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A dynamic simulation of a quadruped with impulsive collision modeling

Kittivatcharapong, Sakon, Ph.D.
The Ohio State University, 1994
A Dynamic Simulation of a Quadruped with Impulsive Collision Modeling

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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To My Parents
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CHAPTER I

Introduction

1.1 Background

At the present time, mainly stationary manipulator robots are used in production tasks in factories. Yet, both inside and outside of industry, many tasks could be performed by a mobile robot. Among these tasks are terrestrial exploration, maintenance or repair in difficult environments and undersea operations. Since these tasks require off-road capabilities similar to those of terrestrial animals, the mobile robot should possess a higher degree of terrain adaptability compared to wheeled or even tracked vehicles. Based on this requirement, legged locomotion seems to be a very promising solution. Due to its various applications and theoretical advantages over others types of vehicles, a large amount of research has been developed in this area.

One important factor which needs to be considered in the design and control of a legged vehicle is its stability. According to this requirement, one may divide the legged locomotion examples into two categories. The first category is statically stable legged vehicles. This type of legged machine has been designed so that it can move with the center of gravity inside the polygon formed by the feet on the ground. Thus, a minimum number of three supporting feet is required at one time. Since
the speed of this legged locomotion is very slow, it can be assumed to be statically stable regardless of the acceleration. This assumption, however, helps simplify the complexity of the control used in the vehicle. A few examples of the machine which utilizes this concept are the OSU hexapod and the six-legged Adaptive Suspension Vehicle (ASV) [1]. In order to increase the speed and maneuverability, the dynamically stable legged machine has been introduced. This type of vehicle usually requires less than three supporting feet at one time. Therefore, it can run with a higher speed compared to the previously discussed legged vehicle. Since the machine is not statically stable, a more sophisticated control scheme is required for both body stabilization and leg coordination. Examples of dynamically stable legged vehicles would be bipeds and a high-speed quadruped.

One of the projects conducted at the Ohio State University is the development of the actively balanced quadruped. This machine is designed in order to improve both speed and maneuverability over that of the ASV. Therefore, both moderate-speed and high-speed gaits, such as trotting and pacing, can be applied. The first stage of this project, which represents the main objective goal in this work, is to develop the dynamic simulation for this quadruped running machine. The function of this simulation is to provide the testing capability for all controllers used in this vehicle. Impact between feet of a running machine and the ground may be an important aspect in the control and is a significant new feature of this simulation. The simulation also incorporates a graphic display which is very convenient for the complete evaluation.
1.2 Scope of this Work

Figure 1 illustrates the flow diagram of the quadruped control and simulation used in this work. The computational structure is basically divided into five different blocks. The first block represents the motion profile which consists of the high level of position and velocity commands for the vehicle. The second block is the control algorithm for the quadruped. Normally, this module computes the joint torque inputs for the dynamic simulation based on current states of the vehicle, a command set point and foot forces from the force distribution routine. This control software as well as the motion profile has already been developed by Aebker [2]. Therefore, it will not be explained in detail as part of this simulation.

The third block represents the force distribution algorithm used in this simulation. This part of computation provides the optimal foot forces based on commanded body forces and torques from the controller plus stability criteria such as friction cone constraints. Although the algorithm has been implemented for statically stable legged vehicles, further development in this area for dynamically stable cases is necessary. This part of the simulation will be studied in this work.

The fourth block in this figure are the dynamic simulation for this quadruped. This module calculates the position and velocity of the vehicle's body and its legged joints due to the input joint torques. Thus, it simulates the reaction of the quadruped according to its control system. Since the quadruped can operate on both hard and soft terrain, the dynamic simulation should be able to model the motion of this machine on both kinds of surface. Therefore, the simulation algorithm with different
foot end conditions will be implemented in this work. Finally, the output from this
dynamic simulation will also display on the graphic system which is designated as the
last block of this flow diagram.

1.3 Organization

The content of this work has been organized as follows: First, the dynamic simulation
of the quadruped is presented in Chapter 2. The formulation is derived based on the
recursive form of the Newton-Euler equations. The foot end conditions for both soft
and hard terrain are also discussed in this chapter. Based on these conditions, the
original recursive equations are simplified into a matrix form where the parallelism
caused by the multilegged structure is preserved. Another approach to update ro­
tational orientation of the body is also presented in this chapter. Unlike the Euler
angle approach, this technique calculates the new body orientation directly from its
rotational rate. Therefore, it can provide more consistent results than that obtained
from the Euler angle approach. Furthermore, a very accurate integration scheme, the
fourth-order Runge-Kutta will be used in this work.

Because of the large number of computations and the parallel structure of the
system equations, a supercomputer is a good candidate for computing the simulation
results. Therefore, a few experiments on special features of the Cray supercomputer
such as vectorization and in-line subroutine expansion will be presented at the end
of this chapter.

In order to verify the correctness of the simulation, two testing schemes based
on the conservation of power and energy are introduced. The applications of these
Figure 1: Block diagram of the quadruped control and simulation system.
techniques to the dynamic simulation will be discussed in Chapter 3. Since each test can work effectively on different parts of algorithms, they are both necessary for verifying correctness of the simulation.

Chapter 4 will discuss the impact dynamics which occur in 2-D and 3-D articulated bodies. This part of the algorithm is necessary for the hard contact simulation since both impulsive forces and the current states of the vehicle after the collision need to be determined. Because the collision in robots is mostly inelastic, a contact model presented is based on this type of collision. The discussion will cover both single contacts which occur in single-chain manipulators and multiple contacts which occurs in legged vehicles and multifingered robots. The contact models used in both cases are illustrated by different examples of single and multiple rigid bodies. These formulations are also implemented in both two and three dimensions.

The force distribution for dynamically stable legged vehicles will be presented in Chapter 5. The presentation involves both a general description and a mathematical model used in this problem. An optimization scheme, which can determine the foot force solution for this type of vehicle, will be presented in this chapter. Its application to a trotting quadruped will be considered as an example in this work.

The overall simulation results will be evaluated and discussed in Chapter 6. The example of a trotting quadruped on the hard terrain is performed to verify the consistency of both the impact model and the foot force allocation developed in this work. Finally, a summary as well as further possible extensions of this research will be presented in Chapter 7.
CHAPTER II

Dynamic Simulation of a Quadruped

2.1 Introduction

Before a robot can be built, all control algorithms need to be tested through a simulation. While kinematic simulations are sufficient for a slow, statically stable legged machine, dynamic simulation is definitely necessary for a high-speed legged vehicle. Many researchers have studied in this area in order to develop efficient simulation algorithms. Based on current computer architectures, it is possible to improve the speed of the simulation to be close to real time.

The dynamic simulation is divided into two parts: direct dynamics and motion integration. Direct dynamics calculates the joint and body accelerations based on the current positions, velocity and joint torque inputs. Motion integration, on the other hand, determines the new positions and velocity of the vehicle according to its velocity and acceleration. Together they simulate the motion of a quadruped in response to the actuated joint torques. Since the quadruped represents a closed chain structure, solving direct dynamics is not possible unless the end conditions at each foot are specified. One technique is to model the contact points as springs and dampers. Based on this suggested model, the direct dynamics can be reduced into a
very compact form which is similar to the open chain mechanism. This approach is quite practical for a soft contacting surface where less stiffness is required. However, once the contact is very stiff, small step sizes become necessary for this contact model. As a result, a large number of computation is required under this circumstance.

In order to minimize this effect and improve the speed of the computation, another approach which can be applied to the hard terrain is presented in this chapter. The algorithm is developed to provide the flexibility where different foot end conditions can be tested. Therefore, the algorithm can be used for both hard and soft terrain. The direct dynamic is derived based on the recursive Newton-Euler equations. Since the final formulation is also simplified into a simple matrix form, it is very suitable for running on a supercomputer. A new way to update the body orientation without using Euler angles is also introduced in this work.

The content in this chapter is divided into different parts as follows: first, the previous work in solving the direct dynamic problem are reviewed in Section 2.2. The model of a quadruped as well as its coordinate system are illustrated in Section 2.3. The recursive Newton-Euler equations together with the foot constraints for both hard and soft terrain are presented in Section 2.4. The modification of this formulation for the CRAY supercomputer is illustrated in Section 2.5. The state determination and the integration scheme used in this work are explained in Section 2.6 and 2.7 respectively. Finally, the results of the computation on the Cray supercomputer are demonstrated in Section 2.8.
2.2 Previous Work on Direct Dynamics

Most of the studies in this area have been first developed for a single open-chain mechanism. One of the early approaches to calculate the joint accelerations for an open-chain manipulator was reviewed by Walker and Orin [3]. In their work, four different methods for computing the joint accelerations based on the Newton-Euler formulation have been presented. Among these techniques, the composite rigid-body approach is the most efficient since it takes into account the symmetric property of the system inertia matrix. The method also utilizes a recursive procedure to compute the mass, the center of mass and the inertia of each link.

Another approach to solve the direct dynamic based on articulated-body inertias was introduced by Featherstone [4]. The algorithm incorporates recursive procedures to specify joint accelerations and link inertia as well as other dynamic quantities. All the variables are expressed in the spatial notation form which provides a compact formulation of kinematics and dynamics. However, it is still less efficient than the composite rigid-body approach unless the number of joints is greater than 12. The similar approach to that of Featherstone is further extended by Brandl, Johanni and Otter [5]. The modified algorithm is able to determine the joint accelerations for the manipulator with multiple-degree-of-freedom joints.

The direct dynamics for a single closed-chain mechanism was introduced by Orin and McGhee [6]. The algorithm is derived based on the Newton-Euler formulation. Based on specified force and torque constraints at the contact, the system can be considered as an equivalent open chain. Thus, the procedures which were previously used
for the open-chain system can be applied. A similar concept was further developed
by Oh and Orin [7]. In their work, the direct dynamics for multiple closed chains
such as legged vehicles and multiple manipulators are considered. By including the
forces and torques constraints at the contacts, the system can be decomposed into
an open-chain mechanisms. The recursive Newton-Euler equations have been applied
to solve for the joint accelerations. Both kinematic and dynamic variables are also
expressed in the spatial notation form to provide a compact formulation.

A dynamic simulation of legged vehicles using a compliant joint model was pre­
sented by Shih, Frank and Ravani [8]. The simulation includes the leg mass, joint
compliance and effect of leg contact with the ground. Both joints and ground contact
point are modeled as three-dimensional springs and dampers. This representation
allows the system to be decoupled into independent rigid bodies. Since the reaction
forces and torques between each body are automatically specified, the Newton-Euler
equations can be derived and solved independently for each link. This approach is
conceptually simple; however, it can lead to an excessive computational algorithm
once both joints and contacts become stiff. This disadvantage is due to small integra­
tion intervals requirements which are necessary for the simulation. Another approach
for a quadruped simulation was illustrated by Freeman and Orin [9]. Unlike the pre­
vious compliant joint model, springs and dampers are only used at contact points in
order to specify the end foot conditions. Each link, however, is still coupled through
the joint which is rigidly constrained. Since the contact forces are already defined as
functions of foot positions and velocities, the whole system can be decoupled into an
open-chain tree structure. Therefore, one is able to solve the direct dynamics in this case based on efficient methods using in the open-chain system. The algorithm also demonstrates the parallel structure which can be implemented on an Intel iPSC/2 Hypercube multiprocessor system.

From this previous work, solving direct dynamics for multiple closed-chain mechanisms is very straightforward once the end conditions are specified. Because of its simplicity, the ground contact model based on springs and dampers has been widely used in most of these work. Since this contact model is only suitable for the soft contact surface, different techniques which can be used for the hard contact mode are therefore necessary.

2.3 The Model of a Quadruped

Figure 2 illustrates the model of a quadruped using in the simulation here. The legs of the vehicle are designed to possess the same configuration as of the ASV in order to avoid singularities in their workspaces. Each leg also possesses three degrees of freedom which are connected by revolute joints. The dimension of the body is 6 by 10 feet and the total weight of the vehicle is approximately 3000 lbs.

A coordinate frame is attached to each joint to describe the parameters associated with that link as shown in Figure 3. The adjacent link transformations are defined according to the modified Denavit-Hartenberg parameters [10]. The world coordinate system is fixed with respect to the ground and the body coordinate system is attached to the body’s center of gravity. All dynamic calculations are performed with respect to the world coordinate system.
Figure 2: A model of the quadruped walking machine used in this work.
Figure 3: The coordinate system assignment based on the modified DH parameters.
2.4 Recursive Newton-Euler Equations for a Quadruped

Various techniques have been used to described the dynamic behavior of a robot. Each formulation possess it own desirable properties that can be utilized in different situations. The Lagrange-Euler method, for example, is suitable for control analysis because of its explicit state equation form. The recursive Newton-Euler formulation, on the other hand, has been developed to improve the computational speed [11] and therefore is more appropriate to use in this work.

