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System identification for $H_\infty$ robust control design

DeVilbiss, Stewart Lee, Ph.D.
The Ohio State University, 1994
SYSTEM IDENTIFICATION FOR \( H_\infty \) ROBUST CONTROL DESIGN

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Stewart L. DeVilbiss, B.S., M.S.

* * * * *

The Ohio State University

1994

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I dedicate this dissertation to the lady that I adore,
the love of my life, my wife Beth.
ACKNOWLEDGEMENTS

I first and foremost thank the United States Air Force for funding this research effort through ESA #F33600-88-A-0271. I also thank the Air Force Institute of Technology, Department of Electrical and Computer Engineering, for selecting me as a future faculty member.

I thank my advisor, Steve Yurkovich, for his advice, labor, and friendship during this research effort. I thank Hitay Özbay and Lee Potter for their technical contributions to my research effort, as well as for promoting an environment wherein graduate students are treated, to an appropriate extent, as peers.

I thank many of the control area graduate students, both past and present, for their influence in my research as well as for their friendship and companionship. I thank Ken Cheung for providing the technical inspiration for my dissertation and Jim Gassman for the outstanding work documented in his Master's thesis. I thank John Watkins for his friendship, his willingness to serve as a sounding board for my ideas, and his technical advice. I am honored to have known Onur Toker, and I am very grateful for his consultation, especially in the area of robust control design. My work was also significantly influenced by helpful discussions with Layne Lenning, Thad Peery, and Wu-Chung Su. I thank Keith Redmill for keeping our computer network healthy and up to date.
I have very fond memories of many good times spent with my colleagues in the control area, especially Layne Lenning, Eric Laukonen, LaMoyne Porter, John Watkins, Lew Fulcher, Dan Clancy, Keith Redmill, Kevin Burgess, Tony Bailey, Wu-Chung Su, Julie Hurtig, Scott Brown, Vivek Moudgal, and Mike Zimmerman.

I thank my wife, Beth, who never seemed overly burdened by my venture through the PhD program. I thank her family for the friendship and perspective that they continually provide me with. I thank my parents and God for providing a secure and loving environment within which to grow and for instilling in me the perseverance required to complete a dissertation.
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CHAPTER I

Introduction

A mathematical model never completely captures the behavior of a physical system; uncertainty is always present, at least on a microscopic scale, as expressed in the Heisenberg Uncertainty Principle. The pragmatic choice of a model is one which has sufficient accuracy to meet the needs of the model's end use. For example, a lumped parameter model of a section of wire (conductor) may be suitable when only low-pass excitation signals are anticipated; whereas, a distributed parameter model becomes necessary to accurately predict the effects of higher frequency excitations.

For end use purposes, the modeling class is often restricted, and behavioral anomalies are treated as disturbances. This can be an issue of philosophical debate. If a disturbance is truly dependent upon normal plant operation, shouldn't it be modeled as such? The answer is, not necessarily. The additional complexity required may add little to model fidelity at the expense of a loss of desirable model properties. Consider, as an example, the effect of finite register lengths in floating point digital computations. Computation error due to truncation or rounding in forming the least significant bit of the resultant is a nonlinear phenomena, yet when modeled as a uniformly distributed random variable, a linear model with an additive random disturbance results. This correctly models the computation for some realization of
the random variable. Model linearity is a desirable property to preserve, allowing the application of techniques from the mature field of linear system theory.

System identification is an experimental procedure performed upon a plant or process with the purpose of generating a mathematical model which adequately describes the process excitation/response behavior. Adequacy of the model is judged in the context of its intended use, for example in control design. If a parametric class of system models has been assumed, the identification procedure reduces to the problem of estimating the model parameters which best fit (in some measure) the observed experimental data within the assumed model structure. We shall restrict our model to be in the class of linear systems and conduct an identification experiment to parameterize our modeling structural choice.

It is imperative that an identification experiment be conducted with the plant's desired operating condition in mind. If the actual system possesses significant nonlinearities, one should excite the system about the desired set point, provided the goal is a local linear model; it would be unwise to use a model identified outside the expected range of operation to serve as the basis for control design. If the effect of environmental changes, for example bounded temperature variations, are unmodeled and unmeasured, their influence can only be incorporated into the plant description as a disturbance whose current influence is unknown, but bounded in magnitude.

Uncertainty is always present in an identification procedure. Signal measurements are never available with infinite precision. We can collect only a finite amount of measured data (observations) from which to infer a plant model; this further limits the
resolution to which modeling parameters may be identified. Classical identification techniques focus on delivering a plant description which is optimal, in some measure, in the face of this uncertainty. Traditional control design techniques then deliver a compensator developed based upon the optimal model. Control design techniques for systems describable by a linear time-invariant (LTI) difference equation are quite mature, yet such a model is never complete in predicting behavior of a physical process. A quantification of how reliable or accurate a nominal model is in predicting the true plant excitation/response behavior is appropriate and perhaps necessary to enable a reliable control design, meeting desired closed loop performance objectives. Behavioral anomalies between a plant and its nominal model have been termed model mismatch, captured either as structured (parametric) or unstructured dynamic uncertainty, or both. A control strategy is termed robust if closed loop performance objectives are met under the prescribed control action for all plant behaviors characterized by a nominal model subject to finite perturbations due to model mismatch.

The emergence of $H_\infty$ robust control design as a major topic of research has driven the need for a new class of robust control-oriented identification algorithms. As previously discussed, the classical identification focus is on generating an "optimal" plant estimate. Existing $H_\infty$ robust control design methodologies assume the availability of a nominal model description of the plant accompanied by a bounded quantification of the possible behavioral mismatch of the model relative to the true plant. We assume that a nominal plant model is not known a priori and that a weighting on model mismatch is also unavailable. We do assume to have collected a finite duration data
record from an identification experiment and to have prior knowledge of the underlying model class describing the true plant. We assume to have some prior knowledge of the nature of the disturbance impacting the true system. In the presence of uncertainty, we shall infer both a nominal plant model and also a frequency domain weighting function, overbounding the magnitude response of the true plant relative to the identified nominal model.

The type of a priori plant knowledge available essentially determines the path that the identification strategy should follow. When the plant disturbance is a stochastic process, information theory can be used to derive a parameter estimator; the resultant parameter estimates are often Gaussian (a consequence of the central limit theorem). If the probability density function of the parameter estimates has an infinite or semi-infinite domain, then a nominal parameter estimate can only be bounded by confidence intervals rather than "hard bounds" certain to be satisfied. If the disturbance is treated merely as unknown, but bounded (UBB), then we can employ set theory to identify the set of model parameters consistent with the assumed model structure and the observed data. If this set of so-called feasible parameters is compact, then we have known (or can infer) parameter bounds.

Our objective of identification for robust control design must be pursued with whatever plant knowledge is available. We can influence this process by our choice of identification experiment free parameters, including such variables as the nominal model class (and any order assumptions), the plant excitation sequence (and experiment duration), and the uniform sampling interval. While pursuing this objective
of identification for $H_\infty$ robust control design, we seek to minimize the conservatism introduced in the uncertainty weighting function description; although, we will introduce conservatism, when needed, to keep computations tractable. Existing $H_\infty$ robust control design techniques inherently introduce conservatism in their uncertainty characterizations to allow general solutions to various robust control problems to be developed; we will accept this conservatism in order to avoid the artistry and intuition needed in a robust control design for a specific plant.

In the second chapter of this dissertation, we summarize the optimal volume ellipsoid (OVE) algorithm [1], a recursive parameter set estimation algorithm for systems describable by an autoregressive, exogenous input (ARX) model structure wherein the additive plant disturbance is UBB. We investigate asymptotic properties of the OVE algorithm and address the optimal choice of plant excitation to be applied in the identification experiment. The third chapter addresses the exploitation of an ellipsoidal bound, on the set of feasible ARX model parametric descriptions, in existing $H_\infty$ robust control design techniques; the focus is on translating the structured uncertainty of the ellipsoid into a perturbation bounding weighting function. The fourth chapter presents several weighting function identification examples, and the final chapter is a summary of the contributions of this dissertation and a discussion of future research directions.
CHAPTER II

Parameter Set Estimation with the OVE Algorithm

2.1 Parameter Set Estimation

2.1.1 Background

The primary difference between traditional system identification schemes and parameter set estimation (PSE) techniques lies in the objective. Traditional techniques (such as least squares approaches) attempt to identify a single point in the model set (we shall refer to these as single point methods) representing the model parameters; whereas, PSE techniques attempt to characterize the uncertainty inherent in the experiment by identifying a set of models (those parameters which are consistent with a priori model assumptions and the observed data record) as the feasible parameter set [1, 2]. Parameter set estimation strategies are thus a means of quantifying the uncertainty of a plant model within an assumed model structure. The field of set membership identification was pioneered by [3] for state estimation; an adaptation to system parameter estimation appears in [2, 4].

Many identification algorithms [2, 5, 6, 7] use a causal time-invariant model, linear
in the parameters, as the assumed model structure:

\[ y_k = \phi_k^\top \theta + v_k. \]  

(2.1)

We shall specifically assume a single-input, single-output (SISO) ARX (also known as equation error) [5] model set,

\[ \theta = [a_1 \cdots a_n \ b_0 \cdots b_m]^\top \]  

(2.2)

\[ \phi_k = [-y_{k-1} \cdots - y_{k-n} \ u_{k-n} \cdots u_{k-m-n}]^\top, \]  

(2.3)

where \( \theta \) is the parameter vector, \( \phi_k \) is the regression vector, \( \{y_k\} \) is the measured output sequence, and \( \{u_k\} \) is the known input sequence. The ARX model corresponds to the LTI difference equation

\[ y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} = b_0 u_{k-n} + \cdots + b_m u_{k-m-n} + v_k \]  

(2.4)

with \( n \) assumed process poles, \( m \) assumed process zeroes, and delay index \( \eta \) (an integer which when multiplied by the sampling interval, \( T \), yields the assumed process delay time). Let \( q \) be a one sample period time delay operator, such that \( u_{k-1} = qu_k \). We can express (2.4) in this operator notation as

\[ y_k = \frac{B(q)}{A(q)} u_k + \frac{1}{A(q)} v_k, \]

where \( B(q) \) and \( A(q) \) are defined as

\[ B(q) = q^n (b_0 + b_1 q + \cdots + b_m q^m) \]  

(2.5)

\[ A(q) = 1 + a_1 q + \cdots + a_n q^n. \]  

(2.6)
We refer to \( \{v_k\} \) as a disturbance sequence; although, we shall allow for a broader physical interpretation as needed. Stochastic estimation schemes assume \( \{v_k\} \) is a random noise sequence of known statistics; this leads to parameter set estimates tagged with confidence intervals or probabilistic bounds [5, 8, 9]. If prior statistics on the disturbance are not available, then finite length data records limit our ability to determine an appropriate statistical model. On the other hand, \( H_\infty \) robust control methodologies require plant uncertainty to be bounded [10]. Our investigations shall treat \( \{v_k\} \) as an unknown, but bounded (UBB) sequence; that is, there exists (a known) \( \gamma^* \in \mathbb{R} \) such that \( |v_k| \leq \gamma^* \) for all \( k \in \mathbb{N} \) (see Appendix A for a notation summary). Set membership identification algorithms have been derived based upon the UBB assumption on \( \{v_k\} \) and deliver bounded set estimates; see [1, 2, 4, 11] for examples.

The minimal set of feasible parameters for the ARX model with bounded noise turns out to be an irregular convex set. Although conservatism is introduced, we choose to maintain a more tractable set description. Ellipsoidal algorithms, a type of set membership identification algorithm, overbound the set of feasible parameters with a higher-dimensional generalization of an ellipse, an ellipsoid. We shall consider recursive parameter set estimation algorithms due to both their reduced computational complexity over batch processing (meaning that all measurements are considered simultaneously) algorithms and their potential for on-line application; for example, the adaptive robust control scheme proposed in [7] requires recursive set estimates for on-line adaptation.
The optimal bounding ellipsoid (OBE) algorithm of [2] and the set membership-weighted recursive least squares (SM-WRLS) algorithm of [11] require the center of an overbounding ellipsoid to be a modified recursive least squares (RLS) estimate. The optimal volume ellipsoid (OVE) algorithm of [1, 12] allows the ellipsoid center to be unconstrained, resulting in the overbounding ellipsoid of minimal hypervolume among recursive ellipsoid algorithms. We shall use the OVE algorithm to characterize the set of feasible parameters, because it introduces the least conservatism among the algorithms in its class. As noted in [12], the OVE algorithm is mathematically equivalent to the ellipsoid algorithm with parallel cuts (EPC) of [13]. The EPC algorithm is in turn (mathematically) equivalent to the modified Fogel-Huang algorithm (MFHA) of [14], as established in [15]. We summarize the work of [12] in the following section.

2.1.2 The OVE Algorithm

The process to be identified, \( P \), is assumed to be SISO, causal, open-loop, and describable by the LTI difference equation (2.4) subject to a disturbance term, \( v_k \), unknown but bounded (UBB) in magnitude,

\[
|v_k| \leq \gamma_k
\]  

(2.7)

where there exists \( \gamma_*, \gamma^* \in \mathbb{R} \) such that

\[
0 < \gamma_* \leq \gamma_k \leq \gamma^* < \infty \quad \forall k \in \mathbb{N};
\]  

(2.8)

the sequence, \( \{\gamma_k\} \), is assumed known. The ARX model structure of (2.4) is assumed correctly chosen such that there exists a parameter vector, \( \theta^o \in \mathbb{R}^r \), describing the
true system. We denote the observed data record, consisting of \( N \gg r \) past inputs and outputs \((r \equiv n + m + 1\), the number of unknown parameters\), as

\[
Z^N = [y_0 \ u_0 \ \cdots \ y_{N-1} \ u_{N-1}].
\]  

Each element of the identified model set must be consistent with \( Z^N \) for some unknown \( u_k \) that satisfies (2.7). Equation (2.1) and the assumption of equation (2.7) on \( u_k \) require that \(-\gamma_k \leq y_k - \phi_k^t \theta \leq \gamma_k \) so that

\[
y_k - \gamma_k \leq \phi_k^t \theta \leq y_k + \gamma_k.
\]  

The equality constraints of (2.10) define two parallel hyperplanes, \( \{ H_* , H^* \} \), between which lies the set of feasible parameters,

\[
\mathcal{F}_k \equiv \{ \theta \in \mathbb{R}^r : |y_k - \phi_k^t \theta| \leq \gamma_k \}.
\]  

due to incorporation of the measurement at time index \( k \). The hyperplanes are parallel, because the gradients of the equality constraints with respect to \( \theta \) coincide as \( \phi_k \).

Assuming \( n, m, \eta, \) and \( \{ \gamma_k \} \) are consistent with \( Z^N \), a batch strategy defines the feasible parameter set at time index \( N \) as

\[
\mathcal{F}^N = \bigcap_{k=1}^N \mathcal{F}_k
\]  

which is an irregular convex set containing up to \( 2N \) facets. The usefulness of this convex polytope is limited by its complexity to compute and describe analytically [16, 17]. When \( \mathcal{F}^N \) is compact, then (conceptually) a more convenient polyhedron (bounded
polytope) of minimal hypervolume can be introduced to tightly overbound $\mathcal{F}^N$, yielding a more tractable analytic set description. Define a matrix, $\Phi$, whose $N$ rows are the regression vectors of (2.3) for time indices $k = 1, 2, \ldots, N$. If the rank of $\Phi$ is greater than or equal to $r$ ($\{\phi_k\}$ spans $\mathbb{R}^r$), then $\mathcal{F}^N$ is compact [18]; hence, it is necessary to choose $N \geq r$. In [4], orthotopes aligned with the parameter axes were used in a batch algorithm to overbound $\mathcal{F}^N$; it is typically the case that an ellipsoid of arbitrary orientation can more tightly overbound a polyhedron, yielding a less conservative set estimate. Changing to a recursive formulation makes the computation of an overbounding ellipsoid tractable, comparable in complexity to a recursive least squares algorithm, and admits the possibility of on-line application in a robust adaptive control algorithm. The ellipsoid resulting from sequential processing of the data pairs of $Z^N$ is not of minimal hypervolume over all possible algorithms (only recursive). As noted in [19], the ellipsoid hypervolume can (in general) be reduced by using the resultant ellipsoid as initialization for another pass with the data record, $Z^N$.

An ellipsoid can be completely characterized by a quadratic form

$$E_k \equiv \{ \theta \in \mathbb{R}^r : (\theta - \theta_k)^T P_k^{-1} (\theta - \theta_k) \leq 1 \}$$  \hspace{1cm} (2.13)

where $P_k$ is a positive definite, symmetric matrix whose eigenvectors are the orthogonal set of ellipsoid semiaxes and whose eigenvalues are the squares of the semiaxes' lengths. The ellipsoid, $E_k$, is centered at $\theta_k$ in the parameter space. The ellipsoid hypervolume, $\text{vol}(E_k)$, is proportional to the square root of the product of the eigen-
values of the ellipsoid matrix, $P_k$; that is,

$$\text{vol}(E_k) \propto \left[\prod_{i=1}^{r} \lambda_i(P_k)\right]^{\frac{1}{2}} = \sqrt{\det(P_k)}. \quad (2.14)$$

The OVE algorithm, due to its recursive nature, must be initialized with an ellipsoid estimate. The initial center is arbitrarily chosen as the origin of the parameter space. It is desirable for the initial ellipsoid, $E_0$, to overbound (the unknown) $\mathcal{F}^N$. A conservative choice is $P_0 = \rho I_r$ where $\rho > 1$. If $E_0$ indeed overbounds $\mathcal{F}^N$, then $\mathcal{F}^N$ will be contained within $E_k$ for all $k$, as proven in [12]. The OVE algorithm measurement update takes place in a transformed coordinate wherein the current ellipsoid is an origin-centered unit radius hypersphere. In [12] it was also proven that at each iteration, the minimal hypervolume ellipsoid must pass through the intersection of the previous ellipsoid, $E_k$, and the parallel hyperplanes defined by the new measurement. This is illustrated in Figure 1. If $E_k \subset \mathcal{F}_{k+1}$, then $E_{k+1} = E_k$ and the measurement pair $(y_{k+1}, u_{k+1})$ is effectively discarded. This property is a selective updating feature of the OVE algorithm. The OVE algorithm will terminate if $\gamma_{k+1}$ is found to be inconsistent ($E_k \cap \mathcal{F}_{k+1} = \emptyset$); that is, there is no intersection between the current feasible set estimate as captured in the ellipsoid $E_k$ and the new hyperplane cut $\mathcal{F}_{k+1}$.

**OVE Algorithm Summary**

1. Initialize with $P_0 = \rho I_r$ where $\rho > 1$ and $\theta_0 = 0$. 

2. For the prediction error, $e_{k+1} = y_{k+1} - \phi_{k+1}^t \theta_k$, let

$$
\alpha_k = \frac{e_{k+1} + \gamma_{k+1}}{(\phi_{k+1}^t P_k \phi_{k+1})^{1/2}} \quad \text{and} \quad \beta_k = \frac{\gamma_{k+1}}{(\phi_{k+1}^t P_k \phi_{k+1})^{1/2}} \quad (2.15)
$$

3. If $\alpha_k \leq -1$ or $\alpha_k - 2\beta_k \geq 1$, then the data pair $(y_{k+1}, u_{k+1})$ is not consistent with the modeling assumptions (stop).

4. If $\alpha_k > 1$, then $\beta_k := \beta_k + \frac{1-\alpha_k}{2}$ and $\alpha_k := 1$; otherwise, if $2\beta_k - \alpha_k > 1$, then $\beta_k := \frac{1+\alpha_k}{2}$.

5. If $|\alpha_k - \beta_k| > \epsilon$ for $0 < \epsilon \ll 1$, then $\tau_k$ is the real solution of

$$(r+1)\tau_k^2 + \left(\frac{(1+\alpha_k)(\alpha_k - 2\beta_k + 1)}{\beta_k - \alpha_k} + 2[1 + r(\beta_k - \alpha_k)]\right) \tau_{k+1} + r[\alpha_k(\alpha_k - 2\beta_k) + 1 - 1] = 0$$

such that $\tau_k \in (\alpha_k - 2\beta_k, \alpha_k)$ and

$$
\delta_k = \frac{(\tau_k + 1)^2(\beta_k - \alpha_k) - \tau_k(1 + \alpha_k)(2\beta_k - \alpha_k - 1)}{\tau_k + \beta_k - \alpha_k} \quad (2.16)
$$
\[
\sigma_k = \frac{-\tau_k}{\beta_k - \alpha_k}; \quad (2.17)
\]

otherwise \(|\alpha_k - \beta_k| \leq \epsilon\), set \(\tau_k = 0\) and
\[
\begin{align*}
\delta_k &= \frac{r(1 - \beta_k^2)}{r - 1} \\
\sigma_k &= \frac{1 - r\beta_k^2}{1 - \beta_k^2}. \quad (2.18)
\end{align*}
\]

6. If \(\sigma_k < 0\), then set \(\theta_{k+1} = \theta_k\) and \(P_{k+1} = P_k\); otherwise, let
\[
\begin{align*}
\theta_{k+1} &= \theta_k + \frac{\tau_k P_k \phi_{k+1}^l}{(\phi_{k+1} P_k \phi_{k+1})^{1/2}} \\
P_{k+1} &= \delta_k \left( P_k - \sigma_k \frac{P_k \phi_{k+1}^l \phi_{k+1}^l \phi_{k+1}^l}{\phi_{k+1} P_k \phi_{k+1}} \right). \quad (2.20)
\end{align*}
\]

2.2 Asymptotic Properties of the OVE Algorithm

The OVE algorithm is defined and derived in [12] in terms of a transformed coordinate system. In the following, we establish the relationship between the ellipsoid hypervolume and its transformed magnitude. This enables a proof of the asymptotic behavior of the prediction error as stated in lemma form. Lemma 2 is subsequently used to prove an asymptotic property of the ellipsoid center.

We shall interchangeably use the singular values of \(P_k\) as the eigenvalues of \(P_k\) as justified in the following lemma.

**Lemma 1** If \(P_k \in \mathbb{R}^{r \times r}\) where \(P_k = P_k^l > 0\), then \(\lambda_i\) is an eigenvalue of \(P_k\) if and only if \(\lambda_i\) is a singular value of \(P_k\).

**Proof:** Perform an singular value decomposition (SVD) of \(P_k\) as \(P_k = U \Sigma V^T\) where \(U\) and \(V\) are unitary matrices and \(\Sigma\) is a diagonal matrix with \(r\) real positive diagonal
elements, $\sigma_i > 0$. Because $P_k = P'_k$, we have $P_k P'_k = P'_k P_k$ (that is, $P_k$ is a normal matrix); furthermore,

$$P_k P'_k = U \Sigma V' V' U' = U \Sigma I \Sigma U' = U \Sigma^2 U'.$$

Real (positive definite) symmetric matrices have real (nonnegative) eigenvalues, $\lambda_i > 0$, and normal matrices are orthogonally diagonalizable [20]; that is, there exists a unitary $W$ such that $P_k = W \Lambda W'$ where $\Lambda$ is a diagonal matrix whose elements are the real $\lambda_i > 0$. The fact that $P_k P'_k = P^2_k = W \Lambda^2 W'$ implies that $\Lambda^2 = (U' W)^T \Sigma^2 (U' W)$ ($\Lambda^2$ and $\Sigma^2$ are similar matrices); hence, they have the same set of eigenvalues, the diagonal elements of $\Sigma^2$. Because $\lambda_i$ are real and nonnegative, we conclude that for each $i \in [1, r]$, $\exists j \in [1, r]$ such that $\lambda_i = \sigma_j$.

\[\Box\]

2.2.1 Ellipsoid Hypervolume Relationships

The ellipsoid, $E_k$, is characterized by the ellipsoid matrix, $P_k$, as in equation (2.13) with hypervolume measure, $\sqrt{\det(P_k)}$, as in (2.14). In [12] the ellipsoid, $E_k$, overbounding the current feasible parameter set is transformed into a unit radius hypersphere, $\hat{E}_k$, using an affine transformation, $\hat{\theta} = J_k^{-1} (\theta - \theta_k)$, where $P_k = J_k J_k'$ ($P_k = P'_k > 0$ ensures that $J_k^{-1}$ exists). Substitution of $\theta - \theta_k = J_k \hat{\theta}$ into (2.13) yields

$$\hat{E}_k \equiv \{ \hat{\theta} \in R^r : \hat{\theta}' \hat{\theta} \leq 1 \}.$$

The analogue of $P_k$ from (2.13) is now evident as $\hat{P}_k = I_r$ (the identity matrix of dimension $r \times r$). The measurement update occurs in the transformed parameter space
(characterized by \( \hat{\theta} \)) prior to being transformed back into the original coordinate space (characterized by \( \theta \)). Following the measurement update, the new ellipsoid matrix in the transformed coordinates is \( \hat{P}_{k+1} = J_k^{-1}P_{k+1}(J_k^{-1})' \) [12]. In the \( \hat{\theta} \) domain, the hypervolume of the updated ellipsoid, \( \hat{E}_{k+1} \), relative to the previous, \( \hat{E}_k \), is clearly established in [12]. The relationship between an ellipsoid's hypervolume in the \( \hat{\theta} \) domain and the \( \theta \) domain as well as the ratio of the hypervolumes of \( E_{k+1} \) and \( E_k \) (each in the untransformed coordinates, characterized by \( \theta \)) are established in the following theorem. The actual hypervolume of the ellipsoid, \( E_k \), is proportional to the square root of the determinant of the ellipsoid matrix, \( P_k \).

**Theorem 1** Let \( |\cdot| \) be the matrix determinant operator. For \( P_k \) as defined in (2.18) and (2.21), we have

\[
|P_{k+1}| = |P_k||\hat{P}_{k+1}|.
\]

**Proof:** Recalling that \( \hat{P}_{k+1} = J_k^{-1}P_{k+1}(J_k^{-1})' \), we have

\[
|\hat{P}_{k+1}| = |J_k^{-1}P_{k+1}(J_k^{-1})'| \\
= |J_k^{-1}||P_{k+1}||(J_k^{-1})'| \\
= |(J_k^{-1})'||J_k^{-1}||P_{k+1}| \\
= |P_k^{-1}||P_{k+1}| \\
= \frac{|P_{k+1}|}{|P_k|}.
\]

\( \square \)
By construction in the OVE algorithm [12], we have that \( |\hat{P}_{k+1}| = \delta_k(1 - \sigma_k) \leq 1 \) where \( \delta_k(1 - \sigma_k) \) is the transformed ellipsoid semiaxis length, squared, along the transformed regressor direction, and \( \delta_k \) is the squared length of each of the \( r - 1 \) remaining semiaxes of the transformed ellipsoid (\( \sigma_k \) and \( \delta_k \) are defined in step 5 of the OVE algorithm). We have

\[
\frac{|P_{k+1}|}{|P_k|} = \frac{|\hat{P}_{k+1}|}{|\hat{P}_k|} = \delta_k^2(1 - \sigma_k) \leq 1. \tag{2.22}
\]

A zero hypervolume results if the set of feasible parameters is empty, or an isolated point. The OVE algorithm terminates updating when \( \mathcal{F}_{k+1} \cap E_k \) is empty or is an isolated point (see OVE algorithm step 3); therefore, if \( \alpha_k > -1 \) and \( \alpha_k - 2\beta_k < 1 \), \( \forall k \in \mathbb{N} \), then we have for all \( k \in \mathbb{N} \), \( |P_k| > 0 \) and \( |\hat{P}_k| > 0 \). We have established that

\[
0 < |P_{k+1}| = |P_k|\delta_k^2(1 - \sigma_k) \leq |P_k|
\]

provided \( \alpha_k > -1 \) and \( \alpha_k - 2\beta_k < 1 \), \( \forall k \in \mathbb{N} \). The sequence, \( \{|P_k|\} \), is decreasing and bounded below, and consequently converges in \( \mathbb{R} \); let the limit be denoted \( V^* \). If \( V^* \neq 0 \), it follows from (2.22) and the fact that \( |P_k| \) is decreasing that \( \{|\hat{P}_{k+1}|\} \) must converge to 1.

