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AN ASYMPTOTIC SERIES FOR NORMS OF POWERS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Michael Robert Snell, B.A.

*****

The Ohio State University
1995

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To My Family
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CHAPTER I

INTRODUCTION

Let $F$ denote the class of functions analytic on the closed unit disk $D$, such that $f(1) = 1$ and $|f(z)| < 1$ for $z \in D, z \neq 1$. In the future, whenever a function is denoted by $f$, it will be automatically assumed that $f \in F$. We distinguish two cases —

the "typical" case: the value that $\frac{d}{dz} |f(e^{it})|$ takes at $t = 0$ is strictly negative, and

the "exceptional" case: that value is zero. We use the notation $\|f\| = \sum_{\nu=0}^{\infty} |a_{\nu}|$

where $f(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^\nu$. The array $(a_{\nu})_{n \geq 1, \nu \geq 0}$ of complex numbers is defined by $f^n(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^\nu$. The behavior of $\|f^n\| = \sum_{\nu=0}^{\infty} |a_{\nu}|$ as $n \to \infty$ has been the subject of many studies (frequently under assumptions more general than $f \in F$, see [1],[4] and [5])

The results are particularly easy to state in the "typical" case.

Baishanski [1] has shown that in the "typical" case:

$$\|f^n\| = O(1), \text{ as } n \to \infty$$

(1.1)

The results (see 1.18 and 1.19 below) obtained by Girard in his dissertation imply that in the typical case

$$\|f^n\| \to (-\cos \arg A)^{-1/2}, \text{ as } n \to \infty$$

(1.2)

and in particular that under the additional hypothesis that $f''(1)$ is real $\|f^n\| \to 1$, as $n \to \infty$

We will show here that in the typical case:

(i) there exists an asymptotic series

$$\|f^n\| \sim \sum_{k=0}^{\infty} \frac{c_k}{n^k}, \text{ as } n \to \infty$$

(1.3)

(ii) the coefficients $c_k$ of this asymptotic expansion can be expressed in terms of the coefficients of the power series expansion of $\log f(e^{it})$ around $t = 0$; a procedure will be described for obtaining such expressions.
(iii) Expressions for the first three coefficients $c_0, c_1$ and $c_2$ will be given.

(iv) It came to us as a surprising observation that if the coefficients $a_n$ in the
Taylor expansion of $f$ are all real, then all the coefficients $c_j$ of the asymptotic
expansion vanish. In other words: if the Taylor coefficients of $f$ are all real, then

$$\|f^n\| = 1 + O(n^{-s}), \quad n \to \infty$$

for every real $s$.

To present the results in the "exceptional" case, we need first to define the five
parameters

$$\alpha, p, q, A, \beta$$

that Baishanski has associated to the maximum point $z = 1$.

$$\alpha \equiv f'(1)$$

$$|f(e^{it})| = 1 - \beta t^q + o(t^q), \quad t \to 0 \quad \text{and} \quad \beta \neq 0 \quad (1.5)$$

$$f(z) - z^\alpha = A(z - 1)^p + o(z - 1)^p, \quad z \to 1 \quad \text{and} \quad A \neq 0 \quad (1.6)$$

Obviously, (1.5) defines both $\beta$ and $q$, and (1.6) defines $A$ and $p$. Instead of (1.6)
we can use, equivalently

$$z^{-\alpha}f(z) = 1 + A(z - 1)^p + o(1)(z - 1)^p, \quad z \to 1 \quad \text{and} \quad A \neq 0. \quad (1.7)$$

or

$$\log f(e^{it}) = i\alpha t + (-1)^p At^p + o(t^p), \quad t \to 0 \quad \text{and} \quad A \neq 0. \quad (1.8)$$

It is easy to see that:

**$\alpha$ is real and positive**

(If this were not the case then it would be possible to find $z$ in the open unit disk,
$D$, with $z$ arbitrarily close to 1, so that $\alpha(z - 1)$ has positive real part; thus,

$$f(z) = 1 + \alpha(z - 1) + o(z - 1), \quad z \to 1 \quad (1.9)$$

would have real part larger than 1 at such points, contradicting the assumption
$|f(z)| < 1$ for all $z \in D$)

Obviously,

**$\beta$ is real and positive**

$q$ is an even positive integer

$p$ is a positive integer and $p \geq 2$
Moreover, by (1.5)
\[ \Re \log f(e^{it}) = \log |f(e^{it})| = -\beta t^p + o(t^p), \quad t \to 0 \] (1.10)
and by (1.8) and the fact that \( \alpha \) is real,
\[ \Re \log f(e^{it}) = (-1)^p(\Re A)t^p + o(t^p), \quad t \to 0 \] (1.11)
so that we obtain
\[ -\beta t^p + o(t^p) = (-1)^p(\Re A)t^p + o(t^p), \quad t \to 0 \] (1.12)
which implies
\[ \Re A \neq 0 \Leftrightarrow p = q \Leftrightarrow \Re A = -\beta < 0 \] (1.13)
and
\[ \Re A = 0 \Leftrightarrow p < q \] (1.14)
so that we have
\[ 2 \leq p \leq q \] (1.15)
Since, obviously, "typical" means \( q = 2 \) we see that in the "typical" case \( p = q \). In the "exceptional" case this equality may not hold. (For the actual examples of functions \( f \) with \( p = q > 2 \) and \( p < q \) see Baishanski [1] and Laying Tam [12,p.9])

The following table presents the results on the behavior of \( \|f^n\| \) (for both the "typical" case \( q = 2 \) and the "exceptional" case \( q \geq 4 \))

If \( p = q \) then \( \|f^n\| = O(1), \quad n \to \infty \) (Baishanski [1]) (1.16)
If \( p \neq q \) then \( \|f^n\| \to \infty, \quad \text{as} \ n \to \infty \) (Vermes-Clunie [4]) (1.17)

(more precisely \( \|f^n\| \geq Cn^{\frac{1}{2}(1-\frac{p}{q})} \) for some \( C > 0 \))

If \( p = q \) then \( \|f^n\| \to \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{At^q + it^q} \, dt \, dx, \quad n \to \infty \) (Girard [5,p.4]) (1.18)
If \( p \neq q \) then \( \|f^n\| \sim Cn^{\frac{1}{2}(1-\frac{p}{q})}, \quad \text{as} \ n \to \infty \) (Girard [5,p.3]) (1.19)

where the constant \( C = (\frac{p}{q})^{\frac{1}{2} + \delta(p)} |\Gamma(\frac{p-1}{q})|^{1/2} \Gamma(p/2q) |A|^{1/2} \beta^{-p/2q} \) and the function \( \delta \) is 1 for \( p \) even and 0 for \( p \) odd. We shall show in this thesis that if \( p = q \) and if in addition
\[ q = 2 \text{ or } \Im A \neq 0 \] (1.20)
then there exists an asymptotic series for \( \|f^n\| \):
\[ \|f^n\| \sim \sum_{k=0}^{\infty} \frac{d_k}{n^{2k/p}}, \quad n \to \infty \] (1.21)
(The condition (1.20) insures that the Fourier transform of $e^{Atq}$ has no real zeros). As one can see from (1.18) and (1.19) the asymptotic behavior of $\|f^n\|$ is completely determined by parameters $p, q, A$ and $\beta$ (respectively by just the two parameters $q$ and $A$ in the case $p = q$). To obtain the entire asymptotic series of $\|f^n\|$ in the case $p = q$ one needs additional information on the behavior of $f$ in the neighborhood of the maximum point $z = 1$. Such additional information is provided by a sequence $(P_j)$ of polynomials attached to the maximum point. We need two ingredients to define $P_j$: auxiliary polynomials $\pi_j$ and the expansion of $\log f(e^{it})$ around $t = 0$. The polynomials $\pi_j, j = 0, 1, 2, 3, \ldots$ depending on $j$ variables: $a_1, a_2, a_3, \ldots, a_j$, are defined formally by

$$\exp(\sum_{m=1}^{\infty} a_m \lambda^m) = \sum_{j=0}^{\infty} \pi_j(a_1, a_2, a_3, \ldots, a_j) \lambda^j$$

so that

$$\pi_j = \sum a_1^{s_1} a_2^{s_2} \cdots a_j^{s_j} s_1! s_2! \cdots s_j!$$

where the sum is taken over the set $\zeta(j)$ of all $j$-tuples $s = (s_1, s_2, \ldots, s_j)$ such that $s_1 + 2s_2 + \ldots + js_j = j$. On the other hand, since $\log f \circ \exp$ is analytic at $w = 0$ we can expand $\log f(e^{it})$ in a series of powers of $t$. In the case $p = q$ that series will have the form

$$\log f(e^{it}) = i\alpha t + At^q + t^q \sum_{j=1}^{\infty} B_j t^j$$

Since this expansion converges in a neighborhood of $t = 0$ we obtain that there exists $R > 0$ such that

$$|B_j| \leq R^j \text{ for } j = 1, 2, 3, \ldots$$

(This fact we shall need only in the proof of theorem 1)

We are now ready to define the polynomials $P_j$:

$$P_j(t) \equiv \pi_j(B_1 t^{q+1}, B_2 t^{q+2}, \ldots, B_j t^{q+j})$$

It follows from (1.23) that

$$P_j(t) = \sum_{s \in \zeta(j)} C_s t^{q(s_1+s_2+\ldots+s_j)+j}$$

Since $q$ is even, we have that $q(s_1 + s_2 + \ldots + s_j) + j$ is of the same parity as $j$, so that $P_j$ is even for $j$ even and odd for $j$ odd. Also, since the maximal value of $s_1 + s_2 + \ldots + s_j$ for $s \in \zeta(j)$ is equal to $j$, we get that

$$\deg P_j \leq (q + 1)j$$
To make the exposition self-contained, we have included, in this thesis, without any change, a crucial lemma and its proof from Baishanski [1]; that appears here as lemma 2. We have also included the statement and proof of the particular version of the Euler-MacLaurin sum formula which appears in Hardy's Divergent Series.

To our knowledge, theorem 1 does not appear anywhere in literature, but no new ideas nor techniques have been required to obtain that result. Theorem 2 and section 3.1 are due to Baishanski, he has also made the observation that if the Taylor coefficients of \( f \) are real, all the coefficients of the asymptotic series vanish. In addition, with several suggestions, he has helped to make the first four chapters of this text much more readable.

The principal contributions of the author are:

(i) lemma 6, which provides a method to derive, for example, from the asymptotic expansion of \( \phi(n, \nu) \) as \( n \to \infty \) the asymptotic expansion of \( \sum_{\nu \in \mathbb{Z}_n} |\phi(n, \nu)| \) as \( n \to \infty \) under very general conditions.

(ii) the idea to restrict investigation when \( p = q > 2 \) to the case when \( \Im A \neq 0 \)

(iii) the idea for the use of Wright's theorem, and

(iv) the proof of the main theorem and the explicit expressions for the first three coefficients.

The main open question is: what kind of asymptotic expansion do we have in the case \( p = q > 2, \quad \Im A = 0? \)
2.1 Asymptotic Expansion of Coefficients \( a_{nv} \)

In this chapter we present an asymptotic series for

\[ a_{nv} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{it}) e^{-i\nu t} dt \] (2.1)

which holds uniformly in \( \nu \) as \( n \to \infty \)

**Lemma 1**

Let the power series \( \sum a_m \lambda^m \) have a positive radius of convergence. Then

(\text{\textit{i}) \quad \exp(\sum_{m=1}^{\infty} a_m \lambda^m) = 1 + \sum_{j=1}^{\infty} \pi_j \lambda^j \] (2.2)

where the coefficients \( (\pi_j) \) are expressed in terms of the coefficients \( (a_m) \) by the formula (1.23):

\[ \pi_j = \sum \frac{a_1^{s_1} a_2^{s_2} \ldots a_j^{s_j}}{s_1! s_2! \ldots s_j!} \] (2.3)

where the sum is taken over the set \( \zeta(j) \).

(\text{\textit{ii}) \quad \left| \exp(\sum_{m=1}^{\infty} a_m \lambda^m) - (1 + \sum_{j=1}^{k} \pi_j \lambda^j) \right| \leq 2|\lambda \rho e|^{k+1} \] (2.4)

for \( |\lambda| \leq 1/2\rho e \) where \( \rho = \sup |a_m|^{1/m} \). (Since \( \sum a_m \lambda^m \) has a positive radius of convergence, we have that \( \limsup |a_m|^{1/m} < \infty \), so \( \rho \) is finite).

**Proof**
(i) Let

\[ \phi(\lambda) = \exp \sum_{1}^{\infty} a_m \lambda^m = 1 + \sum_{1}^{\infty} \pi_j \lambda^j \]  
(2.5)

\[ \phi_k(\lambda) = \exp \sum_{1}^{k} a_m \lambda^m = 1 + \sum_{j=1}^{\infty} \pi_{j,k} \lambda^j \]  
(2.6)

We have then that,

\[ \phi(\lambda) - \phi_k(\lambda) = \sum_{1}^{\infty} (\pi_j - \pi_{j,k}) \lambda^j \]  
(2.7)

and, on the other hand,

\[ \phi(\lambda) - \phi_k(\lambda) = \exp \sum_{1}^{\infty} a_m \lambda^m - \exp \sum_{1}^{k} a_m \lambda^m = (\exp \sum_{1}^{k} a_m \lambda^m)(\exp \sum_{k+1}^{\infty} a_m \lambda^m) - 1 \]  
(2.8)

Since \( g(\lambda) = \sum_{k+1}^{\infty} a_m \lambda^m \) has a zero of order \( k + 1 \) at \( \lambda = 0 \), the same is true for \( \exp g(\lambda) - 1 = [(\exp \sum_{k+1}^{\infty} a_m \lambda^m) - 1] \) and so by (2.8) \( \phi(\lambda) - \phi_k(\lambda) \) has a zero of order \( k + 1 \) at \( \lambda = 0 \), which, by (2.7), implies that \( \pi_j - \pi_{j,k} = 0 \) for \( j = 1, 2, 3, \ldots, k \). In particular, we obtain

\[ \pi_k = \pi_{k,k} \]  
(2.9)

So, by (2.6), \( \pi_k \) is the coefficient of \( \lambda^k \) in the power series expansion of \( \phi_k(\lambda) = \exp(a_1 \lambda) \exp(a_2 \lambda^2) \ldots \exp(a_k \lambda^k) \). Multiplying power series expansions for \( \exp(a_1 \lambda), \exp(a_2 \lambda^2), \ldots, \exp(a_k \lambda^k) \), we obtain that the coefficient \( \pi_k \) of \( \lambda_k \) is given by (1.23).

(ii) To prove the second part of the lemma we observe that

\[ \exp \sum_{1}^{\infty} a_m \lambda^m - (1 + \sum_{1}^{k} \pi_j \lambda^j) = \sum_{k+1}^{\infty} \pi_j \lambda^j \]  
(2.10)

Since \( |a_m| \leq \rho^m \) and \( s_1 + 2s_2 + \ldots + js_j = j \) we have

\[ |\pi_j| \leq \sum \frac{\rho^{s_1} \rho^{s_2} \ldots \rho^{s_j}}{s_1! s_2! \ldots s_j!} \]

\[ \leq \rho^j \sum \frac{1}{s_1! s_2! \ldots s_j!} \]

\[ \leq \rho^j \left( \sum_{0}^{\infty} \frac{1}{s_1!} \right) \left( \sum_{0}^{\infty} \frac{1}{s_2!} \right) \ldots \left( \sum_{0}^{\infty} \frac{1}{s_j!} \right) \]
Thus, since $|\lambda|\rho e \leq 1/2$ we have

$$\left| \exp \sum_{1}^{\infty} a_m \lambda^m - (1 + \sum_{1}^{k} \pi_k \lambda^k) \right| \leq \sum_{k+1}^{\infty} |\pi_j| \lambda^j \leq \sum_{k+1}^{\infty} (|\lambda|\rho e)^j \leq 2|\lambda|\rho e|^{k+1}$$


(2.12)

Let $F(\psi)$ denote the Fourier transform of the function $\psi$. (i.e.

$$F(\psi)(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(it\xi)\psi(t)\,dt$$

**THEOREM 1**

Let $\varepsilon_n = n^{-1/4}$ and $\gamma_{n\nu} = (\alpha n - \nu)\varepsilon_n$. Then for any positive integer

$$a_{n\nu} = \sum_{j=0}^{k} \varepsilon_n^{j+1} F\left[ P_j(u) e^{Au u} \right] (\gamma_{n\nu}) + o(\varepsilon_n^{k+1}), \quad n \to \infty, \quad \text{uniformly in } \nu$$

(2.13)

**PROOF**

Let

$$a_{n\nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{it}) e^{-ivt} \,dt$$

$$= \frac{1}{2\pi} \int_{I_n} f^n(e^{it}) e^{-ivt} \,dt + \frac{1}{2\pi} \int_{(-\pi,\pi)\setminus I_n} f^n(e^{it}) e^{-ivt} \,dt = J_1 + J_2$$

(2.14)

where $I_n = (-\varepsilon_n \phi_n, \varepsilon_n \phi_n)$, and $\phi_n = \log^2 n$. Since $\varepsilon_n \phi_n \to 0$ as $n \to \infty$, by taking $n$ sufficiently large we insure that the series (1.24) converges on $I_n$ so that

$$J_1 = \frac{1}{2\pi} \int_{I_n} \exp[(n\alpha - \nu)t + nAt^q + nt^q \sum_{j=1}^{\infty} B_j t^j] \,dt$$

(2.15)

From (1.5) and the assumption that $|f(z)| < 1$ for $|z| = 1, z \neq 1$ we obtain that there exists $\delta > 0$ such that

$$|f(e^{it})| \leq e^{-\delta t^q} \quad \text{for } |t| \leq \pi$$

(2.16)

so that,

$$|J_2| \leq \int_{(-\pi,\pi)\setminus I_n} |f^n(e^{it})| \,dt \leq \int_{(-\pi,\pi)\setminus I_n} e^{-\delta t^q} \,dt = 2\varepsilon_n \int_{\phi_n} e^{-\delta u^q} \,du$$

(2.17)
But this is $O(n^{-s})$ as $n \to \infty$ uniformly in $\nu$, for every $s > 0$. (In other words, for every $s > 0$ there exists a $K_s$ such that $|J_2| \leq K_s n^{-s}$ for all $n \geq N$.)

