = 10^5$ kPa, $A_x = 1.0$ cm$^2$
- $I_x = 1.0$ cm$^4$, $L_t = 100.0$ cm,
- $P = 10$ N

$\hat{v}_Y = $ Vertical Displacement
At Point $a$

$\hat{v}_X = $ Horizontal Displacement
At Point $a$

Figure (4.6.b) Non-dimensional Load Point Vertical Displacement Versus Applied Load.

Figure (4.6.c) Non-dimensional Load Point Horizontal Displacement Versus The Applied Load.

Figure (4.6) Large Deflection Analysis Of A Cantilever Beam.
$E, I_z = 1.75 \times 10^6 \text{ t.cm}^2$

$E, A = 1.35 \times 10^6 \text{ t}$

$L_t = 100 \text{ cm}$

$M_{zu} = 7.5 \times 10^3 \text{ t.cm}$

$F_{zu} = 3.5 \times 10^3 \text{ t}$

$\delta_{ya} = \text{Vertical Displacement At Point a}$

Figure (4.7.a) Dimensions And Material Properties Of The Problem

<table>
<thead>
<tr>
<th>Case #</th>
<th>Condition</th>
<th>Theoretical Solution</th>
<th>FEM Solution</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
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<td>$\frac{\delta_{ya}}{L_t}$</td>
</tr>
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<td></td>
<td>3.000</td>
<td>.28551</td>
</tr>
</tbody>
</table>

Figure (4.7.b) Vertical Displacement Of The Loaded Point Versus The Applied Load.

Figure (4.7) Elastic-Plastic Analysis Of A Beam Fixed At Both Ends.
4.8. Applications

The simplified finite element model is used to analyze different reinforced-soil systems. These systems are: a soil reinforced by straight bars, a soil reinforced by piles, and a soil reinforced by tree roots. The results of the above analyses are given in the following three subsections.

4.8.1. Analyses Of A Soil Reinforced By Straight Bars

4.8.1.1. Comparison With Chapter (3) Models

In Section (3.5), different soil-reinforcement composites were analyzed by the finite element model constructed using ABAQUS. In section (3.6), the same composites were analyzed by the beam model described in chapter (3). In this subsection, comparison between the results of the above two models and the model developed in this chapter is carried out. In the subsequent representation, ABAQUS’ model is called “model (1)”, the beam model is called “model (2)” and the simplified finite element model developed in this chapter is called “model (3)”.

Table (3.2) gives a summary of the different soil-reinforcement composites which were analyzed using models (1) and (2). The material properties used in the above analyses are given in Tables (3.3) and (3.4). Composites (13) through (36) in Table (3.2) are analyzed again using model (3). In these analyses, the material properties for the reinforcements are taken as given in Table (3.4). The procedure described in Section (4.4.2) is used to represent the soil response curves. The lateral response curves are assumed to have hyperbolic shapes that follow Equation (4.60) with $C_{sf} = 0.0$ and $\omega_f=1.0$. The initial slopes of the lateral response curves are estimated using the equation given in Table (4.1). The ultimate resistance of the lateral response curves are estimated
using Jewell et al's (1984) lower bound curve shown in Figure (2.4). The axial and torsional response curves are assumed to be elastic-perfectly plastic with initial slopes that follow the equations given in Table (4.1). As explained in section (4.4.2), the ultimate resistance of the axial and torsional response curves are computed using Coulomb's frictional law. The coefficients of friction between the soil and the reinforcement materials are taken as those used in the analyses performed in Chapter (3).

Figure (4.8) shows the load-displacement curves obtained for composites (14) and (26) using models (1) and (3). From this figure, the following can be observed. The shear forces in the reinforcement obtained from both models are in good agreement. The ultimate axial forces in the reinforcement obtained from both models are approximately equal. However, the ultimate axial forces reached at smaller displacement in model (3) than in model (1). This is because of the approximations used to compute the confining pressure and the frictional stress on the reinforcement in model (3). As discussed in section (4.4.2), these approximations involve averaging the confining pressure and the frictional stress around the circumference of the reinforcement.

