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Generalized controllability and observability filtrations and the Wedderburn Forney construction

Giust, Steven Joseph, Ph.D.
The Ohio State University, 1994
GENERALIZED CONTROLLABILITY AND OBSERVABILITY FILTRATIONS AND THE WEDDERBURN FORNEY CONSTRUCTION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
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By

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CHAPTER I

Introduction

In 1860 J. C. Maxwell founded the theory of automatic control systems with his paper "On Governors." He studied and recognized that the automatic control system's behavior, while in the neighborhood of an equilibrium point, could be approximated by a linear ordinary differential equation. In this way, dynamic system stability was defined in terms of the location of roots of certain algebraic characteristic equations. These roots were later called the poles of the system. In an effort to make unstable feedback systems stable and to build feedback amplifiers with loops having fast gain cutoff and small phase shifts, Bell Labs studied wave phase and amplitude in the frequency domain as it related to the "transfer function." This new approach of study in the frequency response of systems continued and was utilized heavily in control design. For example in the early 1920's, Minorsky's work based on these concepts allowed him to introduce the PID controller, which was utilized heavily in steering military ships. In the late 1930's there were two distinct methods used to analyze and describe system feedback behavior. One was the "time-domain" method, which used ODE solutions and associated algebraic equations. The other was the "frequency-domain" method, which involved transfer functions, complex analysis, and more use
of abstract frequency response systems known as black box diagrams. These methods and studies as well as their needs increased greatly through world war II. By 1950, the frequency method was the method of choice in feedback control design. But as the space age grew, so did the need to control several variables simultaneously. The earlier methods for the single-input/single-output systems (SISO) of the past needed modification.

In the 1960's Dr. Rudolph E. Kalman began to restudy the time-domain methods as they applied to multiple-input/multiple-output systems (MIMO). He was formalizing the mathematical definition of a control system. In the 1970's Rosenbrock, see [25], studied the zeros and poles of multivariable systems as they related to control design. He formalized ideas in terms of shifts of the characteristic equation. This work led to pole shifting techniques and Wonham's necessary and sufficient condition for arbitrary pole placement in closed loop systems.

The optimality and control work of Kalman and Rosenbrock revitalized the study and theory of frequency response methods for MIMO systems and their control. Kalman sought to find and construct the mathematical model of a system based on experimental observations and to find the mathematically "cheapest" such model (i.e., the fewest number of components needed to retain system characteristics). This problem was studied first in the context of finite dimensional stationary linear systems, then general time-varying continuous systems, and finally in the context of algebraic theory of discrete systems. (This paper focuses primarily on the algebraic theory of discrete-time invariant systems.) He formalized the input/output structure
of a system in terms of a \textit{minimal realization}, where minimal refers to the dimension of a vector space called the \textit{state space} of the system. He also defined and formalized into a module framework notions of controllability and observability. The results, algebraic techniques, and theory and approach used by Kalman were published in Chapter X of a 1969 book by Kalman, Farb, and Arbib called "Mathematical System Theory," see [12].

Following Kalman's original work on the algebraic structure of the state space, Bostwick Wyman and M. K. Sain continued and extended these notions and developed both module theoretic definitions for the state space and multivariate zero structures of a system. The study of these spaces and their interactions as well as the introduction of the Wedderburn Forney construction, see [4, 23, 16], lead to the \textit{Wyman-Sain-Conte-Perdon Fundamental Pole Zero Exact Sequence}. These spaces and the exact sequence are crucial to the results in this work, see [4, 5].

In this dissertation we first review some of the classical notions and definitions of a system. In chapter two we mathematically define the classical discrete-time invariant system. We study the system first in the time domain. We then introduce the notion of a transfer function and study the input/output structure of the system in the so called frequency domain. We also introduce the notion of a general singular system and show how a singular system may be decomposed into its causal and anticausal components, see [1, 2]. In chapter three we introduce the pole and zero module theoretic notions of a system as defined by Wyman and Sain. We introduce the Wedderburn Forney Space and show how these spaces fit together in the Funda-
mental Pole Zero Exact Sequence. Chapter four introduces the notion of a filtration or chain, which used throughout the work. In chapter five we introduce a generalization of the Wedderburn Forney construction. This chapter may be of interest in applied areas and complex analysis but is disjoint from the remaining chapters and so may be skipped without loss of comprehension. In chapter six we review and outline the classical notions of controllability and observability spaces and their respective indices for a system, as introduced by Kalman and illustrated by Kailath, see [7, 12]. We extend these definitions into the vector space framework of a filtration and place filtrations on the Wyman-Sain pole modules and the Wedderburn Forney space of the exact sequence. Chapter six extends and generalizes controllability filtrations to singular systems, unifying the concepts of controllability for classical strictly proper systems, for purely polynomial systems, and then general singular systems via the $\pi_+\text{-Wedderburn Filtration}$. We show how this construction ties in with a definition for generalized global controllability given by Malabre, see [8, 9, 10]. We also extend the techniques of chapter five to notions of global observability for singular systems. A non-singular pairing defined between the Wedderburn spaces that appear in the fundamental pole zero exact sequence is used to define a generalized global observability filtration on the pole space of the system. We also give a Malabre type recursive definition for this generalized observability filtration. The pairing agrees with the classical case and extends to the singular case the notion that in the classical case the controllability filtration and observability filtrations are dual.
CHAPTER II

Notation and Basic Definitions

2.1 Algebraic Conventions

Throughout this paper our methods will use ideas from commutative algebra, number theory, and the theory of modules over principal ideal domains. We do not stress applications, but instead emphasize the underlying algebraic structures. We will focus on connections of classical control and state space definitions to (a) newer more generalized module theoretic definitions and the Wyman-Sain fundamental pole zero exact sequence and (b) the Wedderburn Forney construction.

We denote by $k$ any arbitrary field of coefficients. An arbitrary vector from, say, an $m$ dimensional vector space $k^m$, will be denoted by $\mathbf{k}$. We have then $k[z]$ and $\bar{k}[z]$, which denote the ring of polynomials in $z$ with coefficients from $k$ or $k^m$, respectively. The field of rational functions in the variable $z$ is denoted by $k(z)$, and the local ring at infinity, $\mathcal{O}_\infty$, is just the subring of proper rational functions in $k(z)$.

If $V$ is any finite dimensional vector space of dimension $m$, say, write

$$V(z) = V \otimes_k k(z), \quad (2.1)$$
a vector space over $k(z)$, which may be viewed as the $m$-dimensional vector space of rational functions. Similarly,

$$\Omega V = V \otimes_k k[z], \quad (2.2)$$

a free module over $k[z]$; and

$$\Omega_\infty V = V \otimes_k \mathcal{O}_\infty. \quad (2.3)$$

We occasionally use the module of strictly proper vectors

$$z^{-1}\Omega_\infty V,$$

and write

$$V(z) = \Omega V \oplus z^{-1}\Omega_\infty V. \quad (2.4)$$

Elements in $\Omega V$ are represented by $\bar{v}[z]$. Elements in $\Omega_\infty V$ and $z^{-1}\Omega_\infty V$ are represented $\bar{v}(z)$. If it is necessary, and not clear from context, to indicate that $\bar{v}(z)$ is strictly proper, we will denote this by $\bar{v}_{sp}(z)$. Throughout, the letters s.p. will stand for strictly proper and indicate that the entries of the vector, matrix, etc. are all from the ring $z^{-1}\mathcal{O}_\infty$.

We define the following two projection maps:

(a)

$$\pi_+: V(z) \mapsto \Omega V \text{ polynomial part} \quad (2.5)$$

(b)

$$\pi_- : V(z) \mapsto z^{-1}\Omega_\infty V \text{ s.p. part} \quad (2.6)$$
So given $\bar{v}(z) \in V(z)$, $\bar{v} = \bar{v}[z] \oplus \bar{v}_{sp}(z)$,

$$\pi_{+}(\bar{v}(z)) = \bar{v}[z], \text{ and}$$

$$\pi_{-}(\bar{v}(z)) = \bar{v}_{sp}(z).$$

Throughout, a transfer function will be a $k(\pi)$-linear transformation

$$G(z) : U(z) \Rightarrow Y(z).$$

We will often, through a slight abuse of notation, identify $G(z)$ with a $p \times m$ matrix of rational functions. Remaining notational conventions are introduced as they appear.

### 2.2 Linear Systems and the Transfer Function

There are many types of linear systems and several popular equivalent ways to define each type. Throughout this work we will primarily be interested in one type viewed from two perspectives. One is the time domain perspective, and the other is the transfer function or frequency domain perspective. (All transfer functions in this document are just linear maps $G(z) : U(z) \rightarrow Y(z)$ and so may be viewed as matrices of rational functions.)

Let $k$ be an arbitrary field (e.g., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$).

**Definition 2.2.1** A time-invariant, discrete-time, purely dynamical causal system over $k$ consists of the sextuple $\Sigma = (X, U, Y; A, B, C)$, the elements of which are related by a dynamics equation and an output equation. $X$ is called the state
space and is an $n$-dimensional vector space over $k$. $U$ is the input space and is $m$-dimensional. $Y$ is the output space and is $p$-dimensional. $A$, $B$, and $C$ are the $k$-linear transformations

$$A : X \rightarrow X; \text{ dynamics map}$$
$$B : U \rightarrow X; \text{ input map}$$
$$C : U \rightarrow Y; \text{ output map}.$$

The two difference equations relating the maps and spaces are given as

\begin{align*}
x(t + 1) &= Ax(t) + Bu(t) \\
y &= Cx(t),
\end{align*} \tag{2.10}

where (2.10) is the dynamics equation, (2.11) is the output equation, and $t \in \mathbb{Z}$.

The system is called time-invariant because the matrices are all constant with respect to time. The intuition is that the input space $U$ consists of input vectors $u(t)$ chosen by the operator as inputs to the system at time $t$, the output space $Y$ consists of vectors $y(t)$ that are viewed as outputs of the system at time $t$, and the state space $X$ consists of vectors $x(t)$, which contain information to describe the physical state of the system at time $t$.

The usual classical input/output analysis begins by assuming that at some initial time $t = N \geq 0$, \( x(-N) = 0 \); i.e., the system is initially at rest. We consider a sequence of inputs starting at some time $n \geq N$ given by

$$u(-n), u(-n + 1), \ldots, u(0), u(1), \ldots.$$

As can be verified from the recurrence equations, this sequence produces the corresponding output sequence

\begin{align*}
y(-n) &= 0, \text{ and} \\
\end{align*} \tag{2.12}
\[ y(t) = \]

\[ CBu(t-1) + CABu(t-2) + CA^2Bu(t-3) + \ldots + CA^{t+n-1}Bu(-n) \]

for \( t > -n \). These input strings will be denoted by means of the \( z \)-transform.

**Definition 2.2.2** The \( z \)-transform of a sequence \( u(-n), u(-n+1), \ldots, u(0), u(1), \ldots \) is \( u(z) \overset{def}{=} \zeta(u(t)) = \)

\[ \sum_{t=-\infty}^{\infty} u(t)z^{-t} = z^n u(-n) + z^{n-1} u(-n+1) + \ldots + u(0) + z^{-1} u(1) + \ldots \]  

(2.14)

That is, we will view the input string as a formal Laurent series in \( U((z^{-1})) \), the space of formal Laurent series in powers of \( z^{-1} \) having vector valued coefficients \( \vec{u} \in U \). The time that a vector enters the system is indicated by \(-1\) times the corresponding exponent of \( z \). In this way, the present is marked by \( t = 0 \). We see that the inputs, \( u(t) \), from the past come from the polynomial part of the input string having positive powers of \( z \). Inputs that enter in the future \( (t > 0) \) are indexed by negative powers of \( z \).

**Definition 2.2.3** The time shift operator, \( \sigma \), of a vector \( u(t) \) is given by

\[ \sigma(u(t)) = u(t+1). \]  

(2.15)

**Lemma 2.2.4** Given a sequence \( u(t) \), \( \zeta(\sigma(u)) = z\zeta(u) \).

**Proof:**
\[
\zeta(\sigma(u)) = \sum_{t=-\infty}^{\infty} \sigma(u(t))z^{-t} = \sum u(t+1)z^{-t} = \sum u(t+1)z^{-(t-1)} = z\zeta(u). \tag{2.16}
\]

The \(z\)-transform is the discrete version of the Laplace transform. The \(z\)-transform of equations 2.10 and 2.11 transforms them as follows:

\[
\begin{align*}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*} \quad \rightarrow \quad \begin{align*}
z\zeta(x) &= A\zeta(x) + B\zeta(u) \\
\zeta(y) &= C\zeta(x).
\end{align*} \tag{2.18}
\]

By combining the transformed equations and solving for \(y(z) = \zeta(y)\) we have

\[
y(z) = C(zI - A)^{-1}Bu(z). \tag{2.19}
\]

The matrix \(G(z) = C(zI - A)^{-1}B\) is called the transfer function for the system \(\Sigma\).

The output sequence is \(y(z) = G(z)u(z)\); it is another viewpoint for the input/output structure of the system. It is easy to see that \(G(z)\) is a matrix of strictly proper rational functions. Such a matrix will be called strictly proper and is often denoted, \(G(z)_{sp}\).

On the other hand, polynomial matrices arise naturally from another type of system.

**Definition 2.2.5** A time-invariant, discrete-time, purely anti-causal system over \(k\) consists of the sextuple \(\Sigma' = (X, U, Y; J, B, C)\), the elements of which are related by a dynamics equation and an output equation. The elements \(X, U, Y, J, B, C\)
are all as before, but the two difference equations relating the maps and spaces are different, and are given as

\[ J x(t + 1) = x(t) + Bu(t) \quad (2.20) \]
\[ y(t) = C x(t) \quad (2.21) \]

where \( J \) is a nilpotent matrix of order \( k \), say.

By using the \( z \)-transform, we obtain the transfer function that results from this type of system,

\[ G(z) = C(zJ - I)^{-1}B = -C \left( \sum_{t=0}^{k-1} J^t z^t \right) B. \quad (2.22) \]

This matrix is completely polynomial and is denoted \( G(z)_{pol} \).

Given an arbitrary matrix \( G(z) \) of rational functions, we can break this matrix into two parts. \( G(z) = G(z)_{pol} + G(z)_{sp} \), where

\[ G(z)_{sp} = C_0(zI - A)^{-1}B_0, \quad \text{and} \]

\[ G(z)_{pol} = C_\infty(zJ - I)^{-1}B_\infty. \quad (2.24) \]

We may write this global transfer function as the combination of these two systems by taking the direct sum of the two individual state spaces renamed again as \( X \) with the equation

\[ \begin{align*}
G(z) &= \begin{bmatrix} C_\infty & C_0 \end{bmatrix} \left[ z \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \right]^{-1} \begin{bmatrix} B_\infty \\ B_0 \end{bmatrix} \\
&= \hat{C} z \hat{E} - \hat{A}^{-1} \hat{B}.
\end{align*} \quad (2.25) \]
Therefore the associated dynamics equations for an arbitrary transfer function are given as

\[ \dot{E}x(i+1) = \dot{A}x(i) + \dot{B}u(i) \]  

(2.27)

\[ y = \dot{C}x(i) \]  

(2.28)

where \( \dot{E} \) is singular and \((\dot{E}z - \dot{A})^{-1}\) is assumed to exist. For a more elaborate explanation of these notions see [1].

We have shown how to construct \( G(z) \) given \( \Sigma \) or \( \Sigma' \). In the other direction, a dynamic realization for a given transfer function \( G(z) \) is a choice of \( \Sigma \) or \( \Sigma' \) that satisfies equation (2.26). Given a transfer function \( G(z) \) it is possible to construct infinitely many possible associated dynamic realizations. A realization is called minimal if its state space \( X \) has the smallest possible dimension among all possible realizations. Such realizations are known to be unique up to a change of basis (see, [7, 12]). Unless otherwise expressed, all realizations throughout this document are assumed to be minimal.

**Definition 2.2.6** A realization is reachable if given any arbitrary vector \( \bar{x} \in X \) there exists a finite input string that can drive the system from the zero state to the state \( \bar{x} \) at time \( t = 1 \).

**Definition 2.2.7** A realization is controllable if any state \( x \in X \) can be driven to the zero state in a finite amount of time.

(Note: The above definitions have been a source of confusion in classical control. Some people define controllable to mean that the system can be driven from any
state to any other state in a finite amount of time, and the above definition 2.2.7 is sometimes referred to as controllable to the origin. If the matrix $A$ is invertible then controllability (to the origin) implies reachability. For our purposes, we will use the terms controllable and reachable interchangeably to mean that the rank of a certain matrix is full. This will agree with our above definition for reachability, see [7])

**Definition 2.2.8** A realization is called observable if for each output string,

$$y(1)z^{-1} + y(2)z^{-2} + \ldots$$

there exists a time, $t_k$, $1 \leq t_k < \infty$ such that the state of the system at time $t = 1$ can be uniquely determined by observing that part of the output

$$y(1)z^{-1} + y(2)z^{-2} + \ldots + y(k)z^{-k}.$$ 

It is well known that a realization is minimal iff it is both controllable and observable, see [7].

For a reader unfamiliar with state space and transfer function notions, we end with a small single-input single-output example showing calculations of how the state-space and transfer function approaches capture the same input/output structure of the system. An example of how to construct $\Sigma$ given $G(z)$ is at the end of the next chapter. For more on these techniques see [7].

Suppose a system $\Sigma$ is given by

$$x(t+1) = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$ (2.29)

$$y(t) = \begin{bmatrix} -2 & 1 \end{bmatrix} x(t).$$ (2.30)
Let \( u(z) = -2z + 1 \) be the input sequence and \( x(-1) = 0 \). Table 1 gives the input time, \( t \); the corresponding input; the state created at time \( t+1 \); the output time \( t+1 \); and the output for this example. Note that for a finite input sequence ending at time \( t = 0 \), all future outputs are determined by the state \( x(1) \) created at time \( t = 1 \) via the equation

\[
y(t) = CA^{t-1}x(1), \quad t \geq 1.
\]

(2.31)

Now consider the transfer function interpretation of the input/output structure of the system. The transfer function is obtained from the equation 2.23 and is given by

\[
G(z) = \frac{(z - 2)}{(z^2 - z - 6)},
\]

(2.32)

and the input will still be given as \( u(z) = (-2z + 1) \).
\[ G(z)u(z) = \frac{(z - 2)(-2z + 1)}{(z^2 - z - 6)} = -2 + 3z^{-1} - 11z^{-2} + 7z^{-3} + \ldots \] (2.33)

This gives the same output sequence as the state-space calculations above; the time of each output is indicated by the corresponding exponent of \( z \). The above is an example of the Kalman input/output experiment, see [12, 13]. For now, we will focus on the transfer function approach.
CHAPTER III

Poles and Zeros, The Wedderburn Forney Space, and The Exact Sequence

The topics in this chapter will pervade all sections and are crucial to the understanding of the rest of the work.

3.1 The Pole and Zero Modules

Definition 3.1.1 We will say \( G(z) \) is a strictly proper transfer function if all of its components are strictly proper rational functions. We say \( G(z) \) is proper if all of its components are proper. We say \( G(z) \) is improper if at least one of its components is an improper rational function.

(Note: Recall, only proper transfer functions are associated with state space realizations of the type \( \Sigma \). Improper transfer functions are associated with systems of the type \( \Sigma' = (\hat{E}, \hat{A}, \hat{B}, \hat{C}) \), related by equations 2.28.)

Given a possibly improper transfer function \( G(z) : U(z) \rightarrow Y(z) \), we attach various pole and zero modules to \( G(z) \). The classical pole module (finite pole space)
of $G(z)$ was originally defined by Kalman in a slightly different form in terms of minimal realizations (see [12]). It was later formalized in module theoretic terms by Wyman as

$$X(G) = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}. \quad (3.1)$$

The module $X(G)$ is a finitely generated torsion module over $k[z]$ and can be characterized intuitively as the space of polynomial inputs modulo those polynomial inputs that produce only polynomial outputs. Its structure agrees with the first homomorphism theorem for modules and makes the following diagram commute

where $B$ is the standard coset projection, $C$ is (monomorphic) inclusion into $\Gamma(Y)$, and

$$\Gamma(Y) \overset{\text{def}}{=} \frac{Y(z)}{\Omega Y} \cong \{ z^{-1}y_1 + z^{-2}y_2 + \ldots | y_i \in Y \} \quad (3.2)$$

is the $k[z]$-module of strictly proper power series in $z^{-1}$. This module structure is defined by the following action: For $y_{sp}(z) \in \Gamma(Y)$ and $p[z] \in k[z]$ define the module action $*$ as

$$p[z] * y_{sp}(z) = \pi_-(p[z] * y_{sp}(z)) \in \Gamma(Y). \quad (3.3)$$
That is, simply take the Cauchy product and throw away the polynomial part. This action completes the above commutative diagram. The space \( \Gamma(Y) \) has a \( k[z] \)-module structure defined as follows: Let \( p[z] \in k[z] \) and \( \overline{y(z)} \in \Gamma(Y) \), then

\[
p[z]y(z) \to (p[z]y(z))_{sp}.
\] (3.4)

In this way the above is a commutative diagram of \( k[z] \)-modules and the maps are all \( k[z] \)-linear.

If we have a \textit{minimal} realization \( (A, B, C; U, X, Y) \) then we have a state space and we could, with a slight abuse of notation, view

\[
\tilde{B} = B \otimes i : U[z] \to X, \text{ and}
\] (3.5)

\[
\tilde{C} : X \to \Gamma(Y)
\] (3.6)

defined as

\[
\tilde{B}(u(-n)z^n + \ldots + u_0) = A^nBu(-n) + \ldots + Bu(0) = \]
(3.7)

\[
[A^nB|A^{n-1}B|\ldots|B][u^T(-n),\ldots,u^T(0)]^T \overset{def}{=} x(1) \in X, \text{ and}
\] (3.8)

\[
\tilde{C}(x(1)) = Cx(1)z^{-1} + CAx(1)z^{-2} + CA^2x(1)z^{-3} + \ldots.
\] (3.9)

Observe first that the image of \( \tilde{B} \) is necessarily in the span of the matrix

\[
[A^nB|A^{n-1}B|\ldots|B],
\] (3.10)

and therefore the span of this matrix must be precisely the space of reachable states. Second, the state space \( X \) also makes the diagram commute, so we must have \( X(G) \cong X \) as \( k[z] \)-modules. For \( u[z] \in U[z] \), we see commutativity holds if

\[
\tilde{B}(zu[z]) = z\tilde{B}(u[z]) \equiv Ax(1) \in X.
\] (3.11)
So the $k[z]$ module action on $X$ is given by

$$zx(1) \overset{def}{=} Ax(1).$$

(3.12)

The vector space structure of the finite pole module yields therefore (after a choice of basis) a state space realization for the strictly proper part of $G(z)$. That is with some choice of basis it is possible to create $\Sigma$. (An example of the calculations involved to construct $A$, $B$, and $C$, are in [7] and at the end of this chapter; however, more is required before we get there.) We will denote the elements in $X(G)$ by the notation $u[z]$. The brackets emphasize that the elements are polynomial. The above construction proves the following theorem, (see [12]):

**Theorem 3.1.2** A system is reachable iff the map $\hat{B}$ defined in terms $B$ is surjective and observable iff $\hat{C}$ defined in terms of $C$ is injective. These properties hold precisely when $X \cong X(G)$ which occurs precisely when $X$ is minimal.

The finite transmission zero module (see, [4, 5, 15]) is defined as

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\ker G(z) + \Omega U}.$$  

(3.13)

It is also a finitely generated torsion $k[z]$-module. From an engineering point of view we have the following intuition. Given a non-homogeneous recurrence relation and initial values, the zeros are those inputs that produce outputs identical to those of the corresponding homogeneous equation having the same initial values. They are inputs that in a sense do nothing and produce no output. For a given set of initial conditions, the finite zero module mathematically describes the space of all inputs
which produce no future outputs, (i.e., for time $t \geq 1$) modulo those which produce absolutely no outputs ever. The finite zero module captures the classical notion of zeros for the strictly proper transfer function.

The analogs of these spaces at infinity are given by

$$X_\infty(G) = \frac{z^{-1}\Omega_\infty U}{G^{-1}(z^{-1}\Omega_\infty Y) \cap z^{-1}\Omega_\infty U} \quad (3.14)$$

$$Z_\infty(G) = \frac{G^{-1}(z^{-1}\Omega_\infty Y) + z^{-1}\Omega_\infty U}{\ker G(z) + z^{-1}\Omega_\infty U}, \quad (3.15)$$

see [2, 4, 5, 14, 15, 19]. The infinite pole and zero modules are also finitely generated torsion modules over the local ring at infinity, $\mathcal{O}_\infty$, defined in Chapter 1. The space $X_\infty(G)$ consists of strictly proper inputs modulo strictly proper inputs that produce strictly proper outputs. The strictly proper elements in $X_\infty(G)$ are represented as $v_{sp}(z)$. This, however, is not the correct pole space to realize a polynomial transfer function. The correct realization pole space $\mathcal{O}_\infty$ -module (often called the big pole space at infinity) was described in [1] and is defined as

$$X'_\infty(G) = \frac{\Omega_\infty U}{G^{-1}(z^{-1}\Omega_\infty Y) \cap \Omega_\infty U}. \quad (3.16)$$

Because each module is a finitely generated torsion module over its respective PID, then each is a finite dimensional vector space over $k$. These spaces may be assembled into global spaces by direct summation as

$$\mathcal{X}(G) = X(G) \oplus X_\infty(G), \quad (3.17)$$

$$\mathcal{X}'(G) = X(G) \oplus X'_\infty(G), \quad (3.18)$$
and

\[ \mathcal{Z}(G) = \mathcal{Z}(G) \oplus \mathcal{Z}_\infty(G) \]  \hspace{1cm} (3.19)

These global spaces carry no natural module structure, since they are direct sums of
modules over different rings, but they will play a major part in what follows [1, 2, 4, 5].

The matrix

\[ [B_0|AB_0| \ldots |A^{n-1}B_0] \]

in the classical literature is called the controllability matrix. We saw earlier, page 18,
that its span is precisely the space of reachable states in the minimal realization of
\( G_{sp}(z) \). The span of this matrix over \( k \) will be denoted

\[ \langle [B_0|AB_0| \ldots |A^{n-1}B_0] \rangle_k. \]

Classically, \( X(G) \) \( \overset{v.sp.}{=} \langle [B_0|AB_0| \ldots |A^{n-1}B_0] \rangle_k \) and is the minimal realization space
for \( G_{sp} \). We saw this result when we considered the commutative diagram on page
17. Similarly, \( X'_\infty(G) \) is the classical realization space for \( G_{poly} \). The vector space
structure of this module yields a minimal realization for the polynomial part of an
improper transfer function. These spaces have all been studied extensively in [1, 2,
4, 5, 13, 14, 15, 17].

From the theory of finitely generated modules over PID's (see [28]) there exist
vectors \( \{ \tilde{f}_i \}_{i=1}^m \subseteq \Omega_\infty U \) and integers \( e_1 \geq e_2 \geq \ldots \geq e_m \) such that

\[ X'_\infty(G) \overset{\text{module}}{=} \frac{\tilde{f}_1}{(z^{-1})^{e_1+1}} \oplus \ldots \oplus \frac{\tilde{f}_m}{(z^{-1})^{e_m+1}} \overset{v.sp.}{=} [B_\infty|JB_\infty| \ldots |J^{e_1}B_\infty] \]  \hspace{1cm} (3.20)

where multiplication by \( z^{-1} \) in the module structure is equivalent to multiplication
by the nilpotent matrix \( J \) in the vector space structure [1]. It is this matrix \( J \) which
occurs in equation 2.26. From the definition of $X_\infty(G)$ we see that

$$X_\infty(G) \cong \frac{z^{-1}f_1}{(z^{-1})^{e_1+1}} \oplus \ldots \oplus \frac{z^{-1}f_m}{(z^{-1})^{e_m+1}} \cong JX'_\infty(G). \quad (3.21)$$

That is to say

$$X_\infty(G) \oplus X(G) = \mathcal{E}(X'_\infty(G) \oplus X(G)) \quad (3.22)$$

and so $\mathcal{X}(G) = \text{im}\mathcal{E}$. \quad (3.23)

This result will be used in the final chapter.

### 3.2 Poles, Zeros, and Coprime Factorizations

For an arbitrary single-input/single-output (SISO) system, the transfer function is just a rational function, $G(z) = \frac{p(z)}{q(z)}$. Classically, the poles of $G$ are the roots of $q(z)$ and the zeros are the roots of $p(z)$ over the field $\mathbb{C}$. They are counted according to their multiplicities. We also have poles and zeros at infinity. If $\dim q(z) > \dim p(z)$, infinity is a zero. If $\dim q(z) < \dim p(z)$, infinity is a pole.

Poles capture where the transfer function "blows up," and zeros capture where $G$ disappears. Now from a different perspective, view $p(z)$ and $q(z)$ as $1 \times 1$ matrices and write $G = [p(z)][q(z)]^{-1}$. The poles and zeros of $G$ can be viewed as those values of $z \in \mathbb{C}$ (or more precisely, those primes in the ring $\mathbb{C}[z]$) at which (or modulo which) the matrices $p(z)$ and $q(z)$ drop rank. We now extend this notion to an arbitrary $p \times m$ transfer function.

**Definition 3.2.1** Let $M$ be a polynomial matrix. The column (row) high-order coefficient matrix of $M$, denoted by $[M]_h$, is the matrix over $k$ whose $i$th column (row)
consists of the coefficients of $z^{e_i}$ in the $i$th column (row) of $M$, where $e_i$ is the greatest polynomial degree occurring in the $i$th column of $M$.

We will soon explain this concept further, see page 32.

For now, suppose we are given an arbitrary transfer function $G(z) : U(z) \rightarrow Y(z)$.

**Definition 3.2.2** We say $(N[z], D[z])$ is a polynomial globally right coprime matrix fraction decomposition (RCPMFD) for $G(z) = ND^{-1}$ if

(i) $D[z] : U[z] \rightarrow U[z]$, $N[z] : U[z] \rightarrow Y[z],$

(ii) $\det D[z] \neq 0,$

(iii) there exist polynomial matrices $A[z]$ and $B[z]$ such that $AD + BN = I,$

(iv) $L \overset{def}{=} \begin{bmatrix} D \\ N \end{bmatrix}$ has column full-rank high-order coefficient matrix.

