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The mod-$p$ cohomology of $GL(2p-2,\mathbb{Z})$

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The Ohio State University, 1994
THE MOD-$p$ COHOMOLOGY OF $GL(2p - 2, \mathbb{Z})$

DISSERTATION

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the Requirements for the Degree Doctor of
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By

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*****

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To my
Parents
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INTRODUCTION

Let \( p \) be a prime. The mod-\( p \) cohomology of \( GL(n, \mathbb{Z}) \) and its subgroups of finite index is of importance for several reasons. First is the mysterious connection between modular representations of the Galois group of the algebraic closure of \( \mathbb{Q} \) and the mod-\( p \) cohomology of \( GL(n, \mathbb{Z}) \) (see [Se], [A1], [A2]). There are also relations to the \( K \)-theory of \( \mathbb{Z} \) and the topology of \( O(n) \backslash GL(n, R)/GL(n, \mathbb{Z}) \) (see [Sol]). Unfortunately, it is very hard to grasp the mod-\( p \) cohomology of \( GL(n, \mathbb{Z}) \) for all \( n \) in general. Some attempts have been made for smaller values of \( n \). For example, the cohomology of \( SL(2, \mathbb{Z}) \) is well-known and in [So] Soule worked out the integral cohomology of \( SL(3, \mathbb{Z}) \). But little is known about arbitrary \( n \), beyond Soule's work.

This thesis studies the mod-\( p \) Farrell cohomology of \( GL(2p - 2, \mathbb{Z}) \) where \( p \) is an odd prime. The Farrell cohomology theory is an extension, due to Farrell, of Tate’s cohomology theory of finite groups to the groups of finite virtual cohomological dimension (v.c.d.). Basic properties of Farrell cohomology theory can be found in the last chapter of [Br]. Let \( \Gamma \) be a group with finite v.c.d.. One important feature of the Farrell cohomology theory is that the Farrell cohomology groups \( \hat{H}^i(\Gamma, M) \) of the group \( \Gamma \) with coefficients in any \( \Gamma \)-module \( M \), are same as the ordinary cohomology groups \( H^i(\Gamma, M) \), if \( i > \text{v.c.d.}(\Gamma) \). Below the v.c.d., there is a map from \( H^i(\Gamma, M) \) to \( \hat{H}^i(\Gamma, M) \) which fits into a long exact sequence whose third term is the homology of \( \Gamma \) with coefficients in the Steinberg module.
Also these groups $\hat{H}^i(\Gamma, M)$ are always torsion groups for all $i \in \mathbb{Z}$ (see exercise 2 of section 3 of the last chapter of [Br]). In fact K. Brown (§4 and §5 of [Br]) observed that the Farrell cohomology groups $\hat{H}^i(\Gamma, M)$ reflect many properties related to the finite subgroups of $\Gamma$. In particular we have $\hat{H}^i(\Gamma, M) = 0$ for all $i$ if $\Gamma$ is torsion-free. K. Brown's precise result is as follows.

(0.1) Theorem: Let $\Gamma$ be a group such that $\text{v.c.d.}(\Gamma) < \infty$, let $p$ be a prime, and $A$ be the ordered set of non-trivial elementary abelian $p$-subgroups of $\Gamma$. Let $\Gamma$ act on $A$ by conjugation. Let $M$ be any $\Gamma$ module. Then the canonical map

$$\hat{H}^i(\Gamma, M)_{(p)} \xrightarrow{\cong} \hat{H}^*_\Gamma(A, M)_{(p)}$$

is an isomorphism.

Thus we see that the $p$-torsion in $\hat{H}^*(\Gamma, M)$ depends only on the normalizers of the elementary abelian $p$-subgroups of $\Gamma$ and how they fit together. One special case where $\hat{H}^*_\Gamma(A, M)$ can be computed easily is that when $A$ is discrete. In that case the above theorem yields:

(0.2) Corollary: Suppose that every elementary abelian $p$-subgroup of $\Gamma$ has rank $\leq 1$. Then there is a canonical isomorphism

$$\hat{H}^*(\Gamma, M)_{(p)} \xrightarrow{\cong} \prod_{P \in \mathcal{P}} \hat{H}^*(N(P), M)$$

where $\mathcal{P}$ is a set of representatives for the conjugacy classes of subgroups of $\Gamma$ of order $p$ and $N(P)$ denotes the normalizer of the group $P$ in $\Gamma$. 
Explicit cohomology calculations using $A$ have been made by Ash in [Ash], in the case where $\Gamma = \text{GL}(n, \mathbb{Z})$ and $n < (2p - 2)$. In those cases the elementary abelian $p$-subgroups of $\Gamma$ have rank $\leq 1$ and hence $A$ is 0-dimensional and the above corollary can be applied to compute the mod-$p$ Farrell cohomology of $\text{GL}(n, \mathbb{Z})$, $n < (2p - 2)$. In [AM] Ash and McConnell have shown that the size of $\check{H}^*(\text{GL}(n, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$ grows faster than exponentially as a function of $p$. They have also shown that in positive dimensions below the v.c.d. roughly half of the Farrell cohomology lifts to the ordinary cohomology.

For other concrete examples where (0.1) and (0.2) lead to an explicit cohomology calculation see [Br1]. This normalizer approach has also been used in the $p$-rank 1 case to determine the Farrell cohomology of certain mapping class groups (see [X1] and [X2]). We also mention that recently H.-W. Henn [H1] has figured out a spectral sequence which relates the cohomology of the centralizers of the nontrivial elementary abelian $p$-subgroups of $\Gamma$ to that of $\Gamma$, unlike K. Brown’s spectral sequence which involves the cohomology of the normalizers of the elementary abelian $p$-subgroups of $\Gamma$. Using his centralizer approach in [H2] H.-W. Henn was able to calculate the mod-2 cohomology of the special linear group $\text{SL}(3, \mathbb{Z}[1/2])$.

In this thesis we use K. Brown’s spectral sequence to calculate the Farrell cohomology of $\Gamma = \text{GL}(2p - 2, \mathbb{Z})$. One of our main results says that the dimension of the Farrell (or ordinary) cohomology of $\text{GL}(2p - 2, \mathbb{Z})$ over $\mathbb{Z}/p\mathbb{Z}$ grows at least linearly with respect to $i$, for $i \gg 0$ (see the end of chapter 2). This result also follows from a theorem of Quillen which implies that the Krull dimension of the ring $H^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$ is 2. But in our treatment we can exhibit some nontrivial classes explicitly.

In the case where $\Gamma = \text{GL}(2p - 2, \mathbb{Z})$ every elementary abelian $p$-subgroup of $\Gamma$ has rank $\leq 2$ and hence $A$ has dimension 1. So the spectral sequence associated
with $\tilde{H}^p_\Gamma(A)(p)$ is non-trivial involving two columns and one differential. Thus to compute the spectral sequence we first need to figure out the fundamental domain for the action of $\Gamma$ on $A$. Since $\Gamma$ acts by conjugation on $A$, it is enough to find the conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$. By a theorem of Jordan-Zassenhaus [CR, page 534] the number of conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ is finite. In general for fixed $n$ and $p$, even though the number of conjugacy classes of elementary abelian $p$-subgroups of $GL(n, \mathbb{Z})$ is finite, it is quite hard to get hold on all of them, particularly on the conjugacy classes of elementary abelian $p$-subgroups of rank $> 1$. Moreover, this number could be very large in general. As can be seen from our case where we put $n = 2p - 2$, this number depends on the the Galois module structure of the class group of the $p$-th cyclotomic field.

The conjugacy classes of elementary abelian $p$-subgroups of $GL(n, \mathbb{Z})$ of rank 1 can be easily determined using Reiner's theorem on the integral representations of the cyclic group of order $p$. In our case these classes have been determined in section 1 of the first chapter. They are grouped together in three types (see end of section 1 of the first chapter). The normalizers of $p$-subgroups belonging to type 1 and type 3 have been found in section 2. The normalizers of type 2 subgroups are hard to describe and their isomorphism types remain an open question.

In section 3 of the first chapter we determined the conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ of rank 2. The key to this classification is lemma (3.1) which states that every such subgroup contains a type 1 or type 3 subgroup. As the reader can see from sections 1 and 3, the set of conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ depends not only on the class group of the cyclotomic field $Q(\zeta_p)/Q$, but also on the action of the Galois group $\Delta$ of $Q(\zeta_p)/Q$ on the class group. Here $\zeta_p$ is a primitive $p$-th root of unity. In section 4 we determined the
normalizers of the elementary abelian $p$-subgroups of rank 2. And finally in section 5 we give a description of the fundamental domain $\mathcal{A}/\Gamma$. In the case of primes when the class number of $Q(\zeta_p)$ is 1, this fundamental domain has a nice shape of the English capital letter $\Lambda$.

In the second chapter we first compute the cohomology of the stabilizers of the simplices of $\mathcal{A}/\Gamma$. Although the Farrell cohomology theory shares most of the formal properties analogous to those of the Tate cohomology theory for finite groups, the main tools in computing the cohomology explicitly such as the Hochschild-Serre spectral sequence or Kunneth formula do not carry over verbatim to the Farrell cohomology. These techniques work in Farrell cohomology only under suitable hypotheses (see for example exercises 4 and 5 of section 3 of the last chapter of [Br]). This is one of the reasons why we could not find the Farrell cohomology of the stabilizers explicitly. However to overcome this difficulty one can compute the ordinary cohomology and then use the fact that the Farrell cohomology is the same as the ordinary cohomology above the v.c.d. This is done in section 1 of the second chapter.

In section 2 we calculate the spectral sequence associated to $\hat{H}^*(\mathcal{A})$. These calculations are stymied by the fact that we do not know explicitly the Farrell cohomology in all dimensions of all the stabilizers involved in $\mathcal{A}/\Gamma$. As mentioned earlier we instead use the ordinary cohomology in dimensions greater than the v.c.d. of all the stabilizers and work out the spectral sequence in degrees greater than the v.c.d.'s of all the stabilizers. Thus the calculations are not complete. However in this calculation we are able to get hold of certain classes that survive in that spectral sequence, thus enabling us to get a lower bound on the $\mathbb{Z}/p\mathbb{Z}$-dimension of the space $\hat{H}^*(\text{GL}(2p - 2, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$. This lower bound is quite large if $\ast$ is large. From this lower bound it also follows that the $\mathbb{Z}/p\mathbb{Z}$-dimension of $\hat{H}^*(\text{GL}(2p - 2, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$
grows linearly with respect to $*$ and grows faster than exponentially as $p$ goes to infinity as in the previously mentioned work [AM] of Ash and McConnell.

Finally in the last chapter we compute explicitly the mod-3 Farrell cohomology $\hat{H}^q(GL(4, \mathbb{Z}), \mathbb{Z}/3\mathbb{Z})$ of $GL(4, \mathbb{Z})$ in degrees $q > 4$. In the course of these calculations we also calculate the mod-3 cohomology of $GL(2, \mathbb{Z}[\omega])$ which is now the centralizer of one of the vertex of $\mathcal{A}/\Gamma$. Here $\omega$ denotes a non-trivial cube root of unity. In [Al], Alperin has given a description of a two dimensional simplicial complex which has a natural structure for the action of $GL(2, \mathbb{Z}[\omega])$ and used it to compute the integral homology of $SL(2, \mathbb{Z}[\omega])$. His main result is,

$$H_n(SL(2, \mathbb{Z}[\omega]), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/3\mathbb{Z} & \text{if } n \equiv 1 \mod 4 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \\ \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & \text{if } n \equiv 3 \mod 4 \\ 0 & \text{if } n \equiv 4 \mod 4, n \neq 0. \end{cases}$$

(1)

We use the same complex to compute the mod-3 cohomology of $GL(2, \mathbb{Z}[\omega])$ (see section 3 of the third chapter). The result we got for the mod-3 cohomology of $GL(2, \mathbb{Z}[\omega])$ can be described as follows (see equation (211)). Let $k = \mathbb{Z}/3\mathbb{Z}$, then for $q \geq 0$ we have,

$$H^q(GL(2, \mathbb{Z}[\omega]), k) \cong \begin{cases} k^{(4m+1)} & \text{if } q = 4m \\ k^{(4m+1)} & \text{if } q = 4m + 1 \\ k^{(4m+1)} & \text{if } q = 4m + 2 \\ k^{(4m+3)} & \text{if } q = 4m + 3. \end{cases}$$

(2)

Finally in section 7 of the last chapter we compute K. Brown's spectral sequence $H^*(\mathcal{A})_p$ to obtain the mod-3 cohomology of $GL(4, \mathbb{Z})$.

The main result of this chapter is the following. Let $k = \mathbb{Z}/3\mathbb{Z}$, then for $q > 3$, we have,
\[ \hat{H}^q(GL(4, \mathbb{Z}), k) = \begin{cases} 
 k^{(2m+1)} & \text{if } q = 8m \\
 0 & \text{if } q = 8m + 1 \\
 k^{(2m)} & \text{if } q = 8m + 2 \\
 k^{(4m+1)} & \text{if } q = 8m + 3 \\
 k^{(2m+1)} & \text{if } q = 8m + 4 \\
 0 & \text{if } q = 8m + 5 \\
 k^{(2m+2)} & \text{if } q = 8m + 6 \\
 k^{(4m+3)} & \text{if } q = 8m + 7. 
\] (3)

Thus the mod-3 cohomology \( H^*(GL(4, \mathbb{Z}), k) \) of \( GL(4, \mathbb{Z}) \) increases linearly with respect to \( * \) in the congruence class mod 8, as long as \( * \) is different from 1 or 5 mod 8, in which cases it is identically zero. The only reason we can give for the vanishing of \( H^*(GL(4, \mathbb{Z}), k) \) when \( * \equiv 1 \) or 5 mod 8 is that the mod-3 cohomology of \( D_{12} \) and \( D_6 \) is identically zero in degrees 1 and 5 mod 8, and these groups are involved in the stabilizers of all the simplices in the fundamental domain of \( \mathcal{A} \) mod \( GL(4, \mathbb{Z}) \).

As mentioned earlier such explicit results for higher primes \( p > 3 \), will be harder because we can not easily get hold of the Farrell cohomology of all the stabilizers of vertices of the fundamental domain of \( \mathcal{A} \) mod \( GL(2p - 2, \mathbb{Z}) \). We hope to make some progress in these directions in the near future so that we can find the Farrell cohomology of all the stabilizers and then use the complex \( \mathcal{A} \) of elementary abelian \( p \)-subgroups of \( GL(2p - 2, \mathbb{Z}) \), as described in the section 5 of the first chapter, to compute some more classes in the mod-\( p \) cohomology of \( GL(2p - 2, \mathbb{Z}) \).
CHAPTER I

THE COMPLEX A

§1 Subgroups of order $p$ in $GL(2p - 2, \mathbb{Z})$

Let $\zeta$ be a primitive $p$-th root of unity where $p$ is an odd prime. Set $K = Q(\zeta)$, $\mathcal{O} = \mathbb{Z}[\zeta]$, $U$ the units in $\mathcal{O}$ and $\Gamma = GL(2p - 2, \mathbb{Z})$. Let $\Delta$ be the Galois group of $K/Q$ and $\mu$ the group of order $p$ generated by $\zeta$. Let $\text{Cl}$ denote the class group of $K$. We denote by capital letters $A, B, \ldots$ the elements of $\text{Cl}$ and use the letters $a, b, \ldots$ to denote the $\mathcal{O}$-ideal classes. For each ideal class $A \in \text{Cl}$, we fix an $\mathcal{O}$-ideal $a$ belonging to the ideal class $A$ and also a free $\mathbb{Z}$-basis for $a$.

First we describe all indecomposable $\mathbb{Z}_p$-modules which are free over $\mathbb{Z}$, up to isomorphism. First we have the trivial module $\mathbb{Z}$ and any fractional ideal $a$ of $K$. We can also form an indecomposable module $M$ out of $a$ and an arbitrary element $\alpha \in a$ as follows. Let $Z_\alpha$ be a free $\mathbb{Z}_p$-module of rank 1 and set $M = a \oplus Z_\alpha$ as abelian group. The group $\mu$ acts on $\alpha$ in the usual way, and we set $\zeta_\alpha = a + z$. We denote this module $M$ by the symbol $(\alpha, a)$. Now we need the following theorem due to Reiner, whose proof may be found in [CR page 729].

(1.1) Theorem (Reiner): Let $A$ ranges over a full set of representatives of the $\mathcal{O}$-ideal classes in the cyclotomic field $Q(\zeta)$. Then every $\mathbb{Z}_p$-module $M$ is expressible as a direct sum of indecomposable modules.
\[
M = \prod_{i=1}^{a} (a_i, a_i) \bigoplus \prod_{j=a+1}^{a+b} a_j \bigoplus \mathbb{Z}^{(c)}
\]

The genus invariant of \(M\) are the integers \(a, b, c\). The only additional invariant needed to determine the isomorphism class of \(M\) is the ideal class of the product \(\prod_{i=1}^{a+b} a_i\). Thus it is possible to choose all but one of the ideal \(a_i\)'s to be the trivial ideal \(\mathcal{O}\).

Here we recall that the two \(\mathbb{Z}\mu\)-modules \(M\) and \(N\) are placed in the same genus if \(M_i \cong N_i\) as \(\mathbb{Z}_{(l)}\mu\)-modules for each prime ideal \(l\) of \(\mathbb{Z}\), where \(\mathbb{Z}_{(l)}\) (resp. \(M_i\) and \(N_i\)) denotes the localization of \(\mathbb{Z}\) (resp. \(M\) and \(N\)) at \(l\).

Thus in the case of interest in this thesis where \(\text{rank}_\mathbb{Z} M = 2p - 2\), the modules \(\mathbb{Z}^{(2p-2)}, \mathcal{O} \oplus a, (a, a) \oplus \mathbb{Z}^{(p-2)}\), where \(a\) ranges over our chosen set of representatives of the \(\mathcal{O}\)-ideal classes, form a complete set of representatives of all the \(\mathbb{Z}\mu\)-modules up to isomorphism.

Next we proceed to describe a set of representatives for the conjugacy classes of subgroups of \(\Gamma\) of order \(p\). So from now on we assume \(M\) to be faithful, which eliminates the possibility \(M = \mathbb{Z}^{(2p-2)}\). Taking the fixed basis for \(a\) and the standard \(\mathbb{Z}\)-basis for the free part \(\mathbb{Z}^{p-1}\) or \(\mathbb{Z}^{p-2}\) we obtain an embedding \(\rho_M : \mu \to GL(2p - 2, \mathbb{Z})\). Given two such embeddings, say \(\rho_1\) and \(\rho_2\), possibly associated with different \(\mathbb{Z}\mu\)-modules \(M_1\) and \(M_2\), we wish to determine when their images in \(GL(2p - 2, \mathbb{Z})\) are conjugate.

\[\text{Im } \rho_1 \text{ is conjugate to Im } \rho_2\]

\[\iff \exists \gamma \in \Gamma \text{ such that } \text{Im } \rho_1 = \gamma (\text{Im } \rho_2) \gamma^{-1}\]

\[\iff \rho_1(\zeta^i) = \gamma \rho_2(\zeta) \gamma^{-1} \quad \text{for some } i \in (\mathbb{Z}/p\mathbb{Z})^x\]

\[\iff \rho_1(\sigma(\zeta)) = \gamma \rho_2(\zeta) \gamma^{-1} \quad \text{where } \sigma \in \Delta \text{ is defined by } \sigma(\zeta) = \zeta^i\]

\[\iff M_1^\sigma \text{ is isomorphic to } M_2 \text{ as } \mathbb{Z}\mu\text{-modules.}\]
Here $M_i^\sigma$ denotes $\mathbb{Z}_\mu$-module $M_1$ with $\zeta$ acting on $M_1$ through multiplication by $\sigma(\zeta)$ instead of $\zeta$ itself. Hence, to get a set of representatives for the conjugacy classes of subgroups of $\Gamma$ of order $p$, we restrict the $a$ to run over a set of representatives of the $\Delta$-orbits of $\text{Cl}$ rather than of $\text{Cl}$ itself.

Thus a set of representatives for the conjugacy classes of the subgroups of order $p$ of $\Gamma$ are in one-one correspondence with the following set of $\mathbb{Z}_\mu$-modules, which are free of rank $(2p - 2)$ over $\mathbb{Z}$.

1) $a \oplus \mathbb{Z}^{(p-1)}$

2) $O \oplus a$

3) $(a, a) \oplus \mathbb{Z}^{(p-2)}$

where $a$ ranges over our chosen set of representative ideals for the distinct ideal classes in $\text{Cl}/\Delta$.

(1.2) Notation: If $M = a$ an ideal in $O$, with $\mu$ acting by multiplication and with fixed basis as above we obtain an embedding $\rho_M : \mu \to GL(p - 1, \mathbb{Z})$. We denote by $[A]$ the image of $\zeta$ under $\rho_M$, where $A$ is the ideal class of $a$. Notice that the matrix $[A]$ depends on the choice of the $\mathbb{Z}$-basis for $a$ and also on the $O$-ideal in the ideal class $A$ of $a$. However it is well-defined upto conjugacy. But since we have fixed an $O$-ideal in each ideal class and a free $\mathbb{Z}$-basis for it, our notation $[A]$ is well defined.

Similarly, if $M = (a, a)$ then with respect to our chosen $\mathbb{Z}$-basis for $M$ we obtain an embedding $\rho_M : \mu \to GL(p, \mathbb{Z})$. Let $[(a, a)]$ denote the image of $\zeta$ under this embedding. Then it follows from the definition that

$$[(a, a)] = \begin{pmatrix} [A] & * \\ * & * \\ \vdots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

(5)
where *'s are certain integers. We denote this matrix by \([A, a]\) because by our choices it depends only on the ideal class \(A\) of the ideal \(a\).

So in matrix notation, a set of representatives for the conjugacy classes of subgroups of \(\Gamma\) of order \(p\) can now be written as

\[
\begin{align*}
\text{Type 1)} & \quad \left\langle \begin{pmatrix} [A] & 0 \\ 0 & I_{p-1} \end{pmatrix} \right\rangle \\
\text{Type 2)} & \quad \left\langle \begin{pmatrix} [A] & 0 \\ 0 & [A] \end{pmatrix} \right\rangle \\
\text{Type 3)} & \quad \left\langle \begin{pmatrix} [(A, a)] & 0 \\ 0 & I_{p-2} \end{pmatrix} \right\rangle
\end{align*}
\]

where \(A\) ranges over the set of representatives for \(\text{Cl}/\Delta\) and \(\Lambda\) denotes the trivial ideal class and \(I_n\) denotes the identity matrix of order \(n\).

(1.3) Definition: The above three subgroups of \(\Gamma\) of order \(p\) will be called the subgroups of type 1, 2, and 3 respectively, associated with (or corresponding to) the ideal class \(A\) of \(\text{Cl}\).

For each \(A \in \text{Cl}/\Delta\), the matrix \([A]\) is of order \(p\) in \(GL(p-1, \mathbb{Z})\) with the cyclotomic polynomial of order \(p\) as its minimal polynomial and \(\zeta, \zeta^2, \ldots, \zeta^{p-1}\) as its eigenvalues. Hence in the above list the non-trivial elements of subgroup of type 1 and type 3 have eigenvalue 1, while the non-trivial elements of subgroups of type 2 have no eigenvalue equal to 1. In fact over the field of rational numbers the subgroups of type 1 are conjugate to those of type 3. We will use this fact when we determine the conjugacy classes of elementary abelian \(p\)-subgroups of \(\Gamma\) of rank 2. By abuse of notation we will also refer to a non-trivial element of subgroup of type \(i\) as a type \(i\)-element.
Remark: As a consequence of Reiner's theorem we see that the conjugacy
classes of $p$-subgroups of $GL(p - 1, \mathbb{Z})$ are in one-one correspondence with the $\Delta$-
orbits of $Cl$. In fact they are $\langle [A] \rangle$, where $A \in Cl/\Delta$. Furthermore they have the
same genus invariant.

§2 Normalizers of groups of type 1 and type 3.

Let $M$ be a $\mathbb{Z}_{\mu}$-module free over $\mathbb{Z}$ of rank $(2p - 2)$. Choosing a $\mathbb{Z}$-basis
for $M$ we obtain an embedding $\rho_M : \mu \to \Gamma$. The normalizers of the image of
$\rho_M$ in $\Gamma$ have been found by Ash in [Ash] in cases when $M \cong a \oplus \mathbb{Z}^{p-1}$ or $M \cong
(a, a) \oplus \mathbb{Z}^{(p-2)}$. We restate his results.

Theorem (Ash): Let $M$ be a $\mathbb{Z}_{\mu}$-module free over $\mathbb{Z}$ of rank $n$, where $n <
(2p - 1)$. Assume that $M$ is either isomorphic to $a \oplus \mathbb{Z}^{(n-(p-1))}$ or $(a, a) \oplus \mathbb{Z}^{(n-p)}$.
Let the image of $\rho_M$ have normalizer $N$ and centralizer $C$ in $GL(n, \mathbb{Z})$. Denote by
$S$ the stabilizer of the ideal class $A$ of $a$ in the Galois group $\Delta$. Then there is an
exact sequence

$$1 \to C \to N \to S \to 1$$

To describe this further, we distinguish two cases:

1. $M \cong a \oplus \mathbb{Z}^{(n-(p-1))}$ and

2. $M \cong (a, a) \oplus \mathbb{Z}^{(n-p)}$.

Set $m = n - p + 1$. In Case 1 we set $\gamma = GL(m, \mathbb{Z})$ and, in Case 2, $\gamma =
\gamma(m, p)$. We adopt the notation $\gamma(m, p)$ for the subgroup of $GL(m, \mathbb{Z})$ whose first
row is congruent to $(\ast, 0, \ldots, 0)$ modulo $p$. We let $\lambda$ stand for the homomorphism
$\gamma(m, p) \to (\mathbb{Z}/p\mathbb{Z})^\times$ sending such an element to $\ast$ modulo $p$.

In either cases let $N_0$ be the semidirect product of $U \times \gamma$ and $S$, where $S$ acts
trivially on $\gamma$ and via the Galois action on $U$. Thus, the group law in $N_0$ is given
by the formula \((\xi, \delta, \sigma)(\xi', \delta', \tau) = (\xi\sigma(\xi'), \delta\delta', \sigma\tau)\). Then \(N\) is a subgroup of \(N_0\) and the map from \(N\) to \(S\) is induced by the obvious projection. Finally we have

Case 1: \(N \cong N_0\).

Case 2: \(N \cong \{(\xi, \delta, \sigma) \in N_0 | \xi \equiv \lambda(\delta)s \mod (\zeta - 1)\}\), where \(\sigma\) and \(s\) are related by the formula \(\sigma(\zeta) = \zeta^s\) for \(s \in (\mathbb{Z}/p\mathbb{Z})^\times\).

Although Ash proved this theorem in the cases where \(n < 2p - 2\), his proof is still valid for arbitrary \(n\). We also remark that \(\gamma\) is embedded into the automorphism group of \(M\) in a natural way: in Case 1 as the automorphism group of \(\mathbb{Z}^{(n-(p-1))}\) and in Case 2 as a subgroup of the automorphism group of the \(\mathbb{Z}\)-span of \(z\) and \(\mathbb{Z}^{(n-p)}\). Also under the above isomorphism \(N \cong N_0\) the \(p\)-elements

\[
\begin{pmatrix} [A] & 0 \\ 0 & I_{p-1} \end{pmatrix} \quad \begin{pmatrix} [(A, a)] & 0 \\ 0 & I_{p-2} \end{pmatrix}
\]

of \(\text{Im } \rho_M\) in \(\Gamma\) in Case 1 and Case 2 respectively are mapped to the element \((\zeta, I_{p-1}, 1)\) of \(N_0\).

In the case of type 2 subgroups, we do have the exact sequence like (6), however the centralizers are hard to describe except in the case of \(\begin{pmatrix} [A] & 0 \\ 0 & [A] \end{pmatrix}\), where we have the following lemma.

(2.2) Lemma: The centralizer \(C\) of \(\begin{pmatrix} [A] & 0 \\ 0 & [A] \end{pmatrix}\) in \(\Gamma\) is isomorphic to \(GL(2, \mathbb{Z}[\zeta])\) and we have an exact sequence,

\[
1 \to GL(2, \mathbb{Z}[\zeta]) \to N \to \Delta \to 1
\]  

(7)

with \(\Delta\) acting via its Galois action on \(GL(2, \mathbb{Z}[\zeta])\).
Proof: First we observe that there is a natural isomorphism from the normalizer $N$ to the group $\{\phi \in \text{Isom}_Z(\mathcal{O}, \mathcal{O}) | \phi \cdot \zeta \cdot \phi^{-1} = \sigma(\zeta) \text{ for some } \sigma \in \Delta \}$ which maps the centralizer $C$ to the subgroup of $\phi$'s for which $\sigma = 1$. We identify $N$ and $C$ with these sets for the rest of the proof. Further in the above equation $\phi \cdot \zeta \cdot \phi^{-1} = \sigma(\zeta)$ the $\zeta$ (resp. $\sigma(\zeta)$) stands for the multiplication map by $\zeta$ (resp. $\sigma(\zeta)$) on $\mathcal{O}$.

Since $\sigma(\mathcal{O}) = \mathcal{O}$ for all $\sigma \in \Delta$, we have $\sigma \cdot \zeta \cdot \sigma^{-1} = \sigma(\zeta)$ on $\mathcal{O}$. Thus it follows that $N/C \cong \Delta$. Next we determine the centralizer $C$. Any matrix $\mathbf{T}$ in $\Gamma$ can be written in the form

$$\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

(8)

where $a, b, c, d$ are matrices in $M_{(p-1) \times (p-1)}(\mathbb{Z})$. Then it can be seen that $\mathbf{T} \in C$ if and only if the matrices $a, b, c, d$ commute with the matrix $[\Lambda]$. Again there is a natural isomorphism from the ring consisting of the matrices in $M_{(p-1) \times (p-1)}(\mathbb{Z})$ which commute with $[\Lambda]$ to the ring of homomorphisms $\{\phi \in \text{Hom}_Z(\mathcal{O}, \mathcal{O}) | \phi \cdot \zeta \cdot \phi^{-1} = \zeta\}$. The latter ring is easily seen to be isomorphic to the ring $\mathcal{O}$ of integers. Hence we conclude that $C \cong GL(2, \mathcal{O})$. □

Next we determine the v.c.d. of the stabilizers of type 1, 2 and 3 subgroups of $\Gamma$. Since all the subgroups belonging to type 1 and type 3 are conjugate over $Q$, the v.c.d. of the stabilizers of all type 1 and type 3 subgroups is the same and is equal to the v.c.d. of their centralizer because the centralizer has a finite index in the stabilizers. By theorem (2.1) the centralizers of type 1 are isomorphic to $U \times GL(p-1, \mathbb{Z})$, hence it follows that the v.c.d. of the stabilizers of type 1 and 3 subgroups of $\Gamma$ is equal to $\text{v.c.d.}(U) + \text{v.c.d.}(GL(p-1, \mathbb{Z}))$. By Dirichlet's Unit theorem $U \cong \mathbb{Z}/2p\mathbb{Z} \times \mathbb{Z}^{(p-3)/2}$. It is well-known that the v.c.d.($U$) = $(p-3)/2$ (see example 5 on page 185 of [Br]) and v.c.d.($GL(p-1, \mathbb{Z})$) = $(p-1)(p-2)/2$ (see
example 5 on page 218 of [Br]). Thus we conclude that the v.c.d. of the stabilizers of type 1 and 3 subgroups of $\Gamma$ is equal to

$$v.c.d.(U) + v.c.d.(GL(p - 1, \mathbb{Z})) = \frac{p - 3}{2} + \frac{(p - 1)(p - 2)}{2}. \quad (9)$$

Similarly all the subgroups of type 2 are conjugate to each other over $Q$, hence the v.c.d. of the stabilizers of all type 2 subgroups of $\Gamma$ is the same and again it is equal to the v.c.d. of their centralizer because the centralizer has a finite index in the stabilizer. By lemma (2.2) the centralizer of

$$\begin{pmatrix} [\Lambda] & 0 \\ 0 & [\Lambda] \end{pmatrix}$$

is isomorphic to $GL(2, \mathcal{O})$. To find the v.c.d. of $GL(2, \mathcal{O})$ we proceed as follows. The group $GL(2, \mathcal{O})$ can be considered as a subgroup of $GL_2(Q(\zeta) \otimes \mathbb{R}) \cong (GL_2(\mathbb{C}))(p-1)/2$. If $K$ is a maximal compact subgroup of $GL_2(\mathbb{C})$, then the symmetric space $GL_2(\mathbb{C})/K$ has dimension 4. Thus the dimension $d$ of the corresponding symmetric space for $GL(2, \mathcal{O})$ is equal to $4(p - 1)/2$. Further the $Q$-rank $l$ of $GL(2, \mathcal{O})$ is 2. Hence by the result of Borel and Serre [BS] the v.c.d. of $GL(2, \mathcal{O})$ is equal to $d - l = 2(p - 2)$. Thus we conclude that the v.c.d. of the stabilizers of all the type 2 subgroups of $\Gamma$ is equal to

$$v.c.d.(GL(2, \mathcal{O})) = d - l = 2(p - 2). \quad (10)$$

§ 3 Elementary abelian $p$-subgroups of $\Gamma$ of rank 2.

To find $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroup inside $\Gamma$ we need to find two cyclic subgroups of order $p$ of $\Gamma$ such that their elements commute with each other. The next lemma shows that we can always pick one of the generators of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \subset \Gamma$ to be of type 1 or type 3.
Lemma: Every subgroup $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma = GL(2p - 2, \mathbb{Z})$ contains an element of type 1 or type 3.