### 2.2.2 Asymptotic Prediction Error

From the OVE algorithm summary, the prediction error is \( e_{k+1} \equiv y_{k+1} - \phi'_{k+1}\theta_k \). The following lemma states that if the conditions of step 3 of the OVE algorithm are never satisfied, then \( |e_{k+1}| \) is asymptotically bounded by the assumed disturbance threshold sequence, \( \{\gamma_k\} \).
Lemma 2 Assume \( \alpha_k > -1 \) and \( \alpha_k - 2\beta_k < 1 \), \( \forall k \in \mathbb{N} \) and \( V^* \neq 0 \). There exists \( M(r, V^*) \in \mathbb{N} \) such that \( \forall k \geq M, |e_{k+1}| \leq \gamma_{k+1} \).

**Proof:** For notational convenience, \( V_k \equiv |P_k| \) and \( \hat{V}_k \equiv |\hat{P}_k| \). It is necessary that \( \theta_k \notin \mathcal{F}_{k+1} \) to have \( |y_{k+1} - \phi'_{k+1}\theta_k| = |e_{k+1}| > \gamma_{k+1} \). Geometrically this means that the origin, \( \hat{\theta}_k \), of the hypersphere (the ellipsoid, \( \mathcal{E}_k \), in the transformed coordinates) is not interior to the (transformed) parallel hyperplane cut generated by the new measurement pair, \( (y_{k+1}, u_{k+1}) \) (see Figure 1). For \( |e_{k+1}| > \gamma_{k+1} \), the supremum of \( \hat{V}_{k+1} \) occurs when one (transformed) hyperplane passes through \( \hat{\theta}_k \) and the other is tangent to \( S \) (which is forced in the OVE algorithm, step 4, when there is no intersection by the second hyperplane). These limit cases occur when \( \beta_k = 0.5 \) and \( \alpha_k = 0 \) or 1. In either case, following substitution into the OVE equations, the resulting hypervolume (for \( r > 1 \)) in the transformed coordinates is

\[
v_r \equiv \hat{V}_{k+1}(r) = \delta_k(1 - \sigma_k) = \frac{r^{2r}}{(r - 1)^{r-1}(r + 1)^{r+1}}
\]

which is solely a function of \( r \). Now \( v_r \) can be rewritten as

\[
v_r = \frac{1}{(1 - \frac{1}{r})^r} \cdot \frac{1}{(1 + \frac{1}{r})^r} \cdot \frac{1 - \frac{1}{r}}{(1 + \frac{1}{r})^2} \to \frac{1}{e} \cdot \frac{1}{e} \cdot 1 = 1
\]

as \( r \to \infty \). Consider the function \( f(r) \equiv \ln v_r \), \( \forall r > 1 \). The derivative of \( f \) with respect to \( r \) can be reduced to

\[
\frac{df}{dr} = \frac{df}{dv} \cdot \frac{dv}{dr} = \ln \left( \frac{1}{1 - \frac{1}{r}} \right) > 0,
\]

because \( \frac{1}{1 - \frac{1}{r}} > 1 \), \( \forall r > 1 \). Now \( \frac{df}{dv} = \frac{1}{v} > 0 \) when \( v = \hat{V}_{k+1} > 0 \) which is true for all \( k \) and any \( r > 1 \). The chain rule allows us to conclude that \( \frac{df}{dr} > 0 \), \( \forall r > 1 \); hence, \( v_r \)
is strictly increasing. We conclude \( \nu_r < 1 \) for any given integer \( r > 1 \), because \( \nu_r \) is strictly increasing and has the limit of 1 as \( r \to \infty \). This is consistent with the fact that the OVE algorithm will only update when a measurement reduces the ellipsoid hypervolume. For a given \( r > 1 \), \( \delta \equiv 1 - \hat{V}_{k+1}(r) > 0 \).

Because \( \{V_k\} \) is convergent it is a Cauchy sequence; that is, for all \( \epsilon > 0 \), there exists \( M(\epsilon) \) such that for all \( m, n \geq M \), we have \( |V_m - V_n| < \epsilon \). Consider \( \epsilon = V^* \delta > 0 \). We know there exists \( M \in \mathbb{N} \) such that \( 0 \leq V_k - V_{k+1} < V^* \delta \leq V_k \delta \); that is,

\[
V_k - V_{k+1} < V_k \delta \quad \forall k \geq M
\]

(we twice used the fact that \( \{V_k\} \) is decreasing). Theorem 1 established that \( \hat{V}_{k+1} = \frac{V_{k+1}}{V_k} \), so we have \( 1 - \hat{V}_{k+1} \leq \delta \), for all \( k \geq M \). For \( |e_{k+1}| > \gamma_{k+1} \) to occur, it is necessary that \( 1 - \hat{V}_{k+1} \geq \delta \). Thus \( \forall k \geq M \), we have \( |e_{k+1}| \leq \gamma_{k+1} \).

\[\Box\]

On the other hand if \( V^* = 0 \), the ellipsoid, \( E_k \), converges to, yet can never equal (due to step 3 of the OVE algorithm), an isolated point. A hyperplane must asymptotically pass through this asymptotic isolated point; this is sufficient to ensure that \( |e_{k+1}| \downarrow \gamma_{k+1} \) as \( k \to \infty \).

### 2.2.3 Asymptotic Properties of the Ellipsoid Center

The ellipsoid hypervolume is a convergent quantity, proportional to the square root of \( |P_k| \) as defined in equation (2.14). We do not expect the ellipsoid center to converge, rather that its drift rate decrease asymptotically as the hypervolume converges,
provided that $P_k$ remains well conditioned and the conditions of step 3 of the OVE algorithm are never satisfied.

**Theorem 2** Suppose each of the following conditions hold

1. $u_k \not\rightarrow 0$ and $\sigma_k \rightarrow 0$ as $k \rightarrow \infty$
2. $\alpha_k > -1$ and $\alpha_k - 2\beta_k < 1$, $\forall k \in \mathbb{N}$
3. $\exists B \in \mathbb{R}$ such that $\forall k \in \mathbb{N}$, $\kappa_2(P_k) \equiv \frac{\sigma(P_k)}{\varepsilon(P_k)} \leq B$.

If $V^* \neq 0$, then $\|\theta_{k+1} - \theta_k\|_2 \rightarrow 0$ as $k \rightarrow \infty$.

**Proof:** From the OVE algorithm summary (2.17) ($|\alpha_k - \beta_k| = 0.5$), we see that

$$\tau_k = \sigma_k(\alpha_k - \beta_k) = \sigma_k \frac{e_{k+1}}{(\phi_{k+1}^T P_k \phi_{k+1})^{1/2}}.$$

Substitution of $\tau_k$ into (2.20) yields

$$0_{k+1} - \theta_k = \frac{\tau_k P_k \phi_{k+1}}{(\phi_{k+1}^T P_k \phi_{k+1})^{1/2}} = \frac{\sigma_k P_k \phi_{k+1} e_{k+1}}{\phi_{k+1}^T P_k \phi_{k+1}}$$

so that

$$(0_{k+1} - \theta_k)(0_{k+1} - \theta_k) = \|\theta_{k+1} - \theta_k\|^2_2 = \frac{\sigma_k^2 \phi_{k+1}^T P_k^T P_k \phi_{k+1} \phi_{k+1}^T}{(\phi_{k+1}^T P_k \phi_{k+1})^2}.$$

$P_k = P_k'$ so $P_k' P_k = P_k^2 = (P_k^2)' > 0$ and $\sigma_i(P_k^2) = \sigma_i^2(P_k)$. Application of the Rayleigh-Ritz theorem [20] yields

$$\|\theta_{k+1} - \theta_k\|^2_2 \leq \frac{\sigma_k^2 \phi_{k+1}^T P_k^T P_k \phi_{k+1}}{\sigma^2(P_k) \|\phi_{k+1}\|^2_2} \leq \frac{\sigma_k^2 B^2 \phi_{k+1}^T \phi_{k+1}}{\|\phi_{k+1}\|^2_2} = A_k$$

where $\kappa_2(P_k) \leq B$ for all $k$ as assumed. Lemma 2 requires either that $\exists M \in \mathbb{N}$ such that $e_{k+1}^2 \leq \gamma_{k+1}^2 \leq (\gamma^*)^2$ for all $k \geq M$. (note that while $e_{k+1}$ is bounded, it is not necessarily convergent). The assumption that $u_k \not\rightarrow 0$ ensures that $\|\phi_{k+1}\|_2 \not\rightarrow 0$.
as $k \to \infty$, because past and present inputs are elements of the regressor of equation (2.3). We have assumed $\sigma_k \to 0$ as $k \to \infty$ so that $A_k \to 0$ as $k \to \infty$. We conclude that $\|\theta_{k+1} - \theta_k\|_2 \to 0$ as $k \to \infty$.

2.2.4 Interpretations

The assumption that $P_k$ remains well conditioned requires that the overbounding ellipsoid never collapses to a lower dimension (less than $r$). In practical application, it could occur that $P_k$ becomes rank deficient numerically; the proof would not be applicable. If the conditions of the above theorem are satisfied, it does not follow that $\{\theta_{k+1}\}$ is a convergent sequence. As a counterexample, consider the sequence $\{\ln k\}$ which has $\ln(k+1) - \ln k = \ln \frac{k+1}{k} \to 0$ as $k \to \infty$, yet $\{\ln k\}$ is a divergent sequence.

The ellipsoid hypervolume is a convergent quantity, proportional to the square root of $|P_k|$ as defined in equation (2.14). We conjecture that the consequent of Theorem 2 occurs for a convergent hypervolume ellipsoid; therefore, $\sigma_k \to 0$ as $k \to \infty$ is anticipated to occur. We do not expect the ellipsoid center to converge, but rather that its drift rate decrease asymptotically to zero as the hypervolume converges.
2.3 Input Synthesis for the OVE Algorithm

The more conservative a parameter set estimate (PSE) is, the less likely it is that a controller exists to meet desired closed-loop performance objectives for the ensemble of frequency responses of all plants parameterized by the set estimate. With regard to the input synthesis problem, techniques for traditional single point identification methods are based on information theoretic approaches [8] and are normally cast in an optimization framework. This work, focusing on synthesizing the probing signal for the parameter set estimation problem, represents one of the first approaches of its kind. Input design for a batch set membership experiment is addressed in [18], and a recursive scheme is proposed in [21]. Our primary focus is improving the transient performance of the set membership identification experiment.

The algorithm developed here is based on the OVE algorithm, summarized earlier, and intends to reduce conservatism in the parameter set estimate. Unlike input synthesis procedures developed for traditional single point identification approaches, the algorithm developed here for PSE is set theoretic in nature and derives its effectiveness from geometrical arguments relative to the ellipsoidal bounding characteristics of the OVE algorithm. It is assumed throughout that one desires to conduct a finite duration experiment to yield the least conservative, yet analytically tractable, estimate of the set of feasible parameters. In practice, the allowable set of input signals is usually restricted by the physical system as well as by safety and economic limits on the system operation. While many other factors enter into the overall set membership identification experiment design problem, for example, the choice of sampling sched-
ule (e.g. see [22]) and experiment duration, we consider the problem of designing the input so as to obtain maximal information from the identification experiment.

Given a priori assumptions of \( n, m, \eta, \) and \( \{\gamma_k\} \), we seek the input sequence of length \( N \) yielding the least conservative OVE algorithm parameter set estimate (assume that \( E_0 \) overbounds \( F^N \) as well as the true parameterization, \( \theta^* \)). The OVE algorithm is of minimal hypervolume among recursive algorithms for a given input sequence and initialization [1], yet the choice of which input sequence to apply to achieve the minimal hypervolume \( E_N \) is not obvious. An intuitive input synthesis technique, based upon geometrical insight into the OVE algorithm update, is investigated pursuant to this objective.

### 2.3.1 Preliminaries

The hyperplane separation in the parameter space is

\[
\frac{2\gamma_{k+1}}{\|\phi_{k+1}\|^2}
\] (2.23)

which is inversely proportional to the norm of the regressor, \( \phi_{k+1} \), and is proportional to the disturbance bound, \( \gamma_{k+1} \). The regressor is formed from past inputs and outputs (2.3); thus, greater energy inputs generally lead to OVE algorithm updates of lesser hyperplane separation. Reduced hyperplane separation may lead to reduced hypervolume ellipsoids; hence, for fairness of comparison with other input sequences, we shall seek a minimum hypervolume \( E_N \) over all \((N\text{-length}) \) input sequences of fixed energy. If the disturbance sequence is labeled noise and the input sequence is termed the signal, then the search is conducted over input sequences of fixed signal-to-noise
ratio (SNR).

Input design for batch set membership identification was investigated in [18]. If the space of admissible regressors is a known convex set, \( \varphi \), an experiment minimizing the volume of \( \mathcal{F}^N \) must explore all vertices of \( \varphi \). If \( \varphi \) has infinitely many vertices (i.e., cannot be expressed as the intersection of a finite number of closed half spaces), then no optimal finite duration (batch) experiment exists. The authors of [18] conclude that the analysis has no application to recursive estimation, because the optimal volume batch experiment for \( N \) measurements can differ significantly from that of duration \( N + 1 \). The analysis does underscore the fact that making the norm of the regressor as large as possible (for a chosen direction) is necessary in an optimal policy; this minimizes the hyperplane separation of (2.23).

**Strategy**

As a consequence of equation (2.10), \( \phi_k \) is orthogonal to the parallel set of hyperplanes that it defines. The greatest hypervolume reduction occurs along directions orthogonal to the hyperplane cuts [23]. Although optimality is not implied, it may be useful to choose the principal ellipsoid semiaxis, \( \nu_1 \), as a direction to “place” \( \phi_k \) (parallel to). This idea was also observed in [24] in terms of aligning \( \phi_k \) with an “as long as possible” axis joining two vertices of \( \mathcal{F}^N \). An eigenvalue decomposition (EVD) of \( P_k \) yields \( \nu_1 \) as the eigenvector corresponding to the largest eigenvalue of \( P_k \), which is the ellipsoid semiaxis of greatest length.
Set up

In the initial phase of the experiment, we apply a pseudorandom white (zero mean and uncorrelated) input sequence uniformly distributed over \((-\bar{u}, \bar{u})\). The disturbance sequence, \(v_k\), is a pseudorandom white noise sequence for the duration of the experiment, uniformly distributed over \((-\bar{v}, \bar{v})\). The initial input record is chosen as white noise because white noise has its power spectrum distributed evenly over the frequency interval of \((-\pi, \pi]\), leaving no gap in the spectrum; this is a persistence of excitation condition (see e.g. [25]). Let the initial data record length be \(k_1\). The initial data run allows the OVE algorithm time to pare down the initial ellipsoid (hypersphere) into an ellipsoid which reflects properties of the plant. We abandon this initial phase in section 2.3.4, wherein we conduct a steady state comparison to the algorithm of [21].

To influence the evolution of the process dynamics, we shall exploit an estimate of the plant parameterization. We invoke a certainty equivalence principle [25] and assume \(\theta^* = \theta_{k_1}\), where \(\theta^*\) is the (unknown) parameter vector used to generate the observed data record, \(Z^N\); there is, however, no guarantee that this ellipsoid center, \(\theta_{k_1}\), is a feasible plant description (\(E_{k_1}\) is a conservative estimate of \(\mathcal{F}_{k_1}\)).

2.3.2 Input Design

The desired regression vector direction, \(\phi_d\), at some future time index, \(k_2\), is \(\pm \nu_1\), where \(\|\nu_1\|_2 = c > 0\) is maximized subject to the signal magnitude and energy constraints of the physical system. The sign of \(\nu_1\) is chosen to minimize the synthesized
input energy energy for a given $c$. The regression vector is composed of inputs and
outputs up to time index $k_2$; see equation (2.3). It is clear that we desire the inputs
specified in $\phi_{k_2} = \phi_d$ to produce the outputs also specified therein. Using the nominal
plant transfer function estimate, derived from $\theta_{k_1}$, we shall generate a state space
realization $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$.

The synthesis can be pursued within an input-output framework (as in [23], for
$\eta = 1$) wherein the system state is $\phi_k$. We would have

$$\phi'_{k+1} = \hat{A}_\phi \phi_k + \hat{B}_\phi u_{k-\eta+1}$$ (2.24)

where

$$\hat{A}_\phi = \begin{bmatrix}
-a_1 & \cdots & \cdots & -a_n & -b_0 & \cdots & \cdots & -b_n \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0
\end{bmatrix}, \quad \hat{B}_\phi = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}. \quad (2.25)$$

In the absence of a disturbance ($u_k = 0$), we can transfer the initial state, $\phi_{k_1}$, to any
desired final state, $\phi_d$, in $r$ steps provided the pair, $(\hat{A}_\phi, \hat{B}_\phi)$, is controllable. In the
case where $n = 2$ and $m = 1$, this requires that

$$b_1^2 - a_1 b_0 b_1 + a_2 b_0^2 \neq 0,$$ (2.26)

where the coefficients are from the parameter vector, $\theta_{k_1}$, as in (2.2). There is no guar­
antee that the coefficients of $\theta_{k_1}$ will satisfy (2.26); hence, controllability of $(\hat{A}_\phi, \hat{B}_\phi)$
cannot be assured. Equation (2.26) is satisfied for $\theta_{k_1} = \theta^*$ provided that a control canonical realization is minimal (that is, $n$ and $m$ are chosen correctly). In degenerate cases, when $(\hat{A}_\phi, \hat{B}_\phi)$ is not controllable (e.g. $\theta_{k_1} = 0$), one could choose another element within $E_{k_1}$ as a basis for the realization.

The state space realization of $(\hat{A}_\phi, \hat{B}_\phi)$ is canonical in nature and will not exhibit the numerical robustness of a balanced realization. The input synthesis procedure will require the inversion of the controllability matrix; the fact that $\theta_{k_1} \neq \theta^*$ compounds the influence of any numerical error present when attempting to place the regression vector. Balancing the realization improves the likelihood that we can (approximately) achieve $\phi_{k_2} = \phi_d$. To reduce the burden in computing the balanced state space realization (through reduction in the dimension of the state vector) and to guarantee controllability of the realization, we choose a minimal (balanced) state space realization. We do so at the cost of a loss of the defacto observability of the realization with system state $\phi_k$ ($\hat{C}_\phi = I_r$); it will be necessary to build a state observer for the minimal realization.

For the purpose of input design, we must restrict the model structure. We can only design inputs for the cases wherein $\eta = 0$ or 1 due to causality. Throughout this dissertation, we shall also require that $l \equiv m + \eta - n \leq 0$ for the following reasons. If $l > 0$, then $l$ poles are introduced at the origin of the complex discrete time frequency domain, resulting in a system matrix $\hat{A}$ of increased dimension, $n+l$ (from $n$); we must also arbitrarily choose the output immediately prior to time index $k_2 - n$ (because the corresponding input is specified in $\phi_d$). On the other hand, for $l < 0$, we are free
to choose the inputs not specified in $\phi_d$, whose corresponding outputs are specified in $\phi_d$, to be of minimal energy (zero).

For the purpose of illustration, assume $\eta = 0$ (hence, $\dot{D} \neq 0$), in the following.

For the minimal realization $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$, we have

$$x_{k+1} = \hat{A}x_k + \hat{B}u_k \quad y_k = \hat{C}x_k + \hat{D}u_k. \quad (2.27)$$

This requires that

$$y_{k_2-n} = \hat{C}x_{k_2-n} + \hat{D}u_{k_2-n},$$

$$y_{k_2-n+1} = \hat{C}A x_{k_2-n} + \hat{C}B u_{k_2-n} + \hat{D}u_{k_2-n+1}$$

$$\vdots$$

$$y_{k_2-1} = \hat{C}A^{n-1} x_{k_2-n} + \hat{C}A^{n-2} B u_{k_2-n} + \cdots + \hat{C}B u_{k_2-2} + \hat{D}u_{k_2-1}$$

from which we define the vectors

$$Y_d = [y_{k_2-n} \cdots y_{k_2-1}]' \quad \text{and} \quad U_d = [0 \cdots 0 u_{k_2-m} \cdots u_{k_2-1}]'. \quad (2.28)$$

In vector form, we have

$$Y_d = O x_{k_1-n} + GU_d$$

where $O' = [\hat{C}' (\hat{C}\hat{A})' \cdots (\hat{C}\hat{A}^{n-1})']$ is the observability matrix and

$$G = \begin{pmatrix}
\hat{D} & 0 & 0 & \cdots & 0 \\
\hat{C}\hat{B} & \hat{D} & 0 & \cdots & 0 \\
\hat{C}\hat{A}\hat{B} & \hat{C}\hat{B} & \hat{D} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\hat{C}\hat{A}^{n-2}\hat{B} & \cdots & \cdots & \hat{C}\hat{B} & \hat{D}
\end{pmatrix}. \quad (2.29)$$
The minimal realization is observable $(\mathcal{O}^{-1}$ exists), so we have $x_{k_2-n} = \mathcal{O}^{-1}[Y_d - GU_d]$. If the system state (realized from $\theta_{k_1}$) is initially $x_{k_2-n}$ and the plant representation is exact, then the input sequence of $U_d$ produces the desired output sequence of $Y_d$ and thus achieves $\phi_d$. Our ability to achieve $\phi_{k_2} = \phi_d$ is restricted by the fact that $\theta^0 \neq \theta_{k_1}$ and the presence of the disturbance sequence, $\{v_k\}$. In general, $x_{k_2-n}$ is not the "initial" state of our system. The simulation truth model, captured in $\theta^0$, is not known; if it were, there would be no need for an estimation exercise. The state $x_{k_2-n}$ can approximately be achieved, subject to the above error sources, from the state $x_{k_1}$ in $n$ steps due to the controllability of $(\hat{A}, \hat{B})$.

Iterating the state equation (2.27) with back substitution we obtain

\[
\begin{align*}
    x_{k_1+1} &= \hat{A}x_{k_1} + \hat{B}u_{k_1} \\
    x_{k_2+1} &= \hat{A}^2x_{k_1} + \hat{A}\hat{B}u_{k_1} + \hat{B}u_{k_1+1} \\
    &\vdots \\
    x_{k_1+n} &= \hat{A}^nx_{k_1} + \hat{A}^{n-1}\hat{B}u_{k_1} + \ldots + \hat{B}u_{k_1+n-1}. \\
\end{align*}
\]

(2.29)

If $\hat{U} \equiv [u_{k_1+n-1} \ldots u_{k_1}]'$, then (2.29) can be summarized as

\[
C\hat{U} = x_{k_1+n} - \hat{A}^nx_{k_1}
\]

where $C = [\hat{B} \hat{A}\hat{B} \ldots \hat{A}^{n-1}\hat{B}]$ is the controllability matrix whose inverse exists. The unknown input sequence can thus be solved for as

\[
\hat{U} = C^{-1}[x_{k_1+n} - \hat{A}^nx_{k_1}]. \\
\]

(2.30)

We have $x_{k_1+n} = x_{k_2-n}$ (implying that $k_2 = k_1 + 2n$) which is known, as are $\hat{A}$ and
The "initial" state, \( x_{k_1} \), is not known and must be estimated prior to computing \( \hat{U} \).

**Observer Design**

An observer shall be our vehicle for estimating \( x_{k_1} \). It is now evident that time index \( k_2 = k_1 + 2n \) when delay index \( \eta \neq 0 \). When \( \eta = 0 \), one additional iteration \( (k_2 = k_1 + 2n + 1) \) is required to establish \( \phi_d \) because \( u_{k_2} \) is not an element of \( U_d \) as defined in equation (2.28). Consider the discrete time observer equations

\[
\dot{x}_{k+1} = \hat{A} \dot{x}_k + \hat{L}(y_k - \hat{C} \dot{x}_k - \hat{D} u_k) + \hat{B} u_k
\]

\[
= (\hat{A} - \hat{L}\hat{C}) \dot{x}_k + \hat{L} y_k + (\hat{B} - \hat{L}\hat{D}) u_k \tag{2.31}
\]

where \( \dot{x}_k \) is the state estimate, \( \hat{L} \) is a vector of free parameters, and \( \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\} \) are from the minimal realization using \( \theta_{k_1} \) as in equations (2.27). The state estimation error dynamics are governed by

\[
\dot{\tilde{x}}_{k+1} \equiv x_{k+1} - \hat{x}_{k+1} = (\hat{A} - \hat{L}\hat{C}) \tilde{x}_k
\]

where \( \tilde{x}_k \) is the state estimation error at time index \( k \). The error dynamics are determined by the eigenvalues of the matrix \( (\hat{A} - \hat{L}\hat{C}) \) which coincide with the eigenvalues of \( (\hat{A}' - \hat{C}'\hat{L}') \). Because \( (\hat{A}, \hat{C}) \) is an observable pair, \( (\hat{A}', \hat{C}') \) is controllable and the eigenvalues of \( \hat{A}' - \hat{C}'\hat{L}' \) can be arbitrarily assigned via the free parameters of \( \hat{L}' \). The observer poles are placed in the complex \( z \)-plane at the angles of the \( n \) roots of unity at a radius of \( 0 < \xi \ll 1 \) so that initial state estimation error decays quickly as \( \xi^k \). To obtain \( \tilde{x}_{k_1} \), an estimate of \( x_{k_1} \), we arbitrarily choose \( \tilde{x}_{k_1-\psi} = 0 \) and use
the measurement pairs \((y_k, u_k)\) for \(k \in \{k_1 - \psi, \cdots, k_1 - 1\}\) to update the observer dynamics of equation (2.31). Later, for convenience, we choose \(\psi \equiv k_2 - k_1\).

**OVE-Based Input Synthesis Procedure (OVE-ISp)**

The input synthesis procedure consists of the following steps:

1. Assume equation error model parameters \(n, m, \eta, \{\gamma_k\}\) with \(m + \eta \leq n\) and \(\eta = 0\) or \(1\).

2. Initialize the OVE algorithm with \(\theta_0 = 0\), \(P_0 = \rho I_r, \rho > 1\), and \((y_k, u_k) = 0\) for all \(k \leq 0\).

3. Apply a known persistently exciting input sequence (e.g. white noise) for an interval of length \(k_1\), and simultaneously update the OVE algorithm with the measured data pairs, \((y_k, u_k)\), for \(k \leq k_1\).

4. Take the last OVE algorithm ellipsoid center, \(\theta_{k_1}\), and generate a (minimal) balanced state space realization, \(\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}\).

5. Compute the observer gain, \(\hat{L}\), (having chosen \(0 < \xi \ll 1\)) and use the last \(\psi\) data pairs to estimate \(x_{k_1}\).

6. Perform an EVD on \(P_{k_1}\) to identify \(\phi_d = \pm \nu_1\), where \(\|\phi_d\|_2 = \epsilon\) is specified.

7. Form \(U_d\) and \(V_d\) from \(\phi_d\) and compute \(x_{k_1+n}\).

8. Compute \(\tilde{U}\) to transfer \(x_{k_1}\) to \(x_{k_1+n}\).
9. The desired input sequence is \( u_* = \{\text{rev}(\hat{U}'), U'_d, u_{k_1+2n+1}\}; u_{k_1+2n+1} \) is specified by \( \phi_d \) when \( \eta = 0 \), and \( \text{rev}(\cdot) \) means reverse the order of the elements of the indicated vector.

### 2.3.3 Numerical Example

Let \( z^{-1} \) be the unit delay operator. Consider the linear shift-invariant system with zeros at \( z = \pm 0.5 \) and poles at \( z = 0.5 \pm 0.5i, 0.2, \) and \(-0.3\); hence, \( m = 2, n = 4, \) and \( r = 7 \). Further choose \( \theta_0 = 2 \) and \( \eta = 0 \). The corresponding difference equation is

\[
y_k - 0.9y_{k-1} + 0.34y_{k-2} + 0.11y_{k-3} - 0.03y_{k-4} = 2.0u_k - 0.5u_{k-2} + v_k \tag{2.32}
\]

with simulation parameter vector

\[
\theta^o = [-0.9 \ 0.34 \ 0.11 \ -0.03 \ 2.0 \ 0.0 \ -0.5]^T. \tag{2.33}
\]

For simulation of this truth model, we shall use \( \bar{v} = 0.05, \bar{u} = 100, k_1 = 200, \xi = 0.1, \quad ||\phi_d||_2 = 1, \) and \( P_0 = 100I_7 \) (corresponding to an origin-centered hypersphere of radius 10 in the parameter space). Observe that \( \theta^o \) is contained within \( E_0 \). We assume \( \gamma_k = 5.0, \forall k \leq k_1 = 200 \). The simulation is conducted in Matlab.