Similarly,

**Definition 3.2.3** We say $(N[z], D[z])$ is a polynomial globally left coprime matrix fraction decomposition (LCPMFD) for $G(z) = D^{-1}N$ if

(i) $D[z] : Y[z] \rightarrow Y[z]$, $N[z] : U[z] \rightarrow Y[z],$

(ii) $\det D[z] \neq 0,$

(iii) there exist polynomial matrices $A[z]$ and $B[z]$ such that $DA + NB = I,$

(iv) $L \overset{def}{=} [D|N]$ has row full-rank high-order coefficient matrix.

If only conditions (i), (ii), and (iii) hold, we say the factorization is polynomial coprime. If only conditions (i), (ii), and (iv) hold, we say the factorization is coprime
at infinity. The term global means all four conditions hold and the factorization is coprime everywhere, both in the finite plane and infinity. When this condition is not required we will just say the factorization is polynomial coprime. For more on high-order coefficient matrices see definition 3.4.4 and [18]. Also, the definition for a global polynomial right coprime (left coprime) matrix fraction decomposition is equivalent to saying $L$ is column minimal (row minimal) in the sense of Forney. High-order coefficient matrices and column minimality will be discussed and explained more when we study Wedderburn-Forney spaces, see definition 3.4.4 and see [11]. We would just say here that an important result of Forney in [11] (The Main Theorem) is that such a matrix $L$ has no rank drop modulo any prime $p(z) \in k[z]$. Definitions for a right (left) matrix fraction decomposition at $\textit{infinity}$, denoted by

\begin{align*}
G(z) &= N_\infty D^{-1}_\infty \quad \text{and} \\
G(z) &= D^{-1}_\infty N_\infty,
\end{align*} \tag{3.24} \tag{3.25}

are identical except all matrices have elements from $\mathcal{O}_\infty$ instead of $k[z]$, and condition (iv) always holds. (Note: An example illustrating the algorithm to calculate a right coprime factorization is in the examples at the end of this chapter. It is a necessary step in computing a realization sextuple $\Sigma$ for a given $G(z)$.)

In [15, 19] it is shown that

\begin{align*}
X(G) &\cong \frac{\Omega U}{[D]\Omega U}; \quad Z(G) \cong \frac{\Omega Y}{[N]\Omega U} \bigg|_{\text{torsion}},
\end{align*} \tag{3.26}
over \( k[z] \), and that

\[
X_{\infty}(G) \cong \frac{z^{-1}\Omega_{\infty}U}{[D_{\infty}](z^{-1}\Omega_{\infty}U)}; \quad Z_{\infty}(G) \cong \frac{z^{-1}\Omega_{\infty}Y}{[N_{\infty}](z^{-1}\Omega_{\infty}U)}_{\text{torsion}}
\]

(3.27)

over \( \mathcal{O}_{\infty} \), where the torsion notation refers to the torsion component of the module only. These modules capture the zeros and poles of a transfer function \( G \). The poles and zeros are the roots of the invariant factors of these modules; the corresponding multiplicities of these poles and zeros are the root multiplicities of the primary components. Analogous to the SISO case above, the poles and zeros of \( G \) can be viewed as those values of \( z \in \mathbb{C} \) (or more precisely, those primes in the ring \( \mathbb{C}[z] \)) at which (or modulo which) the matrices \( N[z] \) and \( D[z] \) drop rank. A more general proof of this is given in [5] along with more discussion of various properties of these spaces in [4, 5, 14, 19]. We include here a version of the theorem and proof.

**Lemma 3.2.4** Let \( k \) be an arbitrary field, and let \( G(z) \) be a transfer function over \( k(z) \). Let \( (N, D) \) be a polynomial RCPMFD for \( G(z) \). The matrix \( N \) drops rank at precisely those primes \( p(z) \in k[z] \) for which \( Z(G) \) has \( p(z) \) torsion, and the rank drop is equal to the multiplicity of this primary factor in the module structure of \( Z(G) \).

**Proof:** Let \( p(z) \) be any prime in \( k[z] \) and \( \mathcal{O}_p \) be its local ring. Define \( r = \text{rank } N[z] \) and define \( r_P = \text{rank } N_P \), the rank of \( N \) mod the residue class field \( \kappa_P \) of \( \mathcal{P} \). Using the result that

\[
Z(G) \cong \frac{\Omega Y}{[N]\Omega U}_{\text{torsion}}
\]

(3.28)
consider the following exact sequences

\[ 0 \to \Omega U \xrightarrow{N[z]} \Omega Y \to Z(G) \oplus k[z]^{p-r} \to 0 \quad (3.29) \]

\[ 0 \to \Omega U \xrightarrow{N[z]} \Omega Y \xrightarrow{\frac{\Omega Y}{[N]\Omega U}} \oplus k[z]^{p-r} \to 0 \quad (3.30) \]

Tensoring through by \( \mathcal{O}_p \) over \( k[z] \) yields the right exact sequence

\[ \Omega U \otimes_{k[z]} \mathcal{O}_p \xrightarrow{N \otimes_{k[z]} \mathcal{O}_p} \Omega Y \otimes_{k[z]} \mathcal{O}_p \to (Z(G) \otimes_{k[z]} \mathcal{O}_p) \oplus \mathcal{O}_p^{p-r} \to 0 \quad (3.31) \]

or

\[ \mathcal{O}_p^m \xrightarrow{N \otimes_{k[z]} \mathcal{O}_p} \mathcal{O}_p \to Z_p(G) \oplus \mathcal{O}_p^{p-r} \quad (3.32) \]

where \( Z_p(G) \) is the \( p \)-torsion component of \( Z(G) \), and is the only primary factor of \( Z(G) \) that survives. Then modulo the field \( \kappa P \) we have

\[ \kappa_P^m \xrightarrow{N} \kappa_P^P \to \kappa_P^p \oplus \kappa_P^{p-r} \to 0 \quad (3.33) \]

Then \( p - r_P = j + p - r \) or \( r - r_P = j \). So \( j \) is nonzero iff \( Z(G) \) has \( p(z) \)-torsion iff \( N[z] \) has rank drop mod \( p[z] \). The rank drop is equal to the vector space dimension (or Jordan block size) associated with this primary factor. A similar proof holds for \( X(G) \) and \( D[z] \). Also, since the local ring was arbitrary, we can use the same proofs for the infinite pole and zero modules over \( \mathcal{O}_\infty \) and an \( \mathcal{O}_\infty \) RCPMF D \( (N_\infty, D_\infty) \). From this we can conclude the following corollaries.

**Corollary 3.2.5** If \( L \) is a polynomial matrix and \( L \) has no rank drop modulo any prime \( p(z) \) in \( k[z] \) then \( Z(L) = 0 \) and \( X(L) = 0 \).
The proof follows immediately from lemma 3.2.4 by letting \( L = G(z) \) play the role of \( N \) in the proof (you can define \( D \) to be the identity if you like).

**Corollary 3.2.6** If \( L \) is a polynomial matrix and has a column full-rank high-order coefficient matrix then \( Z_\infty(L) = 0 \).

**Proof:** Define \( D_\infty = \text{diag}[z^{-e_i}] \), where \( e_i \) is the \( i \)th column degree of \( L \). Then \( LD_\infty = N_\infty \) yields \((N_\infty, D_\infty)\) which is an \( O_\infty \) RCPMFD for \( L \). Since the rank\( O_\infty(N_\infty) = \text{rank}(\lim_{z \to \infty} N_\infty) \) which equals the rank of the high order coefficient matrix of \( L \), then \( N_\infty \) has no rank drop modulo the only prime, \( \infty \), in \( O_\infty \) and so \( Z_\infty(L) = 0 \).

Forney shows in [11] that a minimal basis has no rank drop modulo all \( p(z) \in k[z] \) and has a full-rank high-order coefficient matrix. These two corollaries show that a minimal polynomial basis \( M \) has \( Z(M) = 0 \).

**Corollary 3.2.7** If \((N, D)\) is an polynomial RCPMFD and we define \( L = \begin{bmatrix} D \\ N \end{bmatrix} \) then \( Z(L) = 0 \) and \( X(L) = 0 \).

**Corollary 3.2.8** If \((N_\infty, D_\infty)\) is an \( O_\infty \) RCPMFD and we define \( L = \begin{bmatrix} D_\infty \\ N_\infty \end{bmatrix} \) then \( Z_\infty(L) = 0 \) and \( X_\infty(L) = 0 \).

Results for the left coprime factorizations given earlier hold and follow similarly.
3.3 The Wedderburn-Forney Space and The Fundamental Pole-Zero Exact Sequence

A new result called the generalized Wedderburn Forney space appears in Chapter V of this work. It provides a better, fuller understanding of the Wedderburn Forney space by giving greater insight into how one thinks in this space and into how computations are carried out in this construction.

A full accounting of the zeros of a transfer function must take into consideration not only the global zero space $\mathcal{Z}(G)$ but also the “generic zeros,” which arise from the kernel and cokernel of $G$. Following [4, 5] we measure these generic zeros by the dimension over $k$ of finite dimensional Wedderburn-Forney Vector Spaces (WFS) that arise and are constructed from the kernel and cokernel of $G$.

Definition 3.3.1 Let $k$ be an arbitrary field and $k(z)$ be its field of fractions. Let $V$ be an $m$-dimensional ($m < \infty$) vector space over $k$. $V(z) = k^m(z) = V \otimes_{k(z)} k(z)$ is just the vector space of dimension $m$ over $k(z)$ formed from column vectors with coefficients in $k(z)$. Let $C \subset V(z)$ be an arbitrary subspace of $V(z)$. Three equivalent definitions up to isomorphism for the WFS of $C$ are

\[
W(C) = \frac{\pi_{+}(C)}{\pi_{+}(C) \cap (C)}, \tag{3.34}
\]

\[
W(C) = \frac{\pi_{-}(C)}{\pi_{-}(C) \cap (C)} \quad \text{and} \quad \tag{3.35}
\]
W(C) = \frac{C}{(\pi_+(C) \cap (C)) \oplus (\pi_-(C) \cap (C))}, \quad (3.36)

see Chapter V and [4, 5, 11] for more details. The first is called the \( \pi_+ \)- and the second is called the \( \pi_- \)-Wedderburn space. We won't be using definition (3.36) in this paper; it is here for completeness.

For example, consider equation (3.34) and suppose

\[
\tilde{c} = \begin{pmatrix}
-z + 2 + 3z^{-1} \\
3z^2 + z^{-1} + 4z^{-2}
\end{pmatrix} \in C
\]

then \( \pi_+(\tilde{c}) = \begin{pmatrix}
-z + 2 \\
3z^2
\end{pmatrix} \) so \( \begin{pmatrix}
-z + 2 \\
3z^2
\end{pmatrix} \in W(C). \)

That is, definition (3.34) consists of polynomial parts of vectors from \( C \) modulo those vectors in \( C \) that are themselves polynomial. A similar intuition holds for definition (3.35). An important fact from [5] is that the Wedderburn space is a finite dimensional vector space over \( k \). This fact requires a few more definitions and will be shown later.

The following Theorem is the main result of [5] and relates the vector space structures of the global poles and all zeros of a transfer function \( G \).

**Theorem 3.3.2** The global pole and zero modules of a transfer function are related by the following Fundamental Pole-Zero Exact Sequence (FPZES)

\[
0 \to Z(G) \xrightarrow{\alpha} \frac{X(G) \oplus X_{\infty}(G)}{W(\ker[G])} \xrightarrow{\beta} W(\text{im } [G]) \to 0. \quad (3.37)
\]

From this sequence we see that the dimension of the global pole space is equal to the dimension of the global zero space plus the dimension of the \( W(\ker[G]) \) plus the
dimension of the $W(\text{im}G)$. That is,

$$|\mathcal{X}(G)| = |\mathcal{Z}(G)| + |W(\ker[G])| + |W(\text{im}G)|. \tag{3.38}$$

The sum of the dimensions of these two Wedderburn spaces is called the \textit{defect} of the matrix $G$. The defect measures the difference between the number of global poles and global zeros. One or both of the above Wedderburn spaces is nontrivial precisely when the kernel or cokernel of $G$ is nontrivial. The Wedderburn spaces measure the lack of injectivity and surjectivity of $G$ and are said to capture the "generic zeros" of the system. With this correction, equation (3.38) says the total number of poles of a transfer function is always equal to the total number of zeros (counting both global and generic zeros). (See [4, 5].) (Note: The the defect of a rational function, viewed as a transfer function, is always zero. It's kernel and cokernel are always trivial. Thus the well known phrase \textit{The number of poles of a rational function is always equal to the number of zeros, counting those at infinity}. For a matrix $G$, however, the kernel and cokernel must also be considered.) The map $\alpha$ is defined as follows: A typical element in $\mathcal{Z}$ always has a coset representative of the form $(\bar{u}(z), \bar{v}(z))$, such that $G(z)\bar{u}(z) \in \Omega Y$ and $G(z)\bar{v}(z) \in z^{-1}\Omega_{\infty}Y$. That is $\bar{u}(z) \in \mathcal{Z}(G)$ and $\bar{v}(z) \in \mathcal{Z}_{\infty}(G)$. Then

$$\alpha : \mathcal{Z}(G) \rightarrow \frac{X(G) \oplus X_{\infty}(G)}{W(\ker[G])}$$

is given by

$$\alpha((\bar{u}(z), \bar{v}(z))) \overset{\text{def}}{=} \left(\pi_{+}(\bar{u}(z) + \bar{v}(z)), \pi_{-}(\bar{u}(z) + \bar{v}(z))\right). \tag{3.39}$$

In this work our interest rests primarily with the latter half of the sequence and with
the map $\beta$ where if $(\bar{u}[z], \bar{v}_{sp}(z)) \in \mathcal{X}(G)$, then

$$\beta(\bar{u}[z], \bar{v}_{sp}(z)) \overset{\text{def}}{=} \frac{\pi_+(\mu[\bar{G}] (\bar{u}[z] + \bar{v}_{sp}(z))))}{W(\text{im}[G])}. \quad (3.40)$$

The sequence and spaces will play a crucial role in our treatment of global controllability and observability filtrations and indices.

### 3.4 Definitions and Isomorphisms

We are now in a position to apply the FPZES to prove a few lemmas. First, we will need a few definitions.

**Definition 3.4.1** The degree of a polynomial vector $\bar{v} \in V$, denoted $e = \delta(\bar{v})$, is the greatest degree of its polynomial components.

**Definition 3.4.2** Given a polynomial matrix $M$, the sum of the column (row) degrees, $\sum e_i$, of $M$ is called the column (row) order of $M$.

**Definition 3.4.3** If $V$ is an $n$-dimensional vector space over $k(z)$, a minimal polynomial basis for $M$ is a polynomial basis matrix with least order over all possible polynomial bases for $V$. 
Definition 3.4.4 Let $M$ be a polynomial matrix. The column (row) high-order coefficient matrix of $M$, denoted by $[M]_h$, is the matrix over $k$ whose $i$th column (row) consists of the coefficients of $z^e_i$ in the $i$th column (row) of $M$.

Algorithms showing how to construct a minimal basis are available in the fundamental paper by Forney, see [11]. See also [18]. An important result of Forney that we will use here is as follows:

A polynomial basis $M$ is minimal iff both

(a) $M$ is non-singular modulo $p(z)$ for all irreducible polynomials, and

(b) its high-order coefficient matrix $[M]_h$, has full rank.

Because such a matrix has no rank drop modulo any primes in $k[z]$, then by Lemma (3.2.4) the finite pole and zero modules are zero and the infinite zero module is also zero.

Given $C \in V$, then denote by $C^\perp$ the orthogonal complement of $C$ inside $V(z)$, where orthogonality is with respect to the standard inner product space of $V(z)$ over $k(z)$. This is the standard notion of orthogonality. We will have others later.

Isomorphism 1 Let $C \in V(z)$, then $W(C) \cong W(C^\perp)$.

Proof: Given a space $C \subset V(z)$ we may choose a minimal polynomial column basis matrix $[C]$ for $C$ over $k(z)$ and identify the im$[C]$ with $C$. Then we have

$$0 \to \mathcal{Z}([C]) \xrightarrow{\alpha} \frac{X_\infty([C])}{W(\ker[C])} \xrightarrow{\beta} W(\text{im}[C]) \to 0$$
and

$$0 \rightarrow \mathcal{Z}(\mathbb{C}^T) \xrightarrow{\mathcal{A}} \frac{X_{\infty}(\mathbb{C}^T)}{W(\ker(\mathbb{C}^T))} \xrightarrow{\mathcal{B}} W(\text{im}(\mathbb{C}^T)) \rightarrow 0. \quad (3.42)$$

Since the matrix $\mathbb{C}$ is minimal, it has no rank drop in the finite plane or at infinity. By the corollaries to Lemma (3.2.4) we have that $\mathcal{Z}(\mathbb{C}) = 0$ and similarly $\mathcal{Z}(\mathbb{C}^T) = 0$. Since $\mathbb{C}$ has full rank, $\ker(\mathbb{C}) = 0$ and $\text{coker}(\mathbb{C}^T) = 0$; therefore, $W(\mathbb{C}) = W(\text{im}(\mathbb{C}^T)) = 0$. Exactness, and the fact that transposition doesn’t change the dimension of the pole space $X_{\infty}$ of a minimal matrix, yields that

$$W(\mathbb{C}^T) \cong X_{\infty}(\mathbb{C}^T) \cong X_{\infty}(\mathbb{C}) \cong W(\text{im}(\mathbb{C})). \quad (3.43)$$

So $W(\mathbb{C}) \cong W(\mathbb{C}^T)$.

Actually, more is true. Later we develop a non-singular pairing between these two spaces and see that they are dual with respect to a new inner product.

**Isomorphism 2** If $G(z) = D^{-1}[z]N[z]$ is a global polynomial LCPMFD for $G(z)$ and $L = [D|N]$ then $W(\ker L) \cong X_{\infty}(L)$.

**Proof:** First write

$$0 \rightarrow \mathcal{Z}(L) \xrightarrow{\mathcal{A}} \frac{X(L) \oplus X_{\infty}(L)}{W(\ker L)} \xrightarrow{\mathcal{B}} W(\text{im}L) \rightarrow 0. \quad (3.44)$$

Then note that $\mathcal{Z}(L) = 0$ by the corollaries to Lemma (3.2.4). Also, $W(\text{im}L) = 0$ since $L$ is surjective. Finally, $X(L)$ is just $X_{\infty}(L)$ because $L$ is polynomial and so cannot have finite poles. The sequence then implies that

$$\frac{X_{\infty}(L)}{W(\ker L)} = (0) \quad (3.45)$$
and isomorphism 2 follows.

**Isomorphism 3** Suppose that \( G(z) \) is a \( p \times m \) transfer function and that \( H = [G(z)]I_p \). Then there is an isomorphism of vector spaces \( \mathcal{X}(G) \cong W(\ker H) \).

**Proof:** Write
\[
0 \to \mathcal{Z}(H) \xrightarrow{\phi} \frac{X(H) \oplus X_\infty(H)}{W(\ker H)} \xrightarrow{\theta} W(\text{im } H) \to 0. \tag{3.46}
\]
Note, \( W(\text{im } H) = 0 \) since \( H \) is surjective. It is straightforward from the definitions of the pole and zero spaces to see that extending \( G(z) \) by \( I_P \) kills all the zeros and does not affect the pole spaces. That is, \( \mathcal{Z}(H) = (0) \) and \( \mathcal{X}(G) \cong \mathcal{X}(H) \). Therefore \( \mathcal{X}(H) \cong W(\ker H) \), From the exactness of the sequence we have
\[
\frac{\mathcal{X}(H)}{W(\ker H)} = (0), \tag{3.47}
\]
and isomorphism 3 follows.

**Isomorphism 4** If \( G(z) = D^{-1}[z]N[z] \) is a globally left coprime factorization of \( G \) and \( L = [D|N] \) then there is an isomorphism of vector spaces \( \mathcal{X}(G) \cong X_\infty(L) \).

**Proof:** By definition, the Wedderburn space is determined by the space of its argument and is free of the choice of basis for this space. Therefore, since the \( \ker L = \ker H \), we have \( W(\ker L) = W(\ker H) \). By the previous two isomorphisms we have
\[
X_\infty(L) \cong W(\ker L) = W(\ker H) \cong \mathcal{X}(G). \tag{3.48}
\]
Lemma 3.4.5 Given a $p \times m$ matrix $G : U(z) \to Y(z)$ and a Right Globally Coprime Matrix Fraction Decomposition pair of polynomial matrices $(N, D)$ such that $G = ND^{-1}$, let $\tilde{d}_i$ be the $i$th column of $\begin{bmatrix} D \\ N \end{bmatrix}$ and $e_i$ its corresponding column degree. Then we have

(i) $X(G) = X(\begin{bmatrix} I \\ G \end{bmatrix})$ and $X_\infty(G) = X_\infty(\begin{bmatrix} I \\ G \end{bmatrix})$.

(ii) $\ker \begin{bmatrix} I \\ G \end{bmatrix} = 0 \Rightarrow W(\ker \begin{bmatrix} I \\ G \end{bmatrix}) = 0$.

(iii) $\mathcal{E}(\begin{bmatrix} I \\ G \end{bmatrix}) = 0$.

(iv) $W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) = W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix})$ with $\dim_k(\text{im} \begin{bmatrix} D \\ N \end{bmatrix}) = e \overset{\text{def}}{=} \sum_{i=1}^{m} e_i$.

(v) $(i), (ii), (iii)$, and $(iv) \Rightarrow X(G) \oplus X_\infty(G) \cong W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) = W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix})$.

(vi) $W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix}) = \langle B \rangle_k$ where $B = \{ \frac{1}{\pi}(z^{-j} \tilde{d}_i) \, | \, i = 1, \ldots, m; \ j = 1, \ldots, e_i \}$. That is $B$ forms a basis over $k$ for $W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix})$. 
Proof:

(i) Let \( \bar{u} \in \Omega U \). We have \( \bar{u} \equiv 0 \) in \( X(G) \) \( \Leftrightarrow \)

\[
\bar{u} \in (G^{-1}(\Omega Y) \cap \Omega U) \Leftrightarrow [(\bar{u} \in \Omega U) \land (G(z)\bar{u} \in \Omega Y)] \Leftrightarrow \\
\begin{pmatrix}
I \\
G
\end{pmatrix} \bar{u} \in \begin{pmatrix}
\Omega U \\
\Omega Y
\end{pmatrix} \land (\bar{u} \in \Omega U).
\]

(3.49)

An identical proof shows that \( X_{\infty}(G) = X_{\infty}(\begin{pmatrix}
I \\
G
\end{pmatrix}) \). So

\[
X(G) \oplus X_{\infty}(G) = X(\begin{pmatrix}
I \\
G
\end{pmatrix}) \oplus X_{\infty}(\begin{pmatrix}
I \\
G
\end{pmatrix}).
\]

(3.50)

This makes intuitive sense; we don’t expect stacking with the identity matrix to contribute any poles to the matrix \( G \).

(ii) From the definition it is clear that the Wedderburn of an improper subspace is always zero.

(iii) We show \( Z(\begin{pmatrix}
I \\
G
\end{pmatrix}) = 0 \). The proof \( Z_{\infty}(\begin{pmatrix}
I \\
G
\end{pmatrix}) = 0 \) is identical.

Let \( \bar{u}(z) \in Z(\begin{pmatrix}
I \\
G
\end{pmatrix}) \). Then we may choose a coset representative such that

\[
\begin{pmatrix}
I \\
G
\end{pmatrix} \bar{u}(z) = \begin{pmatrix}
\bar{u}(z) \\
\bar{y}[z]
\end{pmatrix} \in \begin{pmatrix}
\Omega U \\
\Omega Y
\end{pmatrix} \Rightarrow \bar{u}(z) \in \Omega U,
\]

(3.51)

because of the identity matrix component. Then \( \bar{u}(z) \equiv \bar{0} \) in \( Z(\begin{pmatrix}
I \\
G
\end{pmatrix}) \). Since \( \bar{u}(z) \) was an arbitrary coset, every coset of \( Z(\begin{pmatrix}
I \\
G
\end{pmatrix}) \) is equivalent to zero.

(iv) Note \( \begin{pmatrix}
I \\
G
\end{pmatrix} = \begin{pmatrix}
D \\
N
\end{pmatrix}D^{-1} \) so that \( \text{im} \begin{pmatrix}
I \\
G
\end{pmatrix} = \text{im} \begin{pmatrix}
D \\
N
\end{pmatrix} \). By definition the Wedderburn construction is a function of the image space of a matrix and is unaffected by the choice of basis for this image space. There is a caveat here. The Wedderburn of a subspace \( C \) does depend on the basis chosen for the overall global space \( V(z) \) in...
which \( C \) sits (which we are assuming here and throughout to be the standard basis) in much the same way the matrix representation of a linear transformation is affected by choice of basis, but after this is set the Wedderburn space does not depend on the particular basis chosen to represent the subspace in its argument. The statement about the dimension over \( k \) will be shown in (vi).

(v) If we plug the matrix \( \begin{bmatrix} I & G \end{bmatrix} \) into equation (3.37) and apply the above results (ii), (iii), and (iv), we get

\[
X(\begin{bmatrix} I & G \end{bmatrix}) \oplus X_{\infty}(\begin{bmatrix} I & G \end{bmatrix}) \cong W(\text{im} \begin{bmatrix} I & G \end{bmatrix}) = W(\text{im} \begin{bmatrix} D & N \end{bmatrix}).
\]

(3.52)

Now apply (i) and the result follows immediately.

(vi) We will prove a more general result for which this is a specific case (i.e., the matrix \( \begin{bmatrix} D & N \end{bmatrix} \) is a minimal basis in the sense of Forney). Let \( V(z) \) be of dimension \( m < \infty \) over \( k(z) \) and \( C \subseteq V(z) \) be a subspace of \( V(z) \). Let \( \{g_1, \ldots, g_r\} \) be a minimal polynomial basis for \( C \) with \( \delta(g_i) = e_i \). We may assume w.l.o.g. that the \( e_1 \geq \ldots \geq e_r \). Define \( G = [g_1, \ldots, g_r] \) to be the matrix of these ordered column vectors and identify the image of \( G \), \( \text{im}G \), with the space \( C \). The \( W(\text{im}G) \) is a finite dimensional vector space over \( k \) of dimension \( e = \sum_{i=1}^{r} e_i \) and a basis for \( W(\text{im}G) \) is given by the set of class representatives \( B = \{ \pi_+(z^{-j}g_i) \mid j = 1, \ldots, e_i; \, i = 1, \ldots, r \} \) [5, 11].

**Proof:** Let \( \bar{c}(z) \in C \). Then there exists \( a_1(z), \ldots, a_r(z) \in k(z) \) such that

\[
\bar{c}(z) = \sum_{i=1}^{r} a_i(z)g_i = [G] \begin{pmatrix} a_1(z) \\ \vdots \\ a_r(z) \end{pmatrix}.
\]

(3.53)
Then

\[ \pi_+(\bar{c}(z)) = \pi_+ \left( \sum_{i=1}^{r} a_i(z) \bar{g}_i \right) = \]

\[ \pi_+ \left\{ [G] (\pi_+ (\bar{a}) + \pi_- (\bar{a})) \right\} = [G] (\pi_+ (\bar{a})) + \pi_+ \left\{ [G] (\pi_- (\bar{a})) \right\}. \]  \hspace{1cm} (3.54)

Consider the term \( \pi_+ \left\{ [G] (\pi_- (\bar{a})) \right\} \). Write

\[ \pi_- (a_i(z)) = A_1^i z^{-1} + A_2^i z^{-2} + \ldots + A_{e_i}^i z^{-e_i} + \ldots. \]  \hspace{1cm} (3.55)

Note that \( \bar{g}_i, A_q z^{-q} \), for \( q > e_i \), is a strictly proper vector. So that \( \pi_+ (\bar{g}_i, A_q z^{-q}) = 0 \) for \( q > e_i \). Then

\[ \pi_+ \left\{ [G] (\pi_- (\bar{a})) \right\} = \pi_+ \left( \sum_{i=1}^{r} \left( A_1^i \bar{g}_i \right) \right) = \pi_+ \left( \sum_{i=1}^{r} \left( A_1^i \bar{g}_i \right) + \ldots + A_{e_i}^i \bar{g}_i \right) = \bar{w}, \]  \hspace{1cm} (3.56)

which is just a linear combination over \( k \) of the vectors from \( B \). Also \( [G] (\pi_+ (\bar{a})) \equiv 0 \) in \( W(\text{im}C) \) since it is polynomial. So by definition, \( \bar{w} \equiv \pi_+ (\bar{c}(z)) \) in \( W(\text{im}C) \). We have then that \( B \) is at least a spanning set over \( k \) for the Wedderburn space. By the construction of \( B \) it is clear that the order of \( B = e \). This implies that \( \dim_k (W(\text{im}C)) \leq e \). To show \( \bar{w} \) is the unique basis representative from \( B \) for \( \pi_+ (\bar{c}(z)) \in W(\text{im}C) \), it suffices to show that \( \dim_k (W(\text{im}C)) = e \). From the FPZES of \( G \) and Corollary 3.2.5 we already have that \( Z(G) = 0, X(G) = 0, \) and \( W(\text{ker}G) = 0 \). Then

\[ X_\infty (G) \cong W(\text{im}G) = W(C) \]  \hspace{1cm} (3.57)

Define \( D_\infty = \text{diag}[z^{-e_i}] \), and \( N_\infty = GD_\infty \). Then by definition, \( (N_\infty, D_\infty) \) is an \( O_\infty \) RCPMFD for \( G \). So

\[ X_\infty (G) \cong \frac{\Omega_\infty U}{[D_\infty](\Omega_\infty U)} \cong \frac{\bar{f}_i}{(z^{-1})^{e_i}} \oplus \ldots \oplus \frac{\bar{f}_r}{(z^{-1})^{e_m}} \]  \hspace{1cm} (3.58)
It is clear then that
\[ \dim_k(\text{imC}) = \dim_k(X_\infty(G)) = \sum(e_i) = e \]  
(3.59)
so \( B \) is a basis and the dimension of the Wedderburn space equals the sum of its minimal indices.

### 3.5 Examples

#### 3.5.1 Coprime Calculations for a Matrix \( G(z) \).

Given \( G \) a rational transfer function, our goal is to find a realization (i.e. \( A, B, C \)) for \( G \). First we need to find a right coprime matrix fraction decomposition for \( G \).