Proof: Let $a, b$ be two generators of $G$. Then both $a$ and $b$ have order $p$ and they commute with each other. Suppose none of them is of type 1 or 3. Then they are of type 2 and have eigenvalues $\zeta, \zeta^2, \ldots, \zeta^{(p-1)}$ repeated twice. Since they commute, we can simultaneously diagonalize them over $\mathbb{C}$, the field of complex numbers. So after diagonalization assume $a$ and $b$ takes the form,

$$a = \begin{pmatrix}
\zeta & \zeta^2 & \cdots & \zeta^{(p-1)} \\
0 & \zeta & \zeta^2 & \cdots & \zeta^{(p-1)} \\
0 & 0 & \zeta & \zeta^2 & \cdots & \zeta^{(p-1)} \\
\end{pmatrix} \quad (11)$$

$$b = \begin{pmatrix}
\zeta^i & 0 \\
0 & \zeta^i & 0 \\
0 & 0 & \zeta^i & 0 \\
\end{pmatrix} \quad (12)$$

where $i \in \mathbb{Z}/p\mathbb{Z}$. Then

$$a^{-i}b = \begin{pmatrix}
1 & 0 \\
* & * \\
0 & * \\
\end{pmatrix} \quad (13)$$

The matrix on the right hand side of the above equation cannot be the identity matrix for that would imply $b = a^i$ and this contradicts the fact that $a$ and $b$ are the generators of $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Thus $a^{-i}b$ is a non-trivial element of $G$ with an eigenvalue equal to 1, hence it must be of type 1 or type 3. \qed

To determine a set of representatives for the conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ of rank 2, we start with a subgroup $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma$. By the previous lemma $G$ contains an element $a$ of type 1 or type 3. We treat these two cases separately.
CASE 1: Assume $G$ has an element $a$ of type 1. Since we are interested only in the conjugacy class of $G$ we can assume that

$$a = \begin{pmatrix} [A] & 0 \\ 0 & I \end{pmatrix}$$

(14)

for some $A$ in the $\Delta$-orbit of $C_l$ and from now on the letter $I$ will denote the identity matrix of order $p - 1$. Since $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, a other generator of $G$ is of order $p$ and it commutes with $a$. Hence to find an another generator for $G$ we must look for an element of order $p$ inside the centralizer of $a$ in $\Gamma$. Again we remark, since we are only interested in the conjugacy class of $G$, it is enough to find the conjugacy classes of subgroups of order $p$ of the centralizer of $a$. Recall from section 2, that the centralizer of $a$ in $\Gamma$ is

$$C(a) \cong \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u \in U, v \in GL(p - 1, \mathbb{Z}) \right\}$$

and under the above isomorphism the element $a$ is mapped onto the element

$$\begin{pmatrix} \zeta & 0 \\ 0 & I \end{pmatrix}.$$

By the remark (1.4) at the end of section 1 the conjugacy classes of subgroup of order $p$ of $GL(p - 1, \mathbb{Z})$ are given by,

$$\{ [B] \mid B \in C_l/\Delta \}$$

Hence, upto conjugacy inside the centralizer of $a$, the choices for the other cyclic subgroup of $G$ of order $p$ are

$$\left\langle \begin{pmatrix} I & 0 \\ 0 & [B] \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} [A] & 0 \\ 0 & [B] \end{pmatrix} \right\rangle$$
where $B$ ranges over the set of $\Delta$-orbits of $\text{Cl}$. Note here that the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ subgroup generated by

$$\begin{pmatrix} [A] & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} [A] & 0 \\ 0 & [B] \end{pmatrix}$$

and

$$\begin{pmatrix} [A] & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & [B] \end{pmatrix}$$

are the same, hence we can always assume that the top-left entry of the matrix of the other generator is $I$.

Thus starting with the assumption that the one generator of the subgroup $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma$ is

$$\begin{pmatrix} [A] & 0 \\ 0 & I \end{pmatrix}$$

we have shown that, the other generator, upto conjugacy inside the centralizer of $a$ could be

$$\begin{pmatrix} I & 0 \\ 0 & [B] \end{pmatrix}$$

where $A, B$ ranges over the set of $\Delta$-orbit of $\text{Cl}$. If we enumerate the set of $\Delta$-orbit of $\text{Cl}$, say, $A_1, \ldots, A_n$, then under the assumption that the subgroup $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma$ contains an element of type 1, we have shown that $G$ is conjugate to one of the subgroup in the following list.

$$\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle \quad i, j = 1, \ldots, n$$

It is possible that some of the subgroups in the above list may be conjugate, however we have at least narrowed down our search to the above finite list in this case. First of all we see that if $i \neq j$, then the two subgroups in the above list

$$\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle$$
are conjugate by the matrix

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

which takes the two generators of the first group to the two generators of the second group under the action of conjugacy. So we further restrict our attention to the following sublist.

\[
G_{ij} = \left\langle \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle \right\rangle \quad 1 \leq i \leq j \leq n \tag{15}
\]

We claim that no two distinct subgroups in this sublist are conjugate in \( \Gamma \). To prove this we first note that any of these subgroups say,

\[
G_{ij} = \left\langle \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle \right\rangle \tag{16}
\]

with \( i \leq j \) have \((p + 1)\) cyclic subgroup of order \( p \), and they are

\[
H_{ij} = \left\langle \left\langle \begin{pmatrix} A_i & 0 \\ 0 & I \end{pmatrix} \right\rangle \right\rangle \tag{17}
\]

\[
K_{ij} = \left\langle \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle \right\rangle \tag{18}
\]

\[
\left\langle \left\langle \begin{pmatrix} A_i & 0 \\ 0 & [A_j]^r \end{pmatrix} \right\rangle \right\rangle, \quad r = 1, \ldots, (p - 1).
\]
The first two subgroups in the above list are of type 1 and the remaining sub-
groups are of type 2. We call the first two subgroups $H_{ij}, K_{ij}$ the special sub-
groups of $G_{ij}$. Since type is preserved under conjugation, the two special subgroups
of $G_{ij}$ must be mapped to the corresponding two special subgroups of another
$G_{ik} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ subgroup if $G_{ij}$ is to be conjugate to $G_{ik}$ in $\Gamma$. But since
$A_1, \ldots, A_n$ form a complete list of $\Delta$-orbits of $\text{Cl}$, none of the distinct $p$-subgroups

$$\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \quad i = 1, \ldots, n$$

are conjugate. Thus, if $G_{ij}$ and $G_{ik}$ with $i \leq j, l \leq k$ are conjugate in $\Gamma$, then
from the above discussion it follows that $i = l$ and $j = k$.

Thus in this case the set

$$G_{ij} = \left\{ \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \right\rangle \mid 1 \leq i < j \leq n \right\}$$

form a complete set of representatives for the conjugacy classes of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-
subgroups of $\Gamma$ which contain an element of type 1.

(3.2) Remark: We note that none of these groups $G_{ij}$ contain an elements of type
3. In fact it is easy to see that the centralizer of any type 1 element does not contain
an element of type 3 because all the elements in the centralizer are in block diagonal
form, with both blocks of size $p - 1$.

CASE 2: In this case we assume that the subgroup $G' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma$
contains an element $a'$ of type 3. We proceed as in the previous case. We can assume that

$$a' = \begin{pmatrix} [(a, a)] & 0 \\ 0 & I_{p-2} \end{pmatrix} = \begin{pmatrix} [A] & * \\ 0 & I \end{pmatrix}$$

(20)
for some $A \in \text{Cl}/\Delta$. Here the $\mathcal{O}$-ideal $a$ is our chosen representative from the ideal class $A$ and the matrix $* \in M_n(\mathbb{Z})$ is uniquely determined in terms of our chosen basis of $a$.

To find another generator of $G'$ we need to find conjugacy classes of subgroups of order $p$ of the centralizer $C(a')$ of $a'$. Recall that the centralizer of $a'$ is

$$C(a') \cong \left\{ (u, \gamma) \mid u \in U, \delta \in \gamma(p - 1, p) \text{ and } u \cong \lambda(\delta) \mod (\zeta - 1) \right\} \quad (21)$$

Under this isomorphism the element $a'$ is mapped to the element $(\zeta, I)$. Thus to find the conjugacy classes of subgroups of order $p$ of $C(a')$ we first need to find the conjugacy classes of subgroups of order $p$ of $\gamma(p - 1, p)$. Recall that $\gamma(p - 1, p)$ is the subgroup of $GL(p - 1, \mathbb{Z})$ whose first row is congruent to $(\ast, 0, \ldots, 0)$ modulo $p$.

We already know that the conjugacy classes of subgroups of order $p$ of $GL(p - 1, \mathbb{Z})$ are in one-one correspondence with the $\Delta$-orbit Cl. We show that the subgroup $\gamma(p - 1, p)$ of $GL(p - 1, \mathbb{Z})$ behaves exactly the same. That is we does not loose or find any old or new conjugacy classes of cyclic $p$-subgroups as we pass from $GL(p - 1, \mathbb{Z})$ to its subgroup $\gamma(p - 1, p)$.

(3.3)Lemma: The conjugacy classes of subgroups of order $p$ of $\gamma(p - 1, p)$ are in one-one correspondence with the $\Delta$-orbits of Cl.

Proof: From remark (1.4) we know that the conjugacy classes of the $p$-subgroups of $GL(p - 1, \mathbb{Z})$ are given by

$$\{\langle [A] \rangle \mid A \in \text{Cl}/\Delta\}$$

where $\langle [A] \rangle$ denotes the image of $\mu$ under the embedding $\rho_A : \mu \to GL(p - 1, \mathbb{Z})$ obtain by taking our fixed $\mathbb{Z}$-basis of the $\mathbb{Z}^\mu$-module $M = a$. It is not at all necessary that the matrix $[A]$ lies in $\gamma(p - 1, p)$. However we shall show that some
conjugate of $[A]$ lies in $\gamma(p - 1, p)$. This will prove that for each conjugacy class of the subgroup of order $p$ of $GL(p - 1, \mathbb{Z})$, there is a representative of that class in $\gamma(p - 1, p)$. To prove this we recall by Reiner's theorem, that all the conjugacy classes of $p$-subgroups of $GL(p - 1, \mathbb{Z})$ are in one genus, i.e., the subgroups $\langle [A] \rangle$ as $A$ ranges over $\text{Cl}/\Delta$ are all conjugate to each other in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$ after reduction modulo $p$. In particular they are all conjugate to the subgroup $\langle [\Lambda] \rangle$ of $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$, where $\Lambda$ denote the trivial ideal class of $\mathcal{O}$ and $[A]$ denote the matrix obtain from the matrix $[A]$ by reducing its entries modulo $p$. By choosing the $\mathbb{Z}$-basis $\zeta, \zeta^2, \ldots, \zeta^{p-1}$ for the trivial ideal class $\mathcal{O}$ we see that the matrix $[A]$ is a cyclic matrix, i.e., it has a cyclic basis and its minimal polynomial is the cyclotomic polynomial $x^{p-1} + x^{p-2} + \ldots + 1$. It follows that the matrix $[\Lambda]$ is also cyclic and its minimal polynomial is $(x - 1)^{p-1} \in \mathbb{Z}/p\mathbb{Z}[x]$. Thus the Jordan Canonical form of $[\Lambda]$ in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$ is

$$
\bar{J} = \begin{pmatrix}
1 & & & & 0 \\
1 & 1 & & & \\
1 & 1 & 1 & & \\
& & & & \\
0 & & & 1 & 1 \\
\end{pmatrix}
$$

and it has only one eigenvalue namely $x = 1$ and the corresponding eigenspace has dimension 1 over $\mathbb{Z}/p\mathbb{Z}$. If we let $\bar{J}$ act from right on the row vectors

$$(x_1, \ldots, x_{p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$$

then the eigenspace corresponding to the eigen value $x = 1$ is given by

$$\{(x, 0, \ldots, 0) | x \in \mathbb{Z}/p\mathbb{Z}\}.$$
Thus all the matrices $\overline{A}$ where $A \in \text{Cl}/\Delta$ are conjugate to $\overline{J}$ in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$, i.e., there exist a matrix $X \in GL(p - 1, \mathbb{Z}/p\mathbb{Z})$ such that $X \overline{A} X^{-1} = \overline{J}$. Let

$$Z = \begin{pmatrix} x^{-1} & 0 \\ 1 & \ddots \\ 0 & \cdots & 1 \end{pmatrix}$$

be a diagonal matrix in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$, where $x = \det(X) \in (\mathbb{Z}/p\mathbb{Z})^\times$. Then the matrix $ZX$ has determinant 1 and we have $ZX \overline{A} (ZX)^{-1} = Z \overline{J} Z^{-1}$.

Since the reduction mod-$p$ map

$$SL(p - 1, \mathbb{Z}) \xrightarrow{\text{mod } p} SL(p - 1, \mathbb{Z}/p\mathbb{Z})$$

is surjective, there exists a matrix $Y \in SL(p - 1, \mathbb{Z})$ such that $\overline{Y} = ZX$. Then we get $Y \overline{A} Y^{-1} = Z \overline{J} Z^{-1}$. It can be easily seen that the first row of the matrix $Z \overline{J} Z^{-1}$ is equal to $(1, 0, 0, \ldots, 0)$. Hence it follows that $Y \overline{A} Y^{-1}$ lies in $\gamma(p - 1, p)$.

Thus starting with the conjugacy class $[A]$ of $p$-subgroups of $GL(p - 1, \mathbb{Z})$ we have shown that there exists a $p$-subgroup $\langle Y[A]Y^{-1} \rangle$ in that conjugacy class which lies in $\gamma(p - 1, p)$.

Next we show that the conjugacy class of the subgroup $\langle Y[A]Y^{-1} \rangle$ in $\gamma(p - 1, p)$ is uniquely determined by that of $[A]$ in $GL(p - 1, \mathbb{Z})$. This will follow if we show that if the two cyclic subgroups of $\gamma(p - 1, p)$ of order $p$ are conjugate in $GL(p - 1, \mathbb{Z})$ then they are in fact conjugate in $\gamma(p - 1, p)$. We will prove in fact a stronger result that if two such subgroups of $\gamma(p - 1, p)$ are conjugate by the matrix $\omega$ in $GL(p - 1, \mathbb{Z})$ then $\omega$ lies $\gamma(p - 1, p)$. This is proved in the following lemma (3.4).

Thus we get a bijection from the set of conjugacy classes of $p$-subgroups of $GL(p - 1, \mathbb{Z})$ to the set of conjugacy classes of $p$-subgroups of $\gamma(p - 1, p)$. This proves the lemma. □
Lemma: Let $\alpha$ and $\beta$ be two elements of $\gamma(p-1, p)$ of order $p$. If there exists a matrix $\omega \in GL(p - 1, \mathbb{Z})$ such that $\omega \alpha \omega^{-1} = \beta$, then $\omega \in \gamma(p-1, p)$.

Proof: We know that under reduction modulo $p$, $\bar{\alpha}$ and $\bar{\beta}$ are conjugate in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$ and have only one eigenvalue namely $x = 1$ with corresponding eigenspace of dimension 1. Since $\alpha, \beta$ lies in $\gamma(p - 1, p)$ it is easy to see that the eigenspace for both $\bar{\alpha}$ and $\bar{\beta}$ is $\{(x, 0, \ldots, 0) | x \in \mathbb{Z}/p\mathbb{Z}\}$ where both $\bar{\alpha}$ and $\bar{\beta}$ acts from right on the row vectors $(x_1, \ldots, x_{p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$. Since $\omega \alpha = \beta \omega$, for $(1, 0, \ldots, 0) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$, we have

\[(1, 0, \ldots, 0)\bar{\omega} \bar{\alpha} = (1, 0, \ldots, 0)\bar{\beta} \bar{\omega} = (1, 0, \ldots, 0)\bar{\omega} \]

This implies $(1, 0, \ldots, 0)\bar{\omega}$ is an eigenvector for $\bar{\alpha}$, hence

\[(1, 0, \ldots, 0)\bar{\omega} = (y, 0, \ldots, 0) \]

for some $y \in \mathbb{Z}/p\mathbb{Z}$, but

\[(1, 0, \ldots, 0)\bar{\omega} = (a_1, a_2, \ldots, a_{p-1}) \]

where $(a_1, a_2, \ldots, a_{p-1})$ is the first row of $\omega$. The last two equations now gives $a_i = 0$ for $i = 2, \ldots, (p - 1)$, which implies $\omega$ lies in $\gamma(p-1, p)$. \qed

Remark: From now on we assume that for each representative $A$ of a $\Delta$-orbit of $\text{Cl}$, the free $\mathbb{Z}$-basis that was chosen in the beginning for our representatives ideal $a$ of the ideal class $A$, is such that the matrix $[A]$ lies in $\gamma(p - 1, p)$. We can do this because of the lemma (3.3). Then the conjugacy classes of $p$-subgroups of $\gamma(p - 1, p)$ are in one-one correspondence with the set

\[\{([A]) | A \in \Delta/\text{Cl}\}\]
Furthermore upon reduction modulo $p$, all these subgroups are conjugate to $\langle \overline{J} \rangle$ in $GL(p - 1, \mathbb{Z}/p\mathbb{Z})$ and in particular we have $\lambda([A]) = 1$ for all $A \in \text{Cl}/\Delta$, where $\lambda$ denotes the homomorphism $\lambda : \gamma(p - 1, p) \to (\mathbb{Z}/p\mathbb{Z})^\times$ described in the theorem (2.1).

Now that we know the conjugacy classes of the $p$-subgroups of $\gamma(p - 1, p)$, we are ready to find the conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ of rank 2. We recall our set-up. Let $G' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ be a subgroup of $\Gamma$ which contain an element $a'$ of type 3. Assume,

$$a' = \begin{pmatrix} [A] & * \\ 0 & 1 \end{pmatrix}$$

for some $A \in \Delta/\text{Cl}$. Then the centralizer of $a'$ is

$$C(a') \cong \left\{ (u, \gamma) \mid u \in U, \gamma \in \gamma(p - 1, p) \text{ and } u \equiv \lambda(\gamma) \mod (\zeta - 1) \right\}$$

Since for any $s \in (\mathbb{Z}/p\mathbb{Z})^\times$ we can always find a unit $u \in U$ such that $u \equiv s \mod (\zeta - 1)$ (for instance take the cyclotomic unit $u = 1 + \zeta + \ldots + \zeta^{s-1}$) we claim that the conjugacy classes of the cyclic $p$-subgroups of $C(a')$ are

$$\langle (\zeta, 1) \rangle, \quad \langle (1, [B]) \rangle, \quad \langle ([\zeta, B]) \rangle \text{ where } B \in \Delta/\text{CL}$$

For if $(G, H)$ is a cyclic subgroup of $C(a')$ of order $p$, then clearly $G = \langle \zeta \rangle$ or $G = \langle \zeta \rangle$. If $H = 1$ then $G$ has to be $\langle \zeta \rangle$, and in that case $(G, H) = \langle (\zeta, 1) \rangle$. So assume $H \not= 1$, then $H$ is a cyclic subgroup of $\gamma(p - 1, p)$ of order $p$ and hence by lemma (3.3) $H$ is conjugate to $\langle [B] \rangle$ in $\gamma(p - 1, p)$ for some $B \in \text{Cl}/\Delta$. Thus there exists a matrix $\gamma \in \gamma(p - 1, p)$ such that $\gamma H \gamma^{-1} = \langle [B] \rangle$. Let $\lambda(\gamma) = s$. 
Let $u = 1 + \zeta + \ldots + \zeta^{n-1}$ be a cyclotomic unit in $U$, then $u = \lambda(\gamma) \mod (\zeta - 1)$.

Thus $(u, \gamma) \in C(a')$ and

$$(u, \gamma)(G, H)(u, \gamma)^{-1} = (G, \langle [B] \rangle)$$  \hspace{1cm} (29)

Since $G = 1$ or $G = \langle \zeta \rangle$ our claim follows.

Thus the choices for the other cyclic $p$-subgroup of $G'$ up to conjugacy in $C(a')$ are

$$\langle 1, [B] \rangle, \quad \langle \zeta, [B] \rangle \text{ where } B \in \text{Cl}/\Delta$$

Since the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroup generated by

$$\langle (\zeta, 1) \rangle \text{ and } \langle (1, [B]) \rangle$$

and

$$\langle (\zeta, 1) \rangle \text{ and } \langle (\zeta, [B]) \rangle$$

are the same, we see that the up to conjugacy $G'$ has the form

$$\langle (\zeta, 1) \rangle \times \langle (1, [B]) \rangle$$

for some $B \in \text{Cl}/\Delta$.

(3.6) Remark: Let a $G'$ be an elementary abelian $p$-subgroup of $\Gamma$ of rank 2 which contains a cyclic $p$-subgroup of type 3, say,

$$\left\langle \begin{pmatrix} [A] & * \\ 0 & I \end{pmatrix} \right\rangle.$$  

If we know that there is another cyclic $p$-subgroup $H'$ of $G'$ which is conjugate to type 3 subgroup, say to

$$\left\langle \begin{pmatrix} [B] & * \\ 0 & I \end{pmatrix} \right\rangle,$$
for some $B \in \text{Cl}/\Delta$ then in fact, we have proved there is a matrix $\phi$ in the centralizer of
\[
\left\langle \begin{pmatrix} [A] & * \\ 0 & I \end{pmatrix} \right\rangle
\]
such that
\[
\phi H' \phi^{-1} = \left\langle \begin{pmatrix} I & * \\ 0 & [B] \end{pmatrix} \right\rangle.
\] (30)
The important point here is that $\phi$ lies in the centralizer of
\[
\left\langle \begin{pmatrix} [A] & * \\ 0 & I \end{pmatrix} \right\rangle.
\]
If we enumerate the $\Delta$-orbits of Cl, say $A_1, \ldots, A_n$ and revert back to our matrix notation, we have proved that if a subgroup $G' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\Gamma$ contain an element of type 3, then $G'$ is conjugate to one of the subgroups in the following list.
\[
G'_{ij} = \left\langle \begin{pmatrix} [A_i] & * \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & * \\ 0 & [A_j] \end{pmatrix} \right\rangle \quad i, j = 1, \ldots, n
\] (31)
It is possible that some of the subgroups in the above list may be conjugate. In fact the two subgroups
\[
G'_{ij} = \left\langle \begin{pmatrix} [A_i] & * \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & * \\ 0 & [A_j] \end{pmatrix} \right\rangle
\] (32)
\[
G'_{ji} = \left\langle \begin{pmatrix} [A_j] & * \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & * \\ 0 & [A_i] \end{pmatrix} \right\rangle
\] (33)
in the above list are conjugate. To see this we note that these subgroups say for example, the group $G'_{ij}$ has $(p + 1)$-subgroups of order $p$, and they are
\[
H'_{ij} = \left\langle \begin{pmatrix} [A_i] & * \\ 0 & I \end{pmatrix} \right\rangle
\] (34)
\[
K'_{ij} = \left\langle \begin{pmatrix} I & * \\ 0 & [A_j] \end{pmatrix} \right\rangle
\] (35)
The first $p$-subgroup $H'_{i,j}$ is of type 3 by assumption. Non-trivial elements of the second subgroup $K'_{i,j}$ have an eigenvalue 1 and hence $K'_{i,j}$ must be of type 1 or 3. It can not be of type 1 for in that case the conjugacy class of $G'_{i,j}$ would have been detected in Case 1. Also we know from Case 1 (see remark (3.2)) that if a rank-2 elementary abelian $p$-subgroup of $\Gamma$ contains an element of type 1 then it can not have an element of type 3. Thus the subgroup $K'_{i,j}$ must be of type 3. Also it is easy to see that the remaining $(p - 1)$ subgroups of $G'_{i,j}$ are of type 2. This also shows that subgroups $G'_{i,j}$ contains only type 2 and type 3 elements. We call the first two subgroups namely $H'_{i,j}$ and $K'_{i,j}$, the special subgroups of $G'_{i,j}$. Since type is preserved under conjugation in $\Gamma$, the special subgroups of $G'_{i,j}$ must be mapped to the corresponding special subgroups of $G'_{i,k}$ if $G'_{i,j}$ have to be conjugate to $G'_{i,k}$ in $\Gamma$. We know by Reiner's theorem that the special subgroup

$$K'_{i,j} = \left\langle \begin{pmatrix} 1 & \ast \\ 0 & [A_i] \end{pmatrix} \right\rangle$$

(36)

of $G'_{i,j}$ is conjugate to the special subgroup

$$H'_{i,j} = \left\langle \begin{pmatrix} [A_i] & \ast \\ 0 & I \end{pmatrix} \right\rangle$$

(37)

of $G'_{i,j}$, so there exists a matrix $\alpha \in \Gamma$ such that

$$\alpha K'_{i,j} \alpha^{-1} = H'_{i,j}$$

(38)

Now $\alpha H'_{j,i} \alpha^{-1}$ is a cyclic $p$-subgroup which together with $H'_{i,j}$ generate a $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroup of $\Gamma$. Hence by the remark (3.5) we get a $\phi$ in the centralizer of $H'_{i,j}$ such that

$$\phi \alpha H'_{j,i} \alpha^{-1} \phi^{-1} = K'_{i,j}.$$  

(39)
If \( q = \phi \alpha \), then

\[
e H'_{ji} e^{-1} = K'_{ij} \quad \text{and} \quad e K'_{ji} e^{-1} = H'_{ij}
\]  

(40)

This proves that the subgroups \( G'_{ij} \) and \( G'_{ji} \) are conjugate. So we further restrict our attention to the following sublist.

\[
G'_{ij} = \left\{ \left. \left( \begin{array}{cc} [A_i] & * \\ 0 & I \end{array} \right) \right\} \times \left( \begin{array}{cc} I & * \\ 0 & [A_j] \end{array} \right) \right\} \quad 1 \leq i < j \leq n
\]  

(41)

As it was done for \( G_{ij} \) in Case 1, we can show that the no two distinct subgroups in the list \( \{ G_{ij} | 1 \leq i < j \leq n \} \) are conjugate. We omit the details.

Thus the groups \( G_{ij} \) with \( 1 \leq i \leq j \leq n \) form a complete set of representatives for the \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \)-subgroups of \( \Gamma \) in this case wherein we are assuming that they contain an element of type 3.

Before concluding this case we remark that the subgroups \( G'_{ij} \) found in this case contains only type 2 and type 3 elements. So they are not conjugate to ones found in Case 1 because the subgroups \( G_{ij} \) of case one do not have an element of type 3. Since every \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \)-subgroup of \( \Gamma \) falls either in Case 1 or Case 2, it follows that the set

\[
\{ G_{ij}, G'_{ij} \mid 1 \leq i \leq j \leq n \}
\]

form a complete set of representatives for the elementary abelian \( p \)-subgroups of \( \Gamma \) of rank 2.
§ 4 Normalizers of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroups of $\Gamma$.

In this section we determine the normalizers of subgroups $G_{ij}$ and $G'_{ij}$ inside $\Gamma$.

First we consider the groups $G_{ij}$.

(4.1) Theorem: Let $C_{ij}$ and $N_{ij}$ denote the centralizer and normalizer of $G_{ij}$ respectively in $\Gamma$. We have an exact sequence

$$1 \to C_{ij} \to N_{ij} \to S_{ij} \to 1$$  \hspace{1cm} (42)

and

$$C_{ij} \cong U \times U$$

$$S_{ij} \cong S_i \times S_j \quad \text{if } i \neq j$$  \hspace{1cm} (43)

$$\cong S_i \wr (\mathbb{Z}/2\mathbb{Z}) \quad \text{if } i = j$$

Here $S_i$ denote the stabilizer of the ideal class $A_i$ in the Galois group $\Delta$ and $S_i \wr (\mathbb{Z}/2\mathbb{Z})$ denote the wreath product of $S_i$ with cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order 2, namely $(S_i \times S_i) \rtimes \mathbb{Z}/2\mathbb{Z}$ where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ flips the two subgroups $S_i$. Furthermore $S_i \times S_j$ acts on $U \times U$ componentwise and $\mathbb{Z}/2\mathbb{Z}$ flips the two subgroups $U \times 1$ and $1 \times U$. Thus we have

$$N_{ij} \cong (U \times S_i) \times (U \rtimes S_j) \quad \text{if } i \neq j$$  \hspace{1cm} (44)

$$\cong (U \times S_i) \wr (\mathbb{Z}/2\mathbb{Z}) \quad \text{if } i = j$$

Proof: We recall some facts about these groups which were proved in the last section. The group

$$G_{ij} = \left\langle \left( \begin{array}{cc} [A_i] & 0 \\ 0 & I \end{array} \right) \right\rangle \times \left\langle \left( \begin{array}{cc} I & 0 \\ 0 & [A_j] \end{array} \right) \right\rangle$$  \hspace{1cm} (45)

has two special cyclic $p$-subgroups, namely,

$$H_{ij} = \left\langle \left( \begin{array}{cc} [A_i] & 0 \\ 0 & I \end{array} \right) \right\rangle \quad \text{and} \quad K_{ij} = \left\langle \left( \begin{array}{cc} I & 0 \\ 0 & [A_j] \end{array} \right) \right\rangle$$  \hspace{1cm} (46)
which consists of type 1 elements while the other \((p-1)\)-subgroups of \(G_{ij}\) consists of type 2 elements. Thus if an element \(\psi \in \Gamma\) belong to \(N_{ij}\) then under the action of conjugation it has to map the subgroups \(H_{ij}, K_{ij}\) of \(G_{ij}\) within themselves or it can flip these subgroups.

First we determine the centralizer \(C_{ij}\) of subgroup \(G_{ij}\) in \(\Gamma\). Clearly \(C_{ij} = C(H_{ij}) \cap C(K_{ij})\). From the section 2 we know that

\[
C(H_{ij}) \cong \left\{ \begin{pmatrix} u & 0 \\ 0 & \gamma \end{pmatrix} \mid u \in U, \gamma \in GL(p-1, \mathbb{Z}) \right\} \quad (47)
\]

\[
C(K_{ij}) \cong \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & u \end{pmatrix} \mid u \in U, \gamma \in GL(p-1, \mathbb{Z}) \right\} \quad (48)
\]

Thus it follows that

\[
C_{ij} \cong \left\{ \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \mid u, u' \in U \right\} \quad (49)
\]

Now we determine the groups \(S_{ij} = N_{ij}/C_{ij}\). Let \(\phi \in S_{ij}\), then \(\phi\) maps the two special subgroups \(H_{ij}\) and \(K_{ij}\) onto themselves. If \(i \neq j\) then \(H_{ij}\) and \(K_{ij}\) are not conjugate, it follows that \(\phi\) has to take \(H_{ij}\) to \(H_{ij}\) and \(K_{ij}\) to \(K_{ij}\). This in particular shows that \(N_{ij} = N(H_{ij}) \cap N(K_{ij})\). From theorem (2.1) in section 2, we have an exact sequences

\[
1 \rightarrow C([A_i]) \rightarrow N([A_i]) \rightarrow S_i \rightarrow 1
\]

\[
1 \rightarrow C([A_j]) \rightarrow N([A_j]) \rightarrow S_j \rightarrow 1
\]

where \(C([A_i]), N([A_i])\) (resp. \(C([A_i]), N([A_i])\)) denote respectively the centralizer and normalizer of the element \([A_i]\) (resp. \([A_j]\)) in \(GL(p-1, \mathbb{Z})\). Identifying the subgroup \(G_{ij}\) as the subgroup of \(GL(p-1, \mathbb{Z}) \times GL(p-1, \mathbb{Z})\) which embeds naturally into \(GL(2p-2, \mathbb{Z})\) as

\[
GL(p-1, \mathbb{Z}) \times GL(p-1, \mathbb{Z}) \rightarrow GL(2p-2, \mathbb{Z})
\]

\[
(X, Y) \rightarrow \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}
\]
we conclude, from the fact $N_{ij} = N(H_{ij}) \cap N(K_{ij})$ that we must have $S_{ij} \cong S_i \times S_j$, with the first factor $S_i \times 1$ acting on the first factor $H_{ij} \times 1$ of $G_{ij}$ and the second factor $1 \times S_j$ acting on the second factor $1 \times K_{ij}$. Thus if $i \neq j$ we have

$$N_{ij} \cong (U \rtimes S_i) \times (U \rtimes S_j).$$

(50)

If $i = j$ then the subgroup $H_{ii}$ and $K_{ii}$ are conjugate in $\Gamma$, because they are type 1 subgroups associated with the same ideal class $A_i$. In fact the element

$$v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \Gamma$$

(51)

takes the subgroups $H_{ii}$ to $K_{ii}$ and vice-versa. Since any automorphism $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is completely determined by its action on any two distinct cyclic $p$-subgroups, it follows that $v$ belongs to $N_{ii}$. Furthermore it can be shown as before that $N(H_{ii}) \cap N(K_{ii})/C_{ii} \cong S_i \times S_i$ which is obviously a subgroup of $N_{ii}/C_{ii} = S_{ii}$. Next we claim that in the case $i = j$ the normalizer $N_{ii}$ is generated by $N(H_{ii}) \cap N(K_{ii})$ and the flip given by $v$, for if $\phi \in N_{ii}$ then it is easy to see that either $\phi \in N(H_{ii}) \cap N(K_{ii})$ or $\phi v \in N(H_{ii}) \cap N(K_{ii})$. This proves that

$$N_{ii} = (U \times S_i) \rtimes \mathbb{Z}/2\mathbb{Z},$$

(52)

where $\mathbb{Z}/2\mathbb{Z}$ denotes the group generated by the element in $S_{ii}$ induced by the $v$.

□

The normalizers of $G_{ij}'$ in $\Gamma$ can be found similarly. We state the results in this case and give a sketch of its proof.

(4.2) Theorem: Let $C_{ij}'$ and $N_{ij}'$ denote the centralizer and normalizer of the group $G_{ij}'$ in $\Gamma$. We have an exact sequence

$$1 \to C_{ij}' \to N_{ij}' \to S_{ij}' \to 1$$

(53)
and
\[ C'_{ij} \cong \{(u, u') | u, u' \in U, u \equiv u' \mod (\zeta - 1)\} \]
\[ S'_{ij} \cong S_i \times S_j \quad \text{if } i \neq j \]
\[ S'_{ij} \cong S_i \wr \mathbb{Z}/2\mathbb{Z} \quad \text{if } i = j. \]

If \( C'_{ij} \) is considered as the subgroup of \( U \times U \), then the action of \( S'_{ij} \) on \( C'_{ij} \) is the same as in the previous theorem.

**Proof (Sketch):** The group \( G'_{ij} \) has two special cyclic \( p \)-subgroup \( H'_{ij} \) and \( K'_{ij} \) which are of type 3 and the remaining \((p - 1)\) of its cyclic \( p \)-subgroups are of type 2. Hence the proof is essentially the same as that of the last theorem.