In (2.15) we make the substitution $t = \epsilon_n u$ to obtain

$$J_1 = \epsilon_n \frac{1}{2\pi} \int_{-\phi_n}^{\phi_n} \exp(\gamma_n u i + Au^q + u^q \sum_{m=1}^{\infty} B_m \epsilon_n^m u^m) du$$

(2.18)

Applying lemma 1 to

$$\exp \sum_{m=1}^{\infty} B_m u^{m+q} \epsilon_n^m$$

(with $a_m = B_m u^{m+q}$, $\lambda = \epsilon_n$) and assuming that $|u| \leq \phi_n$ we obtain from (1.25) that,

$$\rho = \sup |a_m|^{1/m} = \sup |B_m|^{1/m} |\phi_n|^{m+1} \leq R |\phi_n|^{1+q}$$

(2.19)

Using the already introduced notation

$$P_j(t) \equiv \pi_j(B_1 t^{q+1}, B_2 t^{q+2}, \ldots, B_j t^{q+j})$$

(2.20)

we obtain from the lemma: if

$$|\epsilon_n| \leq \frac{1}{2\rho e} = \frac{1}{2eR|\phi_n|^{1+q}}$$

(which certainly holds for $n$ sufficiently large) then

$$\exp \sum_{m=1}^{\infty} B_m u^{m+q} \epsilon_n^m = 1 + \sum_{j=1}^{k} P_j(u) \epsilon_n^j + R_k(u, \epsilon_n)$$

(2.21)

where

$$|R_k(u, \epsilon_n)| \leq 2(\epsilon_n \rho e)^{k+1} = C(\epsilon_n \phi_n^{1+q})^{k+1}$$

(2.22)

Substituting (2.21) into (2.18) we obtain

$$J_1 = \sum_{j=0}^{k} \epsilon_n^{j+1} \frac{1}{2\pi} \int_{-\phi_n}^{\phi_n} \exp(\gamma_n u i + Au^q) P_j(u) du + \tilde{R}_k(\epsilon_n)$$

(2.23)

where

$$\tilde{R}_k(\epsilon_n) = \epsilon_n \frac{1}{2\pi} \int_{-\phi_n}^{\phi_n} \exp(\gamma_n u i + Au^q) R_k(u, \epsilon_n) du$$

(2.24)

Obviously,

$$|\tilde{R}_k(\epsilon_n)| \leq \epsilon_n \frac{1}{2\pi} \int_{-\phi_n}^{\phi_n} \exp(-\delta u^q) |R_k(u, \epsilon_n)| du$$

(2.25)
So that by (2.22)
\[ |\tilde{R}_k(\epsilon_n)| \leq C_1 \epsilon_n^{k+2} \phi_n^{(1+\nu)(k+1)} = o(\epsilon_n^{k+1}) \text{ as } n \to \infty \]  
(2.26)
We observe also that for \( j = 1, 2, 3, \ldots, k \)

\[
\left| \mathcal{F} \left[ P_j(u)e^{Au^q} \right](\gamma_{nu}) - \frac{1}{2\pi} \int_{-\phi_n}^{\phi_n} \exp(\gamma_{nu}ui + Au^q)P_j(u)du \right|
\]

\[
\leq \frac{1}{2\pi} \int_{|u| \geq \phi_n} \exp(-\delta u^q)|P_j(u)|du
\]

\[
\leq C_k e^{-\delta \phi_n} = o(\epsilon_n^{k+1}), \quad n \to \infty
\]  
(2.27)
So that from (2.23) and (2.26) and the previous estimate for \( J_2 \) we obtain finally that for every positive integer \( k \),

\[
a_{nu} = \sum_{j=0}^{k} e^{j+1} \mathcal{F} \left[ P_j(u)e^{Au^q} \right](\gamma_{nu}) + o(\epsilon_n^{k+1}) \text{ as } n \to \infty, \text{ uniformly in } \nu. \]  
(2.28)

2.2 An Estimate For \( a_{nu} \)

**THEOREM 2**
There exists a constant \( C = C(f) \) such that

\[
|a_{nu}| \leq C \epsilon_n \exp(-|\gamma_{nu}|)
\]  
(2.29)
for \( n \) sufficiently large and all \( \nu \).
The proof of the theorem is based on the following two lemmas:

**Lemma 2**
Let

\[
\psi(r, t) = \Re \log z^{-\alpha} f(z) = \log |f(re^{it})| - \alpha \log r
\]  
(2.30)
where \( z = re^{it} \). There exist positive constants \( M, C_m \quad (m = 0, 1, \ldots, q - 1) \) and \( \epsilon \) such that in \( V_\epsilon = \{(r, t) : |\log r| \leq \epsilon, \quad |t| \leq \epsilon\} \) the following inequality holds:

\[
\psi(r, t) \leq \sum_{m=0}^{q-1} C_m |\log r|^{q-m} |t|^m - M |t|^q
\]  
(2.31)

**Lemma 3**
There exists a constant $C = C(f)$ such that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} r_n^{-\alpha n} |f(r_ne^{it})|^n dt \leq C \varepsilon_n
\] (2.32)
for $r_n = \exp(\pm \varepsilon_n)$ and $n$ sufficiently large.

**Proof (Lemma 2)**

By Taylor's formula
\[
\psi(r, t) = \sum_{m=0}^{q-1} C_m(r) t^m + C_q(r, t) t^q, \quad r, t \in V_\varepsilon
\] (2.33)
where
\[
C_m(r) = \frac{1}{m!} \frac{\partial^m \psi(r, t)}{\partial t^m} \bigg|_{t=0}, \quad m = 0, 1, \ldots, q-1
\] (2.34)
\[
C_q(r, t) = \frac{1}{q!} \frac{\partial^q \psi(r, t)}{\partial t^q} \bigg|_{t=r}, \quad |r| < |t|, \quad t, r \in V_\varepsilon
\] (2.35)

Now, $\log z^{-\alpha} f(z) = O(1)(z-1)^\alpha$ as $z \to 1$ so
\[
\psi(r, t) = \Re[O(1)(e^{ir(r-1)} + (e^{i(t-r)})^\alpha)], \quad r \to 1, t \to 0
\] (2.36)
Thus the partial derivatives of $\psi(r, t)$ of order less than $q$ are zero at $r = 1, t = 0$, i.e.
\[
d^n C_m(r) / dr^n |_{r=1} = \frac{1}{m!} \frac{\partial^{n+m} \psi(r, t)}{\partial r^n \partial t^m} \bigg|_{t=0, r=1} = 0
\] (2.37)
for $n + m \leq q - 1$. So $C_m(r)$ has a zero at the point $r = 1$ of order $\geq q - m$ for $m = 0, 1, \ldots, q-1$, and hence,
\[
|C_m(r)| \leq C_m' |r-1|^{q-m} \leq C_m |\log r|^{q-m}
\] (2.38)
for $r, t \in V_\varepsilon$ with $\varepsilon$ sufficiently small. Finally,
\[
d^q \psi(1, t) / dt^q |_{t=0} = -q! \beta < 0
\] (2.39)
which gives:
\[
\partial^q \psi(r, t) / \partial t^q < -M, \quad M > 0
\] (2.40)
again for $r, t \in V_\varepsilon$.

**Proof (Lemma 3)**

Let $\varepsilon > 0$ be chosen so that $f$ is analytic in $|z| \leq 1 + \varepsilon$. 


(2.40) holds in $V_\epsilon$

$$\sup\{ |z|^{-\alpha} |f(z)| : -\epsilon \leq \log |z| \leq \epsilon, |\arg z| \geq \epsilon \} = \delta < 1$$

Let $N$ be such that $\epsilon_n < \log(1 + \epsilon)$ and $\delta^n < \epsilon_n$ for $n \geq N$. Let $r_n = \exp(\epsilon_n)$ or $\exp(-\epsilon_n)$; then

$$\frac{1}{2\pi} \int_{|t|<\pi} r_n^{-\alpha n} |f(r_ne^{it})|^n dt < \delta^n < \epsilon_n \quad (2.41)$$

On the other hand, by lemma 2, for $|t| < \epsilon$

$$\psi(r_n, t) \leq \sum_{m=0}^{q-1} C_m |\log r_n|^{q-m} |t|^m - M(t)^q$$

$$\leq \frac{1}{n} \sum_{m=0}^{q-1} C_m \left| \frac{t}{\epsilon_n} \right|^m - M(t)^q \quad (2.42)$$

so that

$$n\psi(r_n, t) \leq \sum_{m=0}^{q-1} C_m \left| \frac{t}{\epsilon_n} \right|^m - M \left( \frac{t}{\epsilon_n} \right)^q \quad (2.43)$$

and thus

$$\int_{|t|<\epsilon} r_n^{-\alpha n} |f(r_ne^{it})|^n dt = \int_{|t|<\epsilon} \exp(n\psi(r_n, t)) dt$$

$$\leq \int_{-\epsilon}^{\epsilon} \exp \left[ \sum_{m=0}^{q-1} C_m \left| \frac{t}{\epsilon_n} \right|^m - M \left( \frac{t}{\epsilon_n} \right)^q \right] dt$$

$$\leq \epsilon_n \int_{-\infty}^{\infty} \exp \left[ \sum_{m=0}^{q-1} C_m |u|^m - Mu^q \right] du \leq K\epsilon_n \quad (2.44)$$

From the last estimate and from (2.41) we deduce the statement of the lemma.

**Proof of Theorem 2**

By Cauchy's integral formula

$$a_{n\nu} = \frac{1}{2\pi} \int_{|z|=r} f^n(z)z^{-\nu-1} dz \quad (2.45)$$

If $|\nu| < \alpha n$ we choose $r_n = e^{-\epsilon_n}$, and if $\nu \geq \alpha n$ we choose $r_n = e^{\epsilon_n}$. Thus we obtain, using lemma 3,

$$|a_{n\nu}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r_ne^{it})|^n r_n^{-\nu} dt$$

$$\leq r_n^{n\alpha-\nu} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r_ne^{it})|^n r_n^{-n\alpha} dt \leq C\epsilon_n r_n^{n\alpha-\nu} \quad (2.46)$$
which gives the desired result:

\[ |a_{n\nu}| \leq C\varepsilon_n \exp(-\varepsilon_n |n\alpha - \nu|) \]  \hspace{1cm} (2.47)
CHAPTER III
PREPARATION FOR THE PROOF OF THE MAIN THEOREM

We have collected in this chapter three results which will be used in the proof of the main theorem, both in the typical case \( q = 2 \), and in the exceptional case, \( p = q > 2 \).

3.1 Contribution of Non-Central Coefficients

We shall distinguish: the central coefficients \( a_{n\nu} \), those for which \( \nu \) is close to the value of \( an \), more precisely those for which

\[
\nu \in Z_n = \{ \nu : |an - \nu| \leq n^{1/q} \log^2 n \} \quad (3.1)
\]

from the remaining coefficients; to which we refer as "non-central". Our first step is showing that non-central coefficients can be neglected when determining the asymptotic series of \( \sum |a_{n\nu}| \); our second step is to show then, that the asymptotic expansion of \( \sum |a_{n\nu}| \) up to a term of order \( o(e^k) \) is determined by \( A, q \) and just the first \( k \) polynomials \( P_1, P_2, \ldots, P_k \).

To show that the non-central terms of \( \sum |a_{n\nu}| \) may be neglected we consider two parts of the sum separately. First let \( \psi_n = \log^2 n \) and consider the non-central terms with \( \nu \geq an + \psi_n/e_n \). By lemma 2:

\[
|a_{n\nu}| \leq C\epsilon_n \exp(-|\gamma_{n\nu}|) \quad (3.2)
\]

and since for \( \nu \geq an \)

\[
\exp(-|\gamma_{n\nu}|) = \exp([an - \nu]e_n) = \exp(an e_n)(e^{-\epsilon_n})^\nu \quad (3.3)
\]

we obtain that the contribution of the non-central coefficients to the right of \( an \) is

\[
\sum_{\nu \geq an + \psi_n/e_n} |a_{n\nu}| \leq C\epsilon_n \exp(an e_n) \sum_{\nu \geq an + \psi_n/e_n} (e^{-\epsilon_n})^\nu \\
\leq C\epsilon_n \exp(an e_n)(e^{-\epsilon_n})^{an + \psi_n/e_n} \frac{1}{1 - e^{-\epsilon_n}} \\
\leq C_1 e^{-\psi_n} = O(n^{-s}), \text{ as } n \to \infty \text{ for any real } s \quad (3.4)
\]
Similarly, since \( \exp(-|\gamma_{\nu}|) = \exp(-\alpha n \epsilon_n)(\epsilon_n)^\nu \) for \( \nu < \alpha n \), the sum of the non-central terms to the left of \( \alpha n \) may be estimated by:

\[
\sum_{\nu < \alpha n - \psi_n / \epsilon_n} |a_{n\nu}| \leq C_1 e^{-\psi_n} = O(n^{-s}), \text{ as } n \to \infty \quad (3.5)
\]

It follows that

\[
\sum_0^\infty |a_{n\nu}| = \sum_{\nu \in \mathbb{Z}_n} |a_{n\nu}| + O(n^{-s}), \text{ as } n \to \infty \quad (3.6)
\]

for any real \( s \).

Let

\[
a^{(k)}_{n\nu} = \sum_{j=0}^k \epsilon_{n}^{j+1} \mathcal{F}(P_{j}(u)e^{4t^q})(\gamma_{n\nu}) \quad (3.7)
\]

Our second step is to show that

**Proposition 1**

\[
\sum_{\nu = 0}^\infty |a_{n\nu}| = \sum_{\nu \in \mathbb{Z}_n} |a^{(k)}_{n\nu}| + o(\epsilon_n^k), \quad n \to \infty \quad (3.8)
\]

Applying the formula (2.28) with \( k + 1 \) instead of \( k \), we see that the remainder \( o(\epsilon_n^{k+1}) \) of (2.28) can be expressed as

\[
\epsilon_n^{k+2} F(P_{k+1}(u)e^{4t^q})(\gamma_{n\nu}) + o(\epsilon_n^{k+2}) \quad (3.9)
\]

so that, really, for the remainder \( o(\epsilon_n^{k+1}) \) in (2.28) we have the better estimate \( O(\epsilon_n^{k+2}) \). This means that by Theorem 1 we have

\[
a_{n\nu} = d^{(k)}_{n\nu} + O(\epsilon_n^{k+2}) \quad (3.10)
\]

Since \( Z_n \) contains \( 2\psi_n / \epsilon_n = 2n^{1/q} \log^2 n \) summands, we get

\[
\left| \sum_{\nu \in \mathbb{Z}_n} |a_{n\nu}| - \sum_{\nu \in \mathbb{Z}_n} |d^{(k)}_{n\nu}| \right| \leq \sum_{\nu \in \mathbb{Z}_n} |a_{n\nu}| - |d^{(k)}_{n\nu}| \leq \sum_{\nu \in \mathbb{Z}_n} C\epsilon_n^{k+2} \leq C\epsilon_n^{k+2}2\psi_n / \epsilon_n = C(\epsilon_n^{k+1} \log^2 n) = o(\epsilon_n^k) \text{ as } n \to \infty \quad (3.11)
\]
3.2 Riemann Sum Versus Integral

Both in the typical case and in the exceptional case we show that instead of sums

$$\sum_{\nu} |a_{\nu}|$$

we can consider sums of the type

$$\sum_{\nu \in \mathbb{Z}_n} \{ h_1(\gamma_{\nu}) \epsilon_n + \ldots + h_k(\gamma_{\nu}) \epsilon_n^k + \ldots \} \quad (3.12)$$

where $h_k$ are certain entire functions of rapid decay as $|x| \to \infty$. It is easy to see that the sum

$$\sum_{\nu \in \mathbb{Z}_k} h_k(\gamma_{\nu}) \epsilon_n \quad (3.13)$$

is the Riemann sum of an integral. To estimate the error which arises when that Riemann sum is replaced by the corresponding integral, we shall use the following lemma.

