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The structure of a three-dimensional vortex

Orangi, Saeed, Ph.D.
The Ohio State University, 1994
THE STRUCTURE OF A THREE-DIMENSIONAL VORTEX

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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1994

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CHAPTER I

Introduction

Vortical structures, commonly known as vortices, constitute significant regions of fluid motion in all branches of fluid mechanics. Theoretical analysis and numerical computation of complicated flow fields may become more tractable if the essentials of the behavior of the attendant vortical structures are better understood.

In aerodynamics, in which consideration of vortex flows may be regarded as distinctive, three dimensional vortices arise on slender configurations and highly swept wings at large angles of incidence. When fluid passes over the sharp leading-edges of slender delta wings at incidence it separates forming free shear layers. These layers curve upward and inboard, and under the influence of vorticity contained in them, eventually roll up in a spiral fashion to form roughly axisymmetric cores. The presence of these leading-edge vortices in the proximity of the wing surface enhances the lift capability greatly. This substantial nonlinear contribution to lift partially compensates for the loss of the leading-edge suction peak and the consequent increase in the drag.

Vortices trailing an aircraft are quite stable and may persist in tightly rolled up form up to a distance 1000 times the wing span. Because of the persistence of these vortices, an aircraft flying into the swirling vortex wake of another aircraft ahead of
it may be thrown out of control. Therefore it is of practical importance to find means to dissipate them at shorter distances [14].

Other aspects of aerodynamics in which vortex motion is important include: the scarf vortex generated from the boundary layer on the side of the fuselage, inlet vortex, particularly during the ground running of a jet engine, and trapped vortex above the upper surface of an airfoil or high aspect ratio wing.

Vortex cores are also generated in a number of confined flows such as fluid injection into a supersonic combustor to enhance the mixing process and swirl atomizer. The existence of these types of vortices depends on an ambient distribution of vorticity.

Geophysical vortices such as tornadoes and hurricanes are examples of vortical structures that occur in the atmosphere. These vortices cover a wide range of flow phenomena involving the coupling of ambient vorticity and strong buoyant convection [21].

Favorable effects induced by vortex flows cannot be sustained indefinitely. For instance, in the case of a leading-edge vortex, when the angle of attack exceeds a certain value, the extent of the vortex core and its turbulence level increase abruptly. This sudden change in the structure of the vortex core is generally referred to as "vortex breakdown". In a different application, it may be beneficial to enhance the vortex breakdown, as in the case of supersonic combustors, in order to enhance the mixing process.

In all of the applications mentioned, the compressibility of the fluid is important. In spite of recognition of the importance of the compressibility effects for several
decades, theoretical studies of fully three-dimensional flows of compressible fluids with embedded vortices have yet to produce satisfactory results.

1.1 Objectives

The focus of our investigation is the structure of fully three-dimensional vortices in a perfect, viscous, heat-conducting gas. We will investigate a class of isolated vortices in both incompressible and compressible fluids, and present a detailed study of the effects of various flow parameters on the vortex structure. The solutions to be presented are self-similar with specifically power-law behavior of velocity, pressure, and temperature at the vortex edge. Once the vortex structure is properly established, it may be integrated into an overall picture of the complete flow field.

Vortical flows have been the focus of numerous theoretical and experimental studies. Before proceeding with the current study, a review of the relevant literature is in order.

1.2 Background

The major effort of investigations of vortical motions have concentrated on incompressible flows. A number of these studies present exact solutions of the Navier-Stokes equations and are summarized here.

A numerical solution for a viscous vortex in an infinite fluid was given by Long [15]. A boundary-layer approximation is made within the core and the flow outside the core is a potential vortex, so that the circulation approaches a constant as distance from the axis tends to infinity.
Batchelor [2] has given an account of the existence of a strong axial current near the axis of symmetry of a steady line vortex trailing from one side of a wing. A similarity solution for the vortex far downstream was obtained by supplementing the boundary-layer type approximation with the approximation that the axial velocity differs from the free stream speed by a small amount, as in the theory of wakes without swirl.

A theory for the core of incompressible leading-edge vortices over slender wings was developed by Hall [9]. Assuming that the flow is axially symmetric and rotational, with viscous effects confined to a slender inner core, an outer inviscid conical solution is found for the outer structure, which consists of simple logarithmic expressions for the velocity components and pressure. Using this inviscid solution as a starting point, Stewartson and Hall [22] obtained a solution for the viscous inner core in the form of an asymptotic expansion, containing inverse powers of the logarithm of a Reynolds number.

The experimental studies of vortex flows include the works of Earnshaw [7], and more recently Verhaagen and Van Ransbeeck [26] on incompressible leading-edge vortices. The results of Earnshaw agree well with the outer inviscid solution of Hall [9]. However, there is a significant discrepancy in the viscous inner solution, which may be attributed to the fact that the inner core is not laminar as the theory assumes [10]. Verhaagen and Ransbeeck have used a large half model delta wing to measure the flow field in the core of a leading-edge vortex. They measured the velocity and pressure distributions using a thin 5-hole probe, and found the axial velocity inside the vortex
core to possess a maximum value of three times its free stream value.

The theoretical studies of three dimensional compressible vortices are limited to several cases of inviscid flow and even fewer cases of viscous flow.

Mack [16] has studied two dimensional viscous, heat-conducting vortex flows of a perfect gas. He obtained simple relationships for flow quantities of the irrotational vortex for arbitrary Prandtl number. These expressions reduce to the isentropic relationships when Prandtl number is $\frac{1}{2}$, thus making the isentropic solution a special case of the irrotational vortex flow of a viscous, heat-conducting, perfect gas. The irrotational solution is only accurate for small Mach numbers. For high Mach numbers, the temperature variation across the vortex is large and the viscosity coefficient is no longer a constant. With a variable viscosity coefficient the flow becomes rotational. The solution of the rotational vortex is shown to have a strong Prandtl number dependence. However, there is no Prandtl number for which the rotational vortex is isentropic.

Brown [3] has generalized the solution obtained by Hall [9] for an incompressible leading-edge vortex to include the effects of compressibility. The vortex core, which is assumed to be geometrically slender, is divided into an incompressible outer layer and a compressible inner region. Brown concludes that a regular solution can be found throughout the vortex core if the ratio of specific heats lies between 1 and 2. The axial velocity is finite in the neighborhood of the axis, the circumferential velocity falls to zero as the axis is approached, and the density remains non-zero until the axis is reached. In general, compressibility produced a marked reduction in the variation
of the pressure and velocity fields across the vortex. An asymptotic solution by Brown and Mangler [4], which includes second-order terms, confirmed the results obtained by Brown.

A discussion of physical effects of compressibility on vortex structures was given by Hall [10]: The responsiveness of the internal core structures changes radically as the conditions on the outside of the core are changed. As a result the supersonic cores may be totally different from the low speed ones. The supersonic core flow leads to exceedingly low densities near the axis of the vortex.

Moore and Pullin [20] used a hodograph-plane numerical method to obtain solutions for the steady self-propagation with respect to the fluid at infinity of two equal, symmetrically shaped vortices in an inviscid compressible fluid. The flow is assumed to be shock free but may be locally supersonic. Their results indicate that the speed of propagation of a vortex pair of given dimensions and circulation decreases and the distortion of its boundary increases due to compressibility effects. They also developed a perturbation theory for vortices of small core radius and small Mach number.

Barsony-Nagy, Er-El, and Yungster [1] studied flows containing one or more point vortices interacting with a smooth or sharped-edge object in a subsonic inviscid flow using a Rayleigh-Janzen expansion method. The effects of compressibility on the positions of the vortices and on the velocities were shown to be significant.

More recently, Kandil and Kandil [12] and Mayer and Powell [18] have presented solutions for steady, axisymmetric, slender vortices. The location of vortex breakdown was predicted for an isolated vortex in the subsonic regime by Kandil and Kandil.
Mayer and Powell have used a similarity assumption to study the effects of vortex strength and axial pressure gradient on the flow field for incompressible vortices. Some limited results were also presented for "compressible" vortices, but with a constant viscosity coefficient. In particular, Mayer and Powell found that temperature attains its minimum value on the vortex axis as it monotonically decreases from the outer boundary. Furthermore, the temperature in the vortex core decreases for increasing Mach number. Stott and Duck [25] have obtained asymptotic results for a trailing line vortex in a compressible flow. Their solution develops in the same way as the incompressible case of Bachelor [2].

In the following chapters, a numerical simulation of an isolated quasi-cylindrical vortex will be presented. We will consider both the incompressible and compressible vortex flows as follows: We will begin with a discussion of the Navier-Stokes equations that govern the axially symmetrical flow of a perfect, viscous, heat-conducting gas. These equations are simplified by introduction of the quasi-cylindrical approximation. The applicable set of boundary conditions are presented and discussed. The governing equations are then transformed into a set of nonlinear ordinary differential equations by similarity assumptions. The numerical scheme used to discretize the similarity equations and Newton’s method employed to linearize the resulting nonlinear finite-difference equations are described. The linearized equations are then solved by an iterative method. Results are presented to examine the effects of various parameters such as vortex strength and compressibility on the structure of the vortex core. Finally, following a summary, some possible extensions of the current study,
including among others the stability of the vortex core, will be discussed.
CHAPTER II
Mathematical Formulation

The complete Navier-Stokes equations that govern the steady axially symmetric flow of a perfect, viscous, heat-conducting gas expressed in cylindrical polar coordinates, \((r, \theta, z)\), are

continuity:

\[
\frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} + \frac{\partial \rho w}{\partial z} = 0
\] (2.1)

axial momentum:

\[
\rho u \frac{\partial w}{\partial r} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( \mu r \frac{\partial u}{\partial r} + \mu r \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial z} \left( -\frac{2}{3} \mu r \frac{\partial u}{\partial r} - \frac{2}{3} \mu u + \frac{4}{3} \mu r \frac{\partial w}{\partial z} \right)
\] (2.2)

radial momentum:

\[
\rho u \frac{\partial u}{\partial r} + \rho w \frac{\partial u}{\partial z} - \frac{\rho v^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( \mu r \frac{\partial u}{\partial r} + \mu r \frac{\partial w}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{4}{3} \mu r \frac{\partial u}{\partial r} - \frac{2}{3} \mu u - \frac{2}{3} \mu r \frac{\partial w}{\partial z} \right) + \frac{2}{3} \mu r \frac{\partial u}{\partial r} - \frac{4}{3} \mu u - \frac{2}{3} \mu r \frac{\partial w}{\partial z} + \frac{2}{3} \mu w
\] (2.3)
tangential momentum:

\[
\rho u \frac{\partial v}{\partial r} + \rho w \frac{\partial v}{\partial z} + \frac{\rho uv}{r} = \mu \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left[ \mu r^2 \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \right] + \frac{\partial}{\partial z} \left[ \mu r \frac{\partial}{\partial z} \left( \frac{v}{r} \right) \right]
\]  

(2.4)

energy:

\[
\rho C_p w \frac{\partial \Theta}{\partial z} + \rho C_p u \frac{\partial \Theta}{\partial r} = \frac{\partial}{\partial z} \left( K \frac{\partial \Theta}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( K \frac{\partial \Theta}{\partial r} \right) + \left( u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z} \right) + \mu \left( \left( \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right)^2 + \frac{4}{3} \left( \frac{u}{r} \right)^2 \right) + \frac{4}{3} \left[ \frac{\partial u}{\partial r} \frac{\partial w}{\partial z} - \frac{1}{r} \frac{\partial}{\partial z} \left( \frac{u}{r} \frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} \right) \right] + r \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \frac{\partial v}{\partial r} - v \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + r \frac{\partial}{\partial z} \left( \frac{v}{r} \frac{\partial v}{\partial z} \right) \right]  
\]  

(2.5)

where \( r \) denotes the radial, \( z \) the axial, \( \theta \) the azimuthal direction and \( u, w, \) and \( v \) are the corresponding components of the velocity vector, respectively. The static pressure, density, and temperature are represented by \( p, \rho, \) and \( \Theta, \) and \( \mu, C_p, \) and \( K \) are the viscosity coefficient, specific heat at constant pressure, and thermal conductivity, respectively. In these equations, we have used the Stokes’ relation,

\[
\lambda + \frac{2}{3} \mu = 0
\]

where \( \lambda \) is the second coefficient of viscosity. Application of the Stokes’ relation, which states that the bulk viscosity vanishes, causes no loss of generality, since with the approximations to be made the bulk viscosity terms are small, and will be neglected.
The condition of axial symmetry, which is the only restriction on these equations, omits all variations in the azimuthal direction implying that conditions are the same in every meridian plane.

