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A comparison of the effects on student learning of two strategies for teaching the concept of derivative

Fiske, Michael Bryan, Ph.D.
The Ohio State University, 1994
A COMPARISON OF THE EFFECTS ON STUDENT LEARNING
OF TWO STRATEGIES FOR TEACHING
THE CONCEPT OF DERIVATIVE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Michael Bryan Fiske, A.B., M.A., M.A.

* * * * *

The Ohio State University
1994

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College of Education
DEDICATION

To Maureen
and to my students past and present
in
Bosnia, Bulgaria, Canada, Croatia, Egypt, England, Germany, Hungary,
India, Iran, Iraq, Italy, Japan, Lebanon, Libya, Pakistan, Palestine, Poland,
Romania, Serbia, Taiwan, Turkey, and the United States.
ACKNOWLEDGMENTS

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PUBLICATIONS


FIELDS OF STUDY

Major Field: Education

Studies in Mathematics Education
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CHAPTER I
THE PROBLEM

The teaching and learning of calculus are at the forefront of a reform movement within
the mathematics community. Reformers want a leaner calculus curriculum and livelier
calculus instruction. The goals are high quality instruction, increased student analytical and
reasoning ability, and improved student conceptual understanding (Douglas, 1986; Steen,

Concurrent with this call for reform of the calculus curriculum and calculus teaching is
the new role that computational and graphics technology are beginning to play in the
calculus classroom. Powerful mathematical languages, symbol–manipulation systems,
computer programming, computer graphing utilities, supercalculators, and graphic
calculators have been used to reduce computation and manipulation, provide numeric,
graphic, and symbolic insight, facilitate multiple representations, and resequence the
calculus curriculum (Tucker, 1990).

The calculus reform movement has the potential to affect more than one million
students who take calculus in high schools, two–year colleges, four–year colleges, and
universities (Steen, 1988b). While the reform movement has focused on streamlining the
calculus curriculum and incorporating technology in calculus teaching, mathematics
educators have been concerned with student learning of calculus–related concepts such as
functions and limits and their impact on curriculum development, the interaction of
technology with curriculum and learning, and the role of the teacher in instruction and
dissemination (Ferrini–Mundy & Graham, 1991). In order to address the issues of
calculus reform, there is the need to identify effective ways of presenting calculus concepts, to investigate how technology can be used to enhance conceptual development, to examine how students learn the concepts, and to explore the role of the teacher in shaping an appropriate learning environment.

Statement of the Problem

The research described in this study was designed to examine the effects on student learning and on teaching of using two different conceptual frameworks for teaching the concepts of differentiability and differentiation in a technologically–rich environment. One conceptual framework was the idea of local linearity, under uniform magnification a differentiable function looks locally straight. The second framework was the mathematical notion of the derivative as a limiting process. In both instructional frameworks, the software, *A Graphic Approach to the Calculus* (Tall, Van Blokland, & Kok, 1991), was used by the students and the teacher.

The two conceptual frameworks for teaching the concepts were chosen to contrast within a technologically–rich environment a visual approach—local linearity—with a sequential, analytic approach—the derivative as a limiting process—to the concepts of differentiability and differentiation. The importance of this contrast lies in designing curricula to better assist students in developing a deeper understanding of the concepts (Eisenberg & Dreyfus, 1991).

The traditional introduction to differentiation as a limiting process and the local linearity introduction both appeal to visual images. The limiting framework for teaching the concept of the derivative uses the simple idea of a secant line approaching a tangent line. The local linearity framework, however, is relatively more complicated in that a student is initially introduced to functions that are locally straight and functions that are not locally
straight. Because the limiting approach to differentiation is less complicated, it is potentially more accessible to a student than is the local linearity approach.

The development of graphing technology that allows uniform magnification of graphs of functions makes it possible to compare and contrast the limiting process and local linearity frameworks for teaching the concepts of differentiation and differentiability. Before the availability of the graphing technology with the facility for uniform magnification, the comparison and contrasting of these two frameworks could not be done. Thus, the present study contributes to the development of a theoretical framework for the teaching and learning of calculus in a technological environment.

It was hypothesized that students who were introduced to the concepts of differentiability and derivative through the idea of local linearity would develop a different, more complete understanding of differentiability and the derivative than students who learned the derivative and differentiability concepts through the traditional geometric and symbolic approach involving the tangent as the limit of secant lines drawn through a point. Furthermore, it was expected that, through the use of the computer software, the teacher would rely less on a lecture format and would rely more on student discovery and construction for the development of the concepts.

**Theoretical Model**

In order to understand the process by which a student develops knowledge of the concepts of derivative and differentiability, it is necessary to account for both the design of the instructional environment in which the student works and the cognitive structure of the student. Instructional design is important because teaching strategies have a direct influence on student learning and cognition (Tennyson & Cocchiarella, 1986). The cognitive structure of the student provides linkage with prior experience upon which the concept of derivative is built. The theoretical framework begins by considering alternative
frameworks for teaching the concept of derivative. Then, the concept images and concept definitions held by the student and the interaction of these conceptual structures with multiple-linked representations provided in a computer environment are considered (Kaput, 1986, 1989; Vinner & Dreyfus, 1989).

Skemp (1986) proposed that new mathematical ideas cannot be taught by simple definition. Rather, concepts should be taught by providing students with a suitable collection of examples based on prior concepts that are already present in the repertoire of the student. Skemp's view is confirmed by empirical research that suggests concept definitions include both critical attributes of the concept and direct reference to prerequisite information (Tennyson & Park, 1980; Tennyson & Cocchiarella, 1986).

Mathematicians have traditionally introduced the concept of derivative to students through a so-called intuitive approach based on linking geometric and symbolic ideas. The common strategy used to determine the derivative of a curve at a point P on the curve is to choose a second point on the curve, P1. The secant line containing P and P1 is drawn. Then, P1 is allowed to move along the curve toward P. Through a limiting process the secant line approaches the tangent to the curve at P if the tangent exists. The limiting position of the secant line is called the derivative of the curve at P (Figure 1). This geometric approach is algebraically generalized to consider the derivative as a function. Thus, prior student experiences with geometry (secants, tangents and slope), if these experiences exist, are linked with their symbolic representations to enhance learning of the derivative concept.
The local linearity approach to the derivative, proposed by Tall (1991), makes use of the visual notion that a differentiable function under uniform magnification is locally straight (Figures 2 and 3). By magnifying portions of a graph, a student can view the changing slope of the graph. This visual image provides a foundation for exploring the derivative numerically and symbolically. Finally, geometric, numeric, and symbolic representations are linked in a formal definition of the derivative.
Figure 2. The locally linear graph on the right is a magnified portion (shown in the dashed box) of the differentiable graph on the left.

Figure 3. The non-locally linear graph on the right is a magnified portion (shown in the dashed box) of the non-differentiable graph on the left.
Prior experience influences the development of student understanding of a new mathematical concept. A student who has studied Euclidean geometry may retain the image that a tangent is a line that meets a circle in exactly one point. When the concept of derivative is introduced in calculus such a student would find it difficult to visualize a tangent that intersects a graph in more than one point or a tangent that actually crosses the graph. Exclusive exposure to polynomial differentiation can lead students to define the derivative of a function $f(x)$ as $n[f(x)]^{n-1}$. Similarly, students develop images that confuse limit and bound. Commonly a student will say, "$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots$ is equal to 1, but 0.9999 $\cdots$ is always less than 1" (Dreyfus, 1990).

Vinner calls this "set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" the concept image (Vinner & Dreyfus, 1989, p. 356). In contrast, the concept definition is a form of words used to specify the concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to the concept as a whole. It is then the form of words that the student uses for his own explanation of his (evoked) concept image (Tall & Vinner, 1981, p 356).

The computer software, A Graphic Approach to the Calculus (Tall, Van Blokland, & Kok, 1991), is designed around generic organizers, "an environment (or microworld) which enables the learner to manipulate examples and (if possible) nonexamples of a specific mathematical concept" (Tall, 1989, p. 39). The first generic organizer is a magnification program that connects differentiability with the concept of local linearity, a differentiable graph under high uniform magnification looks straight (Figures 2 and 3).

The second generic organizer is a gradient curve program that draws the chords defined by the slope function $f(x + h) - f(x) \over h$ for varying $x$ and fixed $h$ and plots the value of the slope for each $x$ (Figure 4). The third generic organizer is a chord drawing program that
draws a chord through a fixed-point \( x \) and a varying-point that steps closer to \( x \) in order to explore the difference quotient \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) (Figure 5).

Figure 4. A function with several tangents drawn and values of the slopes of the tangents plotted.

Figure 5. The tangent at \((1, 1)\) for \( y = x^2 \) with several secants drawn through the point.
This study was designed to explore how two initially separate approaches to the teaching of the concepts of differentiability and differentiation—local straightness and the mathematical idea of a secant approaching a tangent—affected student learning of the concepts. The generic organizers were used to provide conceptual links with prior student understanding and to construct new student knowledge. Because these organizers involved the use of computer software in a classroom setting, the study also examines how the use of the software affected classroom teaching and learning. The examination of student learning is particularly important because the use of computer software has the potential to develop student misconceptions of graphical information. In particular, a student may perceive a computer drawn graph as a string or necklace with little or no attention given to scale and scale change (Goldenberg & Kliman, 1989; Dunham & Osborne, 1991).

The computer software generic organizers were used to allow a student to explore and manipulate examples and nonexamples of the concepts of derivative and differentiability. The programs provided a student with the opportunity to investigate the derivative concept in graphic, numeric, and symbolic forms. Within the context of this investigation, the student used language to communicate and develop her or his understanding of the derivative concept. The interaction among these multiple, linked representations can facilitate the development of an enhanced concept image (Kaput, 1986, 1989).

The instructional strategies in both treatments attempted to develop appropriate concept images and concept definitions in students by linking the concepts of derivative and differentiability with a prior cognitive framework. The limit approach connected the derivative with notions of slope, secants, and tangents through a limiting process. The local linearity approach tied the derivative concept to the straightness of graphs under uniform magnification. Both strategies were formulated geometrically. Then, the geometric representations were joined with numeric, symbolic, and linguistic representations in order to fully develop the concept of derivative (Figure 6).
Potential Significance of the Study

The traditional presentation of the concepts of differentiability and differentiation has focused on the limiting process of a secant approaching a tangent. While this approach has mathematical appeal, research indicates that this presentation does not articulate well with prior student experiences with the concepts of limit and tangent. A student acquires procedural rather than conceptual knowledge of the concepts of differentiability and differentiation, if knowledge is gained at all.

Through a comparison of an alternative approach to teaching the concepts of differentiability and differentiation, that of local linearity, with the secant–tangent approach to the teaching of these concepts in a quasi-experimental setting, it may be possible to formulate a method of presentation that establishes better conceptual links with prior student experience as well as develops more complete student understanding of the concepts. This research also contributes to knowledge about student understanding of calculus concepts as well as concepts underlying the study of calculus. The use of the
generic organizers provides the opportunity to examine the role of visualization in calculus teaching. Finally, the use of the computer software in both conceptual approaches allows for consideration of how technology shapes calculus teaching and learning.
CHAPTER II
REVIEW OF RELATED LITERATURE

This review of the literature will focus on classroom instruction in a technological environment, student understanding of concepts underlying the study of calculus, student understanding of calculus concepts with and without the use of technology, and the roles of graphical representation and visualization in calculus instruction.

Classroom Instruction in a Technological Environment

While calculators and computers have been in classrooms for over thirty years, few studies have been conducted to examine the effects of the use of this technology on instructional practice (Hembree & Dessart, 1992). In fact, it is not apparent that the introduction of technology in the classroom has changed mathematics teaching at all. Romberg & Carpenter (1986) summarized research on instruction in the secondary school mathematics classroom. They found that typical instruction consisted of a fairly static sequence of events. First, the previous night's homework assignment is reviewed with the teacher or a few students working the more difficult problems. This is followed by a brief, or no, explanation of new material. Finally, students are assigned seatwork for the remainder of the period.

This situation is in the process of change due to the availability and classroom use of low-cost software and hand-held computing devices. Recently, several studies have begun to explore the impact of computer and calculator usage on mathematics instruction. These studies focus both on the ways in which teachers teach and the ways in which students learn.
Lampert (1988) saw a dramatic shift in roles for teachers who used the Geometric Supposer software (Schwartz & Yerushalmy, 1985) in geometry classrooms. Responsibility for learning shifted from teacher to students. Students formed their own hypotheses, tested them, and made generalizations. As this process continued, negotiation was necessary between the teacher's curricular agenda and student generated mathematics.

In a study of learning activities and classroom roles in classrooms in which a microcomputer was in use, Fraser, Burkhardt, Coupland, Phillips, Pimm, and Ridgway (1988) found evidence that suggests the computer can promote a wider variety of teaching roles. That is, more open teaching activities are conducted in which the teacher moves from being a manager and task setter to being a resource, counselor, and friend. At the same time, students working with a microcomputer were more free to assume traditional teacher roles. This assumption of roles led students to take greater responsibility for the learning activity and involved them in higher level skill activities. Fraser and colleagues suggest this change of roles between students and teacher takes place because the computer takes on the traditional teaching roles of managing, explaining, and task setting.

Farrell (1989) studied classroom roles and learning activities in six high school precalculus classes that used graphing calculators and in some cases computers. The study was modeled on that of Fraser, et al. and used a variation of their Systematic Classroom Analysis (SCAN) taxonomy for observation. Farrell found the roles students exhibited when technology was in use differed from roles they exhibited when technology was not in use. When technology was being used students were more frequently task setters, consultants, and explainers. While the primary role of the teacher remained that of manager in both technology in use and technology not in use settings, Farrell observed that the teachers were more frequently consultants and less frequently explainers and task setters in the technological setting. This finding is similar to that of Fraser and colleagues. In the classrooms Farrell studied, students were slightly more likely to be actively engaged when
using technology as compared to when they were not using technology. They did more symbolizing, investigating, and problem solving with technology than they did without technology. Small group work was considerably more in evidence when students used technology.

In a combined study of a first-year algebra curriculum that involved extensive use of computer simulations, graphics, and spreadsheets and of a computer-based applied calculus course, Sheets and Heid (1990) reported that computer lab environments stimulate opportunities for work in small groups. After students become used to the idea of working in groups, they more actively engaged in conjecturing and hypothesizing. Sheets and Heid observed students using multiple representations, choosing appropriate mathematical procedures, and interpreting problem results. There were, however, some students who did not take to the small-group environment, preferring to work through assignments on their own. Anecdotal evidence that supports the work of Sheets and Heid is given in Leinbach, Hundhausen, Ostebee, Senechal, and Small (1991).

In summary, the research literature suggests that use of technology in mathematics and calculus enhances student engagement with course content and with each other. In technology oriented classrooms, researchers have observed that most students become actively involved in doing mathematics. Furthermore, it is claimed that this student involvement can lead to a deemphasis on the roles of the teacher as manager of classroom events and explainer of mathematics.

**Concepts Underlying the Study of Calculus**

Knowledge of the concept of derivative is built on student intuitions and images of the concepts of function, slope, rate of change, limit, and tangent. The typical mathematical presentation of the concept of derivative—either as a problem of constructing the tangent to a curve at a given point or in determining an instantaneous rate of change—is said to appeal
to intuitive geometric notions (Apostol, 1961; Courant, 1937; Lang, 1986; Mac Lane, 1986; Thomas & Finney, 1988). First, a secant line is graphically shown approaching a tangent. The visual representation is linked with the symbolic representation of the difference quotient, $\frac{f(x + h) - f(x)}{h}$, and then the derivative is generalized as a function.

The concept images and concept definitions of function, slope, rate of change, limit, and tangent that students bring to this presentation of the derivative is the subject of the following discussion.

The concept of function is essential to the study of the calculus. Thomas (1975) and Dreyfus and Eisenberg (1982) have suggested that acquisition of the concept of function is developmental. Knowledge of the function concept begins by considering the function as a procedure of assignment—that is, what to do with $x$. At a higher level, the function concept is considered in various contexts—tabular, graphical, and symbolic. The student gains ability to move between these contexts. Finally, a function is considered as a mathematical object that has properties. At this stage, the student is able to work with transformations and composition of functions. Thomas identified student difficulties in progressing through these stages. The difficulties occur when connections are made between the visual and the algebraic and when a function begins to be treated as a mathematical object. Similar conclusions have been reached by Dreyfus and Eisenberg (1982) in their study of Israeli junior high school students and by Dreyfus (1990) and Ferrini-Mundy and Graham (1991) in literature reviews of student knowledge of the concept of function.

Vinner and Dreyfus (1989) looked specifically at the knowledge of function that students bring to the study of calculus. They asked a group of 271 college students and 36 junior high school teachers to define a function. Vinner and Dreyfus placed these definitions into six categories: a correspondence, a dependence relation, a rule, an operation, a formula, and a representation. The majority of student responses fell about
equally into the two categories of correspondence—a function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set—and dependence relation—y depends on x. When students were asked to put their definition to use in determining whether or not specific graphs represented functions and in creating functions with specific properties, several prominent themes emerged. These themes were one-valuedness, discontinuity, split domain, and exceptional point. Of those students who were able to identify correctly a given graph as being a function or not, the overriding justification was one-valuedness, for each element in the domain there is exactly one value in the range. When students falsely rejected graphs as functions, they used arguments about the discontinuity of the graph, that the graph would require a split domain—that is, two or more symbolic statements—, and that of an exceptional point, one which did not fit into a particular formula that describes the rest of the graph.

Orton (1983) used clinical interviews with 110 students, 60 sixth form students (ages 16–22) from four schools and 50 students (ages 18–22) from two colleges, to investigate student understanding of slope and rate of change. Orton found that many students experienced conceptual difficulty with the simple notion of the slope (change in $y$ divided by change in $x$) of a straight line. Similar conceptual difficulties were encountered when the concept of slope was extended to average rate of change over an interval for a non-linear graph. Few of the students Orton interviewed were able to make the distinction between average rate of change over an interval and rate of change at a point.

In regard to the concept of limit, Cornu (1983, cited in Tall, 1986) identified four cognitive obstacles—knowledge that is part of the student's repertoire and that has been used satisfactorily in the past to solve certain types of problems but proves to be unsatisfactory in new situations—that interfere with the development of appropriate images for the concept of limit. First, in calculus limits are no longer calculated algebraically. Second, limits are thought of in terms of both the infinitely small ($\varepsilon$) and the infinitely
large (±∞). Third, language, such as "tends to," suggests that a limit is something never attained. Finally, confusion results from passage from static to dynamic images and back to static images—for example, in the passage of secants to a tangent in defining the derivative.

Williams (1991) used a questionnaire with 341 second-semester college calculus students to identify their informal models of limit. When asked to identify which of six sentences best described a limit, 36% of the students chose "A limit is a number or point the function gets close to but never reaches" (p. 221) and 30% of the students chose "A limit describes how a function moves as x moves toward a certain point" (p. 221). Only 19% of the students identified an approximation of the formal statement—"A limit is a number that the y-values of a function can be made arbitrarily close to by restricting x-values" (p. 221)—as best. In treatments with 10 students, designed to change their understanding of limit toward a more formal definition, Williams found the students willing to accept multiple interpretations of the limit concept as true. This acceptance was contextually determined. Students chose to use a particular definition of limit based on the type of function and the ease with which they could use the definition in solving the problem at hand. In particular, students found graphs of functions that they produced and plugging numbers in a function formula as useful tools in explaining the limit concept.

In a study of student understanding of concepts underlying the study of calculus, Monaghan (1992) found that language used by teachers in their discussion of limits generates everyday meanings for students that are at odds with mathematical meanings of the language. In particular, while tends to and approaches carry the same sense, for students they represent movement toward a goal without reaching the goal. Similarly, Monaghan found that, with regard to functions, converges evokes the sense of two objects coming together and touching. The students Monaghan studied saw the word limit as more specific than the other terms. Yet, this specificity was in terms of a final point or an unreachable boundary. Monaghan did not examine the instructional implications of these
observations. He concluded, however, that students should be encouraged to explore their own conceptions of the meanings of these terms in relation to their mathematical meanings and to realize how ordinary language usage may lead to misconceptions.

The concept image of tangent that students bring to their study of calculus is strongly influenced by the study of geometry. The tangent is most frequently thought of as a line that touches a curve at only one point as in the tangent to a circle (Tall, 1986). Vinner (1982) asked students to draw a tangent to the graph of $y = x^3$ at the origin. Many students placed the tangent line slightly to the side so that it did not pass through the curve. The intuitive notion used in defining the derivative, that the secant line approaches the tangent through a limiting process, is not readily seen by students (Tall, 1985a, 1985b, 1985c, 1987).

The images of function, slope, rate of change, limit, and tangent that a student brings to the study of calculus are frequently naive and contextually determined. The image of a function as a formula, the local and linear perception of slope and rate of change, the static and algorithmic view of limit, and the Euclidean geometric concept of tangent all serve as a shaky foundation for the study of the derivative. Mathematical language used to communicate the concepts underlying the study of calculus is ambiguous for students when perceived in terms of its ordinary contextual usage. The inability to connect symbolic representations of these foundational concepts with visual representations further interferes with the development of student understanding.

**Student Understanding of Calculus Concepts**

In his study, Orton (1983) also examined student understanding of differentiation and applications of the derivative. He found that students were able to perform procedural differentiation with polynomial functions, but they had some arithmetic difficulty in differentiating functions of the form $y = ax^{-2}$. Orton's students were able to explain the
symbols $\delta x$ and $\delta y$ but had difficulty with $\frac{\delta x}{\delta y}$. The symbols $dx$ and $dy$ had little meaning for the students apart from the expression $\frac{dy}{dx}$. Orton also found cases in which student understanding of derivative concepts was contextually determined. For example, while some students could find critical points by setting $\frac{dy}{dx} = 0$, they were unable to explain the meaning of $\frac{dy}{dx} = 0$.

Orton concluded that students interpret differentiation as a mechanical rule to be applied. He recommended that the concept of derivative be introduced informally using a numeric and graphic approach. He suggested that calculators would be of great assistance in quickly producing graphical representations of curves and their tangents.

Several efforts have been made to evaluate calculus instruction in a technological environment. Heid (1988) compared 39 students in a first-semester applied calculus course, that used technology to develop calculus concepts for 12 weeks of the course and developed skills during the last 3 weeks, with 100 students in a similar course, that did not use technology and focused on algorithmic skills as calculus concepts were developed throughout the course. The technology included a computer algebra system, a function grapher, a least-squares-fit program, and table-of-values program. Heid found that the group of students who used technology demonstrated a better understanding of calculus concepts than the students who focused on skills. The technology—using students performed almost as well as the skills students on a final exam of routine skills. The better conceptual understanding was demonstrated through students having a wider range of appropriate associations when explaining the derivative, having the ability to reconstruct facts through the use of basic principles, using their own words to talk about concepts, constructing representations that were not presented in class of concepts, and verbally connecting the derivative concept with its mathematical definition. Heid reported that students in the technology group thought the computer helped by lessening the
manipulative aspects of calculus, giving them confidence in their results, and focusing them on problem solving.

Palmiter (1986, 1991) compared the conceptual and computational performance of 38 students, who used a computer algebra system as a central aspect of their study of engineering integral calculus, with the conceptual and computational performance of 39 students, who were taught engineering integral calculus using a traditional paper-and-pencil approach. The students who used the computer algebra system covered the course content—definition of the integral, the fundamental theorem of calculus, inverse functions, techniques of integration, and applications of the integral—in 5 weeks in contrast to the paper-and-pencil group who spent 10 weeks on the same material. On both conceptual and computational exams administered to the computer algebra students after 5 weeks and to the paper-and-pencil students after 10 weeks, the computer algebra students scored significantly better than the paper-and-pencil students. While the computer algebra students were able to use the computer algebra system when taking the computational exam, Palmiter claimed the results demonstrated the ease with which students learn to effectively use a computer algebra system to perform the routine procedures involved in an integral calculus course. In addition, students in the computer algebra calculus course, who continued their study of calculus, maintained higher grade averages in these calculus courses than students in the traditional paper-and-pencil group.

Beckmann (1990) studied the effectiveness of calculus instruction using levels of graphic representation in developing the concepts of limit, continuity, and derivative. Beckmann examined student facility with the use of graphic representation, student ability to solve applied, symbolic routine, and symbolic nonroutine calculus problems, student attitudes toward mathematics, and student attitudes toward the use and usefulness of graphs. Using graphing software with 163 first-semester university calculus students, Beckmann introduced the concept of derivative through a dynamic visual representation of
the relationship between two quantities and their corresponding rate of change. Beckmann concluded that use of a graphic representation system can positively affect student understanding and interest without necessarily negatively influencing skill acquisition in calculus.

Tuftc (1990) compared approximately 200 students enrolled in an introductory university calculus course to 24 students who were simultaneously enrolled in a one-credit supplemental computational calculus course. The experimental group wrote their own computer programs to find limits, right and left hand derivatives, Riemann integrals, and solutions to equations and used graphing software to graph. On a test of conceptual understanding of calculus topics the experimental group scored significantly higher than the control group. Tuftc concluded that the experimental group outperformed the control group because: the discrete numerical output of the computer belongs to the student world; graphic, dynamic representations provide a new dimension that was not previously present; and the use of computers in an experimental mode produces an environment that is more conducive to learning.

Tall (1986) used a British version of the software A Graphic Approach to the Calculus to introduce the concept of derivative to experimental groups of 42 A-level (ages 16-18) students and 51 college students. Control groups consisted of 67 A-level students and 44 first-year university calculus students. In the experimental classes a single computer was used for teacher demonstration and for student exploration. Tall found the experimental students were better than the control students at sketching the derivative for a given graph, recognizing a derivative, specifying a non-differentiable function, and relating the derivative to the slope and to the slope function. With another set of students, who used the generic organizers in the software, however, there was no significant difference with the control group. These results led Tall to conclude that at least some student improvement was due to teaching.
Research has demonstrated there are cognitive obstacles at each stage of the typical mathematician's presentation of the concept of derivative: (a) an "intuitive" approach to limits, (b) fix $x$ to calculate the limit of $\frac{f(x + h) - f(x)}{h}$ as $h$ gets small and call the limit $f'(x)$, and (c) vary $x$ in $f'(x)$ to get the derivative as a function. This oversimplification gets in the way of student understanding of first principles (Tall, 1989).

A model proposed by Tall (1989) for the teaching of the derivative has been implemented in the School Mathematics Project 16-19 curriculum. This model proposes a sequence of instruction that: (a) explores the notion of local straightness, (b) visualizes the changing slope of the graph as another graph, (c) relates the visual picture of the slope to the numerical algorithm to provide analogue insight into the underlying numerical computer process, and (d) relates these experiences to other representations, including the numerical and algebraic limiting process. Tall (1989) discovered that teachers who used this model were so upset by student difficulties with symbolic differentiation that they postponed (d) until the second year of the syllabus and failed to cement the link to first principles of differentiation.

The literature indicates that the use of calculators and computers may enhance student understanding of calculus concepts. Numeric, graphic, and symbolic representations provide multiple means of introducing the concept of derivative. The degree to which these representations help students to form appropriate concept images and concept definitions in calculus is uncertain as is the role of the teacher in implementing instruction based on technology.

Roles of Graphical Representation and Visualization in Calculus Instruction 

While there is evidence that graphical presentations may lead to a better understanding of mathematical concepts by students (Janvier, 1987; Kaput, 1986, 1989), Goldenberg (1988) and Goldenberg and Kliman (1989) present evidence that students do not always
perceive graphical representations correctly. In a study of 6 eighth graders and 12 high school students who were in courses at or above the precalculus level Goldenberg and Kliman discovered a variety of student misunderstandings when the students were using computer graphing instruments. Goldenberg and Kliman summarized their findings in three metaphors: (a) The computer is treated as an automatic paper and pencil; (b) Scaling is like using a magnifying glass; and (c) A mathematical curve is like a bead necklace. Use of these metaphors by students results in misperception of graphical information. After conducting clinical interviews, Goldenberg and Kliman concluded that while computer graphing reveals student misconceptions, when students are confronted with their own misconceptions they do work out the conflicts between incompatible theories or images and thereby deepen their understanding of mathematics.

In a study of 400 university precalculus students, Dunham and Osborne (1991) identified three types of student behavior that lead to misconceptions about the nature of functions and their graphs. This behavior involved student failure to make connections between symbolic and graphic representations, issues of scale and scaling, and transformations of functions. In regard to linking symbolic and graphic representations, Dunham and Osborne observed that students failed to project critical attributes of a function onto domain and range spaces, that students thought of functions as a single entity—a string or rubber band—rather than as a collection of points, and that students could not infer characteristics of a function from more than one point of a function. Confirming Goldenberg's and Kliman's (1989) observations, Dunham and Osborne also found that students pay little attention to scale. Scale changes were viewed by students as shape transformations in which students treated the axes and the graph as separate entities. Dunham and Osborne concluded that visual representation needs to be taught with particular attention given to the language used in visual and symbolic modes.
Vinner (1989) considered the degree to which visual considerations could be taught in calculus and the extent to which students believe visual thinking in contrast to algebraic thinking is important in calculus. In a calculus course for 67 science students, Vinner strongly emphasized the visual approach. During the course students were asked to formulate the mean value theorem (if a function \( f \) is differentiable on \((a, b)\) and continuous on \([a, b]\), then there is a number \( \xi, a < \xi < b \), such that \( f'(\xi) = \frac{f(b) - f(a)}{b - a} \)) to supply a suitable drawing for the theorem, and then to give a visual or algebraic proof of the theorem. The majority of students chose an algebraic proof that required the creation of or memorization of an auxiliary function. This surprised Vinner. The next year in the same course, Vinner presented the mean value theorem with both a visual and algebraic proof. He then asked the 74 students which proof they found more convincing and to explain their answer. About equal numbers of students found each of the proofs more convincing. Vinner contrasted this result with the choice of the previous year's students of an algebraic proof of the theorem. He concluded that an algebraic proof is preferable for students because of their perception that it is more mathematical and more general and that, in the context of a final exam, meaningful (that is, visual) learning is less preferable to memorized, algebraic techniques.

