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Representation theory of quadratic forms

Shao, You Yu, Ph.D.
The Ohio State University, 1994
To My Wife Dong, and Son Yin
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# TABLE OF CONTENTS

**ACKNOWLEDGMENTS** iii  
**VITA** iv  
**INTRODUCTION** 1  
   §0.1 Basics Of Quadratic Forms 5  
   §0.2 Facts About Haar Measure 8  

**CHAPTER**

<table>
<thead>
<tr>
<th>I RELATIVE SPINOR NORM GROUPS</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>§1.1 Representations By Spinor Genus</td>
<td>13</td>
</tr>
<tr>
<td>§1.2 Formulas Over Non-dyadic Fields</td>
<td>19</td>
</tr>
<tr>
<td>§1.3 Primary Reduction Over 2-adic Fields</td>
<td>22</td>
</tr>
<tr>
<td>§1.4 Secondary Reduction Over 2-adic Fields</td>
<td>32</td>
</tr>
<tr>
<td>§1.5 Formulas Over 2-adic Fields</td>
<td>47</td>
</tr>
<tr>
<td>§1.6 Recovery Of Special Cases</td>
<td>61</td>
</tr>
</tbody>
</table>

| II HAAR MEASURE OF REPRESENTATIONS | 72 |

<table>
<thead>
<tr>
<th>III PRIMITIVE REPRESENTATIONS</th>
<th>83</th>
</tr>
</thead>
<tbody>
<tr>
<td>§3.1 Primitive Representations Over Non-dyadic Local Ring</td>
<td>84</td>
</tr>
<tr>
<td>§3.2 Basic Results Over Z₂</td>
<td>87</td>
</tr>
<tr>
<td>§3.3 Primitive Representations Over Z₂</td>
<td>95</td>
</tr>
</tbody>
</table>

**REFERENCES** 109
INTRODUCTION

The representation problem is one of the most classical and fundamental problems in the arithmetic theory of quadratic forms. It has a long history and is generally viewed to have its origin in the investigations by Fermat of numbers represented by certain binary lattices [J]. Most of the earlier results were concerned with the representations of numbers by very specialized integral quadratic lattices [Di], with these lattices usually being positive definite and often having few variables. Beginning from the 1950s the research focus has gradually shifted towards the most general cases of representation problem: the representations of one lattice $K$ of rank $n$ by another lattice $L$ of rank $m$. On one hand, for representations by positive definite lattice $L$, there is a partial result [Ra] about representations of a binary lattice $K$ by $L$ with rank $m \geq 7$. Then there is the famous HKK theorem in 1978 [HKK] which asserts that when $m \geq 2n + 3$, $L$ represents $K$ provided the following conditions are satisfied:

1. $L$ represents $K$ locally everywhere;
2. the arithmetic minimum of $K$ is sufficiently large.

Naturally there is the direct analog of this theorem for primitive representations. However, such a result was proved in [HKK] for only $m \geq 3n + 3$. Very recently Jöchner and Kitaoka [JK] were able to sharpen this inequality to $m \geq 2n + 3$. But the primitivity can not be asserted at one exceptional prime not dividing $2\det L$ unless an extra assumption is made. With regard to condition (1), the question of local
representations was completely resolved by O'Meara in 1958 [OM2] in the cases when 2 is either a unit (non-dyadic) or a prime (2-adic). It is not difficult to see that condition (2) is also necessary.

On the other hand, there is the problem of representations by an indefinite lattice. It is well known that this is only a special case of the more general question of the so-called representations by a spinor genus in the sense of [Ei] and [Kn1]. Much stronger conclusions for representations are now possible. Indeed, when \( n = 1 \) and \( m \geq 4 \), or more generally when the codimension \( m - n \geq 3 \), then \( K \) is represented by all spinor genera in the genus \( \text{gen}(L) \) of \( L \) provided that condition (1) is satisfied. (See [Wa], [Kn2] and [H2].) In particular when \( L \) is indefinite, we have the desired local global principle for representations. When \( n = 1 \) and \( m = 3 \), then either \( K \) is represented by all spinor genera in \( \text{gen}(L) \) or by exactly half of them, provided (1) holds [JW] and [Kn2]. (Strictly speaking what Kneser did was to quantify the number of representations by spinor genus. We will come to this later.) In 1976 Hsia [H2] extended the above result algebraically to more general codimension two situations. The notion of "spinor exception" was introduced in this codimension two case by Hsia [H3], where \( K \) is called spinor exceptional if it is represented by exactly half of the spinor genera in \( \text{gen}(L) \). In 1980 Schulze-Pillot [Sp1] found the necessary and sufficient conditions for a number \( b \in F^* \) to be spinor exceptional (\( n = 1 \) and \( m = 3 \) case). He achieves this by introducing a new invariant called the "relative spinor norm group" and he computed it. In their recent work [HSX] Hsia, Shao and Xu completely solved the problem of representation by spinor genus without any restriction whatsoever on the ranks of the lattices involved and the codimension. Again the invariant of the relative spinor norm group plays a vital role.

In chapter I of this thesis we try to give explicit formulas for this relative spinor norm group when the local field is either non-dyadic or 2-adic. Naturally our work is
based heavily on the knowledge of the (ordinary) spinor norm group of local integral rotations. These latter groups were computed by Kneser [Kn$_1$] over non-dyadic fields, and by Earnest and Hsia over 2-adic fields [EH]. For other works in this direction over general dyadic fields, one can see (chronologically) [H$_1$], [EH$_1$], [BD], [X$_1$] and [X$_2$]. In the first section we give a very brief account of the recent developments in [HSX] which reveals the usefulness as well as the importance of this relative spinor norm group. For more details, see [HSX]. In the second section we give a formula for this group over a non-dyadic local field; and in sections 3, 4 and 5, we develop several formulas which, taken together, completely determine these relative spinor norm groups over any 2-adic local fields. Finally in the last section, section 6, we recover Satz 3 and Satz 4 of [Sp$_1$] by our formulas developed in earlier sections, and also correcting a minor discrepancy in Satz 4 a(i)(γ) of [Sp$_1$] and in Theorem 3.14(iv) of [EH].

With regard to chapter II, as mentioned earlier, Kneser proved in [Kn$_2$] the following: given any $b \in F^*$, where $F$ is an algebraic number field, and any $R$-lattice $L$ of rank $\geq 3$, where $R$ is the ring of integers of $F$, then there is only one constant measure (defined later) of representations of $b$ by any spinor genus in $\text{gen}(L)$ provided that (i) $\text{rank}(L) \geq 4$, and (ii) $\text{gen}(L)$ represents $b$ (i.e., $b$ is represented by $L$ locally everywhere). Moreover there are at most two different measures when $\text{rank}(L) = 3$. Schulze-Pillot expressed the difference of these two possibly different measures (when $\text{rank}(L) = 3$) as a product of some local factors, "eine Art Siegelsche Maßformel mit Charakter" [Sp$_2$]. He also calculated these local factors under different special circumstances. In chapter II we extend Kneser's above result to representations of lattices by lattices (i.e., the higher dimension cases), and give an upper bound for the number of different (Haar) measures of representations by spinor genera. We shall use results from the chapter I to compute this upper bound.
In comparison with the ordinary representation theory, our knowledge on primitive representation is still rather fragmented and limited. And yet primitive representation certainly has its place in mathematics. For, partial results have already found their applications in the algebraic geometry of $K_3$-surfaces and their singularities (see [D] and [Ni]), and that a general (not necessary primitive) representation of $K$ by $L$ induces a primitive representation by $L$ of some bigger lattice $K' \supseteq K$ in $FK$. Though O'Meara completely solved the problem of local representations some time ago, we are only at the beginning stage in finding the necessary and sufficient conditions of local primitive representations. James in his recent work [J1] [J2] gave necessary and sufficient local conditions for the existence of primitive representation by unimodular lattice. He formulated these local conditions in global terms in the indefinite case by using the strong approximation. In chapter III we extend James local results. Since in general this problem is quite complicated, we restrict ourselves only to the next reasonable step where the bigger lattice is even and "almost" unimodular over $Z_p$ for all $p$. In this setting, we solve this problem over non-dyadic local rings in the first section. In §2 we give some basic results over $Z_2$, and we state, for the sake of completeness, the main theorems of [J1] & [J2], but in a slightly different way. In the third section we solve the problem over $Z_2$.

In the rest of this introductory chapter, we give some notations, terminologies, and basic results that will be used in later chapters.
§ 0.1. Basics Of Quadratic Forms

In this first section we shall introduce the necessary definitions and classical results of the quadratic form theory. We follow closely the notations and terminologies of [OM].

GENERAL DEFINITIONS:

Let F be an algebraic number field, R be the ring of integers of F. A quadratic space V over F of rank n is a n-dimensional vector space equipped with a symmetric bilinear form B: V×V → F which, together with the associated quadratic form Q, satisfies

\[ 2B(x,y) = Q(x+y) - Q(x) - Q(y) \]

for all x, y in V. We will always assume that quadratic space V is regular; i.e., B(x,V) = 0 if and only if x = 0.

Suppose W is another quadratic space. A linear map σ from W to V is called a representation of W into V if Q(σx) = Q(x) for all x in W (where two Qs on both side of the equation are understood to be the quadratic forms of the respective spaces). Further, it is called an isometry if it is also a bijection. O(V), the orthogonal group of V, denotes the group of all isometries of V into V itself. Some isometries deserve special attention. They are the so called symmetries \( \{ S_y \mid y \in V, Q(y) \neq 0 \} \) where \( S_y \) is defined by the formula:

\[ S_y(x) = x - \frac{2B(x,y)}{Q(y)}y \]

for all x ∈ V. The classical theorem of Cartan-Dieudonné asserts that O(V) is generated by symmetries. The proper orthogonal group \( O^+(V) \) is the subgroup of O(V) consisting of all the even products of symmetries.
The spinor norm map \( \Theta: O^+(V) \to F^*/F^{*2} \) is defined as follows. Given \( \sigma = S_{y_1}...S_{y_t} \in O^+(V) \), \( \Theta(\sigma) = Q(y_1)...Q(y_t) \text{ mod } F^{*2} \). This homomorphism is well defined where the kernel is denoted by \( O(V) \).

An \( R \)-submodule \( L \) of \( V \) is called a lattice in \( V \) if there is a base \( \{x_1,\ldots,x_n\} \) for \( V \) such that \( L \subseteq R x_1 + \ldots + R x_n \). \( L \) is called a lattice on \( V \) if further \( FL = \{ \alpha x \mid \alpha \in F, x \in L \} = V \). Let \( K \) be another lattice. An \( R \)-linear map \( \sigma \) from \( K \) to \( L \) is called a representation if \( Q(\sigma x) = Q(x) \) for all \( x \in K \); an isometry if \( \sigma \) is also a bijection. Let \( O(L) = \{ \sigma \in O(V) \mid \sigma L = L \} \), and \( O^+(L) = \{ \sigma \in O^+(V) \mid \sigma L = L \} \).

The scale \( s_L \) of a lattice \( L \) is the \( R \)-module of \( F \) generated by the subset \( B(L,L) \). The norm \( n_L \) of \( L \) is the \( R \)-module of \( F \) generated by the subset \( Q(L) \).

**LOCAL SITUATION:**

Let \( p \) be a prime spot of \( F \), and \( F_p \) the completion of \( F \) at \( p \). If \( p \) is finite, we let \( R_p \) stand for the ring of integers of \( F_p \). \( V \) will be a regular quadratic space over \( F \), and \( L \) is an \( R \)-lattice on \( V \). We let \( V_p \) denote a fixed localization of the quadratic space \( V \) at a prime spot \( p \). The lattice \( L_p \) will be the localization of \( L \) in \( V_p \) at any discrete spot \( p \). The notations and terminologies in the General case above will be carried over. In addition we have a natural topology on \( O^+(V_p) \) as well as on \( O^+(L_p) \), making these groups locally compact.

For any local \( R_p \)-Lattice \( M \) of rank \( n \), there is a base \( \{x_1,\ldots,x_n\} \) such that \( M = R_p x_1 + \ldots + R_p x_n \). \( M \) is called \( \alpha \)-modular if \( sM = \alpha \) and \( (\det(x_1,\ldots,x_n)) = \alpha^n \), where \( \det(x_1,\ldots,x_n) = \det(B(x_i, x_j)) \) is called the determinant of \( M \). \( M \) is said to be unimodular if \( M \) is \( R_p \)-modular, and it is modular if it is \( \alpha \)-modular for some \( R \)-module \( \alpha \) of \( F \). Any local lattice \( M \) can be decomposed as \( M = M_1 \perp \ldots \perp M_t \) in which
each component is modular and \( sM_1 \supseteq sM_2 \supseteq ... \supseteq sM_t \). Any such decomposition is called a **Jordan decomposition** of \( M \). \( M \) is called *almost unimodular* if it is an orthogonal sum of an unimodular lattice and a \( g \)-modular lattice, where \( g \) is the prime ideal of \( R_p \).

Given an \( R_p \)-lattice \( M(p) \) of \( V_p \) at each discrete spot \( p \). Suppose that there is an \( R \)-lattice \( L \) on \( V \) with \( L_p = M(p) \) for almost all \( p \) (almost means for all but finitely many). Then there is an \( R \)-lattice \( M \) on \( V \) with \( M_p = M(p) \) for all discrete spots \( p \).

**GLOBAL SITUATION:**

Let \( \Omega \) be the set of all prime spots of \( F \), and \( \infty \) be the set of all infinite spots. Let \( L \) be a global \( R \)-lattice on \( V \). The group of split rotations \( O^+(V) \) of \( V \) is defined as \( O^+(V) = \{(\sigma_p)_{p \in \Omega} \mid \sigma_p \in O^+(L_p) \text{ for almost all finite spots } p\} \). \( O^+(V) \) is independent of the choice of \( L \). Given any finite subset \( S \) of \( \Omega \) with \( \infty \subseteq S \), define \( O^+_S(V) = \prod_{p \in S} O^+(V_p) \times \prod_{p \in S} O^+(L_p) \). It has the product topology which is also locally compact. It is clear that \( O^+_S(V) = \cup O^+_S(V) \) where the union runs over all such finite set \( S \). We define the topology on \( O^+_S(V) \) in such a way that all \( O^+_S(V) \) will be open. \( O^+_S(V) \) is locally compact. Through identifying \( O^+(V) \) with the diagonal subgroup \( \{(\sigma_p)_{p \in \Omega} \mid \sigma_p = \sigma \text{ for some } \sigma \in O^+(V)\} \) of \( O^+_S(V) \), \( O^+(V) \) becomes a discrete, closed subgroup of \( O^+_S(V) \).

For any element \( \sigma = (\sigma_p) \) of \( O^+_S(V) \), and any global lattice \( L; \sigma L \) is defined to be the global lattice with \( (\sigma L)_p = \sigma_p L_p \) for all discrete spots \( p \). Let \( O^+_S(L) = \{\sigma \in O^+_S(V) \mid \sigma L = L\} \); and \( O^+_S(V) = \prod_{p \in \Omega} O^+(V_p) \). The **proper class** \( \mathrm{cls}^+(L) \) of \( L \) is defined to be \( \mathrm{cls}^+(L) = O^+(V)L \); the **proper spinor genus** \( \mathrm{spn}^+(L) \) of \( L \) is defined as \( \mathrm{spn}^+(L) = O^+(V)O^+_S(V)L \); and the **genus** \( \mathrm{gen}(L) \) of \( L \) is defined to be \( \mathrm{gen}(L) = O^+_S(V)L \). A
lattice \( K \) (global) is said to be represented by the genus of \( L \) (resp. by the proper spinor genus of \( L \)) if there is a lattice in \( \text{gen}(L) \) (resp. in \( \text{spn}^+(L) \)) which represents \( K \).

\[\text{§ 0.2. Facts about Haar Measure}\]

In this section, we will concentrate on those facts concerning the theory of Haar measure, which form the basis for our exposition in chapter II. Most results presented here can be found in [B] and [N].

\[\sigma - \text{RING}:\]

Let \( X \) be a locally compact topological space; it is assumed that \( X \) is Hausdorff. A family \( \mathcal{S} \) of subsets of \( X \) is called a \( \sigma\)-ring: if it is closed under the formation of set theoretic differences and countable unions; in other words, if \( A, B \) and \( A_n (n = 1,2,...) \) are all in \( \mathcal{S} \), then so are \( A - B \) and \( \cup_{n\geq 1} A_n \). The \( \sigma \)-ring generated by all the compact sets in \( X \) consists of the \textit{Borel sets} in \( X \). A subset \( B \) of \( X \) is said to be bounded if there exists a compact set \( C \) such that \( B \subseteq C \); \( B \) is said to be \( \sigma \)-bounded if there exist a sequence \( C_n \) of compact sets such that \( B \subseteq \cup C_n \). Every Borel set is \( \sigma \)-bounded. The \( \sigma \)-ring generated by all the closed sets in \( X \) consists of the \textit{weakly Borel sets}. Evidently it is also the \( \sigma \)-ring generated by all the open sets. The relationship between weakly Borel sets and the Borel sets is the following:
**THEOREM 0.2.1.** The class of Borel sets coincides with the class of all \( \sigma \)-bounded weakly Borel sets. In particular, if \( X \) itself is \( \sigma \)-bounded, then the class of Borel sets is the same as the class of weakly Borel sets. (See [B] page 181)

**MEASURE:**

A *measure* on \( X \) is a set function \( \mu \) from a \( \sigma \)-ring \( \mathcal{S} \) of subsets of \( X \) to the extended non-negative real numbers such that:

1. \( \mu(\emptyset) = 0 \);

2. \( \mu(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i) \); where all \( A_i \) are in \( \mathcal{S} \) and \( A_i \cap A_j = \emptyset \) for all \( i \neq j \).

The sets in \( \mathcal{S} \) are called *measurable* sets.

A *Borel measure* on \( X \) is a measure \( \mu \) defined on the class of all Borel sets such that \( \mu(C) < \infty \) for every compact set \( C \).

Let \( \mathcal{K} \) be the class of all compact sets in \( X \), \( \mathcal{O} \) the class of all open Borel sets, \( \mathcal{S} \) the class of all Borel sets, and \( \mu \) a Borel Measure on \( X \). A set \( E \) in \( \mathcal{S} \) is said to be *inner regular* if \( \mu(E) = \max \{ \mu(C) \mid C \subseteq E, C \in \mathcal{K} \} \); *outer regular* if \( \mu(E) = \min \{ \mu(O) \mid E \subseteq O, O \in \mathcal{O} \} \); and \( E \) is said to be *regular* if it is both inner and outer regular. We say that \( \mu \) is a *regular* measure if every set in \( \mathcal{S} \) is regular.

**HAAR MEASURE:**

Now let \( G \) be a locally compact topological group. A *left Haar measure* on \( G \) is a regular Borel measure \( \mu \) such that (1) \( \mu(sE) = \mu(E) \) for all Borel sets \( E \) and all \( s \) in \( G \); (2) \( \mu \) is not identically zero.
THEOREM 0.2.2. Up to a positive constant factor, there is one and only one left Haar measure on G. (See [B] page 259)

Similarly we can define the right Haar measure on G; and the above theorem also holds for right Haar measure. Let μ be a left Haar measure on G. We define for every \( t \in G \), the set function \( μ' \) on the class \( \mathcal{B} \) of all Borel sets of G by \( μ'(E) = μ(Εt) \) for all \( E \in \mathcal{B} \). Actually \( μ' \) is also a left Haar measure. By the above theorem, there exists for every \( t \) a unique number \( Δ(t) > 0 \), such that \( μ'(E) = μ(Εt) = Δ(t)μ(E) \). It turns out that \( Δ(t) \) is independent of the choice of \( μ \) in the first place, and it is a continuous homomorphism from G to the multiplicative group of positive real numbers. It is called the right-hand module of G. More precisely, we sometimes use \( Δ_τ(t) \) in place of \( Δ(t) \). Similarly, if we start out from a right Haar measure \( ν \) on G, we can define a function \( Δ_ν(t) \) from G to the multiplicative group of positive real numbers such that \( ν(τΕ) = Δ_ν(τ)ν(E) \) for all \( τ \) in G and all \( E \) in \( \mathcal{B} \). \( Δ_ν(t) \) is called left-hand module. In fact \( Δ_τ(t)Δ_ν(τ) = 1 \) for all \( τ \in G \). A locally compact group G is said to be unimodular if \( Δ_τ(t) = Δ_ν(t) = 1 \) for all \( τ \in G \); i.e., left Haar measure is also right Haar measure and vice versa. The group \( O_λ^+(V) \) of split rotations of V and its subgroup \( O^+(V) \) introduced in the first section are in fact unimodular. Since \( O^+(V_p) \) is \( σ \)-bounded, so is \( O_λ^+(V) \) for any finite set of spots \( S \) with \( ∞ \subset S \subset Ω \); and so is \( O_λ^+(V) \).

HAAR MEASURE ON HOMOGENEOUS SPACE:

Let G be a topological group; H be a closed subgroup of G. We denote by \( H \backslash G \) the quotient space (of right cosets of G modulo H) equipped with quotient topology. Naturally G acts topologically on \( H \backslash G \) on the right side. We call \( H \backslash G \) the topological homogeneous space. Further if G is a locally compact group, then \( H \backslash G \) is a locally
compact (Hausdorff) topological space. In this particular setting, we can ask whether there is a regular Borel measure $\mu$ on $H \backslash G$, such that (1) $\mu(ES) = \mu(E)$ for all Borel sets $E$ in $H \backslash G$ and all $s$ in $G$; (2) $\mu$ is not identically zero. Such a measure on $H \backslash G$ is called a $G$-invariant Haar Measure, or sometimes simply a Haar measure on $H \backslash G$.

**THEOREM 0.2.3.** If $G$ and $H$ are both unimodular, then there is a (G-invariant) Haar measure on $H \backslash G$, and it is unique up to a positive factor. (See [N] page 138)

In chapter II, we are going to apply these results to the situation where $G = O^+_A(V)$ and $H$ = some subgroup of $O^+(V)$; and both of them are unimodular.
CHAPTER I

RELATIVE SPINOR NORM GROUPS

In the classification theory of integral quadratic forms over a global field it is important that one classifies the spinor genera within a given genus. In particular, in computing the number of spinor genera of a given genus, it is necessary to compute the spinor norm of the group of local integral rotations at each prime spot. These computations have been performed by Kneser [Kn] when the spot in question is non-dyadic. In the dyadic cases where 2 is prime this problem has been solved by Earnest and Hsia [EH]. For some work on this line over general dyadic fields, see [H1], [EH1], [BD], [X1] and [X2].

On the other hand, recent developments [Kn2], [H2], [Sp1] and [HSX] show that the spinor norm group of some relative local integral rotations (to be defined later) plays an important role in the representation theory of quadratic forms. Schulze-Pillot calculated this group for the cases when a rank 1 lattice is represented by the main lattice of rank 3 and in which the local field is non-dyadic or 2-adic [Sp1]. Here we deal with the problem over the same fields but with no restrictions whatsoever on the ranks of the lattices involved. In the first section we give a very brief account of the most recent developments in [HSX] which reveals the usefulness as well as the importance of this relative spinor norm group. For more details, see [HSX]. In the second section
we give a formula for this group over a non-dyadic local field; and in sections 3, 4 and 5, we develop several formulas which, taken together, completely determine these relative spinor norm groups over any 2-adic local fields. Finally in the last section, section 6, we show how to recover Satz 3 and Satz 4 of [Sp1] by our formulas developed in earlier sections, and also correcting a minor discrepancy in Theorem 3.14(iv) of [EH].

§ 1.1 Representations By Spinor Genus

We use the geometric language of quadratic spaces and lattices. Unexplained terminologies and notations are generally those from [OM]. Let $V$ be a quadratic space, of dimension greater than or equal to 3, defined over an algebraic number field $F$ supporting a $R$-lattice $L$ of full rank, where $R$ is the ring of integers of $F$. Similarly, let $K$ be a $R$-lattice on a subspace $U$ of $V$. All spaces are understood to be non-degenerate and all lattices have integral scales. Let $U^\perp$ be the orthogonal complement of $U$ and $\delta = \dim (U^\perp)$. Suppose that $K$ is $p$-adically representable by $L$ for every prime spot $p$. We ask the following question:

"How many proper spinor genera in the genus of $L$ represent $K$?"

A classical result asserts that at least one lattice in the genus of $L$ actually represents $K$. So for the problem we are concerned with we may, without loss of generality, assume that $K \subseteq L$. Let $O^+(V)$ be the proper orthogonal group of $V$. Let $O^+_\delta(V)$ be the group of split rotations of $V$. $O^+_\delta(V)$ acts on the set of all full lattices of $V$ and $O^+_\delta(L)$ denotes the stabilizer of $L$. The genus $\text{gen}(L)$ of $L$ is the $O^+_\delta(V)$ - orbit of $L$. If $J_F$ is the group of
ideles of $F$ and $O^{\times}_A(V)$ the kernel of the spinor norm map $\Theta: O^{\times}_A(V) \to J_F / J_F^2$, the proper spinor genus $\text{spn}^+(M)$ of $M$ in $\text{gen}(L)$ is the $O^{\times}_A(V)O^{\times}_A(V)$-orbit of $M$ and the number $g^+(L)$ of proper spinor genera in the $\text{gen}(L)$ is given by the group index $[O^{\times}_A(V) : O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L)]$, which equals to the idelic index $[J_F : F^* \Theta(O^{\times}_A(L))]$. See [EsH]. Denote $O^{\times}_A(U^\perp)$ the subgroup of $O^{\times}_A(V)$ which are trivial on $U$. An $u = \{u_p\}$ in $O^{\times}_A(V)$ is called an generator for $L/K$ if $K \subseteq uL$, and let $X_A(L,K)$ be the set of all generators for $L/K$. Let $O^{\times}_A(L,K)$ be the subgroup in $O^{\times}_A(V)$ generated by $X_A(L,K)$. Clearly, $O^{\times}_A(L,L) = O^{\times}_A(L)$. One sees easily that $O^{\times}_A(\rho L, \rho K) = \rho O^{\times}_A(L,K)\rho^{-1}$ for any $\rho \in O^{\times}_A(V)$; and $O^{\times}_A(uL,K) = O^{\times}_A(L,K)$ for any $u \in X_A(L,K)$. Now we list some of the results in [HSX] without proofs.

1.1.1. Let $K' \subseteq L'$ and $K' \subseteq L$ where $L' \in \text{gen}(L)$ and $K' \in \text{gen}(K)$. Then $O^{\times}_A(L',K')$ and $O^{\times}_A(L,K)$ are conjugate in $O^{\times}_A(V)$. So that the subgroup $O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L,K)$ is independent of the choices of $L$ and $K$.

1.1.2. $O^{\times}_A(V) / O^{\times}(V)O^{\times}_A(V)O^{\times}_A(U^\perp) \equiv J_F / F^* \Theta(O^{\times}_A(L))\Theta(O^{\times}_A(U^\perp))$; and $O^{\times}_A(V) / O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L,K) \equiv J_F / F^* \Theta(O^{\times}_A(L,K))$.

1.1.3. Let $K \subseteq L$ and $u \in O^{\times}_A(V)$. Then $K$ is represented by $\text{spn}^+(uL)$ if and only if $u$ belongs to $O^{\times}(V)O^{\times}_A(V)g$ for some $g \in X_A(L,K)$.

With the above, we have the following containment chain

1.1.4. $O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L) \subseteq O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L)O^{\times}_A(U^\perp) \subseteq O^{\times}(V)O^{\times}_A(V)X_A(L,K) \subseteq O^{\times}(V)O^{\times}_A(V)O^{\times}_A(L,K) \subseteq O^{\times}_A(V)$. 
**Theorem 1.1.5.** \( O^+_A(V)X_A(L,K) = O^+_A(V)O^+_A(L,K). \)

So the proper spinor genera in \( \text{gen}(L) \) which represent \( K \) form a subgroup \( O^+_A(V)O^+_A(L,K)/O^+_A(V)O^+_A(V)O^+_A(L) \) of \( O^+_A(V)/O^+_A(V)O^+_A(V)O^+_A(L) \).

By 1.1.2., 1.1.3, and Theorem 1.1.5. we know that the number \( r^+(L, K) \) of proper spinor genera in the \( \text{gen}(L) \) representing \( K \) is given by the following indices:

1.1.6. \([O^+(V)O^+_A(V)O^+_A(L,K) : O^+(V)O^+_A(V)O^+_A(L)] = [F^*\Theta(O^+_A(L,K)) : F^*\Theta(O^+_A(L))].\)

Suppose first that the codimension of \( K \) in \( L \) satisfies either \( \delta \geq 3 \) or \( \delta = 2 \) and \( U \perp \) is isotropic. Then \( F^*\Theta(O^+_A(U \perp)) = J_F \), so that \( K \) is representable by all the proper spinor genera in the \( \text{gen}(L) \).

Consider next \( \delta = 2 \) and \( U \perp \) is anisotropic. The theory in this case had been studied earlier in \([JW],[Kn_2],[H_2],[Sp_1]\) although the computations for \( \Theta(O^+_A(L,K)) \) had only been done for rank \( K = 1 \). Since \([J_F : F^*\Theta(O^+_A(U \perp))] = 2 \) we see that \( J_F \neq F^*\Theta(O^+_A(L,K)) \) if and only if \( \Theta(O^+_A(L,K)) \) is contained in \( \Theta(O^+_A(U \perp)) \). When this is the situation we call \( K \) a **spinor exception** in which case \( K \) is represented by precisely half of all the proper spinor genera. We have

**Theorem 1.1.7.** Suppose \( K \) is represented by \( \text{gen}(L) \) with \( \delta = 2 \) and \( U \perp \) being anisotropic. Then \( K \) is a spinor exception if and only if \( \Theta(O^+_A(L,K)) = \)
\[ \Theta(O^+_\Lambda(U^\perp)). \] A proper spinor genus \( \text{spn}^+(uL) \) represents \( K \) if and only if \( \Theta(u) \in F^*\Theta(O^+_\Lambda(L,K)) \).

We now consider the main case, \( \delta = 1 \). From the above we know that for given lattices \( K \) and \( L \) the set of proper spinor genera in the \( \text{gen}(L) \) which represent \( K \) corresponds to a subgroup of \( O^+_\Lambda(V)/O^+(V)O^+_\Lambda(V)O^+_\Lambda(L) \); namely, \( O^+(V)O^+_\Lambda(V)O^+_\Lambda(L, K)/O^+(V)O^+_\Lambda(V)O^+_\Lambda(L) \). Consider next the converse; i.e. given an arbitrary subgroup \( \Gamma \) of \( O^+_\Lambda(V)/O^+(V)O^+_\Lambda(V)O^+_\Lambda(L) \) are there lattices \( K \) of codimension 1 in \( L \) which are represented by precisely those \( \text{spn}^+(uL) \) where \( u \) (mod \( O^+(V)O^+_\Lambda(V)O^+_\Lambda(L) \)) belongs to \( \Gamma \)?

Following [EsH], the spinor class field \( \Sigma \) of the \( \text{gen}(L) \) is the unique abelian extension associated by class field theory to the open subgroup \( F^*\Theta(O^+_\Lambda(L)) \). Similarly, let \( \Sigma_{(L,K)} \) be the subfield of \( \Sigma \) corresponding to \( F^*\Theta(O^+_\Lambda(L,K)) \). We have the field chain \( F \subseteq \Sigma_{(L,K)} \subseteq \Sigma \). We identify \( O^+_\Lambda(V)/O^+(V)O^+_\Lambda(V)O^+_\Lambda(L) \) with the Galois group \( G \) of the spinor class field \( \Sigma \) over \( F \) via the Artin reciprocity map. Since \( \Sigma/F \) is an elementary abelian 2-extension, every subgroup \( \Gamma \) of \( G \) is also an \( F_2 \)-vector space. Let \{ \sigma_1, ..., \sigma_t \} be a basis for \( \Gamma \). By Tchebotarev's density theorem, choose primes \( p_k \) of \( F \) away from the divisors of \( 2\text{vol}(L) \) — hence, unramified in \( \Sigma \) — such that for each \( k \), \( \sigma_k \) is the Artin symbol \( [\Sigma/F : p_k] \).

We construct the lattice \( K \). For each prime \( p \) dividing \( 2\text{vol}(L) \) choose a local characteristic sublattice \( J_{(p)} \) of \( L_p \) in the sense of [Ki], and for \( p = p_k \) (\( k = 1, ..., t \)) choose a sublattice \( N_{(p)} \) of \( L_p \) of corank 1 and of prime determinant. By the Lemma 1.6 of [HKK] construct a sublattice \( K \) of \( L \) which is close to \( J_{(p)} \)'s and the \( N_{(p)} \)'s and, in addition, having order \( \text{ord}_p(\text{det}K_p) = 0 \) everywhere else except for a single prime spot \( q \).
at which place it has order 1. Next, we assert that there are exactly $2^{t+1}$ distinct lattices in $\text{gen}(L)$ that contain $K$. Indeed, any lattice $M \in \text{gen}(L)$ containing $K$ must satisfy $M_p = L_p$ at every $p$ away from $\{p_1, ..., p_t, \ q\}$. At each of the latter $t+1$ prime spots there are exactly two possibilities. To see this, let $<w>$ be the orthogonal complement of $K$ in $M$. Put $K_p = Y_p \perp <v_p>$ where $Y_p$ is unimodular and $Q(v_p) \in \mathbb{P} \mathbb{R}_p^{*}$. We have then $M_p = K_p + <(v_p + aw)/\pi>$ where $\pi$ is a uniformizer at $p$, $a \in \mathbb{R}_p$ and $Q((v_p + aw)/\pi)$ is integral. Since both $Q(v_p)$ and $Q(w)$ are prime elements and $p$ is a non-dyadic prime, the equation

$$Q(v_p)/\pi + a^2Q(w)/\pi \equiv 0 \pmod{p} \quad (1.1.8.)$$

has exactly two solutions $+ / - a \pmod{p}$. We shall refer to these possibilities as the "q-plus" or "q-minus" members according to the sign at prime $q$. We observe that the global symmetry $S_w$ will map $M$ to a lattice which has the corresponding opposite sign at each of these $t+1$ primes. Thus, these $2^{t+1}$ lattices that contain $K$ are paired off isometrically via $S_w$. Denote by $M_{(1)}, ..., M_{(r)}$, $r = 2^t$, all the q-plus lattices which have undetermined signs elsewhere, and put $M_{(j)} := S_w(M_{(j)})$ for each $j$. It can be shown that there is no proper spinor equivalence relation among the lattices $M_{(1)}, ..., M_{(r)}$. (See [HSX]). Similarly, for the q-minus lattices we have $\text{spn}^+(M_{(i)}) \neq \text{spn}^+(M_{(j)})$ for $i \neq j$. Hence, $2^t \leq r^+(L,K) \leq 2^{t+1}$. Suppose $g^+(L) = g(L)$ (e.g., when rank($L$) is odd) then $\text{spn}(M_{(i)}) = \text{spn}^+(M_{(i)})$ for all $i$ and then $r^+(L,K) = r(L,K) = 2^t$. When $g^+(L) = 2g(L)$ and $\text{spn}^+(M_{(i)}) = \text{spn}^+(M_{(j)})$ for some $i \neq j$; or equivalently, the symbol $[\Sigma/F : q] \in \Gamma$ and is unequal to $[\Sigma/F : p_1...p_t]$ , then $r^+(L,K) = 2^t$ and $r(L,K) = 2^{t-1}$. If $g^+(L) = 2g(L)$ and $[\Sigma/F : q] \notin \Gamma$ then $\text{spn}^+(M_{(i)}) \neq \text{spn}^+(M_{(j)})$ for any $i$ and $j$, in which case $r^+(L,K) = 2^{t+1} = 2r(L,K)$. For a fractional ideal $\mathfrak{I}$ of $F$ let $\mathfrak{I}^*$ denote the part which is
prime to 2\text{vol}(L)$, then $[\Sigma/F : 3^*]$ is defined. Put $\mu(K) = 0$ if $[\Sigma/F : \text{vol}(K)^*] \in \Gamma$, and $= 1$ otherwise. Set $\lambda(K) = -1$ if $1 \not\in [\Sigma/F : \text{vol}(K)^*] \in \Gamma$, and $= 0$ otherwise.

**THEOREM 1.1.9.** Let $L$ be a quadratic lattice of rank $\geq 3$. For a subgroup $\Gamma$ of the group $O_+^+(V)/O_+^+(V)O_+^+(L)$ of proper spinor genera in the $\text{gen}(L)$ there always exists a sublattice $K$ of $L$ of codimension one such that $K$ is represented by precisely those proper spinor genera in $\{ \text{spn}^+(\gamma L), \text{spn}^+(\gamma L^*) \mid \gamma \in \Gamma \text{ and } L^* \in \text{cls}^+(L) \}$, which has cardinality $r^+(L,K) = 2t + \mu(K)$. The spinor genera representing $K$ are exactly those on $\{ \text{spn}(\gamma L) \mid \gamma \in \Gamma \}$ which has cardinality $r(L,K) = 2t + \lambda(K)$. A proper spinor genus $\text{spn}^+(uL)$ represents $K$ if and only if $[\Sigma/F : \mathcal{O}(u)] \in \text{Gal}(\Sigma/\Sigma_{L,K})$.