2.1 Quasi-cylindrical approximation

The quasi-cylindrical approximation which states that variations in the axial direction are much smaller than variations in the radial direction simplifies the Navier-Stokes equations (2.1)-(2.5) considerably.

In a manner analogous to boundary-layer theory, an estimate of the order of magnitude of each term in equations (2.1)-(2.5) is made. Orders of magnitude are assigned to each term by consideration of the boundary conditions and the necessity to balance the various forces in the flow.

The basic assumption is that the radial dimension is much smaller than the axial dimension. Hence, we take

\[ z \sim L_c \]

\[ r \sim \delta_t \]

where

\[ \delta_t \ll L_c \]

Similarly,

\[ w \sim w_e \]

\[ \rho \sim \rho_e \]
Here $w_e$ and $\rho_e$ are reference quantities corresponding to axial velocity and density at some location along the vortex edge. Substituting these orders of magnitude into the continuity equation (2.1) provides an estimate of the magnitude of the radial velocity,

$$u \sim w_e \frac{\delta_t}{L_c}$$

It follows from the axial momentum equation (2.2) that the inertial terms and the highest order viscous term are of comparable order if

$$\frac{w_e^2}{L_c} \sim \nu \frac{w_e}{\delta_t^2}$$

where $\nu$ is the kinematic viscosity. Rewriting this expression gives

$$\delta_t \sim \left( \frac{\nu L_c}{w_e} \right)^{\frac{1}{2}} = \left( \frac{\nu}{w_e L_c} \right)^{\frac{1}{2}} L_c$$

This indicates that $\delta_t$ is proportional to the square root of the inverse Reynolds number. This is also inferred from several exact solutions of the Navier-Stokes equations.

The order of magnitude of the pressure may also be established from equation (2.2) by the observation that at the outer boundary the Navier-Stokes equations reduce to the Euler equations. Therefore, the pressure gradient must balance the inertia terms.

The radial momentum equation (2.3) may be used to estimate the order of magnitude of the azimuthal velocity, $v$. One of the characteristics of the three-dimensional flow is the presence of a radial pressure gradient that must be balanced with the terms representing the inertial, viscous, and centrifugal forces. Clearly, the pressure term is always greater than both the inertia and viscous terms and thus the only remaining term is the centrifugal term.* A balance of these two terms gives

$$\frac{w_e^2}{\delta_t} \sim \frac{\nu^2}{\delta_t}$$
or
\[ v \sim w_e \]

The Navier-Stokes equations (2.1-2.5) are simplified by first estimating the orders of magnitude of all terms and then neglecting terms which are small compared with other terms in each equation, as the Reynolds number goes to infinity.

Finally, these equations are rendered dimensionless by defining nondimensional independent variables as

\[ \tilde{z} = \frac{z}{L_c} \quad \tilde{r} = \frac{r}{L_c} \]

Here \( L_c \) is some characteristic length. Also introducing normalized dependent variables of the form

\[ \tilde{w} = \frac{w}{w_e} \quad \tilde{u} = \frac{u}{w_e} \quad \tilde{v} = \frac{v}{w_e} \]

\[ \tilde{p} = \frac{p}{\rho_e w_e^2} \quad \tilde{\rho} = \frac{\rho}{\rho_e} \quad \tilde{\Theta} = \frac{\Theta}{\Theta_e} \quad \tilde{\mu} = \frac{\mu}{\mu_e} \]

where \( w_e, \rho_e, \Theta_e, \) and \( \mu_e \) are reference quantities corresponding to conditions at some location along the vortex edge.

### 2.2 Incompressible Flow

#### 2.2.1 Governing Equations

The conservation equations of mass and momentum in their non-dimensional boundary-layer form for a steady, viscous, heat-conducting axisymmetric vortex in incompressible flow expressed in cylindrical polar coordinates, \((r, \theta, z)\), are
continuity:

\[ \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \]  

(2.6)

axial momentum:

\[ u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\epsilon}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) \]  

(2.7)

radial momentum:

\[ \frac{\partial p}{\partial r} - \frac{v^2}{r} = 0 \]  

(2.8)

tangential momentum:

\[ u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \frac{\epsilon}{r} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \right] + \epsilon \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \]  

(2.9)

where \( \epsilon \) is the inverse Reynolds number, \( Re \), defined in terms of the characteristic length, \( L_c \), and the axial velocity at the outer boundary, \( w_e \),

\[ Re = \frac{\rho_e w_e L_c}{\mu_e} \]

The bars are omitted for convenience.

These equations are subject to the following boundary conditions on the axis \((r = 0)\),

\[ u = 0, \]  

(2.10)

\[ v = 0, \]  

(2.11)

\[ \frac{\partial w}{\partial r} = 0, \]  

(2.12)
and at the outer boundary,

\[
\begin{align*}
    w &= w_e(z) \\
    v &= v_e(z) \\
    p &= p_e(z)
\end{align*}
\]

(2.13)

Equation (2.10) expresses the fact that there are no sources or sinks on the axis of the vortex, and the symmetrical nature of the axial velocity is expressed by equation (2.12). The functional dependence of the quantities at the vortex edge on the axial coordinate, \(z\), stems from whatever the potential flow happens to be.

In addition, the solution of the quasi-cylindrical equations (2.6)-(2.9), which are of the parabolic type, requires that a set of boundary conditions be specified at some upstream cross-section. In contrast, the solution of the complete Navier-Stokes equations, which are of the elliptic type, requires that a set of boundary conditions be specified at some downstream cross-section as well as some upstream cross-section. The lack of downstream dependence of the quasi-cylindrical equations is a great advantage, numerically and conceptually, over the Navier-Stokes equations.

The governing equations (2.6) through (2.9) together with the boundary conditions, equations (2.10) through (2.13), form a system of four non-linear coupled equations which must be solved for the four unknowns, namely, \(u, v, w,\) and \(p\).

### 2.2.2 Similarity Equations

The governing equations (2.6) through (2.9) are a set of parabolic differential equations in two variables, \((r, z)\), which may be reduced to a set of ordinary differential equations for particular choices of \((w_e, v_e, p_e)\) in (2.13).

The premise that self-similar solutions exist, suggests that a similarity variable,
ξ, be defined and flow variables be expressed in terms of this similarity variable. ξ is taken to be

\[ \frac{d}{dr} \left( \frac{1}{r} \right) = \frac{r}{\xi^n} \quad (2.14) \]

and the flow variables are assumed to have the following forms:

\[
\begin{align*}
  u &= \epsilon^{\frac{1}{2}} z^{n+m-1} U_r(\xi) \\
  w &= z^m W(\xi) \\
  v &= z^m V(\xi) \\
  p &= z^{2m} P(\xi)
\end{align*}
\]

These particular forms reflect the requirement that the variables at the vortex edge in equations (2.13) be proportional to some power of the axial coordinate, \( z \) [8, 17]; therefore, such solutions are valid only for a vortex embedded in such a flow.

Transforming the coordinates from \((r, z)\) to \(\xi\) and substituting the assumed forms for the variables into the governing equations reduce these partial differential equations to a set of four coupled ordinary differential equations, provided that

\[ n = \frac{1}{2} (1 - m) \quad (2.19) \]

A modified radial velocity defined by

\[ U = U_r - n\xi W \]

is introduced to further simplify these equations; equations (2.6) through (2.9) then become
continuity:

\[ \frac{d(\xi U)}{d\xi} + (m + 2n)\xi W = 0 \quad (2.20) \]

radial momentum:

\[ \frac{dP}{d\xi} - \frac{V^2}{\xi} = 0 \quad (2.21) \]

axial momentum:

\[ (mW^2 - nV^2) + (k + 2m)P + U \frac{dW}{d\xi} - \frac{1}{\xi} \frac{d}{d\xi} (\xi \frac{dW}{d\xi}) = 0 \quad (2.22) \]

tangential momentum:

\[ V[(n + m)W + \frac{U}{\xi}] + U \frac{dV}{d\xi} - \frac{1}{\xi} \frac{d}{d\xi} (\xi \frac{dV}{d\xi}) + \frac{V}{\xi^2} = 0 \quad (2.23) \]

The boundary conditions on the axis (\( \xi = 0 \)), expressed in their self-similar form, are

\[ U = 0, \quad (2.24) \]

\[ V = 0, \quad (2.25) \]

\[ \frac{dW}{d\xi} = 0, \quad (2.26) \]

and at the outer boundary,

\[ \begin{align*}
W & = 1.0 \\
V & = V_e \\
P & = 1.0
\end{align*} \quad (\xi \to \infty) \quad (2.27) \]

where \( V_e \) is an arbitrary constant, which represents the swirl at the vortex edge.

Equations (2.27) represent the edge condition at a finite \( z \) value.
The transformation of the quasi-cylindrical equations to a set of ordinary differential equations has removed the requirement that a set of boundary conditions be specified at some upstream cross-section. We expect that for a wide class of upstream boundary conditions such self-similar profiles will develop far downstream [5].