Figures (4.9) and (4.10) show the ultimate forces in the reinforcement obtained from models (1), (2) and (3). In Figure (4.9), the vertical axis is the ultimate axial force written in dimensionless form and the horizontal axis is a dimensionless factor \( \Phi_T \) given by relations (3.57) and (3.58). For convenience, these relations are rewritten here as,

\[
\Phi_T = 0.217 \left( \frac{D}{L_s} \right) \left( \frac{E_s}{E_r} \right)^{0.271} \left( \frac{\sigma_2}{\sigma_1} \right)
\]  

(3.57)
for the cases where the bearing resistance of the soil controls:

\[ \Phi_T = 0.169 \left( \frac{D}{L_n} \right) \sqrt{\frac{\sigma_y \sigma_b}{\sigma_T^2}} \]  

(3.58)

In Figure (4.10), the vertical axis is the ultimate shear force in the reinforcement written in dimensionless form and the horizontal axis is a dimensionless factor \( \Phi_Q \) given by relations (3.59) and (3.60). For convenience, these relations are rewritten here as,

for the cases where the bending resistance of the reinforcement controls:

\[ \Phi_Q = 0.75 \left( \frac{E_b}{E_r} \right)^{0.271} \left( 1 - \left( \frac{T_{el}}{T_p} \right)^2 \right)^{0.5} \]  

(3.59)

for the cases where the bearing resistance of the soil controls:

\[ \Phi_Q = \sqrt{\frac{\sigma_k}{3\sigma_y}} \times \sqrt{1 - \left( \frac{T_{el}}{T_p} \right)^2} \]  

(3.60)

In both Figures, the points represent the results of model (1), the solid line represents model (2) and the dashed line represents the average of the results of model (3). These plots show that the results of the three models are generally in agreement. It can also be observed that the agreement between model (1) and (3), especially for the shear forces, is better than that between model (1) and (2).

As stated before, The analysis by model (3) is much cheaper than that by model (1). For instance, the analysis by model (3) for composite (14) needs a CPU time of about 30 minutes on an IBM-3090 machine. However, the analysis by model (1) needs a CPU time of about 110 minutes on a Cray-YMP8 machine. Note that the Cray-YMP8 machine is much faster than the IBM-3090 machine because the Cray machine has vectorization capability. Note also that, the computer program (ABAQUS) of model (1) uses parallel
processors while the program of model (3) does have this capability. In spite of the computing capabilities used in the analysis performed by model (1), the CPU time of this analysis is more than three times that of model (3). This indicates the significant saving in the computational expanses which is obtained by model (3).

4.8.1.2. Effect Of The Reinforcement's Orientation

The analyses described above are for reinforcements perpendicular on the slip surface. To complete our study of the problem, model (3) is used to analyze reinforcements with different orientations with respect to the slip surface. Figure (4.11) shows the results of the analyses of composite (14) with different initial orientation angle ($\theta_0$) for the reinforcement. From this figure, the following can be observed. The shear forces developed in reinforcements with different orientations are approximately equal. Tensile axial forces are developed in the reinforcement if $\theta_0 \geq 0$. However, compressive axial forces are developed in the reinforcement if $\theta_0 < 0$. The magnitude of the ultimate axial forces obtained for reinforcements with different orientations are approximately equal and they are equal to the frictional resistance along the reinforcement length.

The contribution of the reinforcement to the composite strength is computed by the relation (Wu 1976),

$$\Delta s_r = (T \cos \theta - Q \sin \theta) \tan \phi + (T \sin \theta + Q \cos \theta)$$

(4.87)

where:

$\Delta s_r =$ the increase in the composite shearing resistance due to the reinforcement,

$T =$ the axial force in the reinforcement at the point of intersection with the shear plane,

$Q =$ the shear force in the reinforcement at the point of intersection with the shear plane,

$\phi =$ the angle of shearing resistance of the soil,

$\theta = \theta_0 + \Delta \theta$
\[ \theta_0 = \text{the initial orientation of the reinforcement}, \]

\[ \Delta \theta = \text{the rotation of the reinforcement at the point of intersection with the shear plane}. \]

The most beneficial orientation angle for the reinforcement (\( \theta_m \)) can be obtained by differentiating equation (4.87) with respect to the angle (\( \theta \)) and equating it to zero. Hence, the angle \( \theta_m \) can be written as,

\[ \theta_m = \tan^{-1}\left( \frac{T - Q \tan \phi}{T \tan \phi + Q} \right) \] (4.88)

In each of the analyses performed in this and the previous sections, the values of (\( \Delta s_r \)) at different values of applied soil displacement are computed. Then, the orientation angle (\( \theta_m \)) which gives the maximum value of (\( \Delta s_r \)) is obtained. The obtained values of (\( \theta_m \)) confirm relation (4.88).