First we determine the centralizer \( C'_{ij} \). Clearly \( C'_{ij} = C(H'_{ij}) \cap C(K'_{ij}) \). From the section 2 we have,
\[ N(H'_{ij}) \cong \left\{ (u, \delta, \sigma) \mid u \in U, \delta \in \gamma(p - 1, p), \sigma \in S_i, \text{ and } u \equiv \lambda(\delta)s \mod (\zeta - 1) \right\} \]
\[ (55) \]
where \( \sigma \) and \( s \) are related by the equation \( \sigma(\zeta) = \zeta^s \) and
\[ C(H'_{ij}) \cong \left\{ (u, \delta, 1) \mid u \in U, \delta \in \gamma(p - 1, p), \text{ and } u \equiv \lambda(\gamma) \mod (\zeta - 1) \right\}. \]
\[ (56) \]
The group law in \( N(H_{ij}) \) is given by
\[ (u, \delta, \sigma)(u', \delta', \tau) = (u\sigma(u'), \delta\delta', \sigma\tau) \]
\[ (57) \]
Under the above isomorphism the elements
\[ \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \quad \begin{pmatrix} I & 0 \\ 0 & [A_j] \end{pmatrix} \]
of \( G_{ij} \) are mapped to the element
\[ (\zeta, 1, 1) \quad (1, [A_j], 1) \]
of $N(G_{ij})$ respectively. Thus an element $(u, \delta, 1) \in C(H'_{ij})$ lies in $C(K'_{ij})$ if and only if $\delta$ commutes with $[A_j] \in \gamma(p-1,p)$. To find such an element we first observe by lemma (3.4) that $N_{\gamma(p-1,p)}(\langle [A_j] \rangle)$ (the normalizer of the cyclic $p$-subgroup $\langle [A_j] \rangle$ of $\gamma(p-1,p)$ in $\gamma(p-1,p)$) is equal to $N_{GL(p-1,p)}(\langle [A_j] \rangle)$ (the normalizer of $\langle [A_j] \rangle$ in $GL(p-1,p)$). Similarly we have $C_{\gamma(p-1,p)}(\langle [A_j] \rangle) = C_{GL(p-1,p)}(\langle [A_j] \rangle)$.

Thus by theorem (2.1) in section 2, we have

$$C_{\gamma(p-1,p)}(\langle [A_j] \rangle) = C_{GL(p-1,p)}(\langle [A_j] \rangle) \cong U$$

(58)

$$N_{\gamma(p-1,p)}(\langle [A_j] \rangle) = N_{GL(p-1,p)}(\langle [A_j] \rangle) \cong U \rtimes S_j$$

(59)

Since an element $(u, \delta, 1) \in C(H'_{ij})$ lies in $C(K'_{ij})$ if $\delta$ lies in $C_{\gamma(p-1,p)}(\langle [A_j] \rangle)$, we get

$$C'_{ij} \cong \{(u, u', 1) \mid u, u' \in U, u \equiv \lambda(f(u')) \mod (\zeta - 1)\}$$

(60)

where $f$ denotes the isomorphism $f : U \xrightarrow{\cong} C_{\gamma(p-1,p)}(\langle [A_j] \rangle)$ which has been obtained from theorem (2.1). We recall that under this isomorphism (see [Ash]) an element $a(p_2) \zeta^{p-2} + a(p_3) \zeta^{p-3} + \ldots + a_0$ of $U$, where $a_i$'s are certain integers, is mapped to an element $a(p_2)[A_j]^{p-2} + a(p_3)[A_j]^{p-3} + \ldots + a_0$ of $C_{\gamma(p-1,p)}(\langle [A_j] \rangle)$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{f} & C_{\gamma(p-1,p)}(\langle [A_j] \rangle) \\
\mod (\zeta - 1) \downarrow & & \downarrow \lambda \\
(Z/pZ)^\times & \xrightarrow{id} & (Z/pZ)^\times
\end{array}
$$

Hence equation (60) can be rewritten as,

$$C'_{ij} \cong \{(u, u', 1) \mid u, u' \in U, u \equiv u' \mod (\zeta - 1)\}.$$  

(61)

Next we determine the normalizer $N'_{ij}$ in the case $i \neq j$. If $i \neq j$ then the subgroups $H'_{ij}$ and $K'_{ij}$ are not conjugate in $\Gamma$ because $A_i$ and $A_j$ fall into the different
\( \Delta \)-orbits of Cl. Thus any \( \phi \in N'_{ij} \) maps the subgroups \( H'_i \) and \( K'_i \) on to itself respectively, i.e., \( N'_{ij} = N(H'_i) \cap N(K'_i) \). But it is easy to see that an element \( (u, \delta, \sigma) \in N(H'_i) \) belongs to \( N(K'_i) \) if and only if \( \delta \in N(\gamma(p-1,p)(|A_j|)) \). Since for any \( y \in (\mathbb{Z}/p\mathbb{Z})^\times \) we can find an unit \( u \in U \) such that 

\( u \equiv y \mod (\zeta - 1) \), we deduce that the group \( N'_i/C'_i \) of automorphisms of \( G'_i \) induced by the inner automorphisms by the elements of \( N'_i \), is isomorphic to \( (N(H'_i)/C(H'_i)) \times (N(K'_i)/C(K'_i)) \cong S_i \times S_j \). Hence the theorem follows in this case.

If \( i = j \) then we know that the subgroups \( H'_i \) and \( K'_i \) are conjugate in \( \Gamma \). Hence there exists an element \( \alpha \in \Gamma \) such that \( \alpha K'_i \alpha^{-1} = H'_i \). However we do not know whether \( \alpha \) takes \( H'_i \) to \( K'_i \). But \( \alpha H'_i \alpha^{-1} \) is a cyclic subgroup of order \( p \) which together with \( H'_i \) generate a \( \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \)-subgroup of \( \Gamma \). Hence by the remark (3.5) we get a \( \beta \) in the centralizer of \( H'_i \) such that

\[
\beta \alpha H'_i \beta^{-1} \alpha^{-1} = K'_i
\]  

(62)

If \( \varphi' = \beta \alpha \), then

\[
\varphi' H'_i \varphi'^{-1} = K'_i \quad \text{and} \quad \varphi' K'_i \varphi'^{-1} = H'_i
\]

(63)

Thus \( \varphi' \in N'_i \) and since \( \varphi' \) flips the two subgroups \( H'_i, K'_i \) of \( G'_i \) we have \( \varphi'^2 \in \text{Aut}(H'_i) \times \text{Aut}(K'_i) \). So \( \varphi' = (\psi, \varphi) \) for some \( \psi \in \text{Aut}(H'_i) \) and \( \varphi \in \text{Aut}(K'_i) \). Set \( \varphi = \psi^{-1} \varphi' \), then it can be easily seen that \( \varphi \) flips the two subgroups \( H'_i, K'_i \) and induces an element of order 2 in \( N'_i/C'_i = S_i' \). Also, \( N(H'_i) \cap N(K'_i) \subset N'_i \) and it can be shown as in the case of \( i \neq j \) that

\[
(N(H'_i) \cap N(K'_i))/C_i \cong (N(H'_i)/C(H'_i)) \times (N(K'_i)/C(K'_i)) \cong S_i \times S_i.
\]  

(64)
Furthermore since any $\phi \in N'_{ii}$ maps the special subgroups onto themselves it is easy to see that either $\phi \in N(H'_{ii}) \cap N(K'_{ii})$ or $\phi \in N(H'_{ii}) \cap N(K'_{ii})$. Thus $N'_{ii}$ is generated by $N(H'_{ii}) \cap N(K'_{ii})$ and $\varrho$. Hence

$$S'_{ii} = N'_{ii}/C'_{ii} \cong (S_i \times S_i) \times \mathbb{Z}/2\mathbb{Z} \cong S_i \times \mathbb{Z}/2\mathbb{Z}$$ (65)

where $\mathbb{Z}/2\mathbb{Z}$ denotes the flip given by the element induced by $\varrho$. □

§5 1-simplices of $\mathcal{A}$

Recall that the complex $\mathcal{A}$ is a complex of elementary abelian $p$-subgroups of $\Gamma$, where $\Gamma$ acts on $\mathcal{A}$ by conjugation. Since the elementary abelian $p$-subgroups of $\Gamma$ are of rank $\leq 2$, the complex $\mathcal{A}$ is one dimensional. In this section we describe the 1-simplices or rather the edges in the fundamental domain $\mathcal{A}/\Gamma$. Although we will not use this description of edges explicitly when we compute the spectral sequence later in chapter 3, we give this description for sake of completeness and remark that one can use this description of $\mathcal{A}/\Gamma$ to work out the exact answer for the cohomology of $\Gamma$ if he or she is able to find all the stabilizers of vertices of type 2 and their Farrell cohomology.

In the complex $\mathcal{A}$ each edge is attached to an elementary abelian $p$-subgroup of $\Gamma$ of rank 2 and from section 3 we know that the set $\{G_{ij}, G'_{ij} | 1 \leq i \leq j \leq n\}$ form a set of representatives for the conjugacy classes of elementary abelian $p$-subgroups of $\Gamma$ of rank 2. Hence to find the edges in the fundamental domain $\mathcal{A}/\Gamma$, it is enough to find the edges in $\mathcal{A}/\Gamma$ that have $G_{ij}$ or $G'_{ij}$ as one vertex. From now on we assume that one vertex is $G_{ij}$ for an exactly similar result holds, for the edges that are attached to $G'_{ij}$ and the proof is the same. If $v_1$ and $v_2$ are the elementary abelian $p$-subgroups of $\Gamma$ of rank 1 and 2 respectively, then in this section we denote a vertex
of $\mathcal{A}$ corresponding to $v_1$ (resp. $v_2$) by $(v_1)$ (resp. $(v_2)$) and the edge joining $(v_1)$ and $(v_2)$ by $(v_1, v_2)$.

The $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroup $G_{ij}$ has $(p + 1)$ cyclic subgroups of order $p$, hence in the complex $\mathcal{A}$ there are $(p + 1)$ edges that are attached to the vertex $(G_{ij})$. If $(v_1, G_{ij})$ and $(v_2, G_{ij})$ are two edges attached to $(G_{ij})$, where $v_1$ and $v_2$ are cyclic subgroups of $G_{ij}$ of order $p$, then these edges are equivalent under the action of $\Gamma$ if and only if there is an element $\phi \in N_{ij}$ which carries $v_1$ to $v_2$ under the action of conjugation. Hence the number of distinct edges in a fundamental domain of $\mathcal{A}$ mod $\Gamma$ that are attached to the vertex $(G_{ij})$, is same as the number of orbits under the action of $N_{ij}$ on the set of the $(p + 1)$ cyclic subgroups of $G_{ij}$ of order $p$.

First we consider the case $i \neq j$. Then we have

$$G_{ij} = \left\langle \begin{pmatrix} A_i & 0 \\ 0 & I \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} I & 0 \\ 0 & A_j \end{pmatrix} \right\rangle$$

(66)

$$N_{ij} = N(G_{ij}) \cong (U \rtimes S_i) \times (U \rtimes S_j)$$

(67)

The $(p + 1)$ cyclic subgroups of order $p$ of $G_{ij}$ are

$$H_{ij} = \left\langle \begin{pmatrix} A_i & 0 \\ 0 & I \end{pmatrix} \right\rangle$$

(68)

$$K_{ij} = \left\langle \begin{pmatrix} I & 0 \\ 0 & A_j \end{pmatrix} \right\rangle$$

(69)

$$\left\langle \begin{pmatrix} A_i & 0 \\ 0 & [A_j]^l \end{pmatrix} \right\rangle \quad 1 \leq l \leq (p - 1)$$

The first two subgroups are not conjugate because $i \neq j$. Hence they give two edges in the fundamental domain $\mathcal{A}/\Gamma$. The remaining $(p - 1)$ subgroups are of type 2
and hence they are mapped within themselves by the elements in $N_{ij}$, under the action of conjugation. Before we proceed to find the number of distinct orbits in the action of $N_{ij}$ on this set consisting of the $(p - 1)$ subgroups of $G_{ij}$ of type 2, we first record few basic facts about the elements of order $p$ of $GL(p - 1, \mathbb{Z})$.

By remark (1.4) any element $X \in GL(p - 1, \mathbb{Z})$ of order $p$ is conjugate to $[A]$ in $GL(p - 1, \mathbb{Z})$, for some $A \in \text{Cl}$ and the ideal class $A$ is uniquely determined by the conjugacy class of $X$ in $GL(p - 1, \mathbb{Z})$. We denote this ideal class by $[X]$.

(5.1) Remark: It is obvious that $[[A]] = A$ and if $X$ and $Y$ are two elements of $GL(p - 1, \mathbb{Z})$ of order $p$, then $X$ and $Y$ are conjugate if and only if $[X] = [Y]$.

We recall that with respect to our fix basis for an $\mathcal{O}$-ideal $a$ in the ideal class of $A$ we have an embedding

$$\rho_A : \mu \to GL(p - 1, \mathbb{Z})$$

where $\mu = < \zeta >$ and $\zeta^p = 1$. There is an isomorphism between the Galois group $\Delta$ and $\text{Aut}([[A]])$ given by,

$$f : \Delta \to \text{Aut}([[A]])$$

$$f(\sigma)([A]) = \rho(\sigma(\zeta))$$

Let us denote $\rho(\sigma(\zeta))$ by $\sigma[A]$. This gives the usual action of $\Delta$ on Cl. Also it is clear that

$$\sigma(\zeta) = \zeta^n \iff \sigma[A] = [A]^n.$$  \hspace{1cm} (71)

Thus the $(p - 1)$ type 2 subgroups of $G_{ij}$ can be written as

$$\left\langle \left( \begin{array}{cc} [A_i] & 0 \\ 0 & \sigma[A_j] \end{array} \right) \right\rangle,$$

as $\sigma$ ranges over $\Delta$. 
(5.2) Lemma: If $A \in \text{Cl}$, then $[\sigma[A]] = \sigma^{-1}(A)$.

Proof: For any ideal class $A \in \text{Cl}$ let $M = A$ be the $\mathbb{Z}_\mu$-module where $\zeta \in \mu$ acts on $A$ through multiplication by $\zeta$ and let $M^\sigma$ denote the $\mathbb{Z}_\mu$-module $A$ where $\zeta \in \mu$ now acts on $A$ through multiplication by $\sigma(\zeta)$ instead of $\zeta$ itself. Then it is easy to see that $M^\sigma$ is isomorphic to $\tilde{M} = \sigma^{-1}(A)$ as a $\mathbb{Z}_\mu$-module, in fact the map

$$g : M^\sigma \rightarrow \tilde{M}$$

defined by

$$g(a) = \sigma^{-1}(a)$$

(72)

gives a $\mathbb{Z}_\mu$-isomorphism between $M^\sigma$ and $\tilde{M}$.

With respect to our fixed $\mathbb{Z}$-basis for $M^\sigma$ and $\tilde{M}$ we have two embeddings

$$\rho_1 : \mu \rightarrow GL(p - 1, \mathbb{Z})$$

$$\rho_2 : \mu \rightarrow GL(p - 1, \mathbb{Z})$$

Since $M^\sigma$ is $\mathbb{Z}_\mu$-isomorphic to $\tilde{M}$, $\rho_1(\zeta)$ is conjugate to $\rho_2(\zeta)$ in $GL(p - 1, \mathbb{Z})$. But $\rho_1(\zeta) = \sigma[A]$ and $\rho_2(\zeta) = [\sigma^{-1}(A)]$. Thus $\sigma[A]$ is conjugate to $[\sigma^{-1}(A)]$ in $GL(p - 1, \mathbb{Z})$. Hence by remark (5.1), $[[\sigma[A]] = [[\sigma^{-1}(A)]]$, i.e., $[\sigma[A]] = \sigma^{-1}(A)$. □

As an easy consequence of the above lemma we record an another lemma.

(5.3) Lemma: Let $S_A$ be the stabilizer of the ideal class $A$ in the Galois group $\Delta$. Then, for $\alpha, \beta \in \Delta$, the elements $\alpha[A]$ and $\beta[A]$ are in the same $S_A$ orbit if and only if $\alpha(A) = \beta(A)$. 
Proof: Suppose the elements $\alpha[A]$ and $\beta[A]$ are in the same $S_A$ orbit. Then there exists a $\sigma \in S_A$ such that $\sigma \alpha[A] = \beta[A]$. Thus $[\sigma \alpha[A]] = [\beta[A]]$. By previous lemma this implies $(\sigma \alpha)^{-1}(A) = \beta^{-1}(A)$, i.e., $\alpha^{-1} \sigma^{-1}(A) = \beta^{-1}(A)$. But $\sigma \in S_A$, hence we have $\alpha(A) = \beta(A)$.

Conversely, if $\alpha(A) = \beta(A)$ then $\beta \alpha^{-1} \in S_A$ and $\beta \alpha^{-1} \alpha[A] = \beta[A]$, i.e., $\alpha[A]$ is in the $S_A$ orbit of $\beta[A]$. $\square$

Now we proceed to find the orbits under the action of $N_{ij}$ on the set consisting of type 2 subgroups of the group $G_{ij}$.

(5.4) Theorem: The number of distinct orbits of $N_{ij}$ $(i \neq j)$ in the set of $(p - 1)$ cyclic subgroups of $G_{ij}$ of order $p$ are in one-one correspondence with the set of double cosets $S_i \Delta S_j$.

Proof: First we find when does the two type 2 subgroups, say

$$\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & \delta[A_j] \end{pmatrix} \right\rangle \quad \text{and} \quad \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_j] \end{pmatrix} \right\rangle$$

of $G_{ij}$ are in the same $N_{ij}$ orbits. They are in the same $N_{ij}$ orbits

$$\Leftrightarrow \left\{ \begin{array}{l}
\text{there exits a } n \text{ such that } \\
\left( [A_i]^n \begin{pmatrix} 0 \\ \delta[A_j] \end{pmatrix} \right) \text{ is in the } N_{ij}\text{-orbit of the } \\
\text{element } \\
\begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_j] \end{pmatrix} \cdot
\end{array} \right.$$  

Let $\alpha \in \Delta$ be such that $\alpha [A_i] = [A_i]^n$, then by (71) we also have $\alpha [A_j] = [A_j]^n$.

$$\Leftrightarrow \left\{ \begin{array}{l}
\text{there exists } \alpha \in \Delta \text{ such that } \\
\left( \alpha [A_i] \begin{pmatrix} 0 \\ \alpha \delta[A_j] \end{pmatrix} \right) \text{ is in the } N_{ij}\text{-orbit of the } \\
\text{element } \\
\begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_j] \end{pmatrix} \cdot
\end{array} \right.$$  

Now recall that $N_{ij} \cong S_i \times S_j$ acts componentwise on $G_{ij}$. Hence,
\[
\begin{align*}
\Leftrightarrow \begin{cases}
\text{there exists } \alpha \in \Delta \text{ such that } \alpha[A_i] \text{ is in the } S_i \text{ orbit of the element } [A_i] \\
\text{and } \alpha \delta[A_j] \text{ is in the } S_j \text{ orbit of the element } \sigma[A_j].
\end{cases}
\end{align*}
\]

By lemma (5.3) this is equivalent to,
\[
\Leftrightarrow \text{there exists } \alpha \in \Delta \text{ such that } \alpha(A_i) = A_i \text{ and } \alpha \delta(A_j) = \sigma(A_j)
\]

Since \( \Delta \) is commutative,
\[
\Leftrightarrow \text{there exists } \alpha \in \Delta \text{ such that } \alpha \in S_i \text{ and } \alpha \delta \sigma^{-1} \in S_j
\]
\[
\Leftrightarrow \delta \in S_i \sigma S_j
\]

Thus we see that the number of distinct orbits of \( N_{ij} \) in the set of \( (p - 1) \) cyclic subgroups of \( G_{ij} \) of type 2 are in one-one correspondence with the set of double cosets of \( S_i \Delta/S_j \). \( \Box \)

Now it follows that the number of distinct edges attached to \( (G_{ij}) \) in the fundamental domain \( A/\Gamma \) are
\[
2 + |S_i \Delta/S_j| = 2 + \frac{|\Delta| |S_i \cap S_j|}{|S_i||S_j|} \tag{73}
\]
where the first two edges corresponds to the two special subgroups \( H_{ij}, K_{ij} \) and next term gives the number of edges in \( A/\Gamma \) whose other vertex is a subgroup of \( G_{ij} \) of type 2 which are associated with ideal classes \( A_i \sigma(A_j) \), as \( \sigma \) ranges over the distinct representatives of the double cosets.

Next we consider the case \( i = j \). We have,
\[
G_{ii} = \langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \rangle \times \langle \begin{pmatrix} I & 0 \\ 0 & [A_i] \end{pmatrix} \rangle \tag{74}
\]
\[
N_{ii} = N(G_{ii}) \cong (U \times S_i) \wr \mathbb{Z}/2\mathbb{Z} \tag{75}
\]
The \((p + 1)\) cyclic subgroups of order \(p\) of \(G_{ii}\) are

\[
H_{ii} = \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & I \end{pmatrix} \right\rangle \tag{76}
\]

\[
K_{ii} = \left\langle \begin{pmatrix} 0 & 0 \\ I & [A_i] \end{pmatrix} \right\rangle \tag{77}
\]

\[
\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & [A_i]^l \end{pmatrix} \right\rangle \quad 1 \leq l \leq (p - 1)
\]

Clearly the first two special subgroups which are of type 1 are conjugate by the flip. Hence they give only one edge in the fundamental domain \(\mathcal{A}/\Gamma\). The remaining \((p - 1)\) subgroups are of type 2, and since the type is preserved under conjugation these \((p - 1)\) subgroups are mapped within themselves by \(N_{ii}\). We now find the number of distinct orbits under the action of \(N_{ii}\) on the set consisting of these \((p - 1)\) subgroups of \(G_{ii}\) of type 2.

(5.5) Theorem: The number of orbits in the action of \(N_{ii}\) on the set of the \((p - 1)\) cyclic subgroups of \(G_{ii}\) of type 2 are

\[
\frac{|\Delta|}{2|S_i|} + \frac{|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{\times 2}|}{|S_i|}
\]

Proof: We proceed as in the previous theorem. First we determine when does the two type 2 subgroups, say

\[
\left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & \delta[A_i] \end{pmatrix} \right\rangle \quad \text{and} \quad \left\langle \begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_i] \end{pmatrix} \right\rangle
\]

of \(G_{ii}\) are in the same \(N_{ii}\) orbits. They are in the same \(N_{ii}\) orbits,

\[
\Leftrightarrow \begin{cases} 
\text{there exists a } n \text{ such that } \begin{pmatrix} [A_i]^n & 0 \\ 0 & (\delta[A_i])^n \end{pmatrix} \text{ is in the } N_{ii}-\text{orbit of the} \\
\text{element } \begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_i] \end{pmatrix}.
\end{cases}
\]
Let \( \alpha \in \Delta \) be such that \( \alpha[A_i] = [A_i]^n \), then by (71) we also have \( \alpha[A_i] = [A_i]^n \).

\[
\Leftrightarrow \begin{cases} 
\text{there exists } \alpha \in \Delta \text{ such that } \begin{pmatrix} \alpha[A_i] & 0 \\ 0 & \alpha \delta[A_i] \end{pmatrix} \text{ is in the } N_{\alpha}[A_i] \text{-orbit of the element } \\
\begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_i] \end{pmatrix}.
\end{cases}
\]

Now recall that \( N_{\alpha} / C_{\alpha} \cong (S_i \times S_i) \times \mathbb{Z}/2\mathbb{Z} \), where \( S_i \times S_j \) acts componentwise on \( G_{\alpha} = H_{\alpha} \times K_{\alpha} \) and \( \mathbb{Z}/2\mathbb{Z} \) flips the two subgroups \( H_{\alpha} \) and \( K_{\alpha} \). Hence the above two elements

\[
\begin{pmatrix} \alpha[A_i] & 0 \\ 0 & \alpha \delta[A_i] \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} [A_i] & 0 \\ 0 & \sigma[A_i] \end{pmatrix}
\]

are in the same \( N_{\alpha}[A_i] \)-orbit if and only if either

1) \( \alpha[A_i] \) is in the \( S_i \)-orbit of \( [A_i] \) and \( \alpha \delta[A_i] \) is in the \( S_i \)-orbit of \( \sigma[A_i] \)

or

2) \( \alpha[A_i] \) is in the \( S_i \)-orbit of \( \sigma[A_i] \) and \( \alpha \delta[A_i] \) is in the \( S_i \)-orbit of \( [A_i] \).

By lemma (5.3) the above statements 1) and 2) are equivalent to

\[
\begin{cases} 
1) A_i = \alpha(A_i) \text{ and } \sigma(A_i) = \alpha \delta(A_i) \\
or \\
2) A_i = \alpha \delta(A_i) \text{ and } \sigma(A_i) = \alpha(A_i)
\end{cases}
\]

\[
\Leftrightarrow \begin{cases} 
1) \alpha \in S_i \text{ and } \sigma^{-1} \alpha \delta \in S_i \\
or \\
2) \alpha \delta \in S_i \text{ and } \sigma^{-1} \alpha \in S_i.
\end{cases}
\]

The first possibility is equivalent to \( \sigma = \delta \) in \( \Delta/S_i \) and the second possibility is equivalent to \( \sigma \delta = 1 \) in \( \Delta/S_i \). Thus the type-2 subgroup of \( G_{\alpha} \) corresponding to the ideal class \( A_i \sigma(A_i) \) is equivalent to other type-2 subgroup of \( G_{\alpha} \) corresponding to the ideal class \( A_i \delta(A_i) \) if and only if \( \delta = \sigma \) or \( \delta = \sigma^{-1} \) in \( \Delta/S_i \). Thus the number of elements in the \( N_{\alpha}[A_i] \)-orbit of the type-2 subgroup of \( G_{\alpha} \) corresponding to the ideal
class \( A\sigma(A) \) is equal to \( |\Delta/S_i| \) if \( \sigma^2 = 1 \) in \( \Delta/S_i \) and is equal to \( 2|\Delta/S_i| \) if \( \sigma^2 \neq 1 \) in \( \Delta/S_i \). But the number of \( \sigma \in \Delta/S_i \) of order 2 is equal to \( 2|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{x^2}|/|S_i| \). Hence the number of orbits in the action of \( N_{ii} \) on the set of the \( (p - 1) \) cyclic subgroups of \( G_{ii} \) of type 2 are

\[
\frac{|\Delta|}{2|S_i|} + \frac{|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{x^2}|}{|S_i|}.
\]

This completes the proof. \( \Box \)

Thus we conclude that in the fundamental domain \( \mathcal{A}/\Gamma \) there are

\[
1 + \frac{|\Delta|}{2|S_i|} + \frac{|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{x^2}|}{|S_i|}
\]

number of edges attached to the vertex \( G_{ii} \). The above number can be written as

\[
1 + \frac{1}{2} \left( \frac{|\Delta|}{|S_i|} - 2\frac{|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{x^2}|}{|S_i|} \right) + \frac{2|S_i \cap (\mathbb{Z}/p\mathbb{Z})^{x^2}|}{|S_i|}.
\]

Then the first one correspondes to the edge that has the other vertex \( H_{ii} \) (or \( K_{ii} \)) and the middle term correspondes to the edges whose other vertex is a type 2 subgroup of \( G_{ii} \) corresponding to the ideal classes \( A_i\sigma(A_i) \) where \( \sigma \in \Delta/S_i \) and \( \sigma^2 \neq 1 \), in \( \Delta/S_i \) while the last term correspondes to the edges whose other vertex is a type 2 subgroups of \( G_{ii} \) corresponding to the ideal classes \( A_i\sigma(A_i) \) where \( \sigma \in \Delta/S_i \) and \( \sigma^2 = 1 \), in \( \Delta/S_i \). This completes the description of the complex \( \mathcal{A} \).

\section{Class number one case \( p \leq 19 \).}

If the class number of \( Q(\zeta_p) \) is one then we have only the trivial ideal class \( \Lambda \) with the whole Galois group \( \Delta \) acting trivially on \( \Lambda \). Hence there are only three conjugacy classes of elementary abelian \( p \)-subgroups of \( \Gamma \) of rank 1 and they are respectively the type 1, 2, and 3 subgroups of \( \Gamma \) corresponding to the trivial ideal class. Also from section (3) we see that there are only two conjugacy classes of
elementary abelian $p$-subgroups of $\Gamma$ of rank 2, one containing type 1 and type 2 elements and the other containing only type 2 and type 3 elements. Thus the fundamental domain $\mathcal{A}/\Gamma$ has a shape of the English capital letter $\mathcal{M}$, with five vertices and four edges. The bottom three vertices of $\mathcal{M}$ are the rank 1 elementary abelian $p$-subgroups of $\Gamma$ of type 1, 2 and 3 respectively and the top two vertices are the elementary abelian $p$-subgroups of $\Gamma$ of rank 2. More specifically the bottom three vertices for $\mathcal{M}$ from left to right can be taken as,

\[
\left\langle \begin{pmatrix} [A] & 0 \\ 0 & I_{p-1} \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} [A] & 0 \\ 0 & [A] \end{pmatrix} \right\rangle, \quad \left\langle \begin{pmatrix} [A] & I_{p-1} \\ 0 & I_{p-1} \end{pmatrix} \right\rangle
\]

and then each of the top two vertices is just the $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$-subgroup generated by the two $\mathbb{Z}/p\mathbb{Z}$-subgroups to which it is attached by an edge. We will come back to this complex in the last chapter where we will make some explicit calculations in the case of prime $p = 3$. 
CHAPTER II

SPECTRAL SEQUENCE

§1 Cohomology of the Stabilizers

In this section we determine the Farrell cohomology of the stabilizers of the vertices of $A/\Gamma$ corresponding to the elementary abelian $p$-subgroups of $\Gamma$ of type 1 and type 3, and the elementary abelian $p$-subgroups of $\Gamma$ of rank 2. In all cases we apply the Hochschild-Serre spectral sequence to the short exact sequence (6),(42) and (53) of theorem (2.1), (4.1) and (4.2) respectively, of the previous chapter. We calculate the ordinary cohomology groups and then use the fact that the ordinary cohomology is same as the Farrell cohomology above the v.c.d. Thus our results, in later sections, where we compute the spectral sequence associated with $A$, are valid only in the dimensions greater than the v.c.d of all stabilizers that are involved in $A/\Gamma$.

The cohomology groups of the stabilizers in the type 1 and type 3 cases have been found by Ash in [Ash].

(1.1) Theorem (Ash): Let $M$ be a $\mathbb{Z}_\mu$-module, free over $\mathbb{Z}$ of rank $n$, where $n \geq (p - 1)$. Assume $M$ is either isomorphic to $a \oplus \mathbb{Z}^{n-(p-1)}$ or $(a, a) \oplus \mathbb{Z}^{(n-p)}$ for some $\mathcal{O}$ ideal $a$. Let the rest of the notation be the same as that of the theorem(2.1) of the first chapter.
Then the cohomology of the normalizer $N$ of the image of $\rho_M$ in $GL(n, \mathbb{Z})$, with coefficients in the trivial module $\mathbb{Z}/p\mathbb{Z}$ is given by

$$H^t(N, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{b+c+d=t} W_a(b, c) \otimes H^d(\gamma, \mathbb{Z}/p\mathbb{Z})$$

(78)

where $W_a(b, c)$ denotes a $\mathbb{Z}/p\mathbb{Z}$-vector space whose dimension is the number of subsets $K$ with $c$ elements of $\{2, 4, 6, \ldots, (p-3)\}$ such that $|S_A|$ divides $b' + \sum K$.

Here $b'$ denotes the largest integer contained in $(b+1)/2$.

Here we recall that $\gamma = GL(n-(p-1), \mathbb{Z})$ if $M = a \oplus \mathbb{Z}^{n-(p-1)}$ and $\gamma = \gamma(n-(p-1), p)$ if $M = (a, a) \oplus \mathbb{Z}^{(n-p)}$. Also $|S_A|$ denotes the order of the group $S_A$ which is the stabilizer of the ideal class $A$ of an $\mathcal{O}$-ideal $a$ in the Galois group $\Delta$.

Ash proved this theorem for $n < (2p - 2)$. However his methods works for any $n$. In fact if $n = p - 1$, then by theorem (2.1) of Chapter I we have, $N \cong U \rtimes S_A$, where the order of the Galois Stabilizer $S_A$ is prime to $p$ and thus we have,

$$H^t(N, \mathbb{Z}/p\mathbb{Z}) \cong H^t(U \rtimes S_A, \mathbb{Z}/p\mathbb{Z})$$

$$\cong H^t(U, \mathbb{Z}/p\mathbb{Z})^{S_A}$$

$$\cong \bigoplus_{b+c=t} W_a(b, c). \quad \text{by [Ash]}$$

(79)

So if $M \cong a \oplus \mathbb{Z}^{n-p+1}$ then $N \cong (U \rtimes S_A) \times \gamma$ and the theorem follows from the Kunneth formula and equation (79). If $M \cong (a, a) \oplus \mathbb{Z}^{n-p}$ we first need to observe, as Ash did in his proof, that the centralizer $C$ of a type 3 $p$-subgroup of $\Gamma$ has an index prime to $p$ in the group $U \times \gamma(p-1, p)$ and the coset representatives can be found in $U \rtimes 1$ which acts trivially on $C$. Then the rest of his proof goes exactly the same giving the above result.
Now we move on to the cohomology of the normalizers of the elementary abelian $p$-subgroups of $\Gamma$ of rank 2. From section 3 of the first chapter any such group $v$ is conjugate to $G_{ij}$ or $G'_{ij}$ for some $1 \leq i \leq j \leq n$. Hence we assume without loss of generality that $v$ is either $G_{ij}$ or $G'_{ij}$. For further notations we refer to sections 3 and 4 of the first chapter. To determine the cohomology groups of their normalizers we use the same techniques as Ash used in the above theorem. In either case we have the exact sequence (42) (resp. (53))

$$1 \to C_v \to N_v \to S_v \to 1$$

(80)

where $C_v = C_{ij}$, $N_v = N_{ij}$ and $S_v = S_{ij}$ (resp. $C_v = C'_{ij}$, $N_v = N'_{ij}$, $S_v = S'_{ij}$) have their meaning as in theorem (4.1) (resp. (4.2)). Also from these theorems we know that the order $|S_v|$ of the group $S_v$ is prime to $p$, and hence the mod-$p$ cohomology of $N_v$ with coefficients in the trivial module $\mathbb{Z}/p\mathbb{Z}$ is just the $S_v$ invariants in the mod-$p$ cohomology of $C_v$.

First we consider the groups $G_{ij}, G'_{ij}$ with $i \neq j$. In the case of $G_{ij}$, we have $N_{ij} \cong (U \times S_i) \times (U \times S_j)$ which together with equation (79) and Kunneth formula gives

$$H^t(N_{ij}, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_j(l, m),$$

(81)

where $W_i(b, c) = W_{ai}(b, c)$ and $W_j(b, c) = W_{aj}(b, c)$.