2.3 Compressible Flow

2.3.1 Governing Equations

The non-dimensional boundary-layer equations expressing the conservation of mass, momentum, and energy for a steady, viscous, heat-conducting axisymmetric vortex in a compressible fluid, written in cylindrical polar coordinates, \((r, \theta, z)\), are

continuity:

\[
\frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} + \frac{\partial \rho w}{\partial z} = 0
\]  

(2.28)

axial momentum:

\[
\rho u \frac{\partial w}{\partial r} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\epsilon}{r} \frac{\partial}{\partial r} (\mu r \frac{\partial w}{\partial r})
\]

(2.29)

radial momentum:

\[
\frac{\partial p}{\partial r} - \frac{\rho v^2}{r} = 0
\]

(2.30)

tangential momentum:

\[
\rho u \frac{\partial v}{\partial r} + \rho w \frac{\partial v}{\partial z} + \frac{\rho uv}{r} = \frac{\epsilon}{r} \frac{\partial}{\partial r} [\mu r^2 \frac{\partial (\frac{v}{r})}{\partial r}] + \epsilon \mu \frac{\partial}{\partial r} (\frac{v}{r})
\]

(2.31)
energy:

\[
\rho u \frac{\partial \Theta}{\partial r} + \rho w \frac{\partial \Theta}{\partial z} = (\gamma - 1)M^2(u \frac{\partial p}{\partial r} + w \frac{\partial p}{\partial z}) \\
+ \frac{1}{Pr} \epsilon \frac{1}{r} \frac{\partial}{\partial r} (\mu r \frac{\partial \Theta}{\partial r}) + (\gamma - 1)M^2 \epsilon \mu r (\frac{\partial w}{\partial r})^2 \\
+ (\gamma - 1)M^2 \epsilon \mu r \frac{\partial}{\partial r} \left( \frac{v}{r} \frac{\partial v}{\partial r} - \mu v \frac{\partial}{\partial r} \left( \frac{v}{r} \right) \right)
\]

(2.32)

where \(\gamma\) is the ratio of specific heats, \(\epsilon\) is the inverse Reynolds number, \(Re\), \(Pr\) is the Prandtl number, and \(M\) is the Mach number defined in terms of the edge temperature,

\[
Re = \frac{\rho_e w_e I_e}{\mu_e}
\]

\[
M = \frac{w_e}{\sqrt{\gamma R \Theta_e}}
\]

where \(R\) is the gas constant. This system of equations is closed by utilizing the state equation

state:

\[
p = \frac{1}{\gamma M^2 \rho \Theta}
\]

(2.33)

The bars are omitted from these equations for convenience.

These equations are subject to the following boundary conditions on the axis \((r = 0)\),

\[
u = 0,
\]

(2.34)

\[
v = 0,
\]

(2.35)

\[
\frac{\partial w}{\partial r} = 0,
\]

(2.36)

\[
\frac{\partial \Theta}{\partial r} = 0,
\]

(2.37)
and at the outer boundary,

\[
\begin{align*}
  w &= w_e(z) \\
  v &= v_e(z) \\
  p &= p_e(z) \\
  \Theta &= \Theta_e(z)
\end{align*}
\] 

\( r \to \infty \) \hspace{1cm} (2.38)

The symmetrical nature of the temperature on the vortex axis is expressed by equation (2.37) and equations (2.38) fix the outer edge of the vortex. The functional dependence of the edge condition, equation (2.38), on the axial coordinate, \( z \), again depends on the character of the flow in which the vortex is embedded.

As for the incompressible flow, the solution of the quasi-cylindrical equations, (2.6)-(2.9), which are of the parabolic type, requires that a set of boundary conditions be specified at some upstream cross-section.

Equations (2.28) through (2.34) are a set of six scalar coupled equations which must be solved for the six unknowns, namely, \( u, v, w, p, \rho, \) and \( \Theta \), subject to the boundary conditions, equations (2.35) through (2.38).

### 2.3.2 Similarity Equations

The similarity forms of the governing equations are derived using the same procedure as in the incompressible case, using the same similarity variable, \( \xi \), and the similarity forms of solutions assumed for the velocity components, and pressure. However, the addition of the energy and state equations requires that similarity forms of solutions be also assumed for density and temperature

\[
\rho = z^k \rho(\xi)
\] 

(2.39)
The transformation of the coordinates and subsequent substitution of the assumed similarity forms into the governing equations (2.28)-(2.38) results in a set of six coupled ordinary differential equations if \( k \) is taken to be

\[
k = 1 - 2n + (2\omega - 1)m
\]  

(2.41)

The viscosity coefficient is temperature dependent and is approximated by the power law [22]

\[
\mu = T^{\omega}
\]

Thus the governing equations in self-similar form are

continuity :

\[
\frac{d(\rho \xi U)}{d\xi} + (k + m + 2n)\rho \xi W = 0
\]  

(2.42)

momentum :

\[
\frac{dP}{d\xi} - \frac{\rho V^2}{\xi} = 0
\]  

(2.43)

\[
\rho(mW^2 - nV^2) + (k + 2m)P + \rho U \frac{dW}{d\xi} - \frac{1}{\xi} \frac{d}{d\xi}(\xi \mu \frac{dW}{d\xi}) = 0
\]  

(2.44)

\[
\rho V[(n + m)W + \frac{U}{\xi}] + \rho U \frac{dV}{d\xi} - \frac{1}{\xi} \frac{d}{d\xi}(\mu \mu \frac{dV}{d\xi}) + V \frac{1}{\xi^2} \frac{d}{d\xi} (\xi \mu) = 0
\]  

(2.45)

energy :

\[
2m\rho WT + \rho U \frac{dT}{d\xi} - (\gamma - 1)M^2[(k + 2m)PW + \rho \frac{V^2}{\xi}] =
\]

\[
\frac{1}{Pr} \frac{1}{\xi} \frac{d}{d\xi}(\mu \frac{dT}{d\xi}) + (\gamma - 1)M^2\mu [e_{rz}^2 + e_{r\theta}^2]
\]

\[
e_{rz} = \frac{dW}{d\xi}, \quad e_{r\theta} = \frac{\xi}{d\xi} (\frac{V}{\xi})
\]  

(2.46)
state:

\[ \gamma M^2 P = \rho T \]  \hspace{1cm} (2.47)

and the boundary conditions on the axis \((\xi = 0)\), expressed in their similarity form are

\[ U = 0, \]  \hspace{1cm} (2.48)
\[ V = 0, \]  \hspace{1cm} (2.49)
\[ \frac{dW}{d\xi} = 0, \]  \hspace{1cm} (2.50)
\[ \frac{dT}{d\xi} = 0, \]  \hspace{1cm} (2.51)

and at the outer boundary,

\[
\begin{align*}
W & = 1.0 \\
V & = V_e \\
T & = 1.0 \\
P & = \frac{1}{\gamma M^2}
\end{align*}
\]  \hspace{1cm} \xi \to \infty \hspace{1cm} (2.53)

Equations (2.53) represent the edge condition at a specific axial location which is given by a finite \(z\) value.

As for the incompressible flow, the requirement that a set of boundary conditions be specified at some upstream cross-section has been removed by the transformation of the quasi-cylindrical equations to a set of ordinary differential equations.
2.4 Boundary Condition Modification

The edge swirl velocity, $V_e$, determines the strength of the vortex. Therefore, varying $V_e$ generates vortices with different intensities. However, it is found that by specifying $V_e$ convergence of the numerical scheme cannot be achieved. This boundary condition (2.53b) may be replaced by specifying the slope of the swirl, $V'$, on the vortex axis [19].

The reason for this and other alternatives that have been tested will be discussed in more detail in the next chapter.
CHAPTER III

Method of solution

A finite-difference method is used to solve the governing equations in their self-similar form. The numerical scheme used is referred to as the Box Method [13].

Application of this scheme requires that the governing equations be reduced to a set of first-order equations. This is done by introducing two new dependent variables \( W'(\xi) \) and \( V'(\xi) \) defined by

\[
W' = \frac{dW}{d\xi} \\
V' = \frac{dV}{d\xi}
\]

3.1 Incompressible Flow

3.1.1 Finite-Difference Equations

Substituting \( W' \) and \( V' \) into the governing equations and the corresponding boundary conditions, equations (2.20)-(2.27), yields

**continuity:**

\[
\frac{d(\xi U)}{d\xi} + (m + 2n)\xi W = 0
\]

(3.3)

**radial momentum:**

\[
\frac{dP}{d\xi} - \frac{V^2}{\xi} = 0
\]

(3.4)
axial momentum:

\[(mW^2 - nV^2) + (k + 2m)P + UW' - \frac{1}{\xi} \frac{d}{d\xi}(\xi\mu W') = 0 \]  \hspace{1cm} (3.5)

tangential momentum:

\[V[(n + m)W + \frac{U}{\xi}] + UV' - \frac{1}{\xi} \frac{d}{d\xi}(\xi V') + \frac{V}{\xi^2} = 0 \]  \hspace{1cm} (3.6)

on the axis \((\xi = 0)\),

\[U = 0, \]  \hspace{1cm} (3.7)
\[V = 0, \]  \hspace{1cm} (3.8)
\[V' = V_0', \]  \hspace{1cm} (3.9)
\[W' = 0, \]  \hspace{1cm} (3.10)

and at the outer boundary,

\[
\begin{align*}
W & = 1.0 \\
\frac{P}{\mu} & = 1.0
\end{align*}
\]  \hspace{1cm} \[\xi \to \infty \]  \hspace{1cm} (3.11)

Note that equation (3.9) specifies the slope of the swirl velocity on the vortex axis, which is used to replace the swirl edge condition. Equations (3.3)-(3.11) are still well-posed, with the change in the boundary condition.

To obtain the finite-difference form of these first-order equations, the derivatives are approximated by a central-difference formula. The numerical domain which spans from the vortex axis, \(\xi = 0\), to the edge of the vortex, \(\xi = \xi_e\), is divided into \(N\) equally-spaced intervals. The width of each interval is given by

\[\Delta\xi = \frac{\xi_e}{N}\]
The finite-difference form of equations (3.3), (3.5), and (3.6) are written for the mid-point of each interval designated by $\xi_{j-1/2}$ which is defined by

$$\xi_{j-1/2} = \frac{\xi_j + \xi_{j-1}}{2}$$

Each dependent variable is similarly defined at the mid-points; for example

$$U_{j-1/2} = \frac{U_j + U_{j-1}}{2}$$

The resulting finite-difference equations are a set of nonlinear algebraic equations that must be linearized. The linearization is done using Newton’s method and writing the Newton iterates for all dependent variables (i.e. $U$) as follows:

$$U_j^{(i+1)} = U_j^{(i)} + (\delta U)_j^{(i)}$$

where $U_j^{(i+1)}$ is the $(i+1)$ iterate.

These iterates are then substituted into the equations and terms containing quadratic $\delta$ quantities, $\delta U$, $\delta W$, $\delta W'$, $\delta V$, and $\delta V'$, are dropped. Thus, the linearized finite difference equations are obtained

**continuity** :

$$(\alpha_1)_j(\delta U)_j + (\alpha_2)_j(\delta U)_{j-1} + (\alpha_3)_j(\delta W)_j + (\alpha_4)_j(\delta W)_{j-1} = (R_c)_j$$

(3.13)
axial momentum:

\[(\beta_1)_2(\delta U)_2 + (\beta_2)_2(\delta U)_{j-1} + (\beta_3)_2(\delta W)_2 + (\beta_4)_2(\delta W)_{j-1} + \]

\[(\beta_5)_2(\delta W')_2 + (\beta_6)_2(\delta W')_{j-1} + (\beta_7)_2(\delta V)_2 + (\beta_8)_2(\delta V)_{j-1} = (R_s)_j \]  

(3.14)

tangential momentum:

\[(\theta_1)_2(\delta U)_2 + (\theta_2)_2(\delta U)_{j-1} + (\theta_3)_2(\delta W)_2 + (\theta_4)_2(\delta W)_{j-1} + \]

\[(\theta_5)_2(\delta V)_2 + (\theta_6)_2(\delta V)_{j-1} + (\theta_7)_2(\delta V')_2 + (\theta_8)_2(\delta V')_{j-1} = (R_\theta)_j \]

(3.15)

and equations for \( W' \) and \( V' \) are

\[2(\delta W')_2 - 2(\delta W)_{j-1} - \Delta \xi ((\delta W')_2 + (\delta W')_{j-1}) = (R_w)_j \]

(3.16)

\[2(\delta V')_2 - 2(\delta V)_{j-1} - \Delta \xi ((\delta V')_2 + (\delta V')_{j-1}) = (R_v)_j \]

(3.17)

Similarly, the boundary conditions become

\[(\delta U)_o = 0, \]

(3.18)

\[(\delta W')_o = 0, \]

(3.19)

\[(\delta V)_o = 0, \]

(3.20)

\[(\delta V')_o = 0, \]

(3.21)

\[(\delta W)_N = 0 \]

(3.22)

The foregoing conditions satisfy the requirement that the boundary conditions remain constant throughout the iteration process.
The coefficients and the right-hand side of equations (3.13) through (3.17) are given in appendices A and B, respectively.