The above result indicates that there is an optimum orientation angle for the reinforcement that is most beneficial for the reinforcement system. However, if the reinforcement is oriented at an angle \( \theta_o < 0 \), compressive axial force will develop in the reinforcement and this can reduce the efficiency of the reinforcement system. Also, if the reinforcement is oriented at an angle larger than that given by relation (4.88), the component of the shear force normal to the slip surface causes a significant reduction of the normal stress on the shear plane and this can reduce the efficiency of the reinforcement system.
4.8.1.3. Contribution Of The Reinforcement’s Axial And Bending Stiffnesses To The Composite Strength

There have been much debate and confusion about the relative contribution of the reinforcement’s axial and bending stiffnesses to the composite strength. Jewell (1992) recommended to neglect the reinforcement’s bending stiffness. On the other hand, Juran et al. (1992) stated that it is essential to consider the reinforcement’s bending stiffness. Considering relation (4.87), the contribution of the reinforcement’s axial and bending stiffnesses to the composite strength can respectively be written as,

\[ \Delta s_T = T \sin \theta + T \cos \theta \tan \phi \]  
\[ \Delta s_Q = Q \cos \theta - Q \sin \theta \tan \phi \]

where:
\[ \Delta s_T = \text{the contribution of the reinforcement’s axial stiffness to the composite strength}, \]
\[ \Delta s_Q = \text{the contribution of the reinforcement’s bending stiffness to the composite strength}. \]

ultimately, the values of \( T \) and \( Q \) are equal to the ultimate axial and shear forces (\( T_{\text{max}} \) and \( Q_{\text{max}} \)) which can be carried by the reinforcement. Considering the results shown in Figures (4.12) and (4.13), \( T_{\text{max}} \) and \( Q_{\text{max}} \) can be written as,

\[ T_{\text{max}} = \mu \pi DL_c \sigma_1 (1 + \Phi_T) \]  
\[ Q_{\text{max}} = \sigma_y D^2 (\Phi_Q) \]

Substitute (4.91) and (4.92) in (4.89) and (4.90), then, the relative contribution of the reinforcement’s axial and bending stiffnesses to the composite strength can be written as,
The above relation indicates that the relative contribution of the reinforcement’s axial and bending stiffnesses to the composite strength depends on the following parameters:

1- the relative stiffness between the soil and the reinforcement,
2- the shear strength parameters of the soil,
3- the bearing resistance of the soil,
4- the yield strength of the reinforcement,
5- the diameter and the length of the reinforcement,
6- the reinforcement orientation,
7- the frictional properties between the soil and the reinforcement,
8- the confining pressure on the soil-reinforcement composite.

Therefore, for an accurate analysis of a reinforced soil, both the reinforcement’s axial and bending stiffnesses should be considered with taking account of the above parameters. However, an analysis with neglecting the reinforcement’s bending stiffness can either underestimate or overestimate the contribution of the reinforcement to the composite strength.
Figure (4.8) Comparison Between The Results Of Models (1) And (3).
Figure (4.9) Comparison Between The Ultimate Axial Forces Obtained From Models (1), (2) And (3).

Figure (4.10) Comparison Between The Ultimate Shear Forces Obtained From Models (1), (2) And (3).
Soil Movement

Reinforcements At Different Orientation Angles ($\theta_o$)

Slip Surface

**Figure (4.11) Effect Of The Orientation Angle ($\theta$) On The Developed Forces In The Reinforcement**
4.8.2 Analysis of A Soil Reinforced By Piles

Heyman (1965) described tests on two instrumented piles founded in a soil undergoing lateral movement. The lateral movement of the soil occurred due to the construction of a road embankment. One of the two tested piles (pile I) was located 12 meter from the embankment toe and the other pile (pile II) was located at the embankment toe. Figure (4.12.a) shows a sketch for the embankment and the tested piles. Figure (4.12.b) shows the profile of a boring in the test site.