Eisenberg & Dreyfus (1991) concur with Vinner that students choose a symbolic framework rather than a visual one to process mathematical information. They argue that this choice is not accidental. For many mathematicians, mathematics teachers, and students, mathematics is nonvisual. Furthermore, visual representations are difficult to understand unless a student is initiated in the interpretation process. Finally, there is the perception that efficient teaching of school mathematics requires a sequential, algorithmic, rather than a bundled, visual presentation. Eisenberg & Dreyfus concluded that the powerful benefits of visual thinking in mathematics necessitate the development of appropriate teaching materials for its inclusion in the school mathematics curriculum.
Distinguishing Features of the Present Research

The use of technology in the study of mathematics and calculus appears to have a positive effect on student understanding of concepts (Tucker, 1990). It is not clear, however, whether this positive effect is due to the presence of a changed classroom environment—working in small groups, for example—or to a change in emphasis in the content of beginning calculus—more work with graphic and numeric representations, for example. Moreover, it is not apparent that mere use of technology overcomes cognitive obstacles and helps students to develop appropriate concept images in calculus. Dreyfus (1990) summarized the results of research on calculus learning by saying, "Students learn the procedures of calculus on a purely algorithmic level that is built on very poor concept images" and "visualization [in calculus] is rare, and if it occurs the cognitive link between the visual/graphical and the analytic/algebraic representations is a major point of difficulty" (p. 125).

The present research recognizes the role that incomplete or inaccurate concept images play in developing student conceptual understanding. It attempts to build on Tall's research in a setting within the United States while controlling for teacher variability and student use of computer software. This research also differs from Tall's. In this study students worked at multiple computers in group exploration of the derivative. Tall conducted his research with one computer per classroom used for demonstration and exploration purposes.

The present research incorporated specific teaching strategies that have been designed for concept teaching (Tennyson & Park, 1980; Tennyson & Cocchiarella, 1986). Each instructional strategy developed the derivative concept by presenting critical attributes of the derivative from a particular perspective—either as a limiting process or through local linearity. Both strategies connected the derivative concept to prior learning experiences.
Finally, examples and nonexamples of differentiable functions were presented to both treatment groups.

In designing the instruments to examine student understanding of concepts underlying the study of calculus and calculus concepts, efforts were made to link this study with prior research. Thus, the questions used on the pretest, posttest, and in student interviews overlap, in part, with questions used in previous studies in order to develop and further explore knowledge of student understanding.

On the basis of the literature, this study was designed to examine both the role that technology plays in affecting calculus teaching and learning and alternative conceptual approaches—local linearity and the so-called mathematically intuitive notion of the secant approaching the tangent—to the teaching of calculus. The literature review provided a basis for framing the questions to be explored.
CHAPTER III
RESEARCH METHODS AND PROCEDURES

This study was designed to complement and to build upon prior research on student knowledge of concepts underlying the study of calculus and on student understanding of the concept of derivative. The role of technology use in teaching and learning provided a framework within which the study was conducted. This chapter begins with the research hypothesis and a presentation of the research design. Preparation for the treatments and the treatments are then outlined. This is followed by a discussion of the instruments used in observing classroom teaching and learning. Instruments used for quantitative measures on the pretest and posttest are described. Links with prior research are indicated. The protocol for student interviews is discussed. The process for student evaluation of teaching and learning is presented. The chapter concludes with a description of procedures used in data analysis.

Research Hypothesis

It was hypothesized that students who received instruction on the concepts of differentiability and the derivative organized around the idea of local linearity would achieve a different and more complete understanding of these concepts than students who received instruction on the concepts of differentiability and the derivative based on the limiting idea of the secant approaching the tangent. Furthermore, it was expected that, as students interacted with the computer software in the instructional environment, the role of the teacher would shift from being the presenter of information to one of guide and mentor.
That is, students would become more involved in developing their own knowledge of the differentiability and the derivative.

Research Design

The design was nonequivalent control group. Two existing high school calculus classes with a single teacher were chosen on the basis of availability. One class (N = 27) was randomly assigned to receive instruction on the concepts of derivative and differentiability using local linearity as the organizing principle. This class will be called the local linearity group. The control group (N = 28) received instruction on these concepts with the limiting idea of the secant approaching the tangent as the organizing principle. The control group will be called the secant–tangent group. Both groups used the computer software, *A Graphic Approach to the Calculus* (Tall, Van Blokland, & Kok, 1991), as a central part of the instructional process.

Treatments

Preparation for the Treatments

The researcher made contact with a calculus teacher in a midwestern, suburban high school, in the spring of 1991 in order to determine the feasibility of doing the research. The teacher has a master's degree in mathematics and has taught secondary school mathematics for 27 years, 13 years in the same suburban high school. The teacher has been the only calculus teacher at the high school during these 13 years, but had never used graphing calculators or computer software in teaching mathematics or calculus. At the first meeting with the teacher, the software, *A Graphic Approach to the Calculus* (Tall, Van Blokland, & Kok, 1991), was demonstrated by the researcher. At this time, the calculus
teacher agreed to participate in the study. Permission was obtained from the school's
district office to conduct the research.

During the summer of 1991, the researcher met once with the teacher for
approximately two and one-half hours to give him experience in using the software and to
discuss the syllabus that would be used. It was mutually agreed that the two classes would
follow the College Board, Advanced Placement Mathematics, AB syllabus and would use
normally used in the calculus course. In addition, the researcher gave the teacher the
instructional handbook accompanying the software, *A Graphic Approach to the Calculus:
Teacher's Guide* (Tall, 1991), *Teacher Notes for Use with A Graphic Approach to the
Calculus* (Tall & Kronmeyer, 1991), and *Calculus: Instructors Preliminary Edition,
(Volume 1)* (Dick & Patton, 1991) for reference and guidance. The first two resources
provide an introduction to the software and examples of how the software can be used in a
classroom setting. The Dick and Patton text is designed to incorporate technology in
calculus instruction and it emphasizes graphical representations of the derivative.

Population and Sample

The population was high school students enrolled in a calculus course. The sample
consisted of two existing calculus classes in a midwestern, suburban high school. The
local linearity class consisted of 13 females and 14 males. The secant-tangent class
consisted of 13 females and 15 males. The students in the two calculus classes were
twelfth-graders, except for one student in the secant-tangent class who was a tenth-grader.
Of the 55 students in both classes, one student was of Afro-American ethnic background
and one student of Asian-Indian ethnic background. The other students were white, non-
Hispanic. The school student population was more ethnically diverse than the two calculus
classes. The classes were called Advanced Placement Calculus and given honors standing.
The teacher followed the Advanced Placement syllabus; although, not all students intended to take the College Board Advanced Placement Mathematics, AB, examination.

**Instruction**

Both treatment groups were taught by the same teacher. Instruction took place in the regular classroom, set up in a rectangular array of 5 desks across and 6 desks deep with the teacher's desk at the front left, and in a small computer room specially established for the instruction. The computer room had 5 rectangular tables. At opposite ends of each table was a NEC Powermate SX/16 computer, 10 in all, with a Quadrant Components color monitor. Students worked at the computers in groups of 2 or 3. Because of the small room, it was impossible to arrange the tables so the teacher could see all of the computer monitors at once.

The fundamental difference between the two treatments was the use by the experimental (local linearity) group of local linearity as the organizing principle for the concept of derivative, while the control (secant–tangent) group used the limiting position of a secant approaching the tangent as the organizing principle for the derivative concept. Both treatment groups used their organizing principle to explore examples and nonexamples of the concept of derivative. Table 1 is an outline of the instruction provided for each of the two groups.

The local linearity group met during the first period of the school day, from 7:30 a.m. to 8:25 a.m. with 5 minutes for announcements at the end of the period. The secant–tangent group met during second period, from 8:30 a.m. to 9:20 a.m. The local linearity group received 20 days of instruction. Because of a senior class assembly, the secant–tangent group received 19 days of instruction.
The local linearity instruction consisted of a four stage process:

1. exploring local straightness of graphs,
2. relating the visual image of the slope to the numerical algorithms by fixing \( x \) and calculating the limit as \( h \) gets small of \( \frac{f(x + h) - f(x)}{h} \) for various values of \( h \), both visually and numerically.
3. viewing the changing slope of a graph as another graph by varying \( x \) in \( \frac{f(x + h) - f(x)}{h} \) to get the derivative as a function,
4. linking these experiences to numeric, symbolic and linguistic representations.

The secant–tangent instruction consisted of exploring the limit of the difference quotient \( \frac{f(x + h) - f(x)}{h} \) in four steps:

1. approach the derivative as the limit of the secant approaching the tangent,
2. relating the visual image of the tangent to the numerical algorithms by fixing \( x \) and calculating the limit as \( h \) gets small of \( \frac{f(x + h) - f(x)}{h} \) for various values of \( h \), both visually and numerically.
3. viewing the changing slope of a graph as another graph by varying \( x \) in \( f'(x) \) to get the derivative as a function,
4. linking these experiences to numeric, symbolic and linguistic representations.
Table 1

**Brief Syllabi for Experimental and Control Instruction.**

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Local Linearity Treatment</th>
<th>Lesson</th>
<th>Secant–Tangent Treatment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>Exploration of functions—linear, quadratic, rational, trigonometric, exponential, and absolute value using the Looking for Formula menu from A Graphic Approach.</td>
<td>1–2</td>
<td>Exploration of functions—linear, quadratic, rational, trigonometric, exponential, and absolute value using the Looking for Formula menu from A Graphic Approach.</td>
</tr>
<tr>
<td>3</td>
<td>Use Draw Graphs and Magnify programs to explore local linearity.</td>
<td>3</td>
<td>No class, senior class assembly.</td>
</tr>
<tr>
<td>4</td>
<td>Generate graphs that are examples and nonexamples of local linearity. Wrinkled graphs.</td>
<td>4</td>
<td>Explore difference quotient with fixed ( x ) and varying ( h ) for a variety of functions numerically and symbolically (Gradient program).</td>
</tr>
<tr>
<td>5</td>
<td>Quiz on identifying symbolic expressions from graphs.</td>
<td>5</td>
<td>Quiz on identifying symbolic expressions from graphs.</td>
</tr>
<tr>
<td>6</td>
<td>Explore difference quotient with fixed ( x ) and varying ( h ) for a variety of functions. (regular classroom)</td>
<td>6</td>
<td>Explore difference quotient with fixed ( x ) and varying ( h ) for a variety of functions. (regular classroom)</td>
</tr>
<tr>
<td>7</td>
<td>Use Gradient program to explore ( \frac{f(x + h) - f(x)}{h} ) for fixed ( x ) and varying ( h ) from left and right sides.</td>
<td>7</td>
<td>Use Gradient program to explore ( \frac{f(x + h) - f(x)}{h} ) for fixed ( x ) and varying ( h ) from left and right sides.</td>
</tr>
<tr>
<td>8</td>
<td>Use Gradient program to explore ( \frac{f(x + h) - f(x)}{h} ) for non-differentiable functions. Use Gradient program to explore slope curve for various ( h ).</td>
<td>8</td>
<td>Use Gradient program to explore ( \frac{f(x + h) - f(x)}{h} ) for non-differentiable functions. Use Gradient program to explore slope curve for various ( h ).</td>
</tr>
<tr>
<td>9</td>
<td>Use Gradient program to explore slope curve for various ( h ).</td>
<td>9</td>
<td>Use Gradient program to explore slope curve for various ( h ).</td>
</tr>
<tr>
<td>10</td>
<td>Quiz on difference quotient and slope curves.</td>
<td>10</td>
<td>Quiz on difference quotient and slope curves.</td>
</tr>
<tr>
<td>11</td>
<td>Use visual tangent at a point to approximate slope.</td>
<td>11</td>
<td>Use visual tangent at a point to approximate slope.</td>
</tr>
<tr>
<td>Lesson</td>
<td>Local Linearity Treatment</td>
<td>Lesson</td>
<td>Secant–Tangent Treatment</td>
</tr>
<tr>
<td>--------</td>
<td>--------------------------</td>
<td>--------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>12</td>
<td>Use <strong>Magnify</strong> to connect slope curve with local linearity. Formally define derivative as difference quotient.</td>
<td>12</td>
<td>Use <strong>Gradient</strong> to connect slope curve with difference quotient. Formally define derivative as difference quotient.</td>
</tr>
<tr>
<td>13</td>
<td>Graph functions by hand. Use tangents to visualize derivative function. Graph derivative function. Use <strong>Gradient</strong> to check. Guess symbolic representation of the derivative for these functions.</td>
<td>13</td>
<td>Graph functions by hand. Use tangents to visualize derivative function. Graph derivative function. Use <strong>Gradient</strong> to check. Guess symbolic representation of the derivative for these functions.</td>
</tr>
<tr>
<td>14</td>
<td>Formally derive $y'$ for $y = x^n$ for positive and negative. Differentiate polynomials using difference quotient.</td>
<td>14</td>
<td>Formally derive $y'$ for $y = x^n$ for positive and negative. Differentiate polynomials using difference quotient.</td>
</tr>
<tr>
<td>15</td>
<td>Quiz on finding derivative using the difference quotient and drawing derivative graphs.</td>
<td>15</td>
<td>Quiz on finding derivative using the difference quotient and drawing derivative graphs.</td>
</tr>
<tr>
<td>16</td>
<td>Formally derive derivatives for $\sin x$, $\cos x$, $\ln x$, and $a^x$ using the difference quotient. Investigate derivatives using <strong>Gradient</strong>.</td>
<td>16</td>
<td>Formally derive derivatives for $\sin x$, $\cos x$, $\ln x$, and $a^x$ using the difference quotient. Investigate derivatives using <strong>Gradient</strong>.</td>
</tr>
<tr>
<td>17</td>
<td>Review left- and right-hand derivatives. Use <strong>Gradient</strong> to investigate functions that are not everywhere differentiable. Connect with local linearity using <strong>Magnify</strong>.</td>
<td>17</td>
<td>Review left- and right-hand derivatives. Use <strong>Gradient</strong> to investigate functions that are not everywhere differentiable.</td>
</tr>
<tr>
<td>18</td>
<td>Investigate chain rule using <strong>Gradient</strong> for composite functions.</td>
<td>18</td>
<td>Investigate chain rule using <strong>Gradient</strong> for composite functions.</td>
</tr>
<tr>
<td>19</td>
<td>Informally derive chain rule. Confirm for various functions using <strong>Gradient</strong>.</td>
<td>19</td>
<td>Informally derive chain rule. Confirm for various functions using <strong>Gradient</strong>.</td>
</tr>
<tr>
<td>20</td>
<td>Quiz on finding equations of tangent lines to a curve and drawing derivative graphs.</td>
<td>20</td>
<td>Quiz on finding equations of tangent lines to a curve and drawing derivative graphs.</td>
</tr>
</tbody>
</table>
**Instrumentation**

Instruments were designed to examine learning activities, classroom roles, and student knowledge of the derivative concept.

**Classroom Roles and Learning Activities**

A taxonomy of classroom roles and learning activities, Systematic Classroom Analysis Notation (SCAN) (Table 2), developed by Fraser, Burkhardt, Coupland, Phillips, Pimm, and Ridgway (1988), was used by the researcher to record classroom events. SCAN "provides a framework within which we can discuss the relationship between the roles of the teacher, the pupils, and other resources such as the micro, and the learning activities that take place in the classroom" (Fraser et al., 1988, p. 310). Each classroom event was recorded as a quintuple, indicating classroom role, role player, the level of demand on students, the learning activity students were engaged in, and student behaviors. For example, EsβpT represents a student explanation of a problem solving situation that involves recall and synthesis and where other students are talking. Instruction was videotaped to provide for cross-referencing and review.
Table 2

**SCAN Taxonomy of Classroom Roles and Learning Activities with Codes**

<table>
<thead>
<tr>
<th>Classroom Roles</th>
<th>Role Player</th>
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<tbody>
<tr>
<td>M Manager</td>
<td>t Teacher</td>
</tr>
<tr>
<td>E Explainer</td>
<td>s Student</td>
</tr>
<tr>
<td>T Task setter</td>
<td>c Computer</td>
</tr>
<tr>
<td>C Counselor</td>
<td>g Calculator</td>
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<td>R Resource</td>
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<tr>
<td>F Fellow Pupil</td>
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<tr>
<td>D Disciplinarian</td>
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</table>

<table>
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<tr>
<th>Demand on Students</th>
<th>Student Learning Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>α Recall</td>
<td>d Didactic</td>
</tr>
<tr>
<td>β Recall and synthesis</td>
<td>s Symbolizing</td>
</tr>
<tr>
<td>γ Extending</td>
<td>i Investigating</td>
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<td></td>
<td>p Problem solving</td>
</tr>
<tr>
<td></td>
<td>h Higher level skills</td>
</tr>
</tbody>
</table>

**Student Behaviors**

- P Passive
- A Active
- T Talking
- O Off task
Working definitions used in recording the quintuples of classroom roles and learning activities are modeled after those used by Fraser and colleagues (1988) and Farrell (1989). The definitions and their coding follow.

**Classroom Roles**

- **Manager (M)** – The player is the authority or director.
- **Explainer (E)** – The player knows and tells.
- **Task Setter (T)** – The player is a questioner, goal setter, or strategy setter.
- **Counselor (C)** – The player serves as advisor and is a member of the group.
- **Resource (R)** – The player responds to requests for information.
- **Fellow Pupil (F)** – The player is on equal footing with all members of the class.
- **Disciplinarian (D)** – The player acts to control perceived inappropriate behavior.

**Role Player**

- **Teacher (t)** – The teacher in the classroom.
- **Student (s)** – A student in the calculus class.
- **Computer (c)** – The software, *A Graphic Approach to the Calculus*.
- **Calculator (g)** – A calculator, either graphing or scientific.

**Demand on Students**

- **Recall (α)** – Recall of a single fact or idea.
- **Recall and synthesis (β)** – Recall of several facts and using or interpreting them.
- **Extending (γ)** – Building on previous skills and concepts to develop new knowledge.
Student Learning Activities

Didactic (d) – The activity of basic instructional guidance.

Symbolizing (s) – An activity in which symbolic or graphic representations are used procedurally.

Investigating (i) – A guessing, checking, or exploring activity in which the aim is to develop intuition and the outcome is unknown.

Problem solving – An activity in which a plan is developed and implemented to solve and unknown problem.

Higher level skills (h) – An activity in which concepts are formulated and linked, generalizations are made, or a proof is generated.

Student Behaviors

Passive (P) – When students are simply listening to someone talk.

Active (A) – When students are engaged in note taking, writing, or doing mathematics.

Talking (T) – When students are talking about mathematics.

Off task (O) – When students engage in nonmathematical behavior.

Calculus Concepts Inventories

Two instruments were used to measure and explore student understanding of the derivative concept and concepts underlying the study of calculus. A pretest (Appendix A, Calculus Concepts Inventory) was given to both calculus classes for 30 minutes two days before instruction began. The items on the pretest were designed to articulate with prior research on student understanding of concepts foundational to the study of the derivative and to indicate what knowledge of the derivative students may have retained from previous study, if any.
On the pretest, Item 1, parts i through vi, was used by Tall (1986) to study student understanding of rate of change. Orton (1983) used a similar question to make meaningful the distinction between rate of change over an interval and rate of change at a point. Item 2 (Tall, 1986) examined whether a student was able to generate the concept of a chord approaching a tangent intuitively. Part A asked for the slope of a chord when given two points on the chord. Part B suggested a limiting process to find the slope of the tangent. Part C wanted the student to explain her or his reasoning. The purpose of Item 3 (Tall, 1986) was to explore student understanding of the concept of tangent and the limiting process of a chord approaching a tangent. Item 4 (Tuite, 1990) attempted to determine whether or not a student was able to identify the slope of a function at a point as the slope of the tangent at the point. Item 5 linked the slope of a function at a particular point to the more general slope function. Item 6 sought to identify if a student could identify a function that was given in difference quotient form, a standard way in which the derivative is defined. The six parts of Item 7 were concerned with student ability to do symbolic differentiation.

The posttest consisted of three parts and was administered during the two class periods immediately following the completion of instruction. The first part of the posttest consisted of the pretest items (Appendix B, Calculus Concepts Inventory I). Students had 30 minutes in which to complete this part. The second part of the posttest consisted of 10 graphs of functions (Appendix C). Five of the graphs also included the symbolic representation of the function. Each graph was displayed on the overhead projector for 2 minutes. The student was asked to determine if the function were differentiable at all points for which it was defined. If the function was differentiable, then the student was asked to sketch the graph of the derivative. If the function was not differentiable at all points, then the student was asked to write an explanation of why the function was not differentiable. Students were provided with 10 axes on which to record their responses. The purpose was
to look at student reasons for determining differentiability or non-differentiability of a function.

The third part of the posttest (Appendix D, Calculus Concepts Inventory II) consisted of 8 items. The four parts of Item 8 (Tall, 1986) asked the student to sketch the graph of the derivative of a function when given the graph of the function. The intent was to explore student understanding of the derivative as a function. Item 9 gave the graph of the derivative of a function, \( f'(x) \), and asked for a graph of the function \( f(x) \). Here, a student needed to recognize both global and pointwise properties of the derivative function. Item 10 asked the student to provide a graphical example of a function that was not differentiable at a particular point and an explanation of why the function was not differentiable. Item 11 asked for a symbolic representation and explanation for the same situation. It was expected that students in the local linearity treatment might more easily create these examples and use local linearity as a justification for non-differentiability. Item 12 presented distance versus time data in numeric form and asked the student for the velocity at a particular time. A student who connects velocity with the derivative could more accurately represent the approximate velocity. Items 13 through 15 were aimed at eliciting student conceptual images and definitions for the concepts of slope, tangent, and derivative.
**Student Interviews**

After the posttest was administered, 13 students—7 from the local linearity treatment and 6 from the secant-tangent treatment—were interviewed for approximately 50 minutes each. The students were selected based on a sign-up indicating that they had a free period during the school day. Students representing different ability levels, based on scores from the pretest, were included in the interviews.

A protocol for the interviews (Appendix E) was developed. The protocol explored the student's definition of the derivative both pointwise and as a function, the student's definition of the tangent, differentiable and non-differentiable functions, examples and nonexamples of the differentiability of a piecewise defined function, and the symmetric difference quotient—an alternative representation of the derivative. A computer and the graphing software were available during the interviews. The interviews were tape recorded and transcribed.

**Student Evaluations**

Finally, at the completion of the treatment and after the posttest had been administered, students were asked to evaluate the instruction by responding to two questions: (1) What have you liked or disliked about using the computer and software in your study of calculus? and (2) Do you think using the computer software has given you a better understanding of calculus?

**Data Analysis**

Analyses of data corresponded to the four aspects of the instrumentation: classroom roles and learning activities, interviews with students, pretest-posttest learning, and student evaluation. First, summary descriptive statistics for classroom roles, role player, demand on students, student learning activities, and student behaviors were compiled from the
transcripts of classroom events. These statistics were compared to confirm consistency of
instruction between the two treatment groups and were examined to look for patterns of
instruction. Second, transcripts of the interviews with students were analyzed for concept
images and concept definitions involving the derivative, tangent, and differentiability.
Third, exploratory data analysis was conducted to gain an overview of the pretest–posttest
data. Then, descriptive statistics were compiled on the pretest–posttest items for the two
groups. A multivariate analysis of covariance was performed with the posttest total score
as the dependent variable, local linearity or secant–tangent group and sex as independent
variables, and with pretest total score as the covariate. Finally, student evaluations were
read and categorized based on student perceptions of instruction, technology, and the role
of visualization.
CHAPTER IV
RESULTS

The analysis of data is presented in five parts. First, the local linearity group and the secant–tangent group are compared on the basis of the pretest (Calculus Concepts Inventory). Second, the data collected using the SCAN taxonomy of classroom roles and learning activities (Table 2, p. 33) are summarized and analyzed for patterns of instruction. Consistency of instruction between the two groups is examined. Third, statistical results of the test of the major hypothesis, that the local linearity group would achieve a different and more complete understanding of the concepts of differentiability and differentiation, are discussed. Fourth, analyses of student understanding of calculus concepts are presented using the common items from the pretest (Calculus Concepts Inventory) and the posttest (Calculus Concepts Inventory I), the graphing of derivatives from graphs of functions portion of the posttest, and the items from the posttest (Calculus Concepts Inventory II). Finally, student evaluations of instruction and the use of visualization are presented.

Comparison of Local Linearity and Secant–Tangent Groups on the Basis of the Pretest

The treatments were randomly assigned to the two classes prior to the pretest. The pretest (Calculus Concepts Inventory) was administered before instruction began. On the basis of pretest scores, it is clear the classes were not equivalent at the start of instruction. Figure 7, Calculus Concepts Inventory Scores (Pretest), shows that the top quartile of the secant–tangent group was only slightly above the median level of the local linearity.
group. The median score for the secant–tangent group was at the same level as the bottom quartile of the local linearity group.

![Box plots comparing local linearity group and secant-tangent group](image)

Figure 7. Calculus Concepts Inventory Scores (Pretest)

**Classroom Roles and Learning Activities**

All of the fourteen class sessions in which instruction on differentiability and the derivative took place for the local linearity group and the thirteen class sessions in which instruction on differentiability and the derivative took place for the secant–tangent group were recorded in real-time using the SCAN taxonomy of classroom roles and learning activities. For example, an eighteen minute segment from lesson 9 of the local linearity group was recorded as follows. The left column indicates the time when each episode began. The segment started at 7:38 a.m. Using the SCAN taxonomy each quintuple (between slashes) records the classroom role, the role player, the level of demand on students, the type of learning activity, and student behavior.
During this session, students were using the *Gradient* program to explore the slope curve defined by \( \frac{f(x+h) - f(x)}{h} \) for fixed \( h \) and varying \( x \) for a variety of functions. The teacher began by setting the task and asked if everyone understood the task (T\( \beta \)iA/Q\( \alpha \)iA). A student responded "No" (E\( \alpha \)sA). This was followed by the teacher re-explaining the task (E\( \beta \)iA). Students began working in their groups, using the software to investigate (--\( \beta \)iA). Because the students were working in groups and using the software to explore, classroom roles and role players were undetermined. The investigation lasted 10 minutes. The teacher concluded the computer work by making a writing assignment (T\( \beta \)hA). The teacher then told the students to talk in groups about the relationship between the slope curve and the original function as \( h \) assumed smaller values (T\( \beta \)hA). After 6 minutes of discussion, the teacher asked the students to turn off the computer monitors (M---). During each of the events prior to this, students were actively involved in writing, talking, or interacting with the computer software.

Tables 3 and 4 display the number of minutes per episode, the number of episodes per class period, the average time spent during an episode, and the time spent out of a 50 minute class period on instruction. Episodes consisted of instruction around a similar problem, theme, or topic. When instruction changed its theme, introduced a new problem, or started on a new topic, the researcher recorded the time. The elapsed time between recordings was counted as the duration of an episode.
Tables 3 and 4 demonstrate the teacher was consistent in instruction between the local linearity and the secant–tangent groups. The average length of an episode for both the local linearity group and the secant–tangent group was 8 minutes. Moreover, the teacher made full use of each class period. Class periods were 55 minutes for the local linearity group, including 5 minutes for announcements and 50 minutes for the secant–tangent group. The average number of minutes spent on instruction was 46 minutes for the local linearity group and 47 minutes for the secant–tangent group. It should be noted that this instructional pattern does not follow the static routine reported by Romberg and Carpenter (1986).

There appears to be little difference in instructional patterns between when the two classes met in their regular classroom, when they met in the computer room, and when they moved from the regular classroom to the computer room. Lessons 6, 11, and 16 were conducted in the regular classroom without the aid of graphing technology. Lessons 13, 14, and 18 were begun in the regular classroom and continued in the computer facility. The remainder of the lessons were conducted entirely in the room containing computers; although, the computers were not in use all of the time.

When classroom roles and role players are examined, it is evident that the teacher dominates classroom instruction, being the primary role player between 85% and 87% of the time in both classes (Table 6). Yet, the primary roles exhibited in the classroom were distributed almost equally between explainer and task setter (Table 5). The computer software never assumed a position as a primary role player. This is in contrast to Fraser and colleagues (1988), who found the computer becoming an active role player in the classroom.
Table 3

Length in Minutes of Classroom Episodes and Number of Classroom Episodes for Local Linearity Group

<table>
<thead>
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<th>Lesson</th>
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Number of Minutes of Class time Used: 44 36 45 50 50 45 43 42 44 44 49 50 50 50 50

Average Length of an Episode in Minutes: 4.0 5.1 7.5 10.0 5.6 4.5 5.4 3.8 8.8 10.0 8.2 5.0 5.0 5.0
Table 4

Length in Minutes of Classroom Episodes and Number of Classroom Episodes for Secant-Tangent Group

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Number of Minutes of Class time Used

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<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>50</td>
<td>46</td>
<td>50</td>
<td>50</td>
<td>50</td>
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<td>46</td>
<td>51</td>
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<td>48</td>
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<tr>
<td>6</td>
<td>6.8</td>
<td>6.3</td>
<td>5.6</td>
<td>5.5</td>
<td>6.4</td>
<td>6.1</td>
<td>7.5</td>
<td>5.3</td>
<td>5.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Average Length of an Episode in Minutes
Table 5

Percentage of Classroom Events in Which Each Role Was Observed.

