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The effect of using as many variables as are needed to solve word problems on the problem-solving skills and attitudes of students in Algebra I

Mathews, Susann Miller, Ph.D.

The Ohio State University, 1994

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THE EFFECT OF USING AS MANY VARIABLES AS ARE NEEDED TO SOLVE WORD PROBLEMS ON THE PROBLEM-SOLVING SKILLS AND ATTITUDES OF STUDENTS IN ALGEBRA I

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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ACKNOWLEDGMENTS

I am very fortunate to have an opportunity to work with Dr. Sigrid Wagner, my advisor, who is the mentor that I had previously spent my career looking for. I appreciate the time, guidance, and wisdom she has shared with me. My sincere appreciation also goes to the other members of my committee: Dr. Alan Osborne, who always knew the right source and where to find it and who helped me keep school in perspective to life, and to Dr. Douglas Wolfe, who with his quiet ways and statistical knowledge and understanding have given me just the right guidance. As I have often said to those who would listen, "My committee only gave me guidance when they thought that I needed it, and it was always worth listening to."

I am grateful to Jeri Nichols and Ann Farrell at Wright State University who cheerfully took on extra work and shared their experience with me so that I could do my research and write my dissertation. I appreciate the encouragement given to me by my classmates, Tom Edwards and David Ericson. I thank Vicky Beebe, Karen Johnson, Bill Kramer, and Mary Ellen Place, the four Algebra I teachers who conducted the experiment and shared their classes and valuable time with me.

Finally I thank my parents, who raised me to believe in myself; my sons Andy and Ricky, who have always helped me keep my sense of humor; and my husband, Kirk, without whom this work would not have been possible, would have had much less meaning and would have been much less fun.
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CHAPTER I
INTRODUCTION

Problem Description

One of the main goals in teaching algebra to students is to help them learn to solve word problems. However, often they do not know how to begin solving a problem once they have read it. They flounder when trying to translate the English words in the statement of the problem to algebraic symbols in order to solve the problem algebraically. Mathematics educators have recently become aware of the translation problems students face when trying to solve word problems (Clement, Lochhead, and Monk, 1981; Kirshner, Awtry, McDonald, and Gray, 1991; and Lochhead and Mestre, 1988; ). Traditionally, students are taught to work with a single variable and to represent all of the unknowns in terms of the one variable. The researcher has observed that their worst difficulty seems to come from trying to determine which one unknown (among many) to define as the one allowed variable and what to write as the equation to represent the problem. After solving word problems for most of the year, they are finally taught to solve two equations in two variables, and shortly after to solve three or more equations in three or more unknowns. When they return to problem solving and are now shown how to translate the word problem using as many unknowns as they need, they frequently exclaim, "This is so much easier!" Their ability to translate the English words to algebraic symbols and hence solve the mathematics problems often increases dramatically.

Lazar (1933) reported similar observations. He presents an eloquent argument in favor of using two unknowns rather than only one unknown when solving algebraic word problems (although he cites no research to support his ideas). He notes that traditional
reasons for the usual late introduction of equations involving two unknowns are that there is no use in teaching the solution of two equations in two unknowns before students have mastered solving one equation in one unknown and that many topics such as "parentheses" and "order of operations" are prerequisites. However, current computer technology (with algebraic symbol manipulation) in the classroom eliminates these as objections to early introduction of multiple variables. In classrooms without such technology, using either the method of elimination by addition and subtraction or the method of substitution to solve two equations in two unknowns is essentially no more difficult than solving one equation in one unknown.

After observing this phenomenon, three questions arose. Would the students in algebra learn more quickly and with more success how to solve word problems if they were taught how to solve them with as many variables as they needed to represent the unknowns as soon as they began problem solving? Or is it simply that during the second semester, when they are usually taught how to solve equations in several variables and then taught how to apply that skill to problem solving that they find it easier because they are more mathematically mature? Specifically, will learning first with multiple variables help them approach word problems methodically? They should:

- Identify knowns and unknowns
- Declare all the variables (e.g., "x is length of the side in feet"),
- Label them on a diagram (when appropriate),
- Translate the relationships described in a word problem into equivalent algebraic equations,
- Obtain an algebraic solution,
- Interpret it correctly as the answer to the word problem.
Furthermore, would the ability to solve word problems with as many variables as needed increase their positive attitude toward problem solving in school mathematics?

**Purpose and Research Questions**

The purpose of this study was to explore what effect teaching as many variables as needed to solve word problems as soon as students begin problem solving has on the elementary-algebra students' problem-solving skills and on their attitude toward problem solving in school mathematics.

The research questions are:

1. Does teaching students to work with two variables instead of one variable help them solve word problems more successfully?
2. Does using two variables instead of one variable to solve word problems with two unknowns improve students' attitudes toward word problem solving in school mathematics?

**Variables**

The *Curriculum and Evaluation Standards for School Mathematics* (1989) lists problem solving as the first standard to teach at all levels of school mathematics. Although problem solving is important, students find it difficult, especially the translation from word problems to algebraic equations. In the 1990 National Assessment of Educational Progress, only 14.8% of eighth graders correctly answered a multiple-choice item in which they were asked to write an algebraic expression; the percentage correct for twelfth-graders for the same type of translation item was only 27.5%. The percentage of correct answers was even lower when the item was a constructed response item to write an algebraic expression. Only 8.6% of the twelfth-grade answers were correct (Mullis, Dossey, Owen, and Phillips, 1991). The *Curriculum and Evaluation Standards for School Mathematics* also sets helping students develop a positive disposition towards mathematics as a goal. Therefore, the two dependent variables of the research questions...
that the researcher studied are problem-solving skills, including translation, and attitude towards problem solving in school mathematics.

**Problem-solving Skills**

Problem solving in mathematics is open to different interpretations, which include solving the simple word problems that appear in standard textbooks, solving nonroutine problems or puzzles, applying mathematics to problems of the "real" world, and creating and testing mathematical conjectures. As a goal, learning to solve problems is one of the primary reasons for studying mathematics. As a process, it is the "process of applying previously acquired knowledge to new and unfamiliar situations" (National Council of Supervisors of Mathematics, p. 148). As a basic skill, it must be broken down into its many subskills so that it can be taught (Branca, 1980).

In this study, word problems were considered to be those found in current, standard algebra textbooks, and problem-solving skills were considered to be the practical knowledge of how to solve algebraic word problems. In particular, these skills include being able to

- Identify the knowns and unknowns,
- Declare all the variables (e.g., "x is length of the side in feet"),
- Translate the relationships described in a word problem into equivalent algebraic equations,
- Obtain an algebraic solution,
- Interpret it correctly as the answer to the word problem.

**Attitude Toward Problem Solving in School Mathematics**

Attitude, simply defined, is "positiveness or negativeness toward a psychological object" (Mueller, 1986, p. 3). "Disposition refers not simply to attitudes but to a tendency to think and to act in positive ways" (National Council of Teachers of Mathematics, 1989, p. 233). The tricomponent model of attitude incorporates the affective or evaluative
domain, the cognitive domain containing beliefs, and the behavioral domain (Pettyjohn, Banikart, Fitzgerald, Misovich, Spiegler, and Triplet, 1986). Thus in the expanded definition of the tricomponent model, attitude includes disposition.

From the time children enter kindergarten and have their first mathematics lesson in counting and one-to-one correspondences, they begin to develop attitudes and beliefs toward mathematics. "These beliefs exert a powerful influence on students' evaluation of their own ability, on their willingness to engage in mathematical tasks, and on their ultimate mathematical disposition" (National Council of Teachers of Mathematics, 1989, p. 233).

Mathematical disposition includes:

- Confidence in using mathematics to solve problems, to communicate, and to reason;
- Flexibility in exploring mathematical ideas and trying alternative methods in solving problems;
- Willingness to persevere in mathematics tasks;
- Interest, curiosity, and inventiveness in doing mathematics;
- Inclination to monitor and reflect on their own thinking and performance;
- Valuing of the application of mathematics to situations arising in other disciplines and everyday experiences;
- Appreciation of the role of mathematics in our culture and its value as a tool and as a language. (National Council of Teachers of Mathematics, 1989, p. 233)

In the reported study, attitude toward problem solving in school mathematics was measured on a Thurstone-type scale. The scale measured the following aspects of students' attitude toward problem solving in school mathematics: confidence, flexibility, willingness to persevere, and interest. It measured all three components of attitude: the
evaluative or emotional component, the cognitive or belief component, and the behavioral component.

**Theoretical Rationale**

Traditionally, problem solving in algebra is taught using only one variable as long as possible. Students often find the translation of the word problems into algebraic notation very difficult, if not impossible, when they must "put everything in terms of one variable." In so doing, they must determine all of the relationships needed to solve the word problem in their heads before they write the equation(s). When they are finally taught to use as many variables as there are unknowns in the statement of the problem, they can define each variable explicitly and they can write out each relationship between the variables explicitly. They can proceed one step at a time and succeed in setting up the problem correctly.

**Problem Solving Using Two Variables**

**Piaget's Theory**

Jean Piaget has worked closely with individual children in experimental teaching situations to discover how they learn mathematics. Piaget and his colleagues studied the development of children's ideas of *function* at length, from the initial concept of paired dependencies to the more advanced idea of serial dependencies (Piaget et al., 1977). Throughout his work he refers to *function* as the dependency of one object upon another; in algebra, it would mean the dependency of one variable upon another. He is not referring to the abstract Dirichlet-Bourbaki definition of function, which is that of a correspondence between two nonempty sets that assigns to every element in the first set exactly one element in the second set. Piaget discovered that the notion of a paired dependency between two objects or quantities is a very natural one for children. However, the concept of serial dependency is much more difficult for children to understand.
For example, consider the problem shown in Figure 1 (adapted from Piaget, et al., 1977). In the upper diagram, a pan is supported by a string, which travels around a pulley, to connect to a spring. The length of the string is \( L \); the length of the spring (unstretched) is \( X \); length of the horizontal portion of the string is \( Y \); the length of the vertical portion of the string is \( Z \). The pulley is held a distance \( D \) from the wall by the support brace. In the lower diagram, a weight, \( W \), has been placed in the pan, pulling it down by a distance \( b \), and stretching the spring by a distance \( a \). These distances can be measured using the pointers and scales. The length of the string is now \( l \); with horizontal and vertical portions \( y \) and \( z \), and the pulley is now a distance \( d \) from the wall.

![Spring Balance Diagram](image)

Figure 1. Spring Balance. Adapted from Piaget (1977)
In the first stage of considering the dependencies, a child will establish the initial paired dependency between $x$ and $W$—the stretching of the spring as a function of the weight. The intermediary values are ignored or are fairly unstable, uncoordinated pairs constructed on the basis of observations. In Stage 2, the intermediaries appear and then are made explicit in Stage 3 using serial dependencies, perhaps as follows.

The stretch of the spring, $a$, is the change in its length when the weight is added:

$$a = x - X.$$  \tag{1}

The horizontal part of the string takes up the part of the distance of the pulley from the wall that is not taken up by the length of the spring, both without the weight,

$$Y = D - X,$$  \tag{2}

and with the weight,

$$y = d - x.$$  \tag{3}

So the decrease in the horizontal length of string, $Y - y$, depends upon the movement of the pulley and the stretching of the spring:

$$Y - y = (D - d) - (X - x).$$  \tag{4}

Assuming the support is strong, the pulley does not move, so that

$$Y - y = 0 - (X - x) = x - X.$$  \tag{5}

The vertical part of the string is the whole string less the horizontal part (there being no other parts), so their lengths are similarly related, both with the weight,

$$z = l - y,$$  \tag{6}

and without the weight,

$$Z = L - Y.$$  \tag{7}

Thus, the increase in the length of the vertical part of the string, $z - Z$, depends upon the change in length of the string and the change of the horizontal part:

$$z - Z = (l - L) - (y - Y).$$  \tag{8}

Assuming the string is strong, it does not stretch, so that
The fall of the pan when the weight is added, \( b \), is just the change in the vertical length of string (assuming the support keeps the pulley from moving downward):
\[
b = z - Z.
\] (10)

By observation, the pan moves down proportionally to the weight placed in it:
\[
W \propto b.
\] (11)

The above equations express paired dependencies. The next step in model building is to recognize the sequence of relationships they constitute. The resulting model is one of serial dependencies: the movement of the spring depends upon the movement of the horizontal part of the string (and of the pulley, which is assumed not to move); the movement of the horizontal part of the string depends upon the movement of the vertical part of the string (and the string itself, which is assumed not to stretch); the movement of the pan is determined by the vertical movement of the string (and of the pulley, which is assumed not to move). Algebraically, the serial dependencies can be summarized by gathering the above results together:
\[
W \propto b = z - Z = Y - y = x - X = a.
\] (12)

The final step of model building is to recognize transitive dependencies – the conclusion drawn from the serial dependencies is that the stretch of the spring is proportional to the weight in the pan:
\[
a \propto W.
\] (13)

Whether expressed algebraically or not, understanding is achieved through seriation of paired dependencies. It might be vocalized as "The weight pushes down on the pan, which pulls the lower part of the string down, which pulls around the pulley to pull the upper part of the string to the left, which pulls on the spring, stretching it. So, the stretch of the spring depends upon the weight."
By learning to use as many variables as needed to represent the unknowns in a word problem and thus as many equations as needed to represent the paired dependencies between variables, students can explicitly write out the paired dependencies (as in the above example), which according to Piaget's theory, is easier than having to determine the serial dependencies in their heads.

An Example Word Problem in Two Unknowns

The following word problem is taken from Lazar's 1933 article in The Mathematics Teacher. Although he formulated no theory, and Piaget's theory had not then been developed, Lazar's observations about the teaching of word problem solving are consistent with the theory presented above. Lazar argued in support of using two unknowns and two equations to solve word problems. He observed that when a student is taught to solve word problems with only one unknown, he or she is often confused about which facts are to be used as "constituents of the equation and which ones [are] for the setting up of the equation proper" (p. 177). Lazar believes that with the method of two unknowns, the vocabulary of algebra consists of the initials of the unknowns, and the grammar of the equation follows almost verbatim that of the language of the problem.

Lazar's word problem with two unknowns is solved here in two ways. The first solution uses as many variables as are needed; the second uses only one variable. This example shows how using two variables lets students write out paired dependencies, while using only one variable forces students to determine the serial dependencies in their heads.

Problem statement. "Mr. Field invested $3600, some in bonds that paid 4½ per cent and the rest in stocks that paid 6 per cent. His annual income from both was $207. How much money was invested at each rate?" (Lazar, 1933, p. 179).

Solution using as many variables as needed. The first approach uses the method taught to the experimental group (the experimental treatment, described in detail in Chapter 3).
Step 1 is to determine the known and the unknowns. The knowns are the total amount invested, $3600, the interest rate of the first investment, 4\frac{1}{2}\%$, the interest rate of the second investment, 6\%, and the total interest earned, $207$, in a period of 1 year. The unknowns are the amounts of the first and second investments.

Step 2 is to declare all the variables. Let $x$ represent the principle invested at 4\frac{1}{2}\% and $y$ represent the principle invested at 6\%. It will also help to let $i$ be the interest earned at 4\frac{1}{2}\% and $j$ be the interest earned at 6\%.

Step 3 is to translate the prose relationships expressed in the statement of the problem into algebraic relationships among the variables. The first sentence can be paraphrased as "the sum of the amounts invested is $3600", which translates directly into the paired dependency:

$$x + y = 3600.$$ (14)

The second sentence can be paraphrased as "the sum of the interests earned in a year is $207" which translates directly into another paired dependency:

$$i + j = 207.$$ (15)

The next part of the translation uses knowledge of how simple interest is computed:

$I = p\cdot r\cdot t$, where $I$ represents interest, $p$ represents the principle invested, $r$ represents the rate of simple interest, and $t$ represents the time of the investment. For the first investment, the interest is $i$, the principle is $x$, the rate is 4\frac{1}{2}\%(per year), and the time is 1 year, so

$$i = 4\frac{1}{2}\% \cdot x \cdot 1\text{ year}.$$ (16)

which simplifies to another paired dependency,

$$i = \frac{9/2}{100} x.$$ (17)
Similarly, for the second investment, the interest is $j$, the principle is $y$, the rate is $6\%$ (per year), and the time is 1 year, so

$$j = 6 \frac{\%}{\text{year}} \cdot y \cdot 1 \text{year},$$  

which simplifies to a final paired dependency,

$$j = \frac{6}{100} y.$$  

To solve the problem using the algebraic techniques for two equations in two unknowns, the serial dependencies among equations (15), (17) and (19) are used to obtain a second paired dependency among the desired unknowns, $x$ and $y$:

$$\frac{9/2}{100} x + \frac{6}{100} y = 207.$$  

Note that, even if $i$ and $j$ are not used explicitly, so that equation (20) is written down as the translation of the second sentence of the problem, the logic presented above still underlies the work. After solving many "interest problems", students will see the expression $\frac{6}{100} y$ as "the interest earned in a year by $y$ invested at 6\%" and successfully skip these steps. Either way, the result of the translation phase is two equations in two unknowns, equations (14) and (20).

Step 4 is to solve the algebraic problem for the unknown variables. Equation (14) is easy to solve for $y$ in terms of $x$:

$$y = 3600 - x.$$  

This is substituted into equation (20), recognizing the serial dependency, to obtain one equation in one unknown:

$$\frac{9/2}{100} x + \frac{6}{100} (3600 - x) = 207,$$

which can be solved using previously learned techniques, to obtain the result for $x$:

$$x = 600.$$
Using the serial dependency of this equation and equation (21), the result for $y$ is found:

$$y = $3000. \quad (24)$$

Step 5 is to interpret the algebraic results in terms of the original problem, that is to say, to translate back to prose: the amount invested at $4\frac{1}{2}\%$ is $600$, and the amount invested at $6\%$ is $3000$.

**Solution using only one variable.** The traditional approach starts with deciding what the one variable will represent. For example, let $x$ be the amount invested at $4\frac{1}{2}\%$. Then equation (14) is solved without being explicitly formulated, resulting in the assertion that the amount invested at $6\%$ is $3600-x$. (Unfortunately, a resistant error is often made here: the student asserts that the amount invested at $6\%$ is $x-3600$. The researcher believes the reason this error is hard to correct is that students are doing this part in their heads, without benefit of a methodical, algebraic approach.) Then the interest formula is used, and the remainder of the problem is translated to become equation (22). Then, having solved to get equation (23), students often have difficulty finding the amount of the second investment and interpreting the results. It is no surprise to the researcher that students have trouble solving word problems, when they are taught to do them with only one variable, such that they do most of the steps in their heads.

Although Lazar makes only the claims of (a) simplifying the matter of setting up equations and (b) reducing translational errors by introducing simultaneous equations early in a first course in algebra, these two steps are at the heart of reaching one of the main goals of teaching algebra to students -- helping them solve word problems. Many studies have addressed some of the difficulties that students have when trying to solve word problems (see Chapter 2), including trouble with understanding the concepts of variable and function and translational difficulties. Research has not attacked the question of "how many variables" to introduce as soon as possible in a first course in algebra. The purpose of the present study was to investigate the impact that introducing simultaneous equations
early in the first year of algebra had on the problem-solving skills of the students in that course.

**Information Processing Theory**

Larkin, McDermott, Simon, and Simon (1980a) recorded verbal accounts by expert and novice problem solvers as they thought aloud while solving physics problems. From their observations and analysis of the similarities and differences between experts and novices, they developed theories about human information processing in expert- and novice-problem solvers. They tested these theories with computer simulations developed during the think-aloud protocols of the human problem solvers and believe that "their primary claim to acceptance rests on their solid foundation of data and computer simulation" (p. 1336). The capacity for short term memory is only about four "chunks" of information. The more novice a person is in a given field, the smaller the chunks are, and the more expert a person is, the more information the chunks contain. "Paper and pencil provide an unlimited extension of the problem solver's working memory capacity, but at the cost of writing down the information (which can be done more rapidly than it can be memorized) and of gaining access to it when it is needed" (p. 1340). Thus, if novice problem solvers declare a variable for each unknown and write out their relationships explicitly, they can access all of the information necessary to solve the word problem; whereas, they can only access about four chunks of information in their heads at any given time.

Further analysis of the observations of expert and novice problem solvers led Larkin, McDermott, Simon, and Simon (1980b) to note the difference between the way in which experts and novices approached the solutions and the way in which they connected the information in the problem to the variables in the equation representing the problem.

The following example of a physics problem and its solution by an expert and by a novice (Larkin, 1981) illustrates both the forward and backward approaches taken by the
expert and the novice, respectively, and the novice's explicit connection of the variables in each equation to a known or desired variable specified by the problem. Furthermore, it exemplifies the need for even the novice to be comfortable using as many variables as are needed to solve the problem and shows that the novice skips no steps.

**Problem.** A block of mass $m$ moves from rest down a plane of length $L$. If the coefficient of friction between the block and the plane is $\mu$, what is the block's speed as it reaches the bottom of the plane? (p. 313)

**Expert Solution.** The motion of the block is accounted for by the gravitational force, $F_g = mg \sin \theta$, directed downward along the plane, and the frictional force, $f = \mu mg \cos \theta$, directed upward along the plane. The block's acceleration $a$ is then related to the (signed) sum of the forces by $F = ma$. Knowing the acceleration $a$, it is then possible to find the block's final speed $v$ from the relations $l = \frac{1}{2} at^2$ and $v = at$. (p. 314)

**Novice Solution.** To find the desired final speed $v$ requires a principle with $v$ in it, say $v = v_o + 2at$. But both $a$ and $t$ are unknown, so that seems hopeless. Try instead $v^2 - v_o^2 = 2ax$. In that equation $v_o$ is zero and $x$ is known, so it remains to find $a$. Therefore, try $F = ma$. In that equation, $m$ is given and only $F$ is unknown; therefore, use $F = \sum F's$, which in this case means $F = F_g - f$ where $F_g$ and $f$ can be found from $F_g = mg \sin \theta$, $f = \mu N$, and $N = mg \cos \theta$. With a variety of substitutions, a correct expression for speed, $v = \sqrt{2(g \sin \theta - \mu \cos \theta)}$, can be found. (p. 315)

Experts worked with a bottom-up, forward-moving method. They noted which values for the variable were known and selected a principle that allowed them to find the value of a new related variable. They always worked forward, generating new knowledge with each step, and they sometimes combined several principles into one step. In fact, they often combined the selection and application of the required principle, collecting the necessary information and generating new information in one step. The experts stated equations, but they rarely mentioned the names of the variables they were using; they simply proceeded to solve the equations using any known numerical values of the independent variables. To Larkin and her colleagues, it seemed that the expert solvers
already knew what the independent variables represented and many of the numerical values of these variables as they wrote the equations and did not need to separately bind each variable to the information in the problem. On the other hand, novices focused on the quantity that they ultimately needed to find and worked backwards, selecting expressions for the quantities that remained unknown. They wrote their selected equation and then explicitly bound each variable in it to any known information or desired variable. They skipped no steps.

This example from Larkin's research supports the idea of defining each variable that is needed and writing out explicitly all of the equations necessary. In fact,

The expert apparently had stored directly (perhaps as a production) an entire procedure for obtaining a desired value from related known values; he then applied this procedure for obtaining a desired value and stated only the result he obtained. The novice, in contrast, had stored the knowledge that particular equations can be used to obtain values of certain variables. ... At each step in the path, the novice had to ask herself, "What do I do next?" (Larkin et al., 1980a, p. 1139)

**Attitude Toward Problem Solving in School Mathematics**

Because cognition and affect are interrelated, "mathematics teachers should consider the effects of affective components of learning mathematics in planning effective instruction" (Adams, 1989). Thus, the main goal of research on affective issues in problem solving is improvement of instruction and student performance (McLeod, 1989; Reyes, 1984). Moreover, Fennema (1980) found that there is a positive relationship between attitude toward mathematics and mathematics achievement that seems to increase as learners progress in school. Even given the relationship between attitude and achievement in mathematics, the goal of developing a positive attitude towards mathematics is itself an important outcome of teaching (Adams, 1989; Reyes, 1984).

Charles, Lester, and O'Daffer (1987) defined self-confidence with respect to problem solving as one aspect of the construct of attitude toward problem solving. Self-
confidence relates to a student's expectations of success and failure and thus influences his persistence in problem solving (McLeod, 1989). Reyes (1984) has found that self-concept has a consistent, positive relationship with general academic achievement; in fact, there is some support for a causal effect of self-concept on achievement. In addition to its possible effect on academic achievement, self-concept is an important predictor of mathematical course election. So important is self-confidence in mathematics that Steen (1990), believes "confidence rather than calculation should be a chief objective of school mathematics" (p. 134). If introducing the solution of two equations in two unknowns early in a first course in algebra helps algebra students solve word problems more successfully, it could lead to students' developing more self-confidence with respect to problem solving. This is worth ascertaining.

Kulm (1980) states that "there is no question that data on attitude can be extremely useful in evaluating the effects of instructional or curriculum innovations" (p. 370). Since the introduction of the solution of two equations in two variables early in a first course in algebra could be considered a curriculum innovation, investigating students' attitudes toward problem solving in school mathematics is a worthwhile endeavor as part of an exploratory study of the effects of using as many variables as are needed to solve word problems on the problem-solving skills of students in beginning algebra. Furthermore, if it helps algebra students solve word problems more successfully, it could lead to students' developing more self-confidence with respect to problem solving.
In learning how to solve problems in algebra, students experience many difficulties, including the concepts of variable and function, translation from English to algebraic notation, and the number of variables used in solving a problem. They may not know what variables actually are or represent, let alone how to use them to help solve word problems. They may not understand what an equation is nor how it is constructed using variables. Added to this is the difficulty of solving complicated equations reliably. They may not understand what functions are. Although they do not need to know the Bourbaki definition of function nor be able apply the definition, they do need to understand how one variable can depend on another and how this can aid them in setting up a problem. Finally, in the biggest hurdle of all, they cannot translate from the English words to algebraic symbols. They often do not know what to define as their variables nor how to set up their equations to represent the statement of the problem.

Variables

One of the most important aspects of algebra is the notion of variable. Mathematics teachers have read, written, manipulated, and translated variables for so long that it rarely occurs to them to question what they are or how to use them to help solve word problems. In this section the author reviews studies of what variables are and how and why students have trouble with them in algebra. She then relates these to her inquiry into how many variables to introduce to students in solving word problems.
Student Misunderstandings

As Booth (1988) noted in a report of the algebra strand of the Strategies and Errors in Secondary Mathematics (SESM) project conducted in the United Kingdom from 1980 to 1983, "the meaning of letters and variables" was one of the primary sources of algebraic errors made by students, regardless of age or experience in algebra. The algebra strand of the SESM project researched the kinds of errors commonly made by students in algebra and investigated the reasons for these errors. The study involved students in grades 8 to 10, who had been studying algebra in an integrated mathematics program since grade 7. The researchers gave the students word problems involving variables and interviewed them to determine specifically how they would solve the problems and how they used variables. This study found that the students' errors in algebra could be traced to their misunderstanding of

- the focus of algebraic activity and the nature of answers;
- the use of notation and convention in algebra;
- the meanings of letters and variables;
- the kinds of relationships and methods used in arithmetic.

Booth found that one of the obvious differences between arithmetic and algebra is the use of letters in algebra to represent values rather than the use of letters in arithmetic to represent units. This change in usage results in a lack of numerical referent in students' interpretation of letters in algebra. For example, students find it difficult to distinguish between $a = l \times w$ for area = length $\times$ width and $5y$, thinking that the $y$ must stand for yachts or yams (rather than the number of yachts or yams). Another difference is that arithmetic symbols always signify unique values (such as "3") while algebraic symbols signify numbers in general (such as $x + y = y + x$). Thus students often assume that different letters in algebra must stand for different values.
Literal Symbols versus Numerals and Words

Wagner (1983) describes the difficulties that variables present to students. Variables resemble both numbers and letters, but they do not behave exactly like either one. Wagner has pointed out the similarities and differences between variables and numerals and variables and words. Because of this similarity with, but not duplication of numerals or words, students often find variables "easy to use but hard to understand," and sometimes find variables difficult to use correctly. When teachers introduce literal symbols, it is pedagogically sound to compare them with both numerals and letters with which students are familiar. However, lest the students overgeneralize their knowledge of variables, it is also sound to emphasize the differences that literal symbols have with numerals and letters.

The Many Meanings of Variable

Schoenfeld and Arcavi

Schoenfeld and Arcavi state that "The concept of variable is central to mathematics teaching and learning .... Understanding the concept provides the basis for the transition from arithmetic to algebra" (Schoenfeld and Arcavi, 1988). Most mathematics curricula treat variables as primitive terms that do not warrant explanation. It is assumed that after some practice, all students will understand variables and be able to use them. (As both Wagner (1983) and Booth (1988) observed, this is not the case.) In fact, although mathematicians use variables adroitly, even they may have forgotten the many subtleties and multiplicities of meanings of variable.

Schoenfeld and Arcavi suggest an exercise that the reader might try: give one descriptive noun that gets at the meaning of variable. This is quite difficult because the term is too rich to be labeled with only one noun. Each word, such as symbol, placeholder, or identifier, approaches what a variable is but does not get at the heart of its many meanings. With this in mind, one can move on to try to define variable. Even that
is no simple task. Schoenfeld and Arcavi provide ten definitions, which collectively begin to give the richness of the term and consider many of its major facets. Some of the definitions give the reader a sense of "how it works" — "Latin - variabilis: changeable." Other definitions show how variables behave, using such descriptions as "motion" or "empty space." Still others lead to even more confusing terms, such as the reference to "dependence upon domain" in the modern definition of variable. Some of the definitions get at the use of variable as a tool for making generalizations. With these several definitions now in mind, the reader can begin to see the novice's problem in understanding what variables are and how to use them. "Our multiple uses of the term variable makes it hard for students to understand" (Schoenfeld and Arcavi, 1988, p. 422).

Since teachers use the term variable in many different contexts with different meanings, they need to be sensitive to this multiple usage. They need to realize that not only is the concept of variable hard to describe, but it is also even harder to learn. Schoenfeld and Arcavi make three suggestions related to teaching the concept of variable:

1. One of the main tools of variables is their usefulness for expressing mathematical generalizations. To help students make the transition from arithmetic to algebra, teachers should get into the habit of asking students to summarize some of their observations about arithmetic in their own words.

2. "The dynamic aspects of the variable concept should be stressed whenever it is appropriate and feasible" (p. 426). Computer-based tools now make it much easier to capture the dynamic aspects of variable, such as in time-dependent phenomena (the distance an object falls in $t$ seconds).

3. Students should be exposed to as wide a range of problems as possible in algebra. Thus they can learn "that the language of algebra is a powerful means of capturing the mathematical essence of a wide variety of situations" (p. 426).
Kuchemann

Analyzing the results of a test given to 3000 students in the seventh, eighth, and ninth grades in England in 1976, Kuchemann (1978, 1981) found that students have six ways of using literal symbols: evaluating the literal symbol, ignoring the literal symbol, using the letter as an object, using the letters to represent specific unknowns, using the letters as generalized numbers, or actually using them as variables. He categorized their usage into four developmental levels. The first level includes evaluating the literal symbol immediately or ignoring it, and the second level includes all of the uses of the first plus using letters as objects. These first two levels are at the concrete Piagetian stage of development. Students at the third level use letters as specific unknowns in structurally simple instances, and students at the fourth level can use letters as specific unknowns in complex situations, as generalized numbers, and as variables in which they recognize a systematic relationship between the letters. The third and fourth levels are at the formal Piagetian stage of development. Kuchemann found that the majority of secondary students tested were only at levels one or two and were not able to consistently cope with literal symbols used algebraically in which the use of the letters as unknown numbers could not be avoided, the letters were used as generalized numbers, or the letters were used as variables.

Usiskin

Usiskin (1988) complements both Schoenfeld and Arcavi's and Kuchemann's discussions of the roles of variables when he classifies five uses of variables. Variables can be used in formulas, such as \( A = LW \), where each letter has the feel of a known number. Literal symbols can be used in an equation, \( 40 = 5x \), in which \( x \) is an unknown. A letter can be an argument of a function as in \( \sin x = \cos x \times \tan x \). Variables can be used in an equation that generalizes an arithmetic pattern such as \( 1 = n \times (1/n) \). Finally, variables can be used in an equation that is not to be solved, such as \( y = kx \), which is simply a function
of direct variation. In that equation, \( x \) is an argument of the function, \( y \) is the value of the function, and \( k \) is a parameter. The purposes of algebra are closely related to the conceptions of algebra, and these in turn are closely related to the uses of variables.

**Relation to the Author's Research**

When students enter algebra classes, their teachers must be aware that they will need help in sorting out the similarities and differences between literal symbols and numerals and words and in sorting out the many uses of variables. While teaching algebraic problem solving, teachers must continually help students grapple with their difficulties with variables. Students cannot even begin to solve a problem algebraically if they cannot use the vocabulary of algebra, in other words, if they cannot use variables. Whereas Booth studied the difficulties that high school students have in learning algebra, of which a large component was "the meaning of letters and variables," the researcher inquired specifically into how the introduction of as many variables as needed can help students solve word problems more successfully. At the same time, Wagner suggests that mathematics teachers need to be aware of the many ways literal "symbols are used and recognize the particular characteristics they exhibit in various contexts" (Wagner, 1983, p. 478).

Schoenfeld and Arcavi and Kuchemann also emphasize the need to help students with the concept of variable. "As Alfred North Whitehead declared, 'by relieving the brain of all unnecessary work a good notation sets it free to concentrate on more advanced problems, and in effect, increases the mental power of the race'" (Schoenfeld and Arcavi, 1988, p. 423). Mathematical notation, including the use of variables, is a powerful tool. However, variables are subtle and difficult to learn. Once mathematics teachers have mastered variables, they tend to forget how much work went into their understanding. Usiskin states that algebra "provides the means to analyze and describe relationships" (Usiskin, 1988, p. 18). However, students cannot use algebra to analyze and describe
relationships if they cannot use variables appropriately. The research presented here
determined if teaching students to solve word problems by using as many variables as
there are unknowns helped them use variables appropriately to solve word problems more
successfully.

Equations

To solve word problems algebraically, students need to be able to declare which
unknowns are represented by which variables and to write and solve equations that
represent the relationships between the unknowns and the knowns given in the word
problem. The literature on translation from English to Algebra consists of research on the
writing of equations to correctly represent the relationships between the unknowns and the
knowns of a problem; it will be reviewed in the section entitled Translation. This section
will review some of the research that has been conducted on how students construct
meaning for the concept of equation (Herscovics and Kieran, 1980), on how students
approach solving algebraic equations (Kieran, 1988), and on how teachers can help
students learn to solve equations (Roberts, Carter, Davis, and Feurzeig, 1989).

Constructing Meaning for the Concept of Equation

Herscovics and Kieran (1980) noted that students often do not view \(2 + 3 = 5\) as
an identity but instead interpret it operationally: 2 and 3 make 5. They view the equal sign
operationally rather than as an equivalence, and they often continue holding this
operational view when working with algebraic equations.

Working with students in grades seven and eight, Herscovics and Kieran helped
them construct the concept of algebraic equation. The researchers began by helping the
students construct simple arithmetic identities and then expanded the constructions to first
include multiple arithmetic operations on both sides of the equal sign and then to include
hidden numbers. Only after time was spent constructing equations did the researchers
help the students solve the equations with hidden numbers. From this work, the students
grasped the idea that solving equations meant finding the hidden number. The students found that they could hide more than one number and that they could construct equations without solutions. This progression from arithmetic equations to algebraic equations helped the students in the study grasp the meaning of algebraic equations as expressions of equality containing letters rather than as operational expressions.

**Approaches to Solving Algebraic Equations**

Kieran (1988) found that before students are formally taught algebra, they have two distinct approaches to viewing equations and to solving them: an arithmetic approach and an algebraic approach. Those with an arithmetic approach focus on the given operations and solve equations by trial and error, and those with an algebraic approach focus on the inverses of the given operations and solve equations by transposing to the other side. After teaching six thirteen-year-olds how to solve equations using an equation-solving procedure that made explicit the left-right balance of the equation, Kieran found that the students with an arithmetic approach used performing the same operation to both sides to solve equations, while the students with an algebraic approach continued to solve equations by transposing expressions to the other side, often incorrectly when the equation was complicated.

**Methods to help students learn to solve equations**

Bernard and Cohen (1988) integrated equation-solving methods into a developmental learning sequence that takes students step by step from a trial-and-error method of solving equations to the method of applying balance operations to both sides of the equation. However, Bernard and Cohen cited no research that evaluated their program.

Roberts, Carter, Davis, and Feurzeig (1989) developed the Logo-algebra project to test how much algebra could be taught to sixth graders. They worked with two sixth-grade classes who used interactive computer software to represent equations, first using
picture icons and then progressing to algebraic notation. The students entered their equations into the software and solved them step-by-step in the calculator mode of the software. Roberts and his colleagues found that the children who took the course significantly increased their ability to solve algebraic equations.

Relation to the Author's Research

While the previous research on equations has determined how students construct meaning of equations and how they solve equations, no researcher has studied the effect on problem solving of solving equations mentally. When students are forced to put all unknowns in terms of one variable, they have to set up and solve equations mentally. The author's research looked at the effects on algebraic problem solving of using as many variables as unknowns and thus explicitly writing out and solving all necessary equations.

Functions

When trying to solve word problems, students employ functional relationships in which one variable depends upon another variable, either explicitly or implicitly. Frequently, they do not understand what a relation or function between two variables is. They may not need to know the definition of function in order to solve algebraic word problems; however, they must be able to use the idea of relations between variables.

Review of Research

In a review of the research and theory related to functions, graphs, and graphing, Leinhardt, Zaslowky, and Stein (1990) provide a survey of the literature of the tasks, learning, and teaching of functions and graphs. Most of the existing research has concerned students' interpretation of functions and graphing in which their translation from one representation to another or their moving from a local interpretation to a global interpretation has been studied. In contrast to the much-studied interpretation, research that has focused solely on students' construction of functional equations from graphs or graphs from equations is relatively rare.
The functional tasks that have been researched include: prediction, in which the prediction tasks are those of conjecturing from a given part of the graph where other points of the graph should be located or how other parts should look; classification, in which students' understanding of the formal definition of function and of families of functions is investigated; translation, in which students' ability to recognize the same function in different representations is studied; and students' understanding and construction of scaling and units of graphs.

Leinhardt and her colleagues found that the research on students' learning of functions and graphing consists of research on students' intuitions of functions and on students' misconceptions and difficulties with the concepts of functions and graphing. The research on intuitions deals with students' intuitive understanding of functions and with the development of functional thinking. The investigations about misconceptions and difficulties concern the following:

- What is and is not a function;
- Correspondence;
- Linearity;
- Continuous versus discrete graphs;
- Representations of functions;
- Relative reading and interpretation of graphs that represent situations;
- Concept of variables;
- Notation.

The areas of research about the teaching of functions and graphs were found to concern what should be taught, when it should be taught, and how it should be taught. In other words, how should a teacher introduce the concept of function, what should be the sequence for teaching the various concepts concerning functions, and what types of
explanations of those concepts should be given are all areas that have been studied by researchers.

The author of this dissertation will now provide more detail in the areas of research on functions that apply to her study of teaching students to use as many variables as unknowns early in their first course in algebra. These topics are conservation of equation and function, images and definitions of the concept of function, and intuitive notions of function.

Conservation of Equation and Function

One of the features of variables and functions that students have trouble understanding is the conservation of equation and function under transformations of variables. Conservation of a concept refers to the preservation of critical attributes that remain invariant under transformations of some of its irrelevant attributes. The critical attribute of a mathematical relation is that a set of ordered pairs constitutes the relation. Irrelevant attributes include the representation of the relation (concrete, pictorial, graphical, verbal, or symbolic), and the variables used to define the relation.

In a study whose purpose was to illustrate one way of extending Piaget's conservation methodology from simple concepts to relational concepts and to investigate students' ability to conserve equation and function under alphabetic transformations of literal variables, Wagner (1981) interviewed fifteen junior high school students and fifteen senior high school students. Her instrument consisted of four conservation tasks for each student, one conservation-of-equation task and three conservation-of-function tasks. In the conservation-of-equation task the student was presented with the equation \(7 \times w + 22 = 109\), followed by \(7 \times n + 22 = 109\). Wagner then asked, "Are these two statements the same?" In each task, she followed the student's answer with "Why? or "How can you tell?" All the conservation-of-function tasks included a table of values for an independent variable along with a table of values for a dependent variable. In a free-
response task, for one value of the independent variable, the value for the dependent variable was left blank. The student was asked what the value should be. Then the symbol for the dependent variable was changed, and the student was asked again what the value should be. In two functions with furnished responses for the dependent variable, the dependent variable was changed, in one question from \( z \) to \( y \) and in another question from \( k \) to \( k + 3 \). In each case Wagner again asked the student what the value of the dependent variable should be.

The results obtained in the study suggest that the inability to conserve equation or function varies among students of different ages, of either sex, and of various mathematical background; but fewer than half of all the students gave conserving responses to any of the four tasks used.

Although the small size of the study precluded drawing any generalizable inferences about the ability to conserve based on age, sex, or mathematical background, the study does document two common misconceptions about variable previously noted: that changing the variable symbol implies changing the referent, and that the linear ordering of the alphabet corresponds to the linear ordering of the number system.

Furthermore, the vast majority of students reacted to the task in one of two ways. Some seemed to accept the change of variable with a "shrug" and said that the letter made no difference as long as the numbers stayed the same; while others seemed to regard the change of variable as transforming the original problem into a brand new problem.

**Images and Definitions of the Concept Function**

Vinner and Dreyfus

Vinner and Dreyfus (1989) conducted a study in which they examined aspects of images and definitions for the concept function held by college students and junior-high-school mathematics teachers who had not majored in mathematics. The idea behind their study was that all mathematical concepts except primitive ones have formal definitions,
and many of these are introduced to high-school or college students. However, a student
does not necessarily use the definition when deciding that a given mathematical object is
an example or nonexample of a concept. In most cases, he or she decides on the basis of a
class image, a set of mental pictures associated in the student's mind with the concept
names, together with properties characterizing them. The image is the result of the
student's experience with examples and nonexamples of the concept.

The modern concept of function, also known as the Dirichlet-Bourbaki concept, is
that of a correspondence between two nonempty sets that assigns to every element in the
first set (the domain) exactly one element in the second set (the codomain). This approach
to the definition of function, as many individual correspondences, was not recognized as
function by previous generations of mathematicians. Functions that were not accepted as
functions historically, but are considered as functions in this modern definition, include
discontinuous functions, functions defined over split domains, functions with a finite
number of exceptional points, and functions defined by means of a graph. Mathematics
textbooks often give students the modern definition of function but provide only examples
whose rule of correspondence is given by a formula.

In their study, Vinner and Dreyfus included several groups of first-year college
students in two Israeli institutions and 36 junior-high-school mathematics teachers. The
students had majors that required various amounts of college-level mathematics, from a
low level for 33 students majoring in industrial design, through an intermediate level for
67 students majoring in economics or agriculture and a high level for 113 students
majoring in chemistry, biology, or technological education, to a mathematics level for 58
students majoring in physics or mathematics. The questionnaire administered to the
subjects contained seven questions, six that examined some aspect of the image held by
the respondents, and one that examined their definitions of function.
The definitions given by the students fell into seven categories: correspondence (the Dirichlet-Bourbaki definition), dependence relation, rule, operation, formula, and representation (in which the function is identified, perhaps in a meaningless way, with one of its graphical or symbolic representations). The percentage of students who gave some version of the correspondence definition increased with the level of mathematics course the students were taking. The authors considered compartmentalization as one of the most interesting aspects of the study. In compartmentalization, a student has two different, perhaps conflicting schemes in his or her cognitive structure. In this case, compartmentalization would occur if a student held a Dirichlet-Bourbaki definition of function but also held some other image, such as function as a rule-based relation between two numerical variables. Of the respondents who gave a correspondence definition, 100% of the low and intermediate level mathematics students compartmentalized, while 71% of the high-level students compartmentalized, and only 23% of the mathematics majors did so. The authors suggest, given the students' difficulties in the study, that doubt should be raised whether the Dirichlet-Bourbaki approach to function concept should be taught in a course where it is not intensively needed.

In their questionnaire, Vinner and Dreyfus examined some components of mathematical thought, such as the ability to reason and to apply the definition of function in a coherent way. They concluded that the course level of the student is a good indication of the student's reasoning patterns - the higher the course level, the higher the reasoning patterns.

Orton

Orton conducted a study of eight boys and eight girls from each of the second through the fifth years of secondary school and eight students in the sixth year of secondary school in England (Orton, cited in Lovell, 1971). He provided sixteen tasks dealing with the concepts of functions individually to each student. He discovered several
stages of the growth of idea of function. From the students' responses to the sixteen functional tasks, Orton classified the results into four stages in growth of the idea of function:

1. Essentially intuitive or concrete thinking of function;
2. Grasp of the relational aspects of the concept of function but a nonunderstanding of the basic criteria necessary for a relation to be a function;
3. Ability to identify relations and a mastery of the basic concept of function but carelessness regarding checking the uniqueness of images;
4. Display of the mastery of the basic concept of function with the ability to classify all representations of relations as functions or not functions.

Markovits, Eylon, and Bruckheimer

In keeping with the results of the studies by Vinner and Dreyfus and Orton, when Markovits, Eylon, and Bruckheimer (1988) investigated how ninth- and tenth-grade students understood the concept of function, they found several levels of concept understanding. Markovits and his colleagues identified the stages as the ability to classify relations into functions and nonfunctions, the ability to identify the preimage and the image of a given function, the ability to identify equal functions, and the ability to identify functions satisfying given constraints. The researchers asserted that the definition of function as it is now taught involves many concepts: such as graph, domain, range, image set, and rule of correspondence. Consequently, many students have difficulties with several of the component concepts. These difficulties include identifying images and (preimage, image) pairs, distinguishing between the image set and the range, and ignoring the domain and range of a function. They suggest that with weaker students it is defensible to omit any treatment of equal functions and to choose to play down some aspects of functions.
Intuitive Notions of Function

The definition used widely in textbooks and classrooms today presents function as a special type of relation or correspondence, as a rule that assigns to each member of set $A$ exactly one member of set $B$. (This is equivalent to the Dirichlet-Bourbaki definition presented by Vinner and Dreyfus.) The main focus is on the mapping of elements of one set into elements of another set, on distinguishing between relation and function, and on finding functions of functions. Historically, a function was defined as a rule-based relation between two interconnected numerical variables. The emphasis was on how changes in the dependent variable correlated with changes in the independent variable. It has been argued that the Dirichlet-Bourbaki and historical definitions represent two very different forms of thought and that it is not obvious how understanding one definition helps one to understand the other, particularly at the elementary level.

The pervasive concern with the modern instructional definition is that it is abstract and lacks a familiar, situation-based context. In a review of the research and theory related to teaching and learning functions, graphs, and graphing, Leinhardt, Zaslavsky, and Stein (1990) concur with Vinner and Dreyfus's and Markovits, Eylon, and Bruckheimer's conclusions that perhaps all students do not need to learn the modern definition of function as a correspondence. The authors found evidence when reviewing J. G. Greeno's work (1988) that children possess intuitive ideas about functional relationships developed through "global perceptual observations of the physical phenomena that surround them every day" (Leinhardt, et al., p. 28), such as the change of temperature over time. Furthermore, the children's intuitions include notions about dependence, causality, and variation. They suggest that exposure to the modern formal, instructed definition of function fails to build on the intuitive understandings the students may have.

In a study on intuitive functional concepts possessed by students, Dreyfus and Eisenberg (1982) found that not only do children hold intuitive ideas about functions,
these intuitions on functional concepts grow with pupils' progress through school. These intuitions are independent of sex, but the intuitions of high-level students are more often correct than those of low-level students.

Relation to the Author's Research

Wagner's findings (1981) of the majority of interviewed students' inability to conserve equation or function have important implications in teaching students how to solve algebraic word problems. One of the most important aspects of problem solving is deciding what variables represent when setting up the problem. Thus, mathematics teachers need to help students understand that changing a variable symbol does not change the referent. The set of ordered pairs in the relation formed between the independent variables and the dependent variables is conserved whatever variables they choose in solving a word problem. In working with as many variables as needed to set up a word problem, this becomes even more important.

The conclusion drawn by Vinner and Dreyfus (1989), that the higher the mathematical course level, the higher the reasoning patterns, holds particular implications in teaching students algebraic problem solving. Vinner and Dreyfus's study suggests that Algebra I students at the beginning of higher-level mathematics courses have relatively low-level mathematics reasoning powers and perhaps do not need a rigorous modern treatment of functions. Instead, they may need a rule-based treatment of functions. Furthermore, mathematics teachers must recognize that a complex concept is not acquired in one step. Several stages must precede the complete acquisition and mastery of the concept, during which several cognitive schemes may act in the same person in different situations closely related in time.

In teaching algebraic problem solving and the accompanying concept of relations between variables, teachers should build on the student's notions of function, acquired in childhood, which are based on an implicit sense of variables that have attributes of being
concrete, dynamic, and continuous. They should concentrate on the notion of function as a rule-based relation between variables. By building on the students' intuitive concept of function with its "implicit sense of variables," teachers can help students to overcome the significant difficulty of understanding the relation between a dependent variable and an independent variable.

Translation

Recent mathematics textbooks and courses have increasingly emphasized applications. However, modeling real-world situations is more difficult for students than often thought. One of the students' largest difficulties is translating the real-world situation into mathematical language, the language of algebra. In mathematics textbooks, applications are written as word problems. Thus, in order for a student to solve word problems, he or she must overcome the difficulties of translation.

Categorization of Word Problems

Hinsley, Hayes, and Simon (1977) performed five experiments with college students to determine if their conjectures about the way people approach and work word problems were correct. They found that students recognize various categories of word problems. Furthermore, students often recognize a problem's category early in their reading of the problem. Students have information about these problem categories that they use to help them solve the word problem when they have recognized that the problem belongs to a certain category. However, the researchers observed that their subjects solved some problems using a direct translation process, most frequently when the problems were nonstandard problems or when the students did not recognize a category for the problem. From their study, Hinsley, Hayes, and Simon concluded that people use more than a single approach in comprehending word problems; they use heuristics specifically useful for solving problems in a given category if they recognize the category, or they use direct translation.
Using a Detailed Solution of a Word Problem

The structure mapping model, as used by Reed, is a detailed description of how people translate and solve complex word problems. Reed (1987) conducted four experiments with college students to study how effectively they could use a detailed solution of one word problem to solve a related problem. The purpose of the first two experiments was to examine how aware students were of different problem types. Pairs of word problems were contextually alike or different and were structurally alike or different. If two problems were contextually alike, they had the same story context; and if they were structurally alike, all of the concepts and relations in one solution (algebraic equation) could be mapped onto all of the concepts and relations in the other solution. The purpose of his last two experiments was to evaluate three assumptions of the structure-mapping model:

1. An analogy can be represented as a mapping of concepts treated as wholes and attributes and relations from an example problem to a test problem.
2. An isomorphic structure of a mapping is a particularly important characteristic of word problems. If two word problems are isomorphic, they are structurally alike.
3. The degree to which students understand the structure of the worked problem and the transparency of the problem or the ease with which they can perceive how concepts and relations in one problem correspond to those in the other problem influence the success of the mapping.

He found that students did much better in formulating equations for isomorphic problems, and they were fairly good at matching concepts in isomorphic problems.

The Variable Reversal Error in Translation

Clement, Lochhead, and Monk (1981) became aware of the problem of translating from practical situations to mathematics when they conducted interviews with university freshman engineering students, most of whom were taking calculus. In the interviews, the
students were asked to talk aloud while working on simple word problems. Fewer than 50% of the students could solve the problems correctly, indicating a difficulty with translating into and out of algebra.

The word problems given to the students included problems such as the following: "Write an equation using the variables C and S to represent the following statement: At Mindy's restaurant, for every four people who ordered cheesecake, there were five people who ordered strudel. Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered" (p. 287). Only 39% of the students solved this problem correctly.

Clement and his colleagues were originally greatly surprised at how poorly the engineering students solved the given word problems. After some thought, they realized that science and engineering students are rarely asked to construct a formula. They are usually given a formula or asked to select the appropriate formula from a well-defined list. The one place in secondary school mathematics where translation does play a large role is algebraic word problems.

To investigate the sources of errors, the authors collected data on a simpler problem. "Write an equation for the following statement: There are six times as many students as professors at this university.' Use S for the number of students and P for the number of professors." This problem was given as part of a written test to 150 calculus-level students. Thirty-seven per cent of the students missed the problem, and 2/3 of the errors took the form of reversal of variables: 6S = P instead of S = 6P. In a sample of nonscience majors taking college algebra, the error rate was 57%. Fifteen students were videotaped and asked to think aloud while solving problems similar to the one above. Students vacillated between correct and incorrect solutions and appeared thoroughly confused, not just guilty of making hasty mistakes.
Clement, Lochhead, and Monk found two distinct sources for the students' tendency to reverse variables. The authors called the first source the "word order matching" method of syntactic translation. This is a literal, direct mapping of words of English into the symbols of algebra. For "there are six times as many students as professors," the students wrote "6S = P." The second error source is the "static comparison" method. In this mistake, the students understand that the student population is much bigger than the faculty population, but they still believe the relationship is 6S = P. Here 6S is used to indicate the larger group and P is used to indicate the smaller group. S is not understood as a variable that represents the number of students but is rather treated like a label or unit attached to the number 6. The students' interpretation of the equation is as a literal attempt to symbolize the static comparison between the two groups. This approach stems from a representation of the essential features of the problem, but breaks down in the students' misunderstanding of the nature of variable and equation.

The people who did a wide variety of problems correctly used a markedly different approach of "operative translation." This approach requires the comprehension of the static-comparative approach, together with a richer sense of what a mathematical equation is and of what variables are. The number S is seen as bigger than P; therefore, the number P must be operated on by multiplying it by 6 to produce the number that is the same as S.

The reversal error seems to be a self-generated, stable, and persistent misconception concerning the meaning of variables and equations. Even after a semester or more of calculus, many students still have difficulty expressing relationships algebraically.

**Correcting the Variable Reversal Error**

The translational difficulties that were previously observed among college students have since been observed among students of all ages and of many nationalities. Lochhead and Mestre (1988) found a consistent pattern among the students' errors in problems in
which they were asked to read a sentence stating a relationship between two variables and then write an equation expressing that relationship. As Clement and his colleagues had found, two particularly salient types of errors were the strong proclivity of students to perform left-to-right word-order matching when translating "students and professors" problems and a confusion of the distinction between variables and labels.

Although many studies have been done since 1981 concerning the sources of translational errors (Booth, 1984; Clement, 1982; Cocking and Chipman, 1988; Davis, 1984; Herscovics, 1989; Kaput, 1987; Kuchemann, 1981; MacGregor and Stacey, 1993; Malle, 1985; Mestre, 1988; Spanos, Rhodes, Dale, and Crandall, 1988), many teachers are still perplexed about how to help students correct their errors. To help solve the translational difficulties of students, Lochhead and Mestre suggest that teachers follow three steps. (The steps are illustrated with the "students and professors problem".)

1. Probe for qualitative understanding. Ask the students whether there are more students or more professors.

2. Look for quantitative understanding. Ask "Suppose there were 100 professors at the university. How many students would there be?" (One hopes that they reply, "600.")

3. Strive for conceptual understanding.

After completing steps one and two satisfactorily, perform step three by asking the students to write an equation that expresses the relation in the problem statement. (You will probably get several different equations, such as $6S = P$, $6S/P$, $6S + P = T$, and $6S = 6P$.) Now have those students who wrote $6S = P$ to check their equation by substituting $S = 600$ into the equation, and see what they get for $P$. Some will substitute $S = 600$ appropriately and obtain $P = 3600$. At this point, it is easy to elicit conflict by pointing out that this does not agree with $P = 100$. Using Socratic dialogue with the class and asking probing questions, but not giving the correct answer, you can get to the heart of
the misconception. Be sure that you do not let the discussion end until the correct answer manifests itself through the students' discussion or they may be left with the idea that how they should solve these kinds of problems is "first you write down what makes sense, then you write down the reverse of that" (Lochhead and Mestre, p. 134).

**Syntactic Model of Translation**

In the decade since Clement and others observed the reversal error made by students when solving the students and professors problem, the reversal error ($6S = P$ instead of $6P = S$) has become one of the most highly investigated phenomena in the recent history of mathematics education (Kirshner, Awtry, McDonald, and Gray, 1991). The cognitivist approach to understanding the reversal error has been to closely observe subjects, both successful and unsuccessful, as they grapple with the above and similar problems. Kirshner and others state that the feature of the cognitivist approach that gives rise to concerns is that "the clinical interview is an entirely inappropriate method for investigating phenomena that may be related to syntactic processes -- well-known to be unconscious and inaccessible to introspection. Think aloud protocols only can reveal aspects of thought that are consciously accessible to the informant" (p. 3). Therefore, the conclusion of the cognitivist research (that semantic/conceptual, rather than syntactic, knowledge underlies successful translation) is an artifact of the methods used rather than an implication of the research.

For the majority of school word problems, the syntactic methods of phrase-by-phrase translation can be successfully employed. Kirshner and his colleagues classified four types of sentence translations. The simplest sentences can be translated by a direct "phrase-order-method" (POM); for example "John's weight is equal to five times Sally's weight" could be written algebraically as $J = 5S$ where $J$ represents John's weight and $S$ represents Sally's weight. A second class of sentences must have a "within-phrase-adjustment" (WPA) made prior to applying phrase-by-phrase substitutions; "the number of
diskettes is two less than the number of notebooks" can be adjusted to "the number of
diskettes is equal to the number of notebooks subtract two" before employing a direct left-
to-right translation. A third class of sentences requires a "whole-sentence transformation" (WST) to become phrase-order-matched; "the calf weighs four times as much as the pony" needs to be transformed to "the calf's weight is four times the pony's weight" before it can be directly translated. One can accomplish these sentence changes by syntactic means without accessing the quantitative relationship between the variables that underlies the conceptual structure of the sentence.

On the other hand, the students and professor problem is an exemplar of a fourth type of sentences that cannot be "phase-order-matched" (NPOM) with just syntactic transformations. "There are 6 times as many students as professors" can be rewritten as "there are 6 times the number of students as the number of professors," but there is no syntactic transformation available to relocate the verb between the noun phrases to enable phrase-by-phrase translation. However, the sentence does have a POM counterpart. "The number of students is equal to 6 times the number of professors." But reference to the conceptual structure is needed for translation.

Kirshner and others studied 20 professors, 5 instructors, and 17 graduate students in the mathematics department at Louisiana State University. They gave the subjects a random arrangement of 2 POM, 2 WPA, 5 WST, and 5 NPOM sentences (plus some fillers and warm-ups) which were presented individually on a computer terminal. Preceding each sentence were the definitions of the two variables to be used and instructions to translate the upcoming sentence as quickly and as accurately as possible.

The authors reasoned that if their syntactic model were correct, the amount of time required to translate the sentences of various types ought to vary with the adjustment and transformation required. Indeed, the mean latencies (in seconds) by problem type for the 16 errorless subjects were POM -- 9.2, WPA -- 9.4, WST -- 12.2, and NPOM -- 16.8.
Thus the pattern of response latencies for the different sentence types matched the predictions of syntactic theory. Of the 1050 answers, there were 61 errors: 46 on NPOM, 2 on WST, and 13 on filler items. Of the NPOM errors, all but 2 were reversal errors. The high frequency of reversal errors by mathematicians and mathematics graduate students makes it almost inconceivable that translation is a purely conceptual task. Nothing is conceptually difficult about the students and professors problem.

Despite numerous attempts to remediate the reversal error by attending to the conceptual difficulties identified by cognitivist research, the reversal problem persists. Kirshner, Awtry, McDonald, and Gray propose "syntactic parsings and matchings practiced in routine word problems are the foundations of translational skill in algebra" (p. 6).

**Cognitive Models Underlying Translation**

MacGregor and Stacey (1993) studied the literature to determine the theoretical explanations for the reversal error made by students when they are attempting to translate a word problem from English to algebra. They found five major explanations:

1. Syntactic translation (Clement, 1982; Cocking and Chipman, 1988; Herscovics, 1989; Spanos, Rhodes, Dale, and Crandall, 1988) in which students sequentially translate from left-to-right from the English with little regard for meaning;
2. Static comparison (Clement, Lochhead, and Monk, 1981) in which students write an equation to represent an association of related groups rather than equal numbers;
3. Labels frame versus the numerical-variables equation frame (Booth, 1984; Davis, 1984; Kuchemann, 1981; Malle, 1985; Mestre, 1988) in which students have difficulty selecting between the two frames for their understanding of variables;
4. Misunderstanding of the structure and meaning of equations and of the equal sign (Herscovics and Kieran, 1980);
The purpose of MacGregor and Stacey's research was to test the sufficiency of the existing theories of translation reversal errors. Their study showed that the reversal error was present even when all of the possible theoretical causes for it had been eliminated. Furthermore, an examination of the students' errors lead the authors to suggest that the students constructed cognitive models of compared unequal quantities and formulated equations by trying to represent the cognitive model directly or by taking information from their cognitive model. The cognitive models simulate the semantic features of the natural language, not of the mathematical form. MacGregor and Stacey's theory of cognitive models is supported by research in linguistics and psychology.

Relation to the Author's Research

The existing research on translation of word problems from English to algebra has focused on models of translation and on the errors students make during translation. Rather than focusing on models of translation or on students' errors, the author's research studied the effects of teaching students to use as many variables as unknowns in a word problem. Students cannot use any model of translation if they do not even know how to begin the translation process by declaring variables for the unknowns.

It is important for mathematics teachers to concentrate on translation skills when teaching algebra. By introducing two or more variables to solve word problems as soon as possible in an algebra course, and by concentrating on the meaning of variables, perhaps it will be possible to help students translate reliably between algebra and other symbol systems such as English. Considering the research that has been done on translation of word problems into algebraic notation, the researcher proposed that the teachers teaching the experimental sections emphasize that the variables represent the values of the unknowns (and not their units) and that in many cases, but not all, the grammar of the equation can follow almost verbatim that of the language of the problem.
CHAPTER III
EXPERIMENT DESIGN AND METHODOLOGY

This chapter describes the population and sample of the study, including the teachers. This is followed by a discussion of the treatments given to the experimental and control groups, the development of the word-problem units and three tests, the development of the Thurstone-type attitude scale, the experiment design, and the procedure and time line.

Population and Sample

The target population of the study was students taking Algebra I. The accessible population was nine classes of Algebra I taught by four teachers, three of whom taught two classes each and one of whom taught three classes. The fourth teacher, who taught three classes, had two small classes whose total number of students approximately equaled her one large class. No attempt was made to control for interclass variability between the two small classes; however, when observing the two small classes, the researcher noticed no difference in the treatment given to them by the teacher. These two small classes will be discussed as though they were one class. The researcher called algebra teachers in nine school districts in Southwest Ohio. She described the experiment design and methodology. She then explained the units and tests she was developing so that the teachers could use materials that had been written expressly for this experiment (except for the teaching of solving simultaneous equations which the teachers could teach from the textbooks they were using). She told the teachers the approximate time line and procedures they would be expected to follow. Four algebra teachers were selected to
participate on the basis of their willingness to be involved in this research and their teaching at least two classes of algebra in the 1993-1994 school year.

Ninth graders composed the entire population of four of the eight classes and almost all of the population of two more classes (with a few tenth graders making up the rest), while tenth, eleventh, and twelfth graders composed the population of the other two classes (Bill Kramer's classes). Two teachers teach in junior high schools; Karen Johnson teaches at Kettering Junior High School in Kettering, Ohio, and Mary Ellen Place teaches at Wilson Junior High School in Hamilton, Ohio. The other two teach in high schools; Vicky Beebe teaches at Fenwick High School in Middletown, Ohio, and Bill Kramer teaches at Fairmont High School in Kettering, Ohio. Kettering Junior High, Fairmont High School, and Fenwick High School are suburban schools, while Wilson Junior High School is an urban school. A total of 192 students participated in the experiment, with 96 students in the experimental groups and 96 students in the control groups. In Ohio, the most advanced students in mathematics take Algebra I as eighth graders, the average college-preparatory students take Algebra I in the ninth grade, and the noncollege-preparatory students take Algebra I as tenth, eleventh, or twelfth graders. The researcher wanted to look at the effect of the experiment on students of average ability, those in the ninth grade, and on students who are less well prepared for algebra, those in the tenth, eleventh, or twelfth grades. (The latter students are forced to take algebra. General mathematics is no longer offered at their high schools, yet they still must take mathematics to graduate.) One class taught by each teacher received the experimental treatment, while the teacher's other class received the control treatment. The teachers chose these randomly. The resulting group sizes are shown in Table 1.
Table 1. Size of the Control and Experimental Classes of Algebra I Students

<table>
<thead>
<tr>
<th></th>
<th>Vicky</th>
<th>Karen</th>
<th>Bill</th>
<th>Mary Ellen</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>15 in first period&lt;br&gt;16 in third period</td>
<td>15</td>
<td>22</td>
<td>28</td>
<td>96</td>
</tr>
<tr>
<td>Experimental</td>
<td>28</td>
<td>23</td>
<td>22</td>
<td>23</td>
<td>96</td>
</tr>
<tr>
<td>Totals</td>
<td>59 (almost all 9th graders)</td>
<td>38 (9th graders)</td>
<td>44 (10th, 11th, &amp; 12th graders)</td>
<td>51 (9th graders)</td>
<td>192</td>
</tr>
</tbody>
</table>

Treatments

Both the control and experimental classes received the same instruction, but with a different order of presentation. They both were taught how to solve word problems with one unknown and how to solve word problems with more than one unknown. This section explains the treatments and the differences in their order of presentation.

The control treatment was to teach word problems using the typical presentation sequence, but with modified timing to fit the constraints of the experiment. First, problems with one unknown were solved using one equation and one variable. Then, problems with two unknowns were solved, still using one equation in one variable. Finally, algebraic techniques for solving two (linear) equations in two variables were introduced and practiced; then word problems were solved using two variables and two equations. The typical Algebra I approach would be to cover the final stage (hurriedly) at the end of the year (if at all). The three stages of the control treatment were performed in uninterrupted sequence, without the usual delay. Also, both treatments differed somewhat from typical teaching practice by placing particular emphasis upon following a methodical, five-step approach to solving word problems. The steps are discussed in the next section.
The experimental treatment differed from the control treatment only in reversing the order of the second and third stages. In the experimental group, after doing problems calling for one equation in one variable, techniques for solving two equations in two variables were introduced and practiced, then used to solve word problems. Finally, these students were taught to solve word problems with two unknowns using only one equation in one variable. Thus, at the end, both treatments had covered all the same material, and the classes could finish the course in synchronization.

Instrument Development

All of the materials used during the experiment, with the exception of the lessons used to teach solving simultaneous equations, were developed by the researcher. These materials consisted of three word-problem units, three tests, and two Thurstone-type attitude questionnaires. The experimental and control treatments used the same material for learning to solve word problems, but with different orders of presentation. All participating students received booklets containing the three units of word problems with worked examples to use throughout the experiment. In addition to the booklets, the teachers received the tests and questionnaires to be administered at the appropriate times.

The Development of the Word Problem Units

To facilitate uniform implementation of the treatments, the researcher developed three units of algebraic word problems. Unit 1 contained word problems with only one unknown in each problem. (Algebra I textbooks do not have many complete sets of word problems with only one unknown. They often introduce word problems with two unknowns in beginning sets of word problems. Unit 1 alleviated this problem.) Units 2 and 3 contained word problems with two unknowns. The problems in Unit 2 could all be solved with one variable and one equation, but most of the problems in Unit 3 were most readily solvable using two variables and, hence, two equations. The researcher wrote tests to be given at the end of the units. Test 1 contained word problems with one unknown.
Test 2 contained word problems in two unknowns that could be solved using either one or two variables and thus, one or two equations. Most of the problems in Test 3 were most appropriately solved with two variables and two equations, while some could be solved using either technique.

In this study, the word problems used were taken from current, standard algebra textbooks: Addison-Wesley, 1992; Holt, Rinehart, and Winston, 1986; Macmillan, 1992; Prentice Hall, 1993; and Scott Foresman, 1993. This choice was made because these texts and word problems are typical of the Algebra I texts used in today's classrooms.

The researcher prepared a booklet, "Applications of Algebra", containing the three units she had written. This was reproduced and handed out to the teachers and students participating in the study. The full content of the booklet is presented in Appendix A. Each unit contained worked examples and problems for the students to solve, in class or as homework. Figures 2 through 4 contain one of the worked examples from Units 1 through 3, respectively. The solutions to the worked examples were handwritten, to better serve as examples of what the students' work should look like. These materials were used throughout the experiment, except for the teaching of solving simultaneous equations. The algebra teachers taught both the addition and subtraction method and the substitution method of solving simultaneous equations using their textbooks.

Each unit had three or four sections, each of which had worked examples followed by nine to seventeen problems for the students to work. In the solved examples throughout all three units, the researcher modeled breaking the solution down into the small steps of stating the knowns and unknowns, declaring the variable(s), translating English sentences to algebraic equations, solving the equations algebraically, and stating the answer to the problem. Unit 1 contained only word problems with one unknown so that all of them could be solved using only one variable and no expressions dependent on that variable for other unknowns. The sections of Unit 1 included the following types of
Example 3

A certain car can travel 180 miles on 12 gallons of gasoline. How far can this car travel on 20 gallons of gasoline?

**Knowns:** A car can travel 180 miles on 12 gallons of gasoline. This car has 20 gallons of gasoline.

**Unknowns:** How far can the car travel on 20 gal. of gasoline?

**Variable:** \( n \) = number of miles the car can travel on 20 gal. of gas.

**Translation and Solution:**

\[
\begin{align*}
180 &= 12n \\
20 &= 12n \\
12n &= 3600 \\
n &= 300
\end{align*}
\]

**Answer:** The car can travel 300 miles on 20 gal. of gasoline.

---

**Figure 2.** A Worked Example Problem in One Unknown
Example 3

Lola’s age is 14 more than 6 times Juan’s age. The sum of their ages is 35. How old is each?

Knowns: Lola’s age is 14 more than 6 times Juan’s age.

The sum of Lola’s age and Juan’s age is 35.

Unknowns: Lola’s age and Juan’s age.

Variable: \( x = \) Lola’s age

\( 35 - x = \) Juan’s age

Translation:

\[
7x = 224 \\
x = 32 \quad \quad 35 - x = 3
\]

Answer: Lola is 32 years old and Juan is 3 years old.

Figure 3. A Problem in Two Unknowns, Solved with One Variable
Example 3

A resort hotel offers two weekend specials.

Plan (1): 3 nights with 6 meals $132
Plan (2): 3 nights with 2 meals $109

At these rates what is the cost of one night's lodging and what is the average cost per meal? (Assume there is no discount for 6 meals.)

Knowns: The two weekend specials at a resort hotel are
Plan 1: 3 nights with 6 meals for $132
Plan 2: 3 nights with 2 meals for $109

Unknowns: The cost of one night's lodging and the average cost per meal.

Variables: $a$ = cost of one night's lodging
$b$ = average cost of a meal

Translation and solution:

\[3a + 6b = 132\]
\[3a + 2b = 109\]
\[4b = 23\]
\[b = 5.75\]
\[3a + 6(5.75) = 132\]
\[3a + 34.50 = 132\]
\[3a = 97.5\]
\[a = 32.50\]

Answer: A night's lodging costs $32.50 and the average cost of a meal is $5.75.

Figure 4. A Problem in Two Unknowns, Solved with Two Variables
word problems: percent, ratio, simple interest, age, and mixture. Word problems that had two unknowns and that could be solved by using one variable for one unknown and expressing the other unknown in terms of the first variable comprised Unit 2. It included perimeter, age, two number with sums and differences, coin, mixture, simple interest, and time-rate-distance problems. Most of the word problems in Unit 3 were most readily solved using two variables. The types included were more complicated mixture, age, perimeter, coin, time-rate-distance, and simple interest problems. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) stress the de-emphasis of contrived algebraic word problems. However, the Algebra I textbooks that have recently been published, that schools are presently using, and that teachers are currently teaching out of are still full of "contrived" word problems. Therefore, the author believes that if those are the kinds of word problems that algebra students are being required to learn to solve, they should learn to solve those as well as possible. Furthermore, the skills of translating English sentences to mathematical equations will serve students in good stead for problem solving throughout their academic careers. Since the learning of these three units would constitute the majority of the coverage of problem solving that the algebra students involved in the experiment would get, the units needed to contain as many types of word problems as possible that Algebra I books include. The organization of the problems within the units is indicated in Table 2.

**Development of the Tests**

The testing of problem solving serves three functions: informing instructional decision making, letting students know how they are doing, and letting students know that problem solving is important. Current multiple-choice achievement tests are "not nearly so useful as they might be in providing information to guide problem-solving instruction" (Silver and Kilpatrick, 1989, p. 179). Furthermore, current tests tend to place more
emphasis on the aspects of the mathematics curriculum that are easy to assess than on the aspects that are highly valued by professional mathematics educators. Current tests primarily assess the two lowest levels of Bloom's Taxonomy - knowledge and comprehension - more than the higher levels of application, analysis, and synthesis. Therefore, the author developed achievement tests, in which the Algebra I students solved word problems, showing all of their work, to measure their problem-solving skills.

Using the same five algebra textbooks as used for the development of the units, the researcher wrote twelve word problems for Test 1, with one unknown in each word problem. From these twelve problems, eight to ten problems were used for the test.

Table 2. Contents of "Applications of Algebra" Booklet

<table>
<thead>
<tr>
<th>Unit 1 -- Word Problems With 1 Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a -- Percentage and Money Problems</td>
</tr>
<tr>
<td>1b -- Proportion Problems</td>
</tr>
<tr>
<td>1c -- Simple Interest and Mixture Problems</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Unit 2 -- Word Problems With Two Unknowns Solved With One Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a -- Perimeter, Age, And Two-Number Problems</td>
</tr>
<tr>
<td>2b -- Coin and Mixture Problems</td>
</tr>
<tr>
<td>2c -- Mixture and Simple Interest Problems</td>
</tr>
<tr>
<td>2d -- Distance Problems</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Unit 3 -- Word Problems With Two Unknowns Solved With Two Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>3a -- Perimeter, Age, Two-Number, and Mixture Problems</td>
</tr>
<tr>
<td>3b -- Coin and Mixture Problems</td>
</tr>
<tr>
<td>3c -- Distance and Two-Number Problems</td>
</tr>
<tr>
<td>3d -- Simple Interest and Mixture Problems</td>
</tr>
</tbody>
</table>
depending on the teacher's choice. In writing Test 2, the researcher took great care to choose problems that had two unknowns, that were similar to the problems that both the experimental and control groups had solved in the units they had just completed, and that could be solved using either one or two variables. Test 2 had fifteen problems from which eight to ten could be chosen. For Test 3, operating under the same constraints as in Test 2 except that most word problems required two variables to be solved, the researcher again wrote fifteen problems from which eight to ten could be chosen. The teachers were given an opportunity to decide the number and choice of problems to include to make the experiment as natural as possible for them. Some teachers felt that their students could only solve eight problems during one testing period while others felt that their students could and should solve ten problems during a period. Furthermore, some teachers normally gave their students the chance to decide which problems to solve on a test, so the opportunity to do so was included in the tests' design. Because of this, each teacher gave different Tests 1, 2, and 3. Vicky Beebe and Mary Ellen Place gave their students a choice of ten problems from the twelve written for Test 1, a choice of ten problems from the thirteen written for Test 2, and a choice of ten problems from the fifteen written for Test 3. Karen Johnson gave her students nine specific problems to solve for Tests 1 and 2, but she gave her students a choice of nine problems from the fifteen written for Test 3. Bill Kramer gave his students a choice of eight problems from the total written for each test. However, each teacher gave the same directions for both the control and experimental classes when administering each test so that both their control and experimental classes were given the same choices. This was to prevent different tests from confounding the results when the tests for the pooled control and from the pooled experimental groups were compared. Furthermore, the general linear regression analysis of the data used individual student's data in the regression to control for possible
confounding of the results. Appendix B contains complete copies of the item pools for Tests 1, 2, and 3.

The problem-solving skills considered to be of interest were those needed to solve the algebra word problems just described. These skills were organized into the following five-step approach to problem-solving, which were illustrated in the worked examples from the units.

1. Identify the knowns and unknowns.
2. Declare all the variables (e.g., "x is length of the side in feet").
3. Translate the relationships described in the word problem into equivalent algebraic equations in these variables.
4. Obtain an algebraic solution for the variables (e.g., "x = 12").
5. Interpret the solution correctly as the answer to the word problem (e.g., "the side is 12 feet long").

Although getting the correct answer is very important, the many processes that go toward getting the correct answer are important as well. When teaching word problem solving, the four algebra teachers modeled using each of the five steps. They expected students to show each of the steps both in their homework and on their tests. In this study, problem-solving skills were measured by achievement tests consisting of several (eight to ten) word problems, similar to those taught in the units, that the students were to solve, showing all their work. Each problem solution was measured on a ten-point scale with weights for each skill as shown on the tally sheet used to record grades (Figure 5). Ability to coordinate all of the skills and abilities for problem solving develops gradually over time. Therefore, the ability to get correct answers will develop more slowly than the individual component skills and abilities (Charles, Lester, and O'Daffer, 1987). The tally sheet's matrix method of measuring individual skills allowed the teachers to look for progress within and across each goal.
The Development of the Thurstone-type Attitude Scale

As in any assessment, the first step in designing the instrument to measure attitude toward problem solving in school mathematics was to carefully define attitude and to determine which of its aspects would be measured. The aspects of attitude that were measured are students' confidence, flexibility, willingness to persevere, interest, beliefs, and behavioral tendencies in problem solving in school mathematics.

A Thurstone-type scale measured the students' cognitive and behavioral components of attitude as well as their evaluative or emotional aspects of attitude. To develop a valid and reliable Thurstone-type scale, the researcher wrote 100 items that varied in attitude direction and intensity, including neutral items. They were obtained from students who participated in the pilot study, the literature, a sample of mathematicians, and careful thought and analysis. These items were used for writing parallel forms of the Thurstone-type scale, one of which was the pretreatment questionnaire and one of which was the posttreatment questionnaire in the experiment.

<table>
<thead>
<tr>
<th>Identification of Knowns and Unknowns (1 pt)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Declaration of Variables (2 pts)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Translation into Equations (3 pts)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correct Algebraic Solution (2 pts)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correct Interpretation of Solution (2 pts)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 5. Skill Measures Tally Sheet
Tests for validity included testing for content validity, construct validity, and face validity. A panel of experts was asked to check the Thurstone-type attitude scale for content validity. Since attitude is a construct, validation for content and construct validity were performed simultaneously by the same experts. All of the panel members are experts in attitude toward problem solving in school mathematics. Two members are experts in measurement as well, and three were especially knowledgeable about high school mathematics students. Each member of the panel received copies of the 100 items to be used to form the Thurstone-type Scale and asked if the items represent the desired content. Specifically, they were asked, "Do these items measure students' attitude toward problem solving in school mathematics? Do they measure anything else?" The researcher chose all of the items that were used on the two forms of the Thurstone-type attitude scale from the items that were both content and construct valid.

The researcher obtained scale values for all of the items by securing a panel of 20 judges to assign weights of one to eleven to each item (1 = very negative to 6 = neutral to 11 = very positive). These weights were averaged and their standard deviation found. Two items were chosen for each of the eleven levels of attitude toward problem solving in school mathematics, based on their means and standard deviations, choosing the smallest standard deviations. One item for each level went into one of the two parallel forms of the attitude questionnaire. (The lowest level used the same item in both of the two forms, since only one item was validated at that level.) Appendix C presents the attitude instruments that were developed, and also includes a list of the validation panel members.

The directions for the pretreatment and posttreatment attitude questionnaires told the students to circle the items that they most strongly agreed with at the time they took the questionnaire. To determine the score for each student, the researcher found the mean of the weights of the items that the student agreed with on that questionnaire.
After the panel of experts and the researcher were satisfied that the instrument had content and construct validity, both forms were combined and administered to the students who participated in the pilot study. The items were found to contain face validity. Two formats, both including the two forms, were distributed with one format to each class. They differed only in the order of which alternative form of the scale was written first. Because a Komolgorov-Smirnoff Test performed on the results from the two classes showed no significant difference between the scores on the two formats, the researcher concluded that the order of the items did not affect the reliability of Thurstone-type attitude scale.

**Experiment Design**

A factorial block-randomized design was chosen. Each teacher taught one class using the control treatment and one class using the experimental treatment. Each teacher assigned the treatments to the two classes randomly. This design assured that differences among teachers or schools would affect both the control and experimental groups about equally.

The experiment began when the teachers arrived at the place in their curricula in both classes in which they had completed the initial sections on solving equations in one variable and in which they would normally begin teaching problem solving. As Table 3 shows, in the first stage, they taught Unit 1. After completing Unit 1, all students took Test 1. Through this point, both groups were treated the same. Test 1 was intended to provide data on possible pretreatment differences between the block-randomly assigned control and experimental groups. Test 1 also provided an indication of the relative aptitudes of the individual students.

The control and experimental groups diverged after Test 1. The control group covered Unit 2 (using one variable), while the experimental group studied the solution of simultaneous equations and Unit 3. Then both groups took Test 2 (the same test in each
group), so that their relative performance could be measured. Test 2 was intended to indicate with which method the students better learned to solve word problems with two unknowns.

**Table 3.** Experiment Design

<table>
<thead>
<tr>
<th></th>
<th>Traditional Class</th>
<th>Experimental Class</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pre-Attitude Questionnaire</td>
<td>Unit 1: Word Problems in 1 Unknown</td>
</tr>
<tr>
<td>Phase 1</td>
<td>Unit 1: Word Problems in 1 Unknown</td>
<td>Unit 1: Word Problems in 1 Unknown</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Test 1</td>
</tr>
<tr>
<td>Phase 2</td>
<td>Unit 2: Word Problems in 2 Unknown Solved with 1 Variable</td>
<td>Solve Simultaneous Equations</td>
</tr>
<tr>
<td></td>
<td>Unit 3: Word Problems in 2 Unknown Solved with 2 Variables</td>
<td>Unit 3: Word Problems in 2 Unknown Solved with 2 Variables</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Test 2</td>
</tr>
<tr>
<td>Phase 3</td>
<td>Solve Simultaneous Equations</td>
<td>Unit 2: Word Problems in 2 Unknown Solved with 1 Variable</td>
</tr>
<tr>
<td></td>
<td>Unit 3: Word Problems in 2 Unknown Solved with 2 Variables</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Test 3</td>
</tr>
<tr>
<td></td>
<td>Post-Attitude Questionnaire</td>
<td></td>
</tr>
</tbody>
</table>
After Test 2, the roles were swapped. The control group studied the solution of simultaneous equations and Unit 3 (using two variables), while the experimental group covered Unit 2. Both groups then took Test 3, so that their relative performance could again be measured. This plan is summarized in Table 3.

Although the second phase was the heart of the experiment, the researcher included the third phase, in which the methods for solving word problems were reversed in the two groups, for three reasons. (1) If learning to solve word problems with two unknowns using two variables worked well for the experimental group, then the control group should also be given the opportunity to learn the same method. (2) The researcher conjectured that the experimental group might better understand using one variable to solve word problems with two unknowns after they had been taught to use two variables and could see the relationship between using two variables and using one variable. (3) By having the teachers teach all of their algebra students both methods by the end of the experiment, they would only be teaching their two classes differently during the experiment, when all of the materials had been written for them. They could then return to teaching their two algebra classes in synchronization and, therefore, not have extra lesson planning for the remainder of the school year. This approach reduced their extra workload, making it possible to find teachers who were willing to participate in the study.

Procedure and Time Line

By August of 1993, the researcher completed her search for algebra teachers to participate in the study. Four teachers agreed to follow the experiment design, using the units and tests developed by the researcher and explained in telephone conversations during the summer. They all completed the algebra experiment. Meanwhile, she obtained copies of several current Algebra I textbooks to use in writing three units of word problems and three tests.
In early fall of 1993, after the researcher finished writing the units and tests, she met with each teacher individually to discuss with her or him what the experiment would involve and what would be expected. She gave each teacher a complete copy of the units, the tests, the tally sheets for scoring the word problem tests together with an explanation of how to use them, the letters of information and consent forms for students' parents, and the pretreatment and posttreatment attitude surveys. In September, after they had had time to read the material, the researcher met with all of the teachers to explain the experimental design and to answer any questions they might have. She explained to them that the only way she was attempting to control for the effects of the individual teachers as a possible confounding variable was to make sure that each teacher treated both the experimental and the control groups the same when teaching them the same units. For example, if a teacher were to spend four days on solving simultaneous equations with the experimental group, then he or she should spend four days with the control group; or if a teacher were to give one group the choice of solving eight questions on one test, then he or she should give the other group that same choice for that test. This was to allow the teachers to teach in their own ways, to make the experiment as uncontrived as possible, while controlling for any differences in the way the experimental groups and the control groups were treated. Having them teach as they naturally would except for when they taught the solving of systems of two equations and using that to solve word problems was a goal of this experiment. The researcher wanted to make the experiment as natural as possible, while controlling for possible confounding variables in order to determine the effects of bringing forward the solving of systems of equations and using that skill to solve word problems with more than one unknown on regular students in regular algebra classes.

The researcher wrote a letter of information and a consent form that all of the teachers sent home with their students for the students' parents to sign before the student...
could be considered part of the study. Data taking was limited to only those students whose forms were signed and returned. Furthermore, each student was assigned a number for the study so that his or her work and answers to the attitude surveys remained anonymous. Since the researcher was both a student at the Ohio State University and a faculty member of Wright State University during this research, she complied with the human subjects review processes at both universities. The letter of information and consent form are presented in Appendix D.

When the teachers finished their introductory work with their students on solving equations with one variable, they gave the pretreatment attitude survey to their students and then began teaching Unit 1. Mary Ellen began teaching Unit 1 the third week in October, 1993, and finished all three units and tests by the Christmas break, thus using approximately nine weeks to cover all three units and tests. Bill started the second week of November and completed all three units and tests in three weeks, thus also finishing before the Christmas holidays. Karen began word problems with her classes in December and used twenty seven days of school for teaching and testing word problem solving with both of her classes. Her classes, therefore, completed the units in January, 1994. Vicky began teaching word problem solving the third week of November, but because of reviewing with her classes for their semester examinations, giving examinations, and school missed because of winter storms, she did not complete the units with her classes until February, 1994. Upon completion of the word problem units, the teachers then gave their students the posttreatment attitude survey.

The researcher observed the algebra teachers and their classes during the experiment. She observed the classes of Vicky Beebe and of Mary Ellen Place twice and the classes of Bill Kramer and of Karen Johnson once. She also communicated with them by telephone during the experiment. While observing, she noted that no class behaved markedly different from typical algebra classes and that the teachers were teaching the
units written by the researcher. The units invoked a good deal of consistency in the material taught to the students in both groups. She discussed the experiment plan with the teachers to make sure they were following the plan, and she answered any questions that they had.
CHAPTER IV
RESULTS

Statistical analyses of the students' test scores and attitude questionnaires are presented in this section. Data for these analyses consist of the students' scores on Tests 1, 2, and 3, expressed as percentage points to facilitate comparisons, the students' first-quarter Algebra I grades and their scores on the pretreatment and posttreatment attitude questionnaires. For each student on each test, the teachers filled out a Skill Measures Tally Sheet as described in Chapter III, with a total possible score for each problem of 10 points. The tally sheet provided the opportunity to grade a student's ability to correctly identify the knowns and unknowns, declare any variables to be used, translate the word problems into the algebraic equation(s), solve the equation(s), and translate the numerical result of the solved equation(s) into words. Because points for each test problem measured problem-solving skills and success at solving the given word problems, these test results are appropriate data for this study.

A formal study of validity of the questions in the units and tests was considered unnecessary because of the sources from which they were taken. To test inter-grader reliability of test grades, the researcher graded almost all of the last tests (Test 3) from three of the teachers and 8 of the 45 tests from the fourth teacher. She graded these independently of the teachers' grades, using a Skill Measure Tally Sheet for each student's test. A Spearman's rho test, a distribution-free test of correlation based on rank, was performed on the teachers' grades and the researcher's grades of Test 3. The Spearman's rho correlation was .901.
The pretreatment and posttreatment attitude questionnaires measured the students' attitudes at the beginning of the study and at the end of the study. Therefore, data from the questionnaires can be used to measure the effect of the experiment on the students' attitudes.

First the overall test data from the experimental and control groups are laid out and descriptively explored. The raw data are presented in Appendix E. This is followed by hypotheses testing of the test data from the two groups. In order to quantify the difference that was due to the experimental treatment and to see if teacher was a significant variable, a stepwise regression was done. The results of this are included. Although only one teacher was found to be a significant predictor in the regression model, a teacher-by-teacher analysis of the test data was done from each teacher's experimental and control classes, because different teachers gave different forms of Tests 1, 2, and 3, as described in detail in Development of the Tests in Chapter III. The results of these statistical tests are presented in Appendix F. After the test data are analyzed, the data from the pretreatment and posttreatment attitude questionnaires are explored and analyzed with hypothesis testing.

Problem-Solving Skills

Exploratory Statistics

Test scores were separated into the scores for the experimental group and the scores for the control group. Descriptive statistics of the students' scores on the tests are displayed by group in Table 4.

The tests were intended to be sufficiently challenging that the scores would be spread throughout the possible range. It was desired that the strongest students be able to work all the problems successfully, while even the weakest students could earn some points. This would optimize the sensitivity of the experiment to differences among the groups. An examination of Table 4 shows that these goals were achieved.
Table 4. Descriptive Statistics of the Scores for the Control and Experimental Groups on Tests 1-3

<table>
<thead>
<tr>
<th></th>
<th>Test 1 Control</th>
<th>Test 1 Experimental</th>
<th>Test 2 Control</th>
<th>Test 2 Experimental</th>
<th>Test 3 Control</th>
<th>Test 3 Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>87</td>
<td>82</td>
<td>77</td>
<td>84</td>
<td>83</td>
<td>74</td>
</tr>
<tr>
<td>Highest Score</td>
<td>100.00</td>
<td>100.00</td>
<td>98.00</td>
<td>100.00</td>
<td>98.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Upper Quartile</td>
<td>88.00</td>
<td>87.25</td>
<td>67.50</td>
<td>83.00</td>
<td>71.75</td>
<td>75.00</td>
</tr>
<tr>
<td>Median Score</td>
<td>76.00</td>
<td>74.00</td>
<td>52.00</td>
<td>64.00</td>
<td>53.50</td>
<td>53.50</td>
</tr>
<tr>
<td>Lower Quartile</td>
<td>53.00</td>
<td>57.25</td>
<td>29.00</td>
<td>42.00</td>
<td>36.00</td>
<td>34.75</td>
</tr>
<tr>
<td>Lowest Score</td>
<td>17.00</td>
<td>12.00</td>
<td>6.00</td>
<td>16.00</td>
<td>8.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Mean</td>
<td>71.13</td>
<td>71.16</td>
<td>50.92</td>
<td>62.07</td>
<td>54.20</td>
<td>54.16</td>
</tr>
<tr>
<td>SEMean</td>
<td>2.22</td>
<td>2.23</td>
<td>2.63</td>
<td>2.48</td>
<td>2.27</td>
<td>2.52</td>
</tr>
<tr>
<td>StDev</td>
<td>21.50</td>
<td>21.62</td>
<td>24.79</td>
<td>24.15</td>
<td>21.82</td>
<td>23.92</td>
</tr>
</tbody>
</table>

On Test 1, there appears to be no statistically significant difference in the scores of the control and experimental groups throughout the range of scores, except perhaps for the minimum. The upper quartile, median, and lower quartile scores appear to be statistically the same. Similarly, the scores for Test 3, including the minimum, appear to have no difference between the two groups. However, on Test 2, the experimental group scored consistently higher than the control group: 15.5 points higher at the upper
quartiles, 12 points higher at the medians, 13 points higher at the lower quartiles, and 10 points higher at the minimums.

**Hypothesis Testing**

The researcher wanted to test the following hypotheses.

1. The null hypothesis that the scores from the control and experimental groups are equivalent for Test 1 and for Test 3 versus the alternative hypothesis that the scores from the two groups are not equivalent;

2. The null hypothesis that there is no statistically significant difference in medians from the two groups on Test 2, after the experimental treatment has been given to the experimental group while the control group was taught in the traditional order, versus the alternative hypothesis that there is a statistically significant difference.

A two-sample t-test quantitatively tests the significance of the difference between the means of the control and experimental groups. It requires the assumption of normality for the distributions of the scores for each group on each test and the assumption of equal variances in the two samples. If these assumptions are met, the two-sample t-test is more powerful than the nonparametric Mann-Whitney U-test. However, if these assumptions are not met, the two-sample t-test can be invalid, while the relaxed assumptions of the Mann-Whitney U-test may be met so that its results are valid. Simply put, the t-test is more powerful but the U-test is more robust. The two-sample t-test results are generally consistent with the U-test results which are presented here. Unless there is a reason to assume normality of the distributions, a nonparametric test is more appropriate. Because the U-test results are conclusive, the t-test results are relegated to Appendix F.

To quantify the likelihood that there is no difference in location of scores between the two groups for Test 1 and for Test 3 and that there is a statistically significant difference between the control and experimental groups on Test 2 by formal hypothesis testing without the assumption of normality, the researcher performed Mann-Whitney U-
tests (Wilcoxon Rank Sum tests) comparing the scores of the two groups on each test. The Mann-Whitney U-test is more robust than a two-sample t-test, because it assumes only that the data are mutually independent and that the data from each group come from the same continuous population. The data on the test scores from the control and experimental groups come from the same classes, and the data are independent because students' test scores are independent from each other; therefore, these assumptions are appropriate. The results of the Mann-Whitney U-tests are displayed in Table 5.

### Table 5. Mann-Whitney U-test Results

<table>
<thead>
<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistic (U)</td>
<td>8886</td>
<td>9856</td>
<td>8225.5</td>
</tr>
<tr>
<td>Point Estimate of Median Difference: Experimental - Control</td>
<td>0</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>Critical Level</td>
<td>.9947</td>
<td>.0031</td>
<td>.9798</td>
</tr>
</tbody>
</table>

The results show that one cannot reject the hypothesis that the median scores from the two groups are the same for Test 1 and for Test 3. In fact, for Test 1, the critical level (also known as alpha or p-value) of .9947 says that, under the null hypothesis that the median of the control group equals the median of the experimental group, the chance of getting a U value of at least 8886 due to randomness alone is .9947. Therefore, the null hypothesis cannot be rejected. Similarly for Test 3, the null hypothesis cannot be rejected.

However, the median scores from Test 2 are statistically significantly different for the two groups, with a significance level of p = .0031. Under the null hypothesis that the median of the control group equals the median of the experimental group on Test 2, the
chance of getting a U value of 9856 or greater due to randomness alone is only .0031 or .31%. Thus, the null hypothesis can be rejected.

Because different teachers gave different forms of Tests 1, 2, and 3, the researcher performed Mann-Whitney U-tests on the scores of the experimental class and the control class for each teacher for each test. These were performed to determine if there is a statistically significant difference between the median scores. She also performed two-sample t-tests for each teacher for each test. While generally consistent with the corresponding pooled results, these results are less conclusive due to smaller sample sizes of each class than in the entire experimental or control group. The results of the Mann-Whitney U-tests and of the t-tests are presented in Appendix F.

**Stepwise Regression**

A much stronger way of taking into account each individual teacher and the tests each student took is to perform a stepwise regression. A stepwise regression of Test 2 on the independent variables below was performed to determine if they would be significant as predictors in a general linear regression:

1. Treatment received (control or experimental),
2. Individual performance on Test 1,
3. Individual grade for the first nine weeks in Algebra I,
4. Beebe (an indicator variable: 1 for Beebe's students, 0 for the other teachers' students),
5. Johnson,
6. Kramer,
7. Place.

Stepwise regression is designed to identify the significant predictors among a set of possible predictors. In the first step, the variable which reduces the error sum of squares the most is added to the model. In each subsequent step, having added a new variable to the model, all the variables in the model are tested for significance. If any are not
significant, the single least-significant variable is dropped from the model. Then, of the variables not in the model, the one which reduces the error sum of squares the most is added to the model, if it is significant. When the variables remaining would not be significant, and the variables included are significant predictors, the search stops. The results of the stepwise regression, presented in Table 6, show that the only significant predictors of the independent variables put into the stepwise regression for a general linear regression model are (in the order in which added) Test 1, first-quarter grade, treatment received (control or experimental), and whether or not the student is in Kramer's classes. Therefore, these are the coefficients that will be used in the regression analysis. The other teachers were not statistically significant as predictors. The data set contained 210 cases, of which 35 contained missing data, leaving 175 cases which were fitted in the stepwise regression. The stepwise regression did not include a constant term, because of colinearity with the variables for the four teachers.

**Table 6. Stepwise Regression of Test 2 Scores**

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1 coefficient</td>
<td>0.796</td>
<td>0.594</td>
<td>0.562</td>
<td>0.548</td>
</tr>
<tr>
<td>Test 1 t-ratio</td>
<td>40.34</td>
<td>11.34</td>
<td>10.85</td>
<td>10.58</td>
</tr>
<tr>
<td>Grade coefficient</td>
<td>5.5</td>
<td>4.8</td>
<td>5.5</td>
<td></td>
</tr>
<tr>
<td>Grade t-ratio</td>
<td>4.13</td>
<td>3.65</td>
<td>4.09</td>
<td></td>
</tr>
<tr>
<td>Treatment coefficient</td>
<td>9.1</td>
<td>9.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatment t-ratio</td>
<td>3.43</td>
<td>3.62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kramer coefficient</td>
<td></td>
<td></td>
<td>-6.7</td>
<td></td>
</tr>
<tr>
<td>Kramer t-ratio</td>
<td></td>
<td></td>
<td>-2.04</td>
<td></td>
</tr>
</tbody>
</table>
Bill Kramer taught Algebra I to students in the tenth, eleventh, or twelfth grades who were generally not as well prepared for algebra as those students who take Algebra I in the ninth grade. The researcher believes that this difference in significance, from the other algebra teachers, of Kramer as a predictor in the regression model is due to the fact that his students are from a different population than are those students in the classes of the other three teachers.

**Regression Analysis**

Knowing that there was a difference between the control and experimental groups on Test 2, the researcher wanted to quantify the difference that is due to the experimental treatment, accounting for differences in individual student's abilities and whether or not the student was in Kramer's classes. To do this, the appropriate statistical method is a multivariate linear regression, or general linear model. The response variable, $y$, is the Test 2 score. The predictor variables are: an indicator variable, $x_1$, the treatment received by the student; the student's Test 1 score, $x_2$; the student's first-quarter algebra grade, $x_3$; and whether or not the student was in Kramer's class, $x_4$. The student's first-quarter algebra grade and the grade on Test 1 should account for each student's abilities. The levels of the treatment were $x_1 = 0$ for the students in the control group and $x_1 = 1$ for the students in the experimental group. First-quarter grades were assigned values of $x_3 = 4$ for A and A-, $x_3 = 3$ for B+, B, and B- and so forth. The levels of being in Kramer's classes or not were $x_4 = 0$ for not being in Kramer's classes and $x_4 = 1$ for being in Kramer's classes.

The linear regression model is

$$ y_i = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \varepsilon_i $$

(25)

where

$\beta_1$ = the coefficient of being in the control or experimental group
\[ \beta_2 = \text{the coefficient of the student's score on Test 1} \]
\[ \beta_3 = \text{the coefficient of the student's first-quarter algebra grade.} \]
\[ \beta_4 = \text{the coefficient of being in Bill Kramer's class.} \]

In this multivariate regression model, the \( \beta \)'s are unknown parameters, the \( x \)'s and \( y \)'s are known data, and the \( \epsilon \)'s are unknown error terms. Equation 25 is the model. The coefficients are estimated by the values \( b_1, b_2, b_3, \) and \( b_4, \) which are used to obtain the fitted values, \( \hat{y}_i, \) from the data:
\[ \hat{y}_i = b_1 x_{i1} + b_2 x_{i2} + b_3 x_{i3} + b_4 x_{i4}. \]  
(26)

The residuals, \( r_i, \) are the errors in the fitted values:
\[ r_i = y_i - \hat{y}_i. \]  
(27)

The residuals are estimates of the errors, \( r_i = \epsilon_i; \) the regression chooses values for the coefficients (\( b \)'s) which minimize the sum of the squares of the residuals.

The \( \epsilon_i \)'s are assumed to be (1) mutually independent random errors which are (2) normally distributed and have (3) a constant variance. These assumptions require justification. First, independence of the error terms is justified by the fact that the students' test scores and grades are independent of each other. Second, the researcher tested the normality of the errors with a normal probability plot, in which the residuals are plotted against their expected values under the assumption of normality. This plot is shown in Figure 6. The strong linearity of the plot justifies the assumption that the errors are normally distributed. To quantify the linearity, the correlation of the residuals and their expected values under the assumption of normality was computed. Perfect correlation, \( r^2 = 1, \) indicates exact linearity. The correlation here, \( r^2 = .996, \) is convincing evidence of the normality of the regression errors. Third, a scatter plot of the residuals against the expected values of the response variables (\( r_i \) vs. \( \hat{y}_i \)) is shown in Figure 7. The uniform scatter in the residuals (vertically in the plot), independent of the predicted value, is evidence which justifies the assumption of constant variance of the error terms.
Figure 6. Normal Probability Plot of Residuals

Figure 7. Scatter Plot of Residuals
The specific regression equation, obtained as described above, was

\[ \text{Test2} = 9.71 \text{Con/Exp} + 0.548 \text{Test1} + 5.53 \text{Grade} - 7.77 \text{Kramer}. \]  \hspace{1cm} (28)

Of the 210 data points used, 30 contained missing values, leaving 180 cases which were fitted in the regression. The regression is summarized in Table 7, which provides the estimated coefficients, the \( t \) statistics for testing the significance of the coefficients, and the corresponding critical levels (\( p \) values) for the \( t \)-tests.

<table>
<thead>
<tr>
<th>Predictor</th>
<th>Coef</th>
<th>Stdev</th>
<th>( t )-ratio</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Con/Exp</td>
<td>9.705</td>
<td>2.588</td>
<td>3.75</td>
<td>&lt; 0.0005</td>
</tr>
<tr>
<td>Test1</td>
<td>0.54834</td>
<td>0.05126</td>
<td>10.70</td>
<td>&lt; 0.0005</td>
</tr>
<tr>
<td>Grade</td>
<td>5.533</td>
<td>1.338</td>
<td>4.14</td>
<td>&lt; 0.0005</td>
</tr>
<tr>
<td>Kramer</td>
<td>-7.767</td>
<td>3.162</td>
<td>-2.46</td>
<td>0.015</td>
</tr>
</tbody>
</table>

The \( t \)-ratios for each of the coefficients, except Kramer, are large enough that their \( p \)-values are less than .0005. Kramer as a predictor is significant at the .015 critical level. Therefore, as predictors of a student's score on Test 2, the variables (whether the student is in the control or experimental group, his or her score on Test 1, his or her first-quarter grade, and whether or not the student is in Kramer's classes) are all statistically significant. An F-test of the overall regression was performed. The large F value of 511.82 with 4 and 176 degrees of freedom for regression and for error, respectively, shows that the regression is statistically significant at a critical level of less than .0005.

Interpreting the regression in terms of how each predictor variable can predict what a student might get on Test 2 can best be seen with an example. Consider a student who scored at the median level of the control group on Test 1 with a score of 74 out of 100 and a first-quarter grade of C, or 2.0, and was not in Kramer's classes. The student's
score on Test 2, according to the regression model, is expected to be 9.71(0) + 0.548(74) +5.53(2) - 7.77 (0) = 51.6 out of 100. However, if that student were in the experimental group and had scored 74 on the first test and received the same C for the first-nine-weeks grade and was not in Kramer's classes, the expected score is 9.71(1) +0.548(74) +5.53(2) -7.77(0) = 61.3 out of 100. That's an increase in the Test 2 grade of 18.8% attributable to having been in the experimental group.

Similarly, consider a student who scored at the median level of the control group on Test 1 with a score of 74 out of 100 and a first-quarter grade of C, or 2.0, and was in Kramer's classes. The student's score on Test 2, according to the regression model, is expected to be 9.71(0) + 0.548(74) +5.53(2) - 7.77 (1) = 43.8 out of 100. However, if that student were in the experimental group and had scored 74 on the first test and received the same C for the first-nine-weeks grade and was in Kramer's classes, the expected score is 9.71(1) +0.548(74) +5.53(2) -7.77(1) = 53.6 out of 100. That's an increase in the Test 2 grade of 22.4% attributable to having been in the experimental group.

In fact, the expected overall increase due to the experimental group, whether students are ninth graders or are tenth, eleventh, or twelfth graders, is simply $b_3 = 10$ points, which is consistent with the observed increases in median and quartiles seen in Table 4.

### Attitude Results

#### Exploratory Statistics

Attitude questionnaire scores were separated into scores for the experimental group and scores for the control group. Descriptive statistics of the students' scores on the attitude surveys are displayed in Table 8.

The attitude questionnaires were designed to measure the students' attitude toward problem solving in school mathematics, with a highest possible score of 10.10 (positive
attitude toward problem solving in school mathematics) and a lowest possible score of 1.29 (negative attitude toward problem solving in school mathematics).

**Table 8. Attitude Questionnaire Descriptive Statistics**

<table>
<thead>
<tr>
<th></th>
<th>Pretreatment Attitude Control</th>
<th>Pretreatment Attitude Experimental</th>
<th>Posttreatment Attitude Control</th>
<th>Posttreatment Attitude Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>n</strong></td>
<td>71</td>
<td>82</td>
<td>79</td>
<td>86</td>
</tr>
<tr>
<td><strong>Maximum</strong></td>
<td>9.227</td>
<td>9.955</td>
<td>7.9775</td>
<td>7.9775</td>
</tr>
<tr>
<td><strong>Upper Quartile</strong></td>
<td>7.475</td>
<td>7.345</td>
<td>6.4933</td>
<td>6.1394</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>6.631</td>
<td>6.744</td>
<td>5.8169</td>
<td>5.6335</td>
</tr>
<tr>
<td><strong>Lower Quartile</strong></td>
<td>5.682</td>
<td>5.836</td>
<td>5.1978</td>
<td>5.0610</td>
</tr>
<tr>
<td><strong>Minimum</strong></td>
<td>4.028</td>
<td>4.028</td>
<td>3.7300</td>
<td>3.1900</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>6.641</td>
<td>6.700</td>
<td>5.8493</td>
<td>5.6175</td>
</tr>
<tr>
<td><strong>SEMean</strong></td>
<td>0.140</td>
<td>0.134</td>
<td>0.0903</td>
<td>0.0890</td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td>1.302</td>
<td>1.302</td>
<td>0.8567</td>
<td>0.8718</td>
</tr>
</tbody>
</table>

On the pre-attitude questionnaire and on the post-attitude questionnaire, there appears to be no statistically significant difference in the scores for the control and experimental groups throughout the range of the data. However, there does appear to be a statistically significant difference between the pretreatment and posttreatment attitude questionnaires for both groups. In fact, there is a decrease in the medians of the control and experiment groups of 0.8141 and 1.1105, respectively. There is a drop in their means of .7917 and 1.0825, respectively.
Hypothesis Testing

The researcher wanted to test the following hypotheses.

1. The attitudes toward problem solving in school mathematics of the students in the control group are equivalent to those of the experimental group before and after the experiment versus the attitudes of the two groups are not equivalent.

2. There is no statistically significance difference in the medians of the attitudes of the two groups before compared to after the experiment versus there is a statistically significant difference.

As when comparing the test data, a two-sample t-test would quantitatively test the significance of the difference between the means of the control and experimental groups, but unless there is a reason to assume normality for the distributions of the scores for each group on each attitude questionnaire, it is more appropriate to perform a nonparametric test. Therefore, to quantify the likelihood that there is no difference in location between the two groups for the pretreatment questionnaires and between the two groups for the posttreatment questionnaires by formal hypothesis testing without the assumption of normality, the researcher performed Mann-Whitney U-tests comparing the results of the control group with the results of the experimental group. As with the tests scores for the two groups, the assumptions of mutual independence of the data and that the data from within each group come from the same continuous population are appropriate. To quantify the likelihood that there is a statistically significant difference between the pretreatment and posttreatment questionnaires for both groups, she performed Wilcoxon signed-rank tests comparing the pretreatment scores of the control group with its posttreatment scores and similarly for the experimental group. The results of the Mann-Whitney U-tests are displayed in Table 9, and the results of the Wilcoxon signed-rank tests are displayed in Table 10.
Table 9. Attitude Questionnaire U-tests of Control versus Experimental Groups

<table>
<thead>
<tr>
<th></th>
<th>Pretreatment</th>
<th>Posttreatment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Statistic (U)</td>
<td>7690.5</td>
<td>8362.5</td>
</tr>
<tr>
<td>Point Estimate of</td>
<td>0.0520</td>
<td>-0.2216</td>
</tr>
<tr>
<td>Median Difference:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Experimental - Control</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical Level</td>
<td>0.7922</td>
<td>0.0948</td>
</tr>
</tbody>
</table>

Table 10. Attitude Questionnaire Signed-rank Tests of Pretreatment vs Posttreatment

<table>
<thead>
<tr>
<th></th>
<th>Control</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wilcoxon Statistic</td>
<td>704.0</td>
<td>684.0</td>
</tr>
<tr>
<td>Point Estimate of</td>
<td>-0.8017</td>
<td>-1.062</td>
</tr>
<tr>
<td>Median Difference:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Posttreatment - Pretreatment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical Level</td>
<td>&lt; 0.0005</td>
<td>&lt; 0.0005</td>
</tr>
</tbody>
</table>

The results show that one cannot reject the null hypothesis that the median scores from the two groups are the same for the pretreatment attitude questionnaire. The critical level of .7922 indicates that there is a 79.22% chance that the U-statistic will be at least as high as 7690.5 under the null hypothesis that the median of the experimental group on the pretreatment questionnaire equals the median of the control group on the pretreatment questionnaire. However, there is a marginally statistically significant difference in the median scores of the experimental group and the control group on the posttreatment attitude questionnaire at the critical level of p = .0948.
Furthermore, there is an unquestionably statistically-significant difference in the pretreatment attitude scores versus the posttreatment attitude scores for each of the groups. With p-values less than .0005, one can reject the null hypotheses that the medians of the pretreatment attitude scores of each group equal the medians of their posttreatment attitude scores. What is undoubtedly present is a decrease in the attitudes of the students in both groups. A natural question to ask since the point estimate of the decrease in median scores of the experimental group is -1.062 while the point estimate of the decrease in median scores of the control group is -0.8017 is "Does the decrease in scores of the experimental group differ significantly from the decrease in scores of the control group?" A paired-sample Mann-Whitney U-test comparing the decrease in attitude of the control group against the decrease in attitude of the experimental group had a p-value of .3278. Therefore, the result is only significant at the 32.78% level, so the null hypothesis that the median of the change in attitude score of the control group equals the median of the change in attitude score of the experimental group cannot be rejected at a p-value even as large as .10.
CHAPTER V
CONCLUSIONS
Skills

The data support the hypothesis that teaching Algebra I students how to solve word problems using as many variables as are needed to represent the unknowns as soon as they begin solving problems with more than one unknown helps the students learn more successfully how to solve word problems. There was no statistically significant difference between the experimental and control groups on Test 1, before the difference in treatments. However, after the groups were taught different techniques, in phase 2, there was a statistically significant difference (9.71 points with standard deviation of 2.59 points) on their scores on Test 2, as shown by the regression analysis presented here. This overall ten point advantage for the experimental group over the control group is not only statistically significant, it is also an important improvement. The importance can be judged by considering the improvement as a percentage of the control group's scores at the quartiles and median. The lower quartile score for the experimental group was 45% improved over the corresponding score for the control group (42 vs. 29). This shows that the weaker students benefited especially from the experimental treatment. The median and upper quartile scores were 23% higher for the control group (64 vs. 52 and 83 vs. 67.5, respectively). Thus, the average and above average students also benefited substantially from the experimental treatment.

One of the most difficult parts of learning to solve algebraic word problems is the translation from words to mathematics (in this case, algebraic variables and equations).
Thus, if students can learn to translate from the representation of the written language to the symbolic representation of algebra, they can learn to solve word problems better. The only difference in the teaching of the two groups in the second phase of the experiment was that the experimental group was taught to solve systems of equations with two unknowns and to apply that skill to solving word problems with two unknowns, while the traditional, control group was taught to solve word problems with two unknowns using only one variable and putting each unknown in terms of that variable. The researcher concludes, from the evidence presented here, that the experimental treatment was substantially and significantly more effective at providing the students with workable skills with which to translate from the prose representation to the algebraic representation and back again.

Phase three of the experimental design was conducted directly after the students had completed phase two. During phase three, the groups reversed roles. The experimental group learned to solve word problems with more than one unknown using only one variable, while the control group learned to solve word problems with more than one unknown using two variables. At the end of phase 3, Test 3 was given to compare their performance. There was no statistically significant difference in the performance of the two groups, with a common median of 53.50 out of 100 possible points and a mean of 54.20 for the control group and 54.16 for the experimental group. Noting that the control group median had been 52.00 on Test 2, it appears that the control group did not benefit from the coverage with two variables and two unknowns, while the experimental group lost the gains that it had made. It is possible that the students in the control group were so confused by trying to solve word problems with two unknowns using only one variable that they did not expect to understand how to use two variables and, therefore, did not understand. The researcher considers it likely that the experimental group unlearned their good habits of methodical, step-by-step translation when taught (in phase 3) to skip steps
by doing parts of the problem "in their heads" (which is necessary to solve problems with two unknowns using only one variable). Furthermore, at the end of phase three, many students complained to their teachers of being "totally confused."

On the other hand, Test 3 may have been more difficult than Tests 1 and 2. Therefore, although the scores decreased, the students may have been solving more difficult problems. It is possible that rather than the experimental group's losing ground, the control group caught up with the experimental group after they had learned to solve word problems with two unknowns using two variables. The experiment design does not distinguish between these two interpretations.

The duration of the experiment varied among teachers, from three weeks to three months of concentrated word problems. It would have been better to wait for a few months for phase three of the experiment, in order for the students to have more assimilation time, and to solve word problems throughout the year, not just in one very intensive period.

Attitudes

The researcher had hoped that teaching Algebra I students how to solve word problems with as many variables as are needed to represent the unknowns as soon as they began solving problems with more than one unknown would improve the students' attitude toward problem solving in school mathematics. Instead, the pretreatment and posttreatment attitude questionnaires showed that attitude of the students in both the control and experimental groups decreased significantly during the experiment, and the attitude of the students in the experimental group decreased somewhat more than did the attitudes of the students in the control group.

The researcher considers it likely that, had the post-attitude surveys been given after Test 2 when the students in the experimental group had done relatively well on solving word problems in two unknowns, instead of after Test 3, the attitude scores of the
students in the experimental group would have been more positive. Although the researcher was disappointed with these results, upon reflection, she considered it normal that the attitude scores would have decreased for two reasons. (1) As her son explained, when students begin any new project, they are often excited about it (and thus have more positive attitudes toward it); however, after the project has been continuing "forever," the excitement has worn off and their positive attitudes decrease. (2) Solving word problems is hard work, even when students understand how to translate them from prose to algebraic symbols. Thinking so intensely and working so hard for an extended period of time in mathematics class can undermine students' positive attitudes, especially if they felt confused after participating in phase 3. The experimental group had particular cause for frustration in phase 3, as they saw themselves becoming more confused and losing the ground they had gained in phase 2. This may have contributed to their somewhat larger drop in attitude, as compared to the control group. In retrospect, an additional attitude survey at the end of phase 2 might have clarified this.

**Implications for Teaching**

Important implications for teaching follow from: (1) the understanding that the Algebra I students in the experiment learned to solve word problems with two unknowns better when they were taught how to solve simultaneous equations in two variables and to apply that to solving word problems; and (2) observing that concentrated problem solving with no other mathematics taught for months during Algebra I served to confuse the students. Traditional Algebra I textbooks contain word problems with two unknowns many chapters before simultaneous equations are introduced, and they tend to have large clumps of word problems in "application" chapters. Indeed, word problems with two unknowns often are contained in some of the first sections of Algebra I word problems. As shown by this research, teachers should introduce solving systems of equations in two variables to their students as soon as they are approaching these word problems, rather
than waiting until they get to the sections in the textbooks that contain systems of equations. (Ultimately, textbooks should be revised.) They should then model solving word problems with two unknowns using two variables and two equations, methodically doing each step of the five step approach used here with every word problem they solve. They should emphasize the translation steps, writing out explicitly what each variable represents and how the relationships between the variables described in words can be represented symbolically in equations. Teachers should strive not to teach large units of word problems, but should teach word-problem solving throughout the year.

Further Research

The results obtained here point to some directions for further research. Since the most difficult aspect of solving word problems seems to be translating from the written representations to the symbolic algebraic representations, this experiment could be repeated using the technology of computerized symbolic algebraic manipulation. Using computer software to solve the equations once they have been translated from the written representation would allow students to concentrate on the difficult parts: translating from words to mathematics, using as many variables as unknowns, and translating the mathematical solutions back to prose to answer the problem. This research could be done not only with average students who are studying algebra in the ninth grade, but it could also be done with bright students who are studying algebra in the seventh grade and with slower students who are studying algebra in the tenth, eleventh, or twelfth grades. Thus, one could determine if younger students are ready to understand several equations in as many variables; one could determine if students in high school who have had difficulty with the skills of solving equations (and not often even trying to solve word problems) could solve word problems if they could concentrate on the translation and let their computers take care of the equation solving. Further research could also investigate the effect of using the same experiment design as discussed here, but spreading the design out
over many months throughout the year rather than conducting the entire experiment over
an intensive one to two-and-a-half months.
APPENDIX A

HANDOUT BOOKLET: "APPLICATIONS OF ALGEBRA"

This appendix presents the booklet described in Chapter III. The booklet contained exposition, worked examples, and problems which could be assigned by the teachers. The material is formatted exactly as it was handed out, including the handwritten solutions for the worked examples. The only change has been to number the pages in the sequence of this dissertation document.
Applications of Algebraic Equations: Unit 1a
Solving Word Problems With One Unknown

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable. That means to write down exactly what you want the variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

The number of girls in the band is 6 more than twice the number of boys. There are 88 girls in the band. How many boys are in the band?

**Knowns:** 88 girls in the band
   Number of girls = 6 more than twice number of boys

**Unknowns:** Number of boys in the band

**Variable:** \( x = \text{number of boys in the band} \)

**Translation and Solution:**
\[
88 = 6 + 2x \\
82 = 2x \\
41 = x
\]

**Answer:** There are 41 boys in the band.
Example 2

Jim Byers earns 3% commission on his sales of new cars. If he earned $861 on commissions last week, what was the dollar amount of his total sales?

**Knowns:** Jim earns 3% commission.
- Jim earned $861 on commissions last week.

**Unknowns:** Dollar amount of his total sales.

**Variable:** \( s = \) dollar amount of his total sales.

**Translation and Solution:** \( 0.03 \times s = 861 \)
\[ s = \frac{861}{0.03} = 28,700 \]

**Answer:** Jim sold $28,700 of cars.

---

Example 3

The base price of the 1991 Buyer's Car of the Year was $15,925. The base price of the same car in 1994 is $19,705. Find the percent of increase.

**Knowns:** Base price of 1991 car was $15,925.
- Base price of 1994 car is $19,705.

**Unknown:** Amount of increase in the price of the car.

**Variable:** \( n = \) percent of increase

**Translation and Solution:** Amount of increase = 3780
\[ n \times 15,925 = 3780 \]
\[ n = \frac{3780}{15,925} = 0.24 \text{ or } 24\% \]

**Answer:** The percent of increase in the car price from 1991 to 1994 is 24%.
1. The number of boys in the tennis club is 10 more than half the number of girls. There are 30 boys in the tennis club. All together, how many girls are in the club?

2. A salesman rented a car that could get 35 miles per gallon. He paid $19.50 a day for the car plus $0.18 per mile. He rented the car for 1 day and paid $33. How many miles did he travel?

3. Bowling at Sunset Lanes cost Danny and Zorina $9. This included shoe rental of $0.75 a pair. How much did each game cost if Danny bowled 3 games and Zorina bowled 2 games?

4. Popcorn costs $0.75 a box. Carl and Diane each bought 1 box of popcorn at the ball game. Carl bought 3 cans of juice and Diane bought 2 cans of juice during the game. Each one cost the same. They spent a total of $3.25. What did they pay for each can of juice?

5. In a 180-kilogram sample of ore, there was 3.2% metal. How many kilograms of metal were in the sample?

6. Janice scored 85% on the last test. She answered 34 questions correctly. How many questions were on the test?

7. The sales tax on a $20 purchase was $0.90. What was the rate of the sales tax?

8. Suppose 6% of 8000 people polled regarding an election expressed no opinion. How many people had an opinion?

9. A price decreased from $50 to $40. Find the percent of decrease.

10. A price increased from $40 to $50. Find the percent of increase.

11. A price plus 5% tax is equal to $3.15. Find the price without the tax.

12. An item sells for $36 after a 25% discount. Find the original price.

13. A stereo originally sold for $345.00. It is now on sale at a 12% discount. Find the final price of the stereo.

14. Zoe got a discount of $4.50 on a new radio. The discounted price was $24.65. What was the percent of discount to the nearest percent?
When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable. That means to write down exactly what you want the variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of exercises on proportions and word problems that have been solved using all of the steps that were given above. Then work the exercises on proportion, and solve the given word problems.

**Example 1**

Solve each proportion.

A. \( \frac{x}{5} = \frac{3}{7} \)

B. \( \frac{x-6}{3} = \frac{2}{5} \)

A. \( 7x = 15 \)

\( 7x = 15 \)

\( x = \frac{15}{7} \)

B. \( 5(x-6) = 30 \)

\( 5x - 30 = 6 \)

\( 5x = 36 \)

\( x = \frac{36}{5} \)
Example 2

If 1 out of 6 people buy Elly Fant brand peanuts, how many people can be expected to buy this brand of peanuts in a city of 24,000 inhabitants?

\[
\text{Knowns: } 1 \text{ out of 6 people buy Elly Fant brand peanuts.} \\
\text{There are } 24,000 \text{ inhabitants in the city.} \\
\text{Unknown: How many people buy Elly Fant peanuts in the city?}
\]

Variables: \(x\) = number of people expected to buy Elly Fant peanuts.

\[
\text{Translation and Solution: } \frac{1}{6} = \frac{x}{24,000} \\
6x = 24,000 \\
x = 4,000
\]

Answer: 4,000 people can be expected to buy Elly Fant peanuts.

Example 3

A certain car can travel 180 miles on 12 gallons of gasoline. How far can this car travel on 20 gallons of gasoline?

\[
\text{Knowns: A car can travel 180 miles on 12 gallons of gasoline.} \\
\text{This car has 20 gallons of gasoline.}
\]

\[
\text{Unknown: How far can the car travel on 20 gallons of gasoline?}
\]

Variables: \(n\) = number of miles that the car can travel on 20 gallons of gas.

\[
\text{Translation and Solution: } \frac{180}{12} = \frac{n}{20} \\
12n = 3600 \\
n = 300
\]

Answer: The car can travel 300 miles on 20 gallons of gasoline.
Exercises and Word Problems

Solve the proportions.

1. \( \frac{7}{4} = \frac{a}{3} \)  
2. \( \frac{2}{5} = \frac{x}{4} \)  
3. \( \frac{3}{7} = \frac{m}{5} \)  
4. \( \frac{5}{b} = \frac{3}{11} \)

5. \( \frac{x-5}{7} = \frac{3}{5} \)  
6. \( \frac{2}{x} = \frac{3}{x+6} \)  
7. \( \frac{m-2}{4} = \frac{m+3}{5} \)  
8. \( \frac{x-1}{4} = \frac{3+x}{2} \)

9. \( \frac{a+3}{5} = \frac{14}{10} \)  
10. \( \frac{x-1}{5} = \frac{2x+7}{3} \)  
11. \( \frac{5}{2m+5} = \frac{2}{4m-1} \)  
12. \( \frac{x-6}{2} = \frac{x+4}{3} \)

Solve the word problems using all of the steps given above.

1. In Stantonville, 2 out of 5 people belong to a union. How many union members are there if the population is 70,000?

2. If 7 out of 8 people use Attack toothpaste, how many people use Attack in a city with a population of 40,000?

3. Pam's batting average is 0.250 (250:1,000). During baseball season, how many hits should she get in 120 times at bat?

4. Three out of 5 freshman study algebra. How many study algebra in a freshman class of 500?

5. In a game preserve, 239 deer are caught, marked, and then released. Later, out of 198 deer caught, 42 are marked. Estimate the total deer population in the preserve.

6. The scale on the blueprint for a house is 1 inch to 3 feet. If the living room on the blueprint is \( \frac{5}{2} \) inches by 7 inches, what are the dimensions of the actual room?

7. The scale on a map is 1 centimeter to 57 kilometers. Fargo and Bismarck are 4.7 centimeters apart on the map. What is the actual distance between these cities?

8. A baseball pitcher strikes out an average of 3.6 batters per 9 innings. At this rate, how many batters would the pitcher strike out in 315 innings?

9. A school has a policy that 2 adults must accompany every group of 15 students on school trips. How many adults are needed to take 180 students on a trip?

10. The ratio of foreign students to American students at a college is 2 to 35. How many foreign students attend this college if there are 1575 American students?
11. A television station found that 145 out of 350 people surveyed watched a program on education on Monday night. If this survey is representative of the total viewing area (12,250 people), about how many people watched the television special?
When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable. That means to write down exactly what you want the variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

Kiko Murimoto opened a Life Bank savings account that earns 7% annual interest. After 6 months, she receives $52.50 in interest. How much money had Kiko deposited when she opened the account?

Knows: Kiko's savings account earns 7% annual interest.
In 6 months (\(\frac{1}{2}\) yr.), her account earns $52.50.

Unknowns: How much more, did Kiko deposit?

Variables: \(I = \text{interest} = \$52.50\)
\(p = \text{principal deposited}\)
\(r = \text{rate of annual interest} = 7\%\)
\(t = \text{time in years} = \frac{1}{2} \text{ yr.} = 0.5 \text{ year}\)

Translation: \(I = \text{interest} = \text{principal} \times \text{rate} \times \text{time}\)
\(52.50 = p \times 0.07 \times 0.5\)
\(52.50 = 0.035p\)
\(\text{Answer: Kiko deposited $1500 in her savings account.}\)
Example 2

Kendra is doing a chemistry experiment that calls for a 30% solution of copper sulfate. She has 40 mL of 25% solution. How many milliliters of 60% solution should Kendra add to obtain the required 30% solution?

Knowns: Kendra has 40 mL of 25% solution.
She needs to add some 60% solution to obtain a 30% solution.
Unknowns: How much 60% solution Kendra needs to add.

<table>
<thead>
<tr>
<th>Percent of Copper Sulfate</th>
<th>Amount of Solution</th>
<th>Amount of Copper Sulfate</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>40 mL</td>
<td>0.25(40)</td>
</tr>
<tr>
<td>60%</td>
<td>x</td>
<td>0.60x</td>
</tr>
<tr>
<td>30%</td>
<td>40+x</td>
<td>0.30(40+x)</td>
</tr>
</tbody>
</table>

Translation and Solution:

\[0.25(40) + 0.60x = 0.30(40+x)\]
\[10 + 0.60x = 12 + 0.30x\]
\[0.30x = 2\]
\[x = 6.67\]

Answer: Kendra needs to add about 6.67 mL of the 60% solution.

Example 3

Twenty more than 4 times Jack's age is the same as 6 times his age. How old is he?

Knowns: 20 more than 4 times Jack's age is the same as 6 times his age.

Unknowns: What is Jack's age?

Variable: \(x = \) Jack's age

Translation and Solution:

\[4x + 20 = 6x\]
\[20 = 2x\]
\[10 = x\]

Answer: Jack is 10 years old.
Word Problems

1. In Example 1, suppose Kiko adds the $52.50 in interest to the principal. What will the interest be next year?

2. Suppose an investor invested $5000 at 8% and $2000 at 12%.
   a. What interest would he earn on the $5000 investment after 1 year?
   b. What interest would he earn on the $2000 investment after 1 year?
   c. Why do you think he invested the greater amount at the lower rate?

For problems 3 - 5, use $I = prt$ to find the missing quantity.

3. Find $I$ if $p = $8000, $r = 6\%$, and $t = 1$ year.

4. Find $r$ if $I = $1890, $p = $6000, and $r = 9\%$.

5. Find $r$ if $I = $2430, $p = $9000, and $t = 2$ years, 6 months.

6. Three less than 5 times Wanda's age is the same as 3 times her age increased by 37. How old is she?

7. Peanuts sell for $3.00 per pound. Cashews sell for $6.00 per pound. How many pounds of cashews should be mixed with 12 pounds of peanuts to obtain a mixture that sells for $4.20 per pound?

8. An advertisement for an orange drink claims that the drink contains 10% orange juice. How much pure orange juice would have to be added to 5 quarts of the drink to obtain a mixture containing 40% orange juice?

9. Ann Leibowitz owns "The Coffee Pot," a specialty coffee store. She wants to create a special mix using two coffees, one priced at $6.40 per pound and the other priced at $7.28 per pound. How many pounds of the $7.28 coffee should she mix with 9 pounds of the $6.40 coffee to sell the mixture for $6.95 per pound?

10. A pharmacist has 150 dL of a 25% solution of peroxide in water. How many deciliters of pure peroxide should be added to obtain a 40% solution?

11. Ground chuck sells for $1.75 per pound. How many pounds of ground round selling for $2.45 per pound should be mixed with 20 pounds of ground chuck to obtain a mixture that sells for $2.05 per pound?

12. If Ron's age is increased by 5 years more than twice his age, the result is 35. Find his age.
Applications of Algebraic Equations: Unit 2a
Solving Word Problems With 1 or More Unknowns

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable(s). That means to write down exactly what you want each variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

The length of a rectangle is 4 m less than twice the width. The perimeter is 28 m. Find the length and width.

Knowns: The length of a rectangle is 4 m less than twice its width.
The perimeter is 28 m.

Unknowns: The length and width of the rectangle.

Variable: \( w \) = width of rectangle
\( 2w - 4 \) = length of rectangle

Translation and Solution:

\[ 2w + 2(2w - 4) = 28 \]
\[ 2w + 4w - 8 = 28 \]
\[ 6w = 36 \]
\[ w = 6 \]

Answer: The width is 6 meters and the length is \( 2(6) - 4 = 8 \) m.
Example 2

The sum of two numbers is 14. Twice the larger number, increased by three times the smaller number is 34. Find the two numbers.

Knobs: The sum of 2 numbers is 14. Twice the larger increases by 3 times the smaller is 34.

Unknowns: The 2 numbers.

Variable: \( x = \) larger number
\[ 14 - x = \] smaller number

Translation into Solution: \[ 2x + 3(14 - x) = 34 \]
\[ 2x + 42 - 3x = 34 \]
\[ 42 - x = 3 - \]
\[ -x = -2 \]
\[ x = 2 \]
\[ x = 6 \]

Answer: The 2 numbers are 2 and 6.

Example 3

Lola’s age is 14 more than 6 times Juan’s age. The sum of their ages is 35. How old is each?

Knowns: Lola’s age is 14 more than 6 times Juan’s age.
The sum of Lola’s age and Juan’s age is 35.

Unknowns: Lola’s age and Juan’s age.

Variable: \( x = \) Lola’s age
\[ 35 - x = \] Juan’s age

Translation into Solution: \[ x = 14 + 6(35 - x) \]
\[ 7x = 224 \]
\[ x = 32 \]
\[ 35 - x = 3 \]

Answer: Lola is 32 years old and Juan is 3 years old.
Word Problems

1. The length of a rectangle is 2 m less than 3 times the width. The perimeter is 68 m. Find the length and width of the rectangle.

2. The sum of two numbers is 35. Twice the first number is equal to five times the second number. Find the two numbers.

3. The length of a rectangle is 5 m more than twice the width. The perimeter is 52 meters. Find the length and width of the rectangle.

4. The difference between two numbers is 4. Three times the first number, increased by the second number, is 20. Find the two numbers.

5. The sum of two numbers is 27. One number is 3 more than the other. Find the numbers.

6. Find two numbers whose sum is -1 and whose difference is 5.

7. The sum of the length and width of a rectangle is 19 in. The length is one less than twice the width. Find the length and width of the rectangle.

8. The sum of two numbers is 36. One number is 2 more than the other. Find the numbers.

9. The perimeter of a rectangle is 48 m. The width of the rectangle is 2 more than half the length. Find the length and width.

10. Noah's age is 3 years more than twice Tina's age. The sum of their ages is 24 years. Find each of their ages.

11. Find two numbers whose sum is 58 and whose difference is 16.

12. The difference between two numbers is 16. Three times the larger number is seven times the smaller. What are the numbers?

13. The difference between two numbers is 18. Twice the smaller number plus three times the larger is 74. What are the numbers?

14. Mary is 5 times Jose's age. The sum of their ages is 18. How old is each?
When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable(s). That means to write down exactly what you want each variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

**Example 1**

Paul has 30 coins in dimes and quarters. Their total value is $4.50. How many coins of each type does he have?

**Knowns:** Paul has 30 coins in dimes and quarters with a total value of $4.50 or 450.

**Unknowns:** How many dimes and quarters Paul has.

**Variable:** \( d = \) number of dimes

\[ 30 - d = \text{number of quarters} \]

**Translation and Solution:**

\[ 10d + 25(30 - d) = 450 \]
\[ 10d + 750 - 25d = 450 \]
\[ -15d + 750 = 450 \]
\[ -15d = -300 \]
\[ d = 20 \]

**Answer:** There are 20 dimes and 10 quarters.
Example 2

The cost of an adult ticket for a school play was $2.00. The cost of a student ticket was $1.50. The total income from the sale of the tickets was $550. The number of $2.00 tickets sold was 100 less than 3 times the number of $1.50 tickets. How many tickets of each type were sold?

**Knowns:***
- Cost of adult ticket is $2.00
- Cost of student ticket is $1.50
- Total sales is $550

The number of $2.00 tickets sold was 100 less than 3 times the number of $1.50 tickets.

**Unknowns:** How many $2.00 tickets and how many $1.50 tickets were sold.

**Variable:** \( t = \) number of $1.50 tickets sold.

**Translation and Solution:**
\[
1.50t + 2.00(3t - 100) = 550
\]
\[
1.50t + 6.00t - 200 = 550
\]
\[
7.50t - 200 = 550
\]
\[
7.50t = 750
\]
\[
t = 100
\]

**Answer:** 100 $1.50 tickets were sold and 200 tickets for $2.00 were sold.
Word Problems

1. A jar of quarters and nickels contains $1.25. There are 13 coins in all. How many of each are there?

2. A vending machine takes only nickels and dimes. At the end of the day there were three times as many nickels as dimes and a total of $25. How many of each coin were in the machine?

3. The attendance at a school football game was 350. Tickets for adults cost $2.25, compared to $1.00 for children. If the total receipts were $600, how many children and how many adults attended?

4. George saves nickels and dimes for tolls. If he has 28 coins worth $2.60, how many are nickels and how many are dimes?

5. Rosa has $3.10 in nickels and dimes. She has 10 fewer nickels than she has dimes. How many dimes and nickels does she have?

6. The Taylor family reunion had a record turnout of 38 people last year. For a change of pace, they decided to go ice skating instead of having a picnic. Admission for the group (including the skates) cost $153.50. Regular ticket prices were $2.75 for adults and $1.50 for children. Ticket prices including skate rentals were $4.25 for adults and $3.75 for children. How many adults and how many children were at the reunion?

7. A parking meter contains $6.25 in dimes and quarters. If the number of dimes is 2 more than 3 times the number of quarters, how many of each coin are in the parking meter?

8. Bob had 6 more dimes than quarters. He has $1.65 in all. How many coins of each type does he have?

9. Marie has 24 coins in half-dollars and dimes. Their total value is $3.60. How many coins of each type are there?
When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable(s). That means to write down exactly what you want the variable to represent. Only after you have declared each variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

**Example 1**

A chemist has two alcohol-in-water solutions: a 20% alcohol solution and a 50% alcohol solution. He needs 12 L of a solution that is 45% alcohol. How many liters of the two starting solutions should he mix?

**Solution**

He needs 12 L of a 45% alcohol solution.

He needs 12 L of a 45% alcohol solution.

He needs \( 12 \times \) liters of the 20% solution and \( 12 \times \) liters of the 50% solution.

Variables: \( x = \) amount of 20% solution he needs.

<table>
<thead>
<tr>
<th>Percent of alcohol</th>
<th>Amount of Solution</th>
<th>Amount of alcohol</th>
</tr>
</thead>
<tbody>
<tr>
<td>20%</td>
<td>( x )</td>
<td>.20x</td>
</tr>
<tr>
<td>50%</td>
<td>( 12 - x )</td>
<td>.50(12 - x)</td>
</tr>
<tr>
<td>45%</td>
<td>12 L</td>
<td>.45(12)</td>
</tr>
</tbody>
</table>

Translation and Solution:

\[ .20x + .50(12 - x) = .45(12) \]

\[ .20x + 6 - .50x = 5.4 \]

\[ -.30x = -.6 \]

\[ x = 2 \]

**Answer:** He needs 2 L of 20% alcohol solution and 10 L of 50% alcohol solution.
Example 2

Paul Yu is investing $6000 in two accounts, part at 4.5% and the remainder at 6%. If the total annual interest earned from the two accounts is $279, how much did Paul deposit at each rate?

**Knowns:** Paul invests part of $5,000 at 4.5% and he invests the remainder at $5,000 at 6%.  
The total annual interest earned is $279

**Unknowns:** How much did Paul invest at each rate?

**Variable:** \( x \) = amount Paul invested at 4.5%.

**Translation and Solution:** 
\[
0.045x + 0.06(5000 - x) = 279 \\
0.045x + 300 - 0.06x = 279 \\
-0.015x + 300 = 279 \\
-0.015x = -21 \\
x = 1400
\]

**Answer:** Paul invested $1400 at 4.5% and $3600 at 6%.
Word Problems

1. Fred Furguson invested $5000 for one year, part at 9% annual interest and the rest at 12% annual interest. The interest from the investment at 9% was $198 more than the interest from the investment at 12%. How much money did he invest at 9%?

2. The health food store wishes to blend peanuts that cost $1.20 per pound with raisins that cost $2.10 per pound to make 50 pounds of a mixture that cost $1.47 per pound. How many pounds of peanuts and raisins are needed?

3. Part of an investment of $32,000 earns 7.5% annual interest; the rest earns 9%. If the annual interest from both is $2670, how much is invested at the higher rate?

4. A 100 kg mixture of $0.69 per kilogram pinto beans and $0.89 per kilogram kidney beans is valued at $81. How many kilograms of each does it contain?

5. In a laboratory, a pharmacist has 30% and 80% alcohol solutions. She needs 100 mL of a 50% alcohol solution. How many milliliters of the 30% and 80% solutions should she mix?

6. The total annual interest from two bank accounts is $481. One account earns 5.9% annual interest and the other earns 6.75%. If the two accounts contain a total of $7600, how much in the account at the lower interest?

7. A dairy has milk that is 4% butterfat and cream that is 40% butterfat. To make 36 gallons of a mixture that is 20% butterfat, how many gallons of milk and cream must be used?

8. In order to get a thicker sauce, a restaurant owner combines a sauce that is 70% tomato paste with the original sauce, which is 40% tomato paste. How much of each should be used to make 5 L of the new 60% tomato paste sauce?

9. Mrs. Chavis has twice as much money invested at 7.5% as she has at 6.0%. The yearly income from both investments is $840. How much has she invested at each rate?

10. Walnuts cost $9.95 per pound while peanuts cost $6.50 per pound. If there are three less pounds of peanuts than there are walnuts, how many pounds of each are there in a box which costs $62.75?
Applications of Algebraic Equations: Unit 2d
Solving Word Problems With 1 or More Unknowns

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variable(s). That means to write down exactly what you want each variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into an algebraic equation. Solve the equation and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

A family drives to the beach at an average speed of 45 mi/h and returns home on the same road at an average speed of 54 mi/h. If the trip home takes 20 minutes less than the trip to the beach, how far from the beach is the family's home?

Known: A family drives to the beach at 45 mi/h and returns home.

Traveling at 45 mi/h.

The trip home takes 20 min (3/4 hr) less than the trip to the beach.

Unknowns: How far is the beach from their home?

Variable: \( t \) = time it takes the family to drive to the beach.

<table>
<thead>
<tr>
<th>Distance</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Travel to beach</td>
<td>45t</td>
<td>45 mi/h</td>
</tr>
<tr>
<td>Return</td>
<td>( 54(t - \frac{3}{4}) )</td>
<td>54 mi/h</td>
</tr>
</tbody>
</table>

Translation and Solution:

\[ 45t = 54(t - \frac{3}{4}) \]

\[ 45t = 54t - 18 \]

\[ -9t = -18 \]

\[ t = 2 \]

Answer: Since it takes 2 hours to drive to the beach, \( 45 \times 2 = 90 \) miles from their home.
Example 2

A car leaves Hartford traveling north at 56 km/h. Another car leaves Hartford one hour later traveling north on the same road at 84 km/h. How far from Hartford will the second car overtake the first? (Hint: The cars travel the same distance.)

*Known:* One car travels north from Hartford at 56 km/h.

A second car leaves Hartford one hour later and travels north at 84 km/h.

*Unknown:* How far from Hartford the second car will overtake the first.

Variable: \( t = \) travel time of the first car.

<table>
<thead>
<tr>
<th></th>
<th>Distance</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>First car</td>
<td>56(t)</td>
<td>56 km/h</td>
<td>(t) hours</td>
</tr>
<tr>
<td>Second car</td>
<td>84((t-1))</td>
<td>84 km/h</td>
<td>(t-1) hours</td>
</tr>
</tbody>
</table>

*Translation and Solution:* \(56t = 84(t-1)\)
\[56t = 84t - 84\]
\[-28t = -84\]
\[t = \frac{-84}{-28} = 3\]

*Answer:* After the first car travels for 3 hours, the second car overtakes the first. 56\(\times3 = 168\) km from Hartford.
Word Problems

1. Two cars leave town at the same time going in opposite directions. One travels 44 mi/h and the other travels 55 mi/h. In how many hours will they be 297 miles apart?

2. Two cars leave town at the same time going in the same direction on the same road. One travels 32 mi/h and the other travels 47 mi/h. In how many hours will they be 69 miles apart?

3. It takes a small jet plane 4 hours less time than it takes a propeller-driven plane to travel from Glen Rock to Oakville. The jet plane averages 637 km/h while the propeller plane averages 273 km/h. How far is it from Glen Rock to Oakville?

4. A motorcycle breaks down and the rider has to walk the rest of the way to work. The motorcycle was traveling at 45 mi/h, and the rider walks at a speed of 6 mi/h. The distance from home to work is 25 miles, and the total time for the trip was 2 hours. How far did the motorcycle go before it broke down?

5. The concession stand sells hot-dogs and sodas during Beck High School football games. John bought 6 hot-dogs and 4 sodas and paid $6.70. Jessica bought 4 hot dogs and 3 sodas and paid $4.65. What is the price of a hot-dog? What is the price of a soda?

6. A ship leaves a dock moving at 24 mi/h. Three hours later, a second ship sets out from the same dock at 32 mi/h. How long will it take for the second ship to overtake the first?

7. Robinsport and Titusville are 324 mi apart on a railroad line. Trains leave each of these depots at the same time headed for the other depot. One travels at 43 mi/h, the other at 38 mi/h. How long will it take for the two trains to pass each other?

8. Two cyclists simultaneously start in the opposite directions down a straight road, one at 22 km/h and the other at 28 km/h. How long will they have ridden by the time they are 175 km apart?

9. Two snails are 432 cm apart. If they travel toward each other at rates that differ by 2 cm/min and it takes them 27 min to meet, how fast is each snail going?
Applications of Algebraic Equations: Unit 3a  
Solving Word Problems Using More Than 1 Variable

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variables. That means to write down exactly what you want each variable to represent. Only after you have declared the variable should you now translate the English sentence(s) in the word problem into algebraic equations. Solve the equations and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

Manuel scored 35 points fewer in his second bowling game than in his first game. His total score for the two games was 395. How many points did he score in each game?

\begin{align*}
\text{Knowns:} & \quad \text{Manuel scored 35 points fewer in his 2nd bowling game than in his 1st game.} \\
& \text{His total score for both games was 395.} \\
\text{Unknowns:} & \quad \text{How many points did he score in each game?} \\
\text{Variables:} & \quad x = \text{number of points scored in the 1st game} \\
& \quad y = \text{number of points scored in the 2nd game} \\
\text{Translation and Solution:} & \quad y = x - 35 \quad \Rightarrow \quad x + (x - 35) = 395 \\
& \quad x + y = 395 \quad \Rightarrow \quad 2x - 35 = 395 \\
& \quad 2x = 430 \\
& \quad x = 215 \\
& \quad y = 215 - 35 = 180 \\
\text{Answer:} & \quad \text{He scored 215 points in his 1st game and 215 points in his 2nd game.}
\end{align*}
Example 2

The perimeter of a rectangle is 48 m. The width of the rectangle is 2 more than one-half the length. Find the length and width.

**Knowns:** The perimeter of a rectangle is 48 m.
The width is 2 more than \( \frac{1}{2} \) the length.

**Unknowns:** The length and the width.

**Variables:** \( l = \) length and \( w = \) width

**Translation and solution:**

\[
2l + 2w = 48 \\
\Rightarrow 2l + 2(2 + \frac{1}{2} l) = 48 \\
\Rightarrow 2l + 4 + l = 48 \\
\Rightarrow 3l = 44 \\
\Rightarrow l = 14 \frac{2}{3} m
\]

\[
w = 2 + \frac{1}{2} l = 2 + \frac{1}{2} (14 \frac{2}{3}) = 9 \frac{1}{3} m
\]

**Answer:** The length of the rectangle is 14 \( \frac{2}{3} \) m and the width is 9 \( \frac{1}{3} \) m.
Example 3

A resort hotel offers two weekend specials.

Plan (1): 3 nights with 6 meals $132
Plan (2): 3 nights with 2 meals $109

At these rates what is the cost of one night's lodging and what is the average cost per meal? (Assume there is no discount for 6 meals.)

Knowns: The 2 weekend specials at a resort hotel are
Plan 1: 3 nights with 6 meals for $132,
Plan 2: 3 nights with 2 meals for $109.

Unknowns: The cost of 1 night’s lodging and the average cost per meal.

Variables: a = cost of a night’s lodging
b = average cost of a meal.

Translation and Solution:

\[3a + 6b = 132 \quad \Rightarrow \quad 3a + 6b = 132\]
\[5a + 2b = 109 \quad \Rightarrow \quad -(3a + 2b = 109)\]
\[4b = 23\]
\[b = 5.75\]

\[3a + 6(5.75) = 132\]
\[3a + 34.50 = 132\]
\[3a = 97.5\]
\[a = 32.50\]

Answer: A night's lodging costs $32.50 and the average cost of a meal is $5.75.
Example 4

Kim's father is now \(2\frac{1}{2}\) times as old as Kim. Ten years ago he was 4 times as old as Kim.

How old are Kim and his father now?

**Knowns:** Kim's father is \(2\frac{1}{2}\) or \(\frac{5}{2}\) times as old as Kim. Ten years ago, he was 4 times as old as Kim.

**Unknowns:** The ages of Kim and his father now.

**Variables:**
- \(x\) = Kim's father's age now
- \(y\) = Kim's age now

**Translation and Solution:** Kim's father's age 10 yrs. ago is \(x-10\).

Kim's age 10 yrs. ago is \(y-10\).

\[
\begin{align*}
x &= \frac{5}{2}y \\
x - 10 &= 4(y - 10)
\end{align*}
\]

\[
\begin{align*}
\frac{5}{2}y - 10 &= 4y - 40 \\
\frac{5}{2}y &= 4y - 30 \\
\frac{5}{2}y - 4y &= -30 \\
-\frac{3}{2}y &= -30 \\
y &= 20 \\
x &= \frac{5}{2}(20) = 50
\end{align*}
\]

**Answer:** Kim's father is 50 years old, and Kim is 20 years old now.
Word Problems

1. The perimeter of a rectangle is 310 m. The length is 25 m greater than the width. What are the length and the width of this rectangle?

2. One angle of a triangle is 4 times as large as another. The third angle is equal to the sum of the other two angles. What is the measure of the smallest angle? (Hint: The sum of the measures of the angles of a triangle equals 180 degrees.)

3. The combined lengths of the Nile and Amazon rivers is 13,108 km. If the Amazon were 234 km longer, it would be as long as the Nile. What is the length of each river?

4. In 1984 tennis players John McEnroe and Martina Navratilova earned a total of $3,462,665. If McEnroe had earned $884,447 more, he would have earned the same as Navratilova. How much did each earn?

5. Four oranges and five apples cost $2.00. Three oranges and four apples cost $1.56. Find the cost of an orange and the cost of an apple.

6. A 48-ft wire is cut into three pieces. The second piece is three times as long as the first piece. The third piece is four times as long as the second piece. How long is each piece?

7. One number is 25% of another. The larger number is 12 more than the smaller. Both numbers are positive. What are the numbers?

8. The width of a rectangle is 3/4 the length. The perimeter of the rectangle becomes 50 cm when the length and width are each increased by 2 cm. Find the length and width.

9. A hotel offers the following specials: Plan (1) is two nights and one meal for $106. Plan (2) is 2 nights and 4 meals for $130. What price is the hotel charging per night and per meal?

10. Irving is 8 years younger than Rhoda. Two years ago, she was 3 times as old as he was then. How old is each of them now?
11. Ages of Two Students

<table>
<thead>
<tr>
<th>Age 8 yrs ago</th>
<th>Age 5 yrs ago</th>
<th>Age 2 yrs ago</th>
<th>Present Age</th>
<th>Age in 1 year</th>
<th>Age in 4 yrs</th>
<th>Age in 7 yrs</th>
</tr>
</thead>
</table>

Anne

Bill

Use entries from the table above to write an equation for the statements in 12-16.

12. Anne is 3 years younger than Bill.
13. Eight years ago Bill was twice as old as Anne.
14. In 4 years Anne will be 5/6 as old as Bill.
15. Seven years from now Bill’s age will be 7/6 of Anne’s age.
16. The sum of their present ages is 25.
17. Cordell is twice as old as Beth. Eight years ago he was three times as old. How old are they now?
Applications of Algebraic Equations: Unit 3b
Solving Word Problems Using More Than One Variable

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variables. That means to write down exactly what you want each variable to represent. Only after you have declared the variables should you now translate the English sentence(s) in the word problem into algebraic equations. Solve the equations and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

Kami has some nickels and some dimes. The value of the coins is $1.65. There are 12 more nickels than dimes. How many of each kind of coin does Kami have?

**Knowns:** Kami has some nickels and dimes with a total value of $1.65 or $1.65.
She has 12 more nickels than dimes.

**Unknowns:** How many nickels and how many dimes Kami has.

**Variables:**
- \( n \) = number of nickels Kami has.
- \( 5n \) = value of the nickels.
- \( d \) = number of dimes Kami has.
- \( 10d \) = value of the dimes.

**Translation and Solution:**

\[
5n + 10d = 165 \quad \Rightarrow \quad 5(d + 12) + 10d = 165
\]

\[
n = d + 12
\]

\[
5d + 60 + 10d = 165
\]

\[
15d = 105
\]

\[
d = 7
\]

\[
n = 7 + 12 = 19
\]

**Answer:** Kami has 19 nickels and 7 dimes.
Example 2

There are 411 people at a play. Admission was $1 for adults and $0.75 for children. The receipts were $395.75. How many adults and children attended?

There are 411 people at a play.
The tickets are $1 for adults and $0.75 for children.
The total amount received was $395.75. How many adults and how many children attended?

Unknowns: How many adults and how many children attended?

Variables:

- \( a \) = number of adults
- \( c \) = number of children
- \$1 = cost of all adult tickets
- \$0.75 = cost of all children tickets.

Translation and Solution:

\[
\begin{align*}
a + c &= 411 \\
1.00a + 0.75c &= 395.75\end{align*}
\]

\[
\begin{align*}
1.00a + 1.00c &= 411 \\
-(1.00a + 0.75c &= 395.75)
\end{align*}
\]

\[
0.25c = 15.25
\]

\[
c = 61
\]

\[
a + 61 = 411; a = 350
\]

Answer: 350 adults and 61 children attended the play.
Word Problems

1. A jar of dimes and quarters contains $15.25. There are 103 coins in all. How many of each are there?

2. A vending machine takes only nickels and dimes. There are 5 times as many dimes as nickels in the machine. The face value of the coins is $4.40. How many of each coin are in the machine?

3. There were 429 people at a play. Admission was $1 for adults and $0.75 for children. The receipts were $372.50. How many adults and how many children attended the play?

4. There were 200 tickets sold for a college basketball game. Tickets were $1.50 for students and $3.00 for adults. The total amount collected was $495. How many of each type of ticket were sold?

5. A jar contains 5-gram bolts and 10-gram bolts. The contents of the jar weigh 2.35 kg (1000 g = 1 kg). If there are 300 bolts altogether, how many are there of each kind?

6. Rodolfo has $2.55 in dimes and quarters. He has eight more dimes than quarters. How many quarters does he have?

7. The Martins are going to Funtastic Land, an amusement park. The total cost of tickets for the family of two adults and three children is $79.50. If an adult ticket costs $6.00 more than a child's ticket, find the cost of each.

8. The number of nickels that Christine has is 5 times the number of dimes. Their value is $1.05. How many coins of each type does she have?

9. Jill has $2.30 in dimes and quarters. The number of dimes is 5 less than the number of quarters. How many coins of each type does she have?

10. A cash register contains 15 coins in dimes and nickels. The total value is $1.25. How many coins of each type are there?

11. Angelo has $1.90 in dimes and nickels. If he has 4 less nickels than 5 times the number of dimes, how many dimes does he have?

12. Marie has 24 coins in half-dollars and dimes. Their total value is $3.60. How many coins of each type are there?
Applications of Algebraic Equations: Unit 3c
Solving Word Problems Using More Than One Variable

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variables. That means to write down exactly what you want each variable to represent. Only after you have declared the variables should you now translate the English sentence(s) in the word problem into algebraic equations. Solve the equations and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

A train leaves Slaton traveling east at 80 kilometers per hour. An hour later, another train leaves Slaton on a parallel track at 120 kilometers per hour. How far from Slaton will the trains meet?

Knowns: The 1st train leaves Slaton traveling east at 80 km/hr.
The 2nd train leaves Slaton traveling east at 120 km/hr.

Unknowns: How far from Slaton will the 2nd train catch up to the 1st train?

Variables: d = distance each train travels from Slaton.
t = time the 1st train travels.

<table>
<thead>
<tr>
<th>Distance</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st train</td>
<td>d</td>
<td>80</td>
</tr>
<tr>
<td>2nd train</td>
<td>d</td>
<td>120</td>
</tr>
</tbody>
</table>

Translation and Solution: d = 80t
\[80t = 120(t-1)\]
\[d = 120(t-1)\]
\[80t = 120t - 120\]
\[-40t = -120\]
\[t = 3\]

Answer: After the 1st train travels \[20(3) = 60\] km, the trains will meet.
Example 2

Two trains leave York at the same time, one traveling north, the other traveling south. The first train travels at 40 miles per hour and the second at 30 miles per hour. In how many hours will the trains be 235 miles apart? Round answer to the nearest hundredth.

Knowns: 2 trains leave York at the same time with 1 traveling north and the other traveling south.

The 1st train travels at 40 mph; the second travels at 30 mph

Unknowns: how many hours does it take for the trains to be 235 mi. apart?

Variables: \( t \) = time each train travels,
\( x \) = distance the northbound train travels,
\( y \) = distance the southbound train travels.

Translation and Solution:

\[
\begin{align*}
\begin{cases}
  x = 40t \\
  y = 30t \\
  x + y = 235
\end{cases}
\implies 40t + 30t = 235 \\
70t = 235 \\
\frac{70}{70} \cdot t = \frac{235}{70} \\
t = 3.357 \\
\end{align*}
\]

Conclusion: It will take \( 3.36 \) hours for the trains to be 235 miles apart.
Example 3

A motorboat took 3 hours to make a downstream trip with a current of 6 km/h. The return trip against the same current took 5 hours. Find the speed of the boat in still water.

**Knowns:** A motorboat takes 3 hours to make a downstream trip with a current of 6 km/h. The same trip upstream against the same current takes 5 hrs.

**Unknowns:** The speed of the boat in still water.

**Variables:** \( s \) = speed of boat in still water; \( d \) = distance one way

<table>
<thead>
<tr>
<th>Distance</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip downstream</td>
<td>( d )</td>
<td>( s+6 )</td>
</tr>
<tr>
<td>Trip upstream</td>
<td>( d )</td>
<td>( s-6 )</td>
</tr>
</tbody>
</table>

**Translation and Solution.**

\[
d = 3(s+6) \quad \Rightarrow \quad 3(s+6) = 5(s-6)
\]
\[
d = 5(s-6) \quad \Rightarrow \quad 3s + 18 = 5s - 30
\]
\[
48 = 2s
\]
\[
24 = s
\]

**Answer:** The boat travels at 24 km/h in still water.
Example 4

The sum of two numbers is 56. One third of the first number plus one fourth of the second number is 16. Find the numbers.

Knowns: The sum of 2 numbers is 56.
\[ \frac{1}{3} \text{ of the first number plus } \frac{1}{4} \text{ of the second number is 16.} \]

Unknowns: The 2 numbers.

Variables: \( x = 1^{st} \) number
\( y = 2^{nd} \) number

Translation and Solution:
\[ x + y = 56 \implies 4(x + y) = 224 \]
\[ \frac{1}{3}x + \frac{1}{4}y = 16 \implies -12 \left( \frac{1}{3}x + \frac{1}{4}y \right) = 192 \]
\[ 4x + 4y = 224 \]
\[ -4x - 3y = -192 \]
\[ y = 32 \]
\[ 1232 = 56 \implies x = 24 \]

Results: The 1\(^{st}\) number is 24 and the 2\(^{nd}\) number is 32.
Word Problems

1. Two cars leave town at the same time going in opposite directions. One travels 55 miles
   per hour and the other travels at 48 miles per hour. In how many hours will they be 206
   miles apart?

2. Two cars leave town at the same time going in the same direction on the same road.
   One travels 30 miles per hour, and the other travels 46 miles per hour. In how many hours
   will they be 72 miles apart?

3. A train leaves a station and travels east at 72 km/h. Three hours later a second train
   leaves on a parallel track and travels east at 120 km/h. When will it overtake the first
   train?

4. A private airplane leaves an airport and flies due south at 192 km/h. Two hours later a
   jet leaves the same airport and flies due south at 960 km/h. When will the jet overtake the
   plane?

5. A canoeist paddled for 4 hours with a 6-km/h current to reach a campsite. The return
   trip against the same current took 10 hours. Find the speed of the canoe in still water.

6. It takes a passenger train 2 hours less time than it takes a freight train to make the trip
   from Central City to Clear Creek. The passenger train averages 96 km/h while the freight
   train averages 64 km/h. How far is it from Central City to Clear Creek?

7. It took 3 hours to row a boat 18 km against the current. The return trip with the current
   took 1 1/2 hours. Find the speed of the rowboat in still water.

8. A student walks and jogs to college each day. The student averages 5 km/h walking and
   9 km/h jogging. The distance from home to college is 8 km, and the student makes the trip
   in 1 hour. How far does the student jog?

9. The sum of two numbers is 115. The difference is 21. Find the numbers.

10. The sum of two numbers is 26.4. One of the numbers is five times the other. Find the
    numbers.

11. The sum of two numbers is 92. One eighth of the first number plus one third of the
    second number is 19. Find the numbers.

12. The difference of two numbers is 49. One half of the larger number plus one seventh
    of the smaller number is 56. Find the numbers.

13. An airplane flew for 4 hours with a 20 km/h tail wind. The return flight against the
    same wind took 5 hours. Find the speed of the plane in still air.
14. An airplane took 2 hours to fly 600 km against a head wind. The return trip with the wind took 1\(\frac{3}{4}\) hours. Find the speed of the plane in still air.

15. An airplane took 2.5 hours to fly 625 miles with the wind. It took 4 hours and 10 minutes to make the return trip against the same wind. Find the wind speed and the speed of the plane in still air.
Applications of Algebraic Equations: Unit 3d
Solving Word Problems Using More Than One Variable

When solving word problems, it helps to break up your work into small steps. Therefore, when you work on these problems, first read entire the problem. Next, read it again a phrase at a time and make sure you understand what each phrase means. Determine what the problem wants you to find. Write out explicitly what you know and what you do not know but need to find. Declare your variables. That means to write down exactly what you want each variable to represent. Only after you have declared the variables should you now translate the English sentence(s) in the word problem into algebraic equations. Solve the equations and make sure you answer the question asked in the problem, using the correct units when applicable.

Read the following examples of word problems that have been solved using all of the steps that were given above. Then solve the given word problems using all of the steps given above.

Example 1

Michelle Limotta invested $10,000 for one year, part at 8% annual interest and the rest at 12% annual interest. Her total interest for the year was $944. How much money did she invest at each rate?

**Knowns:** Michelle invested part of $10,000 at 8% annual interest for a year and the rest at 12% annual interest for a year. Her total interest for the year was $944.

**Unknowns:** How much money did she invest at each rate?

**Variables:**
- \( x \): amount invested at 8%
- \( y \): amount invested at 12%

**Translation and Solution:**

\[
x + y = 10,000 \quad \Rightarrow \quad x = 10,000 - y
\]

\[
.08x + .12y = 944
\]

\[
.08(10,000 - y) + .12y = 944
\]

\[
800 - .08y + .12y = 944
\]

\[
.04y = 144
\]

\[
y = 3600
\]

\[
x = 10,000 - 3600 = 6400
\]

**Answer:** Michelle invested $6400 at 8% and $3600 at 12%.
Example 2

A metal alloy is 25% copper. Another metal alloy is 50% copper. How much of each alloy should be used to make 1000 grams of a metal alloy that is 45% copper?

Knowns: 1 metal alloy is 25% copper and the other is 50% copper.
Unknowns: How much of each should be used to make 1000 grams of a metal alloy that is 45% copper?

Variables: \(x = \text{amount of 25\% copper alloy}\)
\(y = \text{amount of 50\% copper alloy}\)

<table>
<thead>
<tr>
<th>Amount of alloy</th>
<th>Percent copper</th>
<th>Amount of copper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st alloy</td>
<td>(x)</td>
<td>25%</td>
</tr>
<tr>
<td>2nd alloy</td>
<td>(y)</td>
<td>50%</td>
</tr>
<tr>
<td>Final</td>
<td>1000</td>
<td>45%</td>
</tr>
</tbody>
</table>

Translation and Solution.

\[x + y = 1000\]
\[0.25x + 0.50y = 0.45(1000)\]

\[\begin{align*}
-0.25x + 0.25y &= -250 \\
0.25x + 0.50y &= 450
\end{align*}\]

\[0.25y = 200\]
\[y = 800\]

\[x + 800 = 1000 \Rightarrow x = 200\]

Answer: 200 grams of 25\% copper alloy and 800 grams of 50\% copper alloy should be used.
Example 3

Delectable Dan's Cookie Company sells two kinds of cookies daily: chocolate chip at $6.50 per dozen and white chocolate macadamia at $9.00 per dozen. On Thursday, Dan sold 85 dozen more chocolate chip than white chocolate macadamia cookies. The total sales for both were $4055.50. How many dozen of each were sold?

Knowns: Dan sells choc. chip cookies for $6.50 per dozen and white choc. macadamia cookies for $9.00 per dozen.
Thursday Dan sold 85 doz. more choc. chip than white choc. macadamia cookies.
The total sales were $4055.50.

Unknowns: How many dozen of each were sold?

Variables: \( x \) = the number of dozens of white chocolate macadamia cookies sold
\( y \) = the number of dozens of chocolate chip cookies sold

<table>
<thead>
<tr>
<th>Number of Dozens</th>
<th>Price Per Dozen</th>
<th>Total Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>White choc.</td>
<td>( x )</td>
<td>$9.00</td>
</tr>
<tr>
<td>Choc. chip</td>
<td>( y )</td>
<td>$6.50</td>
</tr>
</tbody>
</table>

Translation and Solution:
\( y = x + 85 \)
\( 9x + 6.5y = 4055.50 \)
\( 9x + 6.5(x+85) = 4055.50 \)
\( 15.5x + 552.5 = 4055.50 \)
\( 15.5x = 3503 \)
\( x = 225 \)
\( y = 225 + 85 = 310 \)

Answer: Dan sold 225 dozen white chocolate macadamia cookies and 310 dozen chocolate chip cookies.
Word Problems

1. Fina invests $4000, part of it at 10% annual interest and the rest at 12% annual interest. If she earned $460 in interest at the end of one year, how much did Fina invest at each rate?

2. Steve Devine invested $7200 for one year, part at 10% annual interest and the rest at 14% annual interest. His total interest for the year was $960. How much money did he invest at each rate?

3. At the Golden Oldies Theater, tickets for adults cost $5.50 and tickets for children cost $3.50. How many of each kind of ticket was purchased if 21 tickets were bought for $83.50?

4. Paul Yu is investing $6000 in two accounts, part at 4.5% and the remainder at 6%. If the total interest earned from the two accounts is $279, how much did Paul deposit at each rate?

5. A liter of cream has 9.2% butterfat. How much skim milk containing 2% butterfat should be added to the cream to obtain a mixture with 6.4% butterfat?

6. Fred Furguson invested $5000 for one year, part at 9% annual interest and the rest at 12% annual interest. The interest from the investment at 9% was $198 more than the interest from the investment at 12%. How much money did he invest at 9%?

7. Angela Raimondi wants to invest $8500, part at 14% annual interest and part at 12% annual interest. If she wants to earn the same amount of interest from each investment, how much should she invest at 14%? (Round to the nearest cent.)

8. Walnuts cost $9.95 per pound, while peanuts cost $6.50 per pound. If there are 3 less pounds of peanuts than there are walnuts, how many pounds of each are there in a box which costs $62.75?

9. A chemist has one solution that is 60% chlorinated and another that is 40% chlorinated. How much of each solution is needed to make a 100L solution that is 50% chlorine?

10. A 50-gallon barrel of milk is 6% butterfat. How much skim milk (NO butterfat) should be mixed to make milk that is 3% butterfat?

11. Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100 milliliters of a solution that is 68% acid?
APPENDIX B

WORD PROBLEM TESTS

This appendix contains the three tests that were used to measure the performance of the participating students at the ends of the three phases of the study, as described in Chapter III.
Algebra Word Problems With One Unknown

Test 1

Solve the following word problems on your own paper. To receive full credit for each problem solution, you must write out all of the knowns and unknowns (1 point), declare the variables (2 pts.), translate the English sentences into algebraic equations (3 pts.), solve the equations algebraically (2 pts.), and answer the question asked in the word problem, using units when applicable (2 pts.).

1. Ten years more than \( \frac{3}{5} \) of Jim's age is 28. How old is Jim?

2. The Illinois Department of Health reported that for the years 1971 to 1982, of 230 horses tested for rabies only 16 actually had the disease. What percentage of horses tested had rabies?

3. A model car is made to the following scale: 1 inch to 10 inches. If the door of the actual car is 33 inches long, what is the door length of the model?

4. How much whipping cream (9% butterfat) should be added to 1 gallon of milk (4% butterfat) to obtain a 6% butterfat mixture?

5. Mary Bly invested some money last year with a 12% interest rate. Her annual interest was $480. How much did she invest?

6. The property tax on a house is $8 per $1000 assessed valuation. What is the tax on a house assessed at $65,000?

7. The members of a loading crew estimated that they could load 8 boxes in 20 minutes. At this rate, how many boxes could they load in 1 hour?

8. It was reported in 1987 that 55% of the 51.5 million married couples in the U.S.A. had two incomes. Approximately how many couples had two incomes?

9. In 1983, only 43,200 or 2.4% of America's enlisted military personnel were college graduates. How many enlisted personnel were there in all?
10. How many ounces of a 6% iodine solution need to be added to 12 ounces of a 10% iodine solution to create a 7% iodine solution?

11. The television series with the largest audience to date was the final episode of \textit{M*A*S*H}. 77\% of the 162 million people were watching TV that night saw the program. How many people were watching \textit{M*A*S*H}?

12. A price decreased from $40 to $35. Find the percent of decrease.
Algebra Word Problems With One or More Unknowns

Test 2

Solve the following word problems on your own paper. To receive full credit for each problem solution, you must write out all of the knowns and unknowns (1 point), declare the variables (2 pts.), translate the English sentences into algebraic equations (3 pts.), solve the equations algebraically (2 pts.), and answer the question asked in the word problem, using units when applicable (2 pts.).

1. Mona's age is 4 times Billy's age. The sum of their ages is 15. Find the age of each.
2. The sum of the ages of Ron and his sister is 20. Ron's age, decreases by twice his sister's age, is 2. Find the age of each.
3. The perimeter of a rectangle is 22 m. The length of the rectangle is 1 m less than 3 times the width. Find the length and width of the rectangle.
4. A $31.50 box of nuts contains almonds mixed with pecans. Almonds cost $7.50 per kilogram and pecans cost $9.00 per kilogram. The number of kilograms of almonds is 3 times the number of kilograms of pecans. Find the number of kilograms of each kind of nut in the mixture.
5. Find two numbers whose sum is -1 and whose difference is 5.
6. There were 411 people at a play. Admission was $1.00 for adults and $0.75 for children. The receipts were $395.75. How many adults and how many children attended?
7. There were 203 tickets sold for a school wrestling match. For those who held activity cards, the price was $1.25. For those who did not hold activity cards, the price was $2.00. The total amount collected was $310. How many of each type of ticket were sold?
8. The sum of two numbers is 30. Their difference is 40. Find the numbers.
9. Kendra is doing a chemistry experiment that calls for a 30% solution of copper sulfate. She has 40 mL of 25% solution. How many milliliters of 60% solution should Kendra add to obtain the required 30% solution?

10. Two cyclists are traveling in the same direction on the same bike path. One travels at 20 miles per hour and the other at 14 miles per hour. After how many hours will they be 15 miles apart?

11. Ismarelda invested $9000 for one year, part at 4% annual interest and the rest at 5% annual interest. Her total interest for the year was $409.50. How much money did she invest at each rate?

12. The length of a rectangle is 2 feet less than 3 times the width. If the perimeter is 68 feet, what are the dimensions of the rectangle?

13. A store sells cashews for $4.40/lb and peanuts for $1.20/lb. How many pounds of each can be bought to get exactly 3 lb of nuts for $6.00?
Algebra Word Problems With One or More Unknowns

Test 3

Solve the following word problems on your own paper. To receive full credit for each problem solution, you must write out all of the knowns and unknowns (1 point), declare the variables (2 pts.), translate the English sentences into algebraic equations (3 pts.), solve the equations algebraically (2 pts.), and answer the question asked in the word problem, using units when applicable (2 pts.).

1. Six apples and three oranges cost $1.77. Two oranges and five apples cost $1.27. Find the cost of an apple and the cost of an orange.

2. Five gallons of unleaded gas plus eight gallons of regular cost $13.87. Five gallons of unleaded plus two gallons of regular cost $7.93. Find the cost per gallon of each kind of gasoline.

3. In one year Cholena Youngblood earned the same amount of interest from an investment at 8% annual interest as an investment at 12% annual interest. She had invested $1500 more at 8%. How much money did she invest at 12%?

4. The sum of two numbers is 45. Three times the first number plus seven times the second is 115. Find the two numbers.

5. Dawn is 3 years older than Lois. Four years ago Dawn was twice as old as Lois. How old are Dawn and Lois now?

6. The perimeter of a rectangle is 10 m. Twice the width is equal to one-half the length. Find the length and the width.

7. A coal barge on the Ohio River travels 24 miles upstream in 3 hours. The return trip takes the barge only 2 hours. Find the rate of the barge in still water.

8. The sum of the length and width of a rectangle is 19 inches. The length is 1 less than twice the width. Find the length and width of the rectangle.
9. The attendance at a school concert was 578. Admission cost $2.00 for adults and $1.50 for children. The receipts were $985.00. How many adults and how many children attended the concert?

10. The speed of a freight train is 14 km/h slower than the speed of a passenger train. The freight train travels 330 km in the same time that it takes the passenger train to travel 400 km. Find the speed of each train.

11. A retailer sold 400 sweatshirts at $10 retail and has 250 left. For what sale price should each of the remaining sweatshirts be sold to have an overall average price of $9 per sweatshirt? (Think of this as a mixture problem with the number of sweatshirts as the amounts and substituting each of their costs for percents.

12. A chemist has two alcohol-in-water solutions: a 20% alcohol solution and a 50% alcohol solution. He needs 12 L of a solution that is 45% alcohol. If he has 2 L of the 20% alcohol solution, how much does he need of the 50% alcohol solution?

13. Angelo has $1.90 in dimes and nickels. If he has 4 less nickels than 5 times the number of dimes, how many dimes does he have?

14. Mr. Whitecloud is 24 years older than his son. Five years from now, he will be 3 times as old as his son will be then. How old are Mr. Whitecloud and his son now?

15. The difference between two numbers is 6. Twice the first number, decreased by the second number, is 30. Find the two numbers.
APPENDIX C

ATTITUDE QUESTIONNAIRES AND VALIDITY PANEL

This appendix contains the Form A and Form B attitude questionnaires and the membership list for the panel of experts who validated them, as described in Chapter III. The questionnaires are formatted exactly as they were used. The weight for each item is shown in parentheses after the item. These weights were not present on the questionnaires as given to the participating students.
Attitude Toward Problem Solving in School Mathematics

Please indicate whether you agree or disagree with each statement at this time. You may have agreed with many of these statements at one time or another, but please circle agree only for those items that you most strongly agree with now. This is not an examination. There are no right or wrong answers. This is simply a study of students' attitudes toward problem solving in school mathematics.

<table>
<thead>
<tr>
<th>Agree</th>
<th>Disagree</th>
<th>1. I believe it takes time and repetition to completely learn mathematical skills. (7.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>2. Word problems make math interesting, because they are in English and &quot;the numbers aren't just there for you.&quot; (9.00)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>3. Mathematics is exhilarating when I make a &quot;break-through&quot;! (10.10)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>4. If I don't know how to solve a problem, I get mad. (4.85)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>5. I don't ask questions in math class because my questions are always stupid ones. (2.57)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>6. I feel very frustrated when trying to solve word problems. (3.95)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>7. I really enjoy helping other students solve a problem after I have figured out how to solve it. (9.81)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>8. Word problems really make me stop and think. (8.00)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>9. Every time I know word problems are going to be on a test, I get a feeling of doom. (3.00)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>10. I believe that if I could figure out word problems, I could do anything in math. (5.90)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>11. I dread every math class I take, because I'm just no good in math. (1.29)</td>
</tr>
</tbody>
</table>

Form A
Attitude Toward Problem Solving in School Mathematics

Please indicate whether you agree or disagree with each statement at this time. You may have agreed with many of these statements at one time or another, but please circle agree only for those items that you most strongly agree with now. This is not an examination. There are no right or wrong answers. This is simply a study of students' attitudes toward problem solving in school mathematics.

<table>
<thead>
<tr>
<th>Agree</th>
<th>Disagree</th>
<th>1. If one method does not work in solving a problem, I will find another method to solve it. (9.81)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>2. It's fun when you figure out problems on your own and get the right answer. (10.10)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>3. I learn easier when I know why a method works. (8.00)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>4. Going over word problems with the teacher in class is easy, but when I'm not in class, I get lost a lot of times. (5.00)</td>
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<tr>
<td>Agree</td>
<td>Disagree</td>
<td>5. I like to be able to apply algebraic techniques to everyday-type problems. (9.00)</td>
</tr>
<tr>
<td>Agree</td>
<td>Disagree</td>
<td>6. There seems to be no connection between school mathematics and real life. (2.38)</td>
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<tr>
<td>Agree</td>
<td>Disagree</td>
<td>7. The way some story problems are worded confuses me, and then I can't think of an equation for them. (4.10)</td>
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<td>Agree</td>
<td>Disagree</td>
<td>8. All the words in a story problem just confuse me. (3.00)</td>
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<td>Agree</td>
<td>Disagree</td>
<td>9. Graphing equations by hand is not a good way to solve them. (5.90)</td>
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<td>Agree</td>
<td>Disagree</td>
<td>10. Word problems are easy as long as they are easy to translate. (6.95)</td>
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<tr>
<td>Agree</td>
<td>Disagree</td>
<td>11. I dread every math class I take, because I'm just no good in math. (1.29)</td>
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Form B
The Panel of Judges Who Judged Construct and Content Validity of the Questions for the Attitude Questionnaires

1. David Ericson: David has taught high-school mathematics for over 15 years and is currently a doctoral candidate in mathematics education at the Ohio State University.

2. Joy Longfellow: Joy is an Assistant Professor of Mathematics at Capitol University and is currently a doctoral student in mathematics education at the Ohio State University.

3. Armando Martinez-Cruz: Armando earned his Ph.D. in mathematics education at the Ohio State University in 1993. He is now a mathematics education faculty member at the Universidad Nacional Autonoma de Mexico in Mexico City.

4. Cathie Mulligan: Cathie has taught both secondary and college mathematics for over 20 years.

5. Sandy Schroeder: Sandy has taught mathematics at the high school or college level for 14 years and is currently a doctoral candidate in mathematics education at the Ohio State University.

6. Mary Schwartz: Mary has taught high school mathematics for nearly 20 years.
APPENDIX D

HUMAN SUBJECTS DOCUMENTS

The human subject review documents discussed in Chapter III are presented here, including the consent form and the information letter sent to the parents of the participating students.
Consent Form

Consent for Participation in the Algebra I Study Designed by Susann Mathews

I consent to my child's participation in the research entitled "The Effect of Using as Many Variables as Are Needed to Solve Word Problems on the Problem-Solving Skills and Attitudes of Algebra I Students."

Susann Mathews and our child's Algebra I teacher have explained in a letter the purpose of the study, the procedures to be followed, and the expected duration of my child's participation. Possible benefits have been described.

I acknowledge that I have had the opportunity to obtain additional information regarding the study and that any questions I have raised have been answered to my full satisfaction. Further, I understand that my child is free to withdraw consent at any time and to discontinue participation without prejudice to my child.

Finally, I acknowledge that I have read and fully understand the explanatory letter that was sent home with my child and the consent form. I sign it freely and voluntarily.

Date: __________________________ Signed: ________________________________

(Participant)

Signed: __________________________ Signed: ________________________________

(Investigator) (Person Authorized to Consent for Participant)

To better understand how the students are learning, Ms. Mathews would like to interview a few students at the end of the study. The interviewee's name will not be used when reporting this study. In fact, Ms. Mathews will ask the interviewee to give an assumed name for the course of the interview. The interviews will be audiotaped to more accurately record the conversation. Please sign below if you and your child do not object to your child's being interviewed. Thank you.

Signed: ________________________________
28 September 1993

Dear Parents of Algebra I Students,

Your child has a chance to participate in research that could help determine the way in which future Algebra I textbooks are written. As you know, learning to solve word problems is one of the most important aspects covered in Algebra I. We will be working on better ways to learn how to solve word problems. We will be covering the same topics as in a regular Algebra I class; however, we will be covering the topics in a different order to determine which order of learning helps the students to learn to solve word problems the best. Specifically, we will be bringing forward in the curriculum solving two equations in two variables. Solving two equations in two variables will be taught directly after the students learn how to solve one equation in one variable and how to apply that knowledge to solving word problems.

Susann Mathews is a faculty member at Wright State University who teaches mathematics and mathematics education. She has prepared the materials that we will use from five of the Algebra I textbooks being used in the United States today. Your child's participation is voluntary. If you (or your child) decide that you do not want your child to participate in this exciting educational experiment, it will not be detrimental to his or her grade. However, we hope all of the students in the Algebra I class will participate to help us better determine how to teach algebraic problem solving. This study will last approximately four weeks. Your child will be taking three word problem tests throughout the study, and he or she will take a short before and after attitude questionnaire to help determine if learning how to solve word problems increases his or her positive attitude.
toward problem solving in school mathematics. As the teacher, I will assign each student a number so that when Ms. Mathews collects the data from the word problem tests and attitude questionnaires she will not know whose paper belongs to whom. She will be able to track the improvement in your child's problem solving simply by seeing the same number attached to the same student's paper for each test.

If you have any questions, please contact either Susann Mathews at 873-2332 (the phone number of the Department of Teacher Education at Wright State University) or me. Please sign the attached parental consent form to allow your child to participate in this exciting study.

Sincerely,

(Teacher's signature.)
APPENDIX E
DATA SETS

This appendix contains the raw data acquired in this research. The data items are as described in Chapter IV. The data are presented in tables. Missing data points are indicated by asterisks. For each student in the study, there is a row of data in the table for that student's teacher and treatment (experimental or control). The column headings are interpreted as follows:

Grade: the student's first quarter mathematics grade.

PreAtt: the student's score on the pretreatment attitude questionnaire.

Test 1: the student's score on the first word problem test.

Test 2: the student's score on the second word problem test.

Test 3: the student's score on the third word problem test, as graded by the teacher.

Rel. Chk.: the student's score on the third word problem test, as graded by the researcher. (These were compared with the scores as graded by the teachers to evaluate inter-grader reliability.)

PostAtt: the student's score on the posttreatment attitude questionnaire.
Table 11. Teacher: Beebe Treatment: Experiment

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Table 12. Teacher: Beebe Treatment: Control

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### Table 18. Teacher: Place Treatment: Control

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APPENDIX F
ADDITIONAL HYPOTHESIS TESTS

This appendix contains the additional hypothesis test results discussed in Chapter IV.

Two sample t-tests comparing the means of the pooled control group with the means of the pooled experimental group are presented in Table 11. The null hypothesis that there is no difference between the control and experimental group mean scores on Test 1 and Test 3 cannot be rejected. There is, however, a statistically significant difference between the mean scores on Test 2.

Table 19. Two Sample t-test Results for Pooled Data

<table>
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<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
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<tbody>
<tr>
<td>t-statistic</td>
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<td>.0023</td>
<td>.99</td>
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<td>Control Mean</td>
<td>71.1</td>
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<tr>
<td>Experimental Mean</td>
<td>71.2</td>
<td>62.1</td>
<td>54.2</td>
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</table>

Two sample t-tests comparing the mean of the control group for each teacher with the mean of the experimental group for that teacher are presented in Tables 12 through 14. With the smaller sample sizes, by teacher, the results are less definitive, but are generally consistent with the pooled results. The apparently inconsistent result,
that Johnson's control group had the higher mean on Test 2, was not statistically
significant, with a critical level of .25, due to the relatively small sizes of 23 experimental
group and 15 control group students.

**Table 20.** Two Sample t-test Results for Test 1 By Teacher

<table>
<thead>
<tr>
<th>Teacher</th>
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<th>Kramer</th>
<th>Place</th>
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<tr>
<td></td>
<td>1.5</td>
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**Table 21.** Two Sample t-test Results for Test 2 By Teacher

<table>
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<tr>
<th>Teacher</th>
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<th>Kramer</th>
<th>Place</th>
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</thead>
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<td>Degrees of Freedom</td>
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<td>39</td>
<td>40</td>
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<td>.033</td>
<td>.13</td>
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<td>56.7</td>
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Table 22. Two Sample t-test Results for Test 3 By Teacher

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Mann-Whitney U-tests comparing the median of the control group for each teacher with the median of the experimental group for that teacher are presented in Tables 15 through 17. As with the t-tests above, the results are less definitive, but are generally consistent with the pooled results. Again, the apparently inconsistent result that Johnson's control group had the higher median on Test 2 was not statistically significant, with a critical level of .2106.

Table 23. Mann-Whitney U-test Results for Test 1 By Teacher

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Table 24. Mann-Whitney U-test Results for Test 2 By Teacher

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Point Estimate of Median Difference: Experimental - Control

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<th>Place</th>
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Critical Level

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Table 25. Mann-Whitney U-test Results for Test 3 By Teacher

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<th>Place</th>
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Point Estimate of Median Difference: Experimental - Control

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Critical Level

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LIST OF REFERENCES


Lazar, N. (1933). One unknown or two? The Mathematics Teacher, 26, 176-182.