**Lemma 4**

If $g$ is a $2s$ times continuously differentiable function on $[a, b]$ and $w = (b - a)/m$, $m$ a positive integer, and

$$RS[g; a, b; w] = (g(a) + g(a + w) + g(a + 2w) + \ldots + g(b))w$$

then

$$|RS[g; a, a; w] - \int_a^b g(t)dt| \leq M + |w|^{2s} \left( \frac{1}{12} \int_a^b |g^{(2s)}(t)|dt \right) \quad (3.14)$$

Where,

$$M = \max \{|wg(b)|, |wg(a)|, |w^2g'(b)|, |w^2g'(a)|, \ldots, |w^{2s}g^{(2s-1)}(b)|, |w^{2s}g^{(2s-1)}(a)|\}$$

The proof is based on the Euler-MacLaurin sum formula. The usual form of that formula is not the most suitable for our purpose. We shall therefore use the following form of the Euler-MacLaurin formula. (which we have taken from Hardy, [8,p.330])

With the notation as in the lemma:

$$\frac{1}{2}g(a) + g(a + w) + \ldots + \frac{1}{2}g(b)$$

$$= \frac{1}{w} \int_a^b g(t)dt + \sum_{r=1}^{s} (-1)^{r-1} w^{2r-1} \frac{B_r}{(2r)!} (g^{(2r-1)}(b) - g^{(2r-1)}(a)) + W_s \quad (3.15)$$
where

\[ W_s = (-1)^s \frac{w^{2s-1}}{2^{2s-1} \pi^{2s}} \int_a^b g^{(2s)}(t) \sum_{l=1}^{\infty} \frac{1}{l^{2s}} \cos \frac{2\pi l (t-a)}{w} \, dt \quad (3.16) \]

\( B_r \)'s are Bernoulli numbers, defined by \( \frac{t}{e^t - 1} = 1 - t/2 + \sum (-1)^{n-1} \frac{B_n}{(2n)!} t^{2n} \). For the convenience of the reader we give here a proof of (3.15). (Poisson's proof, borrowed also from Hardy, Divergent Series.)

**Proof**

By the ordinary theory of Fourier series,

\[ g(x) = \frac{1}{b-a} \int_a^b g(t) \, dt + \frac{2}{b-a} \sum_{n=1}^{\infty} \int_a^b g(t) \cos \frac{2\pi r (t-x)}{b-a} \, dt \quad (3.17) \]

for \( a < x < b \). For \( x = a \) or \( x = b \) the sum is \( \frac{1}{2} \{ g(a) + g(b) \} \). We take \( w = (b-a)/n \) and substitute in (3.17) to get:

\( \frac{1}{2} g(a) + g(a+w) + \ldots + g(a+(n-1)w) + \frac{1}{2} g(b) \)

\[ = \frac{n}{b-a} \int_a^b g(t) \, dt + \frac{2}{b-a} \sum_{r=1}^{\infty} \int_a^b g(t) \sum_{s=0}^{n-1} \cos \frac{2\pi r (t-a-sw)}{b-a} \, dt \quad (3.18) \]

The sum under the integral sign is

\[ \Re \left[ \exp \frac{2\pi ir(t-a)}{b-a} \sum_{s=0}^{n-1} \exp \left( -\frac{2\pi irs}{n} \right) \right] \quad (3.19) \]

and the sum here is 0 unless \( r = ln \), where \( l \) is a positive integer, and then it is \( n \). Hence, (3.18) is

\[ \frac{1}{2} g(a) + g(a+w) + \ldots + \frac{1}{2} g(b) = \frac{1}{w} \int_a^b g(t) \, dt + \frac{2}{w} \sum_{r=1}^{\infty} \int_a^b g(t) \cos \frac{2\pi l (t-a)}{w} \, dt \quad (3.20) \]

Now

\[ \int g(t) \cos \frac{2\pi l (t-a)}{w} \, dt = -\frac{w}{2\pi l} \int g'(t) \sin \frac{2\pi l (t-a)}{w} \, dt \]

\[ = \sum_{r=1}^{s} (-1)^{r-1} \left( \frac{w}{2\pi l} \right)^{2r} \{ g^{(2r-1)}(b) - g^{(2r-1)}(a) \} + \]

\[ (-1)^s \left( \frac{w}{2\pi l} \right)^{2s} \int g^{(2s)}(t) \cos \frac{2\pi l (t-a)}{w} \, dt \quad (3.21) \]

by repeated integration by parts. Substituting in (3.20), and using the fact that

\[ B_n = \frac{(2n)!}{2^{2n-1} \pi^{2n}} \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \]

(3.22)
we obtain the statement which we wished to prove.

**Proof (Lemma 4)**

Multiplying both sides of (3.15) by $w$, adding $\frac{1}{2}(g(a) + g(b))$ and observing that,

$$
\frac{1}{s^{2s-1}\pi^{2s}} \sum_{l=1}^{\infty} \frac{1}{l^{2s}}
$$

is largest for $s = 1$ when it is $\frac{1}{12}$; and observing that the well known fact

$$
B_n = \frac{(2n)!}{2^{2n-1}\pi^{2n}} \sum_{l=1}^{\infty} \frac{1}{l^{2n}}
$$

implies that $0 \leq B_n/(2n)! \leq 1$, we obtain the statement of the lemma.

**Corollary**

If $F$ is infinitely differentiable on $(-\infty, \infty)$, and if, for $k = 0, 1, 2, \ldots$, $F^{(k)}(\pm \psi_n)$ tends to zero faster than any power of $\epsilon_n$ as $n \to \infty$, then

$$
RS[F; -\psi_n, \psi_n; \epsilon_n] = \int_{-\psi_n}^{\psi_n} F(t) dt = O(\epsilon_n^s)
$$

for any positive $s$.

### 3.3 A Uniform Estimate For Some Taylor Remainders

The function $|1 + q_1x + q_2x^2 + \ldots + q_kx^k|$ is real analytic in a neighborhood of $x = 0$. It follows that the $k$th remainder of its Taylor series around zero will be

$$
\leq C(q_1, q_2, \ldots, q_k)|x|^{k+1}
$$

for $x$ sufficiently small; we shall need an estimate for $C(q_1, q_2, \ldots, q_k)$. That estimate will be provided by lemma 6. First, to any infinite sequence $\{q_j\}_{j=0}^{\infty}$ of complex numbers, with $q_0 = 1$ we shall associate a unique sequence $\tilde{q}_j$ with $\tilde{q}_0 = 1$ defined by:

$$
\sum_{j=0}^{s} \tilde{q}_j \tilde{q}_{s-j} = \sum_{j=0}^{s} q_j q_{s-j} \quad \text{for} \quad s = 1, 2, 3, \ldots
$$

In other words we define $\tilde{q}_j$ recursively by

$$
\tilde{q}_s = -\frac{1}{2} \{q_1 \tilde{q}_{s-1} + \tilde{q}_2 \tilde{q}_{s-2} + \ldots + \tilde{q}_{s-1} q_1\} + \frac{1}{2} \sum_{j=0}^{s} q_j \tilde{q}_{s-j}
$$

It is obvious, since $\sum q_j \tilde{q}_{s-j}$ is real, that $(\tilde{q}_j)$ is a sequence of real numbers.
We deduce easily from (3.26) that for any \( k, k = 1, 2, \ldots \) the following expansion holds:

\[
|1 + q_1 x + q_2 x^2 + \ldots + q_k x^k| = 1 + \tilde{q}_1 x + \ldots + \tilde{q}_k x^k + \sum_{j=k+1}^{\infty} b_{j,k} x^j
\]  

(3.28)

As already mentioned, we need an estimate for the difference:

\[
|1 + q_1 x + \ldots + q_k x^k| - \{1 + \tilde{q}_1 x + \ldots + \tilde{q}_k x^k\} \leq (3.29)
\]

For that purpose we prove first the following lemma.

**Lemma 5**

Let

\[
g(x) = (1 + a_1 x + a_2 x^2 + \ldots + a_l x^l)^{1/2} \quad \text{and} \quad M \equiv \max\{|a_1|, |a_2|, \ldots, |a_l|\}
\]  

(3.30)

Then,

\[
|g^{(j)}(0)| < \frac{j!}{l!}(lM)^{1/2}, \quad \text{for} \quad j = 1, 2, 3, \ldots
\]  

(3.31)

**Proof**

We will show by induction that the \( j \)th derivative of \( g \) is given by an expression of the form:

\[
\sum_{\alpha} c(1 + a_1 x + \ldots + a_l x^l)^{\alpha_0}(a_1 + 2a_2 x + \ldots + la_l x^{l-1})^{\alpha_1}
\]

\[
(2a_2 + 6a_3 x + \ldots + l(l-1)a_l x^{l-2})^{\alpha_2} \cdots (l!a_l)^{\alpha_l}
\]  

(3.32)

where the number of terms in the sum is at most \( l^j \), each constant \( |c_\alpha| \leq j! \) and the exponent \( |\alpha_0| \leq \frac{2j-1}{2} < j \); also \( 0 \leq \alpha_1, \alpha_2, \ldots, \alpha_l \leq j \) are integers. The first derivative of \( g \) is:

\[
\frac{1}{2}(1 + a_1 x + \ldots + a_l x^l)^{-1/2}(a_1 + 2a_2 x + \ldots + la_l x^{l-1})
\]  

(3.33)

which is of the form (3.32) with \( j = 1 \). Assume that (3.32) holds for the \( j \)th derivative of \( g \). The \( j + 1 \) derivative is found by differentiating (3.32), which gives:

\[
\sum_{\alpha} \sum_{r=0}^{l-1} c_\alpha \alpha_r (1 + a_1 x + \ldots + a_l x^l)^{\alpha_0}
\]

\[
\alpha_r \left[r!a_r + (r + 1)!a_{r+1} x + \ldots + (l)_r a_l x^{l-r} \right]^{\alpha_r-1}
\]

\[
[(r + 1)!a_{r+1} + (r + 2)!a_{r+2} x + \ldots + (l)_{r+1}a_l x^{l-r-1}]^{\alpha_r+1} \cdots (l!a_l)^{\alpha_l}
\]  

(3.34)
But since the number of terms in the sum indexed by $\alpha$ was at most $i^j$ and the sum over $r$ had $l$ terms, we see that the total number of terms in this expression is at most $l^{i+1}$. The constants are now $c_{\alpha_r}$, but
\[
|c_{\alpha_r}| = |c_{\alpha}| |\alpha_r| \leq j! j \leq (j + 1)!
\]
and since each exponent $\alpha_i$ either remains the same or is changed by $\pm 1$, in each term we have
\[
|\alpha_0| \leq j + 1, \quad 0 \leq \alpha_i \leq j + 1 \text{ for } j = 1, 2, 3, \ldots
\]
Thus, this derivative is of form (3.32) for $j = j + 1$. We now need only find a bound on the expression (3.32) when $x = 0$. But each factor:
\[
\left|\left(r!a_r + (r + 1)!a_{r+1} x + \ldots + (l)!a_{l} x^{l-r}\right)^{\alpha_r}\right| = |r!a_r|^\alpha_r \leq (l!)^j
\]
Where $1 \leq r \leq l$. Since there are at most $l$ such factors (Note that the $r = 0$ factor is always one when $x = 0$) we may combine this result with the bounds on the number of terms and $|c_{\alpha}|$ to get:
\[
|g^{(j)}(0)| \leq l^j (l!)^j j!
\]
We need lemma 5 only in order to deduce the following estimate.

**Lemma 6**

If $|q_k| \leq L$ for $j = 1, 2, \ldots$ and if $|x| \leq A_k L^{-4k}$ then
\[
|1 + q_1 x + \ldots + q_k x^k| - \{1 + \tilde{q}_1 x + \ldots + \tilde{q}_k x^k\} \leq C_k L^{4k(k+1)} |x|^{k+1}
\]
(Where $A_k$ and $C_k$ depend only on $k$)

**Proof**

The function
\[
|1 + q_1 x + \ldots + q_k x^k| = \left(\sum_{s=0}^{2k} \left(\sum_{0 \leq s-i \leq k} q_i \tilde{q}_{s-i}\right) x^s\right)^{1/2} = g(x)
\]
satisfies the conditions of lemma 5 with
\[
l = 2k; \quad a_s = \sum_{0 \leq i \leq k} q_i \tilde{q}_{s-i}
\]
Since $|q_i| \leq L$ for $i = 1, 2, \ldots, k$ we get

$$|a_s| \leq \sum |q_i| |q_{s-i}| \leq (k + 1)L^2$$

so that we can take $M = (k + 1)L^2$ in lemma 5. Since from (3.28) we have that

$$b_{j,k} = \frac{g_j(0)}{j!}$$

it follows from lemma 5 that,

$$|b_{j,k}| \leq l^j((l!M)^{ij} = (2k)^j[(2k)!(k + 1)L^2]^{2jk} \leq (D_kL^{4k})^j$$

where $D_k$ depends only on $k$. This last estimate implies that

$$\left| \sum_{j=k+1}^{\infty} b_{j,k}x^j \right| \leq \sum_{j=k+1}^{\infty} (D_kL^{4k}|x|)^j$$

$$\leq 2(D_kL^{4k}|x|)^{k+1} = C_kL^{4k(k+1)}|x|^{k+1}$$

provided that $|x| \leq \frac{1}{2D_kL^2}$, which proves the lemma.
CHAPTER IV
THE MAIN THEOREM: THE TYPICAL CASE

4.1 Introduction

In this chapter our assumptions are that \( f \) is a function analytic in the closed unit disk, that \( |f(z)| < 1 \) for \( |z| \leq 1 \), \( z \neq 1 \), \( f(1) = 1 \) and that \( \frac{d^2}{dt^2} |f(e^{it})| \neq 0 \) at \( t = 0 \). These assumptions imply (see (1.24)) that the expansion:

\[
\log f(e^{it}) = iat + At^2 + t^2 \sum_{j=0}^{\infty} B_j t^j
\]

(4.1)

converges in a neighborhood of \( t = 0 \). This expansion defines the coefficients

\( A, B_1, B_2, \ldots \). Our purpose is to establish that there exists an asymptotic series for \( \sum |a_{nv}| \) as \( n \to \infty \) (\( a_{nv}, \nu = 0, 1, 2, \ldots \) are the Taylor coefficients of \( f^n \)),

\[
a_{nv} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{it}) e^{-i\nu t} dt
\]

\[
\sum_\nu |a_{nv}| \sim c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots, \text{ as } n \to \infty
\]

(4.2)

It is worth noting that the asymptotic expansion is in powers of \( \frac{1}{n} \), not in powers of \( \frac{1}{\sqrt{n}} \) as one would naturally assume in view of the fact that the expansion of the coefficients \( a_{nv} \) is in powers of \( \frac{1}{\sqrt{n}} \), uniformly in \( \nu \). (see theorem 1)

We shall give an algorithm for computing the coefficients \( c_0, c_1, \ldots \) It will be seen that \( c_j \) is a polynomial in real and imaginary parts of \( \frac{1}{\sqrt{A}}, B_1, B_2, \ldots, B_j \). We shall also give, in this chapter, the full expression for the first three coefficients; \( c_0, c_1, c_2 \).

The natural question is whether the asymptotic series (4.2) is convergent for large \( n \). We have not been able to answer that question.