<table>
<thead>
<tr>
<th>Role</th>
<th>Local Linearity Group</th>
<th>Secant–Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manager</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Explainer</td>
<td>40</td>
<td>43</td>
</tr>
<tr>
<td>Task Setter</td>
<td>37</td>
<td>32</td>
</tr>
<tr>
<td>Questioner</td>
<td>19</td>
<td>21</td>
</tr>
<tr>
<td>Counselor</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6

Percentage of Classroom Events in Which Each Role Player Was Observed.

<table>
<thead>
<tr>
<th>Role Player</th>
<th>Local Linearity Group</th>
<th>Secant–Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher</td>
<td>87</td>
<td>85</td>
</tr>
<tr>
<td>Student</td>
<td>13</td>
<td>15</td>
</tr>
<tr>
<td>Computer</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Calculator</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The demand placed on students involved synthesizing procedures and concepts more than recall of information (Table 7). This is confirmed by the almost equal levels of symbolizing and investigating learning activity in the local linearity group (Table 8). While the secant–tangent group was engaged in symbolizing more frequently than in investigating, both activities indicate a move away from the traditional didactic style of the mathematics classroom (cf. Romberg & Carpenter, 1986). The lack of evidence for problem solving activity may be attributed to the introductory nature of the course content and to the first-time use by the teacher of technology in the classroom. Noticeable
is the balance between active and passive student behaviors (Table 9). Students were engaged in writing, discussing, and doing mathematics in approximately one-half of the classroom events. Off task behaviors were not observed.

Table 7

Percentage of Classroom Events in Which Each Demand Level Was Observed

<table>
<thead>
<tr>
<th>Demand Level</th>
<th>Local Linearity Group</th>
<th>Secant-Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recall</td>
<td>41</td>
<td>49</td>
</tr>
<tr>
<td>Recall and Synthesis</td>
<td>53</td>
<td>48</td>
</tr>
<tr>
<td>Extending</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 8

Percentage of Classroom Events in Which Each Learning Activity Was Observed

<table>
<thead>
<tr>
<th>Learning Activity</th>
<th>Local Linearity Group</th>
<th>Secant-Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Didactic</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Symbolizing</td>
<td>42</td>
<td>52</td>
</tr>
<tr>
<td>Investigating</td>
<td>42</td>
<td>37</td>
</tr>
<tr>
<td>Problem Solving</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Higher Level Skill</td>
<td>8</td>
<td>7</td>
</tr>
</tbody>
</table>
Table 9
Percentage of Classroom Events in Which Each Student Behavior Was Observed

<table>
<thead>
<tr>
<th>Behavior</th>
<th>Local Linearity Group</th>
<th>Secant–Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>48</td>
<td>56</td>
</tr>
<tr>
<td>Active</td>
<td>52</td>
<td>44</td>
</tr>
</tbody>
</table>

This discussion of classroom roles and teaming activities indicates that instruction for the two groups was equivalent. The number of classroom episodes per class and the time spent on each episode were uniform. The instructional style used in both classes was similar in terms of the roles assumed and the role players. The level of demand on students and the types of learning activities engaged in were approximately the same for both groups. Students in both groups were similarly engaged in both active and passive behaviors.

Test of Major Hypothesis

The major hypothesis was that students in the local linearity group would achieve a different and more complete understanding of the concepts of differentiability and differentiation than students in the secant–tangent group. Quantitative information to test this hypothesis was obtained using a variety of instruments: the Calculus Concepts Inventory (Appendix A), the Calculus Concepts Inventory I (Appendix B), the Calculus Concepts Inventory II (Appendix D), and the graphing of the derivative from the graph of the function section of the posttest (Appendix C). Student scores on the Calculus Concepts Inventory were obtained before instruction was begun. The Calculus Concepts Inventory score was used as a covariate in the statistical analysis. The scores from the Calculus Concepts Inventory I which contained the same items as the Calculus Concepts Inventory,
the scores on graphing of the derivative from the graph of the function section, and the scores from the Calculus Concepts Inventory II were combined to yield a posttest score.

Plots of the posttest score against the covariate, the Calculus Concepts Inventory score, show a clear linear relationship between the covariate and the posttest score for the secant–tangent group (Figure 8). This same relationship is not evident in the plot for the local linearity group (Figure 9).

![Graph of Posttest Score versus Pretest Score for Secant–Tangent Group](image)

Figure 8. **Graph of Posttest Score versus Pretest Score for Secant–Tangent Group**
Figure 9. Graph of Posttest Score versus Pretest Score for Local Linearity Group

The method proposed for data analysis was analysis of covariance, including the Calculus Concepts Inventory Score as the covariate, and including treatment and gender in the model as independent variables. The foregoing analysis suggests the use of the Calculus Concepts Inventory Score as the covariate was appropriate. A preliminary test showed that gender was not significant in the model. The final analysis of covariance with the posttest score as the dependent variable is summarized in Table 10.
The significance of the interaction between the treatment (local linearity group or secant–tangent group) and the pretest score, Calculus Concepts Inventory, confirms the earlier observation that the regression model for the two treatment groups is different—that is, they have different slopes. Non–homogeneity of regression violates the basic assumptions for analysis of covariance (White, 1980). Thus, any further attempt to compare the two groups on the basis of an analysis of covariance is inappropriate.

On the outcome measure, there appears to be little difference between the two treatments. A comparison of pretest and posttest mean scores exhibits little difference between the two groups on the posttest score (Table 11). The similarity between treatment groups on the posttest is also evident in the plot in Figure 10.
It was noted previously, on the basis of pretest scores, that the two groups differed significantly. The evidence suggests the secant–tangent group started with less understanding of the concepts of differentiability and differentiation than the local linearity group. Yet, on the posttest the secant–tangent group demonstrated similar understanding to the local linearity group. Thus, the aspect of the major hypothesis that proposed the local linearity group would achieve a more complete understanding of the concepts of differentiability and differentiation than the secant–tangent group must be rejected.

The rejection of a portion of the major hypothesis requires analyses of the subsections of the posttest in order to explore where and how the groups differed in performance. An initial comparison of the two treatment groups when they are blocked by quartile score on the Calculus Concepts Inventory shows that the largest gains on the posttest were made by the students in the upper quartile of the secant–tangent group while
the relative ranking of the lower quartiles remained stable (Table 12). This suggests that the secant-tangent treatment may have been more effective for students who already possessed a better understanding of concepts underlying the study of calculus.

Table 12. Mean Scores on Pretest and Posttest, Blocked by Pretest Quartile

<table>
<thead>
<tr>
<th></th>
<th>Upper Quartile</th>
<th>Third Quartile</th>
<th>Second Quartile</th>
<th>Bottom Quartile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus Concepts Inventory</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local Linearity Group</td>
<td>23.3</td>
<td>16.8</td>
<td>13.3</td>
<td>5.0</td>
</tr>
<tr>
<td>Secant-Tangent Group</td>
<td>22.6</td>
<td>13.7</td>
<td>10.0</td>
<td>4.7</td>
</tr>
<tr>
<td>Posttest (raw means)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local Linearity Group</td>
<td>64.7</td>
<td>62.8</td>
<td>58.0</td>
<td>57.7</td>
</tr>
<tr>
<td>Secant-Tangent Group</td>
<td>71.3</td>
<td>57.6</td>
<td>57.7</td>
<td>48.9</td>
</tr>
</tbody>
</table>

The Calculus Concepts Inventory and the Calculus Concepts Inventory I consisted of the same set of 7 items. A comparison of scores on the Calculus Concepts Inventory I (Figure 11) shows that the secant-tangent group achieved at almost the same level as the local linearity group on the portion of the posttest that contained the same items as the pretest. This is in contrast to the pretest (Figure 7) where the secant-tangent group scored well below the local linearity group. This result suggests that the secant-tangent treatment enabled students in the group to become more proficient with concepts on which they were initially weaker than the local linearity group. Specific items contributing to the gains from pretest to posttest will be examined in the individual item analysis.
On the sketching the graph of the derivative of a function section of the posttest, the local linearity group performed at a lower level than the secant–tangent group (Figure 12). Thus, it appears that during the course of instruction the secant–tangent group became better at understanding the relationship between the graph of a function and the graph of its derivative.
On the Calculus Concepts Inventory II section of the posttest the secant–tangent group performed at a substantially higher level than the local linearity group (Figure 13). The median score for the secant–tangent group was at the upper quartile for the local linearity group. Specific items that caused the differences in scores will be examined in the item analysis section.
Analysis of the data resulted in the rejection of the hypothesis that the local linearity group would achieve a more complete understanding of the concepts of differentiability and differentiation than the secant–tangent group. In fact, the analysis indicates that the secant–tangent group improved their understanding more substantially than the local linearity group. The secant–tangent group began instruction with less understanding of the concepts as indicated by the pretest. On the posttest the two groups achieved similar results. The secant–tangent group made greater gains in performance than the local linearity group on common items from the pretest. The secant–tangent group demonstrated better ability to sketch the graph of the derivative of a function. On the Calculus Inventory II portion of the posttest the secant–tangent group scored substantially better than the local linearity group.
The second aspect of the major hypothesis claimed that students in the local linearity group would achieve a different understanding of the concepts of differentiability and differentiation than the secant-tangent group. Evidence for this difference in understanding will be examined in the analysis of specific items that follows.

Analysis of Pretest and Posttest Items

Calculus Concepts Inventory and Calculus Concepts Inventory I

Item 1 (Figure 14) on the Calculus Concepts Inventory and the Calculus Concepts Inventory I was designed to study students' knowledge of the concept of slope and rate of change. Students were asked to determine the rate of change between six pairs of points on the graph. Orton (1983) used a similar item. Tall (1986) used the same item. Students received 2 points for a correct response to each part, 1 point for a response with an incorrect sign, and 0 points for an incorrect response, with a possible 12 points overall. Median student scores for the six parts of item 1 are given in Table 13.
1. Find the average rate of change between the following points on the graph:

   Note: the "average rate of change" from \( P \) to \( Q \) means the slope of line segment \( PQ \).

   - i. from \( C \) to \( D \)
   - ii. from \( D \) to \( E \)
   - iii. from \( A \) to \( B \)
   - iv. from \( B \) to \( C \)
   - v. from \( C \) to \( E \)
   - vi. from \( D \) to \( C \)

Figure 14. Item 1 on Calculus Concepts Inventory and Item 1 on Calculus Concepts Inventory I. (See Appendices A and B.)
The distribution of points for item 1 in Table 14 confirms what Orton (1983) and Tall (1985a, 1985b, 1985c, 1986) have found, that students beginning the study of calculus have a weak understanding of the concept of slope. While showing improvement from pretest to posttest, the results indicate that students continue to have difficulties. Primary sources of error on the pretest were not remembering the formula for the slope and inverting the slope formula to change in \( x \) divided by change in \( y \). Arithmetic errors were prevalent. Part (v) caused particular difficulty on both the pretest and posttest. Students were confused over "The slope is zero," "There is no slope," and "The slope is undefined." Part (vi) resulted in numerous sign errors and the inability to determine rate of change from right to left. One student on both the pretest and posttest thought this was a multiple-choice question.

Item 2 (Figure 15) examined student ability to generate the concept of a chord approaching a tangent. Students received 2 points for each correct response to parts A and B. On the pretest, about two-thirds of the local linearity group and one-half of the secant-tangent group were able to write a correct expression for the slope of the secant line through points A (1, 1) and B (\( k, k^2 \)) (Table 15). Slightly more than three-fourths of the students in both groups answered part A correctly on the posttest. The improvement in correct responses for the secant-tangent group showed an increased ability to set up the slope formula.
Table 14

Distribution of Points for Item 1

<table>
<thead>
<tr>
<th>Part</th>
<th>Points</th>
<th>Pretest</th>
<th>Posttest</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>7</td>
<td>5</td>
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<tr>
<td></td>
<td>1</td>
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<td>21</td>
<td>25</td>
<td>19</td>
<td>23</td>
</tr>
<tr>
<td>ii</td>
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<td>4</td>
<td>1</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>1</td>
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<td>22</td>
<td>24</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>iii</td>
<td>0</td>
<td>8</td>
<td>1</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td>18</td>
<td>24</td>
<td>16</td>
<td>20</td>
</tr>
<tr>
<td>iv</td>
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<td>6</td>
<td>5</td>
<td>7</td>
<td>6</td>
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<td>0</td>
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<tr>
<td></td>
<td>2</td>
<td>17</td>
<td>23</td>
<td>13</td>
<td>18</td>
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<tr>
<td>vi</td>
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<td>13</td>
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</tr>
<tr>
<td></td>
<td>1</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>14</td>
<td>19</td>
<td>9</td>
<td>15</td>
</tr>
</tbody>
</table>

Entries are the number of individuals receiving the given number of points.

Part B asked students to find the slope of line AT, the tangent line to the graph. A correct solution to this might indicate that a student let \( k \) approach 1 in the slope formula obtained in part A, thus demonstrating the creation of the notion of the secant approaching the tangent. Fewer than one-half the students in either group gave the slope as 2 on the pretest (Table 16). When students were asked for a mathematical explanation of their response, part C, no student used a limiting argument. This result was consistent with those Tall (1986) found in his use of the problem. The findings suggest the visual image
of the secant approaching the tangent as a foundational image for learning the concept of
derivative may not be intuitive for students.

The mathematical explanations in part C fell into several categories (Table 17):
using the derivative—that is, the derivative of $x^2, 2x$, evaluated at $x = 1$ is 2; the
coordinates of the $y$-intercept of the line $AT$ were read from the graph as $(0, -1)$ and used
to find the slope without making reference to the tangent; setting up but not evaluating a
slope formula; saying that the coordinates of point $T$ were needed to find the slope; and
setting up but not evaluating a difference quotient, $\frac{(1 + h)^2 - 1}{h}$. Even after having
studied the concept of derivative, fewer than one-fourth of the students in either treatment
recognized this situation involved the derivative on the posttest. It appears that many
students, wanting to find the slope of line $AT$, used two points on the line whose
coordinates they thought they knew, $(1, 1)$ and $(0, -1)$. The image of slope at a point used
by most students consisted of finding the slope of a line between two points rather than
appealing to the notion of the slope of a tangent line.
2. On the graph $y = x^2$, the point $A$ is $(1, 1)$, the point $B$ is $(k, k^2)$ and $T$ is a point on the line tangent to the graph at $A$.

A. Find the slope of the straight line through the points $A$ and $B$.

B. Find the slope of the tangent line $AT$.

C. Explain how you might find the slope of the tangent line $AT$ using basic principles.

Figure 15. *Item 2 on Calculus Concepts Inventory and Calculus Concepts Inventory.*
(See Appendices A and B.)
Table 15

Student Responses to Item 2A

<table>
<thead>
<tr>
<th>Group</th>
<th>Pretest</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local Linearity Group</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Response: k + 1</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>Response: k - 1</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>Number of Students</td>
<td>18</td>
<td>21</td>
</tr>
<tr>
<td>Answering k - 1 Correctly</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Other Incorrect Response</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Response</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Secant-Tangent Group</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Response: k + 1</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>Response: k - 1</td>
<td>8</td>
<td>23</td>
</tr>
<tr>
<td>Number of Students</td>
<td>13</td>
<td>36</td>
</tr>
<tr>
<td>Answering k - 1 Correctly</td>
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<td>1</td>
</tr>
<tr>
<td>Other Incorrect Response</td>
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<td>4</td>
</tr>
<tr>
<td>Response</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Tall's Experimental Group</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Response: k + 1</td>
<td>15</td>
<td>13</td>
</tr>
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<td>Response: k - 1</td>
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<td>Number of Students</td>
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<td>Answering k - 1 Correctly</td>
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<td>1</td>
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<tr>
<td>Other Incorrect Response</td>
<td>43</td>
<td>41</td>
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<tr>
<td>Response</td>
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<td>6</td>
</tr>
<tr>
<td>Tall's Control Group</td>
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<td></td>
</tr>
<tr>
<td>Response: k + 1</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Response: k - 1</td>
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<td>37</td>
</tr>
<tr>
<td>Answering k - 1 Correctly</td>
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<tr>
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<tr>
<td>Response</td>
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<td>1</td>
</tr>
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<td>Tall's University Group</td>
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</tr>
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</tr>
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<td>3</td>
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<tr>
<td>Other Incorrect Response</td>
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</tr>
<tr>
<td>Response</td>
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<td>0</td>
</tr>
</tbody>
</table>

The table indicates the number of students giving each response.
Table 16

Student Responses to Item 2B

<table>
<thead>
<tr>
<th>Response:</th>
<th>2</th>
<th>Incorrect</th>
<th>No Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local Linearity Group</td>
<td>Pretest 12</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>Posttest 16</td>
<td>4</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Secant–Tangent Group</td>
<td>Pretest 9</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>Posttest 17</td>
<td>9</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The table indicates the number of students giving each response.
Table 17

**Student Justification of Responses for Item 2B**

<table>
<thead>
<tr>
<th></th>
<th>Justification:</th>
<th>Used Derivative</th>
<th>Used y-intercept</th>
<th>Used slope formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local Linearity Group</td>
<td>Pretest</td>
<td>1</td>
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<tr>
<td></td>
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<td>7</td>
<td>8</td>
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<tr>
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<td>Pretest</td>
<td>3</td>
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<td>2</td>
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<table>
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<tr>
<th></th>
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<th>Need coordinates of T</th>
<th>Difference Quotient</th>
<th>Other</th>
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<tbody>
<tr>
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<tr>
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<td>3</td>
<td>0</td>
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<td>Pretest</td>
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<tr>
<td></td>
<td>Postest</td>
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<table>
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</thead>
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<td></td>
<td>Postest</td>
<td>6</td>
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<tr>
<td>Secant-Tangent Group</td>
<td>Pretest</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Postest</td>
<td>8</td>
</tr>
</tbody>
</table>

The table indicates the number of students giving each response.
Item 3 (Figure 16) explored student understanding of the concept of tangent and language associated with the limiting process as a secant approached a vertical tangent. On the pretest, slightly more than one-half the students in each group thought the function $y = \sqrt{x}$ had a tangent at the point $(0, 0)$ (Table 18). On the posttest, the number of students who thought the tangent existed increased somewhat. There were students, however, who changed from thinking there was a tangent to thinking there was not a tangent (Table 19). Moreover, of the secant-tangent group who thought on the pretest the tangent existed, about one-half did not agree that the tangent was vertical. One-fourth of the local linearity group who thought there was a tangent did not support the existence of a vertical tangent. There was similar disagreement about the slope of the tangent both on the pretest and posttest. Between 30 and 50 percent of the students who thought the tangent existed did not think its slope was infinite.

Among those students who thought the tangent at $(0, 0)$ did not exist there were several common themes between both groups. Students perceived the chord AB as the tangent line and stated a variation of, "A tangent cannot cross at two places." This perception occurred most frequently as an explanation on the pretest. Other students thought the function $y = \sqrt{x}$ was undefined at the origin. Another explanation for there being no tangent at the origin was that the slope of a possible tangent was undefined. Thus, the tangent did not exist. One student suggested the tangent should be on the outside not on the inside of a graph.
The statements in parts (iv), (v), and (vi) use the statements "tends to infinity," "has infinity as its limit," and "increases without limit" in nearly identical senses. Students in the local linearity group tended to maintain their beliefs about the truth or falsity of these statements between pretest and posttest. The proportion of students in the secant–tangent group agreeing with the statement in part (iv) increased from pretest to posttest. Perhaps this was due to developing the derivative through the limiting process. More students in the secant–tangent group also agreed with the statement in (vi).
3. The diagram represents the graph of the function \( y = \sqrt{x} \) (taking the positive square root for \( x \geq 0 \)). A is the point (0, 0) and B is the point \((h, \sqrt{h})\).

Circle the letter of your response for each of the following statements:

i. The graph has a tangent at point A.
   
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

If your response is C or D, explain why in the following space, then omit (ii) and (iii).

ii. The tangent at A is vertical.
   
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

iii. The slope of the tangent at A is infinite.
   
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

Figure 16. Item 3 from Calculus Concepts Inventory and Calculus Concepts Inventory I. (See Appendices A and B.)
Figure 16 (continued).  

Item 3 from Calculus Concepts Inventory and Calculus Concepts Inventory I

The diagram, exactly the same as the one on the previous page, represents the graph of the function $y = \sqrt{x}$ (taking the positive square root for $x \geq 0$). A is the point $(0, 0)$ and B is the point $(h, \sqrt{h})$.

Everyone should answer questions (iv), (v), and (vi). Circle your response.

v. As $B \to A$, the slope of the line AB tends to infinity.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

v. As $B \to A$, the slope of the line AB has infinity as its limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

vi. As $B \to A$, the slope of the line AB increases without limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.
Table 18

Responses to Item 3

<table>
<thead>
<tr>
<th>Part</th>
<th>Graph has a tangent at A.</th>
<th>Absolutely Certain Statement is True</th>
<th>Statement is True</th>
<th>Statement is False</th>
<th>Absolutely Certain Statement is False</th>
<th>Omitted</th>
</tr>
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<tbody>
<tr>
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<td>Local Linearity Group</td>
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<tr>
<td></td>
<td></td>
<td>postest 16</td>
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<td>7</td>
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<td></td>
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<td>9</td>
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<td>Tangent is vertical at A.</td>
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<td></td>
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<td>2</td>
<td>1</td>
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<tr>
<td></td>
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<td>postest 9</td>
<td>6</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>iii</td>
<td>Slope of tangent is infinite</td>
<td>Local Linearity Group</td>
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<td>1</td>
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<td></td>
<td>postest 6</td>
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<td>3</td>
<td>1</td>
</tr>
<tr>
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<td>4</td>
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<tr>
<td></td>
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<td>postest 2</td>
<td>9</td>
<td>4</td>
<td>3</td>
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</table>
Table 18 (continued)

Responses to Item 3

<table>
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<tr>
<th>Part</th>
<th>Absolutely Certain</th>
<th>Statement is True</th>
<th>Statement is False</th>
<th>Absolutely Certain</th>
<th>Statement is False</th>
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<td></td>
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<td></td>
</tr>
<tr>
<td>iv</td>
<td>Slope tends to infinity.</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
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<td>9</td>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>v</td>
<td>Slope has infinity as limit.</td>
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<tr>
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<td>2</td>
<td></td>
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<td>8</td>
<td>9</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>vi</td>
<td>Slope increases without limit.</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Local Linearity Group</td>
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<tr>
<td></td>
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<td>7</td>
<td>12</td>
<td>8</td>
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</table>
on the postest than on the pretest. Students may have developed better understanding of
the undefined slope of a vertical line. The fact that fewer students in both groups agreed
with the statement in (vi) than the statements in (iv) and (v) suggests conflicting images.
Those who view statement (v) positively have a bounded view of approaching infinity.
Those who view statement (vi) positively see infinity as unbounded. Students may,
however, hold both views concurrently (Williams, 1991). The linguistic interpretation of a
limit as a bound may also conflict with the phrasing "increases without limit" as Monaghan
(1992) has noted.

Table 19

Number of Students Who Changed Their Responses to Item 3 From Pretest to Posttest

<table>
<thead>
<tr>
<th>Part</th>
<th>False to True</th>
<th>True to False</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>Local Linearity Group: 9, Secant-Tangent Group: 6</td>
<td>3, 4</td>
</tr>
<tr>
<td>iv</td>
<td>Local Linearity Group: 8, Secant-Tangent Group: 11</td>
<td>6, 2</td>
</tr>
<tr>
<td>v</td>
<td>Local Linearity Group: 4, Secant-Tangent Group: 5</td>
<td>7, 6</td>
</tr>
<tr>
<td>vi</td>
<td>Local Linearity Group: 5, Secant-Tangent Group: 6</td>
<td>4, 2</td>
</tr>
</tbody>
</table>

Item 4 (Figure 17) was designed to determine whether or not a student was able to
relate the slope of a curve at a particular point to the slope of the tangent line at that point.
When correct responses (Table 20) and student explanations (Table 21) are compared, it
becomes clear that the overriding procedure when a student is asked to find slope is to find
two points. In this instance, most students used the point at which they were to find the tangent, (5, 3), and the y-intercept of line L, (0, 1), without connecting slope at a point with the slope of the tangent in their explanation. Several students were distracted by the y-intercept of the curve \( f \), (0, 3.5). On the pretest, three students in the secant–tangent class made specific reference to the slope of the tangent line \( L \) as the slope at the point \( x = 5 \). While this reference may indicate student generated knowledge of the relationship between slope and tangent, it more likely a result of the students' having studied the concept in physics the previous year.

Tuft (1990a) used a similar question with university calculus students, asking for \( f'(5) \) instead of for the slope of \( f \) at \( x = 5 \). Students who had studied computer programming in conjunction with calculus were more likely than students who studied only calculus to answer this question correctly. Tuft suggested this is one indication that the computer programming students develop richer schema for the concepts of calculus. Tuft did not ask students to explain how they obtained their solution. The summary of student explanations in Table 21 suggests a student may be more likely to solve this problem by using the numerical values of two points obtained from visual cues than by making connections with the derivative.

Table 20

<table>
<thead>
<tr>
<th></th>
<th>Correct</th>
<th>Incorrect</th>
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<tr>
<td>Postest</td>
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<td>10</td>
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<tr>
<td>Secant–Tangent Group</td>
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<tr>
<td>Pretest</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>Postest</td>
<td>11</td>
<td>17</td>
</tr>
</tbody>
</table>

Entries are the number of students.
4. Suppose line $L$ is tangent to the curve $y = f(x)$ at the point $(5, 3)$ as indicated in the following graph.

What is the slope of $f(x)$ at $x = 5$?

Explain how you obtained the value of the slope of $f(x)$ at $x = 5$.

Figure 17. Item 4 from Calculus Concepts Inventory and Calculus Concepts Inventory I. (See Appendices A and B.)
Table 21

Student Explanations of Responses in Item 4

<table>
<thead>
<tr>
<th>Justification:</th>
<th>Referred to slope of tangent line</th>
<th>Used y-intercept (0.1)</th>
<th>Used y-intercept (0.3-5)</th>
</tr>
</thead>
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<td>Local Linearity Group</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>Postest 5</td>
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<td>1</td>
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</table>

<table>
<thead>
<tr>
<th>Justification:</th>
<th>No Response</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Local Linearity Group</td>
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<tr>
<td></td>
<td>Postest 3</td>
<td>4</td>
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<tr>
<td>Secant–Tangent Group</td>
<td>Pretest 15</td>
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<tr>
<td></td>
<td>Postest 10</td>
<td>4</td>
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</tbody>
</table>

Entries are the number of students with the given response.

Item 5 (Figure 18) required a student to graph the slope function for a function presented visually. Two points were given for graphing $y = -2$ on the interval $-\infty < x < -1$. Two points were given for graphing $y = 1$ on the interval $-1 < x < \infty$. One point was given if the student graphed a constant function on each of the intervals but had an incorrect constant. One point was given for indicating the discontinuity at $x = -1$. Few students were able to sketch this graph on the pretest (Table 22).

After instruction, it was expected that students in the local linearity group would more accurately draw the graph of the slope function than the secant–tangent group. The point distribution on the posttest (Table 22) indicates both groups achieved approximately equivalent results. The median for the local linearity group was 4 points and the median for the secant–tangent group was 3.5 points. Of those students who received no points, many either omitted the problem or copied the original graph. One–half of the students in each group sketched a relatively accurate graph. Among these students, the principal error was
drawing the slope function as a continuous graph either by drawing a vertical line connecting (-1, -2) to (-1, 1) or a steeply sloped line from just to the left of (-1, -2) to (-1, 1). This corroborates evidence in other studies (Dreyfus, 1990; Dreyfus & Eisenberg, 1982; Ferrini-Mundy & Graham, 1991) that students consider continuity as a defining characteristic of functions. Student difficulty may, however, lie with the graphical representation of the slope function defined by \( \frac{f(x + h) - f(x)}{h} \). At points on a graph where the derivative is undefined because of nonequal left and right derivatives, the slope function is a continuous function. The software used by the students to explore the slope function always drew continuous functions.

Item 6 (Figure 19) examined the ability of a student to recognize the difference quotient representation of a function. It was expected that students in the secant–tangent class, who focused on using the difference quotient in their initial introduction to the derivative, would be better able to answer this question correctly. No student was able to answer this correctly on the pretest (Table 23). Only 4 students in each group gave a correct response on the posttest. These students all recognized the difference quotient representation of the derivative in their explanation. The results suggest student inability to link the general symbolic representation, \( \frac{f(x + h) - f(x)}{h} \), with a particular symbolic expression, \( \frac{(x + .0001)^8 - x^8}{.0001} \).
5. The graph of a function \( y = f(x) \) is shown in the graph below.

A new function \( g(x) \) is defined by: for each \( x \), \( g(x) \) is the slope of \( f(x) \) at the point \( x \).

Sketch a graph of \( g(x) \) on the axes below.

Figure 18. Item 5 from Calculus Concepts Inventory and Calculus Concepts Inventory I. (See Appendices A and B.)
Table 22

**Distribution of Points on Item 5**

<table>
<thead>
<tr>
<th>Number of Points Obtained</th>
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<td><strong>Secant-Tangent Group</strong></td>
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<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Posttest</td>
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<td>1</td>
<td>1</td>
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<td>6</td>
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</tbody>
</table>

Entries are the number of individuals receiving the given number of points.