**Definition 3.5.1** Two polynomial matrices are right coprime if their greatest common right divisor \( \text{GCRD} \) is unimodular (i.e., the det is a constant \( \neq 0 \)). This is equivalent to saying \( \{D, N\} \) are polynomial right coprime if there exist polynomial matrices \( A \) and \( B \) s.t.

\[ AD + BN = I, \]  
(3.60)

see [7, page 379].

**Definition 3.5.2** Given \( \{D, N\} \), we say \( U \) is a greatest common right divisor if, say, \( D = D'A \) and \( N = N'A \), for some polynomial matrices \( \{D', N'\} \), and \( A \), then
$U = U'A$. (That is, if $A$ is a common right divisor of $\{D, N\}$ it is also a right divisor of $U$.)

This is analogous to $6 = (12, 18)$, so if, say, $3 \mid 12$ and $3 \mid 18 \Rightarrow 3 \mid 6$, but since matrices are non-commutative we must indicate right.

The idea of greatest refers to the determinant degree, $\delta$ say, since

$$\delta(\det U) \geq \delta(\det A) \quad (3.61)$$

for such $A$.

Step 1

We illustrate the algorithm for computing the GCRD of $\{D, N\}$, where $GD = N$.

For a first guess let $D$ be a polynomial $m \times m$ invertible matrix over $k(z)$ and $N$ be a polynomial $p \times m$ matrix s.t. $GD = N$. This is probably an insufficient guess. Then adjusting by using elementary row operations we have,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} T \\ O \end{bmatrix} \quad (3.62)$$

where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (3.63)$$

is unimodular and $T$ is $m \times m$ upper triangular. Then

$$AD + BN = T \quad (3.64)$$

and if $T$ were unimodular we would be done. If not consider

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} T \\ O \end{bmatrix} \Rightarrow \begin{bmatrix} D = \alpha T \\ N = \gamma T \end{bmatrix} \quad (3.65)$$
Since $D$ is invertible, $T$ has full rank and so $T^{-1}$ exists over $k(z)$.

\[ DT^{-1} = \alpha \]
\[ NT^{-1} = \gamma \]
\[
\begin{pmatrix}
\alpha \\
\gamma
\end{pmatrix}
\] (poly).

(3.66)

Then since

\[ AD + BN = T \] (3.67)
\[ ADT^{-1} + BNT^{-1} = I \] (3.68)
\[ A\alpha + B\gamma = I. \] (3.69)

It is clear that $\alpha = \alpha I$ and $\gamma = \gamma I$ so $I$ is a common right divisor. Also, if

\[
\begin{align*}
\alpha &= \alpha' a \\
\gamma &= \gamma' a
\end{align*}
\]

then

\[
\begin{align*}
A\alpha' a + B\gamma' a &= I \\
[A\alpha' + B\gamma']a &= I,
\end{align*}
\]

so $A$ is a right divisor of $I$. So $I$ is a GCRD of $\alpha, \gamma$ (and clearly unimodular), and therefore $\alpha, \gamma$ are right coprime. Also, $G = ND^{-1} = (\gamma T)(T^{-1}\alpha^{-1}) = (\gamma \alpha^{-1})$. So $\alpha, \gamma$ form a decomposition of $G$. We use $\alpha, \gamma$ to replace our initial guess $D$ and $N$, respectively. This pair forms a Right Coprime Matrix Fraction Decomposition for $G$ (RCMFD). Previous theory said

\[ X(G) = \frac{\Omega U}{DUU}. \] (3.70)

For example, let

\[ G = \begin{bmatrix}
\frac{1}{z+1} & z & -1 \\
\frac{1}{z+1} & z+2 & -1 \\
\frac{3}{z+2} & -1 & z+1
\end{bmatrix} \] (3.71)

try

\[ D = \begin{bmatrix}
z+1 & 0 & 0 \\
0 & z+2 & 0 \\
0 & 0 & z+1
\end{bmatrix} \] (3.72)
So,

\[ GD = \begin{bmatrix} 1 & z & -1 \\ 1 & 3 & -1 \end{bmatrix} = N. \tag{3.73} \]

\[ \begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} z + 1 & 0 & 0 \\ 0 & z + 2 & 0 \\ 0 & 0 & z + 1 \\ 1 & z & -1 \\ 1 & 3 & -1 \end{bmatrix} \tag{3.74} \]

[Note: These are not coprime since the rank drops at \( z = -1 \).] Find GCRD by unimodular row operations.

\[
\begin{bmatrix} D \\ N \end{bmatrix} \xrightarrow{R_5 \leftrightarrow R_6} \begin{bmatrix} 1 & 3 & -1 \\ 0 & z + 2 & 0 \\ 0 & 0 & z + 1 \\ 1 & z & -1 \\ z + 1 & 0 & 0 \end{bmatrix} \xrightarrow{-1R_1 + R_4} \begin{bmatrix} 1 & 3 & -1 \\ 0 & z + 2 & 0 \\ 0 & 0 & z + 1 \\ 1 & 3 & -1 \\ z + 1 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_4} \begin{bmatrix} 1 & 3 & -1 \\ 0 & z + 2 & 0 \\ 0 & 0 & z + 1 \\ 1 & 3 & -1 \\ z + 1 & 0 & 0 \end{bmatrix} \xrightarrow{3R_2 + R_5} \begin{bmatrix} 1 & 3 & -1 \\ 0 & z + 2 & 0 \\ 0 & 0 & z + 1 \\ 1 & 3 & -1 \\ z + 1 & 0 & 0 \end{bmatrix} \xrightarrow{-R_2 + R_4} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & z + 1 \\ 0 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & z + 1 \end{bmatrix} \xrightarrow{3R_2 + R_5} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & z + 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

So,

\[
T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & z + 1 \end{bmatrix}; T^{-1} = \begin{bmatrix} 1 & 0 & 1/z + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/z + 1 \end{bmatrix} \tag{3.77} \]
The new right coprime choices for $D$ and $N$ are

$$D' \overset{\text{def}}{=} (DT^{-1}) = \begin{bmatrix} z+1 & 0 & 0 \\ 0 & z+2 & 0 \\ 0 & 0 & z+1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/z + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1/z + 1 \end{bmatrix}$$

(3.78)

$$= \begin{bmatrix} z+1 & 0 & 1 \\ 0 & z+2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3.79)

$$N' = N T^{-1} = \begin{bmatrix} 1 & z & 0 \\ 1 & 3 & 0 \end{bmatrix}.$$ 

(3.80)

and we see that $GD' = N'$. In the next section we use this coprime algorithm and result to construct a realization $\Sigma$.

### 3.5.2 Construction of a $\Sigma$ Realization for a s.p. $G(z)$

**Step 2** Now find $(A, B, C)$.

We need the Smith form of $D$ from the section above. Keeping track of unimodular operations we have

$$\begin{bmatrix} z+1 & 0 & 1 \\ 0 & z+2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\sigma_1 \ldots \sigma_3} \begin{bmatrix} 1 & 0 & z+1 \\ 0 & z+2 & 0 \\ 1 & 0 & (z+1) \end{bmatrix}$$

(3.81)

$$\xleftarrow{-1(z+1)\sigma_1 \sigma_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & z+2 & 0 \\ 1 & 0 & (z+1) \end{bmatrix} (3.82)

$$-1R_1 + R_3 \xleftarrow{-1 \sigma_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & z+2 & 0 \\ 0 & 0 & z+1 \end{bmatrix} \xrightarrow{\sigma_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (z+2) & (z+1) \\ 0 & 0 & (z+1) \end{bmatrix} \xrightarrow{-\sigma_2 + \sigma_3}$$

(3.83)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & z+2 & -1 \\ 0 & 0 & z+1 \end{bmatrix} \xrightarrow{\sigma_2 - \sigma_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & z+2 \\ 0 & z+1 & 0 \end{bmatrix}.$$ (3.84)
\[
\begin{pmatrix}
(z+2) & c_2 + c_3 \\
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & (z+1) & (z+2)(z+1)
\end{pmatrix}
\]

\begin{equation}
(3.85)
\end{equation}

\[
\begin{pmatrix}
(z+1)R_2 + R_3 \\
-1R_2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & (z+2)(z+1)
\end{pmatrix}
= S \text{ (the Smith form)}.
\]

\begin{equation}
(3.86)
\end{equation}

Therefore 1, 1, \((z+2)(z+1)\) are the invariant factors for the module \(\frac{\mathfrak{n}_U}{\mathfrak{d}_U}\).

Collecting the row and column operations yields the matrix product \(LDR = S\),

where

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & (z+1) & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

\begin{equation}
(3.87)
\end{equation}

\[
= \begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & -1 \\
-(z+2) & (z+1) & (z+2)
\end{pmatrix}
\]

\begin{equation}
(3.88)
\end{equation}

\[
R = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -(z+1) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\begin{equation}
(3.89)
\end{equation}

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & -(z+2) \\
-(z+2) & (z+1) & (z+2)(z+1)
\end{pmatrix}
\]

\begin{equation}
(3.90)
\end{equation}

So \(S\) is equal to the product

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & -1 \\
-(z+2) & (z+1) & (z+2)
\end{pmatrix}
\begin{pmatrix}
z+1 & 0 & 1 \\
0 & z+2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & -1 & -(z+2) \\
0 & -1 & -(z+1) \\
1 & (z+1) & (z+2)(z+1)
\end{pmatrix}
\]

\[
\begin{equation}
(3.91)
\end{equation}

We view \(D\mathfrak{n}_U\) as a space with \((n_1, n_2, n_3)\) as a basis. Then \(D\) can be viewed as a change of basis matrix from the basis \(\mathcal{N}\) above to the standard basis, denoted \(F\). The
notation for a change of basis matrix will be written as $D_{N'}^F$. The matrix $S$ represents a change of basis also. It tells how to write the space $(D \otimes U)$ with respect to some basis, $N'$ say, in terms of a new basis of $\Omega U$, say $F'$. So write $S_{N'}^F$. Then $LDR = S$ is really $L_{F'}^F D_{N'}^F R_{N'} = S_{N'}^F$. We need to know the basis $F'$ in terms of the standard basis $F$. The matrix $L^{-1}_{F'}$ does this for us, say $F' = (f_1^*, f_2^*, f_3^*)$. One can check that

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1(z + 2) & -1 \\ 1 & z + 2 & 1 \end{bmatrix},$$

(3.91)

and so

$$L^{-1}_{F'} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{F'} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{F},$$

(3.92)

the vector $f_1^*$ in terms of the standard basis. Therefore, $\frac{\Omega_{U}}{D_N^U}$ is isomorphic to

$$\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_F, \begin{bmatrix} 0 \\ -(z + 2) \\ (z + 1) \end{bmatrix}_F, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}_F \rangle^* = \langle \frac{\Omega_{U}}{D_N^U} \rangle$$

$$\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_F, \begin{bmatrix} 0 \\ -(z + 2) \\ (z + 1) \end{bmatrix}_F, \begin{bmatrix} 0 \\ -(z + 2)(z + 1) \\ (z + 2)(z + 1) \end{bmatrix}_F \rangle^* = \frac{\langle \tilde{f}_1^*, \tilde{f}_2^*, \tilde{f}_3^* \rangle}{S(f_1^*, f_2^*, f_3^*)}$$

(3.93)

We have from the Kalman input/output diagram that

$$\tilde{B} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{F'} L \begin{bmatrix} 1 \\ 1 \\ -(z + 2) \end{bmatrix}_{F'} =$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -(z + 2) \\ (z + 1) \end{bmatrix} - (z + 2) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

(3.94)

(3.95)

which up to equivalence is

$$\equiv -(z + 2) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$
Similarly,
\[
\tilde{B} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \left( \begin{array}{c} -1 \\ 0 \\ (z+1) \end{array} \right) \downarrow_{F^*} \left( \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right) = (z+1) \left( \begin{array}{c} 0 \\ 1 \end{array} \right)
\]
(3.96)
\[
\tilde{B}' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (z+2) \left( \begin{array}{c} 0 \\ -1 \end{array} \right).
\]
(3.97)

Now,
\[
\frac{\left\langle \left( \begin{array}{c} 0 \\ 0 \\ (z+1) \end{array} \right), \left( \begin{array}{c} -1 \\ 0 \\ (z+2) \end{array} \right) \right\rangle}{\left\langle \left( \begin{array}{c} 0 \\ 0 \\ (z+1) \end{array} \right), \left( \begin{array}{c} -1 \\ 0 \\ (z+2) \end{array} \right) \right\rangle} \approx \frac{\left\langle \left( \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right) \right\rangle}{\left\langle \left( \begin{array}{c} 0 \\ 1 \\ -1 \end{array} \right) \right\rangle} = X(G).
\]
(3.98)

So
\[
\varphi \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \rightarrow 1
\]
(3.99)

and
\[
\varphi \begin{pmatrix} z \\ 0 \\ -1 \end{pmatrix} \rightarrow z.
\]
(3.100)

For the 2-dimensional space \(X(G)\), choose the basis \((1, z)\).

Then \(\tilde{B}' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \approx -(z+2) = \left( \begin{array}{c} -2 \\ -1 \end{array} \right)\) w.r.t. the \((1, z)\) basis because
\[
\varphi \begin{pmatrix} -(z+2) \\ 0 \\ -1 \end{pmatrix} = -(z+2).
\]
(3.101)

Similarly,
\[
\tilde{B}' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \approx -(z+1),
\]
(3.102)
and is denoted as

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \cong (z + 2) \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix} \cong \begin{pmatrix}
2 \\
1
\end{pmatrix}.
\] (3.103)

So, the \( B \) matrix is

\[
B = \begin{bmatrix}
-2 & 1 & 2 \\
-1 & 1 & 1
\end{bmatrix}
\]

and from \( z^2 + 3z + 2 \) the \( A \) matrix is

\[
A = \begin{bmatrix}
0 & -2 \\
1 & -3
\end{bmatrix}.
\]

Now from the Kalman output map we have

\[
\tilde{C} \left( \begin{array}{c}
-2 \\
-1
\end{array} \right) = G^\# \left( \begin{array}{c}
1 \\
0 \\
0
\end{array} \right) = \left( \frac{1}{z+1} \right) = \left( \frac{1}{z+1} - z^{-2} + z^{-3} - \cdots \right) = \left( \frac{1}{z+1} - z^{-2} + z^{-3} - \cdots \right)
\] (3.104)

\[
= C \left( \begin{array}{c}
-2 \\
-1
\end{array} \right) z^{-1} + CA \left( \begin{array}{c}
-2 \\
-1
\end{array} \right) z^{-2} + CA^2 \left( \begin{array}{c}
-2 \\
-1
\end{array} \right) z^{-3} + \cdots (3.105)
\]

So,

\[
C \left( \begin{array}{c}
-2 \\
-1
\end{array} \right) = \begin{pmatrix}
1 \\
1
\end{pmatrix}.
\] (3.106)

Also,

\[
\tilde{C} \left( \begin{array}{c}
1 \\
1
\end{array} \right) = G^\# \left( \begin{array}{c}
0 \\
1 \\
0
\end{array} \right) = \left( \frac{z}{z+2} \right) = \left( \frac{z}{z+2} - 2z^{-1} + 4z^{-2} + 8z^{-3} + \cdots \right) = \left( \frac{z}{z+2} - 2z^{-1} + 4z^{-2} + 8z^{-3} + \cdots \right)
\] (3.107)

\[
= C \left( \begin{array}{c}
1 \\
1
\end{array} \right) z^{-1} + CA \left( \begin{array}{c}
1 \\
1
\end{array} \right) z^{-2} + CA^2 \left( \begin{array}{c}
1 \\
1
\end{array} \right) z^{-3} + \cdots
\]

So,

\[
C \left( \begin{array}{c}
1 \\
1
\end{array} \right) = \begin{pmatrix}
-2 \\
3
\end{pmatrix}
\] (3.108)

and

\[
\begin{pmatrix}
c_1 & c_2 \\
c_3 & c_4
\end{pmatrix} \begin{pmatrix}
-2 & 1 \\
-1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & -2 \\
1 & 3
\end{pmatrix}
\] (3.109)
and so yields

\[ C = \begin{bmatrix} 1 & -3 \\ -4 & 7 \end{bmatrix}. \] (3.110)

One can verify that

\[ G = C^{-1}(zI - A)^{-1}B. \] (3.111)
CHAPTER IV

Local Filtrations

4.1 General Introduction

The module or vector space structures of the spaces defined so far are not sufficient to capture the complete picture of global observability and controllability spaces for singular systems. Nor are these classical notions able to be generalized and unified by the Wedderburn Forney space without the following underlying idea. The most appropriate algebraic structure for the global space of zeros and poles attached to transfer functions is the notion of a filtered vector space, defined below. The global pole and zero spaces defined earlier were direct sums of finitely generated torsion modules over different PID rings and so have no real well-defined module structure. The notion of a filtered vector space provides a structure between the structure of ordinary vector spaces, which do not provide enough information, and module structures, which are not available in the global case. The general technique will be to construct filtered vector space structures that agree with classical controllability and observability spaces of strictly proper transfer functions and generalize them to singular systems, and to study the relationships between certain sets of integers (e.g.,
column degrees, controllability indices, row degrees, observability indices, invariant factor exponents, and Wedderburn indices). The structure of a filtered vector space attached to the global spaces and Wedderburn spaces will be compatible with the isomorphisms from the FPZES and with inner products to be defined below.

Let $V$ be any finite $n$-dimensional vector space over the field $k$.

**Definition 4.1.1** A decreasing (increasing) filtration on the vector space $V$ is a decreasing (increasing) sequence of subspaces

$$V = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_r = (0).$$

A filtered vector space is a vector space equipped with such a filtration.

Associated to such a filtration is the set of integers

$$\{n_i \mid n_i \overset{\text{def}}{=} \dim \left( \frac{V_{i-1}}{V_i} \right) ; i = 1, \ldots, r \}.\$$

The integers form a partition of $n$, (i.e., $n = \sum_{i=1}^r n_i$).

**Definition 4.1.2** The Ferrers Diagram or Tableau of a partition is an arrangement or grid of dots such that the $i$th row has $n_i$ dots. The corresponding conjugate or dual partition of $n$ is the sequence of integers given by the number of dots in the columns of its Ferrers diagram.

If $\{m_1, \ldots, m_r\}$ is the set of these conjugate indices, then for each $t$ the integer $m_t$ is exactly the number of $n_i \geq t$. Of course analogous definitions can be made for the increasing case.
For example, suppose

\[ V = V_0 = \mathbb{R}^5 = (\tilde{e}_1, \ldots, \tilde{e}_5)_k, \]

the span of the standard basis vectors over the field \( k \). And suppose that the filtration is given as,

\[ V_0 = V, \quad V_1 = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)_k, \quad V_2 = (\tilde{e}_1, \tilde{e}_3)_k, \text{ and } V_3 = (0). \]

Then \( \dim(V_0/V_1) = 2, \dim(V_1/V_2) = 1, \text{ and } \dim(V_2/V_3) = 2 \). The Ferrers diagram is given in Table 2. Note, there are \( m_1 = 3, n'_1 \geq 1 \) and \( m_2 = 2, n'_2 \geq 2 \). This fact will later be used in counting the dimensions of certain Jordan blocks.

### 4.2 The \( \pi_+ \)-Wedderburn Filtration

In this section we define the \( \pi_+ \)-Wedderburn Filtration on a space \( W(\mathcal{C}) \) and show how a certain conjugate partition of the dimension of \( W(\mathcal{C}) \) yields the Wedderburn indices.

Let \( k \) be an arbitrary field and \( k(z) \) be the field of rational functions. Define \( V(z) = k^m(z) \), the vector space of \( \dim m < \infty \) formed as in Chapter 1. Let \( \mathcal{C} \subset V(z) \),

### Table 2: Ferrer's Diagram for the Partition of \( n \).

<table>
<thead>
<tr>
<th></th>
<th>( n_1 = 2 )</th>
<th>( n_2 = 1 )</th>
<th>( n_3 = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 = 3 )</td>
<td>( m_2 = 2 )</td>
<td>( n = 5 )</td>
<td></td>
</tr>
</tbody>
</table>
be a proper subspace of $V(z)$, and define $\{\bar{g}_1, \ldots, \bar{g}_r\}$ be a minimal polynomial basis for $C$ with corresponding column degree $\delta(\bar{g}_i) = e_i$. We may assume w.l.o.g. that the $e_1 \geq \ldots \geq e_r$. We also define $G = [\bar{g}_1, \ldots, \bar{g}_r]$ to be the matrix of these ordered column vectors and identify the image of $G$, im$G$, with the space $C$. At the end of the previous chapter we saw that the $W(\text{im}G)$ is a finite dimensional vector space over $k$ of dimension $e = \sum_{i=1}^{r} e_i$ and a basis for $W(\text{im}G)$ is given by the set of class representatives

$$B = \{\pi_j(z^{-j}\bar{g}_i) \mid j = 1, \ldots, e_i; \ i = 1, \ldots, r\}. \quad (4.1)$$

For example let

$$k = \mathbb{C}, m = 5, G = \begin{bmatrix} z^4 + 1 & 1 & z^2 + 1 \\ z^2 & z & z^3 + z \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & z^4 + 1 & 0 \end{bmatrix}$$

$$[G]_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can check that $[G]_h$ has full rank and that $\mathcal{E}(G) = 0$ (Forney checks that all $3 \times 3$ minors have gcd 1) so $G$ is a minimal basis for its column span. The basis $B$ over $k$ of class representatives for $W(\text{im}G)$ is then given by
Note in general that each vector \( \vec{g}_i \) in the minimal realization for a space \( \mathcal{C} \) always contributes \( e_i \) vectors to the basis \( B \) and the degree of \( \pi_+(z^{-i} \vec{g}_i) \) is \( e_i - j \).

Continuing now with our general discussion, we have the following definition.

**Definition 4.2.1** Let \( n = (e_1 - 1) > 0 \). The \( \pi_+ \)-Wedderburn filtration of \( G (\pi_+ WF) \) is defined as \( W(\text{im}G) = W_n \supseteq \ldots \supseteq W_i \supseteq \ldots W_0 \supseteq (0) = W_{-1} \) where

\[
W_i = \pi_+(\text{im} \begin{bmatrix} D \\ N \end{bmatrix}) \cap (\vec{k}_0 + z\vec{k}_1 + \ldots + z^i\vec{k}_i). \tag{4.2}
\]

That is, \( [w] \in W_i \) iff \( [w] \) has a representative of degree \( i \) or less in \( W(\text{im}G) \). More concretely,

\[
W_i = \{ \overline{w} \in W(\mathcal{C}) | \exists \text{ a coset representative for} \overline{w} \text{ of the form } \vec{k}_0 + z\vec{k}_1 + \ldots + z^s\vec{k}_s \equiv \overline{w}; \ 0 \leq s \leq t; \ \vec{k}_s \neq \vec{0}, \text{ unless } \overline{w} = \vec{0} \in W(\mathcal{C}) \},
\]
**Definition 4.2.2** \( \mathcal{B}_t = \{ \tilde{b} \in \mathcal{B} \mid \deg \tilde{b} \leq t \} \), and \( (\mathcal{B}_t)_k \) to be the span of \( \mathcal{B}_t \) over \( k \).

That is if there are \( r \) columns in the matrix \( G \) then

\[
\mathcal{B}_t = \{ \pi_+(z^{-j} \tilde{d}_i) \mid 0 \leq (e_i - j) \leq t; j, e_i > 0; 1 \leq i \leq r \}. \tag{4.3}
\]

We need to look more closely at the \( W_t \). We wish to get a handle on the dimension of \( W_t \) over \( k \). In fact, we claim that if \( \bar{w} \in W_t \), then

\[
w = \sum_{i=1}^r \sum_{j=0}^{e_i-t} a_{ij} \pi_+(z^j \tilde{g}_i). \tag{4.4}
\]

We will show that \( (\mathcal{B}_t)_k \cong W_t \). Since the vectors in \( (\mathcal{B}_t)_k \) are linearly independent then we will have a measure on the dimension of the spaces in the Wedderburn filtration as well as a means of constructing the filtration. To show this we will first need a lemma. In the following when given an arbitrary vector of the form

\[
\bar{v}(z) = \begin{bmatrix} n_1[z] \\ d_1[z] \\ \vdots \\ n_m[z] \\ d_m[z] \end{bmatrix}, \tag{4.5}
\]

by \( \deg(\bar{v}(z)) \) we will mean \( \max_i (\deg n_i - \deg d_i) \).

**Lemma 4.2.3 (Predictable Degree)** Let \( [G] \) be minimal with \( \dim(W(\text{im}G)) = e \) as above. Let \( \bar{y}[z] \neq 0 \) be an arbitrary vector in \( (\mathcal{B})_k \), say \( \bar{y}[z] = \sum_{s=1}^e a_s \tilde{b}_s, a_s \text{ scalar} \).

Then \( \deg(\bar{y}[z]) = \max_{a_s \neq 0} (\deg \tilde{b}_s) \), \( 1 \leq s \leq e \).

**Proof:** Suppose first that \( \tilde{h}(z) = \sum_{i=1}^r a_i(z) \tilde{g}_i[z] = [G](\tilde{a}) \), \( a_i = n_i/d_i \in k(z) \). Define \( d[z] = \text{lcm}(d_i[z]) \) and \( \deg(d[z]) = l \). Then \( d[z] \tilde{h}(z) = \tilde{h}'[z] = [G]a'[z] \) is polynomial.

By Forney's main theorem [11],

\[
\deg(\tilde{h}'[z]) = \max_i (\deg h'_i[z]) = \max_i (\deg(a'_i[z]) + e_i). \tag{4.6}
\]
Then

\[ l + \deg(\tilde{h}(z)) = \deg(d[z]\tilde{h}(z)) = \deg(h'[z]) = \max_i(\deg(a_i'[z]) + e_i) = \max_i(\deg(a_i(z)) + l + e_i) = \max_i(\deg(a_i(z)) + e_i) + l. \]

(4.7)

In general, \([G]\) minimal and \(\tilde{h}(z) = [G]\tilde{a}(z)\) implies \(\deg(\tilde{h}(z)) = \max_i(\deg(a_i(z)) + e_i)\).

Now given

\[ \tilde{y}[z] = \sum_{s=1}^{c} a_s \tilde{b}_s = \sum_{i=1}^{r} \sum_{j=1}^{c_i} \pi_+(z^{-j} \tilde{g}_i) k_{ij} = \]

(4.8)

\[ \pi_+ \left\{ \left[ G \right] \left( \begin{array}{c} k_{11} z^{-1} + k_{12} z^{-2} + \ldots + k_{1e_1} z^{-e_1} \\ \vdots \\ k_{r1} z^{-1} + k_{r2} z^{-2} + \ldots + k_{rer} z^{-er} \end{array} \right) \right\} = \pi_+ \left\{ \left[ G \right] \left( \begin{array}{c} a_1(z) \\ \vdots \\ a_r(z) \end{array} \right) \right\} , \]

the previous remark and the fact \(\tilde{y}[z] \neq \vec{0}\) implies \(\deg(\tilde{y}[z]) = \max_i(\deg(a_i(z)) + e_i)\). For each \(i, 1 \leq i \leq r\), define \(l_i\) to be the minimum value of \(j\) for which \(k_{ii} \neq 0, 1 \leq j < \infty\). Then

\[ \deg(\tilde{y}[z]) = \max_i(e_i - l) = \max_i(\deg(\pi_+(z^{-l_i} \tilde{g}_i))) = \max_{a_s \neq 0}(\deg \tilde{b}_s). \]

(4.9)

One may argue that due to cancellations of coefficients a linear combination \(\sum a_s \tilde{b}_s\) in \((B)_k\) may result in a vector \(y[z]\) of lesser degree than \(\max_{a_s \neq 0}(\deg \tilde{b}_s)\). The above lemma of predictable degree says this never happens when \([G]\) is minimal.

**Theorem 4.2.4** \((B)_k \cong W_t\) as vector spaces.

**Proof:** Clearly \((B)_k \subseteq W_t\). Suppose \(\bar{x}, \bar{x} \neq \vec{0}\), is a coset representative from \(W_t\).