4.2 Polynomials \( Q_j \) and \( S_j \)

According to theorem 1, we obtain for the coefficients \( a_{nv} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^n(e^{it}) e^{-i\nu t} dt \) the following asymptotic series, which holds uniformly in \( \nu \),

\[
a_{nv} \sim \sum_{j=0}^{\infty} n^{-\frac{j+1}{2}} F(e^{A\mu} P_j(t)) (\gamma_{nv})
\]

(4.3)
where \( \gamma_{nv} = (\alpha n - \nu)n^{-1/2} \). Since,

\[
\int_{-\infty}^{\infty} e^{i\gamma t} e^{it^2} dt = i^{-j} \frac{d^j}{d\gamma^j} \int_{-\infty}^{\infty} e^{i\gamma t} e^{it^2} dt
\]

\[
= i^{-j} \frac{d^j}{d\gamma^j} \left( \sqrt{-\frac{\pi}{A}} e^{\gamma^2/4A} \right)
\]

we obtain that:

\[
\mathcal{F} \left( e^{At^2} \right)(\gamma) = \frac{i^{-j}}{2} \sqrt{\frac{1}{-A\pi}} \left( \frac{1}{2\sqrt{-A}} \right)^j H_j \left( \frac{\gamma}{2\sqrt{-A}} \right) e^{\gamma^2/4A}
\]

(4.5)

where \( H_j \) is the Hermite polynomial of degree \( j \), defined by:

\[
H_j(x) = (-1)^j e^{-x^2} \frac{d^j}{dx^j} e^{-x^2}
\]

(4.6)

Since \( P_j \) is a polynomial of degree at most \( 3j \), (see Introduction) and is even/odd when \( j \) is even/odd, we obtain from (4.5) that the polynomial \( Q_j \) defined by:

\[
\mathcal{F} \left( e^{At^2} P_j(t) \right)(\gamma) = \frac{1}{2\sqrt{-A\pi}} Q_j(\gamma) e^{\gamma^2/4A}
\]

(4.7)

is also of degree \( \leq 3j \), that it is even/odd when \( j \) is even/odd, and that the coefficients of \( Q_j \) are polynomials in \( \frac{1}{\sqrt{-A}}, B_1, B_2, \ldots, B_j \). From the explicit form of the polynomials \( P_j \) (see formulas (1.23) and (1.26)) and from the formula (4.5) it is easy to see that coefficients of \( Q_j \) can also be viewed as polynomials in \( \frac{1}{\sqrt{-A}}, D_1, D_2, \ldots, D_j \). where:

\[
D_m = \frac{B_m}{2^{m+2}A^{m+1}} \text{ for } m = 1, 2, \ldots
\]

(4.8)

It should also be noted that \( Q_0 \) is the constant polynomial 1. We can now rewrite the formula (4.3) as

\[
a_{nv} \sim \sum_{j=0}^{\infty} n^{-\frac{j+1}{2}} \frac{1}{2\sqrt{-A\pi}} e^{\gamma^2/4A} Q_j(\gamma_{nv})
\]

(4.9)

where \( \gamma_{nv} = (\alpha n - \nu)n^{-1/2} \). By theorem 3 we obtain then

\[
\sum_{\nu=0}^{\infty} |a_{nv}| = \sum_{\nu \in \mathbb{Z}_n} \frac{1}{2 \sqrt{\pi |A|}} n^{-1/2} e^{-\frac{\gamma_{nv}^2}{4A}} |Q_1(\gamma_{nv}) + \ldots + n^{-\frac{1}{2}} Q_k(\gamma_{nv})| + o(n^{-\frac{1}{2}})
\]

(4.10)
We define now a sequence of polynomials $(S_j)$ by:
\[
S_0(x) = 1 \\
S_j(x) = -\frac{1}{2} [S_1S_{j-1} + S_2S_{j-2} + \ldots + S_{j-1}S_1] + \frac{1}{2} [Q_0Q_j + Q_1Q_{j-1} + \ldots + Q_jQ_0]
\] (4.11)

Since the polynomials $Q_s$ are of degree at most $3s$, the same holds for the polynomials $S_s$ by (4.11), moreover (4.11) implies that $S_s$ is even/odd if $j$ is even/odd. Furthermore, according to (4.11), the coefficients of the polynomials $S_s$ are polynomials in real and imaginary parts of $Q_s$; letting:
\[
R_s = \Re D_s, \quad I_s = \Im D_s, \quad s = 1, 2, \ldots
\] (4.12)

we obtain that the coefficients of $S_s$ are polynomials in:
\[
\Re \sqrt{A}, \Im \sqrt{A}, R_1, R_2, \ldots, R_s, I_1, I_2, \ldots, I_s
\] Setting $x = n^{-1/2}$ and $q_j = Q_j(\gamma_{\nu})$ we obtain, with the notation of section 3.3, that
\[
\tilde{g}_j = S_j(\gamma_{\nu})
\] (4.13)

and that,
\[
|1 + n^{-\frac{1}{2}}Q_1(\gamma_{\nu}) + \ldots + n^{-\frac{1}{2}}Q_k(\gamma_{\nu})| = |1 + q_1x + \ldots + q_kx^k| 
\] (4.14)

To apply lemma 6 we need an upper bound $L$ for $|Q_j(\gamma_{\nu})|$, $j = 1, 2, \ldots ; \nu \in \mathbb{Z}_n$. Since
\[
\deg Q_j \leq 3j \quad \text{and} \quad |\gamma_{\nu}| = |(\alpha n - \nu)n^{-1/2}| \leq \log^2 n \text{ for } \nu \in \mathbb{Z}_n
\] (4.15) (4.16)

we obtain:
\[
|Q_j(\gamma_{\nu})| \leq M|\gamma_{\nu}|^{3j} \leq M \log^{6j} n
\] (4.17)

with $M$ depending only on $A, B_1, B_2, \ldots, B_j$. We can then take $L = M \log^{6k} n$; then the condition of lemma 6: $|x| \leq A_k L^{-4k}$ translates as
\[
n^{-1/2} \leq A_k \left( M \log^{6k} n \right)^{-4k},
\] which is certainly satisfied for $n$ sufficiently large. By lemma 6, we have then that uniformly for $\nu \in \mathbb{Z}_n$:
\[
|1 + n^{-\frac{1}{2}}Q_1(\gamma_{\nu}) + \ldots + n^{-\frac{1}{2}}Q_k(\gamma_{\nu})| \\
= 1 + n^{-\frac{1}{2}}S_1(\gamma_{\nu}) + \ldots + n^{-\frac{1}{2}}S_k(\gamma_{\nu}) + O \left( (\log n)^m n^{-\frac{k+1}{2}} \right) \text{ as } n \to \infty
\] (4.18)
(with $m = 24k^2(k + 1)$).
Substituting the last expression into (4.10) and observing that:

\[
\left| \sum_{\nu \in \mathbb{Z}_n} \frac{1}{2\sqrt{\pi} |A|} n^{-1/2} e^{\frac{\gamma^2}{4} \Re\left(\frac{1}{A}\right)} O \left((\log n)^m n^{-\frac{k+1}{2}}\right) \right| \\
\leq C \text{card}(Z_n)n^{-1/2}(\log n)^m n^{-\frac{k+1}{2}} \\
\leq C(\log n)^{m+2}n^{-\frac{k+1}{2}} = o(n^{-\frac{k}{2}})
\]  

(4.19)

We obtain:

\[
\sum_{\nu=0}^{\infty} |a_{\nu \nu}| = \frac{1}{2\sqrt{\pi} |A|} \sum_{\nu \in \mathbb{Z}_n} n^{-1/2} e^{\frac{\gamma^2}{4} \Re\left(\frac{1}{A}\right)} \left[1 + n^{-1/2} S_1(\gamma_{\nu \nu}) + \ldots \right. \\
\ldots + n^{-\frac{k}{2}} S_k(\gamma_{\nu \nu}) \right] + o(n^{-\frac{k}{2}})
\]  

(4.20)

We shall now use the notation introduced in section 3.2, namely:

\( R_s [F; a, b; w] \) is the Riemann sum of the function \( F \) corresponding to the partition of interval \([a, b]\) into subintervals of length \( w \). We observe that \( \gamma_{\nu \nu+1} - \gamma_{\nu \nu} = n^{-1/2} \) for every \( \nu \) and \( n \), and that the points \( \gamma_{\nu \nu}, \nu \in \mathbb{Z}_n \) form a partition of the interval \([-\psi_n, \psi_n]\), where \( \psi_n \sim \log^2 n, n \to \infty \). Therefore,

\[
\sum_{\nu \in \mathbb{Z}_n} n^{-1/2} \exp \left( \frac{\gamma_{\nu \nu} \Re\left(\frac{1}{A}\right)}{4} \right) S_j(\gamma_{\nu \nu}) = R_s \left[ F; -\psi_n, \psi_n; n^{-1/2} \right]
\]  

(4.21)

where \( F(\gamma) = \exp \left( \frac{\gamma^2}{4} \Re\left(\frac{1}{A}\right) \right) S_j(\gamma) \). Since, \( S_j \) is a polynomial we obtain that:

\[ F^{(m)}(\gamma) = O(e^{-|\gamma|}), \text{ as } |\gamma| \to \infty \]  

(4.22)

which implies that

\[ F^{(m)}(\pm \psi_n) = O(e^{-\log^2 n}), \text{ as } n \to \infty \]  

(4.23)

so that \( F^{(m)}(\pm \psi_n) \) tends to zero faster (as \( n \to \infty \)) than any power of \( \epsilon_n = n^{-1/2} \). Accordingly, the condition of the corollary to lemma 4 is satisfied, so that by that lemma and by (4.2) we have:

\[
\sum_{\nu \in \mathbb{Z}_n} n^{-1/2} \exp \left( \frac{\gamma_{\nu \nu} \Re\left(\frac{1}{A}\right)}{4} \right) S_j(\gamma_{\nu \nu}) - \int_{-\psi_n}^{\psi_n} \exp \left( \frac{\gamma^2}{4} \Re\left(\frac{1}{A}\right) \right) S_j(\gamma) d\gamma \\
= o(n^{-s}), \text{ as } n \to \infty
\]  

(4.24)

for any real \( s \). On the other hand, since

\[ \exp \left( \frac{\gamma^2}{4} \Re\left(\frac{1}{A}\right) \right) S_j(\gamma) = O(e^{-\gamma}), \text{ as } |\gamma| \to \infty \]  

(4.25)
we obtain that
\[
\left| \int_{\psi_n}^{\infty} \exp \left( \frac{\gamma^2}{4} \mathcal{R}(\frac{1}{A}) \right) S_j(\gamma) \, d\gamma \right| \leq k \int_{\psi_n}^{\infty} e^{-\gamma} \, d\gamma = ke^{-\psi_n} = o(n^{-s}), \text{ as } n \to \infty \tag{4.26}
\]
for every real \( s \). A similar estimate holds for the integral taken over the interval \((-\infty, -\psi_n)\). It follows that in (4.24) we can replace the integral \( \int_{\psi_n}^{\infty} \) by \( \int_{-\infty}^{\infty} \).

Since the polynomials \( S_j \) are odd for odd \( j \), we get that
\[
\int_{-\infty}^{\infty} \exp \left( \frac{\gamma^2}{4} \mathcal{R}(\frac{1}{A}) \right) S_j(\gamma) \, d\gamma = 0, \quad \text{for } \gamma \text{ odd} \tag{4.27}
\]
In view of (4.27) and (4.24) we can write (4.20) in the form:
\[
\sum_{\nu=0}^{\infty} |a_{n\nu}| = c_0 + c_1 n^{-1} + c_2 n^{-2} + \ldots + c_m n^{-m} + o(n^{-m}), \quad n \to \infty \tag{4.28}
\]
where,
\[
c_k = \frac{1}{2\sqrt{\pi|A|}} \int_{-\infty}^{\infty} S_{2k}(\gamma) \exp \left( \frac{\gamma^2}{4} \mathcal{R}(\frac{1}{A}) \right) \, d\gamma \tag{4.29}
\]
Introducing the notation:
\[
S_{2k}(\gamma) = \sum_{0 \leq j \leq 3k} T_{j,k} \gamma^{2j} \tag{4.30}
\]
we obtain that
\[
c_k = \sum_{0 \leq j \leq 3k} T_{j,k} h_j \tag{4.31}
\]
where
\[
h_j = \frac{1}{2\sqrt{\pi|A|}} \int_{-\infty}^{\infty} \gamma^{2j} \exp \left( \frac{\gamma^2}{4} \mathcal{R}(\frac{1}{A}) \right) \, d\gamma
\]
\[
= \frac{2^j |A|^{2j+\frac{1}{2}} (2j-1)!!}{(-\mathcal{R}A)^{j+\frac{1}{2}}} \tag{4.32}
\]
We are ready now to present a complete statement of the theorem in the typical case.

**MAIN THEOREM** (typical case)
Let \( f \) satisfy the conditions:

(i) \( f \) is analytic in the closed unit disk
(ii) \( f(1) = 1 \)

(iii) \( |f(z)| < 1 \) for \( |z| \leq 1, \quad z \neq 1 \)

(iv) \( \frac{d^2}{dz^2} |f(e^{it})| \neq 0 \) at \( t = 0 \)

Let \( (a_{n\nu}) \) be defined by

\[
f^n(z) = \sum_{\nu=0}^{\infty} a_{n\nu} z^\nu \tag{4.33}\]

Then, as \( n \to \infty \), there exists an asymptotic expansion of \( \sum_{\nu=0}^{\infty} |a_{n\nu}| \) in powers of \( \frac{1}{n} \):

\[
\sum_{\nu=0}^{\infty} |a_{n\nu}| \sim c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \ldots, \quad \text{as } n \to \infty \tag{4.34}
\]

The coefficient, \( c_m \) of this asymptotic expansion depends on parameters \( A, B_1, B_2, \ldots, B_{2m} \) where these parameters are defined by

\[
\log f(e^{it}) = i\alpha t + A t^2 + t^2 \sum_{m=1}^{\infty} B_m t^m \tag{4.35}
\]

An expression for \( c_m \) may be obtained in the following manner:

1. Polynomials \( P_j \) are defined by

\[
P_j(t) = \sum_{s_1!s_2! \ldots s_j!} \frac{1}{s_1!s_2! \ldots s_j!} B_1^{s_1} B_2^{s_2} \ldots B_j^{s_j} t^{s_1+s_2+\ldots+s_j+j} \tag{4.36}
\]

where the sum is taken over the set of all \( j \)-tuples \( s = (s_1, s_2, \ldots, s_j) \) such that \( s_1 + 2s_2 + \ldots + js_j = j \)

2. Polynomials \( Q_j \) are then defined by

\[
\mathcal{F} [e^{At^q} P_j(t)] (\gamma) = \frac{1}{2\sqrt{-AA}} Q_j(\gamma) e^{\gamma^2/4A} \tag{4.37}
\]

3. Next, the polynomials \( S_j \) are defined recursively:

\[
S_0 = 1 \tag{4.38}
\]

\[
S_j(\gamma) = -\frac{1}{2} [S_1 S_{j-1} + S_2 S_{j-2} + \ldots + S_{j-1} S_1] \\
+ \frac{1}{2} [Q_0 \overline{Q}_j + Q_1 \overline{Q}_{j-1} + \ldots + Q_j \overline{Q}_0] \tag{4.39}
\]

4. \( T_{j,k} \) is the coefficient of \( \gamma^{2j} \) in the polynomial \( S_{2k}(\gamma) \)

5. \( h_j = \frac{2j|A|^{2j+\frac{1}{2}} (2j-1)!!}{(-1)^{j+\frac{1}{2}}} \) \tag{4.40}

finally,

6. \( c_k = \sum_{0 \leq j \leq 3k} T_{j,k} h_j \) \tag{4.41}
4.3 The Coefficients $c_0$, $c_1$ and $c_2$

Applying the procedure described in the Main Theorem -typical case, we have computed the first three coefficients: $c_0$, $c_1$ and $c_2$. With $\theta$ denoting the argument of $A$ the results are

$$c_0 = (-\cos \theta)^{-\frac{1}{2}}$$

$$c_1 = \frac{6}{(-\cos \theta)^{1/2}}$$

$$\cdot \left[ -2R_2(\sec^2 \theta - 1) + J_2^2(-5\sec^3 \theta - 9\sec^2 \theta + 3\sec \theta + 11) \\
- R_1^2(5\sec^3 \theta - 9\sec^2 \theta - 3\sec \theta + 11) - 3R_1J_1 \sin \theta(5\sec^3 \theta - \sec \theta) \right]$$

and the expression for $c_2$, due to its length, is given in Appendix B. We need to point out that the expression for $c_0$ has been previously obtained by Girard [5,p.4]; see (1.18).

In the case when $A$ is real (in the typical case $A$ is real if and only if $f''(1)$ is real), we have $\cos \theta = -1$ and the expressions for the coefficients of the asymptotic expansion become much simpler:

$$c_0 = 1$$

$$c_1 = 24J_1^2$$

and

$$c_2 = \pi \left( 256076785452R_2^2J_2^2 - 203616R_1J_1J_2 + 7560R_2J_1^2 - 384J_2^2 \right)$$

4.4 Functions With Real Taylor Coefficients

**Corrollary (To The Main Theorem)**

If $f(z) = \sum a_\nu z^\nu$ satisfies all the conditions of the Main Theorem, in the typical case, and if all the coefficients $a_\nu$ are real, then

$$\sum_{\nu=0}^{\infty} |a_{n\nu}| = 1 + O(n^{-s}), \quad n \to \infty$$

for any real $s$.