Table 23

**Student Responses to Item 6**

<table>
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<tr>
<th>Responses</th>
<th>$8x^7$</th>
<th>$x^7$</th>
<th>$x^8$</th>
<th>$\bar{b}$</th>
<th>Other</th>
<th>InCorrect</th>
<th>No</th>
<th>Response</th>
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</thead>
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<td>11</td>
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<td>12</td>
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<td><strong>Secant-Tangent Group</strong></td>
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<td>1</td>
<td>9</td>
<td>12</td>
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</tbody>
</table>

Entries are the number of students with the given response.
6. Find a polynomial function with rational coefficients that would have approximately the same graph as the function

\[ f(x) = \frac{(x + 0.0001)^8 - x^8}{0.0001}. \]

Explain how you obtained your function.

Figure 19. Item 6 from Calculus Concepts Inventory and Calculus Concepts Inventory I.
(See Appendices A and B.)

The six parts of item 7 (Figure 20) examined a student's ability to do symbolic differentiation. This item was included on the pretest because the majority of students had been introduced to some aspects of differentiation during their Honors Algebra III course. On each part 4 points were given for a correct response. Three points were given for a correct form with a single error—for example, neglecting a minus sign. One point was deducted for each additional error. Table 24 shows the distribution of points for each part. Table 25 gives the median scores for item 7.

The pretest data indicate that some students were familiar with polynomial differentiation before beginning instruction. They were, however, unable to consistently differentiate any of the other symbolic expressions. Overall improvement in student performance was seen from pretest to posttest. Yet, the median scores show that most students were only able to differentiate two or three of the expressions. By focusing on graphical representations of the derivative, instruction did not provide students with sufficient experience working with symbolic forms.

On the posttest, students who were unable to differentiate \( \sqrt{x} \) did not write the expression using a fractional power. The most common incorrect response to the derivative of \( \frac{1}{x^3} \) was \( \frac{1}{3x^2} \) an overgeneralization of polynomial differentiation. In
differentiating \( \cos 2x \), student errors involved omitting the negative sign and misapplications of the chain rule, writing \(-\cos 2x\) or \(-2\cos x\). No students were able to apply the product rule to \(x\sin x\). The product rule had been investigated graphically but not formally derived. Few students were able to find the derivative of \(\tan x\). The derivative had been investigated in class and informally derived. Of the students who gave incorrect responses to this question, none set up a difference quotient form of the derivative.

7. Find the derivatives of each of the following:

i. \(x^5 + 4x^3\)  

iv. \(\cos 2x\)

ii. \(\sqrt{x}\)  

v. \(x\sin x\)

iii. \(\frac{1}{x^3}\)  

vi. \(\tan x\)

Figure 20. Item 7 from Calculus Concepts Inventory and Calculus Concepts Inventory I. (See Appendices A and B.)
Table 24

Distribution of Points for Item 7

<table>
<thead>
<tr>
<th>Part</th>
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<th>Local Linearity Group</th>
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Entries are the number of individuals receiving the given number of points.
Table 25

Median Scores on Item 7

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<th>Local Linearity Group</th>
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</tr>
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<td>Posttest</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>11</td>
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</tbody>
</table>

Sketching Derivatives From Graphs of Functions

The second section of the posttest consisted of ten graphs of functions (Figure 21). Each graph was displayed on the overhead projector for 2 minutes. The student was asked to determine whether or not the function was differentiable at all points where it is defined. If the function was differentiable, then the student was told to sketch a graph of the derivative. If the function was not differentiable, then the student was told to explain why it was not. Two points were given for a correct sketch of the derivative or a correct explanation of non-differentiability. One point was given for a partially correct graph of the derivative. In cases where the student drew a correct graph of a non-differentiable function and did not give an explanation, one point was given. The point distribution for each graph is given in Table 26. The median scores for the total of the 10 items for the two groups are almost equivalent, 12 for the secant–tangent group and 11 for the local linearity group.

Graphs 1 and 3 were the graphs for which both groups of students could sketch the derivative most easily. Seven students in the local linearity class, however, did think the constant function (graph 3) was not differentiable. The primary justification was that the graph had no slope. One student suggested that because \( y = 3 \) lacked an \( x \) variable, the function was non-differentiable. For the derivative of graph 1, resembling a normal curve, students recognized the intervals of increase, zero slope, and decrease. Difficulties were
Figure 21. Function Graphs for Sketching Derivatives. (See Appendix C.)
Figure 21 (continued). *Function Graphs for Sketching Derivatives.* (See Appendix C.)

5. $f(x) = (\tan x)^2$

6. $
   \begin{align*}
   &f(x) = |\tan x| \\
   &f(x) = \tan x
   \end{align*}$
Figure 21 (continued). Function Graphs for Sketching Derivatives. (See Appendix C.)

9. \[ f(x) = (\sin x)^2 \]

10. \[ f(x) = |x^2| \]
encountered with the asymptotic end behavior where the derivative approaches zero from above on the left and zero from below on the right. Some students sketched the derivative as continuing below the $x$-axis on the left and continuing above the $x$-axis on the right.

Graph 5, $f = (\tan x)^2$ was given with both visual and symbolic representations. Most students recognized the periodic nature of the derivative, intervals of increase and decrease, and where the derivative was zero.

Graph 9 was given with both visual and symbolic representations. Some students wrote the symbolic form of the derivative, $2\sin x \cos x$. Whether or not this influenced their derivative graph is unknown. Most students drew the derivative as a periodic function. A common mistake involved graphing the derivative as a reflection of the original function through the line $y = 0.5$. Apparently these students recognized maximum points as having a derivative of zero, but they failed to see the negative slopes and the minimum points. Another mistake involved the intervals between points of inflection including a minimum point. The points of inflection were viewed as having a derivative of zero, the interval between the maximum point and the point of inflection a negative slope, the interval between the point of inflection and the minimum point a positive slope, and the minimum point a positive slope.

The derivative of graph 10, $f = |x^3|$, was drawn correctly by slightly more than one-third of the students. The most common error was in sketching the derivative as a linear function. The shape of the graph suggests a parabola, so that a student who did not notice the symbolic expression that was included might think of this as the derivative of a quadratic. Some students wrote $f' = 3x^2$, generalizing from the polynomial case.

The derivative of graph 7, $f = (x^2 - 1)^3$, was also drawn correctly by a little more than one-third of the students. Students who received partial credit for this problem usually recognized the negative slope to the left of zero, the positive slope to the right of zero, and the derivative of zero at zero. They failed to graph the zero derivatives at $x = -1$ and $x = 1$. 


A few students correctly drew the derivative to the left of zero, but failed to notice the symmetry of the graph and sketched the derivative to the right of zero as strictly positive.

Students in the local linearity group who identified graph 8, \( f = |\tan x| \), as being non-differentiable used arguments such as "The curve doesn't flatten out," "It has a vee shape," and "It is not locally linear." Students in the secant-tangent group argued that at \( x = 0, \pi, \) and \(-\pi\) that the slopes from the left and right were different. This is consistent with the instruction received. Students in both groups argued the function was not differentiable because it was a function involving absolute value. Those students who received partial credit primarily sketched an approximately correct graph of the derivative, noting intervals of increase and decrease, discontinuities at \( x = \frac{\pi}{2} \) and \(-\frac{\pi}{2}\), but failing to identify the undefined derivatives at \( x = 0, \pi, \) and \(-\pi\).

In contrast to responses for graph 8, only two students in the local linearity group mentioned local straightness in response to the differentiability of graph 6, \( f = -|x-2|+3\). Differing slopes from the left and right at \( x = 2 \) was the primary justification for nondifferentiability. Many students in both groups, however, drew a correct graph of the derivative without an explanation. This suggests there may have been some confusion over what was meant by a function having a derivative at all points. A few students drew the derivative as a continuous curve with a vertical line connecting the two parts of the derivative function.

Local straightness, coming to a point, and different slopes from the left and right were both mentioned by the local linearity group in explaining why graph 2, \( y = |x|^{2/3} \), was not differentiable at \( x = 0 \). Differing slopes was the argument used by those in the secant-tangent group. Students in both groups received one point for sketching a decreasing curve in the third quadrant and a decreasing curve in the first quadrant without explanation. These students appear to have understood what the derivative of the curve represents but
not the question of differentiability. Several students used absolute value to explain non-differentiability, although the symbolic expression was not given.

Graph 4, $f = |\sin x|$, caused the most difficulty for both groups. Students who were able to explain the non-differentiability used arguments similar to those used for graphs 2, 6, and 8. The majority of students attempted to sketch the graph of the derivative. Several did this correctly without explanation. The remainder of the derivative graphs were continuous functions. Some students drew $|\cos x|$ both using a symbolic derivative and not using the symbolic expression. Others graphed periodic functions, interpreting the derivatives at $x = 0, \pi, -\pi, 2\pi, \text{ and } -2\pi$ as either 0 or as cusps on the derivative graph.
Table 26

**Distribution of Points for Graphing Derivatives of Functions from Their Graphs**

<table>
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<th>Number of Points</th>
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</table>

Entries are the number of individuals receiving the given number of points.
Analysis of Calculus Concepts Inventory II

Item 8 (Figure 22) on the posttest consisted of 4 parts, each a graph of a function for which students were asked to sketch the graph of the derivative. This item was used by Tall (1986). For comparison purposes with Tall, a scoring scale of 5 points based on the essential features of the derivative graph was used. These features were negative slope (2 points), positive slope (2 points), and zero slope (1 point). The point distribution for each graph is given in Table 27.

Part (i) is the graph of $|x|^1$8. All but one student was able to sketch the graph of the derivative. Part (ii) is the graph of $(x - .25)^3 + 1$, slightly different from Tall who used $x^3 + 1$, which students may recognize as a cubic. Again, the vast majority of students drew a correct derivative graph. Mistakes involved identifying the point where the function crosses the x-axis (-0.75, 0) as having zero slope and shifting the derivative curve up one unit to match the inflection point of the original function.

The graph in (iii) does not have an identifiable symbolic expression. Tall suggested it would be difficult for students who focused only on symbolic representations of the derivative. The majority of students graphed a correct derivative. The most common error was viewing the interval on which the function is increasing and negative as having negative rather than positive slope. The graph in (iv) caused some difficulties. Mistakes were made by interpreting the graph to the left of zero as a decreasing function, graphing the derivative in the third quadrant. Some students did not know what to do with the asymptotic relationship at zero. They continued the graph of the derivative on the left into the first quadrant and the graph of the derivative on the right started in the third quadrant, resulting in two values for the derivative at zero.

Overall, the results for the two groups on this item were similar. The median score for the local linearity group was 17 and for the secant–tangent group 19. Both groups became adept at sketching the graphs of derivatives of functions. Both groups also performed at a
level equivalent to the students in Tall's experimental groups and were substantially higher in proficiency than Tall's control groups who did not use graphing software (Table 28). The results suggest that the generic organizer, Gradient program, that draws slope curves enhances student ability to sketch the derivatives of functions.

Table 27

**Distribution of Points for Item 8**

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</table>

Entries are the number of individuals receiving the given number of points.
8. Sketch the graphs of the derivatives of each of the following functions on the axes provided.

i. 

ii.

Figure 22. Item 8 from Calculus Concepts Inventory II. (See Appendix D.)
Figure 22 (continued). Item 8 from Calculus Concepts Inventory II. (See Appendix D.)

iii. iv.
Table 28

Group Means for Item 8

<table>
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<th></th>
<th>Graph 1</th>
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<th>Graph 4</th>
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<td>3.6 (SD 1.7)</td>
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<td>4.6</td>
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In item 9 (Figure 23) students were given the graph of the derivative of a function, $f'(x)$, and were asked to sketch the graph of the function, $f(x)$. One point was given for each significant aspect of the function graph: intervals of nonlinear increase and decrease, linear increase, and zero derivative. While no student indicated the discontinuities in the derivative at $x = 2$ and $x = 3$, about one-third of the students were otherwise able to sketch the graph correctly (Table 29). Student misperceptions involved interpreting the interval $(3, 5)$ as a negative slope, the interval $(2, 3)$ as zero slope, and graphing the derivative of $f'(x)$ instead of $f(x)$. 
9. The graph of the derivative of a function, $f'(x)$, is shown on the graph below.

Sketch a graph of $f(x)$ on the axes provided.

Figure 23. Item 9 from Calculus Concepts Inventory II. (See Appendix D.)
Table 29

Point Distribution for Item 9

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<td>0</td>
</tr>
</tbody>
</table>

| Local Linearity Group | 2 | 7 | 2 | 7 | 7 | 2 |
| Secant-Tangent Group   | 3 | 9 | 2 | 6 | 4 | 4 |

Entries are the number of individuals receiving the given number of points.

Items 10 and 11 (Figure 24) asked students to give graphic and symbolic representations of functions that were not differentiable at \( x = 1 \). Thirteen students in the local linearity group and 11 students in the secant-tangent group were able to sketch the graph. Seven students in the local linearity group and 4 in the secant-tangent group were able to provide a symbolic expression. In the secant-tangent group, the only correct graphic and symbolic response was \( |x - 1| \). Students justified nondifferentiability by appealing to different left- and right-hand slopes. In the local linearity group, graphic responses were more varied. They included functions resembling both \( |x - 1| \) and \( |x - 1|^{2/3}, \sin \pi x \), and a step function with differing slopes at \( x = 1 \). These students used both different left- and right-hand slopes and local linearity to justify their responses. For example, "You can't make it flat." When asked for a symbolic expression, the local linearity students resorted to \( |x - 1| \). Students in both groups suggested \( x = 1 \) was not differentiable because the slope was undefined, failing to recognize that this expression is not a function. Other students tried a constant function, arguing that a constant function has zero or no slope.
10. Sketch the graph of a function which is defined at $x = 1$, but is not differentiable at $x = 1$. Explain why your function is not differentiable at this point.

11. Write a symbolic expression for a function which is defined at $x = 1$, but is not differentiable at $x = 1$. Explain why your function is not differentiable at this point.

Figure 24. Items 10 and 11 from Calculus Concepts Inventory II. (See Appendix D.)
Student ability to transfer knowledge of the derivative concept to velocity was of interest in item 12 (Figure 25). Sixteen students in the secant–tangent group approximated the velocity at 20 seconds using the average velocity between time zero and 20 seconds. Seventeen students in the local linearity group used the same method. Two students in the local linearity group and one student in the secant-tangent group found the average velocity between 15 and 20 seconds. One student in the local linearity group sketched a distance versus time graph and identified the need to find the derivative of this graph at 20 seconds. These results suggest little transference. The connection between the derivative of a distance–time function and velocity was not presented in class. When faced with a new situation, students relied on prior knowledge of velocity as distance divided by time.

12. A bullet shot from a 9-millimeter pistol has a muzzle velocity of 1250 \( \frac{ft}{sec} \). The distance the bullet travels in 5 second intervals is given in the table below. What is the approximate velocity of the bullet when the time is 20 seconds?

<table>
<thead>
<tr>
<th>Time in seconds</th>
<th>Distance traveled in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5850</td>
</tr>
<tr>
<td>10</td>
<td>10900</td>
</tr>
<tr>
<td>15</td>
<td>15150</td>
</tr>
<tr>
<td>20</td>
<td>18600</td>
</tr>
<tr>
<td>25</td>
<td>21250</td>
</tr>
<tr>
<td>30</td>
<td>23100</td>
</tr>
<tr>
<td>35</td>
<td>24150</td>
</tr>
<tr>
<td>40</td>
<td>24400</td>
</tr>
</tbody>
</table>

Figure 25. Item 12 on the Posttest. (See Appendix D.)
Analysis of Student Concept Definitions of Slope, Tangent, and Derivative

Three items on the posttest (13, 14, and 15) explored student concept definitions for slope, tangent, and derivative. Students were asked to write responses to each of the items. In the following discussion, quotations from student writing are cited through the use of two initials followed by LL for the local linearity group and ST for the secant–tangent group. For example, either XX-LL or XX-ST.

Item 13 stated:

You have been asked by a friend who understands the notion of the slope of a straight line to explain what is meant by the slope of a more general function. Write a brief explanation of what you would tell your friend.

Student responses were classified using categories developed by Tall (1986) (Table 30). The most common response given by experimental students in Tall’s study used the concepts of tangent and rate of change. Similar results were observed in this study. For example,

It is the slope of the tangent line at any given point on the function that tells us the slope of that point; (BL-LL)

Any graph is consisted of many tangent lines. Those tangent lines have a slope when applying slope to a graph such as a parabola you have to find a tangent (any pt) & measure its slope to find the slope of the graph at any given pt; (HE-LL)

The slope can be found by using any two points on the graph. Divide the change in the y coordinates by the change in the x coordinates for the slope of the function. (BD-ST)

Students who expressed slope using the idea of two points and two close points stated,

The slope is calculated by putting the function into an equation \( \frac{f(x+h) - f(x)}{h} \) the \( h \) is the distance between the 2 points; (EF-ST)

The slope is equal to the value of each secant line when the distance between the two points are very small, therefore, a general function has many slopes, unlike a line. (LA-ST)

One student used the idea of magnification.

I would say that there are small areas that look more like lines as you closer your view to any graphed function. The slope of each of these lines is what is meant. (CA-LL)
While Tall found few explicit references to the global nature of slope, the local linearity students in this study tended to give explanations that were more global.

The slope of a more general function is determined by slopes of tangent lines to the function. At various points on the function the slope is equal to the slope of the tangent line at that pt. (BD-LL) [BD included a sketch of a slope function];

The slope of a general function is the same as the derivative curve of the original function. (AB-LL)

In summary, the definitions for the slope of a general function given by students in both groups primarily involve the procedural use of two points either using the slope formula or rate of change. A small number of students formulated the slope definition using the idea of tangent. While most students maintained a pointwise view of slope, about one-third of the local linearity students considered the slope of a general function globally as a function.
### Classes of Student Explanations of Slope

<table>
<thead>
<tr>
<th>Explanation</th>
<th>Local Linearity Group</th>
<th>Secant–Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used magnification to explain</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Slope of line through two (not close) points</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Slope of line through two close points</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Slope of line through one point</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Slope of tangent</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Rate of change, ( \Delta y/\Delta x )</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Calculus formula–derivative</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Other</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>No Response</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

Of those responding, held:
- a global view of slope: 8, 4
- a pointwise view of slope: 11, 13

Entries are the number of students using each explanation.
Item 14 asked a student to

Write an explanation of what is meant by a tangent to a graph.

Student responses were classified using categories developed by Tall (1986) (Table 31). The most common response involved the idea that a tangent touches the graph at exactly one point.

A tangent to a graph is a line that touches/intersects the function at exactly 1 point. (BD–ST)

A tangent to a graph is a straight line that touches the point of the graph at one point. (KE–LL)

A few students suggested the tangent does not cross through the graph.

A line that touches the graph at only one place but does not go through the graph. (EF–ST)

One student viewed the tangent as a secant line.

A secant line that only touches in one pt. on a graph. (HE–LL)

All of the above definitions of the tangent evoke images of Euclidean geometry where the tangent to a circle is defined. This is even clearer in the case of the student who stated,

A tangent to a graph is a straight line that is attached at any given pt. at a right angle. (CL–ST)

A few students, primarily from the local linearity group, mentioned the concept of slope in conjunction with their definition of tangent.

A line that touches the graph at a point and that has the same slope as that area where the point is. (CA–LL)

A tangent line to a graph is a line that touches at one point on the graph and is the slope of the graph at that given point. (AB–LL)

The tangent line is a line that hits at one point on the function and has the same $m$ value as that point. (KH–LL)

The last student used the language of algebra, $m$ value, when talking about slope.
Even after having studied the concept of derivative, in which the concept of tangent plays an essential role, few students in either group had developed an understanding of the tangent in terms of calculus. Most definitions remained at the level of Euclidean geometry, involving the image of the tangent to a circle.

Table 31.

Classifications of Student Explanations of Tangent

<table>
<thead>
<tr>
<th>Description</th>
<th>Local Linearity Group</th>
<th>Secant-Tangent Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Touches the graph</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Touches the graph at only one point</td>
<td>15</td>
<td>17</td>
</tr>
<tr>
<td>Touches and does not cross graph</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Uses the idea of slope</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Through two very close points</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Other</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Entries are the number of students using each description.

Item 15 asked the student to

Explain what is meant by the derivative of a function.

Responses were classified according to categories formulated by Tall (1986) (Table 32). In addition, responses were categorized if they referred to the derivative as a function, a graph or curve, or mentioned the idea of the slope of a tangent. The description of the derivative as a function, graph, or curve may suggest that the student holds a global view of the derivative in contrast to a pointwise view.
Almost all of the students used the idea of slope to write about the derivative. For example,

The derivative is the slope of a line which connects two points on a function. (AK1-ST)

A number of students used the idea of the slope of the tangent line to explain the derivative.

The slope of the tangent line at the points of a function. (AK2-ST)

The derivative of a function is the set of points represented by the slopes of the tangent lines to the graph. (AL-LL)

In contrast to the findings of Tall (1986), many students viewed the derivative globally as a function, equation, or graph. These students used the words function and equation as synonyms.

The derivative is the general expression for any slope on the original $f(x)$ graph. The derivative can be referred to as the slope curve as well. (AB-LL)

The derivative of an equation is the plotting of slopes from your original equation to get another function. (BD-LL)

The derivative is an equation, when an $x$ value is "plugged in" to it, will give the slope of a line tangent to the point on the original function at the given $x$ value. (CF-ST)

The derivative of a function is the graph of the slopes of the function at their respective $x$'s. (LA-ST)

Only one student viewed the derivative as the derivative of a polynomial function.

The derivative of a function means that everything is taken to the next lowest power. (JR-ST)

For the students in both groups, slope was the primary image used in writing about the derivative. Slope was considered as either a local or global property of a function. Those students who thought of the derivative globally used the terms function, equation, graph, and curve to describe this global property.

Students in this study saw the slope of a general function in terms of the slope of a tangent line at a point on the function. They were, however, likely to define the slope of the tangent as the slope of a line connecting two points. There was a tendency to view slope as a general property of a function rather than slope being only a pointwise property.
The tangent was defined in terms of Euclidean geometry as a line passing through exactly one point of a function. The derivative was seen as representing the slope of a tangent to a function. This slope was defined locally at a point or globally as a function.

Table 32.

Classifications of Student Explanations of the Derivative

<table>
<thead>
<tr>
<th>Description</th>
<th>Local Linearity</th>
<th>Secant-Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used the word &quot;slope&quot;</td>
<td>20</td>
<td>22</td>
</tr>
<tr>
<td>As a tangent</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>As a function only</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>As a calculus formula</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Other</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>No Response</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Additional Description

<table>
<thead>
<tr>
<th>Description</th>
<th>Local Linearity</th>
<th>Secant-Tangent</th>
</tr>
</thead>
<tbody>
<tr>
<td>As a function</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>As a graph or curve</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>As slope of a tangent</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Entries are the number of students using each description.

Summary Comparison of Local Linearity and Secant-Tangent Groups

The posttest consisted of items that explored student understanding in four areas: the concepts of slope and rate of change (items 1, 12, and 13), the concepts of derivative and differentiability (items 2, 4, 5, 6, 9, 10, and 11), symbolic differentiation (item 7), and student ability to sketch the graphs of derivatives of functions (item 8 and the ten items on the graphing derivatives from graphs of functions section).
For the concepts of slope and rate of change, the local linearity group demonstrated a better knowledge than the secant–tangent group. The local linearity students were more able to calculate slope and rate of change than the secant–tangent students on both the pretest and posttest (item 1). In particular, on the posttest the local linearity students were more proficient in determining the slope between two points as zero and in calculating average rate of change between two points from right to left than the secant–tangent group. Students in the local linearity group were more likely to use a smaller time interval when determining the approximate velocity of an object at a given time from a table than students in the secant–tangent group (item 12). In describing the slope of a general curve, students in the local linearity group referred more frequently to the concepts of the slope of the tangent, the rate of change, and the derivative than students in the secant–tangent group (item 13). This reference suggests that the local linearity students had stronger concept images of slope and average rate of change in a generalized setting than the secant–tangent group.

For the items concerned with the concepts of derivative and differentiability (items 2, 4, 5, 6, 9, 10, and 11), students in the local linearity group were better than the secant–tangent students in determining the exact value for the slope of a function at a point using the slope of the tangent line at the point (item 4), sketching the slope function for a piecewise linear function from the graph of the function (item 5), and in providing both graphical and symbolic representations of a function that fails to have a derivative at a particular point (items 10 and 11). The secant–tangent students were slightly better than the local linearity students in sketching the graph of a function from the graph of the function's derivative (item 9). In recognizing a symbolic difference quotient approximation of the derivative of a function (item 6), the local linearity and secant–tangent groups achieved at the same level. For item 2, which required a student to determine the slope for a secant line containing the points (1, 1) and (k, k\(^2\)) on the graph of \( y = x^2 \), to find the slope of the
tangent line at \( x = 1 \) on the same graph, and to provide a mathematical explanation of how the slope of the tangent line was found, students in both groups performed at approximately the same level. Students in the local linearity group, however, were more likely to use an argument based on the derivative to justify how they found the slope of the tangent line than students in the secant–tangent group. Thus, for the concepts of derivative and differentiability the local linearity students demonstrated a slightly better understanding than the secant–tangent students.

Students in the local linearity group achieved a higher median score on the symbolic differentiation portion of the posttest (item 7) than the secant–tangent group (11 versus 8) while beginning at the same level on the pretest. Thus, it can be concluded that the local linearity students attained a higher level of proficiency with symbolic differentiation than the secant–tangent students.

Students in the secant–tangent group demonstrated a higher proficiency in sketching the graph of the derivative of a function from the graph of the function than students in the local linearity group. On item 8, the median scores were 19 for the secant–tangent group and 17 for the local linearity group. For the ten items on the graphing derivatives from graphs of functions section, the median scores were 12 for the secant–tangent group and 11 for the local linearity group. The difference in performance between the two groups in this area was also marked by a difference in explanations for the nondifferentiability of a function at a particular point. Students in the secant–tangent group tended to use an argument based on differing slopes to the left and right of the point to explain nondifferentiability while students in the local linearity group invoked the non–local straightness property when explaining nondifferentiability.

The preceding discussion confirms that in the areas of concepts of slope and rate of change, the concepts of derivative and differentiability, and symbolic differentiation the local linearity group achieved at a higher level than the secant–tangent group. Only in the
area of sketching the graphs of derivatives of functions did the secant-tangent group achieve at a higher level than the local linearity group.

While these differences in performance may be due to initial group variation, it is worth noting the local linearity group maintained its overall initial advantage. Moreover, the performance of the secant-tangent group in sketching the graphs of derivatives of functions suggests the graphing generic organizers assist in developing a significant visual image of the concept of derivative. Thus, it can be concluded that, regardless of the local linearity or secant-tangent approach to the concept of derivative, the use of the graphing generic organizers promoted greater student understanding of the concept of derivative.

**Student Evaluation of Instruction**

At the conclusion of the study, after the posttest had been administered, students were asked to respond in writing to two questions: "What have you liked and/or disliked about using the computers and software in your study of calculus"? and "Do you think using the computer software has given you a better understanding of calculus? Explain." Student responses to these questions are addressed in this section of the data analysis because, without being specifically asked, the students spoke directly to issues of classroom roles and learning activities.

The ambivalent thoughts that students expressed toward working in small groups and assuming responsibility for their own learning was articulated by BB–LL,

I liked working in small groups and being able to work at our own pace.... I gained more, however, from I's (a student(s) explanations about slope relationships than I did from [the teacher's].... Something that I disliked was having the computer do all the graphing and us not learning how to plot a derivative curve properly first.

BB emphasizes the positive aspects students realized in moving from a teacher-directed, lecture-oriented classroom toward a student-centered, create-your-own
mathematical meaning setting. Many other students expressed similar sentiments. For example,

If I was being lectured to every day about what we were doing on the computers, it would all be over my head (KC-ST); We could also work at our own rate instead of with the rest of the class and could discuss the problems with the rest of our group which helped me learn it better (KH-LL); I liked using the computer because I can discuss my thought with others (CK-ST); I enjoyed working in groups . . . because we could help each other out with what we didn't understand (JE-LL); and I really liked the working in groups . . . because what I didn't understand, someone else in the group could explain to me (JL-LL).

BB's dislike indicates an opinion that there is something inherently wrong with doing mathematics on a computer. AL-LL expressed a similar concern, "My concern is that I'm using the computer as a crutch." IM-LL thought that understanding developed using the computer should be linked to the ability to do paper and pencil mathematics, "The problem I had with this is that I had trouble integrating what I had learned on the computer with actually doing it on paper." Other students spoke negatively of working in groups with the computer rather than being told what to do. "The reason I [don't like working with computers] is we have to learn by the process of seeing and doing. For me personally, I learn by writing and studying" (TZ-ST).

Students wrote about the role of visualization in their learning process. CC-LL commented on the linkage of visual and symbolic representations, "I attached pictures with equations and abstract representations." LA-ST thought that visualization aided in learning, "I was able to correct mistakes by visually seeing the relationship between my mistake and the answer." Some students thought of themselves as visual learners. AL-LL, "As a visual learner, the computers helped me to understand how graphs and their
derivative curves look" and JL–LL, "I liked the fact that we got to see everything happening on the computer because I can learn a lot visually."