Then by definition there exists \(\tilde{c}(z) \in \mathcal{C}\) such that \(\pi_+(\tilde{c}(z)) \equiv \bar{x}\) and \(\pi_+(\tilde{c}(z)) = \)
\( \bar{k}_0 + \ldots + \bar{k}_s \alpha^s, s \leq t \). We assume w.l.o.g. that \( s \) is the least integer such that the above is true. If \( \bar{c}(z) \in \mathcal{C} \) then there exists \( a_1(z), \ldots, a_r(z) \in k(z) \) such that

\[
\bar{c}(z) = \sum_{i=1}^{r} a_i(z) \bar{g}_i = [G] \begin{pmatrix} a_1(z) \\ \vdots \\ a_r(z) \end{pmatrix}.
\]

Then

\[
\bar{k}_0 + \ldots + \bar{k}_s \alpha^s = \pi_+(\bar{c}(z)) = \pi_+\left( \sum_{i=1}^{r} a_i(z) \bar{g}_i \right) = \pi_+ \{ [G](\pi_+(-z^e) + \pi_-(\bar{a})) \} = [G](\pi_+(-\bar{a})) + \pi_+ \{ [G](\pi_-(\bar{a})) \}.
\]

Consider the term \( \pi_+ \{ [G](\pi_-(\bar{a})) \} \). Write

\[
\pi_-(a_i(z)) = A_{i-1}^i z^{-1} + A_{i-2}^i z^{-2} + \ldots + A_{i-e_i}^i z^{-e_i} + \ldots.
\]

Note that \( \bar{g}_i A_{i-q}^i z^{-q} \), for \( q > e_i \), is a strictly proper vector. So that \( \pi_+(\bar{g}_i A_{i-q}^i z^{-q}) = \bar{0} \) for \( q > e_i \). Then

\[
\pi_+ \{ [G](\pi_-(\bar{a})) \} = \pi_+ \left( \sum_{i=1}^{r} \bar{g}_i (A_{i-1}^i z^{-1} + A_{i-2}^i z^{-2} + \ldots + A_{i-e_i}^i z^{-e_i} + \ldots) \right) = \sum_{i=1}^{r} (A_{i-1}^i \pi_+(z^{-1} \bar{g}_i) + A_{i-2}^i \pi_+(z^{-2} \bar{g}_i) + \ldots + A_{i-e_i}^i \pi_+(z^{-e_i} \bar{g}_i)) \overset{def}{=} \bar{w},
\]

which is just a linear combination of basis vectors from \( B \). Also \( [G](\pi_+(\bar{a})) = \bar{0} \) in \( \mathcal{W}(\text{im}\mathcal{C}) \) since it is polynomial. By definition \( \bar{w} \equiv \bar{x} \) and so \( \bar{w} \) is the unique basis representative from \( B \) for \( [\bar{x}] \in \mathcal{W}(\text{im}\mathcal{C}) \). So we have

\[
\bar{k}_0 + \ldots + \bar{k}_s \alpha^s = \]

\[
[G](\pi_+(\bar{a})) + \sum_{i=1}^{r} (A_{i}^i \pi_+(z^{-1} \bar{g}_i) + A_{i}^i \pi_+(z^{-2} \bar{g}_i) + \ldots + A_{i-e_i}^i \pi_+(z^{-e_i} \bar{g}_i)) = [G](\pi_+(\bar{a})) + \bar{w}.
\]
It remains to show $\deg \tilde{w} \leq s \leq t$. Then we would have that $\tilde{w} \in (B_t)_k$ or that $W_t \subseteq (B_t)_k$. Define

$$
\alpha_i = \deg(\pi_a(z) + \pi(b_i(z))),
$$
(4.15)

$$
\beta_i = \deg(\pi_b(z) + \pi(c_i(z))),
$$
(4.16)

$$
m(\alpha_i, \beta_i) = \begin{cases} 
\alpha_i & \text{if } \alpha_i > \beta_i \\
\beta_i & \text{if } \pi_a(z) = 0 \\
-\infty & \text{if } \pi_b(z) = 0
\end{cases}
$$
(4.17)

Then from the lemma of predictable degree,

$$
s = \deg(\pi_a([G]\tilde{a}(z))) = \deg([G]\tilde{a}(z)) = \max[\max(\alpha_i, \beta_i) + \epsilon_i] = \max[\max(\alpha_i + \epsilon_i, \beta_i + \epsilon_i)] = \max[\max(\alpha_i + \epsilon_i), \max(\beta_i + \epsilon_i)] = \max[\deg([G]\pi_a\tilde{a}), \deg([G]\pi_b\tilde{a})] = \max[\deg([G]\pi_a\tilde{a}), \deg([G]\pi_b\tilde{a})].
$$
(4.18)

Which implies

$$
\deg([G]\pi_a\tilde{a}) \leq s \text{ and } \deg([G]\tilde{w}) \leq s.
$$
(4.19)

Then $W_t = (B_t)_k$.

The proof is constructive and illustrates how the Wedderburn filtration is built from a minimal polynomial basis matrix $[G]$ of the space $\mathcal{C}$. So

$$
W_t = \pi_a(\mathcal{C}) \cap (k_0 + z^1k_1 + \ldots + z^tk_t) = \\
(\{\pi_a(z^{-j}\tilde{a}) \mid 0 \leq (\epsilon_i - j) \leq t; j, \epsilon_i > 0; 1 \leq i \leq r\})_k.
$$
(4.20)

Define $n = \epsilon_1 - 1$. Recall we ordered the columns of $[G]$ in order of decreasing degree so that $(\epsilon_1 - 1) = n$ is the largest possible value of $t$. From the preceding theorem we can readily calculate a basis $B_t$ and dimension $d_t = \dim(W_t)$ for each
\( W_t, 0 \leq t \leq n \) in the filtration. From the corresponding chain of quotients and their respective degrees
\[
\frac{W_0}{(0)} \subseteq \frac{W_1}{W_0} \subseteq \ldots \subseteq \frac{W_n}{W_{n-1}}, \quad (4.21)
\]
we see \( \sum_{q=0}^{n} \overline{d}_q = e \), the \( \dim[W(\text{im}C)] \), and so \( \{\overline{d}_q\}_{q=0}^{n} \) forms a partition of the integer \( e \). The \( j^{th} \) quotient space is just
\[
\frac{W_j}{W_{j-1}} = \langle \{\pi_+((z^{-e_i+j})_{x_i}) \mid -e_1 + j \leq -1; 1 \leq i \leq r \} \rangle_k.
\]
Each column vector \( \vec{g}_i \) of degree \( e_i \) in the matrix \( G \) contributes exactly \( e_i \) vectors to \( W(\text{im}C) \). But a vector \( \vec{g}_i \) of degree \( e_i \) will contribute exactly one vector to each quotient space \( \frac{W_j}{W_{j-1}} \) for \( j = 0, \ldots, e_i - 1 \). The \( n \) rows of the Ferrers diagram for the Wedderburn filtration each contain \( \overline{d}_q \) dots and the columns each contain \( e_i \) dots. Therefore the conjugate partition of the Wedderburn filtration yields the column degrees of the minimal basis matrix \( [G] \). These are an invariant of the \( W(C) \) called the Wedderburn indices of \( C \). Continuing with our example we have
\[
\begin{array}{cccc}
\dim \frac{W_0}{(0)} = \overline{d}_0 & = 3 & \cdots & \cdots \\
\dim \frac{W_1}{W_0} = \overline{d}_1 & = 3 & \cdots & \cdots \\
\dim \frac{W_2}{W_1} = \overline{d}_2 & = 3 & \cdots & \cdots \\
\dim \frac{W_3}{W_2} = \overline{d}_3 & = 2 & \cdots & \cdots \\
\end{array}
\]
\[
e_1 = 4 \quad e_2 = 4 \quad e_3 = 3
\]
And the filtration is given by
\[
\begin{align*}
W_{-1} & = (0) \\
W_0 & = \left\langle \pi_+((z^{-4}\vec{g}_1), \pi_+((z^{-4}\vec{g}_2), \pi_+(z^{-2}\vec{g}_3))_k \right\rangle_{B_0} \\
W_1 & = \left\langle \pi_+((z^{-3}\vec{g}_1), \pi_+((z^{-3}\vec{g}_2), \pi_+(z^{-1}\vec{g}_3))_k \cup (B_0) \right\rangle_{B_1} \\
W_2 & = \left\langle \pi_+((z^{-2}\vec{g}_1), \pi_+((z^{-2}\vec{g}_2), \pi_+(z^{-1}\vec{g}_3))_k \cup (B_1) \right\rangle_{B_2} \\
W_3 & = \left\langle \pi_+((z^{-1}\vec{g}_1), \pi_+(z^{-1}\vec{g}_2))_k \cup (B_2) \right\rangle_{B_3}
\end{align*}
\]
So that the quotients $\frac{W_i}{W_{i-1}}$; $i = 0, \ldots, (e_1 - 1)$ are given as

\[
\frac{W_0}{W_{-1}} : \langle \pi_+(z^{-4}g_1), \pi_+(z^{-4}g_2), \pi_+(z^{-3}g_3) \rangle_k = \langle \{\pi_+(z^{-e_1}g_i)\} \rangle_k
\]

\[
\frac{W_1}{W_0} : \langle \pi_+(z^{-3}g_1), \pi_+(z^{-3}g_2), \pi_+(z^{-2}g_3) \rangle_k = \langle \{\pi_+(z^{-e_1+1}g_i)\}(e_i + 1 \leq -1) \rangle_k
\]

\[
\frac{W_2}{W_1} : \langle \pi_+(z^{-2}g_1), \pi_+(z^{-2}g_2), \pi_+(z^{-1}g_3) \rangle_k = \langle \{\pi_+(z^{-e_1+2}g_i)\}(e_i + 2 \leq -1) \rangle_k
\]

\[
\frac{W_3}{W_2} : \langle \pi_+(z^{-1}g_1), \pi_+(z^{-1}g_2) \rangle_k = \langle \{\pi_+(z^{-e_1+3}g_i)\}(e_i + 3 \leq -1) \rangle_k.
\]

The dimensions and the calculations for the partitions are clear.

### 4.3 The Controllability and Observability Filtrations

The classical case always deals with strictly proper transfer functions. Suppose then we have formed a minimal realization $(A, B, C)$ for a strictly proper matrix $G$ of rational functions. (Note: since $G$ is strictly proper the global pole module $\mathcal{X}(G)$ is just the finite pole space $X(G)$.) Recall $X(G)$ is both a torsion $k[z]$-module and a $k$-vector space of, say, dimension $n < \infty$.

Since our realization is reachable, the standard reachability matrix from classical control theory,

\[
[B, AB, \ldots, A^{n-1}B],
\]

has full rank $n$ and so contains a basis for $X(G)$. So

\[
X(G) \cong \langle [B, AB, \ldots, A^{n-1}B] \rangle_k.
\]  (4.22)
Definition 4.3.1 Given the above setting, the following chain of subspaces

\[
\begin{align*}
H_{-1} &= (0) \\
\cap H_0 &= \langle B \rangle \\
\cap H_1 &= \langle B, AB \rangle \\
\cap & \ \ \vdots \\
\cap H_{n_0-1} &= \langle B, AB, \ldots, A^{n_0-1}B \rangle = X(G)
\end{align*}
\]

is called the controllability or reachability filtration.

This filtration is called the controllability filtration because it is inspired by the controllability matrix of a minimal realization of the state space structure of \(X(G)\). The spaces in the chain are used in classical literature for putting a system state-space representation into various controller and observer canonical forms, which simplifies feedback control and observer designs, simplifies pole placement, and helps study various decoupling of the system. By the Cayley-Hamilton theorem, with

\[
\Xi_A(z) = -a_0 - a_1 z + \ldots - a_{n-1} z^{n-1}
\]

as the characteristic polynomial of the matrix \(A\), we see that

\[
A^{n+i}B = a_{n-1}A^{n-1+i}B + \ldots + a_0 A^iB.
\]

Therefore, the matrix must have full rank at least by step \(n - 1\). If at any point \(i\) the rank of the matrix stops increasing, it never picks up again and so the chain stops.

Definition 4.3.2 The least value \(n_0\), \(0 \leq n_0 \leq n\) such that \(H_{n_0-1}\) ends the chain is called the controllability index of the matrix \(G\). When the realization is minimal (or
at least controllable), it is the least value for which the matrix \([B, AB, \ldots, A^{n_0-1}B]\) has full rank.

We realize that the filtration \(\{H_i\}\) seems to depend on the realization \((A, B)\) chosen. If \((A', B')\) is another minimal reachable realization, say with vector state space \(Z\), then \(\text{dim}_k Z = \text{dim}_k X\). Also there exists a vector space isomorphism (a change of basis matrix from \(X\) to \(Z\) denoted \(T^Z_X\), and its inverse denoted \(T^X_Z\)) so that the system in terms of the \(Z\) space

\[
z(t+1) = A'z(t) + B'u(t) \\
y = C'z(t),
\]

can be written in terms of the \(X\) space as

\[
T^Z_X x(t+1) = A'T^Z_X x(t) + B'u(t) \\
y = C'T^Z_X x(t),
\]

or

\[
x(t+1) = T^X_Z A'T^Z_X x(t) + T^X_Z B'u(t) \\
y = C'T^X_Z x(t).
\]

So that \(A = T^X_Z A'T^Z_X, B = T^X_Z B',\) and \(C = C'T^Z_X\). For each \(i\), the following substitution yields

\[
[B, AB, \ldots, A^iB] = [T^X_Z B', (T^X_Z A'T^Z_X)T^X_Z B', \ldots, (T^X_Z A'T^Z_X)^i T^X_Z B']
\]
\[= [T^B_2 B', T^B_2 A'B', \ldots, TA'^i B'] \] \hspace{1cm} (4.32)
\[= T^B_2 [B', A'B', \ldots, A'^i B']. \] \hspace{1cm} (4.33)

Then

\[ \text{rank } B = \text{rank } B' \] \hspace{1cm} (4.34)
\[ \text{rank } AB = \text{rank } A'B' \] \hspace{1cm} (4.35)
\[ \vdots \] \hspace{1cm} (4.36)
\[ \text{rank } A'^i B = \text{rank } A'^i B'. \] \hspace{1cm} (4.37)

It is clear then that via \( T \) we have

\[ H_{i-1} = (B, \ldots, A'^{-1} B) \cong H'_i = (B', \ldots, A'^{-1} B'). \] \hspace{1cm} (4.38)

The particular filtrations are essentially invariant for minimal realizations. Define the dimension of \( H_i \) to be

\[ \dim H_i = d_i, \text{ and } \] \hspace{1cm} (4.39)
\[ \dim \frac{H_i}{H_{i-1}} = \dim H_i - \dim H_{i-1} = \bar{d}_i = \text{rank } \frac{\langle B, \ldots, A'^i B \rangle}{\langle B, \ldots, A'^{-1} B \rangle}; \] \hspace{1cm} (4.40)

for \( i = 0, 1, \ldots, n_0 - 1 \). The set \( \{ \bar{d}_i \} \) forms a partition of \( n \), the dimension of \( X(G) \), and is just a measure of how many columns of \( A'^i B \) are linearly independent from those of \( \langle B, \ldots, A'^{-1} B \rangle \).

Now our immediate goal is to recognize what is given by the corresponding conjugate partition associated with the reachability filtration. Namely, \( \{ \bar{d}_j \} \) given from the Ferrer diagram.
To do this we need to recall a definition from classical control theory. Suppose $G$ is a $p \times m$ matrix so that in a minimal realization $B$ is $n \times m$. We know $[B, AB, \ldots, A^iB]$ has full rank for some $i$. For notational purposes let's suppose $m = 4$, and $i = 3$. We wish to analyze the crate construction given by Kailath [7]. We form an $(i + 1 \times m)$ table and label the rows and columns of the table as follows:

\[
\begin{array}{ccc}
\bar{d}_0 & = & \cdot \\
\bar{d}_1 & = & \cdot \\
\vdots & = & \vdots \\
\bar{d}_{n_i-1} & = & \cdot \\
\end{array}
\]

\[
\{\bar{d}_0, \bar{d}_1, \ldots, \bar{d}_{n_i-1}\}
\]

\[
\begin{array}{cccc}
\vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 \\
B & & & \\
AB & & & \\
A^2B & & & \\
A^3B & & & \\
\end{array}
\]

Where $\vec{b}_i$ symbolizes the $i$th column of the matrix $B$ and the box indexed as, say, row $A^2B$ column $\vec{b}_2$ represents the vector $A\vec{b}_2$.

Starting with row $B$, we systematically go from left to right placing an $x$ in this first row under each $\vec{b}_i$ that is linearly independent from the previous vectors $\vec{b}_1, \ldots, \vec{b}_{i-1}$ having an $x$. Next with row $AB$, we go left to right placing an $x$ in the second row under each $A\vec{b}_i$ that is linearly independent from all previous vectors $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m, A\vec{b}_1, \ldots, A\vec{b}_{i-1}$ having an $x$, continuing until we have analyzed each row. We are guaranteed to have $n$ $x$'s because by assumption the controllability matrix $[B, \ldots, A^iB]$ has full rank. Also the set of vectors selected with an $x$ forms a
basis of $X(G)$.

For example consider the following:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. $$

We are looking for the first 5 linearly independent vectors. The crate is given as

$$\begin{array}{ccc} \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ B & X & X & X \\ AB & X & X \\ A^2B & & & & \end{array}$$

A basis for $X(G)$ is given as

$$\langle \tilde{b}_1, \tilde{b}_2, \tilde{b}_3, A\tilde{b}_2, A\tilde{b}_3 \rangle.$$

If we define $r_i$ to be the number of $X$'s in row $A^iB$, then $r_i$ is precisely the number of vectors in the span of the matrix $A^iB$ that are linearly independent from those in $\langle B, AB, \ldots, A^{i-1}B \rangle$. In classical literature, the number of $X$'s in each column of the Kailath crate are called the reachability indices for a minimal realization. But the $r_i = \bar{d}_i$ as defined for our partition of the reachability filtration. Up to order, the number of $X$'s in the columns under each $\tilde{b}_i$ is the conjugate partition of the reachability filtration defined as $\{\bar{d}_i\}$. In summary, in the classical case when $G$ is strictly proper, the conjugate partition of the reachability filtration is just the classical reachability indices of the system. We see from the above argument that it remains invariant of choice of basis for a minimal realization. So we may think of the filtration, the partition, and the conjugate partition as canonical invariants associated with $G$. 
We therefore have a new definition of the reachability (controllability) indices of $G$, viewed as the conjugate partition associated with the reachability filtration.

**Definition 4.3.3** For a strictly proper transfer function with minimal realization $(A, B, C)$, the observability matrix is given by $O = [C^T, A^T C^T, \ldots, A^{n-1} C^T]^T$.

Definition 2.2.8 defined observability. It is a well-known result of classical control that for a minimal realization the observability matrix has full rank. If so, then given a finite sequence of outputs $\{\bar{y}(1), \ldots, \bar{y}(n)\}$ we have

$$\begin{bmatrix} \bar{y}(1) \\ \vdots \\ \bar{y}(n) \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix} \bar{x}(1) \Leftrightarrow \begin{bmatrix} \bar{y}(1) \\ \vdots \\ \bar{y}(n) \end{bmatrix} = O \bar{x}(1). \quad (4.41)$$

Then because $[O^T O]$ has full rank

$$O^T \begin{bmatrix} \bar{y}(1) \\ \vdots \\ \bar{y}(n) \end{bmatrix} = [O^T O] \bar{x}(1) \Leftrightarrow \bar{x}(1) = [O^T O]^{-1} O^T \begin{bmatrix} \bar{y}(1) \\ \vdots \\ \bar{y}(n) \end{bmatrix}. \quad (4.42)$$

There exists a unique solution to the following observability question: What state at time $t = 1$ created the output sequence $\{\bar{y}(1), \ldots, \bar{y}(n)\}$?

**Definition 4.3.4** Given a minimal realization for a strictly proper transfer function, the observability filtration is given as

$$(L_{-1} \overset{\text{def}}{=} X) \supseteq L_0 \supseteq L_1 \supseteq \ldots \supseteq L_{n-1} = (0),$$
where for $i = 0, 1, \ldots, n_0 - 1$

$$L_i = \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^i \end{pmatrix} \overset{\text{def}}{=} \ker O_i$$

and $n_0$ is the least integer such that $L_{n_0-1} = (0)$.

The set $\{ \bar{d}_i = \dim \frac{L_i}{L_{i+1}} \mid i = -1, 0, \ldots, n_0 - 2 \}$ is a partition of $n = \dim X$. The associated conjugate partition to this set is called the set of observability indices.

More formally,

**Definition 4.3.5** We form the Ferrer's diagram from the dimensions of the quotient spaces of the observability filtration. The conjugate partition associated with this diagram is called the set of observability indices.

Similar to the argument used for the controllable case, the filtration is invariant under a change of basis matrix $T^T$. The observability filtration and the indices are a canonical invariant associated with the transfer function $G$.

Let $r_i = \dim L_i$. Then $r_i = \dim(\ker O_i) = n - \text{rank } O_i^T$. Substitution yields

$$\bar{d}_i = \dim \frac{L_i}{L_{i+1}} = \dim L_i - \dim L_{i+1} = r_i - r_{i+1} =$$

$$\text{rank } [C^T, A^TC, \ldots, A^{Ti}C^T] - \text{rank } [C^T, A^TC, \ldots, A^{Ti-1}C^T] =$$

$$\dim \frac{([C^T, A^TC, \ldots, A^{Ti}C^T])}{([C^T, A^TC, \ldots, A^{Ti-1}C^T])}. \quad (4.44)$$

But we recognize this last term as the $i$th level of the reachability filtration for the transfer function $G^T$ with the system

$$x(t + 1) = A^T x(t) + C^T y(t) \quad (4.45)$$

$$u(t) = B^T x(t). \quad (4.46)$$
Therefore the controllability indices associated with $G$ are the observability indices associated with $G^T$. Moreover, from inspection of the control filtration of $G$ and the observability filtration of $G^T$ we have

$$H_i = \langle B, AB, \ldots, A^i B \rangle \text{ and } L_i = \ker \begin{pmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T A^{T_i} \end{pmatrix}. \quad (4.47)$$

Then $H_i = L_i^\perp$. So the controllability filtration for $G$ is the orthogonal compliment of the observability filtration for $G^T$. Similarly, the controllability filtration for $G^T$ is the orthogonal compliment of the observability filtration for $G$. This fact is important and will be used in the latter sections of the paper.

### 4.4 The Kernel and Image Filtrations

We consider another case in which the $p \times m$ matrix $G$ contains entries from $k[z] \subset k(z)$ (i.e., $G$ is a polynomial transfer function). Such transfer functions are called strictly anti-causal.

Recall that the ring $O_\infty$ is a PID and a local ring with unique maximal ideal generated by $\frac{1}{z}$. We also saw in earlier sections that if $G$ is polynomial then $X(G) = 0$ so that $\mathcal{X}(G) = X(G) \oplus X_\infty(G) = X_\infty(G)$. That is, $G$ only has poles at infinity. Suppose we have a right coprime factorization of $G$ given by $(N_\infty, D_\infty)$ according to definition (3.2.2) and comment (3.24). Then on page 25 we saw that $X_\infty(G) =$

$$\frac{z^{-1} \Omega_\infty U}{G^{-1}(z^{-1} \Omega_\infty Y) \cap z^{-1} \Omega_\infty U} \cong \frac{z^{-1} \Omega_\infty U}{[D_\infty](z^{-1} \Omega_\infty U)}. \quad (4.48)$$
From a fundamental theorem for finitely generated torsion modules over PID's, we can view $X_\infty(G)$ via its invariant factor decomposition as (in this case since $O_\infty$ is local there is only one prime and the invariant factor decomposition is also the primary factor decomposition)

$$\frac{\bar{f}_1 O_\infty}{(z-1)^{e_1} O_\infty} \oplus \frac{\bar{f}_2 O_\infty}{(z-1)^{e_2} O_\infty} \oplus \ldots \oplus \frac{\bar{f}_s O_\infty}{(z-1)^{e_s} O_\infty},$$

where $\{\bar{f}_1, \ldots, \bar{f}_s\}$ is a generating set for $X_\infty(G)$ over $O_\infty$ and $(e_1 \leq e_2 \leq \ldots \leq e_s)$.

Define the $O_\infty$-linear nilpotent map $J$ on $X_\infty(G)$ as follows: If $x \in X_\infty(G)$, then $J(x) \overset{\text{def}}{=} (\frac{1}{z})x \in X_\infty(G)$.

From the invariant factor decomposition we see that $J$ is nilpotent of order $e_s$. Since $X_\infty(G)$ is also a finite dimensional vector space over $O_\infty$, we may construct a matrix representation for $J, M$, with basis

$$B = \langle \bar{f}_1, \frac{1}{z}\bar{f}_1, \ldots, \frac{1}{z^{e_1-1}}\bar{f}_1, \bar{f}_2, \frac{1}{z}\bar{f}_2, \ldots, \frac{1}{z^{e_2-1}}\bar{f}_2, \ldots, \bar{f}_s, \frac{1}{z}\bar{f}_s, \ldots, \frac{1}{z^{e_s-1}}\bar{f}_s \rangle,$$ (4.49)

inspired by the invariant factor decomposition above. Then

$$M = \begin{bmatrix} A_1 & 0 \\ \ldots & \ldots \\ 0 & A_s \end{bmatrix}, \quad \text{where } A_i = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \text{ is } e_i \times e_i.$$

The map $J$ is a module theoretic map and $M$ is the matrix Jordan representation of $J$ dependent on a particular choice of basis. That is, if we define, for $x \in X_\infty(G)$,
to be the vector representation of $\mathbf{x}$ with respect to the basis $B$, then

$$[J(\mathbf{x})]_B = M[\mathbf{x}]_B. \quad (4.50)$$

We now use the above to construct a filtration on $X_{\infty}(G)$.

Define $K_i = \ker J^i$, $i = 1, \ldots, e$, and define $K_0 = (0)$ for completeness. It is clear that if $\mathbf{x} \in K_i$ then

$$J^{i+1}(\mathbf{x}) = J \circ J^i(\mathbf{x}) = J(0) = 0, \quad (4.51)$$

so $K_i \subseteq K_{i+1}$. With the above notation we have the following.

**Definition 4.4.1** The following increasing filtration on $X_{\infty}(G)$ given by

$$(0) = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_{e} = X_{\infty}(G)$$

is called the kernel filtration.

It is inspired by the kernels of the nilpotent map $J^i$, $i = 1, \ldots, e$. The map $M$ with its action on the vector space structure of $X_{\infty}(G)$ and the map $J$ with its action on the module structure of $X_{\infty}(G)$ allow a geometric interpretation of the partitions associated with this filtration formed from the Ferrer's diagram of the set

$$\overline{d}_i = \dim \left( \frac{K_i}{K_{i-1}} \right), \quad i = 1, \ldots, e. \quad (4.52)$$

Since $J$ is nilpotent, the only eigenvalue of $M$ is 0. The $\dim_k (K_i) = \dim(\ker J^i) = \dim_k (\ker (M - 0I)^i)$. So the $\dim_k K_i$ is the number of generalized eigenvectors of length
less than or equal to \( i \). Then

\[
\bar{d}_i = \text{(number of linearly independent eigenvectors of } M) \\
= \text{(number of Jordan blocks of } M) \\
= \text{number of } e_i \geq 1. \tag{4.53}
\]

\[
\bar{d}_2 = \text{(number of linearly independent generalized} \\
\text{eigenvectors of } M \text{ of length 2)} \\
= \text{(number of Jordan blocks of } M \text{ of size greater than or equal to 2)} \\
= \text{number of } e_i \geq 2. \tag{4.54}
\]

\[
\vdots
\]

\[
\bar{d}_{e_i} = \text{(number of linearly independent generalized} \\
\text{eigenvectors of } M \text{ of length exactly } e_i) \\
= \text{(number of Jordan blocks of } M \text{ of size greater than or equal to } e_i) \\
= \text{number of } e_i \geq e_i. \tag{4.55}
\]

It is straightforward to see that the conjugate partition gives the set of dimensions of the Jordan blocks, or precisely the sequence \( e_s, e_{s-1}, \ldots, e_1 \), the powers of the invariant (or primary) factors in the decomposition of \( X_\infty(G) \). Note, since the invariant factors of a module are unique we may consider the conjugate partition associated with the kernel filtration as canonical. For example, if \( e_2 = 3 \) then \( A_2 \) is a block of size 3. In the Ferrer diagram, this block contributes one dot to \( \bar{d}_1 \), since \( e_2 \geq 1 \), one dot to \( \bar{d}_2 \), since \( e_2 \geq 2 \), one dot to \( \bar{d}_3 \), since \( e_2 \geq 3 \), and the block makes no further contributions to the Ferrer diagram. Therefore, this block contributes a column of
An analogue to the kernel filtration is the image filtration.

**Definition 4.4.2** Given the nilpotent map \( J \) in the above setting the image filtration is an increasing filtration defined as

\[
X_\infty(G) = P_0 \supseteq P_1 \supseteq \ldots \supseteq P_{e_*} = (0), \text{ where }
\]

\[
P_i \overset{def}{=} \text{im}J^i. \tag{4.56}
\]

That is, the \( i \)th space in the sequence is just \((e^{-i} G_\infty) X_\infty(G)\). Then the partition associated to the image filtration is also closely related to the Jordan form of the map \( J \). If the dimension of \( X_\infty(G) \) is \( n \), then

\[
\bar{d}_1 = \text{n-rank } (J) = \text{number of eigenvectors of } M
\]

\[
= (\text{number of Jordan blocks of } M)
\]

\[
= \text{number of } e_i \geq 1. \tag{4.57}
\]

\[
\bar{d}_2 = (\text{number of linearly independent generalized eigenvectors of } M \text{ of length 2})
\]

\[
= (\text{number of Jordan blocks of } M \text{ of size greater than or equal to 2})
\]

\[
= \text{number of } e_i \geq 2. \tag{4.58}
\]

\[
\vdots
\]

\[
\bar{d}_{e_*} = (\text{number of linearly independent generalized eigenvectors of } M \text{ of length exactly } e_*)
\]
\[
\begin{align*}
\text{number of Jordan blocks of } M \text{ of size greater than or equal to } e_s & \\
= \quad \text{number of } e_i \geq e_s.
\end{align*}
\] 

(4.59)

The related conjugate partition, \(e_s, e_{s-1}, \ldots, e_1\), gives the sizes of the Jordan blocks of \(J\) in decreasing order. These are again precisely the invariant factor exponents of the prime \(z^{-1}\) in the module structure of \(X_\infty(G)\) over the ring \(\mathcal{O}_\infty\). The image and kernel filtrations are different but yield the same partitions capturing information of the invariant factor structure of the infinite pole space.

Recapitulating, there exists reachability and observability filtrations, \(\{K_i\}\) and \(\{L_i\}\), for \(X(G)\) in the case where \(G\) is a \(p \times m\) matrix of strictly proper rational functions. The conjugate partitions of \(\dim X(G)\) associated with these chains of quotients for the above filtrations yield the classical controllability and observability indices. The control filtration for \(G\) was found to be orthogonal to the observability filtration of \(G^T\). The partitions associated with the kernel and image filtrations, \(\{K_i\}\) and \(\{P_i\}\), for \(X_\infty(G)\) in the case when \(G\) is a matrix of polynomials yield the exponents \(e_s, e_{s-1}, \ldots, e_1\), for the prime \(z^{-1}\) in the invariant factor decomposition of \(X_\infty(G)\) over \(\mathcal{O}_\infty\).
CHAPTER V

The Generalized Wedderburn Forney Space

The content of this chapter offers a break from the previous topics. It covers a new result that generalizes the classical Wedderburn Forney space. Reading this chapter may help readers to understand and better familiarize themselves with computations in the classical Wedderburn space, which we will continue to use throughout. This chapter, however, may be skipped without losing comprehension in the remaining chapters.

Our goal in this work is to show that the generalized Wedderburn space, \( W_{\sigma} \) is a finite dimensional vector space isomorphic to the classical Wedderburn space, \( W_{\pm} \), where \( \sigma \) is any open region of the \( \mathbb{C} \)-plane, \( \infty \notin \sigma \).