**Performance Measure**

As a measure of the normalized accuracy of the center estimate used in the synthesis procedure, we shall use

\[
e_\theta \equiv \frac{||\theta_{k_1} - \theta^o||_2}{||\theta^o||_2}. \]

To determine how well $\phi_d$ is actually achieved, we define the angular placement error as

$$\alpha \equiv \arccos \left( \frac{\phi_d \phi_k}{\|\phi_d\|_2 \|\phi_k\|_2} \right).$$

We shall generate a uniformly distributed pseudorandom white noise sequence, $u_r$, of the same length as $u_s$ as a baseline for comparison of hypervolume reduction. The symmetric interval over which $u_r$ may assume values is chosen as $[-u_o, u_o]$ where $u_o = \sqrt{\frac{3}{\pi}} \|u_s\|_2$; this ensures that $\mathcal{E}(\|u_r\|_2) = \|u_s\|_2$, where $\mathcal{E}(\cdot)$ is the expectation operator. The ellipsoid hypervolume at time index $k$ corresponding to the use of $u_r$ is $volr_k$. The results from five independent trials are summarized in Table 1.

**Table 1: Input Synthesis Performance: SNR = 66 dB, $\gamma_k = 5$**

<table>
<thead>
<tr>
<th>Trial</th>
<th>$\epsilon_d$</th>
<th>$\alpha (^\circ)$</th>
<th>$vol_{k_1}$</th>
<th>$vol_{k_2}$</th>
<th>$vol_{r_k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.8 \times 10^{-4}$</td>
<td>5.2</td>
<td>$3.7 \times 10^{-7}$</td>
<td>$3.2 \times 10^{-10}$</td>
<td>$1.1 \times 10^{-9}$</td>
</tr>
<tr>
<td>2</td>
<td>$2.4 \times 10^{-4}$</td>
<td>5.7</td>
<td>$4.0 \times 10^{-7}$</td>
<td>$8.9 \times 10^{-11}$</td>
<td>$1.5 \times 10^{-8}$</td>
</tr>
<tr>
<td>3</td>
<td>$6.1 \times 10^{-4}$</td>
<td>5.4</td>
<td>$6.1 \times 10^{-7}$</td>
<td>$6.6 \times 10^{-10}$</td>
<td>$4.0 \times 10^{-9}$</td>
</tr>
<tr>
<td>4</td>
<td>$3.8 \times 10^{-4}$</td>
<td>7.6</td>
<td>$5.0 \times 10^{-7}$</td>
<td>$7.0 \times 10^{-11}$</td>
<td>$1.7 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>$6.4 \times 10^{-4}$</td>
<td>6.8</td>
<td>$4.5 \times 10^{-7}$</td>
<td>$4.1 \times 10^{-10}$</td>
<td>$2.7 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

**Observations**

The desired regressor direction was achieved within $8^\circ$ for each of the independent trials. The high SNR and long length of the initial data record led to a center estimate (at time index $k_1$) close to the simulation parameter vector, $\theta^o$, which enabled good performance of the synthesis procedure. Simulation results verified that the synthesized input consistently achieved hypervolume reduction an order of magnitude...
greater than a pseudorandom white noise sequence of equal (expected) energy.

**Steady State Analysis**

The work summarized in Table 1 is of a transient nature. One might inquire as to how the input synthesis algorithm performs when repeatedly applied to the plant for set estimation. The synthesis procedure is unchanged; at the end of a synthesis cycle, the complete data input-output record becomes the "initial" data record to which we apply the synthesis procedure. One minor advantage gained is that the estimated system state, \( \hat{x}_k \), can be carried over from one synthesis interval to the next by updating the state observer for all data pairs in the previous synthesis interval of length \( \psi = 2n \) (\( \psi = 2n + 1 \) when \( \eta = 0 \)). The ellipsoid center (nominal plant estimate) will change at the end of each synthesis interval, thus \( \hat{x}_k \) will always have some initial transient error (which decays over \( 2n \) observer updates prior to its use in synthesis). The baseline for comparison is again a white noise sequence of equal (expected) energy and length. Trial 4 from the previous section was expanded upon for the experiment summarized in Table 2; \( I_s \) is the synthesis interval.

In the steady state, the white noise input consistently achieves greater hypervolume reduction than the synthesized input, \( u_s \). Once the event \( v_{olr_k} < vol_k \) occurred for a fixed \( \gamma_k \), the magnitude of \( \gamma_{k+1} \) was reduced. This step reduction in the assumed disturbance threshold induces a transient in the parameter set estimation process which often leads to \( vol_k < vol_{r_k} \) in the next few synthesis intervals. Success is sensitive to the magnitude of the \( \gamma_k \) reduction; for example, the \( \gamma_k \) reduction in
Table 2: Steady State Input Synthesis Performance

<table>
<thead>
<tr>
<th>$I_s$</th>
<th>$\gamma_k$</th>
<th>$\alpha (^\circ)$</th>
<th>$vol_{k+\psi I_s}$</th>
<th>$volr_{k+\psi I_s}$</th>
<th>$\text{sgn}(volr_k - vol_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.0</td>
<td>7.6</td>
<td>$7.0 \times 10^{-11}$</td>
<td>$1.7 \times 10^{-9}$</td>
<td>$+$</td>
</tr>
<tr>
<td>2</td>
<td>5.0</td>
<td>6.9</td>
<td>$3.1 \times 10^{-11}$</td>
<td>$5.4 \times 10^{-11}$</td>
<td>$+$</td>
</tr>
<tr>
<td>3</td>
<td>5.0</td>
<td>11.4</td>
<td>$2.5 \times 10^{-11}$</td>
<td>$6.3 \times 10^{-12}$</td>
<td>$-$</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>3.7</td>
<td>$1.0 \times 10^{-15}$</td>
<td>$4.9 \times 10^{-16}$</td>
<td>$-$</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>4.6</td>
<td>$6.4 \times 10^{-18}$</td>
<td>$3.8 \times 10^{-17}$</td>
<td>$+$</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>4.8</td>
<td>$3.0 \times 10^{-18}$</td>
<td>$6.0 \times 10^{-18}$</td>
<td>$+$</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
<td>9.1</td>
<td>$2.3 \times 10^{-18}$</td>
<td>$2.7 \times 10^{-18}$</td>
<td>$+$</td>
</tr>
<tr>
<td>8</td>
<td>0.5</td>
<td>4.4</td>
<td>$2.1 \times 10^{-18}$</td>
<td>$1.2 \times 10^{-18}$</td>
<td>$-$</td>
</tr>
<tr>
<td>9</td>
<td>.06</td>
<td>4.3</td>
<td>$2.6 \times 10^{-24}$</td>
<td>$3.8 \times 10^{-24}$</td>
<td>$+$</td>
</tr>
<tr>
<td>10</td>
<td>.06</td>
<td>9.6</td>
<td>$5.3 \times 10^{-25}$</td>
<td>$5.6 \times 10^{-25}$</td>
<td>$+$</td>
</tr>
<tr>
<td>11</td>
<td>.06</td>
<td>6.2</td>
<td>$3.9 \times 10^{-25}$</td>
<td>$2.4 \times 10^{-25}$</td>
<td>$-$</td>
</tr>
<tr>
<td>12</td>
<td>.05</td>
<td>5.2</td>
<td>$1.9 \times 10^{-25}$</td>
<td>$5.5 \times 10^{-20}$</td>
<td>$-$</td>
</tr>
<tr>
<td>13</td>
<td>.05</td>
<td>6.5</td>
<td>$9.7 \times 10^{-26}$</td>
<td>$3.7 \times 10^{-20}$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

The synthesis interval $I_s = 4$ did not succeed in achieving $vol_k \leq volr_k$. Recall that we have assumed $|v_k| \leq \gamma_k$ and $\gamma_k \in [\gamma_*, \gamma^*]$. Typically $\gamma_1$ is chosen conservatively ($\gamma_1 = \gamma^*$) to avoid inconsistency or numerical instability as discussed in [23].

Let $P_k$ and $Pr_k$ be the ellipsoid matrices (at time index $k$) corresponding to the synthesized and random inputs, respectively. Figure 2 depicts the synthesized input in both the time and frequency domains. Upon examination of the time domain plot, it is seen that the synthesized input becomes somewhat periodic. The magnitude spectrum (with points connected) of the Discrete Fourier Transform (DFT) (see Figure 2) of the the steady state excitation, $u_s$, exhibits “spikes” at $\Omega = 0$ (corresponding to the signal DC content) and at four other frequency pairs. The synthesized input is thus persistently exciting [25] of (approximately) order 9, whereas (true) white noise is
persistently exciting of all orders. If \( u \) were periodic with period \( \psi = 9 \), then it can be persistently exciting of at most order \( \psi \) [25]; the orientation of the principal ellipsoid semiaxis becomes relatively fixed after a few cycles of input synthesis, contributing to periodicity in \( u \). Persistence of excitation is likely the cause of the superior steady state performance of the white noise input.

### 2.3.4 Comparison to the Pronzato-Walter Algorithm

The objective of the OVE-ISP is to improve the transient performance of the parameter set estimation experiment; although recursive application of the OVE-ISP demonstrates no steady state advantage over a pseudorandom white noise sequence of comparable expected energy, we shall compare such recursive application of the OVE-ISP to the Pronzato-Walter algorithm of [21]. The Pronzato-Walter algorithm is derived under an input level restriction versus an energy constraint. The inputs appearing in the desired regressor, \( \phi_d \), are constrained by our choice of \( \| \phi_d \|_2 = c \). In our previous example, the inputs of (2.30) required to setup the desired initial state were large in magnitude relative to the inputs of \( U_d \) as can be seen in Figure 2. In an effort to constrain the input signal level, we shall alternatively pursue our input synthesis within an input-output framework \( x_k = \phi'_k \) as in (2.24).

Analogous to (2.29), we have

\[
\phi'_{k_1+r} = \hat{A}_\phi \phi'_{k_1} + \hat{A}_{\phi}^{-1} \hat{B}_\phi u_{k_1-n+1} + \cdots + \hat{B}_\phi u_{k_1-n+r}.
\]

The initial system state, \( \phi_{k_1} \), is available, and we shall pursue \( \phi_{k_1+r} = \phi_d \). The
required input sequence is \( \text{rev}(U) \) where
\[
U = C^{-1}_\phi [\phi_{k_1} + \hat{A} \phi_{k_1}]
\tag{2.34}
\]

\( C_\phi = [\hat{B} \hat{A} \hat{B} \cdots \hat{A}^{-1} \hat{B}] \), and \( \hat{A} \) and \( \hat{B} \) are as in (2.25).

The example appearing in [21] is
\[
y_k + 0.4y_{k-1} + 0.85y_{k-2} = 0.75u_{k-1} + u_k,
\]
where \( \eta = 1 \), \( m = 0 \), \( n = 2 \) and \( r = 3 \). For simulation of this truth model, we shall use zero initial conditions, \( N = 99 \), \( \|\phi_d\|_2 = 1 \), \( \bar{v} = 0.05 \), and \( \gamma_k = .05 \). For this steady state comparison we do not use an initial data record \( (k_1 = 0) \); therefore, the OVE-ISP cannot be initialized (here) with \( \theta_0 = 0 \) (a null system realization). We choose the initial ellipsoid, parameterized by \( P_{k_1} = 2I_3 + 1_3 \) \( (1_3 \text{ is a } k \times k \text{ matrix with a } 1 \text{ in all entries}) \) and \( \theta_0 = [1 \ 1 \ 1]' \), such that \( \theta^o \in E_0 \). After \( r \) time steps, we set \( k_1 := k_1 + r \) and apply our synthesis procedure recursively, until \( k_1 + r = N \) (choose \( N \) as an integer multiple of \( r \)). Let \( u_s \) denote the entire sequence of synthesized inputs.

We again generate a uniformly distributed pseudorandom white noise sequence, \( u_r \), now of length, \( N \), as a baseline for comparison of hypervolume reduction (again \( E(||u_r||_2) = ||u_s||_2 \)). The algorithm of [21] was derived under an input level constraint, \( |u_{p_k}| \leq \bar{u}_p \) for all \( k \). The Pronzato-Walter algorithm chooses the current input, \( u_k \), as
\[
u_k = \text{sign} \left( \frac{P_{k}^{-1} \phi_{k+1}}{p_k} \right) \bar{u}_p
\]
where \( \phi_{k+1} \) is \( \phi_{k+1} \) with \( u_k \) deleted and \( P_k^{-1} \) is row \( n+1 \) of \( P_k \) with the diagonal element, \( p_k \equiv P_k(n+1,n+1) \), deleted; the Pronzato-Walter algorithm cannot be used when \( \eta = \)
0. The constraint, $\bar{u}_p$, was chosen such that $\|u_p\|_2 = \sqrt{N} \bar{u}_p = \|u_\alpha\|_2$. The outcome of four independent trials is shown in Figure 3 in terms of the base 10 logarithm of the hypervolumes (2.14) of the resultant ellipsoids versus the time index, $k$. The corresponding input levels are summarized in Table 3. In Figure 3, the solid curves illustrate the use of the OVE-ISP, the dashed curves are generated via Pronzato-Walter’s algorithm, and the effect of the random input is illustrated by the dash-dot curves. The curves exhibit extreme sensitivity to input energy; lesser ellipsoid hypervolumes result when greater input energy is allowed. The relative placement of the curves was insensitive to an increase in the assumed $\gamma_k$. For steady state application, the OVE-ISP again exhibited no advantage over a pseudorandom white noise input of uniform distribution and equivalent expected energy. Our procedure performed similar to the technique of [21], which is optimal (among recursive input design procedures) under an input level versus energy constraint. When the space of feasible regressors is an origin centered hypersphere, the Pronzato-Walter algorithm attempts to place the regressor parallel with the ellipsoid semimajor axis [21]; a fundamental difference between the algorithms is that the Pronzato-Walter input is designed recursively, whereas the OVE-ISP is designed in batch sections of length $r$.

Table 3: Input Signal Levels: $\gamma_k = .05$

<table>
<thead>
<tr>
<th>Trial</th>
<th>$|u_\alpha|_\infty$</th>
<th>$\bar{u}_p$</th>
<th>$u_\sigma$</th>
<th>SNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>1.21</td>
<td>28.0</td>
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<tr>
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<td>0.68</td>
<td>1.16</td>
<td>27.6</td>
</tr>
<tr>
<td>4</td>
<td>1.64</td>
<td>0.75</td>
<td>1.28</td>
<td>28.4</td>
</tr>
</tbody>
</table>
2.3.5 Summary Remarks

We have presented a geometrical approach to input synthesis for parameter set estimation of an ARX model using the OVE algorithm. The input design is based upon a certainty equivalence principle wherein the OVE generated ellipsoid center is treated as the actual (unknown) plant. For the transient investigation, the high SNR and long length of the initial data record led to a center estimate (at time index $k_1$) close to the simulation parameter vector, $\theta^o$, which enabled good performance of the synthesis procedure relative to a pseudorandom, uniformly distributed, white noise input of equal expected energy. The synthesized input consistently achieved hypervolume reduction an order of magnitude greater than a pseudorandom white noise sequence of equal (expected) energy. This can never be proven rigorously due to the necessity of using a certainty equivalence argument. Although none of the geometrical meaning of the synthesis procedure is lost upon steady state application, a white noise sequence generated superior hypervolume reduction.

This investigation of input synthesis serves as a proof of concept of the utility of a heuristic design rule. The full benefit of this technique (other than as means of concluding the set membership identification experiment under an input energy constraint) is anticipated to be realized in an adaptation for slowly time-varying ARX models using a generalization of the OVE algorithm, the OVETV due to [26]. In this scenario, the synthesis routine would intermittently be applied to rapidly reduce the hypervolume of a "drifting" ellipsoid.
Figure 2: Steady-State Input: Time and Frequency Domain
Figure 3: Ellipsoid Hypervolume Sequences
CHAPTER III

Connecting Parameter Set Estimation to Robust Control Design

A control strategy is termed robust if closed-loop performance objectives are attained for all plant behaviors within a known bounded set. When used as the basis for a $H_\infty$ robust control design, these plant behaviors are typically characterized by a nominal linear model subject to bounded uncertainty. For a bounded set of parametric uncertainty, the behavior of the true plant, $P$, is restricted to lie within the ensemble of the corresponding frequency responses. Our objective is to generate the modeling information necessary to proceed with existing $H_\infty$ robust control design tools.

3.1 Background

The gap existing between the modeling premises of robust control design theory and the capabilities of identification theory was highlighted in a plenary talk at the 9th International Federation of Automatic Control/International Federation of Operational Research Societies Symposium on Identification and System Parameter Estimation, 1991 [27]. In this talk, the point was made that "... much of robust control theory is based on prior descriptions of model uncertainty or model errors which classical identification theory has been incapable of delivering."
The subject of identification for robust control design has been receiving considerable attention in the literature; see [28] for a survey of recent work. The nature of the a priori plant knowledge available establishes the framework within which we may pursue our objective of identifying system modeling information in a format suitable for designing a robust controller. The class of the nominal plant model can be given as either a continuous or discretized function of the independent variable (generically referred to as time in our discussion). There may exist prior knowledge of modeling orders; for example, the number of poles and zeros or the length of a finite impulse response (FIR) sequence may be known. In the event that $P$ is FIR of known length (the number of unknown parameters), and that prior knowledge of an ellipsoid encompassing the Markov parameters of $P$ is available, the existence of a worst-case plant, which minimizes the maximum (finite horizon) linear quadratic regulator cost, is proved in [29].

We have previously discussed the shortcoming of identification for robust control within a stochastic setting: confidence intervals on the parametric uncertainty results, rather than "hard bounds". If it occurs that no specific parameter element from a nominal parametric model structure (with an additive disturbance/noise term) could have accounted for all observed data, then we say that the plant modeling has restricted complexity (undermodeling is present). This is a false presupposition whenever the additive disturbance is a realization of a probability distribution with noncompact support; infinitely many parameter elements could have (with nonzero probability) generated an observed data record.
A novel idea of stochastic embedding was first proposed in [30], wherein under-modeling is assumed present. The idea of stochastic embedding is that one assumes \textit{a priori} probability distributions on the parameters of a nominal model, the residual model, and the additive noise. The observed input/output data record is then used to evaluate an \textit{a posteriori} distribution for the nominal model parameters and the residual model. This is pursued in a Bayesian framework in [9]. The best linear unbiased estimates of the nominal model parameters and the residual model are then used as the nominal system model and uncertainty description, respectively.

When the underlying plant is known to be describable as an autoregressive, moving average with exogenous input (ARMAX [5]) structure, there exists a class of algorithms, inspired by the work of [31, 32], for directly identifying a nominal plant model and an accompanying frequency function bounding the influence of an additive perturbation to the nominal model. The impulse response sequence, $g_k$, of the true plant is assumed to be exponentially stable, $|g_k| \leq K \rho^{-k}$, for known $K > 0$ and $\rho > 1$. A bound, $\gamma > 0$, on the additive disturbance sequence, $v_k$, is also assumed known, $|v_k| \leq \gamma$. Empirical transfer function estimates [5] are generated at a uniformly distributed, finite set of frequencies in the interval $[0, 2\pi]$; the resultant estimates are interpolated to form a nominal FIR transfer function model. Knowledge of the bounding parameters $K$, $\rho$, and $\gamma$ is used to generate a uniform frequency bound on an additive perturbation which, relative to the nominal model, contains the frequency response of the true ARMAX plant. Alternative choices of windowing functions for the interpolation step of [32] are proposed in [33]; related algorithms are
discussed in [34] and [35]. A Lagrange interpolant is investigated in [36].

Identification tailored to deliver models for $\mu$-synthesis is discussed in [37]; $\mu$ is the so-called structured singular value (see [38] for a formal definition). A nominal plant model is assumed known as well as a complex block structure of norm-bounded perturbations. A shortcoming of the structured singular value framework is that current controller synthesis techniques are iterative and have no guarantee of convergence; furthermore, a convergent controller sequence is not guaranteed to be optimal. It is true that each iteration generates a decreasing upper bound on $\mu$ (not $\mu$ itself), which does improve robustness to time varying perturbations [39]. A significant amount of prior knowledge is assumed, the subsystem level descriptions and interconnection structure, which leads to the nominal plant model and the structure of the perturbation block.

Our approach to identification for robust control design is indirect, relative to [32], in the sense that our first step identifies a set of all feasible plant descriptions for an ARX model structure; this set is inferred from our prior plant assumptions and an observed data record. We have freedom of choice as to how this set is used to design a robust controller. If a nominal plant model is chosen to be parameterized by a specific element of the identified feasible parameter set, then the remainder of the set is a form of structured uncertainty.

There is a wealth of literature discussing robust control in the presence of structured (parametric) plant uncertainty; for example, see [40, 41, 42] for surveys. These parameter space methods are primarily analysis rather than synthesis tools; thus, they
are not well suited for general purpose control design. Much of the robustness analysis relies upon Kharitonov's Theorem [43] which provides a necessary and sufficient stability test for a family of interval polynomials in the Laplace variable, $s$. The Edge Theorem [44] enables a generalization to discrete time systems. An excellent summary of the evolution of Kharitonov-type results can be found in [45]. A limitation of parameter space methods is that they only examine the characteristic equation [40]; furthermore, assumptions pertaining to the interdependence of closed-loop characteristic equation coefficients can severely limit control theoretic interpretations.

Continuous time controller synthesis in the presence of transfer function coefficient intervals of uncertainty is addressed in [46] for the case when the controller structure (a transfer function with fixed numerator and denominator orders) is prespecified; by the author's own admission, the iterative algorithm is not trivial to implement and suffers from an exponential growth in computational complexity as a function of the number of uncertain parameters. The synthesis technique further provides no measure of optimality, only existence; a different choice of controller structure may provide superior robustness.

We wish to create an identification procedure which leads to an accessible robust control law. To be accessible, a control design should not require plant-specific tuning (as in loopshaping); rather, general results, applicable to a large class of systems and associated uncertainties, are desirable. A good robust control design should meet or approximate some measure of optimality. $H_{\infty}$ robust control design is both accessible and generates controllers which optimize robustness to uncertainties. We shall direct
our effort toward identification of the modeling information needed to pursue an existing $H_\infty$ robust control design procedure. It will be necessary to transform our structured plant perturbations into unstructured plant uncertainty; this requires a time to frequency domain transformation.

In related work, [6] discusses the employment of set membership identification techniques to overbound, with an ellipsoid, feasible sets of coefficients of (specified length) FIR models; the identification takes place in the presence of unknown (nonzero) initial conditions, the truncated tail of the (ARMAX) plant's true (stable) infinite-length impulse response (unmodeled dynamics), and an additive UBB disturbance/noise. A priori bounds are assumed known on the $l_1[0,\infty)$ norm of the impulse response, the $l_\infty[0,\infty)$ norm of the disturbance, and the $l_\infty(-\infty,0]$ norm of the input. A worst-case bound on the additive error due to the disturbance, the transients, and the restricted complexity modeling is derived and used in performing the set membership identification. The ellipsoid bound of the truncated Markov parameter set is transformed into an ellipse bounding the real and imaginary part of the truncated plant's frequency response at a given frequency; this bound is appropriately increased [6] to account for the unmodeled dynamics.

A similar problem is addressed in [47] for a disturbance free plant, initially at rest. A nominal FIR model is identified, but instead of a bound on the absolute sum of the truncated tail as in [6], a frequency domain magnitude bound on the additive plant perturbation (unmodeled dynamics) is assumed known. Disks of uncertainty, centered at the (complex) DFT coefficients of the least squares nominal (FIR) plant estimate,
are generated to contain the DFT coefficient of the true plant at the corresponding frequency bin.

A related problem is addressed in [48] wherein an ARMAX nominal model (additive disturbance is present) of known structure (number of poles and zeros) is also perturbed by additive unstructured uncertainty; a bound on the magnitude response of the additive perturbation as well as its derivative (with respect to frequency) is assumed known. A magnitude bound on the plant's true impulse response is assumed known for each sampling instant. The DFT of the additive disturbance sequence, a bounded set for the (initial) possible parameterization of the ARMAX structure, and a bound on the $l_\infty$ norm of the input are all assumed known. Using this wealth of prior information, a nominal transfer function model and a bound on the associated possible modeling error are identified.

Prior knowledge of the poles of the nominal model is exploited, in the presence of unstructured uncertainty and an UBB disturbance, to identify an $H_\infty$ compatible model in [49]; a disturbance free case, with parametric uncertainty added, is discussed in [50].

The recurrent theme in these identification techniques is to use the available \textit{a priori} information to identify a nominal parametric transfer function and a perturbation bounding function which, when combined with the nominal transfer function, contains the magnitude response of the true plant. This shall also be our focus for an ARX model type with no restricted complexity and with different \textit{a priori} experiment assumptions. The nature of the plant perturbation will also differ from the works of
[6], [47], [48], [49], and [50].

Our specific interest is in using a feasible set of ARX model parameters, bounded by an ellipsoid, in an $H_\infty$ robust control design. The OVE [1]; OBE [2], MPHA [14], and SM-WRLS [11] algorithms all deliver an ellipsoid, $E_N$, as a conservative estimate of the batch feasible parameter set, $\mathcal{F}^N$ of (2.12). For robust control modeling, we shall consider the LTI transfer function parameterized by the ellipsoid center, $\theta_N$, as a nominal plant description and the remaining set of parameters of $E_N$ as a quantification of the uncertainty in the nominal plant model. Within our class of assumed prior plant knowledge, we shall thus provide a means of identifying the modeling information required to perform $H_\infty$ robust control design.

### 3.2 Uncertainty Characterization

Although our inspiration is the desire to make use of the ellipsoid generated by the OVE Algorithm in an identification experiment, it is not necessary that our knowledge of $E_N$ originated from the OVE algorithm. If the true plant is an ARMAX structure with an UBB output disturbance, then convexity of the set of feasible parameters can be lost [51]; past disturbances become correlated with denominator coefficients in a set-estimation scheme. The output disturbance (of an ARMAX model) is translated into an equation error disturbance in [12] via a linear prediction of the denominator coefficients and subsequent use of conservative bounding relationships; this translation enables usage of the OVE algorithm for the estimation of feasible parameter sets for ARMAX structures.

When using the OVE algorithm as our identifier of $E_N$, we seek the least con-
servatism possible in the ellipsoidal overbounding of the convex set, \( \mathcal{F}^N \), of (2.12); this will not, however, necessarily translate to the minimal uncertainty in the set of frequency responses. For the discrete-time class of FIR models, [52] has shown that minimizing the hypervolume of a parameter space ellipsoid does not (in general) correspond to a complex image (frequency response uncertainty) of minimum area, at a specific frequency. A point frequency response set-theoretic estimator in the complex plane is shown to be an ill-posed problem, because a hyperplane cut in the parameter space corresponds, in general, to the entire complex plane (at any frequency of interest) [52]. A recursive ellipsoid algorithm was derived which chooses the parameter space update minimizing the area of the frequency image, an ellipse in the complex plane, at a specific frequency. If performed recursively, this minimization does not correspond to an optimal sequence of ellipsoid updates leading to a minimum area ellipse over all possible previous ellipsoid updates. If \( N \) point frequency response uncertainty estimates are sought, then \( N \) (recursive) parameter set estimation algorithms must be iterated (in parallel). As a new direction, the ellipses could be recursively intersected and the minimum area ellipse overbounding the intersection could be computed.

The results of [52] do not apply to the ARX structure, but do suggest that an update of the OVE algorithm does not, in general, minimize the area of the complex uncertainty image at an arbitrary frequency of interest. Later in our discussion, we will seek the minimization of the worst-case frequency response uncertainty. An ellipse of minimal area does not necessarily correspond to an ellipse of least worst-
case uncertainty (the semimajor axis length). We shall proceed with the minimization of parameter space uncertainty as an attempt to reduce worst-case frequency image uncertainty, over the entire domain of \([0, 2\pi]\).

Any attempt to represent \(\mathcal{F}^N\) explicitly (for example, via its vertices) would be computationally costly (see [16] and [17]); a subsequent overbounding ellipsoid of minimal hypervolume, if unique, would be computationally intensive to characterize. To approximate such a solution when using the OVE algorithm, we could use the data record, \(Z^N\), repeatedly, initializing each "run" with the final ellipsoid from the previous, until ellipsoid convergence has been achieved (within a specified tolerance) in a hypervolume sense or, alternatively, until the algorithm no longer updates for a "run" using the entire length of the given data record.