While working with the computer software facilitated group interaction, it also created some distractions. RM–ST summarized the difficulty of keeping the class together, "[I]t has been hard to stay together as a class; everybody has their own angle to each problem that they'd like to explore." The expectation that the entire class should be focused on the same thing is, perhaps, not unusual from a student's perspective of having always learned mathematics in a lecture setting. The teacher did not function frequently in a managerial role (Table 5). The problem of gaining students' attention as a class when they were working in groups caused difficulty for the teacher early in lesson 7 with the secant–tangent group. The teacher was setting various tasks for the students to do using the Gradient program. The researcher noted, however, that groups of students were working on at least four different problems. The following sequence of events occurred: TαsA/TαsA/M1--/. The teacher had directed the students to graph the function \( y = \sin x \) on the interval \((-\pi, \pi)\), but then noticed that few of the students were paying attention. Thus, the managerial statement, "Please get with me" (M1--). It took several minutes to gain the students' attention. Having never used technology in over 25 years of teaching, the teacher did not have the strategies or techniques to deal with this situation, where it was desired to focus group attention when students were separately and actively pursuing their own mathematics. The researcher and the teacher discussed methods of gaining class attention later in the day, including the managerial, "Turn off your monitors," which was used in the segment discussed earlier.

Teaching calculus using the computer created uncertainty. The teacher reported anxiety over not being able to anticipate where students would have difficulties, what questions they would ask, and what insights they might develop. The teacher also
indicated that teaching calculus with the computer required more preparation than ever before.

Students in both treatment groups had positive attitudes toward working with computer software in their exploration of the concepts of differentiability and the derivative. These attitudes were developed through visualization of concepts rather than just symbolic presentation and through opportunities to interact with one another in discussing and communicating mathematics. There were, however, underlying currents of thought that the computer was doing all of the mathematics. Teaching in a computer environment created uncertainty and required additional preparation on the part of the teacher. Thus, the introduction of technology to develop concepts in the calculus curriculum requires accommodation to the needs and beliefs of both students and teachers if it is to be successful.
CHAPTER V
ANALYSES OF CONCEPT IMAGES OF SLOPE, TANGENT, AND DERIVATIVE
FROM STUDENT INTERVIEWS

Interviews with 7 students from the local linearity group and 6 students from the secant-tangent group were conducted between one and two weeks after the posttest had been administered. The interviews took place during the school day and lasted approximately 50 minutes. The sample of students to be interviewed was obtained by asking students to indicate periods during which they did not have a scheduled class. Selection was based on availability. The students chosen represent approximately equal numbers of high, middle, and low ability students from both classes as indicated by pretest scores.

The interviews were based on a protocol (Appendix E) designed to explore student understanding of the concept of derivative, differentiability and non-differentiability, the concept of tangent, and the symmetric difference quotient—an alternative representation of the derivative. Computer and graphing software were available during the interviews. Each interview was tape recorded and the tape was transcribed. Student written work during the interviews was kept for reference and analysis.

The following analysis is divided into four sections: concept images and concept definitions for the derivative, differentiability and non-differentiability, the concept of tangent, and generalizing to the symmetric difference quotient. Due to varied student responses to items in the protocol and the time limitation of one class period for the interview, each student was not asked every question. Thus, students selected for citation are representative of categories of response as determined by the researcher. Responses
that are unique are identified and discussed. Each student is identified in the transcripts by using two initials for the student followed by two letters identifying the treatment. For example, either XX–LL or XX–ST, indicating a student in the local linearity treatment (LL) or secant-tangent treatment (ST). The researcher is cited as I (the interviewer).

**Concept Images and Concept Definitions for the Derivative**

The first item in the protocol asked students to tell what a derivative is. The analyses are grouped by treatment, starting with the local linearity group.

**EC–LL.** The following interview took place with a female student, EC, in the local linearity group. Her initial concern was with providing a verbal definition of the derivative. The verbal statement of a definition for the derivative appeared to be a concern with many of the students who were interviewed. They were uncomfortable with putting ideas into words. The researcher suggested a visual image might help. EC drew upon the image of a bell curve, relating this to x and y values of the function. She then offered the example of a cubic function whose derivative graph would be a quadratic function, using the power rule for differentiation. This suggests her image of the derivative was closely tied to polynomial differentiation. EC was quick to respond that the derivative of \( y = x^3 \) is \( 3x^2 \). When asked about the meaning of the derivative, EC replied that the derivative indicates the slope of the curve, whether positive or negative, and was able to indicate that the slope of \( y = x^3 \) at \( x = 1 \) is 3.
I: What do you think a derivative is?

EC-LL: Um, a derivative is—I'm not sure if I can explain it like words—I just know that it's—I just know like how to get it.

I: Can you draw pictures?

EC-LL: Yeah. It's an—the derivative is like related to the x values or the y values of the curve—of the bell curve—I think. I—I'm not real sure.

I: Can you give me an example?

EC-LL: Yeah—like a derivative of a trinomial would be a parabola—umm—because all you do is like take the power and multiply it times the number in front of the x and then like lower the power.

I: Okay.

EC-LL: I mean I know how to get them—it's just that sometimes I get foggy like as to how everything relates together.

I: What does the derivative tell you?

EC-LL: Umm—well it tells you like about—it—hmm. Well it tells you about the slope of the real curve—like if it's negative or positive.

I: Okay, so it tells you about the slope. You said if you have a trinomial—say we choose y equals x cubed—

EC-LL: Uh huh.

I: —and you told me that the derivative of that—

EC-LL: Would be like three x squared.

I: —is three x squared.

EC-LL: Mmm hmm.

I: And you said that tells you the slope.

EC-LL: Mmm hmm.

I: If I wanted to find out the slope at a point—say x equals one. How would I find it—the slope at x equals one?

EC-LL: You just plug it in to the derivative. Like—so it would be like three.
EC's image of the derivative was not entirely tied to polynomials. She was able to provide the following example. Her use of the derivative in finding slope continued to appear to be procedural through the use of "plug in."

I: What about functions which aren't polynomials?
EC-LL: Um, like sine or cosine?
I: Mmm hmm.
EC-LL: Well, like you just take the derivative like of sine which is—or cosine—which is like negative sine, I think—
I: Mmm hmm.
EC-LL: —and you just plug in whatever point you want and that will give you the slope of—
I: At that particular point.
EC-LL: Yeah.

EC appealed to the visual image of the computer software graphing the slope curve when asked for a definition of the derivative. When she was asked to consider other ways of looking at the derivative, she used the slope of the function at maximum and minimum points to identify places where the derivative graph crosses the x-axis. She suggested using the slope of the "real curve" to identify when the derivative graph was above or below the x-axis.

I: How would you define the derivative if you didn't know—for example, how would you find the derivative of sine?
EC-LL: Umm, I can't remember how we did that. I can't remember if we just found that—like if the computer kept trying to match it—I'm not sure.
I: So you graphed functions on the computer, like the sine—
EC-LL: Right.
I: —and then you plotted—

EC-LL: Yeah—when it asked you to like find the derivative we just kept trying like different things.

I: —to see if you could match the graph.

EC-LL: Right.

I: Can you think of any other ways you did that?

EC-LL: Hmm. Um well I know—where—I know how like when—umm—like the—if you have a—umm—like a sine graph like—they vary—the top of the curve, the vertex, or that would like most likely be where it crossed the axis.

I: So when you had a maximum a point, then it crossed the x-axis.

EC-LL: Yeah. Right.

I: Okay.

EC-LL: And then, like other than that—and then like—you could look at like the slopes of the real curve and see if they'd be above or below the x-axis.

EC's image of the derivative was a mixture of visual and procedural representations. She was able to use visual aspects of the slope of a function to describe the behavior of the derivative. She relied on procedural knowledge for finding symbolic expressions for the derivative of a function.

JM-LL. JM, a male student in the local linearity group, used a pointwise visual image to globally define the derivative. He replaced each y-coordinate in the original function with the value of the slope at each x to obtain the derivative graph.

I: What do you think a derivative is?

JM-LL: You want the graph of one—or—

I: Well, you can tell me about the graph of a derivative.

JM-LL: The graph would be—it's if you take the derivative of a graph that change the y-values and the slopes at that particular x-value.
When JM was asked for an example of a differentiable function he provided a quadratic function and visualized the derivative as a linear function whose slope depended on the slope of the original quadratic function. He was able to extend this to the derivative of the sine function.

I: Can you give me an example of a function that has a derivative?

JM-LL: Well, like if you had a parabola.

I: If you have a parabola, then what does the graph of its derivative look like?

JM-LL: Um, it's a straight line going in a—depending on the slope of the parabola—depends on the curve or the angle of the graph.

I: Then it's a straight line. What about something like a sine function?

JM-LL: Well, a sine function would be a cosine function.

When JM was asked how to find the derivative of a general function, he hesitated. The example of \( y = x^2 \) with the derivative \( 2x \) prompted JM to look at the slopes of secant lines or tangent lines to approximate the slope at each point on the graph.

I: Okay. How would you find the derivative of a function in general?

JM-LL: Um,—

I: For example, suppose you didn't know the derivative of \( x \) squared was two \( x \). How would you show me that this is the slope function?

JM-LL: Um, well you could do it with a graph—saying measure approximately the slope at each point with the secant or tangent line.

I: Okay.

JM-LL: And then create the derivative of \( x \) from there.
In the following, JM wanted to go back to the procedural derivative to find the desired slope. JM was given \( y = x^3 \) as an example. JM generated the difference quotient definition of the derivative and let \( h \) go to zero to obtain the slope of the tangent line. Here, JM's use of the words "is almost like zero" to describe \( h \) indicates an incomplete knowledge of the limiting process.

I: So you could look at secant lines or tangent lines on—the original graph and then use that to plot points for the derivative for each particular \( x \). Could you do that in a more formal way? For example, how would you find the slope of a secant line?

JM-LL: Um, you could—uh—well if you had—if you already had the derivative?

I: No. If you just had the function. For example, if I wrote—here's a function \( f \) of \( x \) is equal to \( x \) cubed and we picked a particular point \( x \)—I don't care what it is. How would you find another point and a secant line to show me what the slope is?

[The interviewer drew a graph of the function \( f(x) = x^3 \).]

JM-LL: Well, you could pick another point and then—just anywhere on the curve—like here.

I: Okay. So what would you call that point?

JM-LL: \( x + h \). And then you use the \( f \) of \( x + h \) minus \( f \) of \( x \) over—

I: Okay, so that gives you the slope of the secant line. What does that have to do with the tangent?

JM-LL: Well, if the \( h \) becomes so small that it's almost like zero that gives you the slope of the tangent.

I: That gives you the slope of the tangent?

JM-LL: Mmm hmm.

JM again wanted to use the derivative to find the slope of the function \( f(x) = x^2 \) at the point \( (2, 4) \). He was able to identify another point on the graph \( (k, k^2) \) and set up the difference quotient for the slope of the secant line. This time he did not use a limiting process.
I: Suppose I gave you something like this—f of x equals x squared and we're looking at a point two—four—and I ask you to find the slope of this tangent line. How would you go about doing that?

[Here the interviewer provided a graph of \( f(x) = x^2 \) with the point (2, 4) indicated and the tangent drawn.]

JM-LL: Um, let's see—if you took—let me think. If you took—um—well you could find the—oh, the uh—oh, just the tangent line—okay. Well, that oughta work. You could find the equation for the derivative—

I: Okay.

JM-LL: —using the original equation—x squared.

I: So that you would know that the derivative is—

JM-LL: Two x.

I: What if you didn't know that.

JM-LL: What if you didn't know that? Um, you could do—let's see—you could put x squared—

I: Okay, I'd put another point on there. K and k squared. Could you use that?

JM-LL: Yeah, you could go back to the—the \( f \) of \( x \) plus \( h \) and use that. Put two in for \( x \)—well—well you'd use the \( f \) of \( x \) plus \( h \) with \( x \) squared and then—put two in for \( x \) and that would give you the slope.

While JM used visual aspects to describe the derivative, linking the derivative to the slope of a function, he wanted to consider the derivative as something already in hand. JM saw a connection between the slope of a secant and the slope of a tangent through a limiting process. He was able to construct the formula for the slope of a secant line and let \( h \) approach zero to obtain the slope of the tangent. For JM, however, the formal limit was not clearly defined.

PS-LL. PS, a male student in the local linearity class, exhibited an understanding of the derivative similar to that of JM-LL in terms of replacing y-values with the value of the slope to obtain the derivative function. PS used the example of a quadratic function (a parabola).
I: The first thing I want to ask is what you think a derivative is.

PS-LL: It shows the slopes of the different points at the—like an $x$ coordinate on the regular curve—the slope of that would be on the derivative curve at the same $x$ point.

I: Okay, can you give me an example of that?

PS-LL: Um, in a parabola the vertex is horizontal slope so that—that would be where $y$ cross the $x$-axis cause the slope would be zero so the derivative would be zero. And then once it gets to the ends, depending which you are, then it'd be a more sloped—so it'd be higher—like higher or lower on the $x$-axis.

PS used the slope formula for a secant line to suggest an initial procedure for finding a derivative. He rejected the notion that the slope of this secant line defined the derivative.

He suggested a formula for the slope that involved a limiting process.

I: How do you go about finding a derivative?

PS-LL: Um, you get two points on the regular curve—the $f$ of $x$ curve—and you find the slope with $y$ two minus $y$ one over $x$ two minus $x$ one. And that'd be the $y$ coordinate of the $x$—with the $x$ point. The slope that you get from that equation would be the $y$ coordinate of the derivative graph.

I: So you're saying if I have a curve like this, then I choose two points—$x$ one, $y$ one, $x$ two, $y$ two, and I find the slope and that's the slope of that line connecting the two points is the slope of the derivative curve.

[The interviewer sketches a graph of a function $f$.]

PS-LL: No. Well, like, the number you approach when you move this point closer to $x$—

I: So you have this second point closer to $x$.

PS-LL: Right. And that—whatever that—that would be the slope of the $f$ of $x$ curve—that would be the actual point.

I: That would be the actual point. So you have to let this $x$ two, $y$ two point get closer. How do you write that mathematically?

PS-LL: As it approaches the $x$ two—as it approaches $x$ one—(inaudible)—

I: Mmm hmm. For example, if I gave you a function—say $f$ of $x$ is equal to $x$ squared and you wanted to show me at some point on that curve $f$ of $x$ what the derivative is, what would you need to do?
PS–LL: Okay, you can use the formula $f$ of $x$ plus $h$ minus $f$ of $x$ over $h$ and then have it as $h$ approaches zero.

[PS writes the formula.]

I: So you have that and then you have—

PS–LL: A limit.

I: The limit as $h$ goes to zero. What does that actually mean? What's happening?

PS–LL: Um, the distance between these two points here point zero is getting closer.

PS had a global view of the derivative as a graph or a function obtained by looking at the slopes of tangent lines to the original function graph. He was able to construct a formula for the slope of an arbitrary secant line and by letting the two points get closer obtain a limit. PS appeared to think that there is always some small distance between the two points.

KV–LL. In contrast, KV–LL invoked the power rule, slope, and tangent line in her definition of the derivative. As in the case of EC–LL, KV had difficulty expressing her definition in words, but to volunteered to do try. The function she provided was $f(x) = 2x^2 + 5$. She knew that the derivative was $4x$ and this represented a straight line and that the line represented the slope of the original function. KV correctly sketched the graph of the function and the derivative. She was clear in pointing out that to the left of zero, where the value of the derivative is negative, the slope of the function $f$ is negative. She used the same reasoning to describe where the slope of $f$ is positive. KV realized that the power rule does not apply to all functions. She appealed to matching the gradient graphs provided by the computer software, but did not remember how the software obtained these graphs.
I: My first question is what you think a derivative is.

KV-LL: Okay. I think it's—um—the—wait—okay I know that it's one power less than the original—the $f$ of $x$ equation—and the—like the slope—I've got the definition and everything written down but I can't describe it. It's like the tangent line or something—I don't know—I don't know how to put it into words but I could like do it.

I: Can you draw me a picture?

KV-LL: Like of one?

I: Sure.

KV-LL: Like do you want to give me an original equation.

I: Oh, you give me a function, you can make up your own function—


[Here KV sketches the graph of $f(x) = 2x^2 + 5$.]

I: Okay, so you have two $x$ squared plus five.

KV-LL: Yeah. Then the derivative would be four $x$ and that's a straight line. I think. Kind of like that.

[Here KV sketches the graph of $f'(x) = 4x$.]

I: So what does that straight line tell you?

KV-LL: It's the slope of the original function—like where it's below here is where it's got a negative slope—and then when it goes above the $x$-axis is where the original equation has a positive slope.

I: Okay. You mentioned that it's one power less. Does that work in every case?

KV-LL: Um, it doesn't work for the sine and cosine ones.

I: How do we go about finding derivative for sine and cosine?

KV-LL: Um—first of all, I didn't know it was a negative cosine. I'm not sure. I don't remember.

I: How would we figure out what it is? If we didn't know?

KV-LL: How did we do it on the computers?

I: Well, that's—how did we do it?
KV-LL: We—we graphed the gradient line and then we just tried to—we put in formulas for what we thought the equation of it was.

I: How did the computer come up with the gradient line?

KV-LL: Um, from the secant lines? Um, from the slopes of the original things—I'm not sure exactly—I don't remember.

I: From the slopes of the secant lines?

KV-LL: That's what I thought.

I: How do you find the slope of a secant line?

KV-LL: I don't know. I don't remember.

KV’s initial image of the derivative was procedural, involving use of the power rule. She was able to generate visual images of slopes of secant lines and tangent lines to sketch the graph of a derivative. She was unable to describe how these slopes were obtained.

JG-LL. After her initial shock at being asked a math question, JG-LL suggested her image of the derivative was visual. When the interviewer proposed that she draw a picture, JG described the derivative as a graphing of slopes. JG sketched the graph of a quadratic with its vertex in the first quadrant. The dotted derivative graph had a positive slope and crossed the x-axis vertically below the vertex of the parabola. JG stated that when the slope was negative, the derivative graph was below the x-axis.

I: The first question is what do you think a derivative is?

JG-LL: Oh god, it's a math question.

I: If you had to tell someone—yeah, it's a math question.

JG-LL: What do I think the derivative is. Um, what I think more is like the visual aspect of it now, I think.

I: Okay.

JG-LL: Uh, I would say——

I: If you need to draw pictures, you can draw pictures.
JG-LL: I'd say the graphing of the slopes—that makes sense.
I: So a derivative is graphing of slopes—of what?
JG-LL: Well, it just—it explains slopes I suppose. It has to do with slopes.
I: It has to do with slopes. Can you give me an example?
JG-LL: Of a derivative?
I: Mmm hmm.
JG-LL: I can draw them.
I: Use that.
JG-LL: Hmm. Just make a nice one here—and then it'd be somewhat of a dotted line—close to it.
I: So you drew a curve that looks like a quadratic and then you said the derivative of that looks like a straight line.
JG-LL: Uh huh.
I: That's good.
JG-LL: I understood a lot of—I didn't understand a lot of like the distances away and that kind of stuff—so when—I don't know if you saw the last test—he said be specific on them—I didn't do very good on that cause I can just kind of be general on them. Cause I know like when it's negative—it's below the x-axis—I know that kind of stuff.

JG applied the power rule when asked for the derivative of \( f(x) = x^2 \). Initially, she could not suggest a method for obtaining the power rule.

I: If I gave you a quadratic like \( f \) of \( x \) equals \( x \) squared, could you show me how to get the derivative of that?
JG-LL: Mmm hmm. Two \( x \).
I: Okay, where does that two \( x \) come from?
JG-LL: Put the two—multiply it what's in front and there's a one in front so you get the two and you drop the powers so it's two \( x \) to the first power.
I: How do I know that's true?
JG-LL: God—that's what we were—is that right?

I: It's right.

JG-LL: Um—we explained it—you explained it that one day on the board or our teacher did. How do you know it's true?

[Note, the interviewer did not do any instruction.]

I: Mmm hmm.

JG-LL: I don't know.

I: If you know you have a function $x$ squared, how do you know that its derivative function is two $x$?

JG-LL: How do I know that? I don't know how I know that. I just know the formula.

The interviewer asked JG to find the derivatives of $\tan x$ and $\sin x$. She was unable to give a symbolic form for the derivative of $\tan x$, but indicated she could sketch the derivative. In the case of $\sin x$, she knew the symbolic derivative. JG said she knew the symbolic form from having used the computer to graph the slope function.

I: Suppose I told you that something—that something that wasn't a polynomial—say I gave you $f(x) = \tan x$?

JG-LL: I wouldn't know it.

I: Then you wouldn't know how to do it.

JG-LL: Well, I'd know how to draw it.

I: How about $f(x) = \sin x$?

JG-LL: I know it's a cosine.

I: You'd know it's cosine. Did we show it was cosine?

JG-LL: Can I show it?

I: Did we?

JG-LL: Yeah.

I: How did we show it?

JG-LL: By graphing it on the computers.
JG's response that she knew the derivative from having graphed it on the computer prompted the interviewer to probe how she would use the software to explore the derivative graph. JG also knew that the software was graphing slopes. JG entered the function \( f(x) = x^2 \) and used the gradient curve program to draw the slope curve of the function. It is not clear that she understood the role of choosing \( h \) for the slope formula. JG was able to identify the correspondence between a negative slope and the derivative lying below the \( x \)-axis. She also identified the slope formula as generating the slope graph. She did say, however, that this was the slope of the tangent.

I: Okay, we graphed it. Did we—how does the computer come up with its graph? Do you remember?

JG–LL: It takes the slope—it does—it takes the slope so—it kind of graphs on where the slopes are, doesn’t it? Well—

I: Well, we can look at it. Why don’t you pull the keyboard over. You can type—

JG–LL: (inaudible).

I: Let’s go to function and let’s just put in \( x \) squared.

JG–LL: Is that good?

I: Yeah. Now, if you wanted to look at the slope curve—

JG–LL: Mmm hmm, you’d do it?

I: You’d hit—

JG–LL: Is that two?

I: Yeah. Now what are you choosing there?

JG–LL: The distance that it moves each time.

I: The distance.
JG-LL: The little point—the distance away from the dots that it plots.

[JG identifies the value of $h$ in the slope function.]

I: Okay.

JG-LL: (inaudible)

I: Now how's it getting these points that it drew there?

JG-LL: It has to do with—the slope because the slope's negative—so it's down here and (inaudible)

I: So the slope's negative—it's down below the $x$-axis—slope positive it's above the $x$-axis—

JG-LL: Mmm hmm.

I: How does it calculate the slopes?

JG-LL: With that?

[JG indicates the slope formula displayed on the screen.]

I: With what? This formula that says $f$ of $x$ plus $h$ minus $f$ of $x$ over $h$?

JG-LL: Yeah, I would think. I've never thought about that. I just always kind of did it.

I: Okay. That's the slope formula.

JG-LL: Mmm hmm.

I: Do you agree with me?


I: How—

JG-LL: It's the slope of the tangents and so they—they plot the slope of the tangents.

The interviewer now asked JG to derive the derivative for $f(x) = x^2$. After some hesitation, she correctly wrote $f(x) = x^2$ and $f(x + h) = (x + h)^2$. She then set up the difference quotient for the slope. She did not use a limiting process and was uncertain if the difference form could be used in general.
I: Okay. Could you use that formula to show me that the derivative of \( x \) squared is two \( x \)?

JG-LL: I don't know. I forget that. I'd have a tough time. I don't think I'd do it right, truthfully. (Inaudible) plus \( h \)—

I: That must be what you set up. Okay. Let me see what you get. So you would go through the algebra and you would simplify that.

JG-LL: Mmm hmm.

I: And you would get—You would come up with two \( x \).

JG-LL: Mmm hmm.

I: Okay.

JG-LL: Pushed it through—Yeah, I guess I could do that.

I: Could you always use that formula?

JG-LL: I think so.

For JG, the visual image of the derivative was primary. The computer graphing of slope curves enabled her to see the derivative of a function. This image of the computer drawing slopes was not closely tied to the slope formula. JG seemed unaware of the limiting process used to define the derivative formally.

**IM-LL.** IM-LL began by describing the derivative globally in terms of an expression for the slope of any tangent line to the graph. When asked for an example, she chose a quadratic function and used the power rule to give both symbolic and visual representations. Initially, IM generalized the power rule to all functions, but then realized that it did not apply to trigonometric functions.

I: The first thing I want to ask is what you think a derivative is.

IM-LL: Okay, umm, a derivative umm is umm a well, in terms of an equation, it's a general expression the slope of any line that's tangent to any point on a graph. So, umm, yeah—that's what a derivative is.

I: Okay. Can you give me an example?
IM-LL: Umm, like for a parabola, umm, to graph the slopes of any tangent line on a parabola, it would be a straight line. Umm, just like, umm, where the parabola is a power of two, the derivative would be a power of one and power of one is a straight line.

I: Does that power rule work in all cases?

IM-LL: Yup. As far as I know. At least I haven't gotten in—I haven't seen any cases where it hasn't really worked.

I: Well, what kinds of functions have you worked with?

IM-LL: Umm, like $x$ to the fourth and, umm, the trig functions and the umm...

I: Does it work with the trig functions, when you take the derivative of a sine function?

IM-LL: Oh, Oh, no it doesn't. Umm, because—umm, for a sine—like for sine the actual derivative is actually cosine of $x$. So, no it doesn't work for like trigonometric functions, but, it works for everything else.

IM approached the task of deriving the derivative of $\sin x$ with a variety of tools, using measurement, then a specific point, and finally a general expression. IM was quick to realize the advantages of having a general expression for the derivative.

I: How would you show the derivative of sine as cosine.

IM-LL: Well, what I'd do is I'd pick out certain points along the graph of—of sine of $x$, umm—can I take the derivative of the equation of sine $x$? Okay, well, what I would do is like measure the slope of each of those points and what I would do is graph the slope along the $x$-axis and it will come out looking like a cosine curve so I can prove that it's a cosine.

I: So you can visually show it.

IM-LL: Yeah.

I: Can you show it symbolically?

IM-LL: Oh, like—like on paper?

I: Mmm hmm. On paper.
IM-LL: Umm, yes where it's taking the cosine where I use the umm—the—not the definitions, but the—yeah, I guess you could call them definitions for like adding—adding and subtracting angles. You can use those definitions to come up with a proof to prove that sine of x is, umm...

I: How would you show that? Can you write that down?

IM-LL: Oh my gosh.

I: You don't have to do the whole proof.

IM-LL: Okay.

I: Just show me what you would set up.

IM-LL: Okay. You would start out with your basic—umm—f of x plus h minus f of x over h equation and what you would do is have like the cosine of—umm, just pick out any point or actually sine, rather, and have like a certain point and subtract it from—uh, let's just say, uh, like two point one or something. And for h would be point one and subtract sine of two from that and in order to subtract the sines of the equation you would have to do that—umm—sine of—what—sine of beta plus sine of—sine of beta minus alpha or whatever it goes—and you have to plug all that in and get it to work and then I'll come out like, gee, this is cosine. And it will be really neat.

I: Can you do that with x and not just a point? Like two?

IM-LL: Oh, you mean like put in x plus—just say like x plus h here?

I: Mmm hmm.

IM-LL: Yeah, you can do that.

I: Okay.

IM-LL: Cause you'll come out with—instead of having specific definition for sine, for the slope at x equals two or umm, you would actually have the—a general formula which is what the derivative is.

I: Okay.

IM-LL: Yeah. That'll work.

...IM had a strong conceptual framework for the derivative. She was able to use both visual and symbolic representations in her explanation. At this time, however, she did not suggest using a limiting process to obtain the derivative.
CA-LL. CA-LL began by describing the derivative as a global expression for the slopes of a function. He used the word equation to describe a function. When asked for a visual example, CA provided a cubic function and sketched the derivative graph by observing slope properties of the cubic. He used both procedural clues, the derivative of a cubic is a quadratic, and visual clues, "[I]t's a parabola," to check that his derivative sketch was correct.

I: So we'll start with what you think the derivative is.

CA-LL: I think that the derivative is the equation of all these slopes at particular points on an original equation. Maybe I should try to clear that up.

I: You can give me an example. I can give you a pen to write with. So you have a function that you've sketched.

CA-LL: Yes. This is \( f \) of \( x \), and it's a cubic and it's—I think it's negative—it's probably not relevant—and what the derivative—how the derivative would be gained was—would be to, uh, you notice that in this area the slopes are all going to be negative.

I: Okay.

CA-LL: And thus you would know that these would all be below the \( x \)-axis when you draw the derivative and they would meet the \( x \)-axis where the slope is zero such as right there and right here so you can mark those. We'd be crossing a derivative curve. As for this area in between them, you—the slope there is always positive so you know that will be above the \( x \)-axis and if you know that that's a cubic, you can generally assume that \( x \) squared using the shortcut—I mean \( x \) cubed using the shortcut will give you three \( x \) squared—somewhere around in there and you know that it's a parabola. And so you could do that and (inaudible)—

[CA provided a sketch similar to the following.]
The interviewer asked CA to prove his use of the shortcut for finding the derivative. He said and wrote the expression for the slope of a secant but did not use a limit in his expression.

I: How would you prove that—the shortcut—to me?

CA-LL: Um, the shortcut? I'd probably do the old \( f(x + h) - f(x) \) over \( h \).

I: And expand. What else would you do with it?

CA-LL: Well, could you clarify that?

I: What else would you do with it—besides expand?

CA-LL: You mean to prove that it is a—that that is the derivative? Hmm.

The hesitation prompted the interviewer to probe CA’s image of the derivative. CA described the derivative as a graph of a function where the \( y \)-values were given by the slope of the original function. CA described the slope at a specific point as the slope of the tangent to the graph. When asked how this slope was found, CA wanted to use the derivative.

I: What else is the derivative?