The tested piles were steel box piles, 12.5 m long, with a square cross section of 30 by 30 cm and 6 mm wall thickness. To restrain the movement of the pile heads, the heads of the piles were supported by heavy concrete beams which were founded on concrete piles, Figure (4.12.a). The embankment was constructed in three stages to a maximum height of 4 m. Measurements of the moment in the piles and the reaction at the pile heads were taken for each of the construction stages. The lateral soil movements were measured by inclinometers inserted into tubes placed near the test piles. Heyman (1965) did not describe the measured lateral soil movement in detail but he stated that the soil movements were uniform in all layers to a depth of 10 m below the ground surface. Heyman stated also that, at full embankment height, the measured horizontal soil displacements vary from 1.5 cm at 12 m from the embankment toe to 3 cm at the embankment toe with an accuracy of ±0.5 cm.

The above tests are analyzed using the computer program developed in this chapter. The Young’s modulus and the yield strength for the pile material are assumed to be $2.0 \times 10^8$ kPa and $2.4 \times 10^5$ kPa, respectively. The pile heads are pinned and restrained from moving. The soil response curves are represented by the procedure described in section (4.4.2). The lateral response curves are assumed to have hyperbolic shapes that
follow Equation (4.60), with \( C_y = 0.0 \) and \( \omega_y = 1.0 \). The initial slopes of the lateral response curves are computed using the equation given in Table (4.1). The ultimate resistance for the lateral response curves are assumed to be nine times the cohesion for cohesive soils and three times the passive earth pressure for cohesionless soils (Broms 1964-a and 1964-b). The axial and torsional response curves are assumed to be elastic-perfectly plastic with initial slopes that follow the equations given in Table (4.1). The ultimate resistance of the axial and torsional response curves are computed using Coulomb's frictional law. The adhesion and the coefficient of friction between the soil and the reinforcement materials are assumed to be 0.67c and 0.67 \( \tan(\phi) \), respectively.

In the analyses of both piles, the lateral movement of the top 10 m of the soil is applied as the free field soil displacement. Considering the accuracy within which the lateral soil movements were measured, it is assumed that the soil movement for pile I at the full embankment height was in the range between 1.0 cm and 2.0 cm. Similarly, the lateral soil movement for pile II at the full embankment height is assumed to be in the range between 2.5 cm and 3.5 cm. It is further assumed that the lateral soil movements varied linearly with the embankment height.

Figures (4.12.c) and (4.12.d) show the measured head reactions for the two piles as well as the computed results by employing the assumed ranges. Figures (4.12.e) and (4.12.f) show the measured moment distributions in both piles at the full embankment height as well as the computed results by employing the assumed ranges. From these figures, it can be observed that the measured head reactions and bending moments are within the computed ranges. This indicates that the analytical model can predict the moments and forces in the piles successfully.
Section B-B

Figure (4-12-a) The Road Embankment And The Tested Piles (Heyman 1965)

Figure (4-12-b) Boring Profile In The Test Site (Heyman 1965).
Figure (4.12.c) Reaction At The Top Of Pile I.

Figure (4.12.d) Reaction At The Top Of Pile II.

Figure (4.12.e) Distribution Of The Bending Moment In Pile I.

Figure (4.12.f) Distribution Of The Bending Moment In Pile II.

Figure (4.12) Analyses Of Piles Used To Stabilize A Soil Undergoing Lateral Movement.
4.8.3 Analysis Of A Soil Reinforced By Tree Roots

Wu and Watson (1991) performed in situ direct shear test on a c-φ soil, reinforced by roots of a Pinus Radiata tree. The soil in this site had a cohesion of 20.0 kPa and an angle of shearing resistance of 45°. Figure (4.13) shows a schematic drawing of the in situ test. In this test, the sheared soil block had a cross sectional area of 1.25m x 1.10m and the shearing plane was at a depth of 0.5m. The load on the sheared soil block was applied incrementally by three 10 ton hydraulic rams and the corresponding displacements were measured by a rotary potentiometer connected to the tree stem. Some of the roots were instrumented by strain gauges and the load-displacement curves for these roots were obtained. After the test, the soil was excavated and measurement of the roots’ dimensions and orientations were carried out.

The finite element model developed in this chapter is used to analyze one of the instrumented roots. A sketch for this root is shown in Figure (4.14). This root was a lateral root which extended into the soil outside the shear block as shown in Figure (4.14).