For the final case when $\delta = 0$, we could use the $\delta = 1$ case. For, we may just orthogonally adjoin to $K$ any rank one component $<z>$ where $z$ belongs to every $M(j)$, $j = 1, \ldots, 2^t$. Hence the Theorem 1.1.9 remains valid for $\delta = 0$.

In the rest of this chapter we are going to calculate the group $\Theta(O_+^+(L,K))$, the relative spinor norm group. Since this is really the product of local relative spinor norm groups, throughout below everything is local. Thus, $F$ is either a non-dyadic or an unramified dyadic local field, $R$ its ring of integers and $\varrho = \pi R$ the unique maximal ideal and $R^*$ the group of units. When $F$ is dyadic we let $\pi = 2$. Let $K \subseteq L$ be two regular $R$-lattices. $\Theta$ is the spinor norm function. Let $X(L,K) = \{ \varphi \mid \varphi \in O^+(FL); K \subseteq \varphi(L) \}$. $\Theta(L,K)$ is the group generated by $\Theta(X(L,K))$. Our objective is to calculate this relative spinor norm group. We first make two simple observations about this group:

**1.1.10.** $X(L,K) = O^+(K)X(L,K)O^+(L)$; $\Theta(L,K) \supseteq \Theta(O^+(K))\Theta(O^+(L))$. 
1.1.11. If $K' \subseteq L$ with $K' \cong K$, then $\Theta(L, K) = \Theta(L, K')$.

§ 1.2 Formula Over Non-dyadic Fields

Write $L = L_0 \perp \ldots \perp L_t$, $K = K_0 \perp \ldots \perp K_t$ the Jordan decompositions where the $i$-th component of each denotes either a $q^i$-modular lattice or zero. Let $L_{(i)} := L_0 \perp \ldots \perp L_i$, $K_{(i)} := K_0 \perp \ldots \perp K_i$, $\mu_i := \det(FL_{(i)})$, $\nu_i := \det(FK_{(i)})$. By convention, the determinant of zero lattices is unity. Define for any lattice $M$ the even, resp. odd, rank, denoted by $\text{rank}^+(M)$, resp. $\text{rank}^-(M)$, to be the sum of the ranks of those Jordan components whose scales have even, resp. odd, order. WLOG, we shall assume always $sL = R$. We further define three groups:

$$G_1(L/K) = \begin{cases} F^* & \text{if } \exists i \text{ s.t. either } \text{rank} L_{(i)} - \text{rank} K_{(i)} \geq 3 \text{ or } \text{rank} L_{(i)} - \text{rank} K_{(i)} = 2 \text{ and } \mu_i \nu_i = -1 \\ F^* & \text{otherwise} \end{cases}$$

$$G_2(L/K) = \begin{cases} R^*F^* & \text{if } \exists i \text{ such that either } \text{rank} L_{(i)} - \text{rank} K_{(i-1)} \geq 2 \\ F^* & \text{otherwise} \end{cases}$$

$$G_3(L/K) = \begin{cases} R^*F^* & \text{if } \exists i \text{ s.t. either } \text{rank}^+ L_{(i)} - \text{rank}^+ K_{(i-1)} \geq 2 \text{ or } \text{rank}^- L_{(i)} - \text{rank}^- K_{(i-1)} \geq 2 \\ F^* & \text{otherwise} \end{cases}$$

we have the following result:

**Theorem 1.2.1.** Assume $K \subseteq L$ with arbitrary codimension, then
\( \Theta(L,K) = \Theta(O^+(L))G_1(L/K)G_2(L/K)\Theta(O^+(FK^\perp)) \) if \( \Theta(O^+(L)) \subseteq R^*F^*2 \)
\( \Theta(L,K) = \Theta(O^+(L))G_3(L/K)\Theta(O^+(FK^\perp)) \) otherwise.

It is easy to see that if \( L = K \) then the right hand side of THEOREM 1.2.1. reduces to just \( \Theta(O^+(L)) \). Also, when \( K \) is of rank one and \( L \) is of rank three then Satz3 of [Sp1] is recovered. See section 6 below.

To prove the theorem, it is first necessary to observe three reduction steps.

**Lemma 1.2.2.** Assume that \( s_L = s_K \) and \( x \in K \) where \( Q(x)R = s_L \). Then \( \Theta(L,K) = \Theta(O^+(L))\Theta(L^*,K^*) \) where \( L = Rx_LL^* \) and \( K = Rx_LK^* \).

**Lemma 1.2.3.** Assume that \( L = L_0\perp M \) where \( L_0 \) is modular and \( s_{L_0} \supset s_M \), \( s_{L_0} \supset s_K \). If \( L_0 \) is anisotropic then \( \Theta(L,K) = \Theta(L^*,K) \) where \( L^* = \pi L_0\perp M \). (These containments are strict containments.)

**Lemma 1.2.4.** Assume that \( L = L_0\perp M \) where \( L_0 \) is modular and \( s_{L_0} \supset s_M \), \( s_{L_0} \supset s_K \). If \( L_0 \) is isotropic then \( \Theta(L,K) = F^* \). (These containments are strict containments.)

The proofs of Lemma 1.2.2. and Lemma 1.2.3. are routine and omitted. To see Lemma 1.2.4., we may assume by scaling that \( s_L = s_{L_0} = R \). Suppose first that \( K \subseteq \pi L_0\perp M \). Let \( H = Re_1 + Re_2 \supseteq A(0,0) \) be a hyperbolic plane in \( L_0 \). Then \( \sigma = Se_1-e_2Se_1 \), \( \pi e_2 \in X(L/K) \) with \( \Theta(\sigma) = \pi \). Since \( \Theta(L,K) \supset \Theta(O^+(L)) \supset R^*F^*2 \) we have \( \Theta(L,K) = F^* \). Next, if \( K \) is not contained in \( \pi L_0\perp M \) then there exists \( x \in K \), \( x = w_1+w_2 \) with \( Q(w_1) \in R^* \) and \( B(w_1,w_2) = 0 \). Since \( Q(x) \in \varnothing \) we see that \( x \) may be embedded in a
hyperbolic plane $H = Rw_1 + Rw_2 = Re_1 + Re_2 \equiv A(0,0)$ of a first Jordan component of $L$. By the primitivity of $x$ in $H$, we may suppose that $x = e_1 + ae_2$, $a \in \mathcal{O}$. So, $\text{Proj}_H(K) \subset Re_1 + R\pi e_2$ since $sK \subset \mathcal{O}$ and $H$ splits $L$. Thus $\sigma = S_{e_1,\pi e_2} S_{e_1,e_2} \in X(L/K)$ with $\Theta(\sigma) = \pi$ and again $\Theta(L,K) = F^*$. Q.E.D.

We prove Theorem 1.2.1 by induction on the rank of $L$. We may suppose that $\Theta(O^+(L)) \neq F^*$. Observe that during reduction steps Lemma 1.2.2. and Lemma 1.2.3., the codimension remains unchanged and the same holds for the order-parities of the scales of the Jordan components, $\Theta(O^+(FK^\perp))$, $\text{rank} L_{(i)} - \text{rank} K_{(i)}$ — remembering to upscale $L^*$ back to $R$ after applying Lemma 1.2.3, and $\mu_j v_i$. For convenience, denote $\Theta_L := \Theta(O^+(L))$, $U := FK^\perp$, $X := X(L,K)$, $\Theta(U) := \Theta(O^+(U))$.

(1) Suppose that $sL = sK = R$. Using Lemma 1.2.1. and writing $X^* := X(L^*/K^*)$, we have $\Theta(X) = \Theta_L \Theta(X^*)$. Moreover, $G_1(L^*/K^*) = G_1(L/K)$. Also, $G_j(L^*/K^*) = G_j(L/K)$ for $j = 2, 3$ unless $\text{rank} L_0 = 2$ in which case $\Theta_L G_j(L/K) = \Theta_G L_j(L^*/K^*)$.

If $\Theta_L \subseteq R*F^{*2}$ then, using induction, $\Theta(X) = \Theta_L G_1(L/K) G_2(L/K) \Theta(U)$ since $\Theta_L \subseteq \Theta_L$. The scales of all the Jordan components of $L^*$ have the same order parity. When $\text{ord}_{ps} L^*$ is even, then $\Theta_L \subseteq R*F^{*2}$ and $\Theta(X)$ has the required form. Now, let $\text{ord}_{ps} L^*$ be odd. If $\Theta_L \neq R*F^{*2}$ then already $\Theta_L = F^*$. On the other hand, if $\Theta_L \neq F^{*2}$ then $\Theta_L$ is a mixed subgroup of index two in $F^*$ (i.e., containing elements of different order parities). In this case all Jordan components of $L$ are of one-dimensional and all Jordan components of $L^*$ are of the same (odd-) type. One then observes that $G_1(L/K) G_2(L/K) \neq F^{*2}$ if and only if $G_3(L/K) \neq F^{*2}$. Hence, $\Theta(X) = \Theta L \Theta(X^*) = \Theta_L G_1(L/K) G_2(L/K) \Theta(U) = \Theta_L G_3(L/K) \Theta(U)$ as required.

If $\Theta_L \neq R*F^{*2}$ then $\Theta_L \neq R*F^{*2}$. Hence, $\Theta(X) = \Theta_L G_3(L^*/K^*) \Theta(U)$ by induction hypothesis and this equals $\Theta_L G_3(L/K) \Theta(U)$.  

(II) Now, let $sK \subset R = sL$. If $L_0$ has either rank $\geq 3$ or else rank $= 2$ & $\det L_0 = \mu_0 = -1$ then Lemma 1.2.4 provides $\Theta(X) = F^*$, $G_1(L/K) = F^*$, $G_2(L/K) = G_3(L/K) = R^*F^*2$ and $\Theta(X)$ is as required. If rank of $L_0 = 2$, $\det L_0 \neq -1$ then $G_2(L/K) = G_3(L/K) = R^*F^*2$. Since $\Theta(O^+(L)) = F^*$ has been ruled out, $\Theta_L = uF^*2$. Lemma 1.2.3. gives $\Theta(X) = \Theta(X(L^*/K))$. And $\Theta_L = \Theta_L = uF^*2$, $G_j(L^*/K) = G_j(L/K)$ for $j = 1, 2, 3$. This has the effect of lowering the scale of $L$ by $\varphi^2$, or equivalently, upscaling $K$ by $\varphi^{-2}$. Continuing until the scale of $K$ agrees with that of $L$ in which case we are done by (I). The case of $\text{rank} L_0 = 1$ is analogously argued. Q.E.D.

§ 1.3 Primary Reduction Over 2-adic Fields

In what follows we give a number of reduction steps which will be used in the calculation of later sections. If we write $L$ as $L = [r_1]\cdot L_1 \perp [r_2]\cdot L_2 \perp \ldots$, we mean that it is a Jordan decomposition of $L$ where the $i$-th component is $[r_i]\cdot L_i$ with $L_i$ being unimodular and $r_i$ being the order of the scale of that component. Also we assume $r_1 < r_2 < \ldots$. Similarly we write $K = [s_1]\cdot K_1 \perp [s_2]\cdot K_2 \perp \ldots$. Because of the 1.1.10., we may, if necessary, assume that

$$\Theta(O^+(L)) \neq F^* \text{ and } \Theta(O^+(K)) \neq F^*$$

which imposes lots of conditions on the structure of both $L$ and $K$. (See Theorem 3.14 of [EH] listed below.)

A modular lattice $J$ is of even-type (resp. odd-type, mixed-type) if $Q(P(J))$ is contained in $R^*F^*2$ (resp. in $2R^*F^*2$, in neither subsets), where $P(J) = \{ v \in J \mid S_v \in O(J) \}$. It is not difficult to see that modular lattice of rank $\geq 3$ can only be odd or mixed type. $D(\cdot)$ denotes for the quadratic defect function. Let $\Delta \in R^*$ be a unit with
D(Δ) = 4R. We list four most important results from [EH] which will be used frequently in our work.

**PROPOSITION 1.9 OF [EH]:** Let $L = <1> <2^r\alpha>$ where $r$ is an integer and $\alpha \in \mathbb{R}^*$. Then

$$\Theta(O^+(L)) = \begin{cases} 
\{ \gamma \in \mathbb{F}^* \mid (\gamma, -2\alpha) = 1 \} & \text{if } r = 1 \text{ or } 3 \\
\{ \gamma \in \mathbb{R}^*F^2 \mid (\gamma, -\alpha) = 1 \} & \text{if } r = 2 \\
F^2 \cup \alphaF^2 \cup F^2 \cup \alphaF^2 & \text{if } r = 4 \\
F^2 \cup 2^r\alphaF^2 & \text{if } r \geq 5.
\end{cases}$$

Let $L = <1> \perp <2^{r_2}\alpha_2> \perp ... \perp <2^{r_n}\alpha_n>$, where $\alpha_2, \alpha_3, ..., \alpha_n$ are all units. Define $L_{ij} = <2^{r_i}\alpha_i> \perp <2^{r_{i+1}}\alpha_{i+1}>; and r(L_{ij}) = r_{j+1} - r_j$.

**THEOREM 2.2 OF [EH]:** Suppose $L = <1> \perp <2^{r_2}\alpha_2> \perp ... \perp <2^{r_n}\alpha_n>$ with $1 < r_2 < ... < r_n$. Assume there is at least one $k$ for which $r(L_{k,k+1}) = 1$ or $3$. Then if $r_s - r_t = 2$ or $4$ for any $s, t = 1, ..., n$, we have $\Theta(O^+(L)) = F^*$. 

**THEOREM 2.7 OF [EH]:** Suppose that $L = <1> \perp <2^{r_2}\alpha_2> \perp ... \perp <2^{r_n}\alpha_n>$, with $1 < r_2 < ... < r_n$, does not satisfy the hypotheses of Theorem 2.2. Then $\Theta(O^+(L)) = \{ \Pi Q(v_i), \text{over any even number of } v_i \mid v_i \in \cup \l_1 \l_2 P(L_{ij}) \}$. 

**THEOREM 3.14 OF [EH]:** Suppose $L = \{r_1\} \cdot L_1 \perp \{r_2\} \cdot L_2 \perp ... \perp \{r_i\} \cdot L_i$ is a Jordan splitting for $L$, and rank$L_i \leq 2$ for those $i$ where $L_i$ is of even-type or of mixed-type, assuming further that at least one component is of rank $2$, say $L_{i_0}$ being binary. Then $\Theta(O^+(L))$ is determined as follows.

(i) If all components of $L$ have same type (even or odd), then $\Theta(O^+(L)) = \mathbb{R}^*F^2$. 

(ii) If there is a binary component of odd (even) -type, and a component of even (odd)
-type, then $\Theta(O^+(L)) = F^*$.

(iii) Suppose $L_j \equiv A(a_j, b_j)$ for some $i$ with $a_i, b_i \in R^*$.

If there is another binary component $[r_j] \ast L_j$ such that the associated space $F([r_j] \ast L_j)$ is not isometric to $F([r_j] \ast L_j)$, then $\Theta(O^+(L)) = F^*$.

(iv) Suppose $L_i \equiv A(a_i, 2b_i)$ whenever $\text{rank} L_i = 2$, then $\Theta(O^+(L)) \neq F^*$ iff

(a) all the associated spaces of all components are represented by $F([r_i] \ast L_{i_0})$;

(b) for any $j = 1, 2, ..., t - 1$; either $r_{j+1} \geq 4 + r_j$, or $r_{j+1} = 2 + r_j$ with $\text{rank} L_j = r_{j+1} = 1$ and $\det L_j \det L_{j+1} \det(F([r_i] \ast L_{i_0})) \in \{1, \Delta\}$.

In the exceptional cases described in (iv),

$$\Theta(O^+(L)) = \Theta(O^+(L_i)) = \{c \in F^* \mid (c, -\det L_i) = 1\}$$

For other results of calculating the group $\Theta(O^+(L))$ see [EH], [EH1], [X1] and [X2].

**LEMMA 1.3.1.** If $K = K_0 \perp K'$ with $K_0$ being modular and $sK_0 = sK = sL$, then $\Theta(L, K) = \Theta(L', K') \Theta(O^+(L))$ where $L = K_0 \perp L'$.

Proof: Clearly we have the inclusion that $\Theta(L, K) \supseteq \Theta(L', K') \Theta(O^+(L))$. WLOG we may assume that $K_0$ is unimodular. Now by induction on the rank of $K_0$, and the fact that the cancellation theorem works for the leading unimodular components of odd-type, we may assume that $\text{rank} K_0 = 1$. On the other side if $\Theta(O^+(L)) = F^*$, the reverse inclusion obviously holds. So we may further say that $\Theta(O^+(L)) \neq F^*$. Under these conditions we can see that for any $\sigma \in X(L/K)$, there is a $\tau \in O^+(L)$ such that $\sigma|_{K_0} = \text{id}_{K_0}$. Now $\sigma \tau \in X(L'/K')$ so $\sigma \in X(L'/K')O^+(L)$. Thus we have in general $X(L/K) = X(L'/K')O^+(L)$ which gives us the other side of the inclusion.

Q.E.D.
**Lemma 1.3.2.** Assume $L = L_1 \perp M_1 = L_2 \perp M_2$ with $L_1 \equiv L_2$ begin modular and $sL_1 = sL_2 \supset sM_1 = sM_2$. If $nM_1 = nM_2$ then there is $\sigma \in O^+(L)$ such that $\sigma L_1 = L_2$ or equivalently $M_1 \equiv M_2$.

Proof: After scaling we can assume that $L_1$ and $L_2$ are unimodular. If $sM_1 = sM_2 \neq R$, then $M_1$ and $M_2$ will have the same Jordan invariants. And since $L_1 \equiv L_2$, we may view $M_1$ and $M_2$ are on the same space. Now the conditions of 93.29 of [OM] are easily to be seen satisfied for the pair $M_1$ and $M_2$. So $M_1 \equiv M_2$, i.e., there is a $\sigma \in O^+(L)$ such that $\sigma L_1 = L_2$. Suppose $sM_1 = sM_2 = R$. Under our assumption that $nM_1 = nM_2$, again we see that $M_1$ and $M_2$ will have the same Jordan invariants. And same argument goes through. Q.E.D.

**Lemma 1.3.3.** Let $L = [r_1] \bullet L_1 \perp [r_2] \bullet L_2 \perp \ldots$; and $K = [s_1] \bullet K_1 \perp [s_2] \bullet K_2 \perp \ldots$. Assume $s_1 = r_1$ and $\Theta(O^+(L)) \neq F^*$, then $[r_1] \bullet L_1 \perp [r_2] \bullet L_2$ represents $[s_1] \bullet K_1$.

Furthermore, there is a $\sigma \in O^+(L)$ such that $\sigma([s_1] \bullet K_1) \subseteq [r_1] \bullet L_1 \perp [r_2] \bullet L_2$. And $\Theta(L,K) = \Theta(L',K') \Theta(O^+(L))$ where $K' \equiv [s_2] \bullet K_2 \perp [s_3] \bullet K_3 \perp \ldots$; and $L' = [r_1] \bullet L_1' \perp [r_2] \bullet L_2' \perp [r_3] \bullet L_3 \perp \ldots$ with $[r_1] \bullet L_1' \perp [s_1] \bullet K_1 \perp [r_2] \bullet L_2' \equiv [r_1] \bullet L_1 \perp [r_2] \bullet L_2$.

Proof: First we show that $[r_1] \bullet L_1 \perp [r_2] \bullet L_2$ represents $[s_1] \bullet K_1$. Since we always assume that $K \subseteq L$, rank$K_1 \leq$ rank$L_1$. Suppose rank$L_1 = 1$. If $r_3 < r_1 + 3$, then $\Theta(O^*([r_1] \bullet L_1 \perp [r_2] \bullet L_2 \perp [r_3] \bullet L_3)) = F^*$ by Proposition 1.19 of [EH] or Theorem 3.14 of [EH]. Now $r_3 \geq r_1 + 3$, then obviously $[r_1] \bullet L_1 \perp [r_2] \bullet L_2$ represents $[s_1] \bullet K_1$.

Suppose $L_1$ is of odd-type, then $K_1$ has no other choice but to be of odd-type. We know that $L_1$ is unique (up to isometry) in the Jordan decomposition of $L$ iff $\text{ord}(n([r_2] \bullet L_2)) > r_1 + 1$. So one can see that $[r_1] \bullet L_1 \perp [r_2] \bullet L_2$ represents $[s_1] \bullet K_1$. 
Suppose $L_1$ is of even-type, rank 2. (If $L_1$ is of even-type with rank > 2, then $\Theta(O^+(L)) = F^*$.) Because the type of $L_1$ is unique in the Jordan decomposition of $L$, we have $\text{rank} K_1 \leq 2$ and $K_1$ being of even-type. Write $K_1 = Ry_1 \perp K_1'$ where $K_1'$ is of rank $\leq 1$. Since $L_1$ represents all the units, it can be written as $L_1 = Rx_1 \perp Rx_2$ with $Q(x_1) = Q(y_1)$. By Lemma 1.3.2, there is a $\sigma \in O^+(L)$ such that $\sigma y_1 = x_1$, and $\sigma K = R\sigma y_1 \perp \sigma K_1' \perp [s_2] \cdot \sigma K_2 \perp \ldots$; so $\sigma K_1' \perp [s_2] \cdot \sigma K_2 \perp \ldots \subseteq Rx_2 \perp [r_2] \cdot L_2 \perp \ldots$. By the proved first case where $\text{rank} L_1 = 1$, we see $Rx_2 \perp [r_2] \cdot L_2$ represents $\sigma K_1'$ if it is not empty. Hence $[r_1] \cdot L_1 \perp [r_2] \cdot L_2$ represents $[s_1] \cdot K_1$.

Suppose $L_1$ is of mixed-type, rank 2. Same type of reasoning tells that $K_1$ is of rank 1, or $\text{rank} K_1 = 2$ and $K_1$ is of mixed type. Since $r_2 \geq r_1 + 4$ (see Theorem 3.14(iv) of [EH]), $[r_1] \cdot L_1$ represents $[s_1] \cdot K_1$.

Nor we have $[r_1] \cdot L_1 \perp [r_2] \cdot L_2 = [r_1] \cdot L_1' \perp [s_1] \cdot K_1' \perp [r_2] \cdot L_2'$, with $K_1' \equiv K_1$. By Lemma 1.3.2 (the conditions of the lemma are easily to be seen satisfied under all the above circumstances) there is a $\sigma \in O^+(L)$ such that $\sigma ([s_1] \cdot K_1) = [s_1] \cdot K_1'$. By 1.1.11, $\Theta(L,K) = \Theta(L,\sigma K)$ which, by Lemma 1.3.1, is equal to $\Theta(L',K')\Theta(O^+(L))$ where $K' = \sigma ([s_2] \cdot K_2 \perp [s_3] \cdot K_3 \perp \ldots) \equiv [s_2] \cdot K_2 \perp [s_3] \cdot K_3 \perp \ldots$; and $L' = [r_1] \cdot L_1' \perp [r_2] \cdot L_2' \perp [r_3] \cdot L_3 \perp \ldots$. Q.E.D.

**Lemma 1.3.4.** If $L_1$ is of the mixed-type and $\Theta(O^+(L)) \neq F^*$, then $\Theta(L,K) = F^*$ unless that all ambient spaces of all components of $K$ are represented by $F([r_1] \cdot L_1)$ and $r_2 \geq s_1 + 4$. In these exceptional cases, we have $\Theta(L,K) = \Theta(L',K')\Theta(O^+(L))$ where $K' \equiv [s_2] \cdot K_2 \perp \ldots$; and $L' = [s_1] \cdot (\det L_1 \det K_1) \perp [r_2] \cdot L_2 \perp \ldots$, if rank $K_1 = 1$; or $L' = [r_2] \cdot L_2 \perp \ldots$, if rank $K_1 = 2$.

Before giving a proof we make the following observation:
"Assume \([r_1] \bullet L_1 = \langle e_1, e_2 \rangle \subseteq [r_1] \bullet A(\xi, 2\delta)\) and \(r_1 < s_1\) then \(X(L/K) = X(L'/K)\) with \(L' = [r_1 + 1] \bullet A(2\xi, \delta) \perp [r_2] \bullet L_2 \perp \ldots\), and \(\Theta(O^+(L)) \subseteq \Theta(O^+(L'))\)."

Proof of this observation: because \(Q(x)\) is of order \(r_1\) for any \(x = ae_1 + be_2\) with \(a \in \mathbb{R}^*\). So we have \(X(L/K) = X(L'/K)\) with \(L' = \langle 2e_1, e_2 \rangle \perp [r_2] \bullet L_2 \perp \ldots\); clearly that \(\langle 2e_1, e_2 \rangle \subseteq [r_1 + 1] \bullet A(2\xi, \delta)\) and \(\Theta(O^+(L)) \subseteq \Theta(O^+(L'))\).

Proof of Lemma 1.3.4: Repeatedly apply the above observation we may assume \(s_1 = r_1\). By Th 3.14 (iv)(b) of [EH] we see that \(r_2 \geq s_1 + 4 = r_1 + 4\), otherwise \(\Theta(L, K) \supseteq \Theta(O^+(L)) = F^*\). Now Suppose that there is an ambient space \(F([s_j] \bullet K_j)\) of component \([s_j] \bullet K_j\) of \(K\) which is not represented by \(F([r_1] \bullet L_1)\). If rank \((K_j) = 2\), then already we have \(\Theta(O^+(L)) \supseteq \Theta(O^+([s_j] \bullet K_j)) = F^*\), and so \(\Theta(L, K) \supseteq \Theta(O^+(L)) \Theta(O^+(K)) = F^*\). If rank \((K_j) = 1\), say \([s_j] \bullet K_j = Rv\). Pick any vector \(w\) inside \([r_1] \bullet L_1\) such that the symmetry \(S_w\) is in \(O([r_1] \bullet L_1)\). One sees that \(S_v S_w \in X(L/K)\), but \(\Theta(S_v S_w)\) is not inside \(\Theta(O^+([r_1] \bullet L_1))\). So again \(\Theta(L, K)\) must be \(F^*\). Thus we established all the exceptional cases. The formula in these exceptional cases follows by applying Lemma 1.3.3.

Q.E.D.

**Lemma 1.3.5.** Suppose that \(L_1\) is of rank 1.

(i) If \(s_1 > r_1\), then \(\Theta(L, K) = \Theta(L', K)\) with \(L' = [r_1 + 2] \bullet L_1 \perp [r_2] \bullet L_2 \perp \ldots\).

(ii) If \(s_1 = r_1\), then \(\Theta(L, K) = \Theta(L', K') \Theta(O^+(L))\) with \(K' \equiv [s_2] \bullet K_2 \perp \ldots\); and \(L' = [r_2] \bullet L_2' \perp \ldots\); where \(L_2'\) is the lattice of same rank same type as of \(L_2\) and \(\det L_2' = \det L_1 \det L_2 \det K_1\); and \(L_2 \equiv L_2'\) when \(L_2\) is of mixed-type.

Furthermore \(\Theta(L, K) = F^*\) unless the following conditions are satisfied:
(1) all the components of K with rank ≥ 2 are of non-odd-type when r_1 = even;
(2) all the components of K with rank ≥ 2 are of non-even-type when r_1 = odd;
(3) if r_2 = s_1 + 1 and rank L_2 = 1, then r_3 ≥ s_1+6.

Proof: (i): It is easy to see that σL ⊇ K if and only if L ⊇ σ^{-1}K iff L' ⊇ σ^{-1}K where L' = [r_1+2]*L_1 \perp [r_2]*L_2 \perp ...; so X(L,K) = X(L', K) and Θ(L,K) = Θ(L',K).

(ii): Since we have r_1 = s_1, both L_1 and K_1 are of rank 1. If Θ(O^+(L)) = F^*, then obviously the formula holds. Assume Θ(O^+(L)) ≠ F^*. By Lemma 1.3.3, we get [r_1]*L_1 \perp [r_2]*L_2 = [r_1]*K_1 \perp [r_2]*L_2' with K_1 \equiv K_1, and Θ(L,K) = Θ(L',K')Θ(O^+(L)) with K' \equiv [s_2]*K_2 \perp ...; and L' = [r_2]*L_2' \perp ...; where L_2' is the lattice of same rank, same type as of L_2 and det L_2' = det L_1 det L_2 det K_1. Because Θ(O^+(L)) ≠ F^*, r_2 ≥ r_1 + 4 when L_2 is of mixed-type. So L_2 \equiv L_2' whenever L_2 is of mixed-type.

Now suppose there is an odd-type (resp. even-type) component [s_i]*K_i of K of rank ≥ 2 when r_1 is even (resp. odd). Say [r_1]*L_1 = Rv; and pick any vector w ∈ [s_i]*K_i with S_w ∈ O([s_i]*K_i). We have S_v S_w ∈ X(L,K), but Θ(S_v S_w) ∉ Θ(O^+([s_i]*K_i)) which is a subgroup of F^* of index ≤ 2. So Θ(L,K) = F^*. Finally, suppose r_2 = s_1 + 1, rank L_2 = 1 and r_3 ≤ s_1 + 5. By the first statement of this lemma, we can assume r_1 = s_1. Now Θ(O^+(L)) ⊇ Θ(O^+([r_1]*L_1 \perp [r_2]*L_2 \perp [r_3]*L_3)), and the latter equals F^* by Proposition 1.19 of [EH] or Theorem 3.14. of [EH] according to whether rank L_3 = 1 or rank L_3 ≥ 2. So again Θ(L,K) = F^*. Q.E.D.

**Lemma 1.3.6.** Let L_1 be of even-type of rank 2 and Θ(O^+(L_1)) ≠ F^*. Then

(1) if all the components of K are of the same type as that of [r_1]*L_1, we have

i) if s_1 = r_1 and rank K_1 = 1, then Θ(L,K) = Θ(L',K')Θ(O^+(L)) where K' \equiv [s_2]*K_2 \perp ...; and L' = [r_1]* \perp [r_2]*L_2 \perp ...;
(ii) if $s_1 = r_1$ and $\text{rank}K_1 = 2$, then $\Theta(L,K) = \Theta(L',K')\Theta(O^+(L))$ where

$K' \equiv [s_2] \cdot K_2 \perp \ldots, L' = [r_2] \cdot L_2' \perp [r_3] \cdot L_3 \perp \ldots$, and $L_2'$ is the lattice with the same rank and type as those of $L_2$, but $\det L_2' = \det L_1 \det L_2 \det K_1$;

(iii) if $s_1 > r_1$ then $\Theta(L,K) = \Theta(L',K)$ where $L' = [r_1 + 1] \cdot L_1' \perp [r_2] \cdot L_2 \perp \ldots$, and $L_1'$ is the lattice of odd-type rank 2 of $\det L_1' = \det L_1$.

(2) $\Theta(L,K) = F^*$ unless that all the components of $K$ are of the same type as of $[r_1] \cdot L_1$.

Proof: If there is a component of $K$ with different type as of $[r_1] \cdot L_1$, then clearly $\Theta(L,K) = F^*$. This proves (2). Cases (i) and (ii) are just the reformulations of Lemma 1.3.3. (Note that first $L_1$ represents all units; and second the $L_2$ in case (ii) must be of non mixed-type, for otherwise $\Theta(O^+(L)) = F^*$.). For case (iii), suppose that $[r_1] \cdot L_1 = \langle e_1, e_2 \rangle \equiv [r_1] \cdot A(1,0)$ or $[r_1] \cdot L_1 \equiv [r_1] \cdot A(1,4\rho)$. In both cases one sees that $Q(x)$ is of order $r_1$ for any $x = ae_1 + be_2$ with $a \in \mathbb{R}^*$. So $X(L/K) = X(L'/K)$ with $L' = \langle 2e_1, e_2 \rangle \perp [r_2] \cdot L_2 \perp \ldots$; and $\langle 2e_1, e_2 \rangle = [r_1 + 1] \cdot L_1'$. This proves (iii). Q.E.D.

**Lemma 1.3.7.** Let $L_1$ be of odd-type and $\Theta(O^+(L)) \neq F^*$. Then

(1) if all the components of $K$ are of the same type as that of $[r_1] \cdot L_1$, we have:

$\Theta(L,K) = F^*$ if any one of the first three cases occurs:

(i) $s_1 \geq r_1 + 2$, and $L_1 \supseteq A(0,0)$;

(ii) $s_1 \geq r_1 + 1$, $r_2 = r_1 + 1$;

(iii) $s_1 \geq r_1 + 1$, $\text{rank} L_1 \geq 4$.

(iv) if $s_1 = r_1 + 1$, $\text{rank} L_1 = 2$, $\text{rank} K_1 = 1$, and $r_2 \geq r_1 + 2$; then $\Theta(L,K) = \Theta(L',K')$

$\Theta(O^+(L))$ where $L' = [s_1] \cdot (\det L_1 \det K_1) \perp [r_2] \cdot L_2 \perp \ldots$; and $K' \equiv [s_2] \cdot K_2 \perp \ldots$;

(v) if $s_1 = r_1 + 1$, $\text{rank} L_1 = \text{rank} K_1 = 2$, and $r_2 \geq r_1 + 2$; then $\Theta(L,K) = \Theta(L', K')$

$\Theta(O^+(L))$ where $L' = [r_2] \cdot L_2' \perp [r_3] \cdot L_3 \perp \ldots$; and $L_2'$ is the lattice with the
same rank and type as those of $L_2$ but $\det L_2' = \det L_1 \det L_2 \det K_1$, and $K' \equiv [s_2] \bullet K_2 \perp \ldots$.

(vi) if $s_1 \geq r_1+2$, $r_2 \geq r_1+2$ and $L_1 = A(2,2p)$, then $\Theta(L,K) = \Theta(L',K)$ where $L' = [r_1+2] \bullet L_1 \perp [r_2] \bullet L_2 \perp \ldots$.

(vii) if $s_1 = r_1$, $\text{rank } L_1 > \text{rank } K_1$, then $\Theta(L,K) = \Theta(L',K')$ where $L' = [r_1] \bullet L_1' \perp [r_2] \bullet L_2 \perp \ldots$, and $L_1'$ is an odd-type lattice of rank $= \text{rank } L_1 - \text{rank } K_1, \det L_1' = \det L_1 \det K_1$, and $K' \equiv [s_2] \bullet K_2 \perp \ldots$.

(viii) if $s_1 = r_1$, $\text{rank } L_1 = \text{rank } K_1$, then $\Theta(L,K) = \Theta(L',K') \Theta(O^+(L))$ where $L' = [r_2] \bullet L_2' \perp [r_3] \bullet L_3 \perp \ldots$, and $L_2'$ is the lattice with the same rank and type as those of $L_2$, but $\det L_2' = \det L_1 \det L_2 \det K_1$, and $K' \equiv [s_2] \bullet K_2 \perp \ldots$.

(2) $\Theta(L,K) = F^*$ unless all the components of $K$ are of the same type as that of $[r_1] \bullet L_1$.

Proof: Again (2) is obviously true. Now we see $\Theta(L,K) \supseteq \Theta(O^+(L)) = R^* F^* 2$, so in (i) (ii) (iii) all we need is to find a $\sigma$ in $X(L/K)$ with $\Theta(\sigma) = 2F^* 2$.

(i): Suppose $L = [r_1] \bullet A(0,0) \perp L'$ with $[r_1] \bullet A(0,0) = \langle e_1, e_2 \rangle$. If $K \subseteq \langle 2e_1, e_2 \rangle \perp L'$ then $\sigma = S_{e_1} e_2 S_{e_1} - 2e_2 \in X(L/K)$ and $\Theta(\sigma) = 2F^* 2$. Otherwise there is $x \in K$, $x = e_1 + ae_2 + w$, with $a \in R$ and $w \in L'$. Now sublattice $\langle e_1 + w, e_2 \rangle$ is isometric to $[r_1] \bullet A(0,0)$, and suppose $\{e_1^*, e_2^*\}$ is the bases under which the lattice has the matrix $[r_1] \bullet A(0,0)$. Since $x$ is primitive element in $\langle e_1 + w, e_2 \rangle = \langle e_1^*, e_2^* \rangle$ with order of $Q(x) \geq r_1 + 2$, so WLOG $x = e_1^* + be_2^*$ with $b \in 2R$. Write $L = \langle e_1^*, e_2^* \rangle \perp L''$, then $K \subseteq \langle e_1^*, 2e_2^* \rangle \perp L''$ because $s_1 \geq r_1 + 2$. So $\sigma = S_{e_1^*} e_2^* S_{e_1^*} - e_2^* \in X(L/K)$ and $\Theta(\sigma) = 2F^* 2$. Case (i) is proved.