### 3.2.1 Interpreting Ellipsoidal Bounds

We do assume that the unknown scalar process is describable as an LTI difference equation subject to an additive disturbance, \(v_k\), entering the process as in the ARX model of equation (2.4) with \(\eta\), \(n\), and \(m\) known. Let \(\Delta \theta \equiv \theta - \theta_N\). We assume that the true process parameters, \(\theta^o = \Delta \theta^o + \theta_N\), are contained within a known ellipsoid,

\[
E_N = \{\theta \in \mathbb{R}^r : (\theta - \theta_N)'P_N^{-1}(\theta - \theta_N) \leq 1\}, \tag{3.1}
\]

where \(P_N = P_N^t > 0\) is real-valued. Observe that \(\theta \in E_N\) implies that

\[
\Delta \theta \in E_\Delta \equiv \{\Delta \theta \in \mathbb{R}^r : \Delta \theta'P_N^{-1}\Delta \theta \leq 1\}; \tag{3.2}
\]
$E_{\Delta}$ is an origin-centered ellipsoid bounding the parametric displacement, $\Delta \theta$. The boundary of $E_N$ is

$$\partial E_N = \{ \theta \in \mathbb{R}^r : (\theta - \theta_N)'P_N^{-1}(\theta - \theta_N) = 1 \}.$$  \hspace{1cm} (3.3)

If $\theta \in \partial E_N$, then

$$\Delta \theta \in \partial E_{\Delta} \equiv \{ \Delta \theta \in \mathbb{R}^r : \Delta \theta'P_{\Delta}^{-1}\Delta \theta = 1 \}. \hspace{1cm} (3.4)$$

Let

$$\theta_a \equiv [a_1 \cdots a_n]' \in \mathbb{R}^n \text{ and } \theta_b \equiv [b_0 \cdots b_m]' \in \mathbb{R}^{m+1}; \hspace{1cm} (3.5)$$

$\theta_a, \theta_b$ represent a partition of $\theta$ into "denominator" and "numerator" coefficients. We have

$$\Delta \theta = \theta - \theta_N \equiv [\Delta \theta_a' \Delta \theta_b]' \hspace{1cm} (3.6)$$

Let us denote the parameters corresponding to the partition of the ellipsoid center as

$$\theta_N = [\hat{\theta}_a' \hat{\theta}_b]' \text{ where}$$

$$\hat{\theta}_a' = [\hat{a}_1 \cdots \hat{a}_n] \text{ and } \hat{\theta}_b' = [\hat{b}_0 \cdots \hat{b}_m]. \hspace{1cm} (3.7)$$

If we define $\Delta a_i \equiv a_i - \hat{a}_i, \Delta b_i \equiv b_i - \hat{b}_i$, then we have

$$\Delta \theta_a' = [\Delta a_1 \cdots \Delta a_n] \text{ and } \Delta \theta_b' = [\Delta b_0 \cdots \Delta b_m]. \hspace{1cm} (3.8)$$

Let the $Z$-transform of an $l_2$ (square summable) sequence, $\{g_k\}$, be defined as

$$G(z) = \sum_{k=-\infty}^{\infty} g_k z^k; \hspace{1cm} \text{this is a change from Chapter 2 wherein } z^{-1} \text{ was a delay operator. With } v_k = 0, \text{ a candidate transfer function from } u \text{ to } y \text{ is parameterized as} \hspace{1cm} (3.9)$$

$$G(z, \theta) \equiv \frac{B(z, \theta_b)}{A(z, \theta_a)} = \frac{z^n(b_0 + b_1 z + \cdots + b_m z^n)}{1 + a_1 z + \cdots + a_n z^n}. \hspace{1cm} (3.9)$$
for any \( \theta \in E_N \); the true plant is

$$G^\theta(z) \equiv G(z, \theta^\theta) = \frac{B(z, \theta^\theta)}{A(z, \theta^\theta)} \equiv \frac{B^\theta(z)}{A^\theta(z)}.$$  \hspace{1cm} (3.10)

Further define

$$Z_k(z) \equiv [1 \ z \ \cdots \ z^k]^\top ,$$  \hspace{1cm} (3.11)

and let

$$B_0(z) \equiv z^n \theta_0^\top Z_m(z) \quad \text{and} \quad A_0(z) \equiv [1 \ \theta_0^\top] Z_n(z)$$  \hspace{1cm} (3.12)

be the respective numerator and denominator polynomials of the nominal plant, and

$$\Delta_B(z) \equiv z^n \theta_0^\top Z_m(z) \quad \text{and} \quad \Delta_A(z) \equiv z \Delta \theta_n^\top Z_{n-1}(z)$$  \hspace{1cm} (3.13)

be the allowed perturbations to the numerator and denominator, respectively, of the nominal plant model; we shall suppress the arguments of selected functions, provided the meaning remains clear. The feasible transfer function set is

$$\mathcal{G} \equiv \left\{ G(z, \theta) = \frac{B(z, \theta_0^\top)}{A(z, \theta_0^\top)} = \frac{z^n \theta_0^\top Z_m(z)}{1 + z \theta_n^\top Z_{n-1}(z)} : \theta \in E_N, \ B \text{ and } A \text{ coprime} \right\}. \hspace{1cm} (3.14)$$

The restriction that \( A(z, \theta_0^\top) \) and \( B(z, \theta_0^\top) \) are coprime is required, because the modeling orders are assumed to have been chosen correctly. See Appendix A for a summary of the notation introduced here and throughout the dissertation.

### 3.2.2 Robust Stabilization

A function, \( G(z) \), is in the set of \( \mathcal{H}_\infty \) functions if it is analytic on the (open) unit disk of the complex plane, \( \mathbb{D} \equiv \{ z \in \mathbb{C} : |z| < 1 \} \), and satisfies

$$\|G\|_\infty \equiv \sup_{r \in (0,1)} \max_{\theta \in [0,2\pi]} |G(re^{j\theta})| < \infty. \hspace{1cm} (3.15)$$
We define $\text{RH}_\infty$ as the subset of $\text{H}_\infty$ functions describable as the ratio of finite degree polynomials in $z$, with real-valued coefficients. Any finite degree polynomial in $z$ is an $\text{RH}_\infty$ function (polynomials are entire and are bounded on the domain $\{z \in \mathbb{C} : |z| = 1\}$); hence, $A_o$ and $B_o$ of (3.12) and $\Delta_A$ and $\Delta_H$ of (3.13) are all $\text{RH}_\infty$ functions.

![Figure 4: Standard Feedback Configuration](image)

For robust stability, we require that all bounded energy signals, external to the closed-loop system, give rise to finite energy signals internal to the feedback configuration. In Figure 4, the controller is denoted as $C(z)$, $r_k$ is the reference signal, $e_k$ is the measured tracking error, $d_k$ is an actuator disturbance, and $n_k$ is any measurement noise. For the standard feedback configuration, $v_k$ reflects the effect of the plant disturbance as seen at the output, corresponding to an ARMAX model; an equivalent definition of robust stability is the following (see e.g. [53]):

**Definition 1** A feedback controller, $C$, robustly stabilizes $G$ of (3.14) if $G \in \mathcal{G}$ implies $S$, $1-S$, $CS$, and $GS$ are transfer functions in $\text{RH}_\infty$, where $S \equiv (1+GC)^{-1}$ is the sensitivity function.
If we have used the OVE algorithm to identify an ellipsoid overbounding the parameters of an ARX model, then, relative to the standard feedback configuration, the "output" disturbance, $v_k$, must be relocated as shown in Figure 5. We must ensure that the transfer functions from $v$ to $c$, $u$, and $y$ are in $\mathbb{RH}_\infty$. It can be shown that

$$
\frac{u}{v} = C \frac{e}{v} \quad \text{and} \quad \frac{y}{v} = -\frac{e}{v} = \frac{S}{A}
$$

which are all in $\mathbb{RH}_\infty$ provided that $S$, $GS$, $CS$, and $GCS = 1 - S$ are in $\mathbb{RH}_\infty$ and that no unstable factors of $A$ are cleared when the product $G$ of (3.9) is formed; thus, Definition 1 still applies provided that $A$ and $B$ are coprime on $\mathbb{RH}_\infty$. We assume that modeling orders have been chosen correctly; therefore, any $\theta \in E_N$ which parameterizes $A(z)$ and $B(z)$, of (3.9), that are not coprime, is not a feasible plant description. On the other hand, if we have prior knowledge of an ellipsoidal bound on the coefficients of an ARMAX model, then Definition 1 applies directly without qualification.

We seek to exploit our knowledge of $E_N$ in a robust control design. Our objec-
tive is to extract the information of $E_N$ to form a weighting function that bounds allowed perturbations to the nominal plant of (3.12); such a weighting function will be used to enable characterization of solutions to the robust stabilization problem.

An unaddressed issue is our choice of nominal model from the ellipsoid set; the ellipsoid center may not be the best choice. For a specific ellipsoid, a different choice of nominal model could lead to a smaller (in the sense of any $p$ norm) uncertainty weighting function. This is an issue of future research interest. We shall proceed with the assumption that, due to symmetry, the ellipsoid center is a reasonable choice to parameterize a nominal model.

### 3.2.3 Ellipsoid Information Content

Although it is a digression, we must prove a result concerning the information contained within an ellipsoid, which is needed in our investigation. Recall that an ellipsoid is represented as a quadratic form as in equation (3.1) wherein $P_N$ is a real positive definite, symmetric matrix. Partition $P_N$ for any $l \in \{1, 2, \cdots, r - 1\}$ as

$$
P_N = \begin{bmatrix}
P_1 & P_3 \\
P_3 & P_2
\end{bmatrix}
$$

where $P_1 \in \mathbb{R}^{l \times l}$ and $P_2 \in \mathbb{R}^{(r-l) \times (r-l)}$ are positive definite (principal minors of $P_N$) symmetric matrices. Form the partition $\Delta \theta' = [\Delta \theta_1', \Delta \theta_2']$, wherein $\Delta \theta_1 \in \mathbb{R}^l$ and $\Delta \theta_2 \in \mathbb{R}^{r-l}$.

The following lemma states that for any partition of $P_N$ as in (3.16), the feasible parameter set of the ellipsoid corresponding to $P_N$ (3.1) can be conservatively decomposed into two new ellipsoids, one of dimension $l$ and the other of dimension $r - l$;
these subellipsoids characterize the allowed perturbations of $\Delta \theta_1$ decoupled from the allowed perturbations of $\Delta \theta_2$.

**Lemma 3** If $\Delta \theta' P_N^{-1} \Delta \theta \leq 1$ for $P_N = P_N' > 0$ as in equation (3.16), then

$$
\Delta \theta_1' P_1^{-1} \Delta \theta_1 \leq 1 \quad \text{and} \quad \Delta \theta_2' P_2^{-1} \Delta \theta_2 \leq 1.
$$

**(Proof):** The inverse of $P_N$ can be written as in [20]

$$
P_N^{-1} = 
\begin{bmatrix}
(P_1 - P_3 P_2^{-1} P_3)^{-1} & P_1^{-1} P_3 (P_3 P_1^{-1} P_3 - P_2)^{-1} \\
(P_3 P_1^{-1} P_3 - P_2)^{-1} P_3 P_1^{-1} & (P_2 - P_3 P_1^{-1} P_3)^{-1}
\end{bmatrix}
$$

with

$$(P_1 - P_3 P_2^{-1} P_3)^{-1} = P_1^{-1} + P_1^{-1} P_3 (P_2 - P_3 P_1^{-1} P_3)^{-1} P_3 P_1^{-1}.$$

Following some manipulation, we can write

$$
\Delta \theta' P_N^{-1} \Delta \theta = \Delta \theta_1' P_1^{-1} \Delta \theta_1 + [\Delta \theta_2 - P_3' P_1^{-1} \Delta \theta_1] (P_2 - P_3' P_1^{-1} P_3)^{-1} [\Delta \theta_2 - P_3' P_1^{-1} \Delta \theta_1].
$$

Because $\Delta \theta' P_N^{-1} \Delta \theta \leq 1$, if we can show that the matrix $(P_2 - P_3' P_1^{-1} P_3)^{-1}$ is positive definite, then we have $\Delta \theta_1' P_1^{-1} \Delta \theta_1 \leq 1$ as desired. For any nonsingular $A \in \mathbb{R}^{r \times r}$, we have $A' P_N A > 0$. Consider the nonsingular matrix

$$
A = 
\begin{bmatrix}
I & -P_1^{-1} P_3 \\
0 & I
\end{bmatrix}
$$

for which we have

$$
A' P_N A = 
\begin{bmatrix}
P_1 & 0 \\
0 & P_2 - P_3' P_1^{-1} P_3
\end{bmatrix}
> 0.
$$

(3.18)
Equation (3.18) requires that the principal minor $P_2 - P_3 P_1^{-1} P_3$ (and consequently its inverse) be positive definite. We can similarly show that $\Delta \theta_2 P_2^{-1} \Delta \theta_2 \leq 1$. □

To illustrate the conservatism introduced via application of Lemma 3, consider the ellipse

$$[\Delta b \Delta a] \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \Delta b \\ \Delta a \end{bmatrix} \leq 1. \quad (3.19)$$

Following a partition with $l = 1$ in (3.16), we have subellipsoid boundary values of $(\Delta b, \Delta a) \in \{(-2,-1), (-2,1), (2,-1), (2,1)\}$, which all fail to satisfy (3.19) (are not contained within the original ellipse). 

### 3.3 Robust Stabilization for Additive Plant Perturbations

The characterization of uncertainty as an additive perturbation to the nominal plant is versatile; many types of $H_\infty$ robust control problems are solvable with the knowledge of a magnitude bound on the perturbation. We can exploit knowledge of the nominal plant when generating a bound on such perturbations. Use of an additive plant perturbation uncertainty characterization does require the knowledge (or presumption thereof) that each plant, formed as an allowed (additive) perturbation to the nominal plant, has the same number of unstable poles as the nominal plant. In the absence of such knowledge, we address the characterization of modeling uncertainty as a coprime factor perturbation, to a nominal plant, in Section 3.5.

We wish to characterize the parametric uncertainty of the ellipsoid as an additive perturbation, $\Delta G$, to the nominal plant model, $G_0$. We recall that $G$ of (3.14) characterizes the set of all feasible plant descriptions. Because the true plant, $G^\circ$ of (3.10),
is unknown, we must allow the possibility that any $G \in \mathcal{G}$, of (3.14), is the true plant; thus,

$$G(z, \theta) = G_0(z) + \Delta_{G}(z, \theta), \tag{3.20}$$

where $G_0(z) = B_0(z)/A_0(z)$ and $A_\theta$ and $B_\theta$ are from (3.12). Assume that $G_\theta$ is analytic on the unit circle of the complex plane, $\partial \mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. As a first step towards characterizing the set of robustly stabilizing controllers, we shall identify a minimum phase weighting function, $W_\theta \in \mathbb{R}H_\infty$, such that

$$\sup_{\theta \in \mathbb{R}_N} |\Delta_{G}(e^{j\Omega}, \theta)| < |W_\theta(e^{j\Omega})|, \quad \Omega \in [0, 2\pi]; \tag{3.21}$$

a transfer function, $W(z) \in \mathbb{R}H_\infty$, is minimum phase if $W(\hat{z}) = 0$ implies $|\hat{z}| \geq 1$. If for some $\bar{\theta} \in \mathbb{R}_N$, $G(e^{j\Omega}, \bar{\theta})$ had a pole in $\partial \mathbb{D}$, then $|G(e^{j\Omega}, \bar{\theta})|$ is unbounded for some $\Omega \in [0, 2\pi]$; therefore, $|\Delta_{G}(e^{j\Omega}, \bar{\theta})|$ is unbounded. If it is known that $\mathcal{G}$ is free of plants possessing unit circle poles, a $W_\theta \in \mathbb{R}H_\infty$ satisfying (3.21) exists.

It is possible that the set $\mathcal{G}$ of (3.14) contains no plants which possess poles in $\partial \mathbb{D}$. As a preliminary step, we can easily check whether $A_\theta(z)$ of (3.12) has any zeros on $\partial \mathbb{D}$. If the nominal plant has no poles in $\partial \mathbb{D}$ (as assumed), then there exists a neighborhood about $\hat{\theta}_N$ (3.6) that parameterizes transfer functions possessing no poles in $\partial \mathbb{D}$. This is a consequence of the fact that the $n$ roots of a polynomial of degree $n$ are a continuous function of the coefficients (this can be shown as a corollary to Rouché's Theorem [54]). Thus, if the nominal plant has no poles in $\partial \mathbb{D}$ and $\mathbb{R}_N$ can be contained within a $r$-dimensional ball of sufficiently small radius, then the set $\mathcal{G}$ of (3.14) does not possess any poles in $\partial \mathbb{D}$.

If $\mathcal{G}$ contains no plants with unit circle poles, then the number of unstable poles
of each plant within \( \mathcal{G} \) is the same. The loci of roots of the denominator polynomials, \( A(z) \) of (3.9), are a continuous function of the coefficients, \( \theta_a \). If the number of unstable poles of the plant \( G(z) \) were to change relative to the nominal plant \( G_o(z) \), it would be necessary that there exist a plant in \( \mathcal{G} \) with poles in \( \partial \mathcal{D} \); this is true because at least one root locus must continuously transition across the boundary, \( \partial \mathcal{D} \). When the number of unstable poles of each transfer function within \( \mathcal{G} \) is the same and \( W_a \in \mathcal{RH}_\infty \) satisfies (3.21), then a controller, \( C \in \mathcal{RH}_\infty \), that stabilizes \( G_o \), robustly stabilizes all plants in \( \mathcal{G} \) if \( 55 \)

\[
\gamma_a \equiv ||W_a C(1 + G_o C)^{-1}||_\infty < 1 \quad (3.22)
\]

where \( || \cdot ||_\infty \) is defined in (3.15); this sufficient condition, for \( C \) to be a robustly stabilizing controller, is also necessary in the event that there is no conservatism in the generation of \( W_a \).

### 3.3.1 Frequency Dependent Worst Case Magnitude Bound

We see from (3.20) that \( \Delta_G = G - G_o \). At a given normalized radian frequency, \( \Omega_l \in [0, 2\pi] \), maximization of \(|\Delta_G(e^{i\Omega_l})|\) as in (3.21) is equivalent to the maximization of \(|\Delta_G(e^{i\Omega_l})|^2\). Recall that \( \text{Re}(\cdot) \) denotes the real part of the indicated argument. Let

\[
f(\omega, \Omega_l) \equiv |\Delta_G|^2 = |G|^2 - 2\text{Re}(\hat{G}_o G) + |G_o|^2 \quad (3.23)
\]

and define

\[
g(\theta) \equiv (\theta - \theta_N)' P_N^{-1}(\theta - \theta_N) - 1 \quad (3.24)
\]

where \( P_N \) is the ellipsoid matrix of the identified ellipsoid, \( E_N \), given in (3.1). In comparison to (3.21), we see that \( \theta \in E_N \) implies \( g(\theta) \leq 0 \); consequently, we equivalently
seek to

\[
\begin{align*}
\text{maximize } & \ f(\theta, \Omega) \\
\text{subject to } & \ g(\theta) \leq 0.
\end{align*}
\]

for each \( \Omega \in [0, 2\pi] \).

Our strategy to attack this optimization shall be to seek a solution to (3.25) for a finite set of frequencies in \([0, 2\pi]\) and interpolate the resulting set. We arbitrarily choose the normalized radian frequency set

\[
S_M = \{ \Omega_i = \frac{2\pi l}{M} : l \in \{0, 1, \ldots, M-1\} \}
\]

for some positive integer, \( M \); the elements of \( S_M \) are the arguments of the \( M \) roots of unity. We are, in effect, taking uniformly spaced samples of the magnitude of a continuous-valued discrete-time Fourier transform which overbounds the additive perturbation in magnitude at each frequency in \([0, 2\pi]\). To avoid time domain aliasing, the worst-case weighting function, characterized by a continuum of solutions to (3.25), must be representable as an FIR filter, of length \( L \leq M \) [56]; this gives insight into the choice of an adequate number of uniform samples over the frequency interval \([0, 2\pi]\). The underlying nature of the plant perturbation will ultimately determine whether \( M \) has been chosen adequately; this issue will be revisited, following a discussion of interpolation techniques.

We recall from (3.14) that

\[
G(e^{j\Omega}, \theta) = \frac{\theta_0 e^{j\Omega i} Z_m(e^{j\Omega i})}{1 + \theta_0 e^{j\Omega i} Z_{n-1}(e^{j\Omega i})}
\]

We can easily solve for the complex constant, \( G_0 = G(e^{j\Omega}, \theta_N) \) of (3.23), using (3.27). Nonlinear programming techniques (see for example [57]) may be employed to solve
the constrained optimization (3.25) (at each $\Omega_i \in S_M$) for the unknown $\theta_i^*$, such that

$$\left(W_i^*\right)^2 \equiv f(\theta_i^*, \Omega_i) \geq f(\theta, \Omega_i) \quad \forall \theta \in E_N$$  \hspace{1cm} (3.28)

(that is, for each element of \{ $\theta \in \mathbb{R}^r : g(\theta) \leq 0$ \}). For any $\epsilon > 0$, $(1 + \epsilon)W_i^*$ is a sufficient magnitude bound in (3.21), at the normalized frequency, $\Omega_i$. The interpolation of such magnitude samples will be discussed in Section 3.6.

### 3.3.2 Limitations of the Additive Perturbation Characterization

The large class of type 1 (or greater) plants is, in general, excluded from consideration as a nominal plant subjected to an additive plant perturbation, because the true plant (assumed to be contained within $E_N$) has (at least) one pole at $z = 1 \in \partial \mathbb{D}$. This exclusion generalizes to type 2 and greater plants. In the unlikely event the true (type one or greater) plant, parameterized by $\theta^0$, were to reside on $\partial E_N$, the exclusion of consideration of the additive plant perturbation may not be necessary.

As previously mentioned, if $G_0$ is free of poles in $\partial \mathbb{D}$, then it is possible that all plants in $\mathcal{G}$ are also. If one decides that $E_N$ is sufficiently small in hypervolume and proceeds to solve for the worst-case additive perturbation, there is a risk that a pole in $\partial \mathbb{D}$, for a plant within $\mathcal{G}$, remains undetected. The frequency set, $S_M$ of (3.26), used in sampling the weighting function will most likely not include the exact frequency of a present unit circle pole, but the effect should be qualitatively obvious as a spike in the magnitude of the samples adjacent to the pole frequency (although the possible existence of nearby transfer function zeros could diminish this effect).
3.3.3 Kharitonov Analysis

There is no general analytic procedure for checking an arbitrary ellipsoid, $E_N$, and determining whether the number of poles in $D \cup \partial D$ is the same for each element of the corresponding transfer function set, $\mathcal{G}$. When the nominal plant, $G_{o}$, is in $\mathbb{RH}_\infty$, it is possible that all plants within $\mathcal{G}$ are also $\mathbb{RH}_\infty$ functions. If such a condition can be proven to exist, then the additive perturbation weighting of (3.21) is well-posed. The nominal plant is in $\mathbb{RH}_\infty$ when the characteristic polynomial, $A_o$ of (3.12), is Schur (contains no roots within the closed unit disk). Kharitonov analysis can be used to establish sufficient conditions determining whether all elements of the set of denominators, $A(z, \theta_a)$ of (3.9) are Schur. We apply Lemma 3 with $l = n$ for the partition of equation (3.6) to decouple the parameters, $\theta_b$, of the numerator polynomial from those of the denominator polynomial, $\theta_a$. Define the subellipsoids

$$E_a \equiv \{ \theta_a \in \mathbb{R}^n : (\theta_a - \hat{\theta}_a)'P_a^{-1}(\theta_a - \hat{\theta}_a) \leq 1 \}$$

(3.29)

and

$$E_b \equiv \{ \theta_b \in \mathbb{R}^{m+1} : (\theta_b - \hat{\theta}_b)'P_b^{-1}(\theta_b - \hat{\theta}_b) \leq 1 \}$$

(3.30)

where $P_a = P_1$ and $P_b = P_2$ are as in (3.16) for the partition of (3.6).

The set of feasible characteristic equations,

$$\{ A(z, \theta_a) = 1 + a_1 z + \cdots + a_n z^n = 1 + \theta_a' Z_{n-1} : \theta_a \in E_a \}$$

(3.31)

are polynomials in $z$. A conformal mapping, $z = \frac{s}{s+1}$ (mapping $D$ onto the open right half $s$-plane, $\{ s \in \mathbb{C} : \text{Re}(s) > 0 \}$) of (3.31) into a set of polynomials in $s$ can be accomplished via a linear transformation [58], $\mathcal{T}$, from the coefficients of the
polynomials in $z$, $a \equiv [\theta_a]'$, to the coefficients, $\hat{a} = [\hat{a}_n \cdots \hat{a}_0]'$, of the corresponding polynomials,

$$A_c(s, \hat{a}) \equiv (s + 1)^n A\left(\frac{s - 1}{s + 1}\right) = \hat{a}_n s^n + \hat{a}_{n-1} s^{n-1} + \cdots + \hat{a}_0.$$  (3.32)

The $(n+1) \times (n+1)$ linear transformation, $T$, appearing in [58], has the $i,j$ element given as

$$t_{ij} = (\sqrt{2})^{-n} \sum_{k=0}^{i-1} (-1)^k \binom{j - 1}{k} \binom{n + 1 - j}{i - k - 1}.$$  (3.33)

The coefficients of the respective polynomials are related as

$$\hat{a} = Ta \quad \text{and} \quad a = T\hat{a}.$$  (3.34)

Let $\hat{a}^c \equiv T[1 \; \hat{\theta}_a]'$; this is the coefficient set corresponding to the mapping of the coefficients of the nominal plant denominator, $A_0$ of (3.12). Let $P_a$ be the subellipsoid matrix of (3.29). Examine the set

$$E_a \equiv \{ \hat{a} \in \mathbb{R}^{n+1} : (\hat{a} - \hat{a}^c)' P_a^{-1} (\hat{a} - \hat{a}^c) \leq 1 \},$$  (3.35)

where

$$P_a \equiv T \begin{bmatrix} \epsilon & 0 \\ 0 & P_b \end{bmatrix} T' = P_a'.$$  (3.36)

For any $\epsilon > 0$, $P_a$ is positive definite, because (3.34) implies that $T = T^{-1}$; that is, $T$ is of full rank. For any choice of $\epsilon > 0$, the set (3.35) is an ellipsoid in the space corresponding to the parameters, $\hat{a}$. The set of (3.35) corresponds exactly to the range of the parameters of the ellipsoid, $E_a$ of (3.29), under the linear transformation $a = T^{-1} \hat{a}$ when $\epsilon = 0$. There is no uncertainty in the lead coefficient, 1, of $A_0$ of (3.12); choosing an $\epsilon > 0$ corresponds to introducing uncertainty in this lead coefficient.
(decoupled from uncertainty in the parameters of \( \theta_a \)). Introducing uncertainty where none was previously present adds conservatism to the set, \( E_a \), yet permits \( E_a \) to be interpreted as an ellipsoid of parameters in \( \hat{\alpha} \), centered at \( \hat{\alpha}^c \). We shall arbitrarily set \( \epsilon \) equal to the minimum diagonal element of the final (composite) ellipsoid matrix, \( P_N \) of (3.1).