The root is assumed to be rigidly attached to the root mat under the stem of the tree. At this point of attachment, the root is not allowed to displace in the (Y) and (Z) directions. However, the displacement of that point in the (X) direction (the shearing direction) was specified to be equal to the free-field soil displacement (or the displacement of the soil block). The rotation at the above point is allowed in the three directions.

Waldron et al. (1983) tested the roots of Pinus Ponderosa trees and they found that the Young’s modulus ($E_r$) and yield strength ($\sigma_r$) for Pine’s roots can be expressed by the regression relations,

$$E_r \text{ (kPa)} = 1.41 \times 10^5 \left( D \text{ (mm)} \right)^{-0.389}$$

(4.94)
\[ \sigma_y \text{(kPa)} = 8.82 \times 10^3 (D \text{(mm)})^{-0.116} \] 

(4.95)

Watson (1991) performed tests on some roots of the Pinus Radiata tree. The values of \( E_r \) and \( \sigma_y \) given by equation (4.94) and (4.95) were found to be within the ranges obtained for the Pinus Radiata roots. Therefore, \( E_r \) and \( \sigma_y \) for the roots are represented in the finite element analysis using relations (4.94) and (4.95), respectively.

The soil response curves are represented using the procedure described in section (4.4.2). The lateral response curves are assumed to have hyperbolic shapes that follow Equation (4.60) with \( C_y f = 0.0 \) and \( \omega_y = 1.0 \). The initial slopes and ultimate resistance of the lateral response curves are assumed to follow the equations given in Table (4.1). The bearing capacity factors \( N_c \) and \( N_q \) used to compute the ultimate resistance of the lateral response curves are assumed to be 9.0 and 18.0 respectively (Broms 1964-a, 1964-b).

The axial and torsional response curves are assumed to be elastic-perfectly plastic with initial slopes that follow the equations given in Table (4.1). The ultimate resistance of the axial and torsional response curves are computed using Coulomb's frictional law. The adhesion and the coefficient of friction between the soil and the roots are assumed to be 0.6c and 0.6 tan(\( \phi \)), respectively.

Figure (4.15) shows the computed and measured load-displacement curves for the root. This figure shows that the ultimate axial force computed by the analytical model is in agreement with that measured in the experiment. However, the ultimate axial force is developed at smaller displacement in the analytical model than in the experiment. This can be attributed to uncertainties and approximations involved in this problem. These uncertainties and approximations are:
1- The measurement of the root's dimensions and geometry could not be perfectly accurate.

2- The root had tortuous geometry. Commandeur and Pyles (1991) showed that, for a tortuous root, the load-displacement curve has a sigmoid shape which consists of three portions. In the first portion, the curve shows a low modulus and this portion occurs during the straightening-out of the root. The second portion of the curve starts after the root straightens and this portion has higher modulus. The third portion of the curve occurs when the yield strength of the root is approached and during this portion the modulus reduces. The load-displacement curve obtained from the experiment, Figure (4.15), is similar to the curve described above. However, within the accuracy of measured root geometry, the tortuosity of the root could not accurately measured and hence it is not represented in the finite element analyses. This is the reason that the first portion of the above described curve does not appear in the computed load-displacement curve.

3- The analysis is performed using the measured root geometry. This geometry was the geometry of the root at the end of the test (the displaced geometry). However, the analysis should have been performed using the original geometry of the root (which is unknown). As explained in section (4.8.1.2), the above approximation can affect the results of the analyses.

Nevertheless, the analysis performed in this section show the ability of the simplified finite element model to analyze problems with complicated geometry. Such problems can not be analyzed using the models described in chapter (3). It is believed that better agreement between the analytical model and the actual behavior of the soil-root composite can be obtained if the above approximations can be eliminated.
Figure (4.13) In Situ Direct Shear Test On A c-\(\phi\) Soil Reinforced By Vegetation Roots (Wu And Watson 1991).
Figure (4.14) Plane View For The Geometry Of The Instrumented Root.

Figure (4.15) Load - Displacement Curve For The Instrumented Root.
4.9 Summary And Conclusions

A simplified finite element model for the soil-reinforcement interaction problem was developed in this Chapter. This model has the advantage of being computationally much less expensive than the finite element model described in Chapter (3). Also, this model is more capable than the beam model described in Chapter (3) because it can be used to analyze problems under working loads, see Section (4.8.2), and it can give estimates of the soil and the reinforcement deformations. In addition, the simplified finite element model has the advantage that it can be used to analyze problems with complicated geometry.