**Step 1** Define \( \pi_{\sigma}(C); \pi_{\sigma_{\pm}}(C); \) and \( \pi_{\sigma}(a(z)\bar{c}) \). We also see what \( \pi_{\sigma}(a(z)\bar{c}) \) "looks like"

\[
\text{in } \frac{\pi_{\sigma}(C)}{c \cap \pi_{\sigma}(c)} \equiv W_{\sigma}.
\]

**Step 2** Find a spanning set for \( \frac{\pi_{\sigma}(C)}{c \cap \pi_{\sigma}(c)} \). (This is the most involved step).

**Step 3** Show the proposed spanning set is L.I. and \( \dim W_{\sigma} = \dim W_{\pm} \).
From this chosen basis an obvious isomorphism will exist between $W_0$ and $W_{\pm}$.

Let $\sigma$ be an open region of the complex plane ($\infty \not\in \sigma$), $V$ be an $n$-dimensional vector space of rational functions, and $C$ a subspace of $V$. Given $\vec{v} \in V$ define $\pi_{\sigma}(\vec{v})$ and $\pi_{\sigma^c}(\vec{v})$ as follows:

(a) $\pi_{\sigma}(\vec{v}) = \pi_{+}(\vec{v}) + \text{s.p. part of } \vec{v} \text{ that is holomorphic in } \sigma$. (i.e., components of $\vec{v}$ with poles only in $\sigma^c$).

(b) $\pi_{\sigma^c}(\vec{v})$ = s.p. part of $\vec{v}$ holomorphic in $\sigma^c$. (i.e., components of $\vec{v}$ with poles only in $\sigma$).

[Note: the subscript "0" on $\sigma^c$ is to remind us that if $\vec{w} = \pi_{\sigma^c}(\vec{v})$ then $\vec{w}$ is s.p. and so $\lim_{|z| \to \infty} \vec{w} = \vec{0}$.] (5.1)

From the above definitions we see, given any vector $\vec{v}$, that $\vec{v}$ can be written as $\vec{v} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1 = \pi_{\sigma}(\vec{v})$ and $\vec{w}_2 = \pi_{\sigma^c}(\vec{v})$. This is done simply by breaking each coefficient of $\vec{v}$ into its partial fraction decomposition, placing all polynomial parts and pole parts that lie in $\sigma^c$ into $\vec{w}_1$, and placing into $\vec{w}_2$ all s.p. terms of the coefficients of $\vec{v}$ having poles in $\sigma$.

We also define $\pi_{\sigma}(C)$ and $\pi_{\sigma^c}(C)$ as follows:

(c) $\vec{v} \in \pi_{\sigma}(C)$ if $\exists \vec{c} \in C$ s.t. $\vec{v} = \pi_{\sigma}(\vec{c})$.

(d) $\vec{v} \in \pi_{\sigma^c}(C)$ if $\exists \vec{c} \in C$ s.t. $\vec{v} = \pi_{\sigma^c}(\vec{c})$.

One immediate result is the following analogy to $\vec{c} \in \pi_{+}(C) \oplus \pi_{-}(C)$.

(I) If $\vec{v} \in \pi_{\sigma}(C) \cap \pi_{\sigma^c}(C)$ then $\vec{v} = \vec{c}$. 
Proof: \( \overline{v} \in \pi_\sigma(C) \Rightarrow \exists \overline{c}_1 \text{ s.t. } \overline{v} = \pi_\sigma(\overline{c}_1). \)

\( \overline{v} \in \pi_\sigma(C) \Rightarrow \exists \overline{c}_2 \text{ s.t. } \overline{v} = \pi_\sigma(\overline{c}_2). \)

\( \overline{v} \in \pi_\sigma(C) \Rightarrow \overline{v} \text{ is holomorphic in } \sigma. \) Similarly, \( v \in \pi_{\sigma^c}(C) \Rightarrow \overline{v} \text{ is holomorphic in } \sigma^c \) and \( \lim_{|z| \to \infty} \overline{v} = \overline{0}. \) Then \( \overline{v} \) is entire and bounded so \( \overline{v} \) is a constant vector. Since \( \lim_{|z| \to \infty} \overline{v} = \overline{0}, \overline{v} \equiv \overline{0}. \)

Corollary: From (I) and the previous statement of decomposition

\[ \forall \overline{c} \in C, \quad \overline{c} \in \pi_\sigma(C) \oplus \pi_{\sigma^c}(C). \]

Two Definitions:

(e) A vector \( \overline{v} \) is \underline{fully reduced} if all its coefficients are polynomial and for \( \overline{v} = \begin{pmatrix} p_1(z) \\ \vdots \\ p_n(z) \end{pmatrix}, (p_1(z), \ldots, p_n(z)) = 1. \)

It is clear any vector \( \overline{v} \in V \) can be fully reduced without changing the span of \( \overline{v} \) over the field \( C(z). \)

(f) Given \( a(z) \in C(z) \) a rational function we have a partial fraction decomposition

\[
\begin{align*}
a(z) &= p(z) + \left[ \frac{b_1(\lambda_1)}{(z - \lambda_1)^{s_1}} + \cdots + \frac{b_{s_1}(\lambda_1)}{(z - \lambda_1)^{s_1}} \right] + \\
&\quad + \left[ \frac{b_1(\lambda_2)}{(z - \lambda_2)^{s_2}} + \cdots + \frac{b_{s_2}(\lambda_2)}{(z - \lambda_2)^{s_2}} \right] + \cdots + \\
&\quad + \left[ \frac{b_1(\lambda_m)}{(z - \lambda_m)^{s_m}} + \cdots + \frac{b_{s_m}(\lambda_m)}{(z - \lambda_m)^{s_m}} \right],
\end{align*}
\]

(5.2)

(5.3)

(5.4)

(5.5)

where \( s_i \) is the order of the pole \( \lambda_i \) and \( b_i(\lambda_j)(\text{read } b \text{ sub } i \text{ of } \lambda_j) \) is the coefficient.
of \( \frac{1}{(z-\lambda_j)} \), we define

\[
    f_1^{i}(\lambda_i) = \left[ \frac{b_1(\lambda_i)}{(z-\lambda_i)^1} + \cdots + \frac{b_{s_i}(\lambda_i)}{(z-\lambda_i)^{s_i}} \right];
\]

(5.6)

read the fractional part of \( a(z) \) due to \( \lambda_i \) from 1 to \( s_i \).

So

\[
    a(z) = p(z) + f_1^{i_1}(\lambda_1) + \cdots + f_1^{i_m}(\lambda_m).
\]

(5.7)

Let's now focus our attention on \( \sigma \). Let \( \bar{c} \in (C) \) be fully reduced and \( a(z) \in C(z) \) be a rational function, arbitrary. Recall \( \bar{c} \in \pi_\sigma(C) \) if \( \exists \bar{c} \in C \) s.t. \( \pi_\sigma(\bar{c}) = \bar{c} \). We see \( a(z)\bar{c} \in (C) \), so \( \pi_\sigma(a(z)\bar{c}) \in \pi_\sigma(C) \). So what is in \( \pi_\sigma(a(z)\bar{c}) \)? We have that

\[
    \pi_\sigma(a(z)\bar{c}) = \pi_\sigma([p(z) + f_1^{i_1}(\lambda_1) + \cdots + f_1^{i_m}(\lambda_m)]\bar{c})
\]

(5.8)

\[
    = \pi_\sigma(p(z)\bar{c}) + \pi_\sigma(f_1^{i_1}\bar{c}) + \cdots + \pi_\sigma(f_1^{i_m}\bar{c}).
\]

(5.9)

1. \( p(z)\bar{c} = \pi_\sigma(p(z)\bar{c}) \in \pi_\sigma(C) \) since \( \bar{c} \) is polynomial, and therefore so is \( p(z)\bar{c} \).

   [Note: \( p(z)\bar{c} \in C \cap \pi_\sigma(C) \)].

2. If \( \lambda_i \in \sigma^\circ \) then \( f_1^{i_i}(\lambda_i)\bar{c} \) only has polynomial coefficients and poles in \( \sigma^\circ \), (at \( \lambda_i \)).

   So \( f_1^{i_i}(\lambda_i)\bar{c} \) is holomorphic in \( \sigma \) and so

\[
    f_1^{i_i}(\lambda_i)\bar{c} = \pi_\sigma(f_1^{i_i}(\lambda_i)\bar{c}) \in \pi_\sigma(C).
\]

(5.10)

[Note: \( f_1^{i_i}(\lambda_i)\bar{c} \in C \cap \pi_\sigma(C) \)].
(3) Suppose \( \lambda_i \in \sigma \) and suppose \( \deg \bar{c} \overset{\text{def}}{=} \delta \bar{c} = d \). Consider
\[
\bar{w} \overset{\text{def}}{=} f_{d+1}^\lambda(\lambda_i)\bar{c} = \left[ \frac{b_{d+1}(\lambda_i)}{(z - \lambda_i)^{d+1}} + \cdots + \frac{b_{\nu}(\lambda_i)}{(z - \lambda_i)^\nu} \right] \bar{c}. \tag{5.11}
\]

Clearly, \( \bar{w} \in \mathcal{C} \). Also, since we've chosen only powers of \( \frac{1}{z - \lambda_i} \) that are strictly greater than \( \delta \bar{c}, \bar{w} \) is strictly proper. The only pole of \( \bar{w} \) is \( \lambda_i \), which is in \( \sigma \), so \( \bar{w} \) is holomorphic in \( \sigma^c \). Then \( \bar{w} = \pi_{\sigma^c}(\bar{w}) \in \pi_{\sigma^c}(\mathcal{C}) \). So \( \bar{w} \) is not a part of \( \pi_\sigma(a(z)\bar{c}) \).

We've found some parts of \( \pi_\sigma(a(z)\bar{c}) \), namely 1 and 2 above. Let's continue.

Let \( \lambda_1, \ldots, \lambda_r \) be those poles of \( a(z) \) in \( \sigma^c \) and \( \mu_1, \ldots, \mu_r \) be those poles of \( a(z) \) in \( \sigma \). Then consider
\[
\pi_\sigma(a(z)\bar{c}) - \pi_\sigma([f_1^\mu(\mu_1) + \cdots + f_r^\mu(\mu_r)]\bar{c}) = \pi_\sigma(a(z)\bar{c} - \sum_{i=1}^r f_1^\mu(\mu_i)\bar{c}) = \pi_\sigma([p(z) + f_1^\mu_1(\lambda_1) + \cdots + f_1^\mu_r(\lambda_r) + f_{d+1}^\mu(\mu_1) + \cdots + f_{d+1}^\mu(\mu_r)]\bar{c}) \tag{5.12}
\]

\[
= [p(z) + f_1^\mu_1(\lambda_1) + \cdots + f_1^\mu_r(\lambda_r)]\bar{c} \in \mathcal{C} \cap \pi_\sigma(\mathcal{C}). \tag{5.14}
\]

Then at least so far we have, for a fully reduced vector \( \bar{c} \) in the space \( \frac{\pi_\sigma(\mathcal{C})}{\mathcal{C} \cap \pi_\sigma(\mathcal{C})} \),
\[
\pi_\sigma(a(z)\bar{c}) \sim \pi_\sigma\left( \sum_{i=1}^r f_1^\mu(\mu_i)\bar{c} \right),
\]
where \( \mu_i \) are the poles of \( a(z) \) in \( \sigma \) and \( d = \delta \bar{c} \).
So the class of $\overline{a(z)c}$ in $\frac{\pi_e(c)}{e \cap \pi_e(c)}$ can be represented by the fractional parts of $a(z)$ with poles in $\sigma$ and order less than or equal to $d$. We are really concerned with $\frac{\pi_e(c)}{e \cap \pi_e(c)} \overset{\text{def}}{=} W_\sigma$. So now our job is to analyze $\pi_\sigma \left( \sum_{i=1}^{d} f_i(\mu_i)c \right)$.

Consider first just one term of the form

$$\tilde{w} = \frac{1}{(z-\mu)^m} \tilde{c} \quad (1 \leq m \leq d; \ (\mu \in \sigma), \quad (5.15)$$

where $\tilde{c}$ is fully reduced. For notational purposes, assume

$$\begin{align*}
(\delta p_1) \leq (\delta p_2) \leq \cdots \leq (\delta p_n) &= d, \\
\text{where } \tilde{c} &= \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}. \quad (5.17)
\end{align*}$$

Let $s$ be the first index where $\delta(p_s) = m$. Then $\frac{p_i(z)}{(z-\mu)^m}$ is s.p. for $i = 1, \ldots, (s-1)$. For $p_i (i = s, \ldots, n)$ we use the Euclidean algorithm,

$$\frac{p_i}{(z-\mu)^m} = q_i + a_i(z), \quad (5.18)$$

where $q_i$ is polynomial and $m + \delta q_i = \delta p_i$. More importantly $a_i(z)$ is s.p.; if it has a pole, it must be at $\mu \in \sigma$. So $a_i(z)$ is analytic in $\sigma^c$. Now for $\tilde{w}$, (equation (5.15)), decompose $\tilde{w}$ as described earlier

$$\frac{1}{(z-\mu)^m} \tilde{c} = \pi_{\sigma^c} \left( \frac{1}{(z-\mu)^m} \tilde{c} \right) + \pi_{\sigma^c} \left( \frac{1}{(z-\mu)^m} \tilde{c} \right) \quad (5.19)$$

$$= \begin{pmatrix} 0_1 \\ \vdots \\ 0_{s-1} \\ q_s \\ \vdots \\ q_n \end{pmatrix} + \begin{pmatrix} \frac{p_1}{(z-\mu)^m} \\ \vdots \\ \frac{p_{s-1}}{a_s} \\ a_s \\ \vdots \\ a_n \end{pmatrix} \quad (5.20)$$

(i.e. polynomial)
Then for terms like $\frac{1}{(z-\mu)^m}\bar{c}$ ($\mu \in \sigma; 1 \leq m \leq d$)

$$\pi_{\sigma}\left(\frac{1}{(z-\mu)^m}\bar{c}\right) = \pi_{+}\left(\frac{1}{(z-\mu)^m}\bar{c}\right)$$  \hspace{1cm} (5.21)

or

$$\pi_{\sigma}\left(\sum_{i=1}^{r} f_1^d(\mu_i)\bar{c}\right) = \pi_{+}\left(\sum_{i=1}^{r} f_1^d(\mu_i)\bar{c}\right).$$  \hspace{1cm} (5.22)

So we know what $\pi_{\sigma}(a(z)\bar{c})$ looks like. It equals

$$p(z)\bar{c} + \sum_{i=1}^{t} f_1^{\lambda_i}(\lambda_i)\bar{c} + \pi_{+}\left(\sum_{i=1}^{r} f_1^d(\mu_i)\bar{c}\right),$$

where $\lambda_i$ are poles of $a(z)$ in $\sigma\bar{c}$, $\mu_i$ are poles of $a(z)$ in $\sigma$, and $d = \delta\bar{c}$.

What is important for us though is the following:

Given any fully reduced polynomial vector $\bar{c} \in C$ and any rational function $a(z)$ we can find a representative of $\overline{a(z)\bar{c}}$ in $\frac{\pi_{\sigma}(C)}{\pi_{\sigma}(\bar{c}) \cap C}$ given by the polynomial part of $\sum_{i=1}^{r} f_1^d(\mu_i)\bar{c}$, all terms as described above. Let's do an example.

Let

$$\sigma = \{z|0 < \text{Re}z < 4\},$$  \hspace{1cm} (5.23)

$$C = \langle \bar{c} = \begin{pmatrix} 3z^2 + 1 \\ 2z + 4 \end{pmatrix} \rangle, \text{ and }$$  \hspace{1cm} (5.24)

$$a(z) = \frac{3z - 6}{3z^2 - 8z - 3}.$$  \hspace{1cm} (5.25)

We seek a representative of $\overline{a(z)\bar{c}}$ in $\frac{\pi_{\sigma}(C)}{\pi_{\sigma}(\bar{c}) \cap C}$, $a(z) = \frac{7}{10 z + \frac{1}{3}} + \frac{3}{10 z - 3}$ and $\bar{c}$. So let

$$w = a(z)\bar{c} = \frac{7}{10} \left(\frac{3z^2 + 1}{2z + 4}\right) + \frac{3}{10} \left(\frac{3z^2 + 1}{2z + 4}\right) =$$  \hspace{1cm} (5.26)

$$f_1^{\lambda_i}(-\frac{1}{3})\bar{c} + f_1^{\lambda_i}(3)\bar{c}.$$  \hspace{1cm} (5.27)
We see 3 is the only pole of \( a(z) \) in \( \sigma \). We have that

\[
f_1^1\left(-\frac{1}{3}\right) \bar{c} = \pi_\sigma(f_1^1\left(-\frac{1}{3}\right) \bar{c}) \in \pi_\sigma(C) \cap C,
\]

since it consists of polynomial parts, fractional parts with poles in \( \sigma^c \) (and so is holomorphic in \( \sigma \)), and is in \( C \). Then up to equivalence

\[
\pi_\sigma(f_1^1(3) \bar{c}) \sim \pi_\sigma(a(z) \bar{c})
\]

since

\[
\pi_\sigma(a(z) \bar{c}) - \pi_\sigma(f_1^1(3) \bar{c}) = f_1^1\left(-\frac{1}{3}\right) \bar{c},
\]

as expected. Decomposing \( f_1^1(3) \bar{c} \) yields

\[
f_1^1(3) \bar{c} = \left(\frac{9}{10}z + \frac{27}{10}\right) + \left(\frac{34}{10(z-3)}\right) = \pi_\sigma + \pi_\sigma z.
\]

So a representative of \( a(z) \bar{c} \) in \( \frac{\pi_\sigma(C)}{\pi_\sigma(C) \cap \bar{c}} = W_\sigma(C) \) is

\[
\pi_\sigma(f_1^1(3) \bar{c}) = \pi_+(f_1^1(3) \bar{c}) = \left(\frac{9}{10}z + \frac{27}{10}\right).
\]

(i.e. just the polynomial part of all fractional components with poles in \( \sigma \) up to the degree of \( \bar{c} \).)

**Step 2**

Before we proceed we need a lemma.

**Lemma 1:** Suppose \( \bar{c} \) is fully reduced of degree \( d \). Let \( \lambda \) be fixed. Create the set

\[
\left\{ \pi_+\left[ \frac{1}{(z - \lambda)^i} \right] \bar{c} \mid i = 1, \ldots, d \right\}.
\]
Recall if \( \lambda \in \sigma \), (and it will be) then

\[
\pi_+ \left( \frac{1}{(z - \lambda)^r} \right) = \pi_\sigma \left( \frac{1}{(z - \lambda)^r} \right). \tag{5.32}
\]

Let \( \mu \) be an arbitrary complex number and consider \( \pi_+ \left( \frac{1}{(z - \mu)^r} \right) (1 \leq r \leq \delta) \). Then

\( \exists a_1, \ldots, a_\delta \in \mathbb{C} \) s.t.

\[
\pi_+ \left( \frac{1}{(z - \mu)^r} \right) = a_1 \pi_+ \left( \frac{1}{z - \lambda} \right) + a_2 \pi_+ \left( \frac{1}{(z - \lambda)^2} \right) + \cdots + a_\delta \pi_+ \left( \frac{1}{(z - \lambda)^\delta} \right). \tag{5.33}
\]

**Proof:** The first step is to consider a single polynomial \( p(z) \) and to realize that, division of \( p(z) \) by \( (z - \lambda) \) or \( (z - \mu) \) etc., and stripping off the polynomial part, is a linear operation.

If we represent \( p(z) = a_\delta z^\delta + \cdots + a_0 \) by \( \begin{pmatrix} a_\delta \\ \vdots \\ a_0 \end{pmatrix} \) viewed as an element of a \((\delta + 1)\)-dimensional vector space with basis \( \{z^\delta, z^{\delta-1}, \ldots, 1\} \) then division shows

\[
\frac{a_\delta z^{\delta-1} + (a_{\delta-1} + \lambda a_\delta) z^{\delta-2} + (a_{\delta-2} + \lambda a_{\delta-1} + \lambda^2 a_\delta) z^{\delta-3} + \cdots + (a_0 \lambda^{\delta-1} + \cdots + a_1)}{z - \lambda} = a_\delta z^\delta + a_{\delta-1} z^{\delta-1} + \cdots + a_0 + \text{a remainder in } \mathbb{C}.
\]

This operation can be viewed as

\[
D_{z - \lambda}(p(z)) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\lambda & 1 & 0 & 0 & 0 & \cdots & 0 \\
\lambda^2 & \lambda & 1 & 0 & 0 & \cdots & 0 \\
\lambda^3 & \lambda^2 & \lambda & 1 & 0 & \cdots & 0 \\
\lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^{\delta-1} & \lambda^{\delta-2} & \cdots & \cdots & \cdots & \cdots & \lambda & 1 & 0 \\
\end{bmatrix} \begin{pmatrix} a_\delta \\ \vdots \\ a_0 \end{pmatrix} = \pi_+ \left( \frac{1}{(z - \lambda)p(z)} \right) \tag{5.34}
\]

given in terms of this basis \( \{z^\delta, \ldots, 1\} \). So,
\[
D_{(z-\lambda)}^2p(z) = \pi_+ \left( \frac{1}{(z-\lambda)^2}p(z) \right) \tag{5.35}
\]

\[
\vdots
\]

\[
D_{(z-\lambda)}^d p(z) = \pi_+ \left( \frac{1}{(z-\lambda)^d}p(z) \right) \tag{5.37}
\]

For example, let \( p(z) = 3z^3 + 5z^2 - 7z + 2 \) and \( \lambda = 2 \).

\[
\pi_+ \left( \frac{1}{z-2}p(z) \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 11 \\ 15 \end{bmatrix} = 3z^2 + 11z + 15, \tag{5.39}
\]

as it should be.

\[
\pi_+ \left( \frac{1}{(z-2)}p(z) \right) = D_{z-2}^2 \begin{bmatrix} 3 \\ 5 \\ -7 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ -7 \\ 2 \end{bmatrix} = 3z + 17. \tag{5.41}
\]

Matrices of this form are nilpotent of order \( d \) with each product yielding diagonal 1's marching to zero in the lower left corner. A simple induction proof shows that

\[ \exists a_1, \ldots, a_d, \text{ simply from the structure and } 1's, \text{ s.t.} \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\mu & 1 & 0 & 0 & \cdots & 0 \\
\mu^2 & \mu & 1 & 0 & \cdots & 0 \\
\mu^3 & \mu^2 & \mu & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mu^{d-1} & \mu^{d-2} & \cdots & \cdots & 1 & 0
\end{bmatrix} = a_1 \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 \\
\lambda & 1 & 0 & 0 & \cdots & 0 \\
\lambda^2 & \lambda & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^{d-1} & \lambda^{d-2} & \cdots & \cdots & 1 & 0
\end{bmatrix}
\]
Continuing with our above example. Suppose $\mu = -5$, then

\[
D_{z+5} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 25 & -5 & 1 & 0 \end{bmatrix}
\]

\[
D_{z+5}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -10 & 1 & 0 & 0 \end{bmatrix}
\]

\[
D_{z+5}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\]

\[
D_{z+5}^4 \equiv [0].
\]

The goal is, choosing $r$ say $1 \leq (r - 2) \leq (d - 3)$, can we find constants $a_1, \ldots, a_d$ s.t.

\[
D_{z+5}^2 = a_1 D_{z-2}^1 + a_2 D_{z-2}^2 + a_3 D_{z-2}^3?
\]

Yes, $a_1 = 0 \quad a_2 = 1 \quad a_3 = -14$. Recall the lemma was to show there exist $a_1, \ldots, a_d$ s.t.

\[
\pi_+\left(\frac{1}{(z - \mu)^r}\right) = a_1\pi_+\left(\frac{1}{(z - \lambda)^1}\right) + a_2\pi_+\left(\frac{1}{(z - \lambda)^2}\right) + \cdots + a_d\pi_+\left(\frac{1}{(z - \lambda)^d}\right)
\]
(i.e., that the $a_1, \ldots, a_d$ work for an entire vector not just a single polynomial.) But this is easy now, since it was the map $D_{(x-\mu)}$ we expressed as a linear combination of the maps $D_{(x-\lambda)}^1, \ldots, D_{(x-\lambda)}^d$.

So given $\bar{c}$ fully reduced of deg $d$, expand each entry of $\bar{c}$ into the basis $\{z^d, \ldots, 1\}$, (i.e. view $\bar{c}$ in an $n(d+1)$-dimensional vector space.)

e.g.

$$\bar{c} = \begin{pmatrix} 3z^2 + 1 \\ 2z^3 + 3z + 2 \end{pmatrix}$$

viewed as

$$\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 2 \\ 0 \\ 3 \\ 2 \end{pmatrix}$$

or more simply

$$\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 2 \\ 0 \\ 3 \\ 2 \end{pmatrix}$$

So

$$\pi_+ \left( \frac{1}{(z-\mu)^r} \bar{c} \right) = \begin{pmatrix} D_{(z-\mu)}^r & D_{(z-\mu)}^r & \cdots & 0 \\ 0 & D_{(z-\mu)}^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & D_{(z-\mu)}^r \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \begin{pmatrix} D_{(z-\lambda)}^1 & \cdots & 0 \\ 0 & \cdots & D_{(z-\lambda)}^1 \end{pmatrix} + \cdots + a_d \begin{pmatrix} D_{(z-\lambda)}^d & \cdots & 0 \\ 0 & \cdots & D_{(z-\lambda)}^d \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$a_1 \pi_+ \left( \frac{1}{(z-\lambda)^r} \bar{c} \right) + \cdots + a_d \pi_+ \left( \frac{1}{(z-\lambda)^d} \bar{c} \right). \square$$

So now we have a spanning set of $\frac{\pi_+ (\bar{c})}{\pi_+ (\bar{c}) \cap \bar{c}}$. Let's see how.

Choose and fix $\lambda \in \sigma$, then

$$\tilde{B} = \left\{ \pi_{\sigma} \left( \frac{1}{(z-\lambda)^r} \bar{c} \right) \mid i=1,2,\ldots,d_j, j=1,\ldots,m \right\}$$

(5.50)
is a basis, where \( B = (\vec{c}_1, \ldots, \vec{c}_m) \) is a fully reduced basis of \( C \). Let \( \vec{c} \in C \) be arbitrary.

\[
\vec{c} = \sum_{j=1}^{m} a_j(z) \vec{c}_j, \quad \text{and} \quad \pi_{\sigma}(\vec{c}) = \sum_{j=1}^{m} \pi_{\sigma}(a_j(z) \vec{c}_j).
\] (5.51)

By linearity it suffices to show \( \pi_{\sigma}(a_j \vec{c}_j) \) can be spanned by \( \bar{B} \). We saw before that a representative can be found by focusing on the poles of \( a_j(z) \) (of order \( \leq d_j \)) that are in \( \sigma \). For \( \lambda_1, \ldots, \lambda_r \in \sigma \),

\[
\pi_{\sigma}(a_j \vec{c}_j) = \pi_{\sigma}(\sum_{j=1}^{r} f_{ij}(\lambda_j) \vec{c}_j) = \sum_{j=1}^{r} \pi_{\sigma}(f_{ij}(\lambda_j) \vec{c}_j) = \sum_{j=1}^{r} \pi_{\sigma}\left(\frac{b_{d_j}(\lambda_j)}{(z - \lambda_j)^{d_j}} \vec{c}_j\right)
\] (5.52)

But from lemma 1, each one of these terms above, \( b_j \pi_{\sigma}\left[\frac{1}{(z - \lambda_j)^{d_j}} \vec{c}\right] \), can be written as a linear combination over \( \mathbb{C} \) of elements from \( \mathcal{B} \). So \( \overline{a(z)\vec{c}} \) in \( \pi_{\sigma}(C) \) can be written in terms of \( \bar{B} \).

Note: A fully reduced basis is really a minimal basis. So, \( \frac{\pi_{\sigma}(C)}{\pi_{\sigma}C \cap \vec{c}} \) is a vector space over \( \mathbb{C} \) of dimension less than or equal to \( \sum_{i=1}^{m} (d_i - \delta \vec{c}_i) \). So

\[
\dim W_{\sigma} \leq \dim W_+.
\] (5.53)

Step 3. It remains to show \( \bar{B} \) is L.I. spanning set over \( \mathbb{C} \). This will show \( \dim W_{\sigma} = \dim W_+ \). We need again a lemma.

Aside: From lemma 3.1 of On Zeroes and Poles of a Transfer Function (see, [4, 5]) it was shown that, for a basis \( \{\mu_1, \ldots, \mu_r(z)\} \) of \( C \) consisting of polynomial vectors of
degrees $d_1 \leq \cdots \leq d_r$, respectively,

$$S = \left\{ \pi_-(z^{-i}\mu_j) \big| i = 1, 2, \ldots, d_j \right\}$$  \hspace{1cm} (5.54)

forms a spanning set for $W(C)$.

In the previous chapter, we showed that, if $\{\mu_1, \ldots, \mu_r\}$ is minimal, $S$ is actually a basis for $W(C)$ over $\mathbb{C}$. (i.e., $S$ is L.I. and $\dim W(C) = \sum_{i=1}^r d_i$).

We would now like to generalize this lemma only slightly. The proof is essentially identical.

Lemma 2: Given any fixed $\lambda \in \mathbb{C}$ and minimal polynomial basis $\{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ of $C$,

$$S = \left\{ \pi_-(z+\lambda)^{-i}\mathcal{C}_j \big| j = 1, \ldots, d_j \right\}$$  \hspace{1cm} (5.55)

is also a spanning set of $W_{\pm}(C)$, the classical Wedderburn space.

**Proof:** Let $\mathcal{C}(z) \in \mathcal{C}$. Then $\mathcal{C}(z) = \Sigma a_j \mathcal{C}_j$, $a_j \in k(z)$. Then $\pi_-(\mathcal{C}(z)) = \pi_[- \Sigma \pi_-(a_j) \mathcal{C}_j]$, because $\mathcal{C}_j$ is polynomial and therefore so is $\pi_+(a_j) \mathcal{C}_j$. Now, (instead of expanding $\pi_-\mathcal{C}_j$ into powers of $z^{-1}$ as in Lemma 3.1), expand $\pi_-\mathcal{C}_j$ into powers of $(z+\lambda)^{-1}$, say $\pi_-\mathcal{C}_j = \left( \sum_{i=1}^{d_j} b_i(z+\lambda)^{-i} \right) + \frac{a(z+\lambda)}{(z+\lambda)^{d_j+1}}$ with $a(z+\lambda)$ proper. Then $a(z+\lambda)(z+\lambda)^{-(d_j+1)} \mathcal{C}_j$ is strictly proper and so dies in $W_{\pm}(C)$. So, $S = \left\{ \pi_-[(z+\lambda)^{-i}\mathcal{C}_j] \right\}$ is a spanning set for $W_{\pm}(C)$. There are precisely $\sum_{i=1}^r d_i$ vectors in $S$ so if $B$ is minimal then $S$ is necessarily a basis for $W(C)$.