CA-LL: Just basically the graph of a function that is—whose \( y \) values are the slopes at \( x \) values on the original.

I: How do I find at a specific point what the slope is?

CA-LL: You draw a tangent to that point on the curve and where the slope of that tangent is will be your \( y \) coordinate on the (inaudible) [derivative graph?].

I: How do you find the slope of the tangent?

CA-LL: Um, let me think about this. You can plug your \( x \) value into the derivative function if you've gotten that.
At this juncture, the interviewer proposed that the derivative function was not known. CA was provided with a specific example where \( y = x^2 \). Points \( P (2, 4) \) and \( A (k, k^2) \) were indicated on the graph and a tangent line was drawn at \( P \). CA was asked to find the slope of the tangent line at \( P \). After initial hesitation, CA suggested finding an approximation to the slope using the points \( (2, 4) \) and \( (2.1, 2.1^2) \).

I: But what if I don't know the derivative function?
CA-LL: Hmm. Then there's trouble. Let's figure this out.
I: Suppose I give you a quadratic like this—\( y \) equals \( x \) squared.
CA-LL: Okay.
I: And I want you to find the slope of the line which is tangent at point \( P \).
CA-LL: And I only know this one point on the line?
I: You only know that one point on the line and you want to find the slope of the tangent line.

CA-LL: Shoot. I guess the best thing to do would be to find an approximate value like—(inaudible)—um, what I would do is—see you've got \( y \) equals two squared and then I'd plug in another point that's right next to it like two point one.

CA was then asked to generalize finding the slope. He proposed using the difference quotient with the point \( (0, 0) \) and was redirected by the interviewer to the point \( A (k, k^2) \). CA used a limiting process to let \( k \) approach 2. He concluded that the slope found by this method was "more immediate—more accurate" for the slope of the tangent.

I: Can you make that more general? Can you do this symbolically rather than numerically?
CA-LL: Um, by using the \( f \) of \( x \) plus \( h \) minus \( f \) of \( x \) over \( h \)—
I: Or, for example, do you know another point on that curve?
CA-LL: Uh, zero, zero.
I: Well, have I given you a symbolic point?
CA-LL: Yeah. A.
I: Can you find the slope of AP?
CA-LL: Okay. Yes. I can.
I: Now, how do you get a point closer to P?
CA-LL: Um—Make k smaller.
I: Make k smaller?
CA-LL: Yeah.
I: Like what?
CA-LL: Well, you just make sure that k is closer to P in—numerically.
I: Closer to—
CA-LL: The x value of P.
I: Closer to—
CA-LL: Two.
I: Two. What happens as k goes to two?
CA-LL: Uh, as k goes to two, the distance between them gets smaller, and a lot of the slope becomes more immediate—more accurate for that particular point there.
I: For?
CA-LL: For P.
I: For P. The slope of the tangent? Am I right?
CA-LL: Yes.

CA used both visual and procedural knowledge in working with the derivative. He was able to accurately sketch the graph of the derivative for a cubic function, recognizing intervals of increase and decrease and information provided by maximum and minimum points of the function. When pressed for a definition of the derivative, CA wanted to use the derivative as known—that is, as obtained through a shortcut. CA saw the limiting
process of a secant approaching the tangent as achieving only an approximation for the slope of the tangent.

JR–ST. JR, a woman in the secant—tangent class, began by expressing a visual image of the derivative while apparently hesitating about a symbolic definition. JR sketched a parabola and explained how to sketch the graph of the derivative using positive, negative, and zero slope.

I: The first question I wanted to ask is what you think a derivative is.

JR–ST: Well, see it's hard for me—this is what I wrote on this paper yesterday—it's like I know how to get and I know what it will look like. If it's a certain way I can guess what it will look like. But this is the way it always with math with me is I don't understand what it is. I mean I know it's the derivative of that curve and you can get like lots of stuff from it but I don't know what it is. I mean you know what I'm saying?

I: Can you write down an example?

JR–ST: You mean like draw an example?

I: Yeah. Draw an example. Sure.

JR–ST: It'll be simple, but—like say you have like your little—just a parabola like that, okay? Then your derivative—these are at—since this is a negative slope it's going to be down here. Okay, so this is just going to be a straight line like that cause that's positive.

I: Okay.

JR–ST: And you know, if there's a horizontal there's no slope cause that's where the x—it's going to touch the x-axis.

In response to the question from the interviewer for further clarification of the derivative, JR sketched a cubic function and used the sketch to illustrate her ideas. She had clear ideas about the relationships between the derivative and both the slope and steepness of a function.
I: Okay. How do you find the derivative that you drew?

JR-ST: How do you find it?

I: Mmm hmm.

JR-ST: You look at—you take like certain points on here and you look at if the slope is positive or negative—if it leans this way it's positive—if it goes that way it's negative. Then you say like if this is a great—like say this was not—this is—these are pretty much all the same slope—but if like one was out here that would be less on the slope than that. Cause that isn't as great of a vertical as that is. So that would be further up closer to the x-axis. So it would end up looking like that if that was how it was.

I: Okay, so you're saying where there's a less steep slope the derivative curve is closer to the x-axis.

JR-ST: That's what I think. All right?

When asked how to find the value of the slope, JR suggested the formula for the slope of a line between two points. She was unable to relate this to the slope at a particular point. Her image of the derivative was visual. The y-value for a particular x on the derivative curve was the value of the slope of the original function at that x.

I: Mmm hmm. How do you calculate the value of the slope?

JR-ST: Calculate the value of the slope?

I: If you picked a point on this y equals x squared, how would you calculate the value of the slope?

JR-ST: You got me early in the morning and I'm without my notes. Can you—I think you can do that one thing where you go y minus y one equals—oh, that's the equation, isn't it?

I: What's that the equation for?

JR-ST: Slope of a line or—slope of a line.

I: The slope of a line.

JR-ST: Mmm, for a specific point. I know you can do rise over run for basics like lines and stuff but I don't know about an exact point. I know if you take the y of the derivative, isn't that—this—like the y here is the slope or something—like the y on this is the slope of that point of that line.
JR was able to visually describe the derivative. She recognized the importance of both slope and magnitude of steepness in sketching the derivative. She could identify the slope formula for linear functions, but was unable to generalize this to nonlinear functions.

CB-ST. CB-ST identified the derivative as a representation of slopes. He indicated he had difficulty putting his ideas into words. When asked to draw a picture, CB sketched a parabola in the first quadrant, chose a point, and drew a tangent at the point. He estimated the slope of the tangent and plotted the value of the slope. He estimated the slope at another point and plotted its value. CB stated that the derivative was a visual representation of all of the slopes of tangents on the parabola. He indicated the graph of the derivative for a parabola would be a straight line.

I: What do you think a derivative is?

CB-ST: Uh, it's the representation of the slopes—it's kind of hard to put it in words—I have a problem with that—writing it down.

I: Mmm hmm.

CB-ST: It's because it's where the—on the graph where you have a—

I: Why don't you draw a picture and talk about your picture?

CB-ST: If you have a set of axes there and you have—say a parabola the derivative is going to come down like in the like the corresponding I guess x spots like here—somewhere along here you're going to plot the value of the slope of this right here of the tangent at this point. Whatever that is on the same vertical—I don't know what you'd call that—say the slope here was two—you'd come up and the slope of this tangent at this point would be two—you would plot it at two and up here it might be four—of course it would be moving over this way, so say it was four and over here underneath that one you'd move up and your slope would be at four and then connecting all the points would be a visual representation of all the slopes and all the tangent points on the parabola.

I: So with the parabola you'd get?

CB-ST: A line.
When CB was asked how he would find the slope at a particular point, he suggested using the derivative. He gave an example that the derivative of \( f(x) = x^2 \) is \( 2x \). When asked where the \( 2x \) came from, CB first indicated the shortcut method and then made reference to \( x + h \). He was unable to use the graph he had sketched to generate the slope at a point.

I: Okay. Now, you said several things about calculating the slope at a point and finding the tangent—the slope of the tangent. How do you go about doing that?

CB-ST: Let's see—to find the slope of the—at a point—if you're like given a point—um—the—you're using your derivatives like over there—like of the original equation—so like if you had an equation of the parabola like \( f \) of \( x \) is \( x \) squared and you take the first derivative of that which would be like two \( x \) and then you plug in your—the \( x \) value of wherever you're at in—so that'd be like say if your \( x \) value is two your slope would be four then.

I: How did somebody come up with that two \( x \)?

CB-ST: How—oh—from the original how would you get the first derivative? Well, on something short like that you can use your shortcut for that or you can—

I: Now how does that—

CB-ST: Or you can use the equation of the—\( x + h \)—um, wherever you're adding—let's see—you have your original equation which would be \( f \) of \( x \) and then it goes back to—gosh, it's so hard to.

[Here CB wrote \( (x + h) \).

I: Well, draw a picture of what you're trying to do. Draw a curve.

CB-ST: Okay. The um—

[CB drew a curve in the first quadrant, but did not label any points.]

I: So you're trying to find the derivative at a point. What do you do?

CB-ST: Well, I mean, it's like hard—I mean I know this \( x + h \) you know minus—over the \( h \)—I mean it's just hard to—how to say do you get from the original to that.
The interviewer provided CB with a sketch of \( f(x) = x^2 \) with point \( P(2, 4) \) and point \( A(k, k^2) \) labeled. The tangent at \( P \) was drawn. When he was asked to find the slope of the tangent line at \( P \), CB said he would use the first derivative. The interviewer asked CB what he would do if he did not know the first derivative. CB indicated the secant line. He was able to use the slope formula to find the slope of the secant connecting \( A \) and \( P \). He simplified the slope to \( k + 2 \). When CB was asked how the slope of the secant line was related to the slope of the tangent, he said the tangent line was like the derivative, but the two were not the same.

I: Maybe this will help. Let's look at it in this context. I have \( f \) of \( x \) equals \( x \) squared. And you've used this in your picture. I picked a point two, four. And on the curve, I've written a point \( k, k^2 \). Now, can you tell me the slope of \( AP \)? How would you find the slope of \( AP \)?

CB–ST: You would go right back like if you—what do you mean—since—it's the same as you would find any other slope of a line when you know the two points—would you be—the change in the \( y \) coordinates over the change in the \( x \) coordinates.

I: Okay. Now, I have a tangent line at point \( P \). How do I find the slope of the tangent line at point \( P \)?

CB–ST: You would take the first derivative of the equation, so that would be two \( x \) and then—

I: I don't know the first derivative.

CB–ST: The you—you go—using this—

[CB indicated the secant line \( AP \).]

I: Okay, let's use this specific point. Let's write down the slope of \( AP \). Can you write down the slope of \( AP \)? You can do it right here.

CB–ST: I hope I can do it. Uh, the slope of \( AP \). That would be (inaudible) \( p \) minus two. Wouldn't it?

[CB wrote \( \frac{(k + 2)(k - 2)}{(k - 2)} \).]

I: Can you simplify that?

CB–ST: It would be \( k \) plus two then.
I: So it would be \( k \) plus two. So write down the slope as \( k \) plus two. Now you're trying to find the slope of the tangent line and somehow that's related to the slope of the secant line, isn't it?

CB–ST: Yes.

I: Or am I wrong?

CB–ST: No—the—wait—we found the slope of this right here—

I: Mmm hmm. The slope of the secant line. Now how's the slope of the secant line related to the slope of the tangent? What did we do on the computer?

CB–ST: Um—I keep wanting to say that the—I keep coming into my head that the tangent line is like the derivative but it's not.

I: It's not?

CB–ST: No.

CB had a visual image of the derivative and was able to show how the derivative graph was obtained by estimating the slopes of tangent lines at particular points on a function. He saw the derivative globally as an accumulation of the slopes of all tangent lines. CB was unable to relate his estimation procedure for finding slopes of tangents to the formal process of letting the secant approach the tangent. In fact, CB indicated that the slope of the tangent line and the derivative were somehow different.

JS–ST. JS, in the secant–tangent group, identified the derivative as a collection of all of the tangent points on a curve. When asked for an example, she sketched a cubic function with the derivative as a quadratic function. JS indicated she used visual information—horizontal, increasing, and decreasing slope—to sketch the graph of the derivative. JS was unable to relate the derivative to the limiting process of the secant approaching the tangent.

I: The first thing I want to ask is what do you think a derivative is?

JS–ST: Umm, it's the sum of all the tangent points on a curve.
I: Okay, what do you mean by the sum?
JS–ST: It's—I don't know how to explain it—(inaudible)—that makes like the common point or something.
I: Can you give me an example?
JS–ST: You mean just like draw a derivative?
I: Draw a function and then its derivative.
JS–ST: Well, if you have like—umm—then that would be a parabola.

[JS sketched a cubic function similar to \( y = x^3 \) with the derivative similar to \( y = 3x^2 \).]
I: So if you have a cubic then it would be a parabola.
JS–ST: Right.
I: What points do you look for when you draw the graph of the derivative?
JS–ST: You look where it's horizontal because that's where it crosses the \( x \)-axis.
I: The derivative crosses the \( x \)-axis.
JS–ST: Right. The derivative crosses the \( x \)-axis.
I: Okay.
JS–ST: And then—umm—whether the slope is positive or negative.
I: Okay, so you look for positive and negative slopes.
JS–ST: Right.
I: I think you meant by sum the collection—
JS–ST: Yeah.
I: —take all of the tangent points or slopes of the tangents together.
JS–ST: Mmm hmm.

DS–ST. DS, a student in the secant–tangent class, prefaced his definition of the derivative as a way of finding slope by indicating that he had first learned about the derivative in physics. He indicated, in physics, the derivative was a way of finding
instantaneous speed and in the case of the second derivative, acceleration. DS stated he learned the shortcut method for the derivative first in the form of rules. He referred to the slope formula in the form of \( x + h \) but never actually gave the formula. He suggested the limiting process of letting \( h \) approach zero so that the slope of the secant line would get closer to the actual value of the slope. When asked what was meant by the slope at a point, DS described it as the line tangent to the curve at the point.

I: And what I wanted to start with is what you think a derivative is.

DS–ST: Well, it's just a way of finding the slope of—a slope of any point on a graphic equation. I guess it's a way of finding, you know, like a, like instantaneous speed for physics or you know, acceleration if you take the second derivative. That's how I learned it firstly—I first learned it in physics. I didn't learn it in Algebra III, so—

I: What kinds of things did you learn about the derivative?

DS–ST: Well, we basically learned the shortcut first and then we went over the \( x + h \) part and that kind of confused me and he said, "Don't worry about this—you'll learn about this in algebra." So we just sort of used the rules and applied it first before we really learned anything about it.

I: What's the \( x + h \) part?

DS–ST: That's the—\( x + h \) is finding a very close—or \( h \) is very small and \( x \) is the—\( h \) is very small and then you're just adding that on to \( x \) and that's going to change the slope of the line just a little bit through the point—uh—that's kind of hard to explain cause I learned this conceptually the first time. I've got the whole picture.

I: Mmm hmm.

DS–ST: It's a way of—uh—finding the nearer slope of \( x \)—as \( h \) approaches zero the slope becomes closer to what it actually would be and we can find what it converges on and that would be the exact value of the slope through \( x \).

I: What do you think is meant by the slope at a point? How would you illustrate that?

DS–ST: A line tangent to the curve at that point.
Later DS was asked to find the derivative of \( y = \sin x \). His initial response was to appeal to a memorized form. When prompted, DS was able to construct the difference quotient form for finding the slope of the secant line between two points on the sine graph. He was not, however, able to connect this with the derivative.

**I:** How would I go about doing that (finding the derivative of \( \sin x \))?

**DS-ST:** Well, I was just—I was taught to take the opposite of the cosine of \( x \)—I mean—I forget why, though. I can't tell you. I'm trying to remember.

**I:** Well, you talked a little bit about slopes of secants—slopes of lines, secant lines, \( x + h \)—does that apply to the sine function, too?

**DS-ST:** Well, I guess it applies to everything, I suppose.

**I:** Well, how would you fit that into the sine function?

**DS-ST:** I guess you'd just take \( f \) of \( x \) would be sine of \( x \) and then \( f \) of \( x + h \) would be sine of \( x + h \). That's as far as I can remember taking it.

**I:** So what would you write when you wrote sine of plus—sine of \( x \)—what would you be trying to find?

**DS-ST:** What would I be trying to find? I mean, I'm visual. I can't—

DS began with a procedural knowledge of the derivative learned in physics class. His knowledge of the derivative was apparently limited to symbolic polynomial differentiation. DS had vague images of the derivative connecting with the slope of a tangent line. He could suggest the limiting process of the secant approaching the tangent but could not formally construct this representation.

**RM-ST.** RM, a student in the secant–tangent class, described her definition of the derivative as technical, the slope of a tangent line to a curve at a point. She contrasted this with an applied definition that she did not yet know. In discussing what an applied definition might be, RM offered her understanding of the use of tangent in everyday language as "a tangent to a conversation."
I: And the first question is, what do you think a derivative is?

RM-ST: A derivative? Oh, I didn't know I was going to have to answer questions. A derivative is slope of a tangent line to a curve at a single point. Technical definition.

I: Technical definition.

RM-ST: I don't know yet how it is applied to anything, but—

I: Can you imagine how it's applied to anything?

RM-ST: Well, I figure if you wanted—if you're looking at a graph on a computer screen—I don't know. I never really thought about that. It might—it's got something to do with calculus, but the only way I've ever heard of a tangent being used in anything other than math is just—a tangent to a conversation—in English, you know.

I: Hmm. And it doesn't quite mean the same thing?

RM-ST: Well, whenever I think of it it means the same thing—you know—a point in the conversation that touches on one point to another but is totally unrelated otherwise.

After discussing the meaning of tangent, the interviewer resumed questioning about the derivative. RM suggested a general formulation for the derivative as an expression that "any point you plug into it" yields the slope of the tangent line at that point. She agreed that the derivative could be a function and was able to give an example.

I: You talked about the derivative at the start as being the derivative at a particular point.

RM-ST: Well.

I: Does the derivative tell you anything else?

RM-ST: It tells the slope of a tangent line at that—at any point you plug into it. Any point really.

I: Okay.

RM-ST: If you plug in a specific point it'll the slope at that point.

I: Can the derivative be a function?
RM–ST: Yeah.

I: Could you give me an example?

RM–ST: Of a derivative or a function?

I: A derivative—well—of a function whose derivative is a function.

RM–ST: Well, if you have $y = x^3$, the derivative is $2x^3$—three $x$ squared and that’s a function.

RM maintained both visual and symbolic views of the derivative. She visualized the derivative as the slope of a tangent line. Symbolically, she treated it as a formula used to find slope. She saw the derivative globally as a function. In addition, RM tried to make connections between the mathematical meaning of tangent and its use in everyday language.

CF–ST. CF, a student in the secant-tangent group, began by describing the derivative as the slope of a tangent line to a curve at a particular point. He gave an example by evaluating the derivative of $y = x^2$ at $x = 1$. When asked for a justification of his result, CF referred to the difference quotient for slope and the limiting process of letting the $h$-values get smaller. CF’s use of the word “approaches” suggests the limit is never really achieved. CF suggested that he just got rid of the $h$’s at the final stage of evaluating the derivative. He realized there was a problem with omitting the $h$’s, but was more concerned with division by zero.

I: Tell me what a derivative is.

CF–ST: Well, it’s just basically, uh, you know, the slope of a line tangent to a curve, I guess is the best way to describe it, at one point, so at the same $x$-value it’s...

I: Okay, so you have a slope of a tangent line at a particular point of a curve.

CF–ST: Right.

I: Can you give me an example of that?

CF–ST: Well, like for $y = x^2$ at $x = 1$ the tangent, I think, the slope is two.

I: How would you go about doing that?
CF-ST: Well, actually it kind of approaches two, because if you—you know—as you're calculating the slope and the h's get distance between the two points, gets smaller and smaller, it just gets closer and closer to two.

I: What do you do in your mind with the h's?

CF-ST: Just kind of get rid of them.

I: Make them zero?

CF-ST: Yeah.

I: Do you think there's a problem with that?

CF-ST: Well, there could be. I mean if it's in the denominator you have the undefined problem, so—

When asked to think about the derivative at other than a point, CF responded the derivative is a function, represented as a plot of slopes. He made reference to a positive value on the derivative curve as indicating a greater slope. CF was comfortable with a symbolic representation of the derivative as "f prime of x."

I: Is there something else about the derivative other than the derivative at a point? How else have you thought about the derivative?

CF-ST: Well, it's for the whole function really. It's just a—basically like a plot of the slopes.

I: So, you're saying...

CF-ST: If the slope is greater than the plane—on the derivative curve—I guess you would call it—is a greater—has a greater value.

I: When you write, say, f of x equals x squared and then you write the derivative, you write it as f prime of x equals two x?

CF-ST: Yeah.

I: So you're thinking of the derivative as a function?

CF-ST: Right.

I: I think you said derivative curve?

CF-ST: Right.
CF was able to consider the derivative with both symbolic and graphical representations. He saw the derivative as representing the slope of the tangent line to a curve locally at a point and globally as a function. CF generated the limiting process of the secant approaching the tangent, but considered this to be an approximate rather than an exact value for the derivative.

The above discussion of interviews with students shows few differences between the two treatment groups. The primary view of the derivative held by students who were interviewed was visual—that is, when asked to define the derivative, they sketched a graph. When asked for a definition of the derivative, they had difficulty expressing the definition in words. Yet, they were able to sketch the graph of a function and the graph of its derivative. In sketching the graph of the derivative of a function, students gave particular attention to intervals on which the function was increasing and decreasing or had a horizontal slope. Students saw the derivative as representing the slope of a tangent line to the function at a particular point and generalized the derivative to include all points of the function.

Student concept images and concept definitions of the derivative as expressed in the interviews correspond with data obtained from the posttest. In the interviews, most students used the image of a quadratic or cubic polynomial function when asked about the derivative. They were then able to sketch a graph of the derivative function for the polynomial. On the posttest, students in both treatment groups were generally able to sketch the derivative graph for a function when it resembled the graph of a polynomial function (items 8i, 8ii, Appendix D). Furthermore, on the posttest when a function was continuous and differentiable such as a sine function or a normal curve, students were better able to sketch the graph of the derivative (item 8iii from the Calculus Concepts Inventory II, Appendix D, and items 1, 3, 9, and 10 from the Function Graphs for Sketching Derivatives section of the posttest, Appendix C). It may be hypothesized that
students on the posttest gave the same attention to intervals on which a function was increasing and decreasing or had a horizontal slope as did the students who were interviewed. On the posttest, when a function was discontinuous or nondifferentiable, students were less likely to sketch the function's derivative correctly (item 8iv from the Calculus Concepts Inventory II, Appendix D, and items 2, 4, 6, and 8 from the Function Graphs for Sketching Derivatives section of the posttest, Appendix C). Students, who were interviewed, did not use examples of discontinuous or nondifferentiable functions when defining the derivative.

The identification of the derivative with the slope of a tangent line—either locally at a point or globally for the entire function—by the students who were interviewed also seems to have been present on the posttest (item 4 from the Calculus Concepts Inventory II, Appendix D). While more local linearity students than secant–tangent students were able to find the slope of the function at a point using the slope of the tangent line at the point on this item, the students, who were interviewed almost unanimously described the derivative as the slope of a tangent line. This description was consistent with student definitions of the derivative on the posttest (item 15 from the Calculus Concepts Inventory II, Appendix D) in which the word slope was used consistently to describe the derivative.

While students made reference to the symbolic, difference quotient, representation of the derivative, and suggested the derivative was obtained through a limiting process of letting a secant approach a tangent, none were able to formally derive the derivative as a limit. Among those students who used the limiting process, the limit was seen as an approximation to the slope of the tangent rather than as an exact value. This suggests connections were not made between the visual presentation of the derivative using the Gradient software and the symbolic evaluation of the limit of \( \frac{f(x+h)-f(x)}{h} \). Students were able to find symbolic representations for the derivatives of elementary polynomial and
trigonometric functions, but they could not justify the procedure used to obtain these derivatives.

Similar evidence for student knowledge of the difference quotient representation of the derivative was obtained on the posttest (item 2 from the Calculus Concepts Inventory I, Appendix B). On this item, the majority of students were able to find the slope of the function \( y = x^2 \) at \( x = 1 \) after having found the slope of the secant line between the points \((1, 1)\) and \((k, k^2)\), but none of the students used a limiting argument to justify the value of the derivative they obtained. A primary justification given by the students was that 2 was the value of the derivative of the function at \( x = 1 \). Thus, it appears that while students acknowledge the limit difference quotient definition of the derivative, they fail to use it as mathematical justification, preferring to use a more procedural application of derivative rules. The inability of most students to recognize the difference quotient \( \frac{(x + .0001)^2 - x^2}{.0001} \) (item 6 from the Calculus Concepts Inventory II, Appendix D) as an approximation of the function \( 8x^7 \) is further evidence that students were more concerned with procedure than with definition.
Differentiability and Non-differentiability

The second item of the interview protocol explored student conceptions of the differentiability and non-differentiability of functions. A student was asked to provide examples of functions that were differentiable and functions that were non-differentiable. Not all students were able to provide examples.

IM-LL. IM-LL used a linear function as an example of a differentiable function. She proposed that a parabola was not differentiable at its vertex, arguing that positive and negative \( h \)-values in the difference quotient would yield different left and right derivatives. When the interviewer questioned IM's proposal, she changed her example of a non-differentiable function to the absolute value function.

I: Can you give me examples of functions that have derivatives everywhere?

IM-LL: Oh, yeah. Derivatives everywhere?

I: Mmm hmm. That you can differentiate at every point.

IM-LL: I guess technically you could say like a straight line because a straight line would have a definite slope for everywhere it goes and it would just be a straight horizontal line. You couldn't do that with, say, a parabola, because at the bottom, if you had like a negative \( h \) at one side and a positive \( h \) on the other, you couldn't—umm—you couldn't put them in the slope equation or the \( f \) of \( x \) plus \( h \) equation, and get the same slope because you'd like on one side of the parabola and the other side of the parabola.

I: Are you sure? For the parabola?

IM-LL: At the vertex of a parabola. Maybe it's not a parabola. Maybe it's like a absolute value graph.

Asked to explain the difference between a parabola and the absolute value function, IM appealed to the rounded properties of the parabola. Her response indicated she was looking at the different slopes to the left and right of zero on the absolute value function.
I: What's the difference between the parabola and the absolute value graph?

IM-LL: The absolute value graph doesn't have a rounded end at the bottom whereas like parabola does. So I guess a parabola wouldn't exactly be a good case for that, but an absolute value graph definitely is because you'd have to go—definitely have to go one side or the other. And that would be a really bad case, but a straight is really good.

I: So you're saying that you can find the derivative at the vertex of a parabola, but that you can't find the derivative at the vertex of an absolute value?

In response to the interviewer's clarifying question, IM returned to her original argument. At the vertices of both parabolic and absolute value functions, the difference quotient representation of the derivative would yield a positive slope on one side of the vertex and a negative slope on the other side.

IM-LL: Well, actually, to be quite honest, I don't think you can find it at the vertex of a parabola or an absolute value, because you would have the same scenario at the vertex of a parabola where you would have a negative $h$ on one side and a positive $h$ on the other and you wouldn't get the same results.

IM's response led the interviewer to recall her previous definition of the derivative as the slope of a tangent line. IM stated there was a tangent with a slope of zero at the vertex of a parabola. She indicated that it was not theoretically possible to draw a tangent at the vertex of an absolute value graph. IM's explanation of not theoretically possible to draw a tangent offers a revised perspective on how she viewed the derivative. IM indicated the difference quotient was necessary as a means of finding the slope between two points, two points being required for a line. She suggested the limiting process of the secant approaching the tangent never really reaches the tangent as there are always two points involved. The use of the shortcut method of differentiation—for example, the power rule—overcomes the division by zero error in the limit of the difference quotient. Finally,
IM said the absolute value function was not differentiable at its vertex because it seesawed back and forth.

I: Now you—what did you tell me about the derivative? It was the slope of...

IM–LL: Umm, it's the slope of a tangent line or it's a tangent to any point along a graph.

I: Okay. On the parabola at the vertex, is there a tangent line?

IM–LL: Yeah, there is.

I: What's its slope?

IM–LL: The slope would have to be zero.

I: Okay. Now on the absolute value function at its vertex is there a tangent line at the vertex?

IM–LL: You could draw one, but theoretically, it really—theoretically it's kind of impossible.

I: What do you mean by theoretically?

IM–LL: Because, umm, in order to have—in order to draw any line, you have to have two points. That's why we have the \( f(x) \) plus \( h \) minus \( h \) equation—because you have to have two points and no matter how small you make \( h \), you have to have two points. If you made—um, the derivative is just a way that we've eliminated a division over zero error, where if we made \( h \) equals zero—by you know, the shortcut version—so if you had the bottom of an absolute value graph, you—I don't think that you would be able to differentiate it at that point because it would be like a seesaw. Where you wouldn't know how to—it could go this way or it could go that way and it could still be tangent to that point.

IM's discussion of differentiable and non-differentiable functions primarily involved the use of the difference quotient to look at left and right derivatives at a particular point. A linear function was her image of a differentiable function. The use of the difference quotient suggested to IM that when the slope of a function changed from negative to positive, either gradually as in the case of a parabola or quickly as in the case of an absolute value function, the function was non-differentiable.
This discussion of differentiable and non-differentiable functions clarified IM's understanding of the derivative. She viewed the formal limiting process of a secant approaching a tangent definition of the derivative and the procedural, shortcut, method of finding the derivative as linked representations. The limiting process was used to find the slope of a line connecting two points. The procedural method was used to overcome a division by zero error as \( h \) goes to zero in the limiting process.