The kinematic and dynamic variables which are used in this work are adapted from a recursive Newton-Euler presented by [12]. Here, two subscripts are necessary to identify the leg and link numbers of the vehicle. The leg number will be omitted unless it is explicitly required. All of the variables are expressed with respect to the world coordinate frame and can be summarized as follows:

\[ k \quad = \quad \text{leg number} \]
\[ m_i \quad = \quad \text{total mass of link } i, \]
\[ \bar{s}_i \quad = \quad \text{position of the center of mass of link } i \text{ from} \]
\[ \text{the origin of the coordinate system } i, \]
\[ p^*_i \quad = \quad \text{position of the } i^{th} \text{ coordinate system with} \]
\[ \text{respect to previous coordinate frame}, \]
\[ p^*_0 \quad = \quad \text{the position of the first joint coordinate with} \]
\[ \text{respect to the body coordinate}, \]
\[ \omega_i, \dot{\omega}_i \quad = \quad \text{angular velocity and acceleration of link } i, \]
\[ v_i, \dot{v}_i = \text{linear velocity and acceleration of link } i, \]
\[ f_i = \text{force exerted on link } i \text{ by link } i - 1 \text{ at the coordinate frame } i - 1, \]
\[ n_i = \text{moment exerted in link } i \text{ by link } i - 1 \text{ at the coordinate frame } i - 1, \]
\[ J_i = \text{inertia matrix of link } i \text{ about its center of mass}, \]
\[ \tau_i = \text{joint torques}, \]
\[ g = \text{acceleration of gravity}, \]
\[ \dot{\theta}_i, \ddot{\theta}_i = \text{joint velocity and acceleration}. \]

The Newton-Euler equations of motion consist of a set of recursive equations. Each equation represents the Newton's third law and its rotational version. The kinematic variables of each individual link can be propagated forward from the body coordinate system to the foot. The subscripts 0, 4 and b are used to represent hips, feet and the body, respectively. The Newton-Euler equations for a quadruped can be summarized as follows [13]:

**Forward kinematic equations for each leg:**

\[
\dot{\omega}_i = \omega_{i-1} + z_i \ddot{\theta}_i + \omega_{i-1} \times (z_i \dot{\theta}),
\]
\[
\dot{v}_i = \dot{\omega}_i \times p_i^* + \omega_i \times (\omega_i \times p_i^*) + \dot{v}_{i-1},
\]
\[
\omega_i = \omega_{i-1} + z_i \dot{\theta}_i,
\]
\[
i = 1, 2, 3.
\]

**Recursive dynamic equations for each leg:**

\[
f_{i+1} = f_i - m_i(\dot{\omega}_i \times \dot{s}_i + \omega_i \times (\omega_i \times \dot{s}_i) + \dot{v}_i) + m_i g,
\]
\[ n_{i+1} = n_i - J_i \dot{\omega}_i - \omega_i \times J_i \dot{\omega}_i - (\ddot{s}_i + p_i^*) \times f_i + \ddot{s}_i \times f_{i+1}, \]  
\[ i = 1, 2, 3. \]  

Hip equations:

\[ \dot{\omega}_0 = \dot{\omega}_b, \]

\[ \dot{\mathbf{v}}_0 = \dot{\omega}_b \times p_0^* + \omega_b \times (\omega_b \times p_0^*) + \mathbf{\ddot{v}}_b. \]  

Body dynamic equations:

\[ \sum_{k=1}^{4} f_{1(k)} + m_b \mathbf{\ddot{v}}_b = m_b \mathbf{g}, \]

\[ \sum_{k=1}^{4} n_{1(k)} + p_{0(k)}^* \times f_{1(k)} + J_b \dot{\omega}_b = (J_b \dot{\omega}_b) \times \omega_b. \]  

Joint torques:

\[ \tau_i = n_i^T \mathbf{z}_i. \]

2.4.1 Foot Constraints

The interaction of the foot tip with the environment is an important factor of the closed-chain mechanism. In general, the legged status can be divided into two phases: in the air or on the ground. When legs are moving in the air, there are no forces and torques at the foot tip. Therefore, the foot constraints in this state are simply

\[ f_{ik} = 0, \]  
\[ n_{ik} = 0. \]

Once the legs are put on the ground, different foot constraints need to be specified. For the hard contact model with no slippage, one may assume that there are no translational accelerations occur at the foot tips; therefore,
Furthermore, when the vehicle is moving on a surface such as turf and soil, there are no external torques exerted at the foot tip. This condition is also applied to the hard level terrain as well. Based on this assumption,

\[ n_{4k} = 0. \]  

(2.9)

The constraints in (2.8) and (2.9) can be applied whenever the legs push or contact with the ground. However, once the contacts are broken or the legs start to lift off, the foot constraints must be switched to (2.6) and (2.7). The transition between these two phases needs to be carefully examined. For example, if the leg has been in contact with the ground, but the contact forces shows that the foot is grasping the ground, then physically the leg is starting to lift and the in-the-air conditions must be used. On the other hand, if a leg on the ground is believed to be lifting but instead its acceleration is found to be down into the ground, then the on-the-ground condition must be used. This process can be simply verified through the signs of \( f_{4k} \) and \( \dot{v}_{3k} \).

Another issue involved with the hard contact model is the impact between the foot and the ground. During the collision, the impulsive force generated at foot tips can result in discontinuous changes of joint velocities. Different techniques can be used to update the system state once a collision occurs. One simple approach is to assume that a foot simply stops when it contacts with the ground without any appreciable effect on the body. According to this assumption, the joint velocity \( (\dot{\theta}) \) after impact
can be obtained from the following kinematic equations:

\[
J \dot{\theta} = r \times \omega_b + v_b, \tag{2.10}
\]

where \( r \) is the vector from the body coordinate system to the foot and \( J \) represents the leg Jacobian matrix.

Another approach is to use the dynamic model to calculate impulsive forces and system states after the impact. This technique incorporates all dynamic effects due to the collision and therefore, provides a better approximation compared to the kinematic model. Further discussion in this area will be presented in Chapter 4 of this dissertation.

For the soft contact mode, the ground compliance model of springs and dampers can be applied. In this mode, the force constraint is defined in terms of positions and velocities of the foot tips. The same torque constraint as shown in (2.9) is still applied in this case. Since the contact forces are always specified through the current state of the foot, specific treatment for impact is not necessary for this mode. To identify whether the leg is in the air or on the ground, one may simply examine the vertical positions of the feet with respect to the ground. Because most of the previous simulations have been implemented based on this contact model, further developments in this area may not be necessary; rather, more emphasis should be put on the simulation with the hard contact mode.
2.5 Dynamic Formulation for the Cray Supercomputer

This section illustrates how the recursive formulation of each leg can systematically solved. Since the procedure is similar for each leg, the recursion can be implemented simultaneously. This property provides a parallel structure which is suitable for the supercomputer.

2.5.1 Recursive matrix formulation

Figure 4 displays the kinematic and dynamic variables for each leg based on the Newton-Euler formulation. The orders of these variables are arranged from the body to the foot tip. In this form, the hip coordinate system has been removed and the variables of link 1 are related directly to the body variables. The terms $h_1 \cdots h_{12}$ are associated with the kinematic and dynamic variables for each leg to emphasize the systematic recursive form.

The variables $\dot{v}_b, \omega_b, f_1, n_1$ and $\dot{\theta}$ can be considered as basis variables for each leg. Therefore, the rest of the variables can be recursively determined as

$$h_i = H_i \begin{bmatrix} \dot{v}_b \\ \omega_b \\ f_1 \\ n_1 \\ \dot{\theta} \end{bmatrix} + q_i$$

where $H_i$ is a $3 \times 15$ matrix and $q_i$ is a 3-length vector. The results can be further simplified into a compact form based on the new matrix $G_i$ which is defined as

$$G_i = \begin{bmatrix} H_i & |q_i \end{bmatrix}.$$
\[ h_1 : \quad \dot{\omega}_1 = \dot{\omega}_b + z_1 \dot{\theta}_1 + \omega_b \times z_1 \dot{\theta}_1 \]
\[ h_2 : \quad \dot{v}_1 = \dot{v}_b - p_0^* \times \omega_b - p_1^* \times \omega_1 + \omega_1 \times (\omega_1 \times p_1^*) + \omega_b \times (\omega_b \times p_0^*) \]
\[ h_3 : \quad f_2 = f_1 + (m_1 s_1 \times)\omega_1 - m_1 \dot{v}_1 - m_2(\omega_1 \times (\omega_1 \times s_1)) + m_1 g \]
\[ h_4 : \quad n_2 = n_1 - J_1 \dot{\omega}_1 - \omega_1 \times J_1 \omega_1 - (s_1 + p_1^*) \times f_1 + s_1 \times f_2 \]
\[ h_5 : \quad \dot{\omega}_2 = \dot{\omega}_1 + z_2 \dot{\theta}_2 + \omega_1 \times (z_2 \dot{\theta}_2) \]
\[ h_6 : \quad \dot{v}_2 = -p_2^* \times \omega_2 + \dot{v}_1 + \omega_2 \times (\omega_2 \times p_2^*) \]
\[ h_7 : \quad f_3 = f_2 + (m_2 s_2 \times)\omega_2 - m_2 \dot{v}_2 - m_2(\omega_2 \times (\omega_2 \times s_2)) + m_2 g \]
\[ h_8 : \quad n_3 = n_2 - J_2 \dot{\omega}_2 - \omega_2 \times J_2 \omega_2 - (s_2 + p_2^*) \times f_2 + s_2 \times f_3 \]
\[ h_9 : \quad \dot{\omega}_3 = \dot{\omega}_2 + z_3 \dot{\theta}_3 + \omega_2 \times (z_3 \dot{\theta}_3) \]
\[ h_{10} : \quad \dot{v}_3 = (-p_3^* \times)\omega_3 + \dot{v}_2 + \omega_3 \times (\omega_3 \times p_3^*) \]
\[ h_{11} : \quad f_4 = f_3 + (m_3 s_3 \times)\omega_3 - m_3 \dot{v}_3 - m_3(\omega_3 \times (\omega_3 \times s_3)) + m_3 g \]
\[ h_{12} : \quad n_4 = n_3 - J_3 \dot{\omega}_3 - \omega_3 \times J_3 \omega_3 - (s_3 + p_3^*) \times f_3 + s_3 \times f_4, \]

Figure 4: The dynamic and kinematic variables for each leg.
Based on the definitions of $h_1 \cdots h_{12}$ in Figure 4, $G_i$ in (2.12) can be determined as shown in Figure 5. Since the calculation is formulated in a matrix form, it is well suited for a supercomputer operation.

### 2.5.2 The System Equation

After the foot constraints are specified, the recursive dynamic and kinematic equations can be collected for each leg. This procedure would directly result in a $186 \times 186$ matrix equation for the four leg system. However, with the utilization of the recursion shown in Figure 5, the foot constraints can be directly expressed in terms of the basis variables and therefore result in a smaller $42 \times 42$ matrix equation. One may describe this system equation in a block-structured format as

$$
\begin{bmatrix}
A & B_1 & B_2 & B_3 & B_4 \\
C_1 & D_1 & 0 & 0 & 0 \\
C_2 & 0 & D_2 & 0 & 0 \\
C_3 & 0 & 0 & D_3 & 0 \\
C_4 & 0 & 0 & 0 & D_4 \\
\end{bmatrix}
\begin{bmatrix}
x_b \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
d_b \\
d_1 \\
d_2 \\
d_3 \\
d_4 \\
\end{bmatrix}
$$

(2.13)

where

$$
x_b = [\dot{v}_b \ \dot{\omega}_b]^T,
$$

$$
x_k = [f_{ik} \ n_{ik} \ \dot{\theta}_k]^T,
$$

$$
d_k = \text{constant}.
$$

In this form, the direct dynamics for each leg can be expressed in terms of the basis variables as follows:

$$
C_k x_b + D_k x_k = d_k.
$$

(2.14)
\[
G_1 = \begin{bmatrix} 0 & I & 0 & 0 \\ z_1 & 0 & 0 & 0 \end{bmatrix} \\
\times \begin{bmatrix} \omega_B \times \dot{z}_1 \end{bmatrix}
\]

\[
G_2 = -p_1^* \times G_1 + \begin{bmatrix} I \end{bmatrix} - \begin{bmatrix} -p_5^* \times \omega_3 \times (\omega_1 \times p_1^*) + \omega_b \times (\omega_b \times p_6) \end{bmatrix}
\]

\[
G_3 = (m_1 \bar{g}_1 \times)G_1 - m_1 G_2 + \begin{bmatrix} O_{3 \times 6} & I & O_{3 \times 6} \end{bmatrix} \begin{bmatrix} m_1(-\omega_1 \times (\omega_1 \times \bar{g}_1) + g) \end{bmatrix}
\]

\[
G_4 = -J_1 G_1 + \bar{g}_1 \times G_3 + \begin{bmatrix} O_{3 \times 6} \end{bmatrix} - (\bar{g}_1 + p_1^*) \times \begin{bmatrix} I \end{bmatrix} - \omega_1 \times J_1 \omega_1
\]

\[
G_5 = G_1 + \begin{bmatrix} O_{3 \times 12} & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & \omega_1 \times (z_1 \dot{\theta}_2) \end{bmatrix}
\]

\[
G_6 = G_2 - p_2^* \times G_5 + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} \omega_2 \times (\omega_2 \times p_2^*)
\]

\[
G_7 = G_3 + (m_2 \bar{g}_2 \times)G_5 - m_2 G_6 + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} m_2(-\omega_2 \times (\omega_2 \times \bar{g}_2) + g)
\]

\[
G_8 = -(\bar{g}_2 + p_2^*) \times G_3 + G_4 - J_2 G_5 + \bar{g}_2 \times G_7 + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} - \omega_2 \times J_2 \omega_2
\]

\[
G_9 = G_5 + \begin{bmatrix} O_{3 \times 12} & 0 \\ 0 & z_3 & 0 \\ 0 & 0 & \omega_2 \times (z_3 \dot{\theta}_3) \end{bmatrix}
\]

\[
G_{10} = G_6 + (-p_3^* \times)G_9 + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} \omega_3 \times (\omega_3 \times p_3^*)
\]

\[
G_{11} = G_7 + (m_3 \bar{g}_3 \times)G_9 - m_3 G_{10} + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} m_3(-\omega_3 \times (\omega_3 \times \bar{g}_3) + g)
\]

\[
G_{12} = -(\bar{g}_3 + p_3^*) \times G_7 + G_8 - J_3 G_9 + \bar{g}_3 \times G_{11} + \begin{bmatrix} O_{3 \times 15} \end{bmatrix} - \omega_3 \times J_3 \omega_3
\]

Figure 5: Recursive equations collapsed into a simple matrix form
Recognition of the block structured form of (2.13) allows the solution of the system to be even faster. From (2.14), $x_k$ can be determined by multiplying both sides of this equation with $D_k^{-1}$,

$$x_k = D_k^{-1}(d_k - C_kx_b).$$

(2.15)

Substituting each $x_k$ into (2.13), one then obtains the solution for $x_b$ as follows:

$$(A - \sum_{k=1}^{4} B_kD_k^{-1}C_k)x_b = d_b - \sum_{k=1}^{4} B_kD_k^{-1}d_k,$$

(2.16)

$$x_b = (A - \sum_{k=1}^{4} B_kD_k^{-1}C_k)^{-1}[d_b - \sum_{k=1}^{4} B_kD_k^{-1}d_k].$$

(2.17)

This transformation has simplified the first part of the problem to solving a $6 \times 6$ system. Based on the value of $x_b$ in (2.17), one must calculate the joint accelerations using (2.15).