A computer program was written based on the model developed in this section. This program was written in general format such that it can be used in the analysis of different soil-structure interaction problems (e.g., reinforced-soil slopes, pile foundations, buried pipes, etc.). The above program was used to analyze some sample problems which have well-known theoretical solutions. Comparisons between the results of the computer program and the theoretical solutions were made to check the validity of the program. After confirming the validity of the program, it was used to analyze the different soil-reinforcement composites which were analyzed using the two models developed in Chapter (3). The results of the above three models were in agreement.

The simplified finite element model was used to analyze two practical problems: a soil reinforced by piles and a soil reinforced by tree roots. The computed results for the above reinforced-soil systems were in the same order of magnitude as those observed in field. However, differences between the results from the analytical model and from field observations could not be evaluated because some of the properties used in the model had to be estimated and data from field observations were incomplete.
CHAPTER V

Summary And Conclusions

The purpose of this research has been to develop analytical procedures which can accurately represent the mechanisms of the soil-reinforcement interaction. To this end, three different soil-reinforcement interaction models were developed. A brief description of these models and their results are as follows.

Model (1)

The finite element package ABAQUS was used to perform three dimensional, nonlinear analyses for the problem. These analyses showed that an accurate finite element representation of the problem requires the consideration of the nonlinear behavior of the soil, the dilation behavior of the soil, the nonlinear behavior of the reinforcement, the slip between the soil and the reinforcement, and the geometric nonlinearity of the problem. The finite element model containing the design parameters mentioned above was used to simulate laboratory experiments performed on different soil-reinforcement composites. The results of the analyses were in good agreement with the experimental results. This confirmed the validity of the finite element model developed in this study. After confirming
the validity of the finite element model, it was used to perform a parametric study for the problem. The parameters included in this study were the soil’s relative density, the reinforcement’s stiffness and yield strength, and the confining pressure on the soil-reinforcement composites. The results of this study showed the following.

1- The reinforcement’s contribution to the composite strength increases with increasing the soil’s relative density, the reinforcement’s stiffness, and the confining pressure on the composite.

2- The stress transfer between the soil and the reinforcement depends on the friction between the soil and the reinforcement and the passive soil pressure on the reinforcement. The passive soil pressure on the reinforcement controls the magnitude and the distribution of the normal stress at the soil-reinforcement interface which, in turn, controls the frictional resistance along the reinforcement length.

3- The magnitude and the distribution of the normal pressure at the soil-reinforcement interface depend on the relative stiffness between the soil and the reinforcement, the ultimate bearing resistance of the soil, the yield strength of the reinforcement, and the initial confining pressure on the composite. Therefore, the developed axial and shear forces in the reinforcement are function of the above parameters.

**Model (2)**

A beam model using a modification of the closed-form solution for beams on elastic foundations was developed. In this model, both the reinforcement’s axial and bending stiffnesses were considered. The limiting values for the reinforcement tensile and bending
resistance were taken into account. Also, the interaction between the passive soil pressure on the reinforcement and the frictional resistance along the reinforcement was considered.

The beam model computes the ultimate forces that can be carried by the reinforcement. The results of the beam model were compared with the results of model (1) and good agreement was detected. The results of this model were represented in the form of design charts which can be used to perform limit equilibrium analyses for reinforced soils.

Model (3)

A simplified finite element model for the problem was developed. As in model (1), the reinforcement was represented by beam elements with material and geometric nonlinearity. However, the difference between this model and model (1) is that the soil in this model was represented as nonlinear springs (Winkler type foundation) whereas in model (1) it was represented as a continuum. This simplification makes model (3) computationally much less expansive than model (1). This model is also more capable than the beam model because it can be used to analyze problems under working loads and can give estimates of the soil and the reinforcement deformations. In addition, the simplified finite element model has the advantage that it can be used to analyze problems with complicated geometry.

The simplified finite element model was used to analyze the different soil-reinforcement composites which were analyzed using models (1) and (2). The results of the above three models were in agreement. The simplified finite element model was also used to solve two practical problems: a soil reinforced by piles and a soil reinforced by vegetation roots. The results were in the same order of magnitude as those observed in field. Differences between the results from the analytical model and from field
observations could not be evaluated because some of the properties used in the model had to be estimated and data from field observations were incomplete.
List Of References