**JG-LL.** JG-LL described a non-differentiable function as one that came to a point. She offered a pictorial representation, but was unable to give any explanation.

I: We talked about functions being not differentiable.

JG-LL: I—don't know. I just—oh, gosh—I'm messed up on that.

I: What do you think it means?

JG-LL: I—I don't know—all I remember is that we were talking about one that was like this and when it came—it couldn't because it came to a point or something like that.

[Here, JG sketched a graph.]

I: Okay, it came to a point.

JG-LL: Yeah, that's what I put on my—all those questions—cause it came to a point. I don't know what that would mean.

I: Okay. Why do you think that makes a function not differentiable?

JG-LL: I don't know. I really don't know.

**JM-LL.** JM-LL initially suggested that a vertical line was a function that did not have a derivative. When questioned about this, he corrected his view to a vertical line was not a function. JM did indicate the slope of a vertical line was undefined. When asked for another example, he sketched a circle and realized it was not a function. He then suggested a function with an asymptote would not have a derivative where the function was not defined.
I: Okay. Can you think of functions that don't have derivatives?

JM-LL: Umm—a vertical line.

I: Is that a function?

JM-LL: I guess not.

I: Why not?

JM-LL: Because if you use the straight line test, you go down.

I: So a vertical line's not a function. What would you think about the slope of a vertical line?

JM-LL: Well, it's undefined.

I: It's undefined slope. What about some other functions that might not have derivatives? Not everywhere, but say at a particular point?

JM-LL: Um, well if the graph goes—if a function would be—if it—if the slope approached infinity, wouldn't that—

I: Okay. Can you draw a picture of a function that looks like that—where it might not have a derivative? What might be a function like that?

JM-LL: Okay. That wouldn't be a function.

[JM drew a circle.]

I: What's not a function?

JM-LL: Well, if you had something that had an asymptote in the original graph, there wouldn't be a derivative at that point where the asymptote is.

I: For example, if I graphed a function that looked like that and had an asymptote?

[The interviewer sketched a graph of \( y = \frac{1}{x-1} \).]

JM-LL: Yeah. And then there'd be a derivative—there would be derivative for it every place except where that—

I: Every place except where it's not defined.

JM-LL: Yeah.
When JM was asked for another example, he responded with absolute value functions. He expanded by stating that the derivative did not exist at the vertex of an absolute value graph. After sketching \( y = |x| \), he argued that there were infinitely many tangents at the origin. Thus, the derivative was undefined at the origin. JM correctly drew the derivative graph.

I: Can you think of something that is defined at a point and doesn’t have a derivative there?

JM-LL: Um, on absolute value graphs when it comes to a point.

I: Okay. Absolute value graphs where they—

JM-LL: On that one point it’s not defined because it could be anything. Depending on how—

I: What do you mean it could be anything?

JM-LL: Well, if you had just a normal—like a v-graph.

[JM sketched \( y = |x| \).]

I: Mmm hmm. Absolute value of \( x \).

JM-LL: I mean if you’re drawing tangents it could like go all the way around. If you—I mean you can’t really tell what it would be like.

I: What does the derivative graph look like for absolute value of \( x \)?

JM-LL: It’d be going like this and that’s that one and (inaudible) and isn’t defined.

[JM sketched \( y = -1 \) for \( x < 0 \) and \( y = 1 \) for \( x > 0 \).]

I: Okay, so it’s not defined at \( x \) equals zero.

JM-LL: Yeah.

When questioned, JM suggested that all functions involving absolute value were non-differentiable at some point. He gave as an example the absolute value of the tangent function. His visual image of this function was of a graph coming to a point. He argued that magnification of the graph would leave the graph looking the same, coming to a point.
Thus, the function would not be differentiable at the point because it did not display local linearity.

I: Are all absolute value functions not differentiable someplace?

JM-LL: Um, all the ones I remember seeing don't—

I: Like which ones do you remember seeing?

JM-LL: I think the tangent—absolute value of a tangent of $x$—like we did that—it would come down—I can't really remember—I think it came down to a point and then it came back up. Instead of going down.

I: So you looked at—what'd you do? Look at a particular point?

JM-LL: If you zoom in at that point it stays the same. It's not—it doesn't display local linearity.

JM's image of a non-differentiable function was entirely visual. He made no reference to symbolic representations of the derivative. His initial view of non-differentiability included the idea of undefined slope. JM amended this view to include functions that had vertical asymptotes. JM included functions defined everywhere as being potentially non-differentiable with his example of the absolute value function. In addition, he included local linearity as a property of differentiable functions.

PS-LL. PS-LL stated that all functions that did not come to a point were differentiable. As an example of a non-differentiable function he used the absolute value function. PS stated the function was not differentiable at the origin. He reasoned that the slope was negative one on the left and positive one on the right and sketched the graph of the derivative. PS added a vertical connector at zero between the left and right portions of the derivative sketch even while claiming the derivative at the origin was undefined.
I: What other functions can you find the derivative of?

PS–LL: Um, you can find the derivative of any function besides the ones that come to a point.

I: So things that come to a point aren't differentiable. You said absolute value of x, so if we look at the graph of f of x equals the absolute value of x, where isn't that differentiable?

[The interviewer sketched the absolute value function.]

PS–LL: At x equals zero.

I: At x equals zero. Why not?

PS–LL: Because, well okay, from this side, this would be a negative slope so it would—there would be a straight line down here and this would be positive—be up here and then you have have this whole—the connector run between them.

[PS sketched y = −1 for x < 0 and y = 1 for x > 0 with a vertical line at x = 0.]

I: So you have the negative slope for the derivative graph on this side and you have a positive constant on the right side. And then at zero?

PS–LL: There's nothing there.

PS used the notion of local linearity to describe non-differentiable functions. His argument for the non-differentiability of the absolute value function involved looking at both right- and left-hand slopes. His use of the vertical line to connect the two portions of the derivative graph suggests a concern for the continuity of graphs.

CF–ST: CF–ST also used the absolute value function as an example of a non-differentiable function. He was unwilling to say that all functions involving absolute value were non-differentiable. CF characterized the non-differentiable aspect of absolute value functions as a sudden change in the slope of the graph.
I: What do you think it means that something's not differentiable?

CF-ST: Well, it just means that there are places on it where the slope is undeterminable.

I: Can you give me an example?

CF-ST: Umm, basically the absolute value of just about anything.

I: Just about anything.

CF-ST: Just about, yeah.

I: Are there absolute value functions which are differentiable everywhere?

CF-ST: I don't know. There might be, but I hardly know of any.

I: But you wouldn't generalize the absolute value function as always being not differentiable.

CF-ST: Right.

I: But if you were looking for something to not be differentiable, you might think of absolute value functions on that end.

CF-ST: Yeah.

I: What's the property of absolute value functions that makes them not differentiable?

CF-ST: Well, it's just they—their curve kind of breaks—it's not a curve anymore. It just kind of shifts direction almost—well, instantaneously in a sense.

RM-ST. RM-ST argued the absolute value function was not differentiable because multiple tangents could be drawn at the origin. Her discussion of substituting an x-value suggested she held a symbolic representation of the derivative. Furthermore, she implied the derivative must be a function because substitution of $x = 0$ in the symbolic derivative of the absolute value function would produce multiple y-values, one for each of the possible tangents.
I: Are all functions differentiable?

RM-ST: No. This one isn't.

I: The absolute value of x isn't differentiable. How do you know?

RM-ST: Good guess? I don't know. Because at that point—at x equals zero there's not one single tangent line you can draw to—you could draw any line to it and it would be tangent to it.

I: Tangent in the sense that it only touches at one point?

RM-ST: Yeah. And I guess that's illegal.

I: Why is it illegal?

RM-ST: Because if you can plug in more than one x value and get an answer, it's not a function? No.

DS-ST. DS-ST said that functions might not be differentiable at a particular point. He used the absolute value function as an example. DS used the sketch of the function provided by the interviewer to demonstrate the slope to the left of the origin was negative one and the slope to the right of the origin was positive one. The sudden shift in slopes at the origin indicated non-differentiability. His statement that the slopes went from minus one to zero to one suggested DS might think the derivative at the origin was zero. This was probably the case as DS contrasted zero slope with no slope, using the square root function at the origin as an example of a function with no slope.

I: Are there functions that aren't differentiable?

DS-ST: Well, at a point I don't think they can be differentiable. I mean certain—

I: Okay, what kinds of functions aren't differentiable at a point?

DS-ST: Like absolute value at—

I: Absolute value of x?

DS-ST: —at the bottom of or whatever—you can't—I don't think you can do that.
I: Okay, so here's a sketch. Why isn't that differentiable at $x$ equals zero?

[The interviewer sketched the absolute value function.]

DS–ST: Well, when you're—it's like when you're coming along and you know, when the slopes are all the same and then it's just—it just suddenly stops at zero. I mean you can't—there's no way to lead into it or lead out of it. It just stops. I mean—

I: So you were pointing to the left side and on the left side, the slope is—

DS–ST: It would be negative one or whatever.

I: And what's the slope on the right side?

DS–ST: It's one.

I: One.

DS–ST: And there's nothing in between there—it just goes from [minus?] one to zero to one. You can just—there's no way to—

I: Okay, so you said it went from one—you said the slope was zero here.

DS–ST: Well, well, maybe it's no slope. I mean, I don't know.

I: What's the difference between zero being a slope and no slope?

DS–ST: Well, oh, no slope is up and down like that. That would like if you tried to take the slope of—uh—radical $x$.

I: Radical $x$ at?

DS–ST: Zero.

SB–ST. SB–ST expressed confusion about the concepts of differentiable and non-differentiable functions. He was able to sketch a graph resembling $y = t \, x \, 1^{1/3}$. He indicated this graph was not differentiable at the origin because the slope to the left was negative and the slope to the right was positive.
I: What do you see as the difference between a function being differentiable and being not differentiable?

SB–ST: That was the most confusing part of the whole deal for me. I'm still confused on the differentiable part of the—

I: Do you have any clues?

SB–ST: I have—I remember some it when we were in here the last day is when, um, try and get the slopes to be the same—not in—I mean one couldn’t be plus and one couldn’t be minus—they both have to have the same—

I: So what kind of examples did you see of a non-differentiable function?

SB–ST: Um, the one that came down here and then went like this? Was that one?

[SB sketched a graph that resembled \( y = |x|^{1/3} \).

I: Why do you think that this function was not differentiable?

SB–ST: Because I know that the slopes on this—on this half over here are all going to be positive and these are going to negative—so it doesn’t matter where they—

In summary, students in the local linearity group, who were interviewed, had richer images of a non-differentiable function than students in the secant–tangent group, who were interviewed. Both groups appealed to absolute value type functions to indicate non-differentiability at a point. The local linearity students used local straightness, differing slopes, and multiple tangents to explain why a function was not differentiable. Their image included all types of graphs that contained vee–shapes. The secant–tangent students used differing slopes as a primary argument for non-differentiability.

The responses to the interview question about differentiable and non-differentiable functions are similar to data obtained on the posttest (Calculus Concepts Inventory I, Appendix B and Calculus Concepts Inventory II, Appendix D). In particular, students in the local linearity group were better able to provide graphic and symbolic representations of a function that is not differentiable at \( x = 1 \) (items 10 and 11 from the Calculus Concepts Inventory II, Appendix D). The examples of non-differentiable functions used by students on the posttest, as in the interviews, consisted exclusively of functions involving absolute
value. In justifying why a function was not differentiable at $x = 1$ on the posttest, students in the local linearity group gave more varied responses—including, local straightness and different left- and right-hand slopes—than students in the secant-tangent group.

Few students, who were interviewed, used the difference quotient or left and right derivatives to explain non-differentiability. No student used this argument in response to item 11 on the posttest. The lack of appeal of the formal definition of the derivative is consistent with student interview responses noted earlier. When students used the concept of tangent to look at a non-differentiable point on a function, they frequently spoke of multiple tangents at the point. The concept of tangent was not used at all on the posttest to explain non-differentiability.

PS-LL's graph of the derivative function for $y = |x|$—which took the form of a continuous graph, $y = -1$ for $x < 0$, $y = 1$ for $x > 0$, and a vertical line between $y = -1$ and $y = 1$ at $x = 0$—demonstrates the perception some students had on the posttest in sketching the derivative of a function when the derivative failed to exist at a point or points (items 6 and 8 from Function Graphs for Sketching Derivative, Appendix C). Even when the derivative of a function was undefined, these students drew the derivative graph as a continuous curve. In contrast, JM-LL was able to sketch a correct graph of the derivative function for $y = |x|$ as were other students on the posttest in similar problem situations.

JM-LL also noted that if a function had a vertical asymptote, then the derivative failed to exist. On a related item on the posttest item 8iv from the Calculus Concepts Inventory II, Appendix D) the graph contained a vertical asymptote at $x = 0$. For the sketch of the derivative, many students either connected the positive and negative portions of the graph through $x = 0$ to make a continuous graph or they carried the positive portion of the derivative into the first quadrant and started the negative portion of the derivative in the third quadrant. This difficulty that some students had with continuity is consistent with the observations of Vinner and Dreyfus (1989).
The Concept of Tangent

The third item on the interview protocol asked a student about the concept of tangent. The principal response was that a tangent touched a graph at one point. Students based their conception of the tangent on the Euclidean tangent to a circle. This image included the tangent not crossing the graph or touching the graph at another point. The interviewer presented students with a counterexample, the graph of \( y = x^3 \) and asked students to look at the tangent to the graph at the point (1, 1). The tangent at this point intersects the graph of the function. Some students were asked to look at the tangent to the graph of \( y = |x| \) where the tangent to the function is identical with the graph to the left and the right of the origin.

JM-LL. JM-LL began with a Euclidean definition of the tangent. When confronted with the first counterexample, he revised his definition to say that the tangent intersected the graph only once in an area at which the point of tangency was the center. JM recognized that the tangent to the absolute value function was identical to the function to the left and the right of the origin. He proposed a new definition of the tangent as the slope of a function at a particular point.

I: Well, what is a tangent?

JM-LL: It just touches the graph at one point.

I: Okay. Just touches the graph at one point. Here's a graph of \( y = x^3 \). Here's a graph of \( y = x^3 \). And I have the point one, one there. And there's my tangent line. Does that only touch the graph at one point?

JM-LL: No. Um, well, you could just say it's a secant that has no—I mean the points are in the same spot. So that it—so it could touch anywhere else. But the point is that it only touches the one point at the—where the tangent line's centered around I guess.

I: Touches at only one point where the tangent line's centered around.

JM-LL: So like right here it would be only touching that one point.

I: But it can touch elsewhere.
JM-LL: Yeah.

I: If we look at absolute value of $x$ and I pick a point on the function—

JM-LL: (Inaudible)

I: What were you thinking?

JM-LL: Oh well, when I was saying that I was thinking since it's a straight line it's gonna be exactly the same.

I: So the tangent line is the line.

JM-LL: The tangent line is the line.

I: In other words, if I draw a tangent to that point on absolute value of $x$—

JM-LL: Yeah. Right.

I: Then the tangent line is the line.

JM-LL: Yeah.

I: So it touches it at all places around that point—at least for a while—until it gets down to zero.

JM-LL: Mmm hmm.

I: Can you think of another definition of tangent?

JM-LL: Um—uh, it could be the slope of the function at that particular point. That would be the line.

I: Okay, so the tangent—

JM-LL: The line would have that slope.

I: Has the slope of the curve at a particular point.

JM-LL: Mmm hmm.
JG-LL. JG-LL defined a tangent as touching a curve in one place. When presented with a counterexample, she was unable to reconcile this with her prior experience.

I: What do you know about tangents?
JG-LL: Touches a line in one place.
I: Touches a line in one place.
JG-LL: Or a curve really.
I: A curve. There's a picture of $y = x^3$ cubed.
[Interviewer presents a graph of the function.]
JG-LL: Oh. That kind of a tangent.
I: Is this tangent?
[Interviewer draws a tangent at (1, 1).]
JG-LL: Is this thing a tangent?
I: Mmm hmm.
JG-LL: Mmm hmm. But it touches down here, too.

CA-LL. CA-LL included touching the graph at more than one point in his definition of the tangent. He amended this definition to say a tangent must intersect a graph at at least one point. Finally, he concluded a tangent was parallel to the point of intersection.

I: What's a tangent?
CA-LL: Um, it touches one point. Oh, yeah, there could be a tangent there.
[On the graph of $y = x^3$ at the origin.]
I: Where could it be?
CA-LL: Uh, right—right on the x-axis.
I: Does a tangent only touch a curve at one point?
CA-LL: Well, it can be tangent to more than one point, but—

I: There's a picture of \( y = x^3 \). And I've drawn the tangent at one, one. Looks to me like it intersects down here at negative two, eight.

CA-LL: Yeah. Well, it can touch it at more than one point, but on some curves it only touches at one point.

I: So what's a—is—what's a useful definition of tangent? Touches a curve at only one point?

CA-LL: Uh, it's a straight line drawn so that it's—so that it touches only one point.

I: But I drew a straight line that touches only—that touches two points and it's tangent there.

CA-LL: Well at least one point. That touches—that it touches—

I: Touches at least one?

CA-LL: Touches at least one point.

I: I can draw lots of curves that touch at least one point, can't I?

CA-LL: Yeah.

I: Because now I can draw one this way and that way—

[Interviewer presented an example in which a tangent intersects a graph at multiple points.]

CA-LL: That is parallel to the immediate slope at that point.

I: So a tangent is what?

CA-LL: A tangent is a line parallel to the immediate slope of the point it touches.

I: Parallel to the point it touches.

IM-LL. IM-LL began by claiming a tangent touches a graph at only one point. When presented with the example of a tangent to a point on one of the branches of the absolute value function, she created an image of a shrinking tangent. That is, because the tangent exactly matched the graph of the function, it contracted to a point. Her reference was a point of tangency between two circles. IM later revised this definition in the context of multiple intersection points for a tangent.
I: Maybe we need to clarify what a tangent is.

IM-LL: Oh a tangent is something that touches something at—just at one point.

I: Just at one point.

IM-LL: One point. Yeah.

I: Suppose I pick two points on—umm, I pick a point on this right part of the absolute value. Can you draw a tangent there?

IM-LL: The tangent would actually be—because of the slope, it would actually be just like—that. But it would be just a tiny little...

I: Like what?

IM-LL: —a tiny little line. It couldn't exactly be a line. It would actually be a point and it would be just touching that point. So a tangent doesn't necessarily have to be a line. It can be just a point.

I: It can just be a point?

IM-LL: Right. Cause you can have circles touching each other at one point and—you can have a—you know, at that point, they're tangent to each other. So, yeah. So it could be just a single point, or you could go ahead and say it's a whole line, if you really wanted to, but it would be tangent to that whole side, so...

I: How about if I take $y = x^3$ and I draw the tangent at one one. Does it touch the graph at only one point?

IM-LL: I don't know, but that doesn't mean it's not a tangent line. Umm, you can have cases where the line—like if you drew a line, it would run through the—it could run through the rest of the graph but that doesn't mean it's not tangent at that one point.

I: So it can run through the rest of the graph?

IM-LL: Right.

I: Can that happen on the absolute value function? If we picked a point? Can the tangent run through the rest of the graph?

IM-LL: Yeah, it can.

I: So does it have to be just one point?

IM-LL: No.
CF-ST: CF-ST was slightly startled to see a tangent identical with the graph of the absolute value function. This contradicted his geometric intuition. The same was true for a tangent that intersected a graph at multiple points. CF was, however, willing to revise his definition, stating that a tangent had the same slope as the point of tangency.

I: What does it mean to you to be tangent?
CF-ST: Well, it only intersects at one point, basically.
I: Is that true?
CF-ST: Yeah, I mean—
I: Does that have a tangent? If I draw a point of the left side...

[The interviewer referred to a point on the graph of $y = |x|$.]

CF-ST: Hmm.
I: —is there a tangent there?
CF-ST: Well, you got me. Burned the thing on that one.
I: I'm not trying to burn you. The question is, is there a tangent there?
CF-ST: Oh, yeah, there is, but—
I: What does it do?

CF-ST: Well, it's the same as the line itself.
I: It's the same as the line itself.
CF-ST: Right.
I: If I give you $y$ equals $x$ cubed and I draw the tangent line at one, one—what happens?

CF-ST: It intersects down here again.
I: Could you try and re-define what a tangent is?
CF-ST: Hmm.
I: Or is there a necessity to?
CF–ST: Well, a tangent's just basically a line with the slope of the same value as the slope at that point.

I: Okay, so it might be worthwhile trying to define the tangent in terms of the slope at the particular point.

CF–ST: Yeah.

DS–ST: DS–ST suggested a tangent could intersect a curve more than once. For him, tangency was a local property. He used the analogy of a ball rolling along a table.

I: What's a tangent?

DS–ST: It's a line that hits through the curve at only one point in that area. I mean, it may hit through the equation later on, but at that one—it only hits once in that area, so—

I: How would you define that area?

DS–ST: I don't—I don't know, to tell you the truth. It's kind of like a ball rolling on a table. It only hits at one point.

JS–ST: JS–ST referred to a tangent as touching a graph at only one point. When confronted with a counterexample, she added the condition that a tangent must be perpendicular to the graph. This notion makes reference to the tangent to a circle being perpendicular to the radius.

I: What do you think a tangent is?

JS–ST: Umm, I'm—I'm really not sure what a tangent is. That's like—isn't just a point on the graph where a line touches the graph?

I: Okay, so a line touches the graph. Can it touch more than once?

JS–ST: Not to be tangent. If it's tangent it can only touch once.

I: Here's your cubic.

[The interviewer displayed a graph of $y = x^3$.]
JS-ST: Uh huh.

I: And there's the point one one. If I draw a tangent there at one, one, does it hit the graph at more than one point?

JS-ST: Yeah, it crosses through.

I: It crosses through at another place...

JS-ST: Right.

I: —on the negative side.

JS-ST: Right.

I: So do you think a tangent can hit a curve at more than one point?

JS-ST: I guess it can.

I: So how might you think about a tangent?

JS-ST: Umm, I don't know—is it perpendicular or something?

RM-ST. RM-ST had made prior reference to the tangent in terms of it everyday language use. She defined the mathematical meaning of tangent with reference to the tangent to a circle. She was able to identify a tangent to $y = x^3$ at the origin. When presented with the tangent to this curve at the point (1, 1), she revised her definition so that tangency became a local property.

I: What does a tangent mean in math?

RM-ST: It's a line that touches at one point on a curve. It only touches the curve on one point. And just that one point and no other. It's not a secant or anything.

I: There's $y$ equals $x$ cubed.

RM-ST: Mmm hmm.

I: Where would the tangent be at $x$ equals zero?

RM-ST: It's undefined at $x$ equals zero. Or would it be—is—a line with slope of zero? It's either a line with a slope of zero or it's undefined at zero.

I: You choose.
RM-ST: I think it's a line with a slope of zero.
I: Is it vertical or horizontal.
RM-ST: Horizontal.
I: It's horizontal. Okay. Suppose I pick the point one, one on here.
RM-ST: Mmm hmm.
I: And draw a tangent to it. Does that look about like a tangent?
RM-ST: Yeah.
I: Does it intersect the curve at only one point.
RM-ST: No. Well, it only intersects the curve at one point right there where it's supposed to. That point over there—that's not supposed to be there.
I: Why not?
RM-ST: Well—it does intersect the curve at more than one point.
I: So what might be a better definition of a tangent?
RM-ST: Umm—a line that intersects a curve at only one point in the general area.
I: What's the general area?
RM-ST: This is a trick question.

In summary, the image of a tangent developed in the study of Euclidean geometry, a tangent to a circle, is very strong for students. The study of tangents to functions in developing the concept of derivative did little to revise this image, although, IM-LL's image of a shrinking tangent may have been influenced by the visual presentation of a secant approaching the tangent definition of the derivative. When students were confronted with counterexamples to their definition of a tangent as touching a curve at only one point, they were willing to amend their definition. Tangency was considered as a local property of graphs. Students began to link the tangent with the slope of a function at a particular point.
The image of a tangent in Euclidean terms, as the tangent to a graph at exactly one point, was also strongly evident on the posttest (item 14 from the Calculus Concepts Inventory II, Appendix D). The linkage that a few students from the local linearity group made between the concept of the slope of a function and the tangent on the posttest was not immediately present in the interviews.

**Generalizing to the Symmetric Difference Quotient**

The fourth part of the interview protocol provided students with a function, \( y = f(x) \), sketched in the first quadrant. The points \( x, x + h, \) and \( x - h \) were labeled on the \( x \)-axis. Each student was asked to indicate the \( x \) and \( y \) coordinates of the \( x \)-axis points on the function \( f(x) \). The student was asked to connect the points \( (x + h, f(x + h)) \) and \( (x - h, f(x - h)) \) with a secant. Then the student was asked what would happen to the secant as \( h \) became smaller. The purpose of this question was to explore student understanding of an alternative representation of the derivative, the symmetric difference quotient.

Analyses of two student interviews are presented in the following discussion, one from each of the two treatment groups. The two students, PS–LL and RM–ST, are representative of the interviews with all of the students.

**PS–ST.** After the problem was stated, PS immediately recognized that the secant line approached the tangent line at \( (x, f(x)) \). He stated the secant would have the same slope as the tangent line. When asked for another name for the slope of the tangent line, PS responded it was the derivative.

**I:** I have a graph here of a general function—\( y \) equals \( f \) of \( x \) and I’ve picked a point \( x \) and I’ve labeled its \( y \)-coordinate on the curve. So what would the coordinates of that point be?

**PS–LL:** \( x, f \) of \( x \).
I: Why don't you label that as $x, f$ of $x$. And now I'm going a distance $h$ on either side of $x$. And I've marked those two points and I'm going to indicate the secant line joining the two. Now what happens to that secant line as $h$ gets smaller? What would stay the same?

PS-LL: The slope.

I: The slope would stay the same? Can you show me how that's going to happen?

A. Um, it's going to make the slope of the tangent line at $x, f$ of $x$.

I: It's going to make the slope of the tangent line at $x, f$ of $x$. Do you know anything that's like that? Have we studies anything where we have the slope of the tangent line at a point?

PS-LL: Yeah.

I: What? Do you know another name for the slope of the tangent line at a point?

PS-LL: The derivative.

Now, the interviewer asked PS to write an expression for the slope of the secant line. He derived the symmetric difference quotient. PS compared the symmetric difference quotient with the standard difference quotient, $\frac{f(x+h)-f(x)}{h}$, by explaining how the standard difference quotient was derived using one fixed point.

I: The derivative. Okay, let's go back. Here's my secant line. Why don't you label these two points first? And there's the secant line. Can you write an expression for that secant line-- for the slope of the secant line? And what happens if you simplify the denominator?

PS-LL: It's two $h$.

[Here PS wrote $\frac{f(x+h)-f(x-h)}{2h}$.

I: It's two $h$. So why don't you write that? Now you said when $h$ gets small then the secant line stays parallel. They were all parallel and eventually it becomes the tangent and the slope of that tangent is the derivative. Does that quotient that we have there look like the derivative that we defined before?

PS-LL: Okay, we had $f$ of $x$ (inaudible).
[The interviewer's response indicates PS stated the secant–tangent difference quotient.]

I: \( F \) of \( x+\Delta x \) minus \( f \) of \( x \) over \( \Delta x \). It seems that this is the derivative too. Could you show in a particular case that this gives us a derivative?

PS–LL: The other form that we had had a stationary—had this as a stationary point and this one as like two points coming together to that same point.

PS conjectured that the symmetric difference quotient would yield the slope of the tangent line for the function \( f(x) = x^2 \), specifically \( 2x \). He set up the symbolic representation and algebraically confirmed his conjecture. PS thought the symmetric difference quotient could be used to obtain the derivative in all cases.

I: Mmm hmm. Do they seem to do the same thing? Could you take the function \( f \) of \( x \) equals squared and show me how that would give us the derivative? Or does it? Now we're using this one.

[Interviewer indicates the symmetric difference quotient.]

PS–LL: Okay.

I: What result would you expect it to give us if this gives the derivative of \( f \) of \( x \) equals \( x^2 \)? What should we get as a result when we simplify that?

PS–LL: The tangent line to \( f \) of \( x \).

I: What should we get as the derivative of \( x \) squared?

PS–LL: Two \( x \).

I: We should get two \( x \). So can you simplify that and see what happens?

PS–LL: Two \( x \).

[PS simplified \( \frac{\left(\frac{x+h}{2}\right)^2 - \left(\frac{x-h}{2}\right)^2}{2h} \) to \( 2x \).]

I: So it gave us two \( x \).

I: Do you think there are cases in which this difference quotient—it's called the symmetric difference quotient—does it work all the time?

After the initial presentation of the problem, RM was immediately able to see that the secant approached the tangent at \((x, f(x))\). She physically moved the secant line only after the interviewer's prompting.

I: I've drawn a graph \(y = f(x)\) and I've identified the point \(x\) so how would you label the coordinates of that center point?

RM-ST: \(x, f(x)\).

I: Okay. Now suppose I add \(h\) and I subtract \(h\) from that—then I get two new points on the curve. And I draw their secant line. Now what's going to happen as \(h\) gets smaller?

RM-ST: It's going to approach a tangent at \(x\).

I: How's it going to approach the tangent at \(x\)? Can you move the—

RM-ST: Well, okay. (Inaudible).

[RM used a plastic straightedge to show the movement of the secant.]

I: It's going to approach the tangent at \(x\). Okay, can you write an equation to describe that secant line—the slope of that secant line?