Moreover, instead of computing $D_k^{-1}$ and $(A - \sum_{k=1}^{4} B_kD_k^{-1}C_k)^{-1}$ directly, (2.15) (2.17) can be solved by using Gaussian elimination. This procedure is implemented by first solving the system,

$$D_k\begin{bmatrix} q_k & Y_k \end{bmatrix} = \begin{bmatrix} d_k & C_k \end{bmatrix},$$

(2.18)

by Gaussian elimination to obtain

$$q_k = D_k^{-1}d_k,$$

(2.19)

and

$$Y_k = D_k^{-1}C_k.$$  

(2.20)

From these intermediate variables $q_k$ and $Y_k$, (2.16) can be simplified to
\[(A - \sum_{k=1}^{A} B_k Y_k) x_b = d_b - \sum_{k=1}^{A} B_k q_k\]  
(2.21)

which can also be solved by Gaussian elimination for \(x_b\). After \(x_b\) is specified, \(x_k\) in (2.15) is simply calculated as

\[x_k = q_k - Y_k x_b.\]  
(2.22)

Based on this approach, one can determine \(x_b\) in (2.17) by Gaussian eliminations on four 9 \(\times\) 9 and one 6 \(\times\) 6 matrix equations. Furthermore, since (2.18) can be solve separately for each leg, the Gaussian eliminations at this step could be implemented simultaneously. This property provides the parallel structure which is suitable for a supercomputer.

2.6 State Determination

The system state for this simulation can be defined as joint position and velocity, body translational and rotational velocity, body position and orientation. All of these variables except the body orientation can be simply obtained from direct integration of their acceleration terms. After the new joint velocities and positions are specified, the orientation and position of each leg can be updated through direct kinematics. Since the body orientation cannot be determined by integrating over its rotational rate, a special procedure is required to update this matrix.

One way to describe the body orientation is to use the standard roll(\(\phi\)), pitch(\(\theta\)) and yaw(\(\psi\)) Euler angles [12]. Based on this approach, the body orientation matrix can be described as
\[
R = \begin{bmatrix}
C\psi C\theta & C\psi S\theta \phi - S\psi C\phi & C\phi S\theta C\psi + S\psi S\phi \\
S\phi C\theta & S\psi S\theta \phi + C\psi C\phi & S\phi S\theta C\psi - S\psi S\phi \\
-S\theta & C\theta S\phi & C\theta C\phi
\end{bmatrix}
\]  

(2.23)

where

\[
R = \text{the body orientation matrix,}
\]

\[
\phi, \theta, \psi = \text{standard roll, pitch and yaw Euler angles,}
\]

\[
C\psi, S\psi = \cos \psi, \sin \psi,
\]

\[
C\theta, S\theta = \cos \theta, \sin \theta,
\]

\[
C\phi, S\phi = \cos \phi, \sin \phi.
\]

Once the angular velocity \(\omega_b\) is obtained, it can be transformed into these Euler angle rates as follows [14]:

\[
\dot{\psi} = (\omega_{by} \sin \phi + \omega_{bz} \cos \phi) \sec \theta
\]

(2.24)

\[
\dot{\theta} = \omega_{by} \cos \phi - \omega_{bz} \sin \phi
\]

(2.25)

\[
\dot{\phi} = \omega_{bz} + (\omega_{by} \sin \phi + \omega_{bz} \cos \phi) \tan \theta
\]

(2.26)

As a result, one can calculate the new Euler angles by direct integration of (2.24)-(2.26). Based on the updated \(\psi, \theta, \phi\), the new body orientation matrix is then specified using (2.23).

This approach is very straightforward and has been widely used in the past. However, since finite rotations are not commutative, the new body orientation cannot be obtained by simply updating the value of each Euler angle. This approach is valid when only slight changes in these Euler angles are assumed or small step sizes have
been used in the simulation. Under these conditions, the commutative property of
the rotation still holds [14] and one may apply (2.23) to calculate new orientation
matrix based on these updated Euler angles.

Another way to update the body orientation is developed in this work. The new
orientation matrix is updated based on a single rotation instead of three consecutive
Euler angle rotations as the previous approach. This rotational matrix is directly
obtained from the body angular velocity $\omega_b$ and the elapsed time. Based on this
approach, the new orientation matrix can be determined as follows:

$$R_{new} = R_{w,\phi}R_{old}$$  \hspace{1cm} (2.27)

where $R_{w,\phi}$ represents the rotational matrix due to body angular velocities. To de-
termin $R_{w,\phi}$ in (2.27), one may use the concept of rotation about an arbitrary axis
which is described in [12]:

$$R_{w,\phi} = \begin{bmatrix}
    w_z^2 V \phi + C \phi & w_z w_y V \phi - w_z S \phi & w_z w_z V \phi + w_y S \phi \\
    w_x w_y V \phi + w_z S \phi & w_y^2 V \phi + C \phi & w_y w_z V \phi - w_x S \phi \\
    w_x w_x V \phi - w_y S \phi & w_y w_z V \phi + w_z S \phi & w_z^2 V \phi + C \phi
\end{bmatrix}.$$  \hspace{1cm} (2.28)

The vector $\mathbf{w}$ in this equation represents a unit vector in the direction of $\omega_b$. The
angle $\phi$ in this equation is equal to $||\omega_b|| \Delta t$ and the term $V \phi$ corresponds to $1 - \cos \phi$. According to (2.27), the value of $\phi$ does not have to be small as compared to (2.23). Therefore, this approach will provide the exact body orientation for large time changes
if the rotational rate is constant. This result, however, cannot be obtained from Euler
angles unless the rotation occurs about one of its principal axes.
2.7 Integration Scheme

In order to obtain accurate simulation results without unnecessarily small step sizes, a precise integration scheme, the fourth-order Runge-Kutta, will be used in this work. Unlike Euler integration, this technique requires derivatives at four different points of the function for one complete step. Therefore, the next value of the function can be evaluated based on the weighted average of these derivatives.

The general form of the fourth-order Runge-Kutta for a scalar function $y$ can be expressed as follows [15]:

$$
\begin{align*}
\Delta t &= \text{the step size}, \\
f(t, y(t)) &= \frac{dy}{dt}, \\
p &= f(t_n, y(t_n)), \\
q &= f(t_n + \frac{\Delta t}{2}, y(t_n) + p \frac{\Delta t}{2}), \\
r &= f(t_n + \frac{\Delta t}{2}, y(t_n) + q \frac{\Delta t}{2}), \\
s &= f(t_n + \Delta t, y(t_n) + r \Delta t), \\
y(t_n + \Delta t) &= y(t_n) + \frac{1}{6}(p + 2q + 2r + s)\Delta t. 
\end{align*}
$$

(2.29)

The variables $p, q, r$ and $s$ in this equation represent the change of $y$ which is evaluated at $t_n$, twice at $t_n + \frac{\Delta t}{2}$ and $t_n + \Delta t$. As illustrated in [16], this technique can provide the approximation which is equivalent to using the first five terms of the Taylor series expansion. However, computationally only the first derivative of the function is required. Since the system states in this case are vectors, the variables $y, p, q, r, s$ in (2.29) will be referred to as a vector quantity.
In order to illustrate how this technique is incorporated in the dynamic simulation, one may introduce the intermediate variables \( u, \phi, \dot{u} \) and \( \ddot{u} \) where

\[
\begin{align*}
\mathbf{u} & = \begin{bmatrix} b_p^T & \theta^T \end{bmatrix}^T, \\
\phi & = \text{the net rotational angle of the body due to } \omega_b, \\
\dot{u} & = \begin{bmatrix} v_b^T & \dot{\theta}^T \end{bmatrix}^T, \\
\ddot{u} & = \begin{bmatrix} \ddot{v}_b^T & \ddot{\theta}^T \end{bmatrix}^T, \\
b_p & = \text{body position}, \\
\theta & = \text{the joint angles at each link}.
\end{align*}
\]

Based on these variables, one can further define the state \( y \) and its derivative \( \dot{y} \) as \( [u^T \ u \ \omega_b^T]^T \), and \( [\dot{u}^T \ \ddot{u} \ \ddot{\omega}_b]^T \) respectively. According to this assignment, the system state in the simulation is composed of the state \( y \) and the body orientation matrix \( R \).

Suppose \( t_n \) represents the current state of the vehicle. According to (2.29), the first step is to determine \( p \) which represents \( \dot{y} \) at time \( t_n \),

\[
\mathbf{p} = \begin{bmatrix} \dot{u}(t_n) \\
\ddot{u}(t_n) \\
\omega_b(t_n) \end{bmatrix}. \quad (2.30)
\]

This procedure requires solving the direct dynamics at the present time \( t_n \) in order to obtain \( \dot{u} \) and \( \ddot{u} \). The term \( u(t_n) \) at this stage represents the current body translational velocity and joint velocity of the vehicle.

The second step is to calculate \( q \) which represents \( \dot{y} \) at time \( t_n + \frac{\Delta t}{2} \),

\[
\mathbf{q} = \begin{bmatrix} \dot{u}(t_n + \frac{\Delta t}{2}) \\
\ddot{u}(t_n + \frac{\Delta t}{2}) \\
\omega_b(t_n + \frac{\Delta t}{2}) \end{bmatrix}. \quad (2.31)
\]
Unlike the previous step, this procedure requires the information of the state \( y \) plus the body orientation matrix \( R \) at \( t_n + \frac{\Delta t}{2} \). According to (2.30), the state \( y \) at \( t_n + \frac{\Delta t}{2} \) can be determined as follows:

\[
y(t_n + \frac{\Delta t}{2}) = y(t_n) + p \frac{\Delta t}{2}.
\] (2.32)

As described in Section 2.6, the body orientation matrix \( R \) at \( t_n + \frac{\Delta t}{2} \) can also be calculated as

\[
R_{t_n + \frac{\Delta t}{2}} = R_{\omega, \phi} R_{t_n}
\] (2.33)

where \( \phi \) is equal to \( \| \omega_b(t_n) \| \frac{\Delta t}{2} \) and \( R_{\omega, \phi} \) is defined as shown in (2.28). Once both \( y(t_n + \frac{\Delta t}{2}) \) and \( R(t_n + \frac{\Delta t}{2}) \) are specified, \( \ddot{u} \) and \( \ddot{\omega}_b \) in (2.31) can be obtained from solving system direct dynamics. The term \( \ddot{u}(t_n + \frac{\Delta t}{2}) \) at this stage is computed as

\[
\ddot{u}(t_n + \frac{\Delta t}{2}) = \ddot{u}(t_n) + \ddot{u}(t_n) \frac{\Delta t}{2}.
\] (2.34)

The next step is to compute \( r \) which also represents \( \dot{y} \) at \( t_n + \frac{\Delta t}{2} \),

\[
r = \begin{bmatrix}
\ddot{u}(t_n + \frac{\Delta t}{2}) \\
\ddot{u}(t_n + \frac{\Delta t}{2}) \\
\ddot{\omega}_b(t_n + \frac{\Delta t}{2})
\end{bmatrix}
\] (2.35)

However, both \( y(t_n + \frac{\Delta t}{2}) \) and \( R(t_n + \frac{\Delta t}{2}) \) at this stage are determined based on \( q \) in (2.31). This procedure can be implemented in the same manner as shown in (2.32) and (2.33). As a result, one is able to obtain \( r \) in (2.35) from (2.34) plus the solution of system direct dynamics.