Let \( p_{ii} \) be the \( i \)th diagonal element of \( P_a \) and \( \hat{\alpha}_i \) be the \( i \)th element of the ellipsoid center, \( \hat{\alpha}^c \), for each \( i \in \{1, \cdots, n+1\} \). Because \( E_a \) is (now) an ellipsoid, we can (conservatively) extract intervals of uncertainty for the coefficients, \( \hat{\alpha}_i \), via recursive application of Lemma 3 upon \( P_a \); treat \( P_a \) as \( P_N \) in (3.16) and use \( l = 1 \) to establish that

\[
(\hat{\alpha}_i - \hat{\alpha}^c_i)^2 \leq p_{ii}.
\]

The remaining \( P_2 \) of (3.16) defines a new ellipsoid; interpret this \( P_2 \) as \( P_N \) in (3.16) and once again apply Lemma 3 for \( l = 1 \). This recursive application of Lemma 3 establishes that

\[
\hat{\alpha}_i \in [\hat{\alpha}^c_i - \sqrt{p_{ii}}, \hat{\alpha}^c_i + \sqrt{p_{ii}}]. \tag{3.37}
\]

The uncertainty intervals of (3.37) represent the worst-case variation of each individual parameter; the conservatism introduced into the parametric uncertainty can be visualized geometrically as overbounding the ellipsoid, \( E_a \), with the smallest (in hypervolume) possible orthotope, which is aligned with the parameter axes. Although conservatism has been introduced both in the introduction of \( \epsilon > 0 \) in (3.36) and in the extraction of the parameter uncertainty intervals of (3.37), Kharitonov analysis (see e.g. [59]) provides sufficient conditions for the the set of polynomials parameterized by
$E_\alpha$ to be Hurwitz (free of roots in the closed right half plane, \( \{ s \in \mathbb{C} : \text{Re}(s) \geq 0 \} \)). The Kharitonov test requires that none of the intervals of (3.37) contain the zero element; if the four Kharitonov polynomials constructed from the endpoints of the parameter uncertainty intervals of (3.37) are Hurwitz, then the set of polynomials

\[
\{ A_\alpha(s, \hat{\alpha}) : \hat{\alpha} \in E_\alpha \} \tag{3.38}
\]

are all Hurwitz. If the elements of (3.38) are all Hurwitz, then all elements of the set of polynomials

\[
\{ A(z, \theta_\alpha) : \theta_\alpha \in E_\alpha \} \tag{3.39}
\]

are Schur; this is sufficient to ensure that the number of unstable poles for each plant within $\mathcal{G}$ is identical and enables use of the additive plant perturbation as a representation of the ellipsoidally bounded parametric uncertainty.

3.3.4 Optimal Control and Robust Performance

For $W_\alpha$ satisfying (3.21) given, if there exists a robustly stabilizing controller, $C_\gamma$, achieving $\gamma_\alpha < 1$ in (3.22), then we have increased robustness to uncertainty; $C_\gamma$ is a robustly stabilizing controller for the additive perturbation weighting function, $W_\alpha/\gamma_\alpha$. If an optimal controller exists, define it as

\[
C_{\text{opt}} = \text{arg} \left( \inf_{C \text{ stab } G_\alpha} \| W_\alpha C (1 + G_\alpha C)^{-1} \|_\infty \equiv \gamma_\alpha^* \leq 1 \right), \tag{3.40}
\]

where the infimum is taken over all controllers stabilizing the nominal plant, $G_\alpha$. In the event $C_{\text{opt}}$ does not exist, it may be possible to find a suboptimal controller which comes arbitrarily close to achieving $\gamma_\alpha^*$. 
Recall that the sensitivity function is defined as $S = (1 + GC)^{-1}$; let $S_0 = (1 + G_0 C)^{-1}$. Once the nominal plant and uncertainty weighting function, bounding an additive plant perturbation, have been identified, other types of robust control problems can also be solved. For example, if a stable weighting function, $W_1$, with $W_1$ and $W_1^{-1} \in RH_{\infty}$, is specified as a desired bound on the magnitude of the sensitivity function,

$$|S(e^{j\Omega})| \leq |W_1(e^{j\Omega})| \text{ almost everywhere } \Omega \in [0, 2\pi], \quad (3.41)$$

then a two-block $H_{\infty}$ problem can be solved which provides sufficient conditions for the existence of robustly stabilizing controllers which simultaneously achieve the performance specification of (3.41). This robust performance problem is solvable (see e.g. [53]) if there exists a controller, $C$, which is a ratio of $H_{\infty}$ functions and stabilizes the nominal plant, $G_0(z) \equiv G(z, \theta_N)$, satisfying

$$\left\| \begin{bmatrix} W_1^{-1} S_0 \\ W_1 C S_0 \end{bmatrix} \right\|_\infty \leq \frac{1}{\sqrt{2}}. \quad (3.42)$$

### 3.4 Weighting Function Identification for FIR Plants

The OVE algorithm can also be used to estimate a set of feasible coefficients for an FIR model, a degenerate example of an ARX model. The structure of (2.1) still holds with appropriate modification to the regressor,

$$\phi_k = [u_{k-\eta} \cdots u_{k-m-\eta}]',$$

and to the parameter vector,

$$\theta = \theta_b = [b_0 \cdots b_m]' \quad (3.43)$$
The number of unknown parameters is \( r = m + 1 \). The OVE algorithm uses \textit{a priori} knowledge of \( m \), \( \gamma \), and \( \gamma \geq |\nu_k| \) to generate an ellipsoid,

\[
E_N = \{ \theta_b \in \mathbb{R}^{m+1} : (\theta_b - \theta_N)P_N^{-1}(\theta_b - \theta_N) \leq 1 \},
\]

which bounds the set of \( \theta_b \) that are consistent with the data observed in the identification experiment. The final ellipsoid, \( E_N = E_b \) (see (3.30)), contains the true parameter vector, \( \theta_b = [b_0 \cdots b_m]' \), when the algorithm is initialized with \( E_0 \) containing (the unknown) \( \theta_b \) (a sufficient [1], but not always necessary [23] condition). The final ellipsoid center, \( \theta_N = \hat{\theta}_b = [\hat{b}_0 \cdots \hat{b}_m]' \), is chosen as our nominal plant parameterization,

\[
B_o(z) \equiv B(z, \hat{\theta}_b) = \hat{b}_0 z^n + \cdots + \hat{b}_m z^{m+n}
\]

\[
= \hat{\theta}_b' z^n Z_m(z);
\]  

(3.45)

\( Z_m(z) \) is as defined in (3.11). The true plant, \( B^o(z) = B(z, \theta_b) \) can now be expressed as

\[
B^o(z) = B_o(z) + \Delta_B^o(z)
\]

(3.46)

where \( \Delta_B^o(z) \equiv \Delta_B(z, \theta_b) \) and, for \( \theta_b \in E_b \),

\[
\Delta_B(z, \theta_b) \equiv B(z, \theta_b) - B_o(z)
\]

\[
= (\theta_b - \hat{\theta}_b)' z^n Z_m(z).
\]

(3.47)

The form of (3.46) is of an additive perturbation to a nominal plant. For any FIR plant, parameterized by \( E_b \), no control action is needed for stabilization. To enable characterization of solutions to other types of \( H_\infty \) control problems, for example, the
robust performance problem of (3.42), we must identify a weighting function on the
additive plant perturbation, \( \Delta_B(z, \theta_b) \) of (3.47).

### 3.4.1 Frequency Dependent Worst-Case Magnitude Bound

As in (3.21), we seek to identify a minimum phase weighting function, \( W_a \in \mathbb{RH}_\infty \),
such that

\[
\sup_{\theta_b \in E_b} |\Delta_B(e^{j\Omega}, \theta_b)| < |W_a(e^{j\Omega})| \quad \forall \Omega \in [0, 2\pi].
\] (3.48)

Our strategy to attack this optimization is again to seek worst-case bounds for \( W_a \), in (3.48), at the normalized radian frequency set, \( S_M \) of (3.26), and interpolate the resulting set. At a given frequency, \( \Omega_i \in [0, 2\pi] \), the optimization of (3.48) is equivalent to solving

\[
B_i^* \equiv \sup_{\theta_b \in E_b} |\Delta_B(e^{j\Omega_i}, \theta_b)|^2,
\] (3.49)

for the worst-case parameter vector, \( \theta_b^* \), the argument of the extremum of (3.49).

Recall that \( \Delta \theta_b \equiv \theta_b - \hat{\theta}_b \). The condition of \( \theta_b \in E_b \) is equivalent to \( \Delta \theta_b \in E_\Delta = \{ \Delta \theta_b \in \mathbb{R}^{m+1} : \Delta \theta_b P_\theta^{-1} \Delta \theta_b^t \leq 1 \} \) (see (3.30)). The additive perturbation can now be expressed as

\[
\Delta_B(e^{j\Omega_i}, \Delta \theta_b) \equiv (\Delta b_0 z^n + \Delta b_1 z^{n+1} + \cdots + \Delta b_m z^{n+m})|_{z=e^{j\Omega_i}}
\]

\[
\equiv e^{j\Omega_i} \Delta \theta_b^t Z_m(e^{j\Omega_i}).
\] (3.50)

Observe that \( |\Delta_B(z, \theta)| \) is independent of \( \eta \) for \( z \in \partial \mathbb{D} \). For convenience, we suppress the explicit dependence of \( \Delta_B \) in (3.47) upon \( \Delta \theta_b \) and \( e^{j\Omega_i} \). Let \( (\cdot)^H \) denote the complex conjugate of the transpose of the indicated matrix. Let \( B_i \equiv |\Delta_B|^2 = \)
\[ \Delta_B \Delta^H_B = \Delta \theta_b^t Z_m Z_m^H \Delta \theta_b \text{ and} \]

\[ Q \equiv Z_m Z_m^H; \quad (3.51) \]

\( Q = Q^H \) is a rank one, Hermitian matrix. It follows that

\[ B_l = \Delta \theta_b^t Q \Delta \theta_b \geq 0, \quad (3.52) \]

which is independent of \( \eta \).

The argument, \( \theta_b^* \), of the extremum of (3.49) lies in \( \partial E_b \), the boundary of the ellipsoid; that is, \( \Delta \theta_b^* \in \partial E_\Delta = \{ \Delta \theta_b \in \mathbb{R}^{m+1} : \Delta \theta_b^t P_b^{-1} \Delta \theta_b = 1 \} \). If \( \Delta \theta_b^* \notin \partial E_\Delta \), then traversal along the subspace of the vector from the origin, \( \Delta \theta_b = 0 \), passing through \( \Delta \theta_b^* \neq 0 \) (\( \Delta \theta_b = 0 \) minimizes \( B_l \) and need not be considered) outward to the ellipsoid surface \( \partial E_\Delta \), corresponds to an upward scaling of the magnitude of the inner product of (3.50), by a real-valued scalar; this upward scaling results in a larger \( B_l^* \), which is a contradiction.

Taking into account the relation of (3.52), the problem statement of (3.49) is equivalent to the constrained optimization

\[
\text{maximize } B_l = \Delta \theta_b^t Q \Delta \theta_b \\
\text{subject to } \Delta \theta_b^t P_b^{-1} \Delta \theta_b = 1. \quad (3.53)
\]

As a first step in seeking a solution to (3.53), we form the Lagrangian,

\[ \mathcal{L} \equiv \Delta \theta_b^t Q \Delta \theta_b + \lambda [\Delta \theta_b^t P_b^{-1} \Delta \theta_b - 1]. \]

Let \( \nabla_v (\cdot) \) denote the vector-valued gradient, of the indicated scalar argument, with respect to the vector \( v \). The first order necessary conditions for an extremum are (see e.g. [57]):
1. \[ \nabla_\lambda \mathcal{L} = 0 = \Delta \theta_b P_b^{-1} \Delta \theta_b - 1 \]

2. \[ \nabla_{\Delta \theta_b} \mathcal{L} = 0 = [Q + Q' + 2\lambda P_b^{-1}] \Delta \theta_b \]

where \( P_b \) is the positive definite, symmetric (ellipsoid) matrix. Define the symmetric matrix

\[ \tilde{Q} \equiv \frac{1}{2}(Q + Q'). \quad (3.54) \]

Because \( P_b \) is of full rank, we can rewrite condition 2 as

\[ [P_b \tilde{Q} + \lambda I] \Delta \theta_b = 0. \quad (3.55) \]

Because \( Q \) of (3.51) is a conjugate-symmetric (Hermitian) matrix, we have \( \tilde{Q} = \text{Re}(Q) \) in (3.54). The condition of (3.55) is equivalent to requiring \( \Delta \theta_b \in \nu(P_b \tilde{Q} + \lambda I) \); \( \nu(\cdot) \) denotes the null space of the indicated matrix. This is essentially an eigenvalue problem wherein the Lagrange multiplier, \( \lambda \), is a negative eigenvalue of the matrix \( P_b \tilde{Q} \); \( P_b \tilde{Q} \) is rank 2 or 1, because \( P_b \) is of full rank, \( m + 1 \), and \( \tilde{Q} \) is a sum of rank 1 matrices (see (3.54)). The \( m - 1 \) or \( m \) zero-valued eigenvalues of \( P_b \tilde{Q} \) correspond to a minimization of \( B_1 \), as justified in the following lemma.

**Lemma 4** For \( v \in \mathbb{R}^{m+1} \), \( \tilde{Q} v = 0 \Rightarrow Q v = 0 \); that is, \( \nu(\tilde{Q}) \subset \nu(Q) \).

**Proof:** Recall that \( Z_m(e^{in}) = [1 \ e^{in} \ \cdots \ e^{imn}]' \equiv x, \ Q \equiv xx'' = Q'' \), and \( 2\tilde{Q} = Q + Q' \). Let \( (\cdot)' \) denote the, element by element, complex conjugate of the indicated argument. Observe that

\[ 2\tilde{Q} v = xx'' v + (xx'')'v = xx'' v + \overline{xx'} v = x\alpha + \overline{x\alpha} \]

\[ = 2\text{Re}(x\alpha), \quad (3.56) \]
Because \( x \alpha = Qv \), \( \dot{Q}v = 0 \) implies that \( \text{Re}(Qv) = 0 \) (see (3.56)). It remains to show that \( \text{Im}(Qv) = 0 \) is also implied by \( \dot{Q}v = 0 \). Observe that

\[
\text{Im}(Qv) = \frac{1}{j^2} [Qv - \overline{Qv}] = \frac{1}{j^2} [x\alpha - \overline{x\alpha}] \in \mathbb{R}^{m+1}. \tag{3.57}
\]

Let \(-j\alpha = re^{j\beta} \in \mathbb{C}\). Setting \( \dot{Q}v = 0 \) in equation (3.56) requires that \( x\alpha = -\overline{x\alpha} \); therefore,

\[
\text{Im}(Qv) = \frac{1}{j^2} [2x\alpha] = -j\alpha x = re^{j\beta} x \in \mathbb{R}^{m+1}. \tag{3.58}
\]

Let \( x_i \) be the \( i \)th element of the vector \( x \). We have two cases to consider. If \( \beta = k\pi \) for any integer \( k \), then (3.58) requires that \( x \in \mathbb{R}^{m+1} \), which implies that \( \dot{Q} = Q \) (hence, \( \nu(\dot{Q}) = \nu(Q) \)). Otherwise \( (\beta \neq k\pi) \), we must have \( re^{j\beta} x_i \in \mathbb{R}, \forall i \). In particular, \( re^{j\beta} x_1 = re^{j\beta} \in \mathbb{R} \Rightarrow r = 0 \); i.e., \( \alpha = x^Hv = 0 \). This, in turn, requires that \( xx^Hv = Qv = 0 \).

We need only consider the eigenvectors corresponding to the nonzero eigenvalues of \( P_b\dot{Q} \) to find the worst-case parameter variation, \( \Delta \theta^*_6 \), maximizing \( B_t \). For \( \Delta \theta_b \) in the subspace of \( \mathbb{R}^{m+1} \) spanned by the eigenvectors corresponding to the zero-valued eigenvalues of \( P_b\dot{Q}, P_b\dot{Q}\Delta \theta_b = 0 \); hence, \( \dot{Q}\Delta \theta_b = 0 \). Application of Lemma 4 ensures that for such a \( \Delta \theta_b \), we have \( \Delta \theta^*_6(Q\Delta \theta_b) = \Delta \theta^*_60 = 0 \); this is a minimum of \( B_t \geq 0 \) of (3.52).
Theorem 3  If \( c \) is the largest magnitude eigenvalue of \( P_b \hat{Q} \) and is of geometric multiplicity 1, then \( B^*_i = c \). If \( v \in \mathbb{R}^{m+1} \) is the eigenvector corresponding to \( c \) such that \( v'P_b^{-1}v = 1 \), then \( \Delta \theta^*_b = v \).

Proof: We seek to maximize the quadratic expression \( B_l = v'Qv \in \mathbb{R} \), subject to \( v'P_b^{-1}v = 1 \). The scalar \( B_l = B^*_l \); hence,

\[
B_l = v'Qv = \frac{1}{2}v'Qv + \frac{1}{2}(v'Qv)' = v'\hat{Q}v
\]  
(3.59)

where \( v'P_b^{-1}v = 1 \). We have \( P_b \hat{Q}v = cv \); therefore, \( \hat{Q}v = cP_b^{-1}v \). Premultiplying by \( v' \) yields

\[
v'\hat{Q}v = cv'P_b^{-1}v = c \cdot 1 = c = B_l.
\]  
(3.60)

Observe that \( c \) is a nonnegative real number, because \( \hat{Q} \) is real-valued, positive semidefinite, and symmetric. A necessary condition of the optimization is that the Lagrange multiplier, \( \lambda \), be the negative of an eigenvalue of \( P_b \hat{Q} \) (see (3.55)). We see that (3.60) is maximized when \( \lambda = -c \); hence, \( B^*_l = c \) and \( \Delta \theta^*_b = v \).

\( \Box \)

A (nonzero) repeated eigenvalue, \( c \), of the rank 2 matrix, \( P_b \hat{Q} \), could possibly have geometric multiplicity 2; Theorem 3 does not apply in this case. The necessary condition of (3.55) still applies to the optimization, yet we must now search the intersection of \( \partial E_b \) with the two-dimensional subspace spanned by the eigenvectors corresponding to \( c \), to find the extremum.
Theorem 4 Let $K_1 \equiv v_1^T \hat{Q} v_2, K_2 \equiv v_1^T P_b^{-1} v_2$, and $|K_2| \neq 1$. If $c$ is a nonzero eigenvalue of the matrix $P_b \hat{Q}$ with geometric multiplicity 2 and corresponding eigenvectors $v_1, v_2$ such that $v_1^T P_b^{-1} v_1 = 1, v_2^T P_b^{-1} v_2 = 1$, then $B_i^* = c$ and $\Delta \theta_i^* = \alpha_1 v_1 + \alpha_2 v_2$ where

for $K_1 \geq 0$: $\alpha_1 = \alpha_2 = \pm \sqrt{\frac{1}{2(1+K_2)}}$

for $K_1 < 0$: $\alpha_1 = -\alpha_2 = \pm \sqrt{\frac{1}{2(1-K_2)}}$.

Proof: If $K_2 = \pm 1$, then $v_1 = \pm v_2$ (respectively) must occur; in either case, $\Delta \theta_i^* = \pm v_1$ and $B_i^* = c$. We seek to maximize (3.59), subject to $v^T P_b^{-1} v = 1$, given that $v = \alpha_1 v_1 + \alpha_2 v_2$ and $P_b \hat{Q} v_i = c v_i$ for $v_i^T P_b^{-1} v_i = 1, i = 1, 2$. Expanding (3.59), we see that

$$B_i = (\alpha_1 v_1 + \alpha_2 v_2)^T \hat{Q} (\alpha_1 v_1 + \alpha_2 v_2)$$

$$= \alpha_1^2 v_1^T \hat{Q} v_1 + \alpha_2^2 v_2^T \hat{Q} v_2 + 2 \alpha_1 \alpha_2 v_1^T \hat{Q} v_2$$

$$= c \alpha_1^2 + c \alpha_2^2 + 2 \alpha_1 \alpha_2 K_1. \quad (3.61)$$

From the constraint $v^T P_b^{-1} v = 1$ we have

$$1 = (\alpha_1 v_1 + \alpha_2 v_2)^T P_b^{-1} (\alpha_1 v_1 + \alpha_2 v_2)$$

$$= \alpha_1^2 v_1^T P_b^{-1} v_1 + \alpha_2^2 v_2^T P_b^{-1} v_2 + 2 \alpha_1 \alpha_2 v_1^T P_b^{-1} v_2$$

$$= \alpha_1^2 + \alpha_2^2 + 2 \alpha_1 \alpha_2 K_2.$$
Lagrangian, $\mathcal{L} = f + \lambda g$. Examining the first order necessary conditions for extrema, $\nabla_x \mathcal{L} = 0$ yields $g = 0$, and from $\nabla_{\alpha_i} \mathcal{L} = 0$ we conclude that $\alpha_i^2 = \alpha_2^2$ ($\alpha_1 = \pm \alpha_2$); substitution into $g = 0$ yields
\[
\alpha_i^2 = \frac{1}{2(1 \pm K_2)}.
\] (3.62)

We observe that for a relative maximum of $B_1$, we must have $2\alpha_1 \alpha_2 K_1 > 0$ which requires that $\alpha_1 = \alpha_2$ when $K_1 > 0$, and $\alpha_1 = -\alpha_2$ when $K_1 < 0$. We have $K_1 = cK_2$, because $P_b \hat{Q} v_2 = c v_2$ implies $\hat{Q} v_2 = cP_b^{-1} v_2$ which in turn requires that $v_i' \hat{Q} v_2 = c v_i' P_b^{-1} v_2$; this constrains $K_1$ and $K_2$ to be of the same sign ($c > 0$). A complex $\alpha_1$ would result if $|K_2| > 1$ were possible, which is not the case. Because $\hat{Q}$ is a positive semidefinite (symmetric) matrix, we have $(v_1 - v_2)' \hat{Q} (v_1 - v_2) \geq 0$ from which we conclude $K_1 \leq c$. From $(v_1 + v_2)' \hat{Q} (v_1 + v_2) \geq 0$ we conclude that $K_1 \geq -c$; hence, $|K_1| \leq c$. Because $K_1 = cK_2$, we have $|K_2| \leq 1$. It follows from (3.61) and (3.62) that $B_i^* = 2c\alpha_i^2(1 \pm K_2) = c$, because $\alpha_1 = \pm \alpha_2$ and $K_1 = cK_2$.

\[\square\]

### 3.4.2 Summary Remarks

Theorems 3 and 4 provide an analytic solution to the FIR model worst-case additive perturbation problem statement of (3.53) or, equivalently, of (3.48). We are free to generate magnitude samples at any normalized frequency, in $[0, 2\pi]$, of interest. It is actually only necessary to generate samples in $[0, \pi]$ due to conjugate-symmetry; that is, for $\Omega_l \in [0, \pi]$, $|\Delta_B(e^{j\Omega_l}, \theta_b)| = |\Delta_B(e^{j(2\pi - \Omega_l)}, \theta_b)|$. For any choice of $\epsilon > 0$, the resultant set of magnitude samples, $W_f^* \equiv (1 + \epsilon)\sqrt{B_i^*}$, can be interpolated to form a
minimum phase polynomial, $W(z)$, which now satisfies (see (3.48))

$$\sup_{\theta \in \Theta_i} |\Delta_B(e^{j\Omega_i}, \theta_b)| < |W(e^{j\Omega_i})| \quad \forall \Omega_i \in \mathcal{S}_M.$$  \hspace{1cm} (3.63)

We digress at this point and discuss the relation of this optimization to an earlier result from [52]. For ease in reproducing this result, we draw on the following Lemma, found in [6].

**Lemma 5 (Wahlberg)** Let $x \in \mathbb{R}^n$, $P = P' > 0$, $P \in \mathbb{R}^{n \times n}$, and $E = \{x \in \mathbb{R}^n : x'P^{-1}x \leq 1\}$. Let $A \in \mathbb{R}^{p \times n}$ where $\text{rank}(A) = p \leq n$. If $y = Ax$, then $y'(A^PA')^{-1}y \leq 1$, $\forall x \in E$.

From equation (3.13), we recall that

$$\Delta_B(e^{j\Omega}) = \Delta\theta^*Z_m(e^{j\Omega}) = \Delta R(e^{j\Omega}) + j\Delta I(e^{j\Omega})$$

where $\Delta R(e^{j\Omega}) = Z_R(e^{j\Omega})'\Delta\theta_b$, $\Delta I(e^{j\Omega}) = Z_I(e^{j\Omega})'\Delta\theta_b$, $Z_R = \text{Re}(Z_m)$, and $Z_I = \text{Im}(Z_m)$. Let $A = [Z_R \ Z_I]' \in \mathbb{R}^{2 \times (m+1)}$. Suppressing the dependence upon $e^{j\Omega}$, we have $[\Delta R \ \Delta I]' = A^\Delta\theta_b$. We assume that $\Delta\theta_b P_b^{-1}\Delta\theta_b \leq 1$ as in (3.30). Application of Lemma 5 ensures that

$$\left[ \begin{array}{c} \Delta R \\ \Delta I \end{array} \right]'(AP_b A')^{-1}\left[ \begin{array}{c} \Delta R \\ \Delta I \end{array} \right] \leq 1. \hspace{1cm} (3.64)$$

Equation (3.64) describes an ellipse in $\mathbb{R}^2$, centered at the real and imaginary components of $B_0(e^{j\Omega})$, the nominal numerator of (3.12) evaluated at the normalized frequency $\Omega$, and bounding the real and imaginary components of $B^\infty(e^{j\Omega})$, the true numerator polynomial of (3.10) evaluated at $z = e^{j\Omega}$. This ellipse construction
appears originally in [52] and later in [6]. We will now show that the square of the length of the ellipse semimajor axis (the largest eigenvalue of $A P_b A'$) coincides with $B_f^*$ of (3.49).

For this section, we denote $Z_m(e^{j\Omega})$ as $Z$. Recall that $Q \equiv ZZ^H$ in (3.51). Because $Q' = \overline{Q}$, (3.54) requires that $\hat{Q} = Re(ZZ^H)$; hence,

$$
\hat{Q} = Re(Z)Re(Z^H) - Im(Z)Im(Z^H) = Re(Z)Re(Z') + Im(Z)Im(Z').
$$

We see that

$$
A'A \equiv [Re(Z) Im(Z)] \begin{bmatrix}
Re(Z') \\
Im(Z')
\end{bmatrix} = \hat{Q}.
$$

Let $\lambda(\cdot)$ denote the maximum eigenvalue of the indicated matrix. From Theorem 3, $B_f^* = \lambda(P_b A' A)$. Let $v_i$ be an eigenvector corresponding to an eigenvalue, $\lambda_i$, of $P_b A' A$; that is, $P_b A' A v_i = \lambda_i v_i$. Letting $w_i = A v_i$, we see that $A P_b A' w_i = A P_b A' A v_i = \lambda_i A v_i = \lambda_i w_i$; hence, $\lambda_i \neq 0$ is also an eigenvalue of $A P_b A'$ (when $\lambda_i = 0$, $A v_i = w_i = 0$). We conclude that $B_f^*$ is the largest eigenvalue of $A P_b A' = (A P_b A')' > 0$.

The length of the semimajor axis of the ellipse of (3.64) is the square of the worst-case magnitude perturbation at the frequency $\Omega \in [0, 2\pi]$. The shortcoming of this ellipse representation is that all parameter domain information is lost, because $A$ is not square (hence, is not invertible) for $m > 1$; Theorems 3 and 4 give access to the worst-case parameter perturbation, $\Delta \theta_i^*$. Such access will be needed when Theorems 3 and 4 are used to solve a worst-case analysis of parametric uncertainty interpreted as a coprime factor plant perturbation.
3.5 Robust Stabilization for Coprime Factor Perturbations

The conservatism in our characterization of the plant structured uncertainty as unstructured uncertainty is minimal when expressed as an additive perturbation to the nominal plant model. However, use of the additive perturbation requires the knowledge (or presumption thereof) that each feasible plant, parameterized by the ellipsoid, has the same number of unstable poles. Without such knowledge, we must generalize our uncertainty characterization.