Next in the case of $G'_{ij}$ we observe that the centralizer $C'_{ij}$ has an index $(p - 1)$ in the group $C_{ij} \cong U \times U$. Hence the mod-$p$ cohomology of $C'_{ij}$ is same as that of $C_{ij}$ and since the action of $S'_{ij} = S_{ij}$ on $C'_{ij}$ is same as that induced from its action on $C_{ij}$, we see that the mod-$p$ cohomology of $C'_{ij}$ is same as that of mod-$p$ cohomology of $C_{ij}$. Thus, we have

$$H^t(N'_{ij}, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_j(l, m).$$

(82)
Now we consider the case when $i = j$. In these cases we have an additional action of $\mathbb{Z}/2\mathbb{Z}$ on the right side of equation (81) and (82) which just flips the two tensor product components. Thus we have,

$$H^t(N_{i,i}, \mathbb{Z}/p\mathbb{Z}) \cong \left[ \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_i(l, m) \right]^{\mathbb{Z}/2\mathbb{Z}} \quad (83)$$

$$H^t(N'_{i,i}, \mathbb{Z}/p\mathbb{Z}) \cong \left[ \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_i(l, m) \right]^{\mathbb{Z}/2\mathbb{Z}} \quad (84)$$

§2 Spectral Sequence

In this section we compute $H^t_\ast(A)_p$, using the spectral sequence

$$E_1^{m,q} = \prod_{\sigma \in \sum_m} \tilde{H}^q(\Gamma_\sigma, \mathbb{Z}/p\mathbb{Z}) \Rightarrow \tilde{H}^{m+q}(\Gamma, \mathbb{Z}/p\mathbb{Z}) \quad (85)$$

where $\sum_m$ is a set of representatives for the $m$-cells of $A \mod \Gamma$. Note that this spectral sequence lives in the first and fourth quadrants, but there is no problem with its convergence because $\dim A = 1$ and hence the spectral sequence has only two columns, namely $E_1^{0,q}, E_1^{1,q}$ and only one differential $d_1^{0,q}$, so $E_r = E_\infty$ as soon as $r > 1$. First we describe the differential $d_1^{0,q}$. For this we need to introduce new notation.

(2.1) Notation: First of all, for the rest of this chapter we delete the coefficient module in the mod-$p$ cohomology groups, i.e., we write $\tilde{H}^\ast(G, \mathbb{Z}/p\mathbb{Z})$ as $\tilde{H}^\ast(G)$. It is understood that the cohomology groups are with respect to the trivial $G$-module $\mathbb{Z}/p\mathbb{Z}$, unless otherwise specifically mentioned. Also in this section we denote the 1-simplices of $A$ by $\sigma = (v_0, v_1)$, where $v_0 < v_1$ are the vertices of the 1-simplex $\sigma$ which are ordered by inclusion, i.e., $v_1$ is a elementary abelian $p$-subgroup of $\Gamma$ of rank 2 and $v_0$ is a elementary abelian $p$-subgroup of $\Gamma$ of rank 1 which is also a subgroup of $v_1$. 
The stabilizer $\Gamma_\sigma$ of the 1-simplex $\sigma = (v_0, v_1)$ is just $\Gamma_{v_0} \cap \Gamma_{v_1}$. Thus $\Gamma_\sigma \subset \Gamma_{v_0}$ and $\Gamma_\sigma \subset \Gamma_{v_1}$, and we have two restriction maps

$$\text{res}_{v_0}^{\sigma} : \hat{H}^*(\Gamma_{v_0}) \to \hat{H}^*(\Gamma_\sigma)$$

$$\text{res}_{v_1}^{\sigma} : \hat{H}^*(\Gamma_{v_1}) \to \hat{H}^*(\Gamma_\sigma).$$

Let $A_m$ be the set of $m$-simplices of $A$. Then we can identify $E_1^{m,q}$ with the subgroup of $\prod_{\sigma \in A_m} \hat{H}^q(\Gamma_\sigma, \mathbb{Z}/p\mathbb{Z})$ consisting of those families $(u_\sigma)_{\sigma \in A_m}$ such that $\gamma u_\sigma = u_{\gamma \sigma}$ for all $\gamma \in \Gamma$, $\sigma \in A_m$. Here $\gamma u_\sigma$ denotes the image of $u_\sigma$ under the conjugation isomorphism

$$c(\gamma^{-1})^*: \hat{H}^q(\Gamma_\sigma) \to \hat{H}^q(\Gamma_{\gamma \sigma})$$

induced by the conjugation map $c(\gamma^{-1}) : \Gamma_\sigma \to \Gamma_{\gamma \sigma}$ defined by $x \mapsto \gamma x \gamma^{-1}$ for any $x \in \Gamma_\sigma$.

Now the differential $d_1^{0,q}$ is the restriction to this subgroup of the map

$$d : \prod_{\sigma \in A_0} \hat{H}^q(\Gamma_\sigma) \to \prod_{\tau \in A_1} \hat{H}^q(\Gamma_\tau)$$

defined as follows. Let $(u_\nu) \in \prod_{\nu \in A_0} \hat{H}^*(\Gamma_\nu)$, then the map $d$ is given by (see lemma IX(4.3) of [Br]),

$$(u_\nu) \mapsto (\sigma \mapsto (\text{res}_{v_1}^{\sigma}(u_{v_1}) - \text{res}_{v_0}^{\sigma}(u_{v_0}))). \quad (86)$$

Since we do not know the Farrell cohomology of the stabilizers of rank 1 elementary abelian $p$-subgroups of $\Gamma$ of type 2, and also in case of type 1 and 3, we know only the ordinary cohomology which equals the Farrell cohomology in the dimensions greater than the v.c.d., we will not be able to analyze the spectral sequence (85) completely. However we will find some classes that survive in the spectral...
sequence (85) and try to get a lower bound on the mod $p$-cohomology of $\Gamma$. The classes which we find will be in the kernel of the differential $d^q_1$ for $q$ greater than the v.c.d. of all the stabilizers and hence they will be in $\tilde{H}^q(\Gamma)$. To find them our strategy will be as follows,

**PROPOSITION**: 1) If $\sigma = (v_0, v_1)$ is a 1-simplex in $\mathcal{A}$ with vertices $v_0$ and $v_1$, with $v_0 < v_1$, then we show that the restriction map

$$\text{res}^{v_1}_\sigma : H^*(\Gamma_{v_1}) \to H^*(\Gamma_\sigma)$$

on the ordinary cohomology is injective.

**PROPOSITION**: 2) To each 1-simplex $\sigma$ of $\mathcal{A}/\Gamma$ we determine a subspace $Y_{\sigma}^*$ of $H^*(\Gamma_\sigma)$, such that the following holds.

"Let $v$ be a vertex in $\mathcal{A}/\Gamma$ corresponding to a elementary abelian $p$-subgroup of $\Gamma$ of rank 1. Let $\sigma_1, \ldots, \sigma_{i(v)}$ be the distinct 1-simplices of $\mathcal{A}/\Gamma$, that are attached to the vertex $v$. Let

$$u_j \in Y_{\sigma_j}, \quad j = 1, 2, \ldots, i(v).$$

Then there exist a class $x \in H^*(\Gamma_v)$ such that $\text{res}^{v}_\sigma(x) = u_j$ for each $j = 1, 2, \ldots, i(v)$.”

**PROPOSITION**: 3) For each vertex $\vartheta$ of $\mathcal{A}/\Gamma$ which is an elementary abelian $p$-subgroups of $\Gamma$ of rank 2, we find the classes $u_\vartheta$ in the cohomology group $H^*(\Gamma_\vartheta)$, such that if $\sigma$ is a 1-simplex in $\mathcal{A}/\Gamma$ attached to the vertex $\vartheta$, then the class $\text{res}_\vartheta^\sigma(u_\vartheta) \in H^*(\Gamma_\sigma)$ belongs to the subspace $Y_{\sigma}^*$ of $H^*(\Gamma_\sigma)$, for each $\sigma \in \mathcal{A}/\Gamma$ that has $\vartheta$ as one vertex. Here $Y_{\sigma}^*$ is the subspace of $H^*(\Gamma_\sigma)$ determined in proposition 2. We denote the subspace of $H^*(\Gamma_\vartheta)$ consisting of these classes found in this step by $X_{\vartheta}^*$. 
Let $\sum_{02}$ be a set of representatives for the conjugacy classes of an elementary abelian $p$-subgroups of $\Gamma$ of rank 2. For example we could take

$$\sum_{02} = \{ G_{ij}, G'_{ij} \mid 1 \leq i \leq j \leq n \}, \quad (87)$$

where $G_{ij}$ and $G'_{ij}$ are the elementary abelian subgroups of $\Gamma$ of rank 2 found in Chapter 1 and $n$ is the number of distinct $\Delta$ orbits of $\text{Cl}$, i.e., $n = |\text{Cl}/\Delta|$. Let $q$ be an integer greater than $\text{v.c.d}$ of all the stabilizers of the simplices of $\mathcal{A}/\Gamma$. Then assuming that we have accomplished proposition (1) to (3), it will follow that if $(u_\theta)_{\theta \in \sum_{02}} \in \bigoplus_{\theta \in \sum_{02}} X^q_\theta \subset E_1^{0q}$, then $(u_\theta)_{\theta \in \sum_{02}}$ can be completed to an element $(u_{\nu})_{\nu \in \sum_{0}} \in \bigoplus_{\nu \in \sum_{0}} H^q(\Gamma_{\nu}) = E_1^{0q}$ such that $(u_{\nu})_{\nu \in \sum_{0}}$ lies in the kernel of $d_1^{0,q}$, i.e., it survives in the spectral sequence (85). This in particular implies that the dimension of $H^q(\Gamma)$ is greater than that of $\bigoplus_{\theta \in \sum_{02}} X^q_\theta$.

Now we proceed with the proof of proposition (1).

**Proof of Proposition 1**: Let $\sigma = (v_0, v_1)$ be a 1-simplex in $\mathcal{A}/\Gamma$ with vertices $v_0, v_1$ with $v_0 < v_1$. Recall that we have an exact sequence for $\Gamma_{v_1}$ (see theorem 4.1 and 4.2 of the previous chapter)

$$1 \rightarrow C_{v_1} \rightarrow \Gamma_{v_1} \rightarrow S_{v_1} \rightarrow 1 \quad (88)$$

where $C_{v_1}$ is the centralizer of $v_1$ in $\Gamma$ and $S_{v_1}$ can be identified with the subgroup of automorphisms of $v_1$ induced by the elements of $\Gamma_{v_1}$. Since $\Gamma_\sigma = \Gamma_{v_0} \cap \Gamma_{v_1}$, we also get an exact sequence

$$1 \rightarrow C_{v_1} \rightarrow \Gamma_\sigma \rightarrow S_\sigma \rightarrow 1 \quad (89)$$

where $S_\sigma$ is the subgroup of $S_{v_1}$ consisting of only those automorphisms in $S_{v_1}$ which map the subgroup $v_0$ onto itself. Since the order of the subgroup $S_{v_1}$, hence
also that of $S_\sigma$, is prime to $p$, the mod-$p$ Hochschild-Serre spectral sequence associated with the above exact sequences (88) and (89) degenerates to give

$$H^*(\Gamma_{v_1}) \cong H^*(C_{v_1})^{S_{v_1}}$$

(90)

$$H^*(\Gamma_\sigma) \cong H^*(C_{v_1})^{S_\sigma}.$$  

(91)

Since $S_\sigma \subset S_{v_1}$, it follows that the restriction map

$$\text{res}_{v_1}^{v_1} : H^*(\Gamma_{v_1}) \to H^*(\Gamma_\sigma)$$

is injective. □

**Proof of Proposition 2**: To determine the desired subspace in $H^*(\Gamma_\sigma)$ stated in proposition 2, where $\sigma$ is a 1-simplex of $\mathcal{A}/\Gamma$, we start with the vertex $v$ of $\sigma$ which is an elementary abelian subgroup of $\Gamma$ of rank 1. Let $\Gamma_v$ be the stabilizer (i.e. the normalizer) of $v$ in $\Gamma$. Then applying the Hochschild-Serre spectral sequence to the exact sequence

$$1 \to v \to \Gamma_v \to \Gamma_v/v \to 1$$

(92)

we get

$$E^{m+q}_2 = H^m(\Gamma_v/v, H^q(v)) \Rightarrow H^{m+q}(\Gamma_v).$$

(93)

**Lemma**: The above spectral sequence (93) degenerates to the right of the line $m = \text{v.c.d}(\Gamma_v)$, i.e., $E^{m+q}_2 = E^\infty_{m+q}$, if $m > \text{v.c.d.}(\Gamma_v)$.

**Proof**: First we note that the elementary abelian $p$-subgroups of $\Gamma_v/v$ are of rank $\leq 1$. Let $\Sigma_{1v}$ be the set consisting of all the 1-simplices of $\mathcal{A}/\Gamma$ that are attached to the vertex $v$ in $\mathcal{A}/\Gamma$. Then it is easy to see that a set of representatives for the elementary abelian $p$-subgroups of $\Gamma_v/v$ up to conjugacy in $\Gamma_v/v$ are in one-one correspondence with the set $\Sigma_{1v}$. In fact a set of representatives for the elementary
abelian $p$-subgroups of $\Gamma_v/v$ can be taken as $\{v_{\sigma'}/v \mid \sigma' = (v, v_{\sigma'}) \in \sum_{1,v}\}$ and their stabilizers in $\Gamma_v/v$ are $\{\Gamma_{\sigma'}/v \mid \sigma' \in \sum_{1,v}\}$ respectively. Thus by corollary (0.2) to Brown's theorem mentioned in the introduction we have

$$\hat{H}^m(\Gamma_v/v, H^q(v)) \cong \bigoplus_{\sigma' \in \Sigma_{1,v}} \hat{H}^m(\Gamma_{\sigma'}/v, H^q(v)).$$

(94)

Here $\hat{H}^m(\Gamma_v/v, H^q(v))$ denotes the Farrell cohomology of $\Gamma_v/v$ with coefficients in $\Gamma_v/v$-module $H^q(v)$. Since the Farrell cohomology is same as the ordinary cohomology above the v.c.d., equation (94) is true for the ordinary cohomology if we assume $m > \text{v.c.d.}(\Gamma_v)$, i.e.,

$$H^m(\Gamma_v/v, H^q(v)) \cong \bigoplus_{\sigma' \in \Sigma_{1,v}} H^m(\Gamma_{\sigma'}/v, H^q(v)) \quad \text{if } m > \text{v.c.d.}(\Gamma_v).$$

(95)

Now for each $\sigma = (v, v_{\sigma}) \in \sum_{1,v}$, we have an exact sequence

$$1 \to v \to \Gamma_{\sigma} \to \Gamma_{\sigma}/v \to 1.$$ 

(96)

Applying the Hochschild-Serre spectral sequence we get

$$E_2^{m,q} = H^m(\Gamma_{\sigma}/v, H^q(v)) \Rightarrow H^{m+q}(\Gamma_{\sigma}).$$

(97)

First we show that the above spectral sequence (97) degenerates. In fact for any 1-simplex of $A/\Gamma$ such as $\sigma = (v, v_{\sigma})$ we have an exact sequence (see equation (89) )

$$1 \to C_{v_{\sigma}} \to \Gamma_{\sigma} \to S_{\sigma} \to 1.$$ 

(98)

where $C_{v_{\sigma}}$ is the centralizer of the vertex $v_{\sigma}$ in $\Gamma$ and $S_{\sigma}$ can be identified with the group of automorphisms of $v_{\sigma}$ induced by the elements of $\Gamma_{\sigma}$. Since the order of
$S_{\sigma}$ is prime to $p$, (see theorem 4.1 and 4.3 of the first chapter), the Hochschild-Serre spectral sequence when applied to the exact sequence (98) degenerates to give

$$H^\ast(\Gamma_\sigma) \cong H^\ast((C_{v_\sigma})^{S_{\sigma}}).$$

(99)

From theorem 4.1 and 4.2 of the first chapter we know that either $C_{v_\sigma} \cong U \times U$ or $C_{v_\sigma} \cong \{(u, u') \mid u, u' \in U, u \equiv u' \mod (\zeta - 1)\}$ and in either case the subgroup $v_\sigma$ can be identified with the subgroup $\mu \times 1$ of $U \times U$. In the latter case as observed earlier the group $C_{v_\sigma}$ has an index $p - 1$ in the group $U \times U$, hence the mod-$p$ cohomology of $C_{v_\sigma}$ is same as that of $U \times U$. Thus in either case we have

$$H^\ast(C_{v_\sigma}) \cong H^\ast(U \times U).$$

(100)

Since the subgroup $v_\sigma$ can be identified with the subgroup $\mu \times 1$ of $U \times U$, we have an exact sequence

$$1 \to v \to U \times U \to (U \times U)/v \to 1$$

(101)

which splits, hence the mod-$p$ Hochschild-Serre spectral sequence

$$E_2^{m,q} = H^m(U \times U/v, H^q(v)) \Rightarrow H^{m+q}(U \times U)$$

(102)

degenerates to give

$$H^t(U \times U) \cong \bigoplus_{m+q=t} H^m((U \times U)/v, H^q(v)).$$

(103)

From equation (99) we can think of the spectral sequence (97) as a subsequence of the spectral sequence (102), obtained by taking the $S_{\sigma}$-invariants in the spectral sequence (102). Since the spectral sequence (102) degenerates we conclude that the spectral sequence (97) also degenerates to give

$$H^t(\Gamma_\sigma) \cong \bigoplus_{m+q=t} H^m(\Gamma_\sigma/v, H^q(v)).$$

(104)
Now we compare the two exact sequences (92) and (96),

\[
\begin{align*}
1 & \to v \to \Gamma_v \to \Gamma_v/v \to 1 \\
1 & \to v \to \Gamma_\sigma \to \Gamma_\sigma/v \to 1
\end{align*}
\]

Here the two vertical arrows on the right-side denotes the inclusion maps, which gives the corresponding restriction maps between the spectral sequences (93) and (97),

\[
\begin{align*}
E_2^{m,q} &= H^m(\Gamma_v/v, H^q(v)) \quad \Rightarrow \quad H^{m+q}(\Gamma_v) \\
\text{res} & \downarrow \\
E_2^{m,q} &= H^m(\Gamma_\sigma/v, H^q(v)) \quad \Rightarrow \quad H^{m+q}(\Gamma_\sigma)
\end{align*}
\]

To avoid the conflict in the notation, here we use \( \tilde{E}_2^{m,q} \) instead of \( E_2^{m,q} \) for the second spectral sequence (97). If \( m > \text{v.c.d.}(\Gamma_v) \), then from the equation (95) and the very last exercise in the K. Brown's book [Br] the restriction map on the left in the above diagram (106), can now be described as the component map,

\[
H^m(\Gamma_v/v, H^q(v)) \cong \bigoplus_{\sigma' \in \Sigma_1 v} H^m(\Gamma_{\sigma'}/v, H^q(v)) \overset{\text{res}}{\longrightarrow} H^m(\Gamma_\sigma/v, H^q(v)) \quad (107)
\]

which takes the summand \( H^m(\Gamma_\sigma/v, H^q(v)) \) identically onto \( H^m(\Gamma_\sigma/v, H^q(v)) \) and maps the other summands to zero. We get commutative diagrams like (106) for each \( \sigma \in \Sigma_1 v \). Also for each \( \sigma \in \Sigma_1 v \) we have a commutative diagram between the two spectral sequence (93) and (97),

\[
\begin{align*}
E_2^{(m-2),(q+1)} \quad &d_2^{(m-2),(q+1)} \quad E_2^{m,q} = H^m(\Gamma_v/v, H^q(v)) \\
\text{res} & \downarrow \\
\tilde{E}_2^{(m-2),(q+1)} \quad &d_2^{(m-2),(q+1)} \quad \tilde{E}_2^{m,q} = H^m(\Gamma_\sigma/v, H^q(v))
\end{align*}
\]

Since the spectral sequence (97) degenerates, \( \tilde{d}_2^{(m-2),(q+1)} = 0 \). Furthermore if \( m > \text{v.c.d.}(\Gamma_v) \) then we know that the restriction map on the right in the commutative diagram (108) is just the component map as in (107). Hence from the
commutativity of the diagram (108) which is valid for each \( \sigma \in \Sigma_{1v} \) we conclude that \( d_2^{(m-2),(q+1)} = 0 \) if \( m > \text{v.c.d.}(\Gamma_v) \). Similarly one can show that the higher differentials \( d_r \) of the spectral sequence (93) are all zero if \( m > \text{v.c.d.}(\Gamma_v) \).

This proves that the spectral sequence (93) degenerates to the right of the line \( m = \text{v.c.d.}(\Gamma_v) \), i.e., \( E_2^{mq} = E_\infty^{mq} \). This proves our lemma. \( \square \)

Now we continue with the proof of proposition 2. From the previous lemma we see that,

\[
\bigoplus_{m+q=t, \; m>\text{v.c.d.}(\Gamma_v)} E_2^{mq} = \bigoplus_{m+q=t, \; m>\text{v.c.d.}(\Gamma_v)} \left( \bigoplus_{\sigma' \in \Sigma_{1v}} H^m(\Gamma_{\sigma'}/\nu, H^q(\nu)) \right)
\]

is a subspace of \( H^t(\Gamma_v) \). Passing to the abutments of the spectral sequence (93) and (97) in the commutative diagram (106) we see that the restriction map on this subspace

\[
\bigoplus_{m+q=t, \; m>\text{v.c.d.}(\Gamma_v)} E_\infty^{mq} = \bigoplus_{m+q=t, \; m>\text{v.c.d.}(\Gamma_v)} \left( \bigoplus_{\sigma' \in \Sigma_{1v}} H^m(\Gamma_{\sigma'}/\nu, H^q(\nu)) \right) \rightarrow H^t(\Gamma_v)
\]

of \( H^t(\Gamma_v) \) which appears on the left side of the above commutative diagram is just the component map. Now set

\[
Y^t_\sigma = \bigoplus_{m+q=t, \; m>\text{v.c.d.}(\Gamma_v)} H^m(\Gamma_{\sigma'}/\nu, H^q(\nu)) \]

Recall that \( \nu \) is one of the vertex of \( \sigma \), which is an elementary abelian \( p \)-subgroup of \( \Gamma \) of rank 1. Then it follows immediately that this subspace satisfies \( Y^t_\sigma \) the condition stated in proposition 2. This completes the proof of proposition 2. \( \square \)
Proof of proposition 3): Let \( \vartheta \in A/\Gamma \) be an elementary abelian \( p \)-subgroup of \( \Gamma \) of rank 2. Now we find the classes \( u_\vartheta \in H^*(\Gamma_\vartheta) \) such that if \( \sigma \) is a 1-simplex of \( A/\Gamma \) whose one vertex is \( \vartheta \), then the restriction map

\[
\text{res}^\vartheta_\sigma : H^*(\Gamma_\vartheta) \to H^*(\Gamma_\sigma)
\]
maps \( u_\vartheta \) into the subspace \( Y^*_\vartheta \). To find such classes we need to analyze the proof of theorem (1.1) of this chapter.

Since \( \vartheta \) is an elementary abelian \( p \)-subgroup of \( \Gamma \) of rank 2, it is either conjugate to \( G_{ij} \) or \( G'_{ij} \) for some \( 1 \leq i \leq j \leq n \). First we consider the case when \( \vartheta \) is conjugate to \( G_{ij} \) or \( G'_{ij} \) for some \( 1 \leq i < j \leq n \). In this case, the mod-\( p \) ordinary cohomology of the stabilizer \( \Gamma_\vartheta \) is given by, (see equations (81) and (82))

\[
H^t(\Gamma_\vartheta) \cong \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_j(l, m)
\]

where \( W_i(b, c) \) and \( W_j(l, m) \) have their usual meaning as in section 1. We have derived this formula using Ash's theorem (1.1). We briefly go over its proof. To begin with, we have an exact sequence of theorem (4.1) and (4.2) of the first chapter.

\[
1 \to C_\vartheta \to \Gamma_\vartheta \to S_\vartheta \to 1
\]

(113)

where as usual \( C_\vartheta \) is the centralizer of \( \vartheta \) in \( \Gamma \) and \( S_\vartheta \) can be identified with the group of automorphisms of \( \vartheta \) induced by the elements of \( \Gamma_\vartheta \). See theorems (4.1) and (4.2) of the first chapter for more details. Since the order of \( S_\vartheta \) is prime to \( p \), the mod-\( p \) Hochschild-Serre spectral sequence degenerates to give,

\[
H^t(\Gamma_\vartheta) \cong H^t(C_\vartheta)^{S_\vartheta}.
\]

(114)

Also we have \( C_\vartheta \cong U \times U \) if \( \vartheta \) is conjugate to \( G_{ij} \) and \( C_\vartheta \cong \{(u, u')|u, u' \in U, u \equiv u' \mod (\zeta - 1)\} \) if \( \vartheta \) is conjugate to \( G'_{ij} \). However as
shown during the course of the proof of proposition 2, the mod-$p$ cohomology of $C_\phi$ in the later case is also the same as that of $U \times U$. To compute the cohomology of $U \times U$ we first compute the cohomology of $U$ and then apply the Kunneth formula.

The cohomology of $U$ is calculated in [Ash]. Ash first observes that $U \cong \mu \times U'$, where $U' \cong \mathbb{Z}^{(p-3)/2} \times \mathbb{Z}/2\mathbb{Z}$ and then applies the Kunneth formula to get the cohomology of $U$. Recall that the Galois group $\Delta$ acts on the group $U$ of units and hence on its cohomology. By the formula on the page (334) of [Ash] the $\Delta$-module $H^t(U)$ is isomorphic to,

$$H^t(U) \cong H^t(\mu \times U') \cong \bigoplus_{b+c=t} H^b(\mu) \otimes H^c(U') \cong \bigoplus_{b+c=t} \mathbb{F}[b'] \otimes V_c$$

where

$$V_r = H^r(U') \cong \bigwedge^r \left( \bigoplus_{e=2,4,6,\ldots,(p-1)} \mathbb{F}[e] \right)$$

is the $r$-th exterior product of the $\mathbb{Z}/p\mathbb{Z}$-vectorspace ($\bigoplus_{e=2,4,6,\ldots,(p-1)} \mathbb{F}[e]$) and $\mathbb{F}[e]$ denotes $\mathbb{Z}/p\mathbb{Z}$ viewed as a $\Delta$-module where $\delta \in \Delta$ acts by the formula $\delta \cdot x = s^m x$ when $\delta(\zeta) = \zeta^e$ and for any integer $m$, $m'$ denotes the largest integer contained in $(m + 1)/2$. Thus in the above formula the term $\mathbb{F}[b']$ denotes the classes coming from the cohomology of the subgroup $\mu \subset U$. In fact if we look at Ash's proof we see that $H^b(\mu, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{F}[b']$ as a $\Delta$-module. Now applying the Kunneth formula we get,

$$H^t(C_\phi) \cong H^t(U \times U)$$

$$\cong \bigoplus_{x+y=t} H^x(U) \otimes H^y(U) \cong \bigoplus_{b+c+l+m=t} H^b(\mu) \otimes H^c(U') \otimes H^l(\mu) \otimes H^m(U') \cong \bigoplus_{b+c+l+m=t} \mathbb{F}[b'] \otimes V_c \otimes \mathbb{F}[l'] \otimes V_m$$

(117)
From equation (114) we remark that the mod-$p$ cohomology of $\Gamma_\vartheta$, i.e., the equation (112) can now be derived by taking the $S_\vartheta$-invariants in equation (117). In this case, where we are assuming that $\vartheta$ is conjugate to $G_{ij}$ or $G'_{ij}$ with $i \neq j$, we know from theorem (4.1) and (4.2) of first chapter that $S_\vartheta \cong S_i \times S_j$ which acts on $U \times U$ componentwise, hence by (114) and (117) we get

$$H^t(\Gamma_\vartheta) \cong \left[ \bigoplus_{b+c+l+m=t} F[b'] \otimes V_c \otimes F[l'] \otimes V_m \right]^{S_\vartheta}$$

and by (79)

$$\cong \bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_j(l, m)$$

After this digression we are now ready to find the classes $u_\vartheta$ in $H^*(\Gamma_\vartheta)$ which satisfies the condition stated in proposition 3. Let $\sigma$ be a 1-simplex of $\mathcal{A}/\Gamma$ with one vertex equal to $\vartheta$ where $\vartheta$ is an elementary abelian subgroup of $\Gamma$ of rank 2. Let us write $\vartheta = \alpha \times \beta$, where $\alpha = \langle \bar{\alpha} \rangle$ and $\beta = \langle \bar{\beta} \rangle$ are the special subgroups of $\vartheta$ defined in section 3 of Chapter I. Then the group $U \times U$ can be identified with the group $\alpha \times U' \times \beta \times U'$. We fix this identification for the rest of the chapter, i.e., we write

$$U \times U = \alpha \times U' \times \beta \times U'.$$

Under this identification the cohomology groups $H^t(U \times U)$ can be described as

$$H^t(U \times U) \cong \bigoplus_{b+c+l+m=t} H^b(\alpha) \otimes H^c(U') \otimes H^l(\beta) \otimes H^m(U')$$

$$\cong \bigoplus_{b+c+l+m=t} F[b'] \otimes V_c \otimes F[l'] \otimes V_m$$

We fix this description of $H^t(U \times U)$ for the rest of the chapter.
Suppose now \( \sigma = (\alpha, \vartheta) \), then from the equations (99), (100), (102), (104) and (111), we see that the subspace \( Y^t_\sigma \) of \( H^t(\Gamma_\sigma) \) found in proposition 2, is just the \( S_\sigma \)-invariants in the subspace

\[
\bigoplus_{a+b=t, \ a > v.c.d.(\Gamma_\sigma)} H^a((U \times U)/\alpha, H^b(\alpha))
\]

of \( H^t(U \times U) \). Since \( U \times U \) is abelian it acts trivially on \( H^b(\alpha) \cong \mathbb{Z}/p\mathbb{Z} \) for each \( b \), and under the above identification we have \( (U \times U)/\alpha \cong U' \times \beta \times U' \). Thus we conclude that under the isomorphism of (121) the latter subspace (122) is just

\[
\bigoplus_{a+b=t, \ a > v.c.d.(\Gamma_\sigma)} H^a(U' \times \beta \times U', H^b(\alpha)) \cong \bigoplus_{b+c+l+m=t, \ c+l+m > v.c.d.(\Gamma_\sigma)} F[b'] \otimes V_c \otimes F[l'] \otimes V_m
\]

Now passing to the \( S_\theta \)-invariants in the above equation we conclude from the equations (112) and (114) (see also (79), (81)) that the subspace

\[
\bigoplus_{b+c+l+m=t, \ c+l+m > v.c.d.(\Gamma_\sigma)} W_i(b, c) \otimes W_j(l, m)
\]

of \( H^t(\Gamma_\vartheta) \) is mapped by the injective restriction map \( \text{res}_\vartheta^\sigma : H^*(\Gamma_\vartheta) \to H^*(\Gamma_\sigma) \) into the subspace \( Y^t_\vartheta \) associated with \( \sigma \) in proposition 2.

Similarly if \( \sigma(\beta) = (\beta, \vartheta) \) is an another 1-simplex in \( \mathcal{A}/\Gamma \) with one vertex equal to \( \vartheta \), then we can show as in (121) that the subspace \( Y^t_{\sigma(\beta)} \) of \( H^t(\Gamma_{\sigma(\beta)}) \) found in proposition 2, is just the \( S_{\sigma(\beta)} \)-invariants in the subspace

\[
\bigoplus_{a+b=t, \ a > v.c.d.(\Gamma_\beta)} H^a((U \times U)/\beta, H^b(\beta))
\]

of \( H^t(U \times U) \). Under our identification we now have \( (U \times U)/\beta \cong \alpha \times U' \times U' \), hence the isomorphism of (121) the subspace (125) is just

\[
\bigoplus_{a+b=t, \ a > v.c.d.(\Gamma_\alpha)} H^a(\alpha \times U' \times U', H^b(\beta)) \cong \bigoplus_{b+c+l+m=t, \ b+c+m > v.c.d.(\Gamma_\beta)} F[b'] \otimes V_c \otimes F[l'] \otimes V_m
\]
Now passing to the $S_\theta$-invariants in the above subspace we conclude as in (124) that the subspace

$$\bigoplus_{b+c+l+m=t \atop b+c+m \neq \gcd(1, \ldots, l)} W_i(b, c) \otimes W_j(l, m)$$

(127)
of $H^t(\Gamma_\theta)$ is mapped by the injective restriction map $\text{res}_\sigma^\theta : H^*(\Gamma_\theta) \to H^*(\Gamma_{\sigma(\beta)})$ into the subspace $Y^t_{\sigma(\beta)}$.