(ii): Because $L_1$ is odd-type and $r_2 = r_1 + 1$ we may assume that $[r_1] \bullet L_1 = \langle e_1, f_1 \rangle \perp \langle e_2, f_2 \rangle \perp \ldots \langle e_t, f_t \rangle$; with $2t = \text{rank } L_1$ and $\langle e_i, f_i \rangle \equiv [r_1] \bullet A(0,0)$. Let $K = [s_1] \bullet K_1 \perp K'$; $L = \langle e_1, f_1 \rangle \perp \langle e_2, f_2 \rangle \perp \ldots \langle e_t, f_t \rangle \perp L'$; by (i) we may assume that $s_1 = r_1 + 1$. 


(a) If \( K \subseteq \langle 2e_1,2f_1 \rangle \perp \langle 2e_2,2f_2 \rangle \perp \ldots \perp \langle 2e_t,2f_t \rangle \perp \mathbb{L}' \); then by the same proof of (i) we have \( \sigma \in \mathcal{X}(L/K) \) with \( \Theta(\sigma) = 2F^*2 \).

(b) If \( K' \not\subseteq \langle 2e_1,2f_1 \rangle \perp \langle 2e_2,2f_2 \rangle \perp \ldots \perp \langle 2e_t,2f_t \rangle \perp \mathbb{L}' \), then by the same proof of (i) there is a \( x \in K' \) and \( H = \langle e^*,f^* \rangle = [r_1] \cdot \mathbb{A}(0,0) \subseteq L \) such that \( x = e^* + af^* \) with \( a \in 2\mathbb{R} \). Write \( L = H \perp \mathbb{L}' \). Because \( B(x,K) \subseteq 2^{r_1+1}\mathbb{R} \), we have \( K \subseteq \langle e^*,2f^* \rangle \perp \mathbb{L}' \) and \( \sigma = e_{e^*} \cdot f_{2f^*} \in \mathcal{X}(L/K) \) with \( \Theta(\sigma) = 2F^*2 \).

(c) Finally, \( [s_1] \cdot K_1 \not\subseteq \langle 2e_1,2f_1 \rangle \perp \langle 2e_2,2f_2 \rangle \perp \ldots \perp \langle 2e_t,2f_t \rangle \perp \mathbb{L}' \). Because \( s_1 = r_1 + 1 \) and \( L_1 \) is odd, so \( K_1 \) is even. Let \( [s_1] \cdot K_1 = \langle x_1, x_2, \ldots, x_t \rangle \) be an orthogonal bases. WLOG we may assume there is a \( H = \langle e^*,f^* \rangle = [r_1] \cdot \mathbb{A}(0,0) \subseteq L \) such that \( x_1 = e^* + af^* \) with \( a \in \mathbb{R}^* \). Let \( L'' = \{ z \in L \mid B(x_1, z) = 0 \} \). Then \( K \subseteq \langle x_1 \rangle \perp L'' \subseteq L \); and \( \langle x_1 \rangle \perp L'' \) contains a 3-dim, \( 2^{r_1+1}\mathbb{R} \)-modular sublattice as an orthogonal component, so \( \Theta(L,K) = F^* \). Case (ii) is done.

(iii) Because of (i) and (ii) we assume \( s_1 = r_1 + 1 \) and \( r_2 \geq r_1 + 2 \). So \( L_1 = A(0,0) \perp A(0,0) \perp A(0,0) \perp A(0,0) \perp A(2,2p) \). Because \( L_1 \) is odd-type and \( s_1 = r_1 + 1 \), \( K_1 \) is of even-type. Let \( K = \langle x \rangle \perp K' \) with \( Q(x) \in 2^{s_1}\mathbb{R}^* \). Similar to the proof of (ii)(c) there is a sublattice \( G = \langle e, f \rangle \subseteq L \) with \( \langle e, f \rangle \equiv [r_1] \cdot \mathbb{A}(0,0) \) or \( \equiv [r_1] \cdot \mathbb{A}(2,2p) \) such that \( x \in G \). Write \( L = G \perp L' \), one sees that in any case \( K \subseteq \langle x \rangle \perp \langle y \rangle \perp L' \) for some \( y \in G \) with \( Q(y) \in 2^{s_1}\mathbb{R}^* \). Now the pair of lattices \( \langle y \rangle \perp L' \) and \( K' \) satisfy the conditions of case(ii) and \( \Theta(L,K) \supseteq \Theta(\langle y \rangle \perp L',K') \), so \( \Theta(L,K) = F^* \). Case(iii) is proved.

(iv) and (v): Write \( [r_1] \cdot L_1 = \langle e_1, e_2 \rangle \) which is \( \equiv [r_1] \cdot \mathbb{A}(0,0) \), or \( \equiv [r_1] \cdot \mathbb{A}(2,2p) \); \( r_2 \geq r_1 + 2 \); because \( s_1 = r_1 + 1 \), \( K_1 \) is even. Let \( K = \langle x \rangle \perp K' \) with \( Q(x) \in 2^{s_1}\mathbb{R}^* = 2^{r_1+1}\mathbb{R}^* \). It is not difficult to show the following statement:

"For any \( z \in L \) with \( Q(z) \in 2^{r_1+1}\mathbb{R}^* \), one can find a sublattice \( H_z \subseteq L \) with \( H_z \equiv [r_1] \cdot L_1 \) and \( z \in H_z " \)
Now given \( \sigma \in X(L/K) \), we have \( \sigma^{-1}x \in L \) and because \( Q(\sigma^{-1}x) = Q(x) \in 2^{s_1+1}R^* \), there is \( H_{\sigma^{-1}x} \subseteq L \) and \( H_x \subseteq L \). WLOG we assume \([r_1]\cdot L_1 = H_x \). Write \( L = H_{\sigma^{-1}x} \perp \perp L_{\sigma^{-1}x} = H_x \perp \perp L_x \); we have \( \tau \in O^+(L) \) such that \( \tau H_x = H_{\sigma^{-1}x} \). Because \( \tau x \) and \( \sigma^{-1}x \) are primitive in \( H_{\sigma^{-1}x} \) with same \( Q \)-value of order \( s_1 = r_1+1 \), there is a \( \lambda \in O^+(H_{\sigma^{-1}x}) \subseteq O^+(L) \) such that \( \lambda \tau x = \sigma^{-1}x \). Thus \( \sigma \lambda \tau x = x \), so \( \sigma \lambda \tau \in X(L'/K') \) with \( L' = \{ y \mid y \in L \text{ and } B(x,y) = 0 \} \). So \( \sigma \in X(L'/K')O^+(L) \) which gives the formula in case (iv). And case (v) can be proved by applying Lemma 1.3.5(ii) to \( L' \) and \( K' \).

(vi): This is because \([r_1]\cdot L_1 \) primitively represents only numbers of order \( r_1+1 \).

(vii) and (viii): These are the reformulations of the Lemma 1.3.3.

Q.E.D

§ 1.4 Secondary Reduction Over 2-adic Fields

We need more notations: \( L = [r_1]\cdot L_1 \perp [r_2]\cdot L_2 \perp \ldots \); \( K = [s_1]\cdot K_1 \perp [s_2]\cdot K_2 \perp \ldots [s_t]\cdot K_t \); as in section 3. Define \( \tilde{L}_i = \bigcap_{r_j \leq s_i} [r_j]\cdot L_j \) for \( i=1,2,\ldots, t+1 \), with the convention that \( s_0 = -\infty \) and \( s_{t+1} = \infty \). Also we define \( k_i = \sum_{j=1}^i \text{rank}L_j \); \( k_{t+1} = \sum_{j=1}^t \text{det}K_j \). \( \xi_i = \prod_{r_j \leq s_i} \text{det}L_j \); \( \eta_i = \prod_{s_j \leq \text{det}K_j} \) for all \( i = 1, 2, \ldots, t \); and \( \xi_0 = \eta_0 = 1; k_0 = 0; \xi_{t+1} = \prod_k \text{det}L_k \); \( l_{t+1} = \text{rank}L \); \( \delta_1 = \sum_{r_j \leq s_1} \text{rank}L_j \). \( D(\cdot) \) denotes for the quadratic defect function. We introduce two conditions which will streamline the results presented later.

For some index \( i, 1 \leq i \leq t \); some integer \( s \); some unit \( \xi \) and some lattice \( N \) of rank \( \leq 2 \) ( may be 0 ); we say that \textbf{ConditionA}(\( N, \xi, s, i \)) holds to mean the following:

\begin{itemize}
  \item[A(1):] If \( N \) is of rank zero, then \( D(-\xi) = 4R \) when \( s > s_i + 1 \).
\end{itemize}
A(2): If \( N = [a]*N* \) is of rank one, then \( N' = <\xi\bigcap N* \) is of even-type, and 
\( \det N' \neq -1 \) when \( s > a+2 \).

A(3): If \( N = [a]*N* \) is of rank two, then either 

(i) \( N* \) is of even-type, and \( \xi\det N* \neq -1 \) when \( s > a+2 \), or

(ii) \( N* \) is of odd-type, and \( \xi\det N* \neq -1 \) when \( s > a+1 \).

A(4): If \( N = [a]*N*\bigcap [b]*N** \) with \( a < b \), then \( a \equiv b \pmod{2} \),

\( N' = <\xi \det N*>\bigcap N** \) is of even-type, and \( \det N' \neq -1 \) when \( s > b+2 \).

The motivation for introducing the ConditionA(\(N, \xi, s, i\)) is this: we want to avoid

the invocation of Lemma 1.3.7(i) at the \( i \)-th reduction step, and ConditionA(\(N, \xi, s, i\))

is necessary to achieve that. A(1) deals with the case that the "left-over" after \( i-1 \)

reduction steps is of rank two; A(2) deals with the case that the "left-over" after \( i-1 \)

reduction steps is of rank one; A(3) and A(4) deal with the various situations

associated with the case when the "left-over" after \( i-1 \) reduction steps is of rank zero.

The meaning of "left-over" will be made clear later.

For some index \( i, 1 \leq i \leq t \); some binary space \( W \) of mixed-type; we say

ConditionB(\(i,W\)) holds to mean the following:

When \( l_{i+1} = [a]\bigcap L(i+1, 1) \bigcap...; \) with the leading component been shown.

B(1): When \( l_i - k_i = 0 \) and \( \text{rank} K_i = 1 \), then either

(i) \( a \geq s_i + 4 \); or

(ii) \( a = s_i + 2, \text{rank} L(i+1, 1) = 1 \), and \( \det L(i+1,1)\xi i \eta_{i-1} \det W \in \{ 1, \Delta \} \).
B(2): When \((l_i - k_i = 1 \text{ and } \text{rank} K_i = 1)\) or \((l_i - k_i = 0 \text{ and } \text{rank} K_i = 2)\); then

(i) If \(\tilde{L}_i = [i_1] \ast L(i,1)\) is of rank one; then \(s_j = i_1 \pmod{2}\), \(a > S_j + 4\), and \(F([i_1] \ast <\xi_{i-1} \eta_{i-1}> \perp [i_1] \ast L(i,1)) = W\).

(ii) If \(\tilde{L}_i = [i_1] \ast L(i,1)\) is of rank two; then \(a \geq s_i + 4\), and
\(F([i_1] \ast L(i,1)) = W\).

(iii) If \(\tilde{L}_i = [i_1] \ast L(i,1) \perp [i_2] \ast L(i,2)\) is of rank two, then \(a \geq s_i + 4\), \(i_1 \equiv i_2 \pmod{2}\), and \(F([i_2] \ast <\xi_{i-1} \eta_{i-1}> \det L(i,1) \perp [i_2] \ast L(i,2)) = W\).''

The motivation for introducing ConditionB\((i,W)\) is the following: we want to have \(\Theta(L,K) \neq F^*\) when already \(\Theta(L,K) \supseteq \Theta(O^+(W))\), which is a mixed subgroup on \(F^*\) of index two containing \(\Delta\). For this purpose ConditionB\((i,W)\) is necessary at the \(i\)-th reduction step. \(B(1)\) and \(B(2)\) deal with situations associated with different outcomes after \(i-1\) reduction steps and all possible different structures of \(\tilde{L}_i\).

**PROPOSITION 1.4.1.** If \(\delta_1 \geq 3\), then \(\Theta(L,K) = F^*\).

Proof: First we show that by applying Lemma 1.3.4, Lemma 1.3.5, Lemma 1.3.6, and Lemma 1.3.7 we may assume \(s_1 \leq r_1 + 2\). Suppose \(s_1 > r_1 + 2\). If \(\Theta(O^+(L)) = F^*\), then the Proposition already holds. Assume \(\Theta(O^+(L)) \neq F^*\). If \(L_1\) is of mixed-type rank 2, then by Lemma 1.3.4's observation we have \(\Theta(L,K) = \Theta(L',K)\) where \(L' = [r_1+1] \ast A(2\xi,\delta) \perp [r_2] \ast L_2 \perp \ldots\) (assume \(L_1 = A(\xi,2\delta)\)). If \(L_1\) is of rank 1, then \(\Theta(L,K) = \Theta(L',K)\) by Lemma 1.3.5(i) where \(L' = [r_1+2] \ast L_1 \perp [r_2] \ast L_2 \perp \ldots\). If \(L_1\) is of even-type rank 2, we may further assume that all components of \(K\) are of the same type as of \([r_1] \ast L_1\) (for otherwise, by Lemma 1.3.6, \(\Theta(L,K)\) is already \(F^*\) and Proposition holds). Now Lemma 1.3.6(iii) tells us \(\Theta(L,K) = \Theta(L',K)\) where \(L' = [r_1+1] \ast L'_1 \perp [r_2] \ast L_2 \perp \ldots\),
and \( L'_1 \) is the rank 2 odd-type lattice of \( \det L'_1 = \det L_1 \). Finally, suppose \( L_1 \) is of odd-type. Look at lemma 1.3.7, only (vi) is applicable (if \( \Theta(L, K) = F^* \) then there is nothing to prove), and \( \Theta(L, K) = \Theta(L', K) \) where \( L' = [r_1 + 2] \cdot L'_1 \perp [r_2] \cdot L_2 \perp \ldots \), and \( L_1 = A(2, 2p) \). In any event we have \( \Theta(L, K) = \Theta(L', K) \), where the order of the leading scale of \( L' \) is at least \( r_1 + 1 \) and at most \( r_1 + 2 \). Repeating the above process, we may eventually assume \( s_1 \leq r_2 + 2 \). This leads to the two cases: (a) \( r_3 \geq s_1 \geq r_2 + 1 \), and \( r_2 = r_1 + 1 \); or (b) \( r_2 \geq s_1 \geq r_1 + 1 \). For case (a), because \( r_2 = r_1 + 1 \), we may assume that none of \( L_1 \) and \( L_2 \) are of mixed-type. (See Theorem 3.14 of [EH].) If \( L_1 \) is of odd-type then Lemma 1.3.7(ii) gives \( \Theta(L, K) = F^* \). If \( L_1 \) is of even-type, then \( L_2 \) is of odd-type. In this situation Lemma 1.3.5(i) and Lemma 1.3.7(ii); or Lemma 1.3.6(iii) and Lemma 1.3.7(iii) will ensure that \( \Theta(L, K) = F^* \). In case (b) we have \( \text{rank} L_1 \geq 3 \), so we have to assume that \( L_1 \) is of odd-type, and then by Lemma 1.3.7(iii) we also get \( \Theta(L, K) = F^* \).

**Q.E.D.**

**Proposition 1.4.2.** If \( l_1 - k_1 \geq 3 \), then \( \Theta(L, K) = F^* \).

Proof: Similar to the first part of the proof of Proposition 1.4.1, we may assume \( s_1 \leq r_1 + 1 \), and which leads to either (a) \( r_2 > s_1 \), or (b) \( r_2 = s_1 = r_1 + 1 \). For (a), because \( l_1 - k_1 \geq 3 \) we can only assume \( L_1 \) is odd-type. (After applying Lemma 1.3.7(vii) first if necessary.) Apply Lemma 1.3.7(iii) we get \( \Theta(L, K) = F^* \). For case (b), by Proposition 1.4.1 one can assume that \( \text{rank} L_1 \leq 2 \). If \( L_1 \) is of odd-type then Lemma 1.3.7(ii) applies. If \( L_1 \) is of mixed-type then Theorem 3.14 of [EH] applies. Finally, if \( L_1 \) is of even-type then \( L_2 \) and \( K_2 \) may be assumed to be of odd-type. By Lemma 1.3.5(i), Lemma 1.3.7(vii) and (ii) when \( \text{rank} L_1 = 1 \); or Lemma 1.3.6(iii), Lemma 1.3.7(vii) and (iii) when \( \text{rank} L_1 = 2 \), we still get \( \Theta(L, K) = F^* \). Q.E.D.
PROPOSITION 1.4.3. Let $\delta_1 \leq 2$ and $l_1-k_1 = 2$.

When the following conditions are satisfied:

1. $K_1$ is of odd-type;
2. all the components of $L$ and $K$ are of same type;
3. either (a) $s_1 = r_1$; 
   or (b) $L_1 = N \perp [s_1] \circ M$, $\text{rank} M = \text{rank} K_1$, and ConditionA($N, \xi_0 \eta_0, s_1, 1$) holds;

we have $\Theta(L, K) = \Theta(L', K') R^*$ where $L' = [s_1] \circ L_1 \perp L_2 \perp \ldots, L_1'$ is the odd-type lattice of rank 2 and $\det L' = \xi_1 \eta_1$, and $K' = [s_2] \circ K_2 \perp \ldots$.

Otherwise, we have $\Theta(L, K) = F^*$.

Proof: First we show that $\Theta(L, K) = F^*$ when $K_1$ is of even or mixed type. As what we did in the proof of Proposition 1.4.2, we may assume that (a) $r_2 > s_1$; or (b) $s_1 = r_2 = r_1 + 1$. For case (a) since $l_1-k_1 = 2$, $\text{rank} L_1 \geq 3$, therefore we can only assume that $L_1$ is of odd-type. Because of assumption $\delta_1 \leq 2$ we have $s_1 = r_1$. But then $[r_1] \circ L_1$ and $[s_1] \circ K_1$ would have different parities, so $\Theta(L, K) = F^*$. For case (b) because $r_2 = r_1 + 1$, none of $L_1$ and $L_2$ can be of mixed-type. By Lemma 1.3.7(ii) we may also assume that $L_1$ is even. Then again $[r_1] \circ L_1$ and $[s_1] \circ K_1$ have different parities, and so $\Theta(L, K) = F^*$.

Now we assume that $K_1$ is of odd-type, i.e., (1) holds. Obviously $\Theta(L, K) = F^*$ if there is a component of $L$ or $K$ which has different type as that of $[s_1] \circ K_1$. So we may further assume that (2) is satisfied. We are going to show that either (3a) or (3b) must be satisfied in order that $\Theta(L, K) \neq F^*$. Our line of proof is as follows:

In order for $\Theta(L, K) \neq F^*$, certain necessary conditions will be extracted until (3a) or (3b) is established.
First we note that it is impossible for $\delta_1 = 1$. If this is the case, we would have $r_2 = s_1$, and that rank$L_2 = \text{rank} K_1 + 1$ would be odd since $K_1$ is of odd-type. But this contradicts to our assumption of (2). Hence we have either $\delta_1 = 0$ or $\delta_1 = 2$. $\delta_1 = 0$ is the same as 3(a). When $\delta_1 = 2$ we have $L_1 = N \perp [s_1] \ast M$ with rank$M = \text{rank} K_1$. We show that ConditionA$(N, \xi_0 \eta_0, s_1, 1)$ is necessary in order for $\Theta(L,K) \neq F^*$. There are three possible structures for $N$:

(a) $N = [r_1] \ast L_1$ with $L_1$ being odd-type;
(b) $N = [r_1] \ast L_1$ with $L_1$ being even-type;
(c) $N = [r_1] \ast L_1 \perp [r_2] \ast L_2$.

First, suppose (a). When $s_1 \geq r_1 + 2 > r_1 + 1$ and $\det L_1 = -1$ (equivalently $L_1 \equiv A(0,0)$), then $\Theta(L,K) = F^*$ by Lemma 1.3.7(i). In order for $\Theta(L,K) \neq F^*$, we need $\det L_1 \neq -1$ when $s_1 > r_1 + 1$. This is exactly the A(3)(ii) of ConditionA$(N, \xi_0 \eta_0, s_1, 1)$. (Note $\xi = \xi_0 \eta_0 = 1$, $N^* = L_1$, $s = s_1$ and $a = r_1$). ConditionA$(N, \xi_0 \eta_0, s_1, 1)$ is thus necessary.

Second, suppose (b). To see ConditionA$(N, \xi_0 \eta_0, s_1, 1)$ is necessary, suppose that $\det L_1 = -1$ nad $s_1 > r_1 + 2$. Then $L_1 \equiv A(1,0)$. Apply Lemma 1.3.6(iii) to obtain $\Theta(L,K) = \Theta(L',K)$ where $L' = [r_1 + 1] \ast A(0,0) \perp [r_2] \ast L_2$. Next Lemma 1.3.7(i) applies since $s_1 \geq (r_1 + 1) + 2$, and we get $\Theta(L,K) = \Theta(L',K) = F^*$. So we have to assume that $\det L_1 \neq -1$ when $s_1 > r_1 + 2$. This is the A(3)(i) of ConditionA$(N, \xi_0 \eta_0, s_1, 1)$. (Note $\xi = \xi_0 \eta_0 = 1$, $N^* = L_1$, $s = s_1$ and $a = r_1$). And ConditionA$(N, \xi_0 \eta_0, s_1, 1)$ is necessary.

Last, suppose (c). Obviously $r_1 < r_2 < r_3 = s_1$. Since $r_1 \equiv r_2 \pmod{2}$ (see assumption (2)), repeatedly apply Lemma 1.3.5(i) we get $\Theta(L,K) = \Theta(L',K)$ where $L' = [r_2] \ast (L_1 \perp L_2) \perp [r_3] \ast L_3$. Since we already have $\Theta(L,K) \supseteq \Theta(O^+(K)) \supseteq R^*$. If $L_1 \perp L_2$ is of mixed-type we would have $\Theta(L,K) = F^*$. So naturally $L_1 \perp L_2$ can be
assumed of even-type. Similar to case (β), we have to assume \((\det L_1 \det L_2) \neq -1\) when \(s_1 > r_2 + 2\). A(4) of ConditionA(N, \(\xi_0\eta_0, s_1, 1\)) is then satisfied. (Note \(\xi = \xi_0\eta_0 = 1, N^* = L_1, N^{**} = L_2, s = s_1, a = r_1\) and \(b = r_2\)). Again ConditionA(N, \(\xi_0\eta_0, s_1, 1\)) is necessary.

We have proved that (1), (2) and (3) are all necessary in order for \(\Theta(L,K) \neq F^*\). In other words, if any one of these four conditions is violated then we have \(\Theta(L,K) = F^*\).

For the remainder of the proof, we show the formula for \(\Theta(L,K)\) when all (1), (2), and (3) are satisfied.

First, suppose (1), (2) and 3(a) hold. So \(r_1 = s_1\), both \(L_1\) and \(K_1\) are of odd-type and \(\text{rank} L_1 = \text{rank} K_1 + 2\). Lemma 1.3.7(vii) applies and gives the stated formula.

Now assume that (1), (2) and 3(b) hold. Then \(L_1 = N \perp [s_1] \perp M\) with \(\text{rank} M = \text{rank} K_1\). As what we have just done in above, we divide the problem in to three cases according to the possible structure of \(N\):

- \((\alpha)\) \(N = [r_1] \perp L_1\) with \(L_1\) being odd-type;
- \((\beta)\) \(N = [r_1] \perp L_1\) with \(L_1\) being even-type;
- \((\gamma)\) \(N = [r_1] \perp L_1 \perp [r_2] \perp L_2\).

First, suppose \((\alpha)\). ConditionA(N, \(\xi_0\eta_0, s_1, 1\)) holds means that A(3)(ii) is true. That is \(\det L_1 \neq -1\) when \(s_1 > r_1 + 1\). (Note \(\xi = \xi_0\eta_0 = 1, N^* = L_1, s = s_1\) and \(a = r_1\)). By assumptions (1) and (2), we have \(s_1 \equiv r_1 \pmod{2}\). So naturally \(s_1 \geq r_1 + 2\), and \(L_1 \equiv A(2,2p)\). Repeatedly apply Lemma 1.3.7(vi) we get \(\Theta(L,K) = \Theta(L',K)\) where \(L' = [r_2] \perp (L_1 \perp L_2) \perp [r_3] \perp L_3\ldots\). Now \(L_1 \perp L_2\) is odd-type, \(\text{rank}(L_1 \perp L_2) = \text{rank} K_1 + 2\), and \(r_2 = s_1\). Apply Lemma 1.3.7(vii) to the pair \(L'\) and \(K\), and the formula follows.

Second, suppose \((\beta)\). ConditionA(N, \(\xi_0\eta_0, s_1, 1\)) holds means that A(3)(i) is true. That is, \(\det L_1 \neq -1\) when \(s_1 > r_1 + 2\). (Note \(\xi = \xi_0\eta_0 = 1, N^* = L_1, s = s_1\) and \(a = r_1\)). By Lemma 1.3.6(iii), we have \(\Theta(L,K) = \Theta(L',K)\) where \(L' = [r_1 + 1] \perp L_1 \perp [r_2] \perp L_2\)
Here $L'_1$ is a odd-type rank two lattice of $\det L'_1 = \det L_1$. The problem is now reduced to the corresponding one in 3(a) if $r_1 + 1 = r_2 = (s_1)$, and the formula is obtained accordingly. When $r_1 + 1 < r_2 = (s_1)$, it is reduced to the one in case (a). Note that ConditionA($[r_1 + 1]\bullet L'_1, \xi_0\eta_0, s_1, 1$) holds for the pair $L'$ and $K$. Again the stated formula is obtained.

Finally, assume (γ). Obviously, $r_1 < r_2 < r_3 = s_1$. ConditionA($N, \xi_0\eta_0, s_1, 1$) holds means that $A(4)$ is true. That is, $r_1 \equiv r_2 \pmod{2}$, $L_1 \bot L_2$ is even, and $\det L_1 \det L_2 \neq 1$ when $s_1 > r_2 + 2$. (Note $\xi = \xi_0\eta_0 = 1$, $N^* = L_1, N^{**} = L_2$, $s = s_1, a = r_1$ and $b = r_2$). Repeatedly apply Lemma 1.3.5(i), we get $\Theta(L, K) = \Theta(L', K')R^*$ where $L' = [r_2] \bullet (L_1 \bot L_2) \bot [r_3] \bullet L_3...$. The problem is now reduced to the one in case (β). Note that ConditionA($[r_2] \bullet (L_1 \bot L_2), \xi_0\eta_0, s_1, 1$) holds for the pair $L'$ and $K$. The formula is obtained accordingly.

Q.E.D.

**PROPOSITION 1.4.4.** Let $\delta_1 \leq 2$ and $l_1-k_1 = 1$. Let $K' = [s_2] \bullet K_2 \bot ...$.

1. Suppose $\text{rank} K_1 = 1$. If all components of $L$ and $K$ are of same (odd, or even)
type and ConditionA($L_1, \xi_0\eta_0, s_1, 1$) holds, then $\Theta(L, K) = \Theta(L', K')R^*$
   where $L' = [s_1] \bullet (\xi_1\eta_1) \bot L_2 \bot L_3...$.

2. Suppose $\text{rank} K_1 = 1$. If $D(-\xi_1\eta_0) = 2R$, the ambient spaces of all components
   of $L$ and $K$ are represented by $W := FL_1$ and ConditionB(1, $W$) holds. Then
   $\Theta(L, K) = \Theta(L', K')\Theta(O^+(W))$, where $L'$ is the same as in (1). $\Theta(L, K)$ contains
   a mixed subgroup $\Gamma$ of $F^*$ of index two, and $\Delta \in \Gamma$.

3. Suppose $\text{rank} K_1 = 1$. If $L_1 = [r_1] \bullet L_1 \bot [r_2] \bullet L_2$, $r_1 \equiv r_2+1 \pmod{2}$ and
   $r_3 \geq s_1 + 6$. Then we have
   $\Theta(L, K) = \Theta(L', K')\Theta(O^+([s_1] \bullet \xi_0\eta_0 \det L_1 > \bot [s_1+1] \bullet L_2 \bot [r_3] \bullet L_3...)$,
   where $L' = [s_1+1] \bullet (\xi_1\eta_1) \bot [r_3] \bullet L_3...$. Furthermore, $\Theta(L, K)$ contains a mixed
subgroup $\Gamma$ of $F^*$ of index two, and $\Delta \in \Gamma$.

(4) Suppose $K_1$ is of odd-type. If all components of $L$ and $K$ are of same type,

$r_2 = s_1$ and $\text{rank}L_2 = \text{rank}K_1$, then $\Theta(L,K) = \Theta(L',K')R^*$, where $L' = [s_1+1]\langle \xi \eta_1 \rangle \perp [r_3]\langle L_3 \rangle \cdots$.

If none of the above applies then $\Theta(L,K) = F^*$.

Proof: First we show that $\Theta(L,K) = F^*$ when $K_1$ is of even or mixed-type of rank $\geq 2$. Obviously $\Theta(L,K) \supseteq \Theta(O^+(K_1)) = F^*$ when $\text{rank}K_1 \geq 3$. WLOG we assume $\text{rank}K_1 = 2$. As in the proof of Proposition 1.4.1, we may suppose that (a) $r_2 > s_1$; or (b) $s_1 = r_2 = r_1 + 1$.

For case (a) we have $\text{rank}L_1 = 3$, so $\Theta(L,K) \supseteq \Theta(O^+(L_1)) = F^*$. For case (b) because $s_1 = r_2 = r_1 + 1$, neither $L_1$ or $L_2$ can be mixed-type by Theorem 3.14 of [EH]. Since one of the ambient spaces $F_{L_1}$ and $F_{L_2}$ will not be represented by $FK$ if $K_1$ is of mixed-type, $K_1$ can be assumed of even-type and so $L_1$ is of odd-type. By Lemma 1.3.7(ii), $\Theta(L,K) = F^*$.

First, suppose that $K_1$ is odd-type. Obviously we may assume that all components of $L$ and $K$ are of the same type. If $\delta_1 = 0$, then $l_1 - k_1 = 1$ implies that $r_1 - s_1$ and $\text{rank}L_1 = \text{rank}K_1 + 1$. Because $K_1$ is of odd-type, this contradicts our assumption that all components of $L$ and $K$ are of same type. So $\delta_1$ cannot be zero. If $\delta_1 = 2$, then $l_1 - k_1 = 1$ leads to two possibilities; either $s_1 = r_2$ and $\text{rank}L_2 = \text{rank}K_1 - 1$, or $s_1 = r_3$ and $\text{rank}L_3 = \text{rank}K_1 - 1$. Again because $K_1$ is of odd-type, both of them contradict our assumption that all components of $L$ and $K$ are of same type. So $\delta_1$ can not be two. Thus $\delta_1$ can only be 1, and so $s_1 = r_2$ and $\text{rank}L_2 = \text{rank}K_1$. Conditions in case (4) are fully established. Now assume conditions in (4) are satisfied. Because $\text{rank}L_1 = 1$, $r_1 \equiv r_2 + 1 \pmod{2}$. Applying Lemma 1.3.5(i) several times, we get $\Theta(L,K) = \Theta(L',K)$.
where \( L' = [r_2] \cdot L_2 \cdot [r_2 + 1] \cdot L_1 \cdot L_3 \cdot \ldots \). Apply Lemma 1.3.7(viii) to \( L' \) and \( K \), we get the desired formula. Case (4) is done.

For the remainder of the proof, we let \( \text{rank} K_1 = 1 \). We will show that one set of conditions listed in cases (1), (2) and (3) must hold in order for \( \Theta(L,K) \neq F^* \). Our line of proof is again as follows:

In order for \( \Theta(L,K) \neq F^* \), certain necessary conditions will be extracted until one of these three cases is fully established, and then we prove the formula for \( \Theta(L,K) \) as stated in the corresponding case.

There are six possibilities for \( L_1 \), namely:

1. \((\alpha)\) \( \tilde{L}_1 = [r_1] \cdot L_1 \) with \( L_1 \) being odd-type rank two;
2. \((\beta)\) \( \tilde{L}_1 = [r_1] \cdot L_1 \) with \( L_1 \) being even-type rank two;
3. \((\gamma)\) \( \tilde{L}_1 = [r_1] \cdot L_1 \) with \( L_1 \) being mixed-type rank two;
4. \((\delta)\) \( \tilde{L}_1 = L_1 \cdot L_2 \), \( r_1 \equiv r_2 \pmod{2} \) and \( D(-\xi \eta) \in \{0, 4R\} \);
5. \((\varepsilon)\) \( \tilde{L}_1 = L_1 \cdot L_2 \), \( r_1 \equiv r_2 \pmod{2} \) and \( D(-\xi \eta) = 2R \);
6. \((\phi)\) \( \tilde{L}_1 = L_1 \cdot L_2 \) and \( r_1 \neq r_2 \pmod{2} \).

We shall see that \((\alpha)\), \((\beta)\) and \((\delta)\) correspond to case (1); \((\gamma)\) and \((\varepsilon)\) correspond to case (2); and \((\phi)\) corresponds to case (3).

First, suppose \((\alpha)\). Clearly all components of \( L \) and \( K \) are of the same type. To establish that ConditionA(\( \tilde{L}_1, \xi_0 \eta_0, s_1, 1 \)) holds, it suffices to verify A(3)(ii). When \( s_1 > r_1 + 1 \) and \( \det L_1 = -1 \) (equivalently \( L_1 \equiv A(0,0) \)), then \( \Theta(L,K) = F^* \) by Lemma 1.3.7(i), and hence A(3)(ii) is necessary for \( \Theta(L,K) \neq F^* \). (Note \( \xi = \xi_0 \eta_0 = 1, N^* = L_1, s = s_1 \) and \( a = r_1 \)). So all conditions in case(1) are established. Conversely, to see that the stated formula holds, first note that \( L_1 = A(2,2p) \) when \( s_1 > r_1 + 1 \). Apply
Lemma 1.3.7(iv) to L and K if \( s_1 = r_1 + 1 \). When \( s_1 > r_1 + 1 \), apply Lemma 1.3.7(vi)
(possibly several times) and then Lemma 1.3.7(iv). Case (1) is now proved under (\( \alpha \)).

Next, suppose (\( \beta \)). The argument is similar to (\( \alpha \)). To see that Condition A\((L_1, 
\xi_0 \eta_0, s_1, 1)\) is necessary, suppose we have \( \det L_1 = -1 \) and \( s_1 > r_1 + 2 \). Then \( L_1 \equiv 
A(1,0) \). Apply Lemma 1.3.6(iii) to obtain \( \Theta (L,K) = \Theta (L',K) \) and \( L' = \begin{bmatrix} r_1 + 1 \end{bmatrix} \cdot A(0,0) \cdot \begin{bmatrix} r_2 \end{bmatrix} \cdot L_2 \ldots \). Next Lemma 1.3.7(i) applies since \( s_1 \geq (r_1 + 1) + 2 \), and we
get \( \Theta (L,K) = \Theta (L',K) = F^* \). So Condition A\((L_1, \xi_0 \eta_0, s_1, 1)\) is necessary. Then
stated formula holds by applying Lemma 1.3.6(i) if \( s_1 = r_1 \). When \( s_1 > r_1 \), Lemma
1.3.6(iii) applies, and the problem is reduced to case (\( \alpha \)).

Suppose (\( \chi \)). Clearly \( D(-\xi_1 \eta_0) = D(-\det L_1) = 2R \). Again, in order that \( \Theta (L,K) \neq 
F^* \), the ambient spaces of all components of L and K must be represented by \( W := 
F(r_1) \cdot L_1 \), and \( r_2 \geq s_1 + 4 \) by Lemma 1.3.4. This gives B(2)(ii) of Condition B\((L,W)\).
So all conditions in case (2) are established. The formula for \( \Theta (L,K) \) is seen from
Lemma 1.3.4. This proves Case (2) under (\( \chi \)).

Suppose (\( \delta \)). Since \( r_1 \equiv r_2 \) (mod 2), repeatedly apply Lemma 1.3.5(i), we get
\( \Theta (L,K) = \Theta (L',K) \) where \( L' = \begin{bmatrix} r_2 \end{bmatrix} \cdot \begin{bmatrix} r_1 \end{bmatrix} \cdot L_2 \cdot L_3 \ldots \). Since \( D(-\det L_1 \cdot \det L_2) = D(-
\xi_1 \eta_0) \in \{0, 4R\} \), \( L_1 \perp L_2 \) is of even-type. The problem is now reduced to the finished
case (\( \beta \)). This finishes the case (1) under (\( \delta \)).