The final step is to determine the vector \( s \) which represents \( \ddot{y} \) at \( t_n + \Delta t \),

\[
s = \begin{bmatrix}
\ddot{u}(t_n + \Delta t) \\
\ddot{u}(t_n + \Delta t) \\
\ddot{\omega}_b(t_n + \Delta t)
\end{bmatrix}
\] (2.36)
Besides changing the value of the step size, both \(\mathbf{y}(t_n + \Delta t)\) and \(R(t_n + \Delta t)\) at this stage are determined based on \(r\) in (2.35). The same procedure as described in the previous step can be used to obtain \(\tilde{u}, \dot{\omega}_b\) and \(\dot{u}\) in (2.36).

Based on the vectors \(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}\), the state \(\mathbf{y}\) at \(t_n + \Delta t\) is determined as follows:

\[
\mathbf{y}(t_n + \Delta t) = \mathbf{y}(t_n) + \frac{1}{6}(\mathbf{p} + 2\mathbf{q} + 2\mathbf{r} + \mathbf{s})\Delta t.
\]  

(2.37)

The body orientation matrix \(R\) at this time corresponds to

\[
R_{t_n + \Delta t} = R_w \phi R_{t_n}
\]  

(2.38)

where \(\phi\) is equal to \(\|\omega_b(t_n)\|\Delta t\) and \(R_w \phi\) is specified as shown in (2.28).

In order to apply this integration scheme to the simulation, the direct dynamics needs to be solved four times for one integration interval. Using the same step size, this may result in a large amount of computation time as compared to simple Euler integration. However, since the fourth-order Runge-Kutta is much more accurate than Euler integration, larger step sizes are possible. Because of this factor, the Runge-Kutta method can decrease computation time for the whole simulation.

One factor which needs to be considered before using this technique is the change of the leg status. When the leg starts to lift off the ground, its status is changed from on the ground to in the air. Therefore, one needs to switch the foot constraints in the direct dynamics from (2.8) and (2.9) to (2.6) and (2.7). Since the fourth-order Runge-Kutta requires the derivatives at four different points, it is possible that the function may be integrated over a discontinuity. In order to rectify this problem, a multi-scheme integration composed of the fourth-order Runge-Kutta and a modified
Euler is used in the simulation. Once the leg status changes during computation of these derivatives, the fourth-order Runge-Kutta is switched to the modified Euler or the second order Runge-Kutta scheme. The step size where this situation occurs is also subdivided in order to maintain the same accuracy. Based on this multi-scheme integration, the state of the system can be modeled as closely as possible to its actual value even considering the discontinuities associated with foot status changes.

2.8 Experiments on the Cray Supercomputer

Because of the large computational requirements and the parallel structure implicit in the formulation, a supercomputer is a good candidate for performing the simulation calculations. In order to get the solution as soon as possible, several features such as vectorization and parallelization have been applied to speed up the computation time. Unlike others techniques, these procedures can be implemented at the compiler level without entirely redesigning the algorithm. However, a few modifications need to be made in order to apply those features effectively.

The program has been written in FORTRAN 77 so that it can be fully vectorized on the supercomputer. Since the dynamic equations require many matrix calculations, the functions such as matrix multiplication and addition are written as subroutines where the vectorization can be directly applied.

Several experiments to test the effectiveness of the supercomputer formulation are illustrated in the following sections. In each test, a quadruped is programmed to trot at a constant speed of 10 feet/sec for a 1 second simulation time. This time is considered to be long enough so that the overhead of initialization is a small
percentage of the computation time. The performance for each subroutine will be
analyzed through the Flowtrace facility on the supercomputer.

2.8.1 Scalar Optimization and In-line Subroutines

This experiment is organized to evaluate the computation time for each routine in
order to select the best candidate for the vectorization. The original version of the
program running on the CRAY Y-MP8 without any options takes about 49 seconds.
A similar version running on the SUN 3-140 with one user takes approximately 900
seconds to complete. With the application of scalar optimization on the CRAY, the
computation time can be brought down to 18.898 seconds as shown in Figure 6.

From the Flowtrace analysis in this figure, the percentage of the computation
time for each routine can be easily examined. For example, a combination of the
initialization routine (SKINIT) and others control subroutines (DYNAMICB) use
only 5.6% which is not significant. The Gaussian elimination routines (SOLVERF,
GAUSSN) take a 10.3% share of the overall computation time. The most time con­
suming subroutine is the matrix multiplication (MATMUL) which uses a total of
52.98% of the computation time. Since the routine has been called for 144,745 times,
it is a logical place for testing the vectorization technique. Furthermore, because of
the large number of subroutine calls in this formulation, it is possible that a part of
the computation time has been spent for the overhead of each subroutine. In order
to minimize this effect, the in-line expansion of the low-level utility subroutines has
been implemented. This operation is able to reduce the computation time to 8.311
seconds as shown in Figure 7. However, the matrix multiplication routine which is
embedded into the main routines such as UPD, CSTATE and NEWDYN1, still causes a disproportionate share of the computation time.

2.8.2 Vectorization of the Matrix Multiplication Subroutine

Usually, matrix multiplication is composed of a pair of nested loop structure where the size of each loop depends on the dimension of each matrix. In general, it is convenient to use only one matrix multiplication subroutine where the size is specified as a parameter from the main program. This idea, however, is not suitable for the vectorization because only the inner-most loop will be vectorized. Even though the vectorization can be specified at the compiler level, it may not be effective unless some additional changes are made.

In order to fully utilize the vectorization on a supercomputer, another form of matrix multiplication is introduced. Since the formulation uses only a few sizes of matrices, specific subroutines could be made for each type of multiplication.

For example, the multiplication subroutine for a 3x3 matrix and a 3x15 matrix (mm315) as shown in Figure 8. Instead of using two loops based on the dimension of input matrices to perform the multiplication, the indices of rows and columns for both matrices are tied to recursive form based on parameters ir315 and ic315. According to this new format, the multiplication can be done in only one single loop and all 45 elements can be calculated in parallel. In order to allow this subroutine to be expanded in line, both ir315 and ic315 have to be stored in the calling program as shown in Figure 8. Similar modifications can be made for others matrix multiplication
<table>
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<th>Average T</th>
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**TOTAL** 18.898 1613849 Total calls

Figure 6: Flowtrace analysis of the simulation on Cray Y-MP8 using scalar optimization only.
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<td>2.475 (29.78%)</td>
<td>69496</td>
<td>&gt;</td>
</tr>
<tr>
<td>52 UPDATE2</td>
<td>0.041 (0.49%)</td>
<td>1167</td>
<td>&gt;</td>
</tr>
<tr>
<td>43 UPDF</td>
<td>0.119 (1.43%)</td>
<td>79424</td>
<td>&gt;</td>
</tr>
<tr>
<td>44 VARAD</td>
<td>0.190 (2.28%)</td>
<td>54604</td>
<td>&gt;</td>
</tr>
</tbody>
</table>

** TOTAL 8.311 316082 Total calls**

Figure 7: Flowtrace analysis of the simulation on Cray Y-MP8 using scalar optimization and in-line expansions of subroutines.
routines as well.

Figure 9 illustrates the Flowtrace analysis of the simulation based on new matrix multiplication subroutines, scalar optimization, shortloop vectorization and in-line expansions of subroutines. Unlike the previous experiment, both UPD and GAUSSN are also expanded in line with subroutines NEWDYN1 and SOLVERF. Since UPD requires a lot of matrix multiplications whose size is a $3 \times 3$ and a $3 \times 15$, the new subroutine mm315 as shown in Figure 8 can help improve the vectorization of this subroutine. According to the previous result in Figure 7, the computation time for both UPD and NEWDYN1 is 3.423 sec. After using in-line expansions and the new subroutine mm315, the computation time has been reduced to 1.541 sec. The same result is also obtained from the subroutine CSTATE which requires matrix multiplications whose sizes are $4 \times 4$. With the custom matrix multiplication like mm315, its computation time has been reduced from 0.947 sec to 0.557 sec. Furthermore, due to the in-line expansions of the subroutine GAUSSN, the computation time for both SOLVERF and GAUSSN also decrease from 1.907 sec to 1.846 sec. As a result, the computation time for the whole simulation can be brought down to 5.988 second based on new matrix multiplications and in-line expansions of subroutines. The comparisons of the computation time based on different options in each experiment are displayed in Table 1.

2.8.3 Suggested parallelization

Because the formulation is designed in such a way that computations on individual legs can be done in parallel, it is possible to lessen the computation time by using
c Calling Program
real a(3,3), b(3,15), c(3,15)
integer ir315(45), ic315(45)
data ir315/1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,
 * 1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,
 * 1,2,3/
data ic315/1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,6,6,6,6,7,7,7,
 * 8,8,8,9,9,9,10,10,10,11,11,11,12,12,12,
 * 13,13,13,14,14,14,15,15,15/
call mm315(a,b,c,3,3,15,ir315,ic315)

c SUBROUTINE
Subroutine mm315 amat, bmat, cmat, irow, jcol, kcol, ir, ic
amat(irowxjcol) times bmat(jcolxkcol) goes to cmat(irow x kcol)
c specialized to 3x3 times 3x15
real amat(irow,jcol), bmat(jcol,kcol)
real cmat(45)
integer ir(45), ic(45)
do 10 kk=1,45
k=ir(kk)
i=ic(kk)
cmat(kk)=amat(k,1)*bmat(1,i)+amat(k,2)*bmat(2,i)+
 x amat(k,3)*bmat(3,i)
10 continue
return
end

Figure 8: An example of a new matrix multiplication routine to fully utilize vectorization
### FlowTrace -- Alphabetized Summary

<table>
<thead>
<tr>
<th>Routine</th>
<th>Time Executing</th>
<th>Called</th>
<th>Average T</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 CROSS3</td>
<td>0.068 (1.14%)</td>
<td>34748</td>
<td>&gt;</td>
</tr>
<tr>
<td>37 CSTATE</td>
<td>0.557 (9.31%)</td>
<td>1383</td>
<td>&gt;</td>
</tr>
<tr>
<td>35 DYN</td>
<td>0.019 (0.32%)</td>
<td>200</td>
<td>&gt;</td>
</tr>
<tr>
<td>1 DYNAMICB</td>
<td>0.075 (1.25%)</td>
<td>1</td>
<td>0.075</td>
</tr>
<tr>
<td>2 DYNAMICB</td>
<td>0.002 (0.04%)</td>
<td>1</td>
<td>0.002</td>
</tr>
<tr>
<td>3 DYNAMICB</td>
<td>0.042 (0.70%)</td>
<td>1</td>
<td>0.042</td>
</tr>
<tr>
<td>5 DYNAMICB</td>
<td>0.005 (0.08%)</td>
<td>399</td>
<td>&gt;</td>
</tr>
<tr>
<td>6 DYNAMICB</td>
<td>0.051 (0.85%)</td>
<td>5533</td>
<td>&gt;</td>
</tr>
<tr>
<td>7 DYNAMICB</td>
<td>0.082 (1.36%)</td>
<td>16880</td>
<td>&gt;</td>
</tr>
<tr>
<td>8 DYNAMICB</td>
<td>0.004 (0.06%)</td>
<td>1608</td>
<td>&gt;</td>
</tr>
<tr>
<td>9 DYNAMICB</td>
<td>0.020 (0.34%)</td>
<td>200</td>
<td>&gt;</td>
</tr>
<tr>
<td>13 DYNAMICB</td>
<td>0.059 (0.98%)</td>
<td>18312</td>
<td>&gt;</td>
</tr>
<tr>
<td>19 DYNAMICB</td>
<td>&gt; (0.00%)</td>
<td>4</td>
<td>&gt;</td>
</tr>
<tr>
<td>20 DYNAMICB</td>
<td>0.065 (1.09%)</td>
<td>816</td>
<td>&gt;</td>
</tr>
<tr>
<td>23 DYNAMICB</td>
<td>0.100 (1.67%)</td>
<td>4824</td>
<td>&gt;</td>
</tr>
<tr>
<td>24 DYNAMICB</td>
<td>0.004 (0.06%)</td>
<td>816</td>
<td>&gt;</td>
</tr>
<tr>
<td>26 DYNAMICB</td>
<td>&gt; (0.00%)</td>
<td>4</td>
<td>&gt;</td>
</tr>
<tr>
<td>31 DYNAMICB</td>
<td>0.012 (0.21%)</td>
<td>200</td>
<td>&gt;</td>
</tr>
<tr>
<td>55 DYNAMICB</td>
<td>&gt; (0.01%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>56 DYNAMICB</td>
<td>0.001 (0.02%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>58 DYNAMICB</td>
<td>0.005 (0.09%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>59 DYNAMICB</td>
<td>0.022 (0.36%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>66 DYNAMICB</td>
<td>0.014 (0.23%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>69 DYNAMICB</td>
<td>0.260 (4.35%)</td>
<td>390</td>
<td>&gt;</td>
</tr>
<tr>
<td>74 DYNAMICB</td>
<td>0.004 (0.07%)</td>
<td>2730</td>
<td>&gt;</td>
</tr>
<tr>
<td>75 DYNAMICB</td>
<td>0.057 (0.95%)</td>
<td>3900</td>
<td>&gt;</td>
</tr>
<tr>
<td>76 DYNAMICB</td>
<td>0.016 (0.27%)</td>
<td>1560</td>
<td>&gt;</td>
</tr>
<tr>
<td>77 DYNAMICB</td>
<td>0.008 (0.15%)</td>
<td>4680</td>
<td>&gt;</td>
</tr>
<tr>
<td>78 DYNAMICB</td>
<td>0.078 (1.31%)</td>
<td>390</td>
<td>&gt;</td>
</tr>
<tr>
<td>65 FOOTFORC</td>
<td>0.003 (0.04%)</td>
<td>4</td>
<td>&gt;</td>
</tr>
<tr>
<td>64 HO</td>
<td>0.010 (0.17%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>50 ICMP</td>
<td>&gt; (0.01%)</td>
<td>490</td>
<td>&gt;</td>
</tr>
<tr>
<td>47 IMPACT</td>
<td>&gt; (0.01%)</td>
<td>9</td>
<td>&gt;</td>
</tr>
<tr>
<td>4 IMSLSTOR</td>
<td>&gt; (0.00%)</td>
<td>1</td>
<td>&gt;</td>
</tr>
<tr>
<td>38 INTDRV</td>
<td>0.016 (0.27%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>51 MEULER</td>
<td>0.055 (0.91%)</td>
<td>270</td>
<td>&gt;</td>
</tr>
<tr>
<td>42 NEWDYN1</td>
<td>1.541 (25.73%)</td>
<td>1229</td>
<td>0.001</td>
</tr>
<tr>
<td>62 REALIZEF</td>
<td>0.007 (0.11%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>40 RGN</td>
<td>0.012 (0.21%)</td>
<td>199</td>
<td>&gt;</td>
</tr>
<tr>
<td>36 SKINIT</td>
<td>0.009 (0.15%)</td>
<td>1</td>
<td>0.009</td>
</tr>
<tr>
<td>46 SOLVERF</td>
<td>1.946 (30.94%)</td>
<td>1308</td>
<td>0.001</td>
</tr>
<tr>
<td>65 TWFORCE</td>
<td>0.432 (7.22%)</td>
<td>199</td>
<td>0.002</td>
</tr>
<tr>
<td>49 UPDATE2</td>
<td>0.037 (0.06%)</td>
<td>1167</td>
<td>&gt;</td>
</tr>
<tr>
<td>43 UPDF</td>
<td>0.119 (1.99%)</td>
<td>79424</td>
<td>&gt;</td>
</tr>
<tr>
<td>44 VARAD</td>
<td>0.188 (3.14%)</td>
<td>54604</td>
<td>&gt;</td>
</tr>
</tbody>
</table>