As we pass on to the other 1-simplices $\sigma$ in $\mathcal{A}/\Gamma$ that are attached to $\theta$, we get a similar description of the subspace $Y^t_{\sigma}$ as that of (124) and (127) except that we will be modding out $U \times U$ by other cyclic subgroups of $\theta$ of order $p$, instead of $\alpha$ and $\beta$. Since $\alpha = < \tilde{\alpha} >$ and $\beta = < \tilde{\beta} >$, the cyclic subgroups of $\theta$, other than $\beta$, are $\kappa_r = < \tilde{\alpha} \tilde{\beta}^r >$ for some $r \in \{0, 1, 2, \ldots, (p - 1)\}$. Thus if $\sigma$ is a 1-simplex that is attached to the vertex $\theta$ in $\mathcal{A}/\Gamma$ and is different from $\sigma(\beta)$, then $\sigma = \sigma(\kappa_r) = (\kappa_r, \theta)$, for some $r \in \{0, 1, \ldots, (p - 1)\}$. Let

$$f : U' \times \alpha \times U' \times \beta \to U' \times \alpha \times U' \times \beta$$

(128)
be the automorphism of the group $U \times U$ defined as follows, $f(\tilde{\alpha}) = \tilde{\alpha} \tilde{\beta}^r$, $f(\tilde{\beta}) = \tilde{\beta}$ and let $f$ be identity map on $U' \times U'$. Then via $f$ we can identify $(U \times U)/\kappa_r$ with $U' \times U' \times \beta$. Thus we have a commutative diagram of short exact sequences,

$$\begin{array}{cccccc}
1 & \to & \alpha & \to & U \times U & \to & (U \times U)/\alpha & \to & 1 \\
1 & \downarrow f & \downarrow f & \downarrow \tilde{f} & & & & & \\
1 & \to & \kappa_r & \to & U \times U & \to & (U \times U)/\kappa_r & \to & 1 \\
\end{array}$$

(129)
where $\tilde{f}$ denotes the map induced by $f$. This diagram gives the commutative diagram between the corresponding two spectral sequences obtained by applying the Hochschild-Serre spectral sequence to the above short exact sequences. Since these spectral sequences degenerates we get an isomorphism

$$f^* : \bigoplus_{a+b=t} H^a((U \times U)/\kappa_r, H^b(\kappa_r)) \to \bigoplus_{a+b=t} H^a((U \times U)/\alpha, H^b(\alpha)).$$

(130)
Now as in (122) we can show that the subspace $Y_{\sigma(\kappa_r)}^t$ of $H^t(\Gamma_{\sigma(\kappa_r)})$ found in proposition 2 is just the $S_{\sigma(\kappa_r)}$-invariant in the subspace

$$\bigoplus_{a+b=t} H^a((U \times U)/\kappa_r, H^b(\kappa_r))$$  \hspace{1cm} (131)

of $H^t(U \times U)$. Since we have identified $(U \times U)/\kappa_r$ with $U' \times U' \times \beta$, the above subspace (131) can also be described as the subspace of

$$H^t(U \times U) \cong \bigoplus_{a+b=t} H^a((U \times U)/\kappa_r, H^b(\kappa_r))$$  \hspace{1cm} (132)

$$\cong \bigoplus_{a+b=t} H^a(U' \times U' \times \beta) \otimes H^b(\kappa_r)$$

consisting of those classes for which $a > v.c.d.(\Gamma_{\kappa_r})$.

Now we compare (132) and (121). Since $f$ is identity on $U' \times \beta \times U'$, we see immediately from (131) that under the isomorphism of (121) the subspace (131) of $H^t(U \times U)$ is just

$$\bigoplus_{b+c+l+m=t} F[b'] \otimes V_c \otimes F[l'] \otimes V_m.$$  \hspace{1cm} (133)

Now passing to the $S_\theta$-invariants in the above equation we conclude that the subspace

$$\bigoplus_{b+c+l+m=t} W_i(b, c) \otimes W_j(l, m)$$  \hspace{1cm} (134)

of $H^t(\Gamma_\theta)$ is mapped by the injective restriction map

$$\text{res}^{\theta}_{\sigma(\kappa_r)} : H^t(\Gamma_\theta) \rightarrow H^t(\Gamma_{\sigma(\kappa_r)})$$

into the subspace $Y_{\sigma(\kappa_r)}^t$ of $H^t(\Gamma_{\sigma(\kappa_r)})$.

Now set,

$$\Xi = \max\{v.c.d.(\Gamma_v)|v \text{ is a elementary abelian } p\text{-subgroup of } \Gamma \text{ of rank } 1\}.$$  \hspace{1cm} (135)
Thus from equations (9) and (10) we see that

\[
\Xi = \begin{cases} 
\frac{p - 3}{2} + \frac{(p - 1)(p - 2)}{2} & \text{if } p > 3 \\
\frac{2(p - 2)}{2} & \text{if } p = 3 
\end{cases}
\] (136)

Then from (124), (127) and (134) we see that the subspace

\[
\bigoplus_{b+c+l+m=t} \bigoplus_{c+l+m>\Xi} W_i(b, c) \times W_j(l, m) 
\] (137)

of \( H^t(\Gamma_\vartheta) \) is mapped by the injective restriction map

\( \text{res}_\vartheta : H^t(\Gamma_\vartheta) \rightarrow H^t(\Gamma_\vartheta) \) into the subspace \( Y_\vartheta^t \) of \( H^t(\Gamma_\vartheta) \), for each 1-simplex \( \sigma \) in \( \mathcal{A}/\Gamma \) that is attached to the vertex \( \vartheta \) of \( \mathcal{A}/\Gamma \). Denote this subspace by \( X_\vartheta^t \). Then the classes in this subspace satisfy the condition stated in proposition 3. This completes the argument for proposition 3 in the case when \( \vartheta \) is conjugate to the subgroups \( G_{ij} \) or \( G_{ij}' \) for some \( 1 \leq i < j \leq n \).

Next we consider the case when \( i = j \). In this case the same analysis holds true except that we have one additional element in \( S_\vartheta \) which acts on the centralizer \( C_\vartheta \subset U \times U \) by flipping the two components of \( U \times U \). So if we ignore this flip then we get the same result as in the case when \( i \neq j \). Then taking into account the flip in these cases we derived the same formula as that of (131) with \( i = j \), except that we need to restrict further to the subspace of (131) that is fixed under the action of the flip. So in terms of the notation of section 1 where we have denoted the group generated by the flip by \( \mathbb{Z}/2\mathbb{Z} \), we see that the subspace

\[
\left[ \bigoplus_{b+c+l+m=t} \bigoplus_{c+l+m>\Xi} W_i(b, c) \otimes W_i(l, m) \right] \mathbb{Z}/2\mathbb{Z}
\] (138)
of $H^t(\Gamma_\vartheta)$ is mapped by the injective restriction map $\text{res}_\sigma^\vartheta : H^t(\Gamma_\vartheta) \to H^t(\Gamma_\sigma)$ into the subspace $Y_\sigma^t$ of $H^t(\Gamma_\vartheta)$, for each 1-simplex $\sigma$ in $\mathcal{A}/\Gamma$ that is attached to the vertex $\vartheta$ of $\mathcal{A}/\Gamma$. Denote this subspace by $X_\vartheta^t$. Then the classes in this subspace satisfy the condition stated in proposition 3. This completes the proof of proposition 3. □

As mentioned after the statement of proposition three on page (51) the classes in the subspace

$$\bigoplus_{\vartheta \in \Sigma_{02}} X_\vartheta^t \tag{139}$$

survive in the spectral sequence (85) and we have

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(H^t(\Gamma)) \geq \dim_{\mathbb{Z}/p\mathbb{Z}}\left(\bigoplus_{\vartheta \in \Sigma_{02}} X_\vartheta^t\right).$$

If we take

$$\sum_{\Sigma_{02}} = \{G_{ij}, G'_{ij} | 1 \leq i \leq j \leq n\}, \tag{140}$$

and set

$$X(t, i, j) = X_\vartheta^t, \quad \text{if } \vartheta = G_{ij}$$

$$X'(t, i, j) = X_\vartheta^t, \quad \text{if } \vartheta = G'_{ij} \tag{141}$$

then from (137) and (138), we can rewrite (139) as

$$\bigoplus_{1 \leq i \leq j \leq n} (X(t, i, j) \oplus X'(t, i, j)) \tag{142}$$

From (137) we have

$$X(t, i, j) \cong X'(t, i, j)$$

$$\cong \bigoplus_{b+c+l+m=t \atop c+l+m \geq \Xi} W_i(b, c) \otimes W_j(l, m) \quad \text{for } 1 \leq i < j \leq n \tag{143}$$
and from (138) we have

\[ X(t, i, i) \cong X'(t, i, i) \]

\[
\mathbb{Z}/2\mathbb{Z} \left[ \bigoplus_{\begin{array}{c} b+c+l+m=t \\ c+l+m \geq \Xi \\ b+c+m \geq \Xi \end{array}} W_i(b, c) \otimes W_i(l, m) \right] \]

for \(1 \leq i \leq n\) \hspace{1cm} (144)

From above we immediately see that in general when the total degree \(t\) is much greater than the number \(\Xi\) then the subspaces \(X(t, i, j)\) and \(X'(t, i, j)\) are not vacuous and in fact they are quite large [see §6 of [A-M]].

If the Galois stabilizer of the ideal class \(A_i\) is trivial, then (144) simplifies. In that case \(\dim_{\mathbb{Z}/p\mathbb{Z}} W_i(b, c) = \) number of subsets of \(c\) elements in \(\{2, 4, 6, \ldots, (p-3)\}\) independently of \(b\), because by (79), (115), and (116) (see also page 334 in [Ash]),

\[ W_i(b, c) \cong F[b'] \otimes V_c. \]

Thus in this case (144) reduces to

\[
\mathbb{Z}/2\mathbb{Z} \left[ \bigoplus_{\begin{array}{c} b+c+l+m=t \\ c+l+m \geq \Xi \\ b+c+m \geq \Xi \end{array}} F[b'] \otimes V_c \otimes F[l'] \otimes V_m \right] \]

\hspace{1cm} (146)

Recall here that the non-trivial element of \(\mathbb{Z}/2\mathbb{Z}\) acts by flipping the components

\[ F[b'] \otimes V_c \otimes F[l'] \otimes V_m \]

and

\[ F[l'] \otimes V_m \otimes F[b'] \otimes V_c. \]
Thus we see that the dimension of $X(t, i, i)$ and $X'(t, i, i)$ increases linearly with respect to $t$, if $t \gg \Xi$.

As remarked earlier, for $t > \Xi$, we have

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(H^t(\Gamma)) \geq \dim \left[ \bigoplus_{1 \leq i \leq j \leq n} X(t, i, j) \oplus X'(t, i, j) \right]. \quad (147)$$

Thus we have found a lower bound for the dimension of $H^t(\Gamma)$ over $\mathbb{Z}/p\mathbb{Z}$. As pointed out above by §6 in [A-M] this lower bound is in fact sufficiently large for large values of $t$ and is exponential in $p$ for sufficiently large $p$.

In the next chapter where we set $p = 3$ and calculate the mod-3 cohomology of $GL(4, \mathbb{Z})$, we will see that the dimension of the subspaces $X(t, i, i)$ and $X'(t, i, i)$ of the cohomology increases linearly with respect to $t$ as long as $t \not\equiv 1 \mod 4$.

Before concluding this chapter we recall that the cohomology classes that we found in (143) and (144) are in the kernel of the differential $d_1^{0,q}$. Similar but explicit calculation of mod-3 cohomology of $GL(4, \mathbb{Z})$ in the next chapter where we take $p = 3$, shows that cokernel of $d_1^{0,q}$ is 0 if $q > 3$, i.e., K. Brown’s spectral sequence collapses on the left edge $E_2^{0,q}$ of the $E_2$ page. It is suspected that this might always be true in general, i.e., the cokernel of $d_1^{0,q}$ is zero for other primes. If that is the case then K. Brown’s spectral sequence yields $\tilde{H}^q(\Gamma) \cong \ker (d_1^{0,q})$ and then our lower bound gives a fairly good estimate on the cohomology of $\Gamma$. 
CHAPTER III

THE MOD-3 COHOMOLOGY OF $GL(4, \mathbb{Z})$

§1 $\Lambda$-Complex

In this chapter we set $p = 3$ and compute the mod-3 Farrell cohomology of $GL(4, \mathbb{Z})$ using K.Brown's complex $\mathcal{A}$ of elementary abelian 3-subgroups of $GL(4, \mathbb{Z})$. We will compute $H^t(GL(4, \mathbb{Z}), \mathbb{Z}/p\mathbb{Z})$ for $t > 3$. In this section we describe the fundamental domain for $GL(4, \mathbb{Z})$ in $\mathcal{A}$. To begin with we first fix the notation for the rest of this chapter. Set $\Gamma = GL(4, \mathbb{Z})$ and $\mathbf{k} = \mathbb{Z}/3\mathbb{Z}$. Let $\omega$ be a primitive cube root of unity and $\mathcal{O} = \mathbb{Z}[\omega]$ be the ring of integers in the cyclotomic field $\mathbb{Q}(\omega)$. Let $U = \{\pm 1, \pm \omega, \pm \omega^2\}$ be the groups of units in $\mathcal{O}$, $\mu = \{1, \omega, \omega^2\}$ be the group generated by $\omega$ and $\Delta$ be the Galois group of $\mathbb{Q}(\omega)/\mathbb{Q}$. Also in this chapter we write $H^*(G)$ for the cohomology groups of a group $G$ with coefficients in the trivial $G$-module $\mathbf{k}$.

The class number of $\mathbb{Q}(\omega)$ is 1. So there is only one ideal class namely the trivial ideal class $\Lambda$. Thus by section 6 of the first chapter, the fundamental domain for $\Gamma$ in the complex $\mathcal{A}$ has a shape of the English capital letter $\Lambda$. We denote the bottom three vertices of $\Lambda$ from left to right by $v_1$, $v_2$ and $v_3$ respectively. Similarly the top two vertices from left to right will be denoted by $v_4$ and $v_5$ respectively. From section 6 of the first chapter we know that $v_1$, $v_2$ and $v_3$ are the elementary abelian 3-subgroups of $\Gamma$ of type 1, 2, and 3 respectively. Finally we denote the 1-simplices of $\Lambda$ from left to right by $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\sigma_4$ respectively.
To describe the vertices in terms of matrices in \( \Gamma \) we first choose \( \{1, \omega\} \) as a \( \mathbb{Z} \)-basis for the trivial ideal \( \mathcal{O} \), then we see that the matrix \( \begin{bmatrix} \Lambda \end{bmatrix} \) is just \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \). Thus from section 1 of the first chapter the representatives for the conjugacy classes of rank-1 elementary abelian 3-subgroups of \( \Gamma \) of type 1 and 2 are the groups generated by the matrices \( a \) and \( b \), where

\[
\begin{align*}
  a &= \begin{pmatrix} [\Lambda] & 0 \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
  b &= \begin{pmatrix} [\Lambda] & 0 \\ 0 & [\Lambda] \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.
\end{align*}
\]

Next let \( M = (\mathcal{O}, 1) \oplus \mathbb{Z} \), where \( 1 \in \mathcal{O} \), be the \( \mathbb{Z}_\mu \)-module, free over \( \mathbb{Z} \) of rank 4, as described in section one of the first chapter, then with respect to our \( \mathbb{Z} \)-basis of \( \mathcal{O} \) the image of \( \omega \) under the embedding \( \rho_M : \mu \to \Gamma \) is a matrix

\[
\begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

Thus \( < \hat{c} > \)-the group generated by \( \hat{c} \) is a rank 1 elementary abelian 3-subgroup of \( \Gamma \) of type 3. Since the class number of \( Q(\omega) \) is 1, the groups \( < a >, < b > \) and \( < \hat{c} > \) form a set of representatives for the conjugacy classes of elementary abelian 3-subgroups of \( \Gamma \) of rank 1. We know from section 3 of the first chapter that there are only two conjugacy classes of the elementary abelian 3-subgroups of \( \Gamma \) of rank 2. We also know that one of them contains only type 1 and 2 elements and the other contains only type 2 and 3 elements. The matrices \( a \) and \( b \) commute and hence they together generate one elementary abelian 3-subgroup of \( \Gamma \) of rank 2. However the matrices \( b \) and \( c \) do not commute with each other. Hence to find the...
other conjugacy class of elementary abelian 3-subgroup of $\Gamma$ of rank 2 we need to find a conjugate of $\tilde{c}$ which commutes with $b$. Let
\[
c = \begin{pmatrix}
0 & -1 & 1 & 0 \\
1 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (151)
then it is easy to see that $c$ is conjugate to $\tilde{c}$ and that $c$ commutes with $b$. Hence $b$ and $c$ generate another elementary abelian 3-subgroup of $\Gamma$ of rank 2. Thus $\langle a \rangle \times \langle b \rangle$ and $\langle b \rangle \times \langle c \rangle$ form a set of representatives for the conjugacy classes of elementary abelian 3-subgroups of $\Gamma$ of rank 2. Now we set
\[
u_1 = \langle a \rangle,
u_2 = \langle b \rangle, \quad \nu_3 = \langle c \rangle
\] (152)
and
\[
u_4 = \langle a \rangle \times \langle b \rangle, \quad \nu_5 = \langle c \rangle \times \langle b \rangle
\] (153)
Next we use the same notation as that of section 2 of the second chapter to denote the 1-simplices of $\Lambda$, i.e.,
\[
s_1 = (\nu_1, \nu_4), \quad s_2 = (\nu_2, \nu_4), \quad s_3 = (\nu_3, \nu_5), \quad s_4 = (\nu_4, \nu_5).
\] (154)
This completes the description of complex $\Lambda$. 

(1.1) Notation : If $\sigma_j = (v_i, v_k)$ is a 1-simplex of $A$ with one vertex $v_i$, then we denote the restriction map $\text{res} : H^\ast(\Gamma_{v_i}) \to H^\ast(\Gamma_{\sigma_j})$ by $\rho_{ij}$.

(1.1) Remark : There is also another way to find the second conjugacy class of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Instead of changing $\tilde{c}$ we can find a conjugate of $b$ which commutes with $\tilde{c}$. In fact we let
\[
\tilde{b} = \begin{pmatrix}
0 & -1 & 1 & -2 \\
1 & -1 & 0 & -1 \\
0 & 0 & 1 & -3 \\
0 & 0 & 1 & -2
\end{pmatrix}
\] (155)
\[ \hat{v}_2 = \langle \hat{b} \rangle \quad \text{and} \quad \hat{v}_3 = \langle \hat{c} \rangle \] (156)

then it is easy to see that \( \hat{b} \) is conjugate to \( b \) and it commutes with \( \hat{c} \) and that the group \( \hat{v}_5 = \langle \hat{c} \rangle \times \langle \hat{b} \rangle \) is conjugate to \( v_5 = \langle c \rangle \times \langle b \rangle \). Thus \( v_4 \) and \( \hat{v}_5 \) also forms a set of representatives of abelian 3-subgroups of \( \Gamma \) of rank 2.

Let \( \hat{\sigma}_3 = (\hat{v}_2, \hat{v}_5) \) and \( \hat{\sigma}_4 = (\hat{v}_3, \hat{v}_5) \). Then it can be shown that \( \hat{\sigma}_3 \) and \( \hat{\sigma}_4 \) are \( \Gamma \)-equivalent to the 1-simplices \( \sigma_3 \) and \( \sigma_4 \) respectively.

§2 Cohomology of \( \Gamma_{v_1} \) and \( \Gamma_{v_3} \)

First we compute the mod-3 cohomology of the stabilizer \( \Gamma_{v_1} \) of the vertex \( v_1 \). Since the Galois Stabilizer of the trivial ideal class \( \Lambda \) is the whole Galois group, we get from section 1 of the first chapter an exact sequence

\[ 1 \rightarrow C_{v_1} \rightarrow \Gamma_{v_1} \rightarrow \Delta \rightarrow 1 \] (157)

where the centralizer \( C_{v_1} \) is isomorphic to the group \( U \times GL(2, \mathbb{Z}) \) and the Galois group \( \Delta \) acts trivially on \( GL(2, \mathbb{Z}) \) and via its galois action on \( U \). Since \( U \rtimes \Delta \) is isomorphic to the dihedral group \( D_{12} \) of order 12, we get

\[
\begin{align*}
\Gamma_{v_1} & \cong (U \times GL(2, \mathbb{Z})) \rtimes \Delta \\
& \cong (U \rtimes \Delta) \times GL(2, \mathbb{Z}) \\
& \cong D_{12} \times GL(2, \mathbb{Z})
\end{align*}

(158)

So by the Kunneth formula we get,

\[
\begin{align*}
H^t(\Gamma_{v_1}) & \cong \bigoplus_{i+j=t} H^i(U \rtimes \Delta) \otimes H^j(GL(2, \mathbb{Z})) \\
& \cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(GL(2, \mathbb{Z}))
\end{align*}

(159)

The cohomology of \( D_{12} \) is well-known.
Lemma: The mod-3 cohomology ring of $D_{12}$ is isomorphic to $k[\xi, \eta]$, where $\deg(\xi) = 4$, $\deg(\eta) = 3$ and $\eta^2 = 0$.

Proof: Since $D_{12}$ can be described as $\{a, b \mid a^6 = 1, b^2 = 1, bab^{-1} = a^{-1}\}$, we have a short exact sequence

$$1 \rightarrow \mathbb{Z}/6\mathbb{Z} \rightarrow D_{12} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 1$$

where $\mathbb{Z}/6\mathbb{Z} \cong \langle a \rangle$, $\mathbb{Z}/2\mathbb{Z} \cong \langle b \rangle$ and $bab^{-1} = a^{-1}$. Since the order of the group $\mathbb{Z}/2\mathbb{Z}$ is prime to 3, the mod-3 Hochschild-Serre spectral sequence degenerates to give

$$H^n(D_{12}) \cong H^n(\mathbb{Z}/6\mathbb{Z})^{\mathbb{Z}/2\mathbb{Z}}.$$ \hspace{1cm} (161)

But the cohomology ring of $\mathbb{Z}/6\mathbb{Z}$ is isomorphic to $k[\zeta, \eta]$ where the generators $\zeta$ and $\eta$ satisfy the conditions, $\deg(\zeta) = 2$, $\deg(\eta) = 1$ and $\eta^2 = 0$. It can also be verified that the $b^* \cdot$ acts on these generators as follows, $b^*(\zeta) = 2\zeta$ and $b^*(\eta) = 2\eta$. Thus by (161) we see that the mod-3 cohomology ring of $D_{12}$ is isomorphic to $k[\xi, \eta]$, where $\xi = \zeta^2$ and $\eta = \zeta \cup \eta$ and the lemma follows. \[\square\]

For $GL(2, \mathbb{Z})$ we observe that every elementary abelian 3-subgroup of $GL(2, \mathbb{Z})$ has rank $\leq 1$, so by the corollary (1.2) of the introduction, we have

$$\hat{H}^m(GL(2, \mathbb{Z})) \cong \prod_{P \in \mathcal{P}} \hat{H}^m(N(P)).$$ \hspace{1cm} (162)

where $\mathcal{P}$ is a set of representatives for the conjugacy classes of the subgroups of $GL(2, \mathbb{Z})$ of order 3.

By remark (1.3) of the first chapter, $GL(2, \mathbb{Z})$ has only one conjugacy class of subgroups of order 3 and as a representative for that class we choose

$$P = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle.$$ \hspace{1cm} (163)
The normalizer $N(P)$ of $P$ in $GL(2,\mathbb{Z})$ is isomorphic to $D_{12}$. In fact,

$$N(P) = \{ \alpha, \beta | \alpha^6 = 1, \beta^2 = 1, \beta \alpha \beta = \alpha^{-1} \} \cong D_{12},$$  \hspace{1cm} (164)

where,

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (165)

Thus equation (162) becomes,

$$\hat{H}^m(GL(2,\mathbb{Z})) \cong \hat{H}^m(N(P)) \cong \hat{H}^m(D_{12})$$  \hspace{1cm} (166)

Since the v.c.d of $GL(2,\mathbb{Z})$ is 1, the ordinary cohomology $H^m(GL(2,\mathbb{Z}))$ of $GL(2,\mathbb{Z})$ is same as the Farrell cohomology $\hat{H}^m(GL(2,\mathbb{Z}))$ if $m > 1$. However it is easy to compute the ordinary mod-3 cohomology of $GL(2,\mathbb{Z})$, for example from Serre's tree of $SL_2(\mathbb{Z})$, we can write $GL(2,\mathbb{Z})$ as the amalgamated product $D_{12} \ast \mathbb{Z}/2\mathbb{Z}$ of $D_{12}$ and $\mathbb{Z}/2\mathbb{Z}$ and then one sees immediately that the mod-3 ordinary cohomology of $GL(2,\mathbb{Z})$ is same as that of $D_{12}$. But as can be seen easily that the mod-3 cohomology of $D_{12}$ is same as that of the mod-3 Farrell cohomology of $D_{12}$ in degrees $\geq 0$, hence the same is true for $GL(2,\mathbb{Z})$. From equation (159) and (166) we now get,

$$H^t(\Gamma_{\nu_1}) \cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12})$$  \hspace{1cm} (167)

(2.2) Remark: From equation (166) we see that the natural restriction map

$$\text{res} : \hat{H}^*(GL(2,\mathbb{Z})) \to \hat{H}^*(N(P))$$

induces an isomorphism between mod-3 Farrell cohomology of $GL(2,\mathbb{Z})$ and that of its subgroup $N(P) \cong D_{12}$. For this reason we will say that the mod-3 Farrell cohomology of $GL(2,\mathbb{Z})$ is detected in the subgroup $N(P)$. Similarly from equation (159) and (166) we see that the mod-3 cohomology of $\Gamma_{\nu_1}$ is detected in its subgroup $(U \rtimes \Delta) \times N(P) \cong D_{12} \times D_{12}$. 

Next we compute the mod-3 cohomology of the stabilizer $\Gamma_{\tilde{v}_3}$ of the vertex $\tilde{v}_3$. Since the Galois Stabilizer of the trivial ideal class $\Lambda$ is the whole Galois group, from theorem (1.1) of the first chapter we get an exact sequence

$$1 \rightarrow C_{\tilde{v}_3} \rightarrow \Gamma_{\tilde{v}_3} \rightarrow \Delta \rightarrow 1,$$

where $C_{\tilde{v}_3}$ denotes the centralizer of the group $\tilde{v}_3$ in $\Gamma$ and we have

$$C_{\tilde{v}_3} \cong \{(u, \epsilon) | u \in U, \epsilon \in \gamma(2,3), u \equiv \lambda(\epsilon) \mod (\omega - 1)\}$$

(169)

Recall that $\gamma(2,3)$ is the subgroup of $GL(2,\mathbb{Z})$ consisting of all the matrices whose first row is equivalent to $(\ast, 0)$ modulo 3 and $\lambda$ is the homomorphism $\gamma(2,3) \rightarrow k^\times$ sending such an element to $\star$ modulo 3.

In the above exact sequence (168) the Galois group $\Delta$ acts trivially on $\gamma(2,3)$ and via its Galois action on $U$. Since the order of $\Delta$ is prime to 3, the Hochschild-Serre spectral sequence when applied to the exact sequence (168) degenerates to give

$$H^t(\Gamma_{\tilde{v}_3}) \cong H^t(C_{\tilde{v}_3})^\Delta.$$

(170)

Next we observe that the index of the subgroup $C_{\tilde{v}_3}$ in the group $U \times \gamma(2,3)$ is 2 and the coset representatives can be found in $U \times 1$ which acts trivially on $C_{\tilde{v}_3}$, hence the mod-3 cohomology of $C_{\tilde{v}_3}$ is the same as that of $U \times \gamma(2,3)$. Since $\Delta$ acts trivially on $\gamma(2,3)$ and $U \times \Delta \cong D_{12}$, we get from equation (170) and Kunneth formula

$$H^t(\Gamma_{\tilde{v}_3}) \cong [H^t(C_{\tilde{v}_3})]^\Delta$$

$$\cong \left[ \bigoplus_{i+j=t} H^i(U) \otimes H^j(\gamma(2,3)) \right]^\Delta$$

(171)

$$\cong \left[ \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(\gamma(2,3)) \right].$$
The elementary abelian 3-subgroups of \( \gamma(2, 3) \) are of rank \( \leq 1 \) and hence to determine \( H^m(\gamma(2, 3)) \) we use the corollary (1.2) in the introduction which gives,

\[
\hat{H}^t(\gamma(2, 3)) \cong \prod_{P' \in \mathcal{P}'} \hat{H}^t(N(P')), \tag{172}
\]

where \( \mathcal{P}' \) is a set of representatives for the conjugacy classes of elementary abelian 3-subgroups of \( \gamma(2, 3) \) of rank 1. By lemma (3.3) of Chapter I there is only one conjugacy class of subgroups of order 3. As a representatives for that class we choose

\[
P' = \left\langle \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \right\rangle. \tag{173}
\]

The normalizer \( N(P') \) of \( P' \) in \( \gamma(2, 3) \) is isomorphic to the dihedral group \( D_{12} \). In fact

\[
N(P') = \{ \alpha', \beta' | \alpha'^5 = 1, \beta'^2 = 1, \beta' \alpha' \beta' = \alpha'^5 \} \cong D_{12}, \tag{174}
\]

where

\[
\alpha' = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \beta' = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \tag{175}
\]

Hence equation (172) becomes,

\[
\hat{H}^t(\gamma(2, 3)) \cong \hat{H}^t(N(P')) \cong \hat{H}^t(D_{12}). \tag{176}
\]

The v.c.d of \( \gamma(2, 3) \) is 1 and hence the mod-3 cohomology \( H^t(\gamma(2, 3)) \) is same as the mod-3 Farrell cohomology \( \hat{H}^t(\gamma(2, 3)) \) if \( t > 1 \). However as shown in Appendix A the mod-3 cohomology of \( \gamma(2, 3) \) is same as its mod-3 Farrell cohomology even in degree 0 and 1. Since the mod-3 cohomology of \( D_{12} \) is same as the mod-3 Farrell cohomology in degrees \( \geq 0 \) equation (171) can be rewritten as

\[
H^t(\Gamma_{\delta}) \cong \bigoplus_{i+j=t} H^i(U \times \Delta) \otimes H^j(N(P')) \\
\cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12}). \tag{177}
\]
Remark: From equation (171) and (177) we see that the mod-3 cohomology of $\Gamma_{v_3}$ is detected in the subgroup $C'_0 \rtimes (\Delta \times <\beta'>)$ of $\Gamma_{v_3}$, where

$$C'_0 = \{(u, \epsilon)|u \in U, \epsilon \in <\alpha'>, u \equiv \lambda(\epsilon) \mod (\omega - 1)\}.$$  

(178)

Here $\Delta$ acts trivially on $<\alpha'>$ and via the Galois action on $U$ whereas $\beta'$ acts trivially on $U$ and by conjugation on $\alpha'$. The group $<\alpha'>$ generated by $\alpha'$ is isomorphic to $U$ and similarly the group $<\beta'>$ is isomorphic to $\Delta$. Thus we can rewrite the above equation as

$$C'_0 \cong C_0 = \{(u, u')|u, u' \in U, \text{and } u \equiv u' \mod (\omega - 1)\}$$  

(179)

and thus under this isomorphism we see that the mod-3 cohomology of $\Gamma_{v_3}$ is detected in $C_0 \rtimes (\Delta \times \Delta)$.

§3 Cohomology of $\Gamma_{v_2}$.

In this section we compute the mod-3 Farrell cohomology of $\Gamma_{v_2}$. Since

$$v_2 = \left( \begin{array}{cc} [\Lambda] & 0 \\
0 & [\Lambda] \end{array} \right),$$  

(180)

by lemma (2.2) of Chapter I the centralizer $C_{v_2}$ of $v_2$ in $\Gamma$ is isomorphic to $GL(2, \mathbb{Z}[\omega])$ and that we have an exact sequence

$$1 \to C_{v_2} \to \Gamma_{v_2} \to \Delta \to 1$$  

(181)

where $\Delta$ acts on $GL(2, \mathbb{Z}[\omega])$ via the Galois action on $\mathbb{Z}[\omega]$.

Applying the Hochschild-Serre spectral sequence to (181) we get

$$E_2^{st} = H^s(\Delta, H^t(C_{v_2})) \Rightarrow H^{s+t}(\Gamma_{v_2})$$  

(182)
Since the order of $\Delta$ is prime to 3, this spectral sequence reduces to
\[
H^t(\Gamma_{v_2}) \cong H^0(\Delta, H^t(C_{v_1})) \cong H^t(C_{v_1})^\Delta
\]
(183)
i.e., $H^t(\Gamma_{v_2})$ is just the $\Delta$-invariants in $H^t(C_{v_2})$.

To determine $H^*(GL(2, \mathbb{Z}[\omega])$ we use the Alperin's complex $\mathcal{X}$. First we set $
 \Gamma' = GL(2, \mathbb{Z}[\omega])$. We briefly describe the complex $\mathcal{X}$. See [Al] for more details.

Consider the set $\mathcal{L}$ of free direct product summands of $O^2$. Elements of $\mathcal{L}$ are called lines. The line in $O^2$ passing through the origin and the point $(a, b)$ will be denoted by $(a, b)$. We say that $L_1, L_2 \in \mathcal{L}$ are independent if $L_1 + L_2 = L_1 \oplus L_2 = O^2$.

Then $\mathcal{X}$ is the complex of partially ordered sets (by inclusion) of elements of subsets of $\mathcal{L}$ of type
\[
\{L_0, \ldots, L_q\}, \quad q \geq 1, \quad L_i \in \mathcal{L}
\]
for which $L_i$, $L_j$ are independent for $0 \leq i \neq j \leq q$. The element $g \in \Gamma'$ acts on the line $(a, b)$ through the matrix multiplication, i.e., $g \cdot (a, b) = g \begin{bmatrix} a \\ b \end{bmatrix}$.

By the corollary in section 5 of [Al], the fundamental domain for the action of $\Gamma'$ on $\mathcal{X}$ is a single 2-simplex with vertices
\[
A = \{(0, 1), (1, 0)\}, \\
B = \{(0, 1), (1, 0), (1, 1)\}, \\
C = \{(0, 1), (1, 0), (1, 1), (1, \omega)\}
\]
(184)

Next we determine the stabilizers $\Gamma'_v$ of each vertex $A, B, C$ of $\mathcal{X} \mod \Gamma'$. Each vertex is determined by a collection of pairwise independent lines $\mathcal{L}(v)$. Consequently we have a homomorphism
\[
\Gamma'_v \rightarrow \text{Sym}(\mathcal{L}(v))
\]
with image denoted by $\Sigma_v$ and kernel denoted by $K_v$. Here $\text{Sym}(S)$ denotes the symmetric group on the set $S$. 
If \( v = A \), then \( \Gamma'_A \) contains
\[
\Pi = \begin{pmatrix} -\omega & 0 \\ 0 & -\omega \end{pmatrix}, \quad \Psi = \begin{pmatrix} -\omega & 0 \\ 0 & \omega^2 \end{pmatrix}
\] (185)
in \( K_A \) and
\[
\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (186)
which induces the transposition of the elements of \( A \). Consequently there is an exact sequence
\[
1 \to K_A \to \Gamma'_A \to \Sigma_A \to 1
\] (187)
where \( K_A \cong \langle \Pi > \times < \Psi > \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) and \( \Sigma_A \cong \langle \Theta \rangle \). Here \( \Theta \) commutes with \( \Pi \) and \( \Theta \Psi \Theta^{-1} = \Psi^5 \).