**Proof**

It is sufficient to show that all the coefficients $c_j$ of the asymptotic expansion vanish. We shall compute those coefficients by the procedure described in the statement of the main theorem. We show that

(i) The coefficients $B_j$ are real for $j$ even and purely imaginary for $j$ odd. The coefficient $A$ is real.

(ii) The polynomials $P_j$ are real for $j$ even and purely imaginary for $j$ odd.
(iii) The polynomials \( Q_j \) are real for every \( j \).
(iv) \( S_j = Q_j \) for every \( j \).
(v) If \( j \geq 1 \), then \( S_j \) is a linear combination of \( H_s \left( \frac{x}{2\sqrt{-A}} \right) \), \( s \geq 1 \) where \( H_s \) is the Hermite polynomial of degree \( s \).
(vi) \( c_j = 0 \) for \( j \geq 1 \).

To show (i) we observe that, because the coefficients \( a_\nu \) are real, we have

\[
\overline{f(z)} = f(\overline{z})
\]

which implies that,

\[
\log f(e^{it}) = \log f(e^{-it})
\]

Because of (4.1) we obtain

\[
\frac{i\alpha t + \alpha t^2 + \sum B_j t^{j+2}}{-i\alpha t + \alpha t^2 + \sum (-1)^j t^{j+2}} = 1
\]

which implies that,

\[
\overline{\alpha} = \alpha, \quad \overline{B_j} = (-1)^j B_j
\]

and proves (i). We note that (i) can be phrased as \( B_j = C_j i^j \) where the \( C_j \)'s are real for every \( j = 1, 2, \ldots \). The last expression implies that

\[
B_1 i B_2 i \ldots B_j i = i^{s_1 + s_2 + \ldots + s_j} C_1^{s_1} C_2^{s_2} \ldots C_j^{s_j}
\]

so that we obtain from the expression (4.36) for \( P_j \) that \( P_j = i^j \) (a real polynomial) which proves (ii). Since \( P_j \) is also even/odd if \( j \) is even/odd, we obtain that:

for \( j \) odd, \( e^{i\alpha t} P_j(t) \) is odd and purely imaginary, so its Fourier transform is real, which implies that \( Q_j \) is real
for \( j \) even, \( e^{i\alpha t} P_j(t) \) is even and real, so we again obtain that \( Q_j \) is real.

That proves (iii), and we obtain then from (4.39), by an easy argument that (iv) holds. To prove (v) we observe that from the expression (4.36) for \( P_j \) it follows that the constant term in \( P_j \) is zero; by (4.5) and (4.7) we then obtain that \( Q_j(\gamma) \) is a linear combination of \( H_s \left( \frac{\gamma}{2\sqrt{-A}} \right) \), \( s \geq 1 \) where \( H_s \) are Hermite polynomials. Finally, because of the orthogonality of the Hermite polynomials, we get

\[
\int_{-\infty}^{\infty} H_s \left( \frac{\gamma}{2\sqrt{-A}} \right) e^{\gamma^2/4A} d\gamma = 0 \text{ for } s \geq 1
\]

which proves (vi).
CHAPTER V
SIX LEMMAS FOR THE EXCEPTIONAL CASE

We will now present six lemmas which are necessary (along with the previous lemmas) for the proof of the main theorem in the case $p = q > 2$.

**Lemma 7**
Let $F(t) = e^{\Re t}$ ($\Re A < 0$, $q \geq 2$ an even integer), then for any polynomial $P(t)$

$$F[P(t)F(t)](x) = o(e^{-k|x|}) \text{ as } |x| \to \infty \quad (5.1)$$

for any $k > 0$

**Proof**
Define the contour $C$ to be a rectangle with sides lying on the lines: $\Re z = 0$, $\Re z = k$, $\Re z = R$, and $\Re z = -R$, traversed in the positive direction.

Since $PF$ is entire, we have

$$\int_C e^{ixz} P(z) F(z) dz = 0 \quad (5.2)$$

$$\int_C e^{ixz} P(z) F(z) dz = \int_{-R}^R e^{ixt} P(t) F(t) dt + i \int_0^k e^{ix(R+iy)} P(R+iy) F'(R+iy) dy$$

$$+ i \int_k^0 e^{ix(-R+iy)} P(-R+iy) F(-R+iy) dy + \int_{-R}^R e^{ix(t+ik)} P(t+ik) F'(t+ik) dt \quad (5.3)$$

Now

$$\left| \int_0^k e^{ix(R+iy)} P(R+iy) F(R+iy) dy \right| \leq \int_0^k e^{-xy} |P(R+iy) F'(R+iy)| dy \quad (5.4)$$

If $\deg P(t) = \alpha$ then $|P(R+iy)| = O(R^\alpha)$ as $R \to \infty$ so (5.4) is:

$$\leq ke^{k|x|} \cdot O(R^\alpha) \cdot \max_{y \in [0,k]} |e^{A(R+iy)}| \text{ as } R \to \infty \quad (5.5)$$
But this is $o(1)$ as $R \to \infty$

Similarly,

$$\int_{k}^{0} e^{i\pi(-R+iy)} P(-R+iy) F(-R+iy) dy = o(1) \text{ as } R \to \infty$$  \hspace{1cm} (5.6)

Hence:

$$\mathcal{F}(FP) = \int_{-\infty}^{\infty} e^{ixt} F(t) P(t) dt = \int_{-\infty}^{\infty} e^{ix(t+ik)} F(t+ik) P(t+ik) dt$$

$$= e^{-kx} \int_{-\infty}^{\infty} e^{ix} F(t+ik) P(t+ik) dt$$  \hspace{1cm} (5.7)

Now $F(t+ik) P(t+ik) \in L_1(\mathbb{R}, t)$ so this expression is $o(e^{-kx})$ as $x \to \infty$. (By the Riemann-Lebesgue Lemma) Using the reflection of $C$ in the real line, we can show

$$\mathcal{F}(FP) = o(e^{-kx}) \text{ as } x \to -\infty, \text{ for } k < 0$$  \hspace{1cm} (5.8)

combining these results proves the lemma.

Our next lemma deals with the zeros of the Fourier transform of $e^{At}$ when $A$ has non-zero imaginary part. In the proof of this lemma we will use a theorem of Polya [13], which is that, for $A$ real

$$\mathcal{F}(e^{At}) (\gamma) = \int_{-\infty}^{\infty} e^{A\gamma t + i\gamma t} dt$$  \hspace{1cm} (5.9)

has infinitely many zeros, all of which are real.

**Lemma 8**

Let $F_0(t) = e^{At}$, $\Re A < 0$, $\Im A \neq 0$ and $q \geq 2$ be even. Then $\mathcal{F}(F_0)(\gamma)$ has no real zeros.

**Proof**

Let $B = -A$ so $B = be^{i\phi}$, $\phi \neq 0$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ then

$$B^\frac{1}{q} = b e^{i\frac{\phi}{q}}$$  \hspace{1cm} (5.10)

by the change of variables $x = B^\frac{1}{q} t$ we have

$$\mathcal{F}(F_0)(\gamma) = B^{-\frac{1}{q}} \int_{l} e^{-x^q} e^{i\gamma B^{-\frac{1}{q}} x} dx$$  \hspace{1cm} (5.11)
where \( I \) is the line \( \arg x = \frac{\pi}{q} \). Let \( C \) be the contour formed by taking the part of the lines \( \Re z = 0 \) and \( I \) in the right half-plane, inside the circle of radius \( R \), along with the arc of that circle between those lines. Consider this contour to be traversed in the positive direction. Then:

\[
\int_C e^{-z^q} e^{i\gamma B^{-\frac{1}{q}}x} dz = 0 \quad \text{by Cauchy’s Theorem} \tag{5.12}
\]

Now,

\[
\left| \int_{\mathcal{C} \cap \{|z|=R\}} e^{-z^q} e^{i\gamma B^{-\frac{1}{q}}x} dz \right| \leq \frac{\pi R}{2} e^{-R^q \cos \phi} e^{i\gamma B^{-\frac{1}{q}}|R|} \to 0 \quad \text{as} \ R \to \infty \tag{5.13}
\]

And similarly for the reflection of \( C \) in the origin, thus

\[
\mathcal{F}(F_0)(\gamma) = B^{-\frac{1}{q}} \int_I e^{-z^q} e^{i\gamma B^{-\frac{1}{q}}x} dx = -B^{-\frac{1}{q}} \int_{-\infty}^{\infty} e^{-z^q} e^{i\gamma B^{-\frac{1}{q}}x} dx \tag{5.14}
\]

But by Polya’s Theorem, this integral has infinitely many zeros all of which occur for \( \gamma B^{-\frac{1}{q}} \in \mathbb{R} \), and clearly \( \gamma = 0 \) is not a zero of \( \mathcal{F}(F_0)(\gamma) \). So for \( \Re A \neq 0 \), \( \mathcal{F}(F_0)(\gamma) \) has no real zeros.

**Lemma 9**

Let \( q > 2, s > 0, \Re A < 0 \) and \( \Re A \neq 0 \) \((q \in 2\mathbb{Z}, s \in \mathbb{Z} \text{ and } A \in \mathbb{C})\) Then the integral

\[
\int_{-\infty}^{\infty} e^{At^q} e^{int} dt \tag{5.15}
\]

may be written as a \( _2F_1 \) hypergeometric series:

\[
2(-A)^{-s+\frac{1}{q}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+s+1}{q}\right) \Gamma(n+1)}{\Gamma(2n+1) \Gamma(n+1)} (-w^2)^n \quad \text{for } s\text{-even} \tag{5.16}
\]

\[
2i(-A)^{-s+\frac{1}{q}} q^{-1} w \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+s+2}{q}\right) \Gamma(n+1)}{\Gamma(2n+2) \Gamma(n+1)} (-w^2)^n \quad \text{for } s\text{-odd} \tag{5.17}
\]

**Proof**

\[
\int_{-\infty}^{\infty} e^{At^q} e^{int} dt = (-A)^{-\frac{1}{q}} \int_{-\infty}^{\infty} e^{-u^q} e^{i\gamma u/(-A)^{\frac{1}{q}}} du \quad \text{for } s \text{-odd} \tag{5.18}
\]
\[ = (-A)^{-\frac{1}{q}} \int_{-\infty}^{\infty} e^{-u^q} e^{i\gamma u / (-A)^{\frac{1}{q}}} du \] (by Cauchy's Theorem)

\[ = 2(-A)^{-\frac{1}{q}} \int_{0}^{\infty} e^{-u^q} \cos \left( \frac{u \gamma}{(-A)^{\frac{1}{q}}} \right) du \]

\[ = 2(-A)^{-\frac{1}{q}} \int_{0}^{\infty} e^{-u^q} \cos(u \gamma) du; \text{ where } w = \gamma(-A)^{-\frac{1}{q}} \]

\[ = 2(-A)^{-\frac{1}{q}} q^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+1}{q})}{\Gamma(2n+1)} w^{2n} \] (5.18)

So,

\[ \int_{-\infty}^{\infty} e^{At^q} e^{i\gamma t^s} dt = i^{-s} \frac{d^s}{d\gamma^s} \left( \int_{-\infty}^{\infty} e^{At^q} e^{i\gamma t^s} dt \right) \]

\[ = i^{-s} \frac{d^s}{d\gamma^s} \left( 2(-A)^{-\frac{1}{q}} q^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+1}{q})}{\Gamma(2n+1)} \frac{((-A)^{-\frac{1}{q}} \gamma)^{2n}}{((-A)^{-\frac{1}{q}} \gamma)^{2n}} \right) \] (5.19)

We must now separate the cases \( s \)-even and \( s \)-odd. For \( s \)-even this is:

\[ i^{-s} \left( 2(-A)^{-\frac{1}{q}} q^{-1} \sum_{n=s/2}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+1}{q})}{\Gamma(2n+1)} \frac{((-A)^{-\frac{1}{q}} \gamma)^{2n-s}}{((-A)^{-\frac{1}{q}} \gamma)^{2n-s}} \right) \]

\[ = 2i^{-s}(-A)^{-\frac{s+1}{q}} q^{-1} \sum_{n=s/2}^{\infty} \frac{\Gamma(\frac{2n+1}{q})\Gamma(2n+1)}{\Gamma(2n+1)\Gamma(2n-s+1)} (-1)^n((-A)^{-\frac{1}{q}} \gamma)^{2n-s} \]

\[ = 2(-A)^{-\frac{s+1}{q}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+s+1}{q})\Gamma(n+1)}{\Gamma(n+1)\Gamma(n+s+1)} (-1)^n \]

\[ = 2(-A)^{-\frac{s+1}{q}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+s+1}{q})\Gamma(n+1)}{\Gamma(n+1)\Gamma(n+1)} (-w^2)^n \] (5.20)

which is the required \( {}_2F_1 \) Hypergeometric series in \( z = -w^2 \)

Let us now consider \( s \)-odd

\[ i^{-s} \frac{d^s}{d\gamma^s} \left( 2(-A)^{-\frac{1}{q}} q^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+1}{q})}{\Gamma(2n+1)} \frac{((-A)^{-\frac{1}{q}} \gamma)^{2n}}{((-A)^{-\frac{1}{q}} \gamma)^{2n}} \right) \]

\[ = 2i^{-s} \left( (-A)^{-\frac{1}{q}} q^{-1} \sum_{n=s+1}^{\infty} (-1)^n \frac{\Gamma(\frac{2n+1}{q})}{\Gamma(2n+1)} \frac{((-A)^{-\frac{1}{q}} \gamma)^{2n-s}}{((-A)^{-\frac{1}{q}} \gamma)^{2n-s}} \right) \]
\[= 2i^{-s}(-A)^{-\frac{s+1}{2}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+1}{q})\Gamma(2n+1)}{\Gamma(2n+1)\Gamma(2n-s+1)}(-1)^n(-A)^{-\frac{1}{2}}\gamma^{2n-s}\]
\[= 2i^{-s}(-A)^{-\frac{s+1}{2}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+s+2}{q})}{\Gamma(2n+2)}((-A)^{-\frac{1}{2}}\gamma)^{2n+1}(-1)^n+\frac{s+1}{2}\]
\[= 2i(-A)^{-\frac{s+1}{2}} q^{-1} w \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+s+2}{q})}{\Gamma(2n+2)\Gamma(n+1)}(-w^2)^n\]
\[= 2i(-A)^{-\frac{s+1}{2}} q^{-1} w \sum_{n=0}^{\infty} \frac{\Gamma(\frac{2n+s+2}{q})}{\Gamma(2n+2)\Gamma(n+1)}(-w^2)^n\] (5.21)

which again is the required \(2F_1\) Hypergeometric series in \(z = -w^2\)

Our next result is a theorem of E.M. Wright, on the asymptotic expansion of the generalized hypergeometric function:

**Lemma 10** (Wright[14,p.292])

Let the general \(2F_1\) hypergeometric function be given by:

\[2F_1(\alpha_1, \alpha_2, \beta_1, \beta_2, \mu, \rho; z) = \sum_{n=0}^{\infty} g(n) z^n\] (5.22)

where \(g(n) = \frac{\Gamma(\beta_1+n)\Gamma(\beta_2+n)}{\Gamma(n+1)}\) and define the constants:

\[k = 1 + \rho - \sum \alpha_r\] (5.23)

\[h + \left(\prod \alpha_r^{\sigma_r}\right) \left(\rho^{-\rho}\right)\] (5.24)

\[\sigma = \sum \beta_r - \mu - \frac{1}{2}\] (5.25)

and the asymptotic expression:

\[I(X) \sim A_X X^\sigma e^X\] as \(|X| \to \infty\) for some constant \(A_X\) (5.26)

then if \(g(t)\) has no poles, \(|\arg z| \leq \pi\) and \(|\arg(-z)| \leq \pi\), then the asymptotic expansion of \(2F_1(z)\) is given by:

\[I(z) + I(z_2)\] for \(1 < k < 2\) (5.27)

where \(z_{1,2} = k(h|z|)^{\frac{1}{2}} e^{i(n\pm\eta)/k}\) and \(\eta = \arg(-z)\)