The set of feasible transfer functions, $G$ of (3.14), can be expressed as

$$G(z, \theta) = \frac{B_0 + \Delta_B}{A_0 + \Delta_A}, \quad \theta \in E_N, \ A(z) \text{ and } B(z) \text{ coprime},$$

which is the form of a so-called coprime factor perturbation [60, 61] wherein $A_0$ and $B_0$ characterize the nominal plant and $\Delta_A$ and $\Delta_B$ are allowed perturbations due to parametric uncertainty. As previously discussed, the polynomials $A_0$, $B_0$, $\Delta_A$, and $\Delta_B$ are all elements of $\mathbb{RH}_{\infty}$. Assume we can find a minimum phase weighting function, $W \in \mathbb{RH}_{\infty}$, such that

$$|\Delta_A(e^{j\Omega})|^2 + |\Delta_B(e^{j\Omega})|^2 < |W(e^{j\Omega})|^2 \quad \forall \Omega \in [0, 2\pi].$$

(3.65)

For coprime factor perturbations bounded as in (3.65), the robust stabilization problem is solvable, according to [60], if there exists a controller $C$, which is a ratio of $H_{\infty}$ functions and stabilizes the nominal plant, $G_0(z) \equiv G(z, \theta_N)$, satisfying

$$\gamma_c \equiv \left\| W A_0^{-1} \begin{bmatrix} S_0 \\ CS_0 \end{bmatrix} \right\|_{\infty} \leq 1$$

(3.66)
where \( S_0 \equiv (1 + G_o C)^{-1} \). Let \( \sigma(\cdot) \) denote the maximum singular value of the indicated matrix. The condition of (3.66) is equivalent to

\[
\gamma_c = \text{ess sup}_{\Omega \in [0, 2\pi]} \sigma \left( WA_o^{-1} \left[ \begin{bmatrix} S_o \\ GS_o \end{bmatrix} \right] \right) \leq 1.
\]

For \( W \) satisfying (3.65) given, if there exists a robustly stabilizing controller, \( C_r \), achieving \( \gamma_c < 1 \) in (3.66), then we have increased robustness to uncertainty; \( C_r \) is a robustly stabilizing controller for the coprime factor perturbation weighting function, \( W/\gamma_c \). If an optimal controller exists, define it as

\[
C_{opt} = \arg \left( \inf_{C \in \mathcal{G}_s} \left\| WA_o^{-1} \left[ \begin{bmatrix} S_o \\ GS_o \end{bmatrix} \right] \right\|_\infty \equiv \gamma_c^* \leq 1 \right)
\]

where the infimum is taken over all controllers stabilizing the nominal plant, \( G_o \). In the event \( C_{opt} \) does not exist, it may be possible to find a suboptimal controller which comes arbitrarily close to achieving \( \gamma_c^* \).

The mathematical problem captured in (3.65) is equivalently to finding

\[
\sup_{\theta \in \mathcal{E}_N} \left\{ |\Delta_A(e^{j\Omega})|^2 + |\Delta_B(e^{j\Omega})|^2 \right\}
\]

for each frequency, \( \Omega \in [0, 2\pi] \). The argument, \( \theta^* \), of the extremum of (3.68) at a given frequency, \( \Omega \in [0, 2\pi] \), must reside on \( \partial \mathcal{E}_N \) (see (3.3)). Recall that \( \Delta \theta \equiv \theta - \theta_N \); \( \theta \in \mathcal{E}_N \) implies \( \Delta \theta \in \mathcal{E}_\Delta = \{ \Delta \theta \in \mathbb{R}^r : \Delta \theta \Delta^* P_N^{-1} \Delta \theta \leq 1 \} \). The condition \( \theta^* \in \partial \mathcal{E}_N \) is equivalent to \( \Delta \theta^* \in \partial \mathcal{E}_\Delta \) (see (3.4)). If it were possible that \( \Delta \theta^* \notin \partial \mathcal{E}_\Delta \), then

\[
|\Delta_A(e^{j\Omega})|^2 = |\Delta_\theta_a Z_{n-1}(e^{j\Omega})|^2 \text{ (or } |\Delta_B(e^{j\Omega})|^2 = |\Delta_\theta_a e^{jn\Omega} Z_m(e^{j\Omega})|^2 \text{) could be increased by following the (subspace) vector direction of } \Delta \theta_a (\Delta \theta_b) \text{ outward to the boundary of } \mathcal{E}_\Delta \text{ while leaving } \Delta \theta_b (\Delta \theta_a) \text{ unchanged; this is a contradiction that any interior point of } \mathcal{E}_\Delta \text{ is the argument which maximizes } (3.68) \text{ at } \Omega = \Omega.
\]
Our strategy shall again be to solve (3.68) for each $\Omega_i \in S_M$, from (3.26), and to interpolate the resulting magnitude bounds. The problem statement of (3.68) is equivalent to the optimization

$$\text{maximize } W_i^2 \equiv |\Delta_A(e^{j\Omega_i})|^2 + |\Delta_B(e^{j\Omega_i})|^2$$

subject to $g(\Delta \theta) = \Delta \theta' P_N^{-1} \Delta \theta - 1 = 0$ \hspace{1cm} (3.69)

at the normalized frequency, $\Omega_i$. Let $W_i^*$ denote the extremum of $W_i$ of (3.69), and let $\Delta \theta^*$ denote its maximizing argument. The $W_i$ are symmetric about $\Omega = \pi$; thus, they only need to be evaluated for the $M/2$ ($M$ even) or $(M + 1)/2$ ($M$ odd) frequencies of $S_M$ in $[0, \pi]$. Once we have computed $W_i^*$, for each $\Omega_i \in S_M$, we can construct a continuous interpolating function, $W(e^{j\Omega})$, for use in (3.65); for any choice of $\epsilon > 0$, an interpolant must satisfy

$$|W(e^{j\Omega_i})| = (1 + \epsilon)W_i^* \quad \forall \Omega_i \in S_M. \hspace{1cm} (3.70)$$

### 3.5.1 Frequency Dependent Worst-Case Magnitude Bound

As a first step in seeking a solution to (3.69), we form the Lagrangian,

$$\mathcal{L} \equiv W_i^2(\Delta \theta, e^{j\Omega_i}) + \lambda g(\Delta \theta).$$

For $Z_k(z)$ as defined in (3.11), let

$$Q_k(z) \equiv Z_k(z)Z_k^H(z) \quad \text{and} \quad Q(z) \equiv \begin{bmatrix} Q_{n-1}(z) & 0 \\ 0 & Q_m(z) \end{bmatrix}. \hspace{1cm} (3.71)$$

Suppressing the dependence upon $z = e^{j\Omega_i}$, an equivalent representation of $W_i^2$, as defined in (3.69), is

$$W_i^2 = \Delta \theta' Q \Delta \theta. \hspace{1cm} (3.72)$$

The first order necessary conditions for an extremum are (see e.g. [57]):
1. $\nabla_\lambda \mathcal{L} = 0 = \Delta \theta ' P_N^{-1} \Delta \theta - 1 = g(\Delta \theta)$

2. $\nabla_{\Delta \theta} \mathcal{L} = 0 = [Q + Q' + 2\lambda P_N^{-1}] \Delta \theta$.

Because $Q = Q^H$, the matrix

$$\frac{1}{2} (Q + Q') \equiv \hat{Q}$$

(3.73)

is real-valued ($\hat{Q} = \text{Re}(Q)$); $\hat{Q}$ is also symmetric. Condition 2 can be rewritten as

$$[P_N \hat{Q} + \lambda I] \Delta \theta = 0,$$

(3.74)

because $P_N$ is of full rank; this is analogous to the condition of (3.55). Theorems 3 and 4 apply, with the appropriate substitution of $P_N$ in place of $P_s$ and $\hat{Q}$ of (3.73) in place of (3.54), and ensure that

$$(W_i^*)^2 = \lambda(P_N \hat{Q}).$$

(3.75)

For any $\epsilon > 0$, $(1 + \epsilon)(W_i^*)^2$ is a sufficient bound in (3.65) at the normalized frequency, $\Omega_i$; however, knowledge of component-wise magnitude bounds on $\Delta A$ and $\Delta B$ is not directly available. Separate bounds on $|\Delta A|$ and $|\Delta B|$ are necessary, for example, if loopshaping will be attempted during the control design phase. Theorems 3 and 4 permit the computation of $\Delta \theta^* = [(\Delta \theta^*_a)' (\Delta \theta^*_b)']'$, the argument of the extremum of $W_i^2$ (see (3.72)); with this information, it follows that

$$|\Delta_A(e^{j\Omega_i})| \leq |Z_{n-1}'(e^{j\Omega_i}) \Delta \theta^*_a|,$$

$$|\Delta_B(e^{j\Omega_i})| \leq |Z_m'(e^{j\Omega_i}) \Delta \theta^*_b|$$

at the normalized frequency, $\Omega_i$, of interest.
3.6 Constructing a Continuous Bounding Function

For each type of plant perturbation considered, we have generated magnitude bounds, \( W_f^* \) on the worst-case perturbation for the set of \( M \) frequencies of (3.26). This real-valued sequence is a set of samples of the magnitude response of a weighting function that overbounds the perturbation of either (3.21), (3.48), or (3.65) at the corresponding normalized frequency, \( \Omega \).

For example, when the set of feasible plant descriptions, \( \mathcal{G} \), contains no transfer functions with poles in \( \partial \mathcal{D} \), we can solve for a sampled representation of an uncertainty weighting function, bounding an additive plant perturbation (see (3.28)), as

\[
W_i^* = (1 + c)\sqrt{\mathcal{F}(\Delta \theta_i, \Omega_i)}, \text{ for } l \in \{0, 1, \ldots, M - 1\} \tag{3.76}
\]

where \( c > 0 \) is arbitrarily close to 0; this is a real-valued sequence.

Regardless of the nature of the plant perturbation, we face the same task of generating a continuous-valued weighting function, \( W(e^{j\Omega}) \), that satisfies the magnitude constraints,

\[
|W(e^{j\Omega})| = W_i^*, \tag{3.77}
\]

at the corresponding \( M \) roots of unity, \( z = e^{j\Omega} \), about the unit circle. For purposes of \( H_\infty \) control design, we choose the magnitude interpolant to be a rational, minimum phase weighting function, \( W \), (defined on the complex \( z \)-plane) that is analytic on \( \mathcal{D} \). It is also necessary that the magnitude interpolant satisfy qualitative properties of a "good" bounding function; for example, if the number of samples, \( M \), is chosen large enough, we do not expect significant intersample magnitude variations in the
interpolant. The ramifications of a choice of $M$ will be discussed in the next section.

Splines are not well-suited for $H_{\infty}$ control design wherein a single function description of the weighting function for the entire domain interval $[0,2\pi]$ is desired. There exist well-defined techniques for interpolation in the complex plane [62]; for example, Nevanlinna-Pick interpolation can be modified to allow one to construct a rational transfer function of $M$ poles and $M$ zeros, analytic on $D$, and of minimum maximum modulus over the domain of $\partial D$. Lagrange interpolation can be used to construct a polynomial of degree $M-1$ also accomplishing the desired interpolation in the complex plane (see e.g. [62]). For a stable, worst-case bounding function, $W$, [63] states that intersample error in the Lagrange interpolant, $\hat{W}_L$, (which coincides with an $M$-point inverse discrete Fourier transform) can be bounded as

$$\|\hat{W}_L - W\|_\infty \leq \hat{K} \rho^{-M}$$

where $M$ is the number of uniform samples of $W$, $\rho \in (1,R)$, and $R > 1$ is the radius to which the analyticity of $W$ extends in the complex plane. The constant $\hat{K}$ is independent of $M$. Such techniques accomplish the interpolation task of (3.77), yet can fail to possess qualitative properties of a "good" bounding function, as will be illustrated.

**Fourier Series-Based Interpolant**

The interpolation problem can alternatively be viewed as a problem in the real domain $\Omega$ and real-valued range of the magnitudes, $W^*_i$; a means of mapping the function of a real variable to a function defined on the complex plane, with modulus matching $W^*_i$.
at } z = e^{i\Omega}, \text{ will be required. We are free to choose } M, \text{ the number of samples; choose } M \text{ to be odd, and define } L = (M - 1)/2. \text{ The first } M \text{ terms of the trigonometric Fourier series of a function, } W, \text{ defined over the interval } [0, 2\pi] \text{ can be expressed as }

\[ W(\Omega) = \frac{f_0}{2} + \sum_{k=1}^{L} [f_k \cos(k\Omega) + g_k \sin(k\Omega)]. \quad (3.78) \]

Let

\[ T_L \equiv \begin{bmatrix} 0.5 \cos(\Omega_0) & \cdots & \cos(L\Omega_0) & \sin(\Omega_0) & \cdots & \sin(L\Omega_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0.5 \cos(\Omega_{M-1}) & \cdots & \cos(L\Omega_{M-1}) & \sin(\Omega_{M-1}) & \cdots & \sin(L\Omega_{M-1}) \end{bmatrix}, \quad (3.79) \]

\[ \Psi_L \equiv [f_0 \ f_1 \ \cdots \ f_L \ g_1 \ \cdots \ g_L]', \quad (3.80) \]

and

\[ \tau \equiv [W_0^* \ W_1^* \ \cdots \ W_{M-1}^*]', \quad (3.81) \]

We require that } W(\Omega_l) = W^*_l \text{ for each } l \in \{0, 1, \cdots, M - 1\}; \text{ this is a set of } M \text{ linear equations with } M \text{ unknowns:}

\[ T_L \Psi_L = \tau. \quad (3.82) \]

Due to our uniform spacing of } \Omega_i, \text{ } T_L^{-1} \text{ exists; therefore, } \Psi_L = T_L^{-1} \tau \text{ is the vector of trigonometric Fourier series coefficients accomplishing the interpolation of (3.77).}

To extend the function to the complex domain while achieving the specified magnitude along the unit circle, use the Euler identities:

\[ \cos(k\Omega) = \frac{1}{2}(z^k + z^{-k})|_{z=e^{i\Omega}} \text{ and } \sin(k\Omega) = \frac{1}{j2}(z^k - z^{-k})|_{z=e^{i\Omega}}. \quad (3.83) \]

Following substitution of (3.83) into (3.78), we have
\[ W(z) = 0.5 z^{-L} [(f_L + jg_L) z^{2L} + \cdots + (f_1 + jg_1) z^{L+1} + f_0 z^L + (f_1 - jg_1) z^{L-1} + \cdots + (f_L - jg_L)] \]

\[ \equiv z^{-L} [\tau_{M-1} z^{M-1} + \cdots + \tau_1 z + \tau_0] = z^{-L} \hat{W}(z). \] (3.84)

The leading term, \( z^{-L} \), can be dropped because it does not affect the magnitude of \( W \) along \( \partial D \) (\(|z| = 1\)). For convenience, we choose the interpolant to be minimum phase; this can be accomplished by reflecting the zeros of (the polynomial) \( \hat{W} \), which are in \( D \), to their conjugate reciprocal location in the complex plane. Let \( \hat{W} \) be the minimum phase transfer function with the same modulus as \( \hat{W} \) along \( \partial D \); if \( z - z_0 \) is a factor of \( \hat{W}(z) \) where \(|z_0| < 1\), then (for \( \hat{W} \)) substitute the factor

\[ |z_0|(z - 1/\bar{z}_0). \] (3.86)

We have thus found a suitable \( W \) for (3.77).

**Properties of the Fourier Series-Based Interpolant**

For each of the perturbation types that we have investigated, the perturbation magnitude is symmetric about \( \Omega = \pi \):

\[ |\Delta(e^{i(2\pi - \Omega)})| = |\Delta(e^{i2\pi e^{-i\Omega}})| = |\Delta(e^{i\Omega})| = |\Delta(e^{i\Omega})|. \]

The periodic extension of \(|\Delta(e^{i\Omega})|\) is an even function which implies that the coefficients of the sine terms, \( g_k \) of (3.85), must be zero-valued; consequently, the coefficients, \( \tau_k \) of (3.85) will all be real-valued. The minimum phase equivalent of \( \hat{W} \)
of (3.85) will also have real-valued coefficients, because the roots of $\hat{W}$ occur in complex conjugate pairs.

We have implicitly chosen the phase of the interpolating conditions of (3.77) to be zero, meaning $W(e^{j\Omega})$ is real-valued. The real part of $W(e^{j\Omega})$ is a Discrete Fourier transform pair with the conjugate-symmetric part of the inverse Discrete Fourier transform (IDFT) of $W(e^{j\Omega})$; the fundamental period of the IDFT of $W(e^{j\Omega})$, appearing in (3.84), is indeed conjugate-symmetric.

We shall exclusively use this Fourier Series-based interpolant, because it delivers a weighting function which is periodic with period $2\pi$ (a property of the magnitude spectrum of a discrete-time Fourier transform), is an element of $\text{RH}_\infty$ (other rational interpolants do not necessarily have real-valued coefficients), and delivers an interpolant with "good" qualitative bounding properties.

### 3.6.1 Illustration of Interpolation Methods

To illustrate the qualitative bounding properties of the various interpolation schemes discussed, $M = 17$ points were pseudorandomly generated to have a uniform distribution over $[0.5, 1.5]$ and treated as the $W_t^*$ to be interpolated. This ignores the underlying symmetry of an actual $W_t^*$ sequence yet illustrates the versatility of the interpolation schemes discussed in the sequel.

Nevanlinna-Pick interpolation generates a transfer function description which was found to exhibit magnitude troughs between the $W_t^*$, as seen in Figure 6. If $M$ had been chosen large enough, the worst-case perturbation weighting would not display such drastic changes in magnitude between "samples".
Lagrange interpolation delivers a polynomial interpolant which, as an uncertainty bounding function, is a qualitative improvement (see Figure 7) over Nevanlinna-Pick interpolation. From Figure 7 we also see that the trigonometric Fourier series-based interpolant has the qualitative appeal of most smoothly transitioning between the knots, specified by (3.77).

We are collecting uniformly spaced (in normalized radian frequency) real-valued (magnitude) samples of a periodic function, of period $2\pi$, of a real-valued independent variable, $\Omega$. The sampling period of $2\pi/M$ corresponds to a radian sampling frequency of $M$. Assume that the Fourier transform of the underlying magnitude function is zero for all radian frequencies, $\omega > \omega_{\text{max}}$. A choice of $M \geq 2\omega_{\text{max}}$ uniformly spaced samples is sufficient to permit exact reconstruction of the underlying magnitude function using an ideal lowpass filter, as stated in Shannon's Sampling Theorem [64]. The Fourier series interpolant inherently contains radian frequencies up to $L$. Choosing $L = (M - 1)/2 \geq \omega_{\text{max}}$ ensures that $M > 2\omega_{\text{max}}$, which satisfies the spectral assumptions of Nyquist rate sampling.

An alternative viewpoint is that if the worst-case perturbation weighting function is describable as a polynomial of degree $K - 1$, then the finite set of impulse response coefficients can be recovered without aliasing if $M \geq K$ samples of the frequency function are taken (computed) [56]. A transfer function interpolant, such as Nevanlinna-Pick, is of infinite impulse response and could not be recovered, without aliasing effects, from a finite set of samples of its magnitude response.
3.6.2 Reduced Order Overbounding Functions

There is a direct relationship between the order of the identified uncertainty weighting function and the resolution with which the perturbation weighting is sampled. A high order identified weighting function complicates computation of the set of robust controllers and may cause numerical errors and controller orders to become unacceptably large; therefore, it is desirable to have a methodology to identify reduced order weighting functions that overbound the set of samples of the perturbation weighting. There is thus a tradeoff of increased conservatism in our uncertainty weighting function for a reduction in modeling order.

Let us identify a trigonometric Fourier series-based weighting function of reduced order $K - 1 < M - 1$ ($K$ is odd). Let $\tilde{L} = (K - 1)/2$, $\tau \equiv \tau_L$ as in (3.79), $\Psi \equiv \Psi_L$

Figure 6: Magnitude Response of Nevanlinna-Pick Interpolant
as in (3.80), and $r$ be as in (3.81). We propose three cost functions as alternative metrics to minimize while identifying a reduced order weighting function:

$$J_1 = f_0$$

(3.87)

$$J_2 = \| T \Psi - r \|_2^2$$

(3.88)

$$J_3 = \sum_{l=0}^{M-1} (W(\Omega_l) - W_l^*)$$

(3.89)

The goal is to

minimize $J_i$

subject to $T \Psi \geq r$ (i.e. $W(\Omega_l) \geq W_l^*$).

(3.90)

The area under the overbounding curve is

$$\int_0^{2\pi} W(\Omega) = \pi f_0$$

(3.91)
for $W(\Omega)$ as in (3.78); thus, the minimization of $J_1$ corresponds to minimizing the area under the overbounding polynomial of order $K - 1$. This is a linear programming problem which can be solved numerically. The second cost function is

$$J_2 = \sum_{i=0}^{M-1} |W(\Omega_i) - W_i^*|^2,$$

which is the squared vertical distance of the overbounding curve from each of the $W_i^*$. This is a quadratic programming problem which can be solved numerically as a convex optimization. The final cost function, $J_3$, measures the total vertical distance of the overbounding curve from each of the $W_i^*$. Minimizing $J_3$ (subject to the overbounding constraint) is a linear programming task.

Because the $\Omega_i$ are uniformly distributed about the interval $[0, 2\pi]$, the constrained minimization of $J_3$ yields the identical overbounding curve as the constrained minimization of $J_1$. Observe that $J_3 = [1 \cdots 1](\Upsilon \Psi - \tau)$ and that $[1 \cdots 1]\Upsilon$ is a row vector whose $i^{th}$ element is the sum of the elements of the $i^{th}$ column of $\Upsilon$. From (3.79) we see that

$$[1 \cdots 1]\Upsilon \Psi = [K \ 0 \ \cdots \ 0]\Psi = Kf_0;$$

hence, minimizing $J_3 = Kf_0 - \sum_{i=0}^{M-1} W_i^*$ is equivalent to minimizing $J_1$ ($K$ and the constant, $\sum_{i=0}^{M-1} W_i^*$, are fixed terms which do not change the relative optimization).

### 3.6.3 Illustration of Reduced Order Overbounding Functions

The same data set of $W_i^*$, used in illustration of the interpolation techniques, will also be used to illustrate the overbounding functions resulting from minimization of
$J_1$ of (3.87) and $J_2$ of (3.88). The Matlab 4.1 optimization toolbox was used to solve the linear and quadratic programming tasks. In Figure 8, we used $K = 5$ and in Figure 9 we used $K = 9$, corresponding to fourth and eighth order overbounding polynomials (respectively). The minimum square distance curves (minimizing $J_2$) have the appeal of having smaller peak magnitudes.

![Graph](image)

**Figure 8: Fourth Order Overbounding Function Magnitudes**

**The Choice of $\epsilon$**

The (full order) interpolant, $W$, which satisfies the interpolation conditions of (3.77), must actually be inflated slightly so that the strict inequality of either (3.21), (3.48), or (3.65) is established; for any $\epsilon > 0$, a weighting function of $(1 + \epsilon)W$ suffices. An identified reduced order weighting function meets or exceeds the interpolation conditions of (3.77), yet such a weighting may not overbound $|W(e^{i\theta})|$ (the interpolant magnitude) over all normalized frequencies in $[0, 2\pi]$. An identified reduced order
weighting function should be inflated, via an adequate choice of $c$, to overbound the full order interpolant, because the full order interpolant is presumed to completely characterize the intersample behavior (as previously discussed in terms of sampling theory). This point will be illustrated in examples appearing in Chapter IV.

### 3.6.4 Conservatism in the Weighting Function

Despite the built-in safety factors, the question still arises as to whether it is possible for a transfer function, whose parameterization lies within the ellipsoid, $E_N$, to give rise to a complex-valued perturbation that violates the magnitude bound of the (corresponding) identified weighting function, at some intersample frequency, when combined with the complex-valued frequency response of the nominal plant, at that intersample frequency.
A measure of conservatism, in $W$, is introduced by the coprime factor perturbation weighting, because the phase of the nominal plant, relative to the phase of the perturbation, is ignored. The actual (unknown) plant, parameterized by $\theta^\circ$, is some element of $E_N$; $\theta^\circ$, in general, does not reside on $\partial E_N$. If $\theta^\circ$ were to reside on $\partial E_N$, it would not likely give rise to a worst-case plant perturbation (relative to the nominal plant) at any frequency $\Omega \in [0, 2\pi]$. If $\theta^\circ$ were to coincide with a worst-case parameter variation, $\theta^*_i$, at some $\Omega_k \in S_M$, it is likely that $\theta^*_i \neq \theta^\circ$ at any remaining $\Omega_l \in S_M$ (other than $\Omega_l = 2\pi - \Omega_k$); the magnitude perturbation bound from adjacent samples will be conservative, reducing the import of accuracy in the interpolant.

If the OVE algorithm is used to identify $E_N$, further conservatism is introduced when overbounding the irregular convex set, $\mathcal{F}^N$, with an ellipsoid. The identification of the set of feasible parameters would ideally be carried out for an infinite duration, persistently exciting input sequence of maximum allowable signal to disturbance power (interpreted as SNR); this would further reduce the hypervolume of the identified ellipsoid.

From an engineering viewpoint, we gain confidence that our bounding function contains the plant perturbation by "oversampling" and then identifying a reduced order overbounding function (which, in general, introduces an additional degree of conservatism). We are sampling a worst-case perturbation bounding function wherein the worst-case plant parameterization, residing on the boundary of the ellipsoid, is unique, in general, at each frequency sampled. It is somewhat dubious to assume a priori smoothness conditions on the worst-case bounding function in an attempt to
ensure that the magnitude bound is never violated; it is not generated by a single worst-case plant, but rather by traversal along the boundary of an ellipsoid.

3.7 Summary Comments

We have developed a means of translating the parametric uncertainty of an ellipsoid, via worst-case analysis, into a perturbation bounding function; the so-called weighting function, in tandem with the nominal plant model, can then be used to characterize solutions to various $H_\infty$ robust control problem statements. A sampling strategy was combined with an interpolation step to identify a suitable weighting function. The resulting controller order, and the magnitude of numerical error in the computation of the controller, both increase with increasing perturbation weighting function order. At the expense of increased conservatism in our uncertainty characterization, we developed techniques to identify reduced order perturbation bounding functions.

When we have knowledge that the number of unstable poles of each plant within $G$ of (3.14) is the same, the additive perturbation uncertainty characterization is preferred relative to the coprime factor perturbation uncertainty characterization. An additive perturbation is inherently less conservative, because the phase of the nominal plant is incorporated into the worst-case uncertainty analysis (see (3.23)); whereas, the frequency response of the nominal plant does not affect the determination of a coprime factor perturbation weighting (3.65).

To characterize the set of robustly stabilizing controllers, a one-block $H_\infty$ problem, in the additive perturbation case (3.22), is less complex to solve relative to the two-block $H_\infty$ problem of (3.66). From a control design perspective, the additive plant
perturbation is also more versatile than the coprime factor perturbation in terms of the types of robust control problems which are solvable. Use of the coprime factor perturbation uncertainty characterization is, nevertheless, necessary whenever all plants within $\mathcal{G}$ do not have exactly the same number of unstable poles.
CHAPTER IV

Implementation and Comparative Analysis

The development of the preceding chapter was primarily theoretical. The purpose of this chapter is to illustrate the application of the weighting functions identification techniques, in an experiment. We first examine a finite impulse response (FIR) model and compare our weighting function identification technique to that appearing in [32]. The second example is an investigation of weighting function identification of a sampled data system, in the presence of undermodeling. The final example is an illustration of the identification of both a coprime factor and additive perturbation weighting for the same system, and is also an opportunity to apply the Kharitonov analysis of Section 3.3.3.

4.1 FIR Model Structure

The investigation of weighting function identification for FIR model structures motivated the solution to the coprime factor perturbation case. It is also an avenue with which we can make a direct comparison to a landmark paper in the field of identification for robust control design.
4.1.1 The Helmicki-Jacobson-Nett (HJN) Procedure

The seminal technique to accomplish the task of $H_\infty$ control oriented system identification is summarized in [32] and originally appeared in [31]. Because we wish to perform a comparative analysis with our FIR weighting function identification technique, we summarize the contribution of [32] in the following.

The true plant, $G^0(z)$, is assumed to be LTI and describable by the convolution relation,

$$y_k = \sum_{l=0}^{\infty} g^0_l u_{k-l} + v_k,$$  \hspace{1cm} (4.1)

with plant input $u_k$, measured output $y_k$, and output disturbance $v_k$. The disturbance, $v_k$, is assumed unknown, but bounded (UBB) as $|v_k| \leq \hat{v}$ for all $k \geq 0$. The nature of the disturbance differs from the ARX model structure, except in the event that the pulse response of $G^0(z)$ is of finite-length (an FIR model). We choose an FIR model as the truth model of (4.1) to enable a direct comparison (significant conservatism is introduced when the OVE algorithm is used with ARMAX models [12]) of the weighting function identification techniques of Section 3.4 to the HJN procedure.

Let $G_o(z)$ denote the nominal (estimated) plant model. The HJN procedure generates a uniform (in frequency) magnitude bound on the possible additive perturbation, $\Delta^*_G$, to the (unknown) true plant,

$$G^*(z) = G_o(z) + \Delta^*_G(z).$$  \hspace{1cm} (4.2)

Let $D_\rho \equiv \{z \in \mathbb{C} : |z| < \rho\}$. Define $H_{\infty,\rho}$ such that $G^* \in H_{\infty,\rho}$ implies that $G^*$ is analytic over the domain of $D_\rho$ and essentially bounded on $\partial D_\rho \equiv \{z \in \mathbb{C} : |z| = \rho\}$. 