Knowledge of the classical Wedderburn space from the previous chapter says $S' = \left\{ \pi_+[((z+\lambda)^{-i}\mathcal{C}_j] \right\}$ is a basis for $\frac{\pi_+(\mathcal{C})}{\pi_+(\mathcal{C}) \cap \mathcal{C}}$. That is $\pi_+[(z+\lambda)^{-i}\mathcal{C}_j] \notin \mathcal{C}$ else $\pi_+[(z+\lambda)^{-i}\mathcal{C}_j] = 0$ and $S'$ would not be L.I. So we're almost done.

Recall $\bar{B} = \left\{ \pi_\sigma(1/(z-\lambda)\mathcal{C}_j) \big| i = 1, \ldots, d_j \right\}$. But we saw that if $\lambda \in \sigma$ then
\[ \pi_\sigma\left(\frac{1}{(z-\lambda)^i}c_j\right) = \pi_+(\frac{1}{(z-\lambda)^i}c_j). \]  

(5.56)

Now suppose there exists a dependence of terms in $\tilde{B}$. (Let's call the vector in $\tilde{B}$ $\tilde{w}_1, \ldots, \tilde{w}_m, (m = \sum_{i=1}^r \delta c_i)$. Then $\exists k_1, \ldots, k_m \in \mathbb{C}$ s.t.

\[ k_1 w_1 + \cdots + k_m w_m \in \pi_\sigma(C) \cap C. \]  

(5.57)

But this implies $k_1 w_1 + \cdots + k_m w_m \in C$, and this can't be. We had $w_i \in \pi_+(C)$ and we showed above these are L.I. in $\frac{\pi_+(C)}{\pi_+(C) \cap C}$ and so

\[ k_1 w_1 + \cdots + k_m w_m \in (\pi_+(C) \cap C) \Rightarrow k_i = 0 \forall i. \]  

(5.58)

(i.e., $k_1 w_1 + \cdots + k_m w_m \in \pi_+(C)$ and so if not all $k_i$'s are zero and since $\tilde{w}_1, \ldots, \tilde{w}_m$ are L.I. in $w_\pm(C)$ then

\[ k_1 w_1 + \cdots + k_m w_m \notin C. \]  

(5.59)

Therefore $(k_1 w_1 + \cdots + k_m w_m) \notin \pi_\sigma(C) \cap C$. So $\tilde{B}$ is a L.I. set over $\mathbb{C}$. That's it!

**Corollary I.** One can always find a basis for $W_\sigma(C)$ whose representatives will serve also as a basis for $W_\pm(C)$.

**Corollary II.** Given two open regions $\sigma, \tau \subseteq \mathbb{C}$ s.t. $\sigma \cap \tau \neq \emptyset$ and $\infty \notin \sigma, \tau$, we can find a basis with representatives that "look the same" for both $W_\sigma$ and $W_\tau$. 
CHAPTER VI

The Global Controllability Filtration

Consider the map $\beta$ from the main exact sequence (equation 3.37). In this chapter we prove $\mathcal{X}(G) \cong W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$ yields an isomorphism of filtered vector spaces in the special cases when $G$ is a matrix of strictly proper rational functions and when $G$ is a column proper matrix of polynomials. The ideas are then extended to define the global controllability filtration, which is viewed as a generalization and extension of the classical controllability filtrations defined thus far.

6.1 The Isomorphism

We wish to show that $\mathcal{X}(G) \cong W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$, as vector spaces. Recall the following definitions:

$$X(G) = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U} \text{ so that}$$

$$X\left(\begin{bmatrix} I \\ G \end{bmatrix}\right) = \frac{\Omega U}{\left[\begin{bmatrix} I \\ G \end{bmatrix}\right]^{-1}(\Omega U \oplus \Omega Y) \cap \Omega U})$$

In Lemma 3.4.5(v) we proved that $\mathcal{X}(G) \cong W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$ for an arbitrary transfer function $G$. We now consider two special cases.
CASE I Suppose $G$ is a $p \times m$ matrix of strictly proper rational functions from
the field $k(z)$. We place the reachability filtration given in definition 4.3.1 on $X(G)$.
(Note: Since $G$ is strictly proper $X(G) = X(G)$.) On the $W(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right])$ we place the
Wedderburn filtration given in definition 4.2.1 and equation 4.2. In this special case
we will show that $\beta$ is an isomorphism at each level of this filtrations. That is we
need to show

\[
\begin{align*}
H_{-1} &= (0) & \beta|_{H_{-1}} & \rightarrow & W_{-1} &= (0) \\
\mathcal{V} & \cup & \beta|_{H_0} & \rightarrow & W_0 &= \pi_+(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \cap \bar{k} \\
\mathcal{V} & \cup & \beta|_{H_1} & \rightarrow & W_1 &= \pi_+(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \cap (\bar{k} + z\bar{k}) \\
\mathcal{V} & \cup & \vdots & \vdots & \vdots & \vdots \\
\mathcal{V} & \cup & \beta|_{H_{k-1}} & \rightarrow & W_{k-1} &= \pi_+(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \\
\mathcal{V} & \cup & \beta & \rightarrow & W(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right])
\end{align*}
\]

is an isomorphism of filtered vector spaces. Recall that the conjugate partition of
the reachability filtration gave the controllability indices and that of the Wedderburn
filtration gave the Wedderburn indices of $W(\mathrm{im} \left[ \begin{array}{c} I \\ G \end{array} \right])$. It remains to show $\beta$ preserves
the above filtration.
Theorem 6.1.1 $X(G)$ and $W(\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right])$ are isomorphic as filtered vector spaces.

That is, $\beta(H_i) = W_i$ at each level of the filtration, and $\beta$ is monic.

Proof: We already know $\beta$ is a monomorphism. We only need to show that at each level the image of $\beta(H_i)$ is precisely $W_i$. We do this in two steps.

(I) Show $\beta$ maps $H_i$ into $W_i$. If $x \in \{B|AB| \ldots |A^iB]\} \overset{\text{def}}{=} H_i$ then $\exists \overline{u}[z] \in \Omega U$ such that $\overline{x} = \overline{B}(z^i\overline{u}_i + \ldots + \overline{u}_0)$. (i.e., $\overline{x} \in X(G)$, where $\overline{B}$ is the map from the commutative diagram introduced on page 17, namely,

\[ \Omega U \xrightarrow{G_{\pi(z)}} \Gamma(Y) \]

and $\overline{C} = \pi_- [G\overline{u}[z]]$ as viewed in $\Gamma(Y)$. Now since $G$ is strictly proper, then

\[ \pi_+ [G(\overline{u}[z])] \in \{z^{i-1}\overline{y}_{i-1} + \ldots + \overline{y}_0\} \quad (6.3) \]

and so the polynomial part of this image is of degree $(i - 1)$ or less. Therefore

\[ \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] \overline{u}[z] \right) = \frac{z^i\overline{u}_i + \ldots + \overline{u}_0}{z^{i-1}\overline{y}_{i-1} + \ldots + \overline{y}_0} \quad (6.4) \]

\[ \in \left( \pi_+ (\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \cap \overline{k} + z\overline{k} + \ldots + z^i\overline{k} \right) \quad (6.5) \]

So that

\[ \beta(\overline{x}) = \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] (\overline{u}[z]) \right) \in W_i. \quad (6.6) \]

Therefore $\beta$ maps $H_i$ into $W_i$. 
(II) Show surjectivity at each level. If \( \bar{w} \in W_i \), then there exists \( \bar{x} \in X(G) \) such that \( \beta(\bar{x}) = \bar{w} \). It is important to realize that if \( w \) is any representative for \( \bar{w} \) then \( w = \pi_+(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}([z])) \) for some \( \bar{u}([z]) \in \Omega U \) since \( G \) is strictly proper. We may assume we have chosen \( w \) a representative for \( \bar{w} \in W_i \) such that \( w = \bar{k}_0 + \ldots + z^i\bar{k}_i \). Then we choose an \( \bar{x} \in X(G) \) such that \( \pi_+(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}([z])) = w \) and \( \bar{B}\bar{u}([z]) = \bar{x} \). Because of the presence of the identity matrix and because the vector \( w \) only has powers of \( z \) up to \( z^i \), we have

\[
\bar{u}([z]) = z^i\bar{u}_i + z^{i-1}\bar{u}_{i-1} + \ldots + z\bar{u}_1 + \bar{u}_0. \tag{6.7}
\]

Then

\[
\bar{x} = \bar{B}(\bar{u}([z])) \in \langle |B|AB|\ldots|A^iB| \rangle = H_i. \tag{6.8}
\]

So \( \bar{x} \in H_i \) and in general

\[
\beta(H_i) = W_i, \ i = 0, 1, \ldots, k - 1. \tag{6.9}
\]

So \( \beta \) is an isomorphism of filtered vector spaces. \( \square \)

In this special case when \( G \) is strictly proper \( \beta \) was shown to be an isomorphism of filtered vector spaces. Therefore, the pullback of the Wedderburn filtration via \( \beta \) is the classical reachability filtration and the controllability indices are exactly the Wedderburn indices of \( W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \), the column indices of a RCPMFD, \( \{D,N\} \), for \( G \).

As an aside, it is interesting to note, that if the \( \ker G = (0) \), then

\[
\frac{X(G)}{\alpha[Z(G)]} \overset{\beta}{\cong} W(\text{im}G) \tag{6.10}
\]
by the main exact sequence. Then also

\[ |X(G)| - |\mathcal{Z}(G)| = |W(\text{im}G)|. \]  

(6.11)

But by "increasing" \( W(\text{im}G) \) to the \( W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \), we have

\[ X(G) \cong W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}). \]  

(6.12)

That is

\[ |W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})| - |W(\text{im}G)| = |\mathcal{Z}(G)| = \text{(the number of global zeros of } G). \]  

(6.13)

Exchanging the \( W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \) for \( W(\text{im}G) \) allows us to ignore for now the zeros of \( G \) by increasing the dimension of the Wedderburn space by exactly the number of global zeros of \( G \). This suggests possible future study of filtrations involving \( W(\text{im}G) \) and \( X(G) \), if we also consider \( \mathcal{Z}(G) \).

**CASE II** Suppose \( G \) is a \( p \times m \) column proper matrix with polynomial entries from \( k[z] \). We will place the kernel filtration on \( X_\infty(G) \). (Note: since \( G \) is polynomial \( X_\infty(G) = \mathcal{X}(G) \).) We again place the Wedderburn filtration on \( W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \). Our goal is again to show that in this special case, where \( G \) is column proper and polynomial, \( \beta \) is an isomorphism of these filtered vector spaces. We need to show
Recall on page 70 we showed that the conjugate partition of the kernel filtration on \( X_\infty(G) \), definition (4.4.1), gave the exponents \( \{e_i\}_{i=1}^s \) for the prime \( \frac{1}{2} \) in the invariant factor decomposition of \( X_\infty(G) \) over the local ring \( \mathcal{O}_\infty \). If \( \beta \) proves to be an isomorphism, as indicated above, we will have that the Wedderburn indices of \( W(\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \) are exactly this set \( \{e_i\} \).

**Theorem 6.1.2** \( X_\infty(G) \) is isomorphic to \( W(\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right]) \) as filtered vector spaces. That is \( \beta(K_{i+1}) = W_i \) at each level of the filtration, and \( \beta \) is monic.

**Proof:** (I) Injectivity at each level.

Suppose \( u(z) \in K_i \), then \( u(z) \in z^{-1} \Omega_\infty U \) and \( u(z) \in \ker z^{-i} \). But \( u \notin \ker z^{-i} \) implies
\( z^{-1}\overline{u} \equiv \delta in X_\infty(G) \). Or equivalently,

\[ z^{-1}\overline{u}(z) \in \left( G^{-1}(z^{-1}\Omega_\infty Y) \cap z^{-1}\Omega_\infty U \right). \tag{6.14} \]

Then

\[ z^{-1}\overline{u}(z) \in G^{-1}(z^{-1}\Omega_\infty Y) \Rightarrow G(z^{-1}\overline{u}(z)) \in z^{-1}\Omega_\infty Y \tag{6.15} \]

\[ \Rightarrow z^{-i}G(\overline{u}(z)) \in z^{-1}\Omega_\infty Y \tag{6.16} \]

\[ \Rightarrow G(\overline{u}(z)) \in z^{i-1}\Omega_\infty Y. \tag{6.17} \]

Then

\[ \pi_+\left( \begin{bmatrix} I \\ G \end{bmatrix} \overline{u}(z) \right) = \pi_+\left( \begin{bmatrix} \overline{u}(z) \\ G\overline{u}(z) \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{y}_0 + \ldots + z^{i-1}\bar{y}_{i-1} \end{bmatrix} \right). \tag{6.18} \]

This implies

\[ \beta(\overline{u}(z)) = \pi_+\left( \begin{bmatrix} I \\ G \end{bmatrix} \overline{u}(z) \right) \in W_{i-1}(\text{im} \left[ \begin{bmatrix} I \\ G \end{bmatrix} \right]). \tag{6.19} \]

Therefore \( \beta : H_i \hookrightarrow W_{i-1}. \tag{6.20} \)

(II) Surjectivity at each level.

This argument is a little more subtle. It requires more care because we have to deal with images up to congruence in \( W(\text{im} \left[ \begin{bmatrix} I \\ G \end{bmatrix} \right]). \)

Define \( v_i \) to be the column degree of \( \tilde{g}_i \); the \( i \)th column of \( G \). According to Theorem (4.2.4) and Definition (4.2.1), because \( G \) is polynomial and column proper, we can
form a basis $S$ for $W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$ over the base field $k$ of the form:

$$\left\{ \begin{array}{c}
\pi_+(z^{-1} \bar{g}_1) \\
0 \\
\pi_+(z^{-1} \bar{g}_2) \\
\vdots \\
\pi_+(z^{-1} \bar{g}_m)
\end{array}, \begin{array}{c}
\pi_+(z^{-2} \bar{g}_1) \\
0 \\
\pi_+(z^{-2} \bar{g}_2) \\
\vdots \\
\pi_+(z^{-2} \bar{g}_m)
\end{array}, \ldots, \begin{array}{c}
\pi_+(z^{-\nu_1} \bar{g}_1) \\
0 \\
\pi_+(z^{-\nu_1} \bar{g}_2) \\
\vdots \\
\pi_+(z^{-\nu_1} \bar{g}_m)
\end{array}, \begin{array}{c}
\pi_+(z^{-\nu_2} \bar{g}_1) \\
0 \\
\pi_+(z^{-\nu_2} \bar{g}_2) \\
\vdots \\
\pi_+(z^{-\nu_2} \bar{g}_m)
\end{array}, \ldots, \begin{array}{c}
\pi_+(z^{-\nu_m} \bar{g}_1) \\
0 \\
\pi_+(z^{-\nu_m} \bar{g}_2) \\
\vdots \\
\pi_+(z^{-\nu_m} \bar{g}_m)
\end{array} \right\}.$$ 

So if $\bar{w} \in W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$, then $\bar{w}$ has a representative of the form

$$\begin{bmatrix}
0 \\
\vdots \\
0 \\
y_1[z] \\
y_2[z] \\
\vdots \\
y_p[z]
\end{bmatrix}$$

obtained as a linear combination over $k$ of the vectors in $S$.

Let $\bar{w} \in W_i$. Then by Theorem (4.2.4), we can choose a representative $w$ of $\bar{w}$ of the form above.

{ Note: It is precisely here that we need $G$ to be column proper. If $G$ were not column proper, there could be sufficient cancellation among high-order terms of columns of $G$ so that the representative of $\bar{w}$ is due to an element $u(z) \in U(z)$ giving $w =$
\(\pi_+\left(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}(z)\right) \in W_i\) but not of the form having zeros in the first \(m\) positions of the vector. However if \(G\) is column proper, we may apply theorem (4.2.4) and assume that \(\bar{u}(z) \in z^{-1}\Omega_{\infty}U\). This follows from ideas similar to the Lemma of Predictable Degree. That is, \(G\) being column proper allows us to choose the representative of \(\bar{w}\) from the set \(S\) of the form
\[
\begin{bmatrix}
0 \\
\bar{y}_0 + \ldots + z^i\bar{y}_i
\end{bmatrix}
\]
in which \(\deg w \leq i\) and the first \(m\) entries of \(w\) are zeros. The zeros will be needed for our proof.}

Continuing, since \(\beta\) is an isomorphism from \(X_\infty(G)\) to \(W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})\), then there exists \(\bar{u}(z) \in z^{-1}\Omega_{\infty}U\) such that \(\pi_+\left(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}(z)\right) \equiv w\).

If we show that \(\pi_+\left(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}(z)\right) = w\), not just congruent to \(w\), we will have both that \(\bar{u}(z) \in z^{-1}\Omega_{\infty}Y\) and that
\[
\pi_+(G\bar{u}(z)) = \bar{y}_0 + \ldots + z^i\bar{y}_i = \bar{y}[z] \in z^i\Omega_{\infty}Y = z^{i+1}z^{-1}\Omega_{\infty}Y. \quad (6.21)
\]

Therefore,
\[
z^{-(i+1)}\pi_+(G\bar{u}) \in z^{-1}\Omega_{\infty}Y \Rightarrow \quad (6.22)
\]
\[
G(z^{-(i+1)}\bar{u}) \in z^{-1}\Omega_{\infty}Y \Rightarrow \bar{u}(z) \in \ker z^{-(i+1)} = H_{i+1} \quad (6.23)
\]
and we will be done. So it remains to show we have more than mere congruence.

So suppose for the above \(u(z)\) that
\[
\pi_+\left(\begin{bmatrix} I \\ G \end{bmatrix} \bar{u}(z)\right) = w' \text{ and } w' \equiv w \quad (6.24)
\]
Then by definition, \((w' - w) \equiv 0\) in \(W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})\) implies
\[
\tilde{d} \overset{\text{def}}{=} (w' - w) \in \left( \text{im} \begin{bmatrix} I \\ G \end{bmatrix} \cap \pi_+(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \right).
\] (6.25)

Since \(\tilde{u} \in z^{-1}\Omega_\infty U\), we have
\[
\tilde{d} = w' - w = \pi_+ \begin{bmatrix} I \\ G \end{bmatrix} \tilde{u} - \begin{bmatrix} 0 \\ \tilde{y}[z] \end{bmatrix} = \begin{bmatrix} 0 \\ \pi_+(G\tilde{u}) \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{y}[z] \end{bmatrix}.
\] (6.26)

Since \(\tilde{d}\) is \(\bar{0}\) in the Wedderburn space \(\tilde{d}\) is also in the column span of \(\begin{bmatrix} I \\ G \end{bmatrix}\) so \(\exists \tilde{u}_2(z) \in z^{-1}\Omega_\infty U\) such that \(\tilde{d} = \begin{bmatrix} I \\ G \end{bmatrix} \tilde{u}_2(z)\). Because of the identity matrix and the fact the \(I\tilde{u}_2(z) = 0\), this alone implies that \(\tilde{d}\) is the zero vector and we are done. But this is perhaps too quick so we continue. Viewing \(\tilde{u}_2(z)\) in \(X_\infty(G)\), we have
\[
\beta(\tilde{u}_2) = \tilde{d} = \bar{0} \in W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}).
\] (6.27)

Since \(\beta\) is injective \(\tilde{u}_2(z)\) must be congruent to zero in \(X_\infty(G)\). This implies \(G(\tilde{u}_2) \in z^{-1}\Omega_\infty Y\) so that \(\pi_+ \begin{bmatrix} I \\ G \end{bmatrix} \tilde{u}_2 = \bar{0}\). But then
\[
\tilde{d} = \pi_+ \begin{bmatrix} I \\ G \end{bmatrix} \tilde{u}_2 = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}.
\] (6.28)

So \(\tilde{d}\) is exactly the zero vector and \(w' = w\). □

We have seen that, in the special cases when \(G\) is strictly proper or polynomial and column proper, the Wedderburn space \(W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})\) is isomorphic to \(X(G)\) or \(X_\infty(G)\) as filtered vector spaces. Therefore, the conjugate partition for the Wedderburn filtration yields the same integers as those of the reachability or kernel filtrations in each respective case. These results motivate a new definition.
Definition 6.1.3 Given a matrix $G$ with coefficients from the field $k(z)$, place the Wedderburn filtration on $W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$ and define

$$H_i \overset{\text{def}}{=} \beta^{-1}(W_{i-1}). \quad (6.29)$$

This chain of subspaces in $X(G)$ is called the Generalized Global Controllability Filtration (GGCF) of $G$.

The GGCF is defined exactly so that

- $H_0 = (0) \overset{\beta^{-1}}{\leftarrow} W_0 = \pi_+(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \cap \bar{k}$
- $W_1 = \pi_+(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \cap (k + z\bar{k})$
- $W_{k-1} = \pi_+(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$
- $W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$

is an isomorphism of filtered vector spaces that generalizes our two previous specific cases. Support for this definition is given by the following facts.
In the classical case (Case I) $G$ is strictly proper and $X_\infty(G) = 0$. Lemma (3.4.5)(v) shows $X(G) \cong W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix})$. Then $\{H_i\}_{i=1}^k$ is just the \textit{classical controllability filtration} (CCF). That is $H_i = ([B, AB, \ldots, A^{i-1}B])_k \cong W_{i-1}$, the conjugate partition of the WF gives the classical controllability indices, and the GGCF is just the CCF.

See section 4.3 and observe that the number of $X$'s in the $i^{th}$ row of the Kailath crate construction equals $\dim(\frac{H_i}{H_{i-1}}) = \dim(\frac{W_{i-1}}{W_{i-2}})$. See also [6, 7]. When $G$ is polynomial (Case II) $X(G) = 0$ and so $W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix}) \cong X_\infty(G) \cong [JB, \ldots, J^{i-1}B]$. In this case it was shown that $H_i = (\text{span of linearly independent eigenvectors of } J \text{ of length } \leq i) = \ker J^i \cong W_{i-1}$. Again the spaces match and the conjugate partitions of the WF and GCF gave the controllability indices, see [1]. The corresponding indices and spaces agree with and extend or generalize the notions of controllability subspaces from classical control theory. In general, if $G = ND^{-1}$ is a right coprime factorization and $L = \begin{bmatrix} D \\ N \end{bmatrix}$ (i.e., $L$ is a minimal column proper basis for its column span) then the global controllability indices are exactly the Wedderburn indices of $W(\text{im}L)$, which are the column degrees of $L$. This was shown by the proof of Lemma 3.4.5 and the the fact that GGCF and Wedderburn space are isomorphic as filtered vector spaces. These facts for the controllability case closely parallel the controllability work of Malabre [9, 10, 7, 20, 21], which we are about to introduce in the next section. The philosophy employed here, using the Fundamental Pole-Zero Exact Sequence, appeared in [4] for the strictly proper case. The paper [22] contains an outlined conjecture of the global observability case which turned out to use the wrong filtration. The above methods will allow us to tackle the problem of Global Observability, correcting that error by
using duality theory (to be introduced in the next Chapter) led to the observability results in this work.

We have one last important result before we introduce the next section. At this point the reader should review the results of Lemma (3.4.5), Definition (4.2.2), and Theorem (4.2.4). We have defined $B_t$ in equation (4.3). Let $G$ be a $p \times m$ transfer function. The (GGCF) is the increasing chain $\{H_i\}_{i=1}^n$ defined on $\mathcal{X}(G)$ via the pullback of $\beta$ in Lemma 3.4.5(vi) (i.e., $H_i := \beta^{-1}(W_{i-1})$). From equation (4.3) and the definition of $\beta$ as the map

$$\mathcal{X}(G) \xrightarrow{\beta} W(\text{im} \begin{bmatrix} I \\ G \end{bmatrix})$$

(6.30)

given by

$$\beta(\bar{u}[z], \bar{v}_{sp}(z)) \overset{\text{def}}{=} \pi_+ \left( \begin{bmatrix} I \\ G \end{bmatrix} (\bar{u}[z] + \bar{v}_{sp}(z)) \right) \in W(\text{im} \begin{bmatrix} D \\ N \end{bmatrix}),$$

(6.31)

more concretely we have the following theorem.

**Theorem 6.1.4** A basis for the spaces in the GGCF, $\{H_i\}$, over $k$ is given by

$$\{(\pi_+(D\bar{1}_jz^{-v_j}), \pi_-((\bar{1}_jz^{-v_j}))|j = 1, \ldots, m; 1 \leq v_j = e_j, \ldots, e_j - i + 1\},$$

(6.32)

where $\bar{1}_j$ is the standard basis vector with 1 in the $j^{th}$ position and zeros elsewhere.

**Proof:** Suppose $G = ND^{-1}$ is a RCPMFD for $G$ and $\bar{d}_1$ is the first column of the matrix $\begin{bmatrix} D \\ N \end{bmatrix}$. Consider $W_{i_0-1}$ so that $i = i_0$. This is the subspace of all vectors in the Wedderburn space with representatives of degree less than or equal to $i_0 - 1$. 
From Theorem (4.2.4), \( W_{i_0} = (B_{i_0})_k \). For simplicity and without loss of generality we will just consider the first column vector of \( D \) (i.e., let \( j = 1 \)). Then from the definition of set (6.32) \( v_1 = e_1, \ldots, (e_1 - i_0 + 1) \) and for each value of \( v_1 \) we have

\[
\beta(\pi_+(D\bar{1}z^{-v_1}), \pi_+(D\bar{1}z^{-v_1})) = \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] (\pi_+(D\bar{1}z^{-v_1}) + \pi_+(D\bar{1}z^{-v_1})) \right) \\
= \pi_+ \left[ \left[ \begin{array}{c} I \\ G \end{array} \right] D\bar{1}z^{-v_1} \right] \\
= \pi_+ \left[ \left[ \begin{array}{c} I \\ ND^{-1} \end{array} \right] D\bar{1}z^{-v_1} \right] \\
= \pi_+ \left[ \left[ \begin{array}{c} D \\ N \end{array} \right] \bar{1}z^{-v_1} \right] \\
= \pi_+(\bar{d}_1z^{-v_1}).
\]

As \( v_1 \) ranges over the values \( e_1, \ldots, e_1 - i_0 + 1 \) we get the set

\[ \{ \pi_+(\bar{d}_1z^{-e_1}), \ldots, \pi_+(\bar{d}_1z^{-e_1+i_0-1}) \} \]

According to equation (4.3) these are precisely the vectors in \( B_{i_0-1} \) contributed by the first column of \( \left[ \begin{array}{c} D \\ N \end{array} \right] \). As \( j \) goes from 1 to \( m \), we get precisely the entire set \( B_{i_0-1} \), which according to Theorem (4.2.4) is a basis for \( W_{i_0-1} \). Since \( \beta \) is an isomorphism at each level of the filtration, we have that (6.32) is a basis for \( H_{i_0} \). \( \square \)

Recall that in the case when \( G \) is strictly proper (polynomial), as in the earlier specific cases of this chapter, the inverse image of the Wedderburn filtration gave the classical control filtration (kernel filtration). In each of these special cases then we have as a corollary that the set (6.32) gives a basis for each level of the controllability filtration in terms of the coset representatives of the module structure of \( X(G) \) and \( X_\infty(G) \), respectively.
6.2 The Malabre Filtration and the GGCF

This section offers final support of our module definition of the GGCF. We will prove that a certain image of the Malabre Controllability Filtration, which is defined on $X'(G)$ in terms of the realization matrices $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$, is isomorphic to the GGCF as filtered vector spaces. Our definition of the GGCF is only a function of $G$ and the module structure of any minimal realization pole space. The GGCF is therefore an invariant of the vector space structure of minimal realizations for the dynamical structure of $G$.

Refer to equations (2.23, 2.24, 2.26, and 2.26). In [8], Malabre defined the following filtration, which fills up $X'$:

\begin{equation}
R^0 = X' \cap \ker \hat{E} \tag{6.37}
\end{equation}

\begin{equation}
R^{u+1} = X' \cap \hat{E}^{-1}(\hat{B} + \hat{A}R^u). \tag{6.38}
\end{equation}

If we view $\hat{E}$ as in equation (2.26), we see from equations 3.20 through 3.23 that

$$X'(G) \oplus X(G) \xrightarrow{\hat{E}} JX'_\infty(G) \oplus X(G) =$$

$$z^{-1}X'_\infty(G) \oplus X(G) = X_\infty(G) \oplus X(G) = \text{im} \hat{E}. \tag{6.39}$$

Then $R^u$ is equal to

$$\hat{E}^{-1}[\hat{B} + \hat{A}R^{u-1}] \xrightarrow{\hat{E}} ([\hat{B} + \hat{A}R^{u-1}] \cap \text{im} \hat{E}) \overset{\text{def}}{=} \hat{E}R^u \subseteq X_\infty(G) \oplus X(G) \tag{6.40}$$

And so the image under $\hat{E}$ of the Malabre filtration induces an increasing filtration $\hat{E}R^u$ that fills up the global pole space $X(G)$. 
Theorem 6.2.1 $\hat{E}R^\mu = H_\mu \cong W_{\mu-1}$, $\mu = 1, 2, \ldots, e_1$.

Proof: (Induction) We show $\hat{E}R^\mu = H_\mu$. Note the action of $\hat{A}$ is represented by the module operator $(z, 1)$ and the action of $\hat{E}$ is represented by the module operator $(1, z^{-1})$ on the module structure of $X(G) \oplus X_\infty(G)$. From the set (6.32)

$$H_1 \overset{\text{def}}{=} \left\{ \{(\pi_+(D\bar{I}_jz^{-e_j}), \pi_-(D\bar{I}_jz^{-e_j}))\}_k, \ j = 1, 2, \ldots, m. \right\}$$

and

$$\hat{E}R^1 = \ker \hat{E} + \hat{B}.$$ (6.42)

It suffices to check our result on basis vectors; without loss of generality let $j = 1$ and

$$x = d_1z^{-e_1} = u_{e_1} + u_{e_1-1}z^{-1} + \ldots + u_0z^{-e_1} \in H_1.$$ (6.43)

From equation (6.39), $d_1z^{-e_1} \in X(G)$ implies $d_1z^{-e_1} \in \text{im} \hat{E}$. Rewrite $x$ as

$$(\bar{u}, \bar{v}) \oplus (\bar{0}, \bar{v}(z)) \overset{\text{def}}{=} (\bar{u}_{e_1}, -\bar{u}_{e_1}) \oplus (\bar{0}, \bar{u}_{e_1} + \bar{u}_{e_1-1}z^{-1} + \ldots + \bar{u}_0z^{-e_1}).$$ (6.44)

The vectors $\bar{u}_{e_1}$ and $-\bar{u}_{e_1}$ have state space representatives given, respectively, as $B_0\bar{u}_{e_1} \in X(G)$ and $-B_\infty(-\bar{u}_{e_1}) \in X'_\infty(G)$, see [1]. Then in $X(G)$, $(\bar{u}_{e_1}, -\bar{u}_{e_1})$ has state space representative $\hat{B}\bar{u}_{e_1} \in \text{im} \hat{B}$.