In case \( v = B \), \( \Pi, \Theta \in \Gamma'_B \) and \( \Gamma'_B \) also contain
\[
\Phi = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}
\] (188)
which induces a 3-cycle on the lines in \( B \). We have an exact sequence,
\[
1 \to K_B \to \Gamma'_B \to \Sigma_B \to 1
\] (189)
where, \( K_B \cong \langle \Pi > \cong \mathbb{Z}/6\mathbb{Z} \) and \( \Sigma_B \cong \{ \Phi, \Theta \mid \Phi^3 = 1, \Theta^2 = 1, \Theta \Phi \Theta^{-1} = \Phi^2 \} \). It can be shown that \( \Phi \) and \( \Theta \) both commutes with \( \Pi \) and the above exact sequence splits. Thus we get,
\[
\Gamma'_B \cong \langle \Pi > \times \langle \Phi, \Theta >
\cong \mathbb{Z}/6\mathbb{Z} \times \Sigma_B
\] (190)
\cong \mathbb{Z}/6\mathbb{Z} \times D_6

In case \( v = C \), we again have \( \Pi \in K_C \) and \( K_C \cong \langle \Pi > \) is a cyclic group of order 6. It is not difficult to see that the image of \( \Gamma'_C \) in \( \text{Sym}(\mathcal{L}(C)) \) contains no transpositions, however \( \Phi \in \Gamma'_C \) induces a 3-cycle and
\[
\Upsilon = \Psi \Theta = \begin{pmatrix} 0 & -\omega \\ \omega^2 & 0 \end{pmatrix}
\] (191)
induces a element $\bar{T}$ which gives double transposition on the elements of $C$. The elements $\Phi, \bar{T}$ generate an alternating group $A_4$ of $\text{Sym}(L(C))$. Thus we have an exact sequence,

$$1 \to K_C \to \Gamma'_C \to \Sigma_C \to 1.$$  \hspace{1cm} (192)

where $K_C = < \Pi > \cong \mathbb{Z}/6\mathbb{Z}$ and $\Sigma_C = < \Phi, \bar{T} > \cong A_4$. Both the generators $\Phi$ and $\bar{T}$ of $A_4$ act trivially on $\Pi$, so the above exact sequence (192) is central. However it can be shown that it does not split.

Next we find the stabilizers of the 1-cells $AB$, $AC$ and $BC$.

$$\Gamma'_{AB} = \Gamma'_A \cap \Gamma'_B \hspace{1cm} \Gamma'_{AC} = \Gamma'_A \cap \Gamma'_C \hspace{1cm} \Gamma'_{BC} = \Gamma'_B \cap \Gamma'_C \hspace{1cm} (193)$$

$$=< \Pi > \times < \Theta > \hspace{1cm} =< \Pi > \times < \bar{T} > \hspace{1cm} =< \Pi > \times < \Phi >$$

$$\cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \hspace{1cm} \cong \mathbb{Z}/12\mathbb{Z} \hspace{1cm} \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$$

and finally the stabilizer of the 2-cell $ABC$ is

$$\Gamma'_{ABC} = \Gamma'_A \cap \Gamma'_B \cap \Gamma'_C$$

$$=< \Pi > \hspace{1cm} (194)$$

$$\cong \mathbb{Z}/6\mathbb{Z}.$$  

(3.1) The mod-3 cohomology of $GL(2, \mathbb{Z}[\omega])$.

In this section we compute the Farrell cohomology of $\Gamma' = GL(2, \mathbb{Z}[\omega])$. We use the spectral sequence

$$E_1^{p,q} = \prod_{\sigma \in \Sigma_p} H^q(\Gamma'_\sigma) \Rightarrow H^{p+q}(\Gamma'),$$  \hspace{1cm} (195)

where $\Sigma_p$ is a set of representatives for the $p$-cells of $\mathcal{X}$ mod $\Gamma'$.  

Computation of $d_1^{0,q}$

There are six restriction maps we need to figure out.

\[
\begin{align*}
\rho^A_{AB} : H^*(\Gamma'_A) &\to H^*(\Gamma'_{AB}), \\
\rho^A_{AC} : H^*(\Gamma'_A) &\to H^*(\Gamma'_{AC}), \\
\rho^B_{AB} : H^*(\Gamma'_B) &\to H^*(\Gamma'_{AB}), \\
\rho^B_{BC} : H^*(\Gamma'_B) &\to H^*(\Gamma'_{BC}), \\
\rho^C_{AC} : H^*(\Gamma'_C) &\to H^*(\Gamma'_{AC}), \\
\rho^C_{BC} : H^*(\Gamma'_C) &\to H^*(\Gamma'_{BC}).
\end{align*}
\]

First we determine the cohomology of each stabilizer.

For $\Gamma'_A$ we have an exact sequence (187). Applying the Hochschild-Serre spectral sequence to the exact sequence (187), we get

\[
E_2^{p,q} = H^p(\Sigma_A, H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})) \Rightarrow H^{p+q}(\Gamma'_A)
\]

Since the order of $\Sigma_A$ is prime to 3, this reduces to

\[
\hat{H}^0(\Sigma_A, H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})) \cong H^q(\Gamma'_A)
\]

i.e.

\[
H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z})^{\Sigma_A} \cong H^q(\Gamma'_A)
\]

Here $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \cong < \Pi > \times < \Psi >, \Sigma_A \cong < \Theta >$ and $\Theta$ act trivially on $\Pi$ and $\Theta \Psi \Theta^{-1} = \Psi^5$. The action of $\Theta$ on $\Pi$ is exactly the same as the action of $b$ on $\mathbb{Z}/6\mathbb{Z}$ inside $D_{12} = \{ a, b \mid a^6 = 1 = b^2, bab^{-1} = a^5 \}$, hence after applying the Kunneth formula we get,

\[
H^n(\Gamma'_A) \cong \bigoplus_{p+q=n} H^p(\mathbb{Z}/6\mathbb{Z}) \otimes H^q(D_{12})
\] (196)
From equation (190) we see that $\Gamma_B' \cong \mathbb{Z}/6\mathbb{Z} \times D_6$, hence by the Kunneth formula we have

$$H^n(\Gamma_B') \cong \bigoplus_{p+q=n} H^p(\mathbb{Z}/6\mathbb{Z}) \otimes H^q(D_6) \quad (197)$$

To find $H^*(\Gamma_C')$ we proceed in the same way as we did in the computation of $H^*(\Gamma_A')$. Since the exact sequence (192) is central, it is easy to see that the mod-$p$ Hochschild-Serre spectral sequence when applied to (192) degenerates and further since the mod-3 cohomology of $A_4$ is same as that of $\mathbb{Z}/3\mathbb{Z}$, we get,

$$H^n(\Gamma_C') \cong \bigoplus_{p+q=n} H^p(\mathbb{Z}/6\mathbb{Z}) \otimes H^q(\mathbb{Z}/3\mathbb{Z}) \quad (198)$$

The other stabilizers are easy to deal with, because they are just the direct product of two cyclic groups and one can apply the Kunneth formula to compute their cohomology. So we just state the final results.

$$\Gamma'_{AB} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \Rightarrow H^n(\Gamma'_{AB}) \cong H^n(\mathbb{Z}/3\mathbb{Z})$$

$$\Gamma'_{AC} \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \Rightarrow H^n(\Gamma'_{AC}) \cong H^n(\mathbb{Z}/3\mathbb{Z})$$

$$\Gamma'_{BC} \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \Rightarrow H^n(\Gamma'_{BC}) \cong \bigoplus_{p+q=n} H^p(\mathbb{Z}/6\mathbb{Z}) \otimes H^q(\mathbb{Z}/3\mathbb{Z}) \quad (199)$$

$$\Gamma'_{ABC} \cong \mathbb{Z}/6\mathbb{Z} \Rightarrow H^n(\Gamma'_{ABC}) \cong H^n(\mathbb{Z}/6\mathbb{Z})$$

Thus the cohomology rings of the stabilizers are

$$H^*(\Gamma_A') = k[\xi, \eta, \xi', \eta']$$

$$H^*(\Gamma_B') = k[\xi, \eta, \xi, \eta']$$

$$H^*(\Gamma_C') = k[\xi, \eta', \xi', \eta]$$

$$H^*(\Gamma'_{AB}) = k[\xi, \eta]$$

$$H^*(\Gamma'_{AC}) = k[\xi, \eta]$$

$$H^*(\Gamma'_{BC}) = k[\xi, \eta, \xi', \eta']$$

$$H^*(\Gamma'_{ABC}) = k[\xi, \eta]$$

(200)
Here $\xi, \eta$, (resp. $\xi', \eta'$) are the generators for $H^*(\mathbb{Z}/6\mathbb{Z})$ (resp. $H^*(\mathbb{Z}/3\mathbb{Z})$) and $\hat{\xi}, \hat{\eta}$ are the generators for $H^*(D_{12})$. Thus $\deg(\xi) = \deg(\xi') = 2$, $\deg(\eta) = \deg(\eta') = 1$, $\eta^2 = \eta'^2 = 0$, and similarly $\deg(\hat{\xi}) = 4$, $\deg(\hat{\eta}) = 3$ with $\hat{\eta}^2 = 0$.

The restriction maps $\rho$'s can now be described in terms of the cohomology generators as

$$
\begin{align*}
\rho_{AB}^A &: \quad H^*(\Gamma_A') \rightarrow H^*(\Gamma_{AB}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta] \\
& \quad \xi, \eta, \xi', \eta' \rightarrow \xi, \eta, 0, 0 \\
\rho_{AC}^A &: \quad H^*(\Gamma_A') \rightarrow H^*(\Gamma_{AC}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta] \\
& \quad \xi, \eta, \xi', \eta' \rightarrow \xi, \eta, 0, 0 \\
\rho_{AB}^B &: \quad H^*(\Gamma_B') \rightarrow H^*(\Gamma_{AB}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta] \\
& \quad \xi, \eta, \xi', \eta' \rightarrow \xi, \eta, 0, 0 \\
\rho_{BC}^B &: \quad H^*(\Gamma_C') \rightarrow H^*(\Gamma_{BC}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta, \xi', \eta'] \\
& \quad \xi, \eta, \xi', \eta', \xi'^2, \xi' \cup \eta' \\
\rho_{AC}^C &: \quad H^*(\Gamma_C') \rightarrow H^*(\Gamma_{AC}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta] \\
& \quad \xi, \eta, \xi', \eta' \rightarrow \xi, \eta, 0, 0 \\
\rho_{BC}^C &: \quad H^*(\Gamma_C') \rightarrow H^*(\Gamma_{BC}') \\
& \quad k[\xi, \eta, \xi', \eta'] \rightarrow k[\xi, \eta, \xi', \eta'] \\
& \quad \xi, \eta, \xi', \eta' \rightarrow \xi, \eta, \xi', \eta' \\
\end{align*}
$$

The differential $d_1^{0,q}$ can now be described as,

$$
\begin{align*}
d_1^{0,q} &: H^q(\Gamma_A') \oplus H^q(\Gamma_B') \oplus H^q(\Gamma_C') \rightarrow H^q(\Gamma_{AB}') \oplus H^q(\Gamma_{AC}') \oplus H^q(\Gamma_{BC}') \\
d_1^{0,q} &= (\rho_{AB}^B - \rho_{AB}^A) \oplus (\rho_{AC}^C - \rho_{AC}^A) \oplus (\rho_{BC}^C - \rho_{BC}^B).
\end{align*}
$$
Computation of $d_1^{1,q}$

The three restriction maps $\rho_{ABC}^{AB}$, $\rho_{ABC}^{AC}$, $\rho_{ABC}^{BC}$ are,

\[
\begin{align*}
\rho_{ABC}^{AB} : & H^*(\Gamma_{AB}') \to H^*(\Gamma_{ABC}') \\
& k[\xi, \eta] \xrightarrow{id} k[\xi, \eta] \\
\rho_{ABC}^{AC} : & H^*(\Gamma_{AC}') \to H^*(\Gamma_{ABC}') \\
& k[\xi, \eta] \xrightarrow{id} k[\xi, \eta] \\
\rho_{ABC}^{BC} : & H^*(\Gamma_{BC}') \to H^*(\Gamma_{ABC}') \\
& k[\xi, \eta, \xi', \eta'] \to k[\xi, \eta] \\
\xi, \eta, \xi', \eta' & \to \xi, \eta, 0, 0
\end{align*}
\]

Now we piece together all the above information to compute $d_1^{0,q}$, $d_1^{1,q}$ explicitly in each dimension $q \mod 4$. We consider four cases depending on $q = 0, 1, 2, 3 \mod 4$. In each case we compute $E_2^{0,q}$ and $E_2^{1,q}$.

In each degree we choose the $k$-basis for the cohomology of the stabilizers, $H^q(\Gamma_A')$, $H^q(\Gamma_B')$, e.t.c., as in Appendix B.

**Case 1:** $q \equiv 0 \mod 4$, $q = 4m$

\[
d_1^{0,q} : H^q(\Gamma_A') \oplus H^q(\Gamma_B') \oplus H^q(\Gamma_C') \to H^q(\Gamma_{AB}') \oplus H^q(\Gamma_{AC}') \oplus H^q(\Gamma_{BC}')
\]

\[
: k^{(2m+1)} \oplus k^{(2m+1)} \oplus k^{(4m+1)} \to k \oplus k \oplus k^{(4m+1)}
\]

In terms of generators, $d_1^{0,q}$ can be described as

\[
d_1^{0,q} \begin{pmatrix} (x_0, \ldots, x_{2m}) \\ (y_0, \ldots, y_{2m}) \\ (z_0, \ldots, z_{4m}) \end{pmatrix} = \begin{pmatrix} y_0 - x_0 \\ z_0 - x_0 \\ (z_0 - y_0, z_1, z_2 - y_1, z_3, \ldots, z_{4m} - y_{2m}) \end{pmatrix}
\]  
(203)
Thus \( E_2^{0,4m} = \ker d_1^{0,4m} \cong k^{(4m+1)} \). Next we describe \( d_1^{1,4m} \)

\[
d_1^{1,q} : H^q(\Gamma'_{AB}) \oplus H^q(\Gamma'_{AC}) \oplus H^q(\Gamma'_{BC}) \to H^q(\Gamma'_{ABC})
\]

\[
: k \oplus k \oplus k^{(4m+2)} \longrightarrow k
\]

\[
d_1^{1,q}(x, y, (z_0, \ldots, z_{4m+1})) = x - y + z_0
\]

It is easy see that \( \text{im} d_1^{0,4m} = \ker d_1^{1,4m} \) and \( d_1^{1,4m} \) is a surjective map. Thus, we get \( E_2^{1,4m} = 0 = E_2^{2,4m} \).

**Case 2:** \( q \equiv 1 \mod 4, q = 4m + 1 \)

The map \( d_1^{0,q} \) can be described as

\[
d_1^{0,q} : H^q(\Gamma'_A) \oplus H^q(\Gamma'_B) \oplus H^q(\Gamma'_C) \longrightarrow H^q(\Gamma'_{AB}) \oplus H^q(\Gamma'_{AC}) \oplus H^q(\Gamma'_{BC})
\]

\[
: k^{(2m+1)} \oplus k^{(2m+1)} \oplus k^{(4m+2)} \longrightarrow k \oplus k \oplus k^{(4m+2)}
\]

\[
d_1^{0,q}\begin{pmatrix}
(x_0, \ldots, x_{2m}) \\
(y_0, \ldots, y_{2m}) \\
(z_0, \ldots, z_{4m+1})
\end{pmatrix} = \begin{pmatrix}
y_0 - x_0 \\
z_0 - x_0 \\
(z_0 - y_0, z_1, z_2 - y_1, z_3, \ldots, z_{4m} - y_{2m}, z_{4m+1})
\end{pmatrix}.
\]

Thus,

\[
\ker(d_1^{0,q}) \cong \left\{ \begin{pmatrix}
(t_0, x_1, x_2, \ldots, x_{2m}) \\
(t_0, y_1, y_2, \ldots, y_{2m}) \\
(t_0, 0, y_1, 0, y_2, \ldots, 0, y_{2m}, 0)
\end{pmatrix} \in k^{(2m+1)} \oplus k^{(2m+1)} \oplus k^{(4m+2)} \right\}
\]

\[
\cong k^{(4m+1)}.
\]

Hence \( E_2^{0,4m+1} = \ker d_1^{4m+1} \cong k^{(4m+1)} \).
Similarly,
\[
d_1^{1,q} : H^q(\Gamma'_{AB}) \oplus H^q(\Gamma'_{AC}) \oplus H^q(\Gamma'_{BC}) \to H^q(\Gamma'_{ABC})
\]
\[
: k \oplus k \oplus k^{(4m+3)} \longrightarrow k
\]
\[
d_1^{1,q}(x, y, (z_0, \ldots, z_{4m+2})) = x - y + z_0
\]
and again, \( \text{im } d_1^{0,q} = \ker d_1^{1,4m+1} \) which implies \( E_2^{1,4m+1} = 0 \) and \( d_1^{1,4m+1} \) is surjective which implies \( E_2^{2,4m+1} = 0 \).

**Case 3: \( q \equiv 2 \mod 4, \ q = 4m + 2 \)**

The map \( d_1^{0,q} \) can be described as,
\[
d_1^{0,q} : H^q(\Gamma'_A) \oplus H^q(\Gamma'_B) \oplus H^q(\Gamma'_C) \longrightarrow H^q(\Gamma'_{AB}) \oplus H^q(\Gamma'_{AC}) \oplus H^q(\Gamma'_{BC})
\]
\[
: k^{(2m+1)} \oplus k^{(2m+1)} \oplus k^{(4m+3)} \longrightarrow k \oplus k \oplus k^{(4m+3)}
\]
In terms of generators it looks like
\[
d_1^{0,q} \begin{pmatrix} (x_0, \ldots, x_{2m}) \\ (y_0, \ldots, y_{2m}) \\ (z_0, \ldots, z_{4m+2}) \end{pmatrix}
\]
\[
= \begin{pmatrix} y_0 - x_0 \\ z_0 - x_0 \\ (z_0 - y_0, z_1, z_2 - y_1, z_3, z_4 - y_2, \ldots, z_{2m-1}, z_{2m} - y_m, \\ z_{2m+1}, z_{2m+2}, z_{2m+3} - y_{m+1}, z_{2m+4}, z_{2m+5} - y_{m+2}, \\ z_{2m+6}, \ldots, z_{4m-1} - y_{2m-1}, z_{4m}, z_{4m+1} - y_{2m}, z_{4m+2}) \end{pmatrix}
\]

(207)
Thus,
\[
\ker(d_1^{0,q}) \cong \left\{ \begin{pmatrix} (t_0, x_1, x_2, \ldots, x_{2m}) \\ (t_0, y_1, y_2, \ldots, y_{2m}) \\ (t_0, 0, y_1, 0, y_2, \ldots, y_{m-1}, 0, y_m, 0, \\ 0, y_{m+1}, 0, y_{m+2}, 0, \ldots, y_{2m}, 0) \end{pmatrix} \in k^{2m+1} \oplus k^{2m+1} \oplus k^{4m+3} \right\}
\]
\[
\cong k^{(4m+1)}.
\]
So we get, $\mathcal{E}_{2,4m} = \ker d_1^{0,q} \cong k^{(4m+1)}$.

For $d_1^{1,q}$, we have

$$d_1^{1,q} : H^q(\Gamma_{AB}) \oplus H^q(\Gamma_{AC}) \oplus H^q(\Gamma_{BC}) \rightarrow H^q(\Gamma_{ABC})$$

$$: k \oplus k \oplus k^{(4m+3)} \rightarrow k$$

$$d_1^{1,q}(x, y, (z_0, \ldots, z_{4m+2})) = x - y + z_0 \quad (208)$$

And again we have, $\im d_1^{0,4m+2} = \ker d_1^{1,4m+2}$ which implies $E_{2,4m+2} = 0$, and $d_1^{1,4m+2}$ is surjective which implies $E_{2,4m+2} = 0$.

Case 4: $q \equiv 3 \mod 4$, $q = 4m + 3$

The map $d_1^{0,q}$ can be described as

$$d_1^{0,q} : H^q(\Gamma_A) \oplus H^q(\Gamma_B) \oplus H^q(\Gamma_C) \rightarrow H^q(\Gamma_{AB}) \oplus H^q(\Gamma_{AC}) \oplus H^q(\Gamma_{BC})$$

$$: k^{(2m+2)} \oplus k^{(2m+2)} \oplus k^{(4m+4)} \rightarrow k \oplus k \oplus k^{(4m+4)}$$

$$d_1^{0,q} \begin{pmatrix} (x_0, \ldots, x_{2m+1}) \\ (y_0, \ldots, y_{2m+1}) \\ (z_0, \ldots, z_{4m+3}) \end{pmatrix} = \begin{pmatrix} y_0 - x_0 \\ z_0 - x_0 \\ (z_0 - y_0, z_1 - y_1, z_3, \ldots, z_{2m-1}, z_{2m} - y_m, z_{2m+1}, z_{2m+2}, z_{2m+3} - y_{m+1}, z_{2m+4}, z_{2m+5} - y_{m+2}, \ldots, z_{4m+2}, z_{4m+3} - y_{2m+1}) \end{pmatrix}$$

Therefore,

$$\ker(d_1^{0,q}) \cong \begin{pmatrix} (t_0, x_1, x_2, \ldots, x_{2m+1}) \\ (t_0, y_1, y_2, \ldots, y_{2m+1}) \\ (t_0, 0, y_0, 0, \ldots, y_m, 0, 0, y_{m+1}, 0, \ldots, y_{2m+1}) \end{pmatrix} \in k^{(2m+2)} \oplus k^{(2m+2)} \oplus k^{(4m+4)}$$

$$\cong k^{(4m+3)}$$
Thus, $E_2^{0,4m+3} = \ker(d_1^{0,4m+3}) \cong k(4m+3)$.

Similarly,

\[
d_1^{1,q} : H^q(\Gamma_{AB}) \oplus H^q(\Gamma_{AC}) \oplus H^q(\Gamma_{BC}) \to H^q(\Gamma_{ABC})
\]

\[
: k \oplus k \oplus k^{(4m+5)} \longrightarrow k
\]

\[
d_1^{1,q}(x, y, (z_0, \ldots, z_{4m+4})) = x - y + z_0
\]

Again we have $\text{im}(d_1^{0,4m+3}) = \ker(d_1^{1,4m+3})$ which implies $E_2^{1,4m+3} = 0$, and $d_1^{1,4m+2}$ is surjective which implies $E_2^{2,4m+3} = 0$.

Thus the spectral sequence (195) converges to give $E_2^{0,q} = H^q(\Gamma')$,

\[
H^q(\Gamma') = \begin{cases} 
  k^{(4m+1)} & \text{if } q \equiv 0 \mod 4, \ q = 4m \\
  k^{(4m+1)} & \text{if } q \equiv 1 \mod 4, \ q = 4m + 1 \\
  k^{(4m+1)} & \text{if } q \equiv 2 \mod 4, \ q = 4m + 2 \\
  k^{(4m+3)} & \text{if } q \equiv 3 \mod 4, \ q = 4m + 3. 
\end{cases}
\]  

(3.2) Remark: Since in spectral sequence (195) $E_2^{1,q} = 0 = E_2^{2,q}$, and $H^q(\Gamma') = E_2^{0,q}$, we shall say as before that the cohomology of $\Gamma'$ is detected in $\Gamma_A, \Gamma_B, \Gamma_C$. It is actually equal to the kernel of $d_1^{0,q}$ in $H^q(\Gamma_A) \oplus H^q(\Gamma_B) \oplus H^q(\Gamma_C)$.

(3.3) Remark: Before proceeding to the next section we mentioned that in [Al] Alperin used the complex $\mathcal{X}$ to find the integral homology of $SL_2(\mathbb{Z}[\omega])$. His main result is

\[
H_n(SL_2(\mathbb{Z}[\omega]), \mathbb{Z}) = \begin{cases} 
  \mathbb{Z} & \text{if } n = 0 \\
  \mathbb{Z}/3\mathbb{Z} & \text{if } n \equiv 1 \mod 4 \\
  \mathbb{Z}/4\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \\
  \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & \text{if } n \equiv 3 \mod 4 \\
  0 & \text{if } n \equiv 4 \mod 4, n \neq 0. 
\end{cases}
\]
(3.4) The Farrell cohomology of $\Gamma_{v_2}$:

From equation (183) we know that $H^i(\Gamma_{v_2})$ is equal to the $\Delta = \{1, \delta\}$-invariants in $H^i(GL(2, O))$. To determine $\Delta$-invariants we first find the $\Delta$-invariants in $H^q(\Gamma'_A), H^q(\Gamma'_B)$, and $H^q(\Gamma'_C)$ and then use remark (3.2) to get the $\Delta$-invariants in $H^q(GL(2, O))$.

From exact-sequence (187),

$$1 \to \langle \Pi \rangle \times \langle \Psi \rangle \to \Gamma'_A \to \langle \Theta \rangle \to 1.$$ 

and the fact that $\delta(\Pi) = \Pi^5$, and $\delta(\Psi) = \Psi^5$, we see that the corresponding action of $\delta$ on the generators of the cohomology ring of $H^*(\Gamma'_A)$ is given by,

$$\delta^* : H^*(\Gamma'_A) \to H^*(\Gamma'_A)$$

$$k[\xi, \eta, \bar{\xi}, \bar{\eta}] \to k[\xi, \eta, \bar{\eta}]$$

$$\xi, \eta, \bar{\xi}, \bar{\eta} \to 2\xi, 2\eta, \bar{\xi}, \bar{\eta}.$$ 

Here $\delta^*$ acts trivially on $\bar{\xi}, \bar{\eta}$ because the action of $\delta$ and $\Theta$ on $\Psi$ is same, hence these generators are already fixed under that action. As for $\xi, \eta$, we see that the action of $\delta$ on $\Pi$ is same as the action on the generator of $\mathbb{Z}/6\mathbb{Z}$ inside $D_{12}$, which takes the generators $\xi, \eta$ of $H^* (\mathbb{Z}/6\mathbb{Z})$ to $2\xi, 2\eta$. Hence it follows that

$$H^*(\Gamma'_A)^\Delta = k[\xi^2, \xi \cup \eta, \bar{\xi}, \bar{\eta}] .$$

(214)

Similarly the action of $\delta$ on the cohomology ring of $\Gamma'_B$ can be described as,

$$\delta^* : H^*(\Gamma'_B) \to H^*(\Gamma'_B)$$

$$k[\xi, \eta, \bar{\xi}, \bar{\eta}] \to k[\xi, \eta, \bar{\xi}, \bar{\eta}]$$

$$\xi, \eta, \bar{\xi}, \bar{\eta} \to 2\xi, 2\eta, \bar{\xi}, \bar{\eta}.$$ 

Hence we have

$$H^*(\Gamma'_B)^\Delta = k[\xi^2, \xi \cup \eta, \bar{\xi}, \bar{\eta}] .$$

(216)
Finally for $\Gamma_C'$ we again see as in case of $\Gamma_A'$ that $\delta^*$ takes the generators $\zeta$ and $\eta$ of the cohomology ring $H^*(\Gamma_C') \cong \mathbb{k}[\zeta, \eta, \zeta', \eta']$ of $\Gamma_C'$ to $2\zeta$ and $2\eta$ respectively. However from an equation (188), we see that $\delta$ acts trivially on $\Phi$ and hence from an exact sequence (192) we conclude that $\delta^*$ acts trivially on the cohomology generators $\xi'$ and $\eta'$ of the cohomology ring of $\Gamma_C'$. Hence we now have,

$$\delta^* : H^*(\Gamma_C') \rightarrow H^*(\Gamma_C')$$

$$\begin{align*}
\mathbb{k}[\xi, \eta, \xi', \eta'] & \rightarrow \mathbb{k}[\xi, \eta, \xi', \eta'] \\
\xi, \eta, \xi', \eta' & \rightarrow 2\xi, 2\eta, \xi', \eta'.
\end{align*}$$

(217)

Hence the ring of $\Delta$-invariants in the cohomology ring of $\Gamma_C'$ is generated by $\zeta^2, \zeta \cup \eta, \zeta'$ and $\eta'$, i.e.,

$$H^*(\Gamma_C')^\Delta = \mathbb{k}[\zeta^2, \zeta \cup \eta, \zeta', \eta'].$$

(218)

In the previous subsection (3.1) we have seen that $H^*(\Gamma')$ is equal to the kernel of $d^{0,q}_1$, hence to find $\Delta$-invariants in $H^q(\Gamma')$, it is enough to find the $\Delta$-invariants in kernel of $d^{0,q}_1$. Moreover, in the kernel of $d^{0,q}_1$ the classes which come from $H^q(\Gamma_C')$ are completely determined by $H^q(\Gamma_A')$ and $H^q(\Gamma_B')$. Hence it is enough find the $\Delta$-invariants in the kernel of $d^{0,q}_1$ in $H^q(\Gamma_A')$ and $H^q(\Gamma_B')$. Note also that $H^q(\Gamma_A')^\Delta \cong H^q(\Gamma_B')^\Delta$. We find them in the four cases $q \mod 4$.

Case 1: $q \equiv 0 \mod 4, \; q = 4m$

In this case $x_0, x_1, \ldots, x_m$ are the invariants in $H^q(\Gamma_A')$ and $y_0, y_1, \ldots, y_m$ are the invariants in $H^q(\Gamma_B')$. Thus,

$$H^{4m}(\Gamma')^\Delta \cong \left\{ \begin{pmatrix} (x_0, \ldots, x_m) \\ (x_0, y_1, y_2, \ldots, y_m) \\ (x_0, y_1, y_2, \ldots, y_m, 0, 0, \ldots, 0) \end{pmatrix} \in \mathbb{k}^{(m+1)} \oplus \mathbb{k}^{(m+1)} \oplus \mathbb{k}^{(2m+1)} \right\}$$

$$\cong \mathbb{k}^{(2m+1)}.$$
Case 2: $q \equiv 1 \mod 4$, $q = 4m + 1$

There are no $\Delta$-invariants in $H^{4m+1}(\Gamma'_A)$ and $H^{4m+1}(\Gamma'_B)$. Thus, $H^{4m+1}(\Gamma')^\Delta = 0$.

Case 3: $q \equiv 2 \mod 4$, $q = 4m + 2$

In this case $x_{m+1}, x_{m+2}, \ldots, x_{2m}$ are the $\Delta$-invariants in $H^{4m+2}(\Gamma'_A)$ and $x_{m+1}, y_{m+2}, \ldots, y_{2m}$ are the $\Delta$-invariants in $H^{4m+2}(\Gamma'_B)$.

Thus,

$$H^{4m+2}(\Gamma')^\Delta \cong \left\{ \begin{pmatrix} (x_{m+1}, \ldots, x_{2m}) \\ (y_{m+1}, \ldots, y_{2m}) \\ (0, \ldots, 0, y_{m+1}, y_{m+2}, \ldots, y_{2m}) \end{pmatrix} \in \mathbb{k}^m \oplus \mathbb{k}^m \oplus \mathbb{k}^{2m+1} \right\} \cong \mathbb{k}^{2m}.$$

Case 4: $q \equiv 3 \mod 4$, $q = 4m + 3$

In this case $x_0, \ldots, x_{2m+1}$ are the $\Delta$-invariants in $H^{4m+3}(\Gamma'_A)$, and $y_0, \ldots, y_{2m+1}$ are the $\Delta$-invariants in $H^{4m+3}(\Gamma'_B)$.

Thus,

$$H^{4m+3}(\Gamma')^\Delta \cong \left\{ \begin{pmatrix} (x_0, \ldots, x_{2m+1}) \\ (x_0, y_1, y_2, \ldots, y_{2m+1}) \\ (x_0, y_1, y_2, \ldots, y_{2m+1}) \end{pmatrix} \in \mathbb{k}^{2m+2} \oplus \mathbb{k}^{2m+1} \oplus \mathbb{k}^{2m+1} \right\} \cong \mathbb{k}^{(4m+3)}.$$

Since $H^q(\Gamma_{v_2}) = H^q(\Gamma')^\Delta$, we get

$$H^q(\Gamma_{v_2}) \cong \begin{cases} \mathbb{k}^{(2m+1)} & q \equiv 0 \mod 4, q = 4m \\ 0 & q \equiv 1 \mod 4, q = 4m + 1 \\ \mathbb{k}^{(2m)} & q \equiv 2 \mod 4, q = 4m + 2 \\ \mathbb{k}^{(4m+3)} & q \equiv 3 \mod 4, q = 4m + 3. \end{cases} \quad (219)$$
§4 Cohomology of $\Gamma_{v_4}$ and $\Gamma_{v_5}$

By equations (11) and (12) of Chapter II, the mod-3 cohomology of $\Gamma_{v_4}$ is isomorphic to that of $\Gamma_{v_5}$. Hence it is enough to compute $H^*(\Gamma_{v_4})$. By theorem (4.1) of Chapter I we have an exact sequence

$$1 \to C_{v_4} \to \Gamma_{v_4} \to S_{v_4} \to 1$$

where $C_{v_4} \cong U \times U$ and $S_{v_4} \cong \Delta \times \mathbb{Z}/2\mathbb{Z}$. Further we recall that the group $\Delta \times \Delta$ of $S_{v_4}$ acts on $U \times U$ componentwise and $\mathbb{Z}/2\mathbb{Z}$ flips the two components. Since $U \rtimes \Delta \cong D_{12}$, we have $\Gamma_{v_4} \cong (D_{12} \times D_{12}) \rtimes \mathbb{Z}/2\mathbb{Z}$. Thus the mod-3 cohomology of $\Gamma_{v_4}$ is equal to the $\mathbb{Z}/2\mathbb{Z}$-invariants in the mod-3 cohomology of $D_{12} \times D_{12}$, i.e.,

$$H^i(\Gamma_{v_4}) \cong \left[ H^i(D_{12} \times D_{12}) \right]^{\mathbb{Z}/2\mathbb{Z}}. \quad (220)$$

These invariants have been determined in section 4 of Appendix B. Before concluding this section we recall from proposition 1 of section 2 of Chapter II that the restriction maps $\rho_{41}, \rho_{42}, \rho_{52}$ and $\rho_{54}$ are injective.