---
The HJN procedure assumes that \( G^o \in H_{\infty, \rho} \) and
\[
\| G^o \|_{\infty, \rho} \equiv \text{ess sup}_{z \in D_\rho} |G^o(z)| \leq K, \tag{4.3}
\]
where \( \dot{\gamma}, K > 0 \), and \( \rho > 1 \) are all known. An FIR model is a polynomial, and a polynomial is entire (analytic in \( \mathbb{C} \)); thus, \( \rho \) can be chosen to be arbitrarily large in our comparison.

The HJN identification experiment "applies" complex exponentials,
\[
u_k = \begin{cases} \alpha e^{-j\Omega_l} & \text{for } k = 0, 1, \ldots, \hat{N} - 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}
\]
of duration \( \hat{N} - 1 \) at each normalized radian frequency, \( \Omega_l = 2\pi l/n \), from the set, \( \mathcal{S}_n \) defined in (3.26) (the arguments of the \( n \) roots of unity). We assume that the system under test is initially at rest prior to all experiments. Let \( y_{\hat{N}-1} \) denote the final output measurement of an experiment with input (4.4). Estimates of the frequency response of the plant are computed at each frequency, \( \Omega_l \), of \( \mathcal{S}_n \), as [32]
\[
G_o(e^{j\Omega_l}) = e^{j(\hat{N}-1)\Omega_l} \frac{y_{\hat{N}-1}}{\alpha}, \quad \forall l \in \{0, 1, \ldots, n - 1\}; \tag{4.5}
\]
\( \alpha \) is the magnitude of the complex excitation of (4.4). For our comparison, we choose \( n \) to be the length of the FIR of \( G^o \); this assures that \( G^o \) can be recovered without aliasing [56] in the disturbance free (\( u_k = 0 \)) case. Upon examination of the convolution relation of (4.1) it can easily be shown that if \( \hat{N} \geq n \) for \( u_k \) as in (4.4), then
\[
G_o(e^{j\Omega_l}) = G^o(e^{j\Omega_l}) + e^{j\Omega_l(\hat{N}-1)} \frac{y_{\hat{N}-1}}{\alpha}.
\]
The additive plant perturbation, \( \Delta^o = G^o - G_o \), is shown [32] to be bounded as
\[
|\Delta^o(e^{j\Omega_l})| \leq \epsilon^* \quad \text{for each } \Omega_l \in \mathcal{S}_n
\]
for each
where
\[ \epsilon^* \equiv \frac{K}{(\rho - 1)\rho^{N-1}} + \frac{\tilde{r}}{\alpha}. \] (4.6)

The set of pointwise frequency response estimates, \( G_o(e^{j\Omega}) \) of (4.5), must now be interpolated to yield a continuous (in \( \Omega \)) valued nominal estimate, \( G_o(e^{j\Omega}) \); error bounds for a Lagrange interpolant are derived in [36]. Let \( \csc(\cdot) \) denote the cosecant of the indicated radian angle. The uniform bound on \( \|\Delta_o^c\|_\infty \) from [36] is
\[ \|\Delta_o^c\|_\infty \leq \frac{2K}{(\rho - 1)\rho^{N-1}} + \epsilon^* \varphi_N \] (4.7)
where
\[ \varphi_N = \frac{1}{N} \sum_{k=0}^{N-1} \csc \left( \frac{2k + 1}{2N}\pi \right). \] (4.8)

The bounds of (4.6) and (4.7) each include (a multiple of) the term
\[ \frac{K}{(\rho - 1)\rho^{N-1}} \] (4.9)
which, as we will justify, can be chosen arbitrarily small when (4.1) is known to be an FIR model. Choose an \( L > 0 \) such that
\[ |g_{n-1}^o| \leq L\rho^{-k} \quad \forall k \in \{0, 1, \ldots, n - 1\}; \] (4.10)
\( g_{n-1}^o \) is the last nonzero impulse response coefficient of \( G_o(z) \). For any \( \rho > 1 \), the FIR \( G_o(z) \) is in \( H_{\infty, \rho} \) (\( G_o(z) \) is analytic in \( \mathbb{C} \)). For \( \rho \gg 1 \) sufficiently large, (4.10) is satisfied when \( L\rho^{-(n-1)} \geq |g_{n-1}^o| \); thus, choosing \( L = |g_{n-1}^o|\rho^{N-1} \) \((N \geq n)\) is sufficient for (4.10) to be satisfied. For the FIR truth model, \( G_o \), we see that
\[ |G_o(\rho e^{j\Omega})| = |g_0^o + g_1^o \rho e^{j\Omega} + \cdots + g_{n-1}^o \rho^{n-1} e^{j\Omega(n-1)}| \leq \sum_{k=0}^{n-1} |g_k^o|\rho^k. \] (4.11)
Upon substitution of (4.10) into (4.11), we see that $|G^\circ(\rho e^{j\Omega})| \leq nL$ for all $\Omega \in [0,2\pi]$; consequently, $\|G^\circ\|_{\infty,\rho} \leq K \equiv nL$ (see (4.3)). For the choices of $\rho \gg \max\{1, n|g^\circ_{n-1}|\}$ and $K = n|g^\circ_{n-1}|\rho^{\tilde{N}-1}$, (4.9) becomes $n|g^\circ_{n-1}|/(\rho - 1)$ which is arbitrarily small for arbitrarily large $\rho$; the dominating portion of the bound of (4.7) is the term, $\epsilon^* \varphi_{\tilde{N}}$. From the argument above, $\epsilon^*$ of (4.6) is dominated by the $\tilde{\gamma}/\alpha$ term (which is independent of $\tilde{N}$). The sequence $\varphi_{\tilde{N}}$ of (4.8) is increasing with $\tilde{N}$ [36]; we choose $\tilde{N} = n$ to minimize the bound of (4.7).

The complex exponentials of (4.4) cannot be "applied" to a physical system; this issue is addressed in an appendix to [31]. Such an experiment is accomplished, in practice, by exploiting Euler's identity: $e^{-j\Omega t} = \cos(k\Omega t) - j\sin(k\Omega t)$. Two experiments must be conducted, one with input sequence $u_1(k) = \alpha \cos(k\Omega t)$ and the other with $u_2(k) = \alpha \sin(k\Omega t)$. Linearity of the true system of (4.1) enables superposition of the input and output sequences via the relationships, $u_k = u_1(k) - ju_2(k)$ and $y_k = y_1(k) - jy_2(k)$, where $y_1(k)$ and $y_2(k)$ are the output sequences corresponding to excitation by $u_1(k)$ and $u_2(k)$, respectively. Because $y_k$ is computed in this manner, there are two real-valued disturbances, each bounded in magnitude by $\tilde{\gamma}$, which combine to form a complex disturbance, bounded in magnitude by $\sqrt{2}\tilde{\gamma}$, perturbing the measurement of $y_{\tilde{N}-1}$ in (4.5); this necessitates an expansion of the term $\tilde{\gamma}/\alpha$ to $\sqrt{2}\tilde{\gamma}/\alpha$ in the bound of (4.6). The experiment due to the input of (4.4) consequently consists of two real-valued experiments of $n$ samples each, performed at each of the $n$ frequencies of $S_n$, for a total of $N = 2n^2$ samples.

As a baseline for comparison, we shall conduct an experiment of the same duration,
$N = 2n^2$, with a stationary pseudorandom white input sequence, $\{u_k\}$, which is uniformly distributed over $[-\sigma, \sigma]$. We then apply the OVE algorithm [1] to generate an ellipsoidal bound on the $n$ Markov parameters of the FIR plant, $G^0(z)$. The performance of the OVE algorithm, in terms of ellipsoid hypervolume, improves with increasing input average power (as discussed in Chapter II). For fairness in our comparison, the random input should have the same (expected) average power as the sinusoids required to construct the input of (4.4). The average power of a signal, $u_k$, of duration $n - 1$ is

$$P_{\text{ave}} = \frac{1}{n} \sum_{k=0}^{n-1} u_k u_k^*.$$ 

The average power of a sinusoid of amplitude $\alpha$ (when an integer number of half-cycles are exactly contained in the interval $[0, n - 1]$) is $\alpha^2/2$. The expected average power of a white noise sequence of uniform distribution over the interval $[-\sigma, \sigma]$ is $\sigma^2/3$; hence, we choose $\sigma = \alpha \sqrt{1.5}$.

### 4.1.2 OVE-Based Weighting Function Identification: The FIR Case

The OVE algorithm can be used to estimate a set of feasible coefficients for the FIR model, as discussed in Section 3.4. Comparing (3.43) with (4.1), we see that $b_k^0 = g_{k+n}^0$ and that $n = m + \eta + 1$. Recall that use of the OVE algorithm exploits a priori knowledge of $m$, $\eta$, and $\gamma_k \geq |u_k|$ to generate an ellipsoid,

$$E_N = \{ \theta_b \in \mathbb{R}^{m+1} : (\theta_b - \theta_N)' P_N^{-1} (\theta_b - \theta_N) \leq 1 \},$$

(4.12)

which contains the true parameter vector, $\theta_b^0 = [b_0^0 \cdots b_m^0]' \in E_N$, when properly initialized with $E_0$ containing $\theta_b^0$. The final ellipsoid center, $\theta_N = \hat{\theta}_b$, is chosen as our
nominal plant parameterization as in (3.45). Recall that \( \Delta \theta_n \equiv \theta_n - \theta_N \). The true plant of (4.1), \( B^o(z) = B(z, \theta_n^o) \) can alternatively be expressed as

\[
B^o(z) = B_0(z) + \Delta_B^o(z) \quad (4.13)
\]

where \( \Delta_B^o(z) \equiv \Delta_B(z, \theta_n^o) \) as in (3.47).

Applying Theorems 3 and 4, we will identify worst-case magnitude bounds,

\[
B_i^* = \sup_{\theta \in E_N} |\Delta_B(e^{j\Omega_i}, \theta)|^2 \quad (4.14)
\]

(see also (3.49)) at each \( \Omega_i = 2\pi l/M \in \mathcal{S}_M \) of (3.26); we will later specify a choice of \( M (M = n \) is a possible choice). For any \( \epsilon > 0 \), the resultant set of magnitudes, \( (1 + \epsilon)\sqrt{B_i^*} \), are then interpolated, as in Section 3.6, to form a minimum phase polynomial, \( W(z) \), such that (see (3.48))

\[
\sup_{\theta \in E_N} |\Delta_B(e^{j\Omega_i}, \theta)| < |W(e^{j\Omega_i})| \quad \forall \Omega_i \in \mathcal{S}_M. \quad (4.15)
\]

### 4.1.3 Comparative Analysis

A \( H_2 \) robust control design example exploiting an ellipsoidal bound on the parameters of an FIR model appears in [29]. For comparison to the HJN procedure, we shall use an FIR model consisting of the first \( n = 11 \) coefficients in the pulse response of the transfer function (found in [29]),

\[
\frac{10 + 7 \cos(\frac{\pi}{2}) z}{1 - 1.4 \cos(\frac{\pi}{2}) z + 0.49 z^2};
\]

thus, the true plant of (4.1) is of the form

\[
y_k = \sum_{i=0}^{10} g_i^o u_{k-i} + v_k. \quad (4.16)
\]
Because \( \eta = 0 \), we have \( b_k = g_k^2 \) in (4.1). The disturbance sequence, \( \nu_k \), is chosen to be pseudorandom white noise of uniform distribution over the interval \([-0.1, 0.1]\); hence, \( \gamma_k = \dot{\gamma} = \gamma = 0.1 \) is a sufficient bound on \( |\nu_k| \). Let \( \alpha = 1 \) in (4.4).

**Applying the HJN Procedure**

From (4.8), we have \( \varphi_{11} = 2.4894 \). In the computation of \( \epsilon^* \) of (4.6), for \( \rho = 10^{10} \) we have \( K = 6.2 \times 10^{100} \) and \( K \rho^{-10}/(\rho - 1) = 6.2 \times 10^{-10} \). We also have \( \sqrt{2} \dot{\gamma}/\alpha = 0.1414 \) which indeed dominates in the computation of \( \epsilon^* = 0.1414 \) of (4.6). In a similar manner, the bound of (4.7) is dominated by the product \( \epsilon^* \varphi_{11} = 0.3521 \). We next simulate the truth model of (4.16) with the experiments implied by (4.4) and compute the pointwise frequency response estimates, \( G_o(e^{i\Omega_l}) \), of (4.5). The nominal plant model,

\[
G_o(z) = \sum_{k=0}^{10} \hat{g}_k z^k,
\]

is identified via Lagrange interpolation which is equivalent to the simultaneous solution of the \( n = 11 \) linear equations,

\[
G_o(e^{i\Omega_l}) = \hat{g}_0 + \hat{g}_1 e^{i\Omega_l} + \cdots + \hat{g}_{10} e^{10i\Omega_l},
\]

where

\[
\Omega_l = \frac{2l\pi}{11}, \quad l \in \{0, 1, \cdots, 10\},
\]

for the unknown \( \hat{g}_k \). Define the parametric deviation of the plant estimate from the true plant description as

\[
e_\theta = \left( \frac{\sum_{k=0}^{n-1} |g_k^2 - \hat{g}_k|^2}{\sum_{k=0}^{n-1} |g_k^2|^2} \right)^{\frac{1}{2}}. \tag{4.17}
\]
The Lagrange interpolant has, in general, complex valued coefficients. The parametric deviation of the Lagrange interpolant, for this trial, was $e_\theta = 3.7 \times 10^{-3}$.

**Applying the OVE-Based Identification Procedure**

For the identification experiment using the OVE algorithm, we use the identical disturbance sequence, $v_k$, from the HJN experiment of $N = 2n^2 = 242$ (real-valued) input-output pairs. The input is a white noise sequence of uniform distribution over the interval $[-\sigma, \sigma]$, where $\sigma = \sqrt{1.5}$. The OVE algorithm is initialized with $P_0 = 450I_{11}$ and $\theta_0 = 0$; consequently, $\theta^o \in E_0$. We process the observed data sequences using the OVE algorithm, assuming an FIR model with $m = 10$ and a disturbance bound, $\gamma_k = \hat{\gamma} = \gamma = 0.1$ for all $k$. The final ellipsoid center, $\theta_{242}$, has a parametric deviation of $e_\theta = 1.1 \times 10^{-3}$.

We now use the final ellipsoid matrix, $P_{N=242}$, to identify a weighting function, $W(z)$, overbounding an additive perturbation to the nominal plant as in (3.21). We are free to choose $M$, the number of uniformly spaced samples of $|W|$ of (4.15) along $\partial D$ (the $M$ roots of unity); due to conjugate-symmetry, the samples need only be computed at the roots with argument in the interval $[0, \pi]$. We choose $M$ odd to enable use of our Fourier series-based interpolation technique of (3.85). We desire $M$ to be large so that the resolution of the samples is sufficient to capture the nature of the additive plant perturbation (not the frequency response of the nominal plant, as in the HJN procedure); we choose $M = 77$. The set of magnitude samples, computed using Theorem 3 (Theorem 4 was not needed), appear as the $x$'s in Figure 10.
Reduced Order Weighting

To exactly interpolate the magnitude samples (see Figure 10), a polynomial, in general, must be of degree 76; as a weighting function, such a high order interpolant would overly complicate the control design. We instead choose to overbound the set of magnitude samples with a (reduced order) polynomial of degree 4, $W_4$, which overbounds the additive plant perturbation (4.15). We identify $W_4$ using the reduced order overbounding techniques of Section 3.6.2. We shall minimize the metric of $J_1$ of (3.87); hence, we seek the fourth degree overbounding polynomial with minimum area beneath the magnitude response.

The identification of $W_4$ requires the solution of a linear programming problem as suggested in (3.90). The resultant polynomial,

$$W_4(z) = 6.283 \times 10^{-3}(3.377 + z + 48.648z^2 + z^3 + 3.377z^4),$$

is not minimum phase and must have all zeros interior to $D$ reflected to their conjugate reciprocal location in $\mathbb{C}$ as in (3.86). The minimum phase polynomial, $\tilde{W}_4$, with $|\tilde{W}_4(z)| = |W_4(z)|$ for all $z \in \partial D$ is

$$\tilde{W}_4(z) = (30.404 + 1.175z + 4.255z^2 + 0.082z^3 + 0.148z^4) \times 10^{-2}.$$

Minimization of the metric, $J_2$ of (3.88), generated a polynomial numerically identical to $W_4(z)$. 
A Qualitative Check

We observe that the full order (76th degree polynomial) interpolant exhibits a peak at \( \Omega = \pi \). We evaluate (4.14) at \( \Omega = \pi \) and find that \( \sqrt{B_i^r(\pi)} = 0.3381 \) (this frequency was previously unsampled). The (full order) interpolant, evaluated at \( z = -1 \), has a magnitude of 0.3374, which does not bound \( \sqrt{B_i^r(\pi)} \); we also find that \( |\hat{W}_4(z = -1)| = 0.3355 \) does not overbound 0.3381 in magnitude. To remedy this failure to bound \( \sqrt{B_i^r(\pi)} \), we can multiply \( \hat{W}_4(z) \) by \( 1 + \varepsilon = 1.008 > 0.3381/0.3355 \) (see (3.70)); this sufficiently inflates \( |\hat{W}_4| \) to overbound the previously unsampled peak. An alternative, preferred means of bounding the additive perturbation magnitude, at the normalized radian frequency of \( \pi \), would be to incorporate the knowledge of \( \sqrt{B_i^r(\pi)} \) into the identification of \( W_4 \). The failure of the (full order) interpolant to bound \( \sqrt{B_i^r(\pi)} \) suggests that not enough magnitude samples of the worst-case perturbation were computed; the best remedy would be to collect additional samples, interlaced in frequency within the previous samples, for a total of 154 (uniformly-spaced) samples, which include \( \sqrt{B_i^r(\pi)} \).

To illustrate the capability, we uniformly scale the identified \( W_4 \), upward in magnitude (this introduces additional conservatism). The magnitude response of \((1+0.008)W_4\) appears as the solid line in Figure 10. The uniform HJN bound of 0.3521 in (4.7), overbounds \( |(1 + \varepsilon)W_4| \) for most \( \Omega \in [0, 2\pi] \); we also have \( \|(1 + \varepsilon)W_4\|_\infty = 1.008 \times 0.3606 = 0.3635 \).

The control design is simplified when using a reduced order weighting function, at the expense of increased conservatism in the weighting function. It would be
precarious to simply choose $M$ small, rather than oversample and identify a reduced order overbounding function (see the dashed curve of Figure 10, which illustrates the undersampled case). Some engineering insight should be exercised to ensure that peaks in the magnitude response of the plant perturbation are not missed; the choice of $\epsilon > 0$ adds this flexibility at the expense of a uniform increase in conservatism. In the absence of a disturbance ($v_k = 0$), the identified HJN nominal plant exactly equals the true plant when $n = 11$ point frequency response samples are computed, but not when $n < 11$ (aliasing then occurs). A choice of $n > 11$ improves neither the bound of (4.7), nor the computation of $G_e(e^{j\theta})$ of (4.5). The HJN technique is not at a disadvantage, using only $n = 11$ point frequency response samples (4.5); it has,
in fact, been tuned [32] to account for the prior knowledge that the plant is of FIR of length \( n \). The choice of \( M \) is independent of the length of the impulse response sequence of (4.13).

### 4.1.4 A Delayed FIR Example

We now use the first seven impulse response coefficients of the non-minimum phase transfer function

\[
\frac{z + 2z^2}{1 + 0.5z + 0.81z^2}
\]

as the truth model of (4.1). This transfer function is strictly proper; consequently, \( g_0^\circ = 0 \) and \( \eta = 1 \) (the influence of \( \eta \neq 0 \) is illustrated in (3.45)). So that no advantage is gained over the HJN procedure, we choose to ignore this prior knowledge (that \( \eta = 1 \)) and merely assume that \( m + 1 = 7 \) parameters must be estimated, given \( \eta = 0 \) \( (b_k^\circ = g_k^\circ) \). An OVE initialization with \( \theta_0 = 0 \) and \( P_0 = 10I_7 \) satisfies \( (\theta^\circ)^\intercal P_0^{-1} \theta^\circ = 0.8217 < 1 \); that is, \( E_0 \) contains the true parameter vector, \( \theta^\circ \).

#### Experiment Setup

As in the previous example, we again choose \( \sigma = 1 \) as the magnitude of the complex exponentials of (4.4) and \( \tilde{\gamma} = 0.1 \) as a bound on \( u_k \), which is again uniformly distributed over \([-0.1, 0.1]\); consequently, \([-\sqrt{1.5}, \sqrt{1.5}] \) is the interval over which the input, \( u_k \), is uniformly distributed (\( \sigma = \sqrt{1.5} \)). Because the length of the finite impulse response of the truth model is now seven, we choose the number of HJN unit circle magnitude samples as \( n = 7 \). We again choose \( \hat{N} = n = 7 \), as in (4.4), to minimize \( \varphi_\hat{N} \) subject to \( \hat{N} \geq n \). The total number of input-output data pairs is now
\[ N = 2n^2 = 98. \]

**Experiment Results**

The nominal model identified by the HJN procedure is

\[
G_0(z) = (-0.0007 + 0.0172i) + (1.0056 + 0.0026i)z + (1.5068 + 0.0470i)z^2 \\
+ (-1.5536 + 0.0067i)z^3 + (-0.4407 - 0.0181i)z^4 \\
+ (1.4885 - 0.0538i)z^5 + (-0.3757 + 0.0134i)z^6
\]

corresponding to \( e_\theta = 0.0277 \), as in (4.17). The final ellipsoid center is

\[
\theta_N = [0.0052 \ 1.0116 \ 1.5104 \ -1.5600 \ -0.4381 \ 1.4876 \ -0.3956],
\]

which corresponds to \( e_\theta = 0.0068 \). A choice of \( \rho = 10^{10} \) results in \( K = 2.72 \times 10^{30} \)
and \( K\rho^{-\gamma}/(\rho - 1) = 2.72 \times 10^{-10} \) (see (4.6) and (4.7)); hence, the uniform HJN bound of (4.7) is

\[ \epsilon^* \varphi_\gamma = (0.1414)(2.022) = 0.3114. \]

Using the final ellipsoid matrix, \( P_{N=98} \), and Theorems 3 and 4, magnitude samples at 100 uniformly-spaced normalized frequencies in \([0, 2\pi]\) were generated (denoted as \( \times \)'s in Figure 11). The 6th degree polynomial,

\[
W_6(z) = 0.228(1 + 4.655 \times 10^{-2}z + 1.352 \times 10^{-1}z^2 + 6.221 \times 10^{-2}z^3 \\
+ 5.906 \times 10^{-3}z^4 + 3.976 \times 10^{-3}z^5 + 8.724 \times 10^{-4}z^6),
\]

bounding the magnitude of the samples, was generated by minimization of (3.87) in (3.90). A comparison of \(|W_6(e^{i\Omega})|\) to the HJN bound appears in Figure 11.
**Reducing Conservatism**

In general, we can further reduce conservatism in our additive perturbation weighting by using the final ellipsoid as an initialization for another run with the same observed data record, $Z^{O_8}$ of (2.9); the new (final) ellipsoid is then used to identify the perturbation weighting. To illustrate this point, the entire data record, $Z^{O_8}$, was used for 20 consecutive runs with the OVE algorithm; the ellipsoid initialization for each run was provided by the final ellipsoid of the previous run. The final ellipsoid center is

$$\theta_N = [-0.006 \hspace{0.5cm} 1.002 \hspace{0.5cm} 1.503 \hspace{0.5cm} -1.564 \hspace{0.5cm} -0.4357 \hspace{0.5cm} 1.485 \hspace{0.5cm} -0.390],$$

which corresponds to $e_8 = 3.27 \times 10^{-3}$ (see (4.17)). Using the final ellipsoid matrix and Theorems 3 and 4, magnitude samples at 100 uniformly-spaced normalized frequencies
in $[0, 2\pi]$ were generated (the + symbols in Figure 11). A 2nd degree polynomial magnitude bound is

$$W_2(z) = 1.528 \times 10^{-1} + 2.386 \times 10^{-2}z + 9.314 \times 10^{-4}z^2,$$

which is depicted as the dash-dot line in Figure 11.

### 4.1.5 Summary Remarks

An advantage of our technique, over that of [32], [31], is that a nominal plant with a real-valued (versus complex-valued) impulse response is identified (which is conceptually desirable, but not necessary). Furthermore, in these FIR examples, our identified frequency-dependent weighting function is less conservative (over most of the frequency interval) than the uniform bound of (4.7) on the additive plant perturbation.

The "application" of complex exponentials in the HJN experiment is an inherently off-line procedure. For off-line identification, it is likely that we can significantly reduce conservatism in the identified, OVE-based perturbation weighting via sequentially reprocessing the identified data record ($Z^N$), using the final ellipsoid of the current run as an initialization for the following run; this monotonically decreases the ellipsoid hypervolume. A smaller ellipsoid hypervolume does not necessarily translate into a perturbation weighting which is completely overbounded by the perturbation weighting corresponding to an ellipsoid of greater hypervolume, as discussed in Chapter III; the relative orientation of the ellipsoids is also important. If the special case wherein an ellipsoid, $E_1$, is known to overbound another, $E_2 (E_2 \subset E_1)$, the worst-case
magnitude perturbation, at any normalized frequency, is greater for $E_1$ relative to $E_2$. The HJN procedure does not have the flexibility to benefit from any reprocessing of the observed data.

For any FIR plant, no control action is needed for stabilization; conservatism in the weighting function reduces the likelihood that a robust controller exists to meet a (unspecified) performance criteria, for all possible plant frequency responses. The HJN formulation addresses only stable LTI systems, perturbed by an UBB output disturbance. Our OVE-based identification methodology requires that the UBB disturbance enter as in the ARX model structure and addresses uncertainty weighting functions for systems with unknown stability characteristics. An additive perturbation bounding function can be identified when it is known that the number of unstable poles for each plant within the set of plants, characterized by the identified ellipsoid, is the same. The set of assumptions of the HJN procedure preclude the possibility that the true plant is type 1, whereas our OVE-based identification technique addresses the most general situation (no plant stability knowledge) by identifying a magnitude bound on a coprime factor perturbation.

Let $\Phi$ be a matrix whose $N$ rows are the regression vectors of (2.3) for time indices $k = 1, 2, \ldots, N$. An OVE identification experiment on an unstable system could cause difficulty, yet if a finite duration experiment delivers a full rank $\Phi$, then an ellipsoid overbounding of $\mathcal{F}^N$ of (2.12) is a well-posed problem. If such an experiment were successful, an identified ellipsoid could possibly be too conservative for subsequent use in a robust control design.
4.2 Sampled Data Example: Cargo Ship Steering

The identification of the steering dynamics of a maritime ship is discussed in [65] and summarized in [25]. The control objective is an autopilot design to maintain constant ship heading and velocity. The equations of motion in the horizontal plane are decoupled from the vertical channel and linearized about a nominal operating condition of $u_s = u_s^0$, a nominal forward ship velocity. In a ship-fixed coordinate frame (denoted by the $x_t$-$y_t$ coordinate axes), $u_s$ is the forward (longitudinal) velocity, $v_s$ is the side (lateral or sway) velocity, $\psi_s$ is the heading (yaw) angle, $r_s$ is the turn (yaw) rate, and $\delta_s$ is the rudder deflection; the $x'_t$-$y'_t$ coordinates denote an earth-fixed coordinate frame (see Figure 12). For the linearization, it is assumed that $\delta_s = r_s = v_s = 0$.