Also

$$\hat{E}(\bar{0}, \bar{v}(z)) = (1, z^{-1})(\bar{0}, \bar{u}_{e_1} + \bar{u}_{e_1-1}z^{-1} + \ldots + \bar{u}_0z^{-e_1}) = (\bar{0}, z^{-1}\bar{u}_{e_1} + \ldots + \bar{u}_0z^{-(e_1+1)}) =$$

$$(\bar{0}, d_1z^{-(e_1+1)}) \overset{\beta}{\pi_+} \left[ \begin{array}{c} D \\ N \end{array} \right] D^{-1}d_1z^{-(e_1+1)} = \pi_+ \left[ \begin{array}{c} D \\ N \end{array} \right] \bar{I}_1z^{-(e_1+1)} \cong \bar{0}$$ (6.45)

in $W(\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right])$, since $d_1z^{-(e_1+1)}$ is s.p.
Because $\beta$ is an isomorphism, $(0, \bar{\nu}(z)) \in \ker \hat{E}$. Together, these results imply $H_1 \subseteq \hat{E}R^1$. Proof of the reverse inclusion follows similar methods.

Let $\bar{x} \in \hat{E}R^1$. Then

$$\bar{x} = \bar{x}_1 + \bar{x}_2$$

with

$$\bar{x}_1 \in \ker \hat{E} \text{ and } \bar{x}_2 \in \hat{E}.$$  

We choose module representatives of $\bar{x}_1$ and $\bar{x}_2$ as follows:

For $\bar{x}_1$

$$\bar{x}_1 = (\bar{\nu}[z], \bar{\nu}(z)) \in X(G) \oplus X'(G).$$  

So $\bar{\nu}[z] \in \Omega U$ and $\bar{\nu}(z) \in \Omega_\infty U$, subject to the constraint

$$(1, z^{-1})(\bar{\nu}[z], \bar{\nu}(z)) = \bar{0} \in X'(G)$$  

This implies that

$$G_{sp}(\bar{\nu}[z]) \overset{def}{=} \bar{\nu}_1[z] \in \Omega Y.$$  

So $\bar{\nu}[z] \equiv \bar{0}$ in $X(G)$. Clearly,

$$G_{poly}(\bar{\nu}[z]) \overset{def}{=} \bar{\nu}_2[z] \in \Omega Y.$$  

Also, by equation (6.49)

$$G_{poly}(z^{-1}v(z)) \in z^{-1}\Omega_\infty Y$$  

implies

$$(G_{poly}(v(z)) \in \Omega_\infty Y)$$  

So $G_{poly}(v(z))$ is of the form $\bar{\nu}_0 + z^{-1}\bar{\nu}_1 + \ldots$. 
For $\vec{x}_2$

$$\vec{x}_2 \in \hat{B} \quad (6.54)$$

implies that $\vec{x}_2$ viewed in $X(G)$ is represented as

$$\vec{x}_2 = (u^\tau, -u^\tau) \quad (6.55)$$

for some $u^\tau$ in $\Omega U$. Then

$$\beta(\vec{x}) \overset{def}{=} \beta \left( (u^\tau, -u^\tau) \oplus (\bar{u}[z], \bar{v}(z)) \right) \quad (6.56)$$

$$= \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] (u^\tau + -u^\tau + \bar{u}[z] + \bar{v}(z)) \right) \quad (6.57)$$

$$= \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] (\bar{u}[z] + \bar{v}(z)) \right) \quad (6.58)$$

$$= \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] u[z] \right) + \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] v(z) \right) \quad (6.59)$$

$$= \mathcal{I} + \mathcal{II}. \quad (6.60)$$

$$\mathcal{I} = \pi_+ \left( \left( \begin{array}{c} \bar{u}[z] \\ y_1[z] + y_2[z] \end{array} \right) \right) = \left( y_1[z] + y_2[z] \right) \left[ \begin{array}{c} I \\ G \end{array} \right] \bar{u}[z] \in \text{im} \left[ \begin{array}{c} I \\ G \end{array} \right]. \quad (6.61)$$

So $\mathcal{I}$ is zero in $W(\text{im} \left[ \begin{array}{c} I \\ G \end{array} \right])$.

$$\mathcal{II} = \pi_+ \left( \left[ \begin{array}{c} I \\ G \end{array} \right] \bar{v}(z) \right) = \left( \begin{array}{c} u^0_0 \\ y_0^0 \end{array} \right) \in W_0. \quad (6.62)$$

So $\beta(\vec{x}) \in W_0$. This implies $\vec{x} \in \beta^{-1}(W_0) = H_1$. So $\vec{x} \in H_1$. So $H_1 = \hat{E}R^1$ and $\dim \hat{E}R^1 = \dim H_1$.

(Induction step). Suppose

$$\vec{d}_1 z^{-e_1+1} \in W_{i-1}. \quad (6.63)$$
That is
\[ \beta^{-1}(\dd_1 z^{-\epsilon_1+i-1}) \in \hat{\mathcal{E}} R^i = (\hat{\mathcal{A}} R^{i-1} + \hat{\mathcal{B}}) \cap \hat{\mathcal{E}}. \] (6.64)

We wish to show
\[ \beta^{-1}(\dd_1 z^{-\epsilon_1+i}) \in \hat{\mathcal{E}} R^{i+1} = (\hat{\mathcal{A}} R^i + \hat{\mathcal{B}}) \cap \hat{\mathcal{E}}. \] (6.65)

(Again we have from equation (6.39) that \( \beta^{-1}(\dd_1 z^{-\epsilon_1+i}) \in \text{im}\hat{\mathcal{E}}. \)) Now
\[ \dd_1 z^{-\epsilon_1+i} = \bar{u}_{e_1} z^i + \bar{u}_{e_1-1} z^{i-1} + \ldots + \bar{u}_{e_1-i} z^0 + \bar{u}_{e_1-i-1} z^{-1} + \ldots + \bar{u}_0 z^{-\epsilon_1+i}, \] (6.66)
which viewed in \( \mathcal{X}(G) \) is written as
\[ \beta^{-1}(\dd_1 z^{-\epsilon_1+i}) = \] (6.67)
\[ (\bar{u}_{e_1-i}, -\bar{u}_{e_1-i}) + (z, 1)[(\bar{u}_{e_1} z^{i-1} + \ldots + \bar{u}_{e_1-i+1}, \bar{0}) \oplus (\bar{0}, \bar{u}_{e_1-i} + \ldots + \bar{u}_0 z^{-\epsilon_1+i})]. \]

This is of the form \( \text{im}\hat{\mathcal{B}} + \hat{\mathcal{A}}(*) \). It remains to show \( * \) is in \( R^i \). But
\[ (1, z^{-1}*)(*) = \] (6.68)
\[ (\bar{u}_{e_1} z^{i-1} + \ldots + \bar{u}_{e_1-i+1}, \bar{0}) \oplus (1, z^{-1})(\bar{0}, \bar{u}_{e_1-i} + \ldots + \bar{u}_0 z^{-\epsilon_1+i}) = \]
\[ (\bar{u}_{e_1} z^{i-1} + \ldots + \bar{u}_{e_1-i+1}, \bar{0}) \oplus (\bar{0}, \bar{u}_{e_1-i} z^{-1} + \bar{u}_{e_1-i-1} z^{-2} + \ldots + \bar{u}_0 z^{-\epsilon_1+i-1}) \Rightarrow \]
and \( \beta^{-1}(\dd_1 z^{-\epsilon_1+i-1}) \) is in \( \hat{\mathcal{E}} R^i \) by assumption. So
\[ \hat{\mathcal{E}}(*) \in (\hat{\mathcal{A}} R^{i-1} + \hat{\mathcal{B}}) \cap \hat{\mathcal{E}} \Rightarrow \] (6.69)
\[ (*) \in \hat{E}^{-1}([\hat{\mathcal{A}} R^{i-1} + \hat{\mathcal{B}}] \cap \hat{\mathcal{E}}) \subseteq \hat{E}^{-1}([\hat{\mathcal{A}} R^{i-1} + \hat{\mathcal{B}}] = R^i. \]

\[ \square \]

The image of the Malabre Controllability filtration under \( \hat{\mathcal{E}} \) agrees with the GGCF, which is isomorphic as filtered vector spaces to the Wedderburn filtration via \( \beta \).
CHAPTER VII

Duality and The Global Observability Filtration

7.1 Introduction

The goal of this chapter is to define the Generalized Global Observability Filtration (GGOF) of a of a transfer function $G$. We construct an isomorphism of filtered vector spaces between $X_\infty([D|N])$ and $W(\ker[D|N])$, where $[D|N]$ is a left coprime polynomial matrix fraction decomposition of $G$. The filtration placed on $W(\ker[D|N])$ will be new and is given by a filtration dual to the filtration on $W(\im \begin{bmatrix} D^T \\ N^T \end{bmatrix})$ discussed earlier. The conjugate partition associated with this filtration yields a new set of integers called the global observability indices.

Given a transfer function $G$ of rational functions we have

$$X([G]) \cong X([I|G]) \cong X_\infty([D|N]) \cong W(\ker[D|N]).$$  \hspace{1cm} (7.1)

This filtration on $W(\ker[D|N])$ induces a filtration on the global pole space $X(G)$. This filtration will be called the global observability filtration.

We will proceed as follows:
(A) We start with a lemma showing that $W(\text{im}C)$ and $W(\text{im}C^\perp)$ are naturally dual by constructing a non-singular pairing.

(B) We construct an isomorphism between $W(\ker[D|N])$ and $X_\infty([D|N])$.

(C) We construct a filtration on $W(\ker[D|N])$ induced by the previously constructed polynomial Wedderburn filtration of $W(\text{im} \left[ \begin{array}{c} D^T \\ N^T \end{array} \right])$.

(D) We show $W(\ker[D|N]) \cong X_\infty([D|N])$ as filtered vector spaces. We place the dual filtration on $W(\ker[D|N])$ and the image filtration on $X_\infty([D|N])$.

(E) We show the conjugate partition associated with the dimensions of the chain of quotient spaces of this filtration yields the observability indices.

7.2 (A) Duality

Given any matrix $[G]$ of rational functions we have

\begin{align*}
\mathcal{X}'([G]) & \cong \mathcal{X}'([I|G]) \cong X_\infty([D|N]) \cong W(\ker[D|N]).
\end{align*}

**Proof:** Note the existence of (i), (ii), and (iii) were actually proven in Isomorphisms 2 and 3 on page 34, but we now need to construct the map for step (i). We will need it for the duality.

From equation (3.1) we have

\begin{equation}
X(G) = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}
\end{equation} (7.2)
and
\[ X([I|G]) = \frac{(\Omega Y \oplus \Omega U)}{[I|G]^{-1}(\Omega Y) \cap (\Omega Y \oplus \Omega U)}. \] (7.3)

Let \( \bar{u}[z] \in X(G) \) be arbitrary. We claim the map
\[ \phi : X(G) \leftrightarrow X([I|G]) \]
defined as
\[ \bar{u}[z] \mapsto \begin{pmatrix} \bar{o} \\ \bar{u}[z] \end{pmatrix} \] (7.4)
is the desired isomorphism.

**Well-Defined**: Suppose \( \bar{u}[z] \equiv \bar{o} \). Then by definition \( G\bar{u}[z] \in \Omega Y \), and so
\[ [I|G]\begin{pmatrix} \bar{o} \\ \bar{u}[z] \end{pmatrix} = G\bar{u}[z] \in \Omega Y. \] (7.5)
So \( \phi \) maps zero to zero and is well-defined.

**Injective**: Suppose \( \phi \begin{pmatrix} \bar{o} \\ \bar{u}[z] \end{pmatrix} \equiv \bar{o} \) in \( X([I|G]) \). Then \( G\bar{u}[z] \equiv \bar{o} \) in \( X(G) \). So \( \phi \) is injective.

**Surjective**: Let \( \begin{pmatrix} \bar{y}[z] \\ \bar{u}[z] \end{pmatrix} \in X([I|G]) \) be arbitrary. We claim that
\[ \phi(\bar{u}[z]) \equiv \begin{pmatrix} \bar{y}[z] \\ \bar{u}[z] \end{pmatrix}. \] (7.6)
\[ \phi(\bar{u}[z]) \equiv \begin{pmatrix} \bar{y}[z] \\ \bar{u}[z] \end{pmatrix} \leftrightarrow \begin{pmatrix} \bar{o} \\ \bar{u}[z] \end{pmatrix} \equiv \begin{pmatrix} \bar{y}[z] \\ \bar{u}[z] \end{pmatrix} \] (7.7)
\[ \leftrightarrow [I|G]\begin{pmatrix} \bar{y}[z] \\ \bar{o} \end{pmatrix} \in \Omega Y, \] (7.8)
but this is clear. The proof is identical to show \( X_\infty(G) = X_\infty([I|G]) \). So Isomorphism (3) finishes the proof. \( \Box \)
So there is a very natural isomorphism from $\mathcal{X}(G)$ to $\mathcal{X}([I|G])$, which makes sense because we should not expect concatenation by the identity to change the pole structure. The identity has no poles in either the finite plan or at infinity. The spaces are so similar that we will say, with just a slight abuse of notation, that

$$\mathcal{X}(G) = \mathcal{X}([I|G]). \quad (7.9)$$

Let $\mathcal{C}$ be a $k$-dimensional vector subspace of an $N$-dimensional vector space $V$ over the rational function field $k(z)$. Define $W(\text{im}\mathcal{C})$ to be the Wedderburn of the space $\mathcal{C}$.

**Theorem 7.2.2** $W(\mathcal{C}) \cong W(\mathcal{C}^\perp)$ where $\mathcal{C}^\perp$ is the orthogonal compliment of $\mathcal{C}$ in $V$ with respect to the standard inner product.

**Proof:** Let the columns of $[\mathcal{C}]$ be a minimal polynomial basis matrix for the subspace $\mathcal{C}$ in $V$. Let $\{N_\infty, D_\infty\}$ be a RCPMFD pair at infinity for $[\mathcal{C}]$, (i.e., $[\mathcal{C}] = N_\infty D_\infty^{-1}$ and $D_\infty = \text{diag}[\frac{1}{z^{e_i}}]$, where $e_i$ is $i^{th}$ column degree $[\mathcal{C}]$). Then $[\mathcal{C}^T] = D_\infty^{-1} N_\infty^T$ is a left coprime factorization of $[\mathcal{C}^T]$. But $D_\infty$ is diagonal so $D_\infty = D_\infty^T$. Referring back to equations (3.26 and 3.27)

$$X_\infty([\mathcal{C}^T]) \cong \frac{\Omega_\infty U}{[D_\infty^T] \Omega_\infty U} = \frac{\Omega_\infty U}{[D_\infty] \Omega_\infty U} = X_\infty([\mathcal{C}]). \quad (7.10)$$

From the main exact sequence we have on one hand that

$$0 \to X_\infty([\mathcal{C}]) \cong W(\text{im}[\mathcal{C}]) \to 0 \quad (7.11)$$

and on the other hand that

$$0 \to \frac{X_\infty([\mathcal{C}^T])}{W([\mathcal{C}^T])} \to 0. \quad (7.12)$$
So

\[ W([C^T]) \cong X_\infty([C^T]) \cong X_\infty([C]) \cong W(\text{im}[C]), \]

which implies that

\[ \dim W([C^T]) = \dim W(\text{im}[C]). \]

But \( C^\perp = \ker[C]^T \). So in general \( W(C) \cong W(C^\perp) \). \( \square \)

We would like to show more, namely that \( W(\text{im}[C]) \) is naturally dual to the \( W(\text{im}[C]^\perp) \). We construct a new nonsingular pairing \( W(C) \times W(C^\perp) \). Given a space \( C \) as above consider

\[ W(C) = \frac{\pi_-(C)}{\pi_-(C) \cap C} \]
\[ W(C^\perp) = \frac{\pi_+(C^\perp)}{\pi_+(C^\perp) \cap C^\perp}. \]

Let \( \pi_-(\tilde{c}) \in W(C) \) and \( \pi_+(\tilde{d}) \in W(C^\perp) \).

**Definition 7.2.3** \( \langle \pi_-(\tilde{c}), \pi_+(\tilde{d}) \rangle_{z^{-1}} \) is the coefficient of \( z^{-1} \) in the ordinary dot product of the vectors \( \pi_-(\tilde{c}) \) and \( \pi_+(\tilde{d}) \) viewed as elements in \( V \).

Thus

\[ \langle , \rangle_{z^{-1}} : W(C) \times W(C^\perp) \mapsto k. \]

(Note: If \( \tilde{c} \in C \) and \( \tilde{d} \in C^\perp \) then

\[ 0 = \langle \tilde{c}, \tilde{d} \rangle = \langle \pi_+(\tilde{c}) + \pi_-(\tilde{c}), \pi_+(\tilde{d}) + \pi_-(\tilde{d}) \rangle \]
\[ = \pi_+(\tilde{c})^T \cdot \pi_+(\tilde{d}) + \pi_-(\tilde{c})^T \cdot \pi_+(\tilde{d}) + \pi_+(\tilde{c})^T \cdot \pi_-(\tilde{d}) + \pi_-(\tilde{c})^T \cdot \pi_-(\tilde{d}) \].

But

\[ \langle \pi_+(\tilde{c}), \pi_+(\tilde{d}) \rangle_{z^{-1}} = 0 \]
and

\[
\langle \pi_- (\tilde{c}) , \pi_- (\tilde{d}) \rangle_{z^{-1}} = 0 \tag{7.20}
\]

These results together with the fact that \( \tilde{c} \in C \) and \( \tilde{d} \in C^\perp \) imply that

\[
\langle \pi_- (\tilde{c}) , \pi_+ (\tilde{d}) \rangle_{z^{-1}} = -\langle \pi_+ (\tilde{c}) \pi_- (\tilde{d}) \rangle_{z^{-1}}. \tag{7.21}
\]

So the choice of assigning the \( \pi_+ () \) and \( \pi_- () \) definitions to \( C \) and \( C^\perp \) only changes the value of the inner product \( \langle , \rangle_{z^{-1}} \) by a minus sign.

**Theorem 7.2.4** The above inner product forms a non-singular pairing between \( W(C) \) and \( W(C^\perp) \).

**Proof:** We need to show the following:

(i) The inner product is well-defined.

(ii) The inner product is a non-singular pairing.

(i) **Well-Defined**

Suppose \( \pi_- (\tilde{c}) \in C \) (i.e., \( \pi_- (\tilde{c}) = \tilde{0} \) in \( W(C) \)). Then

\[
\langle \pi_- (\tilde{c}) , \pi_+ (\tilde{d}) \rangle_{z^{-1}} = [\pi_- (\tilde{c})^T] \cdot [\pi_+ (\tilde{d}) + \pi_- (\tilde{d})]_{z^{-1}}, \tag{7.22}
\]

because \( \pi_- (\tilde{c})^T \cdot \pi_- (\tilde{d}) \) begins at \( z^{-2} \) and so contributes nothing to the coefficient for \( z^{-1} \). But

\[
\pi_- (\tilde{c}) + \pi_+ (\tilde{d}) = \tilde{d} \in C^\perp. \tag{7.23}
\]
So in particular the coefficient of $z^{-1}$ is zero and
\[ \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} = 0 \ \forall \ \pi_+(\bar{d}) \in W(C^\perp). \] (7.24)

Similarly, $\pi_+(\bar{d}) \in W(C^\perp)$ arbitrary implies that
\[ \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} = 0 \ \forall \ \pi_-(\bar{c}) \in W(C). \] (7.25)

We have that
\[ \forall \ \pi_+(\bar{d}) = 0 \in W(C^\perp) \] (7.26)
\[ \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} = 0_{\text{map}} \] (7.27)

and
\[ \forall \ \pi_-(\bar{c}) = 0 \in W(C) \] (7.28)
\[ \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} = 0_{\text{map}} \] (7.29)

This shows that inner product is well-defined.

(i) The Pairing is Non-Singular

We must show both

(a) Let $\pi_-(\bar{c}) \in W(C)$ and suppose
\[ \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} = 0 \ \forall \ \pi_+(\bar{d}) \in W(C^\perp). \] (7.30)

Then $\pi_-(\bar{c}) = 0$ in $W(C)$, (i.e., $\pi_-(\bar{c}) \in W(C^\perp)$.) It suffices to show the following:
\[ \pi_-(\bar{c}) \notin C^\perp \Rightarrow \exists \ \pi_+(\bar{d}) \in W(C^\perp) \text{ s.t. } \langle \pi_-(\bar{c}) , \pi_+(\bar{d}) \rangle_{z^{-1}} \neq 0. \] (7.31)
(b) Let \( \pi_+(\vec{d}) \in W(C^\perp) \) and suppose

\[
\langle \pi_-(\vec{c}), \pi_+(\vec{d}) = 0 \rangle_{z^{-1}} = 0 \forall \pi_-(\vec{c}) \in W(C).
\]  

Then \( \pi_+(\vec{d}) = 0 \) in \( W(C^\perp) \), (i.e., \( \pi_-(\vec{c}) \in W(C^\perp) \)). Again, it suffices to show

\[
\pi_+(\vec{d}) \notin C^\perp \Rightarrow \exists \pi_-(\vec{c}) \in W(C) \text{ s.t. } \langle \pi_-(\vec{c}), \pi_+(\vec{d}) \rangle_{z^{-1}} \neq 0.
\]  

Proof of (a): If \( \pi_-(\vec{c}) \notin C \) then there exists a vector \( \vec{d} \in C^\perp \) such that with respect to the standard inner product \( \langle \pi_-(\vec{c}), \vec{d} \rangle \neq 0 \). We may assume without loss of generality that \( \vec{d} \) is polynomial. Now since \( \pi_-(\vec{c}) \in W(C) \) we have

\[
0 = \langle \vec{c}, \vec{d} \rangle = \langle \pi_+(\vec{c}) + \pi_-(\vec{c}), \vec{d} \rangle = \pi_+(\vec{c})^T \cdot \vec{d} + \pi_-(\vec{c})^T \cdot \vec{d}.
\]  

This implies \( \pi_-(\vec{c})^T \cdot \vec{d} \) is polynomial, say

\[
\pi_-(\vec{c})^T \cdot \vec{d} = a_nz^n + \ldots + a_iz^i \in k[z],
\]  

with \( i \) the least integer such that \( a_i \neq 0 \). Then

\[
z^{-i-1}(a_nz^n + \ldots + a_iz^i) = z^{-i-1}\langle \pi_-(\vec{c}), \vec{d} \rangle = \langle \pi_-(\vec{c}), z^{-i-1}\vec{d} \rangle
\]  

\[
= \langle \pi_-(\vec{c}), \pi_+(z^{-i-1}\vec{d}) + \pi_-(z^{-i-1}\vec{d}) \rangle
\]  

\[
= \langle \pi_-(\vec{c}), \pi_+(z^{-i-1}\vec{d}) \rangle + \langle \pi_-(\vec{c}), \pi_-(z^{-i-1}\vec{d}) \rangle
\]  

We make an observation that

\[
z^{-i-1}(a_nz^n + \ldots + a_iz^i) = a_nz^{n-i-1} + \ldots + a_{i+1} + a_iz^{-1}
\]  

with \( a_i \neq 0 \). Also,

\[
\langle \pi_-(\vec{c}), \pi_-(z^{-i-1}\vec{d}) \rangle = b_2z^{-2} + b_3z^{-3} \ldots
\]
and so begins at \( z^{-2} \). This implies that the term \( a_i z^{-1} \) comes from

\[
\langle \pi_-(\vec{c}), \pi_+(z^{-i-1}\vec{d}) \rangle
\]

where \( \pi_+(z^{-i-1}\vec{d}) \in WC^\perp \). So

\[
\langle \pi_-(\vec{c}), \pi_+(z^{-i-1}\vec{d}) \rangle \bigg|_{z^{-1}} = a_i \neq 0, \tag{7.41}
\]

as was to be shown.

Proof of (b): If \( \pi_+(\vec{d}) \not\in C^\perp \) then there exists \( \vec{c} \in C \) such that \( \langle \vec{c}, \pi_+(\vec{d}) \rangle \neq 0 \). We may assume \( \vec{c} \) is a strictly proper vector so that \( \pi_-(\vec{c}) = \vec{c} \). Then

\[
0 = \langle \vec{c}, \pi_+(\vec{d}) + \pi_-(\vec{d}) \rangle = \vec{c}^T \cdot \pi_+(\vec{d}) + \vec{e}^T \cdot \pi_-(\vec{d}). \tag{7.42}
\]

Again, \( \vec{e}^T \cdot \pi_+(\vec{d}) \) must necessarily be strictly proper. We can write

\[
\langle \vec{c}, \pi_+(\vec{d}) \rangle = a_i z^{-i} + \ldots. \tag{7.43}
\]

Then

\[
(a_i z^{-1} + \ldots) = z^{-i-1}\langle \vec{c}, \pi_+(\vec{d}) \rangle = \langle \pi_+(z^{-i-1}\vec{c}) + \pi_-(z^{-i-1}\vec{c}), \pi_+(\vec{d}) \rangle \tag{7.44}
\]

\[
= (\text{polynomial}) + \langle \pi_-(z^{-i-1}\vec{c}), \pi_+(\vec{d}) \rangle \tag{7.45}
\]

\[
= a_{-1} \neq 0 \square \tag{7.46}
\]

So \( W(C) \times W(C^\perp) \) is a non-singular pairing.

### 7.3 (B) The Inclusion

Let \( G \) be a rational transfer function matrix with left coprime polynomial factorization

\[
GD = N \quad \text{so that} \quad [D|N] \quad \text{is row proper and minimal, (i.e., by Corollaries (3.2.5} \]
and 3.2.6) \([D|N]\) has no zeros.) Also from the definition of the Wedderburn space

\[ W(\text{im}[D|N]) = (0) \]

From the main exact sequence

\[ 0 \to \mathcal{E}([D|N]) \overset{\alpha}{\to} \frac{X_\infty([D|N])}{W(\ker[D|N])} \overset{\beta}{\to} W(\text{im}[D|N]) \to 0 \]

we have then that \(X_\infty([D|N]) \cong W([D|N])\). If we define

\[ W([D|N]) = \frac{\pi_-(\ker[D|N])}{\pi_-(\ker[D|N]) \cap (\ker[D|N])} \]

then one can easily check that the map

\[ \alpha : W(\ker[D|N]) \overset{\cong}{\to} X_\infty([D|N]) \]

defined by

\[ \alpha(\pi_-(\tilde{\theta})) = \overline{\pi_-(\tilde{\theta})} \in X_\infty([D|N]) \]

is a well-defined isomorphism.