The radial momentum equation (3.4) is integrated to compute the pressure distribution for a given swirl velocity profile. Integrating equation (3.4), using the second-order accurate trapezoid rule, subject to the edge condition, equation (3.11), gives

\[ P_j = P_{j+1} - \frac{\Delta \xi}{2} \left[ \left( \frac{V^2}{\xi} \right)_j + \left( \frac{V^2}{\xi} \right)_{j+1} \right] \]

\[ j = N - 1, \ldots, 2 \]

on the axis \( r = 0 \),

\[ P_1 = P_2 - \frac{\Delta \xi}{2} \left( \frac{V^2}{\xi} \right)_2 \]

3.1.2 Numerical Procedure

The linear system of equations, (3.13) through (3.17), and the boundary conditions, equations (3.19) through (3.22), are written in matrix-vector form as

\[ A \delta = R \]

where \( \delta \) and \( R \) are column vectors representing the solution and the right-hand side vectors, respectively

\[
\delta = \begin{bmatrix}
\delta_0 \\
\delta_1 \\
\vdots \\
\delta_{N-1} \\
\delta_N 
\end{bmatrix}
\]

\[
\delta_j = \begin{bmatrix}
\delta(U)_j \\
\delta(W)_j \\
\delta(W')_j \\
\delta(V)_j \\
\delta(V')_j 
\end{bmatrix}
\]
\[ R = \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_{N-1} \\ R_N \end{bmatrix} \quad R_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (R_{w1}) \]

\[ R_j = \begin{bmatrix} (R_c)_j \\ (R_z)_j \\ (R_\theta)_j \\ (R_\nu)_j \\ (R_w)_{j+1} \end{bmatrix} \quad R_N = \begin{bmatrix} (R_c)_N \\ (R_z)_N \\ (R_\theta)_N \\ (R_\nu)_N \end{bmatrix} \quad (3.28) \]

The coefficient matrix, \( A \), is given by

\[
A = \begin{bmatrix}
A_0 & C_0 \\
B_1 & A_1 & C_1 \\
B_2 & A_2 & C_2 \\
& & \ddots & \ddots & \ddots \\
& & \ddots & & \ddots \\
& & & B_{N-2} & A_{N-2} & C_{N-2} \\
& & & B_{N-1} & A_{N-1} & C_{N-1} \\
& & & & B_N & A_N
\end{bmatrix} \quad (3.29)
\]

where \( A_j \), \( B_j \), and \( C_j \) are 5x5 matrices

\[
A_0 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -2 & -\Delta \xi & 0 & 0
\end{bmatrix} \quad (3.30)
\]

\[
C_j = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 2 & -\Delta \xi & 0 & 0
\end{bmatrix} \quad 0 \leq j \leq N - 1 \quad (3.31)
\]
There are various ways to set up the coefficient matrix. The order chosen to form \( A \) must ensure that \( A_0 \) is not singular.

The Box scheme used to obtain this linear system of equations has generated a block-tridiagonal system, where each block is a 5x5 matrix. Hence, the block elimination method will be employed to solve this system.

An iterative procedure is utilized to obtain the velocity profiles and the pressure distribution. Initial profiles are defined for the velocity components, and in conjunction with the boundary conditions are used to compute an initial pressure distribution,
all coefficients, and the right-hand side column vectors. A block-tridiagonal solver is then employed to solve for the solution vector, $\delta$. The computed values of $\delta$ are used to update the velocity profiles, compute new pressure distribution, and check the convergence criterion

$$\max |\delta_i| \leq \epsilon$$

where $\epsilon$ is the tolerance permitted. The iteration procedure is repeated until the convergence criterion is satisfied. A flow diagram of the algorithm is depicted in figure 1.

### 3.2 Compressible Flow

#### 3.2.1 Finite-Difference Equations

Finite-difference methods are utilized to solve the governing equations, (2.42) through (2.47), in conjunction with the boundary conditions, equations (2.48) through (2.53).

Substituting the equations for $W'$ and $V'$, (3.1) and (3.2), into equations (2.42)-(2.45) yields the following first-order equations:

**continuity**:

$$\frac{d(\rho U)}{d\xi} + (k + m + 2n)\rho W = 0$$

(3.36)

**radial momentum**:

$$\frac{dP}{d\xi} - \frac{\rho V^2}{\xi} = 0$$

(3.37)
Figure 1: Flow diagram of the algorithm for incompressible flow
axial momentum:

\[ \rho(mW^2 - nV^2) + (k + 2m)P + \rho UW' - \frac{1}{\xi} \frac{d}{d\xi}(\xi\mu W') = 0 \quad (3.38) \]

tangential momentum:

\[ \rho V[(n + m)W + \frac{U}{\xi}] + \rho UV' - \frac{1}{\xi} \frac{d}{d\xi}(\mu \xi V') + V \frac{1}{\xi^2} \frac{d}{d\xi}(\mu \xi) = 0 \quad (3.39) \]

As in the incompressible case, equations (3.36), (3.38), and (3.39) are discretized using a central-difference formula and linearized by Newton's method to obtain

continuity:

\[ (\alpha_1)_j(\delta U)_j + (\alpha_2)_j(\delta U)_{j-1} + (\alpha_3)_j(\delta W)_j + (\alpha_4)_j(\delta W)_{j-1} = (R_c)_j \]

(3.40)

axial momentum:

\[ (\beta_1)_j(\delta U)_j + (\beta_2)_j(\delta U)_{j-1} + (\beta_3)_j(\delta W)_j + (\beta_4)_j(\delta W)_{j-1} + \]
\[ (\beta_5)_j(\delta W')_j + (\beta_6)_j(\delta W')_{j-1} + (\beta_7)_j(\delta V)_j + (\beta_8)_j(\delta V)_{j-1} = (R_\alpha)_j \]

(3.41)

tangential momentum:

\[ (\theta_1)_j(\delta U)_j + (\theta_2)_j(\delta U)_{j-1} + (\theta_3)_j(\delta W)_j + (\theta_4)_j(\delta W)_{j-1} + \]
\[ (\theta_5)_j(\delta V)_j + (\theta_6)_j(\delta V)_{j-1} + (\theta_7)_j(\delta V')_j + (\theta_8)_j(\delta V')_{j-1} = (R_\theta)_j \]

(3.42)
and equations for $W'$ and $V'$ are

$$2(\delta W)_j - 2(\delta W)_{j-1} - \Delta \xi ((\delta W')_j + (\delta W')_{j-1}) = (R_w)_j \tag{3.43}$$

$$2(\delta V)_j - 2(\delta V)_{j-1} - \Delta \xi ((\delta V')_j + (\delta V')_{j-1}) = (R_v)_j \tag{3.44}$$

Similarly, the boundary conditions become

$$(\delta U)_o = 0, \tag{3.45}$$

$$(\delta W')_o = 0, \tag{3.46}$$

$$(\delta V)_o = 0, \tag{3.47}$$

$$(\delta V')_o = 0, \tag{3.48}$$

$$(\delta W)_N = 0 \tag{3.49}$$

The radial momentum equation (3.37) is integrated to compute the pressure distribution for a given swirl velocity profile and density distribution. Integrating equation (3.37), using the second-order accurate trapezoid rule gives

$$P_j = P_{j+1} - \frac{\Delta \xi}{2} \left[ \left( \frac{\rho V'^2}{\xi} \right)_j + \left( \frac{\rho V'^2}{\xi} \right)_{j+1} \right] \tag{3.50}$$

$$j = N - 1, \ldots, 2$$

which is subject to the edge condition,

$$P = \frac{1}{\gamma M^2} \left\{ \begin{array}{c} \xi \to \infty \end{array} \right. \tag{3.51}$$

on the axis ($\xi = 0$),

$$P_1 = P_2 - \frac{\Delta \xi}{2} \left( \frac{\rho V'^2}{\xi} \right)_2 \tag{3.52}$$
The energy equation (2.46) is discretized using a second-order accurate central-difference scheme. Using the same numerical grid, this equation is applied at all interior grid points. The resulting finite-difference equation is

\[(\zeta_1)T_{j-1} + (\zeta_2)T_j + (\zeta_3)T_{j+1} = (R_e)_j\] (3.53)

The boundary condition on the axis is approximated by a forward-difference formula which is second-order accurate

\[-3T_0 + 4T_1 - T_2 = 0\] (3.54)

and at the outer edge,

\[T_N = 1.0\] (3.55)

The coefficients and the right-hand side of equations (3.40), (3.42) through (3.46), and equation (3.53) are given in appendices A and B, respectively.

Finally, the state equation, which is algebraic, is written as

\[\rho_j = \frac{\gamma M^2 P_j}{T_j}\] (3.56)

### 3.2.2 Numerical Procedure

Equations (3.40) through (3.44) and the corresponding boundary conditions, equations (3.46) through (3.49), form a linear system of order five which is expressed in matrix-vector form

\[A \delta = R\] (3.57)
The coefficient matrix, \( A \), and the column vectors, \( \delta \) and \( \mathbf{R} \), have the same form as the ones given for the incompressible flow and will not be repeated here. The energy equation (3.50) is also written in matrix-vector form

\[
a \mathbf{T} = \mathbf{r}
\]

(3.58)

where \( \mathbf{T} \) and \( \mathbf{r} \) are column vectors given by

\[
\mathbf{r} = \begin{bmatrix}
\tau_0 \\
\tau_1 \\
\vdots \\
\tau_{N-1} \\
\tau_N
\end{bmatrix}
\quad \mathbf{T} = \begin{bmatrix}
T_0 \\
T_1 \\
\vdots \\
T_{N-1} \\
T_N
\end{bmatrix}
\]

(3.59)

The elements of these vectors are defined as follows:

\[
\tau_0 = -T_3, \quad \tau_j = (R_e)_{j}, \quad \tau_N = T_e
\]

The coefficient matrix, \( a \), is a tridiagonal matrix given by

\[
a = \begin{bmatrix}
    a_0 & c_0 & & & \\
    b_1 & a_1 & c_1 & & \\
    b_2 & a_2 & c_2 & & \\
    & \ddots & \ddots & \ddots & \\
    & & b_{N-2} & a_{N-2} & c_{N-2} \\
    & & b_{N-1} & a_{N-1} & c_{N-1} \\
    & & b_N & a_N &
\end{bmatrix}
\]

(3.60)

where \( a_j, b_j, \) and \( c_j \) are scalars

\[
a_0 = 3, \quad c_0 = -4, \quad b_N = 0
\]

\[
b_j = (\zeta_1)_j, \quad a_j = (\zeta_2)_j, \quad c_j = (\zeta_3)_j
\]
An iterative method, similar to that used for the incompressible flow, is used to obtain the velocity profiles and the pressure, temperature, and density distributions. The iterative procedure consists of three steps, once initial values are defined for the dependent variables consistent with the governing equations and the boundary conditions. First, the coefficients and the right-hand sides of the equations in the linear system, $A\delta = R$, are computed. Using a block-tridiagonal solver, the solution vector $\delta$ is obtained and used to update $U, W, V$ and also determine the maximum $|\delta_j|$. In the second step, the coefficients and the right-hand side of the energy equation are computed using the updated values of the velocity vector. The linear system $aT = r$ is then solved using the Thomas algorithm to obtain the temperature distribution. Finally, values for pressure, density, and the coefficient of viscosity are computed using equations (3.50), (3.56), and the power law, respectively. To ensure that the computed values of pressure and density satisfy equations (3.50) and (3.56), these values are computed five times for each iteration. This additional step increases the stability of the numerical procedure. This procedure is repeated until the convergence criterion:

$$\max |\delta_j| \leq \epsilon$$

is satisfied. The iterative procedure is shown, in detail, in figure 2.

