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Properties of order statistics from bivariate exponential distributions

Baggs, Maria Geraldine Edra, Ph.D.
The Ohio State University, 1994
PROPERTIES OF ORDER STATISTICS FROM BIVARIATE EXPONENTIAL DISTRIBUTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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To my parents
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With reverence and awe, I acknowledge my Lord and Savior, Jesus Christ, in whom I live and move and have my being.
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Symbol | Meaning
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\( F(x_1, x_2) \) | \( P(X_1 \leq x_1, X_2 \leq x_2) \)
\( \overline{F}(x_1, x_2) \) | \( P(X_1 > x_1, X_2 > x_2) \)
\( F_i(x_i), i=1,2 \) | \( P(X_i > x_i) \)
\( \overline{F}(x_i), i=1,2 \) | \( P(X_i \leq x_i) \)
\( f(x_1, x_2) \) | \( \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2} \)
\( f_i(x_i), i=1,2 \) | \( \frac{\partial F(x_i)}{\partial x_i} \)
\( r_i(t) \) | \( \frac{A_i(t)}{E_i(t)} \), failure rate or hazard rate function
\( c(u, v) \) | \( F(F_1^{-1}(u), F_2^{-1}(v)), (u, v) \in (0, 1) \times (0, 1) \)
and \( F_i^{-1}(t) = \inf\{y : F_i(y) \geq t\}, 0 < t < 1 \)
copula function of \((X_1, X_2)\)
\( T_1 \) | \( \min(X_1, X_2) \)
\( T_2 \) | \( \max(X_1, X_2) \)
\( H(t_1, t_2) \) | \( P(T_1 \leq t_1, T_2 \leq t_2) \)
\( \overline{H}(t_1, t_2) \) | \( P(T_1 > t_1, T_2 > t_2) \)
\( h(t_1, t_2) \) | \( f(t_1, t_2) + f(t_2, t_1), 0 < t_1 < t_2 < \infty \)
\( F_0(t_i) \) | \( P(T_i \leq t_i) \)
\( \overline{F}_0(t_i), i=1,2 \) | \( P(T_i > t_i) \)
\( c_\alpha(u, v) \) | \( H(F_1^{-1}(u), F_2^{-1}(v)), (u, v) \in (0, 1) \times (0, 1) \)
copula function of \((T_1, T_2)\)
\( B(n, p, j) \) | \( \binom{n}{j} p^j (1-p)^{n-j} \)
the BVE | bivariate exponential model
due to Marshall and Olkin (1967)
ACBVE | bivariate exponential model due to Block and Basu (1974)
FBVE | bivariate exponential model due to Freund (1961)
GBVE | bivariate exponential model due to Gumbel (1960)
RBVE | bivariate exponential model due to Raftery (1984)
ACBVE₂ | bivariate exponential model due to Sarkar (1987)
BEE | bivariate exponential model due to Friday and Patil (1977)
GH distribution | generalized hyperexponential distribution (see section 1.3)
PQD | positively quadrant dependent (see section 2.2)
NQD | negatively quadrant dependent (see section 2.2)
IFR | increasing failure rate property (see chapter IV)
DFR  decreasing failure rate property (see chapter IV)
IFRA  increasing failure rate average property (see chapter IV)
CHAPTER I

Introduction

1.1 Basic Notations

We begin by introducing the basic notations used in this dissertation.

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and $\mathbb{R}^+_2 = (0, \infty) \times (0, \infty)$ denote the real line, the positive half of the real line, the real plane, and the first quadrant respectively.

For an absolutely continuous random variable (rv) $X$ on $\mathbb{R}^+$, let its probability density function (pdf) be denoted by $f$, its distribution function (df) by $F$, and its survival function by $\overline{F}$. Then

$$f(x) = \frac{dF(x)}{dx}, \quad F(x) = P(X \leq x), \quad \text{and} \quad \overline{F}(x) = 1 - F(x).$$  \hspace{1cm} (1.1)

We denote the kth moment, mean, and variance of $X$ by $E(X^k)$, $E(X)$, and $\text{Var}(X)$, respectively, where

$$E(X^k) = \int_{\mathbb{R}^+} x^k dF(x) \quad \hspace{1cm} (1.2)$$

$$E(X) = \int_{\mathbb{R}^+} x dF(x) = \int_{\mathbb{R}^+} \overline{F}(x) dx \quad \hspace{1cm} (1.3)$$

and

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$  \hspace{1cm} (1.4)
The median of $X$, denoted $\text{Med}(X)$, is the 50th percentile of the df $F$, that is $\text{Med}(X)$ is the smallest value satisfying $F(\text{Med}(X)) = \frac{1}{2}$.

Denote the failure rate or the hazard rate of $X$ by $r(x)$. Then, $r(x) = \frac{f(x)}{\overline{F}(x)}$, whenever $\overline{F}(x) \neq 0$.

Now, we turn our attention to bivariate variables $X_1$ and $X_2$ on $\mathbb{R}_2^+$. We define their joint df as

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2), \quad (x_1, x_2) \in \mathbb{R}_2^+. \quad (1.5)$$

Then, the marginal df of $X_i$, $i=1,2$ will be

$$F_i(x_i) = \lim_{x_j \to \infty} F(x_1, x_2), \quad x_i \in \mathbb{R}^+, \ i \neq j. \quad (1.6)$$

The survival df can now be defined. It is given by

$$\overline{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

$$= 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}_2^+. \quad (1.7)$$

If the df $F(x_1, x_2)$ is absolutely continuous, then the joint pdf of $(X_1, X_2)$ exists and is obtained by

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}, \quad (x_1, x_2) \in \mathbb{R}_2^+. \quad (1.8)$$

We define the mean of a function $g : \mathbb{R}_2^+ \to \mathbb{R}$ by

$$E[g(X_1, X_2)] = \int_{\mathbb{R}_2^+} g(x_1, x_2) dF(x_1, x_2). \quad (1.9)$$

A measure of dependence between $X_1$ and $X_2$ is the covariance between them given by $\text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2)$. A corresponding measure in standard
units is the Pearson's product moment correlation coefficient defined as

$$\text{Corr}(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}}$$  

(1.10)

We also explore the copula function to describe the dependence between $X_1$ and $X_2$. We write

$$c(u, v) = F(F_1^{-1}(u), F_2^{-1}(v))$$  

(1.11)

where $(u, v) \in (0, 1) \times (0, 1)$ and $F^{-1}(t) = \inf\{y : F(y) \geq t\}, 0 < t < 1$. For example, $c(u, v) = uv$ for independent $(X_1, X_2)$. Finally, we are interested in the conditional distribution of $X_2$ given $X_1 = x_1$. Provided $F(x_1, x_2)$ is absolutely continuous, we have the conditional pdf

$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}, \quad x_2 \in \mathbb{R}^+$$  

(1.12)

where $f_1(.)$ is the marginal pdf of $X_1$.

We use the notation ‘$\sim$’ to say ‘distributed as’. If $X$ is exponentially distributed with pdf $f(x) = \lambda e^{-\lambda x}, x \geq 0$, we write $X \sim \text{e}(\lambda)$.

We denote the order statistics corresponding to the bivariate pair $(X_1, X_2)$ by $T_1 = \min(X_1, X_2)$ and $T_2 = \max(X_1, X_2)$. The spacing between them is the difference $T_2 - T_1$. We use $H(t_1, t_2)$ to denote the survival df of $(T_1, T_2)$, and if it exists, $h(t_1, t_2)$ will be the joint pdf, $0 < t_1 < t_2 < \infty$. The marginal df of $T_i$ is $F_i(t_i), i=1, 2$. We denote the copula function of $(T_1, T_2)$ by $c_0(u, v)$.

For convenience, we denote by $I_A$ or interchangeably $I(A)$ the indicator function

$$I_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$  

(1.13)
We also use \( \log(x) \) to refer to the natural logarithm of \( x \).

Some commonly used notations and abbreviations are collected on page xii for ready reference. We will introduce additional notations and conventions as and when found necessary.

### 1.2 Some Bivariate Exponential Distributions

The exponential distribution occupies a well-established position in survival analyses and reliability theory. A basic study of these areas commence with an assumption of an underlying exponential model. This is possibly due to the lack of memory property (LMP) possessed by the distribution, which allows for an easy solution to mathematical formulations.

This concept of “memorylessness” or LMP has the following definition:

\[
P(X > t + s) = P(X > t)P(X > s) \quad \forall s, t > 0.
\]

The implication is clear. A component that has survived \( t \) time units has the same chance of surviving \( s \) more units as that of a new component. The LMP property is a well-known characterization of an exponential rv.

An obvious way of extending the concept in the bivariate setting is to impose the condition

\[
\overline{F}(t_1 + s_1, t_2 + s_2) = \overline{F}(t_1, t_2)\overline{F}(s_1, s_2)
\]

\( \forall s_1, s_2, t_1, t_2 \geq 0 \), where \( \overline{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2) \) is the bivariate survival function. According to Marshall and Olkin (1967), the unique distribution satisfying (1.15) is the product of exponential marginals and thus the condition imposed by
(1.15) turns out to be too restrictive. So, they propose another definition of bivariate lack of memory property (BLMP). They require that the two components which already survived $t$ units of time be given the same chance of surviving $s_1$ and $s_2$ more units, respectively, as that of new components. Symbolically, the BLMP property holds if the following is true:

$$F(t + s_1, t + s_2) = F(t, t)F(s_1, s_2)$$  \quad (1.16)

$\forall s_1, s_2, t \geq 0$. Now, the solution is a broader class of distributions with survival functions

$$F(x_1, x_2) = \begin{cases} e^{-\theta x_1}F_2(x_2 - x_1), & 0 < x_1 \leq x_2, \\ e^{-\theta x_2}F_1(x_1 - x_2), & 0 < x_2 \leq x_1, \end{cases}$$  \quad (1.17)

where $F(x_1, 0) = F_1(x_1)$ and $F(0, x_2) = F_2(x_2)$ for some $\theta > 0$. Marshall and Olkin (1967) characterize the parameter $\theta$ in the case where $F_j(x)$ is absolutely continuous so that $f_j(x)$ exists. A set of necessary and sufficient conditions for the function $F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$ associated with $F(x_1, x_2)$ to be a df is

(i) \quad $\theta \leq f_1(0) + f_2(0) \leq 2\theta$

(ii) \quad $\frac{d\log f_j(x)}{dx} \geq -\theta$ \quad for all $x \geq 0$, \quad $j=1,2$.

If $F$ itself is absolutely continuous, Block and Basu (1974) show that in the necessary and sufficient condition above (i) is replaced by $\theta = f_1(0) + f_2(0)$. They give a characterization of the BLMP which is later improved by Block (1977). If $F$ is an absolutely continuous df and if $U=\min(X_1, X_2)$, and $V=X_1 - X_2$, then the BLMP holds if and only if (iff) there is a $\theta > 0$ such that
(BB1) $U \sim e(\theta)$, and

(BB2) $U$ and $V$ are independent.

The desire to capture BLMP and/or marginal LMP in the bivariate setting led to the development of several extensions. Block and Basu (1974) showed that the only absolutely continuous distribution that possesses BLMP and exponential marginals is the product of two exponentials. To find a more meaningful extension, then, we need to relax at least one of these criteria. Thus, for example, if we drop absolute continuity, we have the model due to Marshall and Olkin (1967). If we drop BLMP, we have distributions due to Gumbel (1960), Raftery (1984), and Sarkar (1987). Finally, if we don’t require exponential marginals, we find the extensions of Freund (1961), and Block and Basu (1974).

In the interest of exploring inter-relationships and properties of some of the BVE’s, we expand on the work of Hutchinson and Lai (1990, p.149), and Friday and Patil (1977), and present Table 1 and Figure 1, respectively. Altogether, we examine thirteen bivariate exponential models in the literature. In Table 1, we exhibit which of the properties BLMP, exponential marginals, and/or absolute continuity, each of the distributions possesses.

In Figure 1 on p. 25, we show how some distributions may arise as a special case of the more general models. Two main branches are discernible: one, represented by the bivariate exponential extension (BEE) of Friday and Patil (1977); the other, represented by the model due to Arnold (1975). The BEE group of models has as special cases the models by Freund (1961), Marshall and Olkin (1967), and Block and
Table 1: Summary of the properties of some bivariate exponential models

<table>
<thead>
<tr>
<th>Model</th>
<th>Exponential Marginals</th>
<th>BLMP</th>
<th>Absolute Continuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arnold (1975)</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Block and Basu (1974)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Cowan (1987)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Downton (1970)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Freund (1961)</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Friday and Patil (1977)</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Gumbel (1960)</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Hawkes (1972)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Marshall and Olkin (1967)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Raftery (1984)</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Ryu (1993)</td>
<td></td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Sarkar (1987)</td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Weinman (1966)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Basu (1974). These distributions have BLMP and are derived via the shock model formulation.

Arnold's group includes Marshall and Olkin's model, Hawkes' (1972), and Downton's (1970). These models have exponential marginals and are derived via a geometric compounding scheme. For a nice introduction to these methods, we refer the reader to Hutchinson and Lai (1990, chapter 17). Notice that the distribution of Marshall and Olkin falls into both these groups - the unique distribution having BLMP and exponential marginals, as shown in Marshall and Olkin (1967). We refer to Marshall and Olkin's BVE as the BVE. Other models will also be referred to by acronyms given by their originators, if available. Else, we attach the first letter of the author's last name to refer to their models, e.g., FBVE to denote Freund's model.
and so on.

A recent development extends the BVE in yet another light. Ryu (1993) develops an absolutely continuous model that has neither BLMP nor marginal LMP. It exhibits aging in the sense that a pair of new components has greater survival probability than a used pair. Furthermore, the marginal lifetimes have increasing failure rate (IFR) property. We include it in this survey as a bivariate exponential extension in so far as the BVE can be obtained from Ryu’s model by taking appropriate limits. Ryu derives his model by formulating the failure rate as a stochastic process.

The models due to Sarkar (1987), Raftery (1984), Gumbel (1960), and Cowan (1987) fall in neither group. It is true, however, that these distributions have exponential marginals. Sarkar defines a general class of product-type distributions from which he obtains the model by Block and Basu (1974), and his own bivariate exponential distribution by proper choices of parameter values, and marginal distributions. We will elaborate on this issue later on in this discussion (see section 1.2.5). The multivariate exponential distribution proposed by Weinman (1966) reduces to a special case of Freund’s model in the two-component setting: one wherein the marginals are identical. All thirteen models yield independent exponentials when the parameter values are appropriately specified.

The earliest compilation of works on multivariate exponential distributions appears to be that of Johnson and Kotz (1972, chapter 41). Within the past decade or so, interest in these models has been sustained and various authors consolidated much of the material in some survey work. Barnett (1985) writes an informative
review classifying, comparing, and contrasting different bivariate exponential models. He also collates information on their known distributional properties, and inference results. More recently, Hutchinson and Lai (1990, chapters 9 and 17) revisit bivariate exponential distributions which have expanded in number since Barnett's work. They look at the models in reliability context both to summarize known results, and to interrelate them. Their work is commendable in its scope and content. The discussions in Block and Savits (1980) and Basu (1988) are also in the vein of reliability and life testing. Another useful introduction to the subject is given by Block (1985). Papers on characterization results are available from the works of Block and Basu (1974a) and Galambos and Kotz (1978). The latter authors include characterizations based on hazard rate properties. On the latest developments in estimation and hypotheses testing for bivariate exponential models, Klein (1993) gives an update. Finally, it appears appropriate to mention the work of Gross and Miller (1980) who discuss some applications of bivariate exponential models to biological and medical data.

We now present a brief summary of the BLMP group of models and some offshoots, followed by the cluster of distributions at the bottom of Figure 1, and then by Arnold's group of models. The derivation of some of these models is justified by physical motivations. Still, some result from purely mathematical reasons. We attempt to clarify these aspects in our brief summary that follows.

1.2.1 Marshall and Olkin (1967) - the BVE

Marshall and Olkin give a simple set-up. A two-component system is assumed to be subject to fatal shocks coming from three independent sources. Shocks of the first kind
occur according to a Poisson process with parameter $\lambda_1$ and destroy component 1. Shocks of the second kind occur according to a Poisson process with parameter $\lambda_2$ and destroy component 2. Finally, shocks of the third kind occur according to a Poisson process with parameter $\lambda_{12}$ and destroy both components. The component lifetimes $X_1$ and $X_2$ have the bivariate exponential distribution with parameters $\lambda_1, \lambda_2,$ and $\lambda_{12}$. We write $(X_1, X_2) \sim \text{BVE}(\lambda_1, \lambda_2, \lambda_{12})$, as is the convention in the literature. It is not hard to see that $(X_1, X_2) \sim \text{BVE}(\lambda_1, \lambda_2, \lambda_{12})$ iff there exist independent, exponential rv's $Z_1, Z_2,$ and $Z_{12}$ such that $X_1 = \min(Z_1, Z_{12})$, and $X_2 = \min(Z_2, Z_{12})$. This is another characterizing property of the BVE. A singularity occurs along the line $X_1 = X_2(=Z_{12})$, which occurs with probability $\frac{\lambda_{12}}{\lambda} > 0$, where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

Marshall and Olkin extend this model to the $n$-variate case. The key is to assume that now $2^n - 1$ independent Poisson processes govern the arrival of shocks.

1.2.2 Freund (1961) - FBVE

Freund derives his model from a reliability standpoint. His set-up is as follows: two components form a parallel system so that the failure of one either adversely affects the other or enhances its performance. Prior to the first failure, $X_1$ and $X_2$ are independent, exponential rv's with parameters $\alpha$ and $\beta$, respectively. When one component fails, the distribution of the remaining component lifetime is still exponential but the parameter changes. If $X_1 < X_2$, the residual mean of $X_2$ becomes $\frac{1}{\beta}$. Alternatively, if $X_1 > X_2$, the residual mean of $X_1$ changes to $\frac{1}{\alpha}$. When the components are interdependent, as occurs when modelling the lifetimes of a person's pair of organs - kidneys, lungs, etc.- the failure of one will cause a stress on the
other. We model this by requiring $\alpha < \alpha'$ and $\beta < \beta'$. On the other hand, when the components represent competing species, the removal of one will be beneficial to the other. In this case, $\alpha > \alpha'$ and $\beta > \beta'$.

1.2.3 Block and Basu (1974) - ACBVE

Block and Basu derive an absolutely continuous bivariate exponential distribution (ACBVE) which retains the BLMP of the BVE. They forego exponential marginals for generalized hyperexponential distributions (weighted average of exponentials). The result is the absolutely continuous part of the BVE$(\lambda_1, \lambda_2, \lambda_{12})$ distribution. Absolute continuity is desired to account for situations where the simultaneous failure of the two components of a system is rare or even impossible. BLMP is desired for reliability considerations. The ACBVE turns out to be a special case of Freund's model where

\[
\begin{align*}
\alpha &= \lambda_1 + \frac{\lambda_{12}\lambda_1}{\lambda_1 + \lambda_2}, & \alpha' &= \lambda_1 + \lambda_{12} \\
\beta &= \lambda_2 + \frac{\lambda_{12}\lambda_2}{\lambda_1 + \lambda_2}, & \beta' &= \lambda_2 + \lambda_{12}.
\end{align*}
\]

(1.18)

Note that $\alpha' > \alpha$ and $\beta' > \beta$ with this choice of parametrization.

1.2.4 Friday and Patil (1977) - BEE

Friday and Patil give three physical interpretations of their bivariate exponential extension, the BEE. The first derivation assumes a threshold model. Consider a system consisting of two components that are subject to two independent shock sources. Shocks may randomly vary in intensity, independently of their time of occurrence. The first kind of shock always destroys component 1. The second kind always destroys component 2. The two components are assumed to be in close proximity so
that, potentially, either shock may destroy both components. This occurs if the intensity of a shock exceeds the threshold capacities of the two components for absorbing such shocks. The probability of this occurrence is \(1 - \alpha_0\), \(0 < \alpha_0 \leq 1\). If we describe the random shock mechanism according to Freund’s set-up, but allowing for intensity to vary, then the component lifetimes \((X_1, X_2)\)~ BEE.

The second physical interpretation is the gestation model. The system undergoes a period of gestation during which the two components are in close proximity. This may mean sharing the same life support, environment, nutrients, etc. When, with probability \(1 - \alpha_0\), a shock strikes the system between the conception time \((-\theta)\), \(\theta > 0\) to the instant of birth, time \(0\), the parent and offspring components will fail to separate. Succeeding shock from a second source will destroy both components simultaneously. If, however, the system is “shock-free” from \((-\theta, 0)\), then the two components separate, and are now subject to shocks occurring according to Freund’s random shock mechanism.

The third physical interpretation is called the warm-up model. Large systems consisting of two subsystems undergo a warm-up mode prior to operation. The occurrence or non-occurrence of a shock during this period determines the response of the system to subsequent shocks.

Friday and Patil generate rv’s distributed as BEE and its special cases via a piecewise linear transformation over the regions \(\{X_1 < X_2\}\), \(\{X_1 > X_2\}\), and \(\{X_1 = X_2\}\) of two independent standard exponential random variables. This fact is used in the computer simulation of values having a BEE distribution.
1.2.5 Sarkar (1987) - ACBVE$_2$

The ACBVE$_2(\lambda_1, \lambda_2, \lambda_{12})$ proposed by Sarkar is another absolutely continuous offshoot of the BVE. Like the BVE, both the marginals and minimum order statistics are exponential r.v's. Unlike the BVE, it has no BLMP. Instead of the independence between $X_1 - X_2$ and $\min(X_1, X_2)$, Sarkar's distribution assumes independence between $\min(X_1, X_2)$ and $X_1 - X_2 + K(X_1, X_2)$, for some function $K(X_1, X_2)$.

Sarkar's model belongs to a class of distributions satisfying the following general set of conditions:

(S1) exponential marginals,

(S2) exponential minimum, and

(S3) $\min(X_1, X_2)$ independent of $g(X_1, X_2)$.

where $g(X_1, X_2)$ is strictly increasing(decreasing) in $X_1(X_2)$ for fixed $X_2(X_1)$ and has a special form for its distribution.

The first step in the derivation process is to characterize densities possessing condition (S3) above. Sarkar shows that (S3) holds iff the densities $f(x_1, x_2)$ are of the product type on each of the regions $\{x_1 < x_2\}$, and $\{x_1 > x_2\}$ in the first quadrant, and $\min(X_1, X_2)$ is independent of the indicator function $I(X_1 > X_2)$.

The second step is to give a functional equation for the product type densities in terms of the marginal distributions of $X_1$, $X_2$ and $\min(X_1, X_2)$ for any such distributions. The resulting form is indexed by $\alpha = P(X_1 > X_2)$. 
The choice of exponential marginals with parameters $\lambda_1 + \lambda_{12}$ and $\lambda_2 + \lambda_{12}$, and $\alpha = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ yields Sarkar's $ACBVE_2(\lambda_1, \lambda_2, \lambda_{12})$. The choice of generalized hyperexponential marginals (as in Block and Basu (1974)) and $\alpha = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ gives Block and Basu's $ACBVE(\lambda_1, \lambda_2, \lambda_{12})$.

1.2.6 Ryu (1993)

It is apparent that the $ACBVE$ and $ACBVE_2$ have been derived with no physical motivation in mind. Both of these models have been proposed as alternatives to the $BVE$ when absolute continuity is required. Ryu, on the other hand, takes a full behavioral approach in developing his model. He extends Marshall and Olkin’s set-up to include two systems each with two independent components in series. One of the two components in both systems share a common risk and are called common components. The remaining components, however, are system specific, hence, are independent. There are two ways that a system may cease to operate. Either a common component fails or a system specific one does, whichever comes first. Knowledge of a failure of a common component in one system suggests a high chance that this will occur in the other. No new information is gleaned if a system specific component fails. If there is no knowledge of which type of component fails, then some information still proves useful, if there is a chance that a common component failed. Ryu models the dependence of the system lifetimes via the dependence of the failure times of the common components.
1.2.7 Weinman (1966)

Weinman proposes a multivariate generalization of Freund’s model. The physical basis is similar to Freund’s. A parallel system has $n$ identical components whose lifetimes $X_1, X_2, \ldots, X_n$ are independent, identically distributed rv’s with pdf $f_0(x_i)$, $x_i > 0$, when all the components are in operation. Upon the first failure, the system bears a load which is equally shared by the remaining $n-1$ components whose lifetimes are now iid with pdf $f_1(x_i)$, $x_i > t_1$, if $t_1$ is the time at which the first failure occurred. More generally, when $j$ components of the system already have failed and have not been replaced, the remaining $n-j$ components have iid lifetimes with pdf $f_j(x_i)$, $x_i > t_j$, where $t_j$ is the time at which the $j$th failure occurred. Weinman takes the density $f_j(x_i)$ to be $f_j(x_i) = \lambda_j e^{-\lambda_j(x_i-t_j)}$, $x_i > t_j$ which is a shifted exponential with parameter $\lambda_j$.

1.2.8 Raftery (1984) - RBVE

Raftery introduces an absolutely continuous multivariate exponential model. In the bivariate case, his model assumes the full range of correlations between the lower and upper Fréchet bounds. The distribution behaves like the multivariate normal in the sense that its marginals have the same form as itself, the correlation does not depend on the marginal distributions, and it is a linear combination, though random, of exponential rv’s. This multivariate extension has exponential marginals.

Two physical interpretations have been put forth to advance the model. First, consider a system, much like an electric circuit or a road network, having three units
Let the two components $S_1$ and $S_2$ of a system be in one of the three states: normal, unsatisfactory, or failed. At time $t=0$, $S_i$ is in normal condition with probability $\pi_i$ or is unsatisfactory with probability $1 - \pi_i$, $i=1,2$. $S_1$ and $S_2$ are both in normal condition with probability $p_{11}$. The components are subject to three sources of shocks. Shocks of the first kind occur according to a Poisson process with rate $\lambda$ and cause normal components to be unsatisfactory. Shocks of the second kind occur according to a Poisson process with rate $\lambda/(1-\pi_1)$ and cause $S_1$ to fail if it is in an unsatisfactory state. Finally, shocks of the third kind occur according to a Poisson process with rate $\lambda/(1-\pi_2)$ and cause $S_2$ to fail if it is in an unsatisfactory state. If $Z$ denotes the length of time a component is normal given that it started out normal, $(1 - \pi_i)Y_i$ is the time that component $S_i$ is in an unsatisfactory state, and $I_i$ is an indicator that component $S_i$ is normal, $i=1,2$, then (1.19) denotes the lifetime of component $S_i$, $i=1,2$. 

\[ X_i = (1 - \pi_i)Y_i + I_iZ \quad (1.19) \]
The following parsimonious versions of the bivariate model are given by Raftery.

Model 1: \( \pi_1 = \pi_2 = p_{11} = \pi. \)

Model 2: \( p_{11} = \pi \) and \( p_{11} = \begin{cases} 0 & \pi \leq 0.5 \\ 2\pi - 1 & \pi > 0.5 \end{cases} \)

Model 3: \( \pi_2 = p_{11} = \gamma \)

We will refer often to these models in subsequent chapters.

Raftery defines the n-variate model as follows: Let \( Y_1, Y_2, \ldots, Y_n, Z_1, Z_2, \ldots, Z_m \) be iid exponential rv's with parameter \( \lambda \), \( (J_1, J_2, \ldots, J_n) \) a random vector taking values in \( \{0, 1, \ldots, m\}^n \) such that \( P(J_i = 0) = 1 - \pi_i, P(J_i = k) = \pi_{ik}, (i=1,2,\ldots,n ; k=1,2,\ldots,m) \) and \( \pi_i = \sum_{k=1}^{m} \pi_{ik} \). Let \( Z_0 = 0 \). Then, \( (X_1, X_2, \ldots, X_n) \) is defined by

\[
X_i = (1 - \pi_i)Y_i + Z_{J_i}, \quad i = 1,2,\ldots,n.
\]

**1.2.9 Gumbel (1960) - GBVE**

Gumbel explores the analytic properties of two bivariate distributions and finds that with respect to contour curves, regression curves, and correlation values, these differ from the bivariate normal distribution. With his results, he seeks to show patterns of bivariate relations other than the normal which, even now, is so popular. He briefly mentions a third model which along with the first two has exponential marginals. Though theoretical in derivation, recent developments have shown that Gumbel’s third model has a sound physical application. Hougaard (1986) and Lu and Bhattacharyya (1991) show that it can be formulated as a frailty model. This formulation is as follows. Assume that the component lifetimes \( X_1 \) and \( X_2 \) are independent.
Weibull distributed rv’s with survival function $e^{-H_i(t)w}$, $H_i(t) = (\frac{t}{\delta_i})^\beta$, $i=1,2$, in the presence of a random stress $W = w$, where $W$ has a positive stable distribution. Then, $(X_1, X_2)$ will have Gumbel’s bivariate exponential distribution, which we will denote as $\text{GBVE}(\theta_1, \theta_2, \delta)$, following Lu and Bhattacharyya’s notation.

1.2.10 Cowan (1987)

Cowan motivates another bivariate exponential form by looking at the geometry of Poisson line processes. He looks at the joint distribution of a pair of distances from the origin to the nearest lines of a Poisson process in the direction of two angles $\alpha^o$ apart, $0 < \alpha \leq \pi$, and models this as a bivariate exponential distribution. The theory of Poisson line processes shows that marginally the distances have exponential distributions.

1.2.11 Arnold (1975)

Arnold describes the set-up of his hierarchical successive damage models as follows: m components are subject to a hierarchy of shock processes. Basic shocks occur according to a Poisson process. A random number $N^{(1)}_j$ of basic shocks causes a first-order shock to the jth component, $j = 1, \ldots, m$. A random number $N^{(2)}_j$ of first-order shocks causes a second-order shock to the jth component, $j = 1, \ldots, m$. Finally, a random number $N^{(k)}_j$ of (k-1)st order shocks causes a kth order shock to the jth component, $j = 1, \ldots, m$. At each stage, $N^{(\ell)} = (N^{(\ell)}_1, N^{(\ell)}_2, \ldots, N^{(\ell)}_m)$, $\ell = 1, \ldots, k$, has a multivariate geometric distribution defined by Arnold (1975). The assumption of independence among the basic shock process and the random variables $N^{(\ell)}$, $\ell =$
1, ..., k completes the set-up.

1.2.12 Downton (1970)

Pursuing a model based on successive damage, Downton develops the joint distribution of \((X_1, X_2) = (\sum_{i=1}^{N_1} X_{1i}, \sum_{i=1}^{N_2} X_{2i})\), where \((N_1, N_2)\) has a bivariate geometric distribution with probability generating function (pgf)

\[
\Pi(z_1, z_2) = E(z_1^{N_1} z_2^{N_2}) = \frac{z_1 z_2}{\{1 + \alpha + \beta + \gamma - \alpha z_1 - \beta z_2 - \gamma z_1 z_2\}} \tag{1.21}
\]

for \(\alpha, \beta, \gamma \geq 0\), and \((X_{1i}, X_{2i})\), \(i = 1, \ldots, n\), are a random sample from \(F(t_1, t_2)\) such that \(X_{ij} \sim e(\lambda_j)\), \(i = 1, \ldots, n, j = 1, 2\). Downton culled the pgf \(\Pi(z_1, z_2)\) from various works on accident proneness in the literature.

1.2.13 Hawkes (1972)

The motivation for Hawkes' work stems from the same idea of modelling successive damage, but he uses a more general bivariate geometric distribution. Suppose that two events \(A_1\) and \(A_2\) occur with the following joint probabilities:

\[
\begin{array}{c|cc|c|c}
A_2 & A_1 & A_2^c & P_1 & Q_1 \\
\hline
p_{11} & p_{10} & p_{01} & p_{00} & Q_2 \\
\end{array}
\]

Denote by \(N_i\) the number of times up to and including the first occurrence of event \(A_i\), \(i = 1, 2\). Then, \(\Pi\), the pgf of \((N_1, N_2)\), is given by

\[
\Pi(z_1, z_2) = z_1 z_2 \left\{ p_{11} + p_{10} \frac{P_2 z_2}{1 - Q_2 z_2} + p_{01} \frac{P_1 z_1}{1 - Q_1 z_1} + p_{00} \Pi(z_1, z_2) \right\}. \tag{1.22}
\]
We may, for example, regard the event $A_i$ to correspond to a fatal shock on component $i$, $i=1,2$. Downton's model may actually be formulated in this fashion, by taking $p_{00} = \frac{1}{1+\gamma}$ and $p_{10} = p_{01} = 0$.

### 1.3 Generalized Hyperexponential Distributions

The class of generalized hyperexponential (GH) distributions consists of rv's with pdf's that are arbitrary mixtures of non-identical exponential pdf's

$$f(t) = \sum_{i=1}^{n} a_i f_i(t)$$

with $f_i(t) = b_i e^{-b_i t}$, $t > 0$, $b_i > 0$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} a_i = 1$. Without loss of generality, let us assume $b_1 < b_2 < \ldots < b_n$. Since a pdf $f(t) \geq 0$, for all $t$, on letting $t \to \infty$ and $t = 0$, it is easily seen that the conditions (i) $a_1 > 0$, and (ii) $\sum_{i=1}^{n} a_i b_i \geq 0$ are necessary for $f(t)$ in (1.23) to be a pdf (Steutel, 1967). On the other hand, Bartholomew (1969) verifies that a sufficient condition for (1.23) to be a pdf is

$$\sum_{i=1}^{r} a_i b_i \geq 0, \quad (r = 1, \ldots, n).$$

(1.24)

It appears that a simple set of conditions that are both necessary and sufficient has not been found for $n > 2$ (see Botta, Harris, and Marchal, 1987), although, an unwieldy one involving conditions on the Laplace transform of (1.23) was mentioned by Bartholomew (1969).

Botta, Harris, and Marchal (1987) have carried out an extensive investigation of the properties of GH distributions. They present techniques for inverting the distribution function associated with (1.23) and generating rv's with GH df's. They
also cite works on computer-based maximum likelihood estimation of GH df's from sample data, and interrelate this family with other generalizations of the exponential distribution. More recently, Harris, Marchal, and Botta (1992) offer new approaches in assessing whether given functions are true GH pdf's.

Two properties of the class of GH distributions make it an important modelling tool. According to Botta, Harris, and Marchal (1987), distributions belonging to the class have a unique representation or are identifiable. Moreover, this class is dense in the class of all distribution functions with support \((0, \infty)\).

Our interest in this class is sparked by the fact that the ordered lifetimes of components having some of the more important bivariate exponential distributions, marginally, have GH distributions. This will be pursued in chapter 4.

General discussions of finite mixture distributions may be found in Everitt and Hand (1981) and Titterington, Smith, and Makov (1985). The latter reference presents a nice collation of papers dealing with practical applications of such distributions (see Table 2.1.3 there). Exponential mixtures, in particular, have been used to model conception times, and gaps in traffic, among others. Such preponderance of applications gives rise to various techniques of estimation and analysis.

### 1.4 Our Goals

In the context of reliability and life testing, when one deals with lifetime data, distributions such as the exponential, gamma, and Weibull are more appropriate for fitting data and testing hypotheses than the normal distribution. Moreover, many real life systems of two components, including the twin engine of a plane, the pair of kidneys
of a patient, and two adjacent geiger counters, have system lives that are more ade-
quately modelled by incorporating some form of dependence between the component
lifetimes. These facts provide strong motivation for studying multivariate versions
of these distributions. We focus on several bivariate extensions of the exponential
distribution in this work.

Bivariate exponential distributions have been gaining acceptance as applicable
models for describing bivariate data. For example, Gross and Lam (1980) model
pair of responses to two different treatments of patients using the ACBVE. See also
Hutchinson and Lai (1990, chapter 9). We anticipate more exploratory work in fitting
bivariate exponential models to industrial as well as biological and medical data in
the coming years.

The thrust of this dissertation is to explore the properties of order statistics from
some common bivariate exponential models. Order statistics and system lifelengths
coincide for most systems, e.g. k-out-of-n systems have system lifelengths represented
by the (n-k+1)st order statistic. It is also true that, in this setting, most parameter
estimates and test statistics are functions of order statistics. Thus, the insight and
knowledge gained from the study of the properties of these order statistics would play
a prominent role in the investigation of these inference procedures.