Suppose (\( \epsilon \)). Since \( r_1 \equiv r_2 \) (mod 2) repeatedly apply Lemma 1.3.5(i), we get
\( \Theta (L,K) = \Theta (L',K) \) where \( L' = \begin{bmatrix} r_2 \end{bmatrix} \cdot \begin{bmatrix} r_1 \end{bmatrix} \cdot L_2 \cdot L_3 \ldots \). Since \( D(-\det L_1 \cdot \det L_2) = 
D(-\xi_1 \eta_0) = 2R \), \( L_1 \perp L_2 \) is of mixed-type. The problem is now reduced to the finished
case (\( \chi \)). This proves the case (2) under (\( \epsilon \)).

Finally, suppose (\( \phi \)). We have \( r_2 \leq s_1 < r_3 \). Since \( r_1 \neq r_2 \) (mod 2), repeatedly
apply Lemma 1.3.5(i), we get \( \Theta (L,K) = \Theta (L',K) \) where

\[ L' = \begin{bmatrix} s_1 \end{bmatrix} \cdot L_1 \cdot \begin{bmatrix} s_1 + 1 \end{bmatrix} \cdot L_2 \cdot \begin{bmatrix} r_3 \end{bmatrix} \cdot L_3 \ldots \] when \( r_1 \equiv s_1 \) (mod 2);
or \[ L' = [s_1] \circ L_2 \downarrow (s_1 + 1) \circ L_1 \downarrow (r_3) \circ L_3 \ldots \] when \( r_2 = s_1 \) (mod 2).

In both situations \( \Theta(L,K) = \Theta(L',K') \supseteq \Theta(O^+(L')) = F^* \) if \( r_3 < (s_1 + 1) + 5 \). (See Proposition 1.9 of [EH]). In order for \( \Theta(L,K) \neq F^* \), we then have to assume that \( r_3 \geq s_1 + 6 \). Conditions in case (3) are fully established. The desired formula follows by applying Lemma 1.3.5(ii) to \( L' \) and \( K \). Note that if \( r_3 \geq s_1 + 6 \) then \( \Theta(O^+(L')) = \Theta(O^+(s_1) \circ \xi_0 \eta_0 \det L_1 \downarrow (s_1 + 1) \circ L_2 \downarrow (r_3) \circ L_3 \ldots) \) when either \( r_1 \equiv s_1 \) (mod 2) or \( r_2 \equiv s_1 \) (mod 2). This gives case (3).

Q.E.D.

**PROPOSITION 1.4.5.** Let \( \delta_1 \leq 2 \) and \( l_1 - k_1 = 0 \). Let \( K' = [s_2] \circ K_2 \ldots \).

1. Let \( K_1 \) be binary and of even-type. If all components of \( L \) and \( K \) are of the same type and \( \text{ConditionA}(L_1, \xi_0 \eta_0, s_1, 1) \) holds, then \( \Theta(L,K) = \Theta(L',K')R^* \), where

\[
L' = \begin{cases} 
[r_2] \circ L_2 \downarrow (r_3) \circ L_3 \ldots; & \text{when } r_2 > s_1; \\
[r_3] \circ L_3 \downarrow (r_4) \circ L_4 \ldots; & \text{when } r_3 > s_1 \geq r_2; \text{ and}
\end{cases}
\]

\( L_i' \) is a lattice of same rank and type as \( L_i \) but \( \det L_i' = (\det L_1 \ldots \det L_i) \det K_i \).

2. Let \( K_1 \) be of odd-type. If

(i) all components of \( L \) and \( K \) are of same type, and
(ii) either (a) \( \tilde{L}_1 = [s_1] \circ M \),
or (b) \( \tilde{L}_1 = N \downarrow [s_1] \circ M \) with rank \( M = \text{rank} K_i - 2 \) and

\( \text{ConditionA}(N, \xi_0 \eta_0, s_1, 1) \) holds.

Then \( \Theta(L,K) = \Theta(L',K')R^* \) where

\[
L' = \begin{cases} 
[r_2] \circ L_2 \downarrow (r_3) \circ L_3 \ldots; & \text{when } r_2 > s_1; \\
[r_3] \circ L_3 \downarrow (r_4) \circ L_4 \ldots; & \text{when } r_3 > s_1 \geq r_2; \text{ and}
\end{cases}
\]

\( L_i' \) is a lattice of same rank same type as \( L_i \) but \( \det L_i' = (\det L_1 \ldots \det L_i) \det K_i \).

3. Suppose \( K_1 \) is of mixed-type rank 2. If ambient spaces of all components of \( L \) and \( K \) are represented by \( W = F([s_1] \circ K_1) \) and \( \text{ConditionB}(1,W) \) holds. Then

\( \Theta(L,K) = \Theta(L',K') \Theta(O^+(K)) \) where \( L' \) is same as in case (1). \( \Theta(L,K) \) contains
a mixed subgroup \( \Gamma \) of \( F^* \) of index two, and \( \Delta \in \Gamma \).

(4) Suppose \( \text{rank} \ K_1 = 1 \). If either (i) \( r_3 \geq s_1 + 6 \), or (ii) \( \text{rank} \ L_2 \geq 2 \), or (iii) \( r_2 \geq s_1 + 2 \), then \( \Theta(L, K) = \Theta(L', K') \Theta(O^+([s_1]e<\xi_1\eta_0>L[r_2]eL_2...)) \), where \( L' = [r_2]eL_2'\perp[r_3]eL_3... \), \( L_2' \) is a lattice of same rank and type as \( L_2 \) but \( \det L_2' = \det L_1 \det L_2 \det K_1 \).

If none of the above applies then \( \Theta(L, K) = F^* \).

Proof: We divide the problem into four cases as listed in the Proposition. For each case we prove that the conditions there are necessary in order \( \Theta(L, K) \neq F^* \), and then we show that the stated formula holds when all the conditions are satisfied. Our strategy for proving the necessity part is again to keep extracting necessary conditions forced by \( \Theta(L, K) \neq F^* \) until all conditions in each case are fully established.

Since a non-odd-type modular lattice \( K_1 \) of rank \( \geq 3 \) has \( \Theta(O^+(K_1)) = F^* \), we shall assume that \( \text{rank} \ K_1 \leq 2 \) when \( K_1 \) is either even or mixed-type.

Consider first case (1). It is clear that all components of \( L \) and \( K \) can be assumed of same type. We show that ConditionA(\( L_1, \xi_0\eta_0, s_1, 1 \)) is necessary. \( L_1 \) has three possible structures:

(\( \alpha \)): \( L_1 = [r_1]eL_1 \), \( L_1 \) is of odd-type rank two, and \( r_1 < s_1 < r_2 \);

(\( \beta \)): \( L_1 = [r_1]eL_1 \), \( L_1 \) is of even-type rank two, and \( r_1 \leq s_1 < r_2 \);

(\( \gamma \)): \( L_1 = [r_1]eL_1\perp[r_2]eL_2 \), \( \text{rank} \ L_1 = \text{rank} \ L_2 = 1 \), and \( r_2 \leq s_1 < r_3 \).

Suppose (\( \alpha \)). If \( s_1 > r_1 + 1 \) and \( \det L_1 = -1 \) (equivalently \( L_1 \equiv A(0,0) \)), then \( \Theta(L, K) = F^* \) by Lemma 1.3.7(i). Hence A(3)(ii) of ConditionA(\( L_1, \xi_0\eta_0, s_1, 1 \)) is necessary. (Note \( N^* = L_1, \xi = \xi_0\eta_0 = 1, s = s_1, a = r_1 \)). Condition of case (1) are established under (\( \alpha \)). Conversely, suppose all conditions in case (1) are satisfied. Then, the stated formula holds can be seen as follows. If \( s_1 = r_1 + 1 \), apply Lemma 1.3.7(v). If
s₁ > r₁ + 1, then ConditionA(Ł₁, ξ₀η₀, s₁, 1) implies Ł₁ ≡ A(2,2ρ). The formula follows from Lemma 1.3.7(vi) (possibly several times) and then Lemma 1.3.7(v). This proves Case (1) under (α).

Suppose (β). If s₁ > r₁ + 2 and detŁ₁ = -1, then Ł₁ ≡ A(1,0). By Lemma 1.3.6(iii), we have Θ(Ł, K) = Θ(Ł', K) where Ł' = [r₁ + 1]○A(0,0) □ [r₂]○Ł₂... . Since s₁ ≥ (r₁+1) + 2, Lemma 1.3.7(i) gives Θ(Ł, K) = Θ(Ł', K) = F*. Hence, ConditionA(Ł₁, ξ₀η₀, s₁, 1) is necessary. That the stated formula holds may be seen as follows. If s₁ = r₁, use Lemma 1.3.6(i). s₁ > r₁ + 1 is excluded. If s₁ ≥ r₁ + 2, then Lemma 1.3.6(iii) reduces the problem to case (α). This does case(1) under (β).

Suppose (γ). Because of the necessary assumption that all components are of the same type, r₁ ≡ r₂ (mod 2), repeatedly apply Lemma 1.3.5(i)we get Θ(Ł, K) = Θ(Ł', K) where Ł' = [r₂]○(Ł₁⊥Ł₂)⊥[r₃]○Ł₃... . Since we already have Θ(Ł, K) ⊇ Θ(Ł⁺(K)) ⊇ R*. (Ł₁⊥Ł₂) can be assumed to be of even-type. The problem is now reduced to the proved case (β), and the similar proof goes through. Case (1) is completely done.

Consider case (2). Because K₁ is odd, we may again assume that all components of Ł and K are of the same type. We note that δ₁ ≠ 1. For, if it is the case then Ł₁ = [r₁]○Ł₁⊥[r₂]○Ł₂, r₂ = s₁ and rankŁ₂ = rankK₁ - 1. Since K₁ is of odd-type, this contradicts our assumption that all components of Ł and K are of same type. Hence either δ₁ = 0, or δ₁ = 2.

Let δ₁ = 0. We have Ł = [r₁]○Ł₁⊥[r₂]○Ł₂⊥..., r₁ = s₁, rankŁ₁ = rankK₁, Ł is of odd-type, and ConditionA(Ł₁, ξ₀η₀, s₁, 1) is trivial. So the conditions of case (2) are established under the current situation of δ₁ = 0. The formula follows form Lemma 1.3.7(viii).

Let δ₁ = 2. We have Ł₁ = N⊥[s₁]○M and rankM = rankK₁ - 2. There are three possibilities for N, namely:
(α): \( N = [r_1] \bullet L_1 \), \( L_1 \) is of odd-type rank two, and \( s_1 = r_2 \);

(β): \( N = [r_1] \bullet L_1 \), \( L_1 \) is of even-type rank two, and \( s_1 = r_2 \);

(γ): \( N = [r_1] \bullet L_1 \perp [r_2] \bullet L_2 \), rank\( L_1 = \) rank\( L_2 = 1 \), \( r_1 \equiv r_2 \) (mod 2), and \( s_1 = r_3 \).

Arguments similar to those in the proof of case (1) go through accordingly here. Details are omitted. This finishes case (2).

Consider case (3). Since \( K_1 \) is of mixed-type, we can assume that the ambient spaces of all components of \( L \) and \( K \) are represented by \( W := F([s_1] \bullet K_1) \). (See Theorem 3.14 of [EH]). We have two possible structures for \( L_1 \):

(α) \( \tilde{L}_1 = [r_1] \bullet L_1 \) and rank\( L_1 = 2 \);

(β) \( \tilde{L}_1 = [r_1] \bullet L_1 \perp [r_2] \bullet L_2 \), rank\( L_1 = \) rank\( L_2 = 1 \).

Suppose (α). Then obviously \( W = F([s_1] \bullet K_1) \equiv F([r_1] \bullet L_1) \). By Lemma 1.3.4, we have to assume that \( r_2 \geq s_1 + 4 \). Otherwise, \( \Theta(L,K) = F^* \). So \( B(2)(ii) \) of Condition\( B(1,W) \) holds. (Note \( L(1,1) = L_1 \) and \( a = r_2 \)). The stated formula in this situation follows from Lemma 1.3.4.

Suppose (β). Since \( \Theta(L,K) \supseteq \bigcup (O^+(\tilde{L}_1) \Theta(O^+(K_1))) \), in order that \( \Theta(L,K) \neq F^* \), we have to assume \( r_1 \equiv r_2 \) (mod 2). Repeatedly Apply Lemma 1.3.5(i) we get \( \Theta(L,K) = \Theta(L',K) \) where \( L' = [r_2] \bullet (L_1 \perp L_2) \perp [r_3] \bullet L_3 \ldots \). Because \( K_1 \) is of mixed-type, in order for \( \Theta(L,K) \neq F^* \) it is necessary to assume that \( L_1 \perp L_2 \) is also of mixed-type, and \( F([s_1] \bullet K_1) \equiv F([r_2] \bullet (L_1 \perp L_2)) \). The problem is now reduced to (α) about the pair \( L' \) and \( K \). This completes the case (3).

Finally, assume rank\( K_1 = 1 \). Thus \( \tilde{L}_1 = [r_1] \bullet L_1 \) has rank 1 and \( r_1 \equiv s_1 \) (mod 2). Applying Lemma 1.3.5(i), we may assume \( r_1 = s_1 \). If rank\( L_2 = 1 \), \( r_2 = s_1 + 1 \) ( =\( r_1 + 1 \)), and \( r_3 \leq s_1 + 5 \); then \( \Theta(O^+(L)) \supseteq \Theta(O^+([r_2] \bullet L_1 \perp [r_2] \bullet L_2 \perp [r_3] \bullet L_3)) = F^* \) by Proposition 1.9 of [EH]. So we need to assume that either (i) \( r_3 \geq s_1 + 6 \), or (ii)
rank$L_2 \geq 2$, or (iii) $r_2 \geq s_1 + 2$. So, the conditions in (4) were established. The formula follows from Lemma 1.3.5.

Q.E.D.

§ 1.5 Formulas Over 2-adic Fields

After all these preparations in the previous sections we now come to the point to present and prove the main results, which, taken together, determine completely the relative spinor norm groups over the 2-adic fields. First we give some more notations. 

\[ \delta_i = \sum_{r_j < s_i} \text{rank} L_j - k_{i-1} \text{ defined for all index } i; \delta = \max_i \{\delta_i\}; \text{ and } \kappa = \text{rank} L - \text{rank} K, \]

the codimension.

**Theorem 1.5.1.** Suppose $\delta \leq 2$, $\kappa \leq 2$ and $l_i - k_i \leq 2$ for all $i = 1, \ldots, t$; and all the Jordan components of $L$ and $K$ are of the same even (or odd) types. Suppose further that either there is at least one component of rank $\geq 2$, or there is index $j$ where $l_j - k_j = 1$ and $\text{rank} L_j = 2$ with $D(-\xi_j \eta_j) \in \{0, 4R\}$. Then $\Theta(L,K) = R^*F^*2$ under the following conditions: Let $i$ be an arbitrary index.

I  If $l_i - k_i = 2$, then $K_i$ is of odd-type and

(a) $\hat{L}_i = [s_i] M$, and $\text{rank} \hat{L}_i = \text{rank} K_i + 2$; or

(b) $\hat{L}_i = N \perp [s_i] M$, $\text{rank} M = \text{rank} K_i$, and $\text{Condition A}(N, \xi_{i-1} \eta_{i-1}, s_i, i)$ holds.

II  If $l_i - k_i = 1$, then

(a) $\text{rank} K_i = 1$, and $\text{Condition A}(\hat{L}_i, \xi_{i-1} \eta_{i-1}, s_i, i)$ holds; or

(b) $K_i$ is of odd-type, $\hat{L}_i = N \perp [s_i] M$, and $\text{rank} K_i = \text{rank} M$;

III  If $l_i - k_i = 0$, then

(a) $\text{rank} \hat{L}_i \leq \text{rank} K_i = 1$; or
(b) \( K_j \) is of even-type rank 2, and ConditionA(\( \hat{L}_i, \xi_{i-1} \eta_{i-1}, s_i, i \)) holds; or
(c) \( K_j \) is of odd-type, and
   either (i) \( \hat{L}_i = [s_i] \cdot M \), and rank\( M = \text{rank} K_i \);
   or (ii) \( \hat{L}_i = N \perp [s_i] \cdot M \), rank\( M = \text{rank} K_i - 2 \), and
   ConditionA(\( N, \xi_{i-1} \eta_{i-1}, s_i, i \)) holds.

IV If \( \kappa = 2 \) then \( D(\xi_{s+1} \eta_{t}) = 4R \).

Suppose \( l_i - k_i = 2 \) for some \( i \). By the Note 3 below, we see already \( \Theta(L, K) \supsetneq \mathbb{R}^*F^{*2} \). If \( L \) or \( K \) has a mixed-type component, then \( \Theta(L, K) = F^* \). If all components of \( L \) and \( K \) are of rank one, again by Note 3 and Proposition 1.4.3, we have \( \Theta(L, K) = F^* \). So in the next two Theorems we can assume \( l_i - k_i \leq 1 \) for all \( i = 1, \ldots, t \).

**THEOREM 1.5.2.** Suppose \( \delta \leq 2 \), \( \kappa \leq 2 \) and \( l_i - k_i \leq 1 \) for all \( i = 1, \ldots, t \). Assume further that either there is a mixed-type binary component of \( L \) or \( K \) with ambient space \( W \); or there is an index \( j \) where \( l_j - k_j = 1 \) and \( \hat{L}_j = [j_1] \cdot L(j, 1) \perp [j_2] \cdot L(j, 2) \) has rank two, \( j_1 \equiv j_2 \pmod{2} \), and \( D(\xi_j \eta_{j-1}) = 2R \), and let \( W \) be the ambient space of \( [j_2] \cdot \langle \xi_{j-1} \eta_{j-1} \rangle \cdot \text{det} L(j, 1) > \perp [j_2] \cdot L(j, 2) \). Then, \( \Theta(L, K) = \Theta(O^*(W)) \) under then following conditions:

I All the ambient spaces of the Jordan components of \( L \) are represented by \( W \);
   and for \( i = 1, 2, \ldots, t \), either (i) \( r_{i+1} \geq r_i + 4 \),
   or (ii) \( r_{i+1} = r_i + 2 \), rank\( L_i = \text{rank} L_{i+1} = 1 \), and \( \text{det} L_i \text{ det} L_{i+1} \text{ det} W \in \{ 1, \Delta \} \).

II All the ambient spaces of the Jordan components of \( K \) are represented by \( W \);
   and for \( i = 1, 2, \ldots, t-1 \), either (i) \( s_{i+1} \geq s_i + 4 \),
   or (ii) \( s_{i+1} = s_i + 2 \), rank\( L_i = \text{rank} L_{i+1} = 1 \), and \( \text{det} L_i \text{ det} L_{i+1} \text{ det} W \in \{ 1, \Delta \} \).
or (ii) \( s_{i+1} = s_i + 2 \). \( \text{rank} K_i = \text{rank} K_{i+1} = 1 \), and \( \text{det} K_i \text{det} K_{i+1} \text{det} W \in \{1, \Delta\} \).

III If there is another index \( i \) such that \( \tilde{L}_i = \{[i_1] \bullet L(i,1) \perp [i_2] \bullet L(i,2) \} \) of rank two
and \( i_1 \equiv i_2 \pmod{2} \); then \( F([i_2] \bullet \xi_{i-1} \eta_{i-1} \text{det} L(i,1) \perp [i_2] \bullet L(i,2)) \equiv W \)

IV If \( \kappa = 2 \) then \( FL \equiv FK \perp FW \).

And let \( i \) be any index:

V When \( l_i - k_i = 1 \), then \( \text{rank} K_i = 1 \) and ConditionB(i, W) holds;
VI When \( l_i - k_i = 0 \), then \( \text{rank} K_i \leq 2 \) and ConditionB(i, W) holds;

**THEOREM 1.5.3.** Suppose \( \delta \leq 2 \), \( \kappa \leq 2 \) and \( l_i - k_i \leq 1 \) for all \( i = 1, \ldots, t \), and that all Jordan components of \( L \) and \( K \) are of rank one. Assume that for any index \( i \)
I when \( l_i - k_i = 0 \); then either (i) \( r_{i+2} \geq s_i + 6 \), or (ii) \( r_{i+1} \geq s_i + 2 \);
II when \( l_i - k_i = 1 \); then \( r_{i+2} \geq s_i + 6 \) and either (i) \( \tilde{L}_i = \{[r_i] \bullet \text{L}_i \perp [r_{i+1}] \bullet \text{L}_{i+1} \}
\)
\[ r_{i+1} \equiv r_{i+1} \pmod{2} \), or (ii) \( \tilde{L}_i = \{[r_{i+1}] \bullet \text{L}_{i+1}, s_i \equiv r_{i+1} \pmod{2} \).

Then \( \Theta(L,K) = \Theta(O^+(FK \perp ))AB \), where
\[ A = \prod_{l_i - k_i = 0} \Theta(O^+([s_i] \bullet \xi_{i-1} \eta_{i-1} \perp [r_{i+1}] \bullet \text{L}_{i+1} \perp \ldots)), \)
and
\[ B = \prod_{l_i - k_i = 1} \Theta(O^+([s_i] \bullet \xi_{i-1} \eta_{i-1} \text{det} \text{L}_i \perp [s_{i+1}] \bullet \text{L}_{i+1} \perp [r_{i+2}] \bullet \text{L}_{i+2} \perp \ldots)). \]

**THEOREM 1.5.4.** If none of the above theorems applies then \( \Theta(L,K) = F^* \).

From now on, by reduction we mean the invocation of Propositions 1.4.3, 1.4.4, and 1.4.5. Notice that each time we apply Prop 1.4.3, Prop 1.4.4 or Prop 1.4.5, we reduce the problem of calculating the relative spinor norm group by canceling (if \( \Theta(L,K) \neq F^* \)) the first Jordan component of the small lattice and at the same time adjusting the big one. (The small lattice refers to the lattice derived from \( K \) by applying these Propositions, while the big lattice refers to the one derived from \( L \).) Since \( K \) has
initially \( t \) many components, after \( t \) reduction steps we end up with calculating \( \Theta(O^+(FK^\perp)) \). Naturally we may assume that the codimension \( \kappa \leq 2 \). Also during the reduction process, if at a particular stage Prop 1.4.1 or Prop 1.4.2 applies, then we have \( \Theta(L,K) = F^* \). Assume \( L \) and \( K \) become \( L' \) and \( K' \) respectively after \( j - 1 \) reduction steps, then \( l_i' - k_i' = l_j - k_j \) and \( \delta_i' = \delta_j' \). So further we may assume that \( \delta \leq 2 \) (to avoid the application of Prop 1.4.1), and \( l_i - k_i \leq 2 \) for all \( i = 1, \ldots, t \) (to avoid the application of Prop 1.4.2). We have therefore the first of several NOTES below.

**NOTE 1.** In order that \( \Theta(L,K) \not= F^* \), we may assume \( \delta \leq 2 \), \( \kappa \leq 2 \) and \( l_i - k_i \leq 2 \) for all \( i \).

**NOTE 2.** Suppose \( L \) or \( K \) has an even (odd) -type component of rank \( \geq 2 \). In order for \( \Theta(L,K) \not= F^* \), we may assume that all components of \( L \) and \( K \) have the same parity. In this case \( \Theta(L,K) \supseteq R^*F^*2 \).

**NOTE 3.** Suppose there is an index \( i \) such that \( l_i - k_i = 2 \). Then after \( i - 1 \) reduction steps \( L \) becomes \( L' \), \( K \) becomes \( K' \) and \( l_i' - k_i' = l_i - k_i = 2 \). By Prop 1.4.3, \( \Theta(L,K) \supseteq R^*F^*2 \). In order for \( \Theta(L,K) \not= F^* \), we may further assume that all components of \( L \) and \( K \) are of same parity.

**NOTE 4.** Suppose \( L \) or \( K \) has a mixed-type component with supporting space \( W \), then the rank of that component can be assumed to be 2. In this case \( \Theta(L,K) \supseteq \Theta(O^+(W)) \), which is a mixed subgroup of \( F^* \) of index 2 containing \( \Delta \). In order for \( \Theta(L,K) \not= F^* \) we may further assume (See Th.3.14 of [EH]):

1. All the ambient spaces of the Jordan components of \( L \) are represented by \( W \);
   and for \( i = 1, 2, \ldots, t \),
either (i) \( r_{i+1} \geq r_i + 4; \)

or (ii) \( r_{i+1} = r_i + 2, \text{rank}L_i = \text{rank}L_{i+1} = 1, \text{and} \det L_i \det L_{i+1} \det \mathcal{W} \in \{1, \Delta\}. \)

II All the ambient spaces of the Jordan components of \( K \) are represented by \( \mathcal{W} \);

and for \( i = 1, 2, \ldots, t-1, \)

either (i) \( s_{i+1} \geq s_i + 4; \)

or (ii) \( s_{i+1} = s_i + 2, \text{rank}K_i = \text{rank}K_{i+1} = 1, \text{and} \det K_i \det K_{i+1} \det \mathcal{W} \in \{1, \Delta\}. \)

Suppose that all the components of \( L \) and \( K \) are of rank 1. By Prop 1.4.3, we may assume that \( l_j - k_j \leq 1 \) for all \( i \). If there is an index \( j \), such that \( L_j = [a] \cdot [x] \cup [b] \cdot [y] \) with \( a \equiv b \pmod{2} \), then \( l_j - k_j = 1 \). After the first \( j-1 \) reduction steps

\( [a] \cdot [x] \cup [b] \cdot [y] \) will become the leading part of the big lattice. After another several applications of Lemma 1.3.5, \( [b] \cdot [x] \cup [b] \cdot [y] \) becomes the leading Jordan component of the big lattice, while \( [s_j] \cdot K_j \) is the leading Jordan component of the small lattice. Proposition 1.4.4 (1) then gives NOTE 5, and Proposition 1.4.4 (2) gives NOTE 6 below.

NOTE 5. Suppose that all components are of rank 1. In order for \( \Theta(L, K) \neq F^* \), we may assume that all \( l_j - k_j \leq 1 \). If there is an index \( j \), such that \( L_j = [j_1] \cdot L(j, 1) \cup [j_2] \cdot L(j, 2) \) has rank two and \( j_1 \equiv j_2 \pmod{2} \) and \( D(\xi_{j_1} \eta_{j_1}) \in \{0, 1, 2\} \) (or equivalently, that \( \xi_{j_1} \eta_{j_1} \det L(j, 1) \det L(j, 2) \) is of even-type), then we see \( \Theta(L, K) \subseteq \mathbb{R}^* \). So we may further assume that all components have the same type parity.

NOTE 6. Suppose that all components are of rank 1. In order for \( \Theta(L, K) \neq F^* \), we may assume that all \( l_j - k_j \leq 1 \). If there is an index \( j \), such that \( L_j = [j_1] \cdot L(j, 1) \cup [j_2] \cdot L(j, 2) \) has rank two and \( j_1 \equiv j_2 \pmod{2} \) and \( D(\xi_{j_1} \eta_{j_1}) = 2 \mathbb{R} \), (or equivalently, that \( [x] \cup [y] \) is of mixed-type), then \( \Theta(L, K) \subseteq \mathbb{R}^* \).
\[ \Theta(O^+(W)), \text{ where } W = F([j_2] \cdot \langle \xi_{j_1,1} \eta_{j_1,1} \det L(j,1) \rangle \perp [j_2] \cdot L(j,2)). \] \Theta(O^+(W)) \text{ is a mixed subgroup of } F^* \text{ of index } 2 \text{ containing } \Delta. \text{ So we may further assume that}

I  All the ambient spaces of the Jordan components of L are represented by W; and for \( i = 1, 2, \ldots, t, \)

- either (i) \( r_{i+1} \geq r_i + 4; \)
- or (ii) \( r_{i+1} = r_i + 2, \) \( \text{rank} L_i = \text{rank} L_{i+1} = 1, \) and \( \det L_i \det L_{i+1} \det W \in \{1, \Delta\}. \)

II  All the ambient spaces of the Jordan components of K are represented by W; and for \( i = 1, 2, \ldots, t-1, \)

- either (i) \( s_{i+1} \geq s_i + 4; \)
- or (ii) \( s_{i+1} = s_i + 2, \) \( \text{rank} K_i = \text{rank} K_{i+1} = 1, \) and \( \det K_i \det K_{i+1} \det W \in \{1, \Delta\}. \)

**NOTE 7.** Suppose the situation in Note 4 or Note 6 happens. So we have \( \Theta(L, K) \supseteq \Theta(O^+(W)), \) and ambient spaces of all components of L and K are represented by W. Assume further that there is another index \( i \) such that \( \tilde{L}_i = [i_1] \cdot L(i,1) \perp [i_2] \cdot L(i,2) \) has rank two and \( i_1 \equiv i_2 \pmod{2} \) and \( D(\xi_i \eta_{i,1}) = 2R. \) Define \( W' \) to be the ambient space of \( [i_2] \cdot \langle \xi_{i,1} \eta_{i,1} \det L(i,1) \rangle \perp [i_2] \cdot L(i,2). \) By Notes 4 and 6 again we have \( \Theta(L, K) \supseteq \Theta(O^+(W')), \) and all ambient spaces of all components of L and K are represented by \( W'. \) In order that \( \Theta(L, K) \neq F^*, \) we must have \( \det W = \det W'. \) Since both \( W \) and \( W' \) represent a common non-zero element, we have \( W \equiv W'. \)

Suppose that all the components of L and K are of rank 1, and that \( l_i - k_i \leq 1 \) for all \( i. \) If there is an index \( j, \) such that \( \tilde{L}_j = [a] \cdot \langle x \rangle \perp [b] \cdot \langle y \rangle \) with \( a \equiv b + 1 \pmod{2}, \) then \( l_j - k_j = 1. \) After the first \( j-1 \) reduction steps are applied using Prop 1.4.4 and Prop 1.4.5, \( [a] \cdot \langle \xi_{j,1} \eta_{j,1} x \rangle \perp [b] \cdot \langle y \rangle \) becomes the leading part of the big lattice. After another several applications of Lemma 1.3.5, it becomes
either \([s_j] \cdot \xi_{j-1} \eta_{j-1} x > \perp [s_j+1] \cdot y\) when \(a \equiv s_j \mod 2\),
or \([s_j] \cdot y > \perp [s_j+1] \cdot \xi_{j-1} \eta_{j-1} x\) when \(b \equiv s_j \mod 2\).

\([s_j] \cdot K_j\) becomes the leading component of the small lattice. Proposition 1.4.4(3) then gives:

**NOTE 8.** Suppose that the rank of all components of \(L\) and \(K\) is 1. In order for \(\Theta(L,K) \neq F^*\) we may assume that \(l_i - k_i \leq 1\) for all \(i\). If there is index \(j\) such that \(j_1 = [j_1] \cdot L(j,1) \perp [j_2] \cdot L(j,2)\) is of rank 2 with \(j_1 \equiv j_2 + 1 \mod 2\). Then by Proposition 1.19 of [EH], \(\Theta(L,K) \supseteq \Gamma\). \(\Gamma\) is a mixed subgroup of \(F^*\) of index two and \(\Delta \in \Gamma\).

Proof of Theorem 1.5.1:

By Note 2, Note 3 and Note 5, \(\Theta(L,K) \supseteq R^*\). Next, we claim that for any index \(i\), after \(i\) reduction steps we have \(\Theta(L,K) = R^*\Theta(\text{reduced } L, \text{reduced } K)\). This would eventually lead to \(\Theta(L,K) = R^*\Theta(O^*(FK-L))\), which equals \(R^*\) if and only if IV holds.

We shall prove this claim by induction on \(i\).

First, we give the outline of the induction process. At \(i = 1\), the conditions on \(L\) and \(K\) are really the corresponding ones in Prop 1.4.3, Prop 1.4.4 and Prop 1.4.5. Apply these propositions to \(L\) and \(K\) we get \(\Theta(L,K) \subset R^*\Theta(L',K')\) and so \(\Theta(L,K) = R^*\Theta(L',K')\). Suppose at \(i - 1\) all these conditions are satisfied and that after \(i - 1\) reduction steps \(\Theta(L,K) = R^*\Theta(L',K')\). At \(i\), we show that conditions on \(L\) and \(K\) at index \(i\) are equivalent to the corresponding conditions of Prop 1.4.3, Prop 1.4.4 and Prop 1.4.5 on \(L'\) and \(K'\). Apply these Propositions to \(L'\) and \(K'\), we get \(\Theta(L',K') \subset R^*\Theta(L'',K'')\), and so \(\Theta(L,K) = R^*\Theta(L'',K'')\). This completes the induction process.

Now the details of the induction.
(1) At $i = 1$. We see that I(a) and I(b) are the same as 3(a) and 3(b) of Prop 1.4.3 when $l_1 - k_1 = 2$. By this proposition, $\Theta(L, K) = R^*\Theta(L', K')$.

When $l_1 - k_1 = 1$. It is easy to see that II(a) and II(b) are the same as (1) and (4) of the Prop 1.4.4. So $\Theta(L, K) = R^*\Theta(L', K')$ after applying this Proposition to $L$ and $K$.

When $l_1 - k_1 = 0$. III(b) and III(c) are the same as (1) and (2) of Prop 1.4.5. III(a) is nothing but $\text{rank} K_1 = \text{rank} L_1 = 1$, which is the same as (4) of Prop 1.4.5. (Note the rest of the conditions in (4) hold automatically). By Prop 1.4.5, $\Theta(L, K) \subseteq R^*\Theta(L', K')$ and so $\Theta(L, K) = R^*\Theta(L', K')$. (Note when Prop 1.4.5(4) applies in the current situation, we can only assert $\Theta(L, K) \subseteq R^*\Theta(L', K')$).

In any event, we have $\Theta(L, K) = R^*\Theta(L', K')$, where

$$K' = [s_2] \cdot K_2 \perp [s_3] \cdot K_3 \perp \ldots ,$$  
and

(α) $L' = [s_1] \cdot A(2, 2^{-1}(1 + \xi_1 \eta_1)) \perp \hat{L}_2 \perp \hat{L}_3 \ldots$, when $l_1 - k_1 = 2$.

Here $A(2, 2^{-1}(1 + \xi_1 \eta_1))$ is of odd-type.

(β) $L' = [s_1] \cdot \langle \xi_1 \eta_1 \rangle \perp \hat{L}_2 \perp \hat{L}_3 \ldots$, when $l_1 - k_1 = 1$ and $\text{rank} K_1 = 1$.

(γ) $L' = [s_1 + 1] \cdot \langle \xi_1 \eta_1 \rangle \perp \hat{L}_2 \perp \hat{L}_3 \ldots$, when $l_1 - k_1 = 1$ and $K_1$ is odd-type.

(δ) $L' = \hat{L}_2 \perp \hat{L}_3 \ldots$ when $l_1 - k_1 = 0$.

Here $\hat{L}_2 = [a_1] \cdot L'(2, 1) \perp [a_2] \cdot L(2, 2) \perp \ldots$, provided that $\hat{L}_2$ is expressed as $\hat{L}_2 = [a_1] \cdot L(2, 1) \perp [a_2] \cdot L(2, 2) \perp \ldots$.

$L'(2, 1)$ is a lattice of same rank and type as $L(2, 1)$, and $\det L'(2, 1) = \xi_1 \eta_1 \det L(2, 1)$.

(2) Assume that all conditions are satisfied and $\Theta(L, K) = R^*\Theta(L', K')$ after $i - 1$ reduction steps, where $L'$ and $K'$ are as follows:

$$K' = [s_i] \cdot K_i \perp [s_{i+1}] \cdot K_{i+1} \perp \ldots ,$$  
and

(α) $L' = [s_{i-1}] \cdot A(2, 2^{-1}(1 + \xi_{i-1} \eta_{i-1})) \perp \hat{L}_{i-1} \perp \hat{L}_{i+1} \ldots$, when $l_{i-1} - k_{i-1} = 2$.

Here $A(2, 2^{-1}(1 + \xi_{i-1} \eta_{i-1}))$ is of odd-type.
\( \beta \) \ L' = [s_{i-1}] \* \langle \xi_{i-1} \eta_{i-1} \rangle \bot L_i \bot L_{i+1} \ldots, \) when \( l_{i-1} - k_{i-1} = 1 \) and rank \( K_{i-1} = 1. \)

\( \chi \) \ L' = [s_{i-1} + 1] \* \langle \xi_{i-1} \eta_{i-1} \rangle \bot L_i \bot L_{i+1} \ldots, \) when \( l_{i-1} - k_{i-1} = 1 \) and \( K_{i-1} \) is of odd-type.