§5 Cohomology of $\Gamma_{\sigma_1}$ and $\Gamma_{\sigma_4}$

Since $\sigma_1 = (v_1, v_4)$, it is easy to see that $\Gamma_{\sigma_1} = \Gamma_{v_1} \cap \Gamma_{v_4}$ and we have an exact sequence

$$1 \to C_{v_4} \to \Gamma_{v_1} \to S_{\sigma_1} \to 1$$

where $S_{\sigma_1}$ is the subgroup of $S_{v_4}$ which fixes the vertex $v_1$. Recall from theorem (4.1) of Chapter I that $C_{v_4} \cong U \times U$ and under this isomorphism the subgroups $< a >$ and $< a^2b >$ of $v_4 = < a > \times < b >$ (see equation (62)) are mapped onto the subgroups $\mu \times 1$ and $1 \times \mu$ respectively. Also we have $S_{v_4} \cong (\Delta \times \Delta) \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ flips the two subgroups $U \times 1$ and $1 \times U$. Thus it is easy to see that
the subgroup \( \Delta \times \Delta \) of \( S_{v_4} \) fixes the vertex \( v_1 \). Hence \( S_{\sigma_1} \cong \Delta \times \Delta \). So the above exact sequence becomes

\[
1 \to U \times U \to \Gamma_{v_1} \to \Delta \times \Delta \to 1
\]

where \( \Delta \times \Delta \) acts on \( U \times U \) componentwise. Thus we have

\[
\Gamma_{\sigma_1} \cong (U \times U) \rtimes (\Delta \times \Delta)
\]

\[
\cong (U \rtimes \Delta) \times (U \rtimes \Delta)
\]

\[
\cong D_{12} \times D_{12}
\]

Hence

\[
H^i(\Gamma_{\sigma_1}) \cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12})
\]

(222)

(5.1) Remark: From remark (2.1) we know that the mod-3 cohomology of \( \Gamma_{v_1} \) is detected in the subgroup \( (U \rtimes \Delta) \times N(P) \) which is exactly the subgroup \( \Gamma_{\sigma_1} \). Hence it follows that the restriction map

\[
\rho_{11} : H^* (\Gamma_{v_1}) \to H^* (\Gamma_{\sigma_1})
\]

(223)

is the identity map.

Next we consider \( \tilde{\sigma}_4 \). Since \( \tilde{\sigma}_4 = (\tilde{v}_3, \tilde{v}_5) \), it is easy to see that \( \Gamma_{\tilde{\sigma}_4} = \Gamma_{\tilde{v}_3} \cap \Gamma_{\tilde{v}_5} \)

and we have an exact sequence

\[
1 \to C_{\tilde{v}_5} \to \Gamma_{\tilde{\sigma}_4} \to S_{\tilde{\sigma}_4} \to 1
\]

where \( S_{\tilde{\sigma}_4} \) is the subgroup of \( S_{\tilde{v}_5} \) which fixes the vertex \( \tilde{v}_3 \). Recall from theorem (4.2) of Chapter I that \( C_{\tilde{v}_5} \cong C_0 = \{(u, u') | u, u' \in U, u \equiv u' \mod (\omega - 1)\} \) and under this isomorphism the the subgroups \( < \tilde{c} > \) and \( < \tilde{c}^2 \tilde{b} > \) of \( \tilde{v}_5 = < \tilde{c} > \times < \tilde{b} > \) (see remark (1.1) of Chapter II) are mapped onto the subgroups \( \mu \times 1 \) and \( 1 \times \mu \)
respectively. Also we have $S_{\bar{v}_3} \cong (\Delta \times \Delta) \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ flips the two subgroups $U \times 1$ and $1 \times U$. Thus it is easy to see that it is the subgroup $\Delta \times \Delta$ of $S_{\bar{v}_4}$ which fixes the vertex $\bar{v}_3$. Hence $S_{\bar{v}_4} \cong \Delta \times \Delta$. So the above exact sequence becomes

$$1 \rightarrow C_0 \rightarrow \Gamma_{\bar{v}_4} \rightarrow \Delta \times \Delta \rightarrow 1$$

where $\Delta \times \Delta$ acts on $U \times U$ componentwise. As noted in section 1 of the second chapter the mod-3 cohomology of the group $C_0$ is same as $U \times U$. Furthermore since the order of the group $\Delta \times \Delta$ is prime to 3 the Hochschild-Serre spectral sequence when applied to the above exact sequence degenerates to give,

$$H^t(\Gamma_{\bar{v}_4}) \cong H^t(C_0)^{\Delta \times \Delta}$$

$$\cong H^t(U \times U)^{\Delta \times \Delta}$$

$$\cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12})$$

(224)

(5.2) Remark: From equation (177) and remark (2.2) we know that the mod-3 cohomology of $\Gamma_{\bar{v}_3}$ is detected in the subgroup $C_0 \times (\Delta \times \Delta)$ which is exactly the subgroup $\Gamma_{\bar{v}_4}$. Hence it follows that the restriction map

$$\rho_{34} : H^*(\Gamma_{\bar{v}_3}) \rightarrow H^*(\Gamma_{\bar{v}_4})$$

is the identity map.

§6 Cohomology of $\Gamma_{\sigma_2}$ and $\Gamma_{\sigma_3}$.

Since $\sigma_2 = (v_2, v_4)$, it is easy to see that $\Gamma_{\sigma_2} = \Gamma_{v_2} \cap \Gamma_{v_4}$ and we have an exact sequence

$$1 \rightarrow C_{v_4} \rightarrow \Gamma_{\sigma_2} \rightarrow S_{\sigma_2} \rightarrow 1$$

where $S_{\sigma_2}$ is the subgroup of $S_{v_4}$ which fixes the vertex $v_2$. Recall that $C_{v_4} \cong U \times U$ and under this isomorphism we now let the subgroups $< ab > \times 1$ and $1 \times < b >$
of \(v_4 = \langle \alpha \rangle \times \langle b \rangle\) (see equation (153)) be mapped onto the subgroups \(\mu \times 1\) and \(1 \times \mu\) of \(U \times U\) respectively. Also we have \(S_{v_4} \cong (\Delta \times \Delta) \times \mathbb{Z}/2\mathbb{Z}\), where \(\mathbb{Z}/2\mathbb{Z}\) which now flips the two subgroups \(U \times 1\) and \(1 \times U\). Thus it is easy to see that the subgroup \(\Delta \times \Delta\) of \(S_{v_4}\) fixes the vertex \(v_2\). Hence \(S_{\sigma_2} \cong \Delta \times \Delta\). So the above exact sequence becomes

\[
1 \rightarrow U \times U \rightarrow \Gamma_{\sigma_2} \rightarrow \Delta \times \Delta \rightarrow 1
\]

where \(\Delta \times \Delta\) acts on \(U \times U\) componentwise. Thus we have

\[
\Gamma_{\sigma_2} \cong (U \times U) \times (\Delta \times \Delta)
\]

\[
\cong (U \times \Delta) \times (U \times \Delta)
\]

\[
\cong D_{12} \times D_{12}
\]

Hence

\[
H^t(\Gamma_{\sigma_2}) \cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12})
\]

Next we consider \(\sigma_3\). Since \(\sigma_3 = (v_2, v_5)\), it is easy to see that \(\Gamma_{\sigma_3} = \Gamma_{v_2} \cap \Gamma_{v_5}\) and we have an exact sequence

\[
1 \rightarrow C_{v_5} \rightarrow \Gamma_{\sigma_3} \rightarrow S_{\sigma_3} \rightarrow 1
\]

where \(S_{\sigma_3}\) is the subgroup of \(S_{v_5}\) which fixes the vertex \(v_2\). Recall that \(C_{v_5} \cong C_0 = \{(u, u')|u, u' \in U, u \equiv u' \mod (\omega - 1)\}\) and under this isomorphism we now identify the the subgroups \(\langle bc \rangle \times 1\) and \(1 \times \langle b \rangle\) of \(v_5 = \langle c \rangle \times \langle b \rangle\) (see equation (62)) with the subgroups \(\mu \times 1\) and \(1 \times \mu\) respectively. Also we have \(S_{v_5} \cong (\Delta \times \Delta) \times \mathbb{Z}/2\mathbb{Z}\), where \(\mathbb{Z}/2\mathbb{Z}\) flips the two subgroups \(U \times 1\) and \(1 \times U\). Thus it is easy to see that it is the subgroup \(\Delta \times \Delta\) of \(S_{v_5}\) which fixes the vertex \(v_2\). Hence \(S_{\sigma_3} \cong \Delta \times \Delta\). So the above exact sequence becomes

\[
1 \rightarrow C_0 \rightarrow \Gamma_{\sigma_3} \rightarrow \Delta \times \Delta \rightarrow 1
\]
where $\Delta \times \Delta$ acts on $U \times U$ componentwise. As noted in section 1 of the second chapter the mod-3 cohomology of the group $C_0$ is same as $U \times U$. Furthermore since the order of the group $\Delta \times \Delta$ is prime to 3 the Hochschild-Serre spectral sequence when applied to the above exact sequence degenerates to give,

$$H^t(\Gamma_{\sigma_3}) \cong H^t(C_0)^{\Delta \times \Delta}$$
$$\cong H^t(U \times U)^{\Delta \times \Delta}$$
$$\cong H^t(D_{12} \times D_{12})$$
$$\cong \bigoplus_{i+j=t} H^i(D_{12}) \otimes H^j(D_{12})$$

(228)

§7 Computation of $\rho_{22}$ and $\rho_{23}$

In this section we compute the restriction maps $\rho_{22}$ and $\rho_{23}$. Recall that in the fundamental domain $\Lambda$ the vertex $v_2$ is attached to the two 1-simplices, namely $\sigma_2$ and $\sigma_3$ and that we have $\Gamma_{\sigma_2} = \Gamma_{v_2} \cap \Gamma_{v_4}$ and $\Gamma_{\sigma_3} = \Gamma_{v_2} \cap \Gamma_{v_8}$. Thus both the stabilizers $\Gamma_{\sigma_2}$ and $\Gamma_{\sigma_3}$ are subgroups of $\Gamma_{v_2}$ and the maps $\rho_{22}$ and $\rho_{23}$ are the corresponding restriction maps on the cohomology, i.e.,

$$\rho_{22} : \hat{H}^*(\Gamma_{v_2}) \to \hat{H}^*(\Gamma_{\sigma_2})$$
$$\rho_{23} : \hat{H}^*(\Gamma_{v_2}) \to \hat{H}^*(\Gamma_{\sigma_3}).$$

To compute them we first compute the cohomology of $\Gamma_{v_2}$ by a different approach than using Alperin's complex. This approach does not completely determine the cohomology of $\Gamma_{v_2}$, but it does tells us how it is going to restrict to the cohomology of $\Gamma_{\sigma_2}$ and $\Gamma_{\sigma_3}$. As we will see later in this section that using this approach we will be able to completely determine the two maps $\rho_{22}$ and $\rho_{23}$.

Consider the short exact sequence of groups

$$1 \longrightarrow v_2 \longrightarrow \Gamma_{v_2} \longrightarrow \Gamma_{v_2}/v_2 \longrightarrow 1$$
By the Hochschild-Serre spectral sequence, we get
\[ E_2^{st} = H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \Rightarrow H^{s+t}(\Gamma_{v_2}) \]  
(229)

(7.1) Lemma: The above spectral sequence degenerates.

Proof: By lemma (2.2) of section 2 of Chapter II the above spectral sequence (229) degenerates to the right of the line \( s = \text{v.c.d.}(\Gamma_{v_2}) = 2 \). Next we show that \( H^s(\Gamma_{v_2}/v_2, H^t(v_2)) = 0 \) for \( s = 1 \) or \( s = 2 \) and for all \( t \geq 0 \). This will prove the lemma. By equation (183) the mod-3 cohomology of \( \Gamma_{v_2} \) is equal to the \( \Delta \)-invariants in the mod-3 cohomology of \( C_{v_2} \cong GL(2, \mathbb{Z}[\omega]) \). Thus we have
\[ H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \cong H^s(C_{v_2}/v_2, H^t(v_2))^\Delta. \]  
(230)

Since the group \( C_{v_2} \cong GL(2, \mathbb{Z}[\omega]) \) is the centralizers of \( v_2 \) in \( \Gamma \), \( C_{v_2} \) acts trivially on \( H^t(v_2) \), for all values of \( t \). Since \( H^t(v_2) \cong k \), for all \( t \), the above equation reduces to,
\[ H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \cong H^s(C_{v_2}/v_2, k)^\Delta. \]  
(231)

Set \( G = C_{v_2}/v_2 \). Next we observe that the group \( v_2 \) acts trivially on Alperin's complex \( \mathcal{X} \), hence we can use the complex \( \mathcal{X} \) to find \( H^*(G, k) \). The fundamental domain for the \( G \)-action on \( \mathcal{X} \) will again be the same 2-simplex \( ABC \). However the stabilizers of simplices will have to be modded out by \( v_2 \), i.e., for example the stabilizer of the vertex \( A \) in \( G \) is equal to \( \Gamma'_A/v_2 \), and so on, where \( \Gamma'_A \) is the stabilizer of the vertex \( A \) in \( \Gamma' = C_{v_2} \). Next we proceed to calculate the spectral sequence associated with this \( G \)-complex \( \mathcal{X} \) as in subsection (3.1). Since we are now only interested in \( H^1(G, k) \) and \( H^2(G, k) \) and the calculations are essentially the same as that in (3.1) we omit the computational part and state the final result. It can be shown that \( H^1(G, k) = 0 \) and \( H^2(G, k) = 0 \). Thus from equation (231) we get \( H^i(\Gamma_{v_2}/v_2, H^t(v_2)) = 0 \) for \( i = 1 \) and \( i = 2 \) and for all \( t \geq 0 \). As mentioned earlier this proves our lemma. \( \square \)
Since the spectral sequence (229) degenerates we get,

\[ H^q(\Gamma_{v_2}) \cong \bigoplus_{s+t=q} H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \]  

(232)

If \( s > 2 = \text{v.c.d.}(\Gamma_{v_2}) \), then we can replace \( H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \) by \( \hat{H}^s(\Gamma_{v_2}/v_2, H^t(v_2)) \) and further we already showed in the proof of lemma (7.1) that \( H^s(\Gamma_{v_2}/v_2, H^t(v_2)) = 0 \), for \( s = 1 \) or 2 and all \( t \). Hence we rewrite the above equation as,

\[ H^q(\Gamma_{v_2}) \cong \bigoplus_{s+t=q} H^s(\Gamma_{v_2}/v_2, H^t(v_2)) \oplus \bigoplus_{s+t=q} \hat{H}^s(\Gamma_{v_2}/v_2, H^t(v_2)) \]

\( \cong H^0(\Gamma_{v_2}/v_2, H^q(v_2)) \bigoplus \hat{H}^s(\Gamma_{v_2}/v_2, H^t(v_2)) \)  

(233)

To compute the second summand we can use K. Brown's complex of elementary abelian subgroups of \( \Gamma_{v_2}/v_2 \), which are of rank \( \leq 1 \). As seen in the proof of lemma (2.2) of Chapter II, there are only two conjugacy classes of elementary abelian subgroups of order 3 in \( \Gamma_{v_2}/v_2 \) and they are \( v_4/v_2 \) and \( v_5/v_2 \). Their normalizers in \( \Gamma_{v_2}/v_2 \) are \( \Gamma_{\sigma/2}/v_2 \) and \( \Gamma_{\sigma/3}/v_2 \) respectively. Hence by the corollary (0.2) in the introduction we have,

\[ \hat{H}^s(\Gamma_{v_2}/v_2, H^t(v_2)) \cong \hat{H}^s(\Gamma_{\sigma/2}/v_2, H^t(v_2)) \oplus \hat{H}^s(\Gamma_{\sigma/3}/v_2, H^t(v_2)). \]  

(234)

Hence the equation (233) can be written as,

\[ H^q(\Gamma_{v_2}) \cong H^0(\Gamma_{v_2}/v_2, H^q(v_2)) \bigoplus \hat{H}^s(\Gamma_{\sigma/2}/v_2, H^t(v_2)) \bigoplus \hat{H}^s(\Gamma_{\sigma/3}/v_2, H^t(v_2)). \]  

(235)

As discussed in the proof of proposition 2 of §2 of Chapter II the restriction map \( \rho_{22} \) is the identity on the second summand and maps the third summand to zero.
Similarly $\rho_{23}$ is the identity on the third summand and maps the second summand to zero. To see how $\rho_{22}$ and $\rho_{23}$ operates on the first summand we first find the second and third summand in the above equation and then from equation (219) we will be able to find the $k$-dimension of the first summand. To find the second summand we consider the short exact sequence

$$1 \rightarrow \nu_2 \rightarrow \Gamma_{\sigma_2} \rightarrow \Gamma_{\sigma_2}/\nu_2 \rightarrow 1$$

By proposition 2 of Chapter II we know that the Hochschild-Serre spectral sequence when applied to the above spectral sequence degenerates. Recall that the group $\Gamma_{\sigma_2}$ is a finite group with order divisible by 3, hence its mod-3 Farrell cohomology is same as that of the mod-3 ordinary cohomology in non-negative degrees. Thus we get.

$$H^q(\Gamma_{\sigma_2}) \cong \bigoplus_{s+t=q} H^s(\Gamma_{\sigma_2}/\nu_2, H^t(\nu_2))$$

$$= \bigoplus_{0 \leq s \leq 3} H^s(\Gamma_{\sigma_2}/\nu_2, H^t(\nu_2)) \bigoplus H^s(\Gamma_{\sigma_2}/\nu_2, H^t(\nu_2))$$

By equation (226), $\Gamma_{\sigma_2} \cong D_{12} \times D_{12}$, hence it is easy to compute the right-hand side directly. In fact we have

$$\Gamma_{\sigma_2} \cong (U \times U) \rtimes (\Delta \times \Delta)$$

$$\cong D_{12} \times D_{12}. \quad (237)$$

However in this isomorphism we now we identify the subgroup

$$(< -ab > \times < -b >)$$

of $\Gamma_{\sigma_2}$ with the subgroup $U \times U$ of $(U \times U) \rtimes (\Delta \times \Delta)$ by sending the elements $(-ab, 1)$ and $(1, -b)$ to the generators $(-\omega, 1)$ and $(1, -\omega)$ of $U \times 1$ and $1 \times U$ respectively. Since $\nu_2 = < b > \cong 1 \times \mu$ it follows that
Next we fix the coordinates for the $H^*(D_{12} \times D_{12})$ as in Appendix B. We do the same thing with $\Gamma_{\sigma_3}$. In what follows we always assume that $i > 2$ and carry out explicit calculation in each dimension $q \mod 4$.

**Case 1: $q \equiv 0 \mod 4$, $q = 4m$.**

By direct computation, we see that,

\[
\begin{align*}
\hat{H}^0(\Gamma_{\sigma_2}/v_2, H^{4m}(v_2)) &\cong k \\
\hat{H}^1(\Gamma_{\sigma_2}/v_2, H^{4m-1}(v_2)) &\cong 0 \\
\hat{H}^2(\Gamma_{\sigma_2}/v_2, H^{4m-2}(v_2)) &\cong 0 \\
\bigoplus_{s+t=4m, s \geq 2} \hat{H}^s(\Gamma_{\sigma_2}/v_2, H^t(v_2)) &\cong k^m 
\end{align*}
\]

(239)

Recall that $v_2 \subset \Gamma_{\sigma_2} \cap \Gamma_{\sigma_3}$ and it can be easily verified that the above equations also hold if we replace $\Gamma_{\sigma_2}$ by $\Gamma_{\sigma_3}$. Choose the coordinates as in Appendix B, then

\[
\begin{align*}
\bigoplus_{s+t=4m, s \geq 2} \hat{H}^s(\Gamma_{\sigma_3}/v_2, H^t(v_2)) &\cong k^m \cong \{(x_1, x_2, \ldots, x_m)|x_i \in k, \forall i\} \\
\bigoplus_{s+t=4m, s \geq 2} \hat{H}^s(\Gamma_{\sigma_3}/v_2, H^t(v_2)) &\cong k^m \cong \{(y_1, y_2, \ldots, y_m)|y_i \in k, \forall i\}
\end{align*}
\]

(240) (241)

Since $H^{4m}(\Gamma_{v_2}) \cong k^{2m+1}$, we deduce from equation (235) that,

\[
H^0(\Gamma_{v_2}/v_2, H^{4m}(v_2)) \cong k
\]

(242)
Hence we choose coordinates,

\[ H^{4m}(\Gamma_{v_2}) \cong k^{(2m+1)} \cong \left\{ \left( \begin{array}{c} x_0, x_1, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) \mid x_i, y_i \in k, \forall i \right\} \]  

(243)

where the class \( x_0 \) belongs to \( H^0(\Gamma_{v_2}/v_2, H^{4m}(v_2)) \) and it can be easily seen that restriction maps \( \rho_{22} \) and \( \rho_{23} \) maps \( H^0(\Gamma_{v_2}/v_2, H^{4m}(v_2)) \) onto \( H^0(\Gamma_{\sigma_2}/v_2, H^{4m}(v_2)) \) and \( H^0(\Gamma_{\sigma_3}/v_2, H^{4m}(v_2)) \) respectively. The two restriction maps can now be described as,

\[ \rho_{22} : H^{4m}(\Gamma_{v_2}) \rightarrow H^{4m}(\Gamma_{\sigma_2}) \]

\[ \rho_{22} \left( \begin{array}{c} x_0, x_1, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) = (x_0, x_1, \ldots, x_m) \]  

(244)

\[ \rho_{23} : H^{4m}(\Gamma_{v_2}) \rightarrow H^{4m}(\Gamma_{v_2}) \]

\[ \rho_{23} \left( \begin{array}{c} x_0, x_1, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) = (x_0, y_1, y_2, \ldots, y_m) \]  

(245)

**Case 2: \( q \equiv 1 \mod 4, q = 4m + 1 \)**

In this case \( H^q(\Gamma_{v_2}), H^q(\Gamma_{\sigma_2}), H^q(\Gamma_{\sigma_3}) \) are all zeros. Hence the restriction maps are also zero maps.

**Case 3: \( q \equiv 2 \mod 4, q = 4m + 2 \)**

As in case one above we can see by direct computation that,

\[ H^{4m+2}(\Gamma_{\sigma_2}) \cong \bigoplus_{s+t=4m+2} \hat{H}^s(\Gamma_{\sigma_2}/v_2, H^t(v_2)) \]

\[ \cong \bigoplus_{s+t=4m+2} \hat{H}^s(\Gamma_{\sigma_2}/v_2, H^t(\sigma_2)) \]  

(246)

\[ \cong k^m \]
The second equality follows because $\bigoplus_{s+t=m+2} \hat{H}^s(\Gamma_{\sigma_2}/v_2, H^t(v_2)) = 0$. Above equations also holds for $H^{4m+2}(\Gamma_{\sigma_3})$. So again we choose the coordinates as follows,

$$H^{4m+2}(\Gamma_{\sigma_2}) \cong k^m \cong \{(x_1, x_2, \ldots, x_m) | x_i \in k, \forall i\} \quad (247)$$

$$H^{4m+2}(\Gamma_{\sigma_3}) \cong k^m \cong \{(y_1, y_2, \ldots, y_m) | y_i \in k, \forall i\} \quad (248)$$

Since $H^{4m+2}(\Gamma_{v_2}) = k^{2m}$, we deduce from equation (235) that,

$$H^0(\Gamma_{v_2}/v_2, H^{4m+2}(v_2)) \cong 0 \quad (249)$$

Next we choose coordinates as follows,

$$H^{4m+2}(\Gamma_{v_2}) \cong k^{2m} \cong \left\{ \left( \begin{array}{c} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) | x_i, y_i \in k, \forall i \right\} \quad (250)$$

Then the restriction maps $\rho_{22}, \rho_{23}$ can be described as,

$$\rho_{22} : H^{4m+2}(\Gamma_{v_2}) \longrightarrow H^{4m+2}(\Gamma_{\sigma_2})$$

$$\rho_{22} \left( \begin{array}{c} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) = (x_1, x_2, \ldots, x_m) \quad (251)$$

$$\rho_{23} : H^{4m+2}(\Gamma_{v_2}) \longrightarrow H^{4m+2}(\Gamma_{\sigma_3})$$

$$\rho_{23} \left( \begin{array}{c} x_1, x_2, \ldots, x_m \\ y_1, y_2, \ldots, y_m \end{array} \right) = (y_1, y_2, \ldots, y_m) \quad (252)$$

**Case 4: $q \equiv 3 \mod 4, q = 4m + 3$**

By direct computation,

$$\hat{H}^0(\Gamma_{\sigma_2}/v_2, H^{4m+3}(v_2)) \cong k$$

$$\hat{H}^1(\Gamma_{\sigma_2}/v_2, H^{4m+2}(v_2)) = 0$$

$$\hat{H}^2(\Gamma_{\sigma_2}/v_2, H^{4m+1}(v_2)) = 0 \quad (253)$$

$$\bigoplus_{s+t=m+3} \hat{H}^s(\Gamma_{\sigma_2}/v_2, H^t(v_2)) \cong k^{(2m+1)}$$
Above equations also hold if we replace $\Gamma_{\sigma_2}$ by $\Gamma_{\sigma_3}$. Choose the coordinates as follows:

\[
\bigoplus_{s+t=4m+3 \atop s \geq 2} \hat{H}^s(\Gamma_{\nu_2}/v_2, \hat{H}^t(v_2)) \cong k^{(2m+1)} \cong \{(x_1, x_2, \ldots, x_{2m+1}) | x_i \in k, \forall i\} \tag{254}
\]

\[
\bigoplus_{s+t=4m+3 \atop s \geq 2} \hat{H}^s(\Gamma_{\nu_2}/v_2, \hat{H}^t(v_2)) \cong k^{(2m+1)} \cong \{(y_1, y_2, \ldots, y_{2m+1}) | y_i \in k, \forall i\} \tag{255}
\]

Since $H^{4m+3}(\Gamma_{\nu_2}) = k^{(4m+3)}$, we deduce from equation (235) that

\[H^0(\Gamma_{\nu_2}/v_2, H^{4m+3}(v_2)) = k \tag{256}\]

Hence we choose the coordinates as follows,

\[H^{4m+3}(\Gamma_{\nu_2}) \cong k^{(4m+3)} \cong \left\{ \left( \begin{array}{c} x_0, x_1, \ldots, x_{2m+1} \\ y_1, y_2, \ldots, y_{2m+1} \end{array} \right) \bigg| x_i, y_i \in k, \forall i \right\} \tag{257}\]

where the class $x_0$ belongs to $H^0(\Gamma_{\nu_2}/v_2, H^{4m+3}(v_2))$ and it can shown that the restriction maps $\rho_{22}$ and $\rho_{23}$ maps $H^0(\Gamma_{\nu_2}/v_2, H^{4m+3}(v_2))$ onto $H^0(\Gamma_{\sigma_2}/v_2, H^{4m+3}(v_2))$ and $H^0(\Gamma_{\sigma_3}/v_2, H^{4m+3}(v_2))$ respectively.

The two restriction maps can now be described as,

\[\rho_{22} : H^{4m+3}(\Gamma_{\nu_2}) \longrightarrow H^{4m+3}(\Gamma_{\sigma_2}) \tag{258}\]

\[\rho_{22} \left( \begin{array}{c} x_0, x_1, \ldots, x_{2m+1} \\ y_1, y_2, \ldots, y_{2m+1} \end{array} \right) = (x_0, x_1, \ldots, x_{2m+1}) \]

\[\rho_{23} : H^{4m+3}(\Gamma_{\nu_2}) \longrightarrow H^{4m+3}(\Gamma_{\sigma_3}) \tag{259}\]

\[\rho_{23} \left( \begin{array}{c} x_0, x_1, \ldots, x_{2m+1} \\ y_1, y_2, \ldots, y_{2m+1} \end{array} \right) = (x_0, y_1, \ldots, y_{2m+1}) \]
§8 Spectral sequence.

Now we have all the necessary data to compute the Farrell-cohomology $\hat{H}^*(\Gamma)$ of $\Gamma = GL(4, \mathbb{Z})$ when $* > 4$. By K.Brown's theorem we have,

$$\hat{H}^*(\Gamma) \cong \hat{H}^*_p(\mathcal{A})$$ (260)

where $\mathcal{A}$ is the partially ordered set of the non-trivial elementary abelian 3-subgroup of $\Gamma$. To determine the right side, we use the spectral sequence

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} H^q(\Gamma_\sigma) \Rightarrow H^{p+q}(\Gamma),$$ (261)

where $\Sigma_p$ is a set of representatives for $p$-cells of $\mathcal{A}$ mod $\Gamma$. Since $\dim \mathcal{A}=1$, the above spectral sequence is concentrated in the vertical strip $0 \leq p \leq 1$. Hence we only need to compute the differential $d_1^{0,q}$. Recall that the fundamental domain of $\mathcal{A}$ mod $\Gamma$ consists of five vertices $v_i$, $i = 1, \ldots, 5$ and four 1-simplices $\sigma_i, i = 1, \ldots, 4$. The differential $d_1^{0,q}$ has been described in section 2 of the second chapter. It is defined in terms of the restriction maps as follows

$$d_1^{0,q} : \bigoplus_{i=1}^{5} H^q(\Gamma_{v_i}) \rightarrow \bigoplus_{i=1}^{4} H^q(\Gamma_{\sigma_i})$$

$$d_1^{0,q} = (\rho_{41} - \rho_{11}) \oplus (\rho_{42} - \rho_{22}) \oplus (\rho_{53} - \rho_{23}) \oplus (\rho_{54} - \rho_{34}),$$ (262)

where $\rho_{ij}$ are the appropriate restriction maps (see notation (1.1)). Here we mentioned that the restriction maps $\rho_{11}$ and $\rho_{34}$ are identity maps (see remarks (5.1) and (5.2)) and maps $\rho_{41}, \rho_{42}, \rho_{53}, \rho_{54}$ are injective maps (see the end of section 4) and finally we recall that the restriction maps $\rho_{22}$ and $\rho_{23}$ have been described in section 7.
In this section we compute the kernel and cokernel of \( d_1^{0,q} \) when \( q > 2 \) which will give us the \( E_2^{p,q} \)-page and since the spectral sequence converges at \( E_2 \)-page, this will eventually give us the Farrell cohomology \( H^q(\Gamma) \) of \( \Gamma = GL(4, \mathbb{Z}) \) for \( q > 3 \).

Since we are interested only in the dimension of \( H^*(\Gamma) \) over \( k \) it is enough to determine the dimension of the kernel and cokernel of \( d_1^{0,q} \). Furthermore, since

\[
\dim(\text{cokernel}) = \dim(\text{target}) - \dim(\text{source}) + \dim(\text{kernel})
\]

(263)

it is enough to determine the dimension of the kernel of the map \( d_1^{0,q} \).

Recall that the restriction maps \( \rho_{11} \) and \( \rho_{34} \) are the identity maps, hence if \( (X_1, \ldots, X_5) \in \bigoplus_{i=1}^5 H^q(\Gamma_{v_i}) \) lies in the kernel of \( d_1^{0,q} \), then for any given classes \( X_4 \in H^q(v_4) \) and \( X_5 \in H^q(v_5) \) the classes \( X_1 \in H^q(v_1) \) and \( X_3 \in H^q(v_3) \) are uniquely determined in such way that \( \rho_{41}(X_4) = \rho_{11}(X_1) \) and \( \rho_{54}(X_5) = \rho_{34}(X_3) \).

Hence it is enough to study the map

\[
\varphi : H^q(\Gamma_{v_2}) \oplus H^q(\Gamma_{v_4}) \oplus H^q(\Gamma_{v_5}) \longrightarrow H^q(\Gamma_{v_2}) \oplus H^q(\sigma_3)
\]

where,

\[
\varphi = (\rho_{42} - \rho_{22}) \oplus (\rho_{53} - \rho_{23}).
\]

(264)

In the following calculation we always assume that \( q > 2 \), so the Farrell cohomology of all the stabilizers involved in the spectral sequence (261) is same as their ordinary cohomology which have been calculated in the previous sections. Also we use the same coordinatization for the \( k \)-vector spaces \( H^*(\Gamma_{v_2}), H^*(\Gamma_{v_4}) \) and \( H^*(\Gamma_{v_5}) \) as used in the previous section. For \( H^*(\Gamma_{v_4}) \) we first recall that the center \( C_{v_4} \) of \( v_4 \) in \( \Gamma \) is isomorphic \( U \times U \) and under this isomorphism we now identify the subgroups \( < ab > \times 1 \) and \( 1 \times < b > \) of \( C_{v_4} \) with the subgroups \( \mu \times 1 \) and \( 1 \times \mu \) of \( U \times U \) respectively. Next since \( \Gamma_{v_4} \cong D_{12} \wr \mathbb{Z}/2\mathbb{Z} \) we choose the coordinates for \( H^*(\Gamma_{v_4}) \)
as in Appendix B. Since the mod-3 cohomology groups of $\Gamma_v$ are isomorphic to that of $\Gamma_v$ which are in turn isomorphic to that of $D_{12} \wr \mathbb{Z}/2\mathbb{Z}$, we first identify the subgroups $< cb > \times 1$ and $1 \times < b >$ of $\Gamma_v$ with the subgroups $\mathbb{Z}/3\mathbb{Z} \times 1$ and $1 \times \mathbb{Z}/3\mathbb{Z}$ of $D_{12} \wr \mathbb{Z}/2\mathbb{Z}$ and then choose the coordinates for $H^*(D_{12} \wr \mathbb{Z}/2\mathbb{Z})$ as in Appendix B.