The resultant transfer function from rudder deflection to ship heading is of the form

$$G_S(s) = \frac{\psi_s(s)}{\delta_s(s)} = K_o \frac{s + 1/T_3}{s(s + 1/T_1)(s + 1/T_2)}.$$ 

For the cargo ship of [25] (the merchant ship, mariner class in [65]), the transfer function parameters are

$$K_o = -6.162 \times 10^{-5} u_s^2 \quad T_3 = 143.29/u_s$$
$$T_1 = 911.26/u_s \quad T_2 = 61.18/u_s . \quad (4.18)$$

For a nominal forward velocity of $u_s = u_s^0 = 7.7 \text{ m/s}$, $G_S$ has poles at $s = 0$, $s = -0.126$, and $s = -8.45 \times 10^{-3}$ and a zero at $s = -0.0537$. To reduce the likelihood of unrealistic rudder behavior, we introduce (the relatively fast) actuator dynamics
from the rudder command, $\delta_c$, to the rudder deflection, $\delta_r$:

$$G_R(s) = \frac{\delta_r(s)}{\delta_c(s)} = \frac{1}{s+1};$$

observe that the DC gain of this low-pass filter is unity ($G_R(s)|_{s=0} = 1$). We denote the composite transfer function from rudder command to heading angle as

$$G(s) = \frac{\Psi_s(s)}{\delta_c(s)} = G_S(s)G_R(s).$$

\section*{4.2.1 Experiment Setup}

The highest frequency pole of the augmented plant, $G(s)$, is at $1/2\pi$ Hz; a sample rate of 10 Hz (sample period, $T=0.1$ sec) is more than adequate to avoid aliasing
effects in the open-loop identification experiment.

The OVE algorithm estimates the parameters of a difference equation (2.4) of an ARX structure. The model of (4.19) corresponds to a differential equation; therefore, we are identifying a sampled-data description of the plant. There are many techniques for finding the discrete-time equivalent to a continuous-time system (e.g. see [66]). If we assume that the input is held constant over each sample interval (a zero order hold on the input), then a state space realization of $G(s)$ can be integrated over a sample interval to yield an equivalent discrete-time state space realization which can subsequently be converted into a transfer function. The Matlab c2d command, with the zoh option, accomplishes this task. For sample period $T=0.1$ sec, the discrete-time equivalent has

$$B^o(z) = (-0.5928 - 1.7188z^2 + 1.7344z^3 + 0.5586z^4) \times 10^{-6},$$

$$A^o(z) = 1 - 3.8915z + 5.6757z^2 - 3.6770z^3 + 0.8928z^4$$

as in (3.10). We thus have $\eta = 1$, $n = 4$ (poles), $m = 3$ (zeros), and $r = 8$ (unknown parameters). A comparison of the frequency responses of the discrete and continuous models appears in Figure 13; the models match well over the frequency band of interest.

### 4.2.2 The Disturbance Sequence

The underlying continuous-time simulation model is a linearization of the horizontal plane dynamics of a cargo ship; the dynamics are scheduled to the forward ship velocity, $u_s$, as in (4.18). We assume that a zero order hold is utilized at the rudder.
command input to the composite model. When the ship forward velocity is held constant as $u_s = u_s^c = 7.7 \text{ m/s}$, then the sampled output of the discrete model, $G^o(z) = B^o(z)/A^o(z)$, coincides with the output of the continuous plant (within the accuracy of the integrator used to update the dynamics of the continuous plant). We shall vary $u$ with sinusoidal disturbances in an effort to simulate the effect of waves encountered by the ship. This generates a difference in the output of the discrete model versus the scheduled continuous model output; this difference is an ARMAX-like disturbance which must be filtered by $A^o(z)$ prior to interpreting it as the ARX-like disturbance, $v_k$.

4.2.3 The Identification Experiment

The simulation was conducted using Simulink within Matlab 4.1. The continuous dynamics of $G(s)$ of (4.19) were implemented as an S-function, using a controller canonical state space realization. The use of a Runge-Kutta integrator for mixed
continuous and discrete systems is recommended; we chose a fifth order Runge-Kutta integrator with a maximum step size of 0.1 sec (the sample period), a minimum step size of $10^{-6}$ sec, and a tolerance of $10^{-6}$. The pseudorandom rudder command is white with a uniform distribution over $[-140^\circ, 140^\circ]$; a zero order hold, updated each sample interval, on the output of the Simulink random signal generator is necessary to prevent numerical problems when using a variable step size integrator. A saturation block at the output of the rudder dynamics transfer function limits the rudder deflection to $\pm 60^\circ$; this is assumed to be the physical limit of the rudder operation. The ship forward velocity profile is $u_s(t) = 7.7 + 1.5 \sin t - 0.5 \sin 5t$ (m/s). The Simulink block diagram is illustrated in Figure 14. The simulation was conducted for $t \in [0, 90]$ (seconds) resulting in $N = 901$ input-output sample pairs. Time histories of $\delta_r$, $\delta_c$, $\Psi_s$, and $u_k$ (the ARX disturbance) appear in Figure 15; the rudder deflection, $\delta_r$, stays within its saturation limits for the duration of the identification experiment.

The OVE algorithm was initialized with $\theta_0 = 0$ and $P_0 = 65I_6$; hence, the "true" parameter vector, $\theta^*$, is contained within $E_0$. The ARX disturbance sequence is bounded as $u_k \in [-0.0458, 0.0726]$ for all $k$; therefore, $\gamma = 0.0726$ sufficiently bounds $u_k$ in magnitude. To reduce conservatism in the ellipsoid, we reuse the entire data record, with the initialization $\theta_0 = \theta_N$ and $P_0 = P_N$. This recursive usage of the data record was repeated 20 times, until there was less than a 1% change in the final ellipsoid hypervolume between successive runs. The final ellipsoid center, $\theta_N = [\hat{\theta}_r \hat{\theta}_c]'$, 


which characterizes the nominal plant (see (3.12)), has

\[
\hat{\theta}_a = \begin{bmatrix} -3.62 & 5.16 & -3.43 & 0.90 \end{bmatrix}^T, \\
\hat{\theta}_b = 10^{-5} \begin{bmatrix} 1.26 & -0.41 & -1.35 & 0.26 \end{bmatrix}^T.
\]

The true plant is type one; in the face of parametric uncertainty, we must use the coprime factor plant perturbation, as discussed in Section 3.5. Using \( P_N \) of (3.1) and Theorem 3 (the use of Theorem 4 was unnecessary), magnitude samples of a bound on a coprime factor plant perturbation are generated at 99 uniformly spaced normalized frequencies; a polynomial (of reduced degree 2) overbounding the magnitude samples was generated via the constrained minimization of \( J_1 \) (3.87) as in (3.90). Following reflection (see (3.86)) of nonminimum phase zeros (in \( D \)) to the exterior of \( D \), we have a minimum phase weighting function,

\[
W(z) = -3.537 + 6.664z - 3.138z^2, \tag{4.20}
\]
bounding the magnitude samples (see Figure 16).

4.2.4 Control Design

Although the objective of weighting function identification has been accomplished, the question arises as to whether a robustly stabilizing controller exists for the nominal plant, characterized by $\theta_N$ as in (3.12), and the identified weighting function, $W(z)$ of (4.20). An admissible (a ratio of $H_\infty$ functions) controller, $C$, which stabilizes the nominal plant, $G_\alpha$, and satisfies (3.66), is a robustly stabilizing controller. We shall use the state space design procedure of [67], which we shall refer to as the DGKF procedure. We require, as a first step, a conformal mapping of the unit disk of the $z$-plane, $D$, onto the open left half plane of the complex $s$ domain; for example [53],

Figure 15: Identification Experiment: Time Histories
the substitution of \( z = \frac{s - 1}{s + 1} \) into the frequency functions of (3.66) accomplishes the task.

The DGKF procedure of requires the problem statement in the form of a generalized plant as in Figure 17. The DGKF procedure finds a controller \( K \) which stabilizes the generalized plant, \( \Gamma \), and minimizes the infinity norm of the closed-loop transfer function, \( T_{\mathcal{C}w} \), from the external input vector, \( w \), to the output vector, \( \zeta \); we wish to solve the two-block \( \mathcal{H}_\infty \) problem of (3.66). The governing open-loop equations of the generalized plant,

\[
\Gamma = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{bmatrix},
\]

are

\[
\zeta = \Gamma_{11}w + \Gamma_{12}\dot{u}
\]

\[
\dot{\zeta} = \Gamma_{21}w + \Gamma_{22}\dot{u}.
\]
With the choices of $\zeta = [WA_o^{-1}\hat{y} \ W A_o^{-1} \hat{u}]'$ and $\hat{y} = G_o \hat{u} + w$, we have $\Gamma_{21} = 1$, $\Gamma_{22} = G_o$, $\Gamma_{11} = [WA_o^{-1} \ 0]'$, and $\Gamma_{12} = [WA_o^{-1}G_o \ WA_o^{-1}]'$. This yields the closed-loop transfer function (see (3.66))

$$
T_{\zeta w} = \begin{bmatrix} WA_o^{-1}(1 - G_o K)^{-1} \\
-WA_o^{-1}K(1 - G_o K)^{-1} \end{bmatrix},
$$

under the feedback relation $u = K \hat{y}$. Following the inverse (conformal) mapping of $K$ into a function of $z$, $C(z) = -K(z)$ robustly stabilizes $G$ of (3.14) provided that $\gamma_c = \|T_{\zeta w}\|_\infty \leq 1$. The controller order will be of the same order as the generalized plant, $\Gamma(s)$.

Figure 17: Generalized Plant

### 4.2.5 Summary Remarks

For the cargo ship example, the generalized plant has 12 states; each of the two minimal state space realizations of $WA_o^{-1}$ require four states and a minimal realization of $G_o$ requires four states. When the degree of $W$ is greater than or equal to the degree of $A_o$, an increase in the degree of $W$ by one increases the number of states in the DGKF controller by two and exacerbates numerical error in using the DGKF procedure; this highlights the practical necessity of using a reduced order weighting function at the expense of increased conservatism in the uncertainty characterization.
The DGKF procedure delivers a 12-state controller achieving $\gamma_e = 977.59$ in (3.66). By introducing a constant scaling factor of magnitude 584.0 for the nominal plant and perturbation, $\gamma_e = 74.47$ was achieved. By introducing a frequency dependent plant and perturbation scaling, it may be possible to further reduce the magnitude of $\gamma_e$ and find a controller achieving $\gamma_e \leq 1$ (a sufficient condition for robust stabilization); see [68] for a discussion of such loopshaping procedures. A redesign of the identification experiment with a reduced sample rate $(1/T)$ could also possibly result in a controller achieving $\gamma_e \leq 1$; for example, $T = 0.84$ was used with the rudder command limited to $\pm 68^\circ$ (to avoid saturation of the rudder deflection) and $\gamma_e = 15.66$ was achieved. The zero order hold on the rudder command is part of the continuous plant; therefore, a change in the sample period does change the continuous truth model as well as the sampled-data nominal plant.
4.3 Additive Perturbation Example

We now conduct an identification experiment for the system of (2.32), which was previously used to demonstrate input synthesis using the OVE-ISP. The true system is described by the difference equation

\[ y_k - 0.9y_{k-1} + 0.34y_{k-2} + 0.11y_{k-3} - 0.03y_{k-4} = 2.0u_k - 0.5u_{k-2} + v_k, \]

where \( u_k \) is the input, \( y_k \) is the output, and \( v_k \) is the ARX disturbance. The true transfer function, from input to output, is

\[ G^0(z) = \frac{2.0 - 0.5z^2}{1 - 0.9z + 0.34z^2 + 0.11z^3 - 0.03z^4}. \]

We choose the plant excitation as a (known) pseudorandom white noise sequence, uniformly distributed over \([-1.0, 1.0]\). We similarly choose the (unknown) ARX disturbance sequence as a pseudorandom white noise sequence, uniformly distributed over \([-0.1, 0.1]\). The initial ellipsoid, \( E_0 \), with \( P_0 = 9I_7 \) and \( \theta_0 = 0 \), contains the true parameter vector, \( \theta^* \) of (2.33); that is, \( \theta^*P_0^{-1}\theta^* \leq 1 \). We choose to observe \( N = 200 \) input-output sample pairs, as in (2.9). The time histories of the input, \( \{u_k\} \), output, \( \{y_k\} \), and disturbance, \( \{v_k\} \), sequences appear in Figure 18.

The OVE algorithm was used to process a total of 102 consecutive runs with the entire data record, \( Z^N \) of (2.9); the final ellipsoid from each run was used to initialize the next run. The ellipsoid hypervolume (see (2.14)) following the first run was \( 1.96 \times 10^{-7} \) and was \( 2.06 \times 10^{-11} \) upon completion of the 102\(^{nd} \) run. The nominal plant, as characterized by the final ellipsoid center, \( \theta_N \), is

\[ G_o = \frac{2.003 + 0.002z - 0.506z^2}{1 - 0.889z + 0.329z^2 + 0.116z^3 - 0.032z^4}. \] (4.23)
4.3.1 Coprime Factor Perturbation

The final ellipsoid matrix and Theorem 3 (Theorem 4 was not needed) were used to generate 99 uniformly spaced (in normalized frequency) magnitude samples of a worst-case coprime factor perturbation weighting (see (3.65)); the constrained minimization of $J_1$ (3.90) (and subsequent reflection per (3.86)) identified a minimum-phase polynomial, of degree 2, as

$$W(z) = 0.5126(1 - 0.2305z + 0.0133z^2),$$

bounding the magnitude samples (see Figure 19). The DGKF procedure of [67] was again used to generate a 12-state, robustly stabilizing controller achieving $\gamma_c = .2575 < 1$ as in (3.66). The controller has a DC gain of 1.6149 and the poles and
zeros are given in Table 4; many of the poles and zeros are numerically close to being common roots, which suggests that there may exist a reduced order controller which is robustly stabilizing.

Table 4: Robustly Stabilizing Controller Poles and Zeros

<table>
<thead>
<tr>
<th>Zeros</th>
<th>Poles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.49717361389479 + 0.50081852106731i</td>
<td>0.49717361564873 + 0.50081857006774i</td>
</tr>
<tr>
<td>0.49717361389479 - 0.50081852106731i</td>
<td>0.49717361564873 - 0.50081857006774i</td>
</tr>
<tr>
<td>0.49717349438523 + 0.5008185911119i</td>
<td>0.49717349263123 + 0.5008185011069i</td>
</tr>
<tr>
<td>0.49717349438523 - 0.5008185911119i</td>
<td>0.49717349263123 - 0.5008185011069i</td>
</tr>
<tr>
<td>-0.52490049155412 + 0.10908619046228i</td>
<td>0.70336990970692</td>
</tr>
<tr>
<td>-0.52490049155412 - 0.10908619046228i</td>
<td>-0.67009145141614</td>
</tr>
<tr>
<td>0.52548708490008</td>
<td>-0.3114076761506 + 0.00000021395167i</td>
</tr>
<tr>
<td>-0.31140784850017</td>
<td>-0.3114076761506 - 0.00000021395167i</td>
</tr>
<tr>
<td>-0.31140748672979</td>
<td>0.20605117551657 + 0.00000049886102i</td>
</tr>
<tr>
<td>0.20956971362109</td>
<td>0.20605117551657 - 0.00000049886102i</td>
</tr>
<tr>
<td>0.20605117300412 + 0.000000419357047i</td>
<td>-0.000000000000082 + 0.00000051042113i</td>
</tr>
<tr>
<td>0.20605117300412 - 0.000000419357047i</td>
<td>-0.000000000000082 - 0.00000051042113i</td>
</tr>
</tbody>
</table>

4.3.2 Additive Perturbation

The nominal identified plant of (4.23) is stable; that is, all roots of the (nominal) characteristic polynomial, $A_\circ(z)$, are exterior to the closed unit disk ($D \cup \partial D$). We apply the Kharitonov analysis procedure discussed in Section 3.3.3 to identify the following Kharitonov polynomials:

\[
\begin{align*}
    f_1(s) &= 0.1010s^4 + 0.4827s^3 + 1.3724s^2 + 1.6544s + 0.3952 \\
    f_2(s) &= 0.1010s^4 + 0.5670s^3 + 1.3724s^2 + 1.4147s + 0.6397 \\
    f_3(s) &= 0.1010s^4 + 0.5670s^3 + 1.3724s^2 + 1.4147s + 0.3952
\end{align*}
\]
which are all Hurwitz (the real part of all roots, of each $f_i$, are negative). The lead coefficient is in the interval $[0.1010, 0.1612]$, which does not contain the zero element; therefore, all plants parameterized by the final ellipsoid, $E_N$, are stable (see, e.g., [59]). Because the number of unstable poles for each plant within $\mathcal{G}$ of (3.14) is the same, the additive plant perturbation uncertainty characterization of (3.20) can (safely) be used to address solutions to (3.22) for suitable weights, satisfying (3.21).

To compute samples of an additive perturbation weighting (3.21), the nonlinear optimization of (3.25) was solved at 99 uniformly spaced, normalized frequencies in $[0, 2\pi)$ using the Matlab 4.1 `constr` command and the final ellipsoid, $E_N$. A minimum-
phase polynomial (computed via minimization of $J_1$ as in (3.90) with subsequent reflection per (3.86)), of degree 2, bounding the magnitude samples, is (see Figure 20)

$$W_\omega(z) = 0.1158(1 + 1.0141z + 0.2571z^2).$$

(4.24)

Symmetric peaks in the loci of samples are qualitatively apparent in the neighborhood of $0.85$ and $2\pi - 0.85$ radians. Additional magnitude samples were taken in the vicinity of $\Omega = 0.85$ with a resolution of $10^{-3}$ radians. A peak of magnitude less than 0.2216 was sampled at $\Omega = 0.852$ radians. From (4.24), we have $|W_\omega(e^{0.852j})| = 0.2229$; the previously unsampled peak is therefore bounded.

![Figure 20: Additive Perturbation Weighting](image-url)
4.3.3 Summary Comments

When we have knowledge that the number of unstable poles for each plant within $\mathcal{G}$ of (3.14) is identical, the additive perturbation uncertainty characterization is preferred relative to the coprime factor perturbation uncertainty characterization. An additive perturbation is inherently less conservative, because knowledge of the nominal plant is incorporated into the worst-case uncertainty analysis (see (3.23)); whereas, the frequency response of the nominal plant does not affect the determination of a coprime factor perturbation weighting (3.65).

Characterizing the set of robustly stabilizing controllers for the additive perturbation case (3.22) (solving a one-block $H_\infty$ problem) is less complex relative to the coprime factor perturbation case (3.66) (which requires the solution of a two-block $H_\infty$ problem). From a control design perspective, the additive plant perturbation is also more versatile than the coprime factor perturbation in terms of the types of robust control problems which are solvable. Use of the coprime factor perturbation uncertainty characterization is necessary whenever all plants within $\mathcal{G}$ do not possess exactly the same number of unstable poles.
CHAPTER V

Summary Comments

The initial focus of this dissertation was on ramifications of the use of the OVE algorithm, a particular bounding-ellipsoid algorithm, in a parameter set estimation (PSE) experiment. New results on the asymptotic properties of the OVE algorithm were uncovered. An input synthesis procedure for improving the transient performance of a PSE experiment conducted with the OVE algorithm was also presented. The focus then shifted to an investigation of the potential use of knowledge of an ellipsoidal bound on system parameters.

The main results of this dissertation involved the "connection" of such PSE algorithms to the robust control problem. Important and effective techniques for the identification of modeling uncertainty information, vital to the robust control designer, were developed. Several examples uncover important control design issues and serve to illustrate the success of the developed techniques.

The specific contributions of the dissertation research and comments on unsolved problems for future research are briefly summarized in the following.
5.1 Dissertation Contributions

Ellipsoid algorithms are a convenient and computationally efficient means of characterizing the set of feasible parameters in a parameter set estimation (PSE) experiment. The OVE algorithm [1] is the least conservative, in characterization of the set of feasible parameters, among the class of overbounding, recursive ellipsoid algorithms. The limit of achievable performance, of a PSE technique, must be determined via an analysis of its asymptotic properties.

Asymptotic properties of the OVE algorithm were investigated in Chapter II. The OVE algorithm was derived, from geometrical arguments, in a transformed parameter space. The relationship between the ellipsoid hypervolume and its analogue in the transformed space was established in Theorem 1; this result was essential to prove that the prediction error, \( e_{k+1} = y_{k+1} - \phi^T_{k+1} \theta_k \), is asymptotically bounded by the disturbance bounding sequence, \( \{ \gamma_k \} \) (see Lemma 2). Theorem 2 builds upon this result and proves that if, while processing the observed data record, the OVE algorithm does not detect an inconsistency in the observed data and the ellipsoid matrix remains well-conditioned, then the asymptotic drift rate of the ellipsoid center decays to zero. These issues were not addressed in [1] or [12] and represent a new contribution to the results for the OVE algorithm.

Parameter set estimation provides bounded modeling information that is well-suited for exploitation in robust control design. The more conservative a PSE technique is, the less likely it is that a robust controller exists to meet desired closed-loop performance objectives for the ensemble of frequency responses of all plants param-
eterized by the resultant set estimate. The choice of plant excitation for a PSE experiment has significant influence upon the quality of the modeling information resulting from an analysis of the observations. We derived the OVE-Based Input Synthesis Procedure (OVE-ISP) as an effective means of achieving maximal hyper-volume reduction during any transient phase of the identification experiment. The OVE-ISP derives its effectiveness from intuitive geometrical arguments relative to the ellipsoidal bounding and measurement update characteristics of the OVE algorithm. Under a fixed ratio of signal energy to disturbance energy, the steady-state performance of the OVE-ISP was comparable to the performance of a related input synthesis technique.

The main contribution of this dissertation is in providing a consistent means of identifying the modeling information needed to pursue existing $H_\infty$ robust control design techniques. To accomplish this $H_\infty$ robust control design oriented identification task, the ellipsoidal PSE resulting from an identification experiment using the OVE algorithm is translated into a nominal plant model together with an associated worst-case uncertainty bounding function, also referred to as a weighting function. Guidelines for the choice of an appropriate plant perturbation type are provided. A Kharitonov analysis-based procedure was developed to provide sufficient conditions for determining whether an additive perturbation, to the nominal plant, is suitable as an uncertainty characterization.

Identification of a perturbation bounding weighting function is accomplished via a sampling strategy in tandem with an interpolation step. At any normalized frequency
of interest, a worst-case analysis is used to identify a magnitude bound on the plant perturbation. A clever means of interpolating such magnitude samples was devised wherein a real-valued magnitude interpolant over a real-valued domain is identified and subsequently extended to an appropriate complex-valued function defined over the complex domain. The order of the identified weighting function, in general, coincides with the number of worst-case magnitude samples taken. Because the number of states required in a robust controller increases with increasing weighting function order and more numerical error occurs in the computation of a robust controller as the controller order increases, a technique for identifying reduced-order uncertainty bounding functions was also derived. The use of a reduced-order weighting function increases conservatism in the uncertainty characterization and could result in our failure to identify a robust controller, even though such a controller can be identified for the full-order weighting.

5.2 Open Problems

The OVE algorithm can be used to estimate the set of feasible parameters for an ARX model in the presence of an additive, bounded disturbance. It is possible to adapt the OVE algorithm for use with ARMAX structures, as demonstrated in [12]; less conservative means of identifying a consistent set-theoretic ellipsoidal bound for ARMAX structures should be investigated. Application of the OVE to an FIR model presented no difficulty; an extension of the OVE to ARX processes with more than one control input should likewise be possible. A modification of the OVE algorithm to PSE for time-varying systems has already been accomplished in [26]; an adaptation of
the OVE-ISP for input synthesis in the time-varying case remains to be accomplished.

Use of an additive plant perturbation as a characterization of the parametric uncertainty of an ellipsoid bound requires knowledge, or the presumption thereof, that all plants, parameterized by the ellipsoid, have exactly the same number of unstable poles. We provided an analysis procedure to determine whether all plants parameterized by the ellipsoid are stable; an additive plant perturbation can be used in this most common situation. If the identified nominal plant model has exactly \( k > 0 \) unstable poles, it may be quite difficult to develop sufficient conditions to guarantee that every plant, parameterized by the ellipsoid, also has exactly \( k \) unstable poles.

For the additive plant perturbation case, we observed that the argument of extremum of the nonlinear optimization always resided on the ellipsoid boundary; justification as to why this should be expected to occur is an open issue.

Once a consistent ellipsoidal bound on the parameters of a multi-input, single-output (MISO) transfer function has been identified; the exploitation of such an ellipsoidal bound in a robust control design remains a problem open for future investigation. A further generalization would examine robust control design for a multi-input, multi-output system wherein an ellipsoidal bound is available on the parameters of each MISO transfer function element of a transfer function matrix.
Appendix A

Notation

The following is a summary of the notation used throughout this dissertation.

- $\mathcal{P}$: unknown SISO process
- $y_k$: measured output of $\mathcal{P}$
- $u_k$: known input to $\mathcal{P}$
- $\phi_k$: regression vector, regressor
- $n, m$: number of poles, zeros of $\mathcal{P}$
- $\theta = [a_1 \cdots a_n b_0 \cdots b_m]'$: feasible parameter vector
- $\theta^*$: true process parameters
- $r = n + m + 1$: number of unknown parameters
- $T$: sample period
- $\eta, \eta T$: process delay index, time
- $|v_k| \leq \gamma_k$: disturbance sequence and bound
- $\gamma, \gamma^*$: absolute bounds of $\gamma_k$
- $\gamma = \gamma^*$: a constant bound on $|v_k|$ observed data record
- $\mathcal{F}_N = [y_1 u_1 \cdots y_N u_N]$: index of final measurement
- $N$: feasible parameter set at time $k$
- $\mathcal{F}_k$: batch feasible parameter set
- $\partial \mathcal{E}_k = \{ \theta \in \mathbb{R}^r : \Delta \theta P_k^{-1} \Delta \theta \leq 1 \}$: the ellipsoid boundary
- $\mathcal{E}_k$: ellipsoid overbounding $\{ E_{k-1} \cap \mathcal{F}_k \}$
- $\mathcal{E}$: center of $\mathcal{E}_k$
- $P_k = P_k^t > 0$: ellipsoid matrix for $\mathcal{E}_k$
- $\mathcal{N} = \{ 1, 2, 3, \ldots \}$: the set of natural numbers
- $\mathbb{R}$: field of real numbers
- $\mathbb{C}$: field of complex numbers
- $\tilde{\theta} = J_k^{-1}(\theta - \theta_k)$: transformed parameters
- $V_k = ||P_k||, \dot{V}_k = ||\dot{P}_k||$: hypervolume measures
- $E_k = \{ \tilde{\theta} \in \mathbb{R}^r : \tilde{\theta}^T \tilde{\theta} \leq 1 \}$: unit radius hypersphere

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\( \nu_1 \)

principal ellipsoid semiaxis

\( \nu(\cdot) \)

null space of matrix argument

\( \hat{\sigma}(\cdot), \varrho(\cdot) \)

maximum and minimum singular values

\( \text{Re}(\cdot), \text{Im}(\cdot) \)

real and imaginary part

\( \tau_k \)

change in \( \hat{\theta}_k \) along \( +\hat{\phi}_k \)

\( \alpha_k \)

location of \( \hat{H}^* \) cut along \( +\hat{\phi}_k \)

\( 2\beta_k \)

hyperplane separation in \( \hat{\theta} \) domain

\( \delta_k(1 - \sigma_k) \)

semiminor axis length squared in \( \hat{\theta} \) domain

\( \delta_k \)

squared length of remaining semiaxes

\( Z_k(z) \)

\( D = \{ z \in \mathbb{C} : \| z \| < 1 \} \)

the (open) unit disk

\( \partial D = \{ z \in \mathbb{C} : \| z \| = 1 \} \)

the unit circle

\( G \)

feasible transfer function set

\( G(z) = B(z)/A(z) \)

any element of \( G \)

\( B(z), A(z) \)

numerator and denominator polynomials

\( G_o(z) = B_o(z)/A_o(z) \)

nominal plant model

\( G^*(z) = B^o(z)/A^o(z) \)

true plant model

\( \theta_b, \theta_a \)

numerator and denominator coefficients

\( \hat{\theta}_b, \hat{\theta}_a \)

coefficients of nominal plant

\( \Delta B(z), \Delta A(z), \Delta G(z) \)

additive perturbations to \( B, A, G \)

\( W(z) \)

perturbation bounding weighting function

\( S_M = \{ \Omega_l \} \)

the sample frequency set

\( \Omega_l = \frac{2\pi l}{M} \)

\( l \in \{ 0, 1, \cdots, M - 1 \} \)

\( W_l = |W(\omega_m)| \)

samples of magnitude response of \( W \)

\( \bar{x} \)

complex conjugate of \( x \)

\( x^* \)

worst case or optimal \( x \)

\( x' \)

transpose of \( x \)

\( x'' = \bar{x}' \)

conjugate transpose of \( x \)

\( \Phi \)

matrix of regressors

\( \gamma_a, \gamma_c \)

additive, coprime factor performance

\( \nabla(\cdot) \)

the gradient operator
BIBLIOGRAPHY