We have presented a survey of thirteen models in section 1.2, where we gave an
overview of their derivations, and their interrelationships. Of these, we study seven
in detail: Marshall and Olkin’s BVE, Block and Basu’s ACBVE, Freund’s FBVE,
Gumbel’s GBVE, Raftery’s RBVE, Sarkar’s ACBVE2, and Friday and Patil’s BEE.
In chapter 2, we seek to understand the dependence relation between unordered bivariate rv's. We do this by examining graphs of the surfaces \( c(u, v) - uv \) representing the difference between the copula function of the dependent pair \((X_1, X_2)\) and that of an independent pair. Copulas describe that aspect of the joint dependence left after the influence of the marginal distributions not pertinent in describing such a relationship has been eliminated. So by looking at copulas, we are exploring dependence in a new light. We present the copula and its basic properties in section 2.1. Graphs of \( c(u, v) - uv \) are presented in section 2.3. We also study the quadrant dependence properties of some bivariate exponential distributions. This is a way of exploring whether or not \( X_1 \) and \( X_2 \) tend to 'hang' together. (We explain this phrase in section 2.2.) In section 2.4, we give formal proofs establishing the quadrant dependence properties of some of these distributions.

The exploration of order statistics begins in chapter 3. We present their joint distribution, marginal distributions, moments and correlation, graphs of copulas, conditional distribution, regression function, and distribution of the spacing for each of the seven bivariate exponential models.

In chapter 4, we study the reliability properties of the order statistics. Conditions under which the minimum and maximum possess the IFR, decreasing failure rate (DFR), and increasing failure rate average (IFRA) properties are presented in sections 4.1 and 4.2. We observe that the reliability properties of order statistics from dependent samples could differ drastically from the independent model. We also look at their hazard rate ordering. We show in section 4.3 some bivariate exponential
models whose order statistics satisfy this ordering. We formally define these concepts there.

We tackle the issue of predicting the maximum order statistic in chapter 5. We propose two predictors - the conditional mean predictor and the conditional median predictor - under the following set-up. The two components of a parallel system are assumed to be exchangeable and observations are made on n such systems. The minimum lifetime of the (n+1)st sample has just been observed. When will the (n+1)st system fail? We consider this question for the FBVE and ACBVE distributions in sections 5.1 and 5.2, respectively. For the ACBVE model, we show that the expressions for the maximum likelihood estimators (MLE's) of the parameters available in the literature are in error. We obtain the correct expressions for these MLE's and examine the properties of resulting predictors.

Finally, in chapter 6, we summarize our results and provide insight on what has been studied. As we conclude the dissertation, we discuss directions for future work and some open problems.
Sarkar's general class of models includes Block and Basu's distribution as a special case. But, his ACBVE\(_2\) distribution does not.

Figure 1: Interrelationships of some common bivariate exponential distributions
Figure 2: Network describing Raftery's bivariate exponential model
CHAPTER II

Dependence Structure of Bivariate Exponential Distributions

In this chapter, we investigate patterns of dependence among some bivariate exponential variables by examining their copula functions. These functions allow for comparison between the various dependence structures. Also, plots of the copula provide some insight on the quadrant dependence properties of the random variables involved. We define the notion of a copula and the concept of quadrant dependence in sections 2.1 and 2.2, respectively. Graphs will be presented in section 2.3. Finally, we prove the quadrant dependence relations of some bivariate exponential variables in section 2.4.

2.1 Copula and its Properties

The concept of a copula was introduced by Sklar (1959). The underlying idea behind its development is to express the joint distribution of random variables in terms of the one-dimensional marginals. In the bivariate setting, if \( F(x_1, x_2) \) denotes the joint df of \((X_1, X_2)\), and \( F_i(x_i) \), the marginal df of \( X_i \), \( i=1,2 \), then the copula function is given by

\[
c(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)),
\]  

(2.1)
where \(0 \leq u, v \leq 1\) and \(F_i^{-1}(t) = \inf\{y : F_i(y) \geq t\},\) \(0 < t < 1,\) \(i = 1, 2.\)

It is not hard to see that the copula function satisfies the following properties:

\[
\text{(CP1)} \quad c(u, 0) = c(0, u) = 0 \quad \forall \ u \in [0, 1],
\]

\[
\text{(CP2)} \quad c(u, 1) = c(1, u) = u \quad \forall \ u \in [0, 1], \text{ and}
\]

\[
\text{(CP3)} \quad c(u_1, v_1) - c(u_1, v_2) - c(u_2, v_1) + c(u_2, v_2) \geq 0 \quad \forall \ u_1, u_2, v_1, v_2 \in [0, 1] \supseteq u_1 \leq u_2 \text{ and } v_1 \leq v_2.
\]

Thus, we have a function defined on the entire unit square which is itself a df whose marginals are uniform on \([0, 1]\). Kimeldorf and Sampson (1975) refer to equation (2.1) as the uniform representation of \(F\).

It is immediate from equation (2.1) that

\[
F(x_1, x_2) = c(F_1(x_1), F_2(x_2)) \quad (2.2)
\]

\(\forall x_1, x_2 \in \mathbb{R}.\) Sklar (1959) shows that any df can be represented this way. If \(F_i(x_i),\) \(i = 1, 2\) are continuous, then \(c\) is unique. Many other properties of copula functions are given by Schweizer and Wolff (1981), including invariance under almost surely strictly increasing transformations of \(X_1\) and \(X_2.\) Sungur (1990) develops a formula that sheds light on the dependence information carried by parametrized copulas. He gives approximations via Taylor series expansions. He also explores the behavior of the copula for the multivariate normal distribution in detail.
2.2 Quadrant Dependence

The random variable \((X_1, X_2)\), or equivalently, their joint df \(F\) is positively quadrant dependent (PQD) if

\[
F(x_1, x_2) \geq F_1(x_1)F_2(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2
\]

or

\[
\overline{F}(x_1, x_2) \geq \overline{F}(x_1)\overline{F}(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2.
\]  \(2.3\)

Notice what (2.3) suggests. A series system whose components are PQD has a higher system reliability than the one with independent components having the same marginals. Tong (1980, p.78) gives additional insight. When \((X_1, X_2)\) is PQD, then \(X_1\) and \(X_2\) tend to ‘hang’ together in that \(P(X_1 \leq x_1 | X_2 \leq x_2) \geq P(X_1 \leq x_1)\) and \(P(X_2 \leq x_2 | X_1 \leq x_1) \geq P(X_2 \leq x_2)\), \(\forall (x_1, x_2) \in \mathbb{R}^2\). In terms of the covariance structure, Lehmann (1966) shows that PQD holds iff \(\text{Cov}(g(X_1), h(X_2)) \geq 0\), for every pair of increasing functions \(g, h \in \mathbb{R}\). He discusses implications of positive quadrant dependence on the values of some nonparametric measures of dependence. Hutchinson and Lai (1990, chapter 12) give a review of the broader topic of stochastic dependence.

If the inequalities in (2.3) are reversed, we get

\[
F(x_1, x_2) \leq F_1(x_1)F_2(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2
\]

or

\[
\overline{F}(x_1, x_2) \leq \overline{F}(x_1)\overline{F}(x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2.
\]  \(2.4\)

If (2.4) holds, then \((X_1, X_2)\) is negatively quadrant dependent (NQD).
2.3 Graphical Presentations

In this subsection, we explore graphs of the surfaces \( c(u, v) = uv \), which represents the discrepancy between the copula of the dependent pair \((X_1, X_2)\), and that of an independent pair. The rationale behind the choice of \( c(u, v) = uv \) is the fact that any suitably normalized distance measure between the surfaces gives a symmetric, nonparametric dependence measure. Schweizer and Wolff (1981) prove that this is so for the \( \mathcal{L}_p \)-distance measure when \( p = 1, p = 2, \) and \( p \to \infty \).

In most cases, we resort to numerical methods using the MAPLE software to make the plots. The programs have been included in Appendix B. Whenever possible, we will use closed form representations of the copula functions. We devote this chapter to looking at the unordered bivariate exponential rv's. Corresponding plots for order statistics will be presented in the next chapter.

We will look at five distributions in turn: the BVE, Block and Basu's ACBVE, Freund's FBVE, Gumbel's GBVE, and Raftery's RBVE. Before we present any of these pictures, we remark that for two independent exponential rv's, the surfaces \( c(u, v) = uv \) will be flat, precisely the plane \( z = 0 \). Any departure from this pattern reveals the form and extent of dependence between the variables. It is also true that \( \sup_{u,v \in [0,1]} |c(u,v) - uv| \leq 0.25 \) (see, for example, Schweizer and Wolff (1981)). This serves as a reference value for comparing the peak of the surfaces \( c(u,v) = uv \) for the bivariate exponential models.

The top picture in Figure 3 represents the BVE. Notice that the surface behaves like a hill coming to an abrupt peak at the ridge where \( X_1 = X_2 \) (or equivalently,
\[ v = 1 - (1 - u)^{a/b}, \quad 0 < a, b < 1 \] Clearly, the copula function is not differentiable on this curve, where it has a singularity. That the peak occurs on the line \( X_1 = X_2 \) makes sense since we would expect the greatest discrepancy from independence to happen when \( X_2 = f(X_1) \) almost surely, for some non-decreasing function \( f \). The discrepancy, however, is small - no bigger than 0.05 for this situation. The form of the copula function in the symmetric case is given by Conway (1981). In the asymmetric case, the copula function is given by

\[
c(u, v) = u + v - 1 + \min\{(1 - u)(1 - v)^{1-b}, (1 - u)^{1-a}(1 - v)\} \quad (2.5)
\]

where \( 0 < u, v \leq 1, \quad 0 < a, b < 1 \). As this plot shows, \( c(u, v) - uv \geq 0, \forall 0 \leq u, v \leq 1 \) suggesting that the BVE is PQD. Conway draws this conclusion as well from contour plots of the copula function of the BVE. Moreover, Barlow and Proschan (1981) show that if \( (X_1, X_2) \sim BVE \), then \( X_1 \) and \( X_2 \) are associated\(^1\) rv's - a stronger result.

The bottom picture in Figure 3 depicts the curve for the ACBVE. This surface looks like a mound which has a smooth peak around the point (0.7,0.7). Again, small discrepancies are noted between ACBVE variables and independent variables. That \( c(u, v) - uv \geq 0, \forall 0 \leq u, v \leq 1 \) suggests that the ACBVE is PQD. We verify that indeed this is the case in the next section.

The FBVE affords us a view of diverse copula patterns possible for a single distribution. We see in Figure 4 (top picture) an outcome when \( \alpha < \alpha' \) and \( \beta < \beta' \) corresponding to a 'stress situation' where failure of one causes additional strain on the other component. This mound-shaped curve reaches its highest point 0.14 some-

---

\(^1\)\(X_1 \) and \( X_2 \) are associated if \( \text{Cov}(a(X_1, X_2), b(X_1, X_2)) \geq 0 \) for every pair of functions \( a, b \in \mathbb{R}^2 \) which increase in each argument (Esary, Proschan, and Walkup (1967)).
where around (0.7,0.7). It is true that we will always get a surface $c(u,v) - uv$ above the plane $z = 0$ in a 'stress situation'. We defer the proof to the next section. Figure 4 (bottom picture) shows a crater-like curve that dips lowest at -0.05 around (0.5,0.1). This pattern represents an outcome when $\alpha > \alpha'$ and $\beta > \beta'$, an event we have called as an 'enhanced situation' - failure of one improves the survival of the other component. That $c(u,v) \leq uv, \forall 0 \leq u, v \leq 1$ persists in this situation. Again the details are reserved for section 2.4. When we mix these previous situations so that either $\alpha < \alpha'$ or $\beta < \beta'$ but not both, a curve that both dips and climbs can be generated. We see such a pattern in Figure 5 (top picture).

Figure 5 (bottom picture) represents the GBVE of Gumbel. The mound-like shape is now familiar. The highest discrepancy of 0.12 from independence occurs around (0.7,0.7). A closed form of the copula function is available in this case and is given by

$$c(u,v) = u + v - 1 + e^{-(\ln(1-u)^{1/\alpha} + \ln(1-v)^{1/\beta})^{\delta}}$$

(2.6)

$0 \leq u, v \leq 1$. Conway (1981) draws the contours of this distribution and concludes that the GBVE is PQD, though she does not provide a formal proof.

Raftery derives a distribution that exhibits either positive or negative dependence as the relation among the parameters vary. We look at three such relations proposed by Raftery himself as parsimonious versions of his model. As pointed out in section 1.2.8, we refer to the setting $\pi_1 = \pi_2 = p_{11} = \pi$ as model 1 and show in Figure 6 (top picture) one pattern for $c(u,v) - uv$ under this setting. Notice the mound-like shape above the plane $z = 0$ that peaks up to 0.1 around (0.7,0.7). We
prove in the next section that model 1 is PQD. The parametrization for model 2
assumes \( \pi_1 = \pi_2 = \pi \) and \( p_{11} = 0 \) \( (p_{11} = 2\pi - 1) \) according as \( \pi \leq (>) 0.5 \). We show
in Figure 6 (bottom picture) the case \( \pi \leq 0.5 \). Now, the crater-like pattern below
the plane \( z = 0 \) presents itself. The lowest dip at -0.06 occurs around \((0.3,0.1)\). We
establish that this version of model 2 is indeed NQD in section 2.4. Figure 7 (top
picture) is a look at the other possibility: \( \pi > 0.5 \) in model 2. It shows a hill with a
somewhat pointed top around \((0.7,0.7)\). The hill, however, does not lie entirely above
the plane \( z = 0 \). Its base dips down below the plane \( z = 0 \) before curling up again.

This case of model 2 is neither PQD nor NQD, as we show in section 2.4. Finally,
the setting \( \pi_2 = p_{11} = \gamma \) describes model 3. One outcome for this model is given in
Figure 7 (bottom picture). We observe a mound-like pattern which attains a peak at
0.04 around \((0.7,0.7)\). In the following section, we prove that model 3 is PQD. The
closed form of the copula for the RBVE is given by

\[
c(u,v) = u + v - 1 + c_1(1-u)^d_1(1-v)^d_2 + c_2(1-u)(1-v)^d_2
\]

\[+ c_3(1-u)^d_1(1-v) + w(u,v) \quad (2.7)
\]

where

\[
w(u,v) = \begin{cases} (1 - c_3)(1 - u) - (1 - c_3)c_4(1 - u)^d_3(1 - v)^{1-d_3} & , u > v \\ (1 - c_2)(1 - v) - (1 - c_2)c_5(1 - v)^d_4(1 - u)^{1-d_4} & , u < v \end{cases}
\]

\[ (2.8) \]

and

\[
c_1 = p_{11} \left\{ \frac{1}{\pi_1} \right\} - 1 \quad , \quad c_2 = \frac{p_{10}}{\pi_1} = 1 - \frac{p_{11}}{\pi_1} \quad , \quad c_3 = \frac{p_{10}}{p_{21}} = 1 - \frac{p_{11}}{p_{21}} \\
c_4 = \frac{(1-p_{21})^2}{1-\pi_1 \pi_2} \quad , \quad c_5 = \frac{(1-\pi_1)^2}{1-\pi_1 \pi_2} \quad , \quad d_1 = \frac{1}{1-\pi_1} \quad , \quad d_2 = \frac{1}{1-\pi_2} \quad (2.9)
\]

Notice that \(-1 < c_i < \ell < \infty\), \(0 < c_i < 1\), \( i = 2, \ldots, 5\), \(1 < d_1, d_2 < \infty \). We
conjecture that \( \ell = 1/2 \), but we have no formal proof.
Remark: Although the above expression of the copula for the RBVE involves seven constants $c_i$'s, and $d_i$'s, yet in essence, only three parameters are independent: $\pi_1, \pi_2,$ and $p_{11}$. These are not parameters of the marginal distributions; rather, the parameters by which dependence between $X_1$ and $X_2$ is introduced in the RBVE.

2.4 Quadrant Dependence Properties of Some Models

We are now ready to investigate the quadrant dependence properties of Block and Basu's ACBVE, Freund's FBVE, and Raftery's RBVE distributions. For this purpose, we need to check either (2.3) or (2.4) holds for $(x_1, x_2) \in \mathbb{R}_2^+$. To this end, we define a function $G(x_1, x_2)$ that has the form

$$G(x_1, x_2) = C(x_1, x_2) \left( F(x_1, x_2) - F_1(x_1)F_2(x_2) \right),$$

where $C(x_1, x_2) > 0$. This function $G$ defined on $\mathbb{R}_2^+$ is non-negative, i.e.,

$$G(x_1, x_2) \geq 0 \quad \forall \ (x_1, x_2) \in \mathbb{R}_2^+ \quad (2.10)$$

in the case of PQD, or is non-positive, i.e.,

$$G(x_1, x_2) \leq 0 \quad \forall \ (x_1, x_2) \in \mathbb{R}_2^+ \quad (2.11)$$

in the case of NQD.

We now describe our general approaches to establish the PQD or NQD property using $G$ and present them as algorithms for ready reference later in this section.

PQD Algorithm

Partition the domain $\mathbb{R}_2^+$ into $D_1 = \{(x_1, x_2) | x_1 \leq x_2\}$ and $D_2 = \{(x_1, x_2) | x_1 > x_2\}$. On $D_1$, we proceed as follows. Fix $x_1 = x_0$, say, where $x_0 \in (0, \infty)$. Define

$$f(x_2) = G(x_0, x_2), \quad x_2 \in (x_0, \infty) \quad (2.12)$$
and
\[ g(x_0) = G(x_0, x_0). \]  

(2.13)

If we can show that the following two conditions hold, then equation (2.10) holds on \( D_1 \), since the choice of \( x_0 \in (0, \infty) \) is arbitrary.

(PQD1) \( g(x_0) \geq 0, \forall x_0 \in (0, \infty) \), and

(PQD2) \( f(x_2) \) is non-decreasing in \( x_2 \in (x_0, \infty) \)

The procedure can be modified easily to prove that (2.10) holds on \( D_2 \), as well. In this way, one may verify the PQD property.

NQD Algorithm

Suppose now that instead of (PQD1) and (PQD2), the following conditions hold:

(NQD1) \( g(x_0) \leq 0, \forall x_0 \in (0, \infty) \), and

(NQD2) \( f(x_2) \) is non-increasing in \( x_2 \in (x_0, \infty) \)

for some arbitrary \( x_0 \). Then, (2.11) holds on \( D_1 \). The NQD property follows after an analogous procedure is carried out on \( D_2 \).

**Theorem 2.4.1** The ACBVE of Block and Basu is PQD.

**Proof:** We illustrate the PQD algorithm on \( D_2 \).

The condition \( F(x_1, x_2) \geq F_1(x_1)F_2(x_2), \forall (x_1, x_2) \in \mathbb{R}_2^+ \) is equivalent to saying \( G(x_1, x_2) \geq 0, \forall (x_1, x_2) \in \mathbb{R}_2^+ \). On \( D_2 \), \( G \) is defined as

\[
G(x_1, x_2) = (\lambda_1 + \lambda_2)(\lambda_1 e^{\lambda_2(x_1 - x_2)} - \lambda_{12})
- (\lambda_1 e^{\lambda_2 x_1} - \lambda_{12})(\lambda e^{-(\lambda_2 + \lambda_{12})x_2} - \lambda_{12} e^{-\lambda_2 x_2}).
\]  

(2.14)
The expressions for $F, F$, and $F_i$ are given in Appendix A.2.

Fix $x_2 = x_0 \in (0, \infty)$, and define for $x_1 > x_0$

$$f(x_1) = (\lambda_1 + \lambda_2)(\lambda e^{\lambda_2(x_1 - x_0)} - \lambda_1) - \theta(x_0)(\lambda e^{\lambda_2 x_1} - \lambda_1)$$

where

$$\theta(x_0) = e^{-(\lambda_2 + \lambda_1)x_0}(\lambda - \lambda_1 e^{-\lambda_1 x_0}).$$

Now, evaluate $f$ at $x_1 = x_0$, and call it $g$, so that

$$g(x_0) = (\lambda_1 + \lambda_2)^2 - \theta(x_0)(\lambda e^{\lambda_2 x_0} - \lambda_1).$$

Notice that $g(0) = 0$ and $g(\infty) = \lim_{x_0 \to -\infty} g(x_0) = (\lambda_1 + \lambda_2)^2$. For these $f$ and $g$, we want to verify that conditions (PQD1) and (PQD2) hold. Write $g(x_0) = (\lambda_1 + \lambda_2)^2 - g^*(x_0)$, where $g^*(x_0) = \theta(x_0)(\lambda e^{\lambda_2 x_0} - \lambda_1)$. To verify condition (PQD1), it suffices to show that $g$ is non-decreasing in $(0, \infty)$, or equivalently, $g^*$ is non-increasing in $(0, \infty)$. We use the monotonicity property of the natural logarithmic function and verify instead that $h(x_0) = \log g^*(x_0)$ is non-increasing in $(0, \infty)$. If we can show that the derivative $h'(x_0)$ of $h$ wrt $x_0$ is non-positive, we would have shown (PQD1). Now, in view of (2.16),

$$g^*(x_0) = e^{-\lambda_2 x_0}(\lambda - \lambda_1 e^{-\lambda_1 x_0})(\lambda - \lambda_2 e^{-\lambda_2 x_0})$$

Thus,

$$h(x_0) = -\lambda_1 x_0 + \log(\lambda - \lambda_1 e^{-\lambda_1 x_0}) + \log(\lambda - \lambda_2 e^{-\lambda_2 x_0})$$

so that

$$h'(x_0) = -\lambda_1 + \frac{\lambda_1 e^{-\lambda_1 x_0}}{\lambda - \lambda_1 e^{-\lambda_1 x_0}} + \frac{\lambda_2 e^{-\lambda_2 x_0}}{\lambda - \lambda_2 e^{-\lambda_2 x_0}}.$$
Add and subtract \( \frac{\lambda_1}{\lambda - \lambda_{12} e^{-\lambda_{12} x_0}} \) and \( \frac{\lambda_2}{\lambda - \lambda_{12} e^{-\lambda_{12} x_0}} \) to the right-hand side (RHS) of (2.20) to obtain

\[
h'(x_0) = -\left(\lambda_1 + \lambda_2 + \lambda_{12}\right) + \frac{\lambda \lambda_1}{\lambda - \lambda_{12} e^{-\lambda_{12} x_0}} + \frac{\lambda \lambda_2}{\lambda - \lambda_{12} e^{-\lambda_{12} x_0}}. \tag{2.21}
\]

Clearly, \( h'(x_0) \) is decreasing in \((0, \infty)\). Since \( h'(0) = 0 \), then \( h'(x_0) \leq 0, \forall x_0 \in (0, \infty) \), and (PQD1) holds.

To verify condition (PQD2), we want to show that \( f(x_1) \) is non-decreasing on \((x_0, \infty)\). It is enough to show that the derivative \( f'(x_1) \) of \( f \) wrt \( x_1 \) is non-negative.

This derivative is given by

\[
f'(x_1) = (\lambda_1 + \lambda_2) \lambda \lambda_2 e^{\lambda_2 (x_1 - x_0)} - \theta(x_0) \lambda \lambda_2 e^{\lambda_2 x_1}. \tag{2.22}
\]

After some algebraic manipulations, we get

\[
f'(x_1) = 2 e^{\lambda_2 x_1} e^{-\lambda x_0} s(x_0). \tag{2.23}
\]

where

\[
s(x_0) = (\lambda_1 + \lambda_2) e^{(\lambda_1 + \lambda_{12}) x_0} - \lambda e^{\lambda_1 x_0} + \lambda_{12}. \tag{2.24}
\]

It remains to show that \( s(x_0) \geq 0 \). This is immediate since \( s(0) = 0 \) and \( s'(x_0) \geq 0, \forall x_0 \in (0, \infty) \). The PQD algorithm on \( D_2 \) is complete. Repeat this procedure on \( D_1 \) to finish the proof. □

**Theorem 2.4.2**  
(i) If \( \alpha < \alpha' \) and \( \beta < \beta' \), then the FBVE is PQD.

(ii) If \( \alpha > \alpha' \) and \( \beta > \beta' \), then the FBVE is NQD.
**Proof:** The functions $\bar{F}$, $\bar{F}_1$, and $\bar{F}_2$, for Freund's model are given in Appendix A.3. Define

$$G(x_1, x_2) = \left\{ \frac{\beta - \beta'}{\alpha + \beta - \beta'} + \frac{\alpha e^{(\alpha + \beta - \beta')(x_2 - x_1)}}{\alpha + \beta - \beta'} \right\} - e^{-(\alpha + \beta)x_1} \left\{ \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \frac{\beta e^{(\alpha + \beta - \alpha')(x_1)}}{\alpha + \beta - \alpha'} \right\} \left\{ \beta - \beta' + \frac{\alpha e^{(\alpha + \beta - \beta')(x_2)}}{\alpha + \beta - \beta'} \right\}$$

\hspace{1cm} (2.25)

$(x_1, x_2) \in D_1$. Fix $x_1 = x_0$ and consider the function $f$ of $x_2$ given below.

$$f(x_2) = \left\{ \frac{\beta - \beta'}{\alpha + \beta - \beta'} + \frac{\alpha e^{(\alpha + \beta - \beta')(x_2 - x_0)}}{\alpha + \beta - \beta'} \right\} - \theta(x_0) \left\{ \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \frac{\beta e^{(\alpha + \beta - \alpha')(x_0)}}{\alpha + \beta - \alpha'} \right\}$$

\hspace{1cm} (2.26)

where

$$\theta(x_0) = e^{-(\alpha + \beta)x_0} \left\{ \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \frac{\beta e^{(\alpha + \beta - \alpha')(x_0)}}{\alpha + \beta - \alpha'} \right\}.$$ \hspace{1cm} (2.27)

Evaluating $f$ at $x_2 = x_0$, we obtain the function

$$g(x_0) = 1 - \theta(x_0) \left\{ \frac{\beta - \beta'}{\alpha + \beta - \beta'} + \frac{\alpha e^{(\alpha + \beta - \beta')(x_0)}}{\alpha + \beta - \beta'} \right\} = 1 - g^*(x_0).$$ \hspace{1cm} (2.28)

Let us investigate the behavior of $g^*$, or equivalently of $h(x_0) = \log g^*(x_0)$. The function $h$ and its derivative $h'$ are given below.

$$h(x_0) = -(\alpha + \beta)x_0 + \log \left\{ \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \frac{\beta e^{(\alpha + \beta - \alpha')(x_0)}}{\alpha + \beta - \alpha'} \right\} + \log \left\{ \frac{\beta - \beta'}{\alpha + \beta - \beta'} + \frac{\alpha e^{(\alpha + \beta - \beta')(x_0)}}{\alpha + \beta - \beta'} \right\},$$ \hspace{1cm} (2.29)

$$h'(x_0) = -(\alpha + \beta) + \frac{\beta(\alpha + \beta - \alpha')e^{(\alpha + \beta - \alpha')(x_0)}}{(\alpha - \alpha') + \beta e^{(\alpha + \beta - \alpha')(x_0)}} + \frac{\alpha(\alpha + \beta - \beta')e^{(\alpha + \beta - \beta')(x_0)}}{(\beta - \beta') + \alpha e^{(\alpha + \beta - \beta')(x_0)}}.$$ \hspace{1cm} (2.30)
On adding and subtracting \( \frac{(\alpha - \alpha')(\alpha + \beta - \alpha')}{(\alpha - \alpha') + \beta e^{(\alpha + \beta - \alpha')x_0}} \) and \( \frac{(\beta - \beta')(\alpha + \beta - \beta')}{(\beta - \beta') + \alpha e^{(\alpha + \beta - \beta')x_0}} \) to the RHS of (2.30), we get

\[
h'(x_0) = (\alpha - \alpha') + (\beta - \beta') - \frac{(\alpha - \alpha')(\alpha + \beta - \alpha')}{(\alpha - \alpha') + \beta e^{(\alpha + \beta - \alpha')x_0}} \]
\[
- \frac{(\beta - \beta')(\alpha + \beta - \beta')}{(\beta - \beta') + \alpha e^{(\alpha + \beta - \beta')x_0}}.
\]

(2.31)

It is easy to show that \( h'(0) = 0 \). Differentiating \( h \) another time, we get

\[
h''(x_0) = \frac{\beta(\alpha - \alpha')(\alpha + \beta - \alpha')^2 e^{(\alpha + \beta - \alpha')x_0}}{[(\alpha - \alpha') + \beta e^{(\alpha + \beta - \alpha')x_0}]^2}
\]
\[
+ \frac{\alpha(\beta - \beta')(\alpha + \beta - \beta')^2 e^{(\alpha + \beta - \beta')x_0}}{[(\beta - \beta') + \alpha e^{(\alpha + \beta - \beta')x_0}]^2}.
\]

(2.32)

Notice that

\[
h''(x_0) < 0 \quad \text{if } \alpha < \alpha', \beta < \beta'
\]

(2.33)

and

\[
h''(x_0) > 0 \quad \text{if } \alpha > \alpha', \beta > \beta'.
\]

(2.34)

It is also true that

\[
g^*(0) = 1, \quad g^*(\infty) = 0 \quad \text{if } \alpha < \alpha', \beta < \beta'
\]

(2.35)

and

\[
g^*(0) = 1, \quad g^*(\infty) = \infty \quad \text{if } \alpha > \alpha', \beta > \beta'.
\]

(2.36)

Next, let us study the behavior of the derivative \( f' \) of \( f \) wrt \( x_2 \) given by

\[
f'(x_2) = \alpha e^{(\alpha + \beta - \beta')(x_2 - x_0)} - \theta(x_0)\alpha e^{(\alpha + \beta - \beta')x_2}.
\]

(2.37)

After some algebraic manipulations, we get

\[
f'(x_2) = \alpha e^{(\alpha + \beta - \beta')x_2} e^{-(\alpha + \beta)x_0 s(x_0)}
\]

(2.38)
where
\[ s(x_0) = e^{\beta x_0} - \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} \cdot \frac{\beta}{\alpha + \beta - \alpha'} e^{(\alpha + \beta - \alpha')x_0}. \]  (2.39)
Notice that \( s(0) = 0 \) and that
\[ s'(x_0) > 0 \quad \text{provided} \beta < \beta' \text{ and } \alpha + \beta - \alpha' - \beta' < 0 \]  (2.40)
and
\[ s'(x_0) < 0 \quad \text{provided} \beta > \beta' \text{ and } \alpha + \beta - \alpha' - \beta' > 0. \]  (2.41)
Thus, when \( \alpha < \alpha' \) and \( \beta < \beta' \), we see that
a) condition (PQD1) holds from (2.33) and (2.35), and
b) condition (PQD2) holds from (2.40).
Under this parametrization, the FBVE is PQD on \( D_1 \). We repeat the PQD algorithm on \( D_2 \) to complete the proof of (i).
On the other hand, when \( \alpha > \alpha' \) and \( \beta > \beta' \), then
a) condition (NQD1) holds from (2.34) and (2.36), and
b) condition (NQD2) holds from (2.41).
So, the FBVE is NQD on \( D_1 \) with this choice of parameters. We prove the result analogously on \( D_2 \) to show (ii). □

Remark: Under a third situation when \( \alpha < \alpha' \) or \( \beta < \beta' \) but not both, the FBVE is neither PQD nor NQD. In that case, one may generate examples for which \( F(x_1,x_2) \geq F_1(x_1)F_2(x_2) \) for \( (x_1,x_2) \in D_0 \subset \mathbb{R}^2_+ \), but \( F(x_1,x_2) \leq F_1(x_1)F_2(x_2) \) in \( D_0^c \). One such distribution is illustrated in Figure 5 (top picture).
Theorem 2.4.3 Raftcry's parsimonious models possess the following quadrant dependence properties:

(i) when \( \pi_1 = \pi_2 = \pi_{11} = \pi \) (model 1), the distribution is PQD.

(ii) when \( \pi_1 = \pi_2 = \pi \leq 0.5 \), \( \pi_{11} = 0 \) (model 2), the distribution is NQD.

(iii) when \( \pi_1 = \pi_2 = \pi > 0.5 \), \( \pi_{11} = 2\pi - 1 \) (model 2), the distribution is neither PQD nor NQD, unless \( \pi = 1 \). Then, it is PQD.

(iv) when \( \pi_2 = \pi_{11} = \gamma \) (model 3), the distribution is PQD.

Proof: The expressions for \( F, F_1, \) and \( F_2 \) are given in Appendix A.5.

To prove (i), use the PQD algorithm. Define

\[
G(x_1, x_2) = \frac{1}{1 + \pi} \left( 1 - e^{\lambda x_1(1 + \pi)} \right) + e^{\lambda x_1} e^{\frac{\lambda x_1}{1 + \pi}} (1 - e^{-\lambda x_1})
\]  

\( (x_1, x_2) \in D_1 \). On \( D_2 \), use \( G(x_2, x_1) \).

To prove (ii), notice that under this parametrization,

\[
F(x_1, x_2) = e^{-\lambda x_1} e^{-\lambda x_2} + e^{-\lambda x_1} e^{-\lambda x_1} - e^{-\lambda x_1} e^{-\lambda x_2}, \quad (x_1, x_2) \in D.
\]

Since \( F_i(x_i, x_2) = e^{-\lambda x_i},\ x_i > 0, i=1,2, \) then \( F(x_1, x_2) \leq F_1(x_1)F_2(x_2) \) iff \( (e^{-\frac{\lambda x_1}{1 + \pi}} - 1)(1 - e^{-\frac{\lambda x_2}{1 + \pi}}) \leq 0 \). This holds for all \( (x_1, x_2) \in \mathbb{R}^+ \). Thus, the result follows.

To prove (iii), use the PQD algorithm to show that condition (PQD1) holds but (PQD2) fails to hold for small \( \pi_0 \), unless \( \pi = 1 \). Define

\[
G(x_1, x_2) = \left\{ \frac{\lambda x_2}{\pi} \right\} \left\{ (1 - \pi) + (2\pi - 1) e^{\frac{\lambda x_1}{1 + \pi}} \right\}
\]

\[
+ \left\{ \frac{1 - \pi}{\pi} - \left( \frac{2\pi - 1}{\pi(1 + \pi)} \right) e^{\frac{\lambda x_1}{1 + \pi}} \right\} e^{\frac{\lambda x_1}{1 + \pi}}
\]

\[
- \frac{(\pi - 1)(\pi - 2)}{\pi(1 + \pi)}, \quad (2.44)
\]
\((x_1, x_2) \in D_1\). On \(D_2\), use \(G(x_2, x_1)\).

To prove (iv), use the PQD algorithm with

\[
G(x_1, x_2) = \frac{\gamma(1 - \pi_1)^2}{\pi_1(1 - \pi_1\gamma)} + \left\{ \frac{\pi_1 - \gamma}{\pi_1} - e^{\frac{x_2 x_1}{\pi_1}} \right\} e^{\frac{x_1 x_1}{\pi_1}}
\]

\[
+ e^{\frac{x_1 x_1}{\pi_1}} e^{\frac{x_2 x_2}{\pi_1}} - \frac{(1 - \gamma)^2}{1 - \pi_1\gamma} e^{\frac{x_2 x_1}{\pi_1}} e^{\frac{x_2 x_2}{\pi_1}}, \ (x_1, x_2) \in D_1. \tag{2.45}
\]

and

\[
G(x_1, x_2) = \frac{\gamma(1 - \pi_1)^2}{\pi_1(1 - \pi_1\gamma)} + \left\{ \frac{\pi_1 - \gamma}{\pi_1} - e^{\frac{x_2 x_1}{\pi_1}} \right\} e^{\frac{x_2 x_1}{\pi_1}}
\]

\[
+ \frac{\gamma e^{\frac{x_2 x_1}{\pi_1}} e^{\frac{x_2 x_2}{\pi_1}}}{\pi_1} - \frac{\gamma (1 - \pi_1)^2}{\pi_1} \frac{e^{\frac{x_2 x_1}{\pi_1}} e^{\frac{x_2 x_2}{\pi_1}}}{1 - \pi_1\gamma}, \ (x_1, x_2) \in D_2. \tag{2.46}
\]

\(\Box\)
Figure 3: Copula Plots $c(u, v) = uv$. (Top) Plot for the BVE with $\lambda_1 = 0.7, \lambda_2 = 1.8, \lambda_{12} = 0.5$. (Bottom) Plot for the ACBVE with $\lambda_1 = 1.5, \lambda_2 = 1.0, \lambda_{12} = 0.4$. 
Figure 4: Copula Plots $c(u, v) - uv$. (Top) Plot for the FBVE under a 'stress situation' with $\alpha = 0.2$, $\beta = 0.1$, $\alpha' = 0.4$, $\beta' = 0.9$. (Bottom) Plot for the FBVE under an 'enhanced situation' with $\alpha = 0.5$, $\beta = 0.8$, $\alpha' = 0.3$, $\beta' = 0.4$. 
Figure 5: Copula Plots $c(u, v) - uv$. (Top) Plot for the FBVE under a 'mixed situation' with $\alpha = 0.3$, $\beta = 0.9$, $\alpha' = 0.5$, $\beta' = 0.1$ (Bottom) Plot for the GBVE with $\theta_1 = 0.5$, $\theta_2 = 0.8$, $\delta = 0.5$. 
Figure 6: Copula Plots $c(u,v) - uv$. (Top) Plot for model 1 of the RBVE with $\pi_1 = \pi_2 = \pi_{11} = 0.5$, $p_{00} = 0.5$, $p_{10} = p_{01} = 0$, $\lambda = 1$. (Bottom) Plot for model 2 of the RBVE with $\pi_1 = \pi_2 = 0.5$, $\pi_{11} = 0$, $p_{00} = 0$, $p_{10} = p_{01} = 0.5$, $\lambda = 1$. 
Figure 7: Copula Plots $c(u,v) - uv$. (Top) Plot for model 2 of the RBVE with $\pi_1 = \pi_2 = 0.75$, $p_{11} = 0.5$, $p_{00} = 0$, $p_{10} = p_{01} = 0.25$, $\lambda = 1$. (Bottom) Plot for model 3 of the RBVE with $\pi_1 = 0.75$, $\pi_2 = 0.25$, $p_{11} = 0.25$, $p_{00} = 0.25$, $p_{10} = 0.5$, $p_{01} = 0$, $\lambda = 1$. 
CHAPTER III

Distributional Properties of the Order Statistics

In the language of reliability theory and survival analysis, the minimum and maximum order statistics represent the lifetimes of series and parallel systems, respectively. Thus, the marginal distributions of the order statistics coincide with the lifetime distributions of these coherent\(^1\) structures.

We begin our study of the order statistics from bivariate exponential distributions by looking at just such distributions and their dependence properties. In section 3.1, we give the joint pdf and/or survival df of the minimum and maximum. In sections 3.2 and 3.3, we give their marginal distributions, and moments and correlation, respectively. The copula structures are presented graphically in section 3.4. Finally, we look at the regression function and the distribution of the spacing in section 3.5.

3.1 Joint Distribution

We derive here the joint survival distribution of the minimum and maximum from seven bivariate exponential distributions: the BVE, Block and Basu's ACBVE, Fre-
und's FBVE, Gumbel's GBVE, Raftery's RBVE, Sarkar's ACBVE₂, and Friday and Patil's BEE. Whenever the joint distribution is absolutely continuous, we present the joint pdf as well.

One derivation process is employed in all these settings and the following is a brief description. Let \((X_1, X_2)\) be distributed according to the joint cdf \(F(x_1, x_2)\), \((x_1, x_2) \in \mathbb{R}_2^+\). Denote the corresponding survival df by \(F(x_1, x_2)\). Define \(T_1 = \min(X_1, X_2)\) and \(T_2 = \max(X_1, X_2)\) and denote their joint survival df by \(H(t_1, t_2)\). Then,

\[
H(t_1, t_2) = \begin{cases} 
F(t_1, t_1) & , 0 < t_2 < t_1 < \infty \\
F(t_1, t_2) + F(t_2, t_1) - F(t_2, t_2) & , 0 < t_1 \leq t_2 < \infty .
\end{cases}
\]  

(3.1)

Sumita and Kijima (1986) give a proof of this result. If \(f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}\) exists, then the joint pdf of \((T_1, T_2)\), denoted by \(h(t_1, t_2)\) is derived as follows:

\[
h(t_1, t_2) = f(t_1, t_2) + f(t_2, t_1), \ 0 < t_1 < t_2 < \infty .
\]  

(3.2)

**Theorem 3.1.1** Let \((X_1, X_2) \sim BVE(\lambda_1, \lambda_2, \lambda_{12})\) and let \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\). Then, the joint survival df of \((T_1, T_2)\) is given by

\[
H(t_1, t_2) = \begin{cases} 
e^{-\lambda t_1} & , t_2 < t_1 \\
e^{-\lambda t_1 - (\lambda_2 + \lambda_{12})t_2} + e^{-\lambda t_1 - (\lambda_1 + \lambda_{12})t_2} - e^{-\lambda t_2} & , t_1 \leq t_2 .
\end{cases}
\]  

(3.3)

**Proof:** The survival df \(F(x_1, x_2)\) is given in Appendix A.1. Apply equation \(3.1\) to get the result. □

**Theorem 3.1.2** Let \((X_1, X_2) \sim ACBVE(\lambda_1, \lambda_2, \lambda_{12})\) and let \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\). Then,
(i) the joint survival df of \((T_1, T_2)\) is given by

\[
\overline{H}(t_1, t_2) = \begin{cases} 
    e^{-\lambda t_1}, & t_2 < t_1 \\
    \frac{\lambda}{\lambda_1 + \lambda_2} e^{-\lambda_1 t_1 - (\lambda_2 + \lambda_{12}) t_2} \\
    \frac{\lambda}{\lambda_1 + \lambda_2} e^{-\lambda_1 t_1 - (\lambda_2 + \lambda_{12}) t_2} - \frac{\lambda_1 + \lambda_{12}}{\lambda_1 + \lambda_2} e^{-\lambda t_2}, & t_1 < t_2 
\end{cases}.
\]  

(ii) the joint pdf of \((T_1, T_2)\) is given by

\[
h(t_1, t_2) = \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} e^{-\lambda_{12} t_2 - \lambda_1 t_1} + \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} e^{-\lambda_{12} t_2 - \lambda_2 t_1}, \quad 0 < t_1 < t_2 < \infty.
\] 

**Proof:** The survival df and joint pdf of \((X_1, X_2)\) are given in Appendix A.2. Apply equation (3.1) to get (i). Apply equation (3.2) to obtain (ii). \(\square\)

**Theorem 3.1.3** Let \((X_1, X_2) \sim FBVE(\alpha, \beta, \alpha', \beta')\). Partition the parameter space into \(\Omega_1 = \{ (\alpha, \beta, \alpha', \beta') : \alpha + \beta \neq \alpha', \beta' \} \), \(\Omega_2 = \{ (\alpha, \beta, \alpha', \beta') : \alpha + \beta = \alpha', \alpha + \beta \neq \beta' \} \), \(\Omega_3 = \{ (\alpha, \beta, \alpha', \beta') : \alpha + \beta = \beta', \alpha + \beta \neq \alpha' \} \), and \(\Omega_4 = \{ (\alpha, \beta, \alpha', \beta') : \alpha + \beta = \alpha' = \beta' \} \). Then,

(i) the joint survival df of \((T_1, T_2)\) is given by

\[
\overline{H}(t_1, t_2) = \begin{cases} 
    e^{-(\alpha + \beta)t_1}, & t_2 < t_1 \\
    \overline{H}_{\Omega_i}(t_1, t_2), & t_1 < t_2 \quad i=1,2,3,4
\end{cases}
\]  

where

\[
\overline{H}_{\Omega_1}(t_1, t_2) = e^{-\alpha t_1} \left[ \frac{\alpha e^{-\beta t_2 - t_1}}{\alpha + \beta - \beta'} + \frac{\beta e^{-\alpha t_2 - t_1}}{\alpha + \beta - \alpha'} \right] + \frac{\alpha \beta e^{-\alpha t_1}}{(\alpha + \beta - \alpha') (\alpha + \beta - \beta')} e^{-\alpha t_1}
\] 

\[
\overline{H}_{\Omega_2}(t_1, t_2) = \frac{\alpha}{\alpha + \beta - \beta'} e^{-(\alpha + \beta)t_1 - \beta t_2 - t_1} + \left[ \frac{(\beta - \beta')}{\alpha + \beta - \beta'} + \beta (t_2 - t_1) \right] e^{-(\alpha + \beta)t_2}
\] 

\[
\overline{H}_{\Omega_3}(t_1, t_2) = \frac{\beta}{\alpha + \beta - \alpha'} e^{-(\alpha + \beta)t_1 - \alpha t_2 - t_1} + \left[ \frac{(\alpha - \alpha')}{\alpha + \beta - \alpha'} + \alpha (t_2 - t_1) \right] e^{-(\alpha + \beta)t_2}
\] 

\[
\overline{H}_{\Omega_4}(t_1, t_2) = \frac{\alpha + \beta}{\alpha + \beta} e^{-(\alpha + \beta)t_1 - t_1} + \left[ \frac{\alpha - \beta}{\alpha + \beta} + \beta (t_2 - t_1) \right] e^{-(\alpha + \beta)t_2}
\]
\[ \overline{H}_{\Omega_2}(t_1, t_2) = \frac{\beta}{\alpha + \beta - \alpha'} e^{-(\alpha + \beta) t_1 - \alpha'(t_2 - t_1)} + \left( \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \alpha(t_2 - t_1) \right) e^{-(\alpha + \beta) t_2} \]  
(3.9)

and

\[ \overline{H}_{\Omega_4}(t_1, t_2) = [1 + (\alpha + \beta)(t_2 - t_1)] e^{-(\alpha + \beta) t_2}. \]  
(3.10)

(ii) the joint pdf of \((T_1, T_2)\) on \(\Omega_1\) is given by

\[ h(t_1, t_2) = \alpha' e^{-\beta' t_2 - (\alpha + \beta') t_1} + \beta' e^{-\alpha' t_2 - (\alpha + \beta') t_1}, \quad 0 < t_1 < t_2 < \infty. \]  
(3.11)

On \(\Omega_2\), take \(\alpha + \beta - \alpha' = 0\) in \(h(t_1, t_2)\) above. On \(\Omega_3\), take \(\alpha + \beta - \beta' = 0\). Finally, on \(\Omega_4\), take \(\alpha + \beta - \alpha' = 0\) and \(\alpha + \beta - \beta' = 0\).

**Proof:** The expressions \(F(x_1, x_2)\) and \(f(x_1, x_2)\) on \(\Omega_1\) are given in Appendix A.3. Use equation (3.1) to establish the expression \(\overline{H}_{\Omega_1}\). The survival function on \(\Omega_2\) can be obtained as follows:

\[ \overline{H}_{\Omega_2}(t_1, t_2) = \lim_{\alpha'-\alpha\rightarrow(\alpha+\beta)} \overline{H}_{\Omega_1}(t_1, t_2) \]
\[ = \frac{\alpha}{\alpha + \beta - \beta'} e^{-(\alpha + \beta) t_1 - \beta'(t_2 - t_1)} \]
\[ + \lim_{\alpha'-\alpha\rightarrow(\alpha+\beta)} m(\alpha + \beta, \alpha') \]  
(3.12)

where

\[ m(x, y) = \frac{e^{-\alpha t_2}}{x-y} \left\{ \beta e^{(x-y)(t_2-t_1)} + \frac{y(\beta' - \beta) - (x - \beta)\beta'}{x - \beta'} \right\}. \]  
(3.13)

Since \(\lim_{y \rightarrow x} m(x, y)\) is of the form \(\frac{0}{0}\), we apply L’Hospital’s rule. Fix \(x = (\alpha + \beta)\), and differentiate the numerator and denominator of \(m(x, y)\) with respect to \(y\). Proceed
to take the corresponding limit of the ratio of the derivatives as \( y \to x \). This gives us the expression for \( \overline{H}_{\Omega_2}(t_1, t_2) \).

In a similar manner, we can obtain \( \overline{H}_{\Omega_1} \), by taking the limit \( \lim_{y \to (a+\beta)} \overline{H}_{\Omega_1}(t_1, t_2) \).

Finally, we have that

\[
\overline{H}_{\Omega_1}(t_1, t_2) = \lim_{\sigma \to a+\beta} \overline{H}_{\Omega_2}(t_1, t_2) = \lim_{\beta \to (a+\beta)} \overline{H}_{\Omega_2}(t_1, t_2).
\]

(3.14)

This proves result (i). We then use equation (3.2) to show that (ii) holds. \( \square \)

**Theorem 3.1.4** Let \((X_1, X_2) \sim GBVE(\theta_1, \theta_2, \delta)\). Then,

(i) the joint survival df of \((T_1, T_2)\) is given by

\[
\overline{H}(t_1, t_2) = \left\{ e^{-[a(t_1, t_1)]^\delta} - e^{-[a(t_1, t_2)]^\delta} - e^{-[a(t_2, t_2)]^\delta}, \ t_2 < t_1 \right\}.
\]

(ii) the joint pdf of \((T_1, T_2)\) is given by

\[
h(t_1, t_2) = c(t_1, t_2) \left( e^{-[a(t_1, t_1)]^\delta} \right) + \left[ e^{-[a(t_1, t_2)]^\delta} \right] + \left[ e^{-[a(t_2, t_2)]^\delta} \right] - 1,
\]

for \(0 < t_1 < t_2 < \infty\), \(a(t_1, t_2) = \left( \frac{t_1}{\theta_1} \right)^\frac{1}{\delta} + \left( \frac{t_2}{\theta_2} \right)^\frac{1}{\delta} \) and \(c(t_1, t_2) = (\theta_1 \theta_2)^{-\frac{1}{\delta}}(t_1 t_2)^{-\delta-1} \).

**Proof:** The expressions for \( F(x_1, x_2) \) and \( f(x_1, x_2) \) are given in Appendix A.4. Result (i) is a simple consequence of equation (3.1). Result (ii) follows from equation (3.2). \( \square \)

**Theorem 3.1.5** Let \((X_1, X_2)\) have RBVE(\(\lambda, \pi_1, \pi_2, p_{ij}\)). Then,
(i) the joint survival df of $(T_1, T_2)$ is given by

$$H(t_1, t_2) = \left\{ \begin{array}{ll}
\frac{ae^{-\lambda_1(1-\pi_1) + 1-\frac{1}{\pi_2}} + \frac{p_{10}}{\pi_2} e^{-\lambda_1(1-\pi_1) + 1-\frac{1}{\pi_2}}}{\pi_1} + \frac{p_{11} e^{-\lambda_1(1-\pi_1) + 1-\frac{1}{\pi_2}} \pi_2}{\pi_1} & , t_2 < t_1 \\
a e^{-\frac{\lambda_1}{1-\pi_1} - \frac{\lambda_2}{1-\pi_2}} + e^{-\frac{\lambda_1}{1-\pi_1} - \frac{\lambda_2}{1-\pi_2}} & \\
- e^{-\lambda_2(1-\frac{1}{\pi_1} + 1-\frac{1}{\pi_2})} + e^{-\lambda_2} & + \frac{p_{10}}{\pi_2} e^{-\lambda_1(1-\pi_1) - \lambda_2} + e^{-\lambda_1(1-\pi_1) - \lambda_2} - e^{-\lambda_2(1-\frac{1}{\pi_1} + 1-\frac{1}{\pi_2})} & \\
+ \frac{p_{11}}{\pi_2} [e^{-\frac{\lambda_1}{1-\pi_1} - \lambda_2} + e^{-\frac{\lambda_1}{1-\pi_1} - \lambda_2} - e^{-\lambda_2(1-\frac{1}{\pi_1} + 1-\frac{1}{\pi_2})}] & + e^{-\lambda_2} - p_{11}(1-\pi_2)^2 e^{-\frac{1}{1-\pi_2} (t_2 - \pi_2 t_1)} & \\
- p_{11}(1-\pi_2)^2 e^{-\frac{1}{1-\pi_1} (t_2 - \pi_1 t_1)} & , t_1 < t_2 \\
\end{array} \right. $$

(3.17)

(ii) the joint pdf of $(T_1, T_2)$ is given by

$$h(t_1, t_2) = \lambda^2 \left\{ \frac{a}{(1 - \pi_1)(1 - \pi_2)} \left[ e^{-\frac{\lambda_1}{1-\pi_1} - \frac{\lambda_2}{1-\pi_2}} + e^{-\frac{\lambda_1}{1-\pi_1} - \frac{\lambda_2}{1-\pi_2}} \right] + \frac{p_{10}}{(1 - \pi_1) \pi_2} [e^{-\frac{\lambda_1}{1-\pi_1} - \lambda_2} + e^{-\frac{\lambda_1}{1-\pi_1} - \lambda_2}] + \frac{p_{10}}{(1 - \pi_2) \pi_1} [e^{-\frac{\lambda_2}{1-\pi_2} - \lambda_1} + e^{-\frac{\lambda_2}{1-\pi_2} - \lambda_1}] + \frac{p_{11}}{1 - \pi_1 \pi_2} [e^{-\frac{1}{1-\pi_2} (t_2 - \pi_2 t_1)} + e^{-\frac{1}{1-\pi_1} (t_2 - \pi_1 t_1)}] \right\} , 0 < t_1 < t_2 < \infty$$

(3.18)

where $a = p_{11} \left\{ \frac{\pi_1 + \pi_2 - 2 \pi_1 \pi_2}{\pi_1 \pi_2 (1 - \pi_1 \pi_2)} \right\} = 1$.

**Proof:** The expressions for $F(x_1, x_2)$ and $f(x_1, x_2)$ are given in Appendix A.5.

To prove (i), we use equation (3.1) and some algebraic manipulations. To prove (ii), we use equation (3.2). □

**Theorem 3.1.6** Let $(X_1, X_2) \sim ACBVE_2(\lambda_1, \lambda_2, \lambda_{12})$. Define $A(z) = 1 - e^{-z}$, $z > 0$, $\gamma = \frac{\lambda_{12}}{\lambda_1 + \lambda_2}$, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Then,
(i) the joint survival df of \((T_1, T_2)\) is given by

\[
\bar{H}(t_1, t_2) = \begin{cases} 
  e^{-\lambda t_1} & \text{, } t_2 < t_1 \\
  e^{-(\lambda_2+\lambda_1)t_2} \{1 - [A(\lambda_1t_2)]^{-\gamma}[A(\lambda_1t_1)]^{1+\gamma}\} + e^{-(\lambda_1+\lambda_2)t_2} \{1 - [A(\lambda_2t_2)]^{-\gamma}[A(\lambda_2t_1)]^{1+\gamma}\} - e^{-\lambda t_2} & \text{, } t_1 < t_2
\end{cases}
\]

\( \tag{3.19} \)

(ii) the joint pdf of \((T_1, T_2)\) is given by

\[

h(t_1, t_2) = \frac{\lambda_1 \lambda}{(\lambda_1 + \lambda_2)^2} e^{-\lambda_1 t_1 - (\lambda_2 + \lambda_1) t_2} \left\{ \frac{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2) - \lambda_2 \lambda e^{-\lambda_1 t_1}}{[A(\lambda_1t_2)]^{-\gamma}[A(\lambda_1t_1)]^{1+\gamma}} \right\} + \frac{\lambda_2 \lambda}{(\lambda_1 + \lambda_2)^2} e^{-\lambda_2 t_1 - (\lambda_1 + \lambda_2) t_2} \left\{ \frac{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2) - \lambda_1 \lambda e^{-\lambda_2 t_1}}{[A(\lambda_2t_2)]^{-\gamma}[A(\lambda_2t_1)]^{1+\gamma}} \right\}

\]

\( \tag{3.20} \)

for \(0 < t_1 < t_2 < \infty\).

**Proof:** The joint survival df \( \bar{F}(x_1, x_2) \) and joint pdf \( f(x_1, x_2) \) are given in Appendix A.6. Apply equation (3.1) to show (i) and equation (3.2) to show (ii). □

**Theorem 3.1.7** Let \((X_1, X_2) \sim BEE(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')\). Define \( \phi_1 = \frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2' - \alpha_1'} \), and \( \phi_2 = \frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'} \). Partition the parameter space into \( \Lambda_1 = \{ (\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 \neq \alpha_1', \alpha_2' \} \), \( \Lambda_2 = \{ (\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_1', \alpha_1 + \alpha_2 \neq \alpha_2' \} \), \( \Lambda_3 = \{ (\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_2', \alpha_1 + \alpha_2 \neq \alpha_1' \} \), and \( \Lambda_4 = \{ (\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_1' = \alpha_2' \} \). Then, the joint survival df of \((T_1, T_2)\) is given by

\[

\bar{H}(t_1, t_2) = \begin{cases} 
  e^{-(\alpha_1+\alpha_2)t_1} & \text{, } t_2 < t_1 \\
  \bar{H}_{\Lambda_i}(t_1, t_2) & \text{, } t_1 \leq t_2, \ i = 1, 2, 3, 4
\end{cases}
\]

\( \tag{3.21} \)
where

\[
\overline{H}_{A_1}(t_1, t_2) = \phi_1 e^{-(\alpha_1+\alpha_2)t_1-\alpha_2' t_2} + \phi_2 e^{-(\alpha_1+\alpha_2')t_1-\alpha_2' t_2} + (1 - \phi_1 - \phi_2) e^{-(\alpha_1+\alpha_2)t_2} \tag{3.22}
\]

\[
\overline{H}_{A_2}(t_1, t_2) = \phi_1 e^{-(\alpha_1+\alpha_2)t_1-\alpha_2'(t_2-t_1)} + (1 - \phi_1) e^{-(\alpha_1+\alpha_2)t_2} + \alpha_0 \alpha_2 (t_2 - t_1) e^{-(\alpha_1+\alpha_2)t_2} \tag{3.23}
\]

\[
\overline{H}_{A_3}(t_1, t_2) = \phi_2 e^{-(\alpha_1+\alpha_2)t_1-\alpha_1'(t_2-t_1)} + (1 - \phi_2) e^{-(\alpha_1+\alpha_2)t_2} + \alpha_0 \alpha_1 (t_2 - t_1) e^{-(\alpha_1+\alpha_2)t_2} \tag{3.24}
\]

and

\[
\overline{H}_{A_4}(t_1, t_2) = [1 + \alpha_0(\alpha_1 + \alpha_2)(t_2 - t_1)] e^{-(\alpha_1+\alpha_2)t_2}. \tag{3.25}
\]

**Proof:** Refer to Appendix A.7 for \( F(x_1, x_2) \). Apply equation (3.1) to derive the expression \( \overline{H}_{A_1} \). On \( \Lambda_i, i=2,3 \), we obtain the survival function as follows:

\[
\overline{H}_{A_i}(t_1, t_2) = \lim_{b_i \to (\alpha_1+\alpha_2)} \overline{H}_{A_i}(t_1, t_2) \tag{3.26}
\]

where \( b_2 = \alpha_1' \) and \( b_3 = \alpha_2' \). Also, we have that

\[
\overline{H}_{A_4}(t_1, t_2) = \lim_{\alpha_1' \to (\alpha_1+\alpha_2)} \overline{H}_{A_5}(t_1, t_2) = \lim_{\alpha_2' \to (\alpha_1+\alpha_2)} \overline{H}_{A_2}(t_1, t_2). \tag{3.27}
\]

We then apply the L'Hospital's rule in a manner similar to the proof of Theorem 3.1.3.

\[ \square \]
3.2 Marginal Distributions

We now present the marginal distribution of the minimum, $T_1$, and maximum, $T_2$ for the bivariate exponential distributions we listed in section 3.1. These distributions are readily obtained from the joint survival df of $(T_1, T_2)$. If we let $\overline{F}_i(t_i)$ denote the marginal survival df of $T_i$, $i=1,2$, then a basic property of joint df's implies that

$$\overline{F}_i(t_i) = \overline{H}(t_i,0) = \lim_{t_2 \to 0} \overline{H}(t_i,t_2), \quad t_1 \in (0, \infty)$$  \hspace{1cm} (3.28)

and

$$\overline{F}_j(t_j) = \overline{H}(0,t_j) = \lim_{t_1 \to 0} \overline{H}(t_1,t_j), \quad t_2 \in (0, \infty).$$  \hspace{1cm} (3.29)

We find that predominantly the minimum is exponentially distributed, and the maximum has a GH distribution (defined in section 1.3). This holds for the order statistics of independent exponential rv's as well. In general, however, for some bivariate exponential models the minimum has a GH distribution while the maximum is distributed according to a generalized mixture of gamma and exponential distributions.

**Theorem 3.2.1** Let $(X_1, X_2) \sim BVE(\lambda_1, \lambda_2, \lambda_{12})$ and let $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Then,

$$\overline{F}_{(1)}(t_1) = e^{-\lambda_1 t_1}, \quad t_1 > 0$$  \hspace{1cm} (3.30)

and

$$\overline{F}_{(2)}(t_2) = e^{-(\lambda_1+\lambda_{12})t_2} + e^{-(\lambda_2+\lambda_{12})t_2} - e^{-\lambda_2 t_2}, \quad t_2 > 0.$$  \hspace{1cm} (3.31)

**Proof:** Apply equation (3.28) and (3.29) on the joint survival df $\overline{H}(t_1, t_2)$ defined in Theorem 3.1.1. The distribution of the minimum is given by Marshall and Olkin (1967), while that of the maximum by Downton (1970). \qed
**Theorem 3.2.2** Let \((X_1, X_2) \sim ACBVE(\lambda_1, \lambda_2, \lambda_{12})\) and let \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\). Then,

\[
F_{(1)}(t_1) = e^{-\lambda t_1}, \quad t_1 > 0
\]  

and

\[
F_{(2)}(t_2) = \frac{\lambda}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_{12})t_2} + \frac{\lambda}{\lambda_1 + \lambda_2} e^{-(\lambda_2 + \lambda_{12})t_2} + \frac{\lambda + \lambda_{12}}{\lambda_1 + \lambda_2} e^{-\lambda_{12} t_2}, \quad t_2 > 0. 
\]

**Proof:** Use the expression \(H(t_1, t_2)\) given in Theorem 3.1.2(i) and take the limits defined in equations (3.28) and (3.29), respectively. The distribution of the minimum is due to Block and Basu (1974). \(\square\)

**Theorem 3.2.3** Let \((X_1, X_2) \sim FBVE(\alpha, \beta, \alpha', \beta')\). Partition the parameter space into \(\Omega_1 = \{(\alpha, \beta, \alpha', \beta') : \alpha + \beta \neq \alpha', \beta'\}\), \(\Omega_2 = \{(\alpha, \beta, \alpha', \beta') : \alpha + \beta = \alpha', \alpha + \beta \neq \beta'\}\), \(\Omega_3 = \{(\alpha, \beta, \alpha', \beta') : \alpha + \beta = \beta', \alpha + \beta \neq \alpha'\}\), and \(\Omega_4 = \{(\alpha, \beta, \alpha', \beta') : \alpha + \beta = \alpha' = \beta'\}\). Then,

\[
F_{(1)}(t_1) = e^{-(\alpha + \beta)t_1}, \quad t_1 > 0
\]

and

\[
F_{(2)}(t_2) = \frac{\alpha}{\alpha + \beta - \beta'} e^{-\beta't_2} + \frac{\beta}{\alpha + \beta - \alpha'} e^{-\alpha't_2} + \frac{\alpha' \beta' - \alpha' \beta - \alpha \beta'}{(\alpha + \beta - \alpha')(\alpha + \beta - \beta')} e^{-(\alpha + \beta)t_2}, \quad t_2 > 0 
\]

\[
\text{On } \Omega_1: \quad F_{(2)}(t_2) = \frac{\alpha}{\alpha + \beta - \beta'} e^{-\beta't_2} + \frac{\beta}{\alpha + \beta - \alpha'} e^{-\alpha't_2} + \frac{\alpha' \beta' - \alpha' \beta - \alpha \beta'}{(\alpha + \beta - \alpha')(\alpha + \beta - \beta')} e^{-(\alpha + \beta)t_2}, \quad t_2 > 0
\]

\[
\text{On } \Omega_2: \quad F_{(2)}(t_2) = \frac{\alpha}{\alpha + \beta - \beta'} e^{-\beta't_2} + \left[ \frac{\beta - \beta'}{\alpha + \beta - \beta'} + \beta t_2 \right] e^{-(\alpha + \beta)t_2}, \quad t_2 > 0
\]

\[
\text{On } \Omega_3: \quad F_{(2)}(t_2) = \frac{\beta}{\alpha + \beta - \alpha'} e^{-\alpha't_2} + \left[ \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} + \alpha t_2 \right] e^{-(\alpha + \beta)t_2}, \quad t_2 > 0
\]

and

\[
\text{On } \Omega_4: \quad F_{(2)}(t_2) = [1 + (\alpha + \beta)t_2] e^{-(\alpha + \beta)t_2}, \quad t_2 > 0.
\]
Proof: Apply equations (3.28) and (3.29) on $H(t_1, t_2)$ defined in Theorem 3.1.3(i). Hutchinson and Lai (1990) have noted earlier that $T_1$ is exponential. On $\Omega_1$, $T_2$ has a GH distribution, whereas on $\Omega_i$, $i=2,3,4$, $F_2(t_2)$ is a generalized mixture of gamma and exponential distributions. □

**Theorem 3.2.4** Let $(X_1, X_2) \sim GBVE(\theta_1, \theta_2, \delta)$. Then,

$$F_1(t_1) = e^{-\theta_1 [(\frac{t_1}{\theta_1})^\delta + (\frac{t_2}{\theta_2})^\delta]^\delta}, \quad t_1 > 0$$

(3.39)

and

$$F_2(t_2) = e^{-\frac{t_2}{\theta_1}} + e^{-\frac{t_2}{\theta_2}} - e^{-\theta_2 [(\frac{t_1}{\theta_1})^\delta + (\frac{t_2}{\theta_2})^\delta]^\delta}, \quad t_2 > 0.$$ (3.40)

**Proof:** Take the limits defined in equations (3.28) and (3.29) on the expression $H(t_1, t_2)$ given in Theorem 3.1.4(i). Note that $[(\frac{t_1}{\theta_1})^\delta + (\frac{t_2}{\theta_2})^\delta]^\delta = a(1, 1)$, where $a(t_1, t_2)$ is as defined in Theorem 3.1.4. □

**Theorem 3.2.5** Let $(X_1, X_2) \sim RBVE(\lambda, \pi_1, \pi_2, p_{11})$. Then,

$$F_1(t_1) = p_{11} \frac{2 - \pi_1 - \pi_2}{1 - \pi_1 \pi_2} e^{-\lambda t_1} + \left(1 - \frac{p_{11}}{\pi_1}\right) e^{-\lambda(1 + \frac{1}{1 - \pi_2}) t_1}$$

$$+ \left(1 - \frac{p_{11}}{\pi_2}\right) e^{-\lambda(1 + \frac{1}{1 - \pi_1}) t_1} + a e^{-\lambda(1 - \pi_1 + \frac{1}{1 - \pi_2}) t_1}, \quad t_1 > 0$$ (3.41)

where $a = p_{11} \left(\frac{\pi_1 + \pi_2 - 2\pi_1 \pi_2}{\pi_1 \pi_2 (1 - \pi_1 \pi_2)}\right) - 1$, and

$$F_2(t_2) = 2e^{-\lambda t_2} - F_1(t_2), \quad t_2 > 0.$$ (3.42)

**Proof:** Apply equation (3.28) to $H(t_1, t_2)$ defined in Theorem 3.1.5(i) to get $F_1(t_1)$. Use the following facts to find $F_2(t_2)$:

$$F_3(t_2) = F(t_2, 0) + F(0, t_2) - F(t_2, t_2)$$ (3.43)
Theorem 3.2.6 Let \((X_1, X_2) \sim ACBVE_2(\lambda_1, \lambda_2, \lambda_{12})\). Define \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\). Then,

\[
\overline{F}(t_1, t_2) = \overline{F}(t_2) \quad (3.44)
\]

\(\square\)

Theorem 3.2.7 Let \((X_1, X_2) \sim BEE(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')\). Define \(\phi_1 = \frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'}\), and \(\phi_2 = \frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'}\). Partition the parameter space into \(\Lambda_1 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 \neq \alpha_1', \alpha_2'\}\), \(\Lambda_2 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_1', \alpha_1 + \alpha_2 \neq \alpha_2'\}\), \(\Lambda_3 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_2', \alpha_1 + \alpha_2 \neq \alpha_1'\}\), and \(\Lambda_4 = \{(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') : \alpha_1 + \alpha_2 = \alpha_1' = \alpha_2'\}\). Then,

\[
\overline{F}(t_1, t_2) = e^{-(\alpha_1 + \alpha_2) t_1} , t_1 > 0 \quad (3.47)
\]
and

On $A_1$: $\overline{F}_2(t_2) = \phi_1 e^{-\alpha_1 t_2} + \phi_2 e^{-\alpha_1 t_2} + (1 - \phi_1 - \phi_2) e^{-(\alpha_1 + \alpha_2) t_2}$, $t_2 > 0$ (3.48)

On $A_2$: $\overline{F}_2(t_2) = \phi_1 e^{-\alpha_1 t_2} + [(1 - \phi_1) + \alpha_0 \alpha_2 t_2] e^{-(\alpha_1 + \alpha_2) t_2}$, $t_2 > 0$ (3.49)

On $A_3$: $\overline{F}_2(t_2) = \phi_2 e^{-\alpha_1 t_2} + [(1 - \phi_2) + \alpha_0 \alpha_1 t_2] e^{-(\alpha_1 + \alpha_2) t_2}$, $t_2 > 0$ (3.50)

and

On $A_4$: $\overline{F}_2(t_2) = [1 + \alpha_0 (\alpha_1 + \alpha_2) t_2] e^{-(\alpha_1 + \alpha_2) t_2}$, $t_2 > 0$. (3.51)

Proof: These results are immediate consequences of applying equations (3.28) and (3.29) to $H(t_1, t_2)$ given in Theorem 3.1.7. Friday and Patil (1977) give the distribution of the minimum in their paper. Notice that on $A_1$, $T_2$ has a GH distribution, while on $A_i$, $i=2,3,4$, $\overline{F}_2(t_2)$ is a generalized mixture of gamma and exponential distributions. □

3.3 Moments and Correlation

The mean and variance of $T_1$ and $T_2$ and the correlation coefficient between them are derived in a straightforward manner. We present these results below.

Theorem 3.3.1 Let $(X_1, X_2) \sim BVE(\lambda_1, \lambda_2, \lambda_{12})$ and let $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Then,

(i) the mean and variance of $T_1$ are, respectively

$$E[T_1] = \frac{1}{\lambda} \quad \text{and} \quad \text{Var}[T_1] = \frac{1}{\lambda^2}.$$ (3.52)
(ii) the mean and variance of $T_2$ are, respectively

$$E[T_2] = \frac{\lambda^2 - \lambda_1 \lambda_2}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$$

(3.53)

and

$$\text{Var}[T_2] = \frac{(\lambda_2 - \lambda_1)^2}{(\lambda_1 + \lambda_{12})^2(\lambda_2 + \lambda_{12})^2} \cdot \frac{3\lambda_1 \lambda_2 + \lambda \lambda_{12} - 2\lambda^2}{\lambda^2(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$$

$$= \frac{\lambda^2(\lambda_2 - \lambda_1)^2 - (\lambda - \lambda_2)(\lambda - \lambda_1)[3\lambda_1 \lambda_2 - \lambda(\lambda_1 + \lambda_2 + \lambda)]}{\lambda^2(\lambda - \lambda_1)^2(\lambda - \lambda_2)^2}.$$  

(3.54)

(iii) the correlation coefficient between $T_1$ and $T_2$ is given by

$$\text{Corr}(T_1, T_2) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{\sqrt{\lambda^2(\lambda_2 - \lambda_1)^2 - (\lambda - \lambda_2)(\lambda - \lambda_1)[3\lambda_1 \lambda_2 - \lambda(\lambda_1 + \lambda_2 + \lambda)]}}.$$  

(3.55)

Proof: Result (i) follows immediately from the fact that $T_1 \sim e(\lambda)$. Now, we show result (ii).

$$E[T_2] = \int_0^\infty tf(t) \, dt$$

$$= \frac{1}{\lambda_1 + \lambda_{12}} + \frac{1}{\lambda_2 + \lambda_{12}} - \frac{1}{\lambda}$$

$$= \frac{\lambda(\lambda_2 + \lambda_{12}) + \lambda(\lambda_1 + \lambda_{12}) - (\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$$

(3.56)

which simplifies to the expression given above.

$$\text{Var}[T_2] = \frac{2}{(\lambda_1 + \lambda_{12})^2} + \frac{2}{(\lambda_2 + \lambda_{12})^2} - \frac{2}{\lambda^2} \cdot \frac{1}{(\lambda_1 + \lambda_{12})^2} - \frac{1}{(\lambda_2 + \lambda_{12})^2} - \frac{1}{\lambda^2}$$

$$- \frac{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}{2} + \frac{\lambda(\lambda_1 + \lambda_{12})}{2} + \frac{\lambda(\lambda_2 + \lambda_{12})}{2}$$

$$= \frac{1}{(\lambda_1 + \lambda_{12})^2} + \frac{1}{(\lambda_2 + \lambda_{12})^2} - \frac{3}{\lambda^2}.$$
This simplifies to the first form for \( \text{Var}(T_2) \). On noting that \( \lambda - \lambda_1 = \lambda_2 + \lambda_{12} \) and \( \lambda - \lambda_2 = \lambda_1 + \lambda_{12} \), it is easy to see that the alternate expression holds as well.

Next, we derive the Pearson’s product moment correlation coefficient as follows:

\[
E[T_1 T_2] = \int \int \mathcal{H}(u, v) \, dv \, du
\]
\[
= \int_0^\infty \int_0^\infty \left( e^{-\lambda_1 u} e^{-\lambda_2 v} + e^{-\lambda_2 u} e^{-\lambda_1 v} - e^{-\lambda u} \right) dv \, du
\]
\[
+ \int_0^\infty \int_0^\infty e^{-\lambda u} \, du \, dv
\]
\[
= \int_0^\infty \left( \frac{e^{-\lambda_1 u}}{\lambda_2 + \lambda_{12}} + \frac{e^{-\lambda_2 u}}{\lambda_1 + \lambda_{12}} - \frac{e^{-\lambda u}}{\lambda} \right) du + \int_0^\infty \frac{\lambda}{\lambda} e^{-\lambda v} dv
\]
\[
= \frac{1}{\lambda} \left[ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{\lambda} \right] + \frac{1}{\lambda^2}
\]
\[
= \frac{1}{\lambda} \left[ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} \right],
\]
(3.58)

and

\[
\text{Cov}(T_1, T_2) = \frac{1}{\lambda} \left[ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} \right] - \frac{1}{\lambda} \left[ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{\lambda} \right]
\]
\[
= \frac{1}{\lambda^2}.
\]
(3.59)

Hence,

\[
\text{Corr}(T_1, T_2) = \frac{\frac{1}{\lambda^2}}{\sqrt{\frac{\lambda^2 (\lambda_2 - \lambda_1)^2 - (\lambda - \lambda_2) (\lambda - \lambda_1) [\lambda_1 \lambda_2 - \lambda (\lambda_1 + \lambda_2 + \lambda)]}{\lambda^2 (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2}}}
\]
This proves (iii). □

Note that Cov\( (T_1, T_2) = \text{Var}(T_1) \) implies that Cov\( (T_1, T_2 - T_1) = 0 \). In fact, \( T_2 - T_1 \) and \( T_1 \) are independent (See, for example, Theorem 1.4 in Barlow and Proschan (1981), p.131). Also, note that Corr\( (T_1, T_2) \geq 0 \). The upper bound to this correlation is 1, as we show below.

\[
\lim_{\lambda_i \to 0, i=1,2} \text{Corr}(T_1, T_2) = \frac{\lambda_{12}^2}{\lambda_{12}^2} = 1
\] (3.61)

If we take the limit \( \lim_{\lambda_{12} \to 0} \text{Corr}(T_1, T_2) \), we get the correlation value of the order statistics corresponding to that of two independent variables distributed according to the marginal distributions \( e(\lambda_1) \) and \( e(\lambda_2) \), respectively. This expression is as follows:

\[
\lim_{\lambda_{12} \to 0} \text{Corr}(T_1, T_2) = \frac{1}{\sqrt{(1 + \gamma)^2(1 - \frac{1}{\gamma})^2 - \left[3 - 2(1 + \gamma)(1 + \frac{1}{\gamma})\right]}}
\] (3.62)

where \( \gamma = \frac{\lambda_2}{\lambda_1} \).

Remark: A consequence of the identical marginal distributions for the BVE and ACBVE is that results (i) and (ii) of the preceding theorem hold good for the order statistics of Sarkar's distribution.

**Theorem 3.3.2** Let \( (X_1, X_2) \sim ACBVE(\lambda_1, \lambda_2, \lambda_{12}) \) and let \( \lambda = \lambda_1 + \lambda_2 + \lambda_{12} \). Then,

\[
(i) \quad E(T_1) = \frac{1}{\lambda} \quad \text{and} \quad \text{Var}(T_1) = \frac{1}{\lambda^2},
\] (3.63)
\( (ii) \)

\[
E(T_2) = \frac{\lambda + \lambda_{12}}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} - \frac{\lambda_1 \lambda_2 (\lambda + \lambda_{12})}{\lambda (\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}
\]

\[
\text{Var}(T_2) = \frac{G_1(\lambda_1, \lambda_2, \lambda_{12})}{G_2(\lambda_1, \lambda_2, \lambda_{12})}, \tag{3.64}
\]

where \( G_1 \) and \( G_2 \) are defined as follows:

\[
G_1(\lambda_1, \lambda_2, \lambda_{12}) = \lambda^2 \{2(\lambda_1 + \lambda_2)[(\lambda_1 + \lambda_{12})^2 + (\lambda_2 + \lambda_{12})^2] - \lambda(\lambda + \lambda_{12})^2 \}
\]

\[
- (\lambda + \lambda_{12})(\lambda_1 \lambda_2 + \lambda \lambda_{12}) \{2\lambda(\lambda_1 \lambda_2 - \lambda^2) \}
\]

\[
+ (\lambda_1 \lambda_2 + \lambda \lambda_{12})(\lambda_1 + \lambda_2) \} \tag{3.65}
\]

\[
G_2(\lambda_1, \lambda_2, \lambda_{12}) = \lambda^2(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})^2(\lambda_2 + \lambda_{12})^2, \tag{3.66}
\]

and

\( (iii) \)

\[
\text{Corr}(T_1, T_2) = \frac{2\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}{\sqrt{G_1(\lambda_1, \lambda_2, \lambda_{12})}}. \tag{3.67}
\]

**Proof:** Result (i) is immediate from the exponentiality of \( T_1 \). We prove results (ii) and (iii) below.

\[
E(T_2) = \frac{\lambda}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2 + \lambda_{12}} + \frac{\lambda}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1 + \lambda_{12}} - \frac{\lambda + \lambda_{12}}{\lambda (\lambda_1 + \lambda_2)}
\]

\[
= \frac{\lambda}{\lambda_1 + \lambda_2} \left( \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{\lambda} \right) - \frac{\lambda_1 \lambda_2}{\lambda (\lambda_1 + \lambda_2)}
\]

\[
= \frac{\lambda}{\lambda_1 + \lambda_2} \left( \frac{\lambda (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \right) - \frac{\lambda_1 \lambda_2}{\lambda (\lambda_1 + \lambda_2)}
\]

\[
= \frac{\lambda (\lambda^2 - \lambda_1 \lambda_2) - \lambda_1 \lambda_2 (\lambda_1 \lambda_2 + \lambda \lambda_{12})}{\lambda (\lambda_1 + \lambda_2) (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})}
\]

\[
= \frac{\lambda (\lambda^2 - \lambda_1 \lambda_2) - (\lambda + \lambda_{12}) \lambda_1 \lambda_2}{\lambda (\lambda_1 + \lambda_2) (\lambda_1 + \lambda_{12}) (\lambda_2 + \lambda_{12})}. \tag{3.68}
\]
This simplifies readily to the expression given in (ii).

\[
\text{Var}(T_2) = \frac{\lambda}{\lambda_1 + \lambda_2} \left\{ \frac{2}{(\lambda_2 + \lambda_{12})^2} + \frac{2}{(\lambda_1 + \lambda_{12})^2} \right\} - \frac{\lambda + \lambda_{12}}{\lambda_1 + \lambda_2} \frac{2}{\lambda^2}
\]

\[
= \frac{\lambda}{\lambda_1 + \lambda_2} \left\{ \frac{2(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_{12})^2 + 2(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_{12})^2 - \lambda(\lambda + \lambda_{12})^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} \right\}
\]

\[
= \frac{\lambda}{\lambda_1 + \lambda_2} \frac{2(\lambda_1 + \lambda_2)}{(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} \left\{ 2 + \lambda + \lambda_{12} \right\} - \frac{\lambda + \lambda_{12}}{\lambda_1 + \lambda_2} \frac{2(\lambda + \lambda_{12})^2}{(\lambda_1 + \lambda_{12})(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}. \tag{3.69}
\]

The result follows by putting all the pieces under one denominator, using the identity 

\[(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) = \lambda_1 \lambda_2 + \lambda \lambda_{12}, \text{ and grouping together common terms. We derive the correlation expression in (iii) next. Now,}

\[
E(T_1T_2) = \int_0^\infty \int_t^\infty H(t_1, t_2) dt_2 dt_1 + \int_t^\infty \int_0^\infty e^{-\lambda t_1} dt_1 dt_2
\]

\[
= \int_t^\infty \int_t^\infty \frac{\lambda}{\lambda_1 + \lambda_2} \left\{ e^{-\lambda t_1} - (\lambda_2 + \lambda_{12})t_2 + e^{-\lambda t_1} - (\lambda_1 + \lambda_{12})t_2 \right\} dt_2 dt_1
\]

\[
+ \int_0^\infty \frac{1}{\lambda} e^{-\lambda t_2} dt_2
\]

\[
= \int_0^\infty \frac{\lambda}{\lambda_1 + \lambda_2} \left\{ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} \right\} + \frac{1}{\lambda^2}
\]

\[
= \frac{\lambda + \lambda_{12}}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} + \frac{1}{\lambda^2}, \tag{3.70}
\]

\[
\text{Cov}(T_1, T_2) = \frac{1}{\lambda_1 + \lambda_2} \left\{ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} \right\} + \frac{1}{\lambda^2}
\]

\[
- \frac{1}{\lambda} \left\{ \frac{\lambda}{\lambda_1 + \lambda_2} \left\{ \frac{1}{\lambda_2 + \lambda_{12}} + \frac{1}{\lambda_1 + \lambda_{12}} \right\} + \frac{1}{\lambda^2} \right\} - \frac{\lambda + \lambda_{12}}{(\lambda_1 + \lambda_2)\lambda}
\]

\[
= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \frac{\lambda + \lambda_{12}}{\lambda_1 + \lambda_2}.
\]
and consequently,
\[
\frac{1}{\lambda^2} \left( \frac{2\lambda}{\lambda_1 + \lambda_2} \right) = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2}
\]

This proves the theorem. □

**Theorem 3.3.3** Let \((X_1, X_2) \sim FBVE(\alpha, \beta, \alpha', \beta'). Then,

(i) \[E(T_1) = (\alpha + \beta)^{-1} \quad \text{and} \quad Var(T_1) = (\alpha + \beta)^{-2},\] (3.73)

(ii) \[E(T_2) = \frac{\alpha'\alpha + \beta'\beta + \alpha'\beta'}{\alpha'\beta'(\alpha + \beta)}\] (3.74)

and

\[Var(T_2) = \frac{\alpha^2 \alpha^2 + \beta^2 \beta^2 + 2\alpha\beta[\alpha^2 + \beta^2 - \alpha'\beta'] + \alpha^2 \beta^2}{\alpha^2 \beta^2 (\alpha + \beta)^2},\] (3.75)

(iii) \[Corr(T_1, T_2) = \frac{\alpha'\beta'}{\sqrt{\alpha^2 \alpha^2 + \beta^2 \beta^2 + 2\alpha\beta[\alpha^2 + \beta^2 - \alpha'\beta'] + \alpha^2 \beta^2}}.\] (3.76)
Proof: Result (i) is a consequence of the exponentiality of $T_1$. We now prove (ii).

\[
E(T_2) = \frac{\alpha}{\alpha + \beta - \beta' \beta'} \frac{1}{\alpha + \beta - \beta' (\alpha + \beta)^2}
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \frac{1}{\alpha + \beta - \alpha' (\alpha + \beta)^2}
+ \frac{\alpha}{\alpha + \beta - \beta' \beta'} \frac{1}{\beta(\alpha + \beta)^2}
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \frac{1}{\alpha'(\alpha + \beta)^2}
= \frac{\alpha}{\alpha + \beta - \beta'} \left[ \frac{(\alpha + \beta)^2 - \beta^2}{\beta'(\alpha + \beta)^2} \right]
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \left[ \frac{(\alpha + \beta)^2 - \alpha'^2}{\alpha'(\alpha + \beta)^2} \right]
= \frac{\alpha}{\alpha + \beta - \beta'} \left[ \frac{(\alpha + \beta)^2}{\beta'(\alpha + \beta)^2} \right]
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \left[ \frac{(\alpha + \beta)^2}{\alpha'(\alpha + \beta)^2} \right]
= \frac{\alpha'(\alpha + \beta + \beta') + \beta'(\alpha + \beta + \alpha')}{\alpha'(\alpha + \beta)^2}
= \frac{(\alpha' \alpha + \beta' \beta) + \beta' \beta(\alpha + \beta + \alpha')}{\alpha'(\alpha + \beta)^2}
= \frac{\alpha' + \beta' \beta + \alpha' \beta'}{\alpha'(\alpha + \beta)}.
\]

(3.77)

In order to find $\text{Var}(T_2)$, we first evaluate the second moment $E(T_2^2)$.

\[
E(T_2^2) = \frac{\alpha}{\alpha + \beta - \beta' \beta'} \frac{2}{\alpha + \beta - \beta' (\alpha + \beta)^3}
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \frac{2}{\alpha + \beta - \alpha' (\alpha + \beta)^3}
+ \frac{\alpha}{\alpha + \beta - \beta' \beta'} \frac{2}{\beta^2(\alpha + \beta)^3}
+ \frac{\beta}{\alpha + \beta - \alpha' \alpha'} \frac{2}{\alpha'^2(\alpha + \beta)^3}
= 2\frac{\alpha}{\alpha + \beta - \beta'} \left[ \frac{(\alpha + \beta)^3 - \beta^2}{\beta^2(\alpha + \beta)^3} \right]
+ 2\frac{\beta}{\alpha + \beta - \alpha' \alpha'} \left[ \frac{(\alpha + \beta)^3 - \alpha'^2}{\alpha'^2(\alpha + \beta)^3} \right]
= 2\frac{\alpha}{\alpha + \beta - \beta'} \left[ \frac{(\alpha + \beta)^2 + (\alpha + \beta) \beta' + \beta'^2}{(\alpha + \beta - \beta') \beta^2(\alpha + \beta)^3} \right]
+ 2\frac{\beta}{\alpha + \beta - \alpha' \alpha'} \left[ \frac{(\alpha + \beta)^2 + (\alpha + \beta) \alpha' + \alpha'^2}{(\alpha + \beta - \alpha') \alpha'^2(\alpha + \beta)^3} \right]
= \frac{2\alpha(a + \beta - \beta')[(\alpha + \beta)^2 + (\alpha + \beta) \beta' + \beta'^2]}{(\alpha + \beta - \beta') \beta^2(\alpha + \beta)^3}
+ \frac{2\beta(a + \beta - \alpha')[(\alpha + \beta)^2 + (\alpha + \beta) \alpha' + \alpha'^2]}{(\alpha + \beta - \alpha') \alpha'^2(\alpha + \beta)^3}
= \frac{2\alpha \alpha' \alpha + \beta) + 2\alpha \alpha' (\alpha + \beta) \beta' + 2\alpha \alpha' \beta^2}{\alpha' \beta^2(\alpha + \beta)^3}.
\]
\[ \begin{align*}
 &+ \frac{2\beta^2(\alpha + \beta)^2 + 2\beta^2(\alpha + \beta)\alpha' + 2\beta^2\alpha^2}{\alpha^2\beta^2(\alpha + \beta)^3} \\
 &= \frac{2[\alpha\alpha^2 + \beta\beta^2](\alpha + \beta)^2 + [2\alpha\alpha^2\beta' + 2\beta\beta^2\alpha'](\alpha + \beta) + 2\alpha^2\beta^2(\alpha + \beta)}{\alpha^2\beta^2(\alpha + \beta)^3} \\
 &= \frac{2(\alpha + \beta)[\alpha\alpha^2 + \beta\beta^2] + 2\alpha\beta[\alpha\alpha' + \beta\beta'] + 2\alpha^2\beta^2}{\alpha^2\beta^2(\alpha + \beta)^2}. 
\end{align*} \]

Then,

\[
\text{Var}(T_2) = \frac{2(\alpha + \beta)\left[\alpha\alpha^2 + \beta\beta^2\right] + \alpha^2\beta^2 - (\alpha\alpha' + \beta\beta')^2}{\alpha^2\beta^2(\alpha + \beta)^2} \\
= \frac{2\alpha^2\alpha^2 + 2\alpha\beta\alpha^2 + 2\alpha^2\beta^2 + 2\beta^2\beta^2}{\alpha^2\beta^2(\alpha + \beta)^2} \\
+ \frac{\alpha^2\beta^2 - \alpha^2\beta^2 - 2\alpha\beta\beta' - \beta^2\beta^2}{\alpha^2\beta^2(\alpha + \beta)^2} \\
= \frac{\alpha^2\alpha^2 + \beta^2\beta^2 + 2\alpha\beta[\alpha^2 + \beta^2 - \alpha'\beta'] + \alpha^2\beta^2}{\alpha^2\beta^2(\alpha + \beta)^2}. 
\]

This proves (ii). Finally, we verify (iii).

\[
E(T_1T_2) = E(X_1X_2) = \frac{\alpha\alpha' + \beta\beta' + 2\alpha'\beta'}{\alpha\beta'(\alpha + \beta)^2} 
\]

\[
\text{Cov}(T_1, T_2) = \frac{\alpha\alpha' + \beta\beta' + 2\alpha'\beta'}{\alpha\beta'(\alpha + \beta)^2} - \frac{1}{\alpha + \beta} \frac{\alpha\alpha' + \beta\beta' + \alpha'\beta'}{\alpha\beta'(\alpha + \beta)} \\
= \frac{\alpha\beta'(\alpha + \beta)^2}{\alpha\beta'(\alpha + \beta)^2} \\
= (\alpha + \beta)^{-2}. 
\]

These yield the Pearson's product moment correlation as

\[
\text{Corr}(T_1, T_2) = \frac{1}{(\alpha + \beta)^2} \frac{(\alpha + \beta)^2\alpha'\beta'}{\sqrt{\alpha^2\alpha^2 + \beta^2\beta^2 + 2\alpha\beta[\alpha^2 + \beta^2 - \alpha'\beta'] + \alpha^2\beta^2}} \\
= \frac{\alpha'\beta'}{\sqrt{\alpha^2\alpha^2 + \beta^2\beta^2 + 2\alpha\beta[\alpha^2 + \beta^2 - \alpha'\beta'] + \alpha^2\beta^2}}. 
\]

(3.83)
It can be shown that the following limits hold:

\[
\lim_{a',b' \to \infty} \text{Corr}(T_1, T_2) = 1 \tag{3.84}
\]
and that

\[
\lim_{a',b' \to 0} \text{Corr}(T_1, T_2) = 0. \tag{3.85}
\]

Thus, the bounds on the correlation between \( T_1 \) and \( T_2 \) are 0 and 1, that is \( 0 \leq \text{Corr}(T_1, T_2) \leq 1 \).

**Theorem 3.3.4** Let \( (X_1, X_2) \sim GBVE(\theta_1, \theta_2, \delta) \) and define \( \eta = \frac{\theta_1}{\theta_2} \), and \( g(\eta, \delta) = (1 + \eta \delta)^{-\delta} \). Then,

(i)

\[
E[T_1] = \theta_1 g(\eta, \delta) = \theta_2 g(1/\eta, \delta) \quad \text{and} \quad \text{Var}[T_1] = \theta_1^2 [g(\eta, \delta)]^2 = \theta_2^2 [g(1/\eta, \delta)]^2, \tag{3.86}
\]

(ii)

\[
E[T_2] = \theta_1 [1 - g(\eta, \delta)] + \theta_2 = \theta_1 + \theta_2 [1 - g(1/\eta, \delta)] \tag{3.87}
\]

and

\[
\text{Var}[T_2] = \theta_1^2 + \theta_2^2 - 3\theta_2^2 [g(1/\eta, \delta)]^2 + 2(\theta_1 + \theta_2)\theta_2 g(1/\eta, \delta) - 2\theta_1 \theta_2, \tag{3.88}
\]
\[
\text{Corr}(T_1, T_2) = \frac{2\Gamma^2(\delta + 1) - g(\eta, \delta)[1 + \eta - g(1/\eta, \delta)]}{g(\eta, \delta)\sqrt{(\eta - 1)^2 - 3|g(1/\eta, \delta)|^2 + 2(\eta + 1)g(1/\eta, \delta)}}.
\]

**Proof:** Define \(\eta\) and \(g(\eta, \delta)\) as in the theorem. Since \(T_1\) has a marginal exponential distribution with parameter \(\left(\frac{1}{\theta_1}\right)^{\frac{1}{\delta}} + \left(\frac{1}{\theta_2}\right)^{\frac{1}{\delta}}\) \(= a(1, 1)\) of Theorem 3.1.4, the result in (i) follows immediately. The first expression is obtained by factoring out \(\theta_1\), while the alternative expression results by factoring out \(\theta_2\) instead. The mean of \(T_2\) also readily follows from the definition of the expectation \(E[T_2] = \int_0^\infty t f_{(2)}(t)\, dt\) or quite simply as \(E(X_1 + X_2) - E(T_1)\) and by using the two alternate expressions in (i). We turn our attention to the variance of \(T_2\). Since

\[
E[T_2^2] = 2\theta_1^2 + 2\theta_2^2 - \theta_1^2\left[g(\eta, \delta)\right]^2,
\]

using the definition \(\text{Var}[T_2] = E[T_2^2] - (E[T_2])^2\), we obtain the variance expression in (ii). Now,

\[
E[T_1 T_2] = E[X_1 X_2] = \frac{2\theta_1 \theta_2 \Gamma^2(\delta + 1)}{\Gamma(2\delta + 1)}.
\]

Thus,

\[
\text{Corr}(T_1, T_2) = \frac{2\theta_1 \theta_2 \Gamma^2(\delta + 1) - \theta_1 g(\eta, \delta)\left[\theta_1 + \theta_2 - \theta_2 g(1/\eta, \delta)\right]}{\theta_1 g(\eta, \delta)\sqrt{\text{Var}(T_2)}}.
\]

On factoring out \(\theta_1 \theta_2\) from numerator and denominator, we obtain result (iii). \(\square\)

Interpreting the value of the correlation coefficient is often difficult for bivariate models other than the bivariate normal. For the bivariate exponential distributions, a step in this direction is to study the relation between the correlation and the model parameters. We use Gumbel's model in this exploration. Since \(\delta = 1\) yields the
case of independence for Gumbel's model, it seems reasonable to study the rela-
tionship between Corr\(T_1, T_2\) and \(\delta\) for fixed \(\eta\). The actual relationship given in
Theorem 3.2.4(iii) is complicated. We want to find a reasonable approximation of
this relationship that is simpler and easier to interpret. We find that a fifth degree
Taylor series expansion of Corr\(T_1, T_2\) about \(\delta\) for fixed \(\eta\) works rather well. In the
examples that follow, we arrive at the approximation by expanding the Taylor series
of Corr\(T_1, T_2\) around different values of \(\delta\). We then choose one that generally pro-
duces small relative errors. Thus, we do not claim any optimality property for our
approximations.

When \(\eta = 2/3\), we find that the fifth degree expansion about \(\delta = 0.9\) gives a
relative error of approximation that is less than 0.01. This approximation, denoted
by \(r_1(\delta)\), is as follows:

\[
r_1(\delta) = 0.9981737092 - 0.2706214465\delta - 0.5274374868\delta^2
+ 0.2958159470\delta^3 - 0.1051682861\delta^4 + 0.01654410384\delta^5, \quad 0 \leq \delta < 1.
\]

(3.93)

When \(\eta = 2\), we choose the approximation, denoted \(r_2(\delta)\), to be the fifth degree
expansion about \(\delta = 0.1\) when \(0 \leq \delta < 0.2\), and the fifth degree expansion about
\(\delta = 0.8\) when \(0.2 \leq \delta < 1\). This approximation yields a relative error of less than
0.005. We now give the form of \(r_2(\delta)\). On \([0, 0.2)\),

\[
r_2(\delta) = 1.001094621 - 0.6736411805\delta - 1.641683161\delta^2
- 14.86306903\delta^3 + 104.7055517\delta^4 - 194.1272061\delta^5
\]

(3.94)
and on $[0.2, 1)$,

$$r_2(\delta) = 1.037957108 - 0.6860081363\delta + 0.1573499580\delta^2$$

$$- 0.307978000\delta^3 + 0.1897916826\delta^4 - 0.04296064944\delta^5.$$

(3.95)

Upon plotting $\text{Corr}(T_1, T_2)$ and the approximation $r_i(\delta), i=1,2$, we find that these are non-increasing functions of $\delta$. Thus, we see that as the correlation decreases, we get closer and closer to the independence case $\delta = 1$.

**Theorem 3.3.5** Let $(X_1, X_2) \sim RBVE(\lambda, \pi_1, \pi_2, p_{11})$. Then,

(i) $$E[T_1] = \sum_{i=1}^{4} a_i E(V_i)$$

(3.96)

$$\text{Var}[T_1] = \sum_{i=1}^{4} a_i E(V_i^2) - \sum_{i=1}^{4} a_i^2 [E(V_i)]^2 - 2 \sum_{i < j} a_i a_j E(V_i) E(V_j).$$

(3.97)

(ii) $$E[T_2] = \frac{2}{\lambda} - E[T_1]$$

(3.98)

$$\text{Var}[T_2] = \frac{4}{\lambda} E[T_1] - E[T_1^2] - (E[T_1])^2.$$  

(3.99)

(iii) $$\text{Corr}(T_1, T_2) = \frac{\frac{1}{V} (1 - \pi_1 \pi_2 + 2p_{11}) - E(T_1)E(T_1)}{\sqrt{\text{Var}(T_1)(\frac{4}{\lambda} E(T_1) - E(T_1^2) - (E[T_1])^2)}}.$$ 

(3.100)

**Proof:** Write $\overline{F}(t_1) = \sum_{i=1}^{4} a_i G_i(t_1)$, $t_1 > 0$, where

$$a_1 = p_{11} \frac{2 - \pi_1 - \pi_2}{1 - \pi_1 \pi_2}, \quad G_1(t_1) = e^{-\lambda t_1},$$

$$a_2 = 1 - \frac{p_{11}}{\pi_1}, \quad G_2(t_1) = e^{-\lambda(\frac{1}{\pi_1} + 1) t_1},$$

$$a_3 = 1 - \frac{p_{11}}{\pi_2}, \quad G_3(t_1) = e^{-\lambda(\frac{1}{\pi_2} + 1) t_1},$$

$$a_4 = a = p_{11} \left\{ \frac{\pi_1 + \pi_2 - 2\pi_1 \pi_2}{\pi_1 \pi_2 (1 - \pi_1 \pi_2)} \right\} - 1, \quad G_4(t_1) = e^{-\lambda(\frac{1}{\pi_1} + \frac{1}{\pi_2} + 1) t_1}.$$
Note that \( a_j = c_j, j=2,3 \) and \( a_4 = c_1 \) for the constants \( c_1, c_2, c_3 \) we defined in equation (2.9) of section 2.3.

Let \( V_i \) be an exponential rv with survival df \( \overline{G}_i(.) \), \( i = 1, \ldots, 4 \), where \( \overline{G}_i(.) \) is defined in (3.101). Also, for a non-negative continuous rv \( X \) with survival df \( \overline{F}(x) \), it is true that

\[
E(X^k) = k \int_0^\infty x^{k-1} \overline{F}(x) \, dx. \tag{3.102}
\]

Then, the mean of \( T_1 \) follows readily from (3.102) with \( k = 1 \). Also, plugging in \( k = 2 \) in (3.102), we get

\[
E(T_1^2) = \sum_{i=1}^4 a_i E(V_i^2). \tag{3.103}
\]

Hence,

\[
\text{Var}(T_1) = E(T_1^2) - [E(T_1)]^2 = \sum_{i=1}^4 a_i E(V_i^2) - \left[ \sum_{i=1}^4 a_i E(V_i) \right]^2. \tag{3.104}
\]

The result follows by using the identity

\[
\left( \sum a_i x_i \right)^2 = \sum a_i x_i^2 + 2 \sum_{i<i} a_i a_j x_i x_j. \tag{3.105}
\]

The mean of \( T_2 \) follows immediately from Theorem 3.2.5 and (3.102) with \( k = 1 \). Since

\[
E(T_2^2) = \frac{4}{\lambda^2} - E(T_1^2), \tag{3.106}
\]

the variance expression for \( T_2 \) follows readily from the definition. Finally, we use the fact that

\[
E(T_1 T_2) = E(X_1 X_2) = \frac{1}{\lambda^2} (1 - \pi_1 \pi_2 + 2 p_{11}) \tag{3.107}
\]

and the definition of correlation to verify (iii). \( \Box \)
Theorem 3.3.6 Let \((X_1, X_2) \sim BEE(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')\). Define \(\phi_1 = \frac{\phi \alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'}\),
and \(\phi_2 = \frac{\phi \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'}\). Then,

(i) 
\[
E(T_1) = \frac{1}{\alpha_1 + \alpha_2} \quad \text{and} \quad \text{Var}(T_1) = \frac{1}{(\alpha_1 + \alpha_2)^2}.
\]

(ii) 
\[
E(T_2) = \frac{\alpha_0(\alpha_1 \alpha_1' + \alpha_2 \alpha_2') + \alpha_1' \alpha_2'}{\alpha_1' \alpha_2'(\alpha_1 + \alpha_2)} \quad \text{and} \quad \text{Var}(T_2) = \frac{H(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')}{(\alpha_1')^2(\alpha_2')^2(\alpha_1 + \alpha_2)^2}.
\]

(iii) 
\[
\text{Corr}(T_1, T_2) = \frac{\alpha_1' \alpha_2'}{\sqrt{H(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')}}\quad (3.111)
\]

where
\[
H(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') = \alpha_0(2 - \alpha_0)\left\{(\alpha_1 \alpha_1')^2 + (\alpha_2 \alpha_2')^2\right\} + (\alpha_1' \alpha_2')^2
+ 2\alpha_0 \alpha_1 \alpha_2 \left\{(\alpha_1')^2 + (\alpha_2')^2 - \alpha_0 (\alpha_1' \alpha_2')\right\}\quad (3.112)
\]

**Proof:** Result (i) is an immediate consequence of the exponentiality of \(T_1\). We now prove (ii). Using (3.43) and (3.102) with \(k = 1\), we see that

\[
E(T_2) = E(X_1) + E(X_2) - E(T_1)
= \frac{\alpha_0 \alpha_2 + \alpha_1'}{\alpha_1' (\alpha_1 + \alpha_2)} + \frac{\alpha_0 \alpha_1 + \alpha_2'}{\alpha_2' (\alpha_1 + \alpha_2)} - \frac{1}{\alpha_1 + \alpha_2}\quad (3.113)
\]

which simplifies to the expression in (ii). We derive the second moment of \(T_2\) as follows:

\[
E(T_2^2) = \frac{2\phi_1}{(\alpha_2')^2} + \frac{2\phi_2}{(\alpha_1')^2} + \frac{2(1 - \phi_1 - \phi_2)}{(\alpha_1 + \alpha_2)^2}
\]
Using the definition, it is not hard to derive the variance expression in (ii). Now, we turn our attention to (iii).

\[
E(T_1 T_2) = E(X_1 X_2) = \frac{\alpha_0(\alpha_1 \alpha_1' + \alpha_2 \alpha_2') + 2\alpha_1' \alpha_2'}{\alpha_1' \alpha_2'(\alpha_1 + \alpha_2)^2}
\]

\[
\text{Cov}(T_1, T_2) = \frac{\alpha_0(\alpha_1 \alpha_1' + \alpha_2 \alpha_2') + 2\alpha_1' \alpha_2'}{\alpha_1' \alpha_2'(\alpha_1 + \alpha_2)^2} - \frac{1}{\alpha_1' \alpha_2'(\alpha_1 + \alpha_2)}
\]

\[
= \frac{1}{(\alpha_1 + \alpha_2)^2}
= \text{Var}(T_1).
\]

Combining all these, we see that

\[
\text{Corr}(T_1, T_2) = \frac{1}{(\alpha_1 + \alpha_2)^2} \sqrt{\frac{1}{(\alpha_1 + \alpha_2)^2} \frac{H(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')}{\alpha_1 \alpha_2' \alpha_1' \alpha_2'}
\]

which evidently reduces to the correlation expression in (iii). □

### 3.4 Graphs of Copulas

In section 2.3, we presented graphs of copulas for unordered variables having some common bivariate exponential distributions. We complete the discussion of copulas in this section by looking at graphs of the surfaces \(c_0(u, v) - uv\) for the order statistics
and graphs of the copula differences \( c_0(u, v) - c(u, v) \) from these distributions. The graphs are generated using MAPLE and the programs are given in Appendix B. We will look at five distributions in turn: the BVE, Block and Basu's ACBVE, Freund's FBVE, Gumbel's GBVE, and Raftery's RBVE. First, we discuss the patterns of the copula functions of the order statistics.

### 3.4.1 Copulas of Order Statistics

We present one case of the BVE in Figure 8 (top picture). We notice a mound-shaped surface that is skewed towards high values of the maximum. The mound comes to an abrupt peak where \( T_1 = T_2 \), a curve of singularity and non-differentiability of the surface.

The mound-shape pattern is also evident in Figure 9 (top picture), where a surface for the ACBVE is presented. The mound is smooth and attains a peak around \((0.5,0.8)\). Here, we also see a skewness towards large values of the maximum, as in the BVE.

Three situations under the FBVE pertaining to the effect of a component failure on the lifetime of the remaining component are given. In Figure 10 (top picture), the 'stress situation' \((\alpha < \alpha' \text{ and } \beta < \beta')\) is represented. The surface is a steep mound coming to a peak around \( u = v \). We present the 'enhanced situation' \((\alpha > \alpha' \text{ and } \beta > \beta')\) in Figure 11 (top picture). This surface looks like a mound skewed towards small values of the minimum but large values of the maximum. We mix the two previous situations so that \( \alpha < \alpha' \) but \( \beta > \beta' \) in Figure 12 (top picture). The mound pattern is still evident, with the mound skewed towards low values of the minimum.
Figure 13 (top picture) represents a surface from the GBVE. The graph shows a hill that comes to a peak around large values of both the minimum and maximum.

We explore the three parsimonious versions of the RBVE. Figure 14 (top picture) shows model 1 when \( \pi_1 = \pi_2 = p_{11} \). We see the hill-like pattern that peaks around large values of the minimum and maximum. Figure 15 (top picture) represents one version of model 2 when \( \pi_1 = \pi_2 \leq 0.5 \) and \( p_{11} = 0 \). Again, we see a mound that is skewed towards small values of the minimum but large values of the maximum. In Figure 16 (top picture), we show the other version of model 2 \( (\pi_1 = \pi_2 = \pi > 0.5, \ p_{11} = 2\pi - 1) \). The surface is a steep hill that is skewed towards large values of the minimum and maximum. Model 3, when \( \pi_2 = p_{11} \), is represented in Figure 17 (top picture). This surface is mound-shaped and comes smoothly to a peak around large values of the minimum and maximum.

As in section 2.3, the discrepancies between the copula function of the dependent pair \((T_1, T_2)\) and the independent case is small - no bigger than 0.16 in all the graphs presented here. We recall from section 2.3 that \(|c(u,v) - uv|\) can be no bigger than 0.25.

### 3.4.2 Properties of \( c_0(u,v) - c(u,v) \)

When \( X_1 \) and \( X_2 \) are independent, it is true that the resulting order statistics are PQD since they are associated (see, for example, Barlow and Proschan (1981), pp.30-32). This means that \( c_0(u,v) \geq c(u,v) = uv \) in the independent case. A natural question is whether this inequality holds when \((X_1, X_2)\) follows a bivariate exponential distribution. To this end, we compared the pictures obtained in section 2.3 with the
corresponding pictures of the resulting order statistics produced in this section. We find that in all cases but one, the order statistics have bigger copula function than the unordered variables, that is the surfaces $c_u(u, v) - c(u, v)$ lie entirely above the plane $z = 0$. We show these surfaces in Figures 8 - 17 (bottom pictures). We also present a plot of the copula differences for independent exponentials in Figure 18. The surface depicts a mound that is skewed towards small values of the minimum.

Figure 8 (bottom picture) shows one pattern when the observations come from the BVE. The surface lies flat then rises to a mound beyond the ridge $X_1 = X_2$. The bottom picture of Figure 9 shows a mound-like surface that is skewed towards high values of the maximum. It varies little in pattern from the top picture, verifying further, that this choice of parametrization for the unordered rv's vary little from independence.

The 'stress situation' example for the FBVE given in Figure 10 (bottom picture) dips down to -0.01, then rises to make a wave-crest pattern near the diagonal. The 'enhanced situation' example, on the other hand, for the FBVE is given in Figure 11 (bottom picture). We see a big fat mound above the plane $z = 0$. Figure 12 (bottom picture) depicts a 'mixed' FBVE situation. The surface is mound-like but one with two peaks.

We present the difference of the copula surfaces for the GBVE in Figure 13 (bottom picture). We see a hill-like pattern reaching a peak around large values of the minimum and maximum.

The pattern of the copula difference for model 1 of the RBVE show a wave-crest
pattern around the diagonal. We see this in Figure 14 (bottom picture). As for model 2, when \( \pi_1 = \pi_2 \leq 0.5, \rho_{11} = 0 \) as in the bottom picture of Figure 15, the copula difference depicts a mound above the plane \( z = 0 \). Now, the other version of model 2, when \( \pi_1 = \pi_2 = \pi > 0.5, \rho_{11} = 2\pi - 1 \), as in Figure 16 (bottom picture), the surface shows a depression as of the opening of a crater as well as a rise towards a peak around large values of the maximum. Finally, the difference of the copula surfaces for model 3 of the RBVE, shown in Figure 17 (bottom picture) is mound-shaped.

3.5 Regression Function and Spacing

Our object in this section is to find the conditional distribution of \( T_2 \) given \( T_1 = t_1 \), the regression function \( E(T_2|T_1 = t_1) \), and the distribution of the spacing \( T_2 - T_1 \) for the order statistics \((T_1, T_2)\) from some common bivariate exponential models. Our results will involve only distributions with BLMP.

Let \((X_1, X_2)\) with joint survival df \( F(x_1, x_2) \) and absolutely continuous marginal survival df’s \( f_1(x_1) \) and \( f_2(x_2) \), respectively, have the BLMP. Then, as we noted in section 1.2 (see equation (1.17)), if \( F(x_1, x_2) \) is absolutely continuous, it can be expressed as:

\[
F(x_1, x_2) = \begin{cases} 
  e^{-\theta x_2} F_1(x_1 - x_2), & x_1 \geq x_2 \geq 0 \\
  e^{-\theta x_1} F_2(x_2 - x_1), & x_2 \geq x_1 \geq 0 
\end{cases}
\]

(3.118)

where \( \theta = f_1(0) + f_2(0) \), and \( f_i(.) \) is the pdf of \( X_i, i=1,2 \). Otherwise, (3.118) still holds, but \( \theta \leq f_1(0) + f_2(0) \leq 2\theta \).

The joint survival distribution of the order statistics \( T_1 \leq T_2 \) becomes

\[
H(t_1, t_2) = \begin{cases} 
  e^{-\theta t_1}, & 0 < t_2 \leq t_1 \\
  e^{-\theta t_1} F_2(t_2 - t_1) + e^{-\theta t_1} F_1(t_2 - t_1) - e^{-\theta t_2}, & 0 < t_1 < t_2 
\end{cases}
\]

(3.119)
using (3.1) in section 3.1.

**Theorem 3.5.1** Let \((T_1, T_2)\) have the joint survival df \(H(t_1, t_2)\) given in (3.119). Then,

(i) the conditional distribution of \(T_2\) given \(T_1 = t_1\) is given by

\[
P(T_2 > t_2 | T_1 = t_1) = \begin{cases} 1, & t_2 \leq t_1 \\ \frac{\overline{F}_2(t_2 - t_1) + \overline{F}_1(t_2 - t_1)}{-\frac{1}{\theta}(f_1(t_2 - t_1) + f_2(t_2 - t_1))}, & t_2 > t_1 \end{cases}
\]  

(iii) the regression function of \(T_2\) on \(T_1\) is

\[
E(T_2 | T_1 = t_1) = t_1 + [E(X_1) + E(X_2) - \frac{2}{\theta}],
\]

and,

(iii) the df of \(T_2 - T_1\), denoted by \(G(.)\), is

\[
G(t) \equiv P(T_2 - T_1 \leq t) = \frac{1}{\theta} \left[ f_1(t) + f_2(t) \right] - 1, \quad t \geq 0
\]

provided \(F_i(\cdot), i=1,2\) is absolutely continuous.

**Proof:** We derive a general formula for finding the conditional distribution.

\[
P(T_2 > t_2 | T_1 = t_1) = \lim_{\delta \to 0} \frac{P(T_2 > t_2, T_1 \in (t_1, t_1 + \delta t)) / \delta t}{P(T_1 \in (t_1, t_1 + \delta t)) / \delta t}
\]

\[
= \lim_{\delta \to 0} \frac{[P(T_1 > t_1, T_2 > t_2) - P(T_1 > t_1 + \delta t, T_2 > t_2)] / \delta t}{[P(T_1 > t_1) - P(T_1 > t_1 + \delta t)] / \delta t}
\]

\[
= \frac{\partial}{\partial t_1} H(t_1, t_2), \quad 0 < t_1 < t_2 < \infty.
\]

For \(H(t_1, t_2)\) given in (3.119), we find that these derivatives are

\[
\frac{\partial}{\partial t_1} H(t_1, t_2) = -\theta e^{-\theta t_1} \overline{F}_2(t_2 - t_1) + e^{-\theta t_1} f_2(t_2 - t_1)
\]

\[
- \theta e^{-\theta t_1} \overline{F}_1(t_2 - t_1) + e^{-\theta t_1} f_1(t_2 - t_1)
\]

(3.124)
and

$$\frac{\partial}{\partial t_1} \overline{H}(t_1, t_1) = -\theta e^{-\theta t_1}. \quad (3.125)$$

On putting these together in (3.123), we obtain the result (i). Now,

$$E(T_2|T_1 = t_1) = \int_0^\infty P(T_2 > t_2|T_1 = t_1) \, dt_2$$

$$= \int_{t_1}^\infty \left\{ \overline{F}_2(t_2 - t_1) + \overline{F}_1(t_2 - t_1) - \frac{1}{\theta} [f_2(t_2 - t_1) + f_1(t_2 - t_1)] \right\} \, dt_2$$

$$+ \int_0^{t_1} dt_2 \quad (3.126)$$

which reduces to the expression in (ii). To prove (iii), we recall that for distributions with BLMP and absolutely continuous marginals, $T_2 - T_1$ and $T_1$ are independent. (See property (BB2) stated in section 1.2). So,

$$\overline{G}(t) = P(T_2 - T_1 > t)$$

$$= P(T_2 - T_1 > t|T_1 = t_1)$$

$$= P(T_2 > t + t_1|T_1 = t_1). \quad (3.127)$$

We now use (i) and the identity $G(t) = 1 - \overline{G}(t)$ to establish result (iii). □

**Corollary 3.5.2** If $(X_1, X_2)$ has a joint df with BLMP and $c \in \mathbb{R}^+$ is some constant, then the order statistics satisfy $E(T_2 - T_1|T_1 = t_1) = c$, $t_1 > 0$.

**Proof:** This is immediate from Theorem 3.5.1 (ii). □

We believe that the condition given in Corollary 3.5.2 is a possible characterization of bivariate exponential models with BLMP in the family of distributions with absolutely continuous marginals. However, we will not pursue this investigation in this work.
We now look at these results specifically for the BVE, Block and Basu’s ACBVE, Freund’s FBVE, and Friday and Patil’s BEE.

**Theorem 3.5.3** Let \( (T_1, T_2) \) be the order statistics from the BVE(\( \lambda_1, \lambda_2, \lambda_{12} \)). Then,

(i) 
\[
P(T_2 > t_2 | T_1 = t_1) = \left\{ \begin{array}{ll}
\frac{1}{\lambda} e^{-(\lambda_2 + \lambda_{12})(t_2 - t_1)} + \frac{\lambda_2}{\lambda} e^{-(\lambda_1 + \lambda_{12})(t_2 - t_1)} & , t_2 \leq t_1 \\
0 & , t_2 > t_1
\end{array} \right.
\]

(ii) 
\[
E(T_2 | T_1 = t_1) = t_1 + \frac{\lambda_1^2 + (\lambda_1 + \lambda_2)\lambda_{12} + \lambda_2^2}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})} , t_1 > 0.
\]

(iii) 
\[
G(t) = P(T_2 - T_1 \leq t) = 1 - \frac{\lambda_1}{\lambda} e^{-(\lambda_2 + \lambda_{12})t} - \frac{\lambda_2}{\lambda} e^{-(\lambda_1 + \lambda_{12})t} , t \geq 0.
\]

**Proof:** To show these results, we apply Theorem 3.5.1 on the marginal df’s \( F_i(x_i) \), \( i=1,2 \) for the BVE. The corresponding survival df’s are given in Appendix A.1. Marshall and Olkin (1967) show that \( \theta = \lambda \). Also, an alternative proof to (iii) is given by Barlow and Proschan (1981) on pp.131-132. □

**Theorem 3.5.4** Let \( (T_1, T_2) \) be the order statistics from the ACBVE(\( \lambda_1, \lambda_2, \lambda_{12} \)). Then,

(i) 
\[
P(T_2 > t_2 | T_1 = t_1) = \left\{ \begin{array}{ll}
\frac{1}{\lambda} \frac{e^{-(\lambda_2 + \lambda_{12})(t_2 - t_1)}}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda} e^{-(\lambda_1 + \lambda_{12})(t_2 - t_1)} & , t_2 \leq t_1 \\
0 & , t_2 > t_1
\end{array} \right.
\]
(ii)  

$$E(T_2 | T_1 = t_1) = t_1 + \frac{1}{\lambda_1 + \lambda_2} \left\{ \frac{\lambda_2}{\lambda_1 + \lambda_{12}} + \frac{\lambda_1}{\lambda_2 + \lambda_{12}} \right\} , t_1 > 0. \quad (3.132)$$

(iii)  

$$G(t) = P(T_2 - T_1 \leq t) = 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_2 + \lambda_{12})t} - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_{12})t} , t \geq 0. \quad (3.133)$$

**Proof:** These results follow immediately upon applying Theorem 3.5.1 to the marginal df's $F_i(x_i)$ of the ACBVE. The corresponding survival df's are given in Appendix A.2. □

**Theorem 3.5.5** Let $(T_1, T_2)$ be the order statistics from the FBVE($\alpha$, $\beta$, $\alpha'$, $\beta'$). Then,

(i)  

$$P(T_2 > t_2 | T_1 = t_1) = \left\{ \begin{array}{l} \frac{1}{\alpha + \beta} e^{-\beta'(t_2 - t_1)} + \frac{\beta}{\alpha + \beta} e^{-\alpha'(t_2 - t_1)} \quad , t_2 \leq t_1 \\ \frac{\alpha}{\alpha + \beta} e^{-\beta'(t_1 - t_2)} + \frac{\beta'}{\alpha + \beta} e^{-\alpha'(t_1 - t_2)} \quad , t_2 > t_1 \end{array} \right. \quad (3.134)$$

(ii)  

$$E(T_2 | T_1 = t_1) = t_1 + \frac{\alpha \alpha' + \beta' \beta}{\alpha' \beta'(\alpha + \beta)} , t_1 > 0. \quad (3.135)$$

(iii)  

$$G(t) = P(T_2 - T_1 \leq t) = 1 - \frac{\alpha}{\alpha + \beta} e^{-\beta' t} - \frac{\beta}{\alpha + \beta} e^{-\alpha' t} , t \geq 0. \quad (3.136)$$

**Proof:** We apply Theorem 3.5.1 to the marginal df's of the FBVE. The corresponding survival df's are given in Appendix A.3. □
Theorem 3.5.6  Let \((T_1, T_2)\) be the order statistics from the BEE(\(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2'\)).

Then,

(i)  
\[ P(T_2 > t_2 | T_1 = t_1) = \begin{cases} 
\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2} e^{-\alpha_2'(t_2 - t_1)} + \frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2} e^{-\alpha_1'(t_2 - t_1)} & , t_2 \leq t_1 \\
\frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2} e^{-\alpha_2'(t_2 - t_1)} & , t_2 > t_1
\end{cases} \]  
(3.137)

(ii)  
\[ E(T_2 | T_1 = t_1) = t_1 + \frac{\alpha_0}{\alpha_1 + \alpha_2} \left\{ \frac{\alpha_2 \alpha_2' + \alpha_1 \alpha_1'}{\alpha_1' \alpha_2'} \right\} , t_1 > 0. \]  
(3.138)

(iii)  
\[ G(t) = P(T_2 - T_1 \leq t) = 1 - \frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2} e^{-\alpha_2't} - \frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2} e^{-\alpha_1't}, t \geq 0. \]  
(3.139)

Proof: We use Theorem 3.5.1 and the marginal df’s of the RBVE. The corresponding survival df’s are given in Appendix A.7. We note that here \(\theta = \alpha_1 + \alpha_2\). 
\(\Box\)
Figure 8: Copula Plots for the BVE with $\lambda_1 = 0.7$, $\lambda_2 = 1.8$, $\lambda_{12} = 0.5$. (Top) Plot of $c_o(u, v) - uv$. (Bottom) Plot of $c_o(u, v) - c(u, v)$. 
Figure 9: Copula Plots for the ACBVE with $\lambda_1 = 1.5$, $\lambda_2 = 1.0$, $\lambda_{12} = 0.4$. (Top) Plot of $c_0(u, v) - uv$. (Bottom) Plot of $c_0(u, v) - c(u, v)$. 
Figure 10: Copula Plots for the FBVE under a 'stress situation' with $\alpha = 0.2$, $\beta = 0.1$, $\alpha' = 0.4$, $\beta' = 0.9$. (Top) Plot of $c_0(u, v) - uv$. (Bottom) Plot of $c_0(u, v) - c(u, v)$. 
Figure 11: Copula Plots for the FBVE under an 'enhanced situation' with $\alpha = 0.5$, $\beta = 0.8$, $\alpha' = 0.3$, $\beta' = 0.4$. (Top) Plot of $c_0(u,v) - uv$. (Bottom) Plot of $c_0(u,v) - c(u,v)$. 
Figure 12: Copula Plots for the FBVE under a 'mixed situation' with $\alpha = 0.3$, $\beta = 0.9$, $\alpha' = 0.5$, $\beta' = 0.1$. (Top) Plot of $c_c(u, v) - uv$. (Bottom) Plot of $c_c(u, v) - c(u, v)$. 
Figure 13: Copula Plots for the GBVE with $\theta_1 = 0.5$, $\theta_2 = 0.8$, $\delta = 0.5$. (Top) Plot of $c_\phi(u, v) - uv$. (Bottom) Plot of $c_\phi(u, v) - c(uv)$. 
Figure 14: Copula Plots for model 1 of the RBVE with $\pi_1 = \pi_2 = p_{11} = p_{00} = 0.5$, $p_{10} = p_{01} = 0, \lambda = 1$. (Top) Plot of $c_0(u, v) - uv$. (Bottom) Plot of $c_0(u, v) - c(u, v)$. 
Figure 15: Copula Plots for model 2 of the RBVE with $\pi_1 = \pi_2 = 0.5$, $p_{11} = 0$, $p_{00} = 0$, $p_{10} = p_{01} = 0.5$, $\lambda = 1$. (Top) Plot of $c_o(u,v) - uv$. (Bottom) Plot of $c_o(u,v) - c(u,v)$. 
Figure 16: Copula Plots for model 2 of the RBVE with $\pi_1 = \pi_2 = 0.75$, $p_{11} = 0.5$, $p_{00} = 0$, $p_{10} = p_{01} = 0.25$, $\lambda = 1$. (Top) Plot of $c_0(u,v) - uv$. (Bottom) Plot of $c_0(u,v) - c(u,v)$. 
Figure 17: Copula Plots for model 3 of the RBVE with $\pi_1 = 0.75$, $\pi_2 = 0.25$, $p_{11} = 0.25$, $p_{00} = 0.25$, $p_{10} = 0.5$, $p_{01} = 0$, $\lambda = 1$. (Top) Plot of $c_{o}(u,v) - uv$. (Bottom) Plot of $c_{o}(u,v) - c(u,v)$. 
Figure 18: Plot of $c_\rho(u,v) - c(u,v)$ ($= c_\rho(u,v) - uv$) for the joint distribution of independent exponential rv's with parameters $\lambda_1 = 4$, and $\lambda_2 = 0.5$. 
CHAPTER IV

Reliability Properties of the Order Statistics

In this chapter, we explore the reliability properties of the minimum and maximum lifetimes of two-component systems distributed according to the BVE, Block and Basu’s ACBVE, Freund’s FBVE, Gumbel’s GBVE, Raftery’s RBVE, Sarkar’s ACBVE, and Friday and Patil’s BEE distributions. This involves verifying the IFR, DFR, and IFRA properties, as well as the hazard rate ordering of the minimum and maximum.

Let $F(.)$ denote the survival df of a rv $T$, and $f(.)$ the corresponding pdf. We say that $f$ or equivalently $T$ is IFR / DFR if the failure rate $r(t) = \frac{f(t)}{F(t)}$ is increasing / decreasing in $t \in \mathbb{R}^+$. It is said to be IFRA if $-\frac{1}{t} \log F(t)$ is increasing in $t \in \mathbb{R}^+$, where log represents the natural logarithmic function. If $f$ is IFR, then it is IFRA.

Now, for a bivariate pair $(X_1, X_2)$, let $r_i(.)$ and $F_i(.)$ denote the failure rate and survival df of $X_i$, respectively, $i=1,2$. We say that $X_1$ is greater than $X_2$ in hazard rate ordering if $r_1(t) \leq r_2(t) \forall t \geq 0$, or equivalently, $\frac{F_1(t)}{F_2(t)}$ is nondecreasing in $t \in \mathbb{R}^+$. We then write $X_1 \succeq_{hr} X_2$. Boland, El-Neweihi, and Proschan (1994) show that the order statistics $(T_1, \ldots, T_n)$ of $n$ independent, non-identically distributed rv’s with absolutely continuous df’s satisfy the following relation: $T_n \succeq_{hr} T_{n-1} \succeq_{hr} \cdots \succeq_{hr} T_1$. This is an immediate consequence of their result that a $(k+1)$-out-of-$n$ system is more
likely to fail in the near future than a k-out-of n system, \( k = 1, \ldots, n - 1 \) given that both systems are still operating at an arbitrary time \( t \).

We observed in section 3.2 that, predominantly, the order statistics have GH marginals (see section 1.3 for a definition) with three or less components. We begin our investigations in this chapter with a look at GH distributions to find conditions under which the pdf exists and the IFR, DFR, and IFRA properties hold. We present this discussion in section 4.1. In section 4.2, we turn our attention to applying these results to the order statistics of some common bivariate exponential distributions. In section 4.3, we state results on the hazard rate ordering of these order statistics.

4.1 Results for the GH Class of Distributions

The pdf of a generalized mixture of two exponentials takes the form

\[
f(t) = a_1 b_1 e^{-b_1 t} + a_2 b_2 e^{-b_2 t}.
\]

(4.1)

It is easy to see that \( a_1 b_1 + a_2 b_2 \geq 0 \) is a necessary and sufficient condition for (4.1) to be a pdf. Bartholomew (1969) took note of this fact. If \( a_1, a_2 > 0 \), then the pdf in (4.1) is a convex mixture of exponentials, thus, is DFR, since the mixture of DFR distributions is itself DFR. (See Barlow and Proschan, 1981, p.104). Now, the failure rate

\[
r(t) = \frac{a_1 b_1 e^{-b_1 t} + a_2 b_2 e^{-b_2 t}}{a_1 e^{-b_1 t} + a_2 e^{-b_2 t}}
\]

(4.2)

is increasing in \( t \in \mathbb{R}^+ \) iff \( r'(t) \geq 0 \). When \( a_2 < 0 \), it is not difficult to see that this condition is satisfied iff \( (b_1 - b_2)^2 \geq 0 \). Thus, we have shown the following result.
**Theorem 4.1.1** Let \( b_1 < b_2 \). Then, the pdf in (4.1) is IFR, when \( a_2 < 0 \) and is DFR, when \( a_2 > 0 \).

The properties of generalized mixtures of two exponentials are easy to study, as we have shown. But for a generalized mixture of three, the situation is more complicated as we shall now see.

### 4.1.1 The Probability Density Function

Let \( a_1 > 0, a_2, a_3 \in \mathbb{R} \) be such that \( a_1 + a_2 + a_3 = 1 \) and let \( 0 < b_1 < b_2 < b_3 < \infty \).

Consider the function

\[
\frac{1}{a_1 b_1 t^{a_1} + a_2 b_2 e^{-b_2 t} + a_3 b_3 e^{-b_3 t}}. \tag{4.3}
\]

We call any pdf satisfying (4.3) as a GH pdf.

If \( a_1, a_2, a_3 > 0 \), then \( f(t) \) given in (4.3), being a convex mixture of exponential pdf's, is itself a pdf. Otherwise, we state necessary and sufficient conditions for \( f(t) \) to be a pdf in the theorem that follows.

**Theorem 4.1.2** Let \( m = \frac{-a_3 b_3 (b_3 - b_1)}{a_2 b_2 (b_3 - b_1)} \).

(i) If \( a_2 \in \mathbb{R} \) and \( a_3 < 0 \), \( f(t) \) in (4.3) is a pdf iff condition (C1) holds.

(ii) If \( a_2 < 0 \) and \( a_3 > 0 \), \( f(t) \) is a pdf iff either (a) or (b) below holds.

(a) \( m \leq 1 \) and condition (C1)

(b) \( m > 1 \) and condition (C2)

The conditions are:
(C1) \( a_1b_1 + a_2b_2 + a_3b_3 \geq 0 \)

(C2) \( \frac{a_1b_1(b_2-b_1)}{-a_3b_3(b_2-b_3)} \geq m \frac{b_2-b_1}{b_3-b_2} \)

**Proof:** Let \( a_3 < 0 \). When \( a_2 > 0 \), the condition \( f(t) \geq 0 \ \forall t \in \mathbb{R}^+ \) is equivalent to \( g(t) \geq 0 \ \forall t \in \mathbb{R}^+ \) where

\[
g(t) = a_1b_1e^{(b_2-b_1)t} + a_2b_2e^{(b_3-b_2)t} + a_3b_3. \tag{4.4}
\]

Since \( g(t) \) is non-decreasing in \( t \in \mathbb{R}^+ \), an equivalent condition is \( g(0) \geq 0 \), which is nothing but (C1).

When both \( a_2 \) and \( a_3 \) are negative, take the function \( g \) in the above discussion to be the following expression instead:

\[
g(t) = a_1b_1 + a_2b_2e^{(b_2-b_1)t} + a_3b_3e^{(b_1-b_3)t}. \tag{4.5}
\]

This proves (i).

Now, let \( a_3 > 0 \). Non-negativity of the pdf \( f(t) \) requires that

\[
K(t) = a_1b_1 + a_2b_2e^{-(b_2-b_1)t} + a_3b_3e^{-(b_3-b_1)t} \geq 0 \ \forall t \in \mathbb{R}^+. \tag{4.6}
\]

We extend the domain of \( K(t) \) to \( \mathbb{R} \), and note that \( K'(t) < 0 \) if \( t < t_0 \) and \( K'(t) \geq 0 \) if \( t \geq t_0 \) where

\[
t_0 = \frac{1}{b_3 - b_2} \log (m). \tag{4.7}
\]

If \( t_0 \leq 0 \), or equivalently \( m \leq 1 \), \( K(t) \) is increasing in \( (0, \infty) \), and \( K(t) > K(0) = \sum_{i=1}^{3} a_ib_i > 0 \ \forall t > 0 \). If \( t_0 > 0 \), or equivalently \( m > 1 \), \( K(t) \) attains its minimum at \( t_0 \), so that \( K(t) \geq K(t_0) \ \forall t \in \mathbb{R}^+ \). Now, \( K(t_0) \geq 0 \) iff condition (C2) holds. This proves (ii). \( \square \)
4.1.2 The IFR Property

When all of the weights \(a_i\) are non-negative, the pdf \(f\) given in (4.3) is a convex mixture of DFR distributions and is itself DFR. When at least one \(a_i\) is negative, \(f\) can be IFR as we shall see in the next theorem. First, we prove a useful lemma.

**Lemma 4.1.3** If \(l(t)\) and \(m(t)\) are functions of \(t \in \mathbb{R}^+\) satisfying \(l(0) \geq m(0)\), and \(l'(t) \geq m'(t)\) \(\forall t \in \mathbb{R}^+\), then \(l(t) \geq m(t)\) \(\forall t \in \mathbb{R}^+\).

**Proof:** Define the function \(g(t) = l(t) - m(t), t \in \mathbb{R}^+\). Then, \(g'(t) = l'(t) - m'(t) \geq 0 \forall t \in \mathbb{R}^+\), so that \(g(t) \geq g(0)\). But, \(g(0) = l(0) - m(0) \geq 0\). Hence, the result follows. \(\square\)

**Theorem 4.1.4** Let \(s = a_1a_2(b_1 - b_2)^2 + a_1a_3(b_1 - b_3)^2 + a_2a_3(b_2 - b_3)^2\), where the \(a's\) and \(b's\) are associated with the pdf \(f(t)\) given by (4.3).

(i) If \(a_2 > 0\) and \(a_3 < 0\), \(f(t)\) has the IFR property iff \(b_1 = b_2\).

(ii) If \(a_2 < 0\) and \(a_3 < 0\), \(f(t)\) has the IFR property iff \(s \leq 0\). In this case, it cannot be DFR.

(iii) If \(a_2 < 0\) and \(a_3 > 0\), \(f(t)\) has the IFR property if \(s \leq 0\) and \(a_1b_1 + a_2b_2 + a_3b_3 \leq b_1\).

**Proof:** The failure rate

\[
r(t) = \frac{a_1b_1e^{-b_1t} + a_2b_2e^{-b_2t} + a_3b_3e^{-b_3t}}{a_1e^{-b_1t} + a_2e^{-b_2t} + a_3e^{-b_3t}}
\]

(4.8)
is increasing in $t \in \mathbb{R}^+$ iff $r'(t) \geq 0 \ \forall t \in \mathbb{R}^+$. Upon differentiating (4.8) and after some algebraic manipulations, we find that the latter condition is equivalent to

$$g(t) \leq 0 \quad \forall t \in \mathbb{R}^+$$

(4.9)

where

$$g(t) = a_1a_2e^{-(b_1+b_2)t}(b_1 - b_2)^2 + a_2a_3e^{-(b_2+b_3)t}(b_2 - b_3)^2 + a_1a_3e^{-(b_1+b_3)t}(b_1 - b_3)^2. \quad (4.10)$$

Note that (4.9) holds iff

$$a_1a_2(b_1-b_2)^2 + a_1a_3e^{-(b_3-b_2)t}(b_1-b_3)^2 + a_2a_3e^{-(b_3-b_1)t}(b_2-b_3)^2 \leq 0 \quad \forall t \in \mathbb{R}^+. \quad (4.11)$$

When $a_1, a_2 > 0, a_3 < 0$, the left-hand side (LHS) of (4.11) is a non-decreasing function of $t \in \mathbb{R}^+$. Thus, the inequality in (4.11) holds for all $t \in \mathbb{R}^+$ iff it holds as $t \to \infty$, that is, iff $a_1a_2(b_1 - b_2)^2 \leq 0$ or $b_1 = b_2$. This proves (i).

Another equivalent condition to (4.9) is

$$a_1a_2(b_1-b_2)^2e^{-(b_1-b_3)t} + a_1a_3(b_1 - b_3)^2e^{-(b_1-b_2)t} + a_2a_3(b_2 - b_3)^2 \leq 0. \quad (4.12)$$

When $a_1 > 0, a_2, a_3 < 0$, the LHS of (4.12) is a non-increasing function of $t \in \mathbb{R}^+$ so that (4.12) holds for all $t \geq 0$ iff it holds at $t = 0$.

The failure rate $r(t)$ in (4.8) is decreasing in $t \in \mathbb{R}^+$ iff $r'(t) \leq 0$. This condition is equivalent to $g(t) \geq 0 \ \forall t \in \mathbb{R}^+$ where $g(t)$ is given by (4.10). As $t \to \infty$, 

$$\frac{a(t)}{e^{-t(b_1+b_3)}} \to a_1a_3(b_1 - b_3)^2 \leq 0,$$

so that, for large $t$, $r(t)$ is non-decreasing. Thus, $f(t)$ cannot be DFR. This proves (ii).
When $a_2 < 0, a_3 > 0$, we divide both sides of (4.9) by $e^{-(b_2+b_3)t}$ and rearrange terms to get the equivalent inequality

$$(-a_3)\{a_1(b_1 - b_3)^2e^{(b_2-b_1)t} + a_2(b_2 - b_3)^2\} \geq a_1a_2(b_1 - b_2)^2e^{(b_3-b_1)t}. \quad (4.13)$$

Now, take

$$l(t) = (-a_3)\{a_1(b_1 - b_3)^2e^{(b_2-b_1)t} + a_2(b_2 - b_3)^2\}$$
$$m(t) = a_1a_2(b_1 - b_2)^2e^{(b_3-b_1)t} \quad (4.14)$$

in Lemma 4.1.3. Then $l(0) \geq m(0)$ implies

$$(-a_3)\{a_1(b_1 - b_3)^2 + a_2(b_2 - b_3)^2\} \geq a_1a_2(b_1 - b_2)^2 \quad (4.15)$$

which is the condition $s \leq 0$. Also, $l'(t) \geq m'(t)$ implies

$$(-a_3)(b_3 - b_1)e^{(b_2-b_5)t} \geq a_2(b_2 - b_1) \quad (4.16)$$

which holds iff

$$(-a_3)(b_3 - b_1) \geq a_2(b_2 - b_1). \quad (4.17)$$

Using the fact that $a_2 + a_3 = 1 - a_1$, we obtain the condition $\sum_{i=1}^3 a_ib_i \leq b_1$ after some algebraic manipulations. □

**Remarks:** Notice that (4.17) implies $(b_3 - b_1)(a_1 - 1) \geq a_2(b_2 - b_3)$. Thus, if $a_1 < 1$, condition $\sum_{i=1}^3 a_ib_i \leq b_1$ of Theorem 4.1.4(iii) won’t hold.

Further, the first part of Theorem 4.1.4 implies that when $b_3$ is the largest and only the coefficient associated with it, $a_3$, is negative, the pdf in (4.3) is IFR iff it is a mixture of only two distinct distributions.
4.1.3 The IFRA Property

Recall that the pdf $f$ has the IFRA property iff $w(t) = -\frac{1}{t} \log F(t)$ is increasing in $t \in \mathbb{R}^+$. Upon differentiation, we see that $w'(t) \geq 0$ iff

$$-\frac{1}{t} \log F(t) \leq \frac{f(t)}{F(t)} = r(t).$$

That the pdf given by (4.3) is not necessarily IFRA is clear from the following example.

Example 4.1.1:

The function $f(t) = 0.99e^{-t} + 0.022e^{-2t} - 0.003e^{-3t}$ is a pdf since $0.99 + 0.022 - 0.003 = 1.009 > 0$. It is not IFRA since $w(t) = -\frac{1}{t} \log[0.99e^{-t} + 0.011e^{-2t} - 0.001e^{-3t}]$ is not monotonically increasing. For instance, $w(0.1) > w(0.2)$.

One can restrict the choices of the weights $a_i$, however, and obtain the IFRA property. We now state such a result.

Theorem 4.1.5 When $a_1, a_2 \geq 1$, the pdf in (4.3) has the IFRA property. But, it is not IFR.

**Proof:** For the choices $a_1, a_2 \geq 1$, we want to show that (4.18) holds.

First, we will prove that there exists $t_0 \in \mathbb{R}^+$ such that $r(t)$ is increasing in $(0, t_0)$ and decreasing in $(t_0, \infty)$. This is equivalent to showing that $r'(t) \geq (\leq)0$ according as $t < (>)t_0$.

Now define $c_i = b_i - b_i$, $i = 1, 2$ and let

$$g_1(t) = a_1a_2(c_1 - c_2)^2e^{c_1t} \quad \text{and}$$

$$g_2(t) = (-a_3)(a_1c_1^2e^{(c_1-c_2)t} + a_2c_2^2).$$

(4.19)
Note that $c_1 > c_2$ and it is easily seen that $r'(t) \geq 0$ iff $g_2(t) \geq g_1(t)$.

Both functions $g_1(t)$ in (4.19) are differentiable with $g_2(0) \geq g_1(0)$ and $g_2(\infty) < g_1(\infty)$. Since $a_1 \geq 1$ and $b_1 < b_2 < b_3$, we find $t^* = \frac{1}{c_2} \log \left[ \frac{(c_2 - c_1) c_1}{c_2 (c_1 - c_2)} \right] > 0$. Further, from (4.19), it follows that

$$g_2'(t) > g_1'(t) \quad \forall t < t^*$$

$$g_2'(t) < g_1'(t) \quad \forall t > t^*.$$  \hfill (4.20)  \hfill (4.21)

From (4.20) and Lemma 4.1.3, we conclude that $g_2(t) > g_1(t)$, for all $t \leq t^*$. We also have from (4.21) that $g_2(t) - g_1(t)$ is decreasing for all $t > t^*$. Since $g_2(t^*) - g_1(t^*) > 0$ and $g_2(\infty) - g_1(\infty) < 0$, there exists $t_0 > t^*$ for which $g_1(t_0) = g_2(t_0)$. Thus, clearly $g_2(t) - g_1(t) > 0$, for all $t < t_0$ or $r(t)$ is strictly increasing in $(0, t_0)$. Further, for $t > t_0$, $g_2(t) < g_1(t)$ that is, $r(t)$ is strictly decreasing in $(t_0, \infty)$.

Since $r(t)$ increases in $(0, t_0)$, it is clear that (4.18) holds for all $t < t_0$. Now, for $t > t_0$, since $r(t)$ is decreasing and $r(t) \to b_1$, as $t \to \infty$, then $r(t) > b_1$. Further, since the natural log function is monotone and $a_2 e^{-b_1 t} + a_3 e^{-b_1 t} \geq 0$ for $t > t_0 > t^* > \frac{1}{c_2} \log \left[ \frac{(c_2 - c_1) c_1}{c_2 (c_1 - c_2)} \right]$, we see that

$$\log F(t) = \log (a_1 e^{-b_1 t} + a_2 e^{-b_1 t} + a_3 e^{-b_1 t})$$

$$\geq \log e^{-b_1 t}$$

$$= -b_1 t$$  \hfill (4.22)

and

$$-\frac{1}{t} \log F(t) \leq b_1 < r(t) \quad \forall t > t_0.$$  \hfill (4.23)

So, for $t > t_0$, (4.18) holds as well.
Since the failure rate $r(t)$ decreases in $(t_0, \infty)$, clearly, $f(t)$ cannot be IFR. One may also appeal to Theorem 4.1.4(i) to arrive at this conclusion. □

An immediate consequence of Theorem 4.1.5 is given in the following Corollary.

**Corollary 4.1.6** When $a_1 = a_2 = -a_3 = 1$, the pdf

$$f(t) = b_1 e^{-b_1 t} + b_2 e^{-b_2 t} - b_3 e^{-b_3 t} , t > 0$$

(4.24)

has the IFRA property. But, it is not IFR.

**Remark:** Now, if we further suppose that $b_3 = b_1 + b_2$, we can use Theorem 2.6 on p.85 of Barlow and Proschan (1981) to conclude that $f(t)$ is IFRA. This is because, then, $f(t)$ can be identified as the pdf of the maximum of two independent, exponential rv’s with parameters $b_1$ and $b_2$.

### 4.2 IFR, DFR, and IFRA Properties of $T_1$ and $T_2$

If two independent components that make up a system have exponentially distributed lifetimes, the order statistics of the component lifelengths are known to be IFRA. (See, for example, Barlow and Proschan (1981), p.85). In fact, the minimum, being exponential, has a constant failure rate. We now show that the order statistics may not be IFRA when the component lifetimes become dependent.

#### 4.2.1 BVE, ACBVE, FBVE, and GBVE Distributions

First, we present results that conform to the pattern of behavior of order statistics from independent exponentials. For the BVE, ACBVE, GBVE, FBVE, and ACBVE$_2$, ...
the minimum is exponentially distributed and hence has constant failure rate. We now look at the failure rate of the maximum for these distributions.

**Theorem 4.2.1** Let $(X_1, X_2) \sim BVE(\lambda_1, \lambda_2, \lambda_{12})$ and let $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Then, $T_2$ has the IFRA property.

**Proof:** From Theorem 3.2.1, with

$$f^{(2)}(t_2) = \frac{dF^{(2)}(t_2)}{dt_2},$$

we find that the weights of the GH pdf (4.3) are $a_1 = a_2 = -a_3 = 1$. Thus, by Corollary 4.1.6, $T_2$ is IFRA. □

**Remark:** We see in section 3.2 that the order statistics from the ACBVE$_2$ shares the same marginals as those from the BVE. Thus, they also share the same reliability properties given in Theorem 4.2.1.

**Theorem 4.2.2** Let $(X_1, X_2) \sim ACBVE(\lambda_1, \lambda_2, \lambda_{12})$ and let $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Then, $T_2$ has the IFRA property.

**Proof:** We refer to Theorem 3.2.2 for $F^{(2)}$ and use (4.25) to find $f^{(2)}(t_2)$. Then, since $a_1 = a_2 = \frac{\lambda}{\lambda_1 + \lambda_2}$, $T_2$ is IFRA by Theorem 4.1.5. □

**Theorem 4.2.3** Let $(X_1, X_2) \sim GBVE(\theta_1, \theta_2, \delta)$. Then, $T_2$ has the IFRA property.

**Proof:** Take the derivative (4.25) on $F^{(2)}(t_2)$ given in Theorem 3.2.4. We find that $a_1 = a_2 = -a_3 = 1$ are the weights in the GH pdf (4.3). Then, by Corollary 4.1.6, $T_2$ is IFRA. □
We shift our attention, at this point, to a distribution for which the failure rate of the maximum exhibits a variety of behavior depending on the values of the parameters involved.

**Theorem 4.2.4** Let \((X_1, X_2) \sim FBVE(\alpha, \beta, \alpha', \beta')\), where \((\alpha, \beta, \alpha', \beta') \in \Omega_1 = \{\alpha + \beta \neq \alpha', \beta'\}\). Denote the exponential parameters \((\alpha + \beta), \alpha', \text{ and } \beta'\) in the GH pdf (recall equation 4.3) of \(T_2\) by \(b_i\), \(i = 1, 2, 3\). Then,

(i) \(T_2\) has the IFR property if condition (FR1) or (FR2) below holds.

(ii) \(T_2\) has the IFRA property if condition (FR3) or (FR4) below holds.

The conditions are:

(FR1) \(\alpha + \beta = b_1\).

(FR2) \(\alpha + \beta = b_2\), \(\alpha < \alpha', \beta < \beta'\), and \(\alpha'\beta' - \alpha\beta' - \alpha'\beta > 0\).

(FR3) \(\alpha + \beta = b_2\), \(\alpha < \alpha', \beta < \beta'\), and \((\alpha + \beta)^2 - \alpha\alpha' - \beta\beta' \geq 0\).

(FR4) \(\alpha + \beta = b_3\), \(\alpha < \alpha'\), and \(\beta < \beta'\).

**Proof:** We refer to Theorem 3.2.3 for \(F_2(t_2)\) and use (4.25) to find \(f_2(t_2)\). Assume that either (FR1) or (FR2) hold. Then, in the GH pdf of \(T_2\), we have \(a_1 > 0\), \(a_2, a_3 < 0\). By Theorem 4.1.4(ii), we then can conclude that \(T_2\) is IFR. This proves result (i).

Now, suppose that either condition (FR3) or (FR4) holds. Then, the GH weights satisfy \(a_1 > 1\), \(a_2 \geq 1\), and \(a_3 < 0\) and consequently from Theorem 4.1.5, \(T_2\) is IFRA.

This establishes result (ii). □
Remark: It is not difficult to find examples for which the failure rate of the maximum order statistic from the FBVE is neither IFR nor IFRA when $\alpha > \alpha'$ and/or $\beta > \beta'$.

4.2.2 RBVE Distribution

We consider next Raftery's parsimonious models.

Theorem 4.2.5 Let $(X_1, X_2) \sim RBVE(\lambda, \pi_1, \pi_2, p_{11})$. Denote the weights in the GH pdf of $T_1$ by $a_i$ and the corresponding weights for $T_2$ by $a_i^*$, where $a_1^* = 2 - a_1$, and $a_i^* = -a_i$, $i=2,3,4$. Then,

(i) $T_1$ has the IFR property when condition (RAF2) holds.

(ii) $T_1$ has the DFR property when condition (RAF1) or (RAF4) holds.

(iii) $T_2$ has the IFR property when any one of the conditions (RAF1)-(RAF4) holds.

The conditions are:

(RAF1) model 1: $\pi_1 = \pi_2 = p_{11} = \pi$.

(RAF2) model 2: $\pi_1 = \pi_2 = \pi \leq 0.5$, $p_{11} = 0$.

(RAF3) model 2: $\pi_1 = \pi_2 = \pi > 0.5$, $p_{11} = 2\pi - 1$.

(RAF4) model 3: $\pi_2 = p_{11} = \gamma$.

Proof: We refer to equation (3.101) in section 3.3 for the values of the weights $a_i$, $i=1,\ldots,4$ in the pdf of $T_1$ under the fully parametrized RBVE set-up.
Assume that condition (RAF1) holds. Then,

\[ a_1 = \frac{2\pi}{1+\pi}, \quad a_2 = a_3 = 0, \quad a_4 = \frac{1-\pi}{1+\pi} \text{ and } \quad b_1 = \lambda, \quad b_2 = b_3 = \lambda \left(1 + \frac{1}{1-\pi}\right), \quad b_4 = \frac{2\lambda}{1-\pi}. \]  

(4.26)

Notice that \(0 < a_1, a_4 \leq 1\), so that the resulting density for the minimum, namely,

\[ f_{(1)}(t) = \frac{2\pi}{1+\pi} \lambda e^{-\lambda t} + \frac{2\lambda}{1+\pi} e^{-\frac{2\lambda}{1-\pi}t}, \quad t > 0 \]  

(4.27)

is a convex mixture of exponential densities. Thus, we conclude that \(T_1\) is DFR.

The model for \(T_2\), on the other hand, is a GH distribution with two components \(a^*_1 > 0\), and \(a^*_4 < 0\). By Theorem 4.1.1, \(T_2\) is IFR.

Now, let conditions (RAF2) and (RAF3) hold, that is, assume the set-up for model 2. Then,

\[ a_1 = \begin{cases} 0 & 0 \leq \pi \leq 0.5 \\ \frac{2(\pi-1)}{1+\pi} & 0.5 < \pi \leq 1 \end{cases} \]

\[ a_2 = a_3 = \begin{cases} 1 & 0 \leq \pi \leq 0.5 \\ \frac{1-\pi}{\pi} & 0.5 < \pi \leq 1 \end{cases} \]

\[ a_4 = \begin{cases} -1 & 0 \leq \pi \leq 0.5 \\ \frac{-(\pi-2)(\pi-1)}{\pi(1+\pi)} & 0.5 < \pi \leq 1 \end{cases} \]

\[ b_1 = \lambda, \quad b_2 = b_3 = \lambda \left(1 + \frac{1}{1-\pi}\right), \quad b_4 = \frac{2\lambda}{1-\pi}. \]

(4.28)

Thus, since \(a_1 = 0\) and \(a_2 = a_3\), for \(0 \leq \pi \leq 0.5\), \(T_1\) has the IFR property via Theorem 4.1.1. However, when \(0.5 < \pi < 1\), \(T_1\) is neither IFR nor DFR. It is possible to find examples for which the failure rate of \(T_1\) increases for small \(t\) and then decreases. In the extreme case that \(\pi = 1\), however, \(a_2 = a_3 = a_4 = 0\) so that \(T_1\) has constant failure rate.
In contrast, $T_2$ has the IFR property for all values of $\pi$. For $0 \leq \pi \leq 0.5$, $T_2$ has a GH distribution that satisfies $a_1^* > 0$, $a_2^* < 0$, and $a_3^* > 0$. By Theorem 4.1.4 (iii), $T_2$ has the IFR property.

For $0.5 < \pi \leq 1$, $T_2$ also has a distribution that satisfies $a_1^* > 0$, $a_2^* < 0$, and $a_3^* > 0$. Again, by Theorem 4.1.4 (iii), $T_2$ has the IFR property.

Finally, when condition (RAF3) holds, then

$$a_1 = \gamma \frac{2-\pi_1-\gamma}{1-\gamma\pi_1}, \quad a_2 = 1 - \frac{\gamma}{\pi_1}, \quad a_3 = 0, \quad a_4 = \frac{1-\pi_1}{\pi_1(1-\gamma\pi_1)},$$

and

$$b_1 = \lambda, \quad b_2 = \lambda \left(1 + \frac{1}{1-\gamma}\right), \quad b_3 = \lambda \left(1 + \frac{1}{1-\gamma}\right), \quad b_4 = \lambda \left(\frac{1}{1-\pi_1} + \frac{1}{1-\gamma}\right).$$

(4.29)

Notice that in this case $0 < a_1, a_2, a_4 < 1$, so that the pdf (4.3) yields a mixture of DFR distributions and $T_1$ is itself DFR.

On the other hand, the distribution of $T_2$ is GH with three components, two of which have negative weights. Using Theorem 4.1.4(ii), we can conclude that $T_2$ is IFR. □

When $\pi_1 = \pi_2 = \pi$, we summarize the reliability properties of $T_1$ in Figure 19 (See p. 118). This shows the behavior of $T_1$ for varying values of $p_{11}$ and $\pi = (\pi_1 = \pi_2)$.

To facilitate the discussion, we decompose the (restricted) parameter space $\{ p_{11} \leq \pi \leq \frac{p_{11}+1}{2}, 0 \leq p_{11} \leq 1 \}$ into the following seven regions.

(A1) The line $p_{11} = \pi \epsilon(0,1)$. (Model 1)

(A2) The line $p_{11} = 0, 0 < \pi \leq 0.5$. (Model 2)

(A3) The line $p_{11} = 2\pi - 1, 0.5 < \pi < 1$. (Model 2)

(A4) The point $p_{11} = \pi = 1$. (Models 1 & 2)
(A5) The point $p_{11} = \pi = 0$. (Models 1 & 2)

(A6) The region $p_{11} < \pi < \frac{-1 + \sqrt{1 + 8p_{11}}}{2}$.

(A7) The complement of the union of these sets.

The union of the regions A1, A4, and A5 is the parameter space for the distribution of $T_1$ under model 1. The union of the regions A2, A3, A4, and A5 is the parameter space for the distribution of $T_1$ under model 2. On A1, $T_1$ is DFR. On A2, $T_1$ is IFR. On A3, $T_1$ is non-monotone. On A4, $X_1$ and $X_2$ coincide, so the failure rate of $T_1$ is constant. At the origin (A5), the component lives, $X_1$ and $X_2$ are independent so $T_1$, being the minimum of two iid exp($\lambda$) rv's, has constant failure rate. If we take $\pi_1 = \pi_2 = \pi$ and hold $a_4$ to be greater than zero in the fully parametrized model for $T_1$, then the resulting pdf is a convex mixture of four exponential pdf's and thus has the DFR property. The parameter space satisfying this condition is given by A6. We have yet to explore the reliability property of $T_1$ on A7.

In the same manner, when $\pi_1 = \pi_2 = \pi$, we summarize the reliability properties of $T_2$ in Figure 20 (See p. 119). Here, we decompose the (restricted) parameter space into six regions as follows:

(B1) The line $p_{11} = \pi e(0,1)$. (Model 1)

(B2) The line $p_{11} = 0$ and $0 < \pi \leq 0.5$. (Model 2)

(B3) The line $p_{11} = 2\pi - 1$ and $0.5 < \pi < 1$. (Model 2)

(B4) The point $p_{11} = \pi = 1$. (Models 1 & 2)
(B5) The point $p_{11} = \pi = 0$. (Models 1 & 2)

(B6) The complement of the union of these sets.

The parameter space for the distribution of $T_2$ under model 1 is given by the union of the regions B1, B4, and B5. The parameter space for the distribution of $T_2$ under model 2 is the union of the regions B2, B3, B4, and B5. On B1, B2 and B3, $T_2$ is IFR. $X_1$ and $X_2$ coincide on B4, whence $T_2$ has constant failure rate. At the origin, (B5), we obtain independence between $X_1$ and $X_2$ and the survival function $F_2(t) = 2e^{-\lambda t} - e^{-2\lambda t}$, $t > 0$ for $T_2$; whence it has the IFR property. We have not studied the behavior of $T_2$ on B6.

4.2.3 BEE Distribution

Finally, we explore the behavior of order statistics from the BEE. The minimum has the exponential distribution, so its failure rate is constant. We have the following theorem describing the properties of the maximum.

Theorem 4.2.6 Let $(X_1, X_2) \sim BEE(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')$, where $(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2') \in \Lambda_1 = \{\alpha_1 + \alpha_2 \neq \alpha_1', \alpha_2'\}$. Define $\phi_1 = \frac{\alpha_0 \alpha_1}{\alpha_1 + \alpha_2 - \alpha_2'}$ and $\phi_2 = \frac{\alpha_0 \alpha_2}{\alpha_1 + \alpha_2 - \alpha_1'}$. Denote the exponential parameters $(\alpha_1 + \alpha_2), \alpha_1'$, and $\alpha_2'$ in the GH pdf of $T_2$ by $b_i$, $i=1,2,3$. Then,

(i) $T_2$ has the DFR property if condition (BEE5) holds.

(ii) $T_2$ has the IFRA property if condition (BEE3) or (BEE4) holds.

(iii) $T_2$ has the IFR property if condition (BEE1) or (BEE2) holds.
The conditions are:

\((BEE1)\) \(\alpha_1 + \alpha_2 = b_1\).

\((BEE2)\) \(\alpha_1 + \alpha_2 = b_2, \alpha_1 < \alpha_1', \alpha_2 < \alpha_2', \text{ and } (1 - \alpha_0)((\alpha_1 + \alpha_2)^2 - \alpha_1\alpha_1' - \alpha_2\alpha_2') > \alpha_1'\alpha_2 + \alpha_1\alpha_2' - \alpha_1'\alpha_2'\).

\((BEE3)\) \(\alpha_1 + \alpha_2 = b_2, (\alpha_1 + \alpha_2)^2 - \alpha_1\alpha_1' - \alpha_2\alpha_2' \geq 0, \text{ and either (3a) or (3b) below:}\)

\((3a)\) \(\alpha_1 < \alpha_1' \text{ and } \phi_1 > 1\)

\((3b)\) \(\alpha_2 < \alpha_2' \text{ and } \phi_2 > 1\).

\((BEE4)\) \(\alpha_1 + \alpha_2 = b_3, \alpha_1 < \alpha_1', \alpha_2 < \alpha_2', \text{ and } \phi_1, \phi_2 \geq 1.\)

\((BEE5)\) \(\alpha_1 + \alpha_2 = b_3, \alpha_1 > \alpha_1', \alpha_2 > \alpha_2', \text{ and } (1 - \alpha_0)((\alpha_1 + \alpha_2)^2 - \alpha_1\alpha_1' - \alpha_2\alpha_2') > \alpha_1'\alpha_2 + \alpha_1\alpha_2' - \alpha_1'\alpha_2'.\)

**Proof:** We refer to Theorem 3.2.7 for \(F(2)(t_2)\) and apply (4.25) to get \(f(2)(t_2)\) on \(A_1\).

If condition \((BEE5)\) holds, then the corresponding weights satisfy \(0 < a_1, a_2, a_3 < 1\) and \(T_2\) is DFR. This proves (i).

Assume that either \((BEE3)\) or \((BEE4)\) holds. Then, \(a_1, a_2 \geq 1, \text{ and } a_3 > 0.\) By Theorem 4.1.5, \(T_2\) is IFRA. This proves result (ii).

Finally, if condition \((BEE1)\) or \((BEE2)\) holds, then \(a_1 > 0, a_2, a_3 < 0, \text{ and by Theorem 4.1.4(ii), } T_2 \text{ is IFR. We have shown result (iii).}\)

Note that when \(\alpha_0 = 1, \text{ we are back to the FBVE model all over again.} \)
Remark: It is easy to find examples in which the failure rate of \( T_2 \) increases for small \( t \) and then decreases when either \( \alpha_1 > \alpha_1', \alpha_2 < \alpha_2' \) or \( \alpha_1 < \alpha_1', \alpha_2 > \alpha_2' \). While this means that \( T_2 \) can neither be IFR nor DFR, we can also show that \( T_2 \) fails to be IFRA.

4.2.4 Discussion

The bivariate exponential models we have considered in this section provide us with numerous examples that illustrate the contrast between the dependent and independent set-ups. In the case of independence, two exponentially distributed rv's have a minimum whose failure rate is constant, and a maximum which possesses the IFRA property. Among our examples, such reliability properties also hold in the dependent set-up of the BVE, ACBVE, GBVE, and ACBVE$_2$.

The FBVE and the BEE deviate from this pattern in that while their minima have constant failure rates, the maxima have failure rates that may or may not be monotone. In the 'stress situation' when \( \alpha < \alpha' \), \( \beta < \beta' \) for the FBVE, or similarly \( \alpha_1 < \alpha_1' \), \( \alpha_2 < \alpha_2' \) for the BEE and subject to other conditions on the parameters, the corresponding maximum order statistic is either IFRA or IFR. This is intuitively reasonable: upon first failure, the remaining component now bearing an additional load will have increasing chance of instantaneous failure as it ages. Under the 'enhanced situation' when \( \alpha > \alpha' \), \( \beta > \beta' \) and the mixed situation when either \( \alpha > \alpha' \) or \( \beta > \beta' \) for the FBVE, one may find examples in which the failure rate of \( T_2 \) increases for small \( t \) and then decreases. Its distribution may not only fail to be IFR, but also fail to be IFRA. For the BEE, we have found additional conditions on the parameters...
that ensure \( T_2 \) to be DFR in the 'enhanced situation' \( \alpha_1 > \alpha_1', \alpha_2 > \alpha_2'. \)

The parsimonious RBVE models, likewise, show a departure from the reliability pattern under independence. The maximum order statistic turns out to be IFR under the three models. However, the minimum has a failure rate that may be constant, increasing, or decreasing. Under models 1 and 3, \( T_1 \) is DFR. Under model 2 when \( \pi_1 = \pi_2 = \pi \leq 0.5 \) and \( p_{11} = 0 \), \( T_1 \) is IFR. When model 2 holds so that now \( \pi > 0.5 \) and \( p_{11} = 2\pi - 1 \), it is easy to find examples in which the failure rate of \( T_1 \) increases for small \( t \), and then decreases.

### 4.3 Hazard Rate Ordering

In this section, we show that \( T_2 \overset{\text{hr}}{\geq} T_1 \) for the distributions BVE, Block and Basu’s ACBVE, Freund’s FBVE, Sarkar’s ACBVE2, Gumbel’s GBVE, Friday and Patil’s BEE, and Raftery’s RBVE.

We employ the following method of proof. Recall from equation (3.43) in section 3.2 that

\[
F_2(t) = F_1(t) + F_2(t) - F_1(0)
\]

(4.30)

Let \( w(t) = \frac{F_2(t)}{F_1(t)} \). Then, \( T_2 \overset{\text{hr}}{\geq} T_1 \) iff \( w(t) \) is non-decreasing in \( t \in (0, \infty) \) or equivalently,

\[
w'(t) \geq 0 \quad \forall t \in (0, \infty).
\]

(4.31)

It is easily seen that (4.31) holds whenever

\[
f_{(2)}(t)\{F_1(t) + F_2(t)\} \geq F_1(t)\{f_1(t) + f_2(t)\}.
\]

(4.32)
Our object then is to show that (4.32) is satisfied by these bivariate exponential models.

We first prove a general result for distributions with BLMP.

**Theorem 4.3.1** If \((X_1, X_2)\) has a joint df with BLMP and absolutely continuous marginals, then the order statistics satisfy the relation \(T_2 \geq T_1\).

**Proof:** Let \(f_i(.)\) and \(F_i(.)\) denote the marginal pdf and survival df of \(X_i\), \(i = 1, 2\). Then, the minimum order statistic \(T_1\) will be distributed as \(e(\theta)\), where \(\theta = f_1(0) + f_2(0)\). We want to verify that (4.32) holds in this setting, i.e.,

\[
\theta e^{-\theta t} \{F_1(t) + F_2(t)\} \geq e^{\theta t} \{f_1(t) + f_2(t)\}
\]

(4.33)

or equivalently,

\[
\{F_1(t) - \frac{1}{\theta} f_1(t)\} + \{F_2(t) - \frac{1}{\theta} f_2(t)\} \geq 0.
\]

(4.34)

Block and Basu (1974) show that

\[
\frac{1}{\theta} f_i(t) = P(X_i - X_j < t), \quad i \neq j = 1, 2.
\]

(4.35)

This completes the proof. \(\square\)

**Corollary 4.3.2** For the BVE, the ACBVE, the FBVE, and the BEE, \(T_2 \geq T_1\).

**Proof:** All these distributions have BLMP with absolutely continuous marginals. Thus, we invoke Theorem 4.3.1. \(\square\)

**Remark:** Note that the marginal distributions of \(X_1, X_2, \) and \(T_1\) are the same for the BVE and Sarkar's ACBVE2 distributions. (See section 3.2) Thus, we can conclude that \(T_2 \geq T_1\) for the latter distribution.
Theorem 4.3.3  For the GBVE($\theta_1, \theta_2, \delta$), $T_2 \overset{hr}{\geq} T_1$.

Proof: We use the marginal distributions given in Appendix A.4 and the minimum distribution in Theorem 3.2.4 to verify directly that (4.32) holds. Note that this distribution does not have the BLMP. □

Theorem 4.3.4  For the RBVE($\lambda, \pi_1, \pi_2, \rho_11$), $T_2 \overset{hr}{\geq} T_1$.

Proof: We show that (4.32) holds for the expressions $\overline{F}_i(t)$ in the Appendix A.5 and for $\overline{F}_{(1)}(t)$ given in Theorem 3.2.5. The inequality (4.32) reduces to

$$a_2(b_2 - b_1)e^{-b_2t} + a_3(b_3 - b_1)e^{-b_3t} + a_4(b_4 - b_1)e^{-b_4t} \geq 0 \quad (4.36)$$

or equivalently, $g(t) \geq 0$ where

$$g(t) = a_2(b_2 - b_1)e^{(b_4-b_2)t} + a_3(b_3 - b_1)e^{(b_4-b_3)t} + a_4(b_4 - b_1). \quad (4.37)$$

It is easy to see that $g'(t) \geq 0$, $\forall \ t \in (0, \infty)$. If $g(0) \geq 0$, then (4.36) holds. A sufficient condition for $g(0) \geq 0$ is $\sum_{i=1}^{4} a_i b_i \geq b_1$. Using the expressions for $a_i, b_i, i = 1, \ldots, 4$, given in equation (3.101) in section 3.3, we find that this condition is satisfied. □
Figure 19: Reliability properties of $T_1$ under the RBVE for varying values of $p_{11}$ and $\pi = (\pi_1 = \pi_2)$. 
Figure 20: Reliability properties of $T_2$ under the RBVE for varying values of $p_{11}$ and $\pi = (\pi_1 = \pi_2)$
CHAPTER V

Predictors of the Maximum Order Statistic

Inference procedures found in the literature for bivariate exponential distributions almost always assume an 'identified minimum' set-up. This formulation assumes that the lifetimes $X_1$ and $X_2$ are observed, along with an indicator of which component failed first, say $I(X_1 < X_2)$. It is not hard to envision a situation, however, when one keeps track only of the ordered lifetimes. Consider, for example, a parallel system of two-components with exchangeable lifetimes. Given a sample with this kind of information, and suppose that the minimum lifetime of a system has just been observed, can we predict the time to failure of this system? We tackle the problem in this chapter.

Prediction in the context of the univariate exponential distribution have been explored by many authors. Nagaraja (1993) writes a cohesive survey of point predictors and prediction regions derived from various optimality criteria. We now look at some prediction issues concerning two bivariate exponential models, albeit only for the exchangeable set-up.

We propose two predictors of the maximum lifetime, and find their relative efficiencies by comparing mean squared errors. We will refer to these as the conditional mean predictor and the conditional median predictor, respectively. In section 5.1, we
present the details of our exploration for Freund’s FBVE distribution. In section 5.2, we study Block and Basu’s ACBVE distribution. We find that the 'MLE' solution of the parameters given by Mehrotra and Michalek (1976) in the symmetric ACBVE set-up is in error. This lead us to look for the real MLE in this context.

5.1 Freund’s FBVE Distribution

The exchangeable set-up for Freund’s FBVE corresponds to having \( a = a' = \gamma_1 \) and \( b = b' = \gamma_2 \). When this holds, the joint density of \((T_1, T_2)\) is given by

\[
h(t_1, t_2) = 2\gamma_1\gamma_2 e^{-\gamma_2 t_2 - (2\gamma_1-\gamma_2) t_1}, \quad 0 < t_1, t_2 < \infty.
\]  

(5.1)

We refer the reader to Theorem 3.1.3(ii) for the joint density of \((T_1, T_2)\) for general \((a, \beta, a', \beta')\). Furthermore, the conditional density of \(T_2\) given \(T_1 = t_1\) becomes

\[
g(t_2|t_1) = \gamma_2 e^{-\gamma_2 (t_2 - t_1)}, \quad t_1 < t_2 < \infty.
\]  

(5.2)

It immediately follows that the conditional mean of \(T_2\) given \(T_1 = t_1\) is

\[
E(T_2|T_1 = t_1) = \frac{1}{\gamma_2} + t_1
\]  

(5.3)

and the conditional median of \(T_2\) given \(T_1 = t_1\) is

\[
\text{Med}(T_2|T_1 = t_1) = \frac{\log(2)}{\gamma_2} + t_1.
\]  

(5.4)

The expression for the df of \(T_2\) given \(T_1 = t_1\) for general \((\alpha, \beta, \alpha', \beta')\) can be found in Theorem 3.5.5(iii).
5.1.1 Two Predictors of the Maximum

If \( \gamma_2 \) is known, then (5.3) yields the least squares solution and (5.4) the least absolute deviation solution to predicting \( T_2 \). When \( \gamma_2 \) is unknown, it is not unreasonable to propose the following predictors of \( T_2 \):

\[
\hat{T}_2 = \frac{1}{\hat{\gamma}_2} + t_1, \tag{5.5}
\]

the conditional mean predictor, and

\[
\hat{T}_{2M} = \frac{\log 2}{\hat{\gamma}_2} + t_1, \tag{5.6}
\]

the conditional median predictor, where \( \hat{\gamma}_2 \) is the MLE of \( \gamma_2 \). This concept of a conditional median predictor is introduced by Raqab (1992).

**Theorem 5.1.1** Let \( \bar{T}_i = (T_{1i}, T_{2i}), i = 1, \ldots, n \) be a random sample from the pdf (5.1). Suppose that the minimum lifetime, \( T_1 \), of the \((n + 1)st\) sample has just been observed. Then,

\[
\hat{T}_2 = T_1 + \frac{1}{n} \sum_{i=1}^{n} (T_{2i} - T_{1i}) \tag{5.7}
\]

and

\[
\hat{T}_{2M} = T_1 + \frac{\log 2}{n} \sum_{i=1}^{n} (T_{2i} - T_{1i}). \tag{5.8}
\]

**Proof:** In view of (5.5) and (5.6), the result immediately follows if we show that the MLE of \( \gamma_2 \) is given by

\[
\hat{\gamma}_2 = \frac{n}{\sum_{i=1}^{n} (T_{2i} - T_{1i})}. \tag{5.9}
\]
Now, the likelihood function of \( t_i = (t_{1i}, t_{2i}), i = 1, \ldots, n \) is

\[
L(\gamma_1, \gamma_2) = 2^n \gamma_1^n \gamma_2^n e^{-\gamma_2} \sum_{i=1}^n t_{2i} - (2n - \gamma_2) \sum_{i=1}^n t_{1i} \prod_{i=1}^n I_{A_i}
\]

(5.10)

where \( A_i = \{0 < t_{1i} < t_{2i} < \infty\} \). The log likelihood is given by

\[
\log L = c(u, t_i) + n \log \gamma_1 + n \log \gamma_2 - \gamma_2 \sum_{i=1}^n t_{2i} - (2\gamma_1 - \gamma_2) \sum_{i=1}^n t_{1i}.
\]

(5.11)

Differentiating \( \log(L) \) wrt the parameters \( \gamma_1 \), and \( \gamma_2 \), respectively, we get

\[
\frac{\partial \log L}{\partial \gamma_1} = \frac{n}{\gamma_1} - 2 \sum_{i=1}^n t_{1i}
\]

(5.12)

\[
\frac{\partial \log L}{\partial \gamma_2} = \frac{n}{\gamma_2} - \sum_{i=1}^n t_{2i} + \sum_{i=1}^n t_{1i}
\]

(5.13)

Equating the expression (5.13) to zero and solving for \( \gamma_2 \), we obtain the expression \( \gamma_2 \) given in (5.9).

The MLE's \( \hat{\gamma}_2 \) and \( \hat{\gamma}_1 = \frac{n}{\sum_{i=1}^n t_{1i}} \) are due to Hanagal and Kale (1992). \( \square \)

### 5.1.2 Comparison of the MSE of the Predictors

One measure of 'closeness' of a predictor \( \hat{Y} \) to the unknown variable \( Y \) it seeks to predict, is the quantity so-called mean squared error, or MSE. On the average, the MSE gives the magnitude of squared deviations between \( \hat{Y} \) and \( Y \), that is,

\[
\text{MSE}(\hat{Y}) = E(\hat{Y} - Y)^2.
\]

(5.14)

The smaller the MSE of \( \hat{Y} \), the more precisely it predicts \( Y \).

In the previous subsection, we proposed two predictors of the maximum order statistic. We now compare their respective MSE's.
**Theorem 5.1.2** Suppose we have a random sample of size \((n+1)\) from the pdf (5.1), where \(n > 0\), and only \(T_1\) has been observed in the \((n+1)st\) sample. Then,

\[
MSE(\hat{T}_{2M}) \leq MSE(\hat{T}_2), \quad \forall (\gamma_1, \gamma_2) \in \mathbb{R}_2^+.
\]

**Proof:** According to (5.14), the MSE of \(\hat{T}_2\) is obtained by

\[
MSE(\hat{T}_2) = E(\hat{T}_2 - T_2)^2
= E(T_1 + \frac{1}{\gamma_2} - T_2)^2
= E(V^2) + E\left(\frac{1}{\gamma_2^2}\right) - 2E(V)E\left(\frac{1}{\gamma_2}\right).
\]

where \(V = T_2 - T_1\), and \(\frac{1}{\gamma_2} = \sum_{i=1}^{n} \frac{V_i}{n}\). Under the exchangeable set-up of the FBVE, it is not hard to show that \(T_1 \sim e(2\gamma_1)\) and \(V \sim e(\gamma_2)\). By property (BB2) of section 1.2, \(V\) and \(T_1\) are independent. Then,

\[
E\left(\frac{1}{\gamma_2}\right) = \frac{1}{n} \sum_{i=1}^{n} E(V_i) = \frac{1}{\gamma_2}
\]

and

\[
E\left(\frac{1}{\gamma_2^2}\right) = \frac{1}{n^2} \left\{ \sum_{i=1}^{n} E(V_i^2) + 2 \sum_{i < j} E(V_i)E(V_j) \right\}
= \frac{1}{n^2} \left\{ \frac{n}{2\gamma_2^2} + \frac{n(n+1)(n+2)}{6} \right\} \frac{1}{\gamma_2^2}
= \frac{1}{2n\gamma_2^2} + \frac{(n+1)(n+2)}{3n\gamma_2^2}.
\]

Plugging these into (5.16), we find

\[
MSE(\hat{T}_2) = \frac{1}{2\gamma_2^2} + \frac{1}{2n\gamma_2^2} + \frac{(n+1)(n+2)}{3n\gamma_2^2} - \frac{2}{\gamma_2^2}
= \frac{1}{2\gamma_2^2} \left( \frac{2n^2 - 3n + 7}{3n} \right).
\]
Next, we consider the MSE of $T_{2M}$. By (5.14) and (5.18), we have

$$\text{MSE}(T_{2M}) = \text{E}(T_{2M} - T_2)^2 = E(V^2) + (\log 2)^2 E\left(\frac{1}{\gamma_2^2}\right) - 2(\log 2) E(V) E\left(\frac{1}{\gamma_2}\right)$$

$$= \frac{1}{2\gamma_2^2} + (\log 2)^2 \left\{ \frac{1}{2n\gamma_2^2} + \frac{(n+1)(n+2)}{3n\gamma_2^2} \right\} - \frac{2\log 2}{\gamma_2^2}$$

$$= \frac{1}{2\gamma_2^2} \left\{ 1 + (\log 2)^2 \left[ \frac{2n^2 + 6n + 7}{3n} \right] - 4\log 2 \right\}. \quad (5.20)$$

We want to verify that $\frac{\text{MSE}(T_{2M})}{\text{MSE}(T_2)} \leq 1$, $\forall (\gamma_1, \gamma_2) \in \mathbb{R}_2^+$, that is,

$$1 + (\log 2)^2 \left[ \frac{2n^2 + 6n + 7}{3n} \right] - 4\log 2 \leq \frac{2n^2 - 3n + 7}{3n}. \quad (5.21)$$

Upon simplification of (5.21), we find that it is equivalent to the inequality

$$\frac{2(1 - \log 2)}{1 + \log 2} \leq \frac{2n^2 + 7}{3n}, \quad (5.22)$$

where the LHS reduces to 0.362. Let $g(n) = \frac{2n^2 + 7}{3n}$. Then,

$$g'(n) = \frac{3n(4n) - (2n^2 + 7)3}{9n^2} = 1 - \frac{7}{3n^2}. \quad (5.23)$$

A critical point of $g$ is $n_0 = \sqrt{\frac{7}{3}} = 2.33$. Since $g''(n) = \frac{14}{3n^3} > 0$, then $n_0 = 2.33$ yields a minimum. Thus,

$$g(n) \geq g(2.33) = 1.67 \geq 0.362, \quad \forall n > 0 \quad (5.24)$$

and (5.22) holds. □

The above theorem implies that in the exchangeable FBVE set-up, the conditional median predictor is more efficient than the conditional mean predictor in predicting the maximum.
5.2 Block and Basu’s ACBVE Distribution

The symmetric ACBVE distribution corresponds to having $\lambda_1 = \lambda_2$. Denote this common value by $\theta_1$, and let $\theta_2 \equiv \lambda_1$. With these notations, the joint density of $(T_1, T_2)$ becomes (recall Theorem 3.1.3(ii))

$$h(t_1, t_2) = (2\theta_1 + \theta_2)(\theta_1 + \theta_2)e^{-\theta_1 t_1 - (\theta_1 + \theta_2)t_2}, \quad 0 < t_1 < t_2 < \infty. \quad (5.25)$$

From Theorem 3.5.4(iii), the conditional density of $T_2$ given $T_1 = t_1$ in this exchangeable set-up is given by

$$g(t_2|t_1) = (\theta_1 + \theta_2)e^{-(\theta_1 + \theta_2)(t_2 - t_1)}, \quad t_1 < t_2 < \infty. \quad (5.26)$$

So, it follows that the conditional mean of $T_2$ given $T_1 = t_1$ is

$$E(T_2|T_1 = t_1) = t_1 + \frac{1}{\theta_1 + \theta_2} \quad (5.27)$$

and the conditional median of $T_2$ given $T_1 = t_1$ is

$$Med(T_2|T_1 = t_1) = t_1 + \frac{\log 2}{\theta_1 + \theta_2}. \quad (5.28)$$

Let $\bar{T}_i = (T_{1i}, T_{2i}), i = 1, \ldots, n$ be a random sample from the density (5.25), and let $W_1 = \sum_{i=1}^{n} T_{1i}$, and $W_2 = \sum_{i=1}^{n} T_{2i}$. Then, we can write the likelihood function as

$$L(\theta_1, \theta_2) = (2\theta_1 + \theta_2)^n(\theta_1 + \theta_2)^n e^{-\theta_1 W_1 - (\theta_1 + \theta_2) W_2} \prod_{i=1}^{n} I_{A_i} \quad (5.29)$$

where $A_i = \{0 < t_{1i} < t_{2i} < \infty\}$. The routine differentiation approach yields the 'MLEs' as

$$2\theta_1 + \theta_2 = \frac{n}{W_1} \quad \text{and} \quad \theta_1 + \theta_2 = \frac{n}{W_2 - W_1}. \quad (5.30)$$
Some proponents of this 'MLE' solution are Mehrotra and Michalek (1976) and Hanagal and Kale (1992).

A closer investigation of this solution reveals that the 'MLE' of \( \theta_1 \) and \( \theta_2 \), are

\[
\hat{\theta}_1 = \frac{n}{W_1} - \frac{n}{W_2 - W_1}
\]

and

\[
\hat{\theta}_2 = \frac{2n}{W_2 - W_1} - \frac{n}{W_1}
\]

(5.31)

and that \( \hat{\theta}_1 > 0 \iff W_1 < W_2 - W_1 \) and \( \hat{\theta}_2 > 0 \iff \frac{W_2 - W_1}{2} < W_1 \). Thus, the pair \( (\hat{\theta}_1, \hat{\theta}_2) \) given by (5.31) are both non-negative only in the region \( S = \{(W_1, W_2) | \frac{W_2 - W_1}{2} < W_1 < W_2 - W_1 \} \). On \( S^c \), at least one of \( \hat{\theta}_1 \) or \( \hat{\theta}_2 \) is negative. We show below that the probability of \( S^c \) may not be negligible. To this end, partition \( S^c \) into \( S_1 = \{(W_1, W_2) | \frac{W_2 - W_1}{2} > W_1 \} \) and \( S_2 = \{(W_1, W_2) | W_1 > W_2 - W_1 \} \). Then, we have the following result.

**Theorem 5.2.1** Let \( T_i = (T_{1i}, T_{2i}), i = 1, \ldots, n \) be a random sample from the density (5.25). Define \( W_k = \sum_{i=1}^{n} T_{ki}, k = 1, 2, p_1 = \frac{2\gamma + 1}{3\gamma + 3}, \) and \( p_2 = \frac{2\gamma + 1}{3\gamma + 2}, \) where \( \gamma = \frac{\hat{\theta}_1}{\hat{\theta}_2} \). Then,

\[
c_k \equiv P((W_1, W_2) \in S_k) = \sum_{i=n}^{2n-1} \binom{2n-1}{i} p_k^i (1-p_k)^{2n-1-i}, \quad k = 1, 2.
\]

(5.32)

A large sample approximation to (5.32) is given by

\[
c_k \approx \Phi \left( \frac{\mu_k - (n - 0.5)}{\sigma_k} \right)
\]

(5.33)

where \( \Phi(.) \) is the standard normal df, \( \mu_k = (2n-1)p_k \), and \( \sigma_k = \sqrt{(2n-1)p_k(1-p_k)} \).

Before we prove the theorem, we will recall a useful result from the theory of Poisson processes. The following lemma is given by Ross (1985) on pp. 208-209.
Lemma 5.2.2 (Ross, 1985) Let \{N_i(t), \ t \geq 0\} be independent Poisson processes with rates \(\lambda_i, i=1,2\), respectively. Also, let \(X_{n,1}\) denote the time of the \(n\)th event of the first process, and \(X_{m,2}\) the time of the \(m\)th event of the second process. Then,

\[
P \{X_{n,1} < X_{m,2}\} = \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n+m-1-k}.
\]

(5.34)

We are now ready to prove Theorem 5.2.1.

Proof: (of Theorem 5.2.1) Recall that under the symmetric ACBVE distribution, \(T_1 \sim e(2\theta_1 + \theta_2),\ T_2 - T_1 \sim e(\theta_1 + \theta_2),\) and that \(T_1\) and \(T_2 - T_1\) are independent. Then, \(W_1 \sim \text{gamma}(n,2\theta_1 + \theta_2)\) is independent of \(W_2 - W_1 \sim \text{gamma}(n,\theta_1 + \theta_2)\) and \(\frac{W_2 - W_1}{2} \sim \text{gamma}(n,2(\theta_1 + \theta_2))\). This means that \(W_1\) is the waiting time for the \(n\)th arrival of a Poisson process with rate \(2\theta_1 + \theta_2\). Likewise, \(W_2 - W_1\) and \(\frac{W_2 - W_1}{2}\) may be viewed as waiting times for the \(n\)th arrival of a Poisson process with rates \(\theta_1 + \theta_2\) and \(2(\theta_1 + \theta_2)\), respectively.

The expression for the probability \(c_k\) is now apparent from Lemma 5.2.2. To find \(c_1\), we take \(X_{n,1} = W_1,\ X_{n,2} = \frac{W_2 - W_1}{2},\ \lambda_1 = 2\theta_1 + \theta_2,\) and \(\lambda_2 = 2\theta_1 + 2\theta_2\). To find \(c_2\), we take \(X_{n,1} = W_2 - W_1,\ X_{n,2} = W_1,\ \lambda_1 = \theta_1 + \theta_2,\) and \(\lambda_2 = 2\theta_1 + \theta_2\). This establishes (5.32).

To show that the approximation holds, notice that (5.32) may be written as

\[
c_k = 1 - P(Y_k \leq n - 1)
\]

(5.35)

where \(Y_k \sim \text{binomial}(2n-1,p_k)\). We complete the proof by applying the normal approximation to the binomial with continuity correction, and the identity \(\Phi(-x) = 1 - \Phi(x)\).

\(\Box\)
In order to have a grasp on just how big the probabilities $c_k$'s, can be, we present the computations in Table 2. The value of the sample size is fixed at $n = 10$. On examining this table, we find that $c_2$ may be as big as 34%, so that when $n = 10$, and $\gamma = 0.25$, about a third of the time the ‘MLE’ solution $\hat{\theta}_1$ in (5.31) is negative. We also see that $c_1$ may be as large as 40%. Thus when $n = 10$ and $\gamma = 4$, about two out of five times the ‘MLE’ solution $\hat{\theta}_2$ in (5.31) is negative. It is evident, then, that the solution (5.31) is not MLE for $(\theta_1, \theta_2) \in \mathbb{R}_2^+$. Table 2 also shows the remarkable accuracy of the normal approximation.

We will obtain the correct form of the MLE, next.

## 5.2.1 The MLE of the Parameters

We are now ready to derive the MLE of $\theta_1$ and $\theta_2$ in the symmetric ACBVE model.

**Theorem 5.2.3** Let $\overline{X}_i = (T_{1i}, T_{2i}), i = 1, \ldots, n$ have the likelihood function $L(\theta_1, \theta_2)$ given by (5.29) where $W_k = \sum_{i=1}^{n} T_{ki}$, $k=1,2$. Define $\hat{\theta}_1$ and $\hat{\theta}_2$ as in (5.31). Then,
the MLE of \( \theta = (\theta_1, \theta_2) \), denoted \( \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) \), is given by

\[
\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) = \begin{cases} 
(\theta_{c1}, 0), & \text{on } S_1 = \{0 < W_1 < \frac{W_2 - W_1}{2}\} \\
(\hat{\theta}_1, \hat{\theta}_2), & \text{on } S = \{\frac{W_2 - W_1}{2} < W_1 < W_2 - W_1\} \\
(0, \theta_{c2}), & \text{on } S_2 = \{W_2 - W_1 < W_1 < \infty\}
\end{cases}
\] (5.36)

where

\[ \theta_{c1} = \frac{2n}{W_1 + W_2} \text{ and } \theta_{c2} = \frac{2n}{W_2}. \] (5.37)

**Proof**: The solution \( (\hat{\theta}_1, \hat{\theta}_2) \) on \( S \) can be obtained easily by the usual differentiation method. Hence, we will not go into any further details.

On \( S_1 \) where the solution \( \hat{\theta}_1 \) is negative, we use the following general approach in maximizing \( L(\theta_1, \theta_2) \) wrt \( \theta \).

1. Fix \( \theta_1 \geq 0 \) in \( L \) and let \( L_2(\theta_2) = L(\theta_1, \theta_2) \). Then, we solve \( \frac{\partial \log L_2}{\partial \theta_2} = 0 \). This yields two solutions for \( \theta_2 \). We choose the larger solution, which we label \( \hat{\theta}_2 \).

(The smaller solution is always negative.) We find that \( \hat{\theta}_2 \geq 0 \) iff \( 0 < \theta_1 \leq \frac{3n}{2W_2} \equiv \theta_{c3} \). Since \( \frac{\partial^2 \log L_2}{\partial \theta_2^2} < 0 \), \( \forall \theta_2 \in \mathbb{R}^+ \), \( L_2 \) is maximized at \( \hat{\theta}_2 \).

2. Plug \( \hat{\theta}_2 = \hat{\theta}_2(\theta_1) \) into \( L \) and let \( L_1(\theta_1) = L(\theta_1, \hat{\theta}_2) \). Consider \( L_1 \) now as a function of \( \theta_1 \). We find that \( \frac{\partial \log L_1}{\partial \theta_1} \geq 0 \) in a region \( B_1 \supset (0, \theta_{c3}) \); that is, \( L_1 \) increases as \( \theta_1 \) increases in \( (0, \theta_{c3}) \). On putting \( \theta_1 = \theta_{c3} \) in \( \hat{\theta}_2(\theta_1) \), we find that \( \hat{\theta}_2 = 0 \). Hence, \( L(\theta_{c3}, 0) \geq L(\theta_1, \theta_2) \) on \( R_1 = \{(\theta_1, \theta_2) | 0 < \theta_1 \leq \theta_{c3}, \theta_2 \geq 0\} \).

3. On \( R_2 = \{(\theta_1, \theta_2) | \theta_{c3} < \theta_1 < \infty, \theta_2 \geq 0\} \), \( \frac{\partial \log L_2}{\partial \theta_2} < 0 \) for all \( \theta_2 > 0 \) and hence \( L(\theta_1, 0) \geq L(\theta_1, \theta_2) \). Now, we take \( \hat{\theta}_2(\theta_1) = 0 \) and note that \( L(\theta_1, 0) \) increases as
\( \theta_1 \) increases in \((\theta_{c1}, \theta_{c1})\) and then decreases. Thus, \( L(\theta_{c1}, 0) = L(\theta_{c1}, \hat{\theta}_2(\theta_{c1})) \geq L(\theta_1, \theta_2) \) for all \((\theta_1, \theta_2) \in \mathbb{R}_2^+ \). So, the MLE on \( S_1 \) is \( \hat{\theta} = (\theta_{c1}, 0) \).

When \((W_1, W_2) \in S_1\), Figure 21 shows the locus of points \((\theta_1, \hat{\theta}_2(\theta_1))\) for \((\theta_1, \theta_2) \in \mathbb{R}_2^+\).

We now present the details of this approach. Fixing \( \theta_1 \geq 0 \) in \( L \), we get the likelihood

\[
L_2(\theta_2) = (2\theta_1 + \theta_2)^n(\theta_1 + \theta_2)^n e^{-\theta_1(W_2+W_1)-\theta_2W_2}. \tag{5.38}
\]

The log likelihood corresponding to \( L_2 \) is

\[
\log L_2 = n \log (2\theta_1 + \theta_2) + n \log (\theta_1 + \theta_2) - \theta_1(W_2+W_1) - \theta_2W_2 \tag{5.39}
\]

and our goal is to maximize this wrt \( \theta_2 \). For this purpose, consider

\[
\frac{d \log L_2}{d \theta_2} = \frac{n}{2\theta_1 + \theta_2} + \frac{n}{\theta_1 + \theta_2} - W_2. \tag{5.40}
\]

On equating (5.40) to zero and solving for \( \theta_2 \), we obtain the solutions \( \left( \frac{n}{W_2} - \frac{3}{2} \theta_1 \right) \pm B_n^{1/2} \), where \( B_n = \frac{n}{W_2^2} + \frac{(\theta_1)^2}{4} \). Only the solution

\[
\hat{\theta}_2 = \left( \frac{n}{W_2} - \frac{3}{2} \theta_1 \right) + B_n^{1/2} \tag{5.41}
\]

may be non-negative, and \( \hat{\theta}_2 > 0 \) whenever \( 0 < \theta_1 \leq \theta_{c3} \).

Now, we take \( \hat{\theta}_2 \) and plug it back into \( L \). This yields the likelihood

\[
L_1(\theta_1) = \left( \frac{n}{W_2} + \frac{\theta_1}{2} + B_n^{1/2} \right)^n \left( \frac{n}{W_2} - \frac{\theta_1}{2} + B_n^{1/2} \right)^n e^{-n-\theta_1W_1+(\frac{1}{2}\theta_1-B_n^{1/2})W_2} \tag{5.42}
\]

and the log likelihood

\[
\log L_1 = n \log \left( \frac{n}{W_2} + \frac{\theta_1}{2} + B_n^{1/2} \right) + n \log \left( \frac{n}{W_2} - \frac{\theta_1}{2} + B_n^{1/2} \right) - n - \theta_1W_1 + (\frac{1}{2}\theta_1 - B_n^{1/2})W_2. \tag{5.43}
\]
Now, \( \log L_1 \) increases as function of \( \theta_1 \) whenever \( \theta_1 \in B_1 = \left[ 0, \frac{nW_2}{W_1(W_2-W_1)} - \frac{2n}{W_2-W_1} \right] \), where \( B_1 \) is obtained by solving the inequality \( \frac{\partial \log L_1}{\partial \theta_1} \geq 0 \) as an equivalent inequality in \( \theta_1 \). It is easy to show that, since \( \frac{nW_2}{W_1(W_2-W_1)} - \frac{2n}{W_2-W_1} > \frac{3n}{2W_2} \equiv \theta_{c_1} \) on \( S_1 \), \( B_1 \supset (0, \theta_{c_1}) \).

Thus, when \( \theta_1 \) is restricted to \((0, \theta_{c_1})\), \( L_1(\theta_1) \) is maximized at \( \theta_1 = \theta_{c_1} \). Plugging this back in the non-negative solution for \( \theta_2 \), we find that

\[
\hat{\theta}_2 = \frac{n}{W_2} - \frac{3n}{2W_2} + \frac{\sqrt{\frac{n^2}{W_2^2} + \frac{9n^2}{16W_2^2}}}{W_2} \\
= -\frac{5n}{4W_2} + \frac{n}{W_2} \sqrt{\frac{25}{16}} \\
= 0. \tag{5.44}
\]

So on \( R_1 \), \( L(\theta_{c_1}, 0) \geq L(\theta_1, \theta_2) \).

When \( \theta_1 > \theta_{c_1} \), \( L(\theta_1, 0) \geq L(\theta_1, \theta_2), \forall \theta_2 > 0 \). Now,

\[
L(\theta_1, 0) = (2\theta_1^n e^{-\theta_1(W_1+W_2)}) \tag{5.45}
\]

so that

\[
\frac{d \log L(\theta_1, 0)}{d \theta_1} = 2n \frac{\theta_1^n e^{-\theta_1(W_1+W_2)}}{W_1 + W_2}. \tag{5.46}
\]

Notice that \( \frac{\partial \log L(\theta_1, 0)}{\partial \theta_1} < (>) 0 \) depending on whether \( \theta_1 > (<) \theta_{c_1}. \) So, \( L(\theta_1, 0) \) increases as \( \theta_1 \) increases in \((\theta_{c_1}, \theta_{c_1}) \). We then have \( L(\theta_{c_1}, 0) \geq L(\theta_1, \theta_2) \) on \( R_2 \). Furthermore, \( L(\theta_{c_1}, 0) \geq L(\theta_{c_1}, 0) \forall (\theta_1, \theta_2) \in \mathbb{R}_2^+ \).

This yields the MLE solution \( \hat{\theta} = (\theta_{c_1}, 0) \) on \( S_1 \) as we see in (5.36) and (5.37).

We use a similar proof for maximizing \( L(\theta_1, \theta_2) \) wrt \( \theta \) on \( S_2 \), where now the solution \( \hat{\theta}_1 \) is negative. We outline this procedure as follows:

1. Fix \( \theta_2 \geq 0 \) in \( L \) and now let \( L_1(\theta_1) = L(\theta_1, \theta_2) \). Upon solving \( \frac{\partial \log L_1}{\partial \theta_1} = 0 \), we find two solutions. We choose the larger solution, which we denote \( \hat{\theta}_1 \). We show
that $\hat{\theta}_1 \geq 0$ iff $0 < \theta_2 \leq \frac{3n}{W_1 + W_2} \equiv \theta_{c4}$. Since $\frac{\partial^2 \log L_1}{\partial \theta_1^2} < 0$, $\forall \theta_1 \in \mathbb{R}^+$, $L_1$ is maximized at $\hat{\theta}_1$.

2. Plug $\hat{\theta}_1 = \hat{\theta}_1(\theta_2)$ into $L$ and let $L_2(\theta_2) = L(\hat{\theta}_1, \theta_2)$. Now, treat $L_2$ as a function of $\theta_2$. We find that $L_2$ increases in $\theta_2$, that is, $\frac{\partial \log L_2}{\partial \theta_2} \geq 0$ in a region $B_2 \supset (0, \theta_{c4})$.

On putting $\theta_2 = \theta_{c4}$ in $\hat{\theta}_1(\theta_2)$, we find that $\hat{\theta}_1 = 0$. So, $L(0, \theta_{c4}) \geq L(\theta_1, \theta_2)$ on $R_3 = \{(\theta_1, \theta_2)|\theta_1 \geq 0, 0 < \theta_2 \leq \theta_{c4}\}$.

3. On $R_4 = \{(\theta_1, \theta_2)|\theta_1 \geq 0, \theta_{c4} < \theta_2 < \infty\}$, $\frac{\partial \log L_1}{\partial \theta_1} < 0$ for all $\theta_1 > 0$ and hence $L(0, \theta_2) \geq L(\theta_1, \theta_2)$. Next, we note that $L(0, \theta_2)$ increases in $(\theta_{c4}, \theta_{c2})$ and then decreases. Thus, $L(0, \theta_{c2}) \geq L(\theta_1, \theta_2)$, $\forall (\theta_1, \theta_2) \in \mathbb{R}_2^+$.

Let us now look at some details. If we fix $\theta_2 > 0$ in $L$, we get the following log likelihood corresponding to $L_1$:

$$\log L_1(\theta_1) = n \log (2\theta_1 + \theta_2) + n \log (\theta_1 + \theta_2) - \theta_1(W_2 + W_1) - \theta_2W_2.$$  \hfill (5.47)

The solutions obtained by equating $\frac{\partial \log L_1}{\partial \theta_1}$ to zero are $\left(\frac{n}{W_1 + W_2} - \frac{3}{4}\theta_2\right) \pm A_n^{1/2}$, where $A_n = \frac{n^2}{(W_1 + W_2)^2} + \frac{(\theta_2)^2}{16}$. Only one solution given by

$$\hat{\theta}_1 = \left(\frac{n}{W_1 + W_2} - \frac{3}{4}\theta_2\right) + A_n^{1/2}$$ \hfill (5.48)

may be non-negative, and $\hat{\theta}_1 > 0$ when $0 < \theta_2 \leq \theta_{c4}$.

Plugging this solution $\hat{\theta}_1$ now into $L$, we get the likelihood

$$L_2(\theta_2) = \left(\frac{2n}{W_1 + W_2} - \frac{\theta_2}{2} + 2A_n^{1/2}\right)^n \left(\frac{n}{W_1 + W_2} + \frac{\theta_2}{4} + A_n^{1/2}\right)^n e^{-n + (\frac{3}{4}\theta_2 - A_n^{1/2})(W_2 + W_1) - \theta_2W_2}.$$ \hfill (5.49)
Now, as a function of $\theta_2$, $L_2$ or equivalently $\log L_2$ increases in the region where \( \frac{\partial \log L_2}{\partial \theta_2} \geq 0 \). This yields the region $B_2 = \left[ 0, \frac{3n}{W_2 - W_1} - \frac{nW_2}{W_1(W_2 - W_1)} \right]$. On $S_2$, we can show that $\frac{3n}{W_2 - W_1} - \frac{nW_2}{W_1(W_2 - W_1)} > \frac{3n}{W_2 + W_1}$, which implies $B_2 \supset (0, \theta_{c4})$.

When $\theta_2$ is restricted to $(0, \theta_{c4})$ $L_2(\theta_2)$ is maximized at $\theta_2 = \theta_{c4}$. Thus, $\hat{\theta}_1(\theta_2)$ becomes

$$
\hat{\theta}_1 = \frac{n}{W_1 + W_2} - \frac{3}{4} \frac{3n}{W_1 + W_2} + \frac{n^2}{16(W_1 + W_2)^2} \sqrt{(W_1 + W_2)^2 + \frac{9n^2}{16(W_1 + W_2)^2}}
$$

$$
= -\frac{5n}{4W_2 + W_1} + \frac{n}{W_2 + W_1} \sqrt{\frac{25}{16}}
$$

$$
= 0.
$$

(5.50)

So, on $R_3$, $L(0, \theta_{c4}) \geq L(\theta_1, \theta_2)$.

When $\theta_2 > \theta_{c4}$, $L(0, \theta_2) \geq L(\theta_1, \theta_2)$, $\forall \theta_1 > 0$. Since

$$
L(0, \theta_2) = (\theta_2)^{2n}e^{-\theta_2W_2}
$$

(5.51)

then

$$
\frac{\partial \log L(0, \theta_2)}{\partial \theta_2} = \frac{2n}{\theta_2} - W_2.
$$

(5.52)

Notice that $\frac{\partial \log L(0, \theta_2)}{\partial \theta_2} < (>)$ according as $\theta_2 > ( \leq ) \frac{2n}{W_2}$. So, $L(0, \theta_2)$ increases as $\theta_2$ increases in $(\theta_{c4}, \theta_{c2})$. We then have $L(0, \theta_{c2}) \geq L(\theta_1, \theta_2)$ on $R_4$. Moreover, $L(0, \theta_{c2}) \geq L(0, \theta_{c4})$ and hence $L(0, \theta_{c2}) \geq L(\theta_1, \theta_2)$ on $\mathbb{R}_2^+$. Thus, the MLE on $S_2$ is $\hat{\theta} = (0, \theta_{c2})$. This establishes (5.36) and (5.37). □

5.2.2 Two Predictors of the Maximum

The previous discussions pave the way for us to deal with the problem of predicting the maximum in the exchangeable ACBVE set-up. From (5.27), we propose the
conditional mean predictor

\[ \hat{T}_2 = T_1 + \frac{1}{\hat{\theta}_1 + \hat{\theta}_2} \]  \hspace{1cm} (5.53)

and from (5.28), we propose the conditional median predictor

\[ \hat{T}_{2M} = T_1 + \frac{\log 2}{\hat{\theta}_1 + \hat{\theta}_2} \]  \hspace{1cm} (5.54)

where \( \hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2) \) is given by (5.36) and (5.37).

**Theorem 5.2.4** Let \( \bar{T}_i = (T_{1i}, T_{2i}), i = 1, \ldots, n \) be a random sample from the pdf (5.25). Suppose that the minimum lifetime, \( T_1 \), of the \( (n+1) \)st sample has just been observed. Define \( W_k = \sum_{i=1}^{n} T_{ki}, k = 1, 2 \). Then, the conditional mean predictor and the conditional median predictor of the maximum, \( T_2 \), are given by

\[ \hat{T}_2 = T_1 + h(W_1, W_2) \]

and

\[ \hat{T}_{2M} = T_1 + (\log 2)h(W_1, W_2), \]  \hspace{1cm} (5.55)

where

\[ h(W_1, W_2) = \begin{cases} 
\frac{W_1 + W_2}{2n} & \text{on } S_1 = \{0 < W_1 < \frac{W_2 - W_1}{2}\} \\
\frac{W_2 - W_1}{n} & \text{on } S = \{\frac{W_2 - W_1}{2} < W_1 < W_2 - W_1\} \\
\frac{W_2}{2n} & \text{on } S_2 = \{W_2 - W_1 < W_1 < \infty\} 
\end{cases} \]  \hspace{1cm} (5.56)

**Proof:** Evaluate \( \hat{\theta}_1 + \hat{\theta}_2 \) from Theorem 5.2.3 and plug into (5.53) and (5.54).

Then, the result follows. \( \square \)

In order to assess how well the predictors \( \hat{T}_2 \) and \( \hat{T}_{2M} \) predict \( T_2 \), we will compute their MSE's. We begin by collecting necessary tools for that purpose.
The integrals involved in the expressions for the MSE's are linear combinations of integrals of the form

\[
I = \int_0^\infty \int_{c_1}^{c_2} u^k z^\ell f_U(u) f_Z(z) \, dz \, du
\]

(5.57)

where \( k, \ell \geq 0 \) and \( 0 \leq c_1 < c_2 \leq \infty \) and \( f_U(u) \) and \( f_Z(z) \) are gamma\((n,\alpha_1)\) and gamma\((n,\alpha_2)\) pdf's, respectively. We now evaluate \( I \) in terms of binomial probability mass functions in the lemma below. First, we recall two useful basic results.

1. The df of a gamma distributed r.v. with parameters \( r \) and \( \lambda \) is given by

\[
F(x|\lambda) = 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^j}{j!}
\]

(5.58)

(See, for example, Mood, Graybill, and Boes (1974), p.114)

2.

\[
\sum_{j=0}^{n-1} \binom{n+j-1}{j} p^n(1-p)^j = \sum_{j=n}^{2n-1} \binom{2n-1}{j} p^j(1-p)^{2n-1-j}
\]

(5.59)

that is, the chance that the \( n \)th success occurs on the \( (2n-1) \)th trial or earlier in a geometric experiment with success probability \( p \) is the same as the chance of observing at least \( n \) successes in a binomial experiment with \( (2n-1) \) trials and success probability \( p \).

(See, for example, Nagaraja and Chan (1989)).

**Lemma 5.2.5** For \( k, \ell \geq 0 \), \( 0 \leq c_1 < c_2 \leq \infty \), the integral \( I \) in (5.57) can be expressed as

\[
I(c_1, c_2) = \sum_{j=n+k}^{2n+\ell+k-1} \left\{ B(2n + \ell + k - 1, \beta_1, j) - B(2n + \ell + k - 1, \beta_2, j) \right\}
\]

(5.60)
where \( d = \frac{\Gamma(n+\ell)\Gamma(n+k)}{\Gamma(n)\Gamma(n)} (\alpha_1)^k(\alpha_2)^j, \beta_k = \frac{\alpha_1}{\alpha_1+\alpha_2c}, k=1,2, \) and \( B(n,p,j) = \binom{n}{j} p^j (1 - p)^{n-j}, j = 0, 1, \ldots, n. \)

**Proof:** First note that \( I(c_1,c_2) = I(c_1,\infty) - I(c_2,\infty). \) Now,

\[
I(c,\infty) = \int_0^\infty u^k f_U(u) \left( \int_0^\infty z^\ell f_G(z) \, dz \right) \, du
\]

\[
= \int_0^\infty u^k f_U(u) \frac{\Gamma(n+\ell)}{\alpha_2^j} \left( \sum_{j=0}^{n+\ell-1} \frac{e^{-\alpha_2cu} [\alpha_2cu]^j}{j!} \right) \, du
\]

on using (5.58). Hence,

\[
I(c,\infty) = \sum_{j=0}^{n+\ell-1} \frac{\alpha_1^n \Gamma(n+\ell) (\alpha_2c)^j \Gamma(n+k+j)}{\Gamma(n) \alpha_2^j} \frac{\Gamma(n+k+j)}{\alpha_1 (\alpha_1+\alpha_2c)^{n+k+j}}
\]

\[
= d \sum_{j=0}^{n+\ell-1} \binom{n+k+j-1}{j} \left( \frac{\alpha_1}{\alpha_1+\alpha_2c} \right)^{n+k} \left( \frac{\alpha_2c}{\alpha_1+\alpha_2c} \right)^j
\]

\[
= d \sum_{j=0}^{n+\ell-1} \binom{n+k+j-1}{j} \beta^{n+1}(1 - \beta)^j,
\]

where \( \beta = \beta(\alpha_1, \alpha_2, c) = \frac{\alpha_1}{\alpha_1+\alpha_2c}. \) Thus, in view of (5.59), we have

\[
I(c,\infty) = d \sum_{j=n+k}^{2n+\ell+k-1} \binom{2n+\ell+k-1}{j} \beta^j (1 - \beta)^{2n+\ell+k-j-1}
\]

\[
= d \sum_{j=n+k}^{2n+\ell+k-1} B(2n+\ell+k-1, \beta, j).
\]

Consequently, (5.60) holds. □

Notice that \( \beta_1 \to 1 \iff c_1 \to 0 \) and \( \beta_2 \to 0 \iff c_2 \to \infty. \) It follows that

\[
\lim_{c_1 \to 0} \sum_{j=n+k}^{2n+\ell+k-1} B(2n+\ell+k-1, \beta_1, j) = 1
\]

and

\[
\lim_{c_2 \to \infty} \sum_{j=n+k}^{2n+\ell+k-1} B(2n+\ell+k-1, \beta_2, j) = 0.
\]
We now are ready to give closed form expressions for the MSE’s of the two predictors.

**Theorem 5.2.6** Let \( B(n, p, j) = \binom{n}{j} p^j (1 - p)^{n-j}, j = 0, 1, \ldots, n. \) Define \( r_1 = \frac{2\gamma + 2}{4\gamma + 3} \) and \( r_2 = \frac{2\gamma + 1}{3\gamma + 2} \), where \( \gamma = \frac{\theta_1}{\theta_2} \). Given a random sample of size \((n+1)\) from the pdf (5.25), where \( n > 0 \), and only \( T_1 \) has been observed in the \((n+1)\)st sample,

\[
MSE(\hat{T}_2) = \frac{1}{(\theta_1 + \theta_2)^2} + E[h^2(W_1, W_2)] - \frac{2}{\theta_1 + \theta_2} E[h(W_1, W_2)]
\]

and

\[
MSE(\hat{T}_{2M}) = \frac{1}{(\theta_1 + \theta_2)^2} + (\log 2)^2 E[h^2(W_1, W_2)] - \frac{2(\log 2)}{\theta_1 + \theta_2} E[h(W_1, W_2)]
\]

(5.66)

where \( h(W_1, W_2) \) is as defined in (5.56). Further,

\[
E[h(W_1, W_2)] = \frac{1}{2(\theta_1 + \theta_2)} \sum_{j=n}^{2n} B(2n, 1 - r_1, j)
\]

\[
+ \frac{1}{(2\theta_1 + \theta_2)} \sum_{j=n+1}^{2n} B(2n, 1 - r_1, j)
\]

\[
+ \frac{1}{(\theta_1 + \theta_2)} \sum_{j=n+1}^{2n} B(2n, r_1, j)
\]

\[
- \frac{1}{2(\theta_1 + \theta_2)} \sum_{j=n+1}^{2n} B(2n, 1 - r_2, j)
\]

\[
+ \frac{1}{2(2\theta_1 + \theta_2)} \sum_{j=n}^{2n} B(2n, 1 - r_2, j)
\]

(5.67)

and

\[
E[h^2(W_1, W_2)] = \frac{(n + 1)}{4n(\theta_1 + \theta_2)^2} \sum_{j=n}^{2n+1} B(2n + 1, 1 - r_1, j)
\]

\[
+ \frac{1}{(\theta_1 + \theta_2)(2\theta_1 + \theta_2)} \sum_{j=n+1}^{2n+1} B(2n + 1, 1 - r_1, j)
\]
\[ \begin{align*}
+ \frac{(n+1)}{n(\theta_1 + \theta_2)^2} \sum_{j=n+2}^{2n+1} B(2n+1, 1-r_1, j) \\
+ \frac{(n+1)}{n(\theta_1 + \theta_2)^2} \sum_{j=n+2}^{2n+1} B(2n+1, r_1, j) \\
- \frac{3(n+1)}{4n(\theta_1 + \theta_2)^2} \sum_{j=n+2}^{2n+1} B(2n+1, 1-r_2, j) \\
+ \frac{(n+1)}{4n(2\theta_1 + \theta_2)^2} \sum_{j=n}^{2n+1} B(2n+1, 1-r_2, j) \\
+ \frac{1}{2(2\theta_1 + \theta_2)(\theta_1 + \theta_2)} \sum_{j=n+1}^{2n+1} B(2n+1, 1-r_2, j). \quad (5.68)
\end{align*} \]

**Proof:** From (5.14) and (5.55),

\[ \text{MSE}(\hat{T}_2) = E(\hat{T}_2 - T_2)^2 \]

\[ = E(T_1 - T_2)^2 + E[h^2(W_1, W_2)] + 2E(T_1 - T_2)E[h(W_1, W_2)] \quad (5.69) \]

and

\[ \text{MSE}(\hat{T}_{2m}) = E(\hat{T}_{2m} - T_2)^2 \]

\[ = E(T_1 - T_2)^2 + (\log 2)^2 E[h^2(W_1, W_2)] \]

\[ + 2(\log 2)E(T_1 - T_2)E[h(W_1, W_2)]. \quad (5.70) \]

Upon noting that $T_2-T_1 \sim e(\theta_1 + \theta_2)$, we obtain $E(T_1-T_2) = -\frac{1}{\theta_1+\theta_2}$ and $E(T_1-T_2)^2 = \frac{1}{(\theta_1+\theta_2)^2}$. From here, the given MSE expressions follow immediately.

Let us now evaluate the first two moments of $h(W_1, W_2)$. For this purpose, we define $U = W_2 - W_1$, and $Z = W_1$. Then, we can express $h(W_1, W_2)$ as

\[ h(U, U + Z) = \begin{cases} 
\frac{(U+Z)}{2n}, & \text{on } S_1 = \{0 < Z < \frac{U}{2}\} \\
\frac{U}{n}, & \text{on } S_2 = \{\frac{U}{2} < Z < U\} \\
\frac{(U+Z)}{2n}, & \text{on } S_2 = \{U < Z < \infty\}. \quad (5.71) 
\end{cases} \]
Now, $U$ and $Z$ are independent, $U \sim \text{gamma}(n, \alpha_1)$, and $Z \sim \text{gamma}(n, \alpha_2)$, where $\alpha_1 = \theta_1 + \theta_2$ and $\alpha_2 = 2\theta_1 + \theta_2$. If we write the pdf of $U$ as $f_U(u)$ and the pdf of $Z$ as $f_Z(z)$, respectively, then

\begin{align*}
E[h(W_1, W_2)] &= \int_0^{u/2} \int_0^{u/2} \frac{(u + 2z)^2}{2n} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{u/2} \int_{u/2}^{\infty} \frac{u}{2n} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{\infty} \int_{u/2}^{\infty} \frac{(u + z)^2}{2n} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{\infty} \int_0^{u/2} \frac{(u + z)^2}{4n^2} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{\infty} \int_0^{u/2} \frac{(u + z)^2}{4n^2} f_U(u)f_Z(z) \, dz \, du. 
\end{align*}

(5.72)

and

\begin{align*}
E[h^2(W_1, W_2)] &= \int_0^{u/2} \int_0^{u/2} \frac{(u + 2z)^2}{4n^2} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{u/2} \int_{u/2}^{\infty} \frac{u^2}{4n^2} f_U(u)f_Z(z) \, dz \, du \\
&\quad + \int_0^{\infty} \int_{u/2}^{\infty} \frac{(u + z)^2}{4n^2} f_U(u)f_Z(z) \, dz \, du. 
\end{align*}

(5.73)

Each of the six integrals on the RHS of (5.72) and (5.73) can be expressed as a linear combination of integrals of the type 1 given in (5.57). Thus, we can use Lemma 5.2.5 to obtain closed form expressions for $E[h(W_1, W_2)]$ and $E[h^2(W_1, W_2)]$. This leads to the expressions claimed in Theorem 5.2.6. □

Table 3 gives the value of the ratio $R = \frac{\text{MSE}(\hat{T}_{2M})}{\text{MSE}(\hat{T}_2)}$ for various combinations of values of $n$, $\theta_1$, and $\theta_2$. Our computations show that for the combination of parameter values considered, the conditional median predictor, $\hat{T}_{2M}$, is much more efficient than the conditional mean predictor, $\hat{T}_2$, in predicting the maximum in the symmetric ACBVE set-up.
Table 3: Relative efficiency of the conditional median predictor, $T_{2M}$, with respect to the conditional mean predictor, $T_2$.

<table>
<thead>
<tr>
<th>n</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$R$</th>
</tr>
</thead>
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<tr>
<td>10</td>
<td>0.5</td>
<td>1.5</td>
<td>0.66020</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.0</td>
<td>0.56125</td>
</tr>
<tr>
<td>10</td>
<td>1.5</td>
<td>0.5</td>
<td>0.52096</td>
</tr>
<tr>
<td>15</td>
<td>1.25</td>
<td>0.25</td>
<td>0.56679</td>
</tr>
<tr>
<td>20</td>
<td>1.0</td>
<td>1.0</td>
<td>0.58327</td>
</tr>
<tr>
<td>20</td>
<td>1.25</td>
<td>4.25</td>
<td>0.49341</td>
</tr>
</tbody>
</table>
Figure 21: The locus of points $(\theta_1, \tilde{\theta}_2(\theta_1))$ such that $L(\theta_1, \tilde{\theta}_2(\theta_1)) = \max_{\theta_2 \geq 0} L(\theta_1, \theta_2)$. 

\[ \theta_{c3} = \frac{3n}{2w_2} \quad \theta_{c1} = \frac{2n}{w_1 + w_2} \]
In this dissertation, we have examined some properties of order statistics from bivariate exponential distributions. We found a variety of patterns of marginal and joint behaviors that present this collection as a class of diverse modelling tools. We also noted instances of divergence with properties of order statistics from independent pairs of exponential rv's.

We began by presenting a survey of common bivariate extensions of the exponential distribution. We identified which of the LMP, BLMP, and absolute continuity properties these distributions possess. We discussed the physical and mathematical motivations for each, and examined their interrelationships. Also, known technical results for some of these distributions were compiled in the appendix. We completed the survey work in chapter 1 with a discussion of generalized hyperexponential distributions.

In chapter 2, we studied the dependence structure of some bivariate exponential models from a different perspective. We showed plots of the surfaces $c(u,v) - uv$ showing the difference between the copula functions of dependent versus independent pairs of variables. We have found that this discrepancy, though present, is not substantial. We also verified that the BVE, ACBVE, and GBVE are positively quadrant
dependent. On the other hand, the FBVE and RBVE may be PQD, NQD, or neither PQD nor NQD depending on the choice of parameter values.

We studied the distributional properties of order statistics from seven bivariate exponential distributions in chapter 3. These distributions are the BVE, ACBVE, FBVE, GBVE, RBVE, ACBVE$_2$, and BEE. We gave expressions for the joint survival df and the joint pdf whenever it exists, the marginal survival df’s, and moments and correlation. We found that the marginal distributions of the order statistics are predominantly GH distributions. In general, however, these are generalized mixtures of gamma and exponential distributions. We also derived the conditional distribution, regression function and the distribution of the spacing for the models that have the BLMP. We found that for these distributions, $E(T_2 - T_1|T_1 = t_1)$ is constant. We plotted the discrepancy $c_o(u, v) - c(u, v)$ between the copula of the order statistics and the unordered pair. In the independent set-up, the surface $c_o(u, v) - c(u, v)$ is a mound that always lies above the plane $z = 0$. We noticed that plots from the dependent set-ups exhibit a variety of patterns that deviate from this, including one that dips below the plane $z = 0$.

In chapter 4, we examined the reliability properties of the order statistics. The BVE, ACBVE, GBVE, and ACBVE$_2$ have order statistics that behave like those from independent exponentials. Their minima have constant failure rates and their maxima are IFRA. On the other hand, the FBVE, and the BEE have minima with constant failure rates but the maxima may or may not be IFRA. The parsimonious RBVE models have maxima that are IFR, but minima that may or may not have
monotone failure rate. We also proved that $T_2 \geq T_1$ for distributions with BLMP, the GBVE, and the RBVE.

We proposed two predictors for the maximum in the exchangeable ACBVE and FBVE set-ups in chapter 5. The data consists of $n$ complete samples and the minimum of the current sample. We found that for both distributions, the conditional median predictor has smaller MSE than the conditional mean predictor. We also identified an error in the literature in the computation of the MLE of the parameters of the symmetric ACBVE distribution. We presented the correct form of the MLE's.

As we conclude this work, several interesting problems remain. Particularly engaging are the following:

1. Numerically computing the volume of the surfaces $c(u, v) = uv$ as a measure of dependence.

2. Proving the necessity of the property $E(T_2 - T_1 | T_1 = t_1) = c$ for some $c > 0$ of distributions with absolutely continuous marginals and BLMP, hence establishing it as a characterizing property.

3. Predicting the maximum in a Type I censored set-up when the order statistics come from the ACBVE and FBVE distributions.

4. Extending the results on GH distributions studied in section 4.1 to mixtures of four or more exponential components, and

5. Exploring the properties of order statistics from multivariate exponential distributions.
An exploration of these problems promises to be an exciting task.
Appendix A

Bivariate Exponential Distributions - Technical Details

A.1 Marshall and Olkin’s BVE

Let \((X_1, X_2) \sim BVE(\lambda_1, \lambda_2, \lambda_{12})\) and let \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\).

Joint survival df

\[
F(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)}, \quad x_1, x_2 > 0
\]  
(A.1)

Marginal distributions

\(X_i \sim e(\lambda_i + \lambda_{12}), \ i=1,2\). Hence,

\[
F_i(x_i) = e^{-(\lambda_i + \lambda_{12})x_i}, \quad x_i > 0
\]  
(A.2)

and

\[
F_2(x_2) = e^{-(\lambda_2 + \lambda_{12})x_2}, \quad x_2 > 0
\]  
(A.3)

Moments and Correlation

\[
E(X_1) = \frac{1}{\lambda_1 + \lambda_{12}}, \quad \text{Var}(X_1) = \frac{1}{(\lambda_1 + \lambda_{12})^2}
\]  
(A.4)

\[
E(X_2) = \frac{1}{\lambda_2 + \lambda_{12}}, \quad \text{Var}(X_2) = \frac{1}{(\lambda_2 + \lambda_{12})^2}
\]  
(A.5)
A.2 Block and Basu’s ACBVE

Let \((X_1, X_2) \sim ACBVE(\lambda_1, \lambda_2, \lambda_{12})\) and let \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12}\).

Joint survival df

\[
F(x_1, x_2) = \frac{\lambda}{\lambda_1 + \lambda_2} e^{-\lambda_1 x_1 - \lambda_2 x_2} - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} e^{-\max(x_1, x_2)}, \quad x_1, x_2 > 0
\]

Joint pdf

\[
f(x_1, x_2) = \begin{cases} 
\frac{\lambda_1 \lambda_2 \lambda_{12}}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2 + \lambda_{12})x_2} , & 0 < x_1 < x_2 \\
\frac{\lambda_2 \lambda_1 \lambda_{12}}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2 + \lambda_{12})x_1} , & 0 < x_2 < x_1 
\end{cases}
\]

Marginal distributions

\[
F_1(x_1) = \frac{\lambda}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_{12})x_1} - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} e^{-\lambda x_1}, \quad x_1 > 0
\]

\[
F_2(x_2) = \frac{\lambda}{\lambda_1 + \lambda_2} e^{-(\lambda_2 + \lambda_{12})x_2} - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} e^{-\lambda x_2}, \quad x_2 > 0
\]

Moments and Correlation

\[
E(X_1) = \frac{1}{\lambda_1 + \lambda_{12}} + \frac{\lambda_{12} \lambda_2}{\lambda_1 + \lambda_{12}(\lambda_1 + \lambda_2)}
\]

\[
\text{Var}(X_1) = \frac{1}{(\lambda_1 + \lambda_{12})^2} + \frac{\lambda_{12} \lambda_2 (2\lambda_1 \lambda + \lambda_2 \lambda_{12})}{(\lambda)^2(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})^2}
\]

\[
E(X_2) = \frac{1}{\lambda_2 + \lambda_{12}} + \frac{\lambda_{12} \lambda_1}{\lambda_1 + \lambda_2(\lambda_2 + \lambda_{12})}
\]
\[
\begin{align*}
\text{Var}(X_2) &= \frac{1}{(\lambda_2 + \lambda_{12})^2} + \frac{\lambda_{12}\lambda_1(2\lambda_2\lambda + \lambda_1\lambda_{12})}{(\lambda^2)(\lambda_1 + \lambda_2)^2(\lambda_1 + \lambda_{12})^2} \quad (A.16) \\
\text{Cov}(X_1, X_2) &= \frac{(\lambda_1^2 + \lambda_2^2)\lambda_{12}\lambda + \lambda_1\lambda_2\lambda_{12}}{\lambda^2(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})(\lambda_1 + \lambda_2)^2} \quad (A.17) \\
\text{Corr}(X_1, X_2) &= \frac{(\lambda_1^2 + \lambda_2^2)\lambda_{12}\lambda + \lambda_1\lambda_2\lambda_{12}}{\sqrt{[(\lambda_1 + \lambda_{12})^2\xi^2 + \lambda_2\lambda^2(\xi + \lambda_1)][(\lambda_2 + \lambda_{12})^2\xi^2 + \lambda_1\lambda^2(\xi + \lambda_2)]}} \quad (A.18)
\end{align*}
\]

where \( \xi = \lambda_1 + \lambda_2 \).

### A.3 Freund’s FBVE

Let \((X_1, X_2) \sim \text{FBVE}(\alpha, \beta, \alpha', \beta')\) and let \( \zeta = \alpha + \beta \).

**Joint survival df**

\[
\bar{F}(x_1, x_2) = \begin{cases} 
\alpha \beta e^{-\beta x_2 - (\zeta - \beta')x_1} + \frac{\alpha}{\zeta - \alpha'} e^{-\alpha' x_1 - (\zeta - \alpha')x_2}, & 0 \leq x_1 \leq x_2 \\
\alpha \beta e^{-\beta (x_1 - x_2)} + \frac{\alpha}{\zeta - \alpha'} e^{-\alpha' (x_1 - x_2)} \left( 1 - \frac{x_1}{\alpha'} \right), & x_1 \leq x_2 \end{cases} \quad (A.19)
\]

provided \( \zeta \neq \alpha', \beta' \).

**Joint pdf**

\[
f(x_1, x_2) = \begin{cases} 
\alpha \beta e^{-\beta x_2 - (\zeta - \beta')x_1}, & 0 \leq x_1 \leq x_2 \\
\frac{\alpha}{\zeta - \alpha'} e^{-\alpha' x_1 - (\zeta - \alpha')x_2}, & 0 \leq x_2 < x_1 \end{cases} \quad (A.20)
\]

provided \( \zeta \neq \alpha', \beta' \).

**Marginal distributions**

\[
f_1(x_1) = \frac{(\alpha - \alpha') \zeta e^{-\zeta x_1}}{\zeta - \alpha'} + \frac{\beta \alpha' e^{-\alpha' x_1}}{\zeta - \alpha'}, \quad 0 < x_1 < \infty \quad (A.21)
\]

provided \( \zeta \neq \alpha' \).

\[
f_2(x_2) = \frac{(\beta - \beta') \zeta e^{-\zeta x_2}}{\zeta - \beta'} + \frac{\alpha \beta' e^{-\beta' x_2}}{\zeta - \beta'}, \quad 0 < x_2 < \infty \quad (A.22)
\]

provided \( \zeta \neq \beta' \).
Moments and Correlation

\[ E(X_1) = \frac{\alpha' + \beta}{\alpha' \zeta} \quad \text{Var}(X_1) = \frac{(\alpha')^2 + 2\alpha\beta + \beta^2}{(\alpha')^2(\zeta)^2} \]  
(A.23)

\[ E(X_2) = \frac{\beta' + \alpha}{\beta' \zeta} \quad \text{Var}(X_2) = \frac{(\beta')^2 + 2\alpha\beta + \alpha^2}{(\beta')^2(\zeta)^2} \]  
(A.24)

\[ E(X_1X_2) = \frac{2\alpha'\beta' + \alpha\alpha' + \beta\beta'}{\alpha'\beta'(\zeta)^2} \]  
(A.25)

\[ \text{Cov}(X_1, X_2) = \frac{\alpha'\beta' - \alpha\beta}{\alpha'\beta'(\zeta)^2} \]  
(A.26)

\[ \text{Corr}(X_1, X_2) = \frac{\alpha'\beta' - \alpha\beta}{\sqrt{[(\alpha')^2 + 2\alpha\beta + \beta^2][(\beta')^2 + 2\alpha\beta + \alpha^2]}} \]  
(A.27)

### A.4 Gumbel's GBVE

Let \((X_1, X_2) \sim \text{GBVE}(\theta_1, \theta_2, \delta)\) and let \(a(x_1, x_2) = \left(\frac{x_1}{\theta_1}\right)^{\frac{1}{\delta}} + \left(\frac{x_2}{\theta_2}\right)^{\frac{1}{\gamma}}\).

Joint survival df

\[ F(x_1, x_2) = e^{-[a(x_1, x_2)]^\delta}, \quad 0 < x_1, x_2 < \infty \]  
(A.28)

Joint pdf

\[ f(x_1, x_2) = (\theta_1\theta_2)^{-\frac{1}{\delta}}(x_1x_2)^{\frac{1}{\delta}-1} [a(x_1, x_2)]^{\delta-2} \left\{ a(x_1, x_2) \right\}^{\delta} \left[ \frac{1}{\delta} - 1 \right] e^{-[a(x_1, x_2)]^\delta} \]  
(A.29)

Marginal distributions

\(X_i \sim \text{e}(\frac{1}{\theta_i}), i=1,2.\) Hence,

\[ F_1(x_1) = e^{-\frac{x_1}{\theta_1}}, \quad x_1 > 0 \]  
(A.30)

\[ F_2(x_2) = e^{-\frac{x_2}{\theta_2}}, \quad x_2 > 0 \]  
(A.31)
Moments and Correlation

\[ E(X_1) = \theta_1 \quad \text{Var}(X_1) = \theta_1^2 \]  
\[ E(X_2) = \theta_2 \quad \text{Var}(X_2) = \theta_2^2 \]  
\[ E(X_1 X_2) = \frac{2\theta_1 \theta_2 \Gamma^2(\delta + 1)}{\Gamma(2\delta + 1)} \]  
\[ \text{Corr}(X_1, X_2) = \frac{2\Gamma^2(\delta + 1)}{\Gamma(2\delta + 1)} - 1 \]

A.5 Raftery's RBVE

Let \((X_1, X_2) \sim (\lambda, \pi_1, \pi_2, p_{11})\). Let \(x(1) = \min(x_1, x_2), x(2) = \max(x_1, x_2),\) and \(\pi(i) = \pi_j\), if \(x(i) = x_j, i,j = 1,2\).

Joint survival df

\[ F(x_1, x_2) = \sum_{i,j=0}^{1} p_{ij} F_{ij}(x_1, x_2) \]

where

\[ F_{00}(x_1, x_2) = e^{-\frac{\lambda x_1}{\pi_1}} e^{-\frac{\lambda x_2}{\pi_1}} \]

\[ F_{01}(x_1, x_2) = e^{-\frac{\lambda x_1}{\pi_1}} \frac{1 - \pi_2}{\pi_2} - \frac{1 - \pi_2}{\pi_2} e^{-\frac{\lambda x_2}{\pi_1}} \]

\[ F_{10}(x_1, x_2) = e^{-\frac{\lambda x_2}{\pi_2}} \frac{1 - \pi_1}{\pi_1} - \frac{1 - \pi_1}{\pi_1} e^{-\frac{\lambda x_1}{\pi_2}} \]

and

\[ F_{11}(x_1, x_2) = \frac{e^{-\lambda x_2}}{\pi_2} - \frac{(1 - \pi_2)^2}{\pi_2(1 - \pi_1 \pi_2)} e^{-\frac{\lambda x_1}{\pi_1}} e^{-\frac{\lambda x_2}{\pi_1}} \]

\[ - \frac{(1 - \pi_1)(1 - \pi_2)}{1 - \pi_1 \pi_2} e^{-\frac{\lambda x_1}{\pi_2}} e^{-\frac{\lambda x_2}{\pi_2}} \]
Joint pdf

\[ f(x_1, x_2) = \lambda^2 \{ a_1 g_1 g_2 + a_2 g_1 (g_4 - g_2) + a_3 g_2 (g_3 - g_1) + p_{11} g_5 \} \]  

(A.41)

where

\[ a_1 = \frac{p_{00}}{(1-\pi_1)(1-\pi_2)}, \quad a_2 = \frac{p_{01}}{(1-\pi_1)\pi_2}, \quad a_3 = \frac{p_{10}}{\pi_1(1-\pi_2)}, \]

\[ g_i = e^{-\frac{\lambda x_i}{1-\pi_i}}, \quad i = 1, 2, \quad g_3 = e^{-\lambda x_1}, \quad g_4 = e^{-\lambda x_2} \]  

(A.42)

Marginal distributions

\[ X_i \sim \mathcal{E}(\lambda), \quad i = 1, 2. \]  

Hence,

\[ \bar{F}_1(x_1) = e^{-\lambda x_1}, \quad x_1 > 0 \]  

(A.43)

\[ \bar{F}_2(x_2) = e^{-\lambda x_2}, \quad x_2 > 0 \]  

(A.44)

Moments and Correlation

\[ E(X_1) = \frac{1}{\lambda} \quad \text{Var}(X_1) = \frac{1}{\lambda^2} \]  

(A.45)

\[ E(X_2) = \frac{1}{\lambda} \quad \text{Var}(X_2) = \frac{1}{\lambda^2} \]  

(A.46)

\[ E(X_1 X_2) = \frac{1}{\lambda^2} (1 - \pi_1 \pi_2 + 2p_{11}) \]  

(A.47)

\[ \text{Cov}(X_1, X_2) = \frac{1}{\lambda^2} (2p_{11} - \pi_1 \pi_2) \]  

(A.48)

\[ \text{Corr}(X_1, X_2) = 2p_{11} - \pi_1 \pi_2 \]  

(A.49)

A.6 Sarkar's ACBVE₂

Let \((X_1, X_2) \sim \text{ACBVE}_2(\lambda_1, \lambda_2, \lambda_{12}), \) \(\lambda = \lambda_1 + \lambda_2 + \lambda_{12},\) and let \(\gamma = \frac{\lambda_{12}}{(\lambda_1 + \lambda_2)}.\) Define \(A(z) = 1 - e^{-z}, \) \(z > 0.\)
Joint survival df

\[
\tilde{F}(x_1, x_2) = \begin{cases} 
    e^{-(\lambda_2 + \lambda_{12})x_2} \left\{ 1 - [A(\lambda_1 x_2)]^{-\gamma} [A(\lambda_1 x_1)]^{1+\gamma} \right\}, & 0 < x_1 \leq x_2 \\
    e^{-(\lambda_1 + \lambda_{12})x_1} \left\{ 1 - [A(\lambda_2 x_1)]^{-\gamma} [A(\lambda_2 x_2)]^{1+\gamma} \right\}, & x_1 \geq x_2 > 0 
\end{cases} \quad (A.50)
\]

Joint pdf

\[
f(x_1, x_2) = \begin{cases} 
    \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} e^{-\lambda_1 x_1 - (\lambda_2 + \lambda_{12})x_2} \left\{ \lambda_2 + \lambda_1 \lambda_{12} - \lambda_2 \lambda e^{-\lambda_1 x_2} \right\} \\
    [A(\lambda_1 x_2)]^{-(1+\gamma)} [A(\lambda_1 x_1)]^\gamma, & x_1 < x_2 \\
    \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} e^{-\lambda_2 x_2 - (\lambda_1 + \lambda_{12})x_1} \left\{ \lambda_1 + \lambda_2 \lambda_{12} - \lambda_1 \lambda e^{-\lambda_2 x_1} \right\} \\
    [A(\lambda_2 x_1)]^{-(1+\gamma)} [A(\lambda_2 x_2)]^\gamma, & x_1 \geq x_2 
\end{cases} \quad (A.51)
\]

Marginal distributions

\[
X_i \sim e(\lambda_i + \lambda_{12}), \; i=1,2. \quad \text{Hence,}
\]

\[
\tilde{F}_1(x_1) = e^{-(\lambda_1 + \lambda_{12})x_1} \quad (A.52)
\]

\[
\tilde{F}_2(x_2) = e^{-(\lambda_2 + \lambda_{12})x_2} \quad (A.53)
\]

Moments and Correlation

\[
E(X_1) = (\lambda_1 + \lambda_{12})^{-1} \quad \text{Var}(X_1) = (\lambda_1 + \lambda_{12})^{-2} \quad (A.54)
\]

\[
E(X_2) = (\lambda_2 + \lambda_{12})^{-1} \quad \text{Var}(X_2) = (\lambda_2 + \lambda_{12})^{-2} \quad (A.55)
\]

\[
\text{Corr}(X_1, X_2) = \frac{\lambda_{12}}{\lambda} + \frac{\lambda_{12}}{\lambda} \sum_{j=1}^{\infty} \frac{j!}{(\lambda_1 + \lambda_2)^j} \left\{ A_j(\lambda_1) + A_j(\lambda_2) \right\} \quad (A.56)
\]

where \( A_j(x) = x^j \Pi_{k=1}^{\infty} (\lambda + k\lambda)^{-1} \)

A.7 Friday and Patil's BEE

Let \((X_1, X_2) \sim \text{BEE}(\alpha_0, \alpha_1, \alpha_2, \alpha_1', \alpha_2')\). Define \(\nu = \alpha_1 + \alpha_2\), \(\phi_1 = \frac{\alpha_0 \alpha_1}{\nu - \alpha_2}\), and \(\phi_2 = \frac{\alpha_0 \alpha_2}{\nu - \alpha_1'}\).
Joint survival df

\[
F(x_1, x_2) = \begin{cases} 
\phi_1 e^{-(\nu-\alpha_1')x_1-\alpha_2'x_2} + (1 - \phi_1)e^{-\nu x_1}, & x_1 < x_2 \\
\phi_2 e^{-(\nu-\alpha_1')x_2-\alpha_1'x_1} + (1 - \phi_2)e^{-\nu x_1}, & x_2 < x_1 
\end{cases}
\]  

(A.57)

provided \( \nu \neq \alpha_1', \alpha_2' \)

Marginal distributions

\[
\bar{F}_1(x_1) = \phi_2 e^{-\alpha_1'x_1} + (1 - \phi_2)e^{-\nu x_1}, \quad x_1 \geq 0
\]  

(A.58)

\[
\bar{F}_2(x_2) = \phi_1 e^{-\alpha_2'x_2} + (1 - \phi_1)e^{-\nu x_2}, \quad x_2 \geq 0
\]  

(A.59)

Moments and Correlation

\[
E(X_1) = \frac{\alpha_0 \alpha_2 + \alpha_1'}{\alpha_1' \nu}
\]  

(A.60)

\[
\text{Var}(X_1) = \frac{(\alpha_1')^2 + 2\alpha_0 \alpha_2 \nu - (\alpha_0 \alpha_2)^2}{(\alpha_1')^2 \nu^2}
\]  

(A.61)

\[
E(X_2) = \frac{\alpha_0 \alpha_1 + \alpha_2'}{\alpha_2' \nu}
\]  

(A.62)

\[
\text{Var}(X_2) = \frac{(\alpha_2')^2 + 2\alpha_0 \alpha_1 \nu - (\alpha_0 \alpha_1)^2}{(\alpha_2')^2 \nu^2}
\]  

(A.63)

\[
E(X_1 X_2) = \frac{\alpha_0 \alpha_1 + \alpha_2'}{\alpha_2' \nu^2} + \frac{\alpha_0 \alpha_2 + \alpha_1'}{\alpha_1' \nu^2}
\]  

(A.64)

\[
\text{Cov}(X_1, X_2) = \frac{\alpha_1' \alpha_2' - \alpha_0^2 \alpha_1 \alpha_2}{\alpha_1' \alpha_2' \nu^2}.
\]  

(A.65)
Appendix B

Programs for the Copula Calculations

In this chapter, we compile the MAPLE programs that were used in generating the copula plots presented in sections 2.3 and 3.4. The main algorithm used in the calculations is as follows:

1. Assign values to the parameters of the distribution.
2. Define the marginal distributions, $F_i$, of the unordered variables.
3. Solve for the inverse functions of these marginal distributions.
4. Define the copula function, $c(u, v)$, of the unordered pair.
5. Plot the surface $c(u, v) = uv$.
6. Define the marginal distributions, $F_{(i)}$, of the order statistics.
7. Solve for the inverse functions of these marginal distributions.
8. Define the copula function, $c_o(u, v)$, for the ordered pair.
9. Plot the surface $c_o(u, v) = uv$.
10. Plot the surface $c_o(u, v) - c(u, v)$.

We list the symbols used in the programs in Table 4.
Table 4: Symbols used in the MAPLE calculation of the copulas.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_i$</td>
<td>The marginal distributions $F_i(x_i), i=1,2$</td>
</tr>
<tr>
<td>$s_i$</td>
<td>The inverse function of $F_i, i=1,2$</td>
</tr>
<tr>
<td>$F$</td>
<td>The joint distribution $F(x_1, x_2)$</td>
</tr>
<tr>
<td>$c$</td>
<td>The copula function $c(u, v)$</td>
</tr>
<tr>
<td>$H_i$</td>
<td>The marginal distributions $H_i(t_i), i=1,2$</td>
</tr>
<tr>
<td>$s_{si}$</td>
<td>The inverse function of $H_i, i=1,2$</td>
</tr>
<tr>
<td>$H$</td>
<td>The joint distribution $H(t_1, t_2)$</td>
</tr>
<tr>
<td>$co$</td>
<td>The copula function $c_o(u, v)$</td>
</tr>
<tr>
<td>$li$</td>
<td>The parameter $\lambda_i, i=1,2$ and $\lambda_{12}, i=3$</td>
</tr>
<tr>
<td>$l$</td>
<td>The parameter $\lambda$</td>
</tr>
<tr>
<td>$a$, $ap$</td>
<td>The parameters $\alpha$ and $\alpha'$</td>
</tr>
<tr>
<td>$b$, $bp$</td>
<td>The parameters $\beta$ and $\beta'$</td>
</tr>
<tr>
<td>$pi$</td>
<td>The parameter $\theta_i, i=1,2$</td>
</tr>
<tr>
<td>$del$</td>
<td>The parameter $\delta$</td>
</tr>
<tr>
<td>$pi1$, $pi2$</td>
<td>The parameters $\pi_1$ and $\pi_2$, respectively</td>
</tr>
<tr>
<td>$pjk$</td>
<td>The probabilities $p_{jk}, j,k = 0,1$</td>
</tr>
</tbody>
</table>
> I1:=0.7;
> I2:=1.8;
> I3:=0.5;
> F1:=t->(1-exp(-(I1+I3)*t));
> F2:=t->(1-exp(-(I2+I3)*t));
> s1:=u->(solve(F1(t)=u,t));
> s2:=v->(solve(F2(t)=v,t));
> F:=(t1,t2)->( 1- exp(-(I1*I3)*t1)- exp(-(I2+I3)*t2) + exp(-I1*I2*I3*max(t1,t2)) );
> c:=(u,v) -> ( F(s1(u),s2(v)) );
> plot3d(c(u,v)-(u*v),u=0..1,v=0..1,axes=boxed,orientation=[-145,45]);
> I:=I1+I2+I3;
> H1:=t1->(1-exp(-I*t1));
> H2:=t2->( 1 - exp(-(I1+I3)*t2) - exp(-(I2+I3)*t2) + exp(-I*t2) );
> fun:=proc(t1,t2)
> if H1(t1)>H2(t2)
> then H:=F(t2,t2);
> else H:=F(t1,t2) + F(t2,t1) - F(t1,t1)
> fi;
> end;
> ss1:=u->(solve(H1(t1)=u,t1));
> ss2:=v->(fsolve(H2(t2)=v,t2=0..infinity));
> cop:=(u,v) -> (fun(ss1(u),ss2(v)));
> df:=u,v->(cop(u,v)-(u*v));
> plot3d(df,0..1,0..1,axes=boxed,orientation=[-145,45]);
> plot3d((df,0.001...0.999,0.001..0.999,orientation=[-145,45],axes=boxed);
\begin{verbatim}
> \texttt{l1:=1.5;}
> \texttt{l2:=1.0;}
> \texttt{l3:=0.4;}
> \texttt{l:=l1+l2+l3;}
> \texttt{F1:=t1->(1 - (l/(l1+l2))^{*}exp(-(l1+l3)*t1) + (l3/(l1+l2))^{*}exp(-l^{*}t1));}
> \texttt{F2:=t2->(1 - (l/(l1+l2))^{*}exp(-(l2+l3)*t2) + (l3/(l1+l2))^{*}exp(-l^{*}t2));}
> \texttt{s1:=u->(solve(F1(t1)=u,t,0..infinity));}
> \texttt{s2:=v->(solve(F2(t2)=v,t,0..infinity));}
> \texttt{FB:=(t1,t2)->(l/(l1+l2))^{*}exp(-l1^{*}t1-l2^{*}t2-l3^{*}max(t1,t2)) - (l3/(l1+l2))^{*}exp(-l^{*}max(t1,t2));}
> \texttt{FB:=(t1,t2)->(l/(l1+l2))^{*}exp(-l1^{*}t1-l2^{*}t2-l3^{*}max(t1,t2));}
> \texttt{c:=u,v->(F1(s1(u),s2(v)));}
> \texttt{H1:=t1->(1-exp(-l^{*}t1));}
> \texttt{H2:=t2->(1-l/(l1+l2))^{*}exp(-(l2+l3)*t2) - (l/(l1+l2))^{*}exp(-(l1+l3)*t2) + ((l3)/(l1+l2))^{*}exp(-l^{*}t2));}
> \texttt{fun:=proc(t1,t2)}
> \texttt{if t1>t2 then H:=F(t2,t1);}
> \texttt{else H:=F(t1,t2) + F(t2,t1) - F(t1,t1);}
> \texttt{fi;}
> \texttt{end;}
> \texttt{ss1:=u->(solve(H1(t1)=u,t,1));}
> \texttt{ss2:=v->(solve(H2(t2)=v,t,2..infinity));}
> \texttt{cop:=u,v->(fun(ss1(u),ss2(v)));}
> \texttt{df:=u,v->(cop(u,v)-u^{*}v));}
> \texttt{df:=u,v->(cop(u,v)-(u^{*}v));}
> \texttt{plot3d(df,0.001..0.001,0.001..0.001,orientation=[-145,45],axes=BOXED);}
> \texttt{plot3d(df,0.001..0.001,0.001..0.001,orientation=[-145,45],axes=BOXED);}
> \texttt{plot3d(df,0.001..0.001,0.001..0.001,0.001,orientation=[-145,45],axes=boxed,orientation=[-145,45]);}
\end{verbatim}

Figure 23: MAPLE program that plots \(c(u, v) - uv\) ACBVE with \(\lambda_1 = 1.5, \lambda_2 = 1.0,\)
\(\lambda_{12} = 0.4.\)
\begin{verbatim}
> a:=0.3; ap:=0.5;
> b:=0.9; bp:=0.1;
> da:=(a+b-ap); db:=(a+b-bp);
> F1:=t1->( 1 - ((a-ap)/da)*exp(-(a+b)*t1) - (b/da)*exp(-ap*t1));
> F2:=t2->( 1 - ((b-bp)/db)*exp(-(a+b)*t2) - (a/db)*exp(-bp*t2));
> F1B:=t1,t2)->( exp(-(a+b)*t1)*((b-bp)/db)*exp(-(a+b)*(t2-t1)) + (a/db)*exp(-bp*(t2-t1)));
> FB2:=t1,t2)->( exp(-(a+b)*t2)*((a-ap)/da)*exp(-(a+b)*(t1-t2)) + (b/da)*exp(-ap*(t1-t2)));
> if t1<=t2 then FB:=FB1(t1,t2); else FB:=FB2(t1,t2) fi; end;
> s1:=u->(fsolve(F1(t1)=u,t1,0..infinity));
> s2:=v->(fsolve(F2(t2)=v,t2,0..infinity));
> F:=t1,t2)->(1-(1-F1(t1)) - (1-F2(t2)) + fnc1(t1,t2));
> c:=(u, v)->( F(s1(u),s2(v)));
> df:=(u, v)->(c(u, v)-u*v);
> plot3d(df,0.001..0.999,0.001..0.999,orientation=[-145,45],axes=BOXED);
> H1 :=t1->(1-exp(-(a+b)*t1));
> H2:=t2->( 1 -(a/db)*exp(-bp*t2)-(b/da) *exp(-ap*t2) + ((a*bp+b*ap-ap*bp)/(da*db))*exp(-(a+b)*t2));
> if t1>t2 then H:=F(t2,t2); else H:=F(t1,t2) + F(t2,t1) - F(t1,t1) fi; end;
> ss1;=u->(solve(H1(t1)=u,t1));
> ss2:=v->(fsolve(H2(t2)=v,t2,0..infinity));
> cop:=u,v)->(fnc2(ss1(u),ss2(v)));
> deff:=(u, v)->(cop(u, v)-u*v);
> plot3d(deff,0.001..0.999,0.001..0.999,orientation=[-145,45],axes=BOXED);
> if t1>t2 then H:=F(t2,t2); else H:=F(t1,t2) + F(t2,t1) - F(t1,t1) fi; end;
> end;

Figure 24: MAPLE program that plots $c(u,v) - uv$ for the FBVE under a 'mixed situation' with $\alpha = 0.3$, $\beta = 0.9$, $\alpha' = 0.5$, $\beta' = 0.1$.
\end{verbatim}
\begin{verbatim}
> p1:=0.5;
> p2:=0.8;
> delta:=0.5;
> F1:=t->(1-exp(-t/p1));
> F2:=t->(1-exp(-t/p2));
> FB:=(t1,t2)->(exp(-((t1/p1)**(1/delta) + (t2/p2)**(1/delta))**delta));
> s1:=u->(solve(F1(t)=u,t));
> s2:=v->(solve(F2(t)=v,t));
> F:=(t1,t2)->(1-(1-F1(t1))*(1-F2(t2)) + FB(t1,t2));
> c:=(u,v)->(F(s1(u),s2(v))); 
> plot3d(c(u,v)-u*v,u=0..1,v=0..1,axes=boxed,orientation=[-145,45]);
> H1:=t1->(1-exp(-t1*(1/p1)**(1/delta) + (1/p2)**(1/delta)));
> H2:=t2->(1-(1-F1(t2))-(1-F2(t2)) + FB(t2,t2));
> ss1:=u->(solve(H1(t1)=u,t1));
> ss2:=v->(solve(H2(t2)=v,t2,0..infinity));
> fun:=proc(t1,t2)
> if t1>t2 then H:=F(t2,t2);
> else H:=F(t1,t2) + F(t2,t1) - F(t1,t1)
> fi;
> end;
> cop:=(u,v)->(fun(ss1(u),ss2(v)));
> dif:=(u,v)->(cop(u,v) - (u*v));
> plot3d(dif,0..1,0..1,axes=boxed,orientation=[-145,45]);
\end{verbatim}

Figure 25: MAPLE program that plots \( c(u,v) - uv \) for the GBVE with \( \theta_1 = 0.5, \theta_2 = 0.8, \delta = 0.5 \).
> p11 := 0.5; p22 := 0.5; p11 := 0.5;
p00 := 0.5; p10 := 0; p01 := 0; l := 1.0;
> F1 := t -> (1 - exp(-l*t));
> s1 := u -> solve(F1(t) = u, t); s2 := v -> solve(F1(t) = v, t);
> FB00 := (t1, t2) -> (exp(-l*t1)/(1-pi1)) * exp(-l*t2)/(1-pi2);
> FB01 := (t1, t2) -> (exp(-l*t1)/(1-pi1)) * (exp(-l*t2)/pi2) - ((1-pi2)/pi2) * exp(-l*t1)/(1-pi1) * exp(-l*t2)/(1-pi2));
> FB10 := (t1, t2) -> (exp(-l*t1)/(1-pi2)) * (exp(-l*t2)/pi1) - ((1-pi1)/pi1) * exp(-l*t1)/(1-pi1) * exp(-l*t2)/(1-pi2));
> FB := (t1, t2) -> (p00*FB00(t1, t2) + p01*FB01(t1, t2) + p10*FB10(t1, t2) + p11*FB11(t1, t2));
> F := (t1, t2) -> (1 - (1 - F1(t1)) - (1 - F1(t2)) + FB(t1, t2));
c := (u, v) -> (F(s1(u), s2(v)));
> plot3d(1 - c(u, v), 0.001..0.999, 0.001..0.999, axes = boxed, orientation = [-145, 45]);
a1 := (p11*(2-pi1-pi2)/(1-pi1*pi2));
a2 := (1-(p11/pi1));
a3 := (1-(p11/pi2));
a4 := (p11*(pi1+pi2-2*pi1*pi2)/(pi1*pi2*(1-pi1*pi2)) - 1);
b1 := f1/((1/(1-pi2)));
b2 := f2/((1/(1-pi2)));
b3 := f3/((1/(1-pi1)));
b4 := f4/((1/(1-pi1)));
HB1 := t1 -> (a1*exp(-b1*t1) + a2*exp(-b2*t1) + a3*exp(-b3*t1) + a4*exp(-b4*t1));
HB2 := t2 -> (2-a1)*exp(-b1*t2) - a2*exp(-b2*t2) - a3*exp(-b3*t2) - a4*exp(-b4*t2));
> fun := proc(t1, t2)
if t1 > t2 then H := F(t2, t2); else H := F(t1, t2) + F(t2, t1) - F(t1, t1) fi;
end;
s1 := u -> (fsolve(1 - HB1(t1) = u, t1, 0..infinity)); s2 := v -> (fsolve(1 - HB2(t2) = v, t2, 0..infinity));
cop := (u, v) -> (fun(s1(u), s2(v)));
def := (u, v) -> (cop(u, v) - u*v);
> plot3d(1 - def(u, v), 0.001..0.999, 0.001..0.999, axes = boxed, orientation = [-145, 45]);
> df := (u, v) -> (cop(u, v) - u*v);
> plot3d(df(u, v), 0.001..0.999, 0.001..0.999, axes = boxed, orientation = [-145, 45]);
BIBLIOGRAPHY