### 7.4 (C) The Induced Filtration

From the notation of part A let \(C = \ker[D|N]\) and \(C^\perp = \text{im} \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \). Then we have a non-singular pairing between

\[ W(C) = \frac{\pi_-(C)}{\pi_-(C) \cap (C)} \overset{\text{def}}{=} V \quad \text{and} \quad W(C^\perp) = \frac{\pi_+(C^\perp)}{\pi_+(C^\perp) \cap (C^\perp)} \overset{\text{def}}{=} W. \]
We have previously defined the Wedderburn filtration on the space $W$ as follows:

\[
\begin{align*}
    d_0; \quad W_0 &= \pi_+(C^\perp) \cap \bar{k}_0 + z\bar{k}_1 + \ldots + z^{s-1}\bar{k}_{s-1} = W(C^\perp) \\
    d_1; \quad W_1 &= \pi_+(C^\perp) \cap \bar{k}_0 + z\bar{k}_1 + \ldots + z^{s-2}\bar{k}_{s-2} \\
    \vdots & \quad \vdots \\
    d_{s-1}; \quad W_{s-1} &= \pi_+(C^\perp) \cap \bar{k}_0 \\
    d_s; \quad W_s &= (0),
\end{align*}
\]

where

1. $s =$ (largest column degree of $[C^\perp]$) and
2. $d_i = \dim W_i, \quad d_0 = 0.$

Therefore

\[
d_i = \left[ \# \text{ of columns of } [C^\perp] \text{ of degree } \geq (s - i) \right] + d_{i+1} \quad (7.52)
\]

with $(i = s - 1, \ldots, 0).$ So $d_0 = \text{(sum of the column degrees of } \left[ \begin{array}{c} D^T \ \\
N^T \end{array} \right])$, which we defined earlier to be $e.$ The non-singular pairing and the Wedderburn filtration above can be used to induce a filtration on $V = W(C),$ which we call the dual filtration of $W.$ We suggest the following table:
Table 3: Global Observability in $X_\infty([D|N])$

$W(C^+) \xrightarrow{(\cdot)_{d+1}} W(C) \xrightarrow{\alpha} X_\infty([D|N])$; Define $e_j = \delta_j[D|N]$

| Dim.  | $W(C)$ | Dim.  | $W(C)$ | $X_\infty([D|N]) = \bigcap_{i=0}^{d+1} W(C_{\alpha_{d+1-\alpha_{d+1}}} [D|N])$ |
|-------|--------|-------|--------|-----------------------------------------------------|
| $d_s = 0$; | $W_s = (0)$ | $n$; | $W_n = V_0$ | $X_0 = \{ \bar{x} \delta_j([D|N]x) \leq \max(e_j - 1, 0) \}$ |
| $d_{s-1}$; | $W_{s-1} = \pi_+(C^+) \cap k$ | $n - d_{s-1}$; | $W_{n-d_s} = V_1$ | $X_1 = \{ \bar{x} \delta_j([D|N]x) \leq \max(e_j - 2, 0) \}$ |
| $d_{s-2}$; | $W_{s-2} = \pi_+(C^+) \cap k + z_k$ | $n - d_{s-2}$; | $W_{n-d_s} = V_2$ | $X_2 = \{ \bar{x} \delta_j([D|N]x) \leq \max(e_j - 3, 0) \}$ |
| $d_{s-3}$; | $W_{s-3} = \pi_+(C^+) \cap k + z_k + z_{s-3}k$ | $n - d_{s-3}$; | $W_{n-d_s} = V_3$ | $X_3 = \{ \bar{x} \delta_j([D|N]x) \leq \max(e_j - i - 1, 0) \}$ |
| $d_1$; | $W_1 = \pi_+(C^+) \cap k + z_k + \ldots + z_{s-2}k$ | $n - d_1$; | $W_{n-d_s} = V_{s-1}$ | $X_{s-1} = \{ \bar{x} \delta_j([D|N]x) \leq \max(0, 0) \}$ |
| $d_0 = n$; | $W_0 = \pi_+(C^+)$ | $0$; | $W_0 = V_n = (0)$ | $X_s = \{ \bar{x} \delta_j([D|N]x) = 0 \} = (0)$ |
First we need to recall a result from the previous work on

\[
W(C^\perp) = \frac{\pi_+(C^\perp)}{\pi_+(C^\perp) \cap (C^\perp)}. \quad (7.53)
\]

By Definition (4.2.2), Lemma of Predictable Degree, and Theorem (4.2.4) we had the following: If \( e_j = \delta_j \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \) def degree of the \( j \)th column of \( \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \), where \( \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \) is a minimal polynomial basis matrix, then

\[
W_{e-i} = \pi_+(C) \cap k_0 + zk_1 + \ldots + z^{i-1}k_{i-1} \quad (7.54)
\]

is the column span over \( k \) (csk) of the of the matrices

\[
\langle \pi_+ \left\{ \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \right\} \rangle \oplus \langle \pi_+ \left\{ \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \right\} \rangle \oplus \ldots
\]

\[
\oplus \langle \pi_+ \left\{ \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \right\} \rangle \oplus \ldots
\]

\[
\oplus \langle \pi_+ \left\{ \left[ \begin{array}{c} D^T \\ N^T \end{array} \right] \right\} \rangle. \quad (7.55)
\]

For Example

Suppose

\[
[C^\perp] = \left[ \begin{array}{ccc}
z^4 + 1 & 1 & z^2 + 1 \\
z^2 & z & z^3 + z \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & z^4 + 1 & 0
\end{array} \right].
\]
Then
\[ W_4 \cong (0) \]
\[ W_3 \cong \langle \pi_+ \left\{ [C^1] \begin{bmatrix} z^{-4} & 0 & 0 \\ 0 & z^{-4} & 0 \\ 0 & 0 & z^{-3} \end{bmatrix} \right\} \rangle_{csk} = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rangle_k \]
\[ W_2 \cong W_3 \oplus \langle \pi_+ \left\{ [C^1] \begin{bmatrix} z^{-3} & 0 & 0 \\ 0 & z^{-3} & 0 \\ 0 & 0 & z^{-2} \end{bmatrix} \right\} \rangle_{csk} = \langle \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix} \rangle_k \]
\[ W_1 \cong W_2 \oplus \langle \pi_+ \left\{ [C^1] \begin{bmatrix} z^{-2} & 0 & 0 \\ 0 & z^{-2} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix} \right\} \rangle_{csk} = \langle \begin{bmatrix} z^2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ z^2 \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} z^2 + 1 \\ 0 \\ 0 \end{bmatrix} \rangle_k \]
\[ W_0 \cong W_1 \oplus \langle \pi_+ \left\{ [C^1] \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & z^0 \end{bmatrix} \right\} \rangle_{csk} = \langle \begin{bmatrix} z^3 \\ z^2 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ z^3 \end{bmatrix} \rangle_k \]
\[ W(C^1) \]

7.5 (D) The Isomorphism of Filtered Vector Spaces

Theorem 7.5.1 \( \{V_i\} \cong \{X_i\} \) as filtered vector spaces.

Proof: (Compare with the table on Page 118)

Injectivity: Let \( \pi_-([\tilde{v}] \in V_i \). Then \( \pi_-([\tilde{v}] \in W^L_{s-i} \) and so is orthogonal, with respect the inner product \( \langle \cdot, \cdot \rangle_{z^{-1}} \), to all vectors in \( W_{s-i} \), as shown in the table. From the earlier statements, we saw that \( W_{s-i} \) could be thought of as the row span over \( k \) (rsk) of
each matrix in the sum
\[
\langle \pi_+ \left\{ \left[ \begin{array}{cc}
  z^{-e_1} & 0 \\
  \vdots & \ddots \\
  0 & \cdots & z^{-e_p}
\end{array} \right] [D|M] \right\} \rangle_{rsk} \Theta \ldots \Theta \langle \pi_+ \left\{ \left[ \begin{array}{cc}
  z^{-e_1+i-1} & 0 \\
  \vdots & \ddots \\
  0 & \cdots & z^{-e_p+i-1}
\end{array} \right] [D|M] \right\} \rangle_{rsk}
\]

We call this spanning set \( S_i \). (Note: If we ignore those row vectors above that are zero vectors in \( W(C^l) \), namely all terms for which

\[-e_j + k \geq 0, (k = 0, \ldots, i - 1), \] (7.56)

then \( S_i \) is our previously defined basis from Theorem (4.2.4).) For these non-zero rows we have

\[
\begin{align*}
\langle \pi_+(z^{-e_j+i-1} r_j), \pi_-(v) \rangle_{x-1} &= \langle z^{-e_j+i-1} r_j, \pi_-(\vec{v}) \rangle_{x-1} = 0 & (i) \\
\langle \pi_+(z^{-e_j+i-2} r_j), \pi_-(v) \rangle_{x-1} &= \langle z^{-e_j+i-2} r_j, \pi_-(\vec{v}) \rangle_{x-1} = 0 & (i - 1) \\
\vdots & \vdots & \vdots \\
\langle \pi_+(z^{-e_j+1} r_j), \pi_-(v) \rangle_{x-1} &= \langle z^{-e_j+1} r_j, \pi_-(\vec{v}) \rangle_{x-1} = 0 & (2) \\
\langle \pi_+(z^{-e_j} r_j), \pi_-(v) \rangle_{x-1} &= \langle z^{-e_j} r_j, \pi_-(\vec{v}) \rangle_{x-1} = 0 & (1)
\end{align*}
\]

So then working through the above list backwards yields

\[
z^{-e_j} r_j \pi_-(\vec{v}) = a_1 z^{-1} + a_2 z^{-2} + \ldots = a_2 z^{-2} + \ldots \quad (7.57)
\]

because (1) \( \Rightarrow a_1 = 0 \).

\[
z^{-e_j+1} r_j \pi_-(\vec{v}) = z z^{-e_j} r_j \pi_-(\vec{v}) = a_2 z^{-1} + a_3 z^{-2} + \ldots = a_3 z^{-2} + \ldots \quad (7.58)
\]

because (2) \( \Rightarrow a_2 = 0 \). Continuing, we have that

\[
z^{-e_j+i-1} r_j \pi_-(\vec{v}) = z z^{-e_j+i-2} r_j \pi_-(\vec{v}) = a_i z^{-1} + a_{i+1} z^{-2} + \ldots = a_2 z^{-2} + \ldots \quad (7.59)
\]

because (i) \( \Rightarrow a_i = 0 \). So then

\[
z^{-e_j} r_j \pi_-(\vec{v}) = a_{i+1} z^{-i-1} + \ldots \quad (7.60)
\]
This implies that
\[ r_j^i \pi_-(v) = a_{i+1} z^{e_j-i-1} + \ldots \] (7.61)

so
\[ \delta(\pi_+ [r_j^i \pi_-(v)]) \leq e_j - i - 1 \ \forall \ i = 1, \ldots, r. \] (7.62)
\[ \delta_j(\pi_+ ([D|N]\pi_-(v))) \leq \max(e_j - i - 1, 0) \] (7.63)

and so \( \pi_-(v) \in X_i \) defined in the table.

**Surjectivity:** Suppose \( \bar{z}_i \in X_i \) as defined in the table. Since
\[ \alpha : W(C = \ker[D|N]) \xrightarrow{\cong} X_\infty([D|N]) \ \exists \pi_-(v) \in W(C) \ s.t. \ \alpha(\pi_-(v)) \equiv \bar{z}_i. \] (7.64)

**Note:**
\[ \pi_-(v) \equiv \bar{z}_i \Rightarrow [D|N](\pi_-(v) - \bar{z}_i) \] (7.65)
is s.p. so \( \pi_+([D|N]\pi_-(v)) = \pi_+([D|N]\bar{z}_i) \). Then
\[ \delta_k(\pi_+([D|N]\pi_-(v))) \leq \max(e_k - i - 1, 0). \] (7.66)

So
\[ z^{-e_k} r_k^i \pi_-(v) = a_{e_k-i-1} z^{-1} z^{1-i-1} + \ldots \] (1)
\[ z^{-e_k+1} r_k^i \pi_-(v) = a_{e_k-i-1} z^{-2} z^{1-i-1} + \ldots \] (2)
\[ \vdots \] (i)
\[ z^{-e_k+i-1} r_k^i \pi_-(v) = a_{e_k-i-1} z^{-2} + \ldots \] (i)

But then from (i), \ldots, (i) we have for all \( k = 1, \ldots, p \)
\[ \langle \pi_+(z^{-e_k} r_k^i), \pi_-(v) \rangle_{z^{-1}} = 0 \] (7.67)
\[ \langle \pi_+(z^{-e_k+1} r_k^i), \pi_-(v) \rangle_{z^{-1}} = 0 \] (7.68)
\[ \vdots \] (i)
\[ \langle \pi_+(z^{-e_k+i-1} r_k^i), \pi_-(v) \rangle_{z^{-1}} = 0 \] (7.70)
But recall that $W_{s-i}$ equaled the span of the set

$$\{\pi_+(z^e r_k), \ldots, \pi_+(z^{e_k+i-1} r_k^e)\}_{k=1, \ldots, p}.$$  

So $\pi_- (\bar{v})$ is orthogonal to $W_{s-i}$, which implies that $\pi_- (\bar{v}) \in W_{s-i}^\perp = V_i$. □

So $W(\ker[D|N]) \cong X_\infty([D|N])$ as filtered vector spaces. $W(C)$ was given the dual filtration induced by the Wedderburn filtration on $W([D^T N^T])$ and $X_\infty$ was given the filtration in the table. As we will show, the filtration on $X_\infty$ is just the image filtration.

From previous work we showed the conjugate partition of $W(C^\perp)$ gave the column degrees of $\begin{bmatrix} D^T \\ N^T \end{bmatrix}$. In the classical case when $G = D^{-1} N$ is a left coprime polynomial factorization for $G$, these row degrees and indices are the observability indices of $G$. Because of the non-singular pairing between $W(C)$ and $W(C^\perp)$ (or just by looking at the degrees of the spaces in the table) we see the conjugate partition of the induced filtration on $W(C)$ also gives the observability indices, does our filtration on $X_\infty$.

**Lemma 7.5.2** $z^i X_\infty = X_i$, that is, the image filtration is the filtration on $X_i$ from the table on page 118.

**Proof:** It is clear that $z^{-i} X_\infty \subseteq X_i$ since if $\bar{0} \neq \bar{x}_i \in z^{-i} X_\infty$, then $\bar{x}_i = z^{-i} \bar{x}_0$ for some $\bar{x}_0 \in X_\infty$. Also,

$$\pi_+([D|N]\bar{x}_i) = \pi_+(z^i [D|N] \bar{x}_0) = \pi_+ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix} (z^{-i}) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{pmatrix}$$ (7.71)
where each $f_i$ is at most $e_i - 1$ so each row of $\pi_+([D|N] \bar{x}_i)$ is at most $e_i - 1 - i$ or 0. So $\bar{x}_i \in X_i$. The proof $z^{-i}X_\infty = X_i$ is completed by realizing that the dimension of $z^{-i}X_\infty$ is equal to that of $X_i$ for each $i = 0, 1, \ldots, s$. This is true because both give the same partition and conjugate partition of observability indices.

**Definition 7.5.3** The image in $X_\infty$ of the induced Wedderburn filtration is called the generalized global observability filtration in $X_\infty$.

### 7.6 (E) Support for the Definition Generalized Observability Filtration

For the controllability case we had found that the map $\beta^{-1}$ maps the $\pi_+$--Wedderburn filtration $\{W_{s-i}\}$ to $\{H_i\}$, $H_i$ defined in equation (6.32). In the classical case we showed that this filtration agrees with the classical controllability filtration. For a minimal realization of $G$ in the time domain we recall that $G = C(zI - A)^{-1}B$. 
Therefore, $G^T = B^T(zI - A^T)^{-1}C^T$. For $G^T$ we have the controllability filtration

$$
\begin{align*}
H_0 &= (0) & W_S &= (0) \\
\cap & & \cap \\
H_1 &= \langle [C^T] \rangle & W_{S-1} &= \pi_+ (\text{im} \begin{bmatrix} I \\ G^T \end{bmatrix}) \cap \bar{k} \\
\cap & & \cap \\
H_2 &= \langle [C^T|A^TC^T] \rangle & W_{S-2} &= \pi_+ (\text{im} \begin{bmatrix} I \\ G^T \end{bmatrix}) \cap (\bar{k} + z\bar{k}) \\
\cap & & \cap \\
\vdots & & \vdots \\
\cap & & \cap \\
H_k &= \langle [C^T|A^TC^T|\ldots|A^{T^{k-1}}C^T] \rangle & W_{S-k} &= \pi_+ (\text{im} \begin{bmatrix} I \\ G^T \end{bmatrix}) \\
\cap & & \cap \\
X(G^T) & & W(\text{im} \begin{bmatrix} I \\ G^T \end{bmatrix})
\end{align*}
$$

The definition for the classical observability filtration $\{L_i\}$ was given on page (65).

By what we saw in Section 4.3, we had the orthogonality statement

$$
H_1^\perp(G^T) \overset{\text{def}}{=} \langle [C^T|A^TC^T|\ldots|A^{T^{i-1}}C^T] \rangle^\perp = L_i \overset{\text{def}}{=} \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix}
$$

So we must show that, in the case when $G$ is s.p., $X_i$, the GOF of $G$, is equal to $L_i$, the classical observability filtration of $G$. (i.e., We are using the fact that the GOF of $G$ is the orthogonal compliment to the GCF of $G^T$, where orthogonality is induced by the non-singular pairing of $W(\text{im} \begin{bmatrix} D^T \\ N^T \end{bmatrix})$ and $W(\ker[D|N])$. The orthogonality of the controllability and observability filtrations in the global definitions then generalize
the classical results of orthogonality when $G$ is s.p.) So it remains to show

$$X_i = L_i,$$

where $X_i = z^{-i}X_{\infty}(G)$ and $L_i$ is defined in equation (7.72).

**Theorem 7.6.1** Let $G$ be s.p. In this case the GOF agrees with the classical observability filtration for $G$ and the GOF and GCF are then orthogonal with respect to the non-singular pairing.

**Proof:** (Use the table for reference in this proof.) In this case, the GCF for $G^T$ was shown to be the classical control filtration. If we show that the GOF agrees with the classical observability filtration of $G$, orthogonality of GOF for $G$ and GCF for $G^T$ is automatic from the above results.

Let $\vec{x} \in L_i = [\beta^{-1}(W_{s-i})]^t$. Then from $\vec{\beta}$ in the Kalman input/output diagram (page 17) there exists $\vec{u}[z] \in \Omega U$ s.t. $\vec{\beta}(\vec{u}[z]) = \vec{x}$ at time $t = 1$. Since $\vec{x}$ is in $L_i$, the output of $\vec{x}$ for time $t \geq 1$ is of the form

$$G^# = CA^i\vec{x}z^{-(i+1)} + \ldots = \vec{y}_0z^{-(i+1)} + \vec{y}_1z^{-(i+2)} + \ldots$$

That is to say that

$$CA^i\vec{x}z^{-1} + CA^i\vec{x}z^{-2} + \ldots + CA^{i-1}\vec{x}z^{-i} = \vec{0}. \quad (7.74)$$

Consider the vector $\vec{v}$ defined as

$$\begin{pmatrix} \pi_+(G\vec{u}) + \pi_-(G\vec{u}) \\ -\vec{u}[z] \end{pmatrix}$$
It is clear that \( \bar{v} \) is in the kernel of \([I|G]\) and so

\[
\pi_-(\bar{v}) \in W(\ker[D|N]).
\]

Also,

\[
\pi_-(\bar{v}) = \left( \begin{array}{c} \bar{y}_0 z^{-(i+1)} + \bar{y}_1 z^{-(i+2)} + \cdots \\ \bar{u} \end{array} \right). \tag{7.75}
\]

Viewed in

\[
X_\infty([D|N]), \quad \pi_-(\bar{v}) = z^{-i} \left( \begin{array}{c} \bar{y}_0 z^{-(1)} + \bar{y}_1 z^{-(2)} + \cdots \\ \bar{u} \end{array} \right) \in X_i, \tag{7.76}
\]

the image of \( z^{-i} \). So \( L_i \subseteq X_i \). For the reverse inclusion, let \( \bar{x} \in X_i \). Then \( \bar{x} \) is in the image of \( z^{-i} \) and so is equivalent to some vector

\[
\bar{v} = \left( \begin{array}{c} \bar{y}(z) \\ \bar{u}(z) \end{array} \right), \tag{7.77}
\]

such that \( \pi_-(\bar{v}) = z^{-(i+1)} \bar{v}_1 + z^{-(i+2)} \bar{v}_2 + \ldots \) and \( \bar{v} \in \ker[D|N] \). Consider

\[
[I|G]\bar{v} = [I|G] \left( \begin{array}{c} \pi_+(\bar{y}) + \pi-(\bar{y}) \\ \pi_+(\bar{u}) + \pi-(\bar{u}) \end{array} \right) = \pi_+(\bar{y}) + \pi-(\bar{y}) + G\pi_+(\bar{u}) + G\pi-(\bar{u}) = \bar{u}. \tag{7.78}
\]

Then \( G \) s.p. implies

\[
G\pi_-(\bar{u}) = z^{-(i+2)} \bar{u}_1 + z^{-(i+3)} \bar{u}_2 + \ldots \tag{7.80}
\]

\[
pi_-(\bar{y}) = z^{-(i+1)} \bar{y}_1 + z^{-(i+2)} \bar{y}_1 + \ldots \tag{7.81}
\]

So in the term \( G(\pi_+(\bar{u})) \) coefficients for

\[
z^{-1}, z^{-2}, \ldots, z^{-i}
\]
are necessarily zero in order to satisfy the constraint from equation (7.79). So the state vector $\vec{x}$ produces future output with terms starting at $CA^iz^{-(i+1)}$. So $\vec{x}$ is in the level of

$$L_i = \ker \left( \begin{array}{c} C \\ CA \\ \vdots \\ CA^{i-i} \end{array} \right)$$

(7.82)

So $X_i \subseteq L_i$. So for $G$ s.p., we have that the diagram

$$\begin{array}{ccc}
X_i & \xrightarrow{\text{standard}} & \beta^{-1}(W_{s-i}(\text{im} \begin{bmatrix} D^T \\ N^T \end{bmatrix})) \\
\alpha & \xleftarrow{\uparrow} & \uparrow \beta^{-1} \\
W_{s-i}^{-1}(\ker[D|N]) & \xleftarrow{\uparrow \beta^{-1}} & W_{s-i}(\text{im} \begin{bmatrix} D^T \\ N^T \end{bmatrix}) \Box.
\end{array}$$

If we define $C = \text{im} \begin{bmatrix} I \\ G \end{bmatrix}$ and $C^\perp = \ker[I|G^T]$ then from equation (3.37) it was shown that $W(\text{im} C^\perp) \cong W(\text{im} C)$. It was also shown there exists a non-singular pairing $W(\text{im} C^\perp) \times W(\text{im} C) \to k$. The orthogonal compliments of the WF induced by $\times$, namely $\{W_i^\perp\}_{i=0} \subseteq W(C^\perp)$, form a decreasing filtration called the *Global Observability Filtration* (GOF). This GGOF is such that in the classical case when $G$ is strictly proper $W_{i}^\perp \cong [C^T|A^TC^T]\ldots[A^{(i-1)}TC^T]^\perp$ and so agrees with the classical observability filtration. The conjugate partition associated with the chain of quotients yields the generalized global observability indices or row degrees of $[D|N]$; where $G = D^{-1}N$ and $(D, N)$ is a L(Left)GCMFD [6]. These indices and spaces agree with those from classical control in the case when $G$ is s.p.

In the language of Malabre (see, [8, 9, 10]), the GGOF would be defined in terms of the realization matrices $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$. Our definition of the GGOF is only a function
of $G$ and the module structure on the minimal realization pole space. The GGOF is therefore an invariant of the vector space structure of minimal realizations for the dynamical structure of $G$.

So in light of the above discussion we may define the GGOF for $G$ in terms of the Malabre definition for the GGCF of $G^T$

$$R^0 = \mathcal{X}' \cap \text{ker} \tilde{E}^T$$ (7.83)

$$R^{u+1} = \mathcal{X}' \cap \tilde{E}^T (\hat{C}^T + \hat{A}^T R^u).$$ (7.84)

The above argument then shows the $i^{th}$ level of the GGOF is

$$\left(\tilde{E}^T R^i\right)^\perp$$

Refer to equations (2.23, 2.24, and 2.26), and the definition of the Malabre filtration for controllability.

These results strongly support the "correctness" of the definitions for the filtrations in this work as well as the need for more study of the Wedderburn Forney construction. Both Wedderburn filtrations along with their non-singular pairing agree with and extend the classical controllability and observability chains.

### 7.7 Review of Results

There exist non-singular pairings between:

(i) $W(C)$ and $W(C^\perp)$, (ii) $X(G)$ and $X(G^T)$, and (iii) $X_\infty(G)$ and $X_\infty(G^T)$ defined by the following maps:
(1) If \( \tilde{u} \) is a representative of \([\tilde{u}] \in W(C)\) and \( v \) is a representative of \([\tilde{v}] \in W(C^\perp)\) then

\[
\langle [\tilde{u}], [\tilde{v}] \rangle_{z^{-1}} : W(C) \times W(C^\perp) \rightarrow k
\]

is defined as

\[
\tilde{u}^T \tilde{v} |_{z^{-1}}
\]

where we view \( \tilde{u}, \tilde{v} \in V(z) \), and \( \tilde{u}^T \tilde{v} \) is the standard inner product of \( V(z) \times V(z) \) over \( k(z) \), and \( |_{z^{-1}} \) means take the coefficient of \( z^{-1} \) when \( \tilde{u}^T \tilde{v} \) is viewed in \( k((z^{-1})) \).

Then

\[
\langle , \rangle_{z^{-1}} : W(C) \times W(C^\perp) \rightarrow k
\]

is the pairing sought.

(2) Let \( \tilde{u}_1 \) and \( \tilde{u}_2 \) be coset representatives for \([\tilde{u}_1]\) and \([\tilde{u}_2]\) in \( X(G) \) and \( X(G^T) \), respectively. Then the map

\[
\langle , \rangle_{z^{-1}, G} : X(G) \times X(G^T) \rightarrow k
\]

defined as \( (\tilde{u}_1^T G \tilde{u}_2) |_{z^{-1}} \), where computations are as in (1) above.

(3) The map for \( X_\infty(G) \) and \( X_\infty(G^T) \) is the same as that in (2) above.

We introduced an intermediate structure on these spaces that was compatible with our isomorphisms and inner products; the notion of filtered vector spaces was just right for our purposes.

Suppose \( V \) is any finite dimensional vector space over the field \( k \). A filtration on \( V \) is a sequence of subspaces
A filtered vector space is a vector space equipped with a filtration. Associated to such a filtration are $r$ integers $n_1, n_2, \ldots, n_r$ defined by

$$n_i = \dim \left( \frac{V_i - 1}{V_i} \right).$$ (7.88)

The integers $n_i$ form a partition of $n = \dim V$. The Ferrers diagram of a partition is an arrangement of dots such that the $i^{th}$ row has $n_i$ dots. The corresponding conjugate partition of $n$ is the sequence of integers given by the number of dots in the columns of its Ferrers diagram. If $m_1, m_2, \ldots, m_s$ are these conjugate indices, then for each $t$ the integer $m_t$ is exactly the number of $n_i \geq t$.

The filtrations of interest were the classical controllability and observability filtrations, the kernel and image filtrations, the generalized global controllability and observability filtrations, and the Wedderburn and induced Wedderburn filtrations.

The Global Observability Filtration: Given a transfer function $G : U(z) \rightarrow Y(z)$, we define $C^\perp = \ker [G|I]$ and $C = \im \left( \begin{array}{c} G^T \\ I \end{array} \right)$. Isomorphism 1 yields

$$W(\ker [G|I]) \cong W(\im \left( \begin{array}{c} G^T \\ I \end{array} \right)).$$ (7.89)

We placed the $\pi_+$-filtration on $W(\im \left( \begin{array}{c} G^T \\ I \end{array} \right))$ and the induced filtration on $W(\ker [G|I])$. Isomorphism 3 gave

$$W(\ker [G|I]) \cong \mathcal{X}(G),$$ (7.90)
where

$$\alpha([\pi_-(\begin{pmatrix} \bar{u} \\ -Gu \end{pmatrix})]) = (\pi_+(\bar{u}), \pi_-(\bar{u})) \in \mathcal{X}(G)$$  \hfill (7.91)

The *Global Observability Filtration* in terms of the $\alpha$–image of the induced filtration on $W(\ker[G|I])$ is

$$\mathcal{X}(G) = X_s \supseteq X_{s-1} \supseteq \ldots \supseteq X_0 = (0),$$

where $X_i = \alpha(V_i)$ and $V_i$ is $W^\perp_{s-i}$. The GGOF indices are given by the conjugate partition corresponding to this global observability filtration. Support for this definition is given by the following facts:

(i) If $G$ is strictly proper we have the commutative diagram 1 :

\begin{equation*}
\begin{array}{cccc}
W(\ker[G|I]) \times_1 W(\text{im} \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right)) & \rightarrow & k \\
\alpha \downarrow & & \uparrow \pi_+ \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right) \\
X([G]) \times_2 X([G]^T) & \rightarrow & k
\end{array}
\end{equation*}

Diagram 1

The image of $\alpha$ defined above was just the classical observability filtration. We say that $W(\ker[G|I])$ with the induced Wedderburn filtration and $X(G)$ with the classical observability filtration are *isomorphic as filtered vector spaces*. The inner product $\times_2$ is just the inner product defined in (ii) above. The classical controllability filtration on $X(G^T)$ and the $\pi_+$-filtration on $W(\text{im} \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right))$ gave filtered vector space isomorphisms via the map $\beta$,
\[ \beta = \pi_+ \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right). \]  

(7.92)

The commutativity of the above diagrams gave the added corollary that the in the case when \( G \) is s.p. the GGOF on \( X(G) \) is dual to the GGCF on \( X(G^T) \) via \( \times_2 \).

(ii) If \( G \) is polynomial and row minimal, then the global observability filtration is just the image filtration on \( X_\infty(G) \). Once again the induced filtration on \( W(\ker[G|I]) \) and the image filtration on \( X_\infty(G) \) give isomorphic filtered vector spaces.

\[ W([G|I]) \times_1 W(\text{im} \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right)) \rightarrow k \]

\[ \alpha \downarrow \quad \uparrow \pi_+ \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right) \]

\[ X_\infty([G]) \times_2 X_\infty([G]^T) \rightarrow k \]

**Diagram 2**

In this case when \( G \) is polynomial row minimal the kernel filtration on \( X_\infty([G]^T) \) gives an isomorphism with the \( \pi_+ \)-filtration on \( W(\text{im} \left( \begin{pmatrix} G^T \\ I \end{pmatrix} \right)) \) via the map \( \beta \) above.

The added corollary in this case is that the kernel filtration on \( X_\infty([G]^T) \) is dual to the image filtration on \( X_\infty([G]) \).

(iii) The global observability indices are exactly the row degrees of \([D|N]\), where \( G = D^{-1}N \) is a left coprime factorization of \( G \).

The Global Controllability Filtration: Given a transfer function \( G : U(z) \rightarrow Y(z) \), we defined the filtration

\[ \mathcal{X}(G) = X_s \supset X_{s-1} \supset \ldots \supset X_0 = (0), \]  

(7.93)
where

\[ X_i = \beta^{-1}(W_i), \ i = s, s-1, \ldots, 0 \]  

(7.94)

and

\[ W_i := \frac{\pi_+(\text{im} \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \cap (k + \ldots + z^{i-1}k)}{(\pi_+(\text{im} \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \cap \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \cap (k + \ldots + z^{i-1}k)}. \]  

(7.95)

The GGCF indices are given by the conjugate partition corresponding to the global controllability filtration. Here the statement is that the \( \pi_+ \)-filtration on \( W(\text{im} \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \) and the global controllability filtration are isomorphic as filtered vector spaces.

The following facts supported this definition:

(i) If \( G \) is a strictly proper transfer function then \( \mathcal{X}(G) = X(G) \) and filtration on the classical pole module is simply the classical controllability filtration. Thus the Wedderburn filtration on \( W(\text{im} \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \) and the classical controllability filtration were isomorphic as filtered vector spaces via the map \( \beta \) of the main exact sequence.

(ii) When \( G(z) \) was a minimal column proper polynomial transfer function, then the kernel filtration on \( X_\infty(G) \) was isomorphic to \( \pi_+ \)-filtration on \( W(\text{im} \left( \begin{pmatrix} G \\ I \end{pmatrix} \right)) \).

(iii) If \( G = ND^{-1} \) is a right coprime factorization and \( L = \left( \begin{pmatrix} D \\ N \end{pmatrix} \right) \) (i.e., \( L \) is a minimal column basis for its column span) then the global controllability indices are exactly the column degrees of \( L \). These facts for the controllability case agree with the work of Malabre, (see [7, Ch. 6] and [9, 10, 20, 21]). Some of the philosophies and general techniques employed here, using the Fundamental Pole-Zero Exact Sequence, appear also in [4, 5]. The duality theory and the Wedderburn Forney space led to the present results.
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