**Lemma 11**
Let $q > 2$, $s_1, s_2 \geq 0$, $\Re A < 0$ and $\Im A \neq 0$ ($q \in 2\mathbb{Z}$, $s_1, s_2 \in \mathbb{Z}$, $A \in \mathbb{C}$). Then there exists a positive number $C$, such that:

$$\left| \frac{\int_{-\infty}^{\infty} e^{A t} e^{i n t^2} dt}{\int_{-\infty}^{\infty} e^{A t} e^{i n t^2} dt} \right| \sim C \gamma^{\frac{s_1-s_2}{q-1}} \text{ as } |\gamma| \to \infty \hspace{1cm} (5.28)$$

**Proof**

Recall from lemma 9, that each of the integerals above may be written as a hypergeometric series:

$$2(-A)^{-\frac{s+1}{q}} q^{-1} \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+1)}{\Gamma(2n+1)\Gamma(n+1)} (-w^2)^n \text{ for } s \text{-even} \hspace{1cm} (5.29)$$

$$2i(-A)^{-\frac{s+1}{q}} q^{-1} w \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)\Gamma(n+1)}{\Gamma(2n+2)\Gamma(n+1)} (-w^2)^n \text{ for } s \text{-odd} \hspace{1cm} (5.30)$$

Thus, we may apply lemma 10 to each of the integrals. If $s$ is even we have:

$$k = 1 + \rho - \sum \alpha_r = 1 + 2 - 1 - \frac{2}{q} = 2 - \frac{2}{q} \hspace{1cm} (5.31)$$

$$h = \left( \prod_{r=1}^{2} \alpha_r^{\sigma} \right) (\rho^{-\rho}) = \left( \frac{2}{q} \right)^{\frac{1}{q}} \cdot \frac{1}{4} \hspace{1cm} (5.32)$$

$$\sigma = \sum \beta_r - \mu - \frac{1}{2} = s + 1 - \frac{1}{2} \hspace{1cm} (5.33)$$

which gives (for $s$-even):

$$|z| = \left( \frac{2q-2}{q} \right) \left( \frac{1}{4} \left( \frac{2}{q} \right) \left| w^2 \right| \right)^{\frac{1}{2}} = \left( \frac{2q-2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{2}} \left| w \right|^{\frac{2}{q-1}} \hspace{1cm} (5.34)$$

and $\eta = -\frac{2}{q} \text{arg}(-\rho)$ (Note: $0 < |\eta| < \frac{\pi}{4}$). Thus

$$I(z_1) \sim A_s \left[ \left( \frac{2q-2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| w \right|^{\frac{2}{q-1}} \right]^{\frac{2q+2-q}{2q}} \cdot e^{i(\eta+\pi)\left(\frac{2q+2-q}{2q-1}\right)}$$

$$\cdot \exp \left[ \left( \frac{2q-2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| w \right|^{\frac{2}{q-1}} e^{i(\eta+\pi)\left(\frac{q}{q-2}\right)} \right] \hspace{1cm} (5.35)$$

$$I(z_2) \sim A_s \left[ \left( \frac{2q-2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| w \right|^{\frac{2}{q-1}} \right]^{\frac{2q+2-q}{2q}} \cdot e^{i(\eta-\pi)\left(\frac{2q+2-q}{2q-1}\right)}.$$
\[ I(z_1) + I(z_2) \sim A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right) \left| \frac{w}{2} \right| \right] e^{i \frac{s}{2q} \cos \left( \frac{\pi q}{2q - 2} \right)} \]

\[ \cdot \exp \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right) \left| \frac{w}{2} \right| \right] e^{i \frac{s}{2q} \cos \left( \frac{\pi q}{2q - 2} \right)} \]

\[ \cdot 2 \cos \left[ \left( \frac{2s + 2 - q}{4(q - 1)} \right) \cos \left( \frac{\pi q}{2q - 2} \right) \right] \] (5.37)

So,

\[ \int_{-\infty}^{\infty} e^{-At^s} e^{i\tau t^s} dt \]

\[ \sim 2(-A)^{-\frac{s+1}{2}} q^{-1} A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right) \left| \frac{w}{2} \right| \right] e^{i \frac{s}{2q} \cos \left( \frac{\pi q}{2q - 2} \right)} \cdot 2 \]

as \(|\gamma| \to \infty\) for \(s\)-even. Which we may write as

\[ B_1 |\gamma|^{\frac{2q+2-s}{2(q-1)}} \cdot \exp \left[ B_2 |\gamma|^{\frac{s}{2q}} \right] \cdot \cos \left[ B_3 + B_4 |\gamma|^{\frac{s}{2q}} \right] \] (5.39)

Where \(B_1 = B_1(A, s, q), B_2 = B_2(A, q), B_3 = B_3(s, q)\) and \(B_4 = B_4(A, q)\).

Note that \(B_1 \neq 0\); if \(A \notin \mathbb{R}\) then \(B_4 \notin \mathbb{R}\) (But \(B_3 \in \mathbb{R}\) so \(B_3 + B_4 |\gamma|^{\frac{s}{2q}}\) has no zeros for \(\gamma \in \mathbb{R} \setminus \{0\}\).

On the other hand, if \(s\) is odd, the constants are:

\[ k = 1 + \rho - \sum \alpha_r = 1 + 2 - 1 - \frac{2}{q} = 2 - \frac{2}{q} \]

\[ h = \left( \prod_{r=1}^{2} \alpha_r^{\alpha_r} \right) \left( \rho^{-\rho} \right) = \left( \frac{2}{q} \right)^{\frac{1}{4}} \cdot \frac{1}{4} \]

\[ \sigma = \sum \beta_r - \mu = \frac{1}{2} + \frac{s+2}{q} + 1 - \frac{1}{2} = \frac{s+2}{q} - \frac{3}{2} \] (5.40)
Which gives (for $s$-odd):

\[ |z| = \left( \frac{2q - 2}{q} \right) \left( \frac{1}{4} \left( \frac{2}{q} \right) \frac{q}{2} \right)^{\frac{s}{2(q+1)}} = \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{2(q+1)}} \left| \frac{w}{2} \right|^{\frac{q-1}{2(q+1)}} \]  

(5.43)

and $\eta = -\frac{2}{q} \arg(-A)$ (Note: $0 < |\eta| < \pi$). Thus

\[ I(z_1) \sim A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} \right]^{\frac{2s+4-3q}{2q}} e^{i\eta} e^{i(\eta+\pi)(\frac{2s+4-3q}{4(q-1)})} \]

\[ \exp \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i(\eta+\pi)(\frac{q-1}{2q-2})} \right] \]  

(5.44)

\[ I(z_2) \sim A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} \right]^{\frac{2s+4-3q}{2q}} e^{i(\eta+\pi)(\frac{2s+4-3q}{4(q-1)})} \]

\[ \cdot \exp \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i(\eta+\pi)(\frac{q-1}{2q-2})} \right] \]  

(5.45)

\[ I(z_1) + I(z_2) \sim A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} \right]^{\frac{2s+4-3q}{2q}} e^{i\eta} e^{i\pi(\frac{2s+4-3q}{4(q-1)})} \]

\[ \cdot \exp \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i(\frac{q-1}{2q-2})} \cos \left( \frac{\pi q}{2q-2} \right) \right] \cdot 2 \cos \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i\eta} \right] \]  

(5.46)

So,

\[ \int_{-\infty}^{\infty} e^{-At^2} e^{i\pi t^2} dt \]

\[ \sim 2i(A)^{-\frac{s+4}{q}} q^{-1} w A_s \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} \right]^{\frac{2s+4-3q}{2q}} e^{i\eta} \]

\[ \cdot \exp \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i(\frac{q-1}{2q-2})} \cos \left( \frac{\pi q}{2q-2} \right) \right] \cdot 2 \cos \left[ \left( \frac{2q - 2}{q} \right) \left( \frac{2}{q} \right)^{\frac{1}{q-1}} \left| \frac{w}{2} \right|^{\frac{q-1}{q}} e^{i\eta} \right] \]  

(5.47)
as $|\gamma| \to \infty$ for $s$-odd. Which we may write as

$$B_1|\gamma|^{\frac{2s+2-2\epsilon}{2(\epsilon-1)}} \cdot \exp \left[ B_2|\gamma|^{\frac{s}{\epsilon-1}} \right] \cdot \cos \left[ B_3 + B_4|\gamma|^{\frac{s}{\epsilon-1}} \right]$$

(5.48)

Where $B_1 = B_1(A,s,q)$, $B_2 = B_2(A,q)$, $B_3 = B_3(s,q)$ and $B_4 = B_4(A,q)$.

Note that the constants $B_2$ and $B_4$ (which do not depend on $s$) are the same as those for $s$-even (for given $A$ and $q$). The constants $B_1$ and $B_3$ which depend on $s$, have different formulas for $s$-even and $s$-odd. (But again $B_1 \neq 0$ and $\cos \left[ B_3 + B_4|\gamma|^{\frac{s}{\epsilon-1}} \right] \neq 0$ for $A \notin \mathbb{R}, \ \gamma \neq 0$)

Now let $A,q$ be fixed, and $s_1 \neq s_2$ ($A \notin \mathbb{R}$). Since $\cos \left[ B_3 + B_4|\gamma|^{\frac{s}{\epsilon-1}} \right] \neq 0$ for any $s$ with $\gamma \neq 0$ we may write (with $|\gamma|^{\frac{s}{\epsilon-1}} = x$)

$$\frac{\cos \left[ B_3(s_1) + B_4|\gamma|^{\frac{s}{\epsilon-1}} \right]}{\cos \left[ B_3(s_2) + B_4|\gamma|^{\frac{s}{\epsilon-1}} \right]} = \frac{e^{i(B_3(s_1)+\Re B_4x)-x\Im B_4} + e^{-i(B_3(s_1)+\Re B_4x)+x\Im B_4}}{e^{i(B_3(s_2)+\Re B_4x)-x\Im B_4} + e^{-i(B_3(s_2)+\Re B_4x)+x\Im B_4}}$$

(5.49)

which tends to

$$e^{i(B_3(s_2)-B_3(s_1))}, \text{ if } x\Im B_4 \to \infty$$

(5.50)

$$e^{i(B_3(s_1)-B_3(s_2))}, \text{ if } x\Im B_4 \to -\infty$$

(5.51)

Since $B_3(s)$ is real for any $s$, these limits have modulus 1. Hence there exists a constant $C = C(A,q,s_1,s_2)$ (for $A \notin \mathbb{R}$) such that

$$\left| \int_{-\infty}^{\infty} e^{A t} e^{i\gamma t s_1} dt \right| \sim C|\gamma|^{\frac{s_1-s_2}{\epsilon-1}} \text{ as } |\gamma| \to \infty$$

(5.52)

**Lemma 12**

Let $F = e^{At}$ and $P(t)$ be a polynomial. Define:

$$\theta(x) = \arg \left[ F(P(t)) \right]$$

(5.53)

then,

$$\frac{d^j \theta(x)}{dx^j} = O(x^{j-\frac{1}{2}}) \text{ as } x \to \infty; \text{ for } j = 1,2,\ldots$$

(5.54)

**Proof**

Define

$$J_n(x) = \int_{-\infty}^{\infty} e^{ixt} e^{At^2} t^n dt$$

(5.55)

clearly, $iJ_{n+1}(x) = J_n(x)$. So if $P(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0$, then

$$\mathcal{F}(P(t)) = \int_{-\infty}^{\infty} e^{ixt} (a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0) e^{At^2} dt$$
\[ \begin{aligned}
&= a_n J_n(x) + a_{n-1} J_{n-1}(x) + \ldots + a_0 J_0(x) = r(x)e^{i\theta(x)}
\end{aligned} \tag{5.56} \]

Now
\[ \frac{d}{dx} \mathcal{F}(PF)(x) = \frac{i(a_n J_{n+1}(x) + a_{n-1} J_n(x) + \ldots + a_0 J_1(x))}{a_n J_n(x) + a_{n-1} J_{n-1}(x) + \ldots + a_0 J_0(x)} \]
\[ = \frac{r'(x)e^{i\theta(x)} + i r(x)e^{i\theta(x)} \theta'(x)}{r(x)e^{i\theta(x)}} = \frac{r'(x)}{r(x)} + i \theta'(x) \tag{5.57} \]

by lemma 11 the expression (5.57) is \( \sim Cx^{\frac{1}{4}-1} \) (for some constant \( C \)). Since there can be no cancellation between real and imaginary parts, we have that \( \frac{r'(x)}{r(x)} \) and \( \theta'(x) \) are both \( O(x^{\frac{1}{4}-1}) \) as \( x \to \infty \). If we now assume that \( \frac{r^{(n)}(x)}{r(x)} \) and \( \theta^{(n)}(x) \) are \( O(x^{\frac{n}{4}-1}) \) as \( x \to \infty \) for \( n < j \) then
\[ \frac{d^2}{dx^2} \mathcal{F}(PF)(x) = \frac{i(a_n J_{n+j}(x) + a_{n-1} J_{n+j-1}(x) + \ldots + a_0 J_j(x))}{a_n J_n(x) + a_{n-1} J_{n-1}(x) + \ldots + A_0 J_0(x)} \]
\[ = \frac{r^{(j)}(x)e^{i\theta(x)} + i r(x)e^{i\theta(x)} \theta^{(j)}(x)}{r(x)e^{i\theta(x)}} + o(x^{\frac{j}{4}-1}), \text{ as } x \to \infty \tag{5.58} \]

(again by lemma 11) this is \( \sim Cx^{\frac{j}{4}-1} \) which gives the desired result.
CHAPTER VI
THE EXCEPTIONAL CASE

In this chapter, we consider those functions \( f \in F \) with \( p = q \) and the properties:

\[
\frac{d^2}{dt^2} |f(e^{it})| = 0 \text{ at } t = 0 \tag{6.1}
\]

\[
\Im A \neq 0, \quad \Re A \neq 0 \tag{6.2}
\]

Again we note that for \( f \) to be in the class \( F \), it must satisfy the conditions:

(i) \( f \) analytic on the closed unit disk

(ii) \( f(1) = 1 \)

(iii) \( |f(z)| < 1 \) for \( |z| \leq 1, z \neq 1 \)

The equation (6.2) shows that there exists an even integer \( q > 2 \) defined by the expansion

\[
\log f(e^{it}) = i\alpha t + At^q + t^q \sum_{j=0}^{\infty} B_j t^j \tag{6.3}
\]

Furthermore, the above conditions imply that the series in (6.3) converges in a neighborhood of \( t = 0 \). Our purpose is to show that there exists an asymptotic series for \( \sum |a_{nv}| \) as \( n \to \infty \) of the form:

\[
\sum_{\nu} |a_{nv}| \sim c_0 + \frac{c_1}{n^{2/q}} + \frac{c_2}{n^{4/q}} + \ldots \text{ as } n \to \infty \tag{6.4}
\]

6.1 Definition of \( Q_j \) and \( S_j \)

As in the typical case, we use theorem 1 to obtain an asymptotic series for the coefficients \( a_{nv} \), which holds uniformly in \( \nu \),

\[
a_{nv} \sim \sum_{j=0}^{\infty} n^{-\frac{j+1}{q}} \mathcal{F} \left( e^{At^q} P_j(t) \right) \left( \gamma_{nv} \right) \tag{6.5}
\]

where \( \gamma_{nv} = (n\alpha - \nu)n^{-1/q} \). (Recall that \( P_j \) is a polynomial of degree \((q + 1)j\)
We define the expressions $Q_j$ to be:

$$Q_j(\gamma) = \frac{\mathcal{F}\left(e^{A\theta P_j(t)}\right)(\gamma)}{\mathcal{F}(e^{A\theta})(\gamma)}$$

(6.6)

Note that here $Q_j(\gamma)$ is not a polynomial (as in the typical case) but is well-defined, since lemma 8 guarantees that

$$\mathcal{F}(e^{A\theta})(\gamma) \neq 0$$

(6.7)

for any real $\gamma$. Thus, by proposition 1, we have:

$$\sum_{\nu=0}^{\infty} |a_{\nu}| = \sum_{\nu \in \mathbb{Z}_n} n^{-1/q} \left| \sum_{j=0}^{k} n^{-\frac{j}{q}} \mathcal{F}\left(e^{A\theta P_j(t)}\right)(\gamma_{\nu}) \right| + o(n^{-\frac{k}{q}})$$

$$= \sum_{\nu \in \mathbb{Z}_n} n^{-1/q} |Q_0(\gamma_{\nu})| \left[ 1 + n^{-\frac{1}{q}} Q_1(\gamma_{\nu}) + n^{-\frac{2}{q}} Q_2(\gamma_{\nu}) + \ldots ight]$$

$$\ldots n^{-\frac{k}{q}} Q_k(\gamma_{\nu}) + o(n^{-k/q})$$

(6.8)