Case 1: $q \equiv 0 \mod 4$

Subcase 1.1: $q \equiv 0 \mod 4$, $q = 8m$

$$\varphi : H^{8m}(\Gamma_v) \oplus H^{8m}(\Gamma_v) \oplus H^{8m}(\Gamma_v) \rightarrow H^{8m}(\Gamma_{\sigma_2}) \oplus H^{8m}(\Gamma_{\sigma_3})$$

$$k^{(4m+1)} \oplus k^{(m+1)} \oplus k^{(m+1)} \rightarrow k^{(2m+1)} \oplus k^{(2m+1)}$$

$$\varphi \left( \begin{array}{c} (a_0, \ldots, a_{2m}, b_1, \ldots, b_{2m}) \\ (x_0, \ldots, x_m, x_{m-1}, \ldots, x_1, x_0) \\ (y_0, \ldots, y_m, y_{m-1}, \ldots, y_1, y_0) \end{array} \right) = \left( \begin{array}{c} (x_0 - a_0, x_1 - a_1, \ldots, x_{m-1} - a_{m-1}, x_m - a_m, \\ x_{m-1} - a_{m+1}, \ldots, x_1 - a_{2m-1}, x_0 - a_{2m}) \\ (y_0 - a_0, y_1 - b_1, \ldots, y_{m-1} - b_{m-1}, y_m - a_m, \\ y_{m-1} - a_{m+1}, \ldots, y_1 - b_{2m-1}, y_0 - b_{2m}) \end{array} \right)$$

(265)

Therefore it follows that

$$\ker (d_1^{0,8m}) \cong k^{(2m+1)}$$

(266)

and

$$\dim \text{ of } \text{coker} (d_1^{0,8m}) = \dim (\text{target-source+ker}) = 0.$$  

(267)

Therefore

$$\text{coker}(d_1^{0,8m}) = 0.$$  

(268)
Subcase 1.2: $q \equiv 4 \mod 8, q = 8m + 4$

$$\varphi : H^{8m+4}(\Gamma_{v_2}) \oplus H^{8m+4}(\Gamma_{v_4}) \oplus H^{8m+4}(\Gamma_{v_6}) \rightarrow H^{8m+4}(\Gamma_{\sigma_2}) \oplus H^{8m+4}(\Gamma_{\sigma_3})$$

$$k^{(4m+3)} \oplus k^{(m+1)} \oplus k^{(m+1)} \rightarrow k^{(2m+2)} \oplus k^{(2m+2)}$$

$$\varphi \left( \begin{array}{c}
(a_0, \ldots, a_{2m+1}, b_1, \ldots, b_{2m+1}) \\
(x_0, \ldots, x_{m-1}, x_m, x_{m-1}, \ldots, x_0) \\
y_0, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_0
\end{array} \right) = \left( \begin{array}{c}
(x_0 - a_0, x_1 - a_1, \ldots, x_{m-1} - a_{m-1}, x_m - a_m, x_m - a_{m+1}), \\
x_{m-1} - a_{m+2}, \ldots, x_1 - a_{2m}, x_0 - a_{2m+1}) \\
y_0 - a_0, y_1 - b_1, \ldots, y_{m-1} - b_{m-1}, y_m - b_m, y_m - b_{m+1}, \\
y_{m-1} - b_{m+2}, \ldots, y_1 - b_{2m}, y_0 - b_{2m+1}
\end{array} \right) \quad (269)$$

Hence,

$$\ker(d_1^{0,(8m+4)}) \cong k^{(2m+1)} \quad (270)$$

and

$$\dim \text{ of } \ker(d_1^{0,(8m+4)}) = 0. \quad (271)$$

Case 2: $q \equiv 1 \mod 4, q = 4m + 1$

In this case the cohomology of the source and the target are both zero and hence $d_1^{0,(4m+1)} = 0$.

$$\ker(d_1^{0,(4m+1)}) = 0, \quad (272)$$

$$\coker(d_1^{0,(4m+1)}) = 0 \quad (273)$$

Case 3: $q \equiv 2 \mod 4, q = 4m + 2$

From equation (113) recall that

$$d_1^{0,(4m+2)} = (\rho_{41} - \rho_{11}) \oplus (\rho_{42} - \rho_{22}) \oplus (\rho_{53} - \rho_{23}) \oplus (\rho_{54} - \rho_{34}). \quad (274)$$
Let \((X_1, \ldots, X_5) \in \ker(d_1^{0,(4m+2)}),\) then from the description of \( \rho_{22} \) and \( \rho_{23} \) given in the last section, we see that the class \( X_2 \in H^{4m+2}(\Gamma_{v_2}) \) is uniquely determined in terms of the classes \( X_4 \in H^{4m+2}(\Gamma_{v_4}) \) and \( X_5 \in H^{4m+2}(\Gamma_{v_5}) \), for any choice of \( X_4 \) and \( X_5 \) in \( H^{4m+2}(\Gamma_{v_4}) \) and \( H^{4m+2}(\Gamma_{v_5}) \) respectively. Since we already know that the classes \( X_1, X_2 \) are also uniquely determined in terms of \( X_4 \) and \( X_5 \), it follows that

\[
\ker(d_1^{0,(4m+2)}) = H^{4m+2}(\Gamma_{v_4}) \oplus H^{4m+2}(\Gamma_{v_5})
\]

\[
\cong k^m \quad \text{if } m \text{ is even}
\]

\[
\cong k^{(m+1)} \quad \text{if } m \text{ is odd}
\]  

(275)

Hence,

\[
\dim \text{ of coker}(d_1^{0, (4m+2)}) = 0.
\]

(276)

**Case 4: \( q \equiv 3 \mod 4 \)**

**Subcase 4.1:** \( q \equiv 3 \mod 8, \quad q = 8m + 3 \)

\[
\varphi : H^{8m+3}(\Gamma_{v_2}) \oplus H^{8m+3}(\Gamma_{v_4}) \oplus H^{8m+3}(\Gamma_{v_5}) \to H^{8m+3}(\Gamma_{v_2}) \oplus H^{8m+3}(\Gamma_{v_3})
\]

\[
k^{(8m+3)} \oplus k^{(2m+1)} \oplus k^{(2m+1)} \to k^{(4m+2)} \oplus k^{(4m+2)}
\]

\[
\varphi \left( \begin{array}{c}
(a_0, \ldots, a_{4m+1}, b_1, \ldots, b_{4m+1}) \\
(x_0, x_1, \ldots, x_{2m-1}, x_{2m}, x_{2m}, x_{2m-1}, \ldots, x_1, \mu_0) \\
y_0, y_1, \ldots, y_{2m-1}, y_{2m}, y_{2m}, y_{2m-1}, \ldots, y, y_0
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
(x_0 - a_0, \ldots, x_{2m-1} - a_{2m-1}, \ldots, x_{2m} - a_{2m+1}, \ldots, x_0 - a_{4m+1}) \\
y_0 - a_0, y_1 - b_1, \ldots, y_{2m-1} - b_{2m-1}, \ldots, y_{2m} - b_{2m+1}, \ldots, y_0 - b_{4m+1}
\end{array} \right)
\]

(277)

We see that,

\[
\ker(d_1^{0, (8m+3)}) \cong k^{(4m+1)}
\]

(278)
and
\[ \text{coker}(d^{0,(8m+3)}_1) = 0 \] (279)

**Subcase 4.2:** \( q \equiv 7 \mod 8, q = 8m + 7 \)

\[ \varphi : H^{8m+7}(\Gamma_v) \oplus H^{8m+7}(\Gamma_v) \oplus H^{8m+7}(\Gamma_v) \rightarrow H^{8m+7}(\Gamma_{\sigma_2}) \oplus H^{8m+7}(\Gamma_{\sigma_2}) \]
\[ \mathbf{k}^{(8m+7)} \oplus \mathbf{k}^{(2m+2)} \oplus \mathbf{k}^{(2m+2)} \rightarrow \mathbf{k}^{(4m+4)} \oplus \mathbf{k}^{(4m+4)} \]

\[ \varphi = \left( \begin{array}{c}
(a_0, \ldots, a_{4m+3}, b_1, \ldots, b_{4m+3}) \\
(x_0, x_1, \ldots, x_{2m}, x_{2m+1}, x_{2m+1}, x_{2m}, \ldots, x_0) \\
(y_0, y_1, \ldots, y_{2m}, y_{2m+1}, y_{2m+1}, y_{2m}, \ldots, y_0) \\
(x_0 - a_0, x_1 - a_1, \ldots, x_{2m} - a_{2m}, x_{2m+1} - a_{2m+1}, x_{2m+1} - a_{2m+2}, \\
x_{2m} - a_{2m+3}, \ldots, x_1 - a_{4m+2}, x_0 - a_{4m+3}) \\
y_0 - a_0, x_1 - b_1, \ldots, y_{2m} - b_{2m}, y_{2m+1} - b_{2m+1}, y_{2m+1} - b_{2m+2}, \\
y_{2m} - b_{2m+3}, \ldots, y_1 - b_{4m+2}, y_0 - b_{4m+3})
\end{array} \right) \] (280)

Hence,
\[ \text{ker}(d^{0,(8m+7)}_1) \cong \mathbf{k}^{(4m+3)} \] (281)
\[ \text{coker}(d^{0,(8m+7)}_1) = 0. \] (282)

Since the spectral sequence (235) converges at \( E_2 \)-page and
\[ E^{1,q}_2 = \text{coker}(d^{0,q}_1) = 0, \text{ for all values of } q > 2, \text{ we conclude that } H^q(\Gamma) \cong E^{0,q}_2 = \text{ker}(d^{0,q}_1) \text{ if } q > 3. \text{ Thus for } q > 3 \text{ we have,} \]
\[ \hat{H}^q(GL(4, \mathbb{Z}), \mathbb{K}) = \begin{cases} 
\mathbb{K}^{(2m+1)} & \text{if } q = 8m \\
0 & \text{if } q = 8m + 1 \\
\mathbb{K}^{(2m)} & \text{if } q = 8m + 2 \\
\mathbb{K}^{(4m+1)} & \text{if } q = 8m + 3 \\
\mathbb{K}^{(2m+1)} & \text{if } q = 8m + 4 \\
0 & \text{if } q = 8m + 5 \\
\mathbb{K}^{(2m+2)} & \text{if } q = 8m + 6 \\
\mathbb{K}^{(4m+3)} & \text{if } q = 8m + 7. 
\end{cases} \]
APPENDIX A

THE COHOMOLOGY OF $\gamma(2, 3)$

In this appendix we show that the mod-3 ordinary cohomology of $\gamma(2, 3)$ is same as that of its mod-3 Farrell cohomology in degrees $\geq 0$. A similar statement is true for $GL(2, \mathbb{Z})$ and can be proven by using Serre’s tree for $SL_2(\mathbb{Z})$. We use the same tree to prove our result. Recall from equation (176) that the mod-3 Farrell cohomology of $\gamma(2, 3)$ is same as the mod-3 Farrell cohomology of its subgroup $N(P') \cong D_{12}$. Further since $D_{12}$ is a finite group with order divisible by 3, the mod-3 Farrell cohomology is same as its mod-3 ordinary cohomology in degrees $\geq 0$. Hence it is enough to show that the mod-3 ordinary cohomology of $\gamma(2, 3)$ is same as the mod-3 cohomology of the group $D_{12}$. To begin with set $\mathcal{G} = \gamma(2, 3) \cap SL_2(\mathbb{Z})$.

Then we have an exact sequence

$$1 \to \mathcal{G} \to \gamma(2, 3) \to \mathbb{Z}/2\mathbb{Z} \to 1 \quad (284)$$

Since the order of $\mathbb{Z}/2\mathbb{Z}$ is prime to 3, the mod-3 Hochschild-Serre spectral sequence degenerates to give

$$H^q(\gamma(2, 3)) \cong [H^q(\mathcal{G})]^{\mathbb{Z}/2\mathbb{Z}}. \quad (285)$$

Next we proceed to compute $H^q(\mathcal{G})$. First it can be shown that the index $[SL_2(\mathbb{Z}), \mathcal{G}]$ of $\mathcal{G}$ in $SL_2(\mathbb{Z})$ is equal to 4 and the coset representatives for $SL_2(\mathbb{Z})/\mathcal{G}$ are,

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_4 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \quad (286)$$
Now we recall that the group $SL_2(\mathbb{Z})$ acts in a well-known way on the upper half-plane $H = \{z | \text{Im}(z) > 0\}$. Let $y$ be the circular arc consisting of the points $z = e^{i\theta}$ with $\pi/3 \leq \theta \leq \pi/2$; its origin is the point $P = e^{i\pi/3}$ and its terminus is the point $Q = i$. Let $X$ be the union of the transforms of $y$ by $SL_2(\mathbb{Z})$. Then it can be shown that $X$ is a tree (or rather the geometric realization of a tree) on which $SL_2(\mathbb{Z})$ acts with the arc $PQ$ as a fundamental domain. Since $G$ is a subgroup of $SL_2(\mathbb{Z})$ of index 4, $G$ also acts on this tree and a fundamental domain for $G$-action consists of four transforms of $PQ$ by the elements $g_i, i = 1, \ldots, 4$. Thus a fundamental domain for $G$ consists of four arcs and it can be shown that only one vertex $v$, namely, $v = g_2 \cdot P$ has a 3-torsion in $G$ and its stabilizer in $G$ is

$$G_v = \langle \alpha' \rangle \cong \mathbb{Z}/6\mathbb{Z} \quad \text{where} \quad \alpha' = \left\langle \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \right\rangle$$

(287)

Thus the spectral sequence

$$E_1^{pq} = \prod_{\sigma \in \Sigma_p} H^q(G_\sigma) \Rightarrow H^{p+q}(G),$$

(288)

collapses to give

$$H^q(G) \cong H^q(G_v)$$

$$\cong H^q(\langle \alpha' \rangle)$$

(289)

Next in the exact sequence (284) we take $\beta' = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ as a coset representative for $\gamma(2,3)/G (\cong \mathbb{Z}/2\mathbb{Z})$ then it is easy to see that $\beta'^2 = 1$ and $\beta'\alpha'\beta' = \alpha'^5$ hence the group generated by $\alpha'$ and $\beta'$ is isomorphic to $D_{12}$ (in fact this group is same as that of $N(P')$ of section 2, see equations (174) and (175)). Thus from equation (285) and (289) we now get

$$H^q(\gamma(2,3)) \cong [H^q(\langle \alpha' \rangle)]^{<\beta'>}$$

$$\cong H^q(D_{12}).$$

(290)
Hence as mentioned in the beginning of this appendix, we now conclude that the mod-3 cohomology of $\gamma(2,3)$ is same as of its mod-3 Farrell cohomology in degrees $\geq 0$. 
APPENDIX B

Cohomology of \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/6\mathbb{Z} \times D_{12}, D_{12} \times D_{12}, D_{12} \times \mathbb{Z}/2\mathbb{Z} \).

In this appendix we label the coordinates of the mod-3 cohomology of the groups \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/6\mathbb{Z} \times D_{12}, D_{12} \times D_{12} \) and \( D_{12} \times \mathbb{Z}/2\mathbb{Z} \) in each dimension \( q \) (\( q \geq 0 \)). Recall that the mod-3 cohomology group \( H^q(G, k) \) of any group \( G \) is a vector space over \( k \) with a basis vectors which can be written in terms of the generators of the cohomology ring of \( G \) and cup products. In this appendix we first fix the generators for the cohomology rings and then in each dimension \( q \) we identify a basis for \( H^q(G) \) with the standard basis. By the standard basis we mean the following. For the vector space \( k^n \) of dimension \( n \) over \( k \) we denote the standard \( i \)-th basis vector by \( e_i \), i.e., \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) where \( i \)-th coordinate is equal to 1 and all others are equal to 0. In the following tables the left-hand columns contains the basis-vectors of \( H^*(G) \) in terms of the generators of the cohomology ring of \( H^*(G) \) and cup products, and the right-hand column contains the standard basis-vectors \( e_i \)'s for \( k^n \cong H^*(G) \).

§1 \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \)

First we do the case of \( H^*(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \). The cohomology ring of \( \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) has four generators \( \xi_1, \eta_1, \xi_2, \eta_2 \) and is given by,

\[
H^*(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \cong H^*(\mathbb{Z}/6\mathbb{Z}) \otimes H^*(\mathbb{Z}/6\mathbb{Z})
\]

\[
\cong k[\xi_1, \eta_1] \otimes k[\xi_2, \eta_2]
\]

\[
\cong k[\xi_1, \eta_1, \xi_2, \eta_2]
\]

where \( \deg(\xi_1) = 2 = \deg(\xi_2), \deg(\eta_1) = 1 = \deg(\eta_2) \), and \( \eta_1^2 = 0 = \eta_2^2 \).
Case 1: $q = 0 \mod 4, q = 4m$

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{k}^{4m+1} \cong \{ (x_0, \ldots, x_{4m}) | x_i \in \mathbb{k}, \forall i \} \]  

(292)

\[ \xi_1^{2m} = e_0 \]
\[ (\xi_1^{2m-1} \cup \eta_1) \otimes \eta_2 = e_{2m+1} \]
\[ \xi_1^{2m-1} \otimes \xi_2 = e_1 \]
\[ (\xi_1^{2m-2} \cup \eta_1) \otimes (\xi_2 \cup \eta_2) = e_{2m+2} \]
\[ \vdots \]
\[ \xi_1 \otimes \xi_2^{2m-1} = e_{2m-1} \]
\[ (\xi_1 \cup \eta_1) \otimes (\xi_2^{2m-2} \cup \eta_2) = e_{4m-1} \]
\[ \xi_2^{2m} = e_{2m} \]
\[ (\eta_1) \otimes (\xi_2^{2m-1} \cup \eta_2) = e_{4m} \]

Case 2: $q = 1 \mod 4, q = 4m + 1$

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{k}^{4m+2} \cong \{ (x_0, \ldots, x_{4m+1}) | x_i \in \mathbb{k}, \forall i \} \]  

(294)

\[ \xi_{1}^{2m} \cup \eta_{1} = e_{0} \]
\[ \xi_{1}^{2m} \otimes \eta_{2} = e_{2m+1} \]
\[ (\xi_{1}^{2m-1} \cup \eta_{1}) \otimes \xi_{2} = e_{1} \]
\[ (\xi_{1}^{2m-1} \cup \eta_{1}) \otimes (\xi_{2} \cup \eta_{2}) = e_{2m+2} \]
\[ (\xi_{1}^{2m-2} \cup \eta_{1}) \otimes (\xi_{2} \cup \eta_{2}) = e_{2m+3} \]
\[ \vdots \]
\[ \eta_{1} \otimes \xi_{2}^{2m} = e_{2m} \]
\[ \xi_{2}^{2m} \cup \eta_{2} = e_{4m+1} \]

Case 3: $q = 2 \mod 4, q = 4m + 2$

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{k}^{4m+3} \cong \{ (x_0, \ldots, x_{4m+2}) | x_i \in \mathbb{k}, \forall i \} \]  

(296)

\[ \xi_{1}^{2m+1} = e_{0} \]
\[ (\xi_{1}^{2m} \cup \eta_{1}) \otimes \eta_{2} = e_{2m+2} \]
\[ \xi_{1}^{2m} \otimes \xi_{2} = e_{1} \]
\[ (\xi_{1}^{2m-1} \cup \eta_{1}) \otimes (\xi_{2} \cup \eta_{2}) = e_{2m+3} \]
\[ (\xi_{1}^{2m-2} \cup \eta_{1}) \otimes (\xi_{2} \cup \eta_{2}) = e_{2m+4} \]
\[ \vdots \]
\[ \xi_{2}^{2m+1} = e_{2m+1} \]
\[ \eta_{1} \otimes (\xi_{2}^{2m} \cup \eta_{2}) = e_{4m+2} \]
Case 4: $q = 3 \mod 4, q = 4m + 3$

$$H^q(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}) \cong k^{4m+4} \cong \{(x_0, \ldots, x_{4m+4}) | x_i \in k, \forall i\} \quad (298)$$

$$\xi_1^{2m+1} \cup \eta_1 = e_0 \quad \xi_1^{2m+1} \otimes \eta_2 = e_{2m+2}$$

$$(\xi_1^{2m} \cup \eta_1) \otimes \xi_2 = e_1 \quad \xi_1^{2m} \otimes (\xi_2 \cup \eta) = e_{2m+2}$$

$$(\xi_1^{2m-1} \cup \eta_2) \otimes \xi_2 = e_2 \quad \xi_1^{2m-1} \otimes (\xi_2^2 \cup \eta) = e_{2m+3} \quad (299)$$

$$\eta_1 \otimes \xi_2^{2m+1} = e_{2m+1} \quad \xi_2^{2m+1} \cup \eta_2 = e_{4m+3}$$

§2 $\mathbb{Z}/6\mathbb{Z} \times D_{12}$

Next we turn to the mod-3 cohomology of $\mathbb{Z}/6\mathbb{Z} \times D_{12}$. The mod 3 cohomology ring of $\mathbb{Z}/6\mathbb{Z} \times D_{12}$ is

$$H^*(\mathbb{Z}/6\mathbb{Z} \times D_{12}) \cong k[\xi, \eta, \bar{\xi}, \bar{\eta}] \quad (300)$$

where $\deg \xi = 2, \deg \eta = 1, \deg \bar{\xi} = 4, \deg \bar{\eta} = 3$ and $\eta^2 = 0 = \bar{\eta}^2$.

Case 1: $q = 0 \mod 4, q = 4m$

$$H^q(\mathbb{Z}/6\mathbb{Z} \times D_{12}) \cong k^{2m+1} = \{(x_0, \ldots, x_{2m}) | x_i \in k, \forall i\} \quad (301)$$

$$\xi^{2m} = e_0 \quad (\xi^{2(m-1)} \cup \eta) \otimes \bar{\eta} = e_{m+1}$$

$$\xi^{2(m-1)} \otimes \bar{\xi} = e_1 \quad (\xi^{2(m-2)} \cup \eta) \otimes (\bar{\xi} \cup \bar{\eta}) = e_{m+2}$$

$$\xi^{2(m-2)} \cup \bar{\xi}^2 = e_2 \quad (\xi^{2(m-3)} \cup \eta) \otimes (\bar{\xi}^2 \cup \bar{\eta}) = e_{m+3} \quad (302)$$

$$\bar{\xi}^m = e_m \quad \eta \otimes (\bar{\xi}^{m-1} \cup \bar{\eta}) = e_{2m}$$
Case 2: \( q = 1 \mod 4, q = 4m + 1 \)

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times D_{12}) \cong k^{2m+1} \cong \{ (x_0, \ldots, x_{2m}) | x_i \in k, \forall i \} \] (303)

\[ \xi^{2m} \cup \eta = e_0 \]

\[ (\xi^{2(m-1)} \cup \eta) \otimes \xi = e_1 \]

\[ (\xi^{2(m-2)} \cup \eta) \otimes \xi^2 = e_2 \]

\[ \vdots \]

\[ \eta \otimes \xi^m = e_m \]

\[ \xi \otimes (\xi^{m-1} \cup \eta) = e_{2m} \]

Case 3: \( q = 2 \mod 4, q = 4m + 2 \)

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times D_{12}) \cong k^{2m+1} \cong \{ (x_0, \ldots, x_{2m}) | x_i \in k, \forall i \} \] (305)

\[ \xi^{2m+1} = e_0 \]

\[ (\xi^{2(m-1)} + 1 \cup \eta) \otimes \xi = e_1 \]

\[ (\xi^{2(m-2)} + 1 \cup \eta) \otimes (\xi^2 \cup \bar{\eta}) = e_{m+1} \]

\[ (\xi^{2(m-3)} + 1 \cup \eta) \otimes (\xi^3 \cup \bar{\eta}) = e_{m+2} \]

\[ \vdots \]

\[ (\xi \cup \eta) \otimes (\xi^{m-1} \cup \eta) = e_{2m} \]

Case 4: \( q = 3 \mod 4, q = 4m + 3 \)

\[ H^q(\mathbb{Z}/6\mathbb{Z} \times D_{12}) \cong k^{2m+2} \cong \{ (x_0, \ldots, x_{2m+1}) | x_i \in k, \forall i \} \] (307)

\[ \xi^{2m+1} \cup \eta = e_0 \]

\[ (\xi^{2(m-1)} + 1 \cup \eta) \otimes \xi = e_1 \]

\[ (\xi^{2(m-2)} + 2 \cup \eta) \otimes \xi^2 = e_2 \]

\[ \vdots \]

\[ (\xi \cup \eta) \otimes \xi^m = e_m \]

\[ \bar{\xi}^m \cup \bar{\eta} = e_{2m+1} \]
§3 $D_{12} \times D_{12}$

The cohomology ring of $D_{12} \times D_{12}$ is

$$H^*(D_{12} \times D_{12}) \cong H^*(D_{12}) \otimes H^*(D_{12})$$

$$\cong k[\xi_1, \eta_1] \otimes k[\xi_2, \eta_1]$$

where $\deg(\xi_1) = \deg(\xi_2) = 4$, $\deg(\eta_1) = \deg(\eta_2) = 3$ and $\eta_1^2 = \eta_2^2 = 0$.

Case 1: $q \equiv 0 \mod 4, q = 4m$

$$H^{4m}(D_{12} \times D_{12}) \cong k^{m+1} \cong \{(x_0, \ldots, x_m) | x_i \in k, \forall i\}$$

$$\xi_1^m = e_0$$

$$\xi_1^{m-1} \otimes \xi_2 = e_1$$

$$\xi_1^{m-2} \otimes \xi_2^2 = e_2$$

$$\vdots$$

$$\xi_1 \otimes \xi_2^{m-1} = e_{m-1}$$

$$\xi_2^m = e_m$$

Case 2: $q \equiv 1 \mod 4, q = 4m + 1$

$$H^{4m+1}(D_{12} \times D_{12}) = 0$$

Case 3: $q \equiv 2 \mod 4, q = 4m + 2$

$$H^{4m+2}(D_{12} \times D_{12}) \cong k^m \cong \{(x_0, \ldots, x_m) | x_i \in k, \forall i\}$$

$$(\eta_1 \cup \xi_1^{m-1}) \otimes \eta_2 = e_0$$

$$(\eta_1 \cup \xi_1^{m-2}) \otimes \eta_2 = e_1$$

$$\vdots$$

$$(\eta_1 \cup \xi_1^1) \otimes (\eta_2 \cup \xi_2^{m-2}) = e_{m-2}$$

$$(\eta_1 \otimes (\eta_2 \cup \xi_2^{m-1}) = e_{m-1}$$
Case 4: \( q \equiv 3 \mod 4, q = 4m + 3 \)

\[
H^{4m+3}(D_{12} \times D_{12}) \cong k^{2m+2} \cong \{(x_0, \ldots, x_{2m+1}) | x_i \in k, \forall i\}
\]  

(315)

\[
\begin{align*}
\eta_1 \cup \xi_1^m &= e_0 \\
(\eta_1 \cup \xi_1^{m-1}) \otimes \xi_2^1 &= e_1 \\
& \vdots \\
(\eta_1 \cup \xi_1^1) \otimes \xi_2^{m-1} &= e_{m-1} \\
\eta_1 \otimes \xi_2^m &= e_m \\
\xi_1^m \otimes \eta_2 &= e_{m+1} \\
\xi_1^{m-1} \otimes (\eta_2 \cup \xi_2^1) &= e_{m+2} \\
& \vdots \\
\xi_1^1 \otimes (\eta_2 \cup \xi_2^{m-1}) &= e_{2m} \\
\xi_2^m \otimes \eta_2 &= e_{2m+1}
\end{align*}
\]

(316)

§4 \( D_{12} \wr \mathbb{Z}/2\mathbb{Z} \)

Now we find the cohomology of \( D_{12} \wr \mathbb{Z}/2\mathbb{Z} \). Since the order of \( \mathbb{Z}/2\mathbb{Z} \) is prime to 3, it follows that mod-3 cohomology ring of \( D_{12} \wr \mathbb{Z}/2\mathbb{Z} \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \)-invariants in the mod-3 cohomology ring of \( D_{12} \times D_{12} \). Since the generator, say \( \kappa \) of \( \mathbb{Z}/2\mathbb{Z} \) flips the two groups \( D_{12} \times 1 \) and \( 1 \times D_{12} \), it is easy to show that \( \kappa^* \) interchanges the generators in the cohomology ring of \( D_{12} \times D_{12} \), i.e., if,

\[
H^*(D_{12} \times D_{12}) \cong k[\xi_1, \xi_2, \eta_1, \eta_2]
\]

(317)

as discussed earlier, then

\[
\kappa^*(\xi_1) = \xi_2
\]

(318)

\[
\kappa^*(\xi_2) = \xi_1
\]

(319)

\[
\kappa^*(\eta_1) = \eta_2
\]

(320)

\[
\kappa^*(\eta_2) = \eta_1
\]

(321)

Now we proceed to find these invariants in each dimension mod 8. In the following we use the same coordinates for \( H^q(D_{12} \times D_{12}) \) as discussed in the previous section.
Case 1: $q \equiv 0 \mod 4$

Subcase 1.1: $q \equiv 0 \mod 8, \quad q = 8m$

In this case $\kappa^*$ interchanges the generators as follows.

\[
\begin{align*}
  e_0 & \leftrightarrow e_{2m} \\
  e_1 & \leftrightarrow e_{2m-1} \\
  \vdots \\
  e_{m-1} & \leftrightarrow e_{m+1} \\
  e_m & \leftrightarrow e_m
\end{align*}
\]

Hence the invariants are

\[
\begin{align*}
  y_0 &= e_0 + e_{2m} \\
  y_1 &= e_1 + e_{2m-1} \\
  \vdots \\
  y_{m-1} &= e_{m-1} + e_{m+1} \\
  y_m &= e_m
\end{align*}
\]  

Thus

\[
H^{8m}(D_{12} \wr \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{k}^{m+1}
\]

\[
\cong \{(x_0, x_1, \ldots, x_{m-1}, x_m, x_{m-1}, \ldots, x_1, x_0) | x_i \in \mathbb{k}, \forall i \} \subset H^{8m}(D_{12} \times D_{12}).
\]  

(323)

Case 1.2: $q \equiv 4 \mod 4, \quad q = 8m + 4$

In this case $\kappa^*$ operates as follows.

\[
\begin{align*}
  e_0 & \leftrightarrow e_{2m+1} \\
  e_1 & \leftrightarrow e_{2m} \\
  \vdots \\
  e_m & \leftrightarrow e_{m+1}
\end{align*}
\]
Hence the invariants are

\[ y_0 = e_0 + e_{2m+1} \]
\[ y_1 = e_1 + e_{2m} \]
\[ \vdots \]
\[ y_m = e_m + e_{m+1} \]

Thus

\[ H^{8m+4}(D_{12} \rtimes \mathbb{Z}/2\mathbb{Z}) \cong k^{m+1} \]

\[ \cong \{(x_0, x_1, \ldots, x_{m-1}, x_m, x_{m-1}, \ldots, x_1, x_0) | x_i \in k, \forall i \} \subset H^{8m+4}(D_{12} \times D_{12}). \]  

**Case 2: \( q \equiv 1 \text{ mod } 4 \)**

**Subcase 2.1 and 2.2:** \( q \equiv 1, 5 \text{ mod } 8 \)

In this cases the cohomology of \( D_{12} \times D_{12} \) is itself zero. Hence

\[ H^q(D_{12} \rtimes \mathbb{Z}/2\mathbb{Z}) = 0 \]  

**Case 3: \( q \equiv 2 \text{ mod } 4 \)**

**Subcase 3.1:** \( q \equiv 2 \text{ mod } 8, q = 8m + 2 \)

In this case \( \kappa^* \) operates as follows.

\[ e_0 \xrightarrow{\kappa^*} e_{2m-1} \]
\[ e_1 \xrightarrow{\kappa^*} e_{2m-2} \]
\[ \vdots \]
\[ e_{m-1} \xrightarrow{\kappa^*} e_m \]
The invariants are,

\[ y_0 = e_0 + e_{2m-1} \]
\[ y_1 = e_1 + e_{2m-2} \]
\[ \vdots \]
\[ y_{m-1} = e_{m-1} + e_m \] (327)

Thus,

\[ H^{8m+2}(D_{12} \wr \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{k}^m \]
\[ \cong \{(x_0, x_1, \ldots, x_{m-1}, x_m, x_{m-1}, \ldots, x_1, x_0)|x_i \in \mathbb{k}, \forall i\} \subset H^{8m+2}(D_{12} \times D_{12}). \] (328)

Subcase 3.2: \( q \equiv 6 \mod 8, \ q = 8m + 6 \)

The \( \kappa^* \) operates as follows.

\[ e_0 \overset{\kappa^*}{\leftrightarrow} e_{2m} \]
\[ e_1 \overset{\kappa^*}{\leftrightarrow} e_{2m-1} \]
\[ \vdots \]
\[ e_{m-1} \overset{\kappa^*}{\leftrightarrow} e_{m+1} \]
\[ e_m \overset{\kappa^*}{\leftrightarrow} e_m \]

hence the invariants are,

\[ y_0 = e_0 + e_{2m} \]
\[ y_1 = e_1 + e_{2m-1} \]
\[ \vdots \]
\[ y_{m-1} = e_{m-1} + e_{m+1} \]
\[ y_m = e_m \] (329)

Thus

\[ H^{8m+6}(D_{12} \wr \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{k}^{m+1} \]
\[ \cong \{(x_0, x_1, \ldots, x_{m-1}, x_m, x_{m-1}, \ldots, x_1, x_0)|x_i \in \mathbb{k}, \forall i\} \subset H^{8m+6}(D_{12} \times D_{12}). \] (330)
Case 4: \( q \equiv 3 \mod 4 \)

Subcase 4.1: \( q \equiv 3 \mod 8, \ q = 8m + 3 \)

The \( \kappa^* \) operates as follows.

\[

e_0 \overset{\kappa^*}{\rightarrow} e_{4m+1} \\
e_1 \overset{\kappa^*}{\rightarrow} e_{4m} \\
\vdots \\
e_{2m} \overset{\kappa^*}{\rightarrow} e_{2m+1}
\]

Hence the invariants are,

\[
y_0 = e_0 + e_{4m+1} \\
y_1 = e_1 + e_{4m} \\
\vdots \\
y_{2m} = e_{2m} + e_{2m+1}
\]

Thus

\[
H^{8m+3}(D_{12} \wr \mathbb{Z}/2\mathbb{Z}) \cong k^{2m+1}
\]

\[
\cong \{(x_0, \ldots, x_{2m-1}, x_{2m}, x_{2m}, x_{2m-1}, \ldots, x_0) | x_i \in k, \forall i \} \subset H^{8m+3}(D_{12} \times D_{12}).
\]

Subcase 4.2: \( q \equiv 7 \mod 8, \ q = 8m + 7 \)

The \( \kappa^* \) operates as follows,

\[

e_0 \overset{\kappa^*}{\rightarrow} e_{4m+3} \\
e_1 \overset{\kappa^*}{\rightarrow} e_{4m+2} \\
\vdots \\
e_{2m+1} \overset{\kappa^*}{\rightarrow} e_{2m+2}
\]
Hence the invariants are,
\begin{align*}
y_0 &= e_0 + e_{4m+3} \\
y_1 &= e_1 + e_{4m+2} \\
\vdots \\
y_{2m+1} &= e_{2m+1} + e_{2m+2}
\end{align*}
(333)

Thus,
\begin{align*}
H^{8m+7}(D_{12} \wr \mathbb{Z}/2\mathbb{Z}) &\cong \mathbb{k}^{2m+2} \\
&\cong \{(x_0, \ldots, x_{2m+1}, x_{2m+1}, \ldots, x_0) | x_i \in \mathbb{k}, \forall i \} \subset H^{8m+7}(D_{12} \times D_{12}).
\end{align*}
(334)
LIST OF REFERENCES


[H1] H.-W. Henn, Centralizers of elementary abelian $p$-subgroups, the Borel construction of the singular locus and application to the cohomology of discrete groups, preprint.


