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Actively coordinated wheeled vehicle systems

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The Ohio State University, 1994
ACTIVELY COORDINATED WHEELED VEHICLE SYSTEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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* * * *

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To my family
ABSTRACT

This research effort is directed towards actively coordinated wheeled vehicle systems, a class of advanced mobility systems that are suited for operation on unstructured terrain. Actively coordinated vehicles refer to vehicles that possess the ability to influence the contact forces at the vehicle-terrain contact locations, and the ability to vary their configuration to accommodate to terrain obstacles. Unlike legged locomotion systems, wheeled systems do not possess omni-directional motion capability. This leads to nonholonomic kinematic constraints that cause unique complications in wheeled systems.

This dissertation addresses the issues of force planning, motion planning, dynamic simulation, and autonomous navigation of actively coordinated wheeled systems operating on uneven terrain. The work also includes design, fabrication, and preliminary experimental testing of the Wheeled Actively Articulated Vehicle (WAAV) in a laboratory environment. Even though this research effort is primarily directed towards actively coordinated wheeled vehicles, some results obtained during the course of this work are also applicable to other actively coordinated mechanisms.

Inertial sensing is used in robotic systems to obtain orientation and the angular rates of the vehicle body for control purposes, and to obtain absolute vehicle position on the terrain for the purpose of guidance. In the absence of accurate maps of the environment, inertial sensing systems can possess significant drift errors. In this work, a drift-free star sensor based navigation scheme is investigated, and its relative advantages and disadvantages as compared to existing systems are studied.

The redundancy in the force allocation problem of actively coordinated legged and wheeled systems has been studied using geometric reasoning. The nature of the nonlinear optimal force distribution problem has been examined. The optimization schemes developed include globally optimal algorithms that utilize advanced polynomial continuation techniques. On uneven terrain, the configuration of the wheeled vehicle is greatly affected by the local terrain geometry. The kinematic mobility of this configuration on uneven terrain, and the position kinematics of the resulting hybrid series-parallel chain have been studied. These position kinematic solutions have been used as the basis for developing motion planning algorithms.

A dynamic simulator capable of handling the motion of articulated wheeled vehicles on uneven terrain has been developed. A contact model that incorporates a three-dimensional spring/damper system at the wheel-terrain contact locations has been used to simulate the phenomena of rolling and slipping of the wheels.

Preliminary experiments with the Wheeled Actively Articulated Vehicle (WAAV) have been performed. In this work, the mechanical and electrical hardware has been tested, the WAAV has been interfaced to a personal computer, and basic testing of simple mobility maneuvers in a laboratory setting has been attempted.
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# TABLE OF CONTENTS

DEDICATION ............................................................................................................................. ii  
ABSTRACT ............................................................................................................................... iii  
ACKNOWLEDGMENTS ............................................................................................................... iv  
VITA .......................................................................................................................................... v  
LIST OF FIGURES .................................................................................................................... x  
LIST OF TABLES ...................................................................................................................... xiv  

I INTRODUCTION ........................................................................................................ 1  
  1.1 Background and Past Research Efforts ............................................................. 1  
  1.2 The Wheeled Actively Articulated Vehicle (WAAV) ................................ 12  
  1.3 Autonomous Coordination and Control ............................................................. 20  
  1.4 Research Objectives .............................................................................................. 23  
  1.5 Organization ............................................................................................................ 26  

II DRIFT-FREE NAVIGATION ............................................................................. 27  
  2.1 Introduction ......................................................................................................... 28  
  2.2 Kinematics of the SGS .......................................................................................... 31  
    2.2.1 Sensing Vehicle Body Orientation ............................................................... 33  
    2.2.2 Angular Velocities of the Vehicle Body ..................................................... 37  
    2.2.3 Tracking Errors in α, β, and γ ...................................................................... 38  
    2.2.4 Simulating Stars S₁ and S₂, and the Corresponding Cameras ....... 39  
  2.3 Vehicle Terrain Dynamics ...................................................................................... 40  
    2.3.1 Linearized Vehicle dynamics ....................................................................... 42  
    2.3.2 Dynamic Model With Gaussian Noise ..................................................... 49  
    2.3.3 Wheel-Soil Mechanics .................................................................................. 49  

vi
2.3.4 Equivalent Damping ................................................................. 53
2.3.5 Vehicle Simulation Results ..................................................... 58
2.4 Dynamic Tracking of the SGS ....................................................... 62
   2.4.1 Rigid Body Dynamics of the SGS Mount ................................. 62
   2.4.2 Design of the Tracking Controller ......................................... 69
   2.3.3 Simulation of the SGS Tracking .............................................. 73
2.5 Estimation of the Absolute Position of the Vehicle ....................... 74
   2.5.1 Procedure for Estimating the Absolute Position ......................... 81
   2.5.2 Numerical Example of Estimating the Absolute Vehicle
       Position ............................................................................................ 83
2.6 Star Sensing Systems ................................................................. 87
2.7 Conclusions ................................................................................ 88

III FORCE PLANNING ............................................................................. 90
3.1 Introduction ................................................................................. 91
3.2 Characterization of Redundancy ................................................... 93
   3.2.1 Active Mechanisms with Complete Controllability Over
       Contact Conditions ........................................................................... 93
   3.2.2 Active Mechanisms with Partial Controllability Over Contact
       Conditions .......................................................................................... 119
3.3 Optimal Force Allocation ............................................................ 133
   3.3.1 Active Mechanisms with Complete Controllability Over
       Contact Conditions (Four Points of Contact) .................................... 134
   3.3.2 Active Mechanisms with Partial Controllability Over Contact
       Conditions (Four Points of Contact) ................................................... 156
3.4 Conclusions ................................................................................. 161

IV MOTION PLANNING .............................................................................. 165
4.1 Introduction ................................................................................. 166
4.2 Position Kinematics ...................................................................... 168
   4.2.1 Vehicle Geometry and Terrain Model ...................................... 168
   4.2.2 Kinematic Mobility ................................................................. 170
   4.2.3 Wheel-Terrain Contact Locations ............................................. 173
   4.2.4 Terrain Constraint Equations ................................................... 176
4.2.5 Position Kinematic Equations of the Modules .................................... 177
4.2.6 Numerical Issues ..................................................................................... 183
4.2.7 The 'Correct' Vehicle Configuration .................................................... 186
4.2.8 Numerical Example............................................................................. 191
4.3 Autonomous Motion Planning on Uneven Terrain ............................................. 196
  4.3.1 Straight Line Path on Uneven Terrain......................................... 198
  4.3.2 Obstacle Climbing and Ditch Crossing........................................ 205
  4.3.3 Turning Maneuver................................................................................... 210
4.4 Conclusions .............................................................................................................. 222

V DYNAMIC SIMULATION ............................................................................................. 224
  5.1 Introduction........................................................................................................... 225
  5.2 Forward Dynamics ................................................................................................. 228
    5.2.1 Outward Recursion to Perform Velocity Analysis............................. 232
    5.2.2 Inward Recursion to Compute Articulated-Body Inertias and Bias Vectors ................................................................................................... 234
    5.2.3 Outward Recursion to Compute the Accelerations of the System................................................................. 238
  5.3 The Wheel-Terrain Contact Model ......................................................................... 241
    5.3.1 Spring/Damper Attachment Point on the Wheel (CW)...................... 244
    5.3.2 The Displacements of the Three Springs (Vector sw)....................... 246
    5.3.3 Concave and Convex Terrain Edges .................................................... 249
  5.4 A Preliminary Coordination Algorithm ...................................................... 251
  5.5 Simulation Studies .................................................................................................... 253
    5.5.1 Arbitrarily Oriented Even Terrain.......................................................... 254
    5.5.2 Vehicle Configuration in the Presence of Compliance on Uneven Terrain .................................................................................................... 255
  5.6 Conclusions .............................................................................................................. 263

VI THE WHEELED ACTIVELY ARTICULATED VEHICLE (WAAV) ................... 265
  6.1 Introduction........................................................................................................... 265
  6.2 Kinematic Configuration ......................................................................................... 268
  6.3 Description of the Sub-Systems ........................................................................... 271
  6.4 Current Capabilities and Future Experiments.................................................... 282
VI

CONCLUDING REMARKS .............................................................................285

7.1 Summary .................................................................................................285

7.2 Research Extensions .............................................................................288

APPENDIX - A ...............................................................................................293

APPENDIX - B ...............................................................................................295

REFERENCES ...............................................................................................298
# LIST OF FIGURES

1.1 Martin Marietta's Actively Articulated Six Wheeled Vehicle Concept .......... 5
1.2 Martin Marietta's Walking Beam Concept ................................................. 5
1.3 Martin Marietta's Agile Walker Concept ................................................... 7
1.4 Carnegie Mellon University's AMBLER ..................................................... 7
1.5 The Ohio State University's Adaptive Suspension Vehicle ......................... 8
1.6 An Articulated Wheeled Vehicle Configuration on Uneven Terrain ............. 13
1.7 Baseline Configuration of the Wheeled Actively Articulated Vehicle .......... 13
1.8 The Wheeled Actively Articulated Vehicle (WAAV) ..................................... 14
1.9 A Solid Model Graphical Representation of the WAAV .............................. 15
1.10 WAAV Climbing an Obstacle ................................................................. 17
1.11 WAAV Crossing a Ditch ........................................................................... 17
1.12 Self-Recovery of the WAAV ................................................................. 18
1.13 A 'Side-Ways' Maneuver of the WAAV .................................................... 19
1.14 Overview of Autonomous Vehicle Coordination and Control .................. 22
2.1 Camera Mount Containing the Two Cameras Mounted on the Vehicle Body ... 32
2.2 Camera 'i' (i = 1, 2) .................................................................................. 34
2.3 Passive Wheeled Vehicle Interacting With the Terrain .............................. 41
2.4(a) $|H_z(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$) .............................. 45
2.4(b) $|H_z(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$) ................................. 45
2.5(a) $|H_\psi(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$) .............................. 46
2.5(b) $|H_\psi(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$) ............................... 46
2.6(a) $|H_\theta(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$) .............................. 47
2.6(b) $|H_\psi(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$) ............................... 47
2.7 Cylindrical Wheel Rolling Through Soil ................................................................. 52
2.8 Power and Wheel Width Versus Sinkage for the 'Best Case' Soil Parameters ................................................................. 54
2.9 Power and Wheel Width Versus Sinkage for the 'Worst Case' Soil Parameters ................................................................. 54
2.10 Terrain Form Chosen For the Numerical Simulation ....................................................... 55
2.11 Average Power Consumed Versus the Damping Coefficient ............................................... 57
2.12 Variables $\psi, \theta, z, \psi_t, and \psi_{\xi}$ as the Vehicle Traverses a Lava Field ($\zeta = 0.1$) ................................................................. 59
2.13 Variation of $\psi, \theta, z, \psi_t, and \psi_{\xi}$ as the Vehicle Traverses Soil ($\zeta = 2.25$) ................................................................. 60
2.14 Assumed Variation of the Vehicle Yaw $\phi$, and its Derivatives With Time ................................................................. 61
2.15 Variation of $z, \psi_t$ and $\theta$ (Model Includes Gaussian Noise, $\zeta = 2.25$) ................................................................. 63
2.16 Variation of $\delta_{Rt}$ and $\delta_{Lt}$ (Model Includes Gaussian Noise, $\zeta = 2.25$) ................................................................. 63
2.17 Block Diagram of the Star Gazing System (SGS) ................................................................. 64
2.18 Free Body Diagram of the SGS ..................................................................................... 67
2.19 Schematic Diagram of a Constant Field D.C Motor ................................................................. 68
2.20 Reduction in Tracking Error $\alpha_e$ with an Increase in the Controller Sampling Rate ($\zeta = 2.25$) ................................................................. 75
2.21 Tracking Errors $\alpha_e, \beta_e$, and $\gamma_e$ ......................................................................................... 76
2.22 Sensed Values of $\psi_s, \theta_s$, and $\phi_s$, and their Errors $\psi_e, \theta_e$, and $\phi_e$ ($\zeta = 0.1$) ................................................................. 77
2.23 Sensed Values of $\psi_s, \theta_s$, and $\phi_s$, and their Errors $\psi_e, \theta_e$, and $\phi_e$ ($\zeta = 2.25$) ................................................................. 78
2.24 Sensed Angular rate $\psi_t$, and the Corresponding Error $\psi_{\xi}$ ................................................................. 79
2.25 Tracking Errors $\alpha_e, \beta_e$, and $\gamma_e$ ......................................................................................... 80
2.26 Angular rate, $\psi_t$, sensed value $\psi_s$, and sensing error $\psi_{\xi}$ ................................................................. 80
3.1 Articulated Multi-Fingered Gripper ..................................................................................... 94
3.2 Legged Vehicle System on Uneven Terrain ........................................................................ 94
3.3 Force System of a Body Interacting With its Environment .................................................. 99
3.4 $n$ Spatial Points and Associated Vectors ........................................................................ 99
3.5 $n$ Planar Contact Points and the Contact Force Components ............................................. 109
3.6 Orthogonal Force Vector With Four Non-Zero Normal Components .................................. 109
3.7 $n$ Planar Points and Associated Vectors ........................................................................ 109
3.8 Force System at the Wheel-Ground Interface for Contact Point 'i' ....................................... 120
3.9 Infeasible Interaction Force ......................................................................................... 120
3.10 Actively Articulated Wheeled Vehicle Configuration ...................................................... 170
3.11 Two Modules on a Plane Terrain ................................................................. 129
3.12 Normal Force Field ..................................................................................... 129
3.13 Tangential Force Field ................................................................................ 129
3.14 A Simple Null Space Vector for a Spatial Set of Contacts ..................... 132
3.15 Contact Point 'i' ......................................................................................... 135
3.16 Interaction Forces for a Set of Four Spatial Contact Points ................... 139
3.17 A Set of Four Coplanar Contact Points .................................................. 139
3.18 The Six Optimization Variables for a Set of Four Spatial Contact Points ... 141
3.19 Unconstrained Global Optimization for Four Contact Points with Complete
    Force Controllability ...................................................................................... 152
3.20 Optimal Force Distribution for a Four Legged Walking Vehicle ............... 153
3.21 Actively Articulated Legged System Preparing For An Obstacle ............... 162
3.22 Actively Articulated Wheeled System Preparing For An Obstacle .......... 163
3.23 Geometric Parameters of Actively Articulated Wheeled System .............. 163
3.24 Obstacle Negotiation Performance Comparison ........................................ 164
4.1 An Articulated Wheeled Vehicle Configuration on Uneven Terrain .......... 169
4.2 Geometry of the Articulated Wheeled Vehicle ........................................... 173
4.3 Position Kinematics of Module 'i' ............................................................... 174
4.4 The Geometric Parameters of a Two Module Articulated Vehicle ............ 191
4.5 The Eight Configurations of the Master Module .................................... 192
4.6 The Sixteen Configurations of the Follower Module ................................. 193
4.7 The 'Correct' Vehicle Configuration Composed of the 'Correct' Master and
    the 'Correct' Follower Modules .................................................................. 194
4.8 Block Diagram of the Motion Planning Algorithm ..................................... 199
4.9 The Angle $\lambda_R$ Representing the Location of the Right Wheel-Terrain Contact
    .................................................................................................................. 201
4.10 The Search Strategy .................................................................................... 203
4.11 A Three Module Articulated Vehicle Traversing a Simulated Random
    Terrain ......................................................................................................... 206
4.12 Vehicle Traversing a Straight Line Path on Uneven Terrain .................... 207
4.13 Variation of the Kinematic Quantities as the Vehicle Traverses a Straight
    Line Path on Uneven Terrain .................................................................... 208
4.14 Obstacle Climbing and Ditch Crossing by the Use of a Virtual Plane ....... 209
4.15 Automated Obstacle Climbing Using the Virtual Terrain Concept .......... 211
4.16 Variation of Kinematic Quantities During Obstacle Climbing ..................................... 212
4.17 A Two Module Vehicle Performing a 30° Turning Maneuver ................................... 220
4.18 Variation of Kinematic Quantities During a 30° Turning Maneuver .......................... 221
5.1 Geometry of an Articulated Wheeled Vehicle .............................................................. 229
5.2 The Six Rigid Bodies of the Two Module Articulated Vehicle .................................... 233
5.3 The Wheel-Terrain Interaction Model ........................................................................... 243
5.4 A Three-Dimensional Spring/Damper Contact Model ........................................... 245
5.5 Classification of an Edge ............................................................................................... 250
5.6 Convex Edge Traversal ................................................................................................. 251
5.7 Vehicle Simulation on an Arbitrary Even Terrain ....................................................... 254
5.8 Histories of Vehicle States During Nonholonomic Motion on an Arbitrarily Oriented Even Terrain ........................................................................................................... 256
5.9 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 0.4) .................................................................................................................... 259
5.10 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 0.7) ................................................................. 260
5.11 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 10.0) ................................................................................................................ 261
6.1 The Wheeled Actively Articulated Vehicle (WAAV) .................................................... 267
6.2 Kinematic Configuration of the WAAV ........................................................................ 269
6.3 Mechanical Design of the WAAV ................................................................................ 274
6.4 A Block Diagram of the Power Distribution System .................................................... 277
6.5 Wheel Design for Torque Sensing ................................................................................ 280
6.6 The Control Architecture ............................................................................................ 281
LIST OF TABLES

3.1 The Minimum-Norm Solution, the Global optimum, and the Local Optima ..... 156
6.1 Mobility Capabilities of the WAAV ......................................................... 270
B.1 Martin Marietta's Agile Walker Concept .................................................. 296
B.2 Carnegie Mellon University's AMBLER ................................................. 297
CHAPTER I

INTRODUCTION

1.1 Background and Past Research Efforts

High performance vehicle systems are desired in fields which require off-road mobility. These fields include military applications, agriculture, particularly forestry, mining and planetary exploration, and the vehicles may be required to possess varying levels of autonomy. For instance, an unmanned exploration of the Martian surface requires a high level of autonomy, since the round trip time delay of a signal sent from earth averages about 25 minutes [Klein et al., 1986], making teleoperation infeasible. Conventional wheeled vehicle systems are very well suited for motion over prepared surfaces. However, when such vehicles are to perform on uneven terrain, they have limited mobility capability, and they tend to consume large amounts of energy. Further, their designs tend to be bulky since they may be subjected to large, fluctuating structural loads. Some off-road applications require an exceptionally capable locomotion system as compared to the conventional vehicles with which we are accustomed to dealing. The reason is that continuous interaction with a human operator may not be possible in many cases, such as a remotely operated vehicle required in unstructured environments. Thus,
the sensing and intelligent guidance capabilities of a human being may not be available. The current state-of-the-art in computer guidance and control, and the likely sensing capabilities of the system, are greatly inferior to those of a human operator and must be compensated for, at least in part, by the mechanical capabilities of the locomotion system. Studies of the configurations of such off-road mobility concepts have led to two basic responses. One of these is to maximize the size of the vehicle to proportionately reduce the scale of terrain features and obstacles encountered. This is a valid approach since the frequency of encounter of obstacles of given size decreases with their dimension. The second response is to attempt to take advantage of the potential of computer coordination to improve locomotion capability. The potential gains from computer coordination are considerable regardless of the type of locomotion system used. Nevertheless, the experience base with computer coordinated vehicles is, at present, very small. The reason for this is that it only recently became feasible to put sufficiently powerful computers onboard vehicles to perform active coordination. It is likely that, for a given application, a final solution will embody elements of both these approaches.

It has been shown that computer control of actively coordinated vehicle systems can potentially lead to significant improvement in performance and mobility of autonomous and remotely controlled vehicles operating on unstructured terrain [Kumar and Waldron, 1989]. Actively coordinated vehicles refer to vehicles that possess independently controlled actuators for the suspension and the locomotion degrees of freedom. The contact force vectors at the vehicle-terrain contact locations can be directly influenced by using these actuators. These vehicles also have the capability to vary their geometry to accommodate to terrain obstacles. The coordination of these vehicles requires digital integration of all the actuators and associated sensors, and the control strategy requires an understanding of the kinematics and dynamics of spatial hybrid series-parallel chains.
Some of the coordination issues of series-parallel chains with respect to a hybrid manipulator are discussed by Waldron et al. [1989a]. Actively coordinated vehicle systems should possess superior mobility characteristics in unstructured terrains when compared to traditional, passively suspended vehicle systems. Traditional vehicles differ from actively coordinated vehicles since they emphasize the use of systems with a single prime mover, remotely actuating the vehicle degrees of freedom, with the help of a mechanical transmission unit. In addition, they do not possess the capability of varying their configuration to surmount obstacles.

Actively coordinated vehicle systems discussed in this work are different from road vehicles with active suspensions, such as the one discussed by Milliken [1988]. In that work, only the three degrees of freedom which are accommodated by passive suspensions in conventional vehicles, are under integrated active control. These systems essentially function as active vibration dampers. The vehicle systems to be discussed in this work are also distinct from vehicles that possess complex kinematic configurations, but only a subset of their degrees of freedom are controlled actively. Examples of such systems include the CARD [Wilcox and Gennery, 1990], and the Rocker-Bogie [Bickler, 1990; Chottiner, 1992] vehicle configurations. While these machines have relatively large numbers of independently controllable degrees of freedom, they lack the abilities to control the distribution of load among the wheels or to configure themselves to optimally attack obstacles or changing terrain conditions.

Actively coordinated vehicles include walking machines and wheeled locomotion systems. Walking machines have been studied extensively in recent years [McGhee and Ishwandhi, 1979; Hirose and Umetani, 1980; Sutherland and Ullner, 1984; Waldron and McGhee, 1986; Raibert 1986; Song and Waldron, 1988; Kumar and Waldron, 1990]. Actively coordinated wheeled vehicles have not been studied in as much detail. Some of
the coordination issues of active wheeled systems operating in unstructured environments can be found in the literature [Waldron et al., 1991a, 1991b, 1987; Kumar and Waldron, 1989; Waldron, 1989b]. The design of the Wheeled Actively Articulated Vehicle (WAAV), an actively coordinated wheeled vehicle system is described in the literature [Yu and Waldron, 1991; Sreenivasan et al., 1994]. The kinematic configuration, and the mobility features of the WAAV are discussed in some detail in Section 1.2. A vehicle configuration similar to the WAAV is the FMC Attached Scout rover concept [McTamany, 1989]. Another vehicle which is similar to the WAAV is the Articulated Transporter/Manipulator System (ATSM) described recently by Ridgeway et al. [1992] and Chiang et al. [1992]. This vehicle is not fully actively coordinated since the articulations joining adjacent vehicle modules are actively actuated only about two axes (the pitch and the yaw). Of course the ATSM is intended only for operation in a 2D environment with 2D obstacles. Therefore, even though its geometry is similar to that of the WAAV, its coordination and control issues are quite different. The WAAV is almost identical in its geometry to a vehicle that was studied by Martin-Marietta [Spiessbach and Woodis, 1988]. Figure 1.1 is a rendering of such a vehicle concept. Martin-Marietta only studied the concept and did not pursue the configuration any further. A preliminary coordination scheme for such a vehicle has been worked out by Kumar and Waldron [1989]. Martin-Marietta also studied two walking vehicle concepts for off-road applications, particularly for planetary exploration. These concepts are the Walking Beam concept (Figure 1.2) and the Agile Walker concept (Figure 1.3) [Spiessbach et al., 1989].

With respect to the ability to control the contact force distribution, and with respect to the ability to configure themselves to terrain obstacles, actively coordinated wheeled vehicles such as the WAAV are, in many ways, similar to fully terrain adaptive legged vehicles such as the Adaptive Suspension Vehicle (ASV) [Pugh et al., 1990], and the
Figure 1.1 Martin Marietta's Actively Articulated Six Wheeled Vehicle Concept

[Spiessbach and Woodis, 1988]

Figure 1.2 Martin Marietta's Walking Beam Concept [Spiessbach et al., 1989]
AMBLER [Bares et al., 1989]. The configuration of the AMBLER is shown in Figure 1.4, and that of the ASV is shown in Figure 1.5. An important feature of these vehicles is the use of a combination of force and rate control, with independent sensing of vehicle attitude. This mode of control distinguishes them from other walking machines operated on uneven terrain, and this mode of control was very highly developed in the ASV project. The kinematic structure of actively coordinated wheeled vehicles such as the WAAV is more complex when compared to the legged systems. This is due to the presence of wheels that introduce nonholonomic rolling constraints into the kinematic analysis. This makes the coordination, planning, dynamic simulation and the control issues of actively coordinated wheeled vehicles distinct from that of the walking machines.

One of the primary objectives of constructing a fully actively coordinated wheeled vehicle is to fully optimize the contact conditions at each of the wheels. This means, effectively, minimizing the maximum ratio of tangential to normal contact force over all the wheels. This has the effect both of maximizing traction and of minimizing power consumption. The ratio of tangential to normal force, of course, determines the tendency of the wheel to slip which in turn affects power consumption. Power consumed by the vehicles increases rapidly and nonlinearly with this ratio [Waldron et al., 1991a]. Complete active coordination requires a vehicle with an active suspension or variable configuration. Passive distribution of power to the wheels via differentials, as in conventional automotive technology, works well only on surfaces which are close to level. Even then, there is no possibility of optimizing the load distribution among the wheels. The inability of conventional vehicle systems to optimize the load distribution among the wheels leads to large, fluctuating structural loads in these vehicles, especially when they operate on unstructured terrains. This requires the designs of such vehicles to be relatively
Figure 1.3 Martin Marietta's Agile Walker Concept [Spiessbach et al., 1989]

Figure 1.4 Carnegie Mellon University's AMBLER [Bares et al., 1989]
Figure 1.5 The Ohio State University's Adaptive Suspension Vehicle (ASV) [Pugh et al., 1990]
bulky in order to avoid fatigue failures of the structural elements. This in turn leads to increased power consumption. Of course, actively coordinated vehicles have a large number of actuators and sensors which have overheads associated with them. Therefore, the need for active coordination is not justified unless the terrain is 'sufficiently' rugged.

Another capability which comes with the active suspension or actively controllable configuration, is a capability for anticipatory changes in configuration when approaching large obstacles. This is discussed in some detail with respect to the WAAV in Section 1.2. In order to exploit this variable configuration capability when approaching obstacles, a sensing system which can provide a model of the terrain ahead of the vehicle, and the software to react appropriately to that model are required. Such a sensing system is probably essential to an autonomous roving vehicle.

An additional capability which comes with some of the actively coordinated vehicle geometries is a capability for self-recovery from roll-overs or traction failures. This capability can be very useful if an autonomous vehicle mission of high reliability is desired. The self-recovery feature is also discussed in some detail with respect to the WAAV in Section 1.2.

The choice between an actively coordinated wheeled system and a terrain adaptive legged system, for an off-road mobility concept, is a difficult one. Data has been available for a long time showing that legged systems are superior to wheeled or tracked vehicles in rough terrain, at least if one can design a system of comparable efficiency to a biological legged system [Bekker, 1969]. Nevertheless, there are some practical complications. The oscillatory actuator action required by the legged system, and the relatively large number of actuators which must be incorporated result in relatively high internal energy losses. Optimal gaits that minimize the energy consumption, and control schemes that recycle
potential energy during walking are difficult to identify. (Gait is a term used to address the problem of phasing of legs during walking or running.) Most of the legged machines which have been used for full scale experiments, and particularly those used in unimproved terrain, actually have legs which bear no resemblance to biological legs. Even the Adaptive Suspension Vehicle [Pugh et al., 1990] which superficially appears to have some resemblance to biological locomotion characteristics, actually uses a pantograph leg geometry which incorporates sliding joints that are not found in biological systems. The kinematics and mechanics of these legs are also very different from those of biological legs. Therefore, it is difficult to extrapolate from the results obtained from studies of biological systems to legged machines. For the sake of issues such as real-time feasibility and reliability, systems comprised of simple legs are preferred. Figure 1.2 shows the Walking Beam concept from Martin Marietta Corporation [Spiessbach et al., 1989]. In this system the legs simply telescope and forward motion is accomplished by relative motion of the two modules making up the vehicle body. The philosophy in using a rover of this geometry is, once again, to maximize the scale of the vehicle to minimize the relative scale of the obstacles which it has to cross. This works well because the frequency of encounter of obstacles decreases faster than linearly with scale. An agile, active walking vehicle has also been studied (Figure 1.3). This concept has superior agility and is capable of self-recovery and of operating in a degraded mode in the event of failure of one or more legs. The legs have exceptionally large working envelopes providing very large obstacle capability. The front legs are fitted for alternate use as manipulators. Nevertheless, it is an extremely complex machine and at the present stage of technology there are many unanswered questions.

The wheel is a simple and well understood locomotion element. Actively coordinated wheeled systems such as the WAAV have most of the capabilities of legged
systems. In fact, if a large number of modules are used in an articulated chain (such as eighteen in the ATSM [Chiang et al., 1992]), the obstacle negotiation capability of wheeled systems may be better than legged systems. In general wheeled systems inherently have preferential directions of motion, while legged systems such as the ASV have omnidirectional capability. However, actively coordinated wheeled systems can to some extent be made to move 'side-ways' as discussed in Section 1.2. An advantage of actively coordinated wheeled systems appears to be potentially reduced energy consumption. The WAAV is likely to be much more energy efficient that a legged machine on relatively easy terrain, while it still possesses the desired mobility characteristics for operation on difficult terrain. A fully active legged system like the ASV consumes energy merely to stand on its legs. An attempt to establish whether an actively coordinated wheeled system or a terrain adaptive legged system is superior is probably unrealistic. The most important and unpredictable factor that may affect the choice of a mobility system is the set of soil properties. One type of machine may do better on a given type of soil, but worse on others. The choice may also depend on factors such as the particular application, the level of autonomy desired, and the restriction on energy/power consumption to name a few. Further research in the area of advanced mobility systems is required before this question can be resolved.

A detailed survey of the study of mobile robots can be found in the literature [Vijaykumar, 1987]. This reference includes various kinds of systems that were designed to operate both in structured and unstructured environments.

In the next section, the basic kinematic configuration of the WAAV and related vehicles is considered. A brief discussion of some of the mobility features of the WAAV is also provided.
1.2 The Wheeled Actively Articulated Vehicle (WAAV)

A multi-module articulated wheeled vehicle configuration is shown in Figure 1.6. This is a generalization of the WAAV configuration. Each module $M_j$ consists of a body and two wheels and the consecutive modules are connected by articulations $A_j$ that are three degree of freedom ball joints kinematically equivalent to spherical joints. All the wheels and the three degrees of freedom of all the articulations are independently actuated. This particular kinematic configuration has many attractive features that include capabilities of surmounting large obstacles, crossing wide ditches, and self-recovery from an overturn.

During the course of the project "A Comparative Design Study of Locomotion Systems for a Mars Rover" [Burkat, 1987], a three module version of the above kinematic chain was identified as possessing several attractive features. This vehicle configuration is described in detail in the literature [Waldron et al., 1987]. A prototype of this three module vehicle, the WAAV, has been designed and fabricated at Ohio State University. The design of the WAAV is discussed in detail in Yu and Waldron [1991]. The baseline configuration of the WAAV is shown in Figure 1.7. A photograph of the WAAV is shown in Figure 1.8, and a detailed solid model graphical representation of the WAAV that was developed by Pouncy [1993] is shown in Figure 1.9. The WAAV has twelve rotary electric motors, one for each of the six wheels, and three for each of the two articulations. These twelve actuators will be under continuous microcomputer control. The axle of the middle module is mounted on a guide way so that it can translate along this guide way from left to right (see Figure 1.7). This translatory motion allows the center of mass of the system to be located on either sides of the axle of the middle module. This movement of the center of mass is very useful in mobility maneuvers such as step climbing and self-recovery from an accident [Waldron et al., 1987]. A rotary actuator and a power screw are used to obtain
Figure 1.6 An Articulated Wheeled Vehicle Configuration on Uneven Terrain

Figure 1.7 Baseline Configuration of the Wheeled Actively Articulated Vehicle
Figure 1.8 The Wheeled Actively Articulated Vehicle (WAAV)
Figure 1.9 A Solid Model Graphical Representation of the WAAV [Pouny, 1993]
the translatory motion of the middle axle. This actuator is not to be operated under
continuous control. This actuator is to be used in a three position mode: middle axle in the
extreme left position, middle axle in the extreme right position, or middle axle in the central
position. The middle axle in the central position is suited for normal operations involving
motion on all six wheels since, on even terrain, this will lead to low loads on the
articulation actuators. The middle axle in the extreme positions (right or left) are required
during special mobility maneuvers as discussed later. It is assumed that the master module
will eventually house the computing equipment and sensors (including a vision system and
an inertial sensing package). The peak speed of the WAAV is no more than about 1 m/sec.
The design of the WAAV, and the architecture of the associated sensors and electronic
hardware are described in detail in a later chapter.

The WAAV has certain interesting mobility features as indicated above. The
important ones include, step climbing, ditch crossing, self-recovery from an overturn
failure, and the ability to move side-ways. The series of mobility maneuvers required to
climb a step, to cross a ditch, to recover from an overturn failure, and to move side-ways
are shown in Figures 1.10, 1.11, 1.12, and 1.13 respectively. In each of these figures, a
joint motion is indicated in the diagram prior to the one in which the motion is completed.
The indicated articulation joint motions are self-explanatory, and the bold arrows on the
middle module indicate the direction in which the middle axle has to be moved. These
figures demonstrate why the mobility characteristics of an actively coordinated wheeled
vehicle is superior to that of a system that has active control over only a subset of the total
set of degrees of freedom. Also, Figure 1.13 illustrates that an actively coordinated vehicle
can possess omni-directional capability, even though it is not nearly as mobile as legged
machines in the lateral direction. However, actively coordinated wheeled systems are
superior to traditional wheeled systems, if a fine adjustment in the lateral position is
Figure 1.10 WAAV Climbing an Obstacle  Figure 1.11 WAAV Crossing a Ditch
Figure 1.12 Self-Recovery of the WAAV
The vehicle is shown in the top view in frames 3, 4, 7, 8, 13 and 14, and it is shown in the side view in all the other frames.

Figure 1.13 A 'Side-Ways' Maneuver of the WAAV
desired. The coordination problem involved in achieving a desired lateral motion for a conventional wheeled system is a surprisingly difficult one as noted by Murray and Sastry [1992]. It leads to nonholonomic motion planning problems that result in paths requiring several iterative loops before the vehicle can reach the desired position. Another issue that is evident from the mobility maneuvers is that the maximal mobility capability of the WAAV is a function of its geometry. These mobility maneuvers were considered, to some extent, by Burkat [1987] and Yu [1990], and the present configuration of the WAAV has incorporated results from their studies. However, exact relationships between the maximal mobility capabilities of such a vehicle and its geometric design, are not yet known.

1.3 Autonomous Coordination and Control

In this section, a brief overview of autonomous coordination and control of an actively coordinated wheeled vehicle on uneven terrain is presented. The overall control strategy can be understood by studying the block diagram of Figure 1.14. This block diagram represents an advanced coordination scheme, and it is included here to provide a better overall understanding of the autonomous vehicle maneuvers. In this work, only a subset of the issues represented in the block diagram will be addressed in detail.

The supervisor represents the human operator designating terrain goal points. The terrain model is assumed to be obtained from a vision system such as a scanning range-finder. A reasonable geometric model of the terrain can be obtained by planar interpolation among adjacent scanned terrain points. The mechanical properties of the terrain are not very easily established. The available state of the art techniques are not very reliable when it comes to generating terrain strength and sinkage parameters, distribution of rocks etc. In a simplified study, it can be assumed that the terrain parameters lie in one of a finite number
of possible subsets. For instance, the surface soil can be assumed to be weak, nominal, or strong, and approximate values of the strength parameters for each of these groups can be assumed to be available to the control strategy. A control law that is robust with respect to variations of these strength parameters within a given subset is desired. At their highest level of sophistication, the guidance and motion planning algorithms will identify a safe path if one such path exists. In certain situations, the guidance and planning algorithms can be replaced by a human operator. Once a safe path is identified the vehicle is set in motion and the control law is expected to track the desired path. The coordination algorithm generates the desired vehicle kinematics and contact forces. The choice of the coordination algorithm will depend on the level of difficulty of the terrain. For instance on easy terrain (where probability of vehicle failure due to hang-up, static instability etc. is low), the redundancy in actuation can be optimized to minimize energy consumption. On the other hand, on difficult terrain, maximizing vehicle stability margins may be more desirable. As the vehicle tracks the chosen path, the sensed states of the vehicle may be used to check if the articulated configuration is close to a marginal stability situation. If this is true, the vehicle can be stopped and the control can be returned to the supervisor. The control law has no control on the lateral positioning of a module of the wheeled vehicle since the modules do not possess controlled motion capability in the lateral direction. This is a characteristic feature of wheeled systems. On uneven terrain, the vehicle will undergo small amounts of lateral drift from the desired path, and over a reasonable distance this lateral drift may become significant. This problem can be handled by putting an upper limit on the lateral drift of the main module of the vehicle. As soon as the drift exceeds this upper limit, the vehicle can be stopped and a 'fine maneuver', involving lateral movement of the vehicle by appropriately lifting the three modules in a certain sequence, can be performed to eliminate lateral drift. This maneuver is another example of the advantages of vehicles with variable configuration capability. The control law along with lateral
Figure 1.14 Overview of Autonomous Vehicle Coordination and Control
maneuvering can lead to the vehicle reaching the goal point, completing a successful autonomous vehicle motion.

1.4 Research Objectives

The articulated wheeled vehicle configuration shown in Figure 1.6 is complex spatial series-parallel kinematic structure. The coordination issues of such vehicles is, in many ways, similar to terrain adaptive legged systems such as the ASV [Pugh et al., 1990] or the AMBLER [Bares et al., 1989]. In addition, these vehicles inherit certain unique geometric features from conventional wheeled systems. Recent studies with legged systems, particularly with the ASV and the AMBLER, have led to advanced coordination schemes for legged systems. Conventional wheeled systems (including active suspension systems) that operate on relatively even terrain have been studied for several years, and these systems are understood very well. However, actively coordinated wheeled systems with articulated structures that are suited for uneven terrains are being addressed here for the first time.

The vehicle of Figure 1.6 is going to serve as the baseline configuration in this research effort. Some of the general results will be obtained with respect to this geometry. Specific mobility maneuvers will, however, be developed with respect to the WAAV geometry of Figure 1.7. This research effort is primarily directed towards actively coordinated wheeled vehicles. However, some results that will be obtained during the course of this work may be applicable to walking vehicles as well. The vehicle systems addressed in this proposal are typically expected to traverse terrains at the speed of about 0.1 to 1.0 m/sec.
The objective of this research is to address issues such as autonomous navigation, force planning, motion planning, and dynamic simulation of actively coordinated wheeled systems.

Inertial sensing systems are used in current robotic systems to obtain orientation and the angular rates of the body of a robotic vehicle for control purposes, and to obtain absolute vehicle position on the terrain for the purpose of guidance. Inertial sensing systems involve drift errors which can be significant even after the vehicle has traversed only short distances. In the presence of accurate maps of the environment, landmark referencing can be used to accurately sense vehicle body kinematics. In this work, a drift-free star sensor based navigation scheme is to be investigated, and its relative advantages and disadvantages as compared to existing systems is to be studied. This navigation scheme is not limited to actively coordinated wheeled systems; it is applicable to any autonomous mobile system operating on unstructured terrain.

Actively coordinated vehicles possess redundancy in actuation, and this redundancy can be used to improve the mobility of the system by 'optimally' allocating the vehicle-terrain contact forces. Even though the force allocation problem of actively coordinated wheeled systems is, in many ways, similar to that of legged systems, it is distinct from that of legged systems since it does not possess complete controllability over all the components of the contact forces. Of course its force distribution characteristics are superior to that of conventional wheeled systems. The 'force planning' problem of actively coordinated wheeled systems has features from both legged systems and conventional wheeled systems, and this unique problem is addressed in this research. Some general theoretical results characterizing redundancy in force distribution with respect to legged and actively coordinated wheeled systems are also studied.
The configuration of an actively actuated wheeled vehicle system on uneven terrain is greatly affected by the local terrain geometry. Since issues such as vehicle stability, coordination, and control are affected by the configuration the locomotion system assumes on the terrain, it is important to study the position kinematics of actively actuated wheeled vehicles. The proposed work will consider a multi-module articulated wheeled vehicle configuration operating on uneven terrain. The kinematic mobility of this configuration on uneven terrain, and the position kinematics of the resulting hybrid series-parallel chain will be addressed. These position kinematic solutions will be used as the basis for developing motion planning algorithms. Motion planning algorithms will be developed largely with respect to the WAAV geometry of Figure 1.7. Such algorithms are also desired for the more general configuration of Figure 1.6. However, the resulting problems are quite complex, and they will not be addressed in any detail in this work.

A dynamic simulator capable of handling complex vehicle configurations such as the one in Figure 1.6 is highly desirable to investigate control schemes, and to compare performance characteristics of different vehicle configurations. It is also required in order to extrapolate results from experiments on a particular vehicle (such as the WAAV) to other vehicles with similar configurations. Dynamic simulation of such vehicle systems with rolling elements operating on uneven terrain is being attempted here for the first time. Preliminary control strategies will also be tested using the dynamic simulator.

Finally, preliminary experiments with the WAAV system will be attempted. The WAAV is a very complex system with a large number of actuators, sensors and associated interface electronics. In this work, the mechanical and electrical hardware development will be attempted, and basic testing of simple mobility maneuvers in a laboratory setting will be attempted. Some of the relatively advanced coordination schemes that are developed as part
of this work, that require sophisticated hardware for testing, may not be tested on the WAAV as part of this work. However, it is expected that these experiments will be performed at a later stage of the project.

1.5 Organization

The research objectives described in the previous section form the basis of Chapters II through VI. Chapter II addresses the problem of 'drift-free navigation' of a robotic vehicle on uneven terrain. Chapter III discusses the issue of 'force planning' with respect to actively coordinated legged and wheeled systems. Force distribution of legged systems have received some attention in the literature [Kumar and Waldron, 1990; Klein and Kittivatcharapong 1988]. In this chapter, theoretical results characterizing redundancy in legged systems are presented, and these results are extended to actively coordinated wheeled systems. Some optimal force allocation schemes are also discussed. In Chapter IV, the 'motion planning' problem with respect to actively coordinated wheeled systems is addressed by solving the position kinematic problems of such vehicles on uneven terrain. Due to the presence of omni-directional motion capability, the position kinematics of legged systems are relatively very simple. Chapter V investigates the topic of dynamic simulation of actively articulated wheeled systems on uneven terrain. Chapter VI provides a discussion of the design details, the sensing and actuation systems, and the electronic hardware of the WAAV. It also gives a description of the current capabilities of the machine. Chapter VII includes a summary of this research effort. It also provides recommendations for future research in the area of actively coordinated wheeled vehicle systems.
CHAPTER II

DRIFT-FREE NAVIGATION

Abstract

The orientation and the angular rates of the body of a robotic vehicle are required for the guidance and control of the vehicle. In the current robotic systems these quantities are obtained by the use of inertial sensing systems. Inertial sensing systems involve drift errors which can be significant even after the vehicle has traversed only short distances on the terrain. A different approach is suggested in this chapter and this approach guarantees accurate, drift-free sensing of the angular position and rates of the vehicle body. A camera system consisting of two cameras in fixed relationship to one another is made to continuously track two stationary objects (stars or the Sun). The camera system is mounted on the vehicle body through an actuated three degree of freedom joint. The angular positions and rates of these joints can be used to evaluate the angular positions and rates of the vehicle body. An estimate of the absolute position of the vehicle on the terrain can also be obtained from this sensing system. This can serve as the primary system for estimating the position of a vehicle on a planet, or as an inexpensive alternative/backup to a more
accurate Global Positioning System (GPS) for estimating the position of a vehicle on earth.

2.1 Introduction

Robotic vehicles have been studied with great interest in the recent years. When operated in unstructured terrain conditions, these vehicles have been shown to possess several advantages over conventional vehicle systems. The control strategy of a mobile robot on an unstructured terrain requires, among other quantities, the knowledge of the vehicle body orientation and angular rates. Further, autonomous robotic vehicles require the knowledge of the current vehicle position and orientation to update the guidance algorithms of the vehicle. Typically, robotic vehicles sense these quantities using commercially available inertial sensing packages and use the process of dead reckoning to update the sensed values. Drift errors are inherent in the gyroscopes since precession due to bearing friction, and other effects, is, essentially, unpredictable. In addition, since orientations are obtained by integrating angular rates, integration errors are also present. The sensed values may become quite inaccurate after the vehicle has traversed a reasonable distance over the terrain. A periodic update of the vehicle body angular positions and rates, with respect to stationary objects in the surroundings, is necessary to ensure satisfactory performance. If the vehicle is used for the purpose of planetary exploration, external landmark referencing cannot be used since sufficiently accurate terrain maps of most of the regions of interest will not be available.

A different approach can be used to sense the required quantities so that they are free from any drift errors. This approach requires the Star Gazing System (SGS), an active
mount carrying two Charge Transfer Device (CTD) tracking cameras, to be placed on the vehicle body. There are two kinds of CTDs that can be used for tracking purposes, Charge Coupled Devices (CCD) and Charge Injection Devices (CID). Both devices have been used in star tracking applications. However, some characteristics of CIDs make them more attractive for star tracking applications. Design details of these cameras are discussed in some detail in Section 2.6. The SGS has three actively actuated degrees of freedom and it is assumed that these three degrees of freedom can be controlled to track any desired orientation of the mount with respect to the vehicle body. In the environment surrounding the vehicle, two stationary objects (stars, Sun, or distant landmarks) are identified. These two objects have to be fixed in position relative to one another in order to use the SGS described here. If the two objects are the Sun and a distant landmark, an additional actively actuated degree of freedom is required between the two camera systems. As the vehicle body traverses the terrain, the SGS is made to lock on to these stationary objects by controlling the three actuators of the mount. A sensor tracking just one object is incapable of sensing orientation changes about the line joining the object and camera center. The positions and the rates of the actuators along with the data from the two cameras of the SGS are then used to obtain the orientation of the vehicle body. The sensed quantities obtained in this manner are drift-free and can be very accurate. In addition to the orientation information, absolute position information of the vehicle can also be estimated from the SGS. This position information is obtained using the same ideas as those used to locate the position of a ship with the help of a sextant. The position information of the vehicle obtained in this manner is limited in its accuracy by the resolution of the cameras. This accuracy can be improved by improving the accuracy of the sensed orientations using signal processing techniques. Better position updates can be obtained from a stationary reference that is at a finite distance from the vehicle. The position information obtained from the SGS can be used as a backup to a more accurate system. The position of a robotic
vehicle operating on earth can be accurately obtained using a satellite based navigation system known as the Global Positioning System (GPS) [Ferguson Jr. and Kroncke, 1981], [Green, 1989], [Neilan and Melbourne 1988]. However, this method is more expensive as compared to the SGS. When robotic vehicles are used for planetary exploration, the absence of satellite navigation makes the SGS an excellent alternative to inertial navigation systems. Satellite navigation can locate a vehicle with errors of the order of a few meters while the SGS errors tend to be of the order of tens of meters.

The Ohio State University Hexapod [McGhee and Iswandhi, 1979], Titan III [Hirose et al., 1984], CMU Quadruped [Raibert et al., 1983], Adaptive Suspension Vehicle [Pugh et al., 1990], AMBLER [Bares et al., 1989], and WAAV [Yu, 1990] are among some of the robotic vehicle systems that have been studied. Some of these vehicles have an inertial sensing system for the purpose of sensing body position and orientation [Waldron and McGhee, 1986], [Bares et al., 1989]. The sensing of the body orientation and angular rates using SGS is similar to the approach used in sensing satellite attitudes using 'star-sensors' [Gai et al., 1983]. Star tracking sensors have also been used in other astronautical applications [Deters and Gutshall, 1987], [Cleavinger and Mayer, 1975]. The most important difference between the situation of a satellite and a mobile robot is that the body of the robotic vehicle undergoes severe irregular motion due to the interaction between the vehicle and the unstructured terrain over which it travels [Waldron, 1985]. The sensors designed for satellite attitude sensing are controlled in an open loop manner since the satellite motion relative to the stars is not very dynamic. A similar sensor system has also been used in the design of solar cells that are required to be appropriately oriented with respect to the sun at all times [Much et al., 1979]. In the case of the SGS a carefully designed, closed loop digital control of three actuators is needed to be able to track the stars. The inertial forces due to the motion of the vehicle body are treated as disturbances
to the controller. In addition, the angular rate of the vehicle body should be small enough to ensure that the stars do not move out of the view of the SGS within a sampling period. The study here considers robotic systems moving at speed of 0.15-0.25 m s\(^{-1}\) over unstructured terrain. This is representative of the requirements of the mobile robotic systems studied in the Mars Rover Sample Return (MRSR) program [Pivirotto and Dias, 1990]. Further, passive wheeled vehicles are considered here since these vehicles undergo the most dynamic body attitude variations due to the interaction with unstructured terrain [Waldron, 1985]. Hence, they are most critical with respect to the design of the SGS.

Section 2.2 discusses the process of sensing the angular position, and components of angular velocity of the vehicle body. Vehicle dynamics and vehicle-terrain interaction models are discussed in Section 2.3. Dynamic tracking of the SGS is studied in Section 2.4. Estimation of absolute vehicle position is considered in Section 2.5. Section 2.6 provides a brief discussion of some of the practical issues involved in star based navigation. Section 2.7 concludes this chapter.

### 2.2 Kinematics of the SGS

The Star Gazing System (SGS) is mounted on the vehicle body as shown in Figure 2.1. The SGS is connected to the body via a three degree of freedom actuated joint similar in geometry to the wrist of a typical six-axis industrial robot. The SGS consists of a stalk that supports two CTD cameras. The angles between the two cameras can be changed by the rotation of an actuated joint which is not controlled actively. Two stationary objects (S\(_1\) and S\(_2\)) in the surroundings of the vehicle are identified and the vectors directed from the initial frame of reference (the world fixed frame) to these objects are noted. In this
Figure 2.1 Camera Mount Containing the Two Cameras Mounted on the Vehicle Body

- \(X_0, Y_0, Z_0\): frame '0' parallel to a fixed world frame (initial reference frame) and attached to the body at its center of mass \(C_0\).
- \(\psi, \theta, \phi\): Euler angles of the body about the frame '0' such that the orientation of the body is represented by a rotation of \(\psi\) about \(X_0\), followed by a rotation of \(\theta\) about \(Y_0\), followed by a rotation of \(\phi\) about \(Z_0\).
- \(X_1, Y_1, Z_1\): frame '1' attached to the camera body at its center of mass \(C_1\).
- \(x_c^1, y_c^1, z_c^1\): frame 'c' attached to camera 'i' at the center of the lens \(c_c^1\).
- \(s_c^1\): unit vector directed from the center of the lens \(c_c^1\) to the \(i\)th star.
- \(u, v, w\): axes intersecting at \(C_a\). \(u\) is fixed to the body, \(v\) varies with rotation \(\alpha\) about \(u\), and \(w\) varies with rotation \(\alpha\) about \(u\) followed by rotation \(\beta\) about \(v\).
- \(\lambda, \varepsilon\): represent the rotations of the cameras with respect to frame '1' about \(Y_1\).
- \(x, y, z\): frame 'b' attached to the body at \(C_0\), parallel to \(u, v, w\) when \(\alpha, \beta, \gamma = 0\).
study, the stationary objects are assumed to be two bright stars, and their positions are known at all times. The motion of the stars is predictable with high accuracy and requires only a simple rotational transformation to convert to the world frame. The axis of rotation can be inferred from the star bearings by simple trigonometry. In order to initiate the navigation process, the SGS has to have the capability to locate the reference stars. In the presence of an orbiter around the planet, the position and orientation of the SGS can be fixed quite accurately when the SGS is 'visible' to the orbiter. Another approach is to use pattern recognition techniques to identify star constellations with respect to which the locations of the reference stars are known. A system that can rapidly identify star patterns using specialized neural network architectures has been demonstrated in simulations by Alvelda et al. [1988]. In the following sub-sections, the kinematics of the SGS is studied in detail, and the process of sensing the orientation and angular rates of the vehicle body is described.

2.2.1 Sensing Vehicle Body Orientation

The sensing process of the SGS can be understood by examining the operation of the cameras. The basic description of a camera along with its associated vectors and coordinate systems is shown in Figure 2.2. The vector $s_c^i$ is the unit vector directed from the center of the lens $c^i$ to $S_i$. In this and the following section it is assumed that $S_1$ and $S_2$ remain in the field of view of the respective CTD cameras. This requires a closed loop control of the joint actuators and this control issue will be addressed in a later section.

In Figure 2.1, $X_0Y_0Z_0$ is the coordinate frame parallel to a fixed initial frame with its origin at body center of mass $C_0$. The quantities $\psi$, $\theta$, and $\phi$ are the Euler angles of the body about the frame '0'. Frame '1' is attached to the camera body at its center of mass. The vector $s_c^i$ is the unit vector directed from the lens center of the $i^{th}$ camera to the star $S_i$. 
Figure 2.2 Camera 'i' (i = 1, 2)

- $n_c^i$: vector along the unit vector $s_c^i$.
- $O_i$: center of the screen.
- $A_i, B_i$: screen coordinate system.
- $O_i c_c^i$: focal length, $f_i^1$, of the lens.
The angles \( \lambda \) and \( \varepsilon \) (about \( Y_1 \)) represent the orientations of the cameras with respect to frame '1'. The coordinate system \( xyz \) is the body frame 'b' attached to the body at \( C_0 \). The position of the three degree of freedom joint at 'a' is represented by the three angles \( \alpha \), \( \beta \) and \( \gamma \) (rotation of \( \alpha \), followed by \( \beta \), followed by \( \gamma \)).

In Figure 2.2, \( n^i_c \) is a vector along the unit vector \( s^i_c \), \( O_i \) is the screen center, \( A_i B_i \) is the screen coordinate system, and \( O_i c^i_c \) is the focal length \( f' \) of the lens. The screen coordinates of the image of \( S_i \), \(( a_i, b_i)\), are sensed by the camera. The unit vector \( s^i_c \) is given by

\[
\begin{align*}
    s^i_c &= \frac{1}{\sqrt{a_i^2 + b_i^2 + f_i^2}} \begin{bmatrix} -a_i \\ -b_i \\ f_i \end{bmatrix} \\
    s_1^1 &= R_{c} s^1_c, \quad s_1^2 = R_{c-\lambda} s^2_c
\end{align*}
\] (2.1)

In the above and following equations, \( s^i_j \) is the unit vector associated with \( S_i \) in the reference frame 'j' and

\[
R_\eta = \begin{bmatrix} \cos \eta & 0 & \sin \eta \\ 0 & 1 & 0 \\ -\sin \eta & 0 & \cos \eta \end{bmatrix}.
\]

Form
\[
\begin{align*}
    s^3_j &= \frac{s^1_j \times s^2_j}{|s^1_j \times s^2_j|}, \quad s^4_j = \frac{s^1_j \times s^3_j}{|s^1_j \times s^3_j|}.
\end{align*}
\]

Let \( R_{1} \) be the rotational transformation matrix from the sidereal frame to the camera frame.

\[
R_{1} = [s^1_1 \quad s^3_1 \quad s^4_1] \quad (2.2)
\]
The vectors directed from the world fixed frame to the objects \( S_1 \) and \( S_2 \) are assumed to be known as functions of time. These vectors are \( s_0^1 \) and \( s_0^2 \). Again form,

\[
\begin{align*}
\mathbf{s}_0^3 &= \frac{s_0^1 \times s_0^2}{|s_0^1 \times s_0^2|}, \\
\mathbf{s}_0^4 &= \frac{s_0^1 \times s_0^3}{|s_0^1 \times s_0^3|}
\end{align*}
\]

Let \( \mathbf{R}_0^s \) be the rotational transformation matrix from the sidereal frame to the world frame.

\[
\mathbf{R}_0^s = \begin{bmatrix} s_0^1 & s_0^3 & s_0^4 \end{bmatrix}
\]

(2.3)

Let \( \mathbf{R}_0^1 \) be the rotation transformation matrix from frame '1' to frame '0'. Hence,

\[
\mathbf{s}_0^k = \mathbf{R}_0^1 \mathbf{s}_1^k \quad k = 1,2,3,4.
\]

Then, \( \mathbf{R}_0^1 = \mathbf{R}_0^s (\mathbf{R}_1^s)^T \).

(2.4)

Let \( \mathbf{R}_0^b \) be the rotation transformation matrix from frame 'b' to frame '0' (body coordinates xyz) and let \( \mathbf{R}_b^1 \) be the rotation transformation matrix from frame '1' to frame 'b'. \( \mathbf{R}_0^b \) is a function of the body orientation (\( \psi, \theta, \) and \( \phi \), the Roll, Pitch and Yaw Euler angles) and \( \mathbf{R}_b^1 \) is a function of the joint angles (\( \alpha, \beta, \) and \( \gamma \)). These rotation matrices are given below. A detailed discussion of Euler angles and rotation matrices can be found elsewhere [Fu et al., 1987].

\[
\mathbf{R}_0^b = \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\
S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\
- S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix}
\]

(2.5)

\[
\mathbf{R}_b^1 = \begin{bmatrix} C\beta S\gamma & -C\beta C\gamma & S\beta \\
S\alpha S\beta C\gamma + C\alpha S\gamma & C\alpha C\gamma - S\alpha S\beta S\gamma & - S\alpha C\beta \\
S\alpha S\gamma - C\alpha S\beta C\gamma & C\alpha S\beta S\gamma + S\alpha C\gamma & C\alpha C\beta \end{bmatrix}
\]

(2.6)
The angles $\alpha$, $\beta$, and $\gamma$ are assumed to be sensed by encoders on the motor shafts and are available at any instant. Hence, $R_b^1$ can be evaluated. Further, $R_0^1 = R_0^b R_b^1$. Hence,

$$R_0^b = R_0^1 (R_b^1)^T \quad (2.7)$$

Use of Equation 2.4 followed by Equation 2.7 gives the rotation transformation matrix relating the body fixed frame 'b' to the world fixed frame '0'. The Euler angles can then be obtained from Equation 2.5. It should be noted that the Roll ($\psi$) and Pitch ($\theta$) are such that $-\pi/2 < \psi, \theta < \pi/2$. Hence, if $R_0^b = r_{mn}$,

$$\theta = -\sin^{-1}(r_{31}), \quad \psi = \sin^{-1}(r_{32}/\cos \theta), \quad \phi = \text{atan2}(r_{21}/\cos \theta, r_{11}/\cos \theta) \quad (2.8)$$

The Euler angles obtained in Equation 2.8 are free from drift errors and are limited in their accuracy only by the resolution of the CTD camera and the shaft encoders, and by the accuracy of the time varying vectors $s_q^i$. The CTD cameras used in the star-tracker applications have an accuracy of about 1 - 10 $\mu$rads [Gai et al., 1983], [Deters and Gutshall, 1987]. Similar resolutions can be obtained from encoders. The time varying vectors $s_q^i$ can be very accurately obtained as a function of the time of the day, the time of the year, and the current location of the vehicle on the planet. The location of the vehicle is known only approximately and this will introduce some errors into the sensed angles in Equation 2.8. However, these errors are very small and are expected to be of lower order as compared to the sensing errors.

2.2.2 Angular Velocities of the Vehicle Body

The sensing noise present in the sensed joint angles and camera coordinates, limits the accuracy of the angular velocities obtained by numerically differentiating the Euler angles obtained in Equation 2.8. A different approach is to 'assume' that the SGS mount is
exactly tracking the stars at all times. Hence, the angular velocity of the SGS with respect to the fixed frame is zero. The joint angular velocities can be sensed by the use of tachometers on the shafts. The body angular velocities can then be obtained as follows:

Let $\omega$ be the angular velocity of the SGS in the body frame 'b', and $\Omega$ be the angular velocity of the body in the frame 'O'. Let, $l = [1 \ 0 \ 0]^T$, $m = [1 \ \alpha \ \alpha \alpha]^T$, $n = [\beta \ -\alpha \beta \ \alpha \beta]^T$. Then, $\omega + \Omega = e_v$, where $\omega = \dot{\alpha}l + \dot{\beta}m + \dot{\gamma}n$, and $e_v$ is the angular velocity tracking error. Assuming that $e_v$ is negligible, the sensed angular velocities are:

$$\psi_s = -\alpha, \ \theta_s = (\alpha \beta \gamma) \hat{\gamma} - (\alpha \beta) \hat{\beta}, \ \phi_s = - (\alpha \beta \gamma) \hat{\gamma} - (\alpha \beta) \hat{\beta}$$

(2.9)

2.2.3 Tracking Errors in $\alpha$, $\beta$, and $\gamma$

The SGS is required to track the objects $S_1$ and $S_2$. This tracking can be achieved by controlling the three actuators so that the vectors $s_c^i$ are aligned along the axes $z_c^i$ as closely as possible. The tracking errors in the alignment of the unit vectors $s_c^i$ can be converted to equivalent errors in the joint angles $\alpha$, $\beta$, and $\gamma$. These errors are required for the closed loop control of the three joints. Let the errors be denoted by $\delta\alpha$, $\delta\beta$, and $\delta\gamma$. These values can be obtained in the following manner. Let $P_0^1$ be the rotation transformation from frame '1' to frame 'O' at the initial instant, when the unit vectors $s_c^i$ are assumed to be aligned along the axes $z_c^i$ exactly. The current transformation from frame '1' to 'O' is $R_0^1$. The transformation due to the errors $\delta\alpha$, $\delta\beta$, and $\delta\gamma$, $R_\delta$ is

$$R_\delta = P_0^1 (R_0^1)^T$$

(2.10)

$$R_\delta = \begin{bmatrix} C\delta\beta S\delta\gamma & -C\delta\beta C\delta\gamma & S\delta\beta \\ S\delta\alpha S\delta\beta C\delta\gamma + C\delta\alpha S\delta\gamma & C\delta\alpha C\delta\gamma - S\delta\alpha S\delta\beta S\delta\gamma & -S\delta\alpha C\delta\beta \\ S\delta\alpha S\delta\gamma - C\delta\alpha S\delta\beta C\delta\gamma & C\delta\alpha S\delta\beta S\delta\gamma + S\delta\alpha C\delta\gamma & C\delta\alpha C\delta\beta \end{bmatrix}$$

(2.11)
The errors are usually small angles (large errors imply that \( S_1 \) or \( S_2 \) is out of the range of the CTD cameras), and hence it is assumed that \( \delta \alpha, \delta \beta, \delta \gamma \in [-\pi/2, \pi/2] \). Let \( R_\delta \) obtained from Equation 2.10 be \( p_{mn} \), then from Equation 2.11,

\[
\delta \beta = \sin^{-1}(p_{13}), \quad \delta \gamma = \sin^{-1}(-p_{12}/\cos \delta \beta), \quad \delta \alpha = \sin^{-1}(-p_{23}/\cos \delta \beta)
\]  

(2.12)

### 2.2.4 Simulating Stars \( S_1 \) and \( S_2 \), and the Corresponding Cameras

For the purpose of a numerical simulation, the locations of the two stars as sensed by the CTD cameras have to be evaluated. The time varying vectors \( s_0^i \) are known. It is required to evaluate the camera coordinates \( (a_i, b_i) \). In the real system these values are available directly from the CTD elements.

At any instant the angles \( \alpha, \beta, \) and \( \gamma \) are available from the shaft position sensors and the angles \( \theta, \phi, \) and \( \psi \) are taken from the simulation. From Equations 2.5 and 2.6, \( R_0^b \), and \( R_b^1 \) can be evaluated. Then, \( R_0^1 = R_0^b R_b^1 \). If \( R_e \), and \( R_{e-\lambda} \) are the same as before,

\[
s_c^1 = (R_e)^T (R_0^1)^T s_0^1.
\]

\[
s_c^2 = (R_{e-\lambda})^T (R_0^1)^T s_0^2.
\]

Let, the value of \( s_c^i \) obtained above be equal to \([c_i, d_i, e_i] \). Then

\[
a_i = -(c_i / e_i) f_i, \quad b_i = -(d_i / e_i) f_i,
\]

(2.13)

completing the computation of the camera coordinates.
2.3 Vehicle Terrain Dynamics

In this section robotic vehicles, equipped with the SGS, operating on unstructured terrain are considered. The study here considers robotic systems moving at speeds of 0.15-0.25 ms\(^{-1}\) over unstructured terrain. This is representative of the requirements of the mobile robotic systems studied in the Mars Rover Sample Return (MRSR) program [Pivirotto and Dias, 1990]. In general, a vehicle traversing unstructured terrains will undergo significant dynamic motion. However, due to the compliance and the damping losses at the vehicle-ground interface, high frequency components are attenuated from the vehicle response. The vehicle-ground interface acts as a low pass filter and an estimate of the 'cut-off' frequency can be obtained. For a given speed of the vehicle, legged robotic systems have a much lower cut-off frequency as compared to wheeled vehicles [Waldron, 1985]. Further, actively suspended wheeled vehicles can handle difficult terrain situations better than passive systems. Actively coordinated wheeled vehicle systems are discussed in the literature [Waldron et al., 1991a]. Hence, from the point of view of design of a tracking controller for the SGS, passive wheeled vehicle systems are the most critical. For the purpose of this study, a six wheeled passive vehicle configuration is considered (Figure 2.3). Some of the other relevant passive wheeled vehicle configurations can be found elsewhere [NASA, 1991]. In particular, a passive wheeled vehicle system being considered for exploration of planetary surfaces is the Rocker Bogie Concept [Lindermann, 1990]. The Rocker Bogie Concept and other such wheeled vehicle systems have the main body composed of several rigid bodies, or modules, interconnected by actuated joints, or passive joints. The vehicle shown in Figure 2.3 is made up of one rigid body mounted on six wheels. Hence, for a given terrain condition, the kinematic quantities of this vehicle body will contain higher frequency components as compared to that of the other wheeled
vehicle systems. If the SGS tracking system works satisfactorily on this vehicle, it can be assumed that it will work satisfactorily on all the other wheeled and legged vehicle systems that are considered for exploration of planetary surfaces.

In Figure 2.3, each wheel-terrain contact point is modelled as a spring and damper in parallel. Approximate values of spring stiffness can be obtained from the tire material used in such vehicles. The damping coefficients to be used in this model depends on the nature of the terrain. Terrains made up of hard rocks (lava fields on the surface of Mars) lead to low values of damping. Terrains composed of sand and soil involve much higher losses and hence are overdamped. In this section a linearized vehicle dynamic model is developed first (Section 2.3.1), a dynamic model with gaussian noise is investigated next (Section 2.3.2), a study of wheel-soil interaction is discussed next (Section 2.3.3), followed by a method to obtain an equivalent damping due to the losses at the wheel-soil interface (Section 2.3.4). Finally, some simulation results of the motion of the six wheeled
vehicle on difficult terrains are given (Section 2.3.5). Section 2.3 primarily deals with the vehicle dynamics and terrain properties. The dynamics and tracking control of the SGS is the topic of Section 2.4.

2.3.1 Linearized Vehicle dynamics

A simplified dynamic analysis of the vehicle body is performed here to obtain the vehicle response to the terrain. The following assumptions are made for this dynamic analysis:

The pitch angle $\theta$, the roll angle $\psi$, and the displacement in the $z$ direction are primarily affected by the terrain variations. As a first order approximation, the yaw angle $\phi$, the $x$ and the $y$ displacements can be taken to be unaffected by the terrain. The stiffness and damping characteristics are the same for all the six wheel contact points. Finally, it is assumed that the three wheels on a given side of the vehicle experience the same terrain input, the input to the second and the third wheel lagging the input to the first wheel by $\tau_2$ and $\tau_3$ seconds respectively. For small $\theta$ and $\psi$, the deflections of the tires at the contact points, $\delta_{R_i}$ and $\delta_{L_i}$, are obtained as follows (It is important to note that when, $\delta_{R_i}$ or $\delta_{L_i}$ is less than zero this model is nonlinear since one of the wheels is lifted off the terrain and the contact forces become zero for that wheel.):

$$\delta_{R1} = z_{R1} - z + q \psi + p\theta \quad \delta_{L1} = z_{L1} - z - q \psi + p\theta$$
$$\delta_{R2} = z_{R2} - z + q \psi \quad \delta_{L2} = z_{L1} - z - q \psi$$
$$\delta_{R3} = z_{R3} - z + q \psi - p\theta \quad \delta_{L3} = z_{L1} - z - q \psi - p\theta$$

(2.14)

In the above equations, $z_{R_i}$ and $z_{L_i}$ are the $z$ coordinates of the terrain at the six contact points. From the above assumption about the terrain input,
\[ z_{R_i}(t) = z_{R_1}(t-T_i), \quad z_{L_i}(t) = z_{L_1}(t-T_i) \quad i = 2, 3 \]  

Considering the vehicle as a free body, the equations of motion in \( \theta, \psi, \) and \( z \) are:

\[ M \ddot{z} = K \sum_{i=1}^{3} (z_{R_i} + z_{L_i}) - 6Kz + C \sum_{i=1}^{3} (\dot{z}_{R_i} + \dot{z}_{L_i}) - 6Cz \]  

\[ I_x \ddot{\psi} = qK \sum_{i=1}^{3} (z_{L_i} - z_{R_i} - 2q\psi) + qC \sum_{i=1}^{3} (\dot{z}_{L_i} - \dot{z}_{R_i} - 2q\dot{\psi}) \]  

\[ I_y \ddot{\theta} = pK(z_{R_3} + z_{L_3} - z_{R_1} - z_{L_1} - 4p\theta) + pC(\dot{z}_{R_3} + \dot{z}_{L_3} - \dot{z}_{R_1} - \dot{z}_{L_1} - 4p\dot{\theta}) \]

where \( M \) is the mass of the vehicle system, and \( I_x \) and \( I_y \) are the mass moments of inertia of the vehicle about \( X_0 \) and \( Y_0 \) respectively.

A frequency domain analysis of the above system is performed by substituting,

\[ z_{R_1} = A_R e^{i\omega t}, \quad z_{L_1} = A_L e^{i\omega t}, \quad z = Z e^{i\omega t}, \quad \theta = \Theta e^{i\omega t}, \quad \psi = \Psi e^{i\omega t} \]

In the above and the following equations, \( i = \sqrt{-1} \). Substitution of Equations 2.19 and 2.15 into Equations 2.16, 2.17, and 2.18 leads to

\[ Z = H_z(\omega) \{ A_R + A_L \}, \quad \psi = H_{\psi}(\omega) \{ A_L - A_R \}, \quad \Theta = H_{\theta}(\omega) \{ A_R - A_L \} \]

\[ H_z(\omega) = \frac{(K + iC\omega)}{(6K - 6C\omega^2) + i(6C\omega)} \]

\[ H_{\psi}(\omega) = \frac{q(K + iC\omega)}{(6q^2K - I_x\omega^2) + i(6q^2C\omega)} \]

\[ H_{\theta}(\omega) = \frac{p(K + iC\omega)}{(6p^2K - I_y\omega^2) + i(4p^2C\omega)} \]
The functions $|H_x(\omega)|^2$, $|H_y(\omega)|^2$, and $|H_\theta(\omega)|^2$ represent the power spectral densities of the above transfer functions. These functions are proportional to the square of the response of the vehicle, in the frequency domain, to a purely random terrain (white noise) that contains components of equal amplitude at all the frequencies. These power spectral densities can be studied to characterize the high frequency attenuation obtained from the vehicle suspension. As in the case of simple mechanical systems, a small value of $M$, $I_x$, or $I_y$ increases the undamped natural frequency and hence the speed of response, while a small value of $K$ decreases the undamped natural frequency and hence the speed of response. A small value of $C$ (underdamped case) leads to a sharp resonant peak, while a large value of $C$ (overdamped case) leads to a relatively flat response. The power spectral densities for typical underdamped ($\zeta = 0.1$) and overdamped ($\zeta = 2.5$) situations are shown in Figures 2.4, 2.5 and 2.6. Here $M = 400$ kg, $I_x = 16.67 \text{ kg m}^2$, $I_y = 150 \text{ kg m}^2$, $K = 17500 \text{ N m}^{-1}$, $q = 0.25 \text{ m}$, $p = 1.0 \text{ m}$, $\tau_2 = 7 \text{ s}$, and $\tau_3 = 14 \text{ s}$. The mechanical model of the system is shown in Figure 2.3. The cut-off frequency $\omega_c$ is defined to be that frequency above which the power spectral density remains less than one-fifth of its value at $\omega = 0$. It is assumed that the terrain frequency components above $\omega_c$ are attenuated. For the variable $z$, $\omega_c = 27 \text{ rad/s}$, for $\zeta = 0.1$ and $\omega_c = 159 \text{ rad/s}$, for $\zeta = 2.5$ (Figure 2.4). For the variable $\psi$, $\omega_c = 36 \text{ rad/s}$, for $\zeta = 0.1$ and $\omega_c = 238 \text{ rad/s}$, for $\zeta = 2.5$ (Figure 2.5). For the variable $\theta$, $\omega_c = 37 \text{ rad/s}$, for $\zeta = 0.1$ and $\omega_c = 276 \text{ rad/s}$, for $\zeta = 2.5$ (Figure 2.6). In the underdamped case, the resonant frequency ($\omega_r$), and the ratio of the power spectral density at $\omega_r$ to that at zero frequency ($\eta$) are useful quantities. For the variable $z$, for $\zeta = 0.1$, $\omega_r = 15.2 \text{ rad/s}$, $\eta = 18.6$ (Figure 2.4 (a)). For the variable $\psi$, for $\zeta = 0.1$, $\omega_r = 18.0 \text{ rad/s}$, $\eta = 12.0$ (Figure 2.5 (a)). For the variable $\theta$, for $\zeta = 0.1$, $\omega_r = 22.4 \text{ rad/s}$, $\eta = 12.1$ (Figure 2.6 (a)).
Figure 2.4 (a) $|H_x(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$)

Figure 2.4 (b) $|H_x(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$)
Figure 2.5 (a) $|H(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$)

Figure 2.5 (b) $|H(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$)
Figure 2.6 (a) $|H_0(\omega)|^2$ vs $\omega$ (Underdamped, $\zeta = 0.1$)

Figure 2.6 (b) $|H_0(\omega)|^2$ vs $\omega$ (Overdamped, $\zeta = 2.5$)
The terrain spatial wavelength ($\lambda$), the frequency ($\omega$), and the vehicle velocity $v$ are related by the relation, $\lambda = \frac{2\pi v}{\omega}$. For $\zeta = 2.5$, the highest $\omega_c = 238$ rad/s. The vehicle velocity $v$ was taken to be 0.14 m/s. The equivalent $\lambda_c = 3.7$ mm. Any terrain component that is smaller than 3.7 mm will not affect the motion of the vehicle body. For $\zeta = 0.1$, the highest $\omega_c = 37$ rad/s. The equivalent $\lambda_c = 23.8$ mm. Any terrain component that is smaller than that will not affect the motion of the vehicle body. The resonant frequencies range from 15.2 rad/s to 22.4 rad/s. This translates to $\lambda_c$ varying from 39.3 mm to 57.9 mm. Finally, the largest value of $\eta$ is 18.6. This means that the amplitude of response of the vehicle body at $\omega_1$ is about 4.3 times that at zero frequency.

As stated earlier, a passive wheeled vehicle is considered here, since it is the most critical with respect to the design of the SGS. The motion of a passive wheeled vehicle over rocky, hard terrain surfaces involves very small internal energy losses. The losses are mainly due to friction at the wheel axles, structural damping, and losses in the tires. This situation can be studied as an underdamped case of the simplified analysis discussed above. Typical values of $\zeta$ considered here range from 0.05 to 0.1. The other situation considered here is the motion of the vehicle through sand and soil that leads to large energy losses at the wheels. As an example, available projections of terrain characteristics on the Martian surface are used to study the feasibility of the SGS on an autonomous mobile robotic system traversing the Martian surface. An estimate of the 'equivalent damping' for the vehicle traversing soil and sand has been obtained and the analysis is given in Sections 2.3.3 and 2.3.4. The analysis in these sections is based on the study of the wheel-soil interaction on the Martian surface performed by McCullough [McCullough, 1989]. In the next section the linearized dynamic model is modified by adding Gaussian noise to the computed accelerations to account for the simplifications in the linearized model.
2.3.2 Dynamic Model With Gaussian Noise

In the linearized model, the suspension variables $\psi$, $\theta$ and $Z_0$ were assumed to be affected by terrain interactions and the other kinematic quantities, $x$, $y$ and $\phi$ were assumed to be independent of vehicle terrain interaction. These assumptions are reasonable only if the terrain obstacles are small compared to the vehicle geometric parameters. Further, the vehicle terrain interaction modelled by springs and dampers was simplistic. Exact dynamic models of rolling elements operating on complex terrain geometries are not available. The contact force vectors at the wheel-terrain contact points cannot be easily modelled. Also mechanical properties of terrains composed of soil, rock and sand cannot be characterized accurately. To make the simulation of the SGS more realistic, the unmodelled dynamics in the linearized model considered above can be modelled by adding gaussian noise components to the computed values of $\ddot{x}$, $\dot{y}$, $\ddot{z}$, $\psi$, $\dot{\theta}$, and $\dot{\phi}$. These gaussian noise components were taken to be a band limited white noise whose break frequency is $1kHz (= 25$ times the maximum cut-off frequency of the vehicle body response). These gaussian noise components were assumed to have zero mean, and the standard deviation was assumed to be half of the maximum values of $\ddot{x}$, $\dot{y}$, $\ddot{z}$, $\psi$, $\dot{\theta}$, and $\dot{\phi}$ computed from the linearized models. Further, if the value of $z_p$ (Figure 2.10) is increased to a value that is comparable to the vehicle parameters, the resulting vehicle body motion is quite drastic, and can be assumed to be sufficient to verify SGS based navigation of robotic vehicles moving at about 15-0.25 m-s$^{-1}$. Simulation of the vehicle motion using this model and the linearized model of Section 2.3.1 are given in Section 2.3.5.

2.3.3 Wheel-Soil Mechanics

In this section the motion of a wheeled vehicle over a terrain comprised of soil and sand is studied. An approximate value of the equivalent damping of the terrain-wheel
interface is obtained, and this is used with the vehicle model of Figure 2.3 to simulate the vehicle motion on such terrains. A Martian terrain model is used for the purpose of this study. Much of the Martian terrain is thought to be composed of weak and dry soil [Moore et al., 1979]. For wheels rolling slowly over such soil, the rolling resistance is largely due to the vertical loading of the soil by the wheel. The effects of bulldozing and compaction are considered to be negligible. Hence, the energy losses are the same, whether the soil is loaded vertically, or whether a wheel transmitting the same vertical load rolls over the soil.

In this section a wheel rolling over such weak and dry soil is considered. The analysis here uses an equation which was developed for the study of the footings of structures of buildings. This equation is known as the Terzaghi's equation, and it is based on three fundamental soil properties. These properties are density (ρ), cohesion (c), and internal friction angle (φ) of the soil. Typical values of ρ, c, and φ for Martian soil are available [Moore et al., 1982]. Terzaghi's equation is:

\[
\sigma = c N_c + \gamma z N_q + 0.5 w \gamma N_\gamma
\]  

(2.24)

where \(\sigma\) is the ultimate bearing stress, \(\gamma\) is the specific weight of soil (\(\gamma = \rho g, g\) is the acceleration due to gravity), \(w\) is the wheel width, \(N_c, N_q,\) and \(N_\gamma\) are dimensionless parameters depending on \(\phi\), and \(z\) is the sinkage into the soil (see Figure 2.7). The values of \(N_c, N_q,\) and \(N_\gamma\) for a given value of \(\phi\) can be obtained from general shear plots of soil materials [McCarthy, 1982]. The best case parameters of the soil, with respect to sinkage and energy loss, are estimated to be \(\phi = 20^\circ, \rho = 1400 \text{ kg m}^{-3},\) and \(c = 1.0 \text{ kPa}.\) For \(\phi = 20^\circ,\) from the shear plots, \(N_c = 13, N_q = 6.5,\) and \(N_\gamma = 3.0.\) The worst case soil conditions can be characterised by \(\phi = 15^\circ, \rho = 1000 \text{ kg m}^{-3},\) and \(c = 0.\) \(N_c\) drops out of the analysis, and from the shear plots, \(N_q = 3.5,\) and \(N_\gamma = 1.0.\) The above numbers represent extreme values of the soil parameters. The best case parameters represent soil characteristics that lead to the least power consumption and the worst case parameters
represent soil that leads to the most power consumption. The overall best case terrain for
the vehicle is, of course, a terrain composed of smooth hard rock. However, in this and
the following section, only a terrain composed of weak and dry soil is considered. The
wheel-soil mechanics discussed here assumes that the soil parameters range from the worst
case parameters to the best case parameters given above.

A cylindrical wheel traversing a terrain comprising of weak, dry soil is shown in
Figure 2.7. The contact surface between the wheel and the soil has to be determined. The
effective length of contact can be taken to be \( \alpha s \), where \( 0 < \alpha < 2 \), and \( s = (2 R - z^2)^{1/2} \),
since the effective length cannot be greater than \( 2s \).

For the purpose of this analysis it is assumed that the vehicle load is equally
distributed among the six wheels. This load should be equal to bearing stress times the
contact surface area projected onto the horizontal plane. Hence,

\[
\frac{Mg}{6} = (cN_c + \rho g z N_q + 0.5 \rho g N_y) \alpha s w
\]

In the above equation the unknowns are \( z \), \( w \) and \( \alpha \). Writing the above equation as a
quadratic in \( w \),

\[
(0.5 \alpha s \rho g N_y) w^2 + (\alpha s z \rho g N_q + \alpha s c N_c) w - \frac{Mg}{6} = 0 \quad (2.25)
\]

A reasonable value of \( \alpha \) has to be assumed and the results can be significantly affected by
the value chosen for \( \alpha \). This leads to some uncertainty in the analysis. For the purpose of
this study, \( \alpha \) was simply taken to be 1.0. Equation 2.25 thus relates the wheel width \( w \)
and the wheel sinkage \( z \). For a given value of \( z \), \( w \) can be obtained by solving the
quadratic Equation 2.25. However, \( z \) cannot be solved in closed form for a given value of
\( w \), since \( s \) is a function of \( z \) (\( s = (2 R - z^2)^{1/2} \)).
Next, an estimate of the power consumed by the wheel rolling in the soil is desired. At the beginning of this section it was assumed that the power losses due to the rolling of the wheel are mainly due to vertical loading effects, for the kind of soil we are interested in.

\[ z = R (1 - \cos \theta), \quad s = R \sin \theta \]  

(2.26)

The work done on the soil \( W = \int_0^{z_u} F(z) \, dz \), where \( F(z) = w \alpha s \sigma \) (\( \sigma \) is obtained from the Terzaghi's equation). The power consumed \( P \) can be viewed as the rate of new soil work done on the leading half of the wheel. Hence,

\[ P = 0.5 \frac{dW}{dt} = 0.5 \int_0^{z_u} (dF/dz) (dz/dt) \, dz \]

Using Equation 2.26, the above integral can be converted to an integral in the variable \( \theta \).

\[ P = 0.5 \int_0^{z_u} (dF/d\theta) (d\theta/dt) R \sin \theta \, d\theta \]
where \( F(\theta) = w \alpha R^2 \gamma \left[ \sin \theta (1 - \cos \theta) N_q + 0.5 \sin^2 \theta N_y \right] \). For dry, weak soil the cohesive effects are negligible and hence that term does not appear from the Terzaghi Equation in the above relation for \( F(\theta) \). Simplifying,

\[
P = 0.5 w \alpha R^3 \gamma \int_0^\theta (d\theta/dt) \left[ (\sin \theta \cos \theta - \sin \theta \cos 2\theta) N_q + \sin^2 \theta \cos \theta N_y \right] d\theta
\]

The horizontal forward velocity of the vehicle is \((ds/dt)\) and this is assumed to be a constant (about 0.15 m/s). From Equation 2.26,

\[
(ds/dt) = R \cos \theta (d\theta/dt).
\]

Using this in the above equation for \( P \) and performing the integration,

\[
P = 0.5 w \alpha R^2 \gamma (ds/dt) \left[ \left\{ 0.75 - \cos \theta u + 0.25 \cos^2 \theta u - 0.5 \sin^2 \theta u \right\} \ln (\cos \theta u) \right] N_q + \left\{ 0.5 \theta u - 0.25 \sin 2\theta u \right\} N_y
\]

(2.27)

The plots of wheel width versus sinkage and power versus sinkage for the best case parameters is shown in Figure 2.8 and the same plots for the worst case parameters are shown in Figure 2.9. These plots are used below to obtain estimates of equivalent damping coefficients for the model of Figure 2.3, as the vehicle traverses different kinds of soil. This is the topic of the next section.

### 2.3.4 Equivalent Damping

The values of damping coefficients to be used for the model of Figure 2.3, to simulate the vehicle moving through soil, are evaluated in this section. A numerical simulation of the motion of the six wheeled vehicle (Figure 2.3) over a particular terrain (Figure 2.10) was performed. The differential equations of motion of the vehicle coordinates \( z, \psi, \) and \( \theta \) are given by Equations 2.16, 2.17, and 2.18. These equations are
Figure 2.8 Power and Wheel Width Versus Sinkage for the 'Best Case' Soil Parameters

Figure 2.9 Power and Wheel Width Versus Sinkage for the 'Worst Case' Soil Parameters
integrated to obtain the time histories of the vehicle kinematic parameters. The dynamic simulation package ACSL was used for the numerical integration of the differential equations. The terrain was considered to be a periodic triangular function, the terrain under the right wheels being offset by a certain amount with respect to the terrain under the left wheels to allow rotations about the pitch and roll directions. A triangular signal contains frequency components at discrete values that are multiples of the basic frequency. By varying the various terrain parameters shown in Figure 2.10, terrains of different degrees of difficulty can be obtained. Hence, this terrain can be used to study the performance of
the SGS mounted on the vehicle configuration of Figure 2.3. Various values of damping coefficients were used and corresponding values of average power consumed during the vehicle motion were obtained from the simulation. A plot of the average power consumed ($P_{av}$) versus the damping coefficient ($\zeta$) is shown in Figure 2.11. The plot shows a linear variation of $P_{av}$ with respect to $\zeta$ and the relationship obtained from the data is as follows:

$$P_{av} = 23.3360 \zeta + 1.5243$$  \hspace{1cm} (2.28)

The vehicle parameters (Figure 2.3), and the parameters of the terrain geometry (Figure 2.10) that were used for the simulation, are given below.

\begin{align*}
M &= 400 \text{ kg}, ~ I_x = 16.67 \text{ kg m}^2, ~ I_y = 150 \text{ kg m}^2, ~ K = 17500 \text{ N m}^{-1}, ~ q = 0.45 \text{ m}, ~ p = 1.05 \text{ m}, \\
\text{and vehicle speed } ds/dt &= 0.15 \text{ m s}^{-1}; ~ z_p = 0.15 \text{ m}, ~ x_p = 0.5 \text{ m}, ~ f = 0.125 \text{ m}.
\end{align*}

The power consumed as the vehicle traverses different kinds of Martian soil can be obtained from the previous section. For the best case soil parameters ($\phi = 20^\circ$, $\rho = 1400 \text{ kg m}^{-3}$, $c = 1.0 \text{ kPa}$, $N_c = 13$, $N_q = 6.5$, and $N_\gamma = 3.0$), Figure 2.8 shows that a wheel width $w = 0.25 \text{ m}$ leads to a sinkage of about 3.3 cm, and a power consumption per wheel $P = 9.10 \text{ W}$. The average power consumption for the vehicle is 54.6 W and this corresponds to an equivalent damping $\zeta_{eq}$ of about 2.25 using Equation 2.28. The worst case soil parameters ($\phi = 15^\circ$, $\rho = 1000 \text{ kg m}^{-3}$, $c = 0$, $N_q = 3.5$, and $N_\gamma = 10.0$) plots are shown in Figure 2.9. For $w = 0.25 \text{ m}$, the sinkage value becomes greater than the radius of the wheel $R = 0.25 \text{ m}$. The wheel-soil model used here cannot handle $z > R$. Hence $w$ is taken to be 0.30 m. At this value of the wheel width, sinkage $z = 0.21 \text{ m}$, and $P = 261 \text{ W}$. The average power consumption for the vehicle is 1566 W and this corresponds to an equivalent damping $\zeta_{eq}$ of about 66 using Equation 2.28. This is an extremely large value of $\zeta_{eq}$ and can be reduced to some extent by further increasing the width of the wheel. In conclusion, the equivalent damping $\zeta_{eq}$ ranges from about 2.25 to 66 for the wheeled

vehicle traversing different kinds of soil. It is to be noted here that the equivalent damping values obtained depend not only on the soil parameters and the wheel width, but also on the terrain geometry chosen for the numerical simulation. In fact if the terrain geometry is chosen to be a perfectly flat horizontal surface, the vehicle will have zero vertical velocity components at the contact points, and will have no losses due to damping. The energy losses from the soil model will still be present and hence the equivalent damping will become infinite.
2.3.5 Vehicle Simulation Results

This section presents some results from the dynamic simulation of the vehicle of Figure 2.3 over two basic types of terrain (terrain geometry as in Figure 2.10). The simulation was performed using the numerical package Advanced Continuous Simulation Language (ACSL). The first kind of terrain is assumed to be a lava field where the damping is very low ($\zeta = 0.1$) and the plots of $\psi$, $\theta$, and $z$, along with plots showing the variation of $\psi$, and $\dot{\psi}$ with time, are displayed in Figure 2.12. The last two plots show the nature of variations of angular velocity, and angular acceleration with time. The second kind of terrain considered is the best case soil terrain ($\zeta = 2.25$). The corresponding plots are shown in Figure 2.13. These simulations were performed using the linearized model of Section 2.3.1. Soil leading to higher damping coefficients generally lead to responses that are not very critical with respect to the tracking controller of the SGS and hence are not discussed in detail here.

As discussed earlier, the variation of $x$, $y$, and $\phi$, and their first and second derivatives are assumed to be largely unaffected by the terrain characteristics. For the purpose of this study, the variation of these kinematic quantities is assumed to be as required by the desired vehicle trajectory. For instance, the variation of $\phi$ and its derivatives as the vehicle performs a turn to its left by about 10 degrees is shown in Figure 2.14. A smooth trajectory $\phi(t)$ is obtained by the use of a fifth degree polynomial between the time instants 0 and 2.5 seconds. It is assumed that at $t = 0$, and $t = 2.5$, the angular velocity and the angular acceleration are both zero. The quintic polynomial function $\phi(t)$ represented in Figure 2.14 is given below.

$$\phi(t) = 0.01075 t^5 - 0.06720 t^4 + 0.11200 t^3$$ \hspace{1cm} (2.29)
Figure 2.12 Variables $\psi$, $\theta$, $z$, $\dot{\psi}$, and $\ddot{\psi}$ as the Vehicle Traverses a Lava Field ($\zeta = 0.1$)
Figure 2.13  Variation of $\psi$, $\theta$, $z$, $\dot{\psi}$, and $\ddot{\psi}$ as the Vehicle Traverses Soil ($\zeta = 2.25$)
Figure 2.14 Assumed Variation of the Vehicle Yaw $\phi$, and its Derivatives With Time
Similar simulations were performed using the dynamic model of Section 2.3.2 that includes Gaussian noise. Also, the value of $z_p$ during the second simulation was increased to a value of 0.45 m. All the other parameters were kept the same as in the case of the linearized model. The resulting vehicle body motion is quite drastic, and can be assumed to be sufficient to verify SGS based navigation of robotic vehicles moving at about 15-0.25 m-s$^{-1}$. Figure 2.15 shows the variation of the kinematic quantities $z$, $\psi$ and $\theta$, and Figure 2.16 shows the variation of $\delta_{Ri}$ and $\delta_{Li}$. It is seen that the six quantities $\delta_{Ri}$ and $\delta_{Li}$ are less than zero for significant periods of time indicating that all the six wheels are lifted off the terrain at some time or the other. These results are for $\zeta = 2.25$ (overdamped, $C = 4861$).

2.4 Dynamic Tracking of the SGS

The Star Gazing System is mounted on the vehicle body as shown in Figure 2.1. As described in Section 2.1, the SGS is required to track two stars at all times. Given the dynamic nature of the vehicle motion over unstructured terrain, a carefully designed feedback control system is required to achieve the desired tracking. This section discusses this controller in detail and presents some simulation results that were obtained. The body orientation, angular rates, and angular accelerations as sensed by the SGS are presented. A block diagram of the whole process is shown in Figure 2.17.

2.4.1 Rigid Body Dynamics of the SGS Mount

The SGS mount is shown in Figure 2.1 and it is required to control the three actuators at the three degree of freedom joint $C_a$ connecting the body and the mount. The closed loop position control strategy has to achieve the desired tracking of the two stars. In
Figure 2.15 Variation of $z, \psi$ and $\theta$ (Model Includes Gaussian Noise, $\zeta = 2.25$)

Figure 2.16 Variation of $\delta_{Ri}$ and $\delta_{Li}$, (Model Includes Gaussian Noise, $\zeta = 2.25$)
Figure 2.17 Block Diagram of the Star Gazing System (SGS)

\[ q = [\alpha \beta \gamma]^T \]

\( \delta q_d \) the desired variation in \( q \) required to eliminate the tracking error of the SGS

\( \tau_{ff}, \tau_{fb}, \tau \) the feedforward, feedback, and total control torque respectively

\( y, q, z \) obtained from dynamics (affected by terrain interaction)

\( x, y, f \) depends on the desired trajectory (assumed to be unaffected by the terrain)
this section, the dynamic model used for the purpose of simulation as well as the controller
design is presented. The dynamic model of the SGS can be expressed as follows:

\[ M(q) \ddot{q} + V(q, \dot{q}) + G(q) + B \dot{q} + \vartheta + f(r, \dot{r}, \dot{p}) = \tau \]  

(2.30)

Here \( q = [\alpha \ \beta \ \gamma]^T, r = [\psi \ \theta \ \phi]^T, p = [x \ y \ z], M \) is the symmetric 3x3 inertia matrix, \( V \) is the 3x1 vector of coriolis and centripetal torques, \( G \) is the 3x1 vector of gravity torques, \( B \) is the 3x3 damping matrix, \( \vartheta \) if the 3x1 vector of coulomb friction torques, \( f \) is a 3x1 vector of torques caused due to the vehicle body kinematics, and \( \tau \) is a 3x1 vector representing the three actuator torques at the joint \( C_a \) in Figure 2.1. The dynamic Equation 2.30 can be obtained by several means [Fu et al., 1987]. Lagrangian dynamics, Newton-Euler formulation, and Recursive Numerical techniques [Walker and Orin, 1982] are some of the alternatives. Here a combination of the Newton-Euler formulation and numerical techniques was used to obtain the dynamic equation. The functions \( B \), and \( \vartheta \) were obtained relatively easily, and typical values for damping and coulomb friction were used. The variation of the other functions is much more complicated. A brief description of the kinematics and the kinetics of the system is given below.

The variables \( \psi, \theta, \) and \( \phi \) represent the rotation of the vehicle body about the axes \( X_0, Y_0, \) and \( Z_0 \) respectively. The coordinate system \( X_0Y_0Z_0 \) is parallel to the world fixed coordinate system and is attached to the center of gravity of the body \( C_0 \). The variables \( x, y, \) and \( z \) are measured from the origin of the world coordinate frame along the axes \( X_0, Y_0, \) and \( Z_0 \) respectively. Finally, the angles \( \alpha, \beta, \) and \( \gamma \) are measured about the axes \( u, v, \) and \( w \) respectively. The axis \( u \) is fixed to the vehicle body, \( v \) is affected by the rotation about \( u, \) and \( w \) is affected by the rotation about \( u \) and \( v \).
The acceleration of the center of mass of the SGS, \( \mathbf{C}_1 \), in the local coordinate frame '1', can be obtained as follows.

\[
\ddot{\mathbf{c}}_1 \equiv (\mathbf{R}_0^1)\mathbf{c}_k + 2[\{(\mathbf{R}_0^1)^T\mathbf{c}\} \times \{(\mathbf{R}_0^1)^T\omega \times \mathbf{c}_{al}^a\}] + \{(\mathbf{R}_0^1)^T\omega \times [(\mathbf{R}_0^1)^T\omega] \times \mathbf{c}_{al}^a\}
\]

(2.31)

\[
\ddot{\mathbf{c}}_k = \dot{\mathbf{r}} \times (\mathbf{R}_0^1)_{0a}^b + \dot{\mathbf{r}} \times \{\dot{\mathbf{r}} \times (\mathbf{R}_0^1)_{0a}^b]\} + \dot{\mathbf{p}} + \dot{\mathbf{r}} \times (\mathbf{c}_{al}^1)
\]

\[
\omega = \dot{\mathbf{l}} + \dot{\mathbf{m}} + \dot{\mathbf{n}}, \quad \dot{\omega} = \dot{\mathbf{l}} + \dot{\mathbf{m}} + \dot{\mathbf{n}}
\]

where \( \mathbf{l} = [1 \ 0 \ 0] \), \( \mathbf{m} = [0 \ \mathbf{c} \ \mathbf{a}] \), \( \mathbf{n} = [\mathbf{s} \beta - \mathbf{s} \alpha \mathbf{c} \beta \ \mathbf{c} \alpha \mathbf{c} \beta] \) (2.32)

\( c_{0a}^b \) is the vector directed from \( C_0 \) to \( C_a \) in the coordinate frame 'b' and \( c_{al}^1 \) is the vector directed from \( C_a \) to \( C_1 \) in the coordinate frame '1'. The matrices \( \mathbf{R}_0^1 \), \( \mathbf{R}_0^b \), and \( \mathbf{R}_b^1 \), are the rotation matrices discussed in Section 2.2 (see Equations 2.5 and 2.6).

The angular velocity (\( \mathbf{\sigma}^1 \)) and the angular acceleration (\( \mathbf{\dot{\sigma}}^1 \)) of SGS in the local frame '1',

\[
\mathbf{\sigma}^1 = (\mathbf{R}_b^1)^T\omega + (\mathbf{R}_0^1)^T\dot{\mathbf{r}}, \quad \mathbf{\mu}^1 = (\mathbf{R}_b^1)^T\omega + (\mathbf{R}_0^1)^T\dot{\mathbf{r}}
\]

The kinetic equations of the SGS can be written as follows (see Figure 2.18):

\[
\mathbf{F} + m_s\mathbf{g} = m_s\dot{\mathbf{c}}_1, \quad \mathbf{\tau} + (c_{1a}^1 \times \mathbf{F}) = \mathbf{I}\mathbf{\sigma}^1 + \{\mathbf{\mu}^1 \times (\mathbf{I}\mathbf{\mu}^1)\}
\]

\[
\mathbf{g} = g (\mathbf{R}_0^1)^T[0 \ 0 \ -1]^T, \text{ where } g = \text{acceleration due to gravity, } c_{1a}^1 = - c_{al}^1
\]

\( \mathbf{I} \) is the 3x3 inertia matrix of the SGS about frame '1'. Frame '1' forms the principal axes for the SGS and hence \( \mathbf{I} \) is a diagonal matrix.

Eliminating the reaction force \( \mathbf{F} \) from the above equations and solving for \( \mathbf{\tau}^1 \),

\[
\mathbf{\tau}^1 = \mathbf{I}\mathbf{\sigma}^1 + \{\mathbf{\mu}^1 \times (\mathbf{I}\mathbf{\mu}^1)\} - (c_{1a}^1 \times (m_s(\dot{\mathbf{c}}_1 - \mathbf{g}))}
\]

(2.33)
The vector of actuator torques \( \tau = [\tau_y \ \tau_p \ \tau_r]^T \) where \( \tau_y \) is about the axis \( u \), \( \tau_p \) is about the axis \( v \), \( \tau_r \) is about the axis \( w \). It can be shown that \( \tau' = \{(R_b^1)^T U\} \tau \) where the matrix \( U \) is given by \( U = [I \ m \ n] \) (see Equation 2.32 for the vectors \( I \), \( m \), and \( n \)). Solving for \( \tau \) from Equation 2.33 (assuming nonsingularity, or no two of \( u \), \( v \), and \( w \), are coincident),

\[
\tau = \{(R_b^1)^T U\}^{-1} [I (I_1^1 + [\mu^1 \times (I \mu^1)]) - \{e_{1a}^1 \times (m_s (\hat{c}^1_1 - g))\}]
\]  

(2.34)
Figure 2.19 Schematic Diagram of a Constant Field D.C Motor

R_i, L_i are the resistance and inductance of the armature circuit

e_i is the input emf to the armature circuit, i_a the armature current, e_b the back emf

i_f is the field current, T_i is the motor torque, \( \tau_i \) is the load torque

J_{mi} is the rotor inertia, B_{mi} is the damping coefficient, N_i is the reduction ratio

Equation 2.34 can be simplified to obtain the dynamic equations in the form of Equation 2.30. In this study, all the matrices and vectors required for the dynamic simulation were obtained numerically at each time step, and hence a combination of the Newton-Euler formulation and numerical methods was used. The emphasis was on simplicity and error-free formulation of the rigid body dynamics rather than on computational efficiency of the algorithm.

The actuators for the SGS were taken to be electric brushless D.C motors. The motors were modelled as typical constant field motors. The actuator and drive train model is shown in Figure 2.19. Assuming that the inductive effects are negligible (L_i \approx 0), the
torque vector $\tau$ in the dynamic equation can be obtained in terms of the motor voltage vector, $e$ as

$$
\tau_i = (N_i K_T/R_i) e_i - N_i^2 \left( J_{mi} \ddot{q}_i + B_{mi} \dot{q}_i \right) 
$$

where $i = y, r, p$ when $p$ is $\alpha, \beta, \gamma$ respectively. $B_{mi} = B_{mi} + (K_{Ti} K_{Bi}/R_i)$ where $K_{Ti}$ is the torque constant, and $K_{Bi}$ is the back emf coefficient. The actuator torque saturation was also included in the model.

Combining Equations 2.30 and 2.35,

$$
e = M'q + B'q + f_1 + f_2
$$

$$
f_1 = f_1(q, \dot{q}, r, \dot{r}, \beta, \dot{\beta}, \gamma, \dot{\gamma})
$$

$$
f_2 = f_2(q, \dot{q})
$$

$M'$, $B'$ are the inertia and damping matrices that include the effects of the rotor inertias and the motor damping and back emf. The vector functions $f_1$ and $f_2$ include all the nonlinear effects due to coupled velocity term $V$, the gravity torque $G$, and the effects of coulomb friction. $f_2$ is a function which can be estimated from the sensed values of the $\alpha, \beta, \gamma$ angles and their time derivatives, along with the approximate value of the gravity vector. However, $f_1$ cannot be estimated as the body kinematics and the coulomb friction values are not available to the controller. Hence $f_1$ acts as a disturbance function to the closed loop feedback system. Having modelled the SGS dynamics, the control issue is addressed next and the design of the tracking controller is the topic of the next section.

2.4.2 Design of the Tracking Controller

The control issues arising in the design of the tracking controller are very similar to those of closed loop position control in serial chain robots. A serial robot tracking a path is required to have the end effector following the path as closely as possible. This is similar to the SGS tracking the stars as closely as possible to ensure that the stars remain in the
field of view of the cameras. Control of robot manipulators has been studied extensively in literature [Golla et al., 1981], [Luh et al., 1980], [Young, 1978], [Tarn et al., 1984]. A model based control technique known as the 'computed torque method' makes use of an available dynamic model of the robot and uses a nonlinear feedforward law in addition to a feedback control loop [Khosla and Kanade, 1985]. The control law used here is based on this method. The computed torque method has been used to obtain a high performance controller and works well in situations where a fairly accurate model of the system is available. Nonlinear feedforward control accounts for the coupled velocity terms and the gravity terms in the torque equation. There are two classes of robots with respect to the issue of control, robots that have large reduction ratios at the actuators, and 'direct drive robots' [Asada and Kanade, 1983]. The nonlinear torque terms are significant in a direct drive robot and robots with low reduction ratios. In direct drive robots the transmission components are completely eliminated leading to higher performance robots. Problems like backlash, high frequency dynamics due to the flexibility of the transmission components, and losses in the reduction units are completely eliminated. State of the art, high torque to weight ratio rare earth motors are used in these robots. However, the control issues of these robots are more complicated due to a strong dynamic coupling among the various joints. Traditional individual joint control (constant gain) [Luh, 1983] used in the control of robots with large gear ratios leads to large steady state errors and even instability when used for controlling direct drive robots. Hence, the computed torque method, a time varying model based control law with nonlinear compensation, has to be used in such situations. It is to be noted that if an accurate model of the direct drive robot is not available, the control issue becomes complicated and techniques like adaptive control [Dubowsky and DesForges, 1979], and delay control have to be used to obtain an estimate of the unmodelled dynamics. Equation 2.36 can be written in the state space form as follows:
\[
\dot{x} = Ax + Ce + \Lambda_1 + \Lambda_2 \quad x = [q^T \dot{q}^T]^T
\] (2.37)

\[
A = \begin{bmatrix}
0 & 1 \\
0 & -(M')^{-1}B'
\end{bmatrix}, 
C = -(M')^{-1}, 
\Lambda_1 = \begin{bmatrix}
-(M')^{-1}f_1 \\
0
\end{bmatrix}, 
\Lambda_2 = \begin{bmatrix}
-(M')^{-1}f_2 \\
0
\end{bmatrix}
\]

\(\Lambda_1\) is a function of the vehicle body kinematics in addition to SGS kinematics, and friction. The function of the SGS is to sense the vehicle kinematics and hence these values are not available to the controller. On the other hand, \(\Lambda_2\) is a function of the sensed joint kinematic quantities. Hence, it can be estimated and used as part of the nonlinear feedforward compensation. If direct drives are used for the actuation of the joint, the rotor inertias of the motors do not contribute significantly to the inertia matrix \(M'\) (see Equation 2.35).

Hence, the matrix \(M'\) is composed of small elements or the norm of the matrix is relatively small. This leads to an \((M')^{-1}\) that has a large norm. Therefore, the euclidean norm of \(\Lambda_1\) becomes large. The contribution of \(\Lambda_1\) is no longer insignificant and since its estimates are not available to the controller, the computed torque control law that includes compensation only for \(\Lambda_2\) can become inadequate. In the case of larger values of the reduction ratio, the rotor inertias contribute significantly to \(M'\) and hence the norm of \(\Lambda_1\) is small and it can be treated as a disturbance to the control law. The norm of the disturbance vector plays an important role in the closed loop stability of the system [Patel and Toda, 1980]. Of course increasing the gear ratios immediately lead to reduced performance due to additional transmission components. Hence, the design process involves a trade-off. The vehicle kinematics cannot be estimated with any degree of certainty due to the nature of the unstructured terrain on which it operates. A moderate gear ratio can be included and a computed torque scheme can be used without any compensation for \(\Lambda_1\). It was found from simulation studies that a gear ratio of 15 - 25 is sufficient to reduce the effects of \(\Lambda_1\) for the control of the SGS.
Another issue relevant to the tracking controller is the fact that the errors in position of the SGS (discussed in Section 2.1) are also due to the motion of the vehicle body, in addition to the steady state errors of the controller. The SGS is required to point exactly at the stars but as the vehicle body moves, the stars move away from the axes of the camera lenses. These errors were found to be essentially functions of the vehicle angular rates. These errors can be reduced by increasing the sampling rate of the controller. The computed torque method involves the calculation of the time varying feedback control gains along with estimation of the nonlinear terms at each sampling instant. It has been shown that sampling rates of about 400 - 500 Hz can be achieved for such a control law. However, if a simple constant gain individual joint control law is used, the controller can be run at a frequency of 1kHz or more. This reduces the magnitude of the errors significantly (see Figure 2.20). Hence, a gear ratio high enough to allow a constant gain individual joint control law (or periodic updates of the control law) is preferred. For reasonable values of gear ratios, it was found that the nonlinear compensation for the vector $\Lambda_2$ is not required. This reduces the level of computation and leads to a very simple control law allowing a high sampling frequency of the controller. The control law can be written as follows:

$$e = \dot{M}'\ddot{q}_d + \dot{B}'\dot{q}_d + \dot{M}'K_v\dot{E} + \dot{M}'K_pE + \dot{M}'K_i\int_0^t E \, dt \quad (2.38)$$

$\dot{M}'$, $\dot{B}'$ are the nominal values of the inertia and the damping matrix. These matrices are almost entirely dependent on the motor parameters. $E = q_d - q$ is the error vector where $q_d$ is the desired position state vector. $K_v$, $K_p$, $K_i$ are the velocity, position, and the integral feedback gain matrices. Applying the control law to the dynamic Equation 2.36, the error dynamics is obtained:

$$\ddot{E} + K_v\dot{E} + K_pE + K_i\int_0^t E \, dt = d(t)$$

$$d(t) = \{(\dot{M}')^{-1}\dot{M}' - I\}\ddot{q} + \{(\dot{B}')^{-1}\dot{B}' - I\}\dot{q} + (\dot{M}')^{-1}\{f_1 + f_2\}$$
\[ \ddot{E} + K_v \dot{E} + K_p E + K_i E = 0 \]  \hspace{1cm} (2.39)

The gain matrices are evaluated by placing the closed loop poles of Equation 2.39 to suit the performance requirements. The closed loop pole locations are chosen to ensure sufficient accuracy and speed of response in addition to system robustness to parameter perturbation and high frequency dynamics. Lightly damped, unmodelled high frequency poles are due to flexibility of the drive trains, and due to the structural modes of the SGS [Good et al., 1985]. It was found that, for the order of compliance expected in the drive trains, the lightly damped uncontrollable closed loop poles remain stable for the relevant range of gain matrices. The structural mode natural frequencies are expected to be at higher frequencies when compared to that due to the flexibility of the drive trains and are not expected to be critical. Finally, closed loop pole locations were chosen such that the overall system is robust with respect to parameter variations. For the purpose of the simulation study, the SGS kinematic and dynamic parameters, the motor parameters, the closed loop pole locations, and the control law used are listed in Appendix A. The next section discusses the simulation results.

2.3.3 Simulation of the SGS Tracking

This section presents some results from the dynamic simulation of the SGS performed using the software ACSL. The SGS tracks the stars as the vehicle moves over terrains of varying equivalent damping. Again, the vehicle is assumed to be traversing the terrain of Figure 2.10. The tracking errors in the variables \( \alpha, \beta, \) and \( \gamma (\alpha_c, \beta_c, \) and \( \gamma_c) \) are shown in Figure 2.21 for \( \zeta = 0.1 \), and for \( \zeta = 2.25 \). The tracking errors in \( \gamma \) varies much more drastically than the other two because of the presence of high frequency components in the variation of \( \phi \) which affect the variation of \( \gamma \) significantly. The variable \( \phi \) does not
get filtered by the vehicle dynamics. Hence, drastic changes in this variable can lead to instability of the controller. However, in this application, the variation of the angle is slow enough to allow the tracking process. Also, the tracking errors are smaller in magnitude for $\zeta = 2.25$ as compared to the tracking errors for $\zeta = 0.1$. Further, the tracking errors contain higher frequency components for $\zeta = 2.25$. This is consistent with the study of the power spectral densities in Section 2.3. The sensed values of $\psi_s$, $\theta_s$, and $\phi_s$, along with the errors in these sensed values ($\psi_e$, $\theta_e$, and $\phi_e$) are shown in Figures 2.22 and 2.23. The sensed value of the angular rate $\dot{\psi}$, $\dot{\psi_s}$, and the corresponding error $\dot{\psi}_e$, are shown in Figure 2.24.

The simulation results presented in Figures 2.21 through 2.24 were used the linearized vehicle dynamics model of Section 2.3.1. The simulation results that were obtained using the dynamic model with Gaussian noise are shown in Figures 2.25 and 2.26. The tracking errors in the variables $\alpha$, $\beta$, and $\gamma$ ($\alpha_e$, $\beta_e$, and $\gamma_e$) are shown in Figure 2.25 ($\zeta = 2.25$), and the sensed value of the angular rate $\dot{\psi}$, $\dot{\psi_s}$, and the corresponding error $\dot{\psi}_e$, are shown in Figure 2.26 ($\zeta = 2.25$).

### 2.5 Estimation of the Absolute Position of the Vehicle

This section describes the process of estimation of the absolute position of the vehicle on a given planet, by using the data available from the SGS. The absolute position obtained here has some error associated with it, and this error is due to the errors in the data from the encoders and the errors in the data from the two cameras. The accuracy of the absolute position can be improved by improving the quality of these sensors. The geometry of this problem, and a procedure to evaluate the absolute position, are described in the section below. A numerical example is considered in the section that follows.
Figure 2.20 Reduction in Tracking Error $\alpha_e$ with an Increase in the Controller Sampling Rate ($\zeta = 2.25$)
Case 1: $\zeta = 0.1$
Case 2: $\zeta = 2.25$

Figure 2.21 Tracking Errors $\alpha_e$, $\beta_e$, and $\gamma_e$
Figure 2.22 Sensed Values of $\psi_s$, $\theta_s$, and $\phi_s$, and their Errors $\psi_e$, $\theta_e$, and $\phi_e$ ($\zeta = 0.1$)
Figure 2.23 Sensed Values of $\psi_s$, $\theta_s$, and $\phi_s$, and their Errors $\psi_e$, $\theta_e$, and $\phi_e$ ($\zeta = 2.25$)
Figure 2.24  Sensed Angular rate $\psi_s$, and the Corresponding Error $\dot{\psi}_e$
Figure 2.25 Tracking Errors $\alpha_e$, $\beta_e$, and $\gamma_e$

Figure 2.26 Angular rate, $\psi$, sensed value $\psi_s$, and sensing error $\psi_e$
2.5.1 Procedure for Estimating the Absolute Position

In this section, it is assumed that a vehicle is traversing the surface of a planet. The vehicle is equipped with an SGS and tracks two stars at all times. In addition, the vehicle is equipped with an inclinometer that provides the true vertical direction. The direction obtained from the inclinometer is not very accurate in the presence of vehicle dynamics. It is assumed here that the vertical direction is obtained by bringing the vehicle to a complete halt to avoid dynamic effects. The vertical direction can also be obtained by 'weighing' the vehicle using the force sensor data as done on the Adaptive Suspension Vehicle [Pugh et al., 1990]. Assuming that a planet is a perfect sphere, the vehicle position can be specified by stating the latitude ($\sigma$), and the longitude ($\phi$) at that location. A planet undergoes different kinds of movements in space (rotation about its own axis, revolution around the sun, etc.). These movements cause relative motion between the two stars and the planet. The relative motion histories of planets and stars can be predicted very accurately and hence can be assumed to be known. In this simplified analysis, only the relative motion due to the rotation of the planet about its own axis is considered as part of the numerical example. However, the other effects can be included without any major changes in this procedure. The absolute position of the robotic vehicle on a planet is obtained using the following procedure:

1. It is assumed that the time varying unit vectors directed towards the two stars, $s_e^1(t)$ and $s_e^2(t)$, are known in a planet fixed frame 'e'. Therefore, the transformation from the sidereal frame 's' to frame 'e' can be obtained as

$$R_e^s = \begin{bmatrix} s_e^1 & s_e^3 & s_e^4 \end{bmatrix}$$

(2.40)
where \( s_e^3 = \frac{s_e^1 \times s_e^2}{|s_e^1 \times s_e^2|} \), \( s_e^4 = \frac{s_e^1 \times s_e^3}{|s_e^1 \times s_e^3|} \).

(2) The vertical direction in the vehicle body frame 'b', \( \mathbf{g}_b \), is obtained from the inclinometer. \( \mathbf{R}_1^b \) can be evaluated knowing the sensed values of \( \alpha, \beta \) and \( \gamma \), and, \( \mathbf{R}_s^1 \) can be evaluated knowing the unit vectors directed towards the two stars, \( s_1^1(t) \) and \( s_1^2(t) \) in the camera frame '1'. Therefore, the vertical vector in frame 's', \( \mathbf{g}_s \), is obtained.

(3) The vertical vector in frame 'e', \( \mathbf{g}_e \), can be obtained using the transformation \( \mathbf{R}_e^s \).

If frame 'e' is such that its z axis is along the axis of the planet, pointing due north, and its x axis is directed towards the longitude \( \rho = 0 \), it can be shown that,

\[
[C \sigma \ C \rho \ C \sigma \ S \rho \ S \sigma]^T = -\mathbf{g}_e
\]

where \( \sigma \) and \( \rho \) are the latitude and the longitude of the vehicle position respectively.

The values of \( \sigma \) and \( \rho \) can be obtained uniquely from the above equation, since \( \sigma \in [-\pi/2, \pi/2] \). The errors in \( \sigma \) and \( \rho \) will be due to the sensing errors in \( \mathbf{g}_b, s_1^1(t), s_1^2(t), \alpha, \beta \) and \( \gamma \). The vector \( \mathbf{g}_b \) is assumed to be computed from two angular position encoders that are mounted at the axes of the inclinometer. It is assumed that the inclinometer position is given by a rotation about the body frame 'y' direction by an angle \( \nu \), followed by a rotation of an angle \( \mu \) about the rotated 'x'. Therefore, \( \mathbf{g}_b = [C\mu \ C\nu \ S\mu \ -C\mu \ C\nu]^T \)

In the next section, a numerical example is considered. Typical maximum errors expected in the sensed quantities are included, and an estimate of the maximum error in the absolute position is obtained.
2.5.2 Numerical Example of Estimating the Absolute Vehicle Position

In this section a robotic vehicle traveling from point '1' to point '2' on Mars is considered. The distance between these two points is 9.7743 kms, and the vehicle takes $t_f = 10$ hours for this journey at an average speed of 0.2715 m/s. The radius of Mars, $R = 2400$ km, and it is assumed that Mars rotates about its own axis in 24 hours; angular velocity $\omega_p = \frac{2\pi}{24} = 0.2618$ rads/hour. The angle of rotation about the axis, $\chi_p = t_f \omega_p = 2.618$ radians. Only the relative motion between the planet and the stars due to the rotation of the planet about its own axis is considered here. Let 'i' represent the initial position of the planet, and 'f' represent the final position of the planet due to a rotation of $\chi_p$ about its own axis. The objectives of this numerical example is to show that this navigation procedure can be used for long journeys, and to obtain estimates of typical maximum errors in the absolute position values on Mars. These position errors are drift-free, and hence are independent of the distance traveled during the journey. In this example, the exact locations of the vehicle, at the start and the end of the journey are used in an inverse analysis to evaluate the exact values of all the sensed quantities. Then, typical errors are introduced into the sensed values and a forward analysis is performed to obtain the estimated value of the vehicle position. Finally, the exact vehicle position is compared to the estimated position and the resulting error in the absolute position of the vehicle is evaluated.

The latitude and longitude values at the initial point $p_1$ are taken to be $\sigma_1 = 45^\circ$ and $\rho_1 = 45^\circ$, and at the final point $p_2$, they are taken to be $\sigma_2 = 45^\circ$ and $\rho_2 = 45.33^\circ$. The vector direction of the first star is assumed to be directed from center of Mars towards a point where $\sigma = 0^\circ$, and $\rho = -40^\circ$, and that of the second star is directed from center of Mars to a point where $\sigma = 45^\circ$, and $\rho = -40^\circ$. These two star directions were chosen such
that the two stars are visible to the vehicle for the entire 10 hours of its journey. The vectors are

\[
s_{e1} = [0.7660444, -0.6427876, 0.0]^T \\
s_{e2} = [0.5416752, -0.4545194, 0.7071067]^T.
\]

**Exact Analysis (Only for the Purpose of Simulation):**

The exact vertical directions at the two points \( p_1 \) and \( p_2 \) in position 'i' of Mars are

\[
g_{e1i} = [-0.5, -0.5, -0.7071067]^T \\
g_{e2i} = [-0.4971119, -0.5028714, -0.7071067]^T.
\]

The exact values of all the quantities discussed above are now used in an inverse analysis to evaluate the exact values of the inclinometer angles, the camera coordinates, and the SGS joint angles. Evaluating the matrix \( R_{e_s} \) and the vectors \( g_{s1i} \) and \( g_{s2i} \) can be obtained. The transformation due to the rotation of Mars by \( \chi_p \) about its own axis, \( R_f \), that takes vectors from frame 's' in position 'i' to frame 's' in position 'i' can be obtained [Fu et al., 1987]. Using this transformation \( g_{sf} \) can be obtained.

\[
g_{s1i} = [-0.0616284, 0.704416, 0.7071068]^T \\
g_{s2f} = [-0.3025223, -0.6391246, 0.7071067]^T
\]

The exact sensed values at the initial vehicle position '1' are now obtained. Assume that \( \varepsilon = 22.5^\circ \), and \( \lambda = 45^\circ \). Also, assume that the stars are exactly along the lens axis. Hence,

\[
s_c1 = s_c2 = [0, 0, 1]^T
\]

Take \( \alpha_1 = 80^\circ \), \( \beta_1 = \gamma_1 = 0^\circ \), and evaluate \( g_{p1i} \).
\[ g_b^{1i} = [0.6296973 \ 0.2002387 \ -0.7505902]^T \]

Using Equation 2.42, \( \mu_1 = 11.550918^\circ \), \( \nu_1 = -39.994509^\circ \).

Hence, in position '1', the exact values of the sensed quantities are:

1. The SGS joint variables are \( \alpha_1 = 80^\circ \), \( \beta_1 = \gamma_1 = 0^\circ \).
2. The inclinometer angles are \( \mu_1 = 11.550918^\circ \), \( \nu_1 = -39.994509^\circ \).
3. The camera vectors are \( s_c^1 = s_c^2 = [0 \ 0 \ 1]^T \).

Next, the exact sensed values at the final vehicle position '2' are obtained. Again assume that \( \epsilon = 22.5^\circ \), and \( \lambda = 45^\circ \) and that the stars are exactly along the lens axis. Hence,

\[ s_c^1 = s_c^2 = [0 \ 0 \ 1]^T \]

Now take \( \alpha_2 = -80^\circ \), \( \beta_2 = \gamma_2 = 0^\circ \), and evaluate \( g_b^{2f} \).

\[ g_b^{2f} = [0.5375111 \ -0.4307522 \ -0.7249373]^T \]

Using Equation 2.42, \( \mu_2 = -25.515306^\circ \), \( \nu_2 = -36.555403^\circ \).

Hence, in position '2', the exact values of the sensed quantities are:

1. The SGS joint variables are \( \alpha_2 = -80^\circ \), \( \beta_2 = \gamma_2 = 0^\circ \).
2. The inclinometer angles are \( \mu_2 = -25.515306^\circ \), \( \nu_2 = -36.555403^\circ \).
3. The camera vectors are \( s_c^1 = s_c^2 = [0 \ 0 \ 1]^T \).
Forward Analysis and Estimation of Error in the Sensed Absolute Position:

Errors are introduced into the exact sensed values obtained in the above section. Then, the vehicle position is estimated. For instance, the vehicle location at the end of the journey is desired. A commercially available high resolution optical encoder having a maximum error of ±0.0005° is assumed to be available for sensing the angles α₂, β₂, γ₂, μ₂, and ν₂. As stated earlier commercially available cameras have a maximum error of about 10 μrads. These errors were included in the sensed values, and the latitude σ₂, and ρ₂, were evaluated using the procedure described earlier. Several different combinations of errors (for instance negative error in μ₂, and positive error in ν₂, etc.) were incorporated and extensive numerical testing was performed over the whole range of errors using a computer program. For this particular example, the maximum error that was obtained was for the following sensed values:

1. \( \alpha_2 = -79.9995^\circ, \beta_2 = \gamma_2 = 0.0005^\circ \).
2. The inclinometer angles are \( \mu_2 = -25.514806^\circ, \nu_2 = -36.555903^\circ \).
3. The camera vectors are \( s_c^1 = [0.00001 \ 0 \ 0.999999]^T, s_c^2 = [-0.00001 \ 0 \ 0.999999]^T \).

The sensed values of the latitude, and longitude were found to be \( \sigma_2 = 45.00145^\circ, \) and \( \rho_2 = 45.32881^\circ \). The maximum error in the estimated absolute position of the vehicle for this particular example was found to be a distance of 70.34 m. Further, if the maximum error in all the sensors is decreased to half of their original values, the maximum error in the absolute position falls to 35.28 m. These numbers are from a specific example. However, in general it can be concluded that the absolute position of the vehicle can be obtained with a maximum error of the order of 100 m from typical sensor technology.
These errors were obtained for typical commercially available sensors. More accurate and expensive sensing and advanced signal processing will lead to better position information.

2.6 Star Sensing Systems

In this section, some practical issues of star trackers that have to be addressed for the implementation of the ideas described in the previous sections, are discussed.

Two kinds of Charge Transfer Devices (CTD) have been used as imaging systems in star tracking applications: Charge Coupled Devices (CCD) and Charge Injection Devices (CID). The performances of these devices have been discussed in detail by Deters and Gutshall [1987], and Weiss et al. [1987]. If the SGS is composed of two CTD cameras, during the tracking process, very few pixels near the middle of the camera arrays will be used. Therefore, CID cameras are more suited for this application since in a CID, each pixel or a small subarray of pixels can be directly accessed. This leads to reduced processing, and hence to an increased rate of star position updates. Special purpose star tracking CIDs can give position updates at 100 Hz or more depending on the size of the subarray used. The accuracy of CID star trackers have been shown to be about 1-3 % of the angle subtended by a pixel. These accuracies have been demonstrated by obtaining centers of stars of various sizes and intensities. For the commercially available CID camera, ST-256 (array: 256×256 pixels; Field of view: 2.93°×2.93°), the accuracy is in the range of 2-6 μrads. A daytime star tracking system has been studied for aircraft navigation by Duncan [1989]. Daytime star trackers are particularly suited for use during a Martian mission since the brightness of the sky on Mars is much lower due to its greater distance from the sun, and due to a lighter atmosphere leading to lower levels of scattering.
An inertial navigation system can be used as a backup in case of the occurrence of clouds or other obstructions that could lead to a tracking failure. If the obstructions remain only for a short time, the drift errors will be quite small. This means that the fixed objects will still be in the field of view of the cameras. In such a situation, the SGS can resume tracking the fixed objects. If the obstructions remain for a long period of time, the tracking will fail. Then, the SGS will have to search the entire sky for the stars of interest. In such a situation, a different approach can be taken to resume the navigation process. This approach involves recognition of star patterns in the sky, and locking on to a new pair of stars. Updated body orientations can then be obtained from the known transformations between the sidereal frames of the old and the new pairs of stars. A system that can rapidly identify star patterns using specialized neural network architectures has been demonstrated in simulations by Alvelda et al. [1988]. The tracking failure of the SGS due to obstructions is unlikely on Mars due to the absence of clouds in the Martian skies. However, Mars is known to have severe dust storms. During such a storm it is expected that the robotic vehicle will temporarily suspend its mission.

A preliminary design study of the SGS has been completed at this time [Martin et al., 1991]. Fabrication and testing of the SGS mounted on the Wheeled Actively Articulated Vehicle to verify the findings of this study is to be attempted.

2.7 Conclusions

A new approach of navigating an autonomous robotic vehicle has been studied. The continuous tracking of the fixed objects leads to a self-contained drift free sensing system. An inertial navigation system may be required as a backup in case of the occurrence of clouds or other obstructions that could lead to a tracking failure. If the
obstructions remain only for a short time, the drift errors will be quite small. This means that the fixed objects will still be in the field of view of the cameras. In such a situation, the SGS can resume tracking the fixed objects. The absolute position information obtained from the SGS is quite good, and with some improvements, it may be sufficient for most applications. Satellite based position information can be more accurate. However, they are much more expensive, and can only provide periodic updates. The characteristics of various terrain conditions, and their effects on vehicle dynamics has also been addressed. Simplified equivalent models of different terrains have been developed for the purpose of system analysis. Numerical simulation results have been obtained to show the feasibility of the SGS tracking system.
CHAPTER III

FORCE PLANNING

Actively coordinated mechanisms typically possess a redundancy in actuation, and this redundancy can be used to optimize desired system performance characteristics. In the case of actively coordinated vehicle systems, redundancy in actuation translates to an underconstrained set of contact force allocation equations. In other words, a desired kinematic state of the vehicle can be commanded by infinitely many contact force systems. Performance of a vehicle is directly influenced by the nature of the forces at the vehicle-terrain contact locations. The degrees of freedom offered by the underspecified force allocation problem can be used to optimize vehicle performance. The resulting force planning algorithms can be used to identify 'safe' terrain paths for the vehicle, and they can be used to provide inputs to the vehicle controller. In this chapter, the redundancy in actively coordinated mechanisms is first characterized using geometric reasoning. Next, the redundancy is used to optimize the contact conditions. Both these problems are addressed with respect to two kinds of actively coordinated mechanisms: (i) Mechanisms with complete contact force controllability (Walking vehicles, Articulated multi-fingered grippers etc.). (ii) Mechanisms with partial contact force controllability (Actively
3.1 Introduction

The force distribution problem of actively coordinated wheeled systems such as the WAAV (Figure 1.7) and the configuration of Figure 1.6 are addressed in this chapter for the first time. The force allocation problem has been studied with respect to two basic types of actively coordinated mechanisms. The first type includes articulated multi-fingered hands grasping an object. Multi-fingered hands have been studied in the literature [Mason and Salisbury, 1985], [Coe, 1989], [Mirza, 1992]. The optimization of the force conditions at the contact points has been studied for three fingered grasps [Demmel and Lafferriere, 1989], [Ji and Roth, 1988]. The general problem of optimizing frictional contacts for three fingered grasps has been addressed by Mukherjee [1991]. Linear programming techniques have been applied to the force distribution problem by Cheng and Orin [1989], and Mirza [1992]. Chung and Waldron [1993] have used the gradient search method to address the same problem, and they have obtained efficient, real-time solutions.

The second type of actively coordinated mechanisms that have been studied in the literature, with respect to force distribution, is legged vehicle systems. Several legged systems have been studied [Hirose, 1984], [Raibert, 1985], [Pugh et al., 1990]. Linear programming sub-optimal solutions to the force allocation optimization problems for legged systems are available [Kumar and Waldron, 1990]. Klein and Kittivatcharapong [1990] have also addressed the problem of optimal force distribution in legged systems. The characteristics of the minimum norm pseudo-inverse solution to the force allocation problem are found in the reference by Kumar and Waldron [1988]. This reference also addresses the nature of redundancy in closed kinematic chains.
The chapter is composed of two basic topics. The first topic addresses the nature of the redundancy in actively coordinated mechanisms by studying the geometry of the mechanism-environment contact locations (Section 3.2). It has been shown that the redundancy can be characterized effectively using a geometric approach. The second topic is the optimal use of the redundancy to improve contact conditions. The characterization of redundancy using a geometric approach, rather than a numerical approach, leads to efficient and robust optimization schemes.

Legged systems such as the Adaptive Suspension Vehicle [Pugh et al., 1990] and AMBLER [Bares et al., 1989], and articulated multi-fingered hands are examples of actively coordinated mechanisms that possess complete controllability over the force vectors at the mechanism-environment contact locations. Actively coordinated wheeled vehicles such as the WAAV represent mechanisms that possess only partial controllability over the contact force vectors. However, it is important to note that, even though systems such as the WAAV are somewhat inferior to legged systems with respect to contact force controllability, they are far superior to passively suspended vehicles such as the CARD [Wilcox and Gennery, 1987], and the Rocker-Bogie [Bickler, 1990], [Chottiner, 1992] vehicle configurations in this respect. The difference between the force distribution problems of legged systems and actively coordinated wheeled systems is subtle, and it is due to the unique kinematic characteristics of wheeled systems. This makes the nature of redundancy, and the optimization problems of wheeled systems distinct from that of legged systems. In this chapter, the two topics discussed in the previous paragraph are separately investigated for actively coordinated mechanisms with complete controllability over contact conditions, and for actively coordinated mechanisms with partial controllability over contact conditions.
3.2 Characterization of Redundancy

3.2.1 Active Mechanisms with Complete Controllability Over Contact Conditions

A typical articulated multi-fingered gripper grasping an object is shown in Figure 3.1. It is assumed that each of the articulated fingers have three actuators that will allow complete control over the contact force vector at the tip of the fingers. It should be noted that complete control over the contact force vector is present when the multi-fingered hand is used to manipulate an object in a precision grasp; this, in general, is not the case in a power grasp. A precision grasp is characterized by a single contact location between each finger and the object, and this contact location is typically near the tip of the finger. On the other hand, power grasps are characterized by multiple points of contact between the object and the surfaces of the finger and the palm, and they exhibit form-closure [Lakshminarayana, 1978]. A discussion of precision and power grasps can be found in the literature [Cutkosky, 1989]. An example of a grasping system comprised of actively articulated fingers is the DIGITS described by Coe [1989] and by Mirza [1992]. A typical actively coordinated legged vehicle system is shown in Figure 3.2. Such a vehicle is assumed to possess three actuated degrees of freedom in every leg which makes it possible to have complete control over the contact force vectors at the tip of each leg. Examples of such legged systems include the ASV [Pugh et al., 1990] and the AMBLER [Bares et al., 1989]. In both multi-fingered grippers and legged systems, it is important to note that complete controllability over contact forces is present only if the commanded inertial and gravitational forces lead to desired normal contact forces that are positive. This is because the nature of contact between the mechanisms and their environment does not allow for negative normal forces. (Force exerted by the environment on the mechanism has to be
Figure 3.1 Articulated Multi-Fingered Gripper

Figure 3.2 Legged Vehicle System on Uneven Terrain
directed away from the environment.) In this section, the force redundancy in systems such as the ones shown in Figures 3.1 and 3.2, that possess complete control over contact conditions, is investigated.

An active mechanism interacting with its environment is shown in Figure 3.3. There are \( n \) contact points \( C_i \). The contact force vector at the contact point \( C_i \) is \( F_i \) and the surface normal at \( C_i \) is \( n_i \). The force vector \( F_B \) and the moment vector \( T_B \) represent the body force and moment respectively and they include both inertial and gravitational forces.

\[
F_i = [F_{ix}, F_{iy}, F_{iz}]^T
\]

\[
C_i = [x_i, y_i, z_i]^T
\]

\[
F = [F_{ix}, F_{iy}, F_{iz}, F_{2x}, F_{2y}, F_{2z}, ..., F_{nx}, F_{ny}, F_{nz}]^T
\]

\[
Q = [F_B^T, T_B^T]^T
\]

The dynamic equations for the rigid body can be written as

\[
[G] F = Q
\]

\[
[G] = \begin{bmatrix}
I_3 & I_3 & ... & I_3 \\
R_1 & R_2 & ... & R_3
\end{bmatrix}, \quad R_i = \begin{bmatrix}
0 & -z_i & y_i \\
z_i & 0 & -x_i \\
y_i & x_i & 0
\end{bmatrix}
\]

\( I_3 \) is the 3x3 identity matrix. \([G]\) is a matrix purely dependent on the geometry of the contact points. For a given vector \( Q \), the force allocation problem involves solving Equation 3.1 for the contact force vector \( F \). Equation 3.1 represents six linear equations in 3\( n \) unknown force components. The rank of the matrix \([G]\) is six unless all the \( n \) contact points are collinear. This special situation can be regarded as trivial. Hence, throughout
this discussion, the rank of $[G]$ is taken to be equal to six. The matrix $[G]$ is a $3n \times 6$ matrix. This requires the nullity of $[G]$ to be equal to $3n - 6$. $N(G)$ represents the null space of the matrix $[G]$.

Kumar and Waldron [1988] addressed the redundancy in force allocation by defining two force fields; the *equilibrating* force field and the *interaction* force field. The equilibrating force field was shown to be the same as the pseudo-inverse, minimum norm solution to Equation 3.1, and this force field had no contribution from $N(G)$. The interaction force field was shown to belong to $N(G)$. An interaction force was defined as a vector of forces belonging to the solution space of Equation 3.1, that had two equal and opposite contact forces, acting between two contact points and along the line joining the two contact points. The interaction force field consisted of all the possible interaction forces for a given set of $n$ points.

Let $F^*$ be the equilibrating force vector. Then,

$$F^* = [G]^+ Q \quad (3.2)$$

where $[G]^+$ is the pseudo-inverse of the non-square matrix $[G]$ and

$$[G]^+ = [G]^T ([G] [G]^T)^{-1} \quad (3.3)$$

$F^*$ can be obtained more efficiently by the use of the equilibrating force field [Kumar and Waldron, 1988].

Let the vector joining a set of two points of contact, $i$ and $j$, be $v$ ($v$ belongs to $\mathbb{R}^3$). Then the interaction force between these two points, for any scalar $k$, will be of the form
\[ F_{ij} = k \left[ 0^T, 0^T, \ldots, v^T, 0^T, \ldots, -v^T, \ldots, 0^T \right]^T \]

Here \( F_{ij} \) is a 3nx1 vector made of n vectors belonging to \( \mathbb{R}^3 \) and only the \( i^{th} \) and \( j^{th} \) vectors are non-zero.

There are exactly \( nC_2 \) interaction forces that can be obtained for a set of n points. The interaction force field does not exist for the case of n equal to 1. When n takes the value of two or three, \( nC_2 \) is equal to 3n - 6 and it can be easily shown that the interaction force vectors form a basis for \( N(G) \). When n takes values greater than three, obtaining a basis for \( N(G) \) is non-trivial.

For a situation involving four contact points, \( nC_2 \) is equal to 3n - 6. However, as will be discussed in later sections, in certain situations the interaction force field is a subspace of \( N(G) \). For n greater than four, \( nC_2 \) is greater than 3n - 6. Thus the choice of a basis requires an efficient selection procedure that leads to simple basis vectors. Also for any value of n greater than three there always exist degenerate situations in which the interaction force field (even though \( nC_2 \) is much greater than 3n - 6) has a dimension lower than that of \( N(G) \). Some fundamental results leading to the characterization of the null space of the matrix \([G]\) are given next. A finite set of contact points is considered and the nature of the redundancy in force distribution is investigated in detail. Two types of distribution of the contact points are considered. The first type consists of contact points which are spatially distributed and the second type consists of a set of contact points lying on a plane. A set of points lying along a straight line are not considered as this leads to a situation where the matrix \([G]\) has a rank less than six and, as discussed earlier, such situations are rare and are not relevant in this study.
3.2.1.1 Spatial Set of Points

For a spatial set of n contact points the interaction force field and \( N(G) \) are identical and each space has a dimension of \( 3n - 6 \). Further, a subset of the \( nC_2 \) interaction force vectors containing \( 3n - 6 \) vectors forms the basis for \( N(G) \).

A proof of the above statement is given and this proof leads to a systematic way of obtaining a set of interaction forces which form a basis for \( N(G) \).

Consider a set of \( n \) spatial contact points (see Figure 3.4). These points are the contact locations of an active mechanism with its environment. Assume that the xyz coordinates of these points are known with respect to a coordinate system fixed to the mechanism body, and these coordinate vectors are \( C_i \) \((i = 1, 2, \ldots, n)\). Since the set of points is spatial, there always exist four points in this set that do not lie on a plane. Without loss of generality we can assume that these four points are \( C_1, C_2, C_3, \) and \( C_4 \).

Obtain \( (n - 1) \) vectors \( a_i \) such that

\[
 a_i = C_i - C_1. \quad (i = 2, 3, 4, \ldots, n) \tag{3.4}
\]

These vectors \( a_i \) belong to \( \mathbb{R}^3 \) and since the first four points are spatial,

\[
 \text{Span} \{a_2, a_3, a_4\} = \mathbb{R}^3.
\]

Steps to obtain the basis vectors of the null space \( N(G) \)

1) Obtain the three interaction forces among the points \( C_2, C_3, \) and \( C_4 \). It should be noted that each interaction force vector belongs to \( N(G) \) and hence all the interaction force vectors belong to \( \mathbb{R}^{3n} \).
Figure 3.3 Force System of a Body Interacting With its Environment

Figure 3.4 $n$ Spatial Points and Associated Vectors
2) Next, obtain the \((n - 1)\) interaction forces between \(C_1\), and each of the other points:

\[
\begin{align*}
F_{12} &= \begin{bmatrix} a_2 \\ -a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix},
F_{13} = \begin{bmatrix} a_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, F_{1n} = \begin{bmatrix} a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\end{align*}
\]

(3.6)

3) Finally, form the interaction force vectors \(F_{k\ell}\) and \(F_{m\ell}\) where \(\ell\) varies from 5 to \(n\) and the values of \(k\) and \(m\) are decided in the following manner.

First, check if \(\text{Span}\{a_2, a_3, a_\ell\} = \mathbb{R}^3\). Then, set \(k = 2, m = 3\).

Else, check if \(\text{Span}\{a_2, a_4, a_\ell\} = \mathbb{R}^3\). Then, set \(k = 2, m = 4\).

If both the above conditions are not satisfied, then, since \(\text{Span}\{a_2, a_3, a_4\} = \mathbb{R}^3\), vectors \(a_2\) and \(a_\ell\) are linearly independent. Hence, \(\text{Span}\{a_3, a_4, a_\ell\} = \mathbb{R}^3\).

Then, set \(k = 3, m = 4\).

Since \(\ell\) varies from 5 to \(n\), the set of vectors \(\{F_{k\ell}, F_{m\ell}\}\) contains \(2(n - 4)\) vectors. The vectors \(F_{k\ell}\) and \(F_{m\ell}\) can be obtained in the same way as vectors \(F_{23}, F_{24},\) and \(F_{34}\) were in Step 1. A general interaction force vector is comprised of \((n - 2)\) zero three-
dimensional vectors and two non-zero three dimensional vectors.

\[
\mathbf{F}_{ij} = \begin{bmatrix}
0 \\
. \\
a_i - a_j \\
0 \\
. \\
a_j - a_i \\
0 \\
. \\
0
\end{bmatrix}
\]

Here, \(i\) and \(j\) are not equal to 1

(3.7)

4) Form a set containing all the interaction force vectors obtained in Steps 1 through 3.

\[
\mathbf{A} = \{ \mathbf{F}_{23}, \mathbf{F}_{24}, \mathbf{F}_{34}, \mathbf{F}_{12}, \mathbf{F}_{13}, \ldots, \mathbf{F}_{1n}, \mathbf{F}_{k5}, \mathbf{F}_{k6}, \ldots, \mathbf{F}_{kn}, \mathbf{F}_{m5}, \mathbf{F}_{m6}, \ldots, \mathbf{F}_{mn} \}
\]

Set \(\mathbf{A}\) contains a total of \([3 + (n - 1) + 2(n - 4)] = (3n - 6)\) vectors. It is shown below that set \(\mathbf{A}\) forms a basis for the \(\mathbf{N}(\mathbf{G})\).

To show that the set \(\mathbf{A}\) is a basis of \(\mathbf{N}(\mathbf{G})\)

The set \(\mathbf{A}\) consists of \(3n - 6\) vectors that belong to \(\mathbf{N}(\mathbf{G})\). It is also known that the \(\dim(\mathbf{N}(\mathbf{G})) = 3n - 6\). Hence, to show that set \(\mathbf{A}\) forms a basis for \(\mathbf{N}(\mathbf{G})\), it is sufficient to show that set \(\mathbf{A}\) is a set of linearly independent vectors.

Consider the following equation:

\[
\alpha_{23} \mathbf{F}_{23} + \alpha_{24} \mathbf{F}_{24} + \alpha_{34} \mathbf{F}_{34} + \alpha_{12} \mathbf{F}_{12} + \alpha_{13} \mathbf{F}_{13} + \ldots + \alpha_{1n} \mathbf{F}_{1n} + \alpha_{k5} \mathbf{F}_{k5} + \alpha_{k6} \mathbf{F}_{k6} + \ldots + \alpha_{kn} \mathbf{F}_{kn} + \alpha_{m5} \mathbf{F}_{m5} + \alpha_{m6} \mathbf{F}_{m6} + \ldots + \alpha_{mn} \mathbf{F}_{mn} = 0
\]

(3.8)
Here \( \alpha_{ij} \)s are scalars and \( \mathbf{0} \) is the zero vector in the space \( \mathbb{R}^{3n} \). Equation 3.8 represents \( 3n \) scalar equations or \( n \) equations involving three-dimensional vectors. Note that the \( i \)th three-dimensional vector equation obtained from Equation 3.8 has been labelled as Equation 3.8-i in the following discussion. Using Equations 3.4 through 3.8, the last \( n - 4 \) equations involving three-dimensional vectors can be written as follows:

\[
\begin{align*}
-\alpha_{15}\mathbf{a}_5 + \alpha_{k5}(\mathbf{a}_5 - \mathbf{a}_k) + \alpha_{m5}(\mathbf{a}_5 - \mathbf{a}_m) &= \mathbf{0} \\
-\alpha_{16}\mathbf{a}_6 + \alpha_{k6}(\mathbf{a}_6 - \mathbf{a}_k) + \alpha_{m6}(\mathbf{a}_6 - \mathbf{a}_m) &= \mathbf{0} \\
&\quad \cdots \\
-\alpha_{1n}\mathbf{a}_n + \alpha_{kn}(\mathbf{a}_n - \mathbf{a}_k) + \alpha_{mn}(\mathbf{a}_n - \mathbf{a}_m) &= \mathbf{0}
\end{align*}
\]

(3.8-5) \hspace{2cm} (3.8-6) \hspace{2cm} (3.8-n)

Each of the above \( n - 4 \) equations can be written as

\[
(-\alpha_{1\ell} + \alpha_{k\ell} + \alpha_{m\ell})\mathbf{a}_\ell - \alpha_{k\ell}\mathbf{a}_k - \alpha_{m\ell}\mathbf{a}_m = \mathbf{0}
\]

(3.9)

where \( \ell \) varies from 5 to \( n \). In Equations 3.8-5 through 3.8-n and in Equation 3.9, \( \mathbf{0} \) represents the zero vector in the space \( \mathbb{R}^3 \). When the set \( A \) was being formed, in Step 2, it was ascertained that \( \text{Span}\{\mathbf{a}_k, \mathbf{a}_m, \mathbf{a}_\ell\} = \mathbb{R}^3 \). Hence, the set of vectors \( \{\mathbf{a}_k, \mathbf{a}_m, \mathbf{a}_\ell\} \) is a linearly independent set. This requires the scalars in Equation 3.9 to be all equal to zero. Therefore, the \( 3(n - 4) \) scalars,

\[
\alpha_{1\ell} = \alpha_{k\ell} = \alpha_{m\ell} = 0 \quad \text{where } \ell \text{ varies from 5 to } n.
\]

(3.10)

Using Equation 3.10 in Equation 3.8, the first four equation involving three-dimensional vectors can be written as follows:
\[ \alpha_{12}a_2 + \alpha_{13}a_3 + \alpha_{14}a_4 = 0 \quad (3.8-1) \]
\[ -\alpha_{12}a_2 + \alpha_{23}(a_2 - a_3) + \alpha_{24}(a_2 - a_4) = 0 \quad (3.8-2) \]
\[ -\alpha_{13}a_3 + \alpha_{23}(a_3 - a_2) + \alpha_{34}(a_3 - a_4) = 0 \quad (3.8-3) \]
\[ -\alpha_{14}a_4 + \alpha_{24}(a_4 - a_2) + \alpha_{34}(a_4 - a_3) = 0 \quad (3.8-4) \]

Again, using the fact that the \( \text{Span}\{a_2, a_3, a_4\} = \mathbb{R}^3 \), all the six scalars in the above four equations must be equal to zero. Hence,

\[ \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = \alpha_{34} = 0 \quad (3.11) \]

Equations 3.10 and 3.11 require all the \( 3n - 6 \) \( \alpha_{ij} \)s to be equal to zero. Therefore, Equation 3.8 has only a trivial solution. This is true if and only if the vectors in \( A \) are linearly independent. Hence, Set \( A \) forms a basis for \( \mathbf{N}(G) \) and this completes the proof.

Example 3.1:

In this example, the procedure described above is applied to a case of five spatial points, and a set of basis vectors for \( \mathbf{N}(G) \) is identified. The five contact points are specified as follows:

\[ C_1 = [0, 0, 0]^T, \quad C_2 = [0, 1, 0]^T, \quad C_3 = [-1, 2, 0]^T, \]
\[ C_4 = [1, 3, 2]^T, \quad C_5 = [-1, 2, 1]^T \]

Since, \( n = 5 \), the dimension of \( \mathbf{N}(G) \) is equal to \( 3n - 6 = 9 \). Also, the null space vectors belong to \( \mathbb{R}^{15} \).

The points \( C_1, C_2, C_3, \) and \( C_4 \) form a set of four spatial points. Obtain the vector
a_i for i = 2, 3, 4, and 5. In this case since C_1 = [0, 0, 0]^T. Hence, a_i = C_i.

Step 1: Obtain F_{23}, F_{24}, and F_{34}.

F_{23} = [0, 0, 0, 1, -1, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0]^T

F_{24} = [0, 0, 0, -1, -2, -2, 0, 0, 1, 2, 2, 0, 0, 0, 0]^T

F_{34} = [0, 0, 0, 0, 0, -2, -1, -2, 2, 1, 2, 0, 0, 0, 0]^T

Step 2: Obtain F_{12}, F_{13}, F_{14} and F_{15}

F_{12} = [0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T

F_{13} = [-1, 2, 0, 0, 0, 0, 1, -2, 0, 0, 0, 0, 0, 0, 0]^T

F_{14} = [1, 3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -3, -2, 0, 0, 0]^T

F_{15} = [-1, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -2, -1]^T

Step 3: In this example \ell just takes the value of 5. Obtain F_{25} and F_{35}

\text{Det} [a_2, a_3, a_5] = 1 \text{ (non-zero)}. Hence, \text{Span} \{a_2, a_3, a_5\} = \mathbb{R}^3. Set k = 2, m = 3.

F_{25} = [0, 0, 0, 1, -1, -1, 0, 0, 0, 0, 0, -1, 1, 1]^T

F_{35} = [0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1]^T

Step 4: Basis B = \{F_{23}, F_{24}, F_{34}, F_{12}, F_{13}, F_{14}, F_{15}, F_{25}, F_{35}\}

To verify that the vectors are linearly independent, form the matrix

F = [F_{23}, F_{24}, F_{34}, F_{12}, F_{13}, F_{14}, F_{15}, F_{25}, F_{35}]. The rank of this matrix should be
equal to 9. The rank was evaluated using the matrix analysis software MATLAB and it was found to be equal to 9.

### 3.2.1.1 Planar Set of Points

*For a planar set of contact points, the interaction force field (I(G)) has dimension (2n - 3) and forms a subspace of N(G) (dim(N(G)) = 3n - 6). There exists another force field, the orthogonal force field (O(G)), which has dimension (n - 3). Further, I(G) U O(G) = N(G).*

The orthogonal force field is first defined. A proof of the above statement follows and this proof includes a systematic procedure to obtain the basis vectors for the spaces I(G) and O(G). Throughout this discussion, it is assumed that the n contact points lie on a plane and that there exist at least three non-collinear points.

A planar set of contact points C<sub>i</sub> is shown in Figure 3.5. The coordinates of all these points are assumed to be known with respect to a convenient coordinate frame that has its xy plane coincident with the plane containing all the points (the contact plane). Also, the x and y components of the contact force at each contact point lie in the contact plane and the z component is normal to the plane. This situation is shown in Figure 3.5. For this planar case, the six rigid body equations represented by Equation 3.1 decouple into two sets of three equations. The first set only involves the 2n components of the contact forces lying in the contact plane and the second set involves the n components of the contact forces normal to the contact plane. Thus, Equation 3.1 takes the following form:

\[
[G_1]F_{xy} = Q_1 
\]  
(3.12)

\[
[G_2]F_z = Q_2 
\]  
(3.13)
Here $F_{xy}$ is a vector that belongs to $\mathbb{R}^{2n}$. The size of the matrix $[G_1]$ is $3\times 2n$. $F_z$ is a vector that belongs to $\mathbb{R}^n$. The size of the matrix $[G_2]$ is $3\times n$. The vectors $Q_1$ and $Q_2$ belong to $\mathbb{R}^3$. The ranks of the two matrices $[G_1]$ and $[G_2]$ are each equal to three, because it is assumed that there are at least three non-collinear points on the contact plane.

\[
F_{xy} = [F_{x1}, F_{y1}, F_{x2}, F_{y2}, \ldots, F_{xn}, F_{yn}]^T
\]
\[
F_z = [F_{z1}, F_{z2}, F_{z3}, \ldots, F_{zn}]^T
\]
\[
Q_1 = [F_{Bx}, F_{By}, T_{Bz}]^T
\]
\[
Q_2 = [T_{Bx}, T_{By}, F_{Bz}]^T
\]

The vectors $F_B$ and $T_B$ represent the body force and moment vectors respectively. Since $\text{rank}(G_1) = 3$ and the matrix $[G_1]$ is of size $3\times 2n$, $\dim(N(G_1)) = 2n - 3$. Similarly, $\dim(N(G_2)) = n - 3$. Moreover, from Equations 3.12 and 3.13, the null space $N(G_1)$ contributes only to the x and y components of the contact forces and the space $N(G_2)$ contributes only to the z components of the contact forces. As was described earlier, the interaction force field involves equal and opposite non-zero contact forces along the line joining the contact points. Hence, in the planar situation all the non-zero components of the interaction force field lie in the contact plane. Therefore, the space $N(G_1)$ has contributions from the interaction force field while the space $N(G_2)$ has no contributions from the interaction force field. It should be noted that for the cases of $n$ less than four, the space $N(G_2)$ does not exist and the interaction force field spans the null space of $[G]$ completely.
**Orthogonal force field**

Consider a set of four non-collinear planar points among the \( n \) planar contact points. There exists a set of non-zero contact forces normal to the contact plane at these four points, all the other normal contact forces being zero, and the \( x \) and \( y \) components of all the contact forces being zero. This vector belongs to the null space \( N(G) \). Further, any scalar multiple of this vector also belongs to \( N(G) \). This vector is known as an *orthogonal force vector* (\( O(G) \)), and the \( nC^4 \) such orthogonal forces obtained by taking four contact points at a time form the *orthogonal force field*. Four non-collinear contact points, along with other contact points and the orthogonal force vector, are shown in Figure 3.6. The orthogonal force vector corresponding to these four points can be derived in the following manner. Let the coordinates of the four points be \( C_i = [x_i, y_i, 0]^T \), \( C_j = [x_j, y_j, 0]^T \), \( C_k = [x_k, y_k, 0]^T \), and \( C_\ell = [x_\ell, y_\ell, 0]^T \). Since the points are non-collinear, it can be assumed without loss of generality that \( C_i, C_j, \) and \( C_k \) are non-collinear. The normal components of the contact forces at the four points are \( F_{z_i}, F_{z_j}, F_{z_k}, \) and \( F_{z_\ell} \) respectively. These forces should have zero resultant force and moment vectors because they contribute to \( N(G) \). Hence,

\[
\begin{align*}
F_{z_i} + F_{z_j} + F_{z_k} + F_{z_\ell} &= 0 \\
x_iF_{z_i} + x_jF_{z_j} + x_kF_{z_k} + x_\ell F_{z_\ell} &= 0 \\
y_iF_{z_i} + y_jF_{z_j} + y_kF_{z_k} + y_\ell F_{z_\ell} &= 0 
\end{align*}
\]

The above three equations can be rearranged to obtain the following matrix equation:

\[
\begin{bmatrix}
1 & 1 & 1 \\
x_i & x_j & x_k \\
y_i & y_j & y_k
\end{bmatrix}
\begin{bmatrix}
F_{z_i} \\
F_{z_j} \\
F_{z_k}
\end{bmatrix}
= \begin{bmatrix}
-F_{z_\ell} \\
-x_\ell F_{z_\ell} \\
-y_\ell F_{z_\ell}
\end{bmatrix} \tag{3.14}
\]
The determinant of the coefficient matrix in Equation 3.14 can be obtained as

\[
D = \begin{vmatrix}
(x_j - x_i) & (x_k - x_i) \\
(y_j - y_i) & (y_k - y_i)
\end{vmatrix}
\]

(3.15)

Since \( C_i, C_j, \) and \( C_k \) are non-collinear, the vector joining \( C_i \) to \( C_j \) and the vector joining \( C_i \) to \( C_k \) are linearly independent. Hence, \( D \) is non-zero and Equation 3.14 can be solved for \( F_{zi}, F_{zj}, \) and \( F_{zk} \), for any value of \( F_{z\ell} \). \( F_{z\ell} \) can be assumed to be equal to any scalar provided it is non-zero. It denotes the strength of the orthogonal force vector. If \( F_{z\ell} \) is equal to zero, then all the other three normal components also become zero and the orthogonal force vector degenerates to a trivial zero vector. For the sake of simplicity \( F_{z\ell} \) can be assumed to be equal to unity. Then the other three components of the orthogonal force field can be obtained using the following equation:

\[
\begin{bmatrix}
F_{zi} \\
F_{zj} \\
F_{zk}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
x_i & x_j & x_k \\
y_i & y_j & y_k
\end{bmatrix}^{-1} \begin{bmatrix}
-1 \\
-x_{\ell} \\
-y_{\ell}
\end{bmatrix}
\]

(3.16)

Next, detailed procedures for obtaining the basis vectors for the orthogonal force field (\( O(G) \)) and the interaction force field (\( I(G) \)) are given. Consider a set of \( n \) planar contact points (see Figure 3.7). Assume that the xy coordinates of these points are known with respect to a coordinate system fixed to the rigid body on the contact plane, and these coordinate vectors are \( C_i \) \((i = 1, 2, \ldots, n)\). The z coordinates of all the contact points are taken to be zero for convenience. Since the set of points are planar and are assumed to be non-collinear, there always exist three points in this set that do not lie along a line. Without loss of generality we can assume that these points are \( C_1, C_2, \) and \( C_3 \). Let,

\[
b_i = C_i - C_1. \quad (i = 2, 3, 4, \ldots, n)
\]

(3.17)
Figure 3.5 \( n \) Planar Contact Points and the Contact Force Components

Figure 3.6 Orthogonal Force Vector With Four Non-Zero Normal Components

Figure 3.7 \( n \) Planar Points and Associated Vectors
The vectors $b_i \in \mathbb{R}^2$. Since the first three points are non-collinear, $\text{Span}\{b_2, b_3\} = \mathbb{R}^2$.

**Steps to obtain the basis vectors of $I(G)$**

1) Obtain the interaction force between the points $C_2$ and $C_3$. It should be noted that each interaction force vector belongs to $N(G)$ and hence all the interaction force vectors belong to $\mathbb{R}^{3n}$. In the following interaction force vectors, the three-dimensional zero vector is denoted by '0' and '0' represents the scalar zero.

$$F_{23} = \begin{bmatrix} 0 \\ b_2 \cdot b_3 \\ 0 \\ b_3 \cdot b_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ (3.18)

2) Next, obtain the $(n - 1)$ interaction forces between $C_i$, and each of the other points:

$$F_{12} = \begin{bmatrix} b_2 \\ 0 \\ -b_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad F_{13} = \begin{bmatrix} b_3 \\ 0 \\ -b_3 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad F_{1n} = \begin{bmatrix} b_n \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -b_n \end{bmatrix}$$ (3.19)

3) Finally, form the interaction force vectors $F_{k\ell}$ where $\ell$ varies from 4 to $n$ and the value of $k$ is decided in the following manner:
First, check if Span\(\{b_2, b_\ell\} = \mathbb{R}^2\). Then, set \(k = 2\).

If the above condition is not satisfied, then, since Span\(\{b_2, b_3\} = \mathbb{R}^2\), vectors \(b_2\) and \(b_\ell\) are linearly independent. Hence, Span\(\{b_3, b_\ell\} = \mathbb{R}^3\).

Then, set \(k = 3\).

Since \(1\) varies from 4 to \(n\), the set of vectors \(\{F_{k\ell}\}\) contains \((n - 4)\) vectors. The vectors \(F_{kl}\) can be obtained in the same way as the vector \(F_{23}\) was obtained in Step 1. A general interaction force vector is comprised of \((n - 2)\) zero three-dimensional vectors and two non-zero three dimensional vectors both having their \(z\) component equal to zero as shown below.

\[
F_{ij} = \begin{bmatrix}
0 \\
. \\
. \\
. \\
0 \\
0 \\
. \\
b_i - b_j \\
0 \\
0 \\
. \\
0 \\
\end{bmatrix}
\]

Here, \(i\) and \(j\) are not equal to 1 \hspace{1cm} (3.20)

4) Next, form a set \(B_1\) containing all the interaction force vectors obtained in Steps 1 through 3.

\[
B_1 = \{F_{23}, F_{12}, F_{13}, \ldots, F_{1n}, F_{k4}, F_{k5}, \ldots, F_{kn}\}.
\]
Set $B_1$ contains a total of $[1 + (n - 1) + (n - 3)] = (2n - 3)$ vectors. It is shown below that set $B_1$ forms a basis for the $I(G)$.

To show that the set $B_1$ is a basis of $I(G)$

The set $B_1$ consists of 2n - 3 vectors that belong to $N(G)$. It is also known that the $\dim (N(G_1)) = 2n - 3$. Each of the interaction forces in set $B_1$ contains $n$ three-dimensional vectors and each of these three-dimensional vectors has its z component equal to zero. If the z components are left out from the interaction force vector, the resulting vector (in $R^{2n}$) will belong to $N(G_1)$. Hence, the maximum dimension of $I(G)$ is equal to 2n - 3. To show that set $B_1$ forms a basis for $I(G)$, it is sufficient to show that set $B_1$ is a set of linearly independent vectors.

Consider the following equation:

$\beta_{23}F_{23} + \beta_{12}F_{12} + \beta_{13}F_{13} + \ldots + \beta_{1n}F_{1n} + \beta_{k4}F_{k4} + \beta_{k5}F_{k5} + \ldots + \beta_{kn}F_{kn} = 0$ (3.21)

Here $\beta_{ij}s$ are scalars and 0 is the zero vector in the space $R^{3n}$. Equation 3.21 represents 3n scalar equations or n equations involving three-dimensional vectors. Note that the ith three-dimensional vector equation obtained from Equation 3.21 has been labelled as Equation 3.21-i in the following discussion. Using Equations 3.17 through 3.21, the last n - 3 equations involving three-dimensional vectors can be written as follows:

$-\beta_{14}b_4 + \beta_{k4}(b_4 - b_k) = 0$ (3.21-4)

$-\beta_{15}b_5 + \beta_{k5}(b_5 - b_k) = 0$ (3.21-5)
Each of the above $n-3$ equations can be written in the form

$$(-\beta_{1\ell} + \beta_{k\ell})b_{\ell} - \beta_{k\ell}b_k = 0 \quad (3.22)$$

where $\ell$ varies from 4 to $n$. In Equations 3.21-4 through 3.21-n and in Equation 3.22, \(0\) represents the zero vector in the space $\mathbb{R}^2$. When Set $\mathbf{B}_1$ was being formed in Step 2, it was ascertained that $\text{Span}\{b_k, b_\ell\} = \mathbb{R}^2$. Hence, the set of vectors $\{b_k, b_\ell\}$ is a linearly independent set. This requires the scalars in Equation 3.22 to be all equal to zero. Therefore, the $2(n-3)$ scalars,

$$\beta_{1\ell} = \beta_{k\ell} = 0 \quad \text{ where } \ell \text{ varies from 4 to } n. \quad (3.23)$$

Using Equation 3.23 in Equation 3.21, the first three equations involving three-dimensional vectors can be written as follows:

$$\beta_{12}b_2 + \beta_{13}b_3 = 0 \quad (3.21-1)$$

$$-\beta_{12}b_2 + \beta_{23}(b_2 - b_3) = 0 \quad (3.21-2)$$

$$-\beta_{13}b_3 + \beta_{23}(b_3 - b_2) = 0 \quad (3.21-3)$$

Again, using the fact that the $\text{Span}\{b_2, b_3\} = \mathbb{R}^2$, all the three scalars in the above three equations have to be equal to zero. Hence,
Equations 3.23 and 3.24 require all the $2n - 3 \beta_{ij}$s to be equal to zero. Therefore, Equation 3.21 has only a trivial solution. This is true if and only if the set of vectors $B_1$ is a linearly independent set. Hence, Set $B_1$ forms a basis for $I(G)$ and this completes the proof.

Steps to obtain the basis vectors of $O(G)$

The points $C_1$, $C_2$, and $C_3$ were assumed to be non-collinear. Hence, these three points along with any other point $C_m$ will contribute towards an orthogonal force vector. Here $m$ varies from 4 to $n$.

1) Obtain the orthogonal force between the points $C_1$, $C_2$, $C_3$, and $C_m$. Since the first three points are non-collinear, the determinant of Equation 3.15 using $i = 1$, $j = 2$, and $k = 3$ will always be non-zero. Equation 3.16 can then be used to evaluate $F_{1z}$, $F_{2z}$, and $F_{3z}$ assuming $F_{zm} = 1$. It should be noted that each orthogonal force vector belongs to $N(G)$ and hence all the orthogonal force vectors belong to $R^{3n}$. In the following orthogonal vectors, the three-dimensional zero vector is denoted by '0' and '0' represents the scalar zero.

$$F_{123m} = [0, 0, F_{1z}, 0, 0, F_{2z}, 0, 0, F_{3z}, 0^T, \ldots, 0^T, 0, 0, 1, 0^T, \ldots, 0^T]^T \quad (3.25)$$

2) Form a set containing all the $n - 3$ orthogonal force vectors obtained in Step 1.

$$B_2 = \{F_{1234}, F_{1235}, F_{1236}, \ldots, F_{123n}\}$$

It is shown below that Set $B_2$ forms a basis for the $O(G)$. 

$$\beta_{12} = \beta_{13} = \beta_{23} = 0 \quad (3.24)$$
To show that the set $B_2$ is a basis of $O(G)$

Set $B_2$ consists of $n - 3$ vectors that belong to $N(G)$. It is also known that the $\dim(N(G_2)) = n - 3$. Each of the orthogonal forces in Set $B_2$ contains $n$ three-dimensional vectors and each of these three-dimensional vectors has its $x$ and $y$ components equal to zero. If the $x$ and $y$ components are left out from the orthogonal force vector, the resulting vector (in $\mathbb{R}^n$) will belong to $N(G_2)$. Hence, the maximum dimension of $O(G)$ is equal to $n - 3$. To show that Set $B_2$ forms a basis for $O(G)$, it is sufficient to show that Set $B_2$ is a set of linearly independent vectors.

Consider the following equation:

$$\gamma_{1234}F_{1234} + \gamma_{1235}F_{1235} + \ldots + \gamma_{123n}F_{123n} = 0$$

(3.26)

Here the $\gamma_{123m}$'s are scalars and $0$ is the zero vector in the space $\mathbb{R}^{3n}$. Equations 3.25 and 3.26 lead to $3n$ scalar equations. $2n$ of these equations are trivially satisfied. By inspection of the last $n - 3$ non-trivial equations it can be easily seen that all the $n - 3$ scalars $\gamma_{123m}$ must be equal to zero. Therefore,

$$\gamma_{1234} = \gamma_{1235} = \ldots = \gamma_{123n} = 0$$

(3.27)

Therefore, Equation 3.26 has only a trivial solution. This is true if and only if the set of vectors $B_2$ is a linearly independent set. Hence, Set $B_2$ forms a basis for $O(G)$ and this completes the proof.

Finally, the interaction force vectors are made of $n$ three-dimensional vectors each of which has its $z$ component equal to zero. The orthogonal force vectors are made up of $n$ three-dimensional vectors each of which has its $x$ and $y$ components equal to zero. Hence,
the set $B = B_1 \cup B_2$ is also a linearly independent set. The set $B$ has a total of $3n - 6$ vectors. This is the same as the dimension of $N(G)$. Hence, the set of vectors $B$ forms a basis for $N(G)$. Further, since $B_1$ is a basis for $I(G)$ and $B_2$ is a basis for $O(G)$,

$$N(G) = I(G) \cup O(G)$$

**Example 3.2**

In this example, the procedure described above is applied to a case of five planar contact points, and a set of basis vectors for $N(G)$ is identified. The five contact points are given to be as follows:

$$C_1 = [0, 0, 0]^T, \quad C_2 = [0, 1, 0]^T, \quad C_3 = [-1, 2, 0]^T,$$

$$C_4 = [-1, -4, 0]^T, \quad C_5 = [5, -2, 0]^T$$

Here, $n = 5$. The dimension of $N(G)$ is equal to $3n - 6 = 9$. Also, the null space vector belong to $\mathbb{R}^{15}$. Since the points are planar, the basis of $N(G)$ is to be found by finding the basis vectors for the spaces $I(G)$ and $O(G)$.

The basis for $I(G)$, $B_1$, contains $2n - 3 = 7$ vectors. These vectors can be found as follows:

The points $C_1$, $C_2$, and $C_3$ form a set of three non-collinear points. Obtain the vector $b_i$ for $i = 2, 3, 4,$ and $5$.

$$b_2 = [0, 1]^T, \quad b_3 = [-1, 2]^T,$$

$$b_4 = [-1, -4]^T, \quad b_5 = [5, -2]^T$$

**Step 1:** Obtain $F_{23}$
\( \mathbf{F}_{23} = [0, 0, 1, -1, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] \)^T

**Step 2:** Obtain \( \mathbf{F}_{12}, \mathbf{F}_{13}, \mathbf{F}_{14}, \) and \( \mathbf{F}_{15} \).

\( \mathbf{F}_{12} = [0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] \)^T

\( \mathbf{F}_{13} = [-1, 2, 0, 0, 0, 0, 1, -2, 0, 0, 0, 0, 0, 0, 0] \)^T

\( \mathbf{F}_{14} = [-1, 4, 0, 0, 0, 0, 0, 0, 1, -4, 0, 0, 0, 0, 0] \)^T

\( \mathbf{F}_{15} = [5, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -5, 2, 0] \)^T

**Step 3:** In this example \( \ell \) just takes the values of 4 and 5. Obtain \( \mathbf{F}_{k5} \) and \( \mathbf{F}_{k5} \).

\( \text{Det} \{ \mathbf{b}_2, \mathbf{b}_4 \} = 1 \) (non-zero). Hence, \( \text{Span} \{ \mathbf{b}_2, \mathbf{b}_4 \} = \mathbb{R}^2 \). For \( \ell = 4 \), set \( k = 2 \).

\( \text{Det} \{ \mathbf{b}_2, \mathbf{b}_5 \} = -5 \) (non-zero). Hence, \( \text{Span} \{ \mathbf{b}_2, \mathbf{b}_5 \} = \mathbb{R}^2 \). For \( \ell = 5 \), set \( k = 2 \).

\( \mathbf{F}_{24} = [0, 0, 0, 1, -3, 0, 0, 0, -1, 3, 0, 0, 0, 0, 0] \)^T

\( \mathbf{F}_{25} = [0, 0, 0, -5, 3, 0, 0, 0, 0, 0, 0, 0, 5, -3, 0] \)^T

**Step 4:** Basis \( \mathbf{B}_4 = \{ \mathbf{F}_{23}, \mathbf{F}_{12}, \mathbf{F}_{13}, \mathbf{F}_{14}, \mathbf{F}_{15}, \mathbf{F}_{24}, \mathbf{F}_{25} \} \)

Next, the basis for \( \mathbf{O(G)}, \mathbf{B}_2 \), containing \( n - 3 = 2 \) vectors is to be obtained.

These vectors can be found as follows:

**Step 1:** Using Equation 3.16,

for \( \ell = 4 \), \( F_{1z} = -2 \), \( F_{2z} = 2 \), \( F_{3z} = 1 \).

\( \mathbf{F}_{1234} = [0, 0, -2, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0] \)^T
for $\ell = 5$, $F_{1z} = -2$, $F_{2z} = 8$, $F_{3z} = -5$.

$F_{1235} = [0, 0, -2, 0, 8, 0, 0, -5, 0, 0, 0, 0, 0, 0]^{T}$

**Step 2:** Basis $B_2 = \{F_{1234}, F_{1235}\}$

The basis for the $N(G)$ is obtained by using all the vectors in sets $B_1$ and $B_2$.

$B = B_1 \cup B_2 = \{F_{23}, F_{12}, F_{13}, F_{14}, F_{15}, F_{24}, F_{25}, F_{1234}, F_{1235}\}$

To verify that the vectors are linearly independent, form the matrix

$F = [F_{23}, F_{12}, F_{13}, F_{14}, F_{15}, F_{24}, F_{25}, F_{1234}, F_{1235}]$. The rank of this matrix should be equal to 9. The rank was evaluated using the matrix analysis software MATLAB and it was found to be equal to 9.

In this section (Section 3.2.1), the nature of the redundancy in active mechanisms that have complete control over the contact forces has been studied in detail. Efficient algorithms to obtain the null space basis vectors have been developed. Two kinds of contact point sets have been considered, planar contact point sets, and spatial contact point sets. In a particular situation, if it can be clearly established *a priori* whether the point sets are spatial or planar, the algorithms developed here can be used directly. However in some situations, such as a legged vehicle traversing a 'quasi-planar' surface, the contact point set can be spatial, even though all the points lie close to a plane surface. In such a situation, use of the spatial algorithm will lead to basis vectors which are poorly conditioned. These vectors are generally required to be used as part of nonlinear numerical optimization techniques. Hence, poorly conditioned vectors are undesirable. Well conditioned basis vectors can be obtained by using a combination of the planar and the spatial algorithm. In
this modified algorithm, a plane that approximately fits all the points is chosen. This choice could be made based on interpolation or regression techniques. The spatial points are then projected onto this plane. The orthogonal force field corresponding to this plane can be used to obtain $n - 3$ null space vectors. The other null space vectors can be obtained by choosing any $2n - 3$ of the $3n - 6$ vectors that can be obtained from the spatial algorithm. This procedure will lead to an exact, well conditioned set of basis vectors. Therefore, the redundancy can be characterized efficiently and accurately for any spatial closed kinematic chain.

3.2.2 Active Mechanisms with Partial Controllability Over Contact Conditions

In the previous section, the force redundancy in systems that allow complete control over contact conditions was studied. In this section, an attempt is made to characterize the redundancy in force allocation in active mechanisms that allow complete control over only a subset of the contact force components. The study is specifically directed towards actively coordinated wheeled systems, and an analysis of redundancy is performed for the actively articulated wheeled vehicle configuration of Figure 3.10.

Force distribution characteristics of actively articulated wheeled systems can be investigated by considering the contact force system at the wheel-ground interface shown in Figure 3.8. Here the 'normal' force component $F_{ni}$ is assumed to be directed along the line joining the contact point and the wheel center. It is, in general, different from the vector direction normal to the terrain at the contact point. The tangential force component $F_{li}$ is taken to be orthogonal to $F_{ni}$ and it lies on the plane of the wheel, and the lateral component $F_{ji}$ is orthogonal to the plane of the wheel. The directions $t$, $l$, and $n$ are analogous to a right-handed xyz coordinate frame. In Figure 3.9, two wheels and the axle of a module are
Figure 3.8 Force System at the Wheel-Ground Interface for Contact Point 'i'

Figure 3.9 Infeasible Interaction Force

Figure 3.10 Actively Articulated Wheeled Vehicle Configuration
shown. The interaction force component (as defined in Section 3.2.1) between the two wheel contact points is also shown. This represents equal and opposite contact forces that are parallel (or nearly parallel) to the wheel axle. This interaction force component does not exist for vehicles such as the one shown in Figure 3.10. Even though the lateral force components can be influenced by the roll and pitch articulation actuators, they cannot be arbitrarily allocated. This is due to the fact that the axle length is fixed, and therefore the distance between the two wheel contact points cannot be actively varied. This aspect of actively coordinated wheeled systems is different from legged vehicles or articulated multi-fingered systems. This lack of controllability over the contact force in the lateral direction can be mathematically expressed as follows:

\[ (F_L - F_R) \cdot a = 0 \]  

(3.28)

Here, \( F_L \) and \( F_R \) are the contact force vectors at the left and right wheels respectively, and \( a \) is the unit vector directed along the axle of the module.

The normal and the tangential contact force components lie in the plane of the wheel, and the actuation system has complete control over these force components. (Of course, complete control over these force components exists provided the commanded normal forces are positive, and the commanded tangential forces can be supported by the friction coefficient at the wheel-ground interface.) Therefore, the redundancy in force distribution exists only in the allocation of the contact force components in the wheel planes.

The force distribution problem for actively coordinated wheeled systems is further complicated when they are required to move on uneven terrain. The geometry of the vehicle requires a lateral slip of each module in the direction of the wheel axle in order to
allow the vehicle to move. If this slip is not allowed (perhaps due to very large coefficients of friction at the wheel-ground interfaces), the vehicle will lose its mobility and it will become a structure. This topic is discussed in greater detail in the chapter on motion planning (Chapter 4). The lateral slip at the wheels puts further constraints on the lateral forces, and these constraints depend on the contact conditions. For the reasons given in the following discussion, in this study, the effect of lateral slip on the contact forces is not considered during difficult mobility maneuvers. If the vehicle is performing difficult mobility maneuvers, the wheel-terrain contact locations are either stationary (as in the case of two modules stationary on the ground in obstacle climbing, ditch crossing etc.), or they change rather slowly. In these cases, the lateral slip may be negligible, and the force distribution can be investigated assuming the lateral contact forces have to only satisfy Equation 3.28. An optimal force distribution strategy in such a situation would be one that would maximize vehicle stability and minimize the tendency for the vehicle to slip at the wheel-terrain contact locations. If the vehicle is moving relatively faster, it is probably operating on easier terrain, and the optimal force distribution strategy in such a situation may be one that minimizes power consumption. A pseudo-inverse force allocation solution can be obtained very efficiently [Kumar and Waldrón, 1988] and it may be a convenient choice on easy terrain. Irrespective of the strategy used, the vehicle may possess some lateral drift from its desired path. This can be corrected by periodically performing a 'lateral maneuver' as was described in Chapter 1. If the objective is to minimize the power consumed during locomotion, it is desirable to minimize the function

\[ P = \sum_{i=1}^{n} F_i \cdot v_{si} \]

where \( n \) is the number of wheel-terrain contact locations, \( F_i \) is the contact force vector at the \( i^{th} \) wheel contact, and \( v_{si} \) is the slip vector at that wheel contact (the absolute velocity of
the contact point on the wheel). The function $P$ represents the power wasted due to slip at the wheel-terrain contact locations. While $F_i$ can be directly measured, $v_{si}$ is difficult to estimate accurately. Also, in order to minimize $P$ subject to the dynamic equilibrium conditions (Equation 3.1) it is desirable to relate $v_{si}$ to $F_i$. An accurate characterization of this relationship requires a good tire model and the knowledge of the local soil properties. This may not be possible in field conditions. Nevertheless, the approximation

$$v_{si} = a \frac{|F_{ti}|}{|F_{ni}|} t_i + b \frac{|F_{li}|}{|F_{ni}|} l_i,$$

may be used. Here, $F_{ti}$ is the component of contact force tangent to the wheel circumference in the plane of the wheel, $F_{ni}$ is the radial component and $F_{li}$ is the lateral component normal to the plane of the wheel. The vectors $t_i$ and $l_i$ are unit vector parallel to $F_{ti}$ and $F_{li}$ respectively. The parameters $a$ and $b$ are respectively tangential and lateral slip coefficients. It should be understood that this approximation will not be good for large slip values and that $a$ and $b$ are very variable and uncertain. Therefore, the coordination scheme resulting from this force distribution should be used conservatively. Using the above assumptions, the function $P$ takes the form

$$P = \sum_{i=1}^{n} \left\{ a \frac{|F_{ti}|^2}{|F_{ni}|} + b \frac{|F_{li}|^2}{|F_{ni}|} \right\}.$$

The nearest to an optimal force allocation procedure with respect to minimal power consumption is the minimization of the above expression subject to the dynamic constraints of Equation 3.1. On relatively easy terrain with even distribution of weight among the wheels, the $F_{ni}$'s will be approximately the same. Further, if the parameters $a$ and $b$ are approximately equal, the optimal force distribution will be approximated by the minimum norm or the pseudo-inverse solution. A more practical problem is one where $F_{ni}$'s are not
equal and the parameters $a$ and $b$ are different. This problem can be significantly simplified by assuming that $F_{n_i}$'s are constant. This means that the normal force components are unaffected by the redundancy in the system. This may be a reasonable assumption on relatively easy terrain. The optimization problem then reduces to that of a weighted minimum-norm problem. The cost function $P$ takes the form

$$P = \sum_{i=1}^{n} \left\{ w_{t_i} |F_{t_i}|^2 + w_{l_i} |F_{l_i}|^2 + 0 |F_{n_i}|^2 \right\}$$

where $w_{t_i}$ is the weighting factor on the square of the tangential force component $F_{t_i}$, $w_{l_i}$ is the weighting factor on the square of the lateral force component $F_{l_i}$, and the weighting factor on the normal force component is zero.

In the above discussion the power minimization problem has been addressed in a preliminary manner. This problem is not discussed any further in this dissertation. The optimal force distribution in these vehicle during difficult mobility maneuvers is discussed in greater detail, later in this chapter.

The force allocation problem of the actively articulated wheeled vehicle is now described in detail. Let us consider the case of $m$ modules, and $n$ contact points. It is assumed that all the $m$ modules are in contact with the terrain. The $n$ contact points are $C_i$. The contact force vector at the contact point $C_i$ is $F_i$ and the surface normal at $C_i$ is $n_i$. The force vector $F_B$ and the moment vector $T_B$ represent the body force and moment respectively and they include both commanded inertial and gravitational forces.

$$F_i = [F_{t_i}, F_{l_i}, F_{n_i}]^T$$

$$C_i = [x_i, y_i, z_i]^T$$

$$F = [F_{t1}, F_{l1}, F_{n1}, F_{t2}, F_{l2}, F_{n2}, ..., F_{tn}, F_{ln}, F_{nn}]^T$$
Q = \left[F_B^T, T_B^T \right]^T

The dynamic equations for the rigid body can be written as

\[ \left[ G \right] F = Q \quad (3.29) \]

Here \( \left[ G \right] \) = \( \left[ \$_{t1}, \$_{11}, \$_{n1}, \$_{t2}, \$_{12}, \$_{n2}, ..., \$_{tn}, \$_{1n}, \$_{nn} \right] \), where \( \$_j \) is a 6×1 screw vector directed along the contact force component \( F_j \). A discussion of the screw vector can be found in the literature [Hunt, 1978]. The computation of the vectors \( \$_j \) requires the coordinates of the contact points \( C_i \), and the unit vectors along the \( t_i, l_i, \) and \( n_i \) directions. Computation of these vectors requires the knowledge of the contact points on the wheel in the module frame \( x_iy_iz_i \) shown in Figure 3.10. This can be obtained using position kinematic solutions, if the terrain geometry is known, as discussed in Chapter 4. It can also be estimated using contact force sensing as was discussed by Kumar and Waldron [1989]. Here, it is assumed that the contact point coordinates can be obtained using one of the above techniques. \( \left[ G \right] \) is a purely geometry dependent matrix. For a given vector \( Q \), the force allocation problem involves solving Equation 3.29 for the contact force vector \( F \). Equation 3.29 represents six linear equations in \( 3n \) unknown force components. The rank of the matrix \( \left[ G \right] \) is six unless all the \( n \) contact points are collinear. The matrix \( \left[ G \right] \) is a 3nx6 matrix. As was discussed in Section 3.2.1, the pseudo-inverse (minimum norm) solution to Equation 3.29 has no contribution from the null space of \( \left[ G \right] \). If \( F^* \) is the pseudo-inverse solution,

\[ F^* = \left[ G \right]^+ Q \quad (3.30) \]

where \( \left[ G \right]^+ \) is the pseudo-inverse of the non-square matrix \( \left[ G \right] \) and

\[ \left[ G \right]^+ = \left[ G \right]^T \left(\left[ G \right] \left[ G \right]^T\right)^{-1} \quad (3.31) \]
\( \mathbf{F}^* \) can be obtained more efficiently by the use of the equilibrating force field [Kumar and Waldron, 1988]. The computation procedure for \( \mathbf{F}^* \) for wheeled systems is almost identical to that discussed in Section 3.2.1, and it has been given above only for the sake of completeness.

The fact that \( \mathbf{F}^* \) has no component from the null space implies that it has no interaction force components. Therefore, \( \mathbf{F}^* \) automatically satisfies Equation 3.28. This is significant because the lateral components of \( \mathbf{F}^* \) satisfy the constraint of Equation 3.28, even though these forces cannot be arbitrarily allocated, and they are also consistent with the commanded force system \( \mathbf{Q} \) since they are compatible with Equation 3.29. Therefore, the optimal lateral contact forces at each wheel contact are the same as the pseudo-inverse solution values of these forces. Once this has been established, the force allocation problem of Equation 3.29 reduces to the following relationship:

\[
\begin{bmatrix}
\mathbf{G}_f \
\mathbf{G}_w
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}_f^* \\
\mathbf{F}_w
\end{bmatrix}
= \mathbf{Q}
\]  

(3.32)

where the matrix \( \mathbf{G}_f \) = \([g_{11}, g_{12}, \ldots, g_{1n}] \) represents the screw axes associated with the lateral contact force directions, the matrix \( \mathbf{G}_w \) = \([g_{11}, g_{12}, g_{12}, \ldots, g_{tn}, g_{tn}] \) represents the screw axes associated with the wheel plane contact force directions, \( \mathbf{F}_f^* \) is the vector of the pseudo-inverse lateral contact forces, and \( \mathbf{F}_w \) is the vector of the wheel plane contact forces. (It includes \( F_{ti} \) and \( F_{ni} \).) \( \mathbf{F}_w \) is the unknown in Equation 3.32. Therefore, the force allocation equation becomes:

\[
[\mathbf{G}_w] \mathbf{F}_w = \mathbf{Q}_w
\]  

(3.33)

Here \( \mathbf{Q}_w = \mathbf{Q} - [\mathbf{G}_f] \mathbf{F}_f^* \). Equation 3.33 represents six equations in \( 2n \) unknowns. Hence, the redundancy in actuation results in a force allocation null space \( \mathbf{N}(\mathbf{G}_w) \) whose
rank is equal to \(2n - 6\). This null space is investigated in greater detail with respect to a planar set of points, and a spatial set of points in the following sections.

3.2.2.1 Planar Set of Points

For a planar set of \(n\) contact points the space \(N(G_w)\) has a dimension of \((2n - 6)\). Further, there exist two force fields the normal field and the tangential field each of which have a dimension of \((n - 3)\). The union of the basis vectors of these two fields forms a basis for \(N(G_w)\).

The nature of the space \(N(G_w)\) for a planar set of points can be understood by considering a two module vehicle such as the one shown in Figure 3.11. It is assumed that all the four wheels lie on the plane terrain. Here, rank \((N(G_w)) = 2\) since \(n = 4\). The two null space vectors are shown in Figures 3.12 and 3.13. The normal force vector has non-zero contact force components normal to the terrain at the four points such that their resultant is equal to zero. This is identical to the orthogonal force vector in Section 3.2.1.2. The tangential force vector is comprised of non-zero contact force components along the line of intersection of the contact plane and the wheel plane. Further, the four tangential contact forces can be chosen such that all have the same magnitude, and the contact force on the left wheel of a module is directed in the opposite direction to the contact force on the right wheel of the same module as shown in Figure 3.13. The pair of tangential contact forces \(F_{11}\) and \(F_{12}\) form a couple of intensity equal to \((a F_{11})\), where \(a\) is the length of the axle. Similarly, the pair of tangential contact forces \(F_{13}\) and \(F_{14}\) form a couple of intensity equal to \((a F_{13})\). However, this couple opposes the couple due to \(F_{11}\) and \(F_{12}\). Since all four tangential forces are assumed to have the same absolute value, the two couples cancel each other, and the vehicle system remains in static equilibrium. Therefore, this tangential force system belongs to the null space \(N(G_w)\). It is readily seen
that the normal null space vector (Figure 3.12) and the tangential null space vector (Figure 3.13) are linearly independent. Therefore, these two vectors form a basis for $N(G_w)$. This can be readily extended to the general case of $m$ modules with $n$ coplanar contact points. Let $F_{ijk\ell}^n$ represent a normal null space vector obtained by taking non-zero forces normal to the contact plane at the contact points $C_i, C_j, C_k$, and $C_\ell$. Further, let us assume, without loss of generality, that the points $C_1, C_2, \text{ and } C_3$ are non-collinear. The following $n - 3$ normal null space vectors can then be shown to be linearly independent using a proof that is very similar to the one given with respect to the orthogonal force field in Section 3.2.1.2:

$$D^n = \{F_{1234}^n, F_{1235}^n, F_{1236}^n, \ldots, F_{123n}^n\} \quad (3.34)$$

Similarly, let $F_{ijk\ell}^t$ represent a tangential null space vector obtained by taking non-zero forces in the plane of the wheels, and in the contact plane, at the contact points $C_i, C_j, C_k$, and $C_\ell$. These represent four contact forces that have to satisfy three equilibrium equations to make up a null space vector. This is always possible provided at least some three of the four contact points are non-collinear. Again, assuming that the points $C_1, C_2, \text{ and } C_3$ are non-collinear, the following $n - 3$ tangential null space vectors can be shown to be linearly independent using a proof that is very similar to the one given with respect to the orthogonal force field in Section 3.2.1.2:

$$D^t = \{F_{1234}^t, F_{1235}^t, F_{1236}^t, \ldots, F_{123n}^t\} \quad (3.35)$$

Form, $D = D^n \cup D^t$. Set $D$ is a set of $2n - 6$ linearly independent vectors since each tangential null space vector is linearly independent of all the normal null space vectors. Therefore, $D$ is a basis for $N(G_w)$. 
Figure 3.11 Two Modules on a Plane Terrain

Figure 3.12 Normal Force Field

Figure 3.13 Tangential Force Field
3.2.2.2 Spatial Set of Points

For a spatial set of \( n \) contact points the space \( N(G^\infty) \) has a dimension of \( (2n - 6) \). The basis vectors for \( N(G^\infty) \) cannot be obtained in a simple form for a spatial set of points. In general, the null space vectors contain up to seven non-zero entries, and computation of these vectors involves the inversion of a \( 6 \times 6 \) matrix.

Vectors belonging to \( N(G^\infty) \) have non-zero contact force components that are in the plane of the wheel associated with the contact point. For a set of \( n \) spatial points, the modules, and hence the wheels, are oriented arbitrarily with respect to each other. Further, in general, any four contact points are not coplanar. For these reasons obtaining simple null space vectors from the geometry of the contact points is not possible. The only simple null space vector that can be obtained is a normal null space vector similar to the one described with respect to a planar set of contact points in the previous section. However, in this case, all these normal null space vectors cannot span the space \( N(G^\infty) \). Therefore, some of the null space vectors have to be obtained numerically. Nevertheless, even if a subset of the basis vectors is simple, it may lead to more efficient optimization schemes. Hence, these simple null space vectors are investigated below.

A normal null space vector in the case of spatial contact points exists among the four contact points of any two modules. This can be understood by considering the two moduled system in Figure 3.14. The four contact points \( C_1, C_2, C_3 \), and \( C_4 \) are shown. The two modules are assumed to be arbitrarily oriented with respect to one another. A plane that is parallel to the two wheels of Module 1, say 'M', is identified. Similarly, a plane parallel to the two wheels of Module 2, say 'L', is also identified. The line of intersection of planes 'L' and 'M', say \( \ell \), is identified. Four contact forces \( F_{n1}, F_{n2}, F_{n3}, \) and \( F_{n4} \) are identified such that they are parallel to the line \( \ell \). These four contact forces
have the unique property that they are all parallel to each other, and they all lie in the planes of the respective wheels. Therefore, if they do not affect the equilibrium of the vehicle, they belong to the space \( N(G_w) \). Provided that at least three of the four contact points \( C_1, C_2, C_3, \) and \( C_4 \) are non-collinear, which is always true for the system in Figure 3.14, non-zero values of \( F_{n1}, F_{n2}, F_{n3}, \) and \( F_{n4} \) can be obtained such that the system stays in equilibrium. Therefore, for any two modules of an articulated wheeled structure, there always exists one such null space vector. All the normal null space vectors obtained in this manner do not span the space \( N(G_w) \). Therefore, some null space vectors have to be obtained numerically.

The tangential null space vector that was present in the case of a planar set of points does not exist for a spatial point set. This can be understood by noting that the four contact points \( C_1, C_2, C_3, \) and \( C_4 \) of Figure 3.14 are, in general, not coplanar. Therefore, the couples created by the tangential forces are not necessarily parallel and may violate the equilibrium of the system.

Let us assume that some \( k \) null space vectors have been identified using the normal null space vector of Figure 3.14. The other \((2n - 6 - k)\) basis vectors can be numerically computed by considering the null space form of Equation 3.33.

\[
[G_w] F_w = 0
\]

(3.36)

Here \( 0 \) is the \( 2n \times 1 \) zero vector. The above equation can be rearranged to obtain

\[
[G_6] F_6 = -[G_{w-6}] F_{w-6}
\]

(3.37)

Here, \( [G_6] = [s_{t1}, s_{t1}, s_{n1}, s_{t2}, s_{n2}, s_{n3}] \), is a \( 6 \times 6 \) matrix. The six screw vectors that constitute the matrix \( [G_6] \) are chosen so that they represent the screw axes of contact force directions associated with three distinct, non-collinear contact points. This ensures that
Plane 'M' parallel to wheels of module 1

Plane 'L' parallel to wheels of module 2

Module 2 -----

Module 1

Line of intersection of planes 'L' and 'M'

Figure 3.14 A Simple Null Space Vector for a Spatial Set of Contacts

$[G_6]$ has a rank of 6. $[G_{w-6}]$ is a $6 \times (2n - 6)$ matrix and is equal to $[s_{\ell 2}, s_{t 3}, s_{\ell 3}, s_{t 4}, s_{\ell 4}, s_{n 4}, s_{t 5}, s_{\ell 5}, s_{n 5}, ..., s_{t n}, s_{\ell n}, s_{nn}]$. $F_6 = [F_{t 1}, F_{\ell 1}, F_{n 1}, F_{t 2}, F_{n 2}, F_{n 3}]^T$, and $F_{w-6} = [F_{t 2}, F_{t 3}, F_{\ell 3}, F_{t 4}, F_{\ell 4}, F_{n 4}, ..., F_{t n}, F_{\ell n}, F_{nn}]^T$. Since $[G_6]$ is nonsingular,

$$F_6 = -[G_6]^{-1} [G_{w-6}] F_{w-6} \tag{3.38}$$

Let $[H] = -[G_6]^{-1} [G_{w-6}]$. $[H]$ is a $6 \times (2n - 6)$ matrix that depends entirely on the screw vectors. Next, define $F_{6i} = [H] e_i$, where $e_i$ is a $(2n - 6) \times 1$ vector that has its $i$th entry equal to one, and all the other entries equal to zero. Then,

$$F_i = [(F_{6i})^T e_i^T]^T \tag{3.39}$$
is a $2n \times 1$ vector that belongs to the space $N(G_w)$. Assuming k null space vectors have already been identified using the normal null space vector of Figure 3.14, the other $(2n - 6 - k)$ null space vectors can be obtained by using Equation 3.39 for $i = 1, 2, 3, \ldots, (2n - 6 - k)$.

The limitations of the above numerical scheme include the need to invert $[G_g]$ to compute $[H]$, and the presence of seven non-zero entries in the null space vectors that may increase the complexity of optimization schemes.

This concludes the discussion on the characterization of redundancy in force allocation in actively coordinated systems. In the next section discusses the issue of optimization of force conditions at the contact locations is discussed. Again, the optimization schemes are separately investigated for actively coordinated mechanisms with complete controllability over contact conditions, and for actively coordinated mechanisms with partial controllability over contact conditions.

### 3.3 Optimal Force Allocation

In this section, redundancy in force allocation is used to optimize contact conditions. The specific optimization problems that have been solved are the following:

1. **Four-point-of-contact problems for legged systems/multi-fingered grasping**: Minimal ratio of tangential to normal contact force leading to maximal system stability is the desired performance criterion. This problem has been solved in the literature using linear programming techniques and the gradient search method [Cheng and Orin, 1989; Chung and Waldron, 1993; Kumar and Waldron, 1990; Klein and
Kittivatcharapong, 1990]. Here, globally optimal solutions are sought after using a methodology that is similar to the work involving three-fingered grasps by Mukherjee and Waldron [1992]. This is described in detail in Section 3.3.1.

2. Four-point-of-contact problems for articulated wheeled vehicles: The difficult mobility maneuvers of the WAAV that were described in Chapter 1, including obstacle climbing and recovery from an overturn, involve vehicle configurations that have only two modules (or four wheels) resting on the terrain. Optimal force distribution that maximizes vehicle stability in such situations is considered in this work. A performance comparison of legged and wheeled systems with respect to force distribution is also provided. This is described in detail in Section 3.3.2.

3.3.1 Active Mechanisms with Complete Controllability Over Contact Conditions (Four Points of Contact)

In this section, globally optimal force distribution in mechanisms with four contact points is considered. Even though these solutions cannot be obtained in real-time, it is useful to investigate them since they provide a baseline that can be used to evaluate the effectiveness of sub-optimal schemes. Further, in some situations, off-line computation can be performed to identify the best force distribution vectors for a particular task. Also, for particular applications, look-up tables can be generated using these globally optimal techniques. One such example is a task which requires a four-fingered grasping system to grasp an object and move it while keeping the contact locations fixed. In an object fixed coordinate frame, only the gravity vector varies as the object is manipulated. Therefore, optimal contact force vectors can be identified for various values of the gravity vector, and a look-up table can be obtained. Finally, such globally optimal solutions are also useful in characterizing the limits of system performance using redundant actuation.
Since these mechanisms are assumed to possess complete contact force controllability, the rank of the null space is six \((3n - 6, n = 4)\). It is required to minimize the maximum absolute ratio of the tangential to the normal contact forces at the four contact points. Let the unit vector normal to the environment at the contact point 'i' be \(n_i\), and let the contact force vector be \(F_i\) as shown in Figure 3.15. The angle between the vectors \(n_i\) and \(F_i\) is \(\theta_i\), and ideally this angle should be equal to zero. Let \(\phi_i = \cos^2(\theta_i) = \frac{(n_i \cdot F_i)^2}{\|F_i\|^2}\), and let

\[
\phi = \min\{\phi_1, \phi_2, \phi_3, \phi_4\}
\]

The value \(\phi\) represents the square of the worst contact ratio among the four contact locations. The problem of minimizing the maximum absolute ratio of the tangential to the normal contact forces at the four contact points can be stated as follows: Maximize \(\phi\) subject to the constraint \(n_i \cdot F_i > 0\) for \(i = 1, 2, 3\) and 4.

![Figure 3.15 Contact Point 'i'](image)

The constraint \(n_i \cdot F_i > 0\) ensures that all the contact force components normal to
the environment are positive. Additional inequality constraints, such as an upper limit on
the norm of the contact force vectors, or an upper limit on the absolute values of the
actuator torques/forces may also be present. Of course the above optimization problem is
subjected to the equality constraint of Equation 3.1 which ensures that all the contact forces
will combine to generate the commanded inertia/gravity vector \( \mathbf{Q} \). The constraint \( \mathbf{n}_i \cdot \mathbf{F}_i > 0 \)
is required to eliminate solutions that have negative normal contact forces. However, it is
not an active constraint in the optimization problem itself since the maximal values of \( \phi \) are
far away from the limiting equality of the inequality \( \mathbf{n}_i \cdot \mathbf{F}_i > 0 \). If \( \mathbf{n}_i \cdot \mathbf{F}_i = 0 \), \( \phi \) will be
equal to zero. Therefore, the optimization problem reduces to the following:

\[
\text{Maximize } \phi \text{ subject to the equality constraint } [\mathbf{G}] \mathbf{F} = \mathbf{Q}. \tag{3.41}
\]

In the following discussion, the above problem will be referred to as the
'unconstrained problem'. The 'constrained problem' will refer to the optimization problem
of Equation 3.41 subjected to inequality constraints on the absolute values of variables such
as the contact forces/actuator torques.

Let \( \mathbf{F}^* \) be the minimum-norm solution to Equation 3.1, and let \( \mathbf{e}_i \) be the \( 3n - 6 \) null
space vectors of the space \( \mathbf{N}(\mathbf{G}) \). Then, the general solution to Equation 3.1 is

\[
\mathbf{F} = \mathbf{F}^* + \sum_{i=1}^{3n-6} \alpha_i \mathbf{e}_i \tag{3.42}
\]

where \( \alpha_i \) represent arbitrary parameters, \( \mathbf{F} = [F_1, F_2, F_3, \ldots, F_n]^\top \), and \( \mathbf{F}_i = [F_{ix}, F_{iy}, \ldots, F_{iz}]^\top \). A solution vector of the form \( \mathbf{F} \) in Equation 3.42 inherently ensures that the equality
constraint \( [\mathbf{G}] \mathbf{F} = \mathbf{Q} \) is satisfied. The optimization problem involves solving for the
arbitrary parameters \( \alpha_i \) that extremize the objective function of Equation 3.41. It is found
in the following discussion that the complexity of the optimization problem is reduced by
the use of simple null space vectors \( \mathbf{e}_i \) generated from the geometry of the contact points as
described in Section 3.2.

In the case of four contact points, there are six parameters $\alpha_1$ through $\alpha_6$. In general each $\phi_i$ can be a function of all the parameters $\alpha_i$. However, if the null space vectors are chosen from the interaction force field and the orthogonal force field, the functions $\phi_i$ will depend on three to four of the six parameters. The minimax problem of Equation 3.41 will have a locally optimal solution if one of the following conditions are satisfied:

1. One of the $\phi_i$'s, say $\phi_k$, reaches a maximum and the other three $\phi_i$'s are all greater than or equal to $\phi_k$. To investigate this possibility, four ($4C_1$) different problems have to solved by considering the maximum of each of the four $\phi_i$'s.

2. Some two of the $\phi_i$'s, say $\phi_k$ and $\phi_\ell$, are equal, and they together reach a maximum while the other two $\phi_i$'s are both greater than or equal to $\phi_k, \phi_\ell$. To investigate this possibility six ($4C_2$) different problems have to solved by considering the maximum of each pair of $\phi_i$'s.

3. Some three of the $\phi_i$'s, say $\phi_k, \phi_\ell$ and $\phi_m$, are equal, and they together reach a maximum while the other $\phi_i$ is greater than or equal to $\phi_k, \phi_\ell$ and $\phi_j$. To investigate this possibility four ($4C_3$) different problems have to solved by considering the maximum of each set of three $\phi_i$'s.

4. All four $\phi_i$'s are equal, and they together reach a maximum. To investigate this possibility only one problem has to be solved.

The globally optimal solution can be obtained by sorting through all the local optima obtained from the four cases listed above and picking the best solution. The four cases are considered in detail in the following discussion, and methods to obtain the local optima for
each of these cases is described. In general, the four contacts can be spatial or planar. If they are spatial, the interaction forces are sufficient to span the null space of rank equal to six. Figure 3.16 shows the case of four spatial points and their associated interaction forces. Interaction forces can be generated from the contact geometry. For instance, the interaction force $F_{34} = \alpha_{34} [0^T, 0^T, v^T, -v^T]^T$.

Figure 3.17 shows a set of four planar contact points. As seen in the figure, an orthogonal force has to be used along with five interaction forces to span the redundancy. The orthogonal force has the form $F_{1234} = \alpha_{1234} [0, 0, A, 0, 0, B, 0, 0, C, 0, 0, D]^T$.

Case 1:

Here the case of one of the $\phi_i$'s, say $\phi_k$, reaching a maximum and all the other three $\phi_i$'s remaining greater than or equal to $\phi_k$ is considered. For the unconstrained problem, it is shown below that this case is always sub-optimal.

$$\phi_k = \cos^2(\theta_k) = \frac{\{n_k \cdot F_k\}^2}{\|F_k\|^2}.$$ Consider, $\gamma_k = \cos(\theta_k)$. For a set of four spatial contact points, $\gamma_k$ will be a function of three parameters, say $\alpha_1$, $\alpha_2$, and $\alpha_3$. An extremum of $\gamma_k$ requires that

$$\frac{\partial \gamma_k}{\partial \alpha_1} = \frac{\partial \gamma_k}{\partial \alpha_2} = \frac{\partial \gamma_k}{\partial \alpha_3} = 0$$ (3.43)

The function $\gamma_k$ has the form $\ell/q$ where $\ell$ is a linear function of the three parameters $\alpha_i$, and $q$ is a quadratic function of $\alpha_i$. The three equations resulting from Equation 3.43 are

$$c_1 \cdot \ell_1/q = 0$$
$$c_2 \cdot \ell_2/q = 0$$
$$c_3 \cdot \ell_3/q = 0$$ (3.44)
Figure 3.16 Interaction Forces for a Set of Four Spatial Contact Points.

Figure 3.17 A Set of Four Coplanar Contact Points.
where \( c_i = \frac{\partial \ell}{\partial \alpha_i} \) is a constant term and \( \ell_i = \frac{\partial q}{\partial \alpha_i} \) is a linear function of \( \alpha_i \).

The ratio \( \frac{\ell}{q} \) can be solved for from the first equation and substituted into the next two equations. This leads to two linear equations and a quadratic equation:

\[
c_1 \ell_2 - c_2 \ell_1 = 0; \quad c_1 \ell_3 - c_3 \ell_1 = 0; \quad c_1 q - \ell \ell_1 = 0
\] 

Equation 3.45 has exactly two sets of solutions for the variables \( \alpha_1, \alpha_2, \) and \( \alpha_3 \).

Therefore, the function \( \gamma_k \) has two extrema. It can be readily seen that the two extrema of \( \gamma_k \) are \( \gamma_k = 1 \) (maximum), and \( \gamma_k = -1 \) (minimum). The value \( \gamma_k = -1 \) is an infeasible solution, and for \( \gamma_k = 1 \), the values \( \gamma_i, i \neq k \), will have values equal or less than 1.

Therefore, the case of one of the \( \phi_i \)'s, say \( \phi_k \), reaching a maximum is sub-optimal. A similar procedure can be followed for the case of a set of four planar points. From Figure 3.17, it can be seen that some \( \gamma_i \), say \( \gamma_k \), will be a function of either four parameters \( \alpha_i \), or three parameters \( \alpha_i \). It can be shown that there are exactly two sets of parameters \( \alpha_i \) for which \( \gamma_k \) is a maximum, and hence it leads to a sub-optimal solution.

The above discussion is not valid for constrained problems involving inequality constraints on variables such as the actuator torques or the contact forces. This is because the optimal solution may lie on the constraint boundary. Therefore, local optima lying on the constraint boundaries have to be identified to solve the globally optimal problem.

Case 2:

Here the case of two of the \( \phi_i \)'s, say \( \phi_k \) and \( \phi_x \), being equal and both reaching a maximum, and the other two \( \phi_i \)'s remaining greater than or equal to \( \phi_k \) is considered. As was stated earlier, to investigate this case six \( (4C_2) \) different problems have to solved by
considering the maximum of all the different pairs of $\phi_i$'s. In the following discussion, a search method that will allow the computation of all the possible local maxima for this case is described. This search method is valid for both constrained and unconstrained problems. The objective is to minimize $\phi_k$ subject to the constraint that $\phi_k = \phi_\ell$. A resulting solution will be considered to be a local optimum provided the other two $\phi_i$'s are greater than $\phi_k$. The objective function can be expressed as follows using the method of Lagrange multipliers:

$$J = \phi_k + \lambda (\phi_k - \phi_\ell)$$  \hfill (3.46)
the objective function, only five of the six parameters can be optimized. The sixth parameter can be solved for by requiring that the other two \( \phi \)'s should be equal. This is because the sixth parameter \( \alpha_{ij} \) affects only the functions \( \phi_i \) and \( \phi_j \). The minimax problem of maximizing the minimum of \( \phi_i \) and \( \phi_j \) requires them to be equal. Here, this leads to the following equation:

\[
\phi_i = \phi_j
\]  

(3.48)

An extremum of \( J \) requires that

\[
\frac{\partial J}{\partial \alpha_{kj}} = \frac{\partial J}{\partial \alpha_{j\ell}} = \frac{\partial J}{\partial \alpha_{ik}} = \frac{\partial J}{\partial \alpha_{i\ell}} = \frac{\partial J}{\partial \alpha_{k\ell}} = 0
\]  

(3.49)

In the above equation only the first four partial derivative equations are used since the last partial derivative equation leads to the solution of the Lagrange multiplier \( \lambda \) which is not a desired variable. The first four equations lead to the following equation:

\[
\frac{\partial \phi_k}{\partial \alpha_{kj}} = \frac{\partial \phi_{\ell}}{\partial \alpha_{j\ell}} = \frac{\partial \phi_k}{\partial \alpha_{ik}} = \frac{\partial \phi_{\ell}}{\partial \alpha_{i\ell}} = 0
\]  

(3.50)

The functions \( \phi_k \) and \( \phi_{\ell} \) are of the following form:

\[
\phi_k = p_k/q_k, \quad \phi_{\ell} = p_{\ell}/q_{\ell}
\]  

(3.51)

Here, \( p_k, q_k, p_{\ell}, \) and \( q_{\ell} \) are quadratic functions in the parameters \( \alpha_{mn} \). Equations 3.50 and 3.51 lead to equations of the following form:

\[
\ell_1 - \beta \ell_2 = 0 \\
\ell_3 - \beta \ell_4 = 0 \\
\ell_5 - \beta \ell_6 = 0 \\
\ell_7 - \beta \ell_8 = 0
\]  

(3.52)
In the above equations, \( \ell \) are linear functions of the variables \( \alpha_{mn} \), and \( \beta = \phi_k (= \phi_\ell) \). The partial derivative of \( J \) with respect to \( \lambda \) yields the equality constraint, \( \phi_k = \phi_\ell (= \beta) \), which leads to the following two equations:

\[
\begin{align*}
    p_k - \beta q_k &= 0, \\
    p_\ell - \beta q_\ell &= 0
\end{align*}
\]  

(3.53)

All the solutions to the unconstrained problem can be obtained by obtaining all the roots of Equations 3.48 (a quartic), 3.52 (four quadratics) and 3.53 (two cubics) in the seven variables \( \alpha_{kj}, \alpha_{k\ell}, \alpha_{ik}, \alpha_{i\ell}, \alpha_{i_j}, \alpha_{ij} \), and \( \beta \). A polynomial continuation technique can be used to solve these equations to obtain all the roots. A detailed discussion of the polynomial continuation technique can be found in the literature ([Morgan, 1987], [Wampler et al., 1990]). However, the overall degree of these seven equations is very large (576) making it computationally unattractive. Further, in the presence of additional inequality constraints in the problem, the continuation solutions become useless. Therefore, a search algorithm that is much more efficient, and that can be used in the presence of additional inequality constraints is developed here. This search algorithm does not yield all the local optima, but it leads to the 'best' local optimum for the case of maximizing \( \phi_k \) with the constraints that \( \phi_k = \phi_\ell \), and \( \phi_\ell > \phi_k \). Additional inequality constraints can also be handled. The search is a one parameter search involving the parameter \( \beta \). The search starts with \( \beta = 1 \) (largest possible value of \( \phi_k \)), and \( \beta \) is reduced till a satisfactory solution is obtained. If \( \beta \) is taken as a parameter, Equation 3.52 represents four linear equations in \( \alpha_{mn} \), and Equation 3.53 represents two quadratics in \( \alpha_{mn} \). Equations 3.48, 3.52, and one of the two equations of Equation 3.53 can be chosen to solve for a particular value of \( \beta \). The overall degree of these equations now is eight. Equation 3.48 has to be used since it is the only equation that contains the variable \( \alpha_{ij} \). Equation 3.52 can first be used to eliminate four of the five parameters \( \alpha_{kj}, \alpha_{k\ell}, \alpha_{ik}, \alpha_{i\ell}, \).
and \( \alpha_{jk} \) (say \( \alpha_{kj}, \alpha_{kj'}, \alpha_{ki}, \) and \( \alpha_{ij} \)). One of the two quadratics of Equation 3.53 can then be used to solve for the parameter \( \alpha_{ij} \). After these five parameters have been solved for, the sixth parameter \( \alpha_{ij} \) can be obtained using the quartic in Equation 3.48. For a given value of \( \beta \), the procedure requires the solution of a quadratic followed by the solution of a quartic. This is not a serious computational burden. If the solution obtained is the 'best' local optimum, the parameters \( \alpha_{mn} \) have to satisfy the following conditions:

1. All the six solutions \( \alpha_{mn} \) have to be real.

2. The resulting values of \( \phi_j \) and \( \phi_i \) have to be greater than \( \beta \).

3. The resulting normal contact force components have to be positive (\( n_i \cdot F_j > 0 \)).

4. Any additional inequality constraints (such as limits on the actuator torques, norm of contact forces, structural forces etc.) have to be satisfied.

Since the search is performed for decreasing values of \( \beta \), starting with \( \beta = 1 \), the first set of parameters that satisfy the above constraints will be the 'best' parameters for this particular case. A suitable lower limit can be set for \( \beta \) from the nature of the application, and if this lower limit is exceeded, the search can be abandoned. In case of legged vehicles, this can be taken to be the value of the function \( \phi \) (as defined in Equation 3.40) when \( \mathbf{F} = \mathbf{F}^* \), the pseudo-inverse solution.

As was mentioned earlier, a total of six such searches have to be performed to account for all the various possible pairs of \( \phi_i \)'s. The above discussion is for a set of four spatial contact points. Equivalent results can also be obtained for the case of a set of four planar contact points without significant changes in the scheme described here.
Case 3:

Here the case of three of the $\phi_i$'s, say $\phi_k$, $\phi_\ell$ and $\phi_j$, are equal, and they together reach a maximum while the other $\phi_i$ is greater than or equal to $\phi_k$, $\phi_\ell$ and $\phi_j$ (see Figure 3.18). In order to investigate this possibility four ($4C_3$) different problems have to be solved by considering the maximum of each set of three $\phi_i$'s. In the following discussion, a search method that will allow the computation of all the possible local maxima for this case is described. This search method is valid for both constrained and unconstrained problems. The objective is to minimize $\phi_k$ subject to the constraint that $\phi_k = \phi_\ell = \phi_j$. A resulting solution will be considered as a local optimum provided $\phi_i$ is greater than $\phi_k$. The objective function can be expressed as follows using the method of Lagrange multipliers:

$$J = \phi_k + \lambda_1 (\phi_k - \phi_\ell) + \lambda_2 (\phi_k - \phi_j)$$

(3.54)

Here, $J = J(\alpha_{kj}, \alpha_{k\ell}, \alpha_{i\ell}, \alpha_{ij}, \alpha_{ij})$ is function of all the six parameters. An extremum of $J$ requires that the following equations are satisfied:

$$\frac{\partial J}{\partial \alpha_{ik}} = \frac{\partial J}{\partial \alpha_{i\ell}} = \frac{\partial J}{\partial \alpha_{ij}} = 0$$

(3.55)

The above equations lead to

$$\frac{\partial \phi_k}{\partial \alpha_{ik}} = \frac{\partial \phi_\ell}{\partial \alpha_{i\ell}} = \frac{\partial \phi_j}{\partial \alpha_{ij}} = 0$$

(3.56)

The functions $\phi_k$, $\phi_\ell$ and $\phi_j$ are of the following form:

$$\phi_k = p_k/q_k, \quad \phi_\ell = p_\ell/q_\ell, \quad \phi_j = p_j/q_j$$

(3.57)

Here, $p_k$, $q_k$, $p_\ell$, $q_\ell$, $p_j$, and $q_j$ are quadratic functions in the parameters $\alpha_{mn}$. Equations 3.56 and 3.57 lead to equations of the following form:
\[ s_1 - \gamma s_2 = 0 \]
\[ s_3 - \gamma s_4 = 0 \]  \hspace{1cm} (3.58)
\[ s_5 - \gamma s_6 = 0 \]

In the above equations, \( s_i \) are linear functions in the variables \( \alpha_{mn} \), and \( \gamma = \phi_k (= \phi_\ell = \phi_j) \).

Partial derivatives of \( J \) with respect to \( \lambda_1 \) and \( \lambda_2 \) yields the equality constraints, \( \phi_k = \phi_\ell = \phi_j \) (= \( \gamma \)), which leads to the following three equations:

\[ p_k - \gamma q_k = 0, \]
\[ p_\ell - \gamma q_\ell = 0, \]
\[ p_j - \gamma q_j = 0 \]  \hspace{1cm} (3.59)

A search algorithm similar to the one developed for Case 2 is discussed here. The search starts with \( \gamma = 1 \) (largest possible value of \( \phi_k \)), and \( \gamma \) is reduced until a satisfactory solution is obtained. If \( \gamma \) is taken as a parameter, Equation 3.58 represents three linear equations in \( \alpha_{mn} \), and Equation 3.59 represents three quadratics in \( \alpha_{mn} \). Therefore, Equations 3.58 and 3.59 represents six equations in six unknowns with an overall degree of eight. These equations can be reduced to a single octic polynomial in one variable [Salmon, 1885]. For a given value of \( \gamma \), the procedure requires the solution of one parameter from an octic followed by solution for the other parameters. This is not a serious computational burden. If the solution obtained is the 'best' local optimum, the parameters \( \alpha_{mn} \) have to satisfy the conditions the following conditions:

1. All the six solutions \( \alpha_{mn} \) have to be real.

2. The resulting values of \( \phi_1 \) has to be greater than \( \gamma \).

3. The resulting normal contact force components have to be positive (\( n_i \cdot F_i > 0 \)).
4. Any additional inequality constraints (such as limits on the actuator torques, norm of contact forces, structural forces etc.) have to be satisfied.

Since the search is performed for decreasing values of $\gamma$, starting with $\gamma = 1$, the first set of parameters that satisfy the above constraints will be the 'best' parameters for this particular case. As was mentioned earlier, a total of four such searches have to be performed to account for all the various possible combinations of three $\phi_i$'s being equal. The above discussion is for a set of four spatial contact points. Equivalent results can also be obtained for the case of a set of four planar contact points without significant changes in the scheme described here.

Case 4:

Here the case of all four $\phi_i$'s being equal is considered (see Figure 3.18). The complexity of the polynomial equations resulting from this case is much higher than that of the earlier cases. Therefore, the computation time required to solve for this case governs the overall computation time required to solve the optimal force distribution problem. Also, it has been found from experience with such problems that the global optimum of the force distribution problem is most often obtained from this case. For the unconstrained problem, a particular force distribution example that does not have its global optimum resulting from this case has not been encountered. In the following discussion, a search method that will allow the computation of all the possible local maxima for this case is described. This search method is valid for both constrained and unconstrained problems. The objective function can be expressed as one of the following functions using the method of Lagrange multipliers:

$$ J_1 = \phi_1 + \lambda_1 (\phi_1 - \phi_2) + \lambda_2 (\phi_1 - \phi_3) + \lambda_3 (\phi_1 - \phi_4) $$ (3.60)
\[ J_2 = \phi_2 + \mu_1 (\phi_2 - \phi_1) + \mu_2 (\phi_2 - \phi_3) + \mu_3 (\phi_2 - \phi_4) \]  
\[ (3.61) \]

\[ \frac{\partial J_1}{\partial \alpha_{23}} = \frac{\partial J_1}{\partial \alpha_{24}} = \frac{\partial J_1}{\partial \alpha_{34}} = 0 \]  
leads to

**A \lambda = 0**  
\[ (3.62) \]

\[
\begin{bmatrix}
\frac{\partial \phi_2}{\partial \alpha_{23}} & \frac{\partial \phi_3}{\partial \alpha_{23}} & 0 \\
0 & \frac{\partial \phi_4}{\partial \alpha_{24}} & 0 \\
0 & \frac{\partial \phi_3}{\partial \alpha_{34}} & \frac{\partial \phi_4}{\partial \alpha_{34}} 
\end{bmatrix}
\]

where \( A = \begin{bmatrix} \frac{\partial \phi_2}{\partial \alpha_{23}} & \frac{\partial \phi_3}{\partial \alpha_{23}} & 0 \\ 0 & \frac{\partial \phi_4}{\partial \alpha_{24}} & 0 \\ 0 & \frac{\partial \phi_3}{\partial \alpha_{34}} & \frac{\partial \phi_4}{\partial \alpha_{34}} \end{bmatrix} \), \( \lambda = [\lambda_1, \lambda_2, \lambda_3]^T \), and \( 0 \) is the 3x1 zero vector.

If Equation 3.62 is to have a nontrivial solution for \( \lambda \),

\[
\text{Det} (A) = 0 
\]
\[ (3.63) \]

\[ \frac{\partial J_2}{\partial \alpha_{13}} = \frac{\partial J_2}{\partial \alpha_{14}} = \frac{\partial J_2}{\partial \alpha_{34}} = 0 \]  
leads to

**B \mu = 0**  
\[ (3.64) \]

\[
\begin{bmatrix}
\frac{\partial \phi_1}{\partial \alpha_{13}} & \frac{\partial \phi_3}{\partial \alpha_{13}} & 0 \\
0 & \frac{\partial \phi_4}{\partial \alpha_{14}} & 0 \\
0 & \frac{\partial \phi_3}{\partial \alpha_{34}} & \frac{\partial \phi_4}{\partial \alpha_{34}} 
\end{bmatrix}
\]

where \( B = \begin{bmatrix} \frac{\partial \phi_1}{\partial \alpha_{13}} & \frac{\partial \phi_3}{\partial \alpha_{13}} & 0 \\ 0 & \frac{\partial \phi_4}{\partial \alpha_{14}} & 0 \\ 0 & \frac{\partial \phi_3}{\partial \alpha_{34}} & \frac{\partial \phi_4}{\partial \alpha_{34}} \end{bmatrix} \), \( \mu = [\mu_1, \mu_2, \mu_3]^T \), and \( 0 \) is the 3x1 zero vector.

If Equation 3.64 has to have a nontrivial solution for \( \mu \),

\[
\text{Det} (B) = 0 
\]
\[ (3.65) \]
The four functions $\phi_i$ can be expressed as a ratio $p_i/q_i$ where $p_i$ and $q_i$ are quadratic functions of some of the six parameters $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24},$ and $\alpha_{34}$. Since all the four $\phi_i$'s are equal, we can equate all of them to a value, say $\sigma$. This leads to the following four equations:

$$p_1 - \sigma q_1 = 0, \quad p_2 - \sigma q_2 = 0, \quad p_3 - \sigma q_3 = 0, \quad p_4 - \sigma q_4 = 0$$  \hspace{1cm} (3.66)

Also, Equations 3.63 and 3.65 lead to the following equations:

$$(u_1 - \sigma u_2)(u_3 - \sigma u_4)(u_5 - \sigma u_6) - (u_7 - \sigma u_8)(u_9 - \sigma u_{10})(u_{11} - \sigma u_{12}) = 0 \hspace{1cm} (3.67)$$

$$(v_1 - \sigma v_2)(v_3 - \sigma v_4)(v_5 - \sigma v_6) - (v_7 - \sigma v_8)(v_9 - \sigma v_{10})(v_{11} - \sigma v_{12}) = 0 \hspace{1cm} (3.68)$$

In the above equations, $u_i$ and $v_i$ represent linear functions of $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24},$ and $\alpha_{34}$. For a given value of $\sigma$, Equations 3.66, 3.67 and 3.68 represent six equations in the six unknowns. Equation 3.66 represents four quadratics, and Equations 3.67 and 3.68 represent two cubics. The overall degree of the system is 144. Due to the coupled nature of these polynomial equations, a numerical scheme known as polynomial continuation has to be used to solve for all the 144 roots of these six equations. Polynomial continuation is described in great detail in the literature ([Morgan, 1987], [Wampler et al., 1990]).

Polynomial continuation is a numerical scheme that yields all the finite solutions of a system of polynomial equations. The solution of a polynomial system 'B' by polynomial continuation involves three basic elements: a start system of polynomials 'A' whose roots are all known, a schedule for changing the parameters from those of 'A' to those of the target system 'B', and a method of tracking the solutions as the scheduled transformation proceeds from 'A' to 'B'. The minimum number of solution paths to be tracked to ensure that all the finite roots of the target system are obtained, is known as the Bezout number of
the polynomial system. The Bezout number is one of the important parameters that governs the computational efficiency of a numerical scheme. A rigorous definition of the Bezout number can be found elsewhere [Wampler et al., 1990]. Here, for a particular value of $\sigma$, a polynomial system that has a Bezout number of 144 has to be solved. Further, a search procedure has to be followed to obtain the 'best' solution for this case by decreasing the value of $\sigma$ starting from the value of $\sigma = 1$. This situation warrants the use of a specific kind of continuation scheme known as parametric continuation. The scheme used here is a variant of the one used by Wampler et al. [1992]. Using parametric continuation, the search procedure can be set up in the following manner. Let us denote the six polynomial equations represented by Equations 3.66, 3.67 and 3.68 by $F(\alpha, \sigma) = 0$. Here, $\alpha$ denotes the vector of six variables $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24},$ and $\alpha_{34}$, and $0$ is the 6x1 zero vector. The search procedure involves the following two steps:

1. Solve for the 144 roots of $F(\alpha, 1) = 0$ using a generic continuation scheme as described by Wampler et al. [1990]. A generic continuation scheme involves a generic start system that has the same Bezout number as the system $F(\alpha, 1) = 0$. Let $G(\alpha) = 0$ be a generic start system with a Bezout number of 144. The systems of equations $G(\alpha)$ and $F(\alpha, 1)$ have a maximum of 144 finite solutions. Assume the schedule for transforming the start system $G(\alpha)$ into the target system $F(\alpha, 1)$ (the schedule is also known as a homotopy) is as below:

$$H(\alpha) = G(\alpha) (1 - t) e^{i\theta} + t F(\alpha, 1) \quad (3.69)$$

The parameter 't' is varied from 0 to 1 to go from $G(\alpha)$ to $F(\alpha, 1)$. Here, $i^2 = -1,$ and $\theta$ is a random real number that is required to avoid singularity problems in polynomial continuation [Wampler et al., 1990]. The homotopy in Equation 3.69 will lead to all the solutions of $F(\alpha, 1)$. 
2. Next, the search can be performed using the parametric homotopy

\[ H(\alpha, \sigma_{j+1}) = F(\alpha, \sigma_j) (1 - t) e^{i\theta} + t F(\alpha, \sigma_{j+1}) \]  

(3.70)

In the above equation, \( \sigma_{j+1} = \sigma_j - h \), where \( h \) is a 'sufficiently' small number. The choice of \( h \) depends on the desired accuracy. A very small value of \( h \) leads to a very reliable and accurate scheme at the cost of increased computation time. A suitable value of \( h \) has to be chosen keeping this trade-off in mind. The starting value of \( \sigma_j \) is \( \sigma_0 = 1 \), and Equation 3.70 is repeatedly used to solve the system of equations for decreasing values of \( \sigma \). This search is continued until a set of parameters \( \alpha_{mn} \) that satisfy the following three conditions are obtained:

(i) All the six solutions \( \alpha_{mn} \) have to be real.

(ii) The resulting normal contact force components have to be positive.

(iii) Any additional inequality constraints (such as limits on the actuator torques, norm of contact forces, structural forces etc.) have to be satisfied.

The above discussion is for a set of four spatial contact points. Equivalent results can also be obtained for the case of a set of four planar contact points without significant changes in the scheme described here.

That completes the discussion of the procedure for computing the globally optimal force distribution in mechanisms with four contact points. These mechanisms were assumed to be actively coordinated, and they were assumed to possess complete contact force controllability. A block diagram of the global optimization procedure is given in Figure 3.19, and a numerical example is given below. The block diagram shows the optimization procedure for an 'unconstrained' problem. Using the method outlined in the
Case 2: decrement parameter
Maximize with two $\phi$s equal.
Six 1-D searches
(Solve a quadratic & a quartic)

Constraints met?
Yes
'Best' local optimum

No
F, null space vectors $e_i$

Case 3: decrement parameter
Maximize with three $\phi$s equal.
Four 1-D searches
(Solve an octic)

Constraints met?
Yes
'Best' local optimum

No

Constraints met?
Yes
'Best' local optimum

No

Choose largest parameter among $\beta$, $\gamma$, and $\sigma$.

Global Optimum

Case 4: decrement parameter
Maximize with four $\phi$s equal.
Parametric continuation
(Bezout $\# = 144$)

Constraints met?
Yes
'Best' local optimum

Figure 3.19 Unconstrained Global Optimization for Four Contact Points with Complete Force Controllability
above discussion, 'constrained' problems can also be solved by merely checking to see if the constraints are satisfied during the parametric search procedure. Constraints such as limits on the actuator torques, norm of contact forces, and structural forces have to be included in mechanisms exhibiting form closure. Form closure is described in the literature [Lakshminarayana, 1978; Nguyen, 1988]. An example of a mechanism exhibiting form closure is a four-fingered grasping system that is grasping an object so that, theoretically, by squeezing the object, it can generate infinitely large normal contact forces at the object-finger interface. It should be noted that even the optimal solution to the 'unconstrained' problem has to satisfy the constraint that the normal components of the contact forces must be positive.

Example 3.3

\[\text{Figure 3.20 Optimal Force Distribution for a Four Legged Walking Vehicle}\]

In this numerical example, a walking vehicle that has four legs contacting the ground (see Figure 3.20) is considered. The xyz frame has its origin at the center of mass of the system, and the z axis is pointed along the negative direction of gravity. In the xyz frame, the coordinates of the four contact points, \( P_1 \), and the corresponding ground contact
normal unit vectors, \( \mathbf{n}_i \), are:

\[
\mathbf{P}_1 = [0.5, -0.5, -0.5]^T, \quad \mathbf{P}_2 = [1, 0.5, -0.75]^T, \quad \mathbf{P}_3 = [-0.5, -0.7, -0.4]^T, \quad \mathbf{P}_4 = [-1, 0.4, -0.6]^T, \quad \mathbf{n}_1 = [0, 0, 1]^T, \quad \mathbf{n}_2 = [0.17365, 0, 0.98481]^T, \quad \mathbf{n}_3 = [0, 0.12187, 0.99255]^T, \quad \mathbf{n}_4 = [0.17101, -0.17365, 0.96985]^T.
\]

All distance units are assumed to be in meters. The weight of the entire system \( W = 400 \) N. Here, it is assumed that the walking machine is stationary leading to a static analysis. A dynamic analysis merely requires the use of the desired acceleration quantities to identify the appropriate wrench \( \mathbf{Q} \) in Equation 3.1. Here, \( \mathbf{Q} = [0, 0, 400, 0, 0, 0]^T \). In this example, even though the four contact points are spatial, the six null space vectors are computed using a procedure similar to the one used for four coplanar points. This is for the following reason. If the four spatial contact points are near coplanar (in the case of walking machines operating on relatively easy terrain, this is quite likely), the null space vectors obtained from the interaction force field may be poorly conditioned. Therefore, by choosing five interaction force vectors, and one orthogonal force vector, better numerical conditioning can be expected during the optimization procedure. The orthogonal force vector can be chosen to have its force components perpendicular to a suitable plane. For instance, a least-square regression plane that fits the four spatial points can be used to obtain the orthogonal vector. In this example, the orthogonal vector was taken to be perpendicular to the \( xy \) plane. The procedure to obtain the null space field is identical to the one shown in Figure 3.17 even though the four points in this example are spatial contact points.

The optimization function was defined in Equation 3.40 as

\[
\phi = \min \{\phi_1, \phi_2, \phi_3, \phi_4\}
\]

where \( \phi_i = \cos^2(\theta_i) = \frac{\mathbf{n}_i \cdot \mathbf{F}_i}{\|\mathbf{F}_i\|^2} \), and the problem involves maximizing \( \phi \). For this example, the minimum norm solution leads to a \( \phi^* \) equal to 0.93459. The globally optimal value of \( \phi \) obtained using the procedure outlined in Figure 3.19, \( \phi_g = 0.98933 \). The same problem was also solved using a typical local optimization technique, namely the Steepest
Descent Method [Arora, 1989]. Such local optimization techniques have the obvious limitation that the local optimum obtained depends on factors such as the choice of the starting vector and the iteration step size. The Steepest Descent Method involves two basic steps. The computation of the gradient, that represents the search direction, is the first step. The second step involves establishing a suitable step size that represents a scaling along the search direction. In this example, various step sizes were chosen leading to different local optima. The starting vector was chosen to be the zero vector. This is a convenient choice and it essentially represents the minimum norm solution since the minimum norm solution has zero contribution from the null space. Three local optimal values that were obtained using the Steepest Descent Method were $\phi_1 = 0.97984$, $0.98901$ and $0.97052$. These values clearly lie between $\phi^*$ and $\phi^g$. The minimum-norm solution yields the following four contact force vectors ($F^* = [F^*_{ix}, F^*_{iy}, F^*_{iz}]^T$):

$$
F^*_{1} = [-0.53239, -2.39415, 87.9789]^T, 
F^*_{2} = [0.49783, 3.25627, 112.6410]^T, 
F^*_{3} = [-0.63038, -3.39794, 84.99139]^T, 
F^*_{4} = [0.66493, 2.53582, 114.3887]^T.
$$

The force distribution results of three different solution schemes is listed in Table 3.1. The minimum-norm solution refers to the pseudo-inverse solution. This solution is clearly sub-optimal. The parametric search scheme of Figure 3.19 that uses the polynomial continuation technique leads to the global optimum. The Steepest Descent Method was used with three different step sizes and it yielded the three local optima given in the table.

In the above table, $F_{in}$ is the normal component of the contact force at point $P_i$, $\alpha_{ij}$ is the parameter associated with the interaction force null space vector between points $P_i$ and $P_j$, and $\alpha_{1234}$ is the parameter associated with the orthogonal force null space vector. For this example the global optimum occurs when all the four $\phi_i$'s are equal. The three local optima have two of the four $\phi_i$'s equal and they are included in the set of local optima.
computed as part of the procedure outlined in Figure 3.19.

### Table 3.1 The Minimum-Norm Solution, the Global optimum, and the Local Optima

<table>
<thead>
<tr>
<th></th>
<th>Min. Norm $\phi^* = 0.93459$</th>
<th>Global Opt. $\phi_g = 0.98933$</th>
<th>Local Opt. 1 $\phi_{l1} = 0.97984$</th>
<th>Local Opt. 2 $\phi_{l2} = 0.98901$</th>
<th>Local Opt. 3 $\phi_{l3} = 0.97052$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>0.99922</td>
<td>0.98933</td>
<td>0.98567</td>
<td>0.98933</td>
<td>0.98515</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.97050</td>
<td>0.98933</td>
<td>0.97984</td>
<td>0.98901</td>
<td>0.98515</td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>0.97389</td>
<td>0.98933</td>
<td>0.97984</td>
<td>0.98967</td>
<td>0.99583</td>
</tr>
<tr>
<td>$\phi_4$</td>
<td>0.93459</td>
<td>0.98933</td>
<td>0.98389</td>
<td>0.98901</td>
<td>0.99732</td>
</tr>
<tr>
<td>$F_{1n}$</td>
<td>87.9789</td>
<td>80.4948</td>
<td>87.2302</td>
<td>89.3227</td>
<td>87.2547</td>
</tr>
<tr>
<td>$F_{2n}$</td>
<td>111.0162</td>
<td>115.4014</td>
<td>111.8664</td>
<td>113.7637</td>
<td>111.8597</td>
</tr>
<tr>
<td>$F_{3n}$</td>
<td>83.9438</td>
<td>89.0414</td>
<td>84.5899</td>
<td>81.4678</td>
<td>83.3850</td>
</tr>
<tr>
<td>$F_{4n}$</td>
<td>110.6133</td>
<td>116.9302</td>
<td>115.7527</td>
<td>120.1371</td>
<td>118.2735</td>
</tr>
<tr>
<td>$\alpha_{12}$</td>
<td>0</td>
<td>-9.836</td>
<td>0.738</td>
<td>-9.540</td>
<td>0.000</td>
</tr>
<tr>
<td>$\alpha_{13}$</td>
<td>0</td>
<td>-15.570</td>
<td>-2.206</td>
<td>-15.589</td>
<td>0.000</td>
</tr>
<tr>
<td>$\alpha_{14}$</td>
<td>0</td>
<td>2.1222</td>
<td>-4.918</td>
<td>1.623</td>
<td>-9.033</td>
</tr>
<tr>
<td>$\alpha_{23}$</td>
<td>0</td>
<td>8.918</td>
<td>3.170</td>
<td>8.599</td>
<td>0.000</td>
</tr>
<tr>
<td>$\alpha_{34}$</td>
<td>0</td>
<td>-21.333</td>
<td>-6.141</td>
<td>-20.347</td>
<td>-6.348</td>
</tr>
<tr>
<td>$\alpha_{1234}$</td>
<td>0</td>
<td>-2.718</td>
<td>-0.265</td>
<td>0.802</td>
<td>0.072</td>
</tr>
</tbody>
</table>

### 3.3.2 Active Mechanisms with Partial Controllability Over Contact Conditions (Four Points of Contact)

As described earlier, wheeled systems such as the WAAV do not possess complete control over the lateral contact forces in a direction orthogonal to the wheel plane. The actively articulated wheeled vehicle of Figure 3.10 has a null space dimension of $2n - 6$ where $n$ is the number of wheels contacting the ground. Legged systems have a null space
of dimension $3n - 6$. Therefore legged systems can be expected to have better force distribution capability. The difficult mobility maneuvers of the WAAV that were described in Chapter 1, including obstacle climbing and recovery from an overturn, involve vehicle configurations that have only two modules (or four wheels) resting on the terrain. Optimal force distribution that maximizes vehicle stability in such situations is considered here. A performance comparison of legged and wheeled systems with respect to force distribution is also provided.

Consider the two vehicles shown in Figures 3.21 and 3.22. The articulated legged system of Figure 3.21 possesses two three-degree-of-freedom legs supporting each of the three modules, and the modules are connected by active articulations. The system of Figure 3.22 is identical in its configuration to the WAAV, and here the six legs of the earlier configuration are replaced by six wheels. Both systems have redundant actuation. However, the wheeled system has far fewer actuators (one per wheel) as compared to the legged system (three per leg). The design complexity and the cost of the wheeled system can be expected to be lower than the legged system, but the legged system can be expected to have better force distribution capability. Here, the performances of these two systems, as they attempt the task of negotiating an obstacle, is studied. Specifically, one of the sub-tasks that requires the two rear modules to be stationary and the first module to be lifted up is investigated. The translatory motion of the middle module axle allows for the placement of the center of mass of the system on the right of the middle module wheel contact centers. Even though this makes lifting of the middle module possible, the contact force distribution is not very satisfactory since the center of mass is too close to the middle module wheels. Specifically, the ratio of the tangential to normal contact force components at the some of the contact locations may be quite high leading to large, undesirable slip rates. This situation can be, to some extent, alleviated by using the redundancy in the force
distribution.

For the wheeled system with four points of contacts, the dimension of the null space is two, while for the legged system the dimension of the null space is six. Let \( F^* \) be the minimum-norm solution, and let \( \varepsilon_i \) be the null space vectors. Then, the general solution to the force distribution problem is

\[
F = F^* + \sum_{i=1}^{\text{j}} \alpha_i \varepsilon_i
\]

where \( j = 2 \) and 6 for the wheeled and the legged systems respectively. The null space vectors for spatial contacts of the wheeled system will, in general, have seven non-zero entries as discussed in Section 3.2.2.2. Therefore, some of the simplifications obtained by the use of interaction and orthogonal force vectors in the case of systems with complete control over contact conditions cannot be obtained for the wheeled systems.

Here a numerical example is considered where a legged system and a wheeled system both are attempting the same mobility maneuver. The contact locations are assumed to be the same for both these systems, and the local contact normals at these points are as given below:

\[
\begin{align*}
n_1 &= [0.35603, 0.05250, 0.93299]^T, \\
n_2 &= [0.32625, 0.28498, 0.90130]^T, \\
n_3 &= [0.26940, -0.059217, 0.96121]^T, \\
n_4 &= [0.09030, 0.36948, 0.92484]^T.
\end{align*}
\]

The geometric parameters of the articulated vehicle are shown in Figure 3.23, and for this numerical example, the following values were used:

\[
c_1 = 0.7, \ b_2 = 0.9, \ c_2 = 0.5, \ b_3 = 0.7, \ a = 0.4, \ r = 0.25.
\]
Each module is assumed to be composed of a body and a wheel carriage. In the local body frame \( (x_1y_1z_1\ frame \ of \ Figure \ 3.10) \), the centers of mass of the body and the wheel carriage are assumed to be at \([0, 0, 0.4]^T\) and \([0, 0, 0]^T\) respectively. The mass of the body is taken to be 30 kg and that of the wheel carriage is taken to be 10 kg. The origin of the frame of module 1 \( (x_1y_1z_1\ frame) \) is taken to be at \([3, 5, 1]^T\) in the \( X_0Y_0Z_0\ frame\). The orientation of the first module in the world frame is given in terms of the Euler angles \( \alpha, \beta, \) and \( \gamma \). These Euler angles are defined as follows: a rotation of \( \gamma \) (yaw) about \( Z_0 \) followed by a rotation of \( \beta \) (pitch) about the rotated \( Y_0 \) followed by a rotation of \( \alpha \) (roll) about the rotated \( X_0 \). In this example, \( \alpha = 10^\circ \), \( \beta = 20^\circ \), and \( \gamma = 55^\circ \). The orientation of modules 2 and 3 are given by Euler angles \( \psi_i, \theta_i, \) and \( \chi_i \), that represent the motion of the ball joint \( A_i \). These Euler angles are defined as follows: a rotation of \( \psi_i \) (roll) about \( x_{Ai} \) \( (x_{Ai} \ is \ coincident \ with \ x_i) \) followed by a rotation of \( \theta_i \) (yaw) about the rotated \( z_{Ai} \) followed by a rotation of \( \chi_i \) (pitch) about the rotated \( y_{Ai} \) \( (frame \ x_{Ai}y_{Ai}z_{Ai} \ is \ shown \ in \ Figure \ 3.10) \). These Euler angles are consistent with the WAAV design. In this example \( \psi_1 = -16.2305^\circ \), \( \theta_1 = -15.0000^\circ \), \( \chi_1 = -8.8986^\circ \), \( \psi_2 = 0.0^\circ \), \( \theta_2 = 0.0^\circ \), and \( \chi_2 \) varies from 0.0 to 45.0° as the vehicle tries to negotiate the step. The vehicle in Figure 3.23 is assumed to be traversing a piece-wise planar terrain, and the equations of the four planes below the positions of the vehicle wheels in the world frame are as follows:

\[
L_{R1}^0 = 0.35603 \ X_0 + 0.05250 \ Y_0 + 0.93299 \ Z_0 - 2.05088 = 0
\]
\[
L_{L1}^0 = 0.32625 \ X_0 + 0.28498 \ Y_0 + 0.90130 \ Z_0 - 3.08349 = 0
\]
\[
L_{R2}^0 = 0.09030 \ X_0 + 0.36948 \ Y_0 + 0.92484 \ Z_0 - 2.61139 = 0
\]
\[
L_{L2}^0 = 0.26940 \ X_0 - 0.059217 \ Y_0 + 0.96121 \ Z_0 - 1.43285 = 0
\]

In this example, the vehicle position coordinates are assumed to be known. It is assumed that the positions of the three modules are the same for both the wheeled and the
legged system, and the contact locations and the corresponding contact normals are also the same. For a given piece-wise terrain, and a given wheeled vehicle geometry, a position kinematic solution of the vehicle has to be performed to obtain the vehicle coordinates. The position kinematics of these articulated wheeled vehicles is addressed in Chapter 4 in great detail, and the results obtained there have been used to generate the values of the vehicle position coordinates here. The position kinematics of the legged system of Figure 3.21 are relatively much simpler since this configuration has omni-direction motion capability. These issues are also discussed in detail in Chapter 4.

The optimization problem once again involves the function \( \phi = \min\{\phi_1, \phi_2, \phi_3, \phi_4\} \), as defined in Equation 3.40, and the maximal values of \( \phi \) are solved for. In the case of the wheeled system, \( \phi \) is a function of two parameters, and in the case of the legged system \( \phi \) is also a function of two parameters. Let the optimal \( \phi = \Phi \). Both the optimization problems were solved using the method of steepest descent [Arora, 1989]. By choosing a suitable start function, it can be shown that solution schemes that can be used in real-time can be obtained. Since these optimization problems are highly nonlinear, they have a large number of local optima, and care must be taken to obtain a smooth variation of \( \Phi \), and hence the actuator efforts, during these mobility maneuvers. Here, the optimal solution for \( \chi_2 = 0.0^\circ \) (\( \Phi(0.0) \)) was obtained by using the minimum-norm solution as the start vector in the steepest descent scheme. This optimal solution, \( \Phi(0.0) \), was then used to obtain the optimal solution \( \Phi(1.0) \), for \( \chi_2 = 1.0^\circ \) and so on. Figure 3.24 shows the variations of \( \phi^* \) (the value of \( \phi \) obtained using the minimum-norm solution), \( \Phi_w \) (the optimal solution for the wheeled system), and \( \Phi_l \) (the optimal solution for the legged system) with respect to \( \chi_2 \). As stated before, \( \chi_2 \) is the pitch angle of the articulation connecting the module being lifted and the middle module, and in this study it was varied from 0.0 to 45.0°. The plot was obtained for a 'difficult' terrain condition where the
inclinations of the four terrain contact normals with the vertical vary from about 16° to about 26°. The terrain plane equations, and other geometric and mass distribution information of the vehicle was listed earlier. As expected it is seen here that the legged system has the best performance, followed by the wheeled system, followed by the minimum-norm solution which is clearly insufficient particularly for small values of $\chi_2$.

3.4 Conclusions

In this chapter, the redundancy in force distribution of actively coordinated mechanisms has been studied. Further, the use of this redundancy to optimize certain specific contact conditions has been addressed. The above issues have been addressed with respect to mechanisms that interact with the environment via multiple frictional contact points. Two kinds of actively coordinated mechanisms have been investigated, mechanisms that have complete contact force controllability and mechanisms that have partial contact force controllability. The relationship between redundancy in actuation and the contact geometry has been studied in detail, and efficient and numerically stable algorithms that generate the force allocation null space vectors have been developed. The characterization of redundancy in actuation has been performed for the general case of $n$ frictional points of contact.

The use of this redundancy to optimize contact conditions has been addressed for the case of four contact points. This problem has been studied here for the first time with respect to actively coordinated wheeled systems. These systems have unique force distribution characteristics due to the presence of rolling elements. These vehicles are mechanisms that have partial contact force controllability and in that sense they are similar to a multi-fingered hand holding an object in a power grasp [Mirza, 1992]. Even though
the force distribution problem has been studied extensively in the literature with respect to mechanisms that have complete contact force controllability, globally optimal solutions have not been obtained for such force distribution problems with the exception of the work of Mukherjee and Waldron [1992]. In this chapter, parametric homotopy techniques have been extended to handle the polynomial systems resulting from the exact optimization equations. The optimization function has itself been used as a parameter in the homotopy continuation scheme. This leads to efficient search schemes since good estimates of the interval in which the global optimum is expected to lie can be generated from a basic understanding of these force distribution problems. These schemes are not suited for real-time applications. However, they are expected to serve as a baseline to check the effectiveness of sub-optimal schemes, and they can be used in situations where off-line programming is possible. For instance, if a robotic vehicle is faced with a particularly harsh terrain condition, force planning may be performed off-line using globally optimal techniques with the help of geometric information obtained from an imaging system, while the vehicle is stationary. Once the solutions are available, these solutions may be used with minor modifications to accommodate for terrain details that were not captured by the imaging system. Off-line programming may also be possible in an industrial automation setting, where a grasping system has to manipulate an object whose geometry is known.

\[ \chi_2 \]

Figure 3.21 Actively Articulated Legged System Preparing For An Obstacle
Figure 3.22 Actively Articulated Wheeled System Preparing For An Obstacle

Figure 3.23 Geometric Parameters of Actively Articulated Wheeled System
Figure 3.24 Obstacle Negotiation Performance Comparison
CHAPTER IV

MOTION PLANNING

On difficult terrain, actively coordinated vehicle systems offer significant potential for enhanced mobility and improved system performance. The configuration of an actively actuated wheeled vehicle system on uneven terrain is greatly affected by the local terrain geometry. Since issues of vehicle stability, coordination, and control are affected by the configuration the locomotion system assumes on the terrain, it is important to study the position kinematics of actively actuated wheeled vehicles. In this chapter a general multi-module articulated wheeled vehicle configuration operating on a piece-wise planar terrain has been considered. The kinematic mobility of this configuration on such an uneven terrain has been studied and the position kinematics of the resulting hybrid series-parallel chain has been solved. Polynomial equations that represent the constrained motion of the vehicle on the terrain have been obtained. It has been shown that the configuration of the first module of the articulated chain can be obtained by solving a polynomial system whose overall degree is eight and the configuration of each of the subsequent modules of the articulated chain can be obtained by solving a polynomial system whose overall degree is thirty two. It has also been shown that, in general, the first module has eight
distinct geometric configurations, and that for each configuration of a preceding module, a subsequent module, in general, has sixteen distinct configurations. Simple procedures to pick out the particular real solution that represents the 'correct' vehicle configuration on the terrain are also included. Next, these position kinematic solutions have been used to perform geometric motion planning of articulated wheeled vehicles on uneven terrain. Given the terrain geometry, the vehicle geometric parameters and a desired path on the terrain, the planning algorithm identifies intermediate vehicle configurations along the path using the position kinematic solutions. These intermediate vehicle configurations can be used to determine if the vehicle is going to remain stable, and free from interference with itself and with the terrain. Further, when the vehicle is performing difficult mobility maneuvers, these intermediate configurations can be used to generate the commanded input to the control scheme.

4.1 Introduction

Actively actuated vehicles include walking machines and wheeled locomotion systems. Walking machines have been studied extensively in recent years [McGhee and Iswandhi, 1979; Hirose and Umetani, 1980; Sutherland and Ullner, 1984; Waldron and McGhee, 1986; Raibert, 1986; Song, 1984; Kumar and Waldron, 1990]. Actively coordinated wheeled vehicles have not been studied in as much detail. Some of the coordination issues of active wheeled systems can be found in the literature [Kumar and Waldron, 1989; Waldron et al., 1987; Waldron et al., 1991]. The WAAV, an actively articulated wheeled vehicle is described in the literature [Yu and Waldron, 1991]. Unlike the walking vehicles, wheeled vehicle systems do not possess omni-directional motion
capability. Further, the attitude of the main body of legged vehicles can be commanded and is largely independent of the terrain variations, while the geometry of an actively articulated wheeled system is affected by local terrain variations. The position kinematics of a walking vehicle that has three degree of freedom legs can be solved by treating each leg as a serial chain, and by solving a separate inverse kinematic problem for each leg. If a wheeled vehicle is constrained to move on an uneven terrain, the vehicle-terrain combination is kinematically a complex hybrid series-parallel mechanism. This mechanism has a specific kinematic mobility depending on the number of modules present in the vehicle. The problem that is addressed here is: Given the specific values of the mobile degrees of freedom of the mechanism, obtain the geometry of the vehicle configuration on a given uneven terrain. This geometric information is required to study force distribution in actively actuated vehicles [Kumar and Waldron, 1988; Sreenivasan and Waldron, 1992]. Also, such position kinematics solutions can be used to perform autonomous motion planning on uneven terrain in order to ensure that the vehicle will remain stable and free from kinematic failure (interference with the terrain or inability to cross obstacles) before entering difficult terrain. Finally, position kinematics can also be used for dynamic simulation of constrained motion of such hybrid series-parallel mechanisms (see Chapter 9 of [Featherstone, 1987]). Some of the other related research in position kinematics of spatial linkages is included in the literature [Murthy and Waldron, 1990; Nanua et al., 1990; Lin et al., 1990; Waldron et al., 1989; Raghavan, 1991; Sreenivasan and Nanua, 1992].

The theory of numerical polynomial continuation has been used in this work to solve for all the roots of the resulting polynomial systems. A good understanding of polynomial systems of equations is necessary in order to obtain numerically efficient
solution schemes. Numerical continuation methods for polynomial systems are described in great detail in the literature [Wampler et al., 1990; Morgan, 1987].

In this chapter, Section 4.2 deals with solving the position kinematics problem of articulated wheeled vehicles operating on piece-wise planar terrain. In Section 4.3, position kinematic solutions of Section 4.2 are used to develop autonomous motion planning strategies for actively articulated wheeled vehicles on piece-wise planar terrain. Section 4.4 includes some concluding remarks.

4.2 Position Kinematics

4.2.1 Vehicle Geometry and Terrain Model

A multi-module articulated wheeled vehicle configuration is shown in Figure 4.1. Each module \( M_i \) consists of a body and two wheels and the consecutive modules are connected by articulations \( A_i \) that are kinematically equivalent to three degree of freedom ball joints. All the wheels and the three degrees of freedom of all the articulations are independently actuated. As was described in Chapter 1, this particular kinematic configuration has many attractive features that include abilities to surmount large obstacles, cross wide ditches, and to self-recover from an accident. A three module version of this kinematic chain is described in detail in the reference [Waldron et al., 1987]. This three module vehicle, the Wheeled Actively Articulated Vehicle (WAAV), has been designed and fabricated and is an ongoing research project here at Ohio State University [Yu and Waldron, 1991], and will be discussed in detail in Chapter 6.

An uneven terrain is, in general, a complex nonlinear surface. The position kinematics problem becomes extremely cumbersome if the terrain is characterized by
nonlinear functions. A significant simplification can be achieved by assuming that the terrain is piece-wise planar. This assumption is tenable since compliance in the tires at the wheel-terrain contact point increases the effective radius of curvature of local ground variations. Further, vision systems (such as scanning rangefinders) generate information about terrain geometries only at discrete locations. For the purposes of motion planning, a terrain model can be obtained by planar interpolation among adjacent terrain points. In this kinematic analysis it is assumed that the equations of the $2n$ planes on which the $2n$ wheels lie are known from the terrain model. In reality the process of ascertaining the plane on which a wheel lies involves a search. If, after the kinematic analysis, it is found that a wheel does not lie on the assumed plane, the plane equation should be updated and the analysis repeated. For the kinds of density of the sensed terrain points obtained from a typical vision system, it is expected that a few search steps would be sufficient to obtain the equations of the planes below the wheels.

![Side View](image)

![Front View](image)

**Figure 4.1** An Articulated Wheeled Vehicle Configuration on Uneven Terrain
4.2.2 Kinematic Mobility

The number of kinematic degrees of freedom of the articulated vehicle on uneven terrain can be understood by looking at the Kutzbach constraint equation:

\[ M = 6(m - j - 1) + \sum_{i=1}^{j} f_i \]  

Here \( m = 3n + 1 \) is the number of members in the system (3 members per module and the ground), \( j = 5n - 1 \) is the number of joints (2n wheel-terrain contacts, 2n wheel-axle joints, and \( n - 1 \) articulations), and \( \sum f_i = 11n - 3 \) is the sum of the number of degrees of freedom of all the joints (3 each for the wheel-terrain joints assumed to be purely rolling contacts, 1 each for the wheel-axle joints, and three each for the articulations). Using these values in Equation 4.1 results in a value of system mobility \( M = 3 - n \). For \( n = 2 \), the mobility is 1 and for \( n > 2 \), the mobility < 1. That is, the system is a structure for \( n > 2 \) under the above assumptions. This is due to the assumption that the wheel-terrain contacts are purely rolling contacts. On uneven terrain a lateral slip is required at the wheel-terrain contacts along the wheel axles for the vehicle to be mobile. This lateral slip is zero in the special case when the two wheels of a module roll on the same plane. If the inclination between the two planes on which the two wheels of a module roll is small, the slip is also going to be small. With the addition of a sliding joint at each wheel-terrain contact point, \( \sum f_i = 13n - 3 \). The system mobility \( M = n + 3 \). The following argument illustrates that this is the correct value of \( M \). A total of \( 3n + 3 \) coordinates are required to fix the articulated configuration in space (6 coordinates to fix the position and orientation of the first module, and 3 joint angles to fix each of the following \( n - 1 \) modules with respect to the previous module). Since, the vehicle has to lie on the uneven terrain, the \( 3n + 3 \) coordinates are constrained by 2n nonlinear equations (1 constraint for each wheel). Therefore, the system
has a mobility of \(3n + 3 - 2n = n + 3\). It should be noted that the angular position of a wheel with respect to the corresponding module is of no relevance here. However, if a velocity analysis is being performed, the angular rates of the wheels can be computed by using the constraint that the wheel-terrain contacts do not slip in the plane of the wheel. This cannot be directly used to obtain the wheel angles since the constraints are nonholonomic.

The first step in solving the position kinematics problem is to identify \(n + 3\) coordinates that represent the vehicle motion, and the next step is to solve for the other \(2n\) coordinates that represent the vehicle accommodation to the terrain, using the commanded values of the motion coordinates and the \(2n\) constraint equations.

In Figure 4.2, a detailed geometry of the articulated chain is shown. The frame 'O' is a world fixed coordinate system and it is assumed that the terrain model is known in frame 'O'. More specifically the equations of the planes \(L_{R_i}\) and \(L_{L_i}\) are known in frame 'O'. \(L_{R_i}\) and \(L_{L_i}\) represent the planes on which the right and the left wheels of module 'i' lie. The first module can be fixed in frame 'O' if the position vector of \(o_i\) in 'O' (\(o_i^O\)) and the rotation matrix transforming vectors from frame 'I' to frame 'O' (\(R_i^O\)) are known. Let \(o_i^O = [o_{i1}, o_{i2}, o_{i3}]^T\), and let \(R_i^O\) be a function of the Euler angles \(\alpha\), \(\beta\), and \(\gamma\). These Euler angles are defined as follows: a rotation of \(\gamma\) (yaw) about \(Z_0\) followed by a rotation of \(\beta\) (pitch) about the rotated \(Y_0\) followed by a rotation of \(\alpha\) (roll) about the rotated \(X_0\).

The motion coordinates of the first module are chosen to be \(o_{x1}, o_{y1}, \beta\) and \(\gamma\), and therefore the accommodation coordinates are \(o_{1x}\) and \(\alpha\). The commanded values of \(o_{1x}\), \(o_{1y}\) can be obtained from the desired location of the main module on the terrain. The angle \(\gamma\) may be chosen to ensure that the vehicle is headed in the correct direction and the angle \(\beta\) can be chosen, for instance, to follow the local terrain slope so that the vision system mounted on the master module can 'look' at the terrain ahead. Once the motion coordinates are chosen the accommodation coordinates can be obtained using the two constraints that
the right and the left wheels have to roll along the planes \( L_{R_1} \) and \( L_{L_1} \) respectively. The procedure of obtaining \( \alpha_{1z} \) and \( \alpha \) is described in Section 6.1. Once the main module configuration is known, the configurations of the other modules can be obtained by solving for the rotation matrices \( R_{i+1}^i \) where \( i \) varies from 1 to \( n-1 \). \( R_{i+1}^i \) is a function of the Euler angles, say \( \psi_i \), \( \theta_i \) and \( \phi^i \), that represent the motion of the ball joint \( A_i \). These Euler angles are defined as follows: a rotation of \( \psi_i \) (roll) about \( x_{Ai} \) (\( x_{Ai} \) is coincident with \( x_i \)) followed by a rotation of \( \theta_i \) (yaw) about the rotated \( z_{Ai} \) followed by a rotation of \( \phi^i \) (pitch) about the rotated \( y_{Ai} \). These Euler angles are consistent with the WAAV design. Here the motion coordinate is chosen to be \( \theta_i \), and therefore the accommodation coordinates are \( \psi_i \) and \( \phi_i \). The angle \( \theta_i \) is chosen based on the desired orientation of module 'i+1' with respect to module 'i'. Then \( \psi_i \) and \( \phi_i \) can be obtained using the two constraints that the right and the left wheels of module 'i+1' have to roll along the planes \( L_{R_{i+1}} \) and \( L_{L_{i+1}} \) respectively. The procedure of obtain \( \psi_i \) and \( \phi_i \) is described in Section 6.2.

As stated earlier wheeled vehicle systems do not possess omni-directional motion capability. For the vehicle geometry of Figure 4.2, the \( i \)th module cannot be commanded to move along the \( y_i \) direction. In other words, the \( y_i \) component of velocity of point \( o_i \) is required to be zero. For \( i = 1 \) this results in a constraint on \( \dot{o}_{x1} \) and \( \dot{o}_{y1} \), and for \( i > 1 \) this results in a constraint on \( \psi_{i-1}, \theta_{i-1} \) and \( \phi_{i-1} \). These are nonholonomic constraints and hence they cannot be used to obtain constraints on the position coordinates. During motion planning, the chosen values of the motion coordinates \( o_{x1}, o_{y1} \) and \( \theta_i \) should be such that they can be achieved from the previous vehicle state without violating these nonholonomic constraints. A feasible approach is to perform a velocity analysis similar to that of the reference [Kumar and Waldron, 1989], and then to numerically integrate the resulting rates to obtain an approximate estimate of the values of the motion coordinates \( o_{x1}, o_{y1} \) and \( \theta_i \).
This section can be summarized in the following manner. The $3n + 3$ coordinates of the articulated geometry are \{o_{x1}, o_{y1}, o_{z1}, \alpha, \beta, \gamma, \psi_i, \theta_i, \phi_i\} where $i$ varies from 1 to $n-1$. The mobility of the vehicle is $n+3$ and the motion coordinates of the vehicle are \{o_{x1}, o_{y1}, \beta, \gamma, \theta_i\}. The accommodation coordinates of the vehicle (2n in number) are \{o_{z1}, \alpha, \psi_i, \phi_i\}. Once the motion coordinates are specified, the accommodation coordinates can be solved for using the 2n constraints arising from the fact that the 2n wheels have to lie on the respective local planes $L_{Ri}$ and $L_{Li}$.

### 4.2.3 Wheel-Terrain Contact Locations

In this section, the locations of the two wheel-terrain contact points of the $i^{th}$ module are evaluated in the local coordinate frame in terms of the vehicle geometry and the terrain normals. Let the equations of the planes $L_{Ri}$ and $L_{Li}$ in frame 'j' be as follows:

\[
L_{Ri}^j = n_{Rix}^j x_j + n_{Riy}^j y_j + n_{Riz}^j z_j + d_{Ri}^j = 0 \tag{4.2}
\]

\[
L_{Li}^j = n_{Lix}^j x_j + n_{Liy}^j y_j + n_{Liz}^j z_j + d_{Li}^j = 0 \tag{4.3}
\]
where the unit vectors normal to the planes $L_{Ri}$ and $L_{Li}$ in frame 'j' are

$$n_{Ri}^j = [n_{Rix}^j, n_{Roy}^j, n_{Riz}^j]^T, \quad n_{Li}^j = [n_{Lix}^j, n_{Liy}^j, n_{Liz}^j]^T$$

As shown in Figure 4.3, the axle is of length '2a', the radius of the wheels is 'r', and the right and the left contact points are $C_{Ri}$ and $C_{Li}$ respectively. Let the coordinate vector of $C_{Ri}$ in frame 'Ri' be $q_{Ri}$ (frame 'Ri' is assumed to be parallel to frame 'i' with its origin at $P_{Ri}$), and the coordinate vector of $C_{Li}$ in frame 'Li' be $q_{Li}$ (frame 'Li' is assumed to be parallel to frame 'i' with its origin at $P_{Li}$). Let $q_{Ri} = [q_{Rix}, 0, q_{Riz}]^T$, and the unit vector normal to $L_{Ri}$ in frame 'Ri' is $n_{Ri}^i$, and let $q_{Li} = [q_{Lix}, 0, q_{Liz}]^T$, and the unit vector normal to $L_{Li}$ in frame 'Li' is $n_{Li}^i$. The equation of the plane $L_{Ri}$ in frame 'Ri' is
\[ \mathbf{L}_{\text{Ri}}^i = n_{\text{Rix}}^i \mathbf{x}_{\text{Ri}} + n_{\text{Riy}}^i \mathbf{y}_{\text{Ri}} + n_{\text{Riz}}^i \mathbf{z}_{\text{Ri}} + d_{\text{Ri}}^i = 0 \quad (4.5) \]

The line of intersection of the plane of the wheel \((\mathbf{y}_{\text{Ri}} = 0)\) and the plane \(\mathbf{L}_{\text{Ri}}^i\) is

\[ n_{\text{Rix}}^i \mathbf{x}_{\text{Ri}} + n_{\text{Riz}}^i \mathbf{z}_{\text{Ri}} + d_{\text{Ri}}^i = 0 \quad (4.6) \]

The perpendicular distance between the above line and the wheel center (origin of frame 'Ri') is equal to the wheel radius 'r'. Using this \(d_{\text{Ri}}^i\) can be computed and shown to be

\[ d_{\text{Ri}}^i = \sigma r \sqrt{(n_{\text{Rix}}^i)^2 + (n_{\text{Riz}}^i)^2} \quad (4.7) \]

The point \(C_{\text{Ri}}\) lies on the line represented by Equation 4.6. Therefore, using Equations 4.6 and 4.7,

\[ n_{\text{Rix}}^i q_{\text{Rix}} + n_{\text{Riz}}^i q_{\text{Riz}} + \sigma r \sqrt{(n_{\text{Rix}}^i)^2 + (n_{\text{Riz}}^i)^2} = 0 \quad (4.8) \]

where \(\sigma = \pm 1\). Another constraint on \(q_{\text{Rix}}\) and \(q_{\text{Riz}}\) is

\[ (q_{\text{Rix}})^2 + (q_{\text{Riz}})^2 = r^2 \quad (4.9) \]

Equations 4.8 and 4.9 can be used to solve for \(q_{\text{Rix}}\) and \(q_{\text{Riz}}\). Eliminating \(q_{\text{Rix}}\) from the above two equations and simplifying yields

\[ \left(\sqrt{(n_{\text{Rix}}^i)^2 + (n_{\text{Riz}}^i)^2} q_{\text{Riz}} + \sigma n_{\text{Riz}}^i r \right)^2 = 0. \]

Solving for \(q_{\text{Riz}}\) from this equation and then solving for \(q_{\text{Rix}}\) from Equation 4.8 yields

\[ q_{\text{Riz}} = -\frac{\sigma n_{\text{Riz}}^i r}{\sqrt{(n_{\text{Rix}}^i)^2 + (n_{\text{Riz}}^i)^2}}, \quad q_{\text{Rix}} = -\frac{\sigma n_{\text{Rix}}^i r}{\sqrt{(n_{\text{Rix}}^i)^2 + (n_{\text{Riz}}^i)^2}} \quad (4.10) \]

Relations similar to Equations 4.5 through 4.9 can be obtained for computing the location of the contact point on the left wheel to obtain the following equations for \(q_{\text{Lix}}\) and \(q_{\text{Liz}}\):
In Equations 4.11, \( \tau = \pm 1 \). The coordinates of the two contact points in frame 'i' are

\[
\begin{align*}
q_{L_{ix}} &= \frac{-\tau n_{L_{ix}}^i r}{\sqrt{(n_{L_{ix}}^i)^2 + (n_{L_{iz}}^i)^2}}, \quad q_{L_{iz}} = \frac{-\tau n_{L_{iz}}^i r}{\sqrt{(n_{L_{ix}}^i)^2 + (n_{L_{iz}}^i)^2}}
\end{align*}
\]  
(4.11)

Equations 4.10, 4.11 and 4.12 give the locations of the two contact points \( C_{R_i} \) and \( C_{L_i} \) in frame 'i' in terms of the vehicle geometric parameters and the contact normals \( n_{R_i}^i \) and \( n_{L_i}^i \). It should be noted here that the normals \( n_{R_i}^i \) and \( n_{L_i}^i \) are unknowns. These normals in frame '0', \( n_{R_i}^0 \) and \( n_{L_i}^0 \), are known from the terrain model.

### 4.2.4 Terrain Constraint Equations

This section derives the two terrain constraints that arise in each of the n modules due to the fact that \( C_{R_i} \) and \( C_{L_i} \) have to lie on the planes \( L_{R_i} \) and \( L_{L_i} \) respectively. These constraints are the basic kinematic equations that relate the motion coordinates to the accommodation coordinates. Since \( C_{R_i} \) and \( C_{L_i} \) lie on the \( L_{R_i} \) and \( L_{L_i} \) the vectors \( e_{R_i}^i \) and \( e_{L_i}^i \) have to satisfy the plane equations \( L_{R_i}^i \) and \( L_{L_i}^i \) respectively. The plane equations \( L_{R_i}^i \) and \( L_{L_i}^i \) can be obtained from Equations 4.2 and 4.3. Using Equations 4.10, 4.11 and 4.12, in equations for \( L_{R_i}^i \) and \( L_{L_i}^i \) the following two constraints are obtained:

\[
\begin{align*}
\left( \frac{-\sigma n_{Rix}^i r}{\sqrt{(n_{Rix}^i)^2 + (n_{Riz}^i)^2}} \right) n_{Rix}^i + (a n_{Riy}^i) + \left( \frac{-\sigma n_{Riz}^i r}{\sqrt{(n_{Rix}^i)^2 + (n_{Riz}^i)^2}} \right) n_{Riz}^i + d_{Ri}^i &= 0 \quad (4.13) \\
\left( \frac{-\tau n_{Lix}^i r}{\sqrt{(n_{Lix}^i)^2 + (n_{Liz}^i)^2}} \right) n_{Lix}^i + (a n_{Liy}^i) + \left( \frac{-\tau n_{Liz}^i r}{\sqrt{(n_{Lix}^i)^2 + (n_{Liz}^i)^2}} \right) n_{Liz}^i + d_{Li}^i &= 0 \quad (4.14)
\end{align*}
\]
Equations 4.13 and 4.14 can be simplified by cancelling out the factors \(\sqrt{(n_{Rix}^i)^2 + (n_{Riz}^i)^2}\) and \(\sqrt{(n_{Lix}^i)^2 + (n_{Liz}^i)^2}\) respectively. This cancellation is evident only if the equations are derived in the module coordinate frame 'i', and is significant as it leads to polynomial systems that have much lower overall degrees. This decreases the computational cost. Finally, the resulting equations are squared to eliminate the radical terms. In the process, the sign variables, \(\sigma\) and \(\tau\) are also eliminated from the resulting equations. The following two constraints are obtained:

\[
r^2 \left( (n_{Rix}^i)^2 + (n_{Riz}^i)^2 \right) = \left( a_{Riy}^i - d_{Ri}^i \right)^2
\]

\[
r^2 \left( (n_{Lix}^i)^2 + (n_{Liz}^i)^2 \right) = \left( a_{Liy}^i - d_{Li}^i \right)^2
\]

Equations 4.15 and 4.16 represent the two constraints that are needed to solve for the accommodation degrees of freedom of each module. These equations relate the geometric parameters of the vehicle and the parameters of the plane equations \(L_{Ri}^i\) and \(L_{Li}^i\). \(L_{Ri}^i\) and \(L_{Li}^i\) are functions of \(L_{Ri}^0\) and \(L_{Li}^0\) (known from the terrain model) and the vehicle configuration. The position kinematics of the first module (the master module) is different from that of the subsequent modules and these two problems have been addressed separately. In the next section, Equations 4.15 and 4.16 are applied first to the master module and then to the rest of the modules to solve for the vehicle configuration.

### 4.2.5 Position Kinematic Equations of the Modules

This section describes the procedure for solving the kinematic equations of the 'n' modules. The configuration of the entire vehicle is obtained by sequentially solving the position kinematics of all the modules starting from the master module. The position
kinematics of the master module is discussed in Section 6.1 and the position kinematics of
the subsequent modules is discussed in Section 6.2.

4.2.5.1 Master Module

In this section, the commanded value of the motion coordinates \( \beta, \gamma, o_x, \) and \( o_y \)
are assumed to be known and the accommodation coordinates \( o_z \) are to be
evaluated. The equations of the planes \( L_{R1} \) and \( L_{L1} \) are assumed to be known in frame '0'
from the terrain model.

The coordinate transformation taking vectors from frame '1' to frame '0' is
according to the following equation:

\[
v^0 = o_1^0 + R_1^0 v^1
\]

where \( v^0 \) and \( v^1 \) are coordinate representations of a vector \( v \) in frames '0' and '1'
respectively, \( o_1^0 = [o_x, o_y, o_z]^T \) is the location of the origin of frame '1' in '0', and
\( R_1^0 \) is the 3x3 rotation matrix.

\[
R_1^0 = \begin{bmatrix}
c\beta c\gamma & -c\alpha s\gamma + s\alpha s\beta c\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\
c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma + c\alpha s\beta s\gamma \\
-s\beta & s\alpha c\beta & c\alpha c\beta
\end{bmatrix}
\]

The plane equations \( L_{R1}^0 \) and \( L_{L1}^0 \) are known and therefore the vectors \( n_{R1}^0 \) and \( n_{L1}^0 \),
and the constants \( d_{R1}^0 \) and \( d_{L1}^0 \) are known (see Equations 4.2, 4.3). Equations 4.15 and
4.16, taking \( i = 1 \), contain components of the vectors \( n_{R1}^1 \) and \( n_{L1}^1 \), and \( d_{R1}^1 \) and \( d_{L1}^1 \).

\[
n_{R1}^1 = (R_1^0)^T n_{R1}^0, \quad n_{L1}^1 = (R_1^0)^T n_{L1}^0
\]

\[
d_{R1}^1 = (n_{R1}^0)^T o_1^0 + d_{R1}^0, \quad d_{L1}^1 = (n_{L1}^0)^T o_1^0 + d_{L1}^0
\]
Since the motion coordinates are commanded, their values are assumed to be available. Equations 4.15 and 4.16, for \( i = 1 \), represent two equations in the two accommodation coordinates \( \alpha \) and \( \omega_{z1} \). Consider the variable set \( A = \{ \sin(\alpha), \cos(\alpha), \omega_{z1} \} \). \( R_1^0 \) and \( \omega_{z1}^0 \) are linear with respect to \( A \). Further, it can be seen from Equations 4.19 and 4.20 that \( n_{R1}^1, n_{L1}^1, d_{R1}^1 \) and \( d_{L1}^1 \) are all linear with respect to \( A \). Therefore, Equations 4.15 and 4.16, for \( i = 1 \), represents two quadratics in \( A \). The third equation required to solve for the three variables in \( A \) is merely the trigonometric identity \( \sin^2(\alpha) + \cos^2(\alpha) = 1 \). Therefore, to evaluate \( \alpha \) and \( \omega_{z1} \) three quadratic polynomials relating the variables \( \sin(\alpha), \cos(\alpha) \) and \( \omega_{z1} \) have to be solved. The overall degree of these three equations or the Bezout number [Wampler et al., 1990] is 8. That means that the first module can assume a maximum of 8 different configurations. The three quadratics can be readily reduced to one octic in one variable as discussed in the literature [Husain and Waldron, 1992]. The use of a single octic can lead to somewhat improved computational efficiency. However, the use of double-precision complex arithmetic to solve for the octic numerically was found to be unreliable at times due to the fact that high degree polynomials lead to inherently ill-conditioned problems. Therefore, polynomial equations were solved using polynomial continuation and the computational cost was not significant. The numerical aspects of this solution procedure are dealt with in some detail in a later section. By solving several numerical examples, it was found that these equations have no spurious solutions or solutions at infinity (see Sections 4.2.6 and 4.2.7). Hence, these polynomials cannot be reduced any further [Morgan, 1987]. Therefore, these three quadratics represent the minimum order polynomial system for this position kinematic problem.

For typical vehicle parameters and terrain equations, it was found that the polynomial equations have eight real solutions. The eight configurations of the master module (that satisfy the constraints that the two module wheels lie on the two respective
terrain planes), as obtained for one such example, is shown in Figure 4.5. Exactly one of these 8 configurations satisfies all the geometric constraints, and hence there is only one 'correct' configuration. Steps to obtain the 'correct' configuration are listed in Section 4.2.7.

4.2.5.2 Subsequent Modules

In this section, the commanded value of the motion coordinate \( \theta_{i-1} \) is assumed to be known and in order to compute the configuration of module 'i', the accommodation coordinates \( \psi_{i-1} \) and \( \phi_{i-1} \) are to be evaluated \((i > 1)\). The equations of the planes \( L_{R_i} \) and \( L_{L_i} \) \((i > 1)\) are assumed to be known in frame '0' from the terrain model. Since the configurations of the modules are solved for sequentially, it is assumed that the configuration of the \( i-1^{th} \) module is known before solving for configuration of the \( i^{th} \) module. Therefore, \( R_{i-1}^0 \), the rotation matrix relating the \( i-1^{th} \) frame to the world frame is assumed to be known, and the location of the origin of frame 'i-1' in '0', \( o_{i-1}^0 \), is also assumed to be known. The coordinate transformation taking vectors from frame 'i' to frame 'i-1' is described by the following equation:

\[
v_{i-1} = o_{i-1} + R_{i-1}^i v_i
\]

where \( v_{i-1} \) and \( v_i \) are coordinate representations of a vector \( v \) in frames 'i-1' and 'i' respectively, \( o_{i-1} \) is the location of the origin of frame 'i' in 'i-1', and \( R_{i-1}^i \) is the 3x3 rotation matrix. Let the distance from the origin of frame 'i-1' to the articulation \( A_{i-1} \) be \( c_{i-1} \), and the distance from the articulation \( A_{i-1} \) to the origin of frame 'i' be \( b_i \). Then,

\[
o_{i-1} = [-c_{i-1}, 0, 0]^T + R_{i-1}^i [-b_i, 0, 0]^T
\]
\[ R_{i-1} = \begin{bmatrix}
    c\theta_{i-1} c\phi_{i-1} & -s\theta_{i-1} & c\theta_{i-1} s\phi_{i-1} \\
    (c\psi_{i-1} s\theta_{i-1} c\phi_{i-1} + s\psi_{i-1} s\phi_{i-1}) & c\psi_{i-1} c\theta_{i-1} & (c\psi_{i-1} s\theta_{i-1} s\phi_{i-1} - s\psi_{i-1} c\phi_{i-1}) \\
    (s\psi_{i-1} s\theta_{i-1} c\phi_{i-1} - c\psi_{i-1} s\phi_{i-1}) & s\psi_{i-1} c\theta_{i-1} & (s\psi_{i-1} s\theta_{i-1} s\phi_{i-1} + c\psi_{i-1} c\phi_{i-1})
\end{bmatrix} \]

(4.23)

The plane equations \( L_{R_i}^0 \) and \( L_{L_i}^0 \) are known from the terrain model and since the configuration of module 'i-1' is known (\( R_{i-1}^0, o_{i-1}^0 \) known), the plane equations can be obtained in frame 'i-1'. Therefore the vectors \( n_{R_i}^{i-1} \) and \( n_{L_i}^{i-1} \), and the constants \( d_{R_i}^{i-1} \) and \( d_{L_i}^{i-1} \) are known (see Equations 4.2, 4.3). Equations 4.15 and 4.16, taking \( i > 1 \), contain components of the vectors \( n_{R_i}^{i} \) and \( n_{L_i}^{i} \), and \( d_{R_i}^{i} \) and \( d_{L_i}^{i} \).

\[
\begin{align*}
    n_{R_i}^{i} &= (R_i^{i-1})^T n_{R_i}^{i-1}, & n_{L_i}^{i} &= (R_i^{i-1})^T n_{L_i}^{i-1} \\
    d_{R_i}^{i} &= (n_{R_i}^{i-1})^T o_{i}^{i-1} + d_{R_i}^{i-1}, & d_{L_i}^{i} &= (n_{L_i}^{i-1})^T o_{i}^{i-1} + d_{L_i}^{i-1}
\end{align*}
\]

(4.24)

(4.25)

Assuming the motion coordinate \( \theta_{i-1} \) is known, Equations 4.15 and 4.16, for \( i > 1 \), represent two equations in the two accommodation coordinates \( \psi_{i-1} \) and \( \phi_{i-1} \). Consider the variable set \( B_{i-1} = \{\sin(\psi_{i-1}), \cos(\psi_{i-1}), \sin(\phi_{i-1}), \cos(\phi_{i-1})\} \). From Equations 4.22 and 4.23, it is seen that \( R_i^{i-1} \) and \( o_i^{i-1} \) are quadratic with respect to \( B_{i-1} \). Further, it can be seen from Equations 4.24 and 4.25 that \( n_{R_i}^{i}, n_{L_i}^{i}, d_{R_i}^{i} \) and \( d_{L_i}^{i} \) are all quadratic with respect to \( B_{i-1} \). Therefore, Equations 4.15 and 4.16, for \( i > 1 \), represent two quartics in \( B_{i-1} \). Let, \( s_{i-1} = \tan(\psi_{i-1}/2) \) and \( t_{i-1} = \tan(\phi_{i-1}/2) \). Then,

\[
\begin{align*}
    s\psi_{i-1} &= \frac{2 s_{i-1}}{1 + s_{i-1}^2}, & c\psi_{i-1} &= \frac{1 - s_{i-1}^2}{1 + s_{i-1}^2}, & s\phi_{i-1} &= \frac{2 t_{i-1}}{1 + t_{i-1}^2}, & c\phi_{i-1} &= \frac{1 - t_{i-1}^2}{1 + t_{i-1}^2}
\end{align*}
\]

(4.26)

Using Equation 4.26 to substitute for variables in \( B_{i-1} \), two octics in \( s_{i-1} \) and \( t_{i-1} \) are obtained. These octics have a special structure because the highest degree of \( s_{i-1} \) and \( t_{i-1} \),
taken individually, is 4. This is due to the fact that \( R_i^{i-1} \) and \( \phi_i^{i-1} \) do not contain quadratic terms involving the same angle. For instance terms like \( \psi_{i-1} \cos \theta_{i-1} \) or \( s^2 \phi_{i-1} \) are not present in Equation 4.23. Using this information, it can be shown that by making the substitution,

\[
 u_{i-1} = s_{i-1} t_{i-1}
 \]  

(4.27)

the two octics reduce to two quartics in the variables \( u_{i-1}, s_{i-1}, \) and \( t_{i-1} \). These two quartics, along with Equation 27, represent three polynomial equations in \( u_{i-1}, s_{i-1}, \) and \( t_{i-1} \). The overall degree of these three equations or the Bezout number [Wampler et al. 1990] is 32. That means that, given the configuration of the first \( i-1 \) modules, the \( i^{th} \) module can assume a maximum of 32 different configurations. These polynomial equations can be solved using polynomial continuation and the computational cost is greater than that for the polynomial system in Section 6.1, but is still not very significant. Once the values of the variables \( u_{i-1}, s_{i-1}, \) and \( t_{i-1} \) are known, \( \psi_{i-1} \) and \( \phi_{i-1} \) can be readily obtained, and hence the configuration of module 'i' is obtained. It should be noted that for every module other than the main module a separate polynomial system of degree 32 has to be solved. By solving several numerical examples, it was found that this polynomial system has no solutions at infinity (except in the special case when \( \psi_i = \phi_j = \pm \pi \), see Section 4.2.7), but has sixteen spurious solutions (see Sections 4.2.6 and 4.2.7). Since there are no solutions at infinity, it may be possible to obtain a univariate polynomial of degree thirty two using the methods of higher algebra [Salmon, 1885]. However, since such a high degree polynomial leads to ill-conditioned numerical problems, the reliability of the resulting algorithm is expected to be low. Typically, this requires the use of integer arithmetic to obtain accurate solutions, and solution of such a high degree polynomial using integer arithmetic tends to be very cumbersome and computationally expensive [Zhang and Song, 1992]. Therefore, these
three polynomial equations were solved here using polynomial continuation. The numerical aspects of this solution procedure are dealt with in detail in a later section.

The fact that these equations led to sixteen spurious solutions in numerical examples seems to suggest that there is a potential improvement possible in the nonlinear reduction procedure of Sections 4.2.4 and 4.2.5.2. Therefore, the three polynomials of degree thirty-two do not represent a minimum order polynomial system.

For typical vehicle parameters and terrain equations, and for the 'correct' configuration of the master module, it was found that the polynomial equations of the follower module have sixteen real and sixteen spurious solutions. The sixteen configurations of the follower module (that satisfy the constraints that the two module wheels lie on the two respective terrain planes), as obtained for one such example, is shown in Figure 4.6. Exactly one of these 16 configurations satisfies all the geometric constraints, and hence there is only one 'correct' configuration. Steps to obtain the 'correct' configuration are listed in Section 4.2.7.

4.2.6 Numerical Issues

Polynomial continuation is a procedure used to solve for all the roots of a polynomial system of equations. This procedure is described in the literature [Wampler et al. 1990]. The position of the main module can be readily solved using generic polynomial continuation as has been discussed in literature. Generic polynomial continuation can also be used to solve for the configurations of the subsequent modules. However, the numerical efficiency of this procedure can be improved by considering a special polynomial continuation technique that exploits the multi-homogenous structure of the equations of Section 4.2.5.2. In Section 4.2.5.2, the Bezout number of the polynomial system, comprised of three equations, was 32. By using multi-homogenous coordinates this can be
reduced to a system of two polynomial equations whose multi-homogenous Bezout number [Wampler et al. 1990] is also 32. The continuation procedure involves a very large number of inversions of the Jacobian matrix of the polynomial system. This inversion is required as part of the solution by Newton's method of the nonlinear equations required to track the homotopy paths. Therefore, the multi-homogenous system of two polynomial equations can be solved more efficiently since the Jacobian is of dimension 2x2 as compared to the 3x3 Jacobian for the system of three equations.

It was stated in Section 4.2.5.2 that Equations 4.15 and 4.16 lead to two octics in the variables $s_{i-1}$ and $t_{i-1}$. These octics have a special structure because the highest degree of $s_{i-1}$ and $t_{i-1}$, taken individually, is 4. Following the procedure presented by Wampler et al. [1990], the multi-homogenous Bezout number is the coefficient of $\alpha_1 \alpha_2$ in the product $(4\alpha_1 + 4\alpha_2)^2$, and is equal to 32. Hence, the two octics represent two polynomials whose multi-homogenous Bezout number is 32. The significance of this number is that even though the two octics have an overall degree of 64, the upper bound on the number of finite solutions is 32, and these 32 solutions can be obtained by using the following special start system that can be constructed using the procedure in Wampler et al. [1990]:

\begin{align*}
(s_{i-1}^4 + \omega_1 s_{i-1}^3 + \omega_2 s_{i-1}^2 + \omega_3 s_{i-1} + \omega_4)(t_{i-1}^4 + \rho_1 t_{i-1}^3 + \rho_2 t_{i-1}^2 + \rho_3 t_{i-1} + \rho_4) &= 0 \\
(s_{i-1}^4 + \mu_1 s_{i-1}^3 + \mu_2 s_{i-1}^2 + \mu_3 s_{i-1} + \mu_4)(t_{i-1}^4 + \epsilon_1 t_{i-1}^3 + \epsilon_2 t_{i-1}^2 + \epsilon_3 t_{i-1} + \epsilon_4) &= 0
\end{align*}

(4.28)

where $\omega_i, \rho_i, \mu_i$ and $\epsilon_i$ are random complex numbers. Equations 4.28 represents two octics and it can be easily shown that they have exactly 32 finite solutions that can be obtained by solving the following two sets of polynomial equations (each set gives 16 distinct finite solutions):

\begin{align*}
(s_{i-1}^4 + \omega_1 s_{i-1}^3 + \omega_2 s_{i-1}^2 + \omega_3 s_{i-1} + \omega_4) &= (t_{i-1}^4 + \epsilon_1 t_{i-1}^3 + \epsilon_2 t_{i-1}^2 + \epsilon_3 t_{i-1} + \epsilon_4) = 0
\end{align*}
The above start system can be solved 'off-line' and used to track the 32 finite solutions of the two octics of Section 4.2.5.2. A similar multi-homogenous approach to solve polynomial systems was also used by Raghavan [1991].

Tracking the undesirable solution vectors at infinity is a time consuming process. However, as was stated earlier in Section 4.2.5, all the solutions at infinity have been eliminated. This improves the efficiency of the numerical scheme.

Polynomial continuation is a procedure that 'tracks' all the roots of a polynomial system [Wampler et al. 1990]. This tracking of each root can be performed independently of the other roots. This makes polynomial continuation ideally suited for computation in a parallel processing mode. Therefore, with the use of parallel architectures, the computation time for the master module position can be potentially reduced to 12.5% of the time it would take in the absence of parallel computing. Similarly, the computation time for the follower modules can be reduced to 3.125% of the time it would take using serial computing.

In the next section, procedures to obtain the 'correct' vehicle configuration are discussed. Since, the correct vehicle configuration is unique, only one of the several solutions to the polynomial systems will be the right one. As soon as the right root has been obtained, the continuation process can be stopped. In general, this will require tracking of fewer paths than the overall Bezout number, and hence will improve computational efficiency.
Finally, when using this approach for situations like motion planning or dynamic simulations, the updated unknown vehicle configuration is close to the previous known configuration. Therefore, the known configuration can be used to solve for the new vehicle position by using a generic Newton's method. If the scheme converges and the solution is the 'correct' solution, the obtained new position is the right one. In the event that the Newton method diverges or converges to an 'incorrect' configuration, the procedure of polynomial continuation will have to be used. It is reasonable to expect that a large number of vehicle configurations can be obtained from the previous vehicle position, and this adds to the efficiency of the solution procedure.

### 4.2.7 The 'Correct' Vehicle Configuration

The 'correct' vehicle configuration can be obtained by eliminating the solutions that fall under one of the following classes of solutions:

1. **Solutions at Infinity (Representational Singularity):** A particular solution vector of a polynomial system may have values of one or more of the variables that are infinitely large. These solutions could be real or complex. In some situations, as in the present case, solutions at infinity may have geometric significance, while in other cases they appear due to inconsistency in the polynomial equations. The continuation process cannot converge to solutions at infinity. Therefore, during the continuation process, if the norm of the solution vector gets larger than a prespecified large value, that particular continuation path can be abandoned. In this manner solutions at infinity can be handled. As was stated earlier, from typical numerical examples it was
found that the polynomial systems of Section 4.2.5 have no solutions at infinity. Therefore, these equations do not possess any inconsistencies.

However, in the event that any of the angular displacement variables $\psi_i$ and $\phi_i$ are equal to $\pm \pi$, the corresponding variables $s_i$ and $t_i$ will be equal to $\pm \infty$. This is, essentially, a representational singularity since geometrically this angular position is not singular. This singularity appears because of the use of the tangent trigonometric function. Therefore, if the absolute value of the variables $s_i$ or $t_i$ exceeds a certain prespecified large value in the continuation scheme, it means that the corresponding variables $\psi_i$ or $\phi_i$ are equal to $\pi$. This means that the follower module will be flipped over about the roll or the pitch axis and hence will not be 'upright'. Therefore, these solutions can be discarded.

2. Complex Solutions: A real polynomial system has an even number of complex finite solutions. If a complex vector is a solution, then its conjugate is also a solution. These solutions have no significance in position kinematics and hence are discarded.

3. Spurious Solutions: In the process of reducing a set of nonlinear equations, incorrect solutions are sometimes added to the correct solution set. These solutions have been referred to as spurious solutions in the position kinematics literature. Even though these solutions are part of the solution set of the final equations, they do not satisfy the initial constraints that were used to obtain those final equations. In some cases such spurious solutions may be eliminated by reinvestigating the nonlinear reduction procedure. However, there is no general procedure that will guarantee the elimination of spurious solutions. In the solution of the follower modules, it was found that sixteen of the thirty two finite solutions were spurious. However, it was not clear how these spurious solutions could be eliminated from the equations. Of course
these solutions have no geometric significance and can be discarded. Their presence is nevertheless annoying since they lead to an increase in the computational burden.

3. **Infeasible Solutions** : Once all the real solution vectors have been identified, it is required that we identify the 'correct' solution. It can be shown that the position kinematic problem of the vehicle has only one 'correct' solution. The discussion later on in this section will clarify this fact. In general, the polynomial systems that represent the position kinematics of the wheeled vehicle have several real solutions. There are two reasons for this. The first reason is that a squaring operation is performed to eliminate the radicals appearing in Equations 4.10 and 4.11. This leads to solutions that involve wheels that have their centers below the local plane. These solutions are obviously infeasible, and are therefore discarded. The second reason why there are several real solution vectors is because the modules are not constrained to have their local z axes facing 'up'. In other words, we may have modules flipped over and still satisfying the constraint that the wheels lie on the respective planes. These solutions can be avoided by requiring the local z axis directions to have a positive z component in the world frame 'O'. It is assumed that the axis $Z_0$ is pointing vertically up. Summarizing this step, infeasible solutions can be eliminated by ensuring that the two wheels of each module are above the terrain (represented by the respective local planes), and by constraining the module z axis to point 'up'.

We have already noted that the first module has up to eight real solutions, and for every solution of the first module, the following module has up to sixteen real (non-spurious) solutions. The eight real solutions of the first module have the following geometric interpretations:

(a) Module body 'up', wheel centers of both the wheels above the ground.
(b) Module body 'up', wheel centers of both the wheels below the ground.

(c) Module body 'up', wheel center of the right wheel above and that of the left wheel below the ground.

(d) Module body 'up', wheel center of the right wheel below and that of the left wheel above the ground.

(e) Module body 'down', wheel centers of both the wheels above the ground.

(f) Module body 'down', wheel centers of both the wheels below the ground.

(g) Module body 'down', wheel center of the right wheel above and that of the left wheel below the ground.

(h) Module body 'down', wheel center of the right wheel below and that of the left wheel above the ground.

These eight configurations are shown in Figure 4.5 for a particular numerical example. The small polygons below the wheels represent the planes on which the wheels lie. The configurations shown in Figure 4.5 are being viewed from below the terrain. Therefore, the first of the eight configurations represents the 'correct' configuration that is as described in (a) above.

The sixteen configurations of the follower module for every position of the preceding module has a geometric interpretation that is very similar to that of the first module. Four configurations are possible depending on whether zero, one or two wheels are above the ground. There are two such sets of four solutions obtained by flipping the body over about the roll axis corresponding to the 'up' and the 'down' configurations in
The discussion of the master module. Finally, there are two such sets of eight solutions obtained by flipping the body about the pitch axis leading to sixteen solutions. These sixteen configurations are shown in Figure 4.6 for a particular numerical example, assuming that the master module has been fixed in its correct configuration as obtained from Figure 4.5. The small polygons below the wheels represent the planes on which the wheels lie. The configurations in Figure 4.6 are being viewed from below the terrain. The configurations obtained by flipping about the pitch axis lead to interference of the two modules as seen in Figure 4.6. The solution that does not get eliminated by the conditions 1 through 4 listed above is the solution on the third row and the fourth column. A zoomed in view of this configuration is shown in Figure 4.7.

Since the first module has exactly eight solutions and the following modules have sixteen non-spurious solutions, the 'correct' module configuration will be the one which does not get eliminated by the above 4 conditions, and all the 'correct' modules assembled together yield the 'correct' vehicle configuration. It should be noted here that for a given set of vehicle parameters, vehicle motion coordinates and terrain geometry, there may be no good solution. This means that the vehicle cannot be constrained to stay on the terrain for the given values of the motion coordinates. An example of such a situation is the case where the first module is required to be on the edge of a deep ditch, and the wheels of the second module are to lie on the bottom of the deep ditch. If the vehicle geometric parameters are not large enough to allow that, the accommodation coordinates will have to be complex to satisfy the constraints. Therefore, there will be no solution to that position kinematic problem. This may mean that some of the motion coordinates of the vehicle have to be changed (like increasing the pitch angle, $\beta$, of the first module to lower the second module on to the floor of the ditch), or in a practical sense it is likely to mean completely avoiding that location on the terrain to avoid vehicle failure.
4.2.8 Numerical Example

The displacement analysis algorithm developed in the earlier sections is illustrated here by considering an example of an articulated wheeled vehicle consisting of two modules. The vehicle parameters are: \( a = 0.4, r = 0.25, c_1 = 0.7, b_2 = 0.7 \) (see Figure 4.4). The planes below the 4 wheels of the 2 modules in frame 'O' are:

\[
L_{R1}^0 = 0.35603 X_0 + 0.05250 Y_0 + 0.93299 Z_0 - 2.05088 = 0
\]

\[
L_{L1}^0 = 0.32625 X_0 + 0.28498 Y_0 + 0.90130 Z_0 - 3.08349 = 0
\]

\[
L_{R2}^0 = 0.09030 X_0 + 0.36948 Y_0 + 0.92484 Z_0 - 2.61139 = 0
\]

\[
L_{L2}^0 = 0.26940 X_0 - 0.059217 Y_0 + 0.96121 Z_0 - 1.43285 = 0
\]

![Figure 4.4 The Geometric Parameters of a Two Module Articulated Vehicle.](image-url)
Figure 4.5 The Eight Configurations of the Master Module
Figure 4.6 The Sixteen Configurations of the Follower Module
Figure 4.7 The 'Correct' Vehicle Configuration Composed of the 'Correct' Master and Follower Modules
The above plane equations are in the same format as Equations 4.2 and 4.3. The motion coordinates are: \( \beta = 20^\circ, \gamma = 55^\circ, \alpha_1 = 3.0, \alpha_y = 5.0 \), and \( \theta_1 = 15^\circ \). The polynomial equations that are to be solved to obtain the accommodation coordinates, and their solutions are listed in Appendix B. For this particular problem, the three quadratic equations of module 1 have 8 real solutions. The 'correct' solution obtained by applying the conditions of Section 7 was \( (\sin(\alpha), \cos(\alpha), 0_{x_1}) = (0.17365, 0.98481, 1.00000) \), giving \( \alpha = 10^\circ \).

The two octics of module 2 are also included in Appendix B. It should be noted that both these octics have the highest power of \( x_1 \) and \( x_2 \) equal to 4. Therefore, they have only 32 solutions. For this example, there were 16 real and 16 complex solutions as listed in Appendix B. The 'correct' solution was \( (\tan(\psi_1/2), \tan(\phi_1/2)) = (-0.0875, -0.0612) \), giving \( \psi_1 = -10^\circ \) and \( \phi_1 = -7^\circ \). Therefore, the unknown accommodation coordinates are \( (\alpha, 0_{x_1}, \psi_1, \phi_1) = (10^\circ, 1, -10^\circ, -7^\circ) \). The fact that these unknowns take integral values is no accident. The above plane equations \( (L_{R_i}^0, L_{L_i}^0) \) were actually obtained by assuming the values of all the 9 coordinates to be the above values. The kinematic problem was then solved assuming the knowledge of the motion coordinates and the planes, and the accommodation coordinates were obtained. The accommodation coordinates that were obtained matched the originally assumed values, hence verifying the position kinematic equations, and the procedure outlined in Section 4.2.7. The configurations shown in Figures 4.5, 4.6 and 4.7 were obtained using the numerical results from this example. The computation times for obtaining the eight positions of the master module and the sixteen positions of the follower module were found to be 0.17 seconds and 0.82 seconds respectively. These are CPU times on an SGI-R4000 workstation. These times can be significantly reduced with the use of a parallel architecture as discussed in Section 4.2.6.
4.3 Autonomous Motion Planning on Uneven Terrain

In this section, the position kinematic solutions of Section 4.2 are used to develop autonomous motion planning strategies for articulated wheeled vehicles on uneven terrain. It is assumed that the terrain geometry is known, and can be obtained as a set of discrete points by the use of an appropriate vision system. The motion planning algorithms that are discussed here are computationally intensive numerical schemes. Real-time implementation of these schemes may require expensive computer hardware. As was discussed in the Section 4.2, parallel architectures can significantly improve the computational speed of the position kinematic solutions. Nevertheless, these algorithms can be used to perform reliable off-line planning which is very useful in situations in which the vehicle is traversing a difficult terrain, and can periodically stop to execute the planning algorithms.

When a mobile system is operated autonomously, it is necessary to obtain appropriate paths for the system to follow to reach its goal points on the terrain. The selection of a path may be based on considerations such as system safety and minimal energy consumption. In this section, preliminary motion planning results are presented based on system safety considerations. If it is assumed that the initial and the goal positions of the vehicle are known on an uneven terrain, the techniques presented here can be used to check for the safety of the vehicle along straight line paths joining these terrain points. Vehicle failure along the path may be either geometric failure or force failure. Failures such as the interference of the vehicle with the terrain or with itself, and the inability of the vehicle to surmount large obstacles or to cross wide ditches are taken to be geometric failures. Failures such as lifting of a wheel off the ground due to negative contact forces normal to the terrain, or large slip rates due to large tangential to normal contact force ratios at the wheel-terrain contact points are considered force failures. The
strategy involves identifying a set of discrete points on the path and solving for the geometric configurations of the vehicle at these intermediate points. These configurations can be used to check for geometric or force failures. If the vehicle configuration at a point on the terrain is found to be unsafe, the vehicle will be required to avoid this point on the terrain by going 'around it'. This may require the vehicle to perform turning maneuvers. Further, a three module articulated vehicle such as the WAAV can use special mobility maneuvers to climb obstacles and cross ditches. This section presents a robust numerical algorithm that can generate motion histories of the vehicle along terrain paths. Extensions of this strategy can also generate motion histories during special maneuvers such as turning, obstacle climbing and ditch crossing. In addition to checking the safety of the vehicle along terrain paths, these strategies can also be used off-line to generate reference trajectories for the various motion variables during complex mobility tasks such as obstacle climbing. These reference trajectories can be used as an input to a control scheme during such tasks. Force distribution issues of these vehicles on uneven terrain have already been addressed in Chapter 3 and will not be considered here. Motion planning with respect to minimal energy consumption is important when autonomous systems (such as planetary exploration systems) have to operate on limited energy budgets. This problem has not been addressed in this work.

On difficult terrain, computer vision systems such as laser rangefinders or stereo vision systems can be used to generate terrain elevation maps. The terrain is divided into an x-y grid, and the elevation maps have a z coordinate associated with each grid point. The terrain geometry can be approximated by fitting planes through sets of three adjacent points. This leads to a piece-wise planar characterization of the uneven terrain (see Figure 4.11). A motion planning algorithm for straight line paths on a piece-wise planar uneven terrain is described in Section 4.3.1. Special mobility maneuvers including obstacle
climbing and ditch crossing are discussed in Section 4.3.2. An algorithm for turning is discussed in Section 4.3.3.

4.3.1 Straight Line Path on Uneven Terrain

This section describes a planning algorithm that generates motion histories of the vehicle degrees of freedom as it traverses a straight line terrain path. A block diagram of the algorithm is shown in Figure 4.8. As was discussed in Section 4.2, the $3n + 3$ coordinates of the articulated vehicle are $\{x, y, z, \alpha, \beta, \gamma, \psi_j, \theta_j, \phi_j\}$ where $j$ varies from 1 to $n-1$. The mobility of the vehicle is $n+3$ and the motion coordinates of the vehicle are $\{x, y, \psi_j, \theta_j\}$. The accommodation coordinates of the vehicle ($2n$ in number) are $\{z, \alpha, \psi_j, \phi_j\}$. Here $n$ refers to the number of modules. Figure 4.8 describes the algorithm to obtain the configuration of a multi-moduled articulated system at the $i^{th}$ discrete point on a chosen path.

Assuming that the terrain geometry is known, and that the initial and goal points on the terrain are available, the path planner identifies a path joining these two points. In its simplest form this path is a single straight line. In order to avoid expensive segments or unsafe regions on the terrain, the vehicle may have to traverse several straight line segments. The path planner also generates all the motion coordinates at any point on the terrain. Therefore, at the $i^{th}$ point on the path, the outputs of the path planner are $x_i$, $y_i$, $\psi_i$, $\gamma_i$ and $\theta_{ij}$. The variables $x_i$ and $y_i$ are the coordinates of the $i^{th}$ point on the path. The variable $\gamma_i$ represents the heading and it is set to a constant value as the vehicle traverses a straight line path. The variable $\theta_{ij}$ represents the yaw angle of the $j^{th}$ articulation. This angle can be taken to be equal to zero for straight line vehicle motion. In reality, assuming no lateral slip of the following modules, this angle oscillates somewhat about a mean value...
Figure 4.8 Block diagram of the motion planning algorithm
of zero. However, taking it to be equal to zero is a reasonable assumption, and it simplifies the planning algorithm significantly. When the vehicle is required to perform a turn, these angles have to be computed by integrating the nonholonomic constraints that ensure that the modules have zero lateral velocity. Nonholonomic motion planning is described in Section 4.3.3. The variable $\beta_i$ decides the pitch of the master module. This motion variable can be chosen based on various considerations. For instance, during difficult mobility maneuvers such as surmounting obstacles, the choice of $\beta_i$ may decide whether the maneuver is going to be a success or not. The choice of $\beta_i$ is also important when a vehicle is recovering from an overturn failure. In this section, $\beta_i$ is chosen so that the orientation of the master module allows a vision camera mounted on the module to be pointed onto the terrain ahead of the vehicle. At any location 'i' on the path, the elevation (z coordinate) of points $i, i+1, \ldots, i+k$ on the path are obtained. These k+1 points are supposed to represent the terrain elevation ahead of the vehicle. A least-square straight line is fit through these points and the slope of this straight line is taken to be the average slope of the terrain in front of the vehicle. This variable $\beta_i$ is chosen so that the vehicle heading vector is parallel to this straight line. In this manner all the motion coordinates are specified.

The next step is to obtain the accommodation coordinates of the master module. In order to use the position kinematics algorithm developed in Section 4.2.5.1, the equations of the planes below the two wheels of the master module have to be known. Since the terrain is represented by a complex piece-wise planar geometry, the procedure for identifying the planes below the two wheels is non-trivial and involves a search. Initial guesses are obtained for these two plane equations in the following manner: Current values of the motion coordinates as obtained from the path planner and the previous values of the accommodation coordinates of the master module ($\lambda_{i-1,1}, z_{i-1}$ and $\alpha_{i-1}$) are used to obtain an estimate of the coordinates of the wheel-terrain contact points. Here the vector
\( \lambda_{i-1,1} \) contains the two angles (\( \lambda_R \) and \( \lambda_L \) for the left and the right wheel) that are required to locate the two wheel-terrain contact points with respect to the local coordinate frame of the master module (see Figure 4.9). The subscript 'i-1' refers to the i-1\(^{th}\) point on the path. The estimated locations of the contact points are projected onto the terrain and the planes lying below the two wheels are taken as the initial guesses for the position kinematics of the master module.

Knowing the motion coordinates and the two plane equations, the procedure of Section 4.2.5.1 is now used to obtain all the solutions for the accommodation coordinates of the main module (a maximum of eight real solutions). These solutions are all tested to see if any of them satisfy all the conditions listed in Section 4.2.7. If no 'good' solution is obtained, it means that the vehicle geometry is incapable of adhering to the terrain at the i\(^{th}\) point on the path. This represents an unsafe vehicle configuration and it means that the
vehicle should avoid that point on the path. If a good solution is obtained, it has to be ascertained whether this solution leads to a master module configuration that actually lies on the two planes chosen as initial guesses. The outputs of the position kinematics solution \((\lambda_i, z_i, \alpha_i)\) are used to obtain the two contact points, and if these points satisfy the respective plane equations, then the solution of the configuration of the master module is concluded. If one of the two contact points do not satisfy the respective plane equations, a search strategy has to be followed to identify the two planes below the wheels. The search strategy that was used in this work is described below.

Assume that the terrain information is known as a uniformly distributed grid of data points. A plan view of a representative portion of the piece-wise planar terrain that results is shown in Figure 4.10. The search strategy starts with the initial guess of the plane below a wheel. Let use assume that the plane marked '0' (Plane ABC) in Figure 4.10 is the initial guess. If the contact point does not lie on plane 0 at the end of the computation of one set of accommodation variables, it can be assumed to lie on one of the neighboring planes. Plane 0 has twelve neighboring planes represented by planes 1 through 12. Planes 1, 2 and 3 share an edge with plane 0 while planes 4 through 12 share a vertex with plane 0. The first level of the search strategy tests planes 1, 2 and 3 sequentially (since they share an edge with plane 0) to see if one of them is the plane on which the wheel lies. If this level of search fails, then the second level involves testing planes 4 through 12 that share only a vertex. A similar two level search may also be required for the other wheel of the module. Therefore, in general, at the first level of the search, a maximum of nine different combinations of pairs of planes may have to be tested. At the second level of search, a maximum of 144 different combinations of planes may have to be tested. Since every search test involves the solution of the position kinematics, these search algorithms may be computationally expensive. However, for reasonable resolutions of the terrain data
points, it was found that the second level of search was rarely required during the course of this work. If the resolution of the terrain geometric information is high, the search algorithm may be a major bottleneck during motion planning. In such situations, more creative search strategies that are more efficient than the ones used in this work are desirable.

Once the position kinematics of the master module has been solved for, position solutions for the follower modules are to be obtained. The block diagram of Figure 4.8 describes the procedure to solve for the configuration of the $j^{th}$ following module at the $i^{th}$ point on the path, assuming that the positions of all the preceding modules are available.
The geometric information of the preceding modules, the motion coordinates of the $j^{th}$ following module, and the previous $(i-1)^{th}$ point on the path) values of the accommodation coordinates of the $j^{th}$ following module are all used to obtain initial guesses for the plane equations below the two wheels of the $j^{th}$ following module. Knowing the motion coordinates and the two plane equations, the procedure of Section 4.2.5.2 is now used to obtain all the solutions for the accommodation coordinates of the $j^{th}$ following module (a maximum of sixteen real solutions). These solutions are all tested to see if any of them satisfy all the conditions listed in Section 4.2.7. If no 'good' solution is obtained, it means that the vehicle geometry is incapable of adhering to the terrain at the $i^{th}$ point on the path. This represents an unsafe vehicle configuration and it means that the vehicle should avoid that point on the path. If a good solution is obtained, it has to be ascertained whether this solution leads to a master module configuration that actually lies on the two planes chosen as initial guesses. If one of the two contact points do not satisfy the respective plane equations, a search strategy identical to the one described for the master module has to be followed to identify the two planes below the wheels. The search strategy is more expensive for the solution of the following modules as compared to the solution of the main module, because the positions of the follower modules are obtained by solving a polynomial system of overall degree 32 as compared to a polynomial of overall degree 8 for the master module. A successful search strategy for all the modules leads to the solution of the vehicle configuration at the $i^{th}$ point on the path.

The strategy of Figure 4.8 has been incorporated into a general purpose planning software package that can obtain motion histories of a general articulated vehicle comprised of several modules traversing a piece-wise planar uneven terrain. The numerical output of this software can be visualized using a graphical simulator that has been developed on an SGI-R4000 workstation. This graphical simulator incorporates features such as partial
hidden-line rejection, zoom-in, zoom-out, rotation and translation transformations. It is based on the graphics libraries of X-windows and is portable to all the other workstations that support X-windows. The graphical simulator proved to be extremely valuable when dealing with the large amounts of numerical data that form the output of the planning algorithm. The planning algorithm, that includes numerical continuation schemes and search strategies, was found to be very reliable during the course of this study.

Figure 4.11 shows a three-module articulated vehicle traversing a simulated random terrain. The geometric parameters of the vehicle are the same as the ones used in the numerical example of Section 4.2.8. The terrain is a 25m x 25m area and consists of a uniform grid of 41 x 41 points. The elevation of the terrain is a random number that varies between 0 and 0.25m. Figure 4.12 shows the vehicle in the initial, an intermediate and the final positions on this terrain as it traverses a straight line path between the points (2.5, 7.0) and (16.0, 16.0). Figure 4.13 includes the histories of the vehicle coordinates as it traverses the straight line path. The motion coordinates $p_x, p_y, \gamma, \text{and } \theta_i (i = 1, 2)$ are unaffected by the random terrain variations. The motion coordinate $\beta$, and the accommodation coordinates $p_z, \alpha, \psi, \text{and } \phi_i (i = 1, 2)$ are seen to oscillate about a mean value due to the terrain variations.

This concludes the discussion of the motion planning algorithm. Extensions of this algorithm to special mobility maneuvers are discussed in the following sections.

4.3.2 Obstacle Climbing and Ditch Crossing

A preliminary motion planning strategy that allows for automated obstacle climbing and ditch crossing is discussed in this section. These mobility maneuvers require at least three modules. The strategy here involves the inclusion of virtual planes such as the ones shown in Figure 4.14. When the vehicle approaches a step change in the terrain geometry,
Figure 4.11 A Three Module Articulated Vehicle Traversing a Simulated Random Terrain
Figure 4.12 Vehicle Traversing a Straight Line Path on Uneven Terrain
Figure 4.13 Variation of the Kinematic Quantities as the Vehicle Traverses a Straight Line Path on Uneven Terrain
it does not perceive this step change since the geometric information of the virtual terrain is
superimposed onto the actual terrain information. Therefore, the vehicle coordinates vary
in a relatively smooth manner as the vehicle overcomes the obstacle or the ditch. However,
there is one important issue that has to be addressed. When a module is on the virtual
terrain, it cannot transmit loads onto the terrain. Therefore, in the case of a three-module
articulated system, no more than one module can be on the virtual terrain, and the center of
mass of the system should be located so that it is enclosed within the four contact points of
the two modules that are not on the virtual terrain. The WAAV has an actuation unit that
allows the motion of the middle module axle in the fore-aft direction. This motion

Figure 4.14 Obstacle Climbing and Ditch Crossing by the Use of a Virtual Plane
capability is required in order to appropriately place the center of mass of the vehicle during maneuvers such as obstacle climbing, as discussed in Chapter 1. The motion planning algorithm resulting from the use of the virtual terrain concept has been used to demonstrate automated obstacle climbing. A simulated random terrain that has an elevation variation between 0.0 and 0.25 m has an obstacle that is 0.4 m high. The vehicle approaches the obstacle and automatically configures itself to accommodate to the obstacle. The initial, final and some intermediate vehicle configurations of the resulting motion are shown in Figure 4.15. Position histories of the vehicle coordinates during obstacle climbing are shown in Figure 4.16. The variables \( \beta, \phi_1, \phi_2 \) and \( p_{z1} \) are affected by the geometry of the obstacle as shown. The variables \( \alpha, y_1 \) and \( y_2 \) are among the variables that are affected by the random terrain variations, and not by the obstacle geometry.

4.3.3 Turning Maneuver

The assumptions that the heading angle \( \gamma \) is a constant and that the yaw angles \( \theta_i \) are zero leads to a simplified planning algorithm for straight line paths. However, when the vehicle is performing a turn, these assumptions are not valid. As was stated earlier wheeled vehicle systems do not possess omni-directional motion capability. For the vehicle geometry of Figure 4.2, the \( i \) module cannot be commanded to move along the \( y_i \) direction (direction parallel to the axle of the \( i \) module). In other words, the \( y_i \) component of velocity of point \( o_i \) is required to be zero. For \( i = 1 \), this results in a constraint relating the variables \( \dot{x}_1 \) and \( \dot{y}_1 \), and for \( i > 1 \) this results in a constraint relating the variables \( \psi_{i-1}, \theta_{i-1} \) and \( \phi_{i-1} \). These are nonholonomic constraints and hence they cannot be used to obtain constraints on the position coordinates. During turning, the chosen values of the motion coordinates \( o_{x1}, o_{y1} \) and \( \theta_i \) should be such that they can be achieved from the previous vehicle state without violating these nonholonomic constraints. A velocity analysis is performed below to obtain these nonholonomic constraints. These constraints
Figure 4.15 Automated Obstacle Climbing Using the Virtual Terrain Concept
Figure 4.16 Variation of Kinematic Quantities During Obstacle Climbing
are integrated to obtain intermediate values of all the motion coordinates. The accommodation coordinates are then solved for by using the numerical continuation scheme along with the search strategy as discussed in Section 4.3.1.

The velocity analysis involves solution for all the rate quantities of the vehicle given the free rate variables. During position kinematic analysis, the mobility of an n module system was shown to be n+3. In the rate domain, each module is subjected to one nonholonomic constraint (\( \dot{y}_i = 0 \), for \( i = 1, \ldots, n \)). Therefore, for velocity analysis, the mobility of the system is 3. Three free variables have to be identified and, knowing these variables, the other variables can be obtained by performing a velocity analysis. It is assumed that the position kinematic analysis has been completed at the time the velocity analysis is performed.

Let the velocity of the coordinate frame of the main module (frame 1 in Figure 4.2) be \( \mathbf{v}_1 = [v_{x1}, v_{y1}, v_{z1}]^T \), and let the angular velocity of module 1 be \( \mathbf{\omega}_1 = [\omega_{x1}, \omega_{y1}, \omega_{z1}]^T \). The rate variables \( v_{x1}, \dot{\beta}, \) and \( \dot{\gamma} \) are chosen as the free rate variables since these variables allow for control of forward speed, heading and the pitch rate of the master module. These variables are time derivatives of motion variables that were used during the position kinematic analysis, and among the motion variables they are those least affected by the nonholonomic constraints.

4.3.3.1 Velocity Analysis of the Master Module

The free variables \( v_{x1}, \dot{\beta}, \) and \( \dot{\gamma} \) are assumed to be available from a path planner and the variable \( v_{y1} = 0 \) (no lateral motion for the master module). The unknown rate variables of the master module are \( \dot{\alpha}, v_{z1}, \) and the two wheel angular rates \( \dot{\theta}_{R1} \) and \( \dot{\theta}_{L1} \). Four velocity equations are required to solve for these four variables and these equations
are obtained by constraining the components of the velocities of the two contact point in the plane of the wheel to zero. In general, the lateral components of the contact velocities are non-zero on uneven terrain.

The angular velocity of the main module in the world frame in terms of the rate variables $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ (time derivatives of the Euler angles) is given by

$$\Omega_1 = \begin{bmatrix}
0 & -\sin(\gamma) & \cos(\beta)\cos(\gamma) \\
0 & \cos(\gamma) & \cos(\beta)\sin(\gamma) \\
1 & 0 & -\sin(\beta)
\end{bmatrix} \begin{bmatrix}
\dot{\gamma} \\
\dot{\beta} \\
\dot{\alpha}
\end{bmatrix}$$

(4.29)

$$\omega_1 = [R_1^0]^T \Omega_1$$

(4.30)

where $R_1^0$ is the rotation matrix of Equation 4.18. Once the position kinematic analysis has been completed at a given location on the path, the matrices of Equations 4.29 and 4.30 can be computed. For the purpose of rate analysis, the angular rates $\dot{\beta}$, and $\dot{\gamma}$ are assumed to be known since they are free variables. Therefore, Equations 4.29 and 4.30 can be used to obtain the following linear relations ($k_i$ are position dependent constants that can be computed):

$$\omega_{x1} = k_1 \dot{\alpha} + k_2$$

$$\omega_{y1} = k_3 \dot{\alpha} + k_4$$

(4.31)

$$\omega_{z1} = k_5 \dot{\alpha} + k_6$$

Let the angular rates of the right and the left wheels of the master module be $\dot{\alpha}_R$ and $\dot{\alpha}_L$ respectively. These rates are measured about the positive direction of the axes $y_1$. The vehicle geometric parameters are as shown in Figure 4.4. Assume that the contact
angles (as defined in Figure 4.9) of the two wheel-terrain contact points are $\lambda_{R1}$ and $\lambda_{L1}$. These angles are available at the end of the position kinematic analysis. Let the velocity vectors of the left and the right contact points be $v_{CL1}$ and $v_{CR1}$. Using loop equations, these velocities can be written as follows:

\[ v_{CL1} = \begin{bmatrix} v_{x1} - a\omega_{z1} - r\sin(\lambda_{L1})(\omega_{y1} + \dot{\theta}_{L1}) \\ r\omega_{z1}\cos(\lambda_{L1}) + r\omega_{x1}\sin(\lambda_{L1}) \\ v_{z1} + a\omega_{x1} - r\cos(\lambda_{L1})(\omega_{y1} + \dot{\theta}_{L1}) \end{bmatrix} \]

\[ v_{CR1} = \begin{bmatrix} v_{x1} + a\omega_{z1} - r\sin(\lambda_{R1})(\omega_{y1} + \dot{\theta}_{R1}) \\ r\omega_{z1}\cos(\lambda_{R1}) + r\omega_{x1}\sin(\lambda_{R1}) \\ v_{z1} - a\omega_{x1} - r\cos(\lambda_{R1})(\omega_{y1} + \dot{\theta}_{R1}) \end{bmatrix} \]

At the contact points, assuming rolling in the plane of the wheel, and assuming the wheel-terrain contact to be rigid, the following four equations are obtained.

\[ v_{CL1} \cdot i_1 = 0, \quad v_{CR1} \cdot i_1 = 0, \quad v_{CL1} \cdot j_1 = 0, \quad v_{CR1} \cdot j_1 = 0 \]  

(4.34)

where $i_1$ and $j_1$ are unit vectors along the $x_1$ and $y_1$ directions. Equations 4.32, 4.33 and 4.34 along with Equation 4.31 lead to four linear equations relating the four unknown rate quantities $\dot{x}$, $v_{x1}$, and the two wheel angular rates $\dot{\theta}_{R1}$ and $\dot{\theta}_{L1}$. The solution of these four equations completes the velocity analysis of the master module.

### 4.3.3.2 Velocity Analysis of the Subsequent Modules

In this section, the velocity analysis of a following module is presented. It is assumed that the position analysis of the vehicle has been completed and the rate analyses of all the preceding modules have also been completed. The unknown rate quantities of the $i^{th}$ module ($i$-1$^{th}$ following module) are $\dot{\theta}_{Ri}$, $\dot{\theta}_{Li}$, $\psi_{i-1}$, $\dot{\theta}_{i-1}$ and $\dot{\phi}_{i-1}$. There are four
constraints, similar to the ones of Equation 4.34, on these rate quantities and these constraints are listed below:

\[ \mathbf{v}_{CI} \cdot i = 0, \quad \mathbf{v}_{CR} \cdot i = 0, \quad \mathbf{v}_{CI} \cdot j = 0, \quad \mathbf{v}_{CR} \cdot j = 0 \]  

(4.35)

Further, the constraint that the follower module should have zero lateral velocity leads to

\[ v_{yi} = 0 \]  

(4.36)

Equations 4.35 and 4.36 lead to the following five linear equations in five unknowns:

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{25} \\
A_{21} & A_{22} & A_{33} & A_{24} & A_{25} \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \\
\end{bmatrix}
\begin{bmatrix}
\Psi_{i-1} \\
\theta_{i-1} \\
\dot{\theta}_{i-1} \\
\dot{\psi}_{Li} \\
\dot{\psi}_{Ri} \\
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
B_4 \\
B_5 \\
\end{bmatrix}
\]  

(4.37)

where

\[ A_{11} = q_{11} - a_{31} \cdot r \sin(\lambda_{L2}) \cdot n_{21} \]
\[ A_{21} = q_{31} + a_{11} \cdot r \cos(\lambda_{L2}) \cdot n_{21} \]
\[ A_{31} = q_{11} + a_{31} \cdot r \sin(\lambda_{R2}) \cdot n_{21} \]
\[ A_{41} = q_{31} - a_{11} \cdot r \cos(\lambda_{R2}) \cdot n_{21} \]
\[ A_{51} = q_{21} \]
\[ A_{12} = q_{12} - a_{32} \cdot r \sin(\lambda_{L2}) \cdot n_{22} \]
\[ A_{22} = q_{32} + a_{12} \cdot r \cos(\lambda_{L2}) \cdot n_{22} \]
\[ A_{32} = q_{12} + a_{32} \cdot r \sin(\lambda_{R2}) \cdot n_{22} \]
\[ A_{42} = q_{32} - a_{12} \cdot r \cos(\lambda_{R2}) \cdot n_{22} \]
\[ A_{52} = q_{22} \]
\[ A_{13} = q_{13} - a_{33} \cdot r \sin(\lambda_{L2}) \cdot n_{23} \]
\[ A_{23} = q_{33} + a_{n13} - r \cos(\lambda_{L2}) n_{23} \]
\[ A_{33} = q_{13} + a_{n33} - r \sin(\lambda_{R2}) n_{23} \]
\[ A_{43} = q_{33} - a_{n13} - r \cos(\lambda_{R2}) n_{23} \]
\[ A_{53} = q_{23} \]
\[ A_{14} = -r \sin(\lambda_{L2}) \]
\[ A_{24} = -r \cos(\lambda_{L2}) \]
\[ A_{34} = 0.0 \]
\[ A_{44} = 0.0 \]
\[ A_{54} = 0.0 \]
\[ A_{15} = 0.0 \]
\[ A_{25} = 0.0 \]
\[ A_{35} = -r \sin(\lambda_{R2}) \]
\[ A_{45} = -r \cos(\lambda_{R2}) \]
\[ A_{55} = 0.0 \]

\[ B_1 = q_1 - a_{n3} - r \sin(\lambda_{L2}) n_2 \]
\[ B_2 = q_3 + a_{n1} - r \cos(\lambda_{L2}) n_2 \]
\[ B_3 = q_1 + a_{n3} - r \sin(\lambda_{R2}) n_2 \]
\[ B_4 = q_3 - a_{n1} - r \cos(\lambda_{R2}) n_2 \]
\[ B_5 = -q_2 \]

In the above expressions for \( A_{kl} \) and \( B_k \),

\[
\begin{bmatrix} n_{kl} \end{bmatrix} = (R_{j}^{i-1})^{T} \begin{bmatrix} 0 & 0 & -\sin(\theta_{i-1}) \\ 0 & -\sin(\psi_{i-1}) & \cos(\psi_{i-1})\cos(\theta_{i-1}) \\ 1 & \cos(\psi_{i-1}) & \sin(\psi_{i-1})\cos(\theta_{i-1}) \end{bmatrix}
\]
\[
[n_k] = (R_{i}^{i-1})^{T} \omega_1
\]

\[
[q_{kl}] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -b_2 \\
1 & b_2 & 0
\end{bmatrix} [n_{kl}]
\]

In the above expressions, \( v_{Ai-1,i} \) is the velocity of the \( i-1 \)th articulation point in the \( i \)th frame of reference and \( R_{i}^{i-1} \) is the rotation matrix of Equation 4.23. The solution of the five equations in Equation 4.37 completes the velocity analysis of the \( i \)th module.

### 4.3.3.3 Nonholonomic Motion Planning

It is assumed here that a reference value of the linear velocity of the first module in frame 1, \( v_{x1} \), is known. The free variable \( \dot{\beta} \) is obtained as

\[
\dot{\beta} = \frac{(\beta_{I+1} - \beta_I)}{T}
\]

where \( T \) is the time between two adjacent vehicle positions on the path and \( I \) represents the \( I \)th point on the path. Similarly, the variable \( \dot{\gamma} \) is obtained from

\[
\dot{\gamma} = \frac{(\gamma_{I+1} - \gamma_I)}{T}
\]

The three free variables can be used to perform a velocity analysis for a given vehicle position as described in Sections 4.3.3.1 and 4.3.3.2. Next, the velocity of the first module in the world coordinates, \( v_0 \), can be obtained from

\[
v_0 = R_1^0 v_1
\]
The $x$ and $y$ components of $v_0$, $v_{x0}$ and $v_{y0}$ are obtained from Equation 4.40. The velocity analysis of the follower modules also yields $\dot{\theta}_{i-1}$. The desired angles $\beta$ and $\gamma$ can be obtained as discussed in Section 4.3.1. Therefore, all the accommodation coordinates $(x, y, \beta, \gamma, \theta_j)$ can be obtained by integrating the rate variables $\dot{\theta}_{i-1}$, $v_{x0}$ and $v_{y0}$. The integration of these quantities will suffer from drift errors due to the limited accuracy of numerical integration. If accurate nonholonomic motion planning is desired, the time step of the integration will have to be made small. This leads to high resolution of points on the path and hence significantly increases the computational burden. If numerical integration is used for long durations, error correction by using a known landmark as a reference may be required to minimize integration drifts.

In this work, nonholonomic motion planning was used to perform motion planning during turning maneuvers. An example of a turning maneuver is discussed here. A two module articulated vehicle is required to take a turn of $30^\circ$ (change in $\gamma = 30^\circ = 0.5236$ radians) when it is operating on a simulated, random, piece-wise planar terrain. Figure 4.17 shows the two module vehicle in the initial, an intermediate and the final positions as it performs a $30^\circ$ turn. Position histories of the relevant vehicle coordinates during the turning maneuver are shown in Figure 4.18. The variable $\alpha$ is affected by the random terrain variation. The angle $\gamma$ settles at $0.54543$ radians which is different from the desired value. This is due to integration drifts. The variable $\theta_1$ does not follow the angle $\gamma$ very well during the turning maneuver. This is typical of nonholonomic motion.

This concludes the discussion on motion planning strategies developed during the course of this work.
Figure 4.17 A Two Module Vehicle Performing a 30° Turning Maneuver
Figure 4.18 Variation of Kinematic Quantities During a 30° Turning Maneuver
4.4 Conclusions

The position kinematics of an actively articulated wheeled vehicle constrained to move on an uneven terrain has been studied in this work. It has been shown that the configuration of the resulting hybrid series-parallel kinematic chain can be obtained by solving polynomial systems of equations that represent the wheel-terrain contact constraints. The terrain geometry was assumed to be piece-wise planar. For a vehicle comprised of 'n' modules, it has been shown that the mobility of the vehicle is '3+n'. Knowing the values of these '3+n' motion coordinates, the '2n' accommodation coordinates have been solved using the '2n' wheel-terrain contact constraints. A polynomial system of degree eight has to be solved to obtain the configuration of the master module, and a polynomial system of degree thirty two has to be solved to obtain the configurations of each of the other modules. These polynomial systems are 'minimum order' systems with no solutions at infinity. Numerical issues involving the solution of these polynomial systems have been discussed. A simple procedure to obtain the 'correct' configuration of the vehicle on the terrain has been described. Finally a numerical example illustrating the above method has been solved.

The position kinematic solutions presented here have been used to study motion planning of multi-module actively articulated wheeled vehicles on uneven terrain. A reliable motion planning algorithm that generates motion histories of the vehicle on terrain paths has been developed. This planning algorithm has been interfaced with a graphical simulator on an SGI-R4000 workstation. Extensions of a basic straight line planning algorithm to obstacle climbing, ditch crossing and vehicle turning have also been studied.
In this work path planning algorithms based on other considerations such as minimum energy consumption have not been addressed, and this is a topic for future research. There are several situations in which it may be useful to have only some of a large number of modules of a vehicle on the terrain, while the others are lifted off the terrain. Position kinematics of such chains are more complicated, and they tend to depend on the particular control strategy used. In such situations, sub-optimal techniques may have to be investigated to obtain more efficient solutions. These are topics of future research. Obstacle climbing capabilities can be maximized by approaching the obstacle optimally. In this chapter, the approach and the terrain paths were chosen without any considerations of maximizing obstacle climbing capabilities. Therefore, the automated obstacle climbing algorithms developed in this work are preliminary and need improvement. The general problem of tracking arbitrary terrain paths, as closely as possible, is a challenging nonholonomic motion planning problem and it is also a topic of future research.
CHAPTER V

DYNAMIC SIMULATION

This chapter presents a dynamic simulator capable of performing dynamic analysis of actively articulated wheeled vehicle systems. The articulated wheeled vehicle is treated as a free-floating tree structure and its interaction with the terrain is modeled by treating the contact locations as three-dimensional spring/damper systems. The tangential and lateral contact force components are obtained by integrating the nonholonomic constraints at the contact locations. The phenomena of rolling and slipping have been accounted for in the simulator. The forward dynamics of the free-floating tree structure has been performed efficiently using the Articulated-Body Method. The simulator has been used to study nonholonomic vehicle motion on an arbitrarily oriented even terrain. This study involves the use of the simulator in conjunction with a preliminary control law. The simulator has also been used to study the articulated vehicle on uneven terrain. In particular, it has been used to obtain the vehicle configuration on uneven terrain in the presence of lumped compliance at the contact points. The purely kinematic approach that was presented earlier assumes rigid-body analysis, and the vehicle geometry obtained from that analysis can be used as a starting configuration in the dynamic analysis presented here.
5.1 Introduction

The force and motion planning algorithms described in Chapters 3 and 4 lead to coordination schemes for actively articulated wheeled vehicles. The outputs of these coordination schemes are the desired values of various force and motion states of the system. A suitable control law has to be incorporated in order to obtain these desired states (see Figure 1.14). Traditional linear control techniques may not be suited for the vehicle systems addressed in this dissertation. These wheeled vehicles are not capable of controlled motion in the lateral direction due to the presence of nonholonomic constraints. If the vehicle strays away from a desired trajectory on the terrain, the absence of omnidirectional motion capability may not allow for an asymptotic reduction of the error in the position of the vehicle. Control of such vehicles on even terrain has been addressed in the literature [Murray and Sastry, 1992; Gorinevsky et al., 1993]. However, the current state-of-the-art in nonholonomic motion control does not possess the ability to handle a general automated mobility maneuver, particularly on uneven terrain. Since linear systems theory cannot be easily extended to study these systems, and since it is very difficult to obtain any stability results for nonlinear control schemes, any control scheme that is attempted must be tested numerically or experimentally. Numerical dynamic simulations are required to identify suitable control schemes for such vehicles. These simulations are also required to extend experimental results obtained from a prototype to other vehicles that have similar configurations to the prototype. In general, experimental investigation is required to refine the parameters of dynamic simulators. This is necessary since the interaction of the vehicle with unstructured terrain is complex in nature and is not easily modeled. Models of soil strength and sinkage parameters, rolling resistance, traction etc. tend to be highly empirical in nature [McTamany, 1989].
In this chapter, a fully dynamic simulation scheme is developed for articulated wheeled vehicles. This simulator models the interaction between the wheels and the terrain as three-dimensional spring/damper systems. In this study, the spring/damper systems are assumed to be linear and the stiffnesses of the springs and the damping coefficients are chosen based on typical tire compliance and losses expected in the system. In order to obtain a more realistic simulation, it is necessary to develop sophisticated models for the interaction of the tire and the terrain. This may be done by considering nonlinear spring and damper elements. The parameters of these springs and dampers may have to be tuned by comparing the results of the simulation to experimental data.

Dynamic simulation of vehicles that are composed of several modules and that possess rolling/slipping contact points on uneven terrain is a difficult problem. The difficulties include the presence of multiple closed-loop kinematic chains and the need to compute the contact forces at the wheel-terrain contact locations. Freeman and Orin [1991] used the Decoupled Tree-Structure (DTS) approach to perform a full dynamic simulation of a legged system. In this approach, the closed-loop chains are eliminated by modeling the leg-terrain contacts as spring/damper systems. The forward dynamics of the resulting tree-structure is performed by using the efficient Articulated-Body method developed by Featherstone [1987]. This DTS approach has been used in this chapter to develop the dynamic simulator for the articulated vehicle system. The lumped compliances and damping elements at the wheel-terrain contact locations can be assumed to include the effects of tire compliances, compliances in drive trains, losses due to wheel-soil interaction etc. The spring/damper parameters may have to be set from experimental investigation or from simulation studies such as the one discussed in Chapter 2. Freeman and Orin did not have to deal with rolling elements in their investigation of legged systems. The computation of contact forces in wheeled systems is more involved than that of legged
systems. The method used in this chapter is an extension of the scheme used for planar power grasps by Mirza [1992]. In the case of an object performing planar motion (rolling and slipping), there are two contact forces that have to be estimated. If the contact locations are modeled as two-dimensional spring/damper systems, the computation of the tangential force component is more involved than that of the normal force component since the former requires an integration of the rolling velocity constraint. However, in the case of planar motion of a wheel on a plane terrain, rolling constraints are holonomic. This simplifies the computation of the tangential force component significantly. In the case of spatial motion of a wheel on a plane terrain (similar to a compliant disc moving on a plane), the rolling constraints are nonholonomic and these constraints have to be numerically integrated to obtain the tangential and lateral contact force components. Therefore, in this work, the contact force computation is performed differently from the approach described by Mirza [1992].

The dynamic simulator that has been developed here has been tested by performing two different studies: (i) The simulator has been used to study nonholonomic vehicle motion on an arbitrarily oriented even terrain. This study involves the use of the simulator in conjunction with a coordination scheme that investigates straight line and turning maneuvers. (ii) The simulator has also been used to study the articulated vehicle configuration on uneven terrain. It has been used to obtain the vehicle configuration on uneven terrain in the presence of lumped compliances at the wheel-terrain contact points.

In this chapter, Section 5.2 describes the DTS approach applied to the actively articulated wheeled vehicle configuration of Figure 4.1. In this section, the contact forces are assumed to be available, and the forward dynamics problem (computation of system accelerations, velocities and positions, given the initial system state and the input actuator torques) is solved. The wheel-terrain contact model that is required to compute the contact
force components is described in Section 5.3. A preliminary coordination scheme that has been used for the purpose of this study is discussed in Section 5.4. The two numerical studies listed above are presented in Section 5.5. Section 5.6 concludes this chapter.

5.2 Forward Dynamics

An actively articulated wheeled vehicle configuration is shown in Figure 5.1. For the purpose of this study, a two module version of this articulated vehicle will be considered. It can be seen from the following discussion that this dynamic analysis can be readily extended to a larger number of modules. A two module system is sufficient to study motion characteristics of actively articulated wheeled vehicle systems. In Figure 5.1, the first two modules of the vehicle are composed of six rigid bodies (the two module bodies and the four wheels). In the DTS approach this system is modeled as a free-floating tree structure interacting with the terrain. The terrain interaction is represented by the wheel-terrain contact forces. The tree structure has thirteen degrees of freedom (six degrees of freedom of the floating master module body, three degrees of freedom of the articulation $A_1$, and one degree of freedom each of the four wheels). The forward dynamics algorithm is an essential part of the process of dynamic simulation of multi-body systems. The computational complexity of the forward dynamics algorithm for a general tree structure using typical forward dynamics schemes such as the ones described by Walker and Orin [1981] is $O(n^3)$ or $O(n^2)$, where $n$ is the number of degrees of freedom of the system. The $O(n^3)$ method requires the inversion of the $n \times n$ inertia matrix. The method that has been used here does not require this inversion, and it is an $O(n)$ method that was devised by Featherstone [1983]. This method is especially efficient for tree structure systems that have $n \geq 9$ [Featherstone, 1987]. In the case of articulated vehicles
such as the one shown in Figure 5.1, $n \geq 13$. Hence, the $O(n)$ method has been used in this study.

A complete survey of various forward dynamic algorithms is available in the literature [Jain, 1991; McMillan, 1994]. The literature has addressed forward dynamics of serial chain robots that have a fixed base extensively. Efficient dynamic simulation of systems that possess closed chains has been addressed by Lilly and Orin [1994]. Another approach that has been used to obtain dynamic models of serial, parallel, and hybrid rigid kinematic chains is the method of dynamic influence coefficients [Thomas and Tesar, 1982; Freeman and Tesar, 1982; Sklar and Tesar, 1988]. The algorithm presented in this section has been developed by modifying the algorithms of Featherstone [1987]. The algorithm has to be modified in order to handle a free-floating tree structure. The presence of multi-degree-of-freedom joints at the articulations also requires certain changes in the forward dynamics algorithm. In this study, an attempt has been made to use algorithms that lead to
numerically efficient schemes. However, the emphasis of this work is not towards numerical efficiency; it is towards developing a robust dynamic simulator that models the motion of actively articulated wheeled vehicles.

Featherstone’s Articulated Body method uses the spatial vector notation. This is a more natural notation for rigid bodies since they possess six degrees of freedom in space. The kinematic quantities of a rigid body are represented as 6×1 spatial vectors. The velocity state of a rigid body is expressed as \( \dot{\mathbf{v}} = [\mathbf{\omega}^T, \mathbf{v}_0^T]^T \), where \( \mathbf{\omega} \) is the 3×1 angular velocity vector, and \( \mathbf{v}_0 \) is the 3×1 linear velocity vector denoting the velocity of the point on the rigid body that is instantaneously coincident with the origin of the coordinate frame. Similarly, a force system acting on a rigid body can be represented as a 6×1 vector \( \dot{\mathbf{f}} = [\mathbf{f}^T, \mathbf{m}_0^T]^T \), where \( \mathbf{f} \) is the 3×1 resultant force vector, and \( \mathbf{m}_0 \) is the 3×1 vector denoting the resultant moment due to all the applied forces and moments about the origin of the coordinate frame. The equations of motions of a rigid body are expressed as follows:

\[
\dot{\mathbf{f}} = \mathbf{I} \ddot{\mathbf{a}} + \dot{\mathbf{v}} \times \mathbf{I} \dot{\mathbf{v}}
\]  

(5.1)

Here \( \ddot{\mathbf{a}} \) is the spatial acceleration of the rigid body, \( \mathbf{I} \) is the 6×6 spatial rigid body inertia matrix, and \( \times \) is the spatial cross operator defined by

\[
\begin{bmatrix}
\mathbf{a} \\
\mathbf{b}
\end{bmatrix} \times =
\begin{bmatrix}
\mathbf{a} \times& 0 \\
\mathbf{b} \times& \mathbf{a} \times
\end{bmatrix}
\]  

(5.2)

where \( \mathbf{a} \times =
\begin{bmatrix}
0 & -a_z & a_y \\
a_z & 0 & -a_x \\n-a_y & a_x & 0
\end{bmatrix}
\)

The location of a joint axis is defined by a spatial vector called a screw axis. The screw axis for a revolute joint is known as a unit line vector. A unit line vector is a 6×1 vector \( \mathbf{c} = [\mathbf{c}^T, \mathbf{c}_0^T]^T \), where \( \mathbf{c} \) is the unit vector directed along the axis direction, \( \mathbf{c}_0 = \mathbf{a} \times \mathbf{c} \)
and a is the position vector of some point on the revolute joint axis. The screw axis for a
prismatic joint is known as a unit free vector. A unit free vector \( \hat{d} = [0^T, d^T]^T \), where \( d \) is
the unit vector directed along the prismatic axis direction. Multiple degree of freedom joints
are represented by \( 6\times m \) matrices where \( m \) is the number of degrees of freedom of the joint.
A detailed discussion of the spatial vector notation is available in the literature
[Featherstone, 1987]. Freeman and Orin [1989] have used a somewhat different spatial
notation wherein the \( 6\times 1 \) force vector is composed of the moment vector followed by the
force vector. This notation does not require separate definitions for spatial cross and dot
products, and it leads to some simplifications in the algorithm.

A collection of rigid bodies connected by active or passive joints is called an
articulated body. An articulated-body inertia is defined by picking a particular member (say
member 'i') of the articulated body called the handle, and by relating an applied force \( \hat{f} \) to
the handle and the resulting acceleration \( \hat{a} \) of the handle according to the following
relationship:

\[
\hat{f} = \hat{I}_i^A \hat{a} + \hat{p}_i
\]  

\( (5.3) \)

where \( \hat{I}_i^A \) is a \( 6\times 6 \) matrix that represents the articulated-body inertia of the handle, and \( \hat{p}_i \)
is called the bias spatial vector of the handle. The articulated-body dynamics algorithm is
derived from the basic linear force-acceleration relationship of Equation 5.3. It consists of
three steps:

1. An outward recursion from the base rigid body to compute the velocities of all the rigid
   bodies involved.

2. An inward recursion from the outer most members to the base to compute all the
   articulated-body inertias and bias vectors (\( \hat{I}_i^A \) and \( \hat{p}_i \)).
3. An outward recursion to compute all the accelerations of the system.

In this chapter all the $6 \times 1$ spatial vectors are denoted by lower case bold letters with a 'hat' (^). All the $3 \times 1$ vectors are denoted by lower case bold letters, $6 \times 6$ spatial matrices are denoted by upper case bold letters with a 'hat' (^), and all the $3 \times 3$ matrices are denoted by upper case bold letters.

The above method as applied to the articulated wheeled vehicle system is described below.

5.2.1 Outward Recursion to Perform Velocity Analysis

Let the six rigid bodies of the two module version of the actively articulated vehicle be numbered as shown in Figure 5.2. Rigid body '1' is assumed to be the floating base. The articulated-body forward dynamics analysis is performed here in the coordinate frame '1' (see Figure 5.1). The coordinate frames '1' and '2' are attached to the bodies numbered '1' and '4' respectively. The coordinate frames '1' and '2' are shown in Figure 5.1, and these frames have their origins at the centers of mass of the two modules. The numbering scheme of the six rigid bodies is illustrated in Figure 5.2. Unless otherwise mentioned, all the vectors and spatial vectors listed below belong to frame '1'.

Let $\mathbf{s}_i$ be the screw axis of the revolute joint connecting body 'i' to its preceding member. Then,

$$\mathbf{\hat{s}}_2 = \mathbf{\hat{s}}_3 = [\mathbf{J}_1^T, (\mathbf{d}_1 \times \mathbf{j}_1)^T]^T$$

(5.4)

$$\mathbf{\hat{s}}_5 = \mathbf{\hat{s}}_6 = \mathbf{X}_1^2 [\mathbf{J}_2^T, (\mathbf{d}_2 \times \mathbf{j}_2)^T]^T$$

(5.5)

where $\mathbf{j}_i$ is the unit vector along the axle of module 'i' in frame 'i', and $\mathbf{d}_i$ is the position
vector of any point on the axle of module 'i' in frame 'i'. The 6×6 matrix $X_i^2$ transforms spatial vectors from frame '2' to frame '1'.

$$X_i^2 = \begin{bmatrix} R_i^2 & 0 \\ (i_1^2 \times) R_i^2 & R_i^2 \end{bmatrix} \quad (5.6)$$

where $R_i^2$ is the rotation matrix transforming vectors from frame '2' to frame '1' and $i_1^2$ is the position vector of the origin of frame '2' in frame '1'. If the roll, yaw and pitch degrees of freedom of the articulation are assumed to be $\psi$, $\theta$ and $\phi$ respectively, then

$$R_i^2 = \begin{bmatrix} c\theta c\phi & -s\phi & c\theta s\phi \\ (c\psi s\theta c\phi + s\psi s\phi) & c\psi c\theta & (c\psi s\theta s\phi - s\psi c\phi) \\ (s\psi s\theta c\phi - c\psi s\phi) & s\psi c\theta & (s\psi s\theta s\phi + c\psi c\phi) \end{bmatrix} \quad (5.7)$$

The geometric parameters of the vehicle are as shown in Figure 4.4. The articulation is a three degree of freedom joint. Therefore, its screw axes are represented by a 6×3 screw matrix $\hat{S}_4$. It can be shown that
where \( \mathbf{a} = [a_x, a_y, a_z]^T \) is the position vector of the concurrency point of the articulation in frame '1'. If \( \hat{\mathbf{v}}_1 \) is the spatial velocity of body 'i' in frame '1', then the outward velocity recursion leads to

\[
\begin{align*}
\hat{\mathbf{v}}_2 &= \hat{\mathbf{v}}_1 + \hat{\mathbf{s}}_2 \hat{\theta}_{L1} \\
\hat{\mathbf{v}}_3 &= \hat{\mathbf{v}}_1 + \hat{\mathbf{s}}_3 \hat{\theta}_{R1} \\
\hat{\mathbf{v}}_4 &= \hat{\mathbf{v}}_1 + \hat{\mathbf{s}}_4 \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \\
\hat{\mathbf{v}}_5 &= \hat{\mathbf{v}}_4 + \hat{\mathbf{s}}_5 \hat{\theta}_{L2} \\
\hat{\mathbf{v}}_6 &= \hat{\mathbf{v}}_4 + \hat{\mathbf{s}}_6 \hat{\theta}_{R2}
\end{align*}
\]  

where \( \hat{\theta}_{Li} \) and \( \hat{\theta}_{Ri} \) are the relative angular velocities of the left and the right wheel of the \( i^{th} \) module respectively.

### 5.2.2 Inward Recursion to Compute Articulated-Body Inertias and Bias Vectors

For a serial kinematic chain the inward recursive relations that are required to compute articulated-body inertias, \( \hat{\mathbf{I}}^A_i \), and bias vectors, \( \hat{\mathbf{p}}_i \), are the following
[Featherstone, 1987]:

\[
\hat{\mathbf{p}}_i = \hat{\mathbf{p}}_i^y + \hat{\mathbf{p}}_{i+1} + \left( \hat{\mathbf{I}}_i^A \hat{\mathbf{v}}_{i+1} \times \hat{s}_{i+1} \hat{\mathbf{\dot{q}}}_{i+1} \right) + \\
\frac{\hat{\mathbf{I}}_{i+1}^A \hat{s}_{i+1} (Q_{i+1} - \hat{s}_{i+1}^S (\hat{\mathbf{I}}_{i+1}^A \hat{\mathbf{v}}_{i+1} \times \hat{s}_{i+1} \hat{\mathbf{\dot{q}}}_{i+1} + \hat{\mathbf{p}}_{i+1})))}{\hat{s}_{i+1}^S \hat{\mathbf{\dot{s}}}_{i+1}^S}
\]

(5.11)

where \( \dot{q}_i \) and \( Q_i \) are the angular rate and the joint torque about the screw axis \( \hat{s}_i \), respectively. The spatial vector \( \hat{s}_{i}^S \) represents the spatial transpose of \( \hat{s}_i \). The spatial transpose of \( \hat{a} \) is given by \( \hat{a}^S = \begin{bmatrix} a^T \\ a_0 \end{bmatrix} \). The spatial vector \( \hat{\mathbf{p}}_i^y \) is the bias force that includes the effects of the velocity cross product (see Equation 5.1) and any external forces acting on body 'i'. The contact forces that act on the wheels have to be included in \( \hat{\mathbf{p}}_i^y \) \((i = 2, 3, 5 \text{ and } 6)\). The \( 6 \times 6 \) inertia matrix \( \hat{\mathbf{I}}_i \) is the spatial inertia tensor that appears in the equations of motions of the rigid body 'i' (see equation 5.1).

\[
\hat{\mathbf{I}}_i = \begin{bmatrix} \mathbf{H}_i^T & \mathbf{M}_i \\ \mathbf{I}_i^+ & \mathbf{H}_i \end{bmatrix}
\]

(5.12)

Here the matrix \( \mathbf{M}_i = m_i \mathbf{I}_3 \) (\( m_i \) is the mass of body 'i' and \( \mathbf{I}_3 \) is the \( 3 \times 3 \) identity matrix), \( \mathbf{H}_i = (\mathbf{p}_i \times) \mathbf{M}_i \) (\( \mathbf{p}_i \) is the position vector of the center of mass of body 'i' in frame '1'), and \( \mathbf{I}_i^+ = \mathbf{I}_i^* + (\mathbf{p}_i \times) \mathbf{M}_i (\mathbf{p}_i \times)^T \). \( \mathbf{I}_i^* = \mathbf{I}_i \) for \( i = 1, 2 \) and 3 and \( \mathbf{I}_i^* = \mathbf{R}_i^2 \mathbf{I}_i \mathbf{R}_i^2 \) for \( i = 4, 5 \) and 6. \( \mathbf{I}_i \) is the \( 3 \times 3 \) inertia matrix of body 'i' about a local frame, parallel to the module frame, with its origin at the center of mass of the body.

The inward recursive expressions in Equations 5.10 and 5.11 have to be modified to account for the branches present in the kinematic structure of the articulated vehicle system. The articulated-body inertias and the bias vectors of body 'i' are dependent upon
these quantities of all the bodies outward of body 'i' in the branched kinematic chain. Therefore, for the six bodies of the articulated vehicle, the articulated-body inertias are obtained as follows:

\[ \mathbf{I}_i^A = \hat{\mathbf{I}}_i \quad (\text{for } i = 2, 3, 5 \text{ and } 6) \]

Let, \( \hat{\mathbf{h}}_i = \hat{\mathbf{I}}_i^A \hat{\mathbf{s}}_i \), and \( d_i = \hat{s}_i^S \hat{\mathbf{h}}_i \) for \( i = 2, 3, 5 \) and \( 6 \).

\[ \hat{\mathbf{I}}_4^A = \hat{\mathbf{I}}_4 + \hat{\mathbf{I}}_5^A + \hat{\mathbf{I}}_6^A - \frac{\hat{h}_5 \hat{h}_5^S}{d_5} - \frac{\hat{h}_6 \hat{h}_6^S}{d_6} \]

The rotation matrix operation that must be performed on \( \mathbf{I}_i \) to obtain \( \mathbf{I}_i^* \) (for \( i = 4, 5 \) and \( 6 \)) is an numerically expensive operation. Similarly, computation of \( \mathbf{I}_i^* \) for \( i = 4, 5 \) and \( 6 \) is also inefficient. Therefore, an efficient implementation of the above equation can be performed as follows:

\[ \hat{\mathbf{I}}_{4,2}^A = \hat{\mathbf{I}}_{4,2} + \hat{\mathbf{I}}_{5,2}^A + \hat{\mathbf{I}}_{6,2}^A - \frac{\hat{h}_{5,2} \hat{h}_{5,2}^S}{d_5} - \frac{\hat{h}_{6,2} \hat{h}_{6,2}^S}{d_6} \quad (5.13) \]

where the subscript '2' denotes that the computation of the corresponding matrices and vectors is performed in the local coordinate frame '2'. Once \( \hat{\mathbf{I}}_{4,2}^A \) has been obtained, the articulated inertia matrix of body '4' in frame '1', \( \hat{\mathbf{I}}_4^A \), can be obtained as

\[ \hat{\mathbf{I}}_4^A = \mathbf{X}_2^T \hat{\mathbf{I}}_{4,2}^A \mathbf{X}_1 \quad (5.14) \]

where \( \mathbf{X}_2 \) is the 6x6 matrix that transforms spatial vectors from frame '1' to frame '2'. Unlike the rotation matrices, these 6x6 matrices are not orthogonal. Therefore,

\[ \mathbf{X}_2 \neq (\mathbf{X}_1^T) \]

Next, let \( \hat{\mathbf{H}}_4 = \hat{\mathbf{I}}_4^A \hat{\mathbf{s}}_4 \), and \( \mathbf{D}_4 = \hat{s}_4^S \hat{\mathbf{H}}_4 \). Then,
\[ \hat{I}_l^A = \hat{I}_l + \hat{I}_2^A + \hat{I}_3^A + \hat{I}_4^A - \frac{\hat{h}_2^T \hat{h}_2}{d_2} - \frac{\hat{h}_3^T \hat{h}_3}{d_3} - \hat{H}_4(D_4)^{-1} \hat{H}_4^S \]  

(5.15)

The bias vectors are now obtained by using a modified version of Equation 5.11. The effects of the contact force vectors have to be included into the bias vectors for the bodies 2, 3, 5 and 6. The spatial vectors of the four contact force vectors in frame 'l' are

\[ \hat{f}_{Li} = [f_{Li}^T, (c_{Li} \times f_{Li})^T] \]  

\[ \hat{f}_{Ri} = [f_{Ri}^T, (c_{Ri} \times f_{Ri})^T] \]  

(5.16)

where \( f_{Li} \) and \( f_{Ri} \) are the left and the right wheel contact force vectors of the \( i^{th} \) module in frame 'l', and \( c_{Li} \) and \( c_{Ri} \) are the position vectors of the left and the right wheel contact points of the \( i^{th} \) module in frame 'l'. Here, it is assumed that the contact points on the wheels are known. The computation of the location of the contact points is discussed along with the wheel-terrain interaction model in the next section. Let \( \hat{p}_i^y = \hat{v}_i \times \hat{I}_i \hat{v}_i \). Then,

\[ \hat{p}_2 = \hat{p}_2^y - \hat{f}_{L1}, \quad \hat{p}_3 = \hat{p}_3^y - \hat{f}_{R1}, \quad \hat{p}_5 = \hat{p}_5^y - \hat{f}_{L2}, \quad \hat{p}_6 = \hat{p}_6^y - \hat{f}_{R2} \]  

(5.17)

Let, \( \hat{c}_i = \hat{v}_i \times \hat{s}_i \hat{q}_i \) and \( u_i = Q_1 - \hat{h}_1^S \hat{c}_i - \hat{s}_i^S \hat{p}_i \) (\( i = 2, 3, 5 \) and \( 6 \)). Then,

\[ \hat{p}_4 = \hat{p}_4^y + \hat{p}_5 + \hat{p}_6 + \hat{I}_5^A \hat{c}_5 + \hat{I}_6^A \hat{c}_6 + \frac{u_5}{d_5} \hat{h}_5 + \frac{u_6}{d_6} \hat{h}_6 \]  

(5.18)

Next, let \( \hat{c}_4 = \hat{v}_4 \times \hat{s}_4 \), and \( u_4 = \hat{s}_4^S \hat{f}_4 \hat{Q}_4^S - \hat{H}_4^S \hat{c}_4 - \hat{s}_4^S \hat{p}_4 \). Then,

\[ \hat{p}_1 = \hat{p}_1^y + \hat{p}_2 + \hat{p}_3 + \hat{p}_4 + \hat{I}_2^A \hat{c}_2 + \hat{I}_3^A \hat{c}_3 + \hat{I}_4^A \hat{c}_4 + \frac{u_2}{d_2} \hat{h}_2 + \frac{u_3}{d_3} \hat{h}_3 + \hat{H}_4(D_4)^{-1} u_4 \]  

(5.19)

In the above expression for \( u_4 \), \( Q_4^A \) is a 3×1 vector of actuator torques of the articulation.
\( Q_{4}^{a} = [\tau_{r}, \tau_{y}, \tau_{p}]^T \) consists of the roll, yaw and pitch actuator torques. In the above expression for \( u_i \), \( Q_i \) is the torque applied to the wheel. \( Q_2 = \tau_{L1} \), \( Q_3 = \tau_{R1} \), \( Q_5 = \tau_{L2} \) and \( Q_6 = \tau_{R2} \). \( \hat{F}_4 \) is a 6×3 matrix that maps the actuator torques to the force system of the vehicle in frame 'l'. Since the actuators are revolute, they produce pure moments. Hence, \( \hat{F}_4 \) consists of three columns of free vectors as given below:

\[
\hat{F}_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & -s\theta \\
0 & -s\psi & c\psi c\theta \\
0 & c\psi & s\psi c\theta
\end{bmatrix}
\]  

(5.20)

This completes the inward recursion to compute the articulated-body inertias and the bias vectors.

5.2.3 Outward Recursion to Compute the Accelerations of the System

The master module (body 'l') is assumed to be a free-floating member. Therefore,

\[
\hat{I}_l^{\wedge} \hat{a}_l + \hat{p}_l = \hat{I}_l^{\wedge} \hat{g}_l
\]

(5.21)

where \( \hat{g}_l = [0^T, \hat{g}_l^T]^T \) and \( \hat{g}_l \) is a 3×1 vector that has the magnitude and direction of the acceleration due to gravity in frame 'l'. Equation 5.21 can be used to solve for \( \hat{a}_l \) by inverting the 6×6 matrix \( \hat{I}_l^{\wedge} \). This inversion is unavoidable in the presence of a floating base system.

\[
\hat{a}_l = (\hat{I}_l^{\wedge})^{-1}(-\hat{p}_l) + \hat{g}_l
\]

(5.22)

An outward recursion can now be performed to compute all the other system accelerations.
\[
\begin{align*}
\ddot{\theta}_{L1} &= \frac{u_2 - \hat{\mathbf{h}}^2 \hat{\mathbf{a}}_1}{d_2}, \quad \ddot{\theta}_{R1} = \frac{u_3 - \hat{\mathbf{h}}^2 \hat{\mathbf{a}}_1}{d_3} \\
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} &= (D_4)^{-1} \begin{bmatrix}
u_4 - \hat{\mathbf{h}}^2 \hat{\mathbf{a}}_1
\end{bmatrix} \\
\hat{\mathbf{a}}_4 &= \hat{\mathbf{a}}_1 + \hat{\mathbf{c}}_4 + \hat{\mathbf{S}}_4 \begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} + \hat{\mathbf{S}}_4 \begin{bmatrix}
\ddot{\psi} \\
\ddot{\theta} \\
\ddot{\phi}
\end{bmatrix}
\end{align*}
\]

Here, \( \hat{\mathbf{S}}_4 \) is obtained by taking the term-by-term derivative of \( \hat{\mathbf{S}}_4 \) (Equation 5.8). This derivative term is present only for the multi-degree-of-freedom articulation joint since the yaw and the pitch joints outboard of the roll joint have screw axes that are dependent on the angles of the articulation joint. Finally, the angular accelerations of the wheels of module '2' are obtained as
\[
\begin{align*}
\ddot{\theta}_{L2} &= \frac{u_5 - \hat{\mathbf{h}}^5 \hat{\mathbf{a}}_4}{d_5}, \quad \ddot{\theta}_{R2} = \frac{u_6 - \hat{\mathbf{h}}^5 \hat{\mathbf{a}}_4}{d_6}
\end{align*}
\]

The kinematic quantities of the main module, \( \hat{\mathbf{v}}_1 \) and \( \hat{\mathbf{a}}_1 \), cannot be directly integrated. The integration of the angular velocity components of the first module does not yield the orientation of the main module. Therefore, the following procedure is followed. The first module can be fixed in frame '0' (see Figure 5.1), if the position vector of \( \mathbf{o}_1 \) in '0' (\( \mathbf{o}_0 \)) and the rotation matrix transforming vectors from frame '1' to frame '0' (\( \mathbf{R}_0 \)) are known. Let \( \mathbf{e}_0 = [\mathbf{e}_{x1}, \mathbf{e}_{y1}, \mathbf{e}_{z1}]^T \), and let \( \mathbf{R}_0 \) be a function of the Euler angles \( \alpha, \beta, \) and \( \gamma \). These Euler angles are defined as follows: a rotation of \( \gamma \) (yaw) about \( Z_0 \) followed by a rotation of \( \beta \) (pitch) about the rotated \( Y_0 \) followed by a rotation of \( \alpha \) (roll) about the rotated \( X_0 \). The spatial velocity vector of body '1', \( \hat{\mathbf{v}}_1 = [\mathbf{\omega}_1^T \mathbf{v}_1^T]^T \). Here, \( \mathbf{\omega}_1 \) is the angular
velocity of body '1' in frame '1' and \( v_1 \) is the linear velocity of the origin of frame '1' \((o_1)\) in frame '1'. The spatial acceleration vector of body '1', \( \dot{\mathbf{a}}_1 = [\dot{\mathbf{a}}_1^T \quad \mathbf{a}_1^T]^T \). Here, \( \alpha_1 \) is the angular acceleration of body '1' in frame '1' and \( \mathbf{a}_1 \) is the linear acceleration of the origin of frame '1' \((o_1)\) in frame '1'. The six acceleration quantities that have to be computed for body '1' are \( \ddot{\mathbf{o}}_{x1}, \ddot{\mathbf{o}}_{y1}, \ddot{\mathbf{o}}_{z1}, \ddot{\alpha}, \ddot{\beta} \) and \( \ddot{\gamma} \). Let us assume that these quantities are all measured in the fixed frame '0'.

\[
\begin{bmatrix}
\ddot{\mathbf{o}}_{x1} \\
\ddot{\mathbf{o}}_{y1} \\
\ddot{\mathbf{o}}_{z1}
\end{bmatrix} = \mathbf{R}_0^1 \mathbf{a}_1 
\tag{5.27}
\]

Here, \( \mathbf{R}_0^1 = \begin{bmatrix}
c\beta \gamma & (-c\alpha s\gamma + s\alpha s\beta c\gamma) & (s\alpha s\gamma + c\alpha s\beta c\gamma) \\
c\beta s\gamma & (c\alpha c\gamma + s\alpha s\beta s\gamma) & (-s\alpha c\gamma + c\alpha s\beta s\gamma) \\
-s\beta & s\alpha c\beta & c\alpha c\beta
\end{bmatrix} \)

The angular velocity of body '1' in frame '0',

\[
\omega_0 = \mathbf{R}_0^1 \omega_1 = \mathbf{T} \begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix}
\tag{5.28}
\]

where \( \mathbf{T} = \begin{bmatrix}
0 & -s\gamma & c\beta c\gamma \\
0 & c\gamma & c\beta s\gamma \\
1 & 0 & -s\beta
\end{bmatrix} \)

\[
\alpha_0 = \mathbf{R}_0^1 \alpha_1 = \mathbf{T} \begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} + \dot{\mathbf{T}} \begin{bmatrix}
\alpha \\
\beta \\
\gamma
\end{bmatrix} 
\tag{5.28}
\]

Therefore,
The set $A = \{ \dot{\theta}_x, \dot{\theta}_y, \ddot{\theta}_1, \ddot{\theta}, \dddot{\theta}, \dot{\phi}_{L1}, \dot{\phi}_{R1}, \dot{\phi}_{L2}, \dot{\phi}_{R2}, \dot{\psi}, \ddot{\theta} \text{ and } \dddot{\phi} \}$ consists of the thirteen acceleration variables. These variables can be integrated twice to obtain the velocities and position quantities. This completes the discussion of the forward dynamics scheme using the articulated-body inertia method.

### 5.3 The Wheel-Terrain Contact Model

The wheel-terrain contact model that is required to compute the contact force components is described in this section. This section also describes the procedure to identify the location of the wheel-terrain contact point on the wheel circumference. Dynamic simulation of unconstrained open-loop kinematic chains can be performed by assuming rigid body models. However, in the presence of constrained motion that leads to closed-loop kinematic chains, rigid body models are not necessarily the best. Use of rigid body dynamic models in such systems leads to static indeterminacy. If this static indeterminacy is handled by merely including a Coulomb friction for the contact forces, one may encounter situations where the forward dynamics problem has no solution or has multiple solutions [Dupont, 1992]. Also, in systems such as actively coordinated vehicles and articulated multifingered hands, significant compliance is present in the system making rigid body modeling inadequate [Mirza, 1992]. For instance, in actively coordinated vehicles, compliance in the system exists due to the presence of drive trains, elastic structural elements, and soft tires. This compliance is often desirable, particularly when the
vehicle is operating on rough terrain, as observed in the frequency domain study of a vehicle traversing an uneven terrain in Chapter 2. Multiple compliant frictional contacts with respect to planar multi-fingered hands have been studied in the literature [Wang et al., 1992; Mirza, 1992]. Contact models for spatial motion of compliant rolling elements lead to nonholonomic constraints, and this contact model is described here.

In this section, the wheel-terrain contact locations are modeled with three-dimensional spring/damper units. The articulated wheeled vehicle system being considered in this study undergoes lateral slipping on uneven terrain, as shown in the kinematic mobility discussion in Section 4.2.2. Therefore, on uneven terrain, this vehicle motion always involves combined rolling and slipping effects and a realistic simulation should be able to include these effects. The compliant contact model described here can readily incorporate the effects of slipping and rolling.

The outputs of the forward dynamics algorithm are the acceleration quantities of the vehicle system, and successive integrations of these accelerations yields the velocity and position coordinates of the vehicle. Once the vehicle configuration is known, the wheel-terrain interaction is modeled as a three-dimensional compliant contact. The outputs of this model are the three components of the contact forces at each of the wheel-terrain contacts. For the sake of simplicity, it is assumed that the wheels are adequately represented by thin circular discs. A more accurate model will have to consider the wheel to be cylindrical or toroidal in shape. The rest of the analysis presented here can be readily modified to allow for more general wheel shapes. This analysis requires the identification of a spring/damper attachment point on the wheel ($C_w$), and a spring/damper attachment point on the terrain ($C_T$) (see Figure 5.3). The point $C_w$ is taken to be the point on the wheel that has penetrated deepest into the terrain. If a more general wheel shape is used during the modeling process, the only increase in complexity of the algorithm comes from the need to
use a somewhat more cumbersome scheme to compute the point \( C_w \). The computation of the point \( C_T \) is more involved as it requires the integration of nonholonomic constraints. The terrain is assumed to be piece-wise planar and this simplifies the analysis significantly. If the terrain equations are nonlinear, the computational complexity of both the points \( C_w \) and \( C_T \) increases dramatically. In general, one wheel can have multiple terrain contact points as is the case when a wheel encounters a step. Multiple contact points can be modeled by evaluating the number of planes of the piece-wise planar terrain that the wheel penetrates into. Multiple contacts points are discussed in greater detail later in this section.

Once the two points \( C_w \) and \( C_T \) have been evaluated, the contact forces are computed by using the three-dimensional spring/damper model of Figure 5.4. Let \( s_w = \)
[\mathbf{s}_{xw}, \mathbf{s}_{yw}, \mathbf{s}_{zw}]^T$ be equal to $(\mathbf{c}_T - \mathbf{c}_W)$. The vector $\mathbf{s}_w$ is the vector directed from $\mathbf{C}_W$ to $\mathbf{C}_T$ in the wheel frame 'w'. The wheel frame can be assumed to have its origin coincident with the origin of the fixed frame. The axis $z_w$ is normal to the terrain plane and is directed 'up', the axis $x_w$ is parallel to the line of intersection of the wheel plane and the terrain plane (see Figure 5.3), and the axis $y_w$ is chosen normal to $x_w$ and $z_w$ so that it forms a right handed coordinate system. After the computation of $\mathbf{C}_W$ is completed, the rate quantities of the vehicle system as obtained from the forward dynamics can be used to compute the velocity of the point $\mathbf{C}_W$. Let this velocity be $\mathbf{v}_w = [v_{xw}, v_{yw}, v_{zw}]^T$. Then, using the model of Figure 5.4, the contact force components are

\begin{align*}
F_{xw} &= \phi \left( K_{xw} \mathbf{s}_{xw} + B_{xw} \mathbf{v}_{xw} \right) \\
F_{yw} &= \phi \left( K_{yw} \mathbf{s}_{yw} + B_{yw} \mathbf{v}_{yw} \right) \\
F_{zw} &= \phi \left( K_{zw} \mathbf{s}_{zw} + B_{zw} \mathbf{v}_{zw} \right)
\end{align*} 

(5.30)

where $\phi = 1$ if the wheel intersects the terrain plane (or $s_{zw} \geq 0$), and $\phi = 0$ if the wheel is completely above the terrain plane (or $s_{zw} < 0$).

The next two subsections describe the procedure used to compute the spring/damper attachment point on the wheel ($\mathbf{C}_W$) and the vector $\mathbf{s}_w$ that represents the displacements of the three springs.

### 5.3.1 Spring/Damper Attachment Point on the Wheel ($\mathbf{C}_W$)

The spring/damper attachment point on the wheel $\mathbf{C}_W$ is taken to be the point on the wheel that has the deepest penetration into the terrain. Let us assume that all the computations in this section are performed in the world coordinate frame $X_0Y_0Z_0$ (see Figure 5.1). The outputs of the forward dynamics can be used to compute the wheel center
C (see Figure 5.3). Let $c = [c_x, c_y, c_z]^T$ be the position vector of point C in frame 'O'. Let the position vector of the point $C_w$ be $c_w = [c_{wx}, c_{wy}, c_{wz}]^T$. The procedure to compute $c_w$ involves the following steps:

1. Compute the equation of the plane of the wheel ($P_w = 0$) in frame 'O'. The normal to the plane is the vector parallel to the wheel axle (say $n$) and the point C lies on the plane. Therefore, $P_w = n^T [X_0, Y_0, Z_0]^T - n^T c = 0$. It is assumed that the equation of the terrain plane ($P_T = 0$) is known in frame 'O'. Obtain the line of intersection, say $L_1$, of these two planes and compute the direction numbers $L_1, M_1$ and $N_1$ of line $L_1$. Direction numbers of a line are any set of three numbers that are proportional to the direction cosines of that line [Spiegel, 1968].

2. Obtain the direction numbers, $L_2, M_2$ and $N_2$ of the line joining the points C and $C_w$ in terms of the known quantities $c_x, c_y$ and $c_z$, and the unknown quantities $c_{wx}, c_{wy}$ and $c_{wz}$. $L_2 = c_{wx} - c_x, M_2 = c_{wy} - c_y$ and $N_2 = c_{wz} - c_z$. 

Figure 5.4 A Three-Dimensional Spring/Damper Contact Model
3. Obtain the following three equations relating the three unknowns $c_{wx}, c_{wy}$ and $c_{wz}$:

   (i) The point $C_W$ lies on the plane of the wheel. Therefore, $P_W(c_w) = 0$. This equation is linear in the three variables.

   (ii) The point $C_W$ lies on the circumference of the wheel. Therefore, $\|c - c_w\|^2 = r^2$. This equation is a quadratic in the three variables.

   (iii) The line $L_1$ is orthogonal to the line joining the points $C$ and $C_W$. Therefore, $L_1L_2 + M_1M_2 + N_1N_2 = 0$. This equation is linear in the three variables.

   The two linear equations and the quadratic in the three unknowns $c_{wx}, c_{wy}$ and $c_{wz}$ can be solved to obtain two solutions for $c_w$. These two solutions correspond to the points $C_W$ and $C'_W$ (see Figure 5.3). The solution that has a lower value of $c_{wz}$ is taken to be the correct solution, as this corresponds to the lower point, $C'_W$. The next subsection describes the procedure to compute the displacements of the three springs (vector $s_w$).

### 5.3.2 The Displacements of the Three Springs (Vector $s_w$)

The displacement of the spring in the '$z_w$' direction normal to the terrain ($s_{zw}$) can be readily computed. It is given as the distance between the point $C_W$ and the plane $P_T$.

$$\text{distance } (C_W \text{ to plane } P_T) = s_{zw} \quad (5.31)$$

The computation of the other two components of the vector $s_w$ is more involved.

Let $s_w^c$ and $s_w^p$ be the current and the previous spring displacement vectors, and let $v_w^c$ and $v_w^p$ be the current and the previous velocity vectors of the spring attachment point on the wheel, $C_W$. The vectors $s_w^p$ and $v_w^p$ are the values of the displacement and the velocity vectors at a time that is one integration time step prior to the current time. Both these vectors are assumed to be expressed in the current wheel frame 'w'. It should be noted that the output of the forward dynamics algorithm yields kinematic quantities in the
world frame 'O'. An appropriate rotation transformation has to be performed to obtain these kinematic quantities in frame 'w'. The rotation matrix, $R^w_0$, that takes vectors from frame 'w' to frame 'O' is equal to $[n_{wx}, n_{wy}, n_{wz}]$, where $n_{wx}$, $n_{wy}$ and $n_{wz}$ are unit vectors measured in frame 'O' that are directed along the axes $x_w$, $y_w$ and $z_w$, respectively. Using the Euler's integration scheme,

$$
\begin{align*}
    s_{wx}^c &= s_{wx}^p + v_{wx}^p (\delta t) \\
    s_{yw}^c &= s_{yw}^p + v_{yw}^p (\delta t)
\end{align*}
$$

where $\delta t$ is the integration time step. Equations 5.31, 5.32 and 5.33 yields the three components of the spring displacement vector. However, the procedure is not yet complete since slipping adjustments have to be made.

$$
\begin{bmatrix}
    \phi
\end{bmatrix} = \phi \begin{bmatrix}
    [K_{wx} \ 0 \ 0] \\
    [0 \ K_{yw} \ 0] \\
    [0 \ 0 \ K_{zw}]
\end{bmatrix}
\begin{bmatrix}
    s_{wx}^c \\
    s_{yw}^c \\
    s_{zw}^c
\end{bmatrix} + \begin{bmatrix}
    B_{wx} \ 0 \ 0 \\
    0 \ B_{yw} \ 0 \\
    0 \ 0 \ B_{zw}
\end{bmatrix}
\begin{bmatrix}
    v_x^c \\
    v_y^c \\
    v_z^c
\end{bmatrix}
$$

Assuming that $\phi$ is non zero, the first slipping adjustments computation involves evaluating the ratio of the force tangential to the terrain to the force normal to the terrain. If this ratio is greater than the friction coefficient of the tire-terrain interface, say $\mu$, then slipping will occur at the contact point. Let the contact force ratio be $\rho$. Then,

$$
\rho = \{(F_{wx}^2 + F_{wy}^2)^{0.5}/F_{zw}\}
$$

Let the contact force vector after slipping adjustment at the current time be $F_w^c$. If $\rho > \mu$, then,
If \( p \leq \mu \), then,

\[
F_{xw}^c = F_{xw} \tag{5.38}
\]

\[
F_{yw}^c = F_{yw} \tag{5.39}
\]

If \( p \leq \mu \), then the current value of the displacement vector is as given by Equation 5.34. However, if slipping occurs \( (p > \mu) \), then the spring displacements have to be readjusted to account for the lower available tangential force component. Then,

\[
s_{xw}^c = (\mu F_{zw} - b_{xw} v_{xw}^c) / K_{xw} \tag{5.40}
\]

\[
s_{yw}^c = (\mu F_{zw} - b_{yw} v_{yw}^c) / K_{yw} \tag{5.41}
\]

The \( z_w \) component \( s_{zw} \) is unaffected by the slip. It should be noted that, according to this model, a slip along the \( y_w \) direction will in general involve a slip along the \( x_w \) direction also. When the component of contact force that is parallel to the terrain plane \( (F_t = (F_{xw}^2 + F_{yw}^2)^{0.5}) \) exceeds the maximum allowed by the coefficient of friction, both the components of the contact force vector, \( F_{xw} \) and \( F_{yw} \), are scaled down. This is a realistic characterization of the phenomenon of slipping since slipping is only dependent upon two components of the contact force, \( F_t \) and \( F_{xw} \), and it is independent of the choice of the coordinate frame.

When the articulated wheeled vehicle traverses an uneven terrain, it was mentioned earlier (Section 4.2.2) that it needs to undergo lateral slip (\( y_w \) direction) in order to have a kinematic mobility of 3. This means that the wheel-terrain contact point is going to exhibit slipping in the direction parallel to the wheel plane also (\( x_w \) direction) even though this
slipping velocity will in general be much lower than the lateral slipping velocity. Therefore, on uneven terrain, the dynamics of the vehicle makes it undergo motion that involves continuous slipping at the wheel-terrain contact points. This makes the control problem of such vehicles on uneven terrain particularly challenging.

5.3.3 Concave and Convex Terrain Edges

This section addresses the issue of a wheel encountering an edge on a piece-wise planar terrain. The presence of compliance at the contact locations makes it possible to obtain a smooth variation of contact forces when a wheel encounters an edge. A rigid body model of the wheel-terrain interface cannot handle the traversal over an edge as easily. The contact modeling has to be treated differently for concave and convex edges. An edge can be classified as a concave or a convex edge by the following simple analysis. An edge formed by two adjacent planes is shown in Figure 5.5. A point that lies near the center of each plane is identified and lines that are normal to the two planes that pass through these points are considered (see lines L and M in Figure 5.5). Let the common normal to these two lines be line N. This line is parallel to the edge E. If this line N lies below the edge E, then the edge is convex, and if the line N lies above the edge E then the edge is concave.

Once the edge has been classified, it has to be noted that a wheel that encounters a concave edge can have up to two contact points and a wheel that encounters a convex edge will, at a given time have only one contact point. For a concave edge, the procedure described in the previous sections is valid except that the intersections of the wheel with both the planes have to be considered. If the wheel penetrates both the planes, then the wheel will encounter contact forces at both the contact points. The rest of the analysis is exactly the same as before. If the edge is convex, it has only one contact point. The intersection of one of the two planes with the wheel leads to this contact point. The correct
plane can be chosen by following the procedure described here. Consider the two positions (1 and 2) of a wheel traversing an edge formed by planes P and Q (see Figure 5.6). The deepest penetration of the wheel into the two planes, normal to the planes, is identified for a given position of the wheel. The smaller of the two penetrations is the correct one and the corresponding contact point is the unique contact point of the wheel. Therefore, in position 1, the wheel contact point is obtained as the one that results from the intersection of the wheel and plane P, and in position 2, the wheel contact point is obtained as the one that results from the intersection of the wheel and plane Q. If the wheel is completely above even one of the two planes, there exists no contact between the wheel and the terrain and the contact force vector is taken to be zero.

The above procedure leads to a continuous variation of the magnitude of the normal contact force component as the wheel traverses the edge. However, the contact location on the wheel is discontinuous as the wheel goes over the edge. This discontinuity does not
Figure 5.6 Convex Edge Traversal

simulate the edge traversal of a compliant wheel in a realistic manner. However, for a terrain edge formed by two planes whose normals include a relatively small angle between them, the effect of this discontinuity is not expected to be significant. If the included angle is large, an interpolation scheme must be used to simulate the effect of the wheel 'pivoting' about the edge.

5.4 A Preliminary Coordination Algorithm

This section presents a preliminary coordination scheme that is used in conjunction with the forward dynamics algorithms to perform simulation studies. The outputs of this scheme are the seven actuator torques for the two module version of the vehicle, one torque each for each of the four wheels and three torques for the articulation. These torques form the input to the forward dynamics algorithm for the purpose of this simulation study. In Section 4.3.3, the rate variables $\dot{v}_x$, $\dot{\beta}$, and $\dot{\gamma}$ that decide the forward velocity, the pitch motion and the heading of the master module, were chosen as the free variables. A velocity analysis was performed in Section 4.3.3.1 to obtain the values of the other constrained rate variables based on rigid body analysis. Once the velocity analysis is complete, a similar
analysis can also be performed in the acceleration domain by starting with the three free acceleration variables $a_x$, $\dot{\beta}$, and $\ddot{\gamma}$. The values of the free rate and acceleration variables are chosen so as to follow a path on a terrain as closely as possible. This is again similar to the analysis in Section 4.3.3, and to the analysis in the work by Kumar and Waldron [1989]. At the end of the kinematic analysis, the desired values of all the rate and acceleration quantities of all the vehicle bodies are known. An inverse dynamic analysis can now be performed to obtain the desired force system in frame 'l' [Featherstone, 1987]. This force system $w$ is given by

$$w = \sum_{j=1}^{6} (\hat{I}_j \hat{a}_j + \hat{v}_j \times \hat{I}_j \hat{v}_j)$$  \hspace{1cm} (5.42)

where the spatial inertia, acceleration and velocity vectors are as defined in Section 5.2. The effects of gravity can be automatically included in $w$ as described in Section 5.2.3. This vector $w$ is identical to the vector $Q$ in Equation 3.1. The minimum-norm contact forces that will result in the force system $w$ can be obtained from Equation 3.2. Once the contact forces are known and the desired kinematics are available, the actuator torques can be computed as described below.

The net force acting on rigid body 'j', $\hat{f}_j^*$, is given by the spatial equation of motion as the rate of change of momentum of the rigid body.

$$\hat{f}_j^* = (\hat{I}_j \hat{a}_j + \hat{v}_j \times \hat{I}_j \hat{v}_j)$$  \hspace{1cm} (5.43)

Let the total spatial force transmitted from rigid bodies outward of body 'j' to body 'j' through the joint 'j' be $\hat{f}_j$. It is assumed that body 'l' is the base. Therefore, the following relations have to be satisfied:

$$\hat{f}_k = \hat{f}_k^* - \hat{f}_k^c$$  \hspace{1cm} (5.44)
where \( k = 2, 3, 5 \) and \( 6 \), and \( \hat{f}_k \) is the external contact force acting on the wheel represented by the rigid body 'k'. The recursive relations for bodies 'l' and '4' are:

\[
\hat{f}_4 = \hat{f}_5 + \hat{f}_6 + \hat{f}_4^* \quad (5.45)
\]

\[
\hat{f}_1 = \hat{f}_2 + \hat{f}_3 + \hat{f}_4 + \hat{f}_1^* . \quad (5.46)
\]

Once the spatial forces transmitted through all the joints are known, the scalar joint torques can be obtained by taking the spatial dot product of the screw axes of the joints with the spatial forces across the joints. The joint torques for the four wheel joints are given by the equation:

\[
Q_j = \hat{s}_j^T \hat{f}_j . \quad (5.47)
\]

The articulation is a multiple degree of freedom joint and has to be handled differently to the wheel joints (see Section 7.4.2 of [Featherstone, 1987]). The \( 3 \times 1 \) vector \( Q_A \), that represents the three joint torques of the articulation joint, is given by the equation:

\[
Q_A = (\hat{S}_4^T \hat{F}_4)^{-1} \hat{S}_4^T \hat{f}_4 \quad (5.48)
\]

where \( \hat{S}_4 \) and \( \hat{F}_4 \) are given by Equation 5.8 and 5.20 respectively. This completes the discussion of the coordination scheme. The coordination scheme described here has been used to perform two numerical simulation studies in the next section.

### 5.5 Simulation Studies

The dynamic simulator that has been developed here is tested in this section by performing two different studies: (i) The simulator has been used to study nonholonomic
vehicle motion on an arbitrarily oriented even terrain. This study involves the use of the simulator in conjunction with a coordination scheme that investigates straight line and turning maneuvers. (ii) The simulator has also been used to study the articulated vehicle configuration on uneven terrain. It has been used to obtain the vehicle configuration on uneven terrain in the presence of lumped compliance at the wheel-terrain contact points.

5.5.1 Arbitrarily Oriented Even Terrain

In this study, the two module actively articulated wheeled vehicle of Figure 5.2 was operated on an even terrain that is inclined at an angle of 10° with the horizontal plane as shown in Figure 5.7. The vehicle is initially moving with a constant velocity of 0.1 m/s and its heading makes an angle of 45° with the x axis of the fixed frame (X₀). The vehicle then turns to its left by an angle of 30° (making an angle of 75° with X₀) and follows it up with a turn back to its right of an angle of 30°. This simulation was executed for 31.0 seconds of real time. The integration scheme used was Euler's method, and the integration time step was taken equal to 0.001 second. The wheel-terrain interface stiffness was taken to be 25,000 N/m and the damping coefficient was assumed to be equal to 1,000 N-s/m.

Figure 5.7 Vehicle Simulation on an Arbitrary Even Terrain
The desired values of the rate variables $v_{x1}$ and $\dot{\gamma}$ are obtained so as to ensure that the vehicle maintains the desired speed of 0.1 m/s and performs the desired turns. The desired pitch rate $\dot{\beta}$ is set equal to zero.

Some of the outputs of the dynamic simulation are shown in Figure 5.8. The variation of the euler angles of the main module in frame 'O' with time is shown in the first plot of Figure 5.8. The variation of the position coordinates of the origin of the frame 'l' of the main module is shown in the second plot of Figure 5.8. The next three plots in Figure 5.8 show the variation of the articulation angles, the variation of the articulation torques, and the variation of the three components of the contact force vector at the left front contact point, respectively. The contact force components are computed in frame 'l' of the main module. It should be noted (last plot in Figure 5.8) that the trailing module does not follow the main module exactly. This is a typical characteristic of nonholonomic motion. Since the radius of curvature of the chosen desired path is very large, the trailing module follows the master module quite well. In the case of paths with smaller radii of curvature, the path following is not as good. Nonholonomic motion planning on even terrain is being investigated in greater detail by Rachkov [1994].

5.5.2 Vehicle Configuration in the Presence of Compliance on Uneven Terrain

In this section the dynamic simulator has been used to simulate the vehicle system on uneven terrain. The problem that has been addressed here is: Compute the configuration of the vehicle, assuming that it has lumped compliance elements at the wheel-terrain contact points, by performing a dynamic simulation of the system starting from an approximate solution. This approximate solution is the one obtained from the rigid body
Figure 5.8 Histories of Vehicle States During Nonholonomic Motion on an Arbitrarily Oriented Even Terrain. (Angles in Radians; Torque in N-m; Force in N; Position in m.)
position kinematic analysis described in Chapter 4. The example considered here is the same as the one considered in Section 4.2.8. The solutions obtained from that example are taken here as the initial conditions for the dynamic simulation. The free motion variables of the main module, namely the x and y coordinates of the origin of frame '1' of the main module, and the euler angles $\beta$ and $\gamma$ are maintained at the desired value by using the coordination scheme of Section 5.4. The other quantities are obtained by performing the dynamic simulation. Euler's integration technique was used during this simulation with an integration time step of 0.001 second. The wheel-terrain interface stiffness and the damping coefficients were again taken to be 25,000 N/m and 1,000 N-s/m respectively.

It was found in this simulation study that if the free position variables, namely the x and y coordinates of the origin of frame '1', are controlled in a closed loop manner to stay at the desired values the system goes unstable. This observation can be explained in the following manner. On uneven terrain, the vehicle becomes a structure if pure rolling contacts are assumed as discussed in Section 4.2.2. Therefore, even for the vehicle to undergo the slightest perturbation of the motion variables, pure rolling constraints have to be violated. This means that, at the beginning of the dynamic simulation, the transients in the system initiate slipping at the wheels. As was discussed in Section 5.3.2, the slipping is not going to be limited to the lateral direction at the wheel-terrain contact point. In general there will be some slip in the tangential direction (in the plane of the wheel) also. Once the slipping is initiated, the vehicle cannot be controlled in a closed loop manner due to loss of controllability. Therefore, the system will have to take its own course until it stabilizes to a locally stable operating point. Therefore, in this study the position of the main module was not controlled in a closed loop manner. The orientation of the main module was controlled in a closed loop manner without any problems. The eventual stable vehicle configuration is typically a small distance away from the initial state of the system.
It was also observed, as would be expected, that the amount of slip at the wheel-terrain contact points was higher for lower levels of the coefficient of friction at the contact locations. Up to a point, an increase in the coefficient of friction decreased the slip observed in the vehicle. However, very large and unrealistic values of friction coefficient (25.0 or greater) led to numerical instability in the simulation since the simulator could not handle the high frequency components generated during the transients. A different way of explaining this result is that, at very high friction coefficients, the vehicle is 'glued' onto the terrain. The slip levels are negligible. However, small perturbations in the motion variables leads to very large internal forces and hence contact forces. This leads to instability in the numerical scheme.

Results for three different values of the coefficient of friction, $\mu$, equal to a realistic value of 0.4, a somewhat high value of 0.7, and an unrealistic value of 10.0 are listed here. Figures 5.9, 5.10 and 5.11 include the plots of some of the outputs of the dynamic simulator for $\mu = 0.4$, 0.7 and 10.0 respectively. As is seen from these outputs, the position kinematic quantities of the vehicle stabilize rather quickly to their equilibrium values. It should be noted from the plots that the motion variables drift somewhat from their initial values. This is largely due to the presence of lateral slip velocities in the vehicle modules. From these simulations it can be seen that the configuration of the vehicle can be obtained efficiently in the presence of compliance, by starting with the position kinematic solutions obtained from a rigid body analysis, and by performing a dynamic simulation.

The starting values that were obtained from the rigid body kinematic solution procedure in Section 4.2.8 were as follows:

$$\alpha = 10^\circ, \beta = 20^\circ, \gamma = 55^\circ, \alpha_{x_1} = 3.0, \alpha_{y_1} = 5.0, \alpha_{z_1} = 1.0, \psi = -10^\circ, \text{ and } \theta = 15^\circ, \phi = -7^\circ$$
Figure 5.9 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 0.4). (Angles in Radians; Torque in N-m; Force in N; Position in m.)
Figure 5.10 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 0.7). (Angles in Radians; Torque in N-m; Force in N; Position in m.)
Figure 5.11 Histories of Vehicle States During Dynamic Simulation Used to Compute Vehicle Configuration in the Presence of Compliance on Uneven Terrain (Friction Coefficient = 10.0). (Angles in Radians; Torque in N-m; Force in N; Position in m.)
For the three values of the coefficient of friction, $\mu = 0.4, 0.7$ and 10.0, the dynamic simulation converged to the following position kinematic quantities:

**Case 1 ($\mu = 0.4$):**

$$\alpha = 16.74578^\circ, \beta = 20.000001^\circ, \gamma = 54.999995^\circ, o_{x1} = 3.47291, o_{y1} = 4.65532, o_{z1} = 0.96631, \psi = -12.53170^\circ, \theta = -27.99760^\circ, \phi = 2.58833^\circ$$

**Case 2 ($\mu = 0.7$):**

$$\alpha = 12.01479^\circ, \beta = 19.99981^\circ, \gamma = 55.00011^\circ, o_{x1} = 3.14083, o_{y1} = 4.92098, o_{z1} = 1.04422, \psi = -12.49374^\circ, \theta = 6.37362^\circ, \phi = -4.33568^\circ$$

**Case 3 ($\mu = 10.0$):**

$$\alpha = 10.67394^\circ, \beta = 20.00243^\circ, \gamma = 55.00229^\circ, o_{x1} = 3.04346, o_{y1} = 4.99989, o_{z1} = 1.0663619, \psi = -10.31113^\circ, \theta = 13.67031^\circ, \phi = -6.35040^\circ$$

The variation in the accommodation variables such as $\alpha$ and $\psi$ is partly due to the compliance in the system. The variation in the motion variables such as $o_{x1}, o_{y1}$ and $\theta$ are largely due to the presence of slip at the contacts. Variation in variables such as $o_{y1}$ could be, to some extent, due to compliance also since this coordinate will change somewhat depending on the value of the lateral contact force. However, in the first two cases listed above, the fact that $o_{x1}$ changed by about 47 cm and 14 cm respectively, and $o_{y1}$ changed by about 34 cm and 8 cm respectively clearly indicates the presence of slip. The amount of slip the system undergoes is quite high due to the fact that the terrain is very uneven. The motion of the wheeled system on uneven terrain requires a lateral slip velocity, and if the terrain is very uneven the lateral slip velocities are high. On the other hand, on even terrain
a lateral slip velocity will be present only if the lateral forces needed to maintain the vehicle system in equilibrium is higher than that allowed by the coefficient of friction.

A general uneven terrain can be viewed as a surface comprised of several small planes arranged together in a relatively random fashion. A wheeled system operating on such an uneven terrain will have the direction of the lateral velocity components switching often due to the variation of the plane orientations. This has a tendency to even out the lateral drifts in the system. The lateral drift that was observed in the above simulations is large since the planes below the wheels remain the same. Therefore, the system keeps drifting until it reaches a local equilibrium point.

In the case of $\mu = 10.0$, as expected, the slip levels are much lower. In all the three cases it should be noted that the values of $\beta$ and $\gamma$ are very close to the desired values of these free variables. This is because these values are not affected much by the slip at the contact points and therefore, as was the case during the course of these simulations, it is possible to control these quantities in a closed loop manner.

5.6 Conclusions

A dynamic simulator has been developed to simulate the motion of articulated wheeled vehicle configurations on even and uneven terrains. The simulation algorithm presented in this discussion can model various phenomena that are encountered during such vehicle motion. These phenomena include combined rolling and slipping at the wheel-terrain contact points, multiple contact points for each wheel and making and breaking of contact. This simulator incorporates an efficient forward dynamics algorithm that uses the Articulated-Inertia method and the contact forces are modeled by including a three-
dimensional spring/damper combination at each contact location. The simulator has been tested by performing two simulation studies. The first one investigates nonholonomic motion on even terrain, and the second one addresses the problem of solving for the vehicle configuration in the presence of compliance on uneven terrain.

In this simulation study, a preliminary wheel-terrain interaction model was used. In order to obtain a more realistic simulation, it is necessary to develop sophisticated models for the interaction of the tire and the terrain. A model that is capable of simulating the vehicle performance on terrains comprised of various kinds of rocks and soil is desirable. This is a topic for future research.

It is highly desirable that this simulator be extended to handle vehicle traversal on general random piece-wise planar terrains. Such a simulator can be used to study the designs of various articulated wheeled vehicles that possess varying levels of active actuation. Another topic for future research is to investigate nonlinear control techniques that can automate the mobility maneuvers of these vehicles in difficult terrain conditions.
CHAPTER VI

THE WHEELED ACTIVELY ARTICULATED VEHICLE (WAAV)

This chapter contains descriptions of the mechanical systems, the sensing, control and actuation sub-systems of the Wheeled Actively Articulated Vehicle. It also includes a discussion of the current capabilities of the WAAV system. Finally, system design enhancements that are being considered, and future experiments that are being planned are addressed.

6.1 Introduction

The Wheeled Actively Articulated Vehicle (WAAV) shown in Figure 6.1 is an experimental system that has been developed to demonstrate the use of active coordination in terrain adaptive wheeled vehicles. Actively coordinated vehicles refer to vehicles that possess independently controlled actuators for the suspension and the locomotion degrees of freedom. The contact force vectors at the vehicle-terrain contact locations can be directly influenced by using these actuators. These vehicles also have the capability to vary their
geometry to accommodate to terrain obstacles. The coordination of these vehicles requires
digital integration of all the commands to all the actuators and data from associated sensors,
and the control strategy requires an understanding of the kinematics and dynamics of spatial
hybrid series-parallel chains. Experimental testing of such vehicles is required to obtain a
better understanding of the design, coordination and control issues of actively articulated
wheeled vehicles. Issues relating to actively coordinated vehicle systems and relevant
literature have been discussed in detail in Chapter 1.

No working model of an actively coordinated wheeled vehicle with variable
configuration capability such as the WAAV has been operated, either in the laboratory, or
on unstructured terrain. The coordination issues of these vehicles cannot be completely
understood without experimental investigation. This is due to the fact that the interaction of
the vehicle with unstructured terrain is complex in nature and is not easily modeled.
Models of soil strength and sinkage parameters, rolling resistance, traction etc. tend to be
highly empirical in nature [McTamany, 1989]. In addition, special mobility maneuvers
such as step climbing and operation on steep slopes [Waldron et al., 1987] require
sophisticated hybrid position/force control schemes. Experimental verification of these
maneuvers is most desirable to ensure reliable autonomous motion capability.

The experimental data obtained from the experiments performed in the laboratory,
and on off-road terrains, can be used in conjunction with dynamic models of the system, in
order to extend the results to more general terrains, and to varied environments. The
dynamic models of the system that are required for this purpose have, to some extent, been
discussed in Chapter 5. The dynamic simulations can also be used to extend the
experimental results to other similar vehicle configurations. For instance a larger number
of modules in the actively articulated chain, and the presence of unactuated passive joints,
Figure 6.1 The Wheeled Actively Articulated Vehicle (WAAV)
can be investigated. The performance of actively coordinated vehicles can be compared to that of passively suspended vehicle systems.

The kinematic configuration of the WAAV is discussed in some detail in Section 6.2. The mechanical systems, the sensing, the control and the actuation sub-systems of the WAAV are described in Section 6.3. A discussion of the current capabilities of the WAAV system and future WAAV experiments that are being planned is included in Section 6.4.

6.2 Kinematic Configuration

A three-module articulated wheeled vehicle configuration is shown in Figure 6.2. This is the kinematic configuration of the WAAV. Each module consists of a body and two wheels and the consecutive modules are connected by articulations that are three degree of freedom joints consisting of three successively orthogonal revolute joints. All the wheels and the three degrees of freedom of all the articulations are independently actuated. The axle of the middle module is mounted on a guide way so that it can translate along this guide way from left to right (in the fore-aft direction in Figure 6.2). This translatory motion allows the center of mass of the system to be located on either side of the axle of the middle module as appropriate to maintain stability. This movement of the center of mass is required during mobility maneuvers such as step climbing and self-recovery from an accident. The WAAV kinematic configuration has many attractive features that include abilities of surmounting large obstacles, crossing wide ditches, and self-recovery from an overturn.

One of the primary objectives of constructing a fully actively coordinated wheeled vehicle is to fully optimize the contact conditions at each of the wheels. This means,
effectively, minimizing the maximum ratio of tangential to normal contact force over all the wheels. This has the effect both of maximizing traction and of minimizing power consumption [Waldron et al., 1991a]. Passive distribution of power to the wheels via differentials, as in conventional automotive technology, works well only on surfaces which are close to level. Even then, there is no possibility of optimizing the load distribution among the wheels.

Another capability which comes with an actively controllable configuration, is a capability for anticipatory changes in configuration when approaching large obstacles. In order to exploit this variable configuration capability when approaching obstacles, a sensing system which can provide a model of the terrain ahead of the vehicle, and the software to react appropriately to that model are required. Such a sensing system is probably essential to an autonomous roving vehicle. Actively coordinated vehicle
geometries also possess the capability for self-recovery from roll-overs or traction failures. This capability can be very useful if an autonomous vehicle mission of high reliability is desired. Obstacle handling capability is described in [Waldron et al., 1987]. A summary of the mobility capabilities of the WAAV is listed in the table below. A discussion of how the WAAV can achieve these mobility capabilities is included in Yu and Waldron [1991].

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<tr>
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<th>Mobility Capabilities of the WAAV</th>
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<tr>
<td>1</td>
<td>Nominal Dimension</td>
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<td>Articulation joint limits</td>
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<td>Middle Axle Travel Stroke</td>
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<td>Average Anticipated Power Consumption</td>
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<td>8</td>
<td>Obstacle Crossing Ability</td>
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6.3 Description of the Sub-Systems

The WAAV has 12 twelve rotary brushless DC motors, one for each of the six wheels, and three for each of the two articulations. These 12 actuators are under continuous microcomputer control. A rotary actuator (a DC motor) and a power screw are used to obtain the fore-aft translatory motion of the middle axle. This actuator is not to be operated under continuous control. It is to be used in a three position mode: middle axle in the extreme forward position, middle axle in the extreme backward position, or middle axle in the central position. The middle axle in the central position is suited for normal operations involving motion on all six wheels since, on even terrain, this will lead to low loads on the articulation actuators. The middle axle in the extreme positions (forward or aft) are required during special mobility maneuvers such as step climbing or self-recovery.

The middle module will house the onboard computing equipment and some centralized hardware to process the sensor information from the position encoders. The WAAV does not possess a vision system at this time. If a vision system is included, it will be mounted on the 'master' or the first module. The desired kinematic quantities of the master module include the direction of the gravity vector, and the six components of linear and angular velocity. An inertial sensing package that was used in the Adaptive Suspension Vehicle [Pugh et al., 1990] consisted of a vertical gyroscope, rate gyroscopes directed along three orthogonal axes, and accelerometers on the same three axes. A similar inertial package is to be used to obtain the kinematic quantities of the master module. Inertial navigation using 'dead-reckoning' may not be adequate when the vehicle traverses significant distances on the terrain, due to drift problems associated with the gyroscopes and integration errors. During preliminary experiments this is not expected to be a problem, since the vehicle will traverse small distances. Inertial sensing systems involve drift errors which can be
significant when the vehicle traverses relatively large distances particularly in severe terrain. A drift-free star sensor based navigation scheme such as the one investigated by Sreenivasan and Waldron [1994] can be used for long range experiments. This system uses a pair of CCD cameras mounted on an active three-degree-of-freedom mount. These two CCD cameras dynamically track two stars or distant objects as the vehicle moves on uneven terrain, and such a system can be used to obtain adequately accurate, drift-free kinematic information for the first module.

The mechanical design details of the WAAV are shown in Figure 6.3. The vehicle structure has been fabricated using the Aluminum Alloy 6061-T6. This material is relatively light in weight, possesses superior welding characteristics, and has almost temperature invariant mechanical properties. The last property allows the use of data generated at terrestrial ambient temperatures for design purposes over a large range of temperatures. ASTM 4140, a typical high strength alloy steel, is used for high-stress components such as drive shafts in order to keep their dimensions small. Due to the relatively high torque and low speed operational requirements of the WAAV, the DC motors are equipped with appropriate speed reduction units. The reduction units account for a significant part of the vehicle weight. Each of the wheel motors are provided with a commercially available harmonic drive made by the Emhart Machinery Group. These harmonic drives provide a reduction ratio of 160 to 1, and they have a transmission efficiency of 75%. Also, all six articulation actuators also use the same harmonic drive as part of multi-stage reduction units. The pitch actuators of the two articulations are mounted inside the middle module frame as shown in Figure 6.3. The motor shaft is connected to a harmonic drive followed by a single stage spur gear pair that provides a second stage reduction ratio of 3 to 1. Finally, the output of the spur gear pair is connected to the pitch axis by a roller chain and sprocket drive that provides a reduction ratio of 2 to 1 leading to
an overall reduction ratio of 960 to 1, and an overall transmission efficiency of about 57.4%. The yaw actuators are mounted outboard of the pitch actuators. The yaw actuators are provided with a compact planetary gearhead unit of ratio 4 to 1 followed by a harmonic drive. Finally, the output of the harmonic drive is connected to the yaw axis by a spiral bevel gear pair of ratio 2 to 1. The yaw motors are mounted above the pitch axes such that the axes of these motors are parallel to the pitch axes as shown in Figure 6.3. The choice of the transmission units and the locations of the yaw motors were based on geometric interference studies that ensured the desired motion ranges of each of the articulation axes. The overall reduction ratio of the two yaw actuation sub-systems is 1280 to 1, and the transmission efficiency is about 57%. The roll actuators are mounted outboard of the yaw actuators on the front and the rear modules. The roll motor is connected to a harmonic drive followed by a single stage spur gear pair that provides a second stage reduction ratio of 3.43 to 1. The overall reduction ratio is about 549 to 1, and the overall transmission efficiency is about 67.5%.

At present, it is not planned to have an onboard power supply unit. Invertors that work from the 110V AC supply are being used for laboratory testing. Power transformers for the main motors are located off board, and the power supply to the vehicle is via an umbilical cable. Four invertor lines rated at 50V DC @ 10A each are required. Each motor is rated at 50 V DC with a maximum current of 5A. Each line supports a set of three motors. The three motors belonging to a set are chosen so that a current of 10 A will be sufficient to drive the three motors at any given time. This is because all three of these motors are not expected to operate at their peak torque requirements at the same time. A block diagram of the power distribution system is shown in Figure 6.4. The two wheel motors of the front module and the front roll actuator are connected to power line # 1. The left wheel of the middle module and the pitch and the yaw actuators of the front module are
Figure 6.3 Mechanical Design of the WAAV
connected to power line # 2. The right wheel of the middle module and the pitch and the yaw actuators of the rear module are connected to power line # 3, and the two wheel motors of the rear module and the rear roll actuator are connected to power line # 4. The four motor sets contain motors that are located close to each other. This reduces the problem of shielding required to isolate the power lines. The power circuit includes a fuse box that ensures that the current drawn by each of the twelve motors does not exceed the designed maximum value. The power system described here is primarily suited for laboratory experiments. A trailer equipped with gasoline powered generators, which was used during the Adaptive Suspension Vehicle project [Pugh et al., 1990], is available for use during field experiments.

The emphasis of the experimental study of the WAAV is towards understanding the mobility, coordination and control issues of such systems. If a similar vehicle is to be used in an autonomous mode of operation, the system design may have to be changed to accommodate an on-board power supply unit. An on-board power supply unit may employ batteries or, in the case of extra-terrestrial missions, radio-isotope thermoelectric generators. In either case, the mechanical system has to be fabricated using low weight to strength ratio materials. Further, use of high torque to weight ratio rare-earth motors can reduce the required speed reduction ratios to less than 150 to 1 for each of the twelve motors. This will decrease the overall system weight significantly. It is also desirable to minimize the transmission losses as far as possible to conserve energy. A vehicle designed in this manner will have a smaller overall volume, which leads to larger ranges of motion for the articulation joints, and hence to superior mobility. Of course, a system that is designed taking into account the above considerations is bound to be more expensive.
One of the reasons for choosing actively coordinated vehicles instead of conventional vehicle systems is the fact that active coordination possesses the ability to optimize the load distribution among the actuators. In passively suspended systems, poor force distribution leads to large, fluctuating structural loads, especially when these systems operate on unstructured terrains. This requires such vehicles to be relatively bulky in order to avoid fatigue failures of the structural elements. This in turn leads to increased power consumption. Of course, actively coordinated vehicles have a large number of actuators and sensors which have overheads associated with them. Therefore, the need for active coordination is not justified unless the terrain is 'sufficiently' rugged. Further, if the actively coordinated system must use bulky transmission and actuation components, its ability to conserve power will not be properly exploited. As the performance of available actuation and transmission systems gets better, and as more efficient systems design techniques are developed, the option of using active coordination for autonomous off-road vehicles should become a more viable one.

The WAAV's actuator assortment consists of twelve brushless DC motors (Manufactured by Pittman Industries, Harleysville, PA), and a small, high-torque DC motor. The twelve main motors are on four isolated circuits, each serving three motors as shown in Figure 6.4. The thirteenth motor, also known as the carriage-slider motor, is a small, high-torque motor used for moving the middle axle back and forth. This motor is powered by a small switching supply that also supplies power to the drive electronics for controller boards of the main motors. There is a 120V line powering this switching supply via the umbilical cable.

The control architecture is implemented on a layered, networked system of computers. An on-board 80386 computer is responsible for the majority of the localized
Figure 6.4 A Block Diagram of the Power Distribution System
motor and sensor functions. An off-board 80386 computer, equipped with a monitor and keyboard for user interaction, is responsible for generating the desired position, force, and velocity commands. Information exchange between the two computers is made possible via an asynchronous serial link achieved by a null modem connection.

Twelve high resolution incremental optical encoders (Hewlett-Packard series HEDS-5000), each attached to a brushless DC motor, are used to read position information on the motor shaft. Decoding is performed on the middle module of the WAAV in one of the on-board computers. To decode the encoder position, there is a high speed, 16-bit encoder card utilizing the Advanced Micro Devices 9513 programmable IC (available from Omega Electronics). The card is configured for 20 channels of pulse counting, but with the use of a programmable array logic for quadrature decoding, there are 8 quadrature encoder inputs and four additional 16-bit counters. The quadrature inputs coming into the card are converted to an "up" channel and a "down" channel. The current wheel position is the down index subtracted from the up index, which results in a 16-bit position index. This information, combined with the position of the other motors, is transformed into an absolute position column vector and stored in an integer array. This angular position information is sufficient for motion planning laboratory experiments that are to be performed. A detailed discussion of these motion planning experiments is included in the next section.

In order to test the special mobility capabilities of the WAAV, and its performance in unstructured terrain conditions, the vehicle should be operated in a force control mode. The coordination scheme is similar to that of the Adaptive Suspension Vehicle, and is described by Kumar and Waldron [1989]. Control of the force distribution in the vehicle requires the sensing of the contact forces at the wheel-terrain interface. Contact force
information can also be used to estimate the location of the contact center with respect to a vehicle fixed frame [Kumar and Waldron, 1989]. The WAAV design includes a provision for the sensing of the contact force components in the plane of the wheel. The lateral force component normal to the wheel plane cannot be sensed accurately for the wheel-axle design of the WAAV. It may be possible to estimate the lateral force component using other sensed information. The contact force components in the plane of the wheels are sensed using a combination of radial force sensors and wheel torque sensors. One set of radial force sensors and wheel torque sensors have been calibrated for sensing purposes. These sensors will be incorporated into all six wheels. The reaction at each wheel center (in the plane of the wheel) due to the respective contact force is sensed by an array of strain gages mounted on the bearing housing of the wheel axle. The moment applied to the wheels is sensed by an LVDT mounted on the wheel rim that measures the angular deflection of a pointer mounted on the wheel hub. Using the sensed force and moment, the contact forces in the plane of the wheel can be estimated. Linear calibration curves have been obtained for both the radial and torque wheel sensors in static tests. Dynamic testing followed by integration of these sensors with the data acquisition system of the WAAV has yet to be performed. The design of the torque sensing wheel the applied torque is shown in Figure 6.5. In order to obtain linear torque to deflection calibration curves, the sensor shown in Figure 6.5 had to be fabricated such that the five spokes, the hub and the rim were machined from one piece. The rim of the sensor was then keyed to the wheel. Otherwise hysteresis in the welded joints at the ends of the spokes led to nonlinear calibration curves.

The twelve DC motors are controlled by onboard drive electronics which include PWM torque control and high-current field effect transistor three phase motor drivers. The controller boards can track reference current values, and they have a current feedback loop that accounts for the actuator dynamics. The reference motor current values are set by an
analog voltage input of ± 10 volts to the PWM board. Motor torque is a linear function of this controller input voltage. The full scale range is -10V to 10V and is available as the output of a 12-bit (plus sign) D to A converter located in the onboard 80386 computer. The thirteenth motor is not controlled directly by the computer. Instead, it is controlled by a small Motorola 6811 based micro-controller board called the MiniBoard 2.0. (Designed by Fred Martin of the MIT Media Lab. Information is available in PostScript/UNIX compressed form via anonymous ftp to: cher.media.mit.edu in the /pub/miniboard directory.) The MiniBoard 2.0 has on board H-bridge motor driver IC for driving the
Figure 6.6 The Control Architecture
The carriage-slider motor. The MiniBoard has several analog and digital inputs that can be used for control purposes in the future.

The control architecture of the WAAV that includes the position feedback loop from the optical encoders is shown in Figure 6.6. The incorporation of the force feedback loop is a topic of future research.

This concludes the discussion of the various sub-systems of the WAAV.

6.4 Current Capabilities and Future Experiments

The mechanical fabrication and assembly of the WAAV has been completed. Testing of the actuation sub-systems and the control architecture has also been performed. The onboard D to A control card has been interfaced with all the motors allowing for computer control of the actuators. The carriage-slider motor is not under direct control of the onboard computer. However, a local control circuit using the Miniboard allows for the positioning of the carriage-slider in one of three desired positions. The choice of the location of the carriage slider can be done from the offboard computer through the umbilical. The WAAV has been tested in an open-loop fashion to ensure that it can perform some basic maneuvers that may be required during a mobility experiment. The optical encoders on the motors have been connected to a data acquisition board and the motor position feedback is available.

The vehicle system is first expected to be used for position control experiments to test basic motion planning algorithms of the WAAV on even terrain in a laboratory setting. The presence of rolling contacts at the wheel ground interface that do not allow for
controlled lateral motion makes the motion planning problem a nonholonomic one. The coordination problem involved in achieving a desired lateral motion for a conventional wheeled system is a surprisingly difficult one as noted by Murray and Sastry [1992]. The nonholonomic motion planning problems result in paths requiring several iterative loops before the vehicle can reach a desired position. The presence of active articulations in the WAAV makes it distinct from the systems that have been studied in the literature. For instance, the WAAV can lift the middle module off the ground and move laterally using a series of actuator maneuvers. Position control experiments can lead to a better understanding of the fine motion maneuvers of the WAAV.

The second set of experiments that are be performed are force control experiments. This will include the investigation of the performance of the WAAV with respect to geometric objects in a laboratory setting, and comparison of this performance with results obtained from numerical simulations. A simple test that involves the control of all the twelve active actuators is desired. This test can be performed, for instance, by causing the vehicle to attempt to track an arc of a circle on an inclined plane surface. The desired kinematics and contact forces during this test will be spatial, and an active control of all the twelve actuators will be required to track the desired states. The variable configuration capability can be tested by investigating the system performance while attempting various mobility maneuvers such as obstacle climbing, ditch crossing, self-recovery etc.

The final set of experiments involves field testing. The vehicle will be made to traverse in-situ soils and rocks in the fields, and it will be required to track desired paths. The vehicle will also be made to perform mobility maneuvers requiring the variable configuration capability. The coordination and control schemes will have to be tuned in order to make them robust with respect to unmodelled uncertainties introduced by the unstructured environment. Once the WAAV performs satisfactorily in these field
experiments, the experimental data that is obtained can be used to compare the performance of the WAAV with other vehicle configurations. The experimental data can also be used, along with simulations, to predict the performance of other vehicles similar to the WAAV, and to predict the system behavior in various terrain conditions.

The force control experiments will require the contact force sensing systems to be integrated with the control architecture. This has not yet been done on the WAAV system. In addition to force sensing, the field experiments will require an inertial sensing package as described in Section 6.3. The integration of the inertial sensing system into the control scheme of the WAAV will require special hardware and software design, and is a topic of future research.

6.5 Conclusions

A technical description of the electro-mechanical systems of the Wheeled Actively Articulated Vehicle has been provided in this chapter. A discussion of the current capabilities of the WAAV system, and the mobility experiments that are planned for the future is also included.
CHAPTER VII

CONCLUDING REMARKS

This is the concluding chapter of this dissertation. It includes a summary of the research issues addressed in this dissertation. The summary is followed by a section discussing the potential research extensions of this dissertation work.

7.1 Summary

Actively coordinated wheeled vehicle systems with articulated structures that can operate on uneven terrains have been addressed here for the first time. These vehicle systems have several attractive features and are particularly suited for autonomous operation on difficult terrain conditions. In order to exploit the potential offered by these actively coordinated systems, it is necessary to develop coordination schemes that can control these systems effectively.

In this dissertation the issues of force and motion management in actively articulated wheeled vehicle systems have been addressed. A basic dynamic simulator that can be used to numerically study the performance of such vehicle systems has been developed. A drift-
free autonomous navigation system for vehicles operating on unstructured terrain has been studied during the course of this work. Finally, the work also includes the design, fabrication and preliminary experimental testing of the Wheeled Actively Articulated Vehicle (WAAV) in a laboratory environment.

**Autonomous Navigation:** Inertial sensing systems are used in current robotic systems to obtain orientation and the angular rates of the body of a robotic vehicle for control purposes, and to obtain absolute vehicle position on the terrain for the purpose of guidance. In the absence of accurate maps of the environment, inertial sensing systems involve drift errors which can be significant even after the vehicle has traversed only short distances. This is particularly true of a land vehicle in which body attitude changes are substantial and rapid. In this work, a drift-free star sensor based navigation scheme has been investigated, and its relative advantages and disadvantages as compared to existing systems have been studied. This navigation scheme is not limited to actively coordinated wheeled systems; it is applicable to any autonomous mobile system operating on unstructured terrain. The absolute position information obtained from this drift-free system is quite good, and with some improvements, may be sufficient for many applications. Satellite based position information can be more accurate. However, satellite based systems are much more expensive, and can only provide periodic updates. Numerical simulation results have been obtained to show the feasibility of a two camera mount star tracking system for autonomous navigation purposes.

**Force Planning:** Actively coordinated vehicles possess redundancy in actuation, and this redundancy can be used to improve the mobility of the system by 'optimally' allocating the vehicle-terrain contact forces. General theoretical results characterizing redundancy in force distribution with respect to actively coordinated mechanisms such as legged systems, multi-fingered articulated hands, and actively coordinated wheeled systems have been developed.
This characterization of redundancy has been used to optimize the force distribution in these active systems. The natures of the resulting nonlinear optimization problems have been studied. The optimization schemes developed include globally optimal algorithms that utilize advanced polynomial continuation techniques.

**Motion Planning:** On uneven terrain, the configuration of the vehicle system of Figure 4.1 is greatly affected by the local terrain geometry. Since issues such as vehicle stability, coordination, and control are influenced by the configuration the locomotion system assumes on the terrain, it is important to study the position kinematics of these wheeled vehicles. The kinematic mobility of this configuration on uneven terrain, and the position kinematics of the resulting hybrid series-parallel chain has been studied. These position kinematic solutions have been used as a basis for developing motion planning algorithms. Preliminary motion planning algorithms that address straight line vehicle motion, obstacle climbing, ditch crossing and nonholonomic maneuvering have been developed with respect to a general multi-moduled articulated wheeled vehicle system.

**Dynamic Simulation:** A dynamic simulator capable of handling the motions of a multi-module vehicle configuration (such as the one in Figure 4.1), is highly desirable to investigate control schemes, and to compare performance characteristics of different vehicle configurations. Dynamic simulation of partially constrained series-parallel wheeled vehicles on uneven terrain is a complex problem that has been studied here. The simulator developed during the course of this work can handle wheels rolling on piece-wise planar uneven terrain that may result in multiple wheel-terrain contact points for a given wheel. The computation of the contact force vectors at the wheel-terrain contact locations has been addressed. Considerable simplification of the dynamic analysis of these partially constrained series-parallel systems is achieved by using the de-coupled tree structure approach. This approach introduces a compliance at the vehicle-terrain contact locations,
and these contact points are modeled by three-dimensional spring-damper combinations. The articulated-body-inertia method, a novel forward dynamics scheme available in the robotics literature, has then been used to develop efficient simulation algorithms for a multi-module articulated wheeled vehicle systems.

Experimental Testing of the WAAV: Preliminary tests and mobility experiments have been performed with the WAAV system of Figure 6.1. The WAAV is a complex system that includes thirteen brushless DC motors, and associated sensors and interface electronics. In this work, the mechanical and electrical hardware has been tested, the WAAV has been interfaced to a personal computer, and basic testing of simple mobility maneuvers in a laboratory setting has been attempted. These experiments demonstrate interesting geometric features of actively articulated wheeled systems. Since these vehicles do not possess omni-directional motion capability, motion management issues tend to be very challenging.

7.2 Research Extensions

Coordination issues of actively articulated wheeled vehicle systems operating on uneven terrain have been addressed in this dissertation. Unlike legged locomotion systems, wheeled systems do not possess omni-directional motion capability. This leads to some unique complications, and these complications have been brought out throughout this dissertation. Motion planning issues become complicated due to the presence of nonholonomic constraints. Control of the force distribution in these vehicles has to be done keeping in mind that the lateral contact forces cannot be directly influenced since there is a need for lateral scrub at the wheels for the system to perform motion on uneven terrain. Modeling of the contact conditions for dynamic simulation purposes also poses challenges
that are not encountered in legged systems due to the presence of rolling/slipping elements that can have multiple contact points. Finally, the control issues of such wheeled systems are particularly challenging. An absence of good models for contact slipping/rolling phenomena makes traditional model based control techniques infeasible.

In this dissertation, force/motion planning issues of actively articulated wheeled vehicles has been studied and preliminary coordination schemes have been discussed. However, vehicle planning based on other considerations such as minimum energy consumption have not been addressed in any detail, and these are topics for future research. Coordination schemes that lead to minimum energy/power consumption are especially desirable in situations where autonomous operation of the vehicle is desired with strict budgets on the energy/power consumption.

In this research work, even though general schemes that can handle multiple modules were developed for position kinematic analysis, dynamic simulation etc., the issue of effectively utilizing a large number of modules was not addressed. There are several situations in which it may be attractive to have only some of a large number of modules of an articulated vehicle on the terrain, while the others are lifted off the terrain. The position kinematics of such chains are more complicated, and they tend to depend on the particular control strategy used. In such situations, optimal techniques may be too computationally intensive, and sub-optimal techniques may have to be investigated to obtain efficient, real-time solutions. These are topics of future research.

Obstacle climbing capabilities can be maximized by approaching the obstacle optimally. In this work, approach and terrain paths were chosen without any consideration of maximizing obstacle climbing capabilities. Therefore, the automated obstacle climbing algorithms developed in this work are preliminary and need improvement. The general
problem of tracking arbitrary terrain paths, as closely as possible, is a challenging nonholonomic motion planning problem and is a topic of future research. It has been noted in the literature that a nonholonomic motion planning problem is best handled by stabilizing control laws about a nominal feasible trajectory, since such systems cannot be stabilized to a point using static state feedback control schemes [Walsh et al., 1992]. Due to the presence of uncertain slip characteristics, this problem is even more challenging for wheeled systems operating on uneven terrain.

It is desirable that the dynamic simulator that has been developed in this work be extended to handle vehicle traversal on very general, random piece-wise planar terrains. Such a simulator can be used to study the designs of various articulated wheeled vehicles that possess varying levels of active actuation. It is reasonable to believe that varying levels of redundant actuation may be required depending on the severity of the terrain on which the system is expected to operate. Also, the inclusion of redundant actuation in a vehicle may not be justified unless effective, real-time coordination and control schemes can be developed in order to exploit the redundancy in hardware. A topic for future research is to investigate nonlinear control techniques that can automate mobility maneuvers of these vehicles in difficult terrain conditions and exploit the hardware redundancy as effectively as possible. The presence of uncertain contact conditions that are extremely difficult to model makes it necessary to investigate robust control schemes. It is unlikely that traditional linear control/model based techniques can be applied to solve these problems, and control techniques such as sliding mode control and neural networks need to be considered. The dynamic simulator has to be extended to study motion planning strategies based on considerations such as minimum energy consumption. Dynamic simulations should also be used to investigate special mobility maneuvers such as obstacle climbing and self-recovery from an overturn failure. If the dynamic simulator is required to
have the capability to handle realistic off-road simulations, it is necessary that the dynamic simulator parameters be tuned by correlating the outputs of the simulator with experimental data from the WAAV. The dynamic simulator can also be used to extend experimental results obtained from the WAAV to other vehicles that have similar configurations as the WAAV.

The WAAV system has been designed and fabricated, and preliminary testing has been performed. Some of the experiments that are to be performed using the WAAV system are listed below. The vehicle system should be used for position control experiments to test the basic motion planning algorithms of the WAAV on even terrain in a laboratory setting. The presence of rolling contacts at the wheel-ground interface that do not allow for controlled lateral motion makes the motion planning problem a nonholonomic one. The nonholonomic motion planning problems result in paths requiring several iterative loops before the vehicle can reach a desired position. The presence of active articulations in the WAAV makes it distinct from the systems that have been studied in the literature. For instance, the WAAV can lift its middle module off the ground and move laterally using a series of actuator maneuvers. Position control experiments can lead to a better understanding of the fine motion maneuvering capability of the WAAV. A second set of experiments that are to be performed are force control experiments. This will include the investigation of the performance of the WAAV with respect to geometric objects in a laboratory setting, and comparison of this performance with results obtained from dynamic simulations. A simple test that involves the control of all the twelve active actuators is also desired. This test can be performed, for instance, by requiring the vehicle to attempt to track an arc of a circle on an inclined plane surface. The desired kinematics and contact forces during this test will be spatial, and an active control of all twelve actuators will be required to track the desired states. The variable configuration capability needs to be tested
by investigating the system performance while attempting various mobility maneuvers such as obstacle climbing, ditch crossing, self-recovery etc. The final set of experiments that are to be performed involve field testing. The vehicle must be made to traverse in-situ soils and rocks in the field, and it must attempt to track desired terrain paths. The vehicle should also be made to perform mobility maneuvers requiring the variable configuration capability. The coordination and control schemes will have to be tuned in order to make them robust with respect to unmodelled uncertainties introduced by the unstructured environment. Once the WAAV performs satisfactorily in these field experiments, the experimental data that is obtained can be used to compare the performance of the WAAV with other vehicle configurations. The experimental data should also be used, along with simulations, to predict the WAAV system behavior on other terrain conditions that may not be easily accessible for experimentation.
APPENDIX - A

TYPICAL PARAMETERS USED IN SGS SIMULATIONS

Star unit vectors: $s_0^1 = [0 \ 0.1736481 \ 0.9848077]^T$, $s_0^2 = [0 \ -0.1736481 \ 0.9848077]^T$.

SGS parameters: Stalk mass $m_s = 0.25$ kg, camera angles $\varepsilon = 10^\circ$ and $\lambda = 20^\circ$, SGS stalk and camera link length = 0.1 m, $C_a$ in frame 'b' = $[0.25 \ 0 \ 0.25]^T$, camera link radius = 0.025 m, lens focal length $f_l = 0.5$ m.

Motor parameters: motor inertia = 0.00011 kg m$^2$, viscous coefficient = 0.001 N ms/rad, current limit = 20 Amps, gear ratio = 40.0, resistance = 1.2 ohms, torque constant and back emf constant = 0.2578.

Closed loop poles: $-15\pm11i, -7; -18\pm13i, -5.5; -21\pm15i, -4$.

Control law matrices:

$$\hat{M}_b = \begin{bmatrix} 0.0225034 & 0 & 0 \\ 0 & 0.0225057 & 0 \\ 0 & 0 & 0.0204923 \end{bmatrix} \quad K_p = \begin{bmatrix} 586 & 0 & 0 \\ 0 & 754 & 0 \\ 0 & 0 & 918 \end{bmatrix}$$
\[ K_v = \begin{bmatrix} 37.0 & 0 & 0 \\ 0 & 41.5 & 0 \\ 0 & 0 & 46.0 \end{bmatrix} \quad K_i = \begin{bmatrix} 2768 & 0 & 0 \\ 0 & 3451 & 0 \\ 0 & 0 & 3996 \end{bmatrix} \]

CCD camera sensing noise: Gaussian with zero mean, standard deviation = 0.00002°.

Angular position sensing noise: Gaussian with zero mean, standard deviation = 0.00032°.
APPENDIX - B

POSITION KINEMATIC EQUATIONS AND THEIR SOLUTIONS FOR A TWO MODULE ARTICULATED VEHICLE EXAMPLE

The polynomial systems of equations for the numerical example in Section 4.2.8 are listed here along with their roots.

Module 1:

Let, $x_1 = \sin(\alpha)$, $x_2 = \cos(\alpha)$ and $x_3 = o_2$. The three quadratic equations relating $x_1$, $x_2$, $x_3$ are:

\[-0.870489 x_3^2 - 0.195202 x_2 x_3 + 0.717504 x_1 x_3 + 1.34404 x_3 + 0.046811 x_2^2 + 0.111873 x_1 x_2 + 0.150696 x_2 - 0.143577 x_2^2 - 0.553915 x_1 - 0.51833 = 0 \]  
\[-0.812347 x_3^2 + 0.074842 x_2 x_3 - 0.714402 x_1 x_3 + 1.225492 x_3 + 0.05963 x_2^2 + 0.045764 x_1 x_2 - 0.056453 x_2 - 0.156393 x_2^2 + 0.538867 x_1 - 0.461716 = 0 \]  
\[x_1^2 + x_2^2 = 1\]

For this problem, all the roots of the above 3 equations are real. These 8 roots are...
listed in the table below:

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.17365</td>
<td>-0.69035</td>
<td>-0.19244</td>
<td>-0.38458</td>
<td>-0.21184</td>
<td>0.35230</td>
<td>0.15003</td>
<td>0.66481</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.98481</td>
<td>-0.72348</td>
<td>-0.98131</td>
<td>0.92309</td>
<td>-0.97730</td>
<td>-0.93589</td>
<td>0.98686</td>
<td>0.74701</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1.00000</td>
<td>0.80449</td>
<td>1.06998</td>
<td>0.72211</td>
<td>0.52668</td>
<td>0.80450</td>
<td>0.45678</td>
<td>0.72210</td>
</tr>
</tbody>
</table>

Module 2:

Let, $x_1 = \tan(\psi/2)$, $x_2 = \tan(\phi/2)$. The two octic equations relating $x_1$ and $x_2$ are:

\[-0.189457 x_1^4 x_2^4 + 0.456782 x_1^3 x_2^4 - 0.594609 x_1^2 x_2^4 + 0.221536 x_1 x_2^4 - 0.079711 x_2^4 + 1.321838 x_1^4 x_2^3 - 0.184451 x_1^3 x_2^3 - 1.009477 x_1^2 x_2^3 + 1.677578 x_1 x_2^3 - 0.985346 x_2^3 - 2.240844 x_1^4 x_2^2 - 0.628602 x_1^3 x_2^2 + 1.14453 x_1^2 x_2^2 + 3.144121 x_1 x_2^2 - 1.833856 x_2^2 + 1.558547 x_1^4 x_2 - 1.867562 x_1^3 x_2 - 2.792058 x_1^2 x_2 - 3.282537 x_1 x_2 - 0.627861 x_2 - 0.285555 x_1^4 + 1.332934 x_1^3 - 1.51686 x_1^2 + 0.441748 x_1 + 0.001572 = 0 \tag{1}
\]
\[
0.046953 x_1^4 x_2^4 - 0.296431 x_1^3 x_2^4 - 1.309397 x_1^2 x_2^4 - 0.037888 x_1 x_2^4 + 0.062063 x_2^4 + 0.32405 x_1^4 x_2^3 + 2.928387 x_1^3 x_2^3 - 0.810008 x_1^2 x_2^3 - 2.996742 x_1 x_2^3 - 0.054047 x_2^3 - 1.814537 x_1^4 x_2^2 - 0.371141 x_1^3 x_2^2 + 2.45862 x_1^2 x_2^2 - 1.465651 x_1 x_2^2 - 1.73152 x_2^2 + 1.078949 x_1^4 x_2 + 0.884888 x_1^3 x_2 - 0.292861 x_1^2 x_2 - 1.257359 x_1 x_2 - 0.981329 x_2 - 0.096970 x_1^4 - 0.34998 x_1^3 - 0.539446 x_1^2 - 0.281733 x_1 - 0.068238 = 0 \tag{2}
\]
Both real and complex roots were obtained for the above two equations. The 32 finite solutions of the above two octics are listed in the table below (complex root pairs are listed in the form (a ± ib):

**Table B.2 Sixteen Solutions of Module 2**

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>x₁</td>
<td>2.7687</td>
<td>9.1378</td>
<td>-16.410</td>
<td>0.2657</td>
<td>0.3295</td>
<td>-0.1086</td>
<td>9.5774</td>
<td>-0.0298</td>
</tr>
<tr>
<td>x₂</td>
<td>0.4099</td>
<td>-0.1582</td>
<td>3.9921</td>
<td>-0.1956</td>
<td>-7.6441</td>
<td>-8.4040</td>
<td>-1.0921</td>
<td>-0.4859</td>
</tr>
<tr>
<td>N</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>x₁</td>
<td>6.8539</td>
<td>-0.4437</td>
<td>2.3491</td>
<td>-0.0875</td>
<td>4.5473</td>
<td>0.2215</td>
<td>-8.0684</td>
<td>5.5879</td>
</tr>
<tr>
<td>x₂</td>
<td>0.7811</td>
<td>-0.4428</td>
<td>1.8539</td>
<td>-0.0612</td>
<td>8.6130</td>
<td>-1.8902</td>
<td>0.4844</td>
<td>1.3984</td>
</tr>
<tr>
<td>N</td>
<td>17, 18</td>
<td>19, 20</td>
<td>21, 22</td>
<td>23, 24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₁</td>
<td>-0.8586 ± 0.3997i</td>
<td>0.6770 ± 0.2641i</td>
<td>-0.8982 ± 0.4127i</td>
<td>0.6611 ± 0.1774i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₂</td>
<td>-0.2314 ± 0.8374i</td>
<td>-0.5928 ± 1.6556i</td>
<td>-0.2066 ± 0.8476i</td>
<td>-0.8158 ± 1.4219i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>25, 26</td>
<td>27, 28</td>
<td>29, 30</td>
<td>31, 32</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₁</td>
<td>-0.8503 ± 0.3879i</td>
<td>0.6720 ± 0.2233i</td>
<td>-0.9042 ± 0.3910i</td>
<td>0.6918 ± 0.1899i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x₂</td>
<td>-0.2215 ± 0.7439i</td>
<td>-0.5300 ± 1.2047i</td>
<td>-0.1995 ± 0.7842i</td>
<td>-0.4970 ± 1.1561i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
REFERENCES


Martin, J., O'Kane, K., Scott, D., Stone, D., and Waldron, K. J., Spring, 1991, "A Feasibility Design: Drift-Free Three-Axis Tracking Camera Mount," Report for ME564, Department of Mechanical Engineering, The Ohio State University, Columbus, OH.