Next, we define $S_j(x)$ by:

$$S_0(x) = 1$$

$$S_j(x) = -\frac{1}{2} [S_1(x)S_{j-1}(x) + S_2(x)S_{j-2}(x) + \ldots + S_{j-1}(x)S_1(x)]$$

$$+ \frac{1}{2} \left[ Q_0(x)\overline{Q}_j(x) + Q_1(x)\overline{Q}_{j-1}(x) + \ldots + Q_j(x)\overline{Q}_0(x) \right]$$

(6.9)

Again, unlike the typical case, $S_j(x)$ is not a polynomial, however, lemma 11 shows that

$$|Q_j(\gamma)| \sim C|\gamma|^{\frac{j+1}{q-1}}$$

(6.11)

for some constant $C$, and thus we have same property for $S_j(x)$:

$$|S_j(x)| \sim C|x|^{\frac{j+1}{q-1}} \text{ as } |x| \rightarrow \infty$$

(6.12)

Using the notation of section 3.3 with $x = n^{-1/q}$ and $q_j = Q_j(\gamma_{\nu})$ we obtain

$$\tilde{q}_j = S_j(\gamma_{\nu})$$

(6.13)

and

$$\left| 1 + n^{-1/q} Q_1(\gamma_{\nu}) + \ldots + n^{-k/q} Q_k(\gamma_{\nu}) \right| = \left| 1 + q_1 x + \ldots + q_k x^k \right|$$

(6.14)

To apply lemma 6 we use (6.11) and

$$|\gamma_{\nu}| = |(\alpha - \nu) n^{-1/q}| \leq \log^2 n \text{ for } \nu \in \mathbb{Z}_n$$

(6.15)
to obtain:

\[ |Q_j(Tt)\| < M\log^2 n \leq M \log^{2(k+1)} n \] (6.16)

where \( M \) depends only on \( A, B_1, B_2, \ldots, B_j \). Thus, taking \( L = M \log^{(k+1)} n \) in lemma 6 the condition \( \|x\| \leq A_k L^{-4k} \) becomes \( \|x\| \leq A_k \left( M \log^{(k+1)} n \right)^{-4k} \) which is satisfied for all sufficiently large \( n \). So, we have, uniformly for \( \nu \in \mathbb{Z}_n \):

\[
1 + n^{-1/q} Q_1(\gamma_{\nu}) + n^{-2/q} Q_2(\gamma_{\nu}) + \ldots + n^{-k/q} Q_k(\gamma_{\nu})
\]

\[
= 1 + n^{-1/q} S_1(\gamma_{\nu}) + n^{-2/q} S_2(\gamma_{\nu}) + \ldots + n^{-k/q} S_k(\gamma_{\nu}) + O(\log^m n \cdot n^{-\frac{k+1}{q}}) \quad (6.17)
\]

Substituting this expression in (6.8) and observing that:

\[
\sum_{\nu \in \mathbb{Z}_n} n^{-1/q} |Q_0(\gamma_{\nu})| \cdot O(\log^m n \cdot n^{-\frac{k+1}{q}}) \quad n \to \infty
\]

\[
\leq C \text{card}(\mathbb{Z}_n) n^{-1/q} \cdot \log^m n \cdot n^{-\frac{k+1}{q}}
\]

\[
\leq C \log^{m+2} n \cdot n^{-\frac{k+1}{q}} = o(n^{-\frac{k}{q}}) \quad \text{as} \quad n \to \infty
\] (6.18)

we obtain

\[
\sum_{\nu=0}^{\infty} |a_{\nu}| = \sum_{\nu \in \mathbb{Z}_n} n^{-1/q} |Q_0(\gamma_{\nu})| \left[ 1 + n^{-1/q} S_1(\gamma_{\nu}) + \ldots \right.
\]

\[
\left. \ldots + n^{-k/q} S_k(\gamma_{\nu}) \right] + o(n^{-k/q}) \quad n \to \infty
\] (6.19)

We now use the notation of section 3.2: \( RS[F; a, b; w] \) is the Riemann sum of \( F \) on the partition of \([a, b]\) into subintervals of length \( w \). Now, \( \gamma_{\nu+1} - \gamma_{\nu} = n^{-1/q} \) for any \( \nu \) and \( n \), so the points \( \gamma_{\nu}, \nu \in \mathbb{Z}_n \) partition \([a, b]\). (where \( n \sim \log^2 n \))

Thus,

\[
\sum_{\nu \in \mathbb{Z}_n} n^{-1/q} |Q_0(\gamma_{\nu})| S_j(\gamma_{\nu}) = RS[F; -\psi_n, \psi_n; n^{-1/q}]
\] (6.20)

where \( F(\gamma) = |Q_0(\gamma)| S_j(\gamma) \).

We wish to use the corollary to lemma 4, to estimate the difference between this Riemann sum and the corresponding integral. In order to do so we must first determine the behavior of the derivatives of \( F(\gamma) \). By the recursive definition of \( S_j(\gamma) \) we have:

\[
S_j(\gamma) = \sum_{\zeta(j)} c(j_1, j_2, \ldots, j_r) R_1^{j_1}(\gamma) R_2^{j_2}(\gamma) \ldots R_r^{j_r}(\gamma)
\] (6.21)

where

\[
R_j(\gamma) = \sum_{i=0}^{j} |Q_i(\gamma)||Q_{j-i}(\gamma)| \cos[\theta_i(\gamma) - \theta_{j-i}(\gamma)]
\] (6.22)
\[ \theta_j(\gamma) = \arg \left[ Q_j(\gamma)/Q_0(\gamma) \right] \] (6.23)

Consider the derivatives:

\[ \frac{d^m}{dx^m} \left( |Q_l(x)||Q_{j-l}(x)| \cos [\theta_l(x) - \theta_{j-l}(x)] \right) \]

\[ = \sum_{\alpha_1 + \ldots + \alpha_7 = m} c(\alpha_1, \ldots, \alpha_7) \left( \frac{d^{\alpha_1}}{dx^{\alpha_1}} Q_0(x) Q_l(x) \right) \left( \frac{d^{\alpha_2}}{dx^{\alpha_2}} Q_0(x) Q_{j-l}(x) \right) \]

\[ \left( \frac{d^{\alpha_3}}{dx^{\alpha_3}} Q_0^{-2}(x) \right) \left( \frac{d^{\alpha_4}}{dx^{\alpha_4}} e^{-i\theta_l(x)} \right) \left( \frac{d^{\alpha_5}}{dx^{\alpha_5}} e^{-i\theta_{j-l}(x)} \right) \]

\[ \left( \frac{d^{\alpha_6}}{dx^{\alpha_6}} e^{2i\theta_0(x)} \right) \left( \frac{d^{\alpha_7}}{dx^{\alpha_7}} \cos [\theta_l(x) - \theta_{j-l}(x)] \right) \] (6.24)

Now,

\[ \frac{d^{\alpha_1}}{dx^{\alpha_1}} [Q_0(x) Q_l(x)] = \frac{d^{\alpha_1}}{dx^{\alpha_1}} \int_{-\infty}^{\infty} P_l(t) e^{At^q} e^{ixt} dt \]

\[ = i^{\alpha_1} \int_{-\infty}^{\infty} t^{\alpha_1} P_l(t) e^{At^q} e^{ixt} dt \]

\[ \sim C \int_{-\infty}^{\infty} t^{\alpha_1+(q+1)t} e^{At^q} e^{ixt} dt \sim C x^{\alpha_1+(q+1)t} Q_0(x) \text{ as } |x| \to \infty \] (6.25)

Similarly, we have the estimates,

\[ \frac{d^{\alpha_2}}{dx^{\alpha_2}} [Q_0(x) Q_{j-l}(x)] \sim C x^{\alpha_2+(q+1)(j-l)} Q_0(x) \text{ as } |x| \to \infty \] (6.26)

\[ \frac{d}{dx} [Q_0^{-2}(x)] = -2[Q_0(x)]^{-3} Q_0'(x) \sim C x^{\frac{1}{q-1}} Q_0^2(x) \text{ as } |x| \to \infty \] (6.27)

\[ \frac{d^2}{dx^2} [Q_0^{-2}(x)] = 6[Q_0(x)]^{-4} [Q_0'(x)]^2 - 2[Q_0(x)]^{-3} Q_0''(x) \]

\[ \sim C x^{\frac{2}{q-1}} Q_0^2(x) \text{ as } |x| \to \infty \] (6.28)

and so forth so in general:

\[ \frac{d^{\alpha_3}}{dx^{\alpha_3}} [Q_0^{-2}(x)] \sim C x^{\frac{\alpha_3}{q-1}} Q_0^2(x) \text{ as } |x| \to \infty \] (6.29)

By lemma 12:

\[ \frac{d^{\alpha_4}}{dx^{\alpha_4}} [e^{-i\theta_l(x)}] = O(x^{\frac{\alpha_4}{q-1}}) \text{ as } |x| \to \infty \] (6.30)
and similarly,
\[
\frac{d^{n_0}}{dx^{n_0}}[e^{-i\theta_j(x)}] = O(x^{\frac{n_0}{q-1}}) \text{ as } |x| \to \infty
\]  \hfill (6.31)
\[
\frac{d^{n_r}}{dx^{n_r}}[\cos(\theta_i(x) - \theta_{j-1}(x))] = O(x^{\frac{n_r}{q-1}}) \text{ as } |x| \to \infty
\]  \hfill (6.32)

Together, these seven estimates imply that:
\[
\frac{d^m}{dx^m} R_j(x) = O \left( x^{\frac{(q+1)+m}{q-1}} \right) \text{ as } |x| \to \infty
\]  \hfill (6.33)

Now the \(m-1\) derivative of \(S_j(x)\) is a sum of terms of the form:
\[
g(x) = c \prod_{k=0}^{m-1} \prod_{l=1}^r \left[ R_j^k(x) \right]^{\alpha_{k,l}}
\]  \hfill (6.34)

The order of such an expression will be \(O(x^{\alpha_{m-1}})\) as \(|x| \to \infty\) where
\[
\alpha_{m-1} = \sum_{k,l} \frac{\alpha_{k,l}[l(q+1)+k]}{q-1}
\]  \hfill (6.35)

We wish to show that for the \(m\)th derivative of \(S_j(x)\), \(\alpha_m = \frac{j(q+1)+m}{q-1}\). For \(m = 0\) this is clear by the definition of \(S_j(x)\). Assume it is also true for all derivatives upto the \(m-1\) derivative. Then when (6.34) is differentiated, all \(\alpha_{k,l}\)'s will remain the same except for two in each term (i.e. there will be particular values of \(l\) and \(k\) such that \(\alpha_{k,l}\) will decrease by 1, and \(\alpha_{k+1,l}\) will increase by 1). Now,
\[
\frac{(\alpha_{k,l} - 1)[l(q_1 + k)]}{q-1} + \frac{(\alpha_{k+1,l} + 1)[l(q_1 + k + 1)]}{q-1}
= \frac{\alpha_{k,l}[l(q+1)+k]}{q-1} + \frac{\alpha_{k+1,l}[l(q+1)+k+1]}{q-1} + \frac{1}{q-1}
\]  \hfill (6.36)

so \(\alpha_m = \alpha_{m-1} + \frac{1}{q-1}\) which gives:
\[
\frac{d^m}{dx^m} S_j(x) = O \left( x^{\frac{j(q+1)+m}{q-1}} \right) \text{ as } |x| \to \infty
\]  \hfill (6.37)

combining this result with the fact that:
\[
\frac{d^m}{dx^m} |Q_0(x)| \sim C x^{\frac{m}{q-1}} Q_0(x)
\]  \hfill (6.38)

gives the behavior of the derivatives of \(F\):
\[
F^{(m)}(\gamma) = O \left( x^{\frac{j(q+1)+m}{q-1}} Q_0(x) \right) \quad |x| \to \infty
\]  \hfill (6.39)
But, lemma 7 shows that

\[ Q_0(x) = o(e^{-k|x|}) \quad |x| \to \infty \]  

(6.40)

for any real \( k > 0 \). Thus,

\[ F^{(m)}(\gamma) = O(e^{-|\gamma|}) \quad |\gamma| \to \infty \]  

(6.41)

which implies that

\[ F^{(m)}(\pm \psi_n) = O(e^{-\log^2 n}) \quad n \to \infty \]  

(6.42)

and thus \( F^{(m)}(\pm \psi_n) \) tends to zero faster than any power of \( e_n = n^{-1/q} \). So the condition of the corollary to lemma 4 is satisfied and we have:

\[ \sum_{\nu \in \mathbb{Z}_n} n^{-1/q} |Q_0(\gamma_{n\nu})|S_j(\gamma_{n\nu}) - \int_{-\psi_n}^{\psi_n} |Q_0(\gamma)|S_j(\gamma)d\gamma = o(n^{-s}) \quad n \to \infty \]  

(6.43)

for any \( s > 0 \). Also by lemma 7:

\[ \left| \int_{-\psi_n}^{\psi_n} |Q_0(\gamma)|S_j(\gamma)d\gamma \right| \leq k \int_{-\psi_n}^{\psi_n} e^{-\gamma}d\gamma \]

\[ = ke^{-\psi_n} = o(n^{-s}) \quad n \to \infty \]  

(6.44)

and similarly for \( -\psi_n \). Thus we may replace the integral over \((-\psi_n, \psi_n)\) with an integral over \((-\infty, \infty)\). Finally, the definition of \( S_j(x) \) shows that \( S_j \) is odd for odd \( j \), so

\[ \int_{-\infty}^{\infty} |Q_0(x)|S_j(x)dx = 0 \]  

(6.45)

for odd \( j \), since \( |Q_0(x)| \) is an even function. And thus,

\[ \sum_{\nu=0}^{\infty} |a_{n\nu}| = c_0 + \frac{c_1}{n^{1/q}} + \frac{c_2}{n^{2/q}} + \ldots + \frac{c_m}{n^{m/q}} + o(n^{-m/q}) \text{ as } n \to \infty \]  

(6.46)

where:

\[ c_j = \int_{-\infty}^{\infty} |Q_0(x)|S_{2j}(x)dx \]  

(6.47)
APPENDIX A

THE FORM OF THE COEFFICIENTS

Although the method stated in the previous section of this chapter does not lead directly to a general formula for the coefficients of the series, we may use this method to develop certain results about the form of these coefficients, in the typical case. First we will define an indexed set of products of real and imaginary parts of the coefficients of log $f$. Let

$$T = \{R_1, R_2, J_2, R_3, J_3, \ldots\} \text{ and}$$

$$T_n = \{\text{products of elements of } S \text{ such that the sum of the indices is } n\}$$

where $R_j = \Re D_j$ and $J_j = \Im D_j$. Next we will define some integer-valued functions on $T_n$, which we will need to express a simple form for the coefficients.

$$u = u(\sigma) = \text{the number of factors of } \sigma \in T_n$$

$$v = v(\sigma) = \text{the number of } J^* \text{ factors of } \sigma \in T_n$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

Each coefficient, $c_j$, will have the following form:

**Claim:**

$$c_j = \frac{\pi}{(- \cos \theta)^{1/2}} \sum_{\sigma \in S_j} \sigma(\sin \theta)^{\delta(v)} P_v(\sec \theta)$$

where $P_v(\sec \theta)$ is a polynomial of degree at most $j + u$.

**Proof**

In the proof of the main theorem we introduced polynomials $P_j, Q_j$ and $S_j$. In order to prove the claim we now introduce an intermediate set of polynomials $R_j$ defined by the expansion:

$$R_j(x) = \sum_{k=0}^{j} [\Re Q_k(x) \Re Q_{j-k}(x) + \Im Q_k(x) \Im Q_{j-k}(x)]$$
for each $1 \leq j \leq 2k$. Which gives:

$$S_j(x) = \frac{1}{2} R_j(x) - \sum_{k=1}^{j-1} (j-1)!(j-k)S_k(x)S_{j-k}(x) \quad (A.8)$$

Now

$$P_j(t) = B_j t^{j+2} + \sum_{l=1}^{j-1} \frac{1}{(l+1)!} \sum_{\sigma=i_1+i_2+\ldots+i_l, i_l \geq 1} \left( \prod_{m=1}^{l} B_{i_m} \right) B_{j-\sigma} t^{j+2l+2}$$

$$\equiv \sum_{l=0}^{j-1} g(l, j) t^{j+2l+2} \quad (A.9)$$

and so we have that

$$Q_j(x) = \sum_{l=0}^{j-1} \frac{(-1)^j}{(l+1)!} G(l, j) H_{j+2l+2} \left( \frac{ix}{2A^{1/2}} \right) \quad (A.10)$$

Where $G(l, j)$ is the same expression as $g(l, j)$ with each $B_j$ replaced by the corresponding $D_j$. Thus, $R_j$ is a sum of eight expressions, the first of which is:

$$(-1)^j \sum_{k=0}^{j-1} \sum_{l=0}^{k-1} \sum_{l' = 0}^{j-k-1} \sum_{l'' = 0}^{l} \frac{R_{H_{k+2l+2}} \cdot R_{H_{j-k+2l'+2}} \cdot R_{G(l, k)} \cdot R_{G(l', j-k)}}{(l+1)!(l'+1)!} \quad (A.11)$$

The remaining seven expressions are similar to the first with some of the real parts replaced by imaginary parts (each combination with an even number of real parts, i.e. 4,2 or 0, will appear) Now since,

$$\left( \frac{ix}{A^{1/2}} \right)^{n-2k} = r^{n-2k} e^{i(\theta+\pi)/2} x^{n-2k} = r^{n-2k} e^{i(\alpha-2\theta)(\theta+\pi)/2} \quad (A.12)$$

we have:

$$\prod_{\substack{j=0 \ldots k-1 \ldots j-k-1 \ldots l \ldots l'=0 \ldots l'' \ldots l'}} \frac{1}{(\alpha+k+2l+2)!} x^{k+2l+2-k}$$

$$\cdot \cos \left[ \frac{k}{2} + l + 1 - \alpha \right](\theta + \pi)$$

$$\prod_{\substack{j=0 \ldots k-1 \ldots j-k-1 \ldots l \ldots l'=0 \ldots l'' \ldots l'}} \frac{1}{(\alpha+k+2l+2)!} x^{k+2l+2-k}$$

$$\cdot \cos \left[ \frac{j-k}{2} + l' + 1 - \alpha' \right](\theta + \pi) \quad (A.13)$$
Now
\[ \cos \left[ \left( \frac{n}{2} - k \right)(\theta + \pi) \right] = \begin{cases} \pm \cos\left( \frac{n}{2} - k \right)\theta & ; \ n \text{-even} \\ \pm \sin\left( \frac{n}{2} - k \right)\theta & ; \ n \text{-odd} \end{cases} \] (A.14)

Applying this to the above formula gives:
\[
\sum_{\alpha=0}^{j/2} \sum_{\alpha'=0}^{l+1} \frac{\pm (k + 2l + 2)(j - k + 2l' + 2)!r^{-\frac{1}{2}(n+1)}(j+2l+2l'+4-2\alpha-2\alpha')}{\alpha!\alpha'!(j - k + 2l' + 2 - 2\alpha)!}(k + 2l + 2 - 2\alpha)! \]
\[\cdot \frac{1}{2} \left\{ \cos \left[ \left( \frac{j}{2} + l + l' + 2 - \alpha - \alpha' \right)\theta \right] + \cos \left[ \left( \frac{j}{2} - k + l - l' + \alpha - \alpha' \right)\theta \right] \right\} \]
\[.x^{j+2l+2l'+4-2\alpha-2\alpha'} \] (A.15)

so for each appropriate value of \( j, k, l, l', \alpha, \alpha' \) there are four coefficients for \((x^{-1/2})^{j+2l+2l'+4-2\alpha-2\alpha'}\) which have the forms:
\[ d_1 \left( \cos \left[ \left( \frac{j}{2} + l + l' + 2 - \alpha - \alpha' \right)\theta \right] \pm \cos \left[ \left( \frac{j}{2} - k + l - l' + \alpha - \alpha' \right)\theta \right] \right) \]
\[ \cdot [R G(l, k)R G(l', j - k) + S G(l, k)S G(l', j - k)] \] (A.16)

and
\[ d_2 \left( \sin \left[ \left( \frac{j}{2} + l + l' + 2 - \alpha - \alpha' \right)\theta \right] \pm \sin \left[ \left( \frac{j}{2} - k + l - l' + \alpha - \alpha' \right)\theta \right] \right) \]
\[ \cdot [R G(l, k)S G(l', j - k) + S G(l, k)R G(l', j - k)] \] (A.17)

Now \( \cos(n\theta) \) can be written as a polynomial of degree \( n \) in \( \cos \theta \), and \( \sin(n\theta) \) can be written as a polynomial of degree \( n \) in \( \cos \theta \) for \( n \) even, or a polynomial of degree \( n - 1 \) multiplied by \( \sin \theta \) for \( n \) odd. So, using the formulas:
\[ \cos \left[ \left( n + \frac{1}{2} \right)\theta \right] = \cos(n\theta) \cos(\theta/2) - \sin(n\theta) \sin(\theta/2) \] and (A.18)
\[ \sin \left[ \left( n + \frac{1}{2} \right)\theta \right] = \cos(n\theta) \sin(\theta/2) + \sin(n\theta) \cos(\theta/2) \] (A.19)

we have, for \( j \) even:
\[ P_{\frac{j}{2}+l+l'-\alpha-\alpha'}(\cos \theta) \cdot [R G(l, k)R G(l', j - k) + S G(l, k)S G(l', j - k)] \]
\[ + \left[ \sin \theta \cdot P_{\frac{j}{2}+l+l'+1-\alpha-\alpha'}(\cos \theta) + P_{\frac{j}{2}+l+l'+2-\alpha-\alpha'}(\cos \theta) \right] \]
\[ \cdot [R G(l, k)S G(l', j - k) + S G(l, k)R G(l', j - k)] \] (A.20)

While for \( j \) odd, this becomes:
\[ P_{\frac{j-1}{2}+l+l'+2-\alpha-\alpha'}(\cos \theta) \left[ \cos(\theta/2) + \sin(\theta/2) \right] \]
\[ + P_{l-1+l'+1-a-a'}(\cos \theta) \sin \theta \sin(\theta/2) \]

\[ [\mathcal{R}G(l, k)\mathcal{R}G(l', j - k) + \mathcal{S}G(l, k)\mathcal{S}G(l', j - k)] \]

\[ + [P_{l-1+l'+2-a-a'}(\cos \theta) \cos(\theta/2) \sin(\theta/2)] \]

\[ + P_{l-1+l'+1-a-a'}(\cos \theta) \sin \theta \sin(\theta/2) \]

\[ [\mathcal{R}G(l, k)\mathcal{S}G(l', j - k) + \mathcal{S}G(l, k)\mathcal{R}G(l', j - k)] \quad (A.21) \]

Let us denote such terms by:

\[ P_m A_{j,k} + (P_m + \sin \theta P_{m-1}) B_{j,k}, \text{ for } j \text{ even} \quad (A.22) \]

\[ [P_{m-\frac{1}{2}}(\sin(\theta/2) + \cos(\theta/2)) + P_{m-\frac{1}{2}} \sin \theta \sin(\theta/2)] A_{j,k} \]

\[ + [P_{m-\frac{1}{2}}(\sin(\theta/2) + \cos(\theta/2)) + P_{m-\frac{1}{2}} \sin \theta \cos(\theta/2)] B_{j,k}, \text{ for } j \text{ odd} \quad (A.23) \]

where \( m = \frac{1}{2} + l + l' + 2 - \alpha - \alpha' \). (Note the \( P_m \) simply denotes a polynomial of degree \( m \), but not a particular polynomial. Hence, many different expressions may have the same representation in this notation. Note also that in equations (A.21) and (A.22) the argument of the polynomials in question is \( \cos \theta \).)

Now the coefficient of \( \left(x^{r-1/2}\right)^{j+l+2l'+4-2a-2a'} = \left(x^{r-1/2}\right)^{2m} \) will be given by a sum of terms of the form (A.15) (resp. (A.16)) for \( j \) even (resp. odd). Recall that

\[ S_j(x) = \frac{1}{2} R_j(x) - \sum_{k=1}^{\infty} j - 1(j - 1)! S_k(x) S_{j-k}(x) \quad (A.24) \]

and hence the coefficient of \( \left(x^{r-1/2}\right)^{2m} \) in \( S_j(x) \) is given by a sum of products of terms of the forms (A.21) and (A.22), which have degrees: \( m_1, m_2, \ldots, m_r \) such that \( m_1 + m_2 + \ldots + m_r = m \). Now \( c_j \) is the integral of \( S_2j \) which is an even polynomial of degree \( 6j \). Hence, we need only concern ourselves with even order terms (i.e. integral values of \( m \), although some (an even number) of the \( m_i \)'s may be half-integers). Clearly, in such a product, since we must have an even number of half-integer orders, we must also have an even number of terms of type (A.22). The rest of the terms being of type (A.21). Thus, such a product must have the form:

\[ \sum P_m E_{j_1,k_1} E_{j_2,k_2} \cdots E_{j_r,k_r} + \sum P_{m-1} E'_{j_1,k_1} E'_{j_2,k_2} \cdots E'_{j_r,k_r} \quad (A.25) \]

where each \( E_{j,k} \) and \( E'_{j,k} \) represents either \( A_{j,k} \) or \( B_{j,k} \) with the appropriate indices. The sums in this expression are indexed in the following way: The first sum will consist of all possible combinations with an even number of \( B_{j,k} \) terms, while the second consists of all combinations with an odd number of such terms. This, then, represents the general term in \( S_j(x) \). We need now only show that when integrated by the appropriate function, such a term will have the form stated
in the claim. First, \( E_{j_1, k_1} E_{j_2, k_2} \cdots E_{j_r, k_r} \) and \( E'_{j_1, k_1} E'_{j_2, k_2} \cdots E'_{j_r, k_r} \) are elements of \( T_j \). This is clear since each \( E_{j_i, k_i} \in T_{j_i} \), and \( T_{j_1} \cdot T_{j_2} \subseteq T_{j_1+j_2} \). Next, consider that the terms which have a factor of \( \sin \theta \) are precisely those which have an odd number of \( B_{j, k} \) terms. Now each term of \( \Re G(l, k) \) and \( \Re G(l', j - k) \) contains an even number of \( J_* \) terms, whereas each term of \( \Im G(l, k) \) and \( \Im G(l', j - k) \) contains an odd number of such terms. Hence, each \( A_{j, k} \) has an even number of \( J_* \) terms and each \( B_{j, k} \) and an odd number. Therefore, only those terms which contain an odd number of \( B_{j, k} \)'s (and in fact all such terms) have a factor of \( \sin \theta \). Finally,

\[
c_j = \sqrt{\pi} \int_{-\infty}^{\infty} e^{\frac{x^2 \cos \theta}{4r}} S_{2j}(x) \, dx
\]

(A.26)

where \( r = |A| \) and \( \theta = \arg A \). Since the expression

\[
\sum P_m E_{j_1, k_1} E_{j_2, k_2} \cdots E_{j_r, k_r} + \sum P_{m-1} \sin \theta E'_{j_1, k_1} E'_{j_2, k_2} \cdots E'_{j_r, k_r}
\]

(A.27)

is the coefficient of \( (xr^{-1/2})^{2m} \) in \( S_j(x) \) we have:

\[
\int_{-\infty}^{\infty} (xr^{-1/2})^{2m} e^{\frac{x^2 \cos \theta}{4r}} \, dx = \frac{\sqrt{\pi} 2^{m+1}(2m - 1)(2m - 3) \cdots (1)}{(-\cos \theta)^{m+\frac{1}{2}}}
\]

(A.28)

now let

\[
\sum P_m E_{j_1, k_1} E_{j_2, k_2} \cdots E_{j_r, k_r} + \sum P_{m-1} \sin \theta E'_{j_1, k_1} E'_{j_2, k_2} \cdots E'_{j_r, k_r}
\]

\[
= \sum_{\sigma \in S_j} \sum_{k, l, l'} \sum_{\alpha=0}^{[\frac{j}{2}]+1} \sum_{\alpha'=0}^{[\frac{j}{2}]+1} \sigma q_\sigma(\cos \theta)(\sin \theta)^{\delta(v)}
\]

(A.29)

Then from what we have shown \( q_\sigma \) has order at most \( m \) (note that each value of \( \sigma \) determines a set of possible values for \( l, l', k \)). So,

\[
c_j = \sqrt{\pi} \sqrt{\pi r} \sum_{\sigma \in S_j} \sum_{k, l, l'} \sum_{\alpha=0}^{[\frac{j}{2}]+1} \sum_{\alpha'=0}^{[\frac{j}{2}]+1} \left[ \sigma \frac{2^{m+1}(4m - 1)(4m - 3) \cdots (1)}{(-\cos \theta)^{2m+1/2}} \right] q_\sigma(\cos \theta)(\sin \theta)^{\delta(v)}
\]

(A.30)
Since \( q_\sigma \) is a polynomial of degree at most \( 2m \) (since we have now doubled the indices) with argument \( \cos \theta \) we may let:

\[
\sum_{k,l,l'} \sum_{\alpha} \sum_{\alpha'} 2^{2m+1} (4m - 1) (4m - 3) \cdots (1) q_\sigma (\cos \theta) \frac{(\cos \theta)^{2m}}{(\cos \theta)^{2m}}
\]

(A.31)

be \( P_\sigma (\sec \theta) \) which will have degree at most \( 2m \). But

\[2m = j + 2l + 2l' + 4 - 2\alpha - 2\alpha' \leq j + 2l + 2l' = j + u,\]

and so the polynomial has the correct order which proves the assertion.
APPENDIX B

THE GENERAL TERM $c_2$

We present here, the complete form for $c_2$ in the typical case:

$$c_2 = \frac{1}{2(-\cos \theta)^{1/2}} \left[ J_4 (\sec^3 \theta - \sec \theta) \right] 1440 \sin \theta$$

$$+ R_1 R_3 (161700 \sec^4 \theta - 324120 \sec^2 \theta + 162420)$$

$$- R_1 J_3 (420 \sec^4 \theta + 900 \sec^3 \theta - 180 \sec^2 \theta - 1140 \sec \theta) \sin \theta$$

$$+ R_2^2 (80640 \sec^4 \theta - 160224 \sec^2 \theta + 79584)$$

$$- R_3 J_1 (420 \sec^4 \theta - 900 \sec^3 \theta - 180 \sec^2 \theta + 1140 \sec \theta) \sin \theta$$

$$- J_1 J_2 (161700 \sec^4 \theta - 324120 \sec^2 \theta + 162420)$$

$$- J_2^2 (78960 \sec^4 \theta - 157152 \sec^2 \theta + 77808)$$

$$- R_1^2 R_2 (15120 \sec^5 \theta + 15120 \sec^4 \theta - 11520 \sec^3 \theta$$

$$+ 21888 \sec^2 \theta - 17424 \sec \theta - 50832)$$

$$+ R_2^2 J_2 (21742560 \sec^5 \theta - 21060 \sec^4 \theta - 43408800 \sec^3 \theta$$

$$- 10368 \sec^2 \theta + 21636720 \sec \theta) \sin \theta$$

$$R_1 R_2 J_1 (43606080 \sec^5 \theta - 85325760 \sec^3 \theta$$

$$+ 42570432 \sec \theta) \sin \theta$$

$$- R_1 J_1 J_2 (40320 \sec^5 \theta + 10080 \sec^4 \theta$$

$$- 40320 \sec^3 \theta - 137088 \sec^2 \theta - 10080 \sec \theta + 229824)$$

$$- R_2 J_1^2 (45360 \sec^5 \theta + 55368 \sec^4 \theta - 69120 \sec^3 \theta$$

$$- 207360 \sec^2 \theta + 17424 \sec \theta + 138096)$$

$$- J_1^2 J_2 (21742560 \sec^5 \theta - 20160 \sec^4 \theta - 43506720 \sec^3 \theta$$

$$+ 20160 \sec^2 \theta + 21755376 \sec \theta) \sin \theta$$

$$+ R_1^4 (957850740 \sec^6 \theta + 517860 \sec^5 \theta - 2874666430 \sec^4 \theta$$

$$+ 52$$
\[-703920 \sec^3 \theta + 2877094386 \sec^2 \theta + 198312 \sec \theta - 960246444) \\
+ R_1^3 J_1 (55440 \sec^6 \theta - 3311280 \sec^5 \theta - 40320 \sec^4 \theta \\
+ 16587369 \sec^3 \theta - 1986778 \sec^2 \theta - 9746640 \sec \theta) \sin \theta \\
- R_2^4 J_2 (86232071880 \sec^6 \theta + 544320 \sec^5 \theta - 258661308660 \sec^4 \theta \\
- 783360 \sec^3 \theta + 2583510272 \sec^2 \theta + 212112 \sec \theta - 86231085868) \\
- R_1 J_1 (388080 \sec^6 \theta - 3311280 \sec^5 \theta + 40320 \sec^4 \theta \\
+ 16587360 \sec^3 \theta + 3888 \sec^2 \theta - 9746640 \sec \theta) \sin \theta \\
+ J_1 (958072500 \sec^6 \theta + 26460 \sec^5 \theta - 2873695650 \sec^4 \theta \\
- 79440 \sec^3 \theta + 2873839950 \sec^2 \theta + 13800 \sec \theta - 958431036) \]
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