\( \delta \) \ L' = \( \tilde{L}_i \bot L_{i+1} \ldots, \) when \( l_{i-1} - k_{i-1} = 0. \)

Here \( \tilde{L}_i = [a_1] \* L'(i,1) \bot [a_2] \* L(i,2) \bot \ldots, \) provided that \( \tilde{L}_i \) is expressed as

\[ \tilde{L}_i = [a_1] \* L(i,1) \bot [a_2] \* L(i,2) \bot \ldots. \]

\( L'(i,1) \) is a lattice of same rank and type as \( L(i,1), \) and

\[ \det L'(i,1) = \xi_{i-1} \eta_{i-1} \det L(i,1). \]

Since reductions do not change the parity, all components of \( L' \) and \( K' \) still have the same parity (even, or odd).

(3) At \( i. \) When \( l_i - k_i = 2, \) we show that condition (I) on \( L \) and \( K \) at index \( i \) is equivalent to (1) and (3) of the Prop 1.4.3 on \( L' \) and \( K' \). (Note that (2) of the Prop 1.4.3 on \( L' \) and \( K' \) holds automatically). Certainly Prop 1.4.3(1) on \( L' \) and \( K' \) is the same thing as saying that \( K_i \) is of odd-type, which is true because of (I). We have Prop 1.4.3(3) left to prove. We do that according to the different outcomes of the first \( i - 1 \) reduction steps.

Suppose (\( \alpha \)). We have \( \tilde{L}_i = N \bot [s_i] \* M, \) rank \( M = \text{rank} K_i \) and rank \( N = 0. \)

Condition \( A(N, \xi_{i-1} \eta_{i-1}, s_i, i) \) holds for \( L \) and \( K \) means that \( A(1) \) is satisfied. That is:

\[ D(- \xi_{i-1} \eta_{i-1}) = 4R \text{ when } s_i > s_{i-1} + 1. \]

This is exactly the \( A(3)(ii) \) of Condition \( A([s_{i-1}] \* A(2,2^{-1}(1 + \xi_{i-1} \eta_{i-1})), \xi_0 \eta_0, s'_1, 1) \) on \( L' \) and \( K'. \) (Note \( D(- \xi_{i-1} \eta_{i-1}) = 4R \) is equivalent to say \( A(2,2^{-1}(1 + \xi_{i-1} \eta_{i-1})) \equiv A(2,2 \rho); \) and \( s'_1 = s_i \).)

Suppose (\( \beta \)). We have \( \tilde{L}_i = N \bot [s_i] \* M, \) rank \( M = \text{rank} K_i \) and rank \( N = 1. \)

Condition \( A(N, \xi_{i-1} \eta_{i-1}, s_i, i) \) holds for \( L \) and \( K \) means that \( A(2) \) is satisfied. That is:

Let \( N = [a] \* N^*, \) \( N' = \langle \xi_{i-1} \eta_{i-1} \rangle \bot N^* \) is of even-type, \( \det N' \neq -1 \) when \( s_i > a + 2. \)
One can check that $A(3)(i)$ of ConditonA([s_{i-1}]<\xi_{i-1}\eta_{i-1}>\perp N, \xi'_0\eta'_0, s'_i, 1)$ on $L'$ and $K'$ is satisfied.

The remaining cases $(\chi)$ and $(\delta)$ can be treated in a similar way. We proved that (1) and (3) of the Proposition 1.4.3 on $L'$ and $K'$ are satisfied. Apply Prop 1.4.3 to $L'$ and $K'$, we get $\Theta(L',K') = R^\ast \Theta(L''',K''')$, and so $\Theta(L,K) = R^\ast \Theta(L''',K''')$ after i reduction steps.

Similarly, we can prove that II is equivalent to conditions (1) or (4) of Prop 1.4.4 on $L'$ and $K'$ when $l_i - k_i = 1$; and III is equivalent to conditions (1), (2) or (4) of Prop 1.4.5 on $L'$ and $K'$ when $l_i - k_i = 0$. In all these situations, apply the appropriate proposition to $L'$ and $K'$, we get $\Theta(L',K') \supseteq R^\ast \Theta(L''',K''')$, and so $\Theta(L,K) = R^\ast \Theta(L''',K''')$ after i reduction steps.

This completes the induction. Q.E.D.

Proof of Theorem 1.5.2: By Note 4, Note 6 and Note 7, we see $\Theta(L,K) \supseteq \Theta(O^+(W))$, which is a mixed subgroup of $F^\ast$ of index two containing $\Delta$. On the other hand we will show, by induction on i, that after i reduction steps $\Theta(L,K) = \Theta(O^+(W)) \Theta(\text{reduced } L, \text{reduced } K)$. This eventually leads to $\Theta(L,K) = \Theta(O^+(W))\Theta(O^+(FK^{-1}))$, which equals $\Theta(O^+(W))$ if and only if condition IV holds (See Theorem 3.14 of [EH]).

(1) At $i = 1$. The statement V implies, (combined with I and II), that the conditions of Prop1.4.4(2) hold when $l_1 - k_1 = 1$. So we have $\Theta(L,K) = \Theta(L',K')\Theta(O^+(W))$ after applying Prop 1.4.4(2) to $L$ and $K$. If $l_1 - k_1 = 0$, then I, II and VI imply that either conditions of Prop 1.4.5(3), or conditions of Prop 1.4.5(4) are satisfied according to whether rank$K_2$ is two or one respectively. In the first case, when Prop 1.4.5(3) applies, we have $\Theta(L,K) = \Theta(L',K')\Theta(O^+(W))$. In the second case, when Prop
1.4.5(4) applies, we have $\Theta(L,K) = \Theta(L',K')\Theta(O^+([s_1] \cdot <\xi_1 \eta_0> \perp [r_2] \cdot L_2 \ldots))$, which is contained inside $\Theta(L',K')\Theta(O^+(W))$ because of ConditionB(1,W) from VI. (See Theorem 3.14 of [EH]. Note in this case $B(1)$ holds where $a = r_2$.) In any event, 
$\Theta(L,K) = \Theta(L',K')\Theta(O^+(W))$, where $K' = [s_2] \cdot K_2 \perp \ldots$; and

$$L' = \begin{cases} 
(s_1) \cdot <\xi_1 \eta_1> \perp L_2 \perp \ldots; & \text{when } l_1 - k_1 = 1; \\
\hat{L}_2 \perp \hat{L}_3 \perp \ldots; & \text{when } l_1 - k_1 = 0 \text{ and } \text{rank} K_1 = 2; \\
\hat{L}_2' \perp \hat{L}_3' \perp \ldots; & \text{when } l_1 - k_1 = 0 \text{ and } \text{rank} K_1 = 1.
\end{cases}$$

Here $\hat{L}_2 = [a_1] \cdot L(2,1) \perp [a_2] \cdot L(2,2) \ldots$, provided that $\hat{L}_2$ is expressed as $\hat{L}_2 = [a_1] \cdot L(2,1) \perp [a_2] \cdot L(2,2) \ldots$.

$L'(2,1) = L(2,1)$, when rank$L(2,1) = 2$;
$L'(2,1) = <\xi_1 \eta_1 \det L(2,1)>$, when rank$L(2,1) = 1$.

(2) Assume that after $i - 1$ reduction steps, $\Theta(L,K) = \Theta(O^+(W))\Theta(L',K')$ where 
$K' = [s_i] \cdot K_i \perp \ldots$;

$\alpha \ L' = [s_{i-1}] \cdot <\xi_{i-1} \eta_{i-1}> \perp \hat{L}_i \perp \ldots$, when $l_{i-1} - k_{i-1} = 1$;

$\beta \ L' = \hat{L}_i \perp \hat{L}_{i+1} \perp \ldots$, when $l_{i-1} - k_{i-1} = 0$ and rank$K_{i-1} = 2$;

$\chi \ L' = \hat{L}_i \perp \hat{L}_{i+1} \perp \ldots$, when $l_i - k_i = 0$ and rank$K_i = 1$;

Here $\hat{L}_i = [i_1] \cdot L(i,1) \perp [i_2] \cdot L(i,2) \ldots$, provided that $\hat{L}_i$ is expressed as $\hat{L}_i = [i_1] \cdot L(i,1) \perp [i_2] \cdot L(i,2) \ldots$.

$L'(i,1) = L(i,1)$, when rank$L(i,1) = 2$;
$L'(i,1) = <\xi_{i-1} \eta_{i-1} \det L(i,1)>$, when rank$L(i,1) = 1$.

It is not difficult to see that (I), (II), (III) and (IV) are still satisfied for the $L'$ and $K'$. 

(3) At i. Suppose \( l_i - k_i = 1 \). We show that condition V on \( L \) and \( K \) at index i implies that conditions of Prop 1.4.4(2) are satisfied for the pair \( L' \) and \( K' \). We do that according to the different outcomes after the first \( i - 1 \) reduction steps.

Suppose (α). Statement V says that \( \text{rank} K_i = 1 \) and ConditionB(i,W) holds. We have \( \tilde{L}_i = [i_1] \cdot L(i,1) \) is of rank one, \( s_{i-1} = i_1 \mod 2 \), \( a \geq s_i + 4 \), \( F([i_1] \cdot \langle \xi_{i-1} \eta_{i-1} \rangle \perp \tilde{L}_i) \equiv W \) and \( \tilde{L}_{i+1} = [a] \cdot L(i+1,1) \perp ... \) (See B(2)(i)). These are exactly B(2)(iii) of ConditionB(1,W) on \( L' \) and \( K' \). So conditions of Prop 1.4.4(2) are all satisfied for \( L' \) and \( K' \).

Suppose (β). Then V implies the following (See B(2)(ii) and B(2)(iii)):

- \( \text{rank} K_i = 1 \), \( \tilde{L}_i \) is of rank two, \( a \geq s_i + 4 \). If \( \tilde{L}_i = [i_1] \cdot L(i,1) \), then \( F(\tilde{L}_i) \equiv W \).
- If \( \tilde{L}_i = [i_1] \cdot L(i,1) \perp [i_2] \cdot L(i,2) \), then \( i_1 \equiv i_2 \mod 2 \) and \( F([i_2] \cdot \langle \xi_{i-1} \eta_{i-1} \rangle \cdot \text{det}(L(i,1)) \perp [i_2] \cdot L(i,2)) \equiv W \).

(Note, in the last condition \( \xi_{i-1} \eta_{i-1} = 1 \). This can be seen as follows. Just prior to the cancellation of \( [s_{i-1}] \cdot K_{i-1} \), the big lattice is of the form \( [s_{i-1}] \cdot \tilde{L}_{i-1} \perp \tilde{L}_i \perp \tilde{L}_{i+1} \perp ... \), where \( \tilde{L}_{i-1} \) is of mixed-type rank two. Hence the order of the scale of \( \tilde{L}_i \) is larger than or equal to \( s_{i-1} + 4 \) by the induction hypothesis on \( i-1 \) stage.)

The above are either the B(2)(ii) or the B(2)(iii) of the ConditionB(1,W) on \( L' \) and \( K' \). So conditions of Prop 1.4.4(2) are all satisfied for \( L' \) and \( K' \).

Case (χ) can be treated in the similar way. We proved that conditions of Prop 1.4.4(2) are satisfied for \( L' \) and \( K' \) when \( l_i - k_i = 1 \). Apply this proposition to \( L' \) and \( K' \) and we get \( \Theta(L',K') = \Theta(L'',K'') \Theta(O^+(W)) \), and so \( \Theta(L,K) = \Theta(L'',K'') \Theta(O^+(W)) \) after i reduction steps.

Similarly, when \( l_i - k_i = 0 \), we can prove that, if \( \text{rank} K_i = 2 \) in VI, then all the hypothesis of Prop 1.4.5(3) are satisfied for the pair \( L' \) and \( K' \). Hence we get \( \Theta(L',K') = \Theta(L'',K'') \Theta(O^+(W)) \). If \( \text{rank} K_i = 1 \) in VI, then all the hypothesis of Prop 1.4.5(4) are satisfied for the pair \( L' \) and \( K' \), and we obtain \( \Theta(L',K') = \)...
\[ \Theta(L'', K'') \Theta(O^+(Z)), \text{ where } Z = [s_i] \circ \xi_i \eta_{i,1} \perp L_{i+1} \perp \ldots \]  

Because of the Condition B(1,W) on L' and K', we have \( \Theta(O^+(Z)) \subseteq \Theta(O^+(W)) \). In either case, \( \Theta(L,K) = \Theta(L'', K'') \Theta(O^+(W)) \) after i reduction steps.

This completes the induction. Q.E.D.

In the proof of this Theorem, we showed that conditions on \( L \) and \( K \) at index \( i \) imply that the corresponding conditions of Proposition 1.4.4 and 1.4.5 are satisfied for the pair L' and K'. In fact, the converse also holds.

Proof of Theorem 1.5.3: We prove, by induction on \( i \), the following:

After \( i \) reduction steps we have
\[
\Theta(L,K) = \Theta(L', K') \prod_{1 \leq j \leq i} \Theta(O^+([s_j] \circ \xi_j \eta_{j,1} \perp [r_{j+1}] \circ L_{j+1} \perp \ldots)) \prod_{1 \leq j \leq i} \Theta(O^+([s_j] \circ \xi_j \eta_{j,1} \circ \det L_j \perp [s_{j+1}] \circ L_{j+1} \perp [r_{j+2}] \circ L_{j+2} \perp \ldots)).
\]

Where L' and K' are the reduced lattices of L and K respectively.

(1) At \( i = 1 \). If \( l_1 - k_1 = 0 \), (I) implies that the conditions of Prop 1.4.5(4) are satisfied. Apply Prop 1.4.5(4) to L and K, we get
\[
\Theta(L,K) = \Theta(O^+([s_1] \circ \xi_1 \eta_1 \perp [r_2] \circ L_2 \perp \ldots)) \Theta(L', K'), 
\]
\[
L' = [r_2] \circ \xi_1 \eta_1 \circ \det L_2 \perp [r_3] \circ L_3 \perp \ldots; K' = [s_2] \circ K_2 \perp [s_3] \circ K_3 \perp \ldots.
\]

If \( l_1 - k_1 = 1 \), then (II) implies that the conditions in Prop 1.4.4(3) are satisfied. Apply this Proposition to L and K, we have
\[
\Theta(L,K) = \Theta(O^+([s_1] \circ \xi_0 \eta_0 \circ \det L_1 \perp [s_1 + 1] \circ L_2 \perp [r_3] \circ L_3 \perp \ldots)) \Theta(L', K'), 
\]
\[
L' = [s_1 + 1] \circ \xi_1 \eta_1 \circ \det L_2 \perp [r_3] \circ L_3 \perp \ldots; K' = [s_2] \circ K_2 \perp [s_3] \circ K_3 \perp \ldots.
\]

So in any case, the above formula for \( \Theta(L,K) \) holds after the first reduction step.

(2) Suppose that up to \( i-1 \), the above formula for \( \Theta(L,K) \) holds, i.e.,
\[
\Theta(L,K) = \Theta(L', K') \prod_{1 \leq j \leq i-1} \Theta(O^+([s_j] \circ \xi_j \eta_{j,1} \perp [r_{j+1}] \circ L_{j+1} \perp \ldots))
\]
\[
\prod_{l_j \cdot k_j = 1, j \leq i - 1} \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \cdot \text{det} L_j \rangle > \downarrow [s_j + 1] \cdot L_{j+1} \cdot \downarrow [r_{j+2}] \cdot L_{j+2} \cdot \downarrow \ldots)).
\]

Where \( K' = [s_j] \cdot K_i \downarrow [s_{j+1}] \cdot K_{i+1} \ldots \), and

\[
L' = \begin{cases} [s_{i-1}] \cdot \langle \xi_{i-1} \cdot \eta_{i-1} \cdot \text{det} L_i \rangle > \downarrow [r_{i+1}] \cdot L_{i+1} \cdot \downarrow \ldots; & \text{when } l_{i-1} - k_{i-1} = 0 \\
[s_{i-1} + 1] \cdot \langle \xi_{i-1} \cdot \eta_{i-1} \rangle > \downarrow [r_{i+1}] \cdot L_{i+1} \cdot \downarrow \ldots; & \text{when } l_{i-1} - k_{i-1} = 1.
\end{cases}
\]

We can see that all components of \( L' \) and \( K' \) are again of rank one.

(3) At \( i \). If \( l_i - k_i = 0 \), then (I) implies that conditions in Prop 1.4.5.4 are satisfied for the pair \( L' \) and \( K' \). By applying this proposition to \( L' \) and \( K' \), we get,

when \( l_{i-1} - k_{i-1} = 0 \):

\[
\Theta(L', K') = \Theta(L'', K'') \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \rangle > \downarrow [r_{i+1}] \cdot L_{i+1} \cdot \downarrow \ldots),
\]

with \( L'' = [r_{i+1}] \cdot \langle \xi_i \cdot \text{det} L_{i+1} \rangle > \downarrow [r_{i+2}] \cdot L_{i+2} \cdot \downarrow \ldots \); and \( K'' = [s_{i+1}] \cdot K_{i+1} \cdot \downarrow \ldots \);

when \( l_{i-1} - k_{i-1} = 1 \) (so rank \( \hat{L}_i = 0 \)):

\[
\Theta(L', K') = \Theta(L'', K'') \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \rangle > \downarrow [r_{i+1}] \cdot L_{i+1} \cdot \downarrow \ldots),
\]

with \( L'' = [r_{i+1}] \cdot \langle \xi_i \cdot \text{det} L_{i+1} \rangle > \downarrow [r_{i+2}] \cdot L_{i+2} \cdot \downarrow \ldots \); and \( K'' = [s_{i+1}] \cdot K_{i+1} \cdot \downarrow \ldots \).

In both cases we have

\[
\Theta(L, K) = \Theta(L'', K'') \prod_{l_j \cdot k_j = 0, j \leq i} \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \cdot \text{det} L_j \rangle > \downarrow [s_j + 1] \cdot L_{j+1} \cdot \downarrow [r_{j+2}] \cdot L_{j+2} \cdot \downarrow \ldots)).
\]

If \( l_i - k_i = 1 \), then (II) implies that the conditions of Prop 1.1.4.4(3) are satisfied for the pair \( L' \) and \( K' \). One can check that after applying Prop 1.4.4(3) to \( L' \) and \( K' \) we always have (whether \( l_{i-1} - k_{i-1} = 0 \), or \( l_{i-1} - k_{i-1} = 1 \)):

\[
\Theta(L', K') = \Theta(L'', K'') \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \cdot \text{det} L_j \rangle > \downarrow [s_j + 1] \cdot L_{j+1} \cdot \downarrow [r_{j+2}] \cdot L_{j+2} \cdot \downarrow \ldots)),
\]

with \( L'' = [s_{i+1}] \cdot \langle \xi_{i+1} \cdot \eta_{i+1} \rangle > \downarrow [r_{i+2}] \cdot L_{i+2} \cdot \downarrow \ldots \); and \( K'' = [s_{i+1}] \cdot K_{i+1} \cdot \downarrow [s_{i+2}] \cdot K_{i+2} \cdot \downarrow \ldots \).

So \( \Theta(L, K) = \Theta(L'', K'') \prod_{l_j \cdot k_j = 0, j \leq i} \Theta^*(\{s_j\} \cdot \langle \xi_j \cdot \eta_j \cdot \text{det} L_j \rangle > \downarrow [s_j + 1] \cdot L_{j+1} \cdot \downarrow [r_{j+2}] \cdot L_{j+2} \cdot \downarrow \ldots)).
\]
This completes the whole induction. Q.E.D.

In the proof of this Theorem, we showed that conditions on L and K at index i imply that the corresponding conditions of Proposition 1.4.4 and 1.4.5 are satisfied for the pair L' and K'. In fact, the converse also holds.

Proof of Theorem 1.5.4: By all these Notes, and by the proofs of Theorems 1.5.1, 1.5.2 and 1.5.3, we can see that all the conditions listed in these theorems are necessary in order for \( \Theta(L,K) \neq F^* \). In other words, \( \Theta(L,K) = F^* \) if none of these Theorems applies. Q.E.D.

§ 1.6 Recovery Of Special Cases

In this section we are going to recover Satz3 and Satz4 of [Sp] by using the formulas developed in section 2 and section 5; and also to correct a minor discrepancy in [EH](Th3.14(iv)).

The corrected version of the Theorem 3.14(iv) of [EH] is already stated in page 23. Here it is again:

"Let \( L = [r_1] \bullet L_1 \downarrow [r_2] \bullet L_2 \downarrow \ldots \downarrow [r_t] \bullet L_t \) be a Jordan splitting for L, and rank \( L_i \leq 2 \) for every i; Suppose that for some \( i_0 \), \( L_{i_0} \) is of mixed-type. Then \( \Theta(O^+(L)) \neq F^* \) iff

(a) all the associated spaces of all components are represented by \( F([r_{i_0}] \bullet L_{i_0}) \);

(b) for any \( j = 1, 2, \ldots, t-1 \), either (i) \( r_{j+1} \geq r_j + 4 \); or (ii) \( r_{j+1} = r_j + 2 \), rank\( L_j = \text{rank}L_{j+1} \), and

\[
\det([r_1] \bullet L_1 \downarrow [r_{i_0}] \bullet L_{i_0} \downarrow [r_{i_0+1}] \bullet L_{i_0+1}) \det([r_{i_0}] \bullet L_{i_0}) \in \{1, 1, \Delta\}.
\]
In the exceptional cases Θ(O+(L)) = Θ(O+([r_1]∗L_{1_0})) = Θ(O+(L_{1_0})) =
\{c ∈ F^∗ | (c, -\text{det}(L_{1_0})) = 1\}

The differences between this version and the one stated in Th3.14(iv) of [EH] are:
first, the condition (a) here combines both statements Theorem 3.14(iv)(a) and (b); and
second, the condition (b) here is almost the same as of Th3.14(iv)(c) but with the

correction that when the corresponding components are of rank 1, their difference of

the scales can be two provided that the determinants satisfy some extra conditions. This

might be best explained by the following example: Let L = [0]∗L_1 ⊥ [2]∗L_2 ⊥ [10]∗L_3

with rank L_1 = rank L_2 = 1; and L_3 is of rank 2 mixed-type. Assume that all F([r_i]∗L_i)

are represented by F([10]∗L_3). Then we know that Θ(O+(L)) = Θ(O+(L_3))
Θ(O+([0]∗L_1 ⊥ [2]∗L_2)). Since Θ(O+(L_3)) = \{c ∈ F^∗ | (c, -\text{det}(L_3)) = 1\} and
Θ(O+([0]∗L_1 ⊥ [2]∗L_2)) = \{c ∈ R^∗ F^∗ 2 | (c, -\text{det}(L_1 ∥ L_2)) = 1\}, Θ(O+(L)) \neq F^∗ iff
Θ(O+(L_3)) ⊇ Θ(O+([0]∗L_1 ⊥ [2]∗L_2)) which is equivalent to
det([0]∗L_1 ⊥ [2]∗L_2)det([10]∗L_3) ∈ \{1, ∆\}. This is exactly the condition stated in the

new version.

Next we are going to prove Satz3 and Satz4 of [Sp] by using our formulas. First
we restate Satz3 which deals with non-dyadic field. Let K ⊆ L be two lattices of rank 1
and 3 over a non-dyadic local field F. Assume Θ(O+(L)) ⊆ Θ(O+(FK⊥)) which is a
index 2 subgroup of F^∗. Let K = <a>.

SATZ 3.

(a) Let Θ(O+(FK⊥)) = R^∗ F^∗ 2, and L = <b_1> ∥ <π^2 r b_2> ∥ <π^2 s b_3> where b_i's are units
and 0 ≤ r ≤ s. Then Θ(L, K) = F^∗ if and only if
(i) \(a \in \mathcal{O}^{2r+1}\), when \(-b_1b_2 \in F^*\);

(ii) \(a \in \mathcal{O}^{2s+1}\), otherwise.

(b) Let \(\Theta(O^+(FK^\perp))\) be a mixed subgroup of index 2, and \(L = \langle b_1 \rangle \perp \langle \pi^r b_2 \rangle \perp \langle \pi^s b_3 \rangle\) where \(b_i\)'s are units and \(0 < r < s\). Then \(\Theta(O^+(L)) = F^*\) if and only if

(i) \(a \in \mathcal{O}^r\), when \(r\) is even;

(ii) \(a \in \mathcal{O}^s\), when \(r\) is odd.

proof of Satz 3: By Theorem 1.2.1, of section 2 we know that

\[\Theta(L,K) = \Theta(O^+(L))G_1(L/K)G_2(L/K)\Theta(O^+(FK^\perp)) \quad \text{if} \quad \Theta(O^+(L)) \subseteq R^*F^2\]

\[\Theta(L,K) = \Theta(O^+(L))G_3(L/K)\Theta(O^+(FK^\perp)) \quad \text{if otherwise.}\]

Suppose case (a). \(\Theta(L,K) = F^*\) if and only if \(G_1(L/K)G_2(L/K) \not\subseteq R^*F^2\) iff \(G_1(L/K) \not\subseteq R^*F^3\) iff \(G_1(L/K) = F^*\) which, by the definition of \(G_1(L/K)\), is equivalent to (i) and (ii).

Suppose case (b). If \(\Theta(O^+(L)) \not\subseteq R^*F^2\) then \(r =\) even; and \(\Theta(L,K) = F^*\) iff \(G_1(L/K) \not\subseteq R^*F^2\) iff \(G_2(L/K) = R^*F^2\) which, by the definition of \(G_2(L/K)\), is equivalent to (i). If \(\Theta(O^+(L)) \not\subseteq R^*F^2\) then \(\Theta(L,K) = F^*\) iff \(G_3(L/K) = R^*F^2\) which, by the definition of \(G_3(L/K)\), is equivalent to (i) and (ii) accordingly. Q.E.D.

Now let \(K \subseteq L\) be two lattices of rank 1 and 3 respectively over 2-adic field \(F\). Assume \(\Theta(O^+(L)) \not\subseteq \Theta(O^+(FK^\perp))\) which is a index 2 subgroup of \(F^*\). Write \(K = \langle a = [s_1] \rangle \langle a' \rangle\), where \(a'\) is a unit. \(sL = R\) and \(\mathcal{O} = (\pi)\).

**Lemma 1.6.1.** If \(l_1-k_1 = 2\), then \(\Theta(L,K) = F^*\).
Proof: Certainly Theorem 1.5.2 and Theorem 1.5.3 are not applicable. By checking condition I of Theorem 1.5.1 (the only case dealing with \( l_1 - k_1 = 2 \)), we see that Theorem 1.5.1 is also not applicable. Thus \( \Theta(L, K) = F^* \) by Theorem 1.5.4. Q.E.D.

**Lemma 1.6.2.** Let \( l_1 - k_1 = 1 \). Then \( \Theta(L, K) = F^* \) if and only if one of the following occurs:

(a) \( L = [r_1] \circ L_1 \downarrow [r_2] \circ L_2 \), \( \text{rank } L_1 = 2 \), \( \det L_1 = -1 \), and

either \( a \in \wp^{r_1+2} \) when \( L_1 \) is odd,

or \( a \in \wp^{r_1+3} \) when \( L_1 \) is even.

(b) \( L = [r_1] \circ L_1 \downarrow [r_2] \circ L_2 \), \( L_1 \) is of mixed-type rank two, and \( a \in \wp^{r_2-3} \).

(c) \( L = [r_1] \circ L_1 \downarrow [r_2] \circ L_2 \downarrow [r_3] \circ L_3 \), \( r_1 \equiv r_2 \mod 2 \), \( L_1 \downarrow L_2 \) is even-type, and

either \( \Theta(O^+([r_2] \circ L_1 \downarrow [r_2] \circ L_2 \downarrow [r_3] \circ L_3)) \not\subset \Theta(O^+(FK^\downarrow)) \),

or \( \det(L_1 \downarrow L_2) = -1 \) and \( a \in \wp^{r_2+3} \).

(d) \( L = [r_1] \circ L_1 \downarrow [r_2] \circ L_2 \downarrow [r_3] \circ L_3 \), \( r_1 \equiv r_2 \mod 2 \), \( L_1 \downarrow L_2 \) is mixed-type, and

either \( \Theta(O^+([r_2] \circ L_1 \downarrow [r_2] \circ L_2 \downarrow [r_3] \circ L_3)) \not\subset \Theta(O^+(FK^\downarrow)) \),

or \( a \in \wp^{r_3-3} \).

(e) \( L = [r_1] \circ L_1 \downarrow [r_2] \circ L_2 \downarrow [r_3] \circ L_3 \), \( r_1 + 1 \equiv r_2 \mod 2 \), and

\( \Theta(O^+([s_1] \circ L_1 \downarrow [s_1+1] \circ L_2 \downarrow [r_3] \circ L_3)) \not\subset \Theta(O^+(FK^\downarrow)) \).

Proof: By our notations introduced in previous sections, \( L_1 \) has five possibilities as listed in the Lemma. We prove this Lemma case by case.

Suppose \( L_1 = [r_1] \circ L_1 \), where \( L_1 \) is of either even- or odd-type and \( \text{rank } L_1 = 2 \). Only two Theorems are possibly applicable here. They are Theorem 1.5.1 and Theorem 1.5.4. We see that in order for \( \Theta(L, K) = F^* \), the condition II of Theorem 1.5.1 has to be violated, i.e., ConditionA(\( L_1, \xi_0 \eta_0, s_1, 1 \)) does not hold. That is to
say det$L_1 = -1$, and either $a \in \varphi^{r_1+2}$ when $L_1$ is odd, or $a \in \varphi^{r_1+3}$ when $L_1$ is even. This proves case (a).

Let $L_1 = [r_1] \ast L_1$, where $L_1$ is of mixed-type and rank$L_1 = 2$. In order for $\Theta(L,K) = F^*$, this time condition V of theorem 1.5.2 has to be violated. By the definition of ConditionB(1,W), this is exactly the same as saying $a \in \varphi^{r_2-3}$. (See B(2)(ii)). Case (b) is proved.

Case (c) and Case (d) can be proved in a similar way as in case (a) and case (b) respectively.

Finally, assume $L_1 = [r_1] \ast L_1 \perp [r_2] \ast L_2$, where $r_1 + 1 \equiv r_2 \pmod{2}$ and rank$L_1 = \text{rank} L_2 = 1$. Suppose $r_3 \geq s_1 + 6$. Then by Theorem 1.5.3, $\Theta(L,K) = \Theta(O^+(FK\perp)) \Theta(O^+([s_1] \ast L_1 \perp [s_1+1] \ast L_2 \perp [r_3] \ast L_3))$, which equals $F^*$ if and only if $\Theta(O^+([s_1] \ast L_1 \perp [s_1+1] \ast L_2 \perp [r_3] \ast L_3)) \preceq \Theta(O^+(FK\perp))$. If $r_3 \leq s_1 + 5$, then $\Theta(O^+([s_1] \ast L_1 \perp [s_1+1] \ast L_2 \perp [r_3] \ast L_3)) \preceq \Theta(O^+(FK\perp))$. Again we have $\Theta(L,K) = F^*$. Case (e) is proved. Q.E.D.

**Lemma 1.6.3.** Let $l_1 - k_1 = 0$. Then $\Theta(L,K) = F^*$ if and only if one of the following occurs:

(a) $L = [r_1] \ast L_1 \perp [r_2] \ast L_2$, $L_2$ is of mixed-type rank two, and $a \in \varphi^{r_2-3}$.

(b) $L = [r_1] \ast L_1 \perp [r_2] \ast L_2 \perp [r_3] \ast L_3$, and

$$\Theta(O^+([s_1] \ast L_1 \perp [r_2] \ast L_2 \perp [r_3] \ast L_3)) \preceq \Theta(O^+(FK\perp)).$$

Proof: Suppose $L = [r_1] \ast L_1 \perp [r_2] \ast L_2$ and rank$L_2 = 2$. If $L_2$ is of even (odd) -type, then by assumption that $\Theta(O^+(L)) \subseteq \Theta(O^+(FK\perp)) \neq F^*$, we see that $[r_1] \ast L_1$ and $[r_2] \ast L_2$ are of the same type. So $\Theta(L,K) = R^*$ by Theorem 1.5.1. This implies that, in order for $\Theta(L,K) = F^*$, $L_2$ has to be of mixed-type and the condition VI of Theorem
1.5.2 has to be violated. By the definition of Condition B(1, W), it is equivalent to say that \( L_2 \) is of mixed-type and \( a \in \mathcal{O}_{12}^3 \). Case (a) is proved.

Now suppose \( L = \langle [r_1] \rangle \uparrow L_1 \downarrow [r_2] \rangle \downarrow L_2 \downarrow [r_3] \rangle \downarrow L_3 \rangle \). \( \Theta(L, K) = F^* \) if and only if either condition I of Theorem 1.5.3 has to be violated, i.e., \( r_2 = s_1 + 1 \) and \( r_3 < s_1 + 5 \); or \( \Theta(O^+(\langle [s_1] \rangle \uparrow L_1 \downarrow [r_2] \rangle \downarrow L_2 \downarrow [r_3] \rangle \downarrow L_3 \rangle) \) \( \subset \Theta(O^+(F K^\perp)) \). When \( r_2 = s_1 + 1 \) and \( r_3 < s_1 + 5 \), we also have \( \Theta(O^+(\langle [s_1] \rangle \uparrow L_1 \downarrow [r_2] \rangle \downarrow L_2 \downarrow [r_3] \rangle \downarrow L_3 \rangle) \) \( \subset \Theta(O^+(F K^\perp)) \). This proves case (b).

Q.E.D.

Now we shall prove Satz 4 of [Sp1] by these lemmas. First we recall this Satz.

**Satz 4.** \( \Theta(L, K) = F^* \) if and only if one of the following occurs:

(a) If \( \Theta(O^+(F K^\perp)) = R^*F^*2 \), all the Jordan components of \( L \) have the same type (odd or even), and

(i) if \( L = \langle b_1 \rangle \downarrow \langle \pi^{2r}b_2 \rangle \downarrow \langle \pi^{2s}b_3 \rangle \rangle \) where \( b_i \)'s are units and \( 0 \leq r \leq s \), then

(\( \alpha \)) \( a \in \mathcal{O}^2r \), when \( D(-b_1b_2) = 2R \);

(\( \beta \)) \( a \in \mathcal{O}^{2s} \), when \( D(-b_1b_2) = 4R \);

(\( \gamma \)) \( a \in \mathcal{O}^{2r+3} \cup \mathcal{O}^{2r+1} \), when \( D(-b_1b_2) = 0 \).

(ii) if \( L \) can not decomposed as an orthogonal sum of 1-dim sublattices, then

(\( \alpha \)) \( a \in \mathcal{O}^{2r+1} \cup \mathcal{O}^{2r+1} \), when \( L = A(0,0) \downarrow \langle \pi^{2r+1}l \rangle \rangle \);

(\( \beta \)) \( a \in \mathcal{O}^{2r+1} \), when \( L = A(2,2p) \downarrow \langle \pi^{2r+1}l \rangle \rangle \);

(\( \gamma \)) \( a \in \mathcal{O}^{2r+2} \), when \( L = \langle b \rangle \downarrow \langle \pi^{2r+1}l A(0,0) \rangle \), or \( L = \langle b \rangle \downarrow \langle \pi^{2r+1}l A(2,2p) \rangle \).

(b) If \( \Theta(O^+(F K^\perp)) \) is a mixed subgroup of \( F^* \) of index 2, \( A \in \Theta(O^+(F K^\perp)) \), and

\( L = \langle b_1 \rangle \downarrow \langle \pi^r b_2 \rangle \downarrow \langle \pi^s b_3 \rangle \rangle \) where \( b_i \)'s are units and \( 0 \leq r \leq s \).

Let \( M = \langle \pi^{r-2}b_1 \rangle \downarrow \langle \pi^r b_2 \rangle \downarrow \langle \pi^s b_3 \rangle \rangle \), and \( N = \langle \pi^r b_1 \rangle \downarrow \langle \pi^r b_2 \rangle \downarrow \langle \pi^s b_3 \rangle \rangle \).

Then, we have
(i) \(a \in \varphi r^{-2}\), when \(r\) is even and \(\Theta(O^+(M)) \subset \Theta(O^+(FK^\perp))\);

(ii) \(a \in \varphi r\), when \(r\) is even, \(\Theta(O^+(M)) \subset \Theta(O^+(FK^\perp)), \Theta(O^+(N)) \subset \Theta(O^+(FK^\perp))\);

(iii) \(a \in \varphi s^{-2}\), when \(r\) is even and \(\Theta(O^+(M)) \Theta(O^+(N)) \subset \Theta(O^+(FK^\perp))\);

(iv) \(a \in \varphi r^{-3}\), when \(r\) is odd.

(c) If \(\Theta(O^+(FK^\perp))\) is a mixed subgroup of \(F^*\) of index 2, \(\Delta \in \Theta(O^+(FK^\perp))\), and

\[L = \langle b_1 \rangle \perp \langle \pi^r b_2 \rangle \perp \langle \pi^s b_3 \rangle,\]

where \(b_1\)'s are units and \(0 < r < s\).