3.3 Boundary Condition Modification

As was pointed out in section (2.3), specifying the edge swirl, $V_z$, causes the numerical method to become unstable and convergence cannot be achieved. The cause of this
Figure 2: Flow diagram of the algorithm for compressible flow
numerical instability appears to be the high degree of sensitivity to the initial guess used to initiate the iteration process.

To remedy this convergence problem, the behavior of the similarity equations (2.20)-(2.23) and (2.42)-(2.47) was investigated in the limit as $\xi \to \infty$ using asymptotic methods. However, no conclusions could be drawn from these investigations that would explain how the swirl and axial velocities behave as the outer edge is approached.

Another approach was to specify the edge conditions in terms of $\xi$. For example

$$V_e = \xi^\lambda$$

where different values of $\lambda$ (i.e. 1, -1, and 2) were used. This same condition was utilized differently by applying the functional form at the last two grid points, $N$ and $N - 1$, and using the ratio as a boundary condition

$$\frac{V_N}{V_{N-1}} = \frac{(\xi_{N-1})^\lambda}{(\xi_N)^\lambda}$$

The advantage of the latter approach is that the functional form is built into the numerical scheme and the solution is thus made to fit a particular behavior. Despite considerable effort, none of these approaches were successful.

The convergence problem was resolved by replacing the edge swirl condition with

$$V' = V'_0$$

where $V'_0$ is a constant value representing the slope of the swirl velocity on the vortex

$^1$The majority of this work was done by Prof. M. R. Foster
axis. To obtain a particular edge swirl, $V_e$ is varied until the converged swirl velocity has the desired edge value.

### 3.4 Alternative Numerical Approaches

The numerical method, described in the preceding sections, represents one of several methods that have been tested. In this section, we will discuss other numerical schemes and procedures that were utilized to solve the governing equations, but failed to produce converged results.

Two other numerical schemes were used to discretize the governing equations. In the first scheme, the axial and tangential momentum equations in their second-order form were approximated by a central-difference formula. The continuity and radial momentum equations were integrated directly to obtain the radial velocity and pressure, respectively. The second scheme is similar to that described in the preceding sections; the only difference is that the discretization was done at each grid point, rather than the mid-points. As a result, the finite difference equations contain fewer terms, and there is less algebra involved. Both of these schemes were numerically unstable and no convergence was achieved.

Application of the governing equations at the outer edge and on the vortex axis provides the additional equations that are needed to complete the coefficient matrix. On the vortex axis, the governing equations are singular; therefore, the singularity must be removed by consideration of the limit behavior of each equation as the vortex axis is approached. The governing equations may also be applied at the vortex edge. Although these equations are not singular there, they must be used in a manner that
does not make the coefficient matrix singular. Our experience shows that application of the governing equations on the axis is problematic, which may partially be due to the loss of tridiagonality of the coefficient matrix.
CHAPTER IV

Results and Discussion

The structure of an isolated vortex is investigated by computing the various flow variables for a range of parameters that appear in the governing equations, (2.20)-(2.27) for incompressible flow and (2.42)-(2.53) for compressible flow. These parameters may be categorized into two groups: 1) similarity parameters, 2) physical parameters. Here similarity parameters are those parameters that appear in the governing equations as a result of the coordinate transformation and the explicit $z$-dependence of self-similar forms representing the flow quantities. In addition, a third group, referred to as the numerical parameters, must also be considered.

The numerical parameters are the number of grid points, $N$, that determines the grid spacing, and the convergence tolerance, $\varepsilon$. It is necessary that the numerical solutions be made independent of the numerical mesh to within the allowed tolerance. The discretization error, defined as the difference between the solutions of the differential equations and the finite-difference equations, should decrease as the mesh is refined. Therefore, one expects that the difference in solutions computed using two successive meshes to be less than the assigned tolerance [23].

Tables 1 and 2 show the axial velocity and temperature distributions, respectively, for different values of $\varepsilon$ for a particular vortex ($M = 2.0, m = 0.0, n = 0.5$). Small
changes are observed as $\varepsilon$ is changed from $10^{-3}$ to $10^{-5}$. However, as $\varepsilon$ is reduced to $10^{-6}$ the distributions remain the same up to 5 significant figures.

The effect of the grid spacing is shown in figures 3 and 4 for the same vortex. As the number of grid points, $N$, is increased from 250 to 500 an upward shift is observed in both the axial velocity and the temperature profiles. However, when $N$ is increased to 1000 changes observed in either of the profiles are less than 0.04%.

Based on these results, $\varepsilon$ and $N$ are taken to be $10^{-5}$ and 500, respectively, for all solutions obtained. The number of iterations required to obtain converged results depends on how close the initial guesses are to the actual solution, and also the physical parameters involved. For example, the number of iterations needed to obtain converged results increases for increasing $M$, and more significantly, the magnitude of swirl velocity. The number of iterations ranged from 10, for an incompressible vortex, to 250, for several compressible vortex solutions. The C.P.U. time required to achieve convergence on the Cray Y-MP supercomputer ranges from approximately 12 to 250 seconds.

To generate solutions for vortices of differing strengths, the derivative of the swirl velocity on the axis, $V_e'$, is specified. However, for comparison purposes and also the ability to study the effect of all parameters on the vortex structure individually, solutions for vortices of similar strength must be generated. This is accomplished by an iterative procedure in which $V_e'$ is systematically changed until the desired strength in terms of $V_e$ is obtained.

Furthermore, the outer edge of the numerical grid, $\xi_e$, is set equal to 20 for all
Table 1: Axial velocity distribution for various $\varepsilon$

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Table 2: Temperature distributions for various $\epsilon$

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Figure 3: Axial velocity distribution for various $N$

($m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0$)

--- $N = 250$, --- $N = 500$, --- $N = 1000$
Figure 4: Temperature distribution for various $N$

$\left( m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0 \right)$

- - - $N = 250$, \hspace{1em} \cdots \hspace{1em} N = 500, \hspace{1em} \linebreak[0] \hline \hspace{1em} N = 1000
solutions [18]. The effect of ξe on the solution and the choice of a particular value will be discussed later.

4.1 Incompressible Flow

The similarity parameters that appear in the governing equations for the incompressible flow are m and n, and the physical parameter is $V_e$. The vortex core z-dependence given by $n$ is uniquely determined by the choice of $m$ through equation (2.19). The z-dependence of the flow quantities is expressed by $m$, and $V_e$ determines the vortex strength.

Solutions are presented for vortices of various strengths for $m = 0$. This value of $m$ corresponds to uniform conditions in the external flow field. Figure 5 shows the swirl velocity profiles. All profiles reach their maximum value in the vicinity of the vortex axis. As the vortex strength increases the peak becomes more pronounced and the slope becomes steeper as it decays toward the edge value.

The axial velocity distributions are depicted in figure 6. These profiles are jet-like and are increasingly intense for growing $V_e$. The radial velocity profiles which are shown in figure 7 always indicate inward flow which one would expect on physical grounds, since the radial inflow confines the vorticity. These profiles are nearly linear for weaker vortices and deviate from this only slightly for stronger vortices. The pressure coefficient defined by

$$C_p = \frac{p - p_e}{\rho w_e^2}$$

is shown in figure 8. It has its minimum value on the axis, and becomes as low as -9
Figure 5: Swirl velocity distribution for $m=0$

$V_0' = 0.2$, $V_0' = 0.4$, $V_0' = 0.8$, $V_0' = 1.0$
for a vortex with an edge swirl of about 1. The corresponding axial velocity on the axis attains a value nearly 3 times that of the external flow.

The solutions presented thus far agree qualitatively with the theoretical results given by Hall [9], Stewartson and Hall [24], and the experimental results of Verhaagen and Van Ransbeeck [26]. There is also good agreement with the numerical solutions of Mayer and Powell [18]. Figure 9 compares our calculation of the swirl velocity with Mayer and Powell for $V_e = 0.82$. The small discrepancy may be due to reproduction of their results from an existing plot.

To examine the effect of $m$ on the vortex structure, solutions are presented for both positive and negative values of this parameter. These cases correspond to vortex edge conditions which are not uniform in $z$, i.e. a vortex in an axial pressure gradient.

Figures 10-13 illustrate the velocity and pressure coefficient distributions for $m = 0, 0.1, 0.2$ for vortices of equal strength as measured by $V_e$. Note that an increase in $m$ causes a drop in swirl and axial velocities while maintaining essentially the same profile qualitatively. There is a significant change in the radial velocity compared with results presented in figure 7 for $m = 0$. These profiles are linear only in the vicinity of the axis. This effect is due to a significant increase in the vortex intensity and not due to the $m$ value. A similar trend was observed in figure 7 for $m = 0$. The pressure coefficients have the same qualitative shape but show a significant decrease in magnitude.

Solutions for $m = -0.6, -0.4, -0.2, 0.0$ are shown in figures 14-17. The changes in velocity profiles are much more pronounced than those for positive $m$. 
Figure 6: Axial velocity distribution for m=0

\[ V'_0 = 0.2, \quad V'_0 = 0.4, \quad V'_0 = 0.8, \quad V'_0 = 1.0 \]
Figure 7: Radial velocity distribution for $m=0$

$\cdots V_0' = 0.2, \quad \cdots V_0' = 0.4, \quad \cdots V_0' = 0.8, \quad \cdots V_0' = 1.0$
Figure 8: Pressure coefficient distribution for $m=0$

$\cdot \cdot \cdot V_0' = 0.2, \quad \ldots \quad V_0' = 0.4, \quad \ldots \quad V_0' = 0.8, \quad V_0' = 1.0$
Figure 9: Swirl velocity distribution for $m = 0$

- - Mayer and Powell [18], — present results
Figure 10: Swirl velocity distribution for $m \geq 0$

\[ \cdots m = 0.0, \quad -\- m = 0.1, \quad -\-\- m = 0.2 \]
Figure 11: Axial velocity distribution for $m \geq 0$

\[ \cdots m = 0.0, \quad \cdots m = 0.1, \quad \cdots m = 0.2 \]
Figure 12: Radial velocity distribution for $m \geq 0$

\[ \cdots m = 0.0, \cdot \cdot m = 0.1, \quad - m = 0.2 \]
Figure 13: Pressure coefficient distribution for $m \geq 0$

- - - $m = 0.0$, - - $m = 0.1$, — $m = 0.2$
The peak values of axial and swirl velocities increase significantly as \( m \) becomes more negative. For \( m = -0.6 \) the peak values are more than 15 and 50 times, respectively, that of \( m = 0 \). As a result the decay to the edge value is quite steep. The radial velocity profiles tend to become nonlinear away from the vortex axis as \( m \) decreases.

Results presented by Mayer and Powell [18] for \( m \neq 0 \) are incomplete. Profiles of axial and radial velocities are given for vortices with different strengths. Furthermore, no swirl velocities are presented, thus making any comparisons impossible.

In general, it is more difficult computationally to obtain solutions for non-zero \( m \). There seems to be a strong connection between the vortex intensity and \( m \). It is not possible to obtain solutions for positive \( m \) unless the vortex intensity is greatly increased. Finally, in this study we have not found it possible to obtain solutions for \( m \) smaller than -0.6 and larger than 0.4.

### 4.2 Compressible Flow

The similarity parameters that appear in the governing equations, (2.42) through (2.53), are a triplet of real numbers \( \{m, n, k\} \) which are chosen subject to equation (2.41). As this equation shows, once a value for the viscosity temperature-dependent exponent, \( \omega \), is specified, choice of the real numbers \( (m, n) \) determines \( k \).

Hence, \( (m, n) \) are the only similarity parameters that must be specified.

The physical parameters involved are Mach number, \( M \), Prandtl number, \( Pr \), ratio of specific heats, \( \gamma \), viscosity exponent, \( \omega \), and the strength of the vortex as determined by the swirl velocity at its edge, \( V_e \). Furthermore, the axial velocity,
Figure 14: Swirl velocity distribution for $m \leq 0$

$\cdots m = 0.0$, $\cdots \cdots m = -0.2$, $\cdots \cdots m = -0.4$, $\cdots m = -0.6$
Figure 15: Axial velocity distribution for $m \leq 0$

$\cdots m = 0.0$, $\cdots\cdots m = -0.2$, $\cdots\cdots m = -0.4$, $\cdots\cdots m = -0.6$
Figure 16: Radial velocity distribution for $m \leq 0$

- $\cdots m = 0.0$, $\cdots\cdots m = -0.2$, $\cdots\cdots m = -0.4$, $\cdots\cdots m = -0.6$
Figure 17: Pressure coefficient distribution for $m \leq 0$

- - - $m = 0.0$, --- $m = -0.2$, - - - $m = -0.4$, --- $m = -0.6$
pressure, and temperature at the vortex edge are

\[ W_e = 1.0 \quad P_e = \frac{1}{\gamma M^2} \quad T_e = 1.0 \quad (4.1) \]

This is justified by the fact that all flow variables have been nondimensionalized using the edge conditions. Of course, \( V_e \) takes various values determined by \( V'(0) \) values.

All results presented in this section are for \( M = 2, m = 0, n = 0.5, \omega = 1, \gamma = 1.4, \) and \( Pr = 0.72, \) unless otherwise specified, and so represent a quadratically growing vortex in a uniform flow.

Figures 18-23 illustrate the effect of intensity, \( V_e, \) on vortex structure. The swirl velocity, shown in figure 18, gradually increases until it reaches its maximum value at the vortex edge, in sharp contrast to the overshoot profiles for incompressible cases. As the vortex strength increases the peak value moves away from the axis and toward the vortex edge indicating an increase in the diffusion of circumferential shear. Figure 19 shows the axial velocity distribution. As the vortex strength increases so does the axial current while maintaining the same character. The axial flow always peaks on the vortex axis and decays to the edge value. The decay is always steep toward the vortex edge. The radial velocity profiles which are shown in figure 20 are always negative as before. For weak vortices they are nearly linear, but become nonlinear for strong vortices. In addition, large edge swirl has the effect of reducing the magnitude of the radial flow in the vicinity of the axis. The pressure coefficient distribution is plotted in figure 21. A higher pressure drop is observed as the intensity is increased. The pressure attains its minimum value on the axis while maintaining a positive gradient throughout the vortex core as expected from the radial momentum
equation (2.29).

Temperature and density distributions are shown in figures 22 and 23. The effect of the vortex strength on the temperature distribution is similar to that of the axial velocity. The density has its minimum value on the axis which decreases with increasing intensity. It should be noted that increasing the vortex strength does not change the character of the flow quantities except for the radial velocity as pointed out earlier.

Figures 24-26 show the effect of varying $M$ on the flow pattern. All velocity components decrease for increasing $M$. The swirl velocity, plotted in figure 24, reaches its maximum value in the vicinity of the vortex axis for small $M$. When $M$ becomes larger the swirl velocity increases from its axis value of zero until it reaches the edge value. As $M$ becomes larger the slope of the swirl velocity becomes steeper and remains positive throughout the core. The axial velocity, shown in figure 25, exhibits a jet like character for all $M$ values. Figure 26 shows the radial velocity. For small $M$ the profiles are nearly linear. They become nonlinear as $M$ increases. In fact, for $M = 2.0$ the radial velocity is nonlinear even in the neighborhood of the vortex axis. Unlike the inviscid results of Brown [3], the radial velocity is always negative.

The pressure coefficient distribution is presented in figure 27. The pressure coefficient and gradient decrease for increasing $M$ in the vortex core except near the outer edge where it is nearly unchanged.

The temperature and density distributions are plotted in figures 28 and 29. The temperature rises and the density falls as $M$ increases. The change in temperature
Figure 18: Swirl velocity distribution for various strength

\( (m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0) \)

\[
\begin{align*}
\cdots V'' & = 0.04, & \cdots V'' & = 0.06, & \cdots - V'' & = 0.07, & --- V'' & = 0.075
\end{align*}
\]
Figure 19: Axial velocity distribution for various strength

(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, γ = 1.40, ω = 1.0)

• • • V₀ = 0.04, ---- V₀ = 0.06, --- V₀ = 0.07, —— V₀ = 0.075
Figure 20: Radial velocity distribution for various strength

($m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0$)

$\cdots V'_{o} = 0.04$, $\ldots V'_{o} = 0.06$, $\cdots V'_{o} = 0.07$, $\cdots V'_{o} = 0.075$
Figure 21: Pressure coefficient distribution for various strength

\( m = 0.0, \; n = 0.5, \; M = 2.0, \; Pr = 0.72, \; \omega = 1.0 \)

- \( V'_0 = 0.04 \), ---- \( V'_0 = 0.06 \), - - - \( V'_0 = 0.07 \), --- \( V'_0 = 0.075 \)
Figure 22: Temperature distribution for various strength

\((m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)\)

\(\cdots V'' = 0.04\), \(\cdots V' = 0.06\), \(- - - V' = 0.07\), \(\cdots V'' = 0.075\)
Figure 23: Density distribution for various strength

\( m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0 \)

\(- - - \, V'_0 = 0.04, \, --- \, V'_0 = 0.06, \, - - - \, V'_0 = 0.07, \, ----- \, V'_0 = 0.075 \)
Figure 24: Swirl velocity distribution for various $M$

$m = 0.0, \ n = 0.5, \ Pr = 0.72, \ \gamma = 1.40, \ \omega = 1.0$

$\cdots \ M = 0.0, \ \cdots \cdots \ M = 0.5, \ \cdots \cdots \ M = 1.0, \ \cdots \cdots \ M = 2.0$
Figure 25: Axial velocity distribution for various $M$

$m = 0.0$, $n = 0.5$, $Pr = 0.72$, $\gamma = 1.40$, $\omega = 1.0$)

- - - - - $M = 0.0$, ----- $M = 0.5$, - - - $M = 1.0$, ---- $M = 2.0$
Figure 26: Radial velocity distribution for various \( M \)

\[ m = 0.0, \quad n = 0.5, \quad Pr = 0.72, \quad \gamma = 1.40, \quad \omega = 1.0 \]

\[ \cdots \ M = 0.0, \quad - - - - \ M = 0.5, \quad - - - - \ M = 1.0, \quad - - - - - - - - - - - - - - - - \ M = 2.0 \]
Figure 27: Pressure coefficient distribution for various $M$

$m = 0.0$, $n = 0.5$, $Pr = 0.72$, $\gamma = 1.40$, $\omega = 1.0$

- $M = 0.0$, $M = 0.5$, $M = 1.0$, $M = 2.0$
depends on the balance between the rate at which heat is generated by viscous dissipation and the rate at which heat is transported by convection and conduction. It is evident from the energy equation that viscous dissipation increases by a factor of the order of magnitude of $M^2$. For all $M$ values, the temperature has its maximum and the density its minimum value on the vortex axis. As shown by the negative temperature gradients, heat that is generated by friction is diffused in the direction of the external flow causing the temperature to rise in the core.

Solutions presented by Mayer and Powell [18] for various $M$ show similar swirl and axial velocity, pressure, and density distributions. However, the radial velocity and temperature profiles are completely different. It is not clear why their radial velocities which are inward (negative) in the incompressible vortex become outward (positive) causing a significant reduction in the entrainment of fluid into the vortex core. The temperature is shown to have its minimum value on the vortex axis, which is surprising because this would require heat to be conducted from the core at a rate larger than the combined rate of heat generation by viscous dissipation in the vortex core and the rate at which heat is transported to the core by convection. We were not able to obtain solutions that exhibit cold cores. For example, we specified temperature distributions representing cold cores to initiate the iterative procedure; however, following several iterations the temperature distribution changed radically. All solutions presented here show "hot" cores, i.e. the axis temperature is higher than the edge temperature.

Further results for $M = 2$ are now presented to examine the effect of $\omega$ on the flow field. The case of constant viscosity is represented by $\omega = 0$. The swirl velocity,
Figure 28: Temperature distribution for various $M$

$(m = 0.0, n = 0.5, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

$\cdots M = 0.5, \quad - - - M = 1.0, \quad - - - M = 2.0$
Figure 29: Density distribution for various $M$

$(m = 0.0, n = 0.5, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

--- $M = 0.5$, - - - $M = 1.0$, --- $M = 2.0$
shown in figure 30, remains nearly unchanged as \( \omega \) changes from 0 to 1. A drop is seen in both the axial velocity, figure 31, and the temperature, figure 32, in the vicinity of the vortex axis. At larger values of \( \omega \), corresponding to higher viscosity, the axial velocity gradient decreases resulting in a reduction of energy dissipation by friction. On the other hand, the viscosity coefficient appears in the energy equation as a multiplying factor in the dissipation term partially canceling the effect of lower velocity gradient. The changes observed in both the velocity and temperature are quite small because of small variation in temperature values across the vortex core.

To study the vortex cores generated in monatomic gases and their differences with vortices generated in diatomic gases, results are presented in figures 33-38 for \( M = 2.0 \). The Prandtl number, \( Pr \), which has a weak dependence on temperature and the ratio of specific heats, \( \gamma \), are 0.72 and 1.40 for diatomic gases (i.e. air), and 0.6667 and 1.6667 for monatomic gases. For monatomic gases all velocity magnitudes are smaller than the values for diatomic gases. Although the difference is small for the swirl and radial velocities, the change in the axial velocity is more pronounced, especially in the vicinity of the vortex axis. The pressure coefficient, figure 36, shows a smaller drop for monatomic gases. The change in both the temperature and density distributions are more pronounced. An increase of about 10% in temperature and an even larger decrease in density is observed for monatomic gases. A smaller \( Pr \) could mean a greater conduction of heat from the vortex core which contributes to a reduction of temperature, but the change in \( Pr \) is too small to cause a significant change in the conduction of heat. The rise in temperature is partially due to increase in \( \gamma \). It
Figure 30: Swirl velocity distribution for various $\omega$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40)$

$\cdots \omega = 0.0, \quad \cdots \omega = 0.5, \quad \cdots \omega = 1.0$
Figure 31: Axial velocity distribution for various $\omega$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40)$

$\cdots \omega = 0.0, \ldots \omega = 0.5, \ldots \omega = 1.0$
Figure 32: Temperature distribution for various $\omega$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40)$

--- $\omega = 0.0$, --- $\omega = 0.5$, --- $\omega = 1.0$
is clear from the energy equation that viscous dissipation increases for increasing $\gamma$. In addition, the specific heat at constant pressure, $C_p$, for diatomic gases is larger than that for monatomic gases; therefore, the rise in temperature will be smaller for diatomic gases.

Solutions are also presented for nonzero values of $m$ which correspond to a nonuniform external flow field. It should be noted that these solutions correspond to very weak vortices as indicated by the small magnitude of $V_e$. Figures 39-41 are plots of the swirl, axial, and radial velocity distributions for $m = -0.1, 0.0, 0.1$. It is clear that for $m = 0.1$ the character of the solution changes drastically. The swirl velocity peaks near the vortex axis and then drops slowly until it reaches the edge value, a behavior similar to that of the incompressible vortex. The axial velocity decreases in the core and reaches its minimum value, about 40% of the edge value, on the axis. For $m = -0.1$, both the swirl and axial velocity profiles are similar to those for $m = 0$ except for a change in their magnitudes. The swirl velocity becomes smaller while the axial velocity is more than twice as large. The radial velocity distributions are nonlinear for all values of $m$ with the degree of nonlinearity increasing significantly for nonzero values of $m$. In addition, each profile takes on a different character. The pressure coefficient distribution, shown in figure 42, is nearly flat for $m = -0.1$. The pressure coefficient drops sharply for $m = 0.1$ in the neighborhood of the axis, and is about three times that of $m = 0$ in magnitude. Figures 43 and 44 show the temperature and density distributions. The temperature profiles are similar to the axial velocity profiles. The density distribution for $m = 0.1$ increases in the core as it
Figure 33: Swirl velocity distribution for various gases

\[ (m = 0.0, n = 0.5, M = 2.0, \omega = 1.0) \]

- - - - monatomic, --- diatomic
Figure 34: Axial velocity distribution for various gases

\[(m = 0.0, n = 0.5, M = 2.0, \omega = 1.0)\]

- - - monatomic , — diatomic
Figure 35: Radial velocity distribution for various gases

\((m = 0.0, n = 0.5, M = 2.0, \omega = 1.0)\)

- - - monatomic , — diatomic
Figure 36: Pressure coefficient distribution for various gases

\[(m = 0.0, n = 0.5, M = 2.0, \omega = 1.0)\]

- - - monatomic, --- diatomic
Figure 37: Temperature distribution for various gases

\[(m = 0.0, n = 0.5, M = 2.0, \omega = 1.0)\]

- - - monatomic , — diatomic
Figure 38: Density distribution for various gases

\[ (m = 0.0, \; n = 0.5, \; M = 2.0, \; \omega = 1.0) \]

--- monatomic , --- diatomic
reaches its maximum on the axis.

4.3 The vortex edge

To date little or no attention has been given by other authors to the edge location of a vortex and its influence on the flow quantities.

Figures 45-50 illustrate the changes that occur in the solution when $\xi_e$ is increased from 20 to 40 while maintaining the same grid spacing. Although the character of the solution does not change its magnitude does. The swirl velocity decreases for increasing $\xi_e$. This is expected since the swirl has a wider domain until it reaches the same edge value. Similarly, the pressure coefficient shows a smaller drop in magnitude. The axial and radial velocities, and temperature increase for increasing $\xi_e$ while the density decreases.

In classical boundary layer theory, transition from the flow quantities in the boundary layer to those outside it takes place asymptotically [22]. Hence, the solution becomes independent of the boundary layer edge location, provided that it is taken sufficiently far from the inner boundary.

Solutions presented by Hall [9], Stewartson and Hall [24], Brown [3], and Mayer and Powell [18] are not invariant with respect to the edge location and do not asymptote to constant edge values. The assertion made by Mayer and Powell [18] that solutions become independent of $\xi_e$ cannot be verified.

Hall and Stewartson [24] presented an analytical solution for inviscid incompressible flow which is logarithmic in character. The axial and swirl velocities are given
Figure 39: Swirl velocity distribution for various $m$

($n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0$)

- $m = -0.1$, - - - $m = 0.0$, — $m = 0.1$
Figure 40: Axial velocity distribution for various $m$

$(n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

\[ \ldots, m = -0.1, \ldots, m = 0.0, \ldots, m = 0.1 \]
Figure 41: Radial velocity distribution for various $m$

$(n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

--- $m = -0.1$, --- $m = 0.0$, -- $m = 0.1$
Figure 42: Pressure coefficient distribution for various $m$

$(n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

$\cdots \ m = -0.1, \ -\ -\ m = 0.0, \ -\ -\ m = 0.1$
Figure 43: Temperature distribution for various $m$

$(n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

- $\cdots m = -0.1$, $\cdots m = 0.0$, $m = 0.1$
Figure 44: Density distribution for various $m$

$(n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

- - - $m = -0.1$, - - - $m = 0.0$, - - - - - - $m = 0.1$
Figure 45: Swirl velocity distribution for various $\xi_e$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

$\cdash\cdash\cdash\xi_e = 20, \quad \cdash\cdash\cdash\xi_e = 40$
Figure 46: Axial velocity distribution for various $\xi_e$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

- - - $\xi_e = 20$,  --- $\xi_e = 40$
Figure 47: Radial velocity distribution for various $\xi_e$

($m = 0.0$, $n = 0.5$, $M = 2.0$, $Pr = 0.72$, $\gamma = 1.40$, $\omega = 1.0$)

--- $\xi_e = 20$, — $\xi_e = 40$
Figure 48: Pressure coefficient distribution for various $\xi_e$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

- - - $\xi_e = 20$, --- $\xi_e = 40$
Figure 49: Temperature distribution for various $\xi_e$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

$\xi$ $\xi_e = 20$, $\xi_e = 40$
Figure 50: Density distribution for various $\xi_e$

$(m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0)$

- - - $\xi_e = 20$, $\xi_e = 40$
by

\[ w = w_e (1 - \alpha \log \left( \frac{r}{z} \right)) \quad (4.2) \]

\[ v = \left[ v_e^2 - w_e^2 \alpha^2 \log \left( \frac{r}{z} \right) \right]^\frac{1}{2} \quad (4.3) \]

where

\[ \alpha = (1 + \frac{2v_e^2}{w_e^2})^\frac{1}{2} - 1 \geq 0 \]

It can be shown that the present results also have a logarithmic behavior in the neighborhood of the vortex edge. Figures 51 and 52 illustrate this behavior for the incompressible vortex. Although no analytical solutions are available for the compressible vortex, it is observed that both the axial velocity and temperature are logarithmic in the vicinity of the outer edge. This is shown in figures 53 and 54. However, the swirl velocity, unlike the incompressible vortex, is not logarithmic near the outer edge.

Although it is not clear if the solution should become independent of \( \xi_e \) or what the choice of a particular value for \( \xi_e \) means, the following explanation may contribute to an eventual resolution of this problem: It is true that the outer region of the present solution must match to the inner limit of an outer inviscid solution in a larger domain [6]. It may be that there are subtleties to this matching that will eventually resolve the difficulty.
Figure 51: Axial velocity distribution

\[ W = 3.143 - 0.72 \log(\xi), \quad \text{present results (} M = 0.0 \text{)}\]
\[ V = (2.281 - 0.43 \log(\xi))^{\frac{1}{2}}, \quad \text{— present results (} M = 0.0 \text{)} \]
Figure 53: Axial velocity distribution

\( m = 0.0, \ n = 0.5, \ M = 2.0, \ Pr = 0.72, \ \gamma = 1.40, \ \omega = 1.0 \)

\( W = 1.537 - 0.18 \log(\xi), \quad \text{present results} \)
Figure 54: Temperature distribution

\[ T = 1.410 - 0.136 \log(\xi), \quad \text{present results} \]

\[ (m = 0.0, n = 0.5, M = 2.0, Pr = 0.72, \gamma = 1.40, \omega = 1.0) \]
CHAPTER V

Conclusions

In the preceding study, we have presented a numerical solution for three-dimensional isolated vortices in perfect, viscous, heat-conducting gases. The vortices considered here were assumed to be quasi-cylindrical.

The complete Navier-Stokes equations were simplified by introduction of axial symmetry and quasi-cylindrical approximations. Using a similarity assumption, these equations were reduced to a set of nonlinear ordinary differential equations, which were solved numerically.

The numerical procedure consists of approximating the ordinary differential equations by a central-difference formula, linearizing the finite-difference equations using Newton's method, and utilizing an iterative method to obtain a class of self-similar solutions.

Solutions were presented for incompressible vortices showing the effects of vortex strength and external axial pressure gradient. The results may be summarized as follows:

- Increasing the vortex strength results in a significant increase in the magnitudes of all velocity components and the overall pressure drop.
• Favorable (positive) axial pressure gradients do not change the character of the flow, but diminish the magnitude of the velocity components and the drop in the pressure.

• Adverse (negative) axial pressure gradients increase the velocity magnitudes sharply, and produce very large velocity gradients in the vicinity of the vortex axis; consequently, the character of the flow changes drastically.

The solutions presented for incompressible flow agree qualitatively with the existing theoretical [9, 24] and experimental [26] results. There is also good agreement with the existing numerical solutions [18].

The effects of various flow parameters on the structure of compressible vortices were studied extensively. The vortex strength, Mach number, Prandtl number, ratio of specific heats, viscosity coefficient, and external axial pressure gradient must be specified to characterize a single vortex.

The compressible results may be summarized as follows:

• Increasing the vortex strength results in a significant increase in the magnitudes of all velocity components, the pressure coefficient drop, and temperature.

• As the flow Mach number increases, the magnitudes of the velocity components, their gradients, and the drop in the pressure coefficient decrease. However, the temperature rises with Mach number.

• All solutions show "hot" cores, i.e. the axis temperature is higher than the edge temperature. This is in contrast to what was found by Mayer and Powell [18].
• A small decrease in both the axial velocity and temperature is observed, when the viscosity coefficient is increased. Because of small variations in the temperature values, the effect of the temperature-dependent viscosity coefficient on the flow is not significant.

• To study the effects of the Prandtl number and ratio of specific heats, solutions were computed for monatomic and diatomic gases. In monatomic gases, the velocities are slightly smaller than those for diatomic gases as is the drop in the pressure coefficient, but the temperature is significantly higher.

• Axial pressure gradient changes the character of the flow drastically. Favorable pressure gradients result in much larger swirl, but a significantly smaller axial velocity. The temperature drops as the vortex axis is approached attaining an axis value less than half the edge value. These trends are reversed for adverse pressure gradients. The radial velocity becomes highly nonlinear throughout the core and temperature increases up to nearly four times its free stream value.

By far, the most striking feature of these vortical structures is the temperature distribution that reaches its maximum on the vortex axis. These so called "hot" cores have not been previously observed. The only previous computation of three-dimensional vortices [18] has shown "cold" cores; however, we were not able to generate such cores. The presence of these "hot" cores could have serious implications in the design of aerodynamic vehicles.

Another important parameter affecting the vortex structure is the location of the vortex edge. We have found that
• All solutions, both incompressible and compressible, vary with the edge location. Although the character of the flow remains the same, the magnitudes of all flow quantities change as the extent of the vortex core is changed.

We have presented results that describe the structure of three-dimensional viscous, heat-conducting vortices, taking into account the effects of compressibility. These solutions are of fundamental interest and practical importance. They contain features, i.e. "hot" cores, that have not been previously observed and represent a significant contribution to establishment of a lexicon of three-dimensional vortical flows that may be utilized in the development of computational techniques, such as the "fitted vortex" method [11], for more complicated flow fields. They also form the basis for the study of the stability of isolated vortices and consequently will ultimately contribute to our knowledge of the vortex breakdown phenomena.
Appendix A

Coefficients:

A.1 Incompressible Flow

A.1.1 continuity

\[(\alpha_1)_j = \xi_j\]
\[(\alpha_2)_j = \xi_{j-1}\]
\[(\alpha_3)_j = d_1(\xi)_j\]
\[(\alpha_4)_j = d_1(\xi)_{j-1}\]

A.1.2 momentum

axial

\[(\beta_1)_j = W'_j\]
\[(\beta_2)_j = \rho W'_{j-1}\]
\[(\beta_3)_j = 2m\rho W_j\]
\[(\beta_4)_j = 2m W_{j-1}\]
\[(\beta_5)_j = U_j - \frac{2\mu}{\Delta\xi} - d_2(\mu)\]
\[(\beta_6)_j = (\rho U)_{j-1} + \frac{2\mu}{\Delta\xi} - d_2(\mu)\]
\[(\beta_7)_j = -2mV_j \quad (A.11)\]
\[(\beta_8)_j = -2mV_{j-1} \quad (A.12)\]

**circumferential**

\[(\theta_1)_j = d_3(V)_j + V'_j \quad (A.13)\]
\[(\theta_2)_j = d_3(V)_{j-1} + V'_{j-1} \quad (A.14)\]
\[(\theta_3)_j = (n + m)V_j \quad (A.15)\]
\[(\theta_4)_j = (n + m)V_{j-1} \quad (A.16)\]
\[(\theta_5)_j = (n + m)W_j + d_3(U)_j + (d_3)^2\mu \quad (A.17)\]
\[(\theta_6)_j = (n + m)W_{j-1} + d_3(U)_{j-1} + (d_3)^2\mu \quad (A.18)\]
\[(\theta_7)_j = (\rho U)_j - \frac{2\mu}{\Delta \xi} - d_3(\mu) \quad (A.19)\]
\[(\theta_8)_j = (\rho U)_{j-1} + \frac{2\mu}{\Delta \xi} - d_3(\mu) \quad (A.20)\]

**Constants:**

\[d_1 = (k + m + 2n)\frac{\Delta \xi}{2}\]
\[d_3 = \frac{2}{(\xi_j + \xi_{j-1})}\]
A.2 Compressible Flow

A.2.1 continuity

\[ (a_1)_j = (\rho_\xi)_j \]  
\[ (a_2)_j = (\rho_\xi)_{j-1} \]  
\[ (a_3)_j = d_1(\rho_\xi)_j \]  
\[ (a_4)_j = d_1(\rho_\xi)_{j-1} \]

A.2.2 momentum

axial:

\[ (\beta_1)_j = (\rho W')_j \]  
\[ (\beta_2)_j = (\rho W')_{j-1} \]  
\[ (\beta_3)_j = 2m(\rho W)_j \]  
\[ (\beta_4)_j = 2m(\rho W)_{j-1} \]  
\[ (\beta_5)_j = (\rho U)_{j} - \frac{\mu_j + \mu_{j-1}}{\Delta \xi} - d_2([\xi\mu]_j - [\xi\mu]_{j-1}) \]  
\[ (\beta_6)_j = (\rho U)_{j-1} + \frac{\mu_j + \mu_{j-1}}{\Delta \xi} - d_2([\xi\mu]_j - [\xi\mu]_{j-1}) \]  
\[ (\beta_7)_j = -2m(\rho V)_j \]  
\[ (\beta_8)_j = -2m(\rho V)_{j-1} \]
circumferential:

\[
\begin{align*}
(\theta_1)_j &= d_3(\rho V)_j + (\rho V')_j \\
(\theta_2)_j &= d_3(\rho V)_{j-1} + (\rho V')_{j-1} \\
(\theta_3)_j &= (n + m)(\rho V)_j \\
(\theta_4)_j &= (n + m)(\rho V)_{j-1} \\
(\theta_5)_j &= (n + m)(\rho W)_j + d_3(\rho U)_j + d_4[(\xi \mu)_j - (\xi \mu)_{j-1}] \\
(\theta_6)_j &= (n + m)(\rho W)_{j-1} + d_3(\rho U)_{j-1} + d_4[(\xi \mu)_j - (\xi \mu)_{j-1}] \\
(\theta_7)_j &= (\rho U)_j - \frac{\mu_j + \mu_{j-1}}{\Delta \xi} - d_2[(\xi \mu)_j - (\xi \mu)_{j-1}] \\
(\theta_8)_j &= (\rho U)_{j-1} + \frac{\mu_j + \mu_{j-1}}{\Delta \xi} - d_2[(\xi \mu)_j - (\xi \mu)_{j-1}]
\end{align*}
\]

A.2.3 energy

\[
\begin{align*}
(\zeta_1)_j &= e_2 \mu_j - \{e_1[\mu'_j + (\frac{\mu}{\xi})_j] - (\rho U)_j\} \\
(\zeta_2)_j &= -2[e_2 \mu_j + m(\rho W)_j] \\
(\zeta_3)_j &= e_2 \mu_j + \{e_1[\mu'_j + (\frac{\mu}{\xi})_j] - (\rho U)_j\} \\
\mu'_j &= \frac{\mu_{j+1} - \mu_{j-1}}{2\Delta \xi}
\end{align*}
\]

Constants:

\[
\begin{align*}
d_1 &= (k + m + 2n)\frac{\Delta \xi}{2} \\
d_2 &= \frac{2}{(\xi_j + \xi_{j-1})\Delta \xi} \\
d_3 &= \frac{2}{(\xi_j + \xi_{j-1})}
\end{align*}
\]
\[ d_4 = \frac{4}{(\xi_j + \xi_{j-1})^2 \Delta \xi} \]
\[ e_1 = \frac{1}{Pr(\Delta \xi)^2} \]
\[ e_2 = \frac{1}{2Pr(\Delta \xi)} \]
Appendix B

Right-hand Sides:

B.1 Incompressible Flow

B.1.1 continuity

\[ (R_c)_j = -(\xi U)_j + (\xi U)_{j-1} - d_1[(\xi W)_j - (\xi W)_{j-1}] \] (B.1)

B.1.2 momentum

axial:

\[ (R_v)_j = -m[(W_i^2)_j + (W_i^2)_{j-1}] + n[(V^2)_j + (V^2)_{j-1}] \]

\[ -(k + 2m)(P_j + P_{j-1}) - (U W')_j - (U W')_{j-1} \]

\[ + \frac{2\mu}{\Delta \xi} (W'_{j} - W'_{j-1}) + d_3(\mu)(W'_{j} + W'_{j-1}) \] (B.2)

circumferential:

\[ (R_\theta)_j = -(n + m)[(V W)_j + (V W)_{j-1}] - d_3[(V U)_j + (V U)_{j-1}] \]

\[ -[(U V')_j + (U V')_{j-1}] + \frac{2\mu}{\Delta \xi} (V'_{j} - V'_{j-1}) \]

\[ + d_3(\mu)(V'_{j} + V'_{j-1}) - d_3^2(\mu)(V_{j} + V_{j-1}) \] (B.3)
Constants:

\[
d_1 = (k + m + 2n) \frac{\Delta \xi}{2}
\]

\[
d_3 = \frac{2}{(\xi_j + \xi_{j-1})}
\]
B.2 Compressible Flow

B.2.1 continuity

\[ (R_c)_j = -\rho U_j + (\rho U)_{j-1} - d_1[(\rho UW)_j - (\rho UW)_{j-1}] \quad (B.4) \]

B.2.2 momentum

axial:

\[ (R_z)_j = -m[(\rho W^2)_j + (\rho W^2)_{j-1}] + n[(\rho V^2)_j + (\rho V^2)_{j-1}] \]
\[ - (k + 2m)(P_j + P_{j-1}) - (\rho U'W')_j - (\rho UW')_{j-1} \]
\[ + \frac{\mu_j + \mu_{j-1}}{\Delta \xi}(W'_j - W'_{j-1}) \]
\[ + d_2[(\xi \mu)_j - (\xi \mu)_{j-1}](W'_j + W'_{j-1}) \quad (B.5) \]

circumferential:

\[ (R_\theta)_j = -(n + m)[(\rho VW)_j + (\rho VW)_{j-1}] - [(\rho UV')_j + (\rho UV')_{j-1}] \]
\[ - d_3[(\rho VU)_j + (\rho VU)_{j-1}] + \frac{\mu_j + \mu_{j-1}}{\Delta \xi}(V'_j - V'_{j-1}) \]
\[ + [(\xi \mu)_j - (\xi \mu)_{j-1}](d_2(V'_j + V'_{j-1}) - d_4(V_j + V_{j-1})) \quad (B.6) \]

B.2.3 \( W' \) and \( V' \) equations

\[ (R_{w})_j = -2W_j + 2W_{j-1} + \Delta \xi(W'_j + W'_{j-1}) \quad (B.7) \]
\[ (R_{v})_j = -2V_j + 2V_{j-1} + \Delta \xi(V'_j + V'_{j-1}) \quad (B.8) \]
B.2.4 energy

\[ (R_e)_j = (\gamma - 1)M^2\{(k + 2m)(PW)_j + \left(\frac{\rho UV^2}{\xi}\right)_j\} + \mu_j[(e_{rz})_j^2 + (e_{r\theta})_j^2] \]  

\[ (e_{r\theta})_j = \frac{V_{j-1} + V_j}{2\Delta \xi} - \left(\frac{V}{\xi}\right)_j \]

\[ (e_{rz})_j = \frac{W_{j-1} + W_j}{2\Delta \xi} \]

Constants:

\[ d_1 = (k + m + 2n)\frac{\Delta \xi}{2} \]

\[ d_2 = \frac{2}{(\xi_j + \xi_{j-1})\Delta \xi} \]

\[ d_3 = \frac{2}{(\xi_j + \xi_{j-1})} \]

\[ d_4 = \frac{4}{(\xi_j + \xi_{j-1})^2\Delta \xi} \]
BIBLIOGRAPHY