**TOTAL** 8.988

**24037 Total calls**

Figure 9: Flowtrace analysis of the simulation on Cray Y-MP8 with new matrix multiplication, scalar optimization, shortloop vectorization and in-line expansions of subroutines.
microtasking on the CRAY. This feature will take advantage of the multiprocessors on the CRAY Y-MP8 which allow each processor to take a job separately. However, a major difficulty in running a program in this mode is the lack of a mechanism for running dedicated jobs at the Ohio Supercomputer Center. Without this facility, jobs are run on the number of available processors and the overhead of starting and stopping a process seems to overcome the potential advantage. More work remains to be done in this direction. Perhaps it would be possible to implement a special job queue for jobs needing a test in dedicated mode to be run only in the middle of the night.

2.9 Summary

The dynamic simulation of a quadruped walking machine is illustrated in this chapter. The direct dynamics of a quadruped is derived based on the recursive Newton-Euler equations. The formulation also provides flexibility where different foot constraints can be applied. This capability allows the simulation to perform in both hard and
soft contact mode. The original recursive formulation is modified into a compact matrix form which is very suitable to run on the supercomputer. As a result, the size of matrix system equation is reduced from $186 \times 186$ to $42 \times 42$ where its solution can be specified by Gaussian elimination of four $9 \times 9$ plus one $6 \times 6$ matrix equations. An adaptive integration scheme composed of the fourth-order Runge-Kutta and the modified Euler is utilized in order to provide both accuracy and flexibility when there is a discontinuous change in the legged status. Integration of rotational rates is also implemented from angular velocities to rotation matrices without resorting to Euler angles. Finally, a simulation of a trotting quadruped with the hard contact mode is tested on the CRAY supercomputer. With the applications of scalar optimization, vectorization and in-line expansions of subroutines the computation time for the whole simulation becomes eight times faster than its original run time without these options.
CHAPTER III

Verifying the Correctness of Dynamic Simulations

3.1 Introduction

Once simulation software is completely developed, the next step is to verify its correctness. This question has been mostly ignored since a simulation is assumed to be correct unless it obviously provides illogical performance. Unfortunately, some errors in the formulation can still exist even though the simulation may provide reasonable performance. This situation results from the fact that not all terms in the formulation are activated or are numerically dominant in a given situation. Similar questions can be made for other iterative robotic calculations such as solving inverse kinematics by using the Jacobian control. Since the Jacobian provides only a first-order approximation, it is not obvious how much error should be presented in the solution. Several principles which are applied to test such iterative programs have been proposed by Klein, Chu and Kittivatcharapong [17].

In this chapter, two techniques which can be used to justify the correctness of the dynamic simulation are presented. These tests are derived based on the conservation of energy and power in the system and knowing how these and kinematic quantities should converge with the order of the solution technique. Since the simulation
consists of direct dynamics and motion integration, both tests are necessary for debugging these algorithms. For instance, the change of power in the system represents instantaneous quantities. Thus, it is suitable for testing the correctness of direct dynamics.

The testing scheme based on conservation of energy is presented in Section 3.2. The application of this technique to the quadruped simulation is demonstrated in Section 3.3. The results obtained from different integration schemes are also considered in this section. Another testing scheme based on conservation of power is discussed in Section 3.4 and finally, the application of this test to the quadruped simulation is illustrated in Section 3.5.

### 3.2 The Energy Conservation Test

One way to verify the correctness of the simulation is by observing the change of energy in the system. According to the conservation of mechanical energy, the summation of potential and kinetic energies for a conservative system is constant. In other words, the total energy calculated at two different times should be the same if the energy is conserved. For a quadruped, the total energy should be conserved unless there is an impact between the foot and the ground in the hard contact model or any losses due to springs and dampers in the soft contact model. Therefore, one may apply this concept to the simulation with the hard contact mode by computing the change of system energy before any foot strikes the ground. The differences of system energy within this period of time as compared to its starting value should be equal to zero. The same principle is also applied in the soft contact mode once
the loss of energy in springs and dampers is included. However, since both positions and velocities are updated through numerical integration, the actual results obtained from the simulation may possess some error. If the simulation is implemented correctly, this error will only depend on the accuracy of the integration scheme which is predictable. Therefore, one is able to verify the correctness of dynamic simulations based on this error.

To illustrate how this concept is applied, first one needs to specify the energy function for this quadruped. In general, the energy of each link can be described as the summation of potential and kinetic energies as follows:

\[
E_i = m_i g^T h_{ci} + \frac{1}{2} m_i v_{ci}^T v_{ci} + \frac{1}{2} \omega_{ci}^T J_i \omega_{ci}
\]

(3.1)

where

\[
E_i = \text{the total energy of link } i,
\]

\[
v_{ci} = \text{translational velocities at center of gravity of link } i,
\]

\[
\omega_{ci} = \text{rotational velocities at center of gravity of link } i,
\]

\[
h_{ci} = \text{positions of the center of gravity of link } i.
\]

If the foot is on the ground in the soft contact mode, one must take into account the loss of energy due to springs and dampers at the contact. Furthermore, the change of energy caused by input joint torques needs to be included in (3.1). Because both positions and velocities are obtained from direct integration, the accuracy of (3.1) will depend on the integration scheme used in the simulation.
According to the Taylor series expansions, one may describe positions and velocities of each link at time $t_0 + \Delta t$ as follows:

$$h_{ci}(t_0 + \Delta t) = h_{ci}(t) + \frac{1}{1!} h'_{ci}(t)(\Delta t) + \frac{1}{2!} h''_{ci}(t)(\Delta t)^2 + \cdots,$$  \hspace{1cm} (3.2)

$$v_{ci}(t_0 + \Delta t) = v_{ci}(t) + \frac{1}{1!} v'_{ci}(t)(\Delta t) + \frac{1}{2!} v''_{ci}(t)(\Delta t)^2 + \cdots,$$  \hspace{1cm} (3.3)

$$\omega_{ci}(t_0 + \Delta t) = \omega_{ci}(t) + \frac{1}{1!} \omega'_{ci}(t)(\Delta t) + \frac{1}{2!} \omega''_{ci}(t)(\Delta t)^2 + \cdots.$$  \hspace{1cm} (3.4)

Unfortunately, the results obtained from the integration subroutine only represent a certain number of terms of these expansions. Therefore, (3.2)–(3.4) can be rewritten as

$$h_{ci}(t_0 + \Delta t) = h'_{ci}(t + \Delta t) + O(\Delta t^r),$$  \hspace{1cm} (3.5)

$$v_{ci}(t_0 + \Delta t) = v'_{ci}(t + \Delta t) + O(\Delta t^r),$$  \hspace{1cm} (3.6)

$$\omega_{ci}(t_0 + \Delta t) = \omega'_{ci}(t + \Delta t) + O(\Delta t^r)$$  \hspace{1cm} (3.7)

where the superscript $i$ indicates the quantity obtained from the integration. The term $O(\Delta t^r)$ in (3.5)–(3.7) represents the truncation error caused by the integration where $r$ can be considered as the order of this error. As mentioned earlier, if there are no mistakes in other algorithms, the error showing in the conservation of energy test will only come from the integration. Theoretically, this type of error can be specified through its rate of convergence. Therefore, unless the same convergence rate is obtained from the test, the dynamic simulation does not work correctly.

In order to determine the convergent rate of this error, one needs to examine how $O(\Delta t^r)$ incorporates into the system energy. For example, the kinetic energy at time $t_0 + \Delta t$ based on (3.6) may be written as follows:
Provided \( v_{ci} \) is not zero, for a small step size \( \Delta t \), the contribution from \( O(\Delta t^{2r}) \) is less significant compared to \( 2v_{ci}^{T}O(\Delta t^{r}) \). Therefore, (3.8) is simplified to

\[
\frac{1}{2}m_{i}v_{ci}^{T}v_{ci} = \frac{1}{2}m_{i}v_{ci}^{T}v_{ci} + R_{2}(\Delta t^{r})
\]

where \( R_{2}(\Delta t^{r}) \) represents \( 2v_{ci}^{T}O(\Delta t^{r}) \) in (3.8). Similar results can be specified for the potential energy and kinetic energy due to the rotation as follows:

\[
m_{i}g^{T}h_{ci} = m_{i}g^{T}h_{ci} + R_{1}(\Delta t^{r}),
\]

\[
\frac{1}{2}\omega_{ci}^{T}J_{i}\omega_{ci} = \frac{1}{2}\omega_{ci}^{T}J_{i}\omega_{ci} + R_{3}(\Delta t^{r}).
\]

As a result, the total energy for each link at \( t_{0} + \Delta t \) can be specified as

\[
E_{i} = E_{i}^{'} + (R_{1} + R_{2} + R_{3})\Delta t^{r},
\]

\[
E_{i} = E_{i}^{'} + K_{i}\Delta t^{r}.
\]

The term \( E_{i}^{'} \) in this equation is the actual value of \( E_{i} \) as calculated from the simulation. At this point, it is clear that the same order of error as appeared in the integration is found in the energy as well.

To verify the conservation of energy, one needs to subtract both sides of (3.12) from the initial energy \( E_{i0} \),

\[
E_{i0} - E_{i} = E_{i0} - E_{i}^{'} - K_{i}\Delta t^{r},
\]

\[
E_{i0} - E_{i}^{'} = K_{i}\Delta t^{r}.
\]
Since the system is conservative, the value of $E_{i0} - E_i$ should be equal to zero. According to (3.13), the difference of the system energy in one step can be described as

$$\Delta E = \sum_{i=1}^{n} K_i \Delta t^r = K \Delta t^r.$$  

(3.14)

Based on (3.14), one should expect the same order of error as appeared in the integration if the simulation functions correctly. The variable $r$ is sometimes known as the convergent rate. According to (3.14), $r$ can be determined from the slope of $\Delta E$ which is a straight line on the log-log scale. Based on this testing scheme, any error caused by direct dynamics and integration subroutines can be detected by examining the value of $r$.

### 3.3 Experiments on Energy Conservation Test

Some experiments on the conservation of energy test are illustrated in this section. Since an error in the formulation may appear as a disturbance, it is possible that this effect is partially removed by the actual controller. In order to prevent this misleading result and allow any form of error to occur, simple routine which sets the joint torques to zero is used instead of the normal control. This, on the other hand, simulates the situation where the vehicle simply collapses. Three integration methods are utilized in these experiments. Therefore, one is able to examine the convergent rate of each integration scheme as well as its accuracy.

The first integration scheme is the primitive Euler integration. This technique integrates the function using only first order approximation. The general form of this
integration scheme can be expressed as follows:

\[
\frac{dy}{dt} = f(t, y),
\]

\[
y_{j+1} = y_j + \hat{y}(t_j, y_j)\Delta t
\]  

(3.15)

where \( y \) is an arbitrary function and \( \Delta t \) is equal to \( t_{j+1} - t_j \). Since this technique can match only the first two terms of the Taylor series, both positions and velocities as obtained from the simulation should have the error of order 2. Therefore, if the simulation works correctly, the value of \( \Delta E \) in (3.14) will also possess the error of order 2.