Let \(M := \langle \pi^{-3} b_1 \rangle \perp \langle \pi^r b_2 \rangle \perp \langle \pi^s b_3 \rangle\). Then, we have

(i) \(a \in \varphi r^{-4}\), when \(r\) is even;

(ii) \(a \in \varphi r^{-3}\), when \(r\) is odd and \(\Theta(O^+(M)) \subset \Theta(O^+(FK^\perp))\);

(iii) \(a \in \varphi s^{-4}\), when \(r\) is odd and \(\Theta(O^+(M)) \subset \Theta(O^+(FK^\perp))\).

Proof of (a)(i): Since \(\Theta(O^+(L)) \subset \Theta(O^+(FK^\perp)) = R^*F^{*2}\), if \(s_1 < 2r\) then \(\Theta(L,K) = R^*F^{*2}\) by Lemma 1.6.3. Thus, \(|l_1 - k_1| \geq 1\) in order for \(\Theta(L,K) = F^*\).

Assume \(|l_1 - k_1| = 1\). Lemma 1.6.2 then claims that \(\Theta(L,K) = F^*\) if and only if Lemma 1.6.2(a), or Lemma 1.6.2(c) or Lemma 1.6.2(d) applies. Conditions of Lemma 1.6.2(a) and Lemma 1.6.2(c) are equivalent to (\(\gamma\)). Conditions of Lemma 1.6.2(b) are equivalent to (\(\alpha\)). (\(\beta\)) does not apply in this case because \(l_1 - k_1 = 1\).

Assume \(|l_1 - k_1| = 2\). Certainly \(\Theta(L,K) = F^*\) by Lemma 1.6.1. On the other hand, one of the conditions in (a)(i) will surely be satisfied. This proves (a)(i).

Proof of (a)(ii): Similar to (a)(i), we have \(|l_1 - k_1| \geq 1\) in order for \(\Theta(L,K) = F^*\).

Suppose \(L = A(0,0) \perp \langle \pi^{2r+1} b \rangle\). By Lemma 1.6.1 and Lemma 1.6.2, we see that \(\Theta(L,K) = F^*\) if and only if \(a \in \varphi 2r+1 \cup \varphi 2\). This proves (\(\alpha\)).

Assume \(L = A(2,2p) \perp \langle \pi^{2r+1} b \rangle\). If \(|l_1 - k_1| = 1\), then \(\Theta(L,K) \neq F^*\) by Lemma 1.6.2. Hence, \(\Theta(L,K) = F^*\) if and only if \(|l_1 - k_1| = 2\), i.e., \(a \in \varphi 2r+1\). (\(\beta\)) is done.
Finally, assume $L = \langle b \rangle \perp \pi^{2r+1}A(0,0)$, or $L = \langle b \rangle \perp \pi^{2r+1}A(2,2\rho)$. We already know that $l_1 - k_1 \geq 1$ is necessary in order for $\Theta(L,K) = F^*$. In this particular situation, $l_1 - k_1 \geq 1$ is equivalent to $l_1 - k_1 = 2$. This proves $(\gamma)$.

Proof of (b)(i): Assume $r$ is even and $\Theta(O^+(M)) \subseteq \Theta(O^+(FK^\perp))$. We show that $\Theta(L,K) = F^*$ if and only if $a \in \wp^{r-2}$.

It is not difficult to see that this condition is sufficient by our three lemmas.

Now we show the necessity. First, we assume $l_1 - k_1 = 0$. Since $\Theta(O^+(FK^\perp))$ is a mixed subgroup of $F^*$ of index 2 and $\Delta \in \Theta(O^+(FK^\perp))$, $s_1$ and $s$ must both be even. $\Theta(L,K) = F^*$ only if conditions of Lemma 1.6.3 are satisfied. That is to say either $\Theta(O^+\langle \pi^s_b_1 \rangle \perp \langle \pi^s_b_2 \rangle \perp \langle \pi^s_b_3 \rangle) \subset \Theta(O^+(FK^\perp))$, or $\langle \pi^s_b_2 \rangle \perp \langle \pi^s_b_3 \rangle$ is of mixed-type $\pi^r$-modular $(r=s)$ and $a \in \wp^{r-3}$. Because $\Theta(O^+(L)) \subseteq \Theta(O^+(FK^\perp))$, the first can happen only if $s_1 = r - 2$ ($a \in \wp^{r-2}$) by Proposition 1.9 of [EH]. The second implies that $a \in \wp^{r-2}$ because both $r$ and $s_1$ are even. So we proved that $a \in \wp^{r-2}$ is necessary when $l_1 - k_1 = 0$. When $l_1 - k_1 \geq 1$, then obviously $a \in \wp^{r-2}$. This proves (b)(i).

Proof of (b)(ii): Assume $r$ is even, $\Theta(O^+(M)) \subseteq \Theta(O^+(FK^\perp))$ and $\Theta(O^+(N)) \subset \Theta(O^+(FK^\perp))$. We show that $\Theta(L,K) = F^*$ if and only if $a \in \wp^r$.

Again, it is easy to see that the condition is sufficient by Lemma 1.6.1 and 1.6.2. (Note only Lemma 1.6.2(c) or (d) applies when $l_1 - k_1 = 1$).

Now the necessity part. Suppose $l_1 - k_1 = 0$. We see, by Lemma 1.6.3, that $\Theta(L,K) = F^*$ only if $\langle \pi^s_b_2 \rangle \perp \langle \pi^s_b_3 \rangle$ is of mixed-type $\pi^r$-modular $(r=s)$ and $a \in \wp^{r-3}$, or $\Theta(O^+\langle \pi^s_b_1 \rangle \perp \langle \pi^s_b_2 \rangle \perp \langle \pi^s_b_3 \rangle) \subset \Theta(O^+(FK^\perp))$. Because $\Theta(O^+(L))\Theta(O^+(M)) \subseteq \Theta(O^+(FK^\perp))$, none of these two can actually happen. (See Proposition 1.9, and
Theorem 3.14 of [EH]). So \( l_1 - k_1 \geq 1 \) when \( \Theta(L,K) = F^* \). But this means \( s_1 \geq r \), so that \( a \in \varrho^r \) is clear. This proves (b)(ii).

Proof of (b)(iii): Assume \( r \) is even, and \( \Theta(O^+(M))\Theta(O^+(N)) \subseteq \Theta(O^+(FK^{-1})) \). We show that \( \Theta(L,K) = F^* \) if and only if \( a \in \varrho^{s-2} \).

By the similar arguments as in the proof of (b)(ii), we have \( l_1 - k_1 \geq 1 \) in order for \( \Theta(L,K) = F^* \).

Suppose \( l_1 - k_1 = 1 \). Because \( \Theta(O^+(M))\Theta(O^+(N)) \subseteq \Theta(O^+(FK^{-1})) \), which is a mixed subgroup of \( F^* \) of index 2 and contains \( \Delta \), \( \Theta(L,K) = F^* \) if and only if Lemma 1.6.2(b) or (d) applies. That is to say, if and only if \( <b_1> \perp <b_2> \) is of mixed-type, and \( a \in \varrho^{s-3} \). As we can see that \( s_1 \) and \( s \) are of the same parity, so \( a \in \varrho^{s-2} \).

Conversely, assume \( a \in \varrho^{s-2} \). Condition \( \Theta(O^+(L))\Theta(O^+(M))\Theta(O^+(N)) \subseteq \Theta(O^+(FK^{-1})) \) forces \( <b_1> \perp <b_2> \) to be of mixed-type. By Lemma 1.6.2(d), we get \( \Theta(L,K) = F^* \).

The case when \( l_1 - k_1 = 2 \) is obvious. Case (b)(iii) is proved.

Proof of (b)(iv): Assume \( r \) is odd. We show that \( \Theta(L,K) = F^* \) if and only if \( a \in \varrho^{r-3} \).

Suppose \( l_1 - k_1 = 0 \). Then \( \Theta(L,K) = F^* \) only if the conditions of Lemma 1.6.3 are satisfied. When \( L = <b_1> \perp <\pi^rb_2> \perp <\pi^rb_3> \) \((r = s)\), the corresponding condition of Lemma 1.6.3 is that \( a \in \varrho^{r-3} \). Note that \( <b_2> \perp <b_3> \) is of mixed-type since \( \Theta(O^+(L)) \subseteq \Theta(O^+(FK^{-1})) \). When \( L = <b_1> \perp <\pi^rb_2> \perp <\pi^sb_3> \) \((1 \leq r < s)\), the corresponding condition is \( \Theta(O^+(<\pi^sb_1> \perp <\pi^rb_2> \perp <\pi^sb_3>) \subseteq \Theta(O^+(FK^{-1})) \). It leads to \( s_1 \geq r - 3 \) and so \( a \in \varrho^{r-3} \). (See Proposition 1.9 of [EH]).

Conversely, assume \( a \in \varrho^{r-3} \). If \( L = <b_1> \perp <\pi^rb_2> \perp <\pi^sb_3> \), then \( <b_2> \perp <b_3> \) is of mixed-type since \( \Theta(O^+(L)) \subseteq \Theta(O^+(FK^{-1})) \). By Lemma 1.6.3, we get \( \Theta(L,K) = F^* \). If \( L = <b_1> \perp <\pi^rb_2> \perp <\pi^sb_3> \) \((1 \leq r < s)\), then
\( \Theta(O^+(<\pi^s b_1> \perp <\pi^r b_2> \perp <\pi^s b_3>)) \subset \Theta(O^+(FK^\perp)) \). Again we have \( \Theta(L,K) = F^* \) by the same Lemma.

Suppose \( l_1 - k_1 \geq 1 \), then \( a \in \varphi^r \subseteq \varphi^{r-3} \). By Lemma 1.6.1(e) and Lemma 1.6.2, we can see that \( \Theta(L,K) = F^* \). This proves case (b)(iv).

Proof of (c)(i): Assume \( r \) is even. We prove that \( \Theta(L,K) = F^* \) if and only if \( a \in \varphi^{r-4} \).

If \( l_1 - k_1 = 0 \), then \( \Theta(L,K) = F^* \) if and only if Lemma 1.6.3(b) applies. That is to say \( \Theta(O^+(<\pi^s b_1> \perp <\pi^r b_2> \perp <\pi^s b_3>)) \subset \Theta(O^+(FK^\perp)) \), which, by Proposition 1.9 of [EH], is the case if and only if \( s_1 \geq r - 4 \) (\( a \in \varphi^{r-4} \)). Note that \( \Theta(O^+(FK^\perp)) \) is a mixed subgroup of \( F^* \) of index 2, \( \Delta \in \Theta(O^+(FK^\perp)) \), and \( \Theta(O^+(L)) \subset \Theta(O^+(FK^\perp)) \).

If \( l_1 - k_1 = 1 \), then \( a \in \varphi^{r-4} \). On the other hand, since the conditions of lemma 1.6.2(c) or Lemma 1.6.2(d) are satisfied, we have \( \Theta(L,K) = F^* \).

The case \( l_1 - k_1 = 2 \) is obvious. This finishes case (c)(i).

Proof of (c)(ii): Suppose \( r \) is odd, and \( \Theta(O^+(M)) \subset \Theta(O^+(FK^\perp)) \). We prove that \( \Theta(L,K) = F^* \) if and only if \( a \in \varphi^{r-3} \).

Let \( l_1 - k_1 = 0 \). Then \( \Theta(L,K) = F^* \) if and only if Lemma 1.6.3(b) applies, i.e., if and only if \( \Theta(O^+(<\pi^s b_1> \perp <\pi^r b_2> \perp <\pi^s b_3>)) \subset \Theta(O^+(FK^\perp)) \), which, by Proposition 1.9 of [EH], is the case if and only if \( s_1 \geq r - 3 \) (\( a \in \varphi^{r-3} \)). (\( s_1 \) is even and \( r \) is odd).

If \( l_1 - k_1 = 1 \), then \( a \in \varphi \subseteq \varphi^{r-3} \). On the other hand, since the conditions of lemma 1.6.2(e) is satisfied, we have \( \Theta(L,K) = F^* \).

The case \( l_1 - k_1 = 2 \) is obvious. This finishes case (c)(ii).

Proof of (c)(iii): Suppose \( r \) is odd, and \( \Theta(O^+(M)) \subset \Theta(O^+(FK^\perp)) \). We prove that \( \Theta(L,K) = F^* \) if and only if \( a \in \varphi^{s-3} \).
If $l_1 - k_1 = 0$, then $\Theta(L, K) = F^*$ if and only if Lemma 1.6.3(b) applies if and only if $\Theta(O^+ (\langle \pi^{s_1} b_1 \rangle \perp \langle \pi^{s_2} b_2 \rangle \perp \langle \pi^{s_3} b_3 \rangle )) \subset \Theta(O^+ (FK^\perp))$, which, by Proposition 1.9 of [EH], is the case if and only if $s_1 \geq s - 4 \left( a \in \mathcal{O}^{s-4} \right)$. Note this time $\Theta(O^+(M)) \subset \Theta(O^+(FK^\perp))$, which is a mixed subgroup of $F^*$ of index two and $\Delta \notin \Theta(O^+(FK^\perp))$.

If $l_1 - k_1 = 1$, then $\Theta(L, K) = F^*$ if and only if Lemma 1.6.2(e) applies if and only if $\Theta(O^+ (\langle \pi^{s_1} b_1 \rangle \perp \langle \pi^{s_2+1} b_2 \rangle \perp \langle \pi^{s_3} b_3 \rangle )) \subset \Theta(O^+ (FK^\perp))$ if and only if $s_1 \geq s - 4 \left( a \in \mathcal{O}^{s-4} \right)$.

The case when $l_1 - k_1 = 2$ is trivial.

Q.E.D.
In his 1961 paper [Kn2], Kneser proved the following: Given any $b \in F^*$, where $F$ is an algebraic number field; given any $R$-lattice $M$ of rank $\geq 3$, where $R$ is the ring of integers of $F$. If $\text{gen}(M)$ represents $b$, then there is only one constant Haar measure (defined later) of the representations of $b$ by any spinor genus in $\text{gen}(M)$ if $\text{rank}(M) \geq 4$; and there are at most two different Haar measures if $\text{rank}(M) = 3$. Schulze-Pillot expressed the difference of these two possible different measures (in the case when $\text{rank}(M) = 3$) as a product of some local factors, "eine Art Siegelsche Maßformel mit Charakter" [Sp2]. He also calculated these local factors under different circumstances.

In this chapter we extend Kneser's above result to representations of lattice by lattice (i.e., the higher dimension cases), and give an upper bound of the number of different Haar measures of representations by different spinor genera. In giving this upper bound we shall use some results from the previous chapter.

Let $V/F$ be a regular quadratic space over algebraic number field $F$, with rank($V$) $\geq 3$; $K$ a $R$-lattice in $V$ where $R$ is the ring of integers of $F$. A pair of $R$-lattices $(M, N)$ with $N \subseteq M$ and $FM = V$ is said to be a representation of $K$ if $K \cong N$. Two representations $(M_1, N_1)$ and $(M_2, N_2)$ are said to be in the same class (or representation class) iff $\exists \sigma \in O^*(V)$, the proper orthogonal group of $V$, such that
\[ \sigma N_1 = N_2, \text{ and } \sigma M_1 = M_2. \] We write \((M_1, N_1) \equiv (M_2, N_2)\). By Witt's Theorem, for any representation \((M, N)\) of \(K\), there is a lattice \(M' \supseteq K\) such that \((M, N) \equiv (M', K)\). Two representations \((M_1, N_1)\) and \((M_2, N_2)\) are said to be in the same genus (or representation genus) if locally at each prime spot they are in the same class; i.e., \(\exists \sigma = (\sigma_p) \in O_\lambda^+(V)\), where \(O_\lambda^+(V)\) is the group of split rotations of \(V\), such that \(\sigma_p M_1 p = M_2 p\) and \(\sigma_p N_1 p = N_2 p\) for all prime spot \(p\).

In the following we assume that \(K \subseteq L\). We define: \(X_A(L, K) = \{ \sigma \in O_\lambda^+(V) \mid \sigma L \supseteq K \}\); \(O^+(V/K) = \{ \sigma \in O^+(V) \mid \sigma K = K \}\); \(O_\lambda^+(V/K) = \{ \sigma \in O_\lambda^+(V) \mid \sigma K = K \}\); and \(O_\lambda^+(L/K) = \{ \sigma \in O_\lambda^+(L) \mid \sigma K = K \}\). Now given any lattice \(M \in \text{gen}(L)\), if \(M\) represents \(K\), then there is a \(\sigma \in O^+(V)\) such that \(\sigma K \subseteq M\). So \(K \subseteq \sigma^{-1}M \in \text{gen}(L)\), and so there is a \(\tau \in X_A(L, K)\) with \(\tau L = \sigma^{-1}M\), i.e., \(\sigma \tau L = M\). On the other hand, any lattice of the form \(\sigma \tau L\), with \(\sigma \in O^+(V)\) and \(\tau \in X_A(L, K)\), represents \(K\). We have:

\[
\{ M \in \text{gen}(L) \mid M \text{ represents } K \} = O^+(V) X_A(L, K) L. \tag{2.1}
\]

Given any representation class\((M, K)\) with \(M \in \text{gen}(L)\) and \(M \supseteq K\), there is a \(\sigma \in X_A(L, K)\) such that \(M = \sigma L\). Assume \(\text{class}(\sigma L, K) = \text{class}(\tau L, K)\) for some other \(\tau \in X_A(L, K)\). Then there is a \(\rho \in O^+(V)\) such that \(\rho \sigma L = \tau L\) and \(\rho K = K\). (Thus \(\rho \in O^+(V/K)\)). So \(\sigma^{-1} \rho^{-1} \tau L = L\), and so \(\sigma^{-1} \rho^{-1} \tau \in O_\lambda^+(L)\). This is to say \(\tau \in \rho \sigma O_\lambda^+(L)\). On the other hand, we have \(\text{class}(\sigma L, K) = \text{class}(\tau L, K)\) for any \(\tau \in O^+(V/K) \sigma O_\lambda^+(L)\). Thus we have one-one correspondence of the following:

\[
O^+(V/K) \backslash X_A(L, K) / O_\lambda^+(L) \leftrightarrow \{ \text{class}(M, K) \mid M \in \text{gen}(L), M \supseteq K \} \tag{2.2}
\]

where \(O^+(V/K) \sigma O_\lambda^+(L)\) maps to \(\text{class}(\sigma L, K)\) with \(\sigma \in X_A(L, K)\).
Similarly we have:

\[ O^+_A(V/K) \backslash X_A(L,K)/O^+_A(L) \leftrightarrow \{ \text{gen}(M, K) \mid M \in \text{gen}L, M \supseteq K \} \]  

(2.3)

where \( O^+_A(V/K) \sigma O^+_A(L) \) maps to \( \text{gen}(\sigma L, K) \) with \( \sigma \in X_A(L,K) \), and

\[ O^+(V/K) \backslash O^+_A(V/K)/O^+_A(L/K) \leftrightarrow \{ \text{class}(M,K) \mid (M, K) \in \text{gen}(L, K) \} \]  

(2.4)

where \( O^+(V/K) \sigma O^+_A(L/K) \) goes to \( \text{class}(\sigma L, K) \) with \( \sigma \in O^+_A(V/K) \).

Because \( O^+_A(V) \) is a locally compact group which is unimodular; and \( O^+(V/K) \) is a discrete subgroup of \( O^+_A(V) \). By the standard results of Haar Measure Theory, we have right invariant Haar measure on the homogeneous space \( O^+(V/K) \backslash O^+_A(V) \), call it \( \mu \).

Given any spinor genus \( \text{spn}(\sigma L) \), define the measure of the portion of this spinor genus which represents \( K \) by:

\[ \text{measure}(\text{spn}(\sigma L), K) = \mu(O^+(V/K) \backslash O^+_A(V) \sigma O^+_A(L) \cap X_A(L,K)) \]  

(2.5)

Note that \( X_A(L,K) \) is an open subset of \( O^+_A(V) \).

Since \( \text{spn}(\sigma L) = O^+(V)O^+_A(V)\sigma O^+_A(L)L \); so the formula 2.5. is well defined, i.e., \( \text{measure}(\text{spn}(\sigma L), K) = \text{measure}(\text{spn}(\tau L), K) \) when \( \text{spn}(\sigma L) = \text{spn}(\tau L) \). Also we see that \( \text{spn}(\sigma L) \) represents \( K \) iff \( X_A(L,K) \cap O^+(V)O^+_A(V)\sigma O^+_A(L) \neq \emptyset \); i.e., \( \sigma \in X_A(L,K)O^+(V)O^+_A(V)O^+_A(L) \). So the number of spinor genera in \( \text{gen}(L) \) which represent \( K \) is equal to the group index: \([O^+(V)O^+_A(V)O^+_A(L)L_X_A(L,K) : O^+(V)O^+_A(V)O^+_A(L)]\),
which by Estes and Hsia, is equal to \([F^*\Theta(X_A(L,K)) : F^*\Theta(O_A^+(L))], \) and by Theorem 1.1.5, is equal to \([F^*\Theta(O_A^+(L,K)) : F^*\Theta(O_A^+(L))]. \) The number of spinor genera in \(\text{gen}(L)\) is equal to the index \([J_F: F^*\Theta(O_A^+(L))]. \) (See [EsH])

Our main concern here is to provide an upper bound for the following number

\[
\# \{ \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in X_A(L,K) \} \text{ of distinct possible Haar measures of representations of } K \text{ by spinor genera in } \text{gen}(L). \text{ When } \text{rank}(L) = 3 \text{ and } \text{rank}(K) = 1, \text{ Kneser [Kn2] proved that the above number is at most 2; and Schulze-Pillot [Sp1] expressed the difference of these two possible different measures as a product of some local factors.}
\]

**Theorem 2.6.** \(\text{measure}(\text{spn}(\tau L), K) = \text{measure}(\text{spn}(\sigma L), K); \text{ whenever } \tau \in O_A^+(V/K)O^+(V)O_A^+(V)\sigma O_A^+(L).\)

**Proof:** We have \(O_A^+(V/K)O^+(V)O_A^+(V)\sigma O_A^+(L) = O^+(V)O_A^+(V)\sigma O_A^+(L)O_A^+(V/K).\) Now let \(\tau = ab\sigma c\rho \) with \(a \in O^+(V), b \in O_A^+(V), c \in O_A^+(L)\) and \(\rho \in O_A^+(V/K).\) Because \(\text{spn}(ab\sigma cL) = \text{spn}(\sigma L),\) we may, after renaming \(ab\sigma c\) as \(\sigma,\) assume WLOG that \(\tau = \sigma \rho\) with \(\rho \in O_A^+(V/K).\) Thus:

\[
\text{measure}(\text{spn}(\tau L), K) = \mu(O^+(V/K)\backslash O^+(V)O_A^+(V)\sigma \rho O_A^+(L) \cap X_A(L,K)).
\]

For any given \(\alpha, \beta \in X_A(L,K),\) if \(O_A^+(V/K)\alpha O_A^+(L) \cap O_A^+(V/K)\beta O_A^+(L) \neq \emptyset,\) then one can see that these two are really the same. So \(X_A(L,K) = \cup O_A^+(V/K)\phi_i O_A^+(L)\) as a disjoint union with \(\phi_i \in X_A(L,K).\) Now for each \(i,\) by the right-invariance
Now we further decompose $\gamma_i \in O_\lambda^*(\phi_i L)$. For each $j$:

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}\rho O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)\gamma_j^i) =$$

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}\rho O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)\gamma_j^i)$$

which, by right-invariance of the measure, is equal to

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}\rho O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K))$$

which, by right-invariance again, (note that $\rho \in O_\lambda^*(V/K)$), is equal to

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)).$$

On the other hand:

$$\text{measure}(\text{spn}(\sigma L), K) = \mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma O_\lambda^*(L) \cap X_\lambda(L, K)).$$

Again $X_\lambda(L, K) = \cup O_\lambda^*(V/K)\phi_i O_\lambda^*(L)$ and for each $i$,

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}\rho O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)\phi_i O_\lambda^*(L) ) =$$

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}\rho O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)\phi_i O_\lambda^*(L) ) =$$

$$\mu(O^+(V/K)\backslash O^+(V)O_\lambda^*(V)\sigma\phi_i^{-1}O_\lambda^*(\phi_i L) \cap O_\lambda^*(V/K)O_\lambda^*(\phi_i L)).$$
Furthermore with $O^+(V/K)O^+(\phi_iL) = \cup O^+(V/K)\gamma_j^i$, we have (for each $j$):

\[
\mu(O^+(V/K)\setminus O^+(V)O^+(V)\sigma\phi_i^{-1}O^+(\phi_iL) \cap O^+(V/K)\gamma_j^i) = \\
\mu(O^+(V/K)\setminus O^+(V)O^+(V)\sigma\phi_i^{-1}O^+(\phi_iL) \cap O^+(V/K)).
\]

This last one is exactly the corresponding portion in the measure($\text{spn}(\sigma\rho L),K) = \text{measure}(\text{spn}(\tau L),K)$. Thus we proved that $\text{measure}(\text{spn}(\tau L),K) = \text{measure}(\text{spn}(\sigma L),K)$.

Q.E.D.

**COROLLARY 2.7.**  

\[
\#\{\text{measure}(\text{spn}(\sigma L),K) \mid \sigma \in X_L(L,K)\} \leq \\
[F^*\Theta(O^+_L(L,K)) : F^*\Theta(O^+_L(V/K))\Theta(O^+_L(L))]
\]

Proof: By Theorem 2.6. we see that $\#\{\text{measure}(\text{spn}(\sigma L),K) \mid \sigma \in X_L(L,K)\} \leq [O^+(V)O^+_L(V)O^+_L(L)X_L(L,K) : O^+_L(V/K)O^+(V)O^+_L(V)O^+_L(L)]$, which by the map introduced by Estes and Hsia in their work [EsH], equals $[F^*\Theta(X_L(L,K)) : F^*\Theta(O^+_L(V/K))\Theta(O^+_L(L))]$, which is $[F^*\Theta(O^+_L(L,K)) : F^*\Theta(O^+_L(V/K))\Theta(O^+_L(L))]$ by Theorem 1.1.5.  

Q.E.D.

**COROLLARY 2.8.**  

When rank($L$) - rank($K$) $\geq 3$ there is only one constant measure; When rank($L$) - rank($K$) = 2 there are at most two different measures.

Proof: Let $U = FK^\perp$ in $V$, then rank($U$) $\geq 2$. We have $\Theta(O^+_L(V/K)) \supseteq \Theta(O^+_L(U))$ and $[J_F : F^*\Theta(O^+_L(U))] = 1$ when rank($U$) $\geq 3$; and $[J_F : F^*\Theta(O^+_L(U))] \leq 2$ when rank($U$) = 2. By the Corollary 2.7 we get the conclusion.  

Q.E.D.
In the following we are going to give some examples where there is always one constant measure. We shall use the formulas developed in Chapter 1 in the calculations of the upper bound in Corollary 2.7.

**Example 2.9.** Let \( L = \langle -1 \rangle \times \langle (p_1p_2 \ldots p_r)^2 \rangle \times \langle (p_1p_2 \ldots p_r)^3 \rangle \times \langle (p_1p_2 \ldots p_r)^4 \rangle \) where \( p_i \equiv 1 \pmod{8} \) and \( \frac{p_i}{p_j} = 1 \ \forall \ i \neq j \). Let \( K = \langle (p_1p_2 \ldots p_r)^3 \rangle \times \langle (p_1p_2 \ldots p_r)^3 \rangle \).

Both are lattices over \( \mathbb{Z} \). Then \( \text{gen}(L) \) has \( 2^r \) spinor genera. It is obvious that \( L_p \) represents \( K_p \) locally at all prime spots \( p \), including infinite. So \( \text{gen}(L) \) represents \( K \).

Now for any prime \( p \notin \{2, p_1, p_2, \ldots, p_r\} \), \( L_p = \langle -1 \rangle \times \langle 1 \rangle \times \langle 1 \rangle \times \langle -1 \rangle \) and \( K_p = \langle 1 \rangle \times \langle -1 \rangle \times \langle 1 \rangle \times \langle -1 \rangle \). By Theorem 1.2.1, we have \( \Theta(L_p, K_p) \supseteq \mathbb{Z}_p^* \mathbb{Q}_p^{*2} \). At \( p = p_i \), \( L_p = \langle -1 \rangle \times \langle p_i^2 \rangle \times \langle p_i^3 \rangle \times \langle -p_i^4 \rangle \) and \( K_p = \langle p_i^3 \rangle \times \langle -p_i^3 \rangle \). By Theorem 1.2.1, \( \Theta(L_p, K_p) = \mathbb{Q}_p^* \).

Finally when \( p = 2 \), \( L_p = \langle -1 \rangle \times \langle 1 \rangle \times \langle 1 \rangle \times \langle -1 \rangle \); \( K_p = \langle 1 \rangle \times \langle -1 \rangle \); so \( \Theta(L_p, K_p) = \mathbb{Q}_p^* \) by Theorem 1.5.4. So \( F^* \Theta(O^+_A(L, K)) = \mathbb{Q}_Q \), and there are \( 2^r \) spinor genera in \( \text{gen}(L) \) which represent \( K \); i.e., all spinor genera in \( \text{gen}(L) \) represent \( K \).

On the other hand when \( p \notin \{2, p_1, p_2, \ldots, p_r\} \) \( \Theta(O^+(L_p)) = \mathbb{Z}_p^* \mathbb{Q}_p^{*2} \). When \( p = p_i \), \( \Theta(O^+_A(V_p/K_p)) \supseteq \Theta(O^+(K_p)) = \mathbb{Z}_p^* \mathbb{Q}_p^{*2} \). When \( p = 2 \), \( \Theta(O^+(L_p)) = \mathbb{Q}_p^* \). So we have \( F^* \Theta(O^+_A(V/K)) \Theta(O^+_A(L)) = \mathbb{Q}_Q \). By Corollary 2.7. \#\{ measure(spn(\sigma L), K) \mid \sigma \in X_A(L, K) \} \leq [F^* \Theta(O^+_A(L, K)) : F^* \Theta(O^+_A(V/K)) \Theta(O^+_A(L))] = [\mathbb{Q}_Q : \mathbb{Q}_Q] = 1. \) So all spinor genera in \( \text{gen}(L) \) represent \( K \) with equal measure.

**Corollary 2.10.** Let \( L' \) be another lattice on \( V \) with the property that \( \sigma L \supseteq K \) iff \( \sigma L' \supseteq K \) for \( \sigma \in O^+_A(V) \). Then \( \text{measure}(\text{spn}(\tau L), K) = \text{measure}(\text{spn}(\sigma L), K) \) when \( \tau \in \sigma O^+_A(L') \). Under this condition, there are at most \( [F^* \Theta(O^+_A(L, K)) : F^* \Theta(O^+_A(V/K)) \Theta(O^+_A(L)) \Theta(O^+_A(L'))] \) many different measures.
Proof: By the condition, we have \( X_A(L,K) = X_A(L',K) \). Let \( \rho \in O^*_A(L') \) and \( \tau = \sigma \rho \).

\[
\text{measure}\left(\text{spn}(\tau L), K\right) = \mu(\Omega^+(V/K)\Omega^+(V)\Omega_A^+(V)\sigma\rho O^*_A(L) \cap X_A(L,K)) = \\
\mu(\Omega^+(V/K)\Omega^+(V)\Omega_A^+(V)\sigma O^*_A(L) \cap X_A(L',K)) = \\
\mu(\Omega^+(V/K)\Omega^+(V)\Omega_A^+(V)\sigma O^*_A(L) \cap X_A(L',K)) = \\
\mu(\Omega^+(V/K)\Omega^+(V)\Omega_A^+(V)\sigma O^*_A(L) \cap X_A(L,K)) = \text{measure}(\text{spn}(\sigma L), K).
\]

The formula follows obviously. Q.E.D.

**Example 2.11.** \( L = \langle 1 \rangle \langle (2p_1 p_2 \ldots p_r)^6 \rangle \langle 2(2p_1 p_2 \ldots p_r)^{11} \rangle \langle (2p_1 p_2 \ldots p_r)^{18} \rangle \) where \( p_i \equiv 1 \mod(8) \) and \( \frac{2^i}{p_j} = 1 \) \( \forall \ i \neq j \). Let \( K = \langle (2p_1 p_2 \ldots p_r)^{11} \rangle \langle 2(2p_1 p_2 \ldots p_r)^{11} \rangle \). Then \( \text{gen}(L) \) has \( 2^{r+2} \) spinor genera, and it is clear that \( \text{gen}(L) \) represents \( K \). Now for any prime \( p \in \{ 2, p_1, p_2, \ldots, p_r \} \), by Theorem 1.2.1. we have \( \Theta(L_p, K_p) \supseteq Z_p^*Q_p^{*2} \).

At \( p = p_i \), \( L_p = \langle 1 \rangle \langle p_i^6 \rangle \langle 2p_i^{11} \rangle \langle 2p_i^{18} \rangle \) and \( K_p = \langle p_i^{11} \rangle \langle 2p_i^{11} \rangle \). By Theorem 1.2.1. \( \Theta(L_p, K_p) = Q_p^* \). Finally when \( p = 2 \), \( L_p = \langle 1 \rangle \langle 2^6 \rangle \langle 2^{12} \rangle \langle 2^{18} \rangle ; \) \( K_p = \langle 2^{11} \rangle \langle 2^{12} \rangle ; \) so \( \Theta(L_p, K_p) = Q_p^* \) by Theorem 1.5.2. and Theorem 1.5.4. Thus \( F^*\Theta(O_A^+(L,K)) = J_Q \), and all spinor genera in \( \text{gen}(L) \) represent \( K \).

Let \( L' = \langle 2^6 \rangle \langle (2p_1 p_2 \ldots p_r)^6 \rangle \langle 2(2p_1 p_2 \ldots p_r)^{11} \rangle \langle (2p_1 p_2 \ldots p_r)^{18} \rangle \) be a sublattice of \( L \), then certainly that \( \sigma L \subseteq K \) iff \( \sigma L' \supseteq K \) for \( \sigma \in O^*_A(V) \). When \( p \in \{ 2, p_1, p_2, \ldots, p_r \} \) \( \Theta(O^+(L_p)) = Z_p^*Q_p^{*2} \). When \( p = p_i \), \( \Theta(O_A^+(V_p/K_p)) \supseteq \Theta(O^+(K_p)) = Z_p^*Q_p^{*2} \). When \( p = 2 \), \( \Theta(O_A^+(V_p/K_p)) \Theta(O^+(L_p')) \supseteq \Theta(O^+(K_p)) \Theta(O^+(L_p')) = Q_p^* \). So we have \( F^*\Theta(O_A^+(V/K))\Theta(O_A^+(L))\Theta(O_A^+(L')) = J_Q \). By Corollary 2.10.

\[
\left\lfloor \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in X_A(L,K) \right\rfloor \leq \\
[F^*\Theta(O_A^+(L,K)) : F^*\Theta(O_A^+(V/K))\Theta(O_A^+(L))\Theta(O_A^+(L'))] = [J_Q : J_Q] = 1. \text{ So all spinor genera in } \text{gen}(L) \text{ represent } K \text{ with equal measure.}
\]
**Corollary 2.12.** Let $L'$ be another lattice on $V$ with the property that $\sigma L \supseteq K$ iff $\sigma L' \supseteq K$ for $\sigma \in O_\lambda^+(V)$. Suppose $O_\lambda^+(L') \supseteq O_\lambda^+(L)$ then $\exists n$ such that

$$\text{measure}(\text{spin}(\sigma L'), K) = n \times \text{measure}(\text{spin}(\sigma L), K),$$

for all $\sigma \in O_\lambda^+(V)$.

Proof: We decompose $O_\lambda^+(V)O_\lambda^+(L') = \cup O_\lambda^+(V)O_\lambda^+(L')\phi_i$ as a disjoint union with $i$ from 1 to $n$ and $\phi_i \in O_\lambda^+(L')$. Then for any $\sigma \in O_\lambda^+(V)$ we have

$$O_\lambda^+(V)O_\lambda^+(L') = \cup O_\lambda^+(V)O_\lambda^+(L')\sigma O_\lambda^+(L)\phi_i$$

as a disjoint union with $i$ from 1 to $n$.

Now because $X_A(L', K) = X_A(L, K)$ and for each $i$,

$$\mu(O_\lambda^+(V/K)O_\lambda^+(V)\sigma O_\lambda^+(L)\phi_i \cap X_A(L, K))$$

$$= \mu(O_\lambda^+(V/K)O_\lambda^+(V)\sigma O_\lambda^+(L) \cap X_A(L, K))$$

Because $\phi_i \in O_\lambda^+(L')$ and $X_A(L, K)O_\lambda^+(L') = X_A(L, K).$ So $\text{measure}(\text{spin}(\sigma L'), K) = n \times \text{measure}(\text{spin}(\sigma L), K)$. Q.E.D.