When asked to write an expression for the slope of the secant line, RM thought she was being asked to write the standard difference quotient form of the derivative,

\[
\frac{f(x + h) - f(x)}{h}
\]

Her prior response apparently convinced her that this was another form of the derivative. Yet, when RM was asked for clarification, she restricted use of this method to when \(h\) was not close to zero.
RM-ST: The derivative equation?

I: Well, to describe this secant line. Is that the derivative equation that we've been using?

RM-ST: Well, it could be the slope equation. Cause it's only the derivative equation as \( x \) gets closer to zero. So if we just say that \( x - h \) gets closer, so if we just say that \( h \) isn't close to zero, then we could use that formula.

I: But when we did the derivative difference quotient, we fixed one point like that and then we let \( h \) get small so the secant line came like this. Now I'm taking these two points on either side of \( x \).

RM proceeded to derive a correct form for the symmetric difference quotient. She explained the \( 2h \) in the denominator as the result of moving toward \( x \) from both sides in contrast to a one-sided derivative.

RM-ST: Would it be \( \frac{f(x+h) - f(x-h)}{2h} \)?

I: Okay. So why don't you write that down?

RM-ST: I'm gonna be over something. Okay there's the delta \( y \) and the delta \( x \). And delta \( x \) is—ahh—delta \( x \).

I: Is delta \( x \)—what happens when you subtract the \( x \) coordinates? What do you get?

RM-ST: \( x \). No. \( x \) plus \( h \) minus \( x \) minus \( h \). Two \( h \). So—

[Here RM wrote \( \frac{f(x+h) - f(x-h)}{2h} \).]

I: Two \( h \).

RM-ST: Because you're moving from both points.
RM agreed the symmetric difference quotient could be used as a definition of the derivative, although she suggested it was not widely used.

I: Now you said that sort of reminded you of the derivative.
RM–ST: Mmm hmm.
I: Because you said the secant line approached the tangent line and that's what you said the derivative was, right?
RM–ST: Mmm hmm.
I: Could this be a definition of the derivative?
RM–ST: Yeah. Not a widely used one, probably. But it probably could.

RM thought she had already shown that the symmetric difference quotient was a representation of the derivative when she was asked for a particular case. The interviewer provided \( f(x) = x^2 \) as an example. RM proposed using a calculator, using small values of \( h \) to derive the result arithmetically, as a method of proof. She finally agreed to derive the result symbolically. She concluded the result she obtained using the symmetric difference quotient was the derivative of \( f(x) = x^2 \).

I: Okay. Could you show me in a particular case that that's the derivative function?
RM–ST: Well, I just showed you with that formula right there.

[RM indicated the symmetric difference quotient.]
I: You've described it in words, that it's exactly the same thing. Could you show me for \( y \) equals squared or \( f \) of \( x \) equals \( x \) squared? That this gives us the derivative?
RM–ST: If I had a calculator.
I: Why do you need a calculator?
RM–ST: Well, I would make \( h \) a very small point like point zero zero zero zero zero zero zero one.
I: Let's just leave it $h$. Why would you do that?

RM-ST: So that it would be approaching the point at which $h$ equals zero, but not quite $h$ equals zero because otherwise you would have an undefined denominator.

I: Okay. Why don't you try it just leaving $h$ in?

RM-ST: Leaving it in.

I: Mmm hmm.

RM-ST: So $x$ squared—that would be $x$ plus $h$ squared—

$$\frac{(x+h)^2 - (x-h)^2}{2h}$$

[RM wrote and simplified this to $2x$.]

I: What does that reduce to?

RM-ST: Two $x$.

I: Which is?

RM-ST: Twice that. Well—

I: What were we trying to show?

RM-ST: That this formula is equal to the derivative.

I: Is it?

RM-ST: Well, yeah, uh huh. Cause the derivative of $x$ squared is two $x$.

Both RM-ST and PS-LL were quick to recognize from the visual representation of the symmetric difference quotient that the secant line in this representation becomes tangent to the graph through the limiting process. They conjectured the symmetric difference quotient was an alternative representation of the derivative. The visual presentation appeared to be convincing proof to both students as they had to be prompted to demonstrate the correctness of the symmetric difference quotient with a particular example. With $f(x) = x^2$ the students constructed a correct algebraic derivation.
Summary of Interview Analyses

The analyses of student interviews on the concepts of derivative, differentiability and non-differentiability, and tangent, and the ability of students to generalize the derivative to an alternative representation of the derivative—the symmetric difference quotient—have shown patterns of student understanding consistent with the analyses of the pretest and posttest data. On the posttest, the local linearity students demonstrated greater facility with the concepts of slope and rate of change, the concepts of derivative and differentiability, and symbolic differentiation than the secant-tangent students. This difference in performance and understanding, however, was not as evident in the interviews. The distinguishing feature between the local linearity and the secant-tangent students in the interviews was in the area of differentiability and non-differentiability. Here, students in the local linearity group used richer images—including local straightness, differing slopes, and multiple tangents—to explain why a function was not differentiable. The secant-tangent students used differing slopes as a primary argument for non-differentiability. Because the treatments provided the local linearity students with local straightness as the initial, defining characteristic of differentiability, it may be suggested that this characteristic contributed to their having richer images of the concept.

In both treatment groups, the visual representation of the derivative as a graph of a function was strongest. A few students were able to make connections between the visual representation and the symbolic difference quotient representation of the derivative. The limiting process used to obtain the formal definition of the derivative remained an enigma for students.

Part of student difficulty with understanding the limiting process of the secant approaching a tangent is due to their image of the tangent intersecting a curve at only one point. The identification of the derivative with the slope of the tangent at a particular point contradicts student images that two points are necessary to obtain the slope of a line. In
this case, the visual presentation of the symmetric difference quotient form of the derivative—in which two points equidistant from the desired point of tangency are used to obtain the limit—appeared to be more accessible and more convincing to students than the standard one-sided difference quotient representation of the derivative.
CHAPTER VI
SUMMARY AND CONCLUSIONS

This study was designed to examine and explore the effects on student learning of two different conceptual approaches to teaching the concepts of differentiability and the derivative in a technological environment. One conceptual approach was local linearity, under uniform magnification a differentiable function looks locally straight. The second approach was the traditional presentation of the derivative as a limiting process. Both conceptual approaches used computer software, *A Graphic Approach to the Calculus* (Tall, Van Blokland, & Kok, 1991), arranged around three generic organizers that were designed to provide conceptual links with prior student understanding and to construct new student knowledge. The generic organizers were: a magnification program for connecting the concept of differentiability with local straightness, a secant approaching a tangent program for visualizing the derivative as a limiting process, and a gradient curve program for connecting a pointwise understanding of the derivative with the global view of the derivative as a function. Results and conclusions are presented in three sections: a summary of findings related to the major hypothesis, the role of teaching and learning in a technological environment, and a discussion of the limitations and generalizability of the study. The concluding section presents directions for further research on the teaching and learning of the concepts of differentiability and the derivative.
Summary of Findings

The major hypothesis of this study, that students who were introduced to the concepts of differentiability and differentiation using the organizing principle of local linearity would achieve a different and more complete understanding of the concepts than students who were introduced to the concepts using the organizing principle of the secant approaching the tangent, was not confirmed. In fact, after starting from a lesser understanding, as exhibited on the pretest, of calculus concepts and concepts underlying the study of calculus, the students in the secant–tangent group achieved results similar to those of the students in the local linearity group.

Several factors may have contributed to the failure to confirm the major hypothesis: the lack of random assignment to the two treatment groups and the use of the computer software generic organizers. The lack of random assignment resulted in significant differences between the two groups on the pretest. The interaction between the treatment and the pretest is an indication that instruction enhanced the understanding of students in the secant–tangent group in areas in which they had initially weaker performance than the local linearity group. Thus, it was not the difference in treatment, but the instruction itself, on concepts such as slope, that enabled the secant–tangent students to achieve gains. The use of the three generic organizers—the magnification program, the secant approaching a tangent program, and the gradient curve program—by both groups had a strong effect on students. This is evidenced by the predominantly visual image of the derivative that students in both treatment groups held on the posttest and in the interviews.

Analysis of pretest data has shown that students in both treatment groups held concept images and concept definitions similar to those of other beginning calculus students prior to the instruction. When using the concept of slope to find rate of change over an interval, a portion of students in this study were unable to construct the form, "change in y divided by change in x." Orton (1983) saw similar difficulties. Furthermore, concept images for zero
slope and no slope were confused. The language of limits used on the Calculus Concept Inventories generated conflicting images of bounded and unbounded infinity as Williams (1991) and Monaghan (1992) found. No student in this study was able to construct the limiting argument for the definition of derivative as the secant approaching the tangent even when they could write an expression for the slope of an arbitrary secant line connecting two points on a curve. This confirms the findings of Tall (1986). In fact, a common explanation was that a secant line could not approach a tangent line because a tangent could not pass through two points on a curve. The recurring image of a tangent was geometric, the tangent to a circle, as seen by Vinner (1983) and Tall (1986).

The instruction in the two treatments was the same except for the initial presentation of the derivative concept. The local linearity instruction began by approaching the derivative through exploring the local straightness of graphs. The secant–tangent instruction began by approaching the derivative as the limit of the secant approaching the tangent. Then, in both groups, instruction visually and numerically related the visual image of slope—in the local linearity instruction—or tangent—in the secant–tangent instruction—to the difference quotient \( \frac{f(x+h)-f(x)}{h} \) by fixing \( x \) and letting \( h \) get small. Thirdly, the changing slope of a graph was viewed globally by varying \( x \) in \( \frac{f(x+h)-f(x)}{h} \) to obtain the derivative graph as a function. Finally, these three stages of instruction were linked to numeric, symbolic, and linguistic representations of the derivative.

After instruction, students in both treatment groups had developed a definition of the derivative concept that involved the concept of slope. The derivative was either a global property of a function, representing the slope for all points of a function, or it represented the slope of a function at a particular point. In addition, about one-fourth of the students in each treatment group linked the derivative with the slope of a tangent. The local linearity students were three times as likely as the secant–tangent students to state the derivative was a function. More than 35\% of the students stated the derivative was a graph or curve in
their definition of the derivative. Few students in either group thought of the derivative as simply a calculus formula or procedure.

Students from both groups became adept at sketching the graph of the derivative of a function from the graph of the function. Students in the secant–tangent group were clearly better at this task than the local linearity group. This might suggest the secant–tangent students were more visual learners than the local linearity students. In sketching the graph of the derivative from a graph of the function, students interpreted continuity of the function as continuity of the derivative function. That is, if a function were piecewise continuous such as \( y = \frac{1}{x} \), then most students were able to sketch the derivative as piecewise continuous. If the graph was continuous such as \( y = \text{tx} \), then most students sketched the derivative as a continuous graph, including the segments \( y = -1 \) for \( x < 0 \), \( x = 0 \) for \( x = 0 \), and \( y = 1 \) for \( x > 0 \).

Analyses of the interviews with students confirm the previous observations. Students interviewed from both treatment groups had a strong visual image of the derivative as the graph of the slopes of a function. These students perceived the derivative as a global object, applying to the entire function. They made reference to the slope at a point being identical with the slope of a tangent at that point on the function. While some students used the limiting process of the secant approaching the tangent in their arguments, none were able to explain this process formally. In fact, finding the derivative of a function in symbolic form appeared to be a separate activity from finding the derivative of a function given in a graphical representation. At least one student suggested symbolic differentiation avoided a division by zero error which would occur as \( h \) approached zero in the difference quotient definition of the derivative. The difference quotient representation of the derivative suggested to the students that the secant line never really reached the tangent line as two points were required to find slope.
Students, who were interviewed in the local linearity group, had a richer and more varied image of the concepts of differentiability and non-differentiability than students in the secant-tangent group. Students in the local linearity group used a wider variety of examples to illustrate non-differentiability and made reference to local straightness as a property of differentiable functions. They used left- and right-hand derivatives and slopes in their arguments. Students in the secant-tangent group used only absolute value functions to illustrate non-differentiability. The key for them in determining non-differentiability was a sudden change in slope. The richer and more varied image of the concepts of differentiability and non-differentiability held by students in the local linearity group may be attributed to the initial phase of instruction. That is, the use of local linearity as a defining characteristic of the derivative provided a student with an additional geometric, visual, image that enhanced the multiple linked representations of the derivative.

In the interviews with students, continuity of the derivative was an issue when graphing the derivative of a function that had a point where the derivative was not defined. Students either wanted to connect the two parts of the derivative graph separated by the point of non-differentiability or to define the derivative as zero at the non-differentiable point.

Students, who were interviewed, were identical in their understanding of the concept of tangent with students' definitions of tangent on the posttest. For them, a tangent was defined in Euclidean geometric terms—as with a circle—as a line intersecting a graph at exactly one point. When presented with counterexamples of a tangent that intersected a graph in multiple points, students redefined the tangent as a line that had the same slope at the point of tangency. The transition from the idea of tangency developed in the study of geometry to the use of tangents in developing the concept of derivative remained incomplete.
In general, students in the local linearity group completed this study with a better knowledge of slope and rate of change, derivative concepts, and symbolic differentiation than the secant–tangent group. The secant–tangent group excelled in sketching the graphs of derivatives of functions. The visual aspects of the derivative as a graph or function were predominant. The differences between groups may be attributed to the use of local straightness in the initial phase of the local linearity instruction. While complete connections between the graphical representation of the derivative and the symbolic representation of the limit of the difference quotient were not made, it is clear that both treatment groups benefited from the use of the software generic organizers—the magnification, the secant approaching a tangent, and the gradient sketching programs—in their study of the concepts of differentiability and the derivative. The use of the generic organizers appears to have facilitated and enhanced the multiple linked representations—numeric, geometric, symbolic, and linguistic—of the concepts of differentiability and the derivative.

The Roles of Teaching and Learning in a Technological Environment

Analyses of the transcripts provided by the SCAN taxonomy demonstrated that instruction was consistent between the two treatment groups. While the classroom was teacher–centered, in that the teacher was the primary role player, instruction did not follow the traditional static pattern found in many mathematics classrooms (Romberg & Carpenter, 1986). The teacher functioned primarily as explainer or task setter and not as a manager. Emphasis was placed on students' synthesizing procedures and concepts; students spent much less time on pure recall of facts. Students were almost equally engaged in symbolizing and investigating activities. They were actively doing mathematical tasks more than one–half of class time. These results are similar to those found in other classrooms in which technology was used actively (Farrell, 1989). While the computer software was
used frequently as a tool to explore the derivative concept, the computer was not a major role player in setting tasks for students. This finding contrasts with those of Fraser and colleagues (1988).

Teaching calculus using the computer created some uncertainty for the teacher. This was the first time he had taught using the local linearity approach to the derivative. He indicated lessons took more preparation than they had previously because this was his initial experience using technology in the classroom and because of the concept of local linearity. Student difficulties with both the content and the technology, questions students would ask, and insights gained were not easily anticipated on the basis of experience.

Management techniques for working in small groups with computers had to be developed. Apart from these concerns, the teacher expressed the positive desire for software that could be used in the general mathematics classes he taught. He felt all students would benefit from using technology to explore mathematics.

In general, students had a positive response to working with the computer software in a small–group setting. Students felt the opportunities to explore the derivative concept using the computer software and to interact and communicate with one another about their thoughts and insights in small–group activities were beneficial. Students thought the visual presentation of the derivative concept assisted them in learning. There was some perception, however, that the computer was doing all of the mathematics.

These observations suggest appropriate technology can be introduced in the secondary school calculus classroom with relative ease. The use of technology changed the nature of teaching calculus. The teacher became less of a manager and more of a task setter. The use of technology facilitated group learning and encouraged students to explore concepts and to communicate with one another. Students reacted positively to working in a group setting where the responsibility for conjecturing and communicating was with them rather than the
teacher. The dynamic, visual aspect of the presentation was appreciated by students who considered themselves visual learners.

The use of technology was not without difficulties. With the use of computers, the teacher had to change the way in which he conducted class. The visual representation of the derivative concept required changes in the presentation of the content. Providing the teacher with additional training on structuring classroom instruction, discourse, and content in a computer environment might have eased some of the difficulties.

Generalizability and Limitations of Study

Every effort was made to insure that the results of this study would be generalizable to other secondary school calculus classrooms. The generalizability of the results of this study to the teaching of the differentiability and derivative concepts in college calculus teaching environments is limited because of the setting in a secondary school classroom. Secondly, the inability to assign students randomly to the treatments that resulted in substantial group differences on the pretest limits generalizing conclusions. Because the teacher did not have prior experience in using technology in the calculus classroom, different results might have obtained with a more technologically-experienced and technologically-trained secondary school calculus teacher. Furthermore, this was the first time the teacher had used local linearity to teach the concept of derivative. While he had been given a teacher's guide accompanying the software that emphasized the local linearity approach to the derivative (Tall, Van Blokland, & Kok, 1989) and a calculus text that emphasized a graphical approach to the derivative (Dick & Patton, 1991), additional training might have improved his ability to teach the local linearity concept of derivative. The teacher's previous experience teaching the derivative involved using the limiting process of the secant approaching the tangent. It might be expected that with more
experience and training in using the concept of local linearity the teacher would have obtained different results.

**Directions for Further Research**

The students in this study developed a visual understanding of the derivative as a graphical object. Solid connections were not made with the derivative as a symbolic object. The transition from graphical representation to symbolic representation remains a point of concern. Students in the local linearity group did not develop the relationship between local straightness and the slope of the tangent either in graphical or symbolic form. Students in the secant–tangent group viewed the derivative as generalized slope, but did not relate this to the limiting process of a secant approaching a tangent.

Two obstacles are evident in the failure to make connections between the derivative as a graphic object and the derivative as a symbolic object. The first obstacle is student understanding of the concept of tangent in geometric terms as a tangent to a circle. The second obstacle is student understanding of the concept of slope. Students in this study thought two points were necessary to determine the slope of a line, carrying an image developed during their study of algebra. When the concepts of slope and tangent were linked in the limiting process of defining the derivative at a point, neither treatment group was able to see how the secant became the tangent. Furthermore, the students in the local linearity group did not see how the magnification process identified the slope of a function with the slope of the tangent.

These observations suggest several research questions for curricular development prior to and for the teaching of calculus. First, how should the concept of tangent be presented so that a student develops appropriate images for the use of the tangent in calculus? While local straightness appears to be one possible answer to this question, the results of this study did not demonstrate that the requisite connections were made. Second, how should
the concept of slope be presented so that the concept can be generalized from the slope of a line segment between two points to the slope of an arbitrary curve at a particular point? Evidence from this study suggests students were able to visualize the generalized slope of a function through the use of the generic organizer Gradient program. They were not able to mathematically justify how this generalized slope was obtained. Third, how can the concept of limit be presented so that a student successfully links the concepts of slope and tangency in defining the derivative? For students in this study, the concept of limit carried multiple meanings as an approximation and a bound.

Each of the three concepts—slope, tangent, and limit—involves the use of words that have meaning in both everyday and mathematical contexts. Further investigation is required into how students perceive these words in their everyday use and how students then make the transition to developing appropriate mathematical language.

The significant gains made by the secant-tangent group in comparison to the local linearity group suggest that the secant-tangent students may have been more visual learners than the local linearity students. Research that establishes a student's learning style as either visual or symbolic before a graphical approach to calculus concepts is used would be valuable.

Finally, while this study has demonstrated that the use of technology assisted in facilitating student learning of calculus concepts and contributed to active student participation and communication of mathematics, teaching in a technological environment remains a relatively new endeavor. In this study, teaching calculus with computers was not without its difficulties. Research is needed to investigate more effective ways of using technology to present mathematical concepts in a classroom environment, how to organize classroom instruction to use technology effectively, and the types of conceptual and pedagogical questions that arise in a technological setting. Simultaneous with these investigations and on the basis of them is the need to explore ways to enhance the training
of mathematics teachers in the use of technology and in new approaches to mathematical concepts.
Calculus Concepts Inventory

The following questions are designed to study how you understand mathematical concepts that form a basis for the calculus.

Please answer each question to the best of your ability. Show the mathematical work you do to arrive at an answer, and supply explanations of your thinking when they are asked for.

This is not a test. Your responses will not be used to determine your grade in this course.

Name: ___________________________
1. Find the average rate of change between the following points on the graph:

Note: the "average rate of change" from $P$ to $Q$ means the slope of line segment $PQ$.

- i. from $C$ to $D$
- ii. from $D$ to $E$
- iii. from $A$ to $B$
- iv. from $B$ to $C$
- v. from $C$ to $E$
- vi. from $D$ to $C$
2. On the graph \( y = x^2 \), the point A is (1, 1), the point B is \((k, k^2)\) and T is a point on the line tangent to the graph at A.

A. Find the slope of the straight line through the points A and B.

B. Find the slope of the tangent line AT.

C. Explain how you might find the slope of the tangent line AT using basic principles.
3. The diagram represents the graph of the function \( y = \sqrt{x} \) (taking the positive square root for \( x \geq 0 \)). A is the point (0, 0) and B is the point \((h, \sqrt{h})\).

Circle the letter of your response for each of the following statements:

i. The graph has a tangent at point A.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

If your response is C or D, explain why in the following space, then omit (ii) and (iii).

ii. The tangent at A is vertical.
    A. I am absolutely certain the statement is true.
    B. I think the statement is true.
    C. I think the statement is false.
    D. I am absolutely certain the statement is false.

iii. The slope of the tangent at A is infinite.
    A. I am absolutely certain the statement is true.
    B. I think the statement is true.
    C. I think the statement is false.
    D. I am absolutely certain the statement is false.
The diagram, exactly the same as the one on the previous page, represents the graph of the function $y = \sqrt{x}$ (taking the positive square root for $x \geq 0$). A is the point $(0, 0)$ and B is the point $(h, \sqrt{h})$.

Everyone should answer questions (iv), (v), and (vi). Circle your response.

v. As $B \to A$, the slope of the line $AB$ tends to infinity.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

v. As $B \to A$, the slope of the line $AB$ has infinity as its limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

vi. As $B \to A$, the slope of the line $AB$ increases without limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.
4. Suppose line $L$ is tangent to the curve $y = f(x)$ at the point $(5, 3)$ as indicated in the following graph.

What is the slope of $f(x)$ at $x = 5$?

Explain how you obtained the value of the slope of $f(x)$ at $x = 5$. 

5. The graph of a function \( y = f(x) \) is shown in the graph below.

A new function \( g(x) \) is defined by: for each \( x \), \( g(x) \) is the slope of \( f(x) \) at the point \( x \).

Sketch a graph of \( g(x) \) on the axes below.
6. Find a polynomial function with rational coefficients that would have approximately the same graph as the function

\[ f(x) = \frac{(x + .0001)^8 - x^8}{.0001}. \]

Explain how you obtained your function.

7. If you have studied the calculus before, find the derivatives of each of the following:

i. \( x^5 + 4x^3 \)

ii. \( \sqrt{x} \)

iii. \( \frac{1}{x^3} \)

iv. \( \cos 2x \)

v. \( x \sin x \)

vi. \( \tan x \)
APPENDIX B

CALCULUS CONCEPTS INVENTORY I
Calculus Concepts Inventory

The following questions are designed to study how you understand mathematical concepts that form a basis for the calculus.

Please answer each question to the best of your ability. Show the mathematical work you do to arrive at an answer, and supply explanations of your thinking when they are asked for.

This is not a test. Your responses will not be used to determine your grade in this course.

Name: _______________________
1. Find the average rate of change between the following points on the graph:

Note: the "average rate of change" from P to Q means the slope of line segment PQ.

i. from C to D

ii. from D to E

iii. from A to B

iv. from B to C

v. from C to E

vi. from D to C
2. On the graph $y = x^2$, the point A is (1, 1), the point B is (k, $k^2$) and T is a point on the line tangent to the graph at A.

A. Find the slope of the straight line through the points A and B.

B. Find the slope of the tangent line AT.

C. Explain how you might find the slope of the tangent line AT using basic principles.
3. The diagram represents the graph of the function \( y = \sqrt{x} \) (taking the positive square root for \( x \geq 0 \)). A is the point (0, 0) and B is the point \((h, \sqrt{h})\).

Circle the letter of your response for each of the following statements:

i. The graph has a tangent at point A.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

If your response is C or D, explain why in the following space, then omit (ii) and (iii).

ii. The tangent at A is vertical.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

iii. The slope of the tangent at A is infinite.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.
The diagram, exactly the same as the one on the previous page, represents the graph of the function $y = \sqrt{x}$ (taking the positive square root for $x \geq 0$). A is the point $(0, 0)$ and B is the point $(h, \sqrt{h})$.

Everyone should answer questions (iv), (v), and (vi). Circle your response.

v. As $B \to A$, the slope of the line AB tends to infinity.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

v. As $B \to A$, the slope of the line AB has infinity as its limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.

vi. As $B \to A$, the slope of the line AB increases without limit.
   A. I am absolutely certain the statement is true.
   B. I think the statement is true.
   C. I think the statement is false.
   D. I am absolutely certain the statement is false.
4. Suppose line $L$ is tangent to the curve $y = f(x)$ at the point $(5, 3)$ as indicated in the following graph.

What is the slope of $f(x)$ at $x = 5$?

Explain how you obtained the value of the slope of $f(x)$ at $x = 5$. 
5. The graph of a function $y = f(x)$ is shown in the graph below.

A new function $g(x)$ is defined by: for each $x$, $g(x)$ is the slope of $f(x)$ at the point $x$.

Sketch a graph of $g(x)$ on the axes below.
6. Find a polynomial function with rational coefficients that would have approximately the same graph as the function

\[ f(x) = \frac{(x + .0001)^8 - x^8}{.0001}. \]

Explain how you obtained your function.

7. If you have studied the calculus before, find the derivatives of each of the following:

i. \( x^5 + 4x^3 \)  
iv. \( \cos 2x \)

ii. \( \sqrt{x} \)  
v. \( x \sin x \)

iii. \( \frac{1}{x^3} \)  
vi. \( \tan x \)
APPENDIX C

FUNCTION GRAPHS FOR SKETCHING DERIVATIVES
Functions and Their Derivatives

You will be shown the graphs of 10 functions for 2 minutes each. Determine whether or not the function is differentiable at all points where it is defined. If the function is differentiable, then sketch the graph of its derivative. If the function is not differentiable, then write a brief explanation of why it is not.

NAME ____________________________
1.

2.

3.

4.

\[ f(x) = |\sin x| \]
5. $f(x) = (\tan x)^3$

6. 

7. 

8. $f(x) = |\tan x|$
9. \( f(x) = (\sin x)^2 \)

10. \( f(x) = |x^3| \)
APPENDIX D

CALCULUS CONCEPTS INVENTORY II
Calculus Concepts Inventory
II

The following questions are designed to study how you understand mathematical concepts that form a basis for the calculus.

Please answer each question to the best of your ability. Show the mathematical work you do to arrive at an answer, and supply explanations of your thinking when they are asked for.

This is not a test. Your responses will not be used to determine your grade in this course.

Name: ____________________________
8. Sketch the graphs of the derivatives of each of the following functions on the axes provided.

i.    

ii.   

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9. The graph of the derivative of a function, $f'(x)$, is shown on the graph below.

Sketch a graph of $f(x)$ on the axes provided.
10. Sketch the graph of a function which is defined at $x = 1$, but is *not differentiable* at $x = 1$. Explain why your function is not differentiable at this point.

11. Write a symbolic expression for a function which is defined at $x = 1$, but is *not differentiable* at $x = 1$. Explain why your function is not differentiable at this point.

12. A bullet shot from a 9-millimeter pistol has a muzzle velocity of $1250 \frac{ft}{sec}$. The distance the bullet travels in 5 second intervals is given in the table below. What is the
approximate velocity of the bullet when the time is 20 seconds?

<table>
<thead>
<tr>
<th>Time in seconds</th>
<th>Distance traveled in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5850</td>
</tr>
<tr>
<td>10</td>
<td>10900</td>
</tr>
<tr>
<td>15</td>
<td>15150</td>
</tr>
<tr>
<td>20</td>
<td>18600</td>
</tr>
<tr>
<td>25</td>
<td>21250</td>
</tr>
<tr>
<td>30</td>
<td>23100</td>
</tr>
<tr>
<td>35</td>
<td>24150</td>
</tr>
<tr>
<td>40</td>
<td>24400</td>
</tr>
</tbody>
</table>

13. You have been asked by a friend who understands the notion of the slope of a straight line to explain what is meant by the slope of a more general function. Write a brief explanation of what you would tell your friend.
14. Write an explanation of what is meant by a tangent to a graph.

15. Explain what is meant by the derivative of a function.
APPENDIX E

INTERVIEW PROTOCOL
1. What is the derivative?

2. What is a differentiable function?
   a. Give examples of differentiable functions.
   b. Give examples of non-differentiable functions.
   c. Is \( f(x) = \text{constant} \) differentiable?
   d. Is \( y = \begin{cases} x^2 & x \leq 0 \\ x^3 & x > 0 \end{cases} \) differentiable at \( x = 0 \)?
   e. Is \( y = \begin{cases} x^2 & x \leq 1 \\ x^3 & x > 1 \end{cases} \) differentiable at \( x = 1 \)?

3. What is a tangent?

4. Have student investigate the symmetric difference quotient,
   \[ y = \frac{f(x + h) - f(x - h)}{2h} \]
   a. First visually by looking at the secant line as \( h \) goes to zero.
   b. Does this remind you of anything?
   c. Demonstrate validity of symmetric difference quotient for \( y = x^2 \).
LIST OF REFERENCES