The next integration scheme is based on the modified Euler method or the second-order Runge-Kutta. This technique uses an average of derivatives at two adjacent points \( t_j \) and \( t_{j+1} \) to calculate the change of \( y_j \). Therefore, it requires derivatives at two different points of the function. The formulation for this integration scheme can be summarized as follows:

\[
p = f(t_j, y_j),
\]

\[
q = f(t_{j+1}, y_j + p\Delta t),
\]

\[
y_{j+1} = y_j + \frac{1}{2}(p + q)\Delta t.
\]  

(3.16)

It has been shown in [16] that the truncation error of this integration is equal to \( O(\Delta t^3) \). Therefore, if this technique is used to update both positions and velocities in the simulation, it will generate the truncated error of order 3. Consequently, the same order of error should be obtained from (3.14) if the simulation is correct.

The last integration scheme is the fourth-order Runge-Kutta. This technique pro-
vides the best approximation of positions and velocities with higher order convergence rates as compared to the Euler and modified Euler methods. As mentioned in Section 2.7, the truncation error of this integration scheme is equal to $O(\Delta t^5)$. Thus, for the correct simulation, the change of energy as defined in (3.14) should possess the error of order 5.

Two non-trivial situations are presented in order to verify the correctness of the dynamic simulation. Instead of simply dropping the vehicle, the quadruped's initial conditions are falling and spinning around the y axis at the same time. This experiment allows the integration of all kinematic quantities including the orientation matrix to be tested simultaneously. The leg configuration as well as the initial angular velocity are listed in Figure 10. The time step size ($\Delta t$) used in this experiment is assigned as $\frac{1}{n}$ where $n$ is selected as 2, 4, 8 and 20. According to (3.14), one may describe $\Delta E$ in terms of $n$ as follows:

$$\Delta E = K \left( \frac{1}{n} \right)^r.$$  \hspace{1cm} (3.17)

After a number of corrections, the value of $\Delta E$ based on Euler integration, modified Euler and the fourth-order Runge-Kutta are displayed in Figure 10. Since these results are plotted on the log-log scale, the slope of these curves then represents the convergence rate $r$ as defined in (3.17). According to these graphs, the value of $r$ can be calculated as 2 for Euler integration, 3 for modified Euler and 5 for fourth-order Runge-Kutta. In this case, each rate of convergence is the same as specified in each integration scheme. The fastest convergence rate provides the best opportunity to detect any low order error generated from an incorrect simulation.
Figure 10: Changing of energy after one step integration for Euler, modified Euler and the fourth-order Runge-Kutta. The corresponding initial conditions are also listed in the legend.

<table>
<thead>
<tr>
<th>Leg</th>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( \theta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-90^\circ)</td>
<td>(0^\circ)</td>
<td>(-90^\circ)</td>
</tr>
<tr>
<td>2</td>
<td>(90^\circ)</td>
<td>(0^\circ)</td>
<td>(-90^\circ)</td>
</tr>
<tr>
<td>3</td>
<td>(-90^\circ)</td>
<td>(0^\circ)</td>
<td>(-90^\circ)</td>
</tr>
<tr>
<td>4</td>
<td>(90^\circ)</td>
<td>(0^\circ)</td>
<td>(-90^\circ)</td>
</tr>
</tbody>
</table>

\[ \mathbf{v}_b^r = (0 \ 0 \ 0) \]

\[ \mathbf{\omega}_b^r = (0 \ 0.2 \ 0) \]

\[ \mathbf{g}_b^r = (0 \ 0 \ 10) \]
Figure 10 also illustrates the accuracy of each integration scheme using in the simulation. According to these graphs, one can compare the step size which is required for each technique under the same accuracy. For example, the error caused by the fourth-order Runge-Kutta with step size of $\frac{1}{2}$ sec is approximately equal to the error generated by modified Euler with step size of $\frac{1}{20}$ sec. This implies that the number of steps for modified Euler are 10 times greater than those of the fourth-order Runge-Kutta with the same accuracy. If one compares how often direct dynamics need to be solved between the fourth-order Runge-Kutta and modified Euler, the corresponding ratio will be 1:5 in this case. Similar comparison can be made with the Euler integration as well.

The next experiment is to verify the correctness of the simulation with different foot constraints. In this case, the quadruped is programmed to have two front feet resting on the ground and two back feet in the air. The leg configuration is randomly assigned as shown in Figure 11. The simulation is set to operate in the hard contact mode where there are no accelerations for the foot on the ground. Since there are no initial velocities, the vehicle is simply falling down under gravity. Unlike the previous experiment, the error which accumulates within the entire period of simulation is considered instead. According to (3.14), this amount of error can be specified as

$$\Delta E(t_s) = nK\left(\frac{t_s}{n}\right)^r = K\left(\frac{t_s}{n}\right)^{r-1}$$

(3.18)

where $t_s$ represents the simulation time and $n$ is the number of steps. Therefore, the rate of convergence after integrating an entire interval will be one order less than the rate for a single step. Figure 11 displays the change of system energy as obtained
Figure 11: Changing of energy after $n$ steps using modified Euler and the fourth-order Runge-Kutta. The vehicle has two front feet resting on the ground and two back feet in the air. The initial conditions used in this experiment are also displayed in the legend.
from modified Euler and the fourth-order Runge-Kutta at the end of the simulation. The value of \( n \) is varied from 10 to 100 whereas the simulation time \( t_s \) is specified as 0.5 sec in this case. The convergent rates based on the slope of each curve can be seen as 2 for modified Euler and 4 for the fourth-order Runge-Kutta.

Another quantity which can be monitored is the position of a foot on the ground. According to the hard contact model, while a foot is on the ground, its acceleration is equal to zero. Because both feet are initially resting on the ground, their positions should be constant based on this contact model. Using this assumption, one can apply the same principle as described in the conservation of energy to verify the correctness of foot positions. However, since the position of a foot on the ground is specified by the joint positions, the result of this test can help verifying the correctness of kinematic calculations which use the body as a base.

Figure 12 illustrates the change in vertical positions of foot 1 as obtained from modified Euler and the fourth-order Runge-Kutta. Although the position of this foot is supposed to be constant for the entire simulation, integration errors from both integration schemes still occur. Both curves also represent straight lines on the log-log scale which are similar to those obtained from the conservation of energy test. Using the same principle, one may identify the convergent rate of each integration scheme based on the slope of these curves. In this case, both curves illustrate the correct slope of 2 for modified Euler and 4 for the fourth-order Runge-Kutta.

Before the correct results as shown in Figures 10, 11 and 12 were obtained, several mistakes in the formulation were detected in the early version of the program. For
Figure 12: Changing of the vertical foot position of leg 1 after \( n \) steps using modified Euler and the fourth-order Runge-Kutta.
instance, the convergence rate of a foot on the ground based on the fourth-order Runge-Kutta was only 1 instead of 5 for a single step. This error came from a computation of the foot velocity that was inconsistent with the body rotation rate.

3.4 The Power Conservation Test

Even though many errors in the direct dynamics and motion integration can be detected by examining the conservation of energy, they simply represent the accumulated quantities which occur in different parts of the algorithms. As a result, one may not be able to identify the source of these errors based on merely the conservation of energy. Since solving direct dynamics involves a very complex formulation as well as the matrix inversion, another testing scheme, which can be applied to this part of the simulation, is therefore necessary.

One principle which can be used is the conservation of power. According to this concept, the power flowing in and out of the conservative system is the same. Since power is instantaneous and not accumulated, one is able to use this concept for detecting any error from solving direct dynamics.

For a quadruped, the power input comes directly from the joint torques which then results in changing of the system kinetic and potential energies. Based on this definition, one is able to determine the power for each link in terms of its energy function as follows:

\[
  p_i = \frac{dE_i}{dt} = m_i g \mathbf{T} \mathbf{h}_{ci} + m_i \mathbf{v}_{ci}^T \mathbf{v}_{ci} + \frac{d}{dt} \left( \frac{1}{2} \mathbf{\omega}_{ci}^T \mathbf{J}_i \mathbf{\omega}_{ci} \right). \tag{3.19}
\]
The last term on the right-hand side can be rewritten as

\[
\frac{d}{dt} \left( \frac{1}{2} \omega_{ci}^T J_i \omega_{ci} \right) = \frac{1}{2} \omega_{ci}^T J_i \omega_{ci} + \frac{1}{2} \omega_{ci}^T \left( \frac{d}{dt} (J_i \omega_{ci}) \right),
\]

\[
= \frac{1}{2} \omega_{ci}^T J_i \omega_{ci} + \frac{1}{2} \omega_{ci}^T (J_i \omega_{ci} + \omega_{ci} \times J_i \omega_{ci}).
\] (3.20)

Since gravitational forces are in the z direction, the first term on the right-hand side of (3.19) can be written as

\[
m_i g^T h_{ci} = m_i g \dot{h}_{ci} = -m_i g v_{ci}.
\] (3.21)

where the minus sign indicates the opposition between \( h_{ci} \) and \( v_{ci} \) as specified in this simulation.

When a foot is on the ground in the soft contact mode, one needs to consider the power loss due to springs and dampers at the foot tip. This portion of power can be determined as

\[
p_{fi} = f_{ti}^T v_{3i}
\] (3.22)

where \( f_{ti} \) and \( v_{3i} \) represent the contact forces and velocities of foot \( i \). For the hard contact mode where a foot is simply resting on the ground, there will be no power loss at the foot tip. The power input at each joint can be simply defined as follows:

\[
p_{ti} = \tau_i \dot{\theta}_i.
\] (3.23)

One way to apply the power conservation test is to consider the difference between the power input and its result for the whole system. This procedure requires computing the summation of (3.19)–(3.23) for all mechanisms. If the direct dynamics
is formulated and solved correctly, the system power input and its rate of change in energy should be the same.

The other way to apply this concept is to consider the flow of power in each mechanism. This approach requires the information of all forces and torques applied to each link. Since the system is conservative, the contribution of power due to these forces and torques should be the same as the changing rate of kinetic and potential energies of that link. Using this approach, one is able examine the correctness of kinematic and dynamic variables of each link. Therefore, any mistake in the formulation can be localized.

For the real application, one may use the first approach to verify the correctness of the system direct dynamics and then utilize the second technique to identify any possible error in each mechanism.

3.5 Experiments on Power Conservation Test

Similar situations as implemented in the energy conservation test will be repeated in this section. The first experiment is to verify the correctness of direct dynamics when all feet are in the air. In this experiment, the quadruped is simply falling down with random translational and rotational body velocities as shown in Table 2. The leg configurations used in this experiment are also listed in this table. Random joint torques have been assigned to generate the input power to the system.

Table 3 displays the power input and the changing rate of energy for each leg as well as the vehicle’s body. These quantities are calculated right after the direct dynamics is solved in the simulation. Since the power input and the changing rate
Table 2: Input Parameters for testing of power conservation.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-90°</td>
<td>0°</td>
<td>-90°</td>
<td>1.0</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>90°</td>
<td>0°</td>
<td>-90°</td>
<td>2.0</td>
<td>2.0</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>-90°</td>
<td>0°</td>
<td>-90°</td>
<td>3.0</td>
<td>0.8</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>90°</td>
<td>0°</td>
<td>-90°</td>
<td>0.3</td>
<td>5.0</td>
<td>0.9</td>
</tr>
</tbody>
</table>

$v_f^T$ = (0.2 0.5 1.0)  
$\omega_f^T$ = (0.3 0.2 0.1)  
g$^T$ = (0 0 10)

Table 3: The power input and the changing rate of energy in a quadruped.

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Power input</th>
<th>Changing rate of energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body</td>
<td>0.0</td>
<td>-2.161325</td>
</tr>
<tr>
<td>leg 1</td>
<td>5.409E-4</td>
<td>0.399046</td>
</tr>
<tr>
<td>leg 2</td>
<td>2.025E-3</td>
<td>-0.103035</td>
</tr>
<tr>
<td>leg 3</td>
<td>7.260E-3</td>
<td>1.474355</td>
</tr>
<tr>
<td>leg 4</td>
<td>2.832E-3</td>
<td>2.832379</td>
</tr>
<tr>
<td>Total</td>
<td>1.266E-2</td>
<td>1.266E-2</td>
</tr>
</tbody>
</table>

of energy are the same, the system is indeed conservative. Therefore, the direct dynamics are correctly formulated and solved in this case.

The second experiment is set up to test the correctness of the formulation with different foot constraints. The vehicle has two front feet on the ground and two back feet in the air. Springs and dampers have been used in order to simulate the soft contact mode. The joint torques and other input parameters are randomly assigned as displayed in Table 4.
Table 4: Input Parameters for testing of power conservation with springs and dampers attached to the front feet.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0°</td>
<td>70.16°</td>
<td>-92.12°</td>
<td>0.2</td>
<td>0.7</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0°</td>
<td>68.59°</td>
<td>-94.14°</td>
<td>3.0</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0°</td>
<td>16.66°</td>
<td>-124.21°</td>
<td>0.1</td>
<td>2.0</td>
<td>0.9</td>
</tr>
<tr>
<td>4</td>
<td>0°</td>
<td>-94.19°</td>
<td>107.52°</td>
<td>1.0</td>
<td>1.1</td>
<td>0.8</td>
</tr>
</tbody>
</table>

$v_b^T$ = (0 0 0)

$\omega_b^T$ = (0 0.1 0)

$g_T$ = (0 0 10)

Spring constant 9000
Damper constant 464

The power input and the changing rate of energy for each mechanism are listed in Table 5. Unlike the previous experiment, the power loss due to springs and dampers needs to be included in this case. The system power input and its changing rate of energy are also equal in this case. Therefore, one may conclude that the direct dynamic with different foot constraints has been formulated and solved correctly in this experiment.

3.6 Summary

Two testing schemes which can help verifying the correctness of dynamic simulation are presented in this chapter. The first test is developed based on the conservation of energy whereas the second test is derived using the conservation of power. The energy conservation test deals with accumulated quantities such as velocities and
Table 5: The power input and the changing rate of energy in a quadruped with springs and dampers attached to the front feet.

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Power input</th>
<th>Changing of energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body</td>
<td>0.0</td>
<td>-5.616</td>
</tr>
<tr>
<td>leg 1</td>
<td>9.212E-3</td>
<td>-3.069</td>
</tr>
<tr>
<td>leg 2</td>
<td>0.139</td>
<td>-2.991</td>
</tr>
<tr>
<td>leg 3</td>
<td>-2.237E-3</td>
<td>-5.054E-3</td>
</tr>
<tr>
<td>leg 4</td>
<td>8.086E-5</td>
<td>0.116</td>
</tr>
<tr>
<td>Spring/Damper</td>
<td>0</td>
<td>11.7123</td>
</tr>
<tr>
<td>Total</td>
<td>.14613</td>
<td>.14613</td>
</tr>
</tbody>
</table>

positions. Therefore, it is very effective for testing the correctness of the motion integration. The power conservation test, on the other hand, is involved with forces and accelerations which represent instantaneous quantities. Therefore, it can be used to verify the correctness of direct dynamic as well as its solution. Since the simulation is composed of solving direct dynamic and doing motion integration, both tests are necessary for a complete examination.

General starting joint rotation rates and body translation and rotation rates are recommended. For instance, simply letting the vehicle fall from the air without rotation has an obvious answer but it generates no effects due to rotational terms in the simulation. The formulation also results in no high-order terms whose convergence rate can be checked.
CHAPTER IV

Impact Dynamics

4.1 Introduction

When a robot makes contact with the ground, the impulse generated at the contact may cause discontinuities in joint velocities. This problem is not only found in legged vehicles but also in general-purpose manipulators and multifingered robots as well. In order to simulate the motion of these robots once the impact occurs, an explicit relation between joint velocities and impulsive forces needs to be defined.

A simple way to include the impact into the simulation is by replacing each contact with springs and dampers. Based on this contact model, the force at the end effector is always specified in terms of its position and velocity. Therefore, the original closed chain can be decoupled into an open-chain system where the simulation can be done regardless of the impact. Although this technique is not complicated, it may lead to a slow algorithm when applied to a stiff surface. This disadvantage results from the integration subroutines requiring small step sizes for a large coefficient for springs and dampers used in the simulation.

In order to improve the computation time, a dynamic model which determine the effect of impulsive forces and joint velocities after impact is necessary for the hard
contact surface. According to this approach, the relationship of impulsive forces and joint velocities during impact can be expressed in a closed form which is called an impulsive formulation. Since the joint velocity after impact is directly obtained from this formulation, the step sizes used in the simulation can be selected according to integration accuracy requirement of the non-impulsive dynamics. However, because the dynamic equations of articulated bodies are very complicated, more effort needs to be put into the analysis as compared to the approach using springs and dampers.

In general, impact problems can be classified into two categories. The first category is the impact caused by a single contact. This type of collision occurs in most single-chain manipulators. The second category is impact which involves more than one contact point. This type of collision is found in legged vehicles and multifingered robots. The impulsive formulation for both types of collisions will be studied in this work. Since collision with a robot is mostly inelastic, the discussion will be restricted to this particular situation for both single and multiple contacts. For simplicity, sliding friction is not included in the model.

The content in this chapter is organized as follows: first, the previous work in this area is reviewed in Section 4.2. The impulsive formulation which can be used in a single collision is discussed in Section 4.3. The applications of this concept to a single rigid body and multiple rigid bodies are demonstrated in Section 4.3.1 to 4.3.5. The impact model which can be applied to multiple contacts are presented in Section 4.4. The first application of this impact model to a 2-D single rigid body is illustrated in Section 4.4.1. Section 4.4.2 and 4.4.3 illustrate the numerical examples
and sensitivity of the model under this circumstance. Section 4.4.4 explains how this principle can be applied to a 3-D single rigid body. The application of this impact model in a 3-D quadruped is presented in Section 4.4.5. Finally, numerical examples and sensitivity of the model in this case are demonstrated in Section 4.4.6 and 4.4.7.

4.2 Previous Work

The impact effect on a planar biped has been introduced by Zheng and Hemami [18]. The expression of joint velocities while the biped is landing on its heels and toes is presented. The formulation is then extended to a 3-D single-chain manipulators in their later work [19]. In both cases, one can determine the joint velocity directly from the impulsive formulation once the contact velocity after impact is specified. Different examples of the inelastic contact are also illustrated in both papers. However, it is still not clear whether the assumption of an inelastic collision is always applied to a single contact. The effect of the impact on a planar biped in a back somersault maneuver was also studied by Khosravi, Yurkovich and Hemami [20], [21]. The contact velocity after impact is assumed to be equal to zero. Therefore, the same approach as presented in [18] can be used to determine the impulsive forces during the landing phase of this biped.

Another application of this impulsive formulation in two robot arms was also presented by Zheng [22]. In this case, the impulsive forces are used to detect the alignment errors between two coordinating robot arms. The impulsive formulation for general manipulators is also proposed by Featherstone [23]. Unlike the form previously suggested by Zheng and Hemami [19], a general contact model based on the
The coefficient of restitution was utilized. According to this approach, one can determine the impulsive forces and joint velocities based on the value of this parameter.

Although the coefficient of restitution provides the relationship of contact velocities before and after impact, its application is still limited to simple systems such as single particles or symmetric objects. In other words, a contact model based on the coefficient of restitution may generate inconsistent results once it is applied to a more arbitrarily shaped system.

To illustrate this phenomena, one can consider a 2-D impact of a distributed mass system as shown in Figure 13. The system is composed of two particles connected by a rigid, massless rod. The mass of each particle is specified as $m_1$ and $m_2$ where $m_2$ is very much larger than $m_1$. The mass $m_1$ strikes the ground with an initial velocity of $v_1$ as shown in Figure 13a. The mass $m_2$, on the other hand, is simply rotating at the time of collision. The value of $v_1$ in this case is simply specified as $[1\ 1]^T$ m/sec.

Suppose the collision is perfectly elastic and the value of coefficient of restitution is equal to 1. Using the simple coefficient of restitution model [24], the velocity normal to the surface will be opposite the original normal component, and the lateral velocity component will be unchanged. Therefore, the velocity of both masses after impact will be in the direction of $v'_1$ as shown in Figure 13b. For this particular case, $v'_1$ can be determined as $[1\ -1]^T$ m/sec. Based on this velocity, one can calculate the kinetic energy of this system after impact as follows:

$$\frac{1}{2}m_1(v'_{1x}^2 + v'_{1y}^2) + \frac{1}{2}m_2(v'_{1x}^2 + v'_{1y}^2) = m_1 + m_2. \quad (4.1)$$

For an elastic impact, the kinetic energy of the system before and after impact should
Figure 13: A 2-D example of the impact in a distributed mass system showing the velocity of both masses. (a) before the collision, (b) after the collision.
be conserved. Because $m_2$ does not possess any initial velocity, the kinetic energy of this system before the collision is simply

$$\frac{1}{2} m_1 (v_{ix}^2 + v_{iy}^2) = m_1.$$  \hfill (4.2)

According to (4.1) and (4.2), $v'_1$ as obtained from the coefficient of restitution model can lead to inconsistent results in the system kinetic energy. For a consistent result, the velocity of $m_1$ after impact should be in the direction of $v''_1$ as shown in Figure 13b. In other words, $m_1$ simply bounces back with a velocity of $[-1 \ -1]^T$ m/sec while $m_2$ is simply rotating backwards.

This last problem, therefore, shows the inadequacy of a simple coefficient of restitution model when applied to a distributed mass system and motivates the development of a more justifiable technique.

Another aspect of the impact in a redundant manipulator is presented by Walker [25]. The impulsive force at the contact is expressed in terms of the manipulator's configuration. Due to the extra degree of freedom, different configuration can be selected to minimize the effect of the impulsive force.

An impulsive model for a quadruped running machine is proposed by Agrawal and Waldron [26]. The impact at each leg is modeled as impulses delivered to the vehicle's body. Based on the assumption that all the body's angles are small and there are no impulses in the lateral direction, the dynamic equation during impact can be simplified into a linear system. This approach is suitable when the running speed increases and the duration of the support phase is small. However, it requires an active control scheme to minimize the change of body's angles.
The impulsive formulation for a quadruped with the hard contact model is introduced by Rehman [27]. Unlike the model used by Agrawal and Waldron [26], the leg's mass and its moment of inertia are included. An inelastic contact is assumed once a foot strikes the ground. According to this impact model, the joint velocity after the collision can be determined by using the Newton-Euler formulation. However, due to the effect of impulsive forces, each foot may or may not stay on the ground after impact. Therefore, Rehman's assumption does not always provide a consistent solution. An example of a rigid body with two contact points in Section 4.4 will illustrate this phenomena. Another approach to determine impulsive forces to the multiple contacts case will also be discussed in Section 4.4.

4.3 The Impulsive Formulation for the Inelastic Impact with Single-Point Contact

The inelastic impact which occurs from a single-point contact is considered in this section. Unlike most of previous work, the consistency of the resulting motion for models of inelastic contact is also considered. In order to show that this type of impact is always possible for rigid bodies including the articulated bodies, five different examples in both 2-D and 3-D are demonstrated in the next sections. The examples are also classified from a very simple case to a complex one. Different techniques have been applied to demonstrate the results for different cases.
4.3.1 A Single Rigid Body in Two Dimensions

A 2-D example of a single rigid body with one-point contact is considered in this section. The system can be described as a single rigid body colliding with the ground as shown in Figure 14. The mass and the moment of inertia of this rigid body are equal to $m_A$ and $J_A$ whereas the linear and angular velocities at its center of gravity are represented by $v_A$ and $\dot{\theta}$.

First consider the dynamics where a collision is assumed to take a finite amount of time,

$$ F = m_A \dot{v}_A - m_A g, $$

(4.3)
\[ \tau = J_A \ddot{\theta} \hat{z} \quad (4.4) \]

where \( \hat{z} \) is a unit vector in the \( z \) direction. In converting to an impulsive formulation, one may integrate both sides of the above equations in an infinitesimally short time interval \( \Delta t \) and obtain [19]

\[
\begin{align*}
\lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} F \, dt &= \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} \dot{v}_A dt - \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} m_A g \, dt, \\
\lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} \tau \, dt &= \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} J_A \ddot{\theta} \hat{z} \, dt.
\end{align*}
\]

Because gravitational forces are finite during impact, the second term on the right of (4.5) becomes zero as \( \Delta t \to 0 \). Hence, Eqs. (4.5) and (4.6) can be simplified as follows:

\[
\begin{align*}
\lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} F \, dt &= m_A \left( v_A(t_0^+) - v_A(t_0^-) \right), \\
\lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} \tau \, dt &= J_A \left( \dot{\theta}(t_0^+) - \dot{\theta}(t_0^-) \right) \hat{z}.
\end{align*}
\]

The quantities on the left hand side of (4.7) and (4.8) represent the impulsive force \( (F_I) \) and torque \( (\tau_I) \) at the center of gravity of this object. By substituting \( F_I \) and \( \tau_I \) in (4.7) and (4.8), the impulsive formulas for this object are

\[
\begin{align*}
F_I &= m_A \Delta v_A, \\
\tau_I &= J_A \Delta \dot{\theta} \hat{z},
\end{align*}
\]

where \( \Delta v_A \) and \( \Delta \dot{\theta} \hat{z} \) are changes of linear and angular velocities. The velocity at \( B \) can be expressed in terms of the velocity at center of gravity as

\[ v_B = v_A + \dot{\theta} \hat{z} \times \mathbf{r}. \quad (4.11) \]
Therefore, the change of velocity at B during impact is

\[ \Delta \mathbf{v}_B = \Delta \mathbf{v}_A + (\Delta \dot{\theta} \hat{z}) \times \mathbf{r}. \]  

(4.12)

To determine an expression of the impulsive forces which generate these changes in velocities, the term \( \Delta \mathbf{v}_A \) in (4.9) can be replaced with its solution in (4.12) as follows:

\[ \mathbf{F}_I = m_A (\Delta \mathbf{v}_B - \Delta \dot{\theta} \hat{z} \times \mathbf{r}). \]  

(4.13)

Substituting for \( \Delta \dot{\theta} \hat{z} \) using (4.10), one can rewrite the above expression as

\[ \mathbf{F}_I = m_A (\Delta \mathbf{v}_B - \frac{\tau_I \times \mathbf{r}}{J_A}). \]  

(4.14)

Since impulsive torques at the center of gravity are

\[ \tau_I = \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} \mathbf{r} \times \mathbf{F} \, dt, \]

\[ = \mathbf{r} \times \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} \mathbf{F} \, dt, \]

\[ = \mathbf{r} \times \mathbf{F}_I, \]  

(4.15)

(4.14) can be written in terms of impulsive forces and contact velocities as follows:

\[ \mathbf{F}_I + \frac{m_A (\mathbf{r} \times \mathbf{F}_I) \times \mathbf{r}}{J_A} = m_A \Delta \mathbf{v}_B. \]  

(4.16)

In two dimensions, one can replace a triple vector product with an outer product as follows:

\[ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \hat{a})^T \mathbf{b} \]  

(4.17)

where the vectors \( \hat{a}, \hat{c} \) are defined as \([a_y - a_z]^T\) and \([c_y - c_z]^T\). According to (4.17), the impulsive force can be further simplified to

\[ \left( \frac{I}{m_A} + \frac{\bar{r} \bar{r}^T}{J_A} \right) \mathbf{F}_I = \Delta \mathbf{v}_B \]  

(4.18)
where $I$ represents a $2 \times 2$ identity matrix.

For an inelastic collision, the contact velocity after impact is equal to zero. Therefore, the change of velocity at $B$ due to this impulsive force is

$$
\Delta v_B = v_B(t_0^+) - v_B(t_0^-) = -v_B(t_0^-) \quad (4.19)
$$

In order for the contact point to stop after an impact, impulsive forces need to be in an opposite direction to contact velocities or the inner product between $F_I$ and $v_B(t_0)$ has to be less than zero. This condition can be verified by multiplying both sides of (4.18) with a vector $F_I^T$,

$$
F_I^T \left( \frac{J_A}{m_A} + \frac{\ddot{r}^T}{J_A} \right) F_I = F_I^T \Delta v_B. \quad (4.20)
$$

Substituting for $\Delta v_B$ in (4.20), one gets

$$
F_I^T v_B(t_0) = -F_I^T \left( \frac{J_A}{m_A} + \frac{\ddot{r}^T}{J_A} \right) F_I. \quad (4.21)
$$

In order for this expression to be negative for all $F_I$, $(\frac{J_A}{m_A} + \frac{\ddot{r}^T}{J_A})$ must be a positive definite matrix. By examining the principal minors of this matrix which are equal to $\frac{J_A}{m_A}$ and $\frac{\ddot{r}^T\ddot{r}}{J_A m_A}$, one can conclude that this matrix is positive definite. This condition, therefore, verifies the consistency of the inelastic contact which is used in this rigid body. Therefore, the result in (4.18) is always consistent for the impulsive force in this case.

### 4.3.2 Two Rigid Bodies in Two Dimensions

Before going to three-dimensional cases, another example of an impact of two rigid bodies is illustrated in this section. Unlike the previous example, both rigid bodies
are connected by a hinge to model an articulated structure in two dimensions as shown in Figure 15. These rigid bodies are moving toward the ground where rigid body A contacts with the ground at B. The mass and the inertia of each rigid bodies are equal to $m_A$, $J_A$, $m_D$ and $J_D$ in this case. The linear and angular velocities at their centers of gravity are represented by $v_A$, $v_D$ and $\dot{\theta}_A\hat{z}$, $\dot{\theta}_D\hat{z}$.

Based on the same assumptions as described in the previous example, the impulsive formulas for each object at the time of collision can be expressed as

$$F_{IB} - F_{IC} = m_A \Delta v_A, \quad (4.22)$$

$$F_{IC} = m_D \Delta v_D, \quad (4.23)$$

$$\tau_{IB} + \tau_{ICA} = J_A \Delta \dot{\theta}_A\hat{z}, \quad (4.24)$$

$$\tau_{ICD} = J_D \Delta \dot{\theta}_D\hat{z} \quad (4.25)$$

where

$F_{IB}$ = impulsive forces at the contact B,

$F_{IC}$ = transferred impulsive forces at the hinge,

$\tau_{IB}$ = impulsive torques at the center of gravity A due to $F_{IB}$,

$\tau_{ICA}$ = impulsive torques at the center of gravity A due to $F_{IC}$,

$\tau_{ICD}$ = impulsive torques at the center of gravity D due to $F_{IC}$,

The kinematic equations which describe changes of velocities in this case are

$$\Delta v_A = \Delta v_B - \Delta \dot{\theta}_A\hat{z} \times r_1, \quad (4.26)$$

$$\Delta v_C = \Delta v_A + \Delta \dot{\theta}_A\hat{z} \times r_3, \quad (4.27)$$

$$\Delta v_D = \Delta v_C - \Delta \dot{\theta}_D\hat{z} \times r_2. \quad (4.28)$$
Figure 15: A 2-D example of two rigid bodies with one-point contact. (a) Geometry of the problem, (b) the free body diagram.
where $\Delta v_C$ represents changing velocities at the hinge. By eliminating $\Delta v_C$ from (4.28), the relative changes of velocities between these rigid bodies can be expressed as follows:

$$\Delta v_D - \Delta v_A = \Delta \dot{A} \hat{z} \times r_3 - \Delta \dot{D} \hat{z} \times r_2.$$  

(4.29)

To derive an expression for impulsive forces in this case, one can replace $\Delta v_A, \Delta v_D$ in (4.26) and (4.29) with the impulsive forces described in (4.22) and (4.23),

$$\frac{F_{IB}}{m_A} - \frac{F_{IC}}{m_A} = \Delta v_B - \Delta \dot{A} \hat{z} \times r_1,$$

(4.30)

$$\frac{F_{IC}}{m_A} + \frac{F_{IC}}{m_D} - \frac{F_{IB}}{m_A} = \Delta \dot{A} \hat{z} \times r_3 - \Delta \dot{D} \hat{z} \times r_2.$$  

(4.31)

Next, substituting for $\Delta \dot{A} \hat{z}$ and $\Delta \dot{D} \hat{z}$ in (4.31) using the results in equations (4.24) and (4.25),

$$\frac{F_{IB}}{m_A} - \frac{F_{IC}}{m_A} = \Delta v_B - \frac{\tau_{IB}}{J_A} \times r_1 - \frac{\tau_{ICA}}{J_A} \times r_1,$$

(4.32)

$$\frac{F_{IC}}{m_A} + \frac{F_{IC}}{m_D} - \frac{F_{IB}}{m_A} = \frac{\tau_{IB}}{J_A} \times r_3 + \frac{\tau_{ICA}}{J_A} \times r_3 - \frac{\tau_{ICD}}{J_D} \times r_2.$$  

(4.33)

According to the same definition as described the previous example, impulsive torques $\tau_{IB}, \tau_{ICA}$ and $\tau_{ICD}$ can be described in terms of impulsive forces $F_{IB}$ and $F_{IC}$. Therefore, by substituting these impulsive torques in (4.32) and (4.33) the impulsive formulation for this compound object is given by

$$\frac{F_{IB}}{m_A} - \frac{F_{IC}}{m_A} = \Delta v_B - \frac{(r_1 \times F_{IB})}{J_A} \times r_1 + \frac{(r_3 \times F_{IC})}{J_A} \times r_1,$$

(4.34)

$$\frac{F_{IC}}{m_A} + \frac{F_{IC}}{m_D} - \frac{F_{IB}}{m_A} = \frac{(r_1 \times F_{IB})}{J_A} \times r_3 - \frac{(r_3 \times F_{IC})}{J_A} \times r_3 - \frac{(r_2 \times F_{IC})}{J_D} \times r_2.$$  

(4.35)

These equations can be simplified into a matrix form as
where vectors $\bar{r}_1, \bar{r}_2$ and $\bar{r}_3$ are defined as shown in (4.17). The relationship of impulsive forces and contact velocities in this case can be specified by the solution of $F_{IB}$ in (4.36). Therefore, by eliminating the term $F_{IC}$ in this equation, one can find the expression for $F_{IB}$ as follows:

\[
(A - BD^{-1}C)F_{IB} = \Delta v_B, \tag{4.37}
\]

\[
F_{IB} = (A - BD^{-1}C)^{-1}\Delta v_B. \tag{4.38}
\]

Since $\Delta v_B$ is equal to $-v_B(t_0)$, the impulsive force in (4.38) becomes

\[
F_{IB} = -(A - BD^{-1}C)^{-1}v_B(t_0). \tag{4.39}
\]

As described in the previous example, this impulsive force needs to be in the opposite direction to the contact velocity. This implies that an inner product between $F_{IB}$ and $v_B(t_0)$ must be negative or the matrix $(A - BD^{-1}C)^{-1}$ in (4.39) has to be positive definite. Since the inverse of a positive definite matrix is also positive definite, the matrix $(A - BD^{-1}C)$ will be examined to verify this property.

To show that $(A - BD^{-1}C)$ is a positive definite matrix, one can first express this matrix in terms of the adjoint and determinant of $D$ as follows:

\[
(A - BD^{-1}C) = \frac{(\det(D)A - B \ adj(D)C)}{\det(D)}. \tag{4.40}
\]
According to this expression, \((A - BD^{-1}C)\) will be positive definite if the nominator on the right hand side of (4.40) is positive definite and denominator on the right hand side of (4.40) is greater than zero.

Maple has been used to expand the denominator, \(\det(D)\) as shown in Figure 16. The terms can be recognized as a sum of squares and is always greater than zero which is displayed in the second half of this figure.

The next step is to verify that \((\det(D)A - B \adj(D)C)\) is a positive definite matrix. Because this matrix is complicated, to illustrate this property by calculating all principal minors as before might not be appropriate. Another way to demonstrate this property is by examining the quadratic form of \(y^T(\det(D)A - B \adj(D)C)y\) where \(y\) represents a general unit vector in two dimensions,
If this quadratic form is greater than zero for any non-zero vector \( y \), then
\[
\text{Figures 17 and 18 display the result of this quadratic form calculated by Maple. By grouping and factoring the terms with similar denominators, this expression can be rewritten as a sums of squares as shown in Figure 19. Once the quadratic form was expanded by Maple, it was necessary to do the grouping and factorizing manually. Since Maple is better at expanding than recognizing patterns, it was used to verify that the manual grouping was correct, however, by expanding the factored form.}
\]

In this form, it is clear that the quadratic form \( y^T(\text{det}(D)A - B \adj(D)C)y \) is always greater than zero. Therefore, the matrix \( (\text{det}(D)A - B \adj(D)C) \) is positive definite. This condition indicates an existence of impulsive forces which generate a consistent inelastic impact for this compound object. Therefore, the inelastic model can still be applied and the result in (4.39) can be used to calculate the impulsive force in this problem.

4.3.3 Three or More Rigid Bodies in Two Dimensions

A 2-D example of three articulated bodies with one-point contact is considered in this section. It will be shown that this generalizes to any number of rigid bodies. The system is composed of three rigid bodies A, B and C where rigid body A makes contact with the ground as shown in Figure 20. To determine the impulsive formulation for this system, one would like to apply the results obtained in Section 4.3.1 and
Figure 17: The quadratic form between \([\cos(x) \sin(x)]^T\) and \([\det(D)A - B \ adj(D)C]\) obtained from Maple.
Figure 18: The continuation of the result in Figure 17.
\[ \frac{[r_{1x}r_{2x}\sin(x) - r_{1y}r_{2y}\cos(x)]^2 + [r_{1x}r_{2y}\sin(x) - r_{1y}r_{2y}\cos(x)]^2}{i_1i_2m_2} \]

\[ \frac{[r_{1x}r_{2x}\sin(x) - r_{1y}r_{2y}\cos(x) + r_{2x}r_{3y}\cos(x) - r_{2y}r_{3x}\cos(x)]^2}{i_1i_2m_1} \]

\[ \frac{[r_{1x}r_{2y}\sin(x) - r_{1y}r_{2y}\cos(x) + r_{2x}r_{3y}\sin(x) - r_{2y}r_{3x}\sin(x)]^2}{i_1i_2m_1} \]

\[ \frac{[r_{1x}\sin(x) - r_{1y}\cos(x) - r_{3x}\sin(x) + r_{3y}\cos(x)]^2}{m_1^2i_1} \]

\[ \frac{[r_{1x}\sin(x) - r_{1y}\cos(x)]^2}{i_1m_2^2} + \frac{[r_{1x}\sin(x) - r_{1y}\cos(x) - r_{3x}\sin(x)]^2}{i_1m_1m_2} \]

\[ \frac{[r_{1x}\sin(x) - r_{1y}\cos(x) + r_{3y}\cos(x)]^2r_{3x}^2\cos^2(x) + r_{3y}^2\sin^2(x)}{i_1m_1m_2} \]

\[ \frac{r_{2x}^2 + r_{2y}^2 + [r_{2x}\sin(x) - r_{2y}\cos(x)]^2}{i_2m_1m_2} + \frac{1}{i_2m_1^2} + \frac{1}{m_1^2m_2} \]

Figure 19: The quadratic form between $[\cos(x)\sin(x)]^T$ and $[\det(D)A - B\quad \text{adj}(D)C]$ displayed as a sums of squares.
4.3.2. The idea is to simplify this system into a form of two articulated bodies whose impulsive formulation, as well as the consistency of the inelastic contact, is already determined.

One could imagine two ways to combine the three bodies of Figure 20 into 2 bodies. One could either combine the upper two bodies B and C into one body or combine the lower two bodies A and B into an equivalent body. It will be shown that the upper two bodies can be combined and will generate the same results as in the last section on two bodies hitting a surface. The other possibility of the lower two bodies being treated as a