**Example 2.13.** Let $L = \langle -l \rangle \oplus \langle p_1p_2...p_r \rangle \oplus \langle p_1p_2...p_r \rangle^7$ where $p_i \equiv 1 \pmod{8}$ and $(\frac{p_i}{p_j}) = 1 \forall i \neq j.$ Let $K = \langle p_1p_2...p_r \rangle^4.$ Then $\text{gen}(L)$ has $2^r$ spinor genera and $\text{gen}(L)$ represents $K.$ Now for any prime $p \notin \{2, p_1, p_2, \ldots, p_r\}$, by Theorem 1.2.1. we have $\Theta(L_p, K_p) \supseteq Z_p^*Q_p^{*2}.$ At $p = p_i$, $L_p = \langle -l \rangle \oplus \langle p_i^2 \rangle \oplus \langle p_i^7 \rangle$ and $K_p = \langle p_i^4 \rangle.$ By Theorem 1.2.1. $\Theta(L_p, K_p) = Q_p^*$ When $p = 2$, $L_p = \langle -l \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle \oplus \langle 1 \rangle$; $K_p = \langle 1 \rangle$; so $\Theta(L_p, K_p) = Q_p^*$ by Theorem 1.5.2. and Theorem 1.5.4. Thus $F^*\Theta(O_\lambda^+(L, K)) = 1_Q$, and all spinor genera in $\text{gen}(L)$ represent $K.$

Let $L' = \langle -p_1^2 \rangle \oplus \langle p_1p_2...p_r \rangle \oplus \langle p_1p_2...p_r \rangle^7.$ Then one can check to see that the conditions of Corollary 2.12 are satisfied; and $O_\lambda^+(L')$ strictly contains $O_\lambda^+(L).$
Calculation tells that \( \#\{ \text{measure}(\text{spn}(\sigma L'), K) \mid \sigma \in X_A(L', K) \} = 1. \) By Corollary 1.12, we have \( \#\{ \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in X_A(L, K) \} = 1. \)

(Note: That the lattice \( K \) is a splitting lattice, in the sense of \([H_3]\), in all the above examples but in no cases it is exceptional.)

**Example 2.14.** (See [HSX]). Let \( L \) be a \( R \)-lattice of rank \( \geq 3 \). Let \( \{ p_1, p_2, \ldots, p_t \} \) be a set of prime spots not dividing \( 2\text{Vol}(L) \). For each prime \( p \) dividing \( 2\text{Vol}(L) \) choose a local characteristic sublattice \( J(p) \) of \( L_p \) in the sense of \([K_i]\); for \( p = p_k \) (\( k = 1, \ldots, t \)) choose a sublattice \( N(p) \) of \( L_p \) of prime determinant and \( \text{rank}(N(p)) = \text{rank}(L) - 1 \).

By the Lemma 1.6 of \([HKK]\), we construct a sublattice \( K \) of \( L \) which is close to \( J(p) \)'s and \( N(p) \)'s and, in addition, \( K_p \) is unimodular everywhere except for a single prime spot \( q \) at which \( \text{ord}_p(\det K_p) = 1 \). We claim that \( \#\{ \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in X_A(L, K) \} = 1 \), i.e., for any spinor genera \( \text{spn}(\sigma L) \) and \( \text{spn}(\tau L) \), if they both represent \( K \) then \( \text{measure}(\text{spn}(\sigma L), K) = \text{measure}(\text{spn}(\tau L), K) \). Thus for the lattice \( K \) constructed in the proof of Theorem 1.1.9, we also have \( \#\{ \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in X_A(L, K) \} = 1. \)

Now for any \( p \) dividing \( 2\text{Vol}(L) \), because \( J(p) \) is a characteristic lattice and that \( K_p \) is very close to \( J(p) \), so \( K_p \) itself is a characteristic lattice. We have \( \Theta(L_p, K_p) = \Theta(O^+(L_p)) \). For \( p = p_k \) (\( k = 1, \ldots, t \)), since \( \text{ord}_p(\det K_p) = 1 \) and \( L_p \) being unimodular, we have \( \Theta(O^+(V_p/K_p))\Theta(O^+(L_p)) \supseteq \Theta(O^+(K_p))\Theta(O^+(L_p)) = F_p^* \). Similarly, \( \Theta(O^+(V_q/K_q))\Theta(O^+(L_q)) \supseteq \Theta(O^+(K_q))\Theta(O^+(L_q)) = F_q^* \). Finally when \( p \) is none of the above then \( L_p \) and \( K_p \) are both unimodular and \( \Theta(L_p, K_p) = \Theta(O^+(L_p)) \). By
Corollary 2.7, $\#\{ \text{measure}(\text{spn}(\sigma L), K) \mid \sigma \in \mathcal{X}_A(L,K) \} \leq$

$$[F^* \Theta(O_A^+(L,K)) : F^* \Theta(O_A^+(V/K)) \Theta(O_A^+(L)) \Theta(O_A^+(L'))] = [J_Q : J_Q] = 1.$$
CHAPTER III

LOCAL PRIMITIVE REPRESENTATIONS

Primitive representations (both local and global) have been studied in [C], [Kn3], [Ni], [J1], [J2] and [J3]. Though they are by no mean complete, results from these works have already found their applications in the algebraic geometry of K3-surfaces and their singularities (see [D] and [Ni]). In his recent work [J1] and [J2], James gave the necessary and sufficient local conditions for the existence of the primitive representation by unimodular lattice, and using strong approximation he formulated these local conditions in the global terms in the indefinite case. In this chapter we try to extend James' local results. Since in general the problem is quite complicated, we restrict ourselves only to the next reasonable step where the bigger lattice is even and almost unimodular over $\mathbb{Z}_p$ for all $p$. In the first section we solve this problem over non-dyadic local ring. In § 2 we give some basic results over $\mathbb{Z}_2$, and for completeness sake we state the main theorems of [J1] & [J2], but in a slightly different way. In the third section we solve this problem over $\mathbb{Z}_2$.

First we fix some notations. Let $L$ be a $\mathbb{Z}_p$-lattice of rank $n$. When $L$ is almost unimodular, we write $L = L(0)\perp pL(1)$ where $L(0)$ is unimodular and $pL(1)$ is $p$-modular. Similarly let $M$ be a $\mathbb{Z}_p$-integral lattice of rank $m$. We may write $M = M(0)\perp pM(1)$ where $M(0)$ is unimodular (possibly zero) and $M(1)$ is integral with $s(pM(1)) \subseteq (p)$; or we may write $M = M(0)\perp pM(1)\perp p^2M(2)$ where $M(0)$ is
unimodular, \( pM(1) \) is \( p \)-modular (possibly zero), and \( M(2) \) is integral with \( s(p^2M(2)) \subseteq (p^2) \). The above decompositions are unique up to an isometry when \( p \) is odd (non-dyadic) But when \( p=2 \), these decompositions are not unique. Nevertheless, the norms, the scales and the ranks of these \( L(i) \) and \( M(j) \) are uniquely determined. We call \( \text{rank}(L(i)) = n(i) \), and \( \text{rank}(M(j)) = m(j) \). So \( n = n(0) + n(1) \), and \( m = m(0) + m(1) \) or \( m = m(0) + m(1) + m(2) \). For two lattices \( L \) and \( M \), \( M \rightarrow L \) denotes that \( L \) primitively represents \( M \), i.e., there is a length preserving linear map \( \phi \) form \( M \) to \( L \) such that \( \phi(M) \) is a direct summand of \( L \). Let \( H \) stand for the \( \mathbb{Z}_p \)-unimodular lattice with matrix \( A(0, 0) \); and \( B \) for the \( \mathbb{Z}_p \)-unimodular lattice with lattice \( A(2, 2) \). (See § 93B of [OM]). \( K^r \), for some lattice \( K \), denotes the lattice consisting of \( r \) orthogonal copies of \( K \). \( p*K \) stands for the sublattice \( \{ pv \mid v \in K \} \). \( L = Hs \) means that \( L \) consists of several orthogonal copies of \( H \).

§3.1. Primitive Representations Over Non-Dyadic Local Ring

Throughout this section \( p \) is an odd prime. First we state Theorem 1 of James [J1] which deals with the case when \( L \) is a unimodular lattice of rank \( n \). Assume \( M \) is \( \mathbb{Z}_p \)-integral of rank \( m \). We write \( M = M(0) \perp pM(1) \). Since \( p \) is odd, the above decomposition of \( M \) is unique up to isometry. With these we have:

**THEOREM 3.1.1.** (See Theorem 1 of [J1]) \( L \) represents \( M \) primitively if and only if \( M(0) \rightarrow L \) and \( \text{Wittindex}(M(0) \perp \text{in } L) \geq m(1) \).
Now let \( L \) be a \( \mathbb{Z}_p \) almost unimodular lattice of rank \( n \); \( M \) be a \( \mathbb{Z}_p \) integral lattice of rank \( m \). Write \( L = L(0) \perp pL(1) \) and \( M = M(0) \perp pM(1) \perp p^2M(2) \) as before.

Question: What is the necessary and sufficient condition for \( M \rightarrow L \) ? (primitive representation).

Because \( p \) is odd, \( L(i) \) and \( M(j) \) are all unique up to an isometry. One of the necessary conditions is that \( L(0) \) represents \( M(0) \). We can obviously cancel the \( M(0) \) portion from \( L(0) \) without affecting the representability, WLOG assume that \( M(0) \) is 0.

**Theorem 3.1.2.** Assume \( L = L(0) \perp pL(1) \) and \( M = pM(1) \perp p^2M(2) \), then \( M \rightarrow L \) if and only if

1. \( \text{wittindex}(L(0)) + n(1) \geq m \), when \( n(1) < m(1) \); or
2. \( M \rightarrow \mathbb{p} H^{\text{wittindex}(L(0))} \perp pL(1) \), when \( n(1) \geq m(1) \).

Proof of (1): \( \Rightarrow \): Assume that \( M \) is primitively contained inside \( L \). Then \( \text{Proj}_{L(0)}(M) \) is a sublattice of \( L(0) \) with primitive rank of at least \( m - n(1) \), i.e., \( \text{Proj}_{L(0)}(M) \) contains a rank \( m - n(1) \) sublattice which is primitive inside \( L(0) \). Because the scale \( s(\text{Proj}_{L(0)}(M)) \subseteq p \), so by Theorem 3.1.1 we have \( \text{wittindex}(L(0)) \geq m - n(1) \).

(1)\( \Leftarrow \): It is obvious because that \( M(1) \) represents \( L(1) \).

Before giving a proof of the second part we have a lemma:

**Lemma 3.1.3.** Let \( L = H \perp L' \) and \( s(M) \subseteq (p) \). If \( M \rightarrow L \) then \( M \rightarrow \mathbb{p} H \perp L' \).

Proof: Assume that \( M \) is contained inside \( L \) primitively. Say \( H = Z_p e_1 + Z_p e_2 \) with \( Q(e_1) = Q(e_2) = 0 \) and \( B(e_1, e_2) = 1 \). If \( M \subseteq (Z_p e_1 + Z_p e_2) \perp L' \), then obviously we
have $M \rightarrow pH \bot L'$. Otherwise there is a $x \in M$ such that $x = e_1 + ae_2 + x'$ where $a \in Z_p$, $x' \in L$. Consider the sublattice $Z_p(e_1 + x') + Z_pe_2$. It is isometric to $H$ so we can write $L = H \bot L' = (Z_p(e_1 + x') + Z_pe_2) \bot L''$. Say $Z_p(e_1 + x') + Z_pe_2 = Zpf_1 + Zpf_2$ with $Q(f_1) = Q(f_2) = 0$ and $B(f_1, f_2) = 1$. Since $x$ is contained primitively inside $Zpf_1 + Zpf_2$, we may assume $x = bf_1 + cf_2$ with $b \in Z_p^*$ and $c \in (p)$. Because $B(x, M) \subseteq (p)$, that for any $y \in M$, we have $y = b'f_1 + c'f_2 + y''$ with $b' \in Z_p$, $c \in (p)$ and $y'' \in L''$. So

$$M \subseteq (Zpf_1 + Zpf_2) \bot L''$$

and $M \rightarrow pH \bot L'' \equiv pH \bot L'$, for $L'' \equiv L'$. Q.E.D.

Proof of (2): "$\Rightarrow"$: By the above lemma we may assume that $M$ is contained primitively inside $A \bot pH^{\text{wittindex}(L(0))} \bot pL(1)$ where $L(0) = A \bot pH^{\text{wittindex}(L(0))}$ and $A$ is anisotropic. Say $K = pH^{\text{wittindex}(L(0))} \bot pL(1)$. Now $\text{Proj}_K(M)$ must be primitive in $K$ because $A$ is anisotropic. Careful calculation reveals that $\text{Proj}_K(M) \equiv pM(1) \bot p^2M'(2)$ of rank $m$. By Theorem 3.1.1 we have $M \rightarrow K$.

$(2)^{\Leftarrow}$: If $n(1) > m(1)$ then $L(1)$ represents $M(1)$. By Theorem 3.1.1 we get $\text{wittindex}(L(0)) + \text{wittindex}(M(1) \bot L(1)) \geq m(2)$. So $p^2M(2) \rightarrow H^{\text{wittindex}(L(0))} \bot (m(1) \bot L(1))$, and $M \rightarrow L$. Similarly we can prove $M \rightarrow L$ when $L(1) \equiv M(1)$. Finally, assume $n(1) = m(1)$ but $L(1) \not\equiv M(1)$. By Theorem 3.1.1 $\text{wittindex}(L(0)) - 1 \geq m(2)$. Now $p^2M(2) \rightarrow H^{\text{wittindex}(L(0))} - 1$, and $pM(1) \rightarrow H \bot pL(1)$, so again $M \rightarrow L$.

Q.E.D.
§ 3.2. Basic Results Over $\mathbb{Z}_2$

**Proposition 3.2.1.** The followings hold over $\mathbb{Z}_2$:

1. $2\mathbb{Z}_2^* \rightarrow B$;
2. $2\mathbb{Z}_2 \rightarrow H$;
3. $1+4\mathbb{Z}_2, 2+8\mathbb{Z}_2 \rightarrow <1> \perp <1>$;
4. $1+4\mathbb{Z}_2, 6+8\mathbb{Z}_2 \rightarrow <1> \perp <5>$;
5. $\mathbb{Z}_2^*, 4\mathbb{Z}_2^* \rightarrow <1> \perp <3>$;
6. $\mathbb{Z}_2^*, 8\mathbb{Z}_2 \rightarrow <1> \perp <7>$;

The left side of the above are all that can be represented primitively by the right side.

Proof: It is straightforward. Q.E.D.

**Proposition 3.2.2.** The followings hold over $\mathbb{Z}_2$:

1. $<2c> \perp <4d> \rightarrow H \perp <2a> \perp <2b>$;
2. $<2c> \perp 2H \rightarrow H \perp <2a> \perp <2b>$, if $<c> \rightarrow <a> \perp <b>$;
3. $<2c> \perp 2B \rightarrow H \perp <2a> \perp <2b>$, if $<a> \perp <b>$ does not represent $<c>$;
4. $<2b> \perp <2c> \perp <4d> \rightarrow H \perp <2a> \perp 2H$;
5. $<2b> \perp <2c> \perp <4d> \rightarrow H \perp <2a> \perp 2B$;
6. $<a> \perp H = <a> \perp <b> \perp <b>$;
7. $<a> \perp B = <b> \perp <3a> \perp <b>$, if $a \neq b$ (mod 4);
8. $<2b> \perp <2c> \perp <4d> \rightarrow H \perp H \perp <2a>$;
9. $<2a> \perp <2c> \perp 2H \rightarrow H \perp H \perp <2a> \perp <2b>$;

Where $a,b,c,d$ are all units.
Proof: Because $<2a> \perp <2b> \perp <2c>$ represents $<4d>$ primitively, so (1) is true. For (2), if $<c> \to <a> \perp <b>$; then $<a> \perp <b> = <c> \perp <d>$ for some unit $d$. Now $2H \to <2c> \perp <2d> \perp <2c>$, with primitivity comes from $<2c> \perp <2d>$. Thus (2) holds. For (3), since $<a> \perp <b>$ does not represent $<c>$ so $2B \to <2d> \perp <2a> \perp <2b>$ with primitivity comes from $<2a> \perp <2b>$. For (4) and (5), if $<a> \to <b> \perp <c>$, then these two hold. On the other hand because $d$ is a unit so we can always change the decomposition of the left side so that $<a> \to <b> \perp <c>$. (6) is obvious. For (7), because $a^{-1}(<a> \perp B) = <1> \perp B = <1> \perp A(a^{-1}b-1, 2) = <a^{-1}b> \perp <3> \perp <a^{-1}b>$. So $<a> \perp B = <b> \perp <3a> \perp <b>$. (8) is straight forward. Finally we can get (9) from (2).

Q.E.D.

PROPOSITION 3.2.3. The following are true over $Z_2$:

(1) $Z_2^* \to <a> \perp <b> \perp <2c>$;

(2) $<2b> \perp <2c> \perp <8d> \to H \perp 2B \perp <2c>$, when $<b> \perp <c>$ does not represent $<a>$.

Where $a,b$ and $c$ are units; $d$ is in $Z_2$.

Proof: The first part is obvious. For the second part, because $<b> \perp <c>$ does not represent $<a>$, so $a \not\equiv c \pmod{4}$. By (7) of Proposition 3.2.2 $<2a> \perp 2B = <2c> \perp <6a> \perp <2c>$. Because $<8d> \to <2b> \perp <6a> \perp <2c>$ (which contains a sublattice of matrix $2H$) with primitivity comes from $<6a> \perp <2c>$, so (2) holds.

Q.E.D.

PROPOSITION 3.2.4. Let $L = M \perp N$; $L \supseteq K$ be two $Z_p$-lattices (any prime $p$):

(1) If $\text{Proj}_N(K)$ is primitive in $N$ and the map $\text{Proj}_N \mid K$ is injective, then $K$ is primitive in $L$;
(2) If $L$ contains $K$ primitively, and $\text{Proj}_M(K) \subseteq p^*M$, then the map $\text{Proj}_N \mid K$ is injective, and $\text{Proj}_N(K)$ is primitive in $N$.

Proof of (1): Let $x \perp y$ be a vector in $L = M \perp N$. Assume $px \perp py$ is in $K$, then $py$ is in $\text{Proj}_N(K)$. Because $\text{Proj}_N(K)$ is primitive in $N$, so $y$ itself is in $\text{Proj}_N(K)$. Thus there is a $z$ in $M$ such that $z \perp y$ is a vector in $K$. Now $p(x - z) \in K \cap M = 0$ (because $\text{Proj}_N \mid K$ is injective). So $x = z$, and $x \perp y$ is in $K$. That is to say $K$ is primitive in $L$.

Proof of (2): because $L$ contains $K$ primitively and $\text{Proj}_M(K) \subseteq p^*M$, so the map $\text{Proj}_N \mid K$ is injective. Now for any $x$ in $N$, if $px$ is in $\text{Proj}_N(K)$ then there is $py$ in $\text{Proj}_M(K)$ such that $px \perp py$ is in $K$. So $x \perp y$ is in $K$, and $x$ is in $\text{Proj}_N(K)$. $\text{Proj}_N(K)$ is primitive in $N$. Q.E.D.

**Proposition 3.2.5.** If $U \perp V$ is primitively contained inside $L$, then $V$ is primitively contained inside $(U \perp$ in $L)$.

Proof: Obvious. Q.E.D.

**Proposition 3.2.6.** If $Z_2U \perp K \subseteq 2^\alpha H \perp L$ primitively with $u \in 2^\alpha H$ and $Q(u) \in 2^{\alpha+2}$. Then $\text{Proj}_L(K)$ is a primitive sublattice in $L$ of the same rank as of $K$.

Proof: Suppose $K = Z_2x_1 + Z_2x_2 + \ldots + Z_2x_k$. Then $x_i = h_i \perp y_i$, for all $i$, with $h_i \in 2^\alpha H$ and $y_i \in L$. Let $2^\alpha H = Z_2e_1 + Z_2e_2$ with $Q(e_1) = Q(e_2) = 0$ and $B(e_1, e_2) = 2^\alpha$. WLOG, we may assume that $u = e_1 + ae_2$ with $a \in \langle 2 \rangle$. Because $u$ is perpendicular to all $x_i$s, so we have $h_i = b_ie_1 + c_ie_2$ with $c_i \in \langle 2 \rangle$. Thus $\text{Proj}_L(K) = Z_2y_1 + Z_2y_2 + \ldots + Z_2y_k$ is primitive inside $L$, since $Z_2U \perp K \subseteq 2^\alpha H \perp L$ primitively. Q.E.D.
The following are the main results of [J1] and [J2]. The statements are slightly different than that of the corresponding ones in James' paper. A $Z_2$ integral lattice is said to be an even lattice if the norm of that lattice is contained in (2); otherwise it is called an odd lattice.

**Theorem 3.2.7** (See Theorem 1 of [J1]) Assume $L = L(0)$ and $M = M(0) \perp 2M(1)$ both being even. Then $M \to L$ if and only if $M(0) \to L(0)$ and

1. $n(0) - m(0) \geq 2m(1)$, when $M(1)$ is odd; or
2. $\text{wittindex}(M(0)^\perp \text{in } L(0)) \geq m(1)$, when $M(1)$ is even.

When $L$ is an odd unimodular $Z_2$ lattice, we can always write $L = \langle a \rangle \perp L(0)$ when $\text{rank}(L) = n = \text{odd}$; and $L = \langle a \rangle \perp \langle b \rangle \perp L(0)$ when $\text{rank}(L) = n = \text{even}$. Here $L(0)$ is an even unimodular $Z_2$ lattice, $a$ and $b$ are units. Such decomposition of $L$ is unique when $n$ is odd. When $n$ is even $L(0)$ is unique if $a + b \equiv 0 \pmod{4}$. A primitive vector $w$ of $L$ is said to be characteristic if $B(w, v) \in (2)$ for all $v$ of $L$ with $Q(v) \in (2)$. It can be shown that the orthogonal complement of a primitive characteristic vector in $L$ is an even sublattice. A primitive representation $\varphi$ from $M$ to $L$ is called characteristic if $\varphi(M)$ contains a primitive characteristic vector of $L$. When $L$ is an odd rank, odd unimodular lattice then $L = \langle a \rangle \perp L(0) = Z_2 w \perp L(0)$. In this case it is easy to see that $w$ is characteristic and that all characteristic vectors of $L$ have the form $xw + 2u$, where $x \in Z_2^*$, and $u \in L(0)$. When $L$ is an even rank, odd unimodular $Z_2$ lattice then $L = \langle a \rangle \perp \langle b \rangle \perp L(0) = A(a + b, b^{-1})L(0) = A(2\eta, \xi)L(0)$ with $2\eta = a + b$, and $\xi = b^{-1}$. One can show that, say $A(2\eta, \xi) = Z_2w + Z_2v$ with $Q(w) = 2\eta$, $Q(v) = \xi$ and $B(w, v) = 1$; $w$ is characteristic and that all primitive characteristic vectors of $L$ have the form $xw + 2yv + 2u$, where $x \in Z_2^*$, $y \in Z_2$ and $u \in L(0)$. When $\text{rank}(L) = n = \text{even}$ and $L$
= A(2\eta, \xi) \perp L(0), we define \( Q(\eta, \xi) = \{ Q(v) \mid v \text{ is primitive characteristic in } L \} \). By above description of the characteristic vectors, \( Q(\eta, \xi) = \{ Q(xw + 2yv + 2u) \mid x \in \mathbb{Z}_2^*, y \in \mathbb{Z}_2 \text{ and } u \in L(0) \} = \{ 2\eta x^2 + 4xy + 4\xi y^2 + 4Q(u) \mid x \in \mathbb{Z}_2^*, y \in \mathbb{Z}_2 \text{ and } u \in L(0) \} \subseteq 2\eta \mathbb{Z}_2^{*2} + 8\mathbb{Z}_2. \)

**Proposition 3.2.8.** If \( \eta \in \mathbb{Z}_2^* \), then \( Q(\eta, \xi) = 2\eta + 8\mathbb{Z}_2. \) If \( \eta \in 2\mathbb{Z}_2^* \), then \( Q(\eta, \xi) = 4\mathbb{Z}_2^* \). If \( \eta \in 4\mathbb{Z}_2 \), then \( Q(\eta, \xi) = 8\mathbb{Z}_2. \)

Proof: The proof is straightforward. Also see Lemma 1 of [J]. Q.E.D.

Actually that \( Q(\eta, \xi) \) has nothing to do with the \( L(0) \) portion of the \( L \). That is why the notation does not reflect it.

**Theorem 3.2.9.** (See Theorem 3 of [J]) Assume \( L = <a> \perp L(0), M = M(0) \perp 2M(1) \) with \( L(0) \) and \( M \) both being even, and \( a \in \mathbb{Z}_2^* \). Then \( M \to L \) if and only if \( M \to L(0) \). (There is no characteristic representation in this case.)

**Theorem 3.2.10.** (See Theorem 8 of [J]) Assume \( L = <a> \perp <b> \perp L(0), M = M(0) \perp 2M(1) \) with \( L(0) \) and \( M \) both being even; and \( a, b \) being units. Then \( M \to L \) non-characteristically if and only if \( \exists L(0) \), such that \( M \to L(0) \). \( M \to L \) characteristically if and only if there \( \exists x \), a primitive vector in \( M(1) \), such that \( 2Q(x) \in Q(2^{-1}(a+b), b^{-1}) \) and

1. \( n(0) - m(0) \geq 2m(1) \); or
2. \( n(0) - m(0) = 2m(1) -2; \) and \( \text{wittindex}(M(0) \perp \text{in } L(0)) = m(1) - 1, \) or
   \[ \text{wittindex}(M(0) \perp \text{in } L(0)) = m(1) - 2 \] with \( M(1) \) being odd.
(Note when \( n(0) = m(0) \) and \( L(0) \neq M(0) \) we define \( \text{wittindex}(M(0)^{\perp} \text{ in } L(0)) = -1 \))

**Corollary 3.2.11.** Assume \( L = \langle a \rangle \perp L(0), M = \langle b \rangle \perp M(0) \perp 2M(1) \) with \( L(0) \) and \( M(0) \) being even, and \( a, b \) being units. Then

\( M \rightarrow L \) non-characteristically if and only if

1. \( L(0) \supseteq H \) and \( M(0) \perp 2M(1) \rightarrow \langle a \rangle \perp \langle -b \rangle \perp (H^{\perp} \text{ in } L(0)) \) non-characteristically; or
2. \( L = \langle a \rangle \perp B, M = \langle b \rangle \) and \( a \neq b \) (mod 4).

\( M \rightarrow L \) characteristically if and only if

1. \( a \equiv b \) (mod 8) and \( M(0) \perp 2M(1) \rightarrow L(0) \); or
2. \( L(0) \supseteq H \) and \( M(0) \perp 2M(1) \rightarrow \langle a \rangle \perp \langle -b \rangle \perp (H^{\perp} \text{ in } L(0)) \) characteristically; or
3. \( L = \langle a \rangle \perp B, M = \langle b \rangle \perp \langle c \rangle, a \neq b \) (mod 4) and \( c \in Q(2^{-1}(3a+b), b^{-1}) \).

**Proof:** First we consider the non-characteristic representation. It is easy to see that both (1) and (2) under this category are sufficient. Now assume \( M \subseteq L \) non-characteristically. Certainly \( L(0) \neq 0 \). If \( L(0) \supseteq H \), we have \( L \equiv \langle a \rangle \perp \langle -b \rangle \perp (H^{\perp} \text{ in } L(0)) \). Because \( M \subseteq L \) non-characteristically, \( \langle b \rangle^{\perp} \) in \( L \) is not even, and so \( M(0) \perp 2M(1) \rightarrow \langle a \rangle \perp \langle -b \rangle \perp (H^{\perp} \text{ in } L(0)) \) non-characteristically. This proves (1).

If \( L = \langle a \rangle \perp B \), then obviously \( a \equiv b \) (mod 8) or \( a \neq b \) (mod 4). It is impossible for \( M \subseteq L \) non-characteristically when \( a \equiv b \) (mod 8), so \( a \neq b \) (mod 4). Now by Proposition 3.2.2(7), \( L = \langle b \rangle \perp \langle 3a \rangle \perp \langle b \rangle \). By the same reason mentioned above, \( M(0) \perp 2M(1) \rightarrow \langle 3a \rangle \perp \langle b \rangle \) non-characteristically. This leads to \( M(0) \perp 2M(1) = 0 \) by Theorem 3.2.10. This proves (2).

Second, we consider the characteristic representation. Again, it is not difficult to see that (1), (2) and (3) are all sufficient. (Note in (3), \( L = \langle b \rangle \perp \langle 3a \rangle \perp \langle b \rangle \). Now
suppose $M \subseteq L$ characteristically. If $<b>^\perp$ in $L$ is even, then $<b>$ is characteristic. So $a \equiv b \pmod{8}$ and $M(0)_L 2 M(1) \rightarrow L(0)$. This proves case (1).

Assume $<b>^\perp$ in $L$ is odd. There are two possibilities. First $L(0) \supseteq H$. In this case $L \equiv <a>^\perp <b>^\perp <b>^\perp (H^\perp \text{in } L(0))$, and so $M(0) 2 M(1) \rightarrow <a>^\perp <b>^\perp (H^\perp \text{in } L(0))$ characteristically. This proves case (2). Second $L(0) = B$, i.e., $L = <a>^\perp B$. Because $<b>^\perp$ in $L$ is odd, we have $a \not\equiv b \pmod{4}$, and $L \equiv <b>^\perp 3a^\perp <b>^\perp A(3a, b^{-1})$. Now $M(0) 2 M(1) \rightarrow A(3a, b^{-1})$ characteristically. So $M(0)_L 2 M(1) = <c>$ with $c \in \mathbb{Q}(2^{-1}(3a+b), b^{-1})$. This is the case (3).

(Also see [J2]).

Q.E.D.

**Corollary 3.2.12.** Assume $L = <a>^\perp <b>^\perp L(0), M = <c>^\perp M(0) 2 M(1)$ with $L(0)$ and $M(0)$ being even, and $a, b, c$ being units. Then $M \rightarrow L$ non-characteristically if and only if

1. $M(0)_L 2 M(1) \rightarrow L(0)$, if $<c> \rightarrow <a>^\perp <b>$; or
2. $L(0) \supset H$, $<c> \not\rightarrow <a>^\perp <b>$ and $M(0) 2 M(1) \rightarrow B^\perp (H^\perp \text{in } L(0))$; or
3. $L = <a>^\perp <b>^\perp B$, $<c> \not\rightarrow <a>^\perp <b>$ and $M(0) 2 M(1) \rightarrow H$.

There is no characteristic representation.

Proof: If $M \subseteq L$ primitively, because $<c>^\perp$ in $L$ is always odd unimodular and of odd rank, that containment can not be characteristic.

Assume $M \subseteq L$ primitively (and so characteristically). If $<c> \rightarrow <a>^\perp <b>$, then $L = <a>^\perp <b>^\perp L(0) = <c>^\perp <d>^\perp L(0)$ for some unit $d$. Thus $M(0) 2 M(1) \rightarrow <d>^\perp L(0)$ and so $M(0) 2 M(1) \rightarrow L(0)$ by Theorem 3.2.9. This does the case (1).

If $<c> \rightarrow <a>^\perp <b>$ and $L \supset H$, then $a + b \not\equiv 0 \pmod{4}$ and $L = <a>^\perp <b>^\perp H^\perp (H^\perp \text{in } L(0)) = <c>^\perp <d>^\perp B^\perp (H^\perp \text{in } L(0))$ for some unit $d$. So
Finally, if $a \rightarrow \langle a \rangle \perp \langle b \rangle$ and $L = B$. Then $L = \langle a \rangle \perp \langle b \rangle \perp B = \langle c \rangle \perp \langle d \rangle \perp B$ for some unit $d$. Thus $M(0) \perp 2M(1) \rightarrow \langle d \rangle \perp H$, and so $M(0) \perp 2M(1) \rightarrow H$ by Theorem 3.2.9. This finishes the necessary part of all cases.

The sufficient part is obvious. (Observe that $\langle a \rangle \perp \langle b \rangle \perp H = \langle c \rangle \perp \langle d \rangle \perp B$, and $\langle a \rangle \perp \langle b \rangle \perp B = \langle c \rangle \perp \langle d \rangle \perp H$ for some unit $d$ whenever $a + b \neq 0 \pmod{4}$, which holds if $a \rightarrow \langle a \rangle \perp \langle b \rangle$). Q.E.D.

**Corollary 3.2.13.** Assume $L = \langle a \rangle \perp L(0)$, $M = \langle b \rangle \perp \langle c \rangle \perp M(0) \perp 2M(1)$ with $L(0)$ and $M(0)$ being even, and $a$, $b$, $c$ being units. Then $M \rightarrow L$ non-characteristically if and only if

1. $\langle c \rangle \perp M(0) \perp 2M(1) \rightarrow \langle a \rangle \perp \langle b \rangle \perp (H \perp \text{ in } L(0))$, if $L(0) \supseteq H$; or
2. $L = \langle a \rangle \perp B$, $M = \langle b \rangle \perp \langle c \rangle$, $a \neq b \pmod{4}$ and $\langle c \rangle \rightarrow \langle 3a \rangle \perp \langle b \rangle$.

There is no characteristic representation.

Proof: Because $(\langle b \rangle \perp \langle c \rangle)^\perp$ in $L$ is always odd unimodular and of odd rank, so there will not be any primitive representation. The rest of the proof goes the similar way as that of Corollary 3.2.11 and Corollary 3.2.12. Q.E.D.

Finally we assume that $L = \langle a \rangle \perp \langle b \rangle \perp L(0)$, $M = \langle c \rangle \perp \langle d \rangle \perp M(0) \perp 2M(1)$ with $L(0)$ and $M(0)$ being even, and $a$, $b$, $c$, $d$ being units. When $L(0)$ is not 0, then we can always assume that $a = c$. For, if necessary we can always change the decomposition of $L$ by adjusting $L(0)$.
**COROLLARY 3.2.14.** Assume \( L = \langle a \rangle \bot \langle b \rangle \bot L(0) \), \( M = \langle c \rangle \bot \langle d \rangle \bot M(0) \)

\( \bot 2M(1) \) with \( L(0) \) and \( M(0) \) being even, and \( a, b, c, d \) being units. Then

\( M \rightarrow L \) non-characteristically if and only if

\( L(0) \) is not 0, and \( \langle d \rangle \bot M(0) \bot 2M(1) \rightarrow \langle b \rangle \bot L(0) \) non-characteristically.

\( M \rightarrow L \) characteristically if and only if

1. \( \langle d \rangle \bot M(0) \bot 2M(1) \rightarrow \langle b \rangle \bot L(0) \) characteristically, if \( L(0) \) is not 0; or

2. \( n = m = 2, \) and \( \langle a \rangle \bot \langle b \rangle \equiv \langle c \rangle \bot \langle d \rangle \).

Proof: Similar to the proofs of previous Corollaries Q.E.D.

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**§3.3. Primitive Representations Over \( \mathbb{Z}_2 \)**

In this section we consider the case where \( L \) is even and almost unimodular, and \( M \) is even lattices over \( \mathbb{Z}_2 \). We try to give the necessary and sufficient conditions for the existence of primitive representation \( M \rightarrow L \). Since the problem is quite complicated and has many different cases, we achieve our goal by reducing all these cases to the known results of section 3.2. Only in this sense we have solved the problem completely. Let \( L = L(0) \bot 2L(1) \) be almost unimodular, and \( M = M(0) \bot 2M(1) \bot 2^2M(2) \). One of the necessary conditions for \( M \rightarrow L \) is that there is a decomposition of \( L \) so that \( L(0) \) will represent \( M(0) \). Because of the evenness, we can cancel \( M(0) \) from \( L(0) \) without affecting the representability. For the rest of this section we assume \( L = L(0) \bot 2L(1) \) and \( M = 2M(1) \bot 2^2M(2) \).

**First Necessary Condition:**
Assume \( m(1) \geq n(1) \). Then \( M \rightarrow L \) only if \( n(0) \geq 2(m - n(1)) \) \ (#I \)

Proof: Assume that \( M \) is primitively contained inside \( L \). Then \( \text{Proj}_{L(0)}(M) \) is a sublattice of \( L(0) \) with primitive rank of at least \( m - n(1) \). Now \( n(\text{Proj}_{L(0)}(M)) \) and \( s(\text{Proj}_{L(0)}(M)) \) are both in \( 2\mathbb{Z}_2 \), so by Theorem 3.2.7 we have \( n(0) \geq 2(m - n(1)) \). Q.E.D.

**Second Necessary Condition:**

Assume \( m(1) < n(1) \). Then \( M \rightarrow L \) only if \( n \geq m(1) + 2m(2) \) \ (#II \)

Proof: First it is not difficult to show the following fact:

"Let \( L = H \bot K \), or \( L = B \bot L \). If \( M \rightarrow L \) then \( M \rightarrow 2C \bot K \), where \( C \) is a rank 2 unimodular lattice. (Note when \( L = H \bot K \), then the \( C \) above is isotropic.)."

Applying this fact repeatedly we may eventually assume that the scale of the big lattice becomes \( 2\mathbb{Z}_2 \). By Proposition 1 of \([J_1]\) we get \( n \geq m(1) + 2m(2) \). Q.E.D.

**Theorem 3.3.1.** Assume \( L(1) \) and \( M(1) \) are both even. Then \( M \rightarrow L \) if and only if

1. \( \text{ Wittindex}(L(0)) + n(1) \geq m \), when \( n(1) < m(1) \); or

2. \( M \rightarrow 2H^{\text{ Wittindex}(L(0)) \bot 2L(1)} \), when \( n(1) \geq m(1) \).

(Note: The statements above are exactly the same as that of Theorem 3.1.2; and so goes the proof.)
Proof: (1)\(\Rightarrow\): Assume that \(M\) is primitively contained inside \(L\). Then \(\text{Proj}_{L(0)}(M)\) is a sublattice of \(L(0)\) with primitive rank of at least \(m-n(1)\). Because the \(\text{s}(\text{Proj}_{L(0)}(M))\subseteq2\mathbb{Z}_2\) and the \(\text{n}(\text{Proj}_{L(0)}(M))\subseteq4\mathbb{Z}_2\) (since both \(L(1)\) and \(M(1)\) are even), we have \(\text{wittindex}(L(0))\geq m- n(1)\) by Theorem 3.2.7.

(1)\(\Leftarrow\): It is obvious since \(M(1)\) represents \(L(1)\).

**Lemma 3.3.2.** Let \(L = H \perp L'\) with \(L'\) being even; and \(n(M) \subseteq (2^2)\). If \(M \rightarrow L\) then \(M \rightarrow 2H \perp L'\).

Proof: Same arguments as in the proof of Lemma 3.1.3. Q.E.D.

(2)\(\Rightarrow\): By Lemma 3.3.2 we may assume that \(M\) is contained primitively inside \(A \perp 2H^{\text{wittindex}(L(0))} \perp 2L(1)\) where \(L(0) = A \perp H^{\text{wittindex}(L(0))}\) and \(A\) is either \(B\) or \(0\).

Say \(K = 2H^{\text{wittindex}(L(0))} \perp 2L(1)\). Consider \(\text{Proj}_K(M)\), which must be primitive in \(K\) (since \(A\), if not 0, primitively only represents \(2\mathbb{Z}_2^*\), and \(n(M) \subseteq 4\mathbb{Z}_2\)). Careful calculation (see remark below) shows that \(\text{Proj}_K(M) \equiv 2M(1) \perp 2^2M'(2)\) of rank \(m\), and \(M'(2)\) being odd if and only if \(M(2)\) being odd. By Theorem 3.2.7 we have \(M \rightarrow 2H^{\text{wittindex}(L(0))} \perp 2L(1)\).

(2)\(\Leftarrow\): Obvious. (Keep in mind that \(2H \rightarrow H \perp 2B\); and \(2B \rightarrow H \perp 2H\).) Q.E.D.

Remark: Let \(L = A \perp 2K\), where both \(A\) and \(K\) are even unimodular; let \(N = 2H\) or \(N = 2B\). If \(N \subseteq L\) with \(\text{Proj}_A(N) \subseteq 2^*A\), then it is not difficult to show \(\text{Proj}_{2K}(N) \equiv N\).

Let \(L = L(0) \perp 2L(1)\). When \(L(1)\) is odd unimodular, we can write \(L(1) = L'(1) \perp \perp '\) with \(L'(1)\) being odd unimodular of rank \(n'(1) \leq 2\), and \(L''(1)\) being even.
unimodular of rank $n'(1)$. Similarly for $M = 2M(1) \perp 2^2M(2)$, when $M(1)$ is odd unimodular, we have $M(1) = M'(1) \perp M''(1)$ of corresponding ranks $m'(1)$ and $m''(1)$.
When $n'(1) \neq 0$, we can always assume that $L(0) = Hs$. In the following we tackle the problem case by case according to all the possible combinations of $n'(1)$ and $m'(1)$.

**Theorem 3.3.3.** Assume that $L(1)$ is odd and $M(1)$ is even; and $L = L(0) \perp 2L'(1) \perp 2L''(1)$ with $L(0) = Hs$. Then $M \rightarrow L$ if and only if

1. When $n'(1) = 1$, then $M \rightarrow L(0) \perp 2L''(1)$.
2. When $n'(1) = 2$ and $n''(1) \geq m(1)$, then $M \rightarrow 2L(0) \perp 2L'(1) \perp 2L''(1)$.
3. When $n'(1) = 2$ and $n''(1) < m(1)$, then $n(0) \geq 2(m - n''(1)) - 2$.

Further if $n(0) = 2(m - n''(1)) - 2$, then either

(i) $M(1) = L''(1) \perp H \perp ?$, or
(ii) $M(2)$ is odd, or
(iii) $a + b \equiv 0 \pmod{4}$, or
(iv) $\exists$ a primitive $x \in M(2)$ such that $2Q(x) \in Q(2^{-1}(a+b), b^{-1})$.

where $L'(1) = \langle a \rangle \perp \langle b \rangle$.

Proof: (1)$\Rightarrow$: Because $M \rightarrow L \rightarrow B \perp L(0) \perp 2L''(1)$, so we have $M \rightarrow L(0) \perp 2L''(1)$ so by Theorem 3.3.1

(1)$\Leftarrow$: Obvious!

(2)$\Rightarrow$: By the fact in the proof of (#II)

(2)$\Leftarrow$: By Theorem 3.2.10.

(3)$\Rightarrow$: Because $M \rightarrow L \rightarrow B \perp H \perp L(0) \perp 2L''(1)$, by Theorem 3.3.1 we have $M \rightarrow H \perp L(0) \perp 2L''(1)$ and so $n(0) \geq 2(m - n''(1)) - 2$. Further If $n(0) = 2(m - n''(1)) - 2$, $M(1) = L''(1) \perp B$, $M(2)$ is even and $a + b \equiv 0 \pmod{4}$. By the fact in the proof of (#II) we get
M \to 2L(0) \perp 2L'(1) \perp 2L''(1). \text{ So } B \perp 2M(2) \to L(0) \perp <a> \perp <b> \text{ and } n(0) = 2m(2) + 2.

There is a primitive vector x in M(2) such that \(2Q(x) \in Q(2^{-1}(a+b),b^{-1})\) by Theorem 3.2.10. Thus (i), (ii), (iii) and (iv) can not be all violated.

(3)"\Rightarrow": If \(n(0) \geq 2(m-n''(1))\) then certainly we have \(M \to L\) because we have \(M \to L(0) \perp 2L''(1)\) already. Suppose \(n(0) = 2(m-n''(1)) - 2\). If (i) holds, i.e., \(M(1) = L''(1) \perp H \perp \), then because \(2H \to H \perp <2a> \perp <2b>\), we have \(M \to L\). If (ii) is true, that is \(M(2)\) being odd, then we can always change the decomposition of \(M\) so that (i) will be true. If (iii) holds, i.e., \(a + b \neq 0 \pmod{2}\), then by using \(2B \to H \perp <2a> \perp <2b>\) we have \(M \to L\). Finally there is a primitive vector x in M(2) with \(2Q(x) \in Q(2^{-1}(a+b),b^{-1})\). Write \(2^2M(2) = Z \leftrightarrow x + Z \leftrightarrow x_2 + \ldots + Z \leftrightarrow x_m(2)\). We can see easily that \(2^2M(2) \to 2(<a> \perp <b>) \perp H \leftrightarrow m(2)\), and again we have \(M \to L\). Q.E.D.

**Theorem 3.3.4.** Assume \(L(1)\) is even and \(M = 2M'(1) \perp 2M''(1) \perp 2^2M(2)\). Then \(M \to L\) if and only if

1. When \(m'(1) = 1\), then \(2M''(1) \perp 2^2M(2) \to H \leftrightarrow m(0) \perp 2L(1)\).
2. When \(m'(1) = 2\) and \(m''(1) \geq n(1)\), then \(n(0) \geq 2(m-n(1))\). Further if \(n(0) = 2(m-n(1))\); then either
   1. \(m''(1) > n(1)\); or
   2. there is \(M''(1)\) with \(M''(1) \equiv L(1)\).
3. When \(m'(1) = 2\) and \(m''(1) < n(1)\), then \(n \geq m(1) + 2m(2) + 2\). Further if \(n(0) = m(1) + 2m(2) + 2\), then either
   1. \(M''(1)\) with \((M''(1) \perp \in L(1))\) being Hs, or
   2. \(M(2)\) is odd; or
   3. \(a + b \neq 0 \pmod{4}\), where \(M'(1) = <a> \perp <b>\).
(Note: when $m'(1) = 2$, set $M'(1) = \langle a \rangle \perp \langle b \rangle$. The $M''(1)$ is unique if and only if $a + b \equiv 0 \pmod{4}$ and $n(M(2)) \subseteq 2\mathbb{Z}_2$.)

Proof: (1)"$\Rightarrow"$: Assume that $M$ is contained inside $L$ primitively. Then $2M''(1) \perp 2^2M(2)$ is a primitive sublattice of $(2M'(1) \perp \text{in } L)$, which is isometric to $H^{1/2n(0)-1} \perp \langle 2c \rangle \perp 2L(1)$ for some unit $c$. By Theorem 3.3.3 we get $2M''(1) \perp 2^2M(2) \rightarrow H^{1/2n(0)-1} \perp 2L(1)$.

(1)"$\Leftarrow"$: Obvious.

**Lemma 3.3.5.** Let $L = L(0) \perp 2L(1)$ with $L(i)$ being even unimodular; $M = \langle 2a \rangle \perp \langle 2b \rangle \perp 2N \perp 4K$ with $N$ being even unimodular, $K$ be $\mathbb{Z}_2$-integral, and $a, b$ be units.

1. Assume $M \perp \langle 8c \rangle = M \perp \mathbb{Z}_2e \subseteq L$ primitively. Then $L = J \perp G$ with $J \equiv H$ (or 2H) such that $e \in J$, $\text{Proj}_G(M) \subseteq G$ primitively, $\text{Proj}_G(M) = \langle 2a' \rangle \perp \langle 2b' \rangle \perp 2N \perp 4K'$ with $a', b'$ being units and $a' + b' \equiv 0 \pmod{4}$ if $a + b \equiv 0 \pmod{4}$, $K'$ being $\mathbb{Z}_2$-integral of same rank as of $K$, and $n(K') = \mathbb{Z}_2$ if and only if $n(K) = \mathbb{Z}_2$.

2. Assume $M \perp 4cA \subseteq L$ primitively with $A = B$ (or $H$) and $c \in \mathbb{Z}_2$, then $L = J \perp G$ with $J \equiv H \perp H$ (or $H \perp 2H$, or $2H \perp 2H$) such that $4cA \subseteq J$, $\text{Proj}_G(M) \subseteq G$ primitively, $\text{Proj}_G(M) = \langle 2a' \rangle \perp \langle 2b' \rangle \perp 2N \perp 4K'$ with $a', b'$ being units and $a' + b' \equiv 0 \pmod{4}$ if $a + b \equiv 0 \pmod{4}$, $K'$ being $\mathbb{Z}_2$-integral of same rank as of $K$, and $n(K') = \mathbb{Z}_2$ if and only if $n(K) = \mathbb{Z}_2$.

Proof of (1): Since $M \perp \langle 8c \rangle = M \perp \mathbb{Z}_2e \subseteq L$ primitively, we have $e = u_0 + u_1$ with $u_i$ in $L(i)$. If $u_0$ is primitive in $L(0)$ then there is a $v_0$ in $L(0)$ such that $B(u_0, v_0) = 1$. One can check that $Z_2(u_0 + u_1) + Z_2v_0 \equiv H$ is an orthogonal component of $L$. We choose $J = Z_2(u_0 + u_1) + Z_2v_0$. Otherwise, there is a $v_1$ in $L(1)$ with $B(u_1, v_1) = 2$. This time we pick $J = Z_2(u_0 + u_1) + Z_2v_1$, which is isometric to $2H$ and an orthogonal component of...
Thus we have \( L = J \perp G \) with \( J \equiv H \) (or \( 2H \)) and \( e \in J \). By Proposition 3.2.6, we see that \( \text{Proj}_G(M) \) is a primitive sublattice of \( G \) with \( \text{rank}(\text{Proj}_G(M)) = \text{rank}M \). In the rest we are going to show that \( \text{Proj}_G(M) \) has the described structure stated in the Lemma. Assume \( Z_2 f = (e^{-1} \text{ in } J) \) with \( Q(f) = -8c \). Then \( M \subseteq Z_2 f \perp \text{Proj}_G(M) \) primitively, and \( n(\text{Proj}_G(M)) = s(\text{Proj}_G(M)) = 2Z_2 \). Say \( M = <2a> \perp <2b> \perp 2N \perp 4K = Z_2 w_1 \perp Z_2 w_2 \perp 2N \perp 4K \). Then \( w_1 = x_1 f \perp u_1 \) where \( x_1 \in Z_2 \), and \( u_1 \in \text{Proj}_G(M) \). Since \( Q(w_1) = 2a \) and \( Q(f) = -8c \), we get \( Q(u_1) = 2a' \) with \( a' \equiv a \pmod{4} \). Thus \( Z_2 u_1 \) splits \( \text{Proj}_G(M) \), and \( \text{Proj}_G(M) = Z_2 u_1 \perp D_1 \). Now \( w_2 = x_2 f \perp y_2 u_1 \perp u_2 \) with \( x_2 \) and \( y_2 \) inside \( Z_2 \), and \( u_2 \) in \( D_1 \). Because \( B(w_1, w_2) = 0 \) and \( Q(w_2) = 2b \), we have \( y_2 \in 4Z_2 \) and \( Q(u_2) = 2b' \) with \( b' \equiv b \pmod{4} \). Again \( Z_2 u_2 \) splits \( D_1 \), and \( \text{Proj}_G(M) = Z_2 u_1 \perp Z_2 u_2 \perp D_2 \). Certainly \( a' + b' \equiv 0 \pmod{4} \) if \( a + b \equiv 0 \pmod{4} \). It is not difficult to see that \( n(D_2) = 4 \) and \( s(D_2) = 2 \). Now we write \( 2N = (Z_2 e_3 + Z_2 f_3) \perp \ldots \perp (Z_2 e_i + Z_2 f_i) \), where \( (Z_2 e_i + Z_2 f_i) \equiv 2H \) (or \( 2B \)). Say \( e_3 = x_3 f \perp y_3 u_1 \perp z_3 u_2 \perp z_3' \) and \( f_3 = x_3 f \perp y_3 u_1 \perp z_3' u_2 \perp f_3' \), where \( x_3, y_3, z_3, x_3', y_3', z_3' \) are all in \( Z_2 \) and \( e_3', f_3' \) are in \( D_2 \). Since \( (Z_2 e_3 + Z_2 f_3) \) is perpendicular to \( (Z_2 w_1 \perp Z_2 w_2) \), all \( x_3, y_3, z_3, x_3', y_3', z_3' \) are in \( 4Z_2 \) and so \( (Z_2 e_3 + Z_2 f_3') \equiv (Z_2 e_3' + Z_2 f_3') \), which splits \( D_2 \). We have \( \text{Proj}_G(M) = Z_2 u_1 \perp Z_2 u_2 \perp (Z_2 e_3' + Z_2 f_3') \perp D_3 \). Keep doing in this way, eventually we have \( \text{Proj}_G(M) = Z_2 u_1 \perp Z_2 u_2 \perp (Z_2 e_3' + Z_2 f_3') \perp (Z_2 e_i' + Z_2 f_i') \ldots \perp (Z_2 e_i' + Z_2 f_i') \perp D_i \), with \( (Z_2 e_i + Z_2 f_i) \equiv (Z_2 e_i' + Z_2 f_i') \); i.e., \( \text{Proj}_G(M) \equiv <2a'> \perp <2b'> \perp 2N \perp 4K \). Straight forward checking tells that \( n(D_i) \subseteq 4Z_2 \) and \( n(D_i) = 4Z_2 \) if \( n(K) = 4Z_2 \). So we can write \( D_i = 4K' \), and Lemma 3.3.5(1) is proved.

(2) can be proved in the similar way. Q.E.D.

**Lemma 3.3.6.** Let \( L = L(0) \perp 2L(1) \) with \( L(i) \) being even unimodular; \( M = <2a> \perp <2b> \perp 2N \perp 4K \) with \( N \) being even unimodular, \( K \) be \( Z_2 \)-integral, \( a, b \) be units.
If $M \perp Z_2 e \subseteq L$ primitively with $c$ being unit, then $L = J \perp G$ with $J \equiv H$ (or $2H$, or $2B$) such that $e \in J$, $\text{Proj}_G(M) \subseteq G$ primitively, $\text{Proj}_G(M) \equiv <2a'> \perp <2b'>$ $\perp 2N' \perp 4K'$ with $a'$, $b'$ being units, $N'$ being even unimodular of same rank as of $N$; and $K'$ being $Z_2$-integral of same rank as of $K$.

Proof of Lemma 3.3.6: Similar to the proof of Lemma 3.3.5(1). Q.E.D.

Proof of Theorem 3.3.4. continue:

(2)"$\Rightarrow$": By The Condition (#I) we get $n(0) \geq 2(m-n(1))$. Suppose $n(0) = 2(m-n(1))$, and none of (i) and (ii) holds. Then we can assume that $a+b \equiv 0 \pmod{4}$ and $M(2)$ is even. Because $M \rightarrow L$, so $2B \perp M \rightarrow 2B \perp L$. WLOG we say $M'(1)$ is $Hs$, and $L(1)$ is not $Hs$. So $<2a> \perp <2b> \perp 2M(2) \perp 16^{1/m''(1)} \rightarrow M \rightarrow L$. Apply Lemma 3.3.5 several times we get $<2a'> \perp <2b'> \rightarrow B \perp 2B$, or $<2a'> \perp <2b'> \rightarrow H \perp 2B$ with $a'+b' \equiv 0 \pmod{4}$. But both of them are impossible. So either (i) or (ii) must be true.

(2)"$\Leftarrow$": If $n(0) > 2(m-n(1)) + 2$, then use $H \perp H$ (or $H \perp B$) to represent $<2a> \perp <2b>$; and use the rest of $L$ to represent $2M''(1) \perp 2^2M(2)$. Now assume that $n(0) = 2(m-n(1)) + 2$, and either (i) $m''(1) > n(1)$, or (ii) there is $M''(1)$ such that $M''(1) \equiv L(1)$. In either this case we first cancel the $L(1)$ part of $M''(1)$, and then use $L(0)$ to represent the rest of $M$.

(3)"$\Rightarrow$": By Conditions (#I) and (#II) we have $n \geq m(1) + 2m(2)$. Assume $n = m(1) + 2m(2)$. Apply Lemma 3.3.5 and Lemma 3.3.6 several times we may assume that $<2a'> \perp <2b'> \perp 2M''(1) \rightarrow L'(0) \perp 2L'(1)$, where both $L'(0)$ and $L'(1)$ are even unimodular and both sides have that same rank, but which is impossible.
Next we assume that \( n(0) = m(1) + 2m(2) + 2 \), and none of the condition (i), (ii), (iii) and (iv) is true. Apply Lemma 3.3.5 several times we get \( <2a'> \perp <2b'> \rightarrow H \perp 2B \) or \( <2a'> \perp <2b'> \rightarrow B \perp 2B \) with \( a',b' \) being units and \( a' + b' \equiv 0 \pmod{4} \). But both are impossible. So one of these conditions holds.

(3)' \( " \Rightarrow " : \) If \( n(0) \geq m(1) + 2m(2) + 4 \) then it is easy to see \( M \rightarrow L \) by canceling \( M''(1) \) form \( L(1) \). Assume \( n(0) = m(1) + 2m(2) + 2 \). If there is an \( M''(1) \) with \( (M''(1) \perp \text{in } L(1)) \) being \( Hs \). Then the same proof above works. If \( M''(1) \neq 0 \) and (ii) or (iii) holds, then we can change the decomposition of \( M \) so that (i) will hold. Now assume that \( M''(1) = 0 \) and either (ii) \( n(M(2)) = Z, \) or (iii) \( a + b \equiv 0 \pmod{4} \). If \( a + b \equiv 0 \pmod{4} \), use \( H \perp H \) (or \( H \perp B \), or \( B \perp 2B \), or \( B \perp 2H \), or \( H \perp 2H \), or \( H \perp 2B \)) to represent \( <2a> \perp <2b>, \) and the rest to represent \( 2M''(1) \perp 2^2M(2). \) If \( n(M(2)) = Z, \) then use \( H \perp H \) (or \( H \perp B \)) to represent \( <2a> \perp <2b>, \) and the rest to represent \( 2M''(1) \perp 2^2M(2). \) In any case we have \( M \rightarrow L. \) Q.E.D.

**THEOREM 3.3.7.** Assume \( L = L(0) \perp 2L' (1) \perp 2L''(1), L(0) = Hs, L' (1) = <a>, \) and \( M = 2M'(1) \perp 2M''(1) \perp 2^2M(2) \) with \( M'(1) = <b> \). Then \( M \rightarrow L \) if and only if

1. **When** \( n''(1) \geq m''(1), \) then \( M \rightarrow 2L(0) \perp 2L'(1) \perp 2L''(1). \)

2. **When** \( n''(1) < m''(1), \) then \( n(0) \geq 2(m''(1) - n''(1) + m(2)). \)

Further if \( n(0) = 2(m''(1) - n''(1) + m(2)). \)

(i) \( a \equiv b \pmod{8}; \) or

(ii) \( 2M''(1) \perp 2^2M(2) \rightarrow H^{\frac{1}{2}n(0)-1} \perp <2a> \perp <2b> \perp 2L''(1). \)

**Proof:** (1)' \( \Rightarrow " : By the fact in the proof of Condition (#II)\)

(1)' \( \Leftarrow " : By Corollary 3.2.11. \)
(2)\(\Rightarrow\): By Condition (\#I) we have \(n(0) \geq 2(m''(1) - n''(1) + m(2))\). Assume \(n(0) = 2(m''(1) - n''(1) + m(2))\) and \(a \equiv b \pmod{8}\). Then certainly we have \(2M''(1) \perp 2M(2) \rightarrow H^{1/2n(0)-1} \perp <2a> \perp <-2b> \perp 2L''(1)\).

(2)\(\Leftarrow\): If \(n(0) \geq 2(m''(1) - n''(1) + m(2)) + 2\), then obviously we have \(M \rightarrow L\), by using \(H\) to represent \(<2b>\) and the rest of \(L\) to represent \(2M''(1) \perp 2M(2)\). If \(n(0) = 2(m''(1) - n''(1) + m(2))\) and \(a \equiv b \pmod{8}\), then again \(M \rightarrow L\) by using \(<2b> \rightarrow <2a>\). Finally if \(n(0) = 2(m''(1) - n''(1) + m(2))\) and \(2M''(1) \perp 2M(2) \rightarrow H^{1/2n(0)-1} \perp <2a> \perp <-2b> \perp 2L''(1)\), we still have \(M \rightarrow L\) by Theorem 3.3.3. Q.E.D.

**THEOREM 3.3.8.** Assume \(L = L(0) \perp 2L'(1) \perp 2L''(1)\), \(L'(1) = <a> \perp <b>\), \(L(0) = Hs\) and \(M = 2M'(1) \perp 2M''(1) \perp 2^2M(2)\) with \(M'(1) = <c>\). Then \(M \rightarrow L\) if and only if

1. When \(n''(1) \geq m''(1)\), then \(M \rightarrow 2L(0) \perp 2L'(1) \perp 2L''(1)\).
2. When \(n''(1) < m''(1)\), then \(n(0) \geq 2(m''(1) - n''(1) + m(2)) - 2\).

Further if \(n(0) = 2(m''(1) - n''(1) + m(2)) - 2\), then either

(i) \(M''(1) = L''(1) \perp H\perp?\) when \(<c> \rightarrow <a> \perp <b>\); or

(ii) \(M''(1) = L''(1) \perp B\perp?\) when \(<c> \leftrightarrow <a> \perp <b>\); or

(iii) \(M(2)\) is odd.

**Proof:**

(1)\(\Rightarrow\): By the fact in the proof of the Condition (\#II).

(1)\(\Leftarrow\): If \(L'(1)\) represents \(M'(1)\) then \(2M''(1) \perp 2^2M(2) \rightarrow 2L(0) \perp 2L''(1)\). By Theorem 3.2.7, we have \(2M''(1) \perp 2^2M(2) \rightarrow L(0) \perp 2L''(1)\) and \(M \rightarrow L\). If \(L''(1)\) is not \(0\) then we can always change the decomposition so that resulting \(L'(1)\) will represent \(M'(1)\). Again under this condition we have \(M \rightarrow L\). Now we assume that \(L'(1)\) does not represent \(M'(1)\) and \(L''(1)\) is \(0\) (so \(M''(1)\) is \(0\)). Because \(2L(0) \perp 2L'(1) =
<2c>⟨<2d⟩⊥<2B⟩⊥2H^{1/2n(0)-1}, for some unit d, so \(2^2M(2) \rightarrow 2B ⊥ 2H^{1/2n(0)-1}\). That is to say either \(1/2n(0) - 1 > m(2)\); or \(1/2n(0) - 1 = m(2)\) and \(M(2)\) is odd. Now with <2c>⟨<4e⟩ → <2a>⟨<2b⟩⊥H for any unit e, we have \(M \rightarrow L\).

\((2)''\Rightarrow"\): By Condition (1) we have \(n(0) \geq 2(m - n(1)) = 2(m''(1) + 1 + m(2) - n'(1) - 2) = 2(m''(1) - n''(1) + m(2)) - 2\). Assume \(n(0) = 2(m''(1) - n''(1) + m(2)) - 2\). Assume further that none of the condition listed is satisfied. So \(m''(1) = n''(1) + 2\), and \(M(2)\) is even. Since \(M \rightarrow L\), we have \(M \rightarrow 2L(0) \perp 2L'(1) \perp 2L''(1)\). So \(<2c>⟨<2B⟩ \perp 2^2M(2) \rightarrow 2L(0) \perp <2a>⟨<2b⟩ \text{ when } <c> \rightarrow <a>⟨<b⟩; or } <2c>⟨<2B⟩ \perp 2^2M(2) \rightarrow 2L(0) \perp <2a>⟨<2b⟩. \text{ We get } 2B \perp 2^2M(2) \rightarrow 2L(0) \text{ in the first case, and } 2H \perp 2^2M(2) \rightarrow 2B \perp 2H^{1/2n(0)-1} \text{ in the second case. But both of them are impossible with the given rank restriction, i.e., } n(0) = 2m(2) + 2.\n
\((2)''\Leftarrow"\): Assume \(n(0) \geq 2(m''(1) - n''(1) + m(2))\). If either \(L'(1)\) represents \(M'(1)\) or \(L''(1)\) is not 0, then obviously we have \(M \rightarrow L\). Suppose \(L'(1)\) does not represent \(M'(1)\) and \(L''(1)\) is 0. One can check that \(4Z_2^* \rightarrow 2L(1) \perp -<2c>\). On the other hand because \(m''(1) > 0\), so there is a primitive vector \(x\) in \(M''(1)\) with \(2Q(x) = 4Z_2^*\), and so \(M \rightarrow L\). Now assume that \(n(0) = 2(m''(1) - n''(1) + m(2)) - 2\). If \(M''(1)\) and \(L''(1)\) are such that \(M''(1) = L''(1) \perp H \perp ?\) and \(<c> \rightarrow <a>⟨<b⟩, by Proposition 3.2.2(2) we use \(H \perp <2a>⟨<2b⟩\) to represent \(<2a>⟨<2b⟩\) and the rest of \(L\) to represent the rest of \(M\). If \(M''(1)\) and \(L''(1)\) are such that \(M''(1) = L''(1) \perp B \perp ?\) and \(<c> \not\rightarrow <a>⟨<b⟩, then By Proposition 3.2.2(3) we use \(H \perp <2a>⟨<2b⟩\) to represent \(<2a>⟨<2b⟩\) and the rest of \(L\) to the rest of \(M\). Finally, when (iii) is true we can always change the decomposition of \(M\) so that either (i) or (ii) will hold. 

\textbf{Q.E.D.}

\textbf{Theorem 3.3.9.} Assume \(L = L(0) \sqcup 2L'(1) \sqcup 2L''(1), L(0) = Hs, L'(1) = <a>, and M = 2M'(1) \sqcup 2M''(1) \sqcup 2^2M(2)\) with \(M'(1) = <b>⟨<c>\). Then \(M \rightarrow L\) if and only if
(1) When $M'(1)$ represents $L'(1)$ and $n''(1) \geq m''(1)$, then $2M''(1) \bot 2^2 M(2) \rightarrow 2H^{1/2} n(0) - 1 \bot 2^2 L''(1)$.

(2) When $m''(1) = 0$, $M'(1)$ does not represent $L'(1)$ and $M(2)$ is even, then $n(0) + n''(1) \geq 2m(2) + 2$. Further if $n(0) + n''(1) = 2m(2) + 2$, then $L''(1)$ is not Hs.

(3) When $n''(1) < m''(1)$, then $n(0) \geq 2(m''(1) + m(2) - n''(1)) + 2$.

(Note when $m''(1) \neq 0$ or $M(2)$ is odd, we can change the decomposition of $M$ so that $M'(1)$ will represent $L'(1)$.)

Proof: (1)"$\Rightarrow$": By the fact in the proof of Condition (#II), we have $M \rightarrow L(0) \bot 2L'(1) \bot 2L''(1) \rightarrow 2c \rightarrow 2^2M'(1) \rightarrow 2^2M(2) \rightarrow H^{1/2} n(0) - 1 \rightarrow 2a \rightarrow 2b \rightarrow 2c \rightarrow 2^2L''(1)$. Since $2c \rightarrow 2c$ represents $2a$, $2a \rightarrow 2b$ represents $2c$ and so $2M''(1) \rightarrow 2^2M(2) \rightarrow H^{1/2} n(0) - 1 \rightarrow 2^2L''(1)$.

(1)"$\Leftarrow$": Since $2M''(1) \rightarrow 2^2M(2) \rightarrow 2H^{1/2} n(0) - 1 \rightarrow 2^2L''(1)$, we have $2M''(1) \rightarrow 2^2M(2) \rightarrow H^{1/2} n(0) - 1 \rightarrow 2^2L''(1)$ by Theorem 3.2.7. With $2b \rightarrow 2c \rightarrow H \rightarrow 2a$ we get $M \rightarrow L$.

(2)"$\Rightarrow$": By Conditions (#I) and (#II) we have $n(0) + n''(1) \geq 2m(2) + 2$. Because $2b \rightarrow 2c$ does not represent $2a$, so $2a \rightarrow 2b$ does not represent $2c$. Similar to the proof in (1)"$\Rightarrow$", we have $2c \rightarrow 2^2M(2) \rightarrow 2a \rightarrow 2b \rightarrow 2H^{1/2} n(0) - 1 \rightarrow 2L''(1)$, so $2c \rightarrow 2^2M(2) \rightarrow 2c \rightarrow 2d \rightarrow 2B \rightarrow 2H^{1/2} n(0) - 2 \rightarrow 2L''(1)$. If $n(0) + n''(1) = 2m(2) + 2$, then $L''(1)$ can not be Hs because that $M(2)$ is even and of the Theorem 3.2.7

(2)"$\Leftarrow$": If $n(0) + n''(1) \geq 2m(2) + 4$ then certainly we have $M \rightarrow L$. If $n(0) + n''(1) = 2m(2) + 2$ and $L''(1)$ is not Hs. Then by Proposition 3.2.3 (2) we again have $M \rightarrow L$.

(3)"$\Rightarrow$": By Condition (#I)
Because M''(1) is not 0, we can always find a decomposition so that M'(1)
represents L'(1); so M→L. Q.E.D.

**Theorem 3.3.10.** Assume \( L = L'(0) \perp 2L'(1) \perp 2L''(1) \), \( L'(1) = \langle a \rangle \perp \langle b \rangle \), \( L(0) = Hs \) and \( M = 2M'(1) \perp 2M''(1) \perp 2M'(2) \) with \( M'(1) = \langle c \rangle \perp \langle d \rangle \). Then \( M \rightarrow L \) iff

1. When \( n''(1) \geq m''(1) \), then \( M \rightarrow 2L'(0) \perp 2L'(1) \perp 2L''(1) \).

2. When \( n''(1) < m''(1) \), then \( n(0) \geq 2(m''(1) - n''(1) + m(2)) \).

Further if \( n(0) = 2(m''(1) - n''(1) + m(2)) \), then either

(i) \( M'(1) = L''(1) \perp H \perp ? \), or
(ii) \( b \equiv d \) (mod 8), or
(iii) \( b \not\equiv d \) (mod 4), or
(iv) \( n(M(2)) \geq 2Z_2 \).

(Note: When \( m''(1) \) is not 0, we can always assume that, if necessary by changing
decomposition of \( M \), that \( L'(1) \) and \( M'(1) \) represent a same unit. So in (2) we always
assume \( <a> = <c> \).)

Proof: \( (\Rightarrow) \): By the fact in the proof of Condition (#II).

(1)\( \Rightarrow \): If \( L'(1) \) and \( M'(1) \) represent a same unit then the condition is certainly
sufficient by Theorem 3.3.7(1). If \( L'(1) \) and \( M'(1) \) do not represent a same unit, then
we may assume that \( n''(1) = m''(1) = 0 \) and \( M(2) \) is even. (For otherwise we can change
the decompositions so that the resulting \( L'(1) \) and \( M'(1) \) will represent a same unit ).
Check case by case, again we see the condition is sufficient. (Not many cases here,
because after scaling, \( L'(1) \) has two possibilities, they are \( <1> \perp <1> \) and \( <1> \perp <5> \).
\( M'(1) \) has three possibilities, namely \( <3> \perp <3> \), \( <3> \perp <7> \) and \( <7> \perp <7> \).)
(2) \Rightarrow ": By First Necessary Condition we have \( n(0) \geq 2 (m''(1) - n''(1) + m(2)) \).
Assume that \( n(0) = 2(m''(1) - n''(1) + m(2)) \), \( M'(1) = L''(1) \perp B \), \( b \equiv d \) \((\text{mod } 4)\) but \( b \neq d \) \((\text{mod } 8)\), and \( n(M(2)) \neq \mathbb{Z}_2 \). By the fact in the proof of the Condition (II) we have
\[
<2c> \perp <2d> \perp 2M''(1) \perp 2^2 M(2) \rightarrow 2L(0) \perp <2a> \perp <2b> \perp 2L''(1).
\]
Thus \( <2d> \perp 2B \perp 2^2 M(2) \rightarrow 2L(0) \perp <2b> \), ans so \( <2d> \perp 2^2 M(2) \rightarrow 2B \perp 2H^{1/2n(0)} \perp <2b> \). By examining the conditions of Theorem 3.2.11 carefully we see that there is a primitive \( x \) in \( M(2) \) with \( Q(x) \in 2\mathbb{Z}_2^* \). That is \( n(M(2)) = 2\mathbb{Z}_2^* \).

(2) \Leftarrow ": If \( n(0) \geq 2(m''(1) - n''(1) + m(2)) + 2 \), then certainly we have \( M \rightarrow L \). Assume that \( n(0) = 2(m''(1) - n''(1) + m(2)) \). If \( M''(1) = L''(1) \perp H \perp ? \), then by Proposition 3.2.2(9), we use \( H \perp H \perp <2a> \perp <2b> \) to represent \( <2c> \perp <2d> \perp 2H \), and the rest of \( L \) to represent the rest of \( M \). If \( M(2) \) is odd then we can change the decomposition of \( M \) (only affecting \( M''(1) \) and \( M(2) \) ) so that the resulting \( M''(1) \) satisfies \( M''(1) = L''(1) \perp H \perp ? \). If \( b \equiv d \) \((\text{mod } 8)\), then obviously \( M \rightarrow L \) (keep in mind \( a = c \)). If \( b \equiv d \) \((\text{mod } 4)\) but \( b \neq d \) \((\text{mod } 8)\), and there is an \( x \) in \( M(2) \) with \( Q(x) \in 2\mathbb{Z}_2^* \), then we have
\[
<2d> \perp 2^2 M(2) \equiv <2d'> \perp 2^2 M'(2) \text{ with } b \equiv d' \text{ (mod } 8)\).
\]
Finally if \( b \neq d \) \((\text{mod } 4)\), then use \( H \perp H \perp <2a> \perp <2b> \) to represent \( <2c> \perp <2d> \perp 2B \), and the rest of \( L \) to represent the rest of \( M \).

Q.E.D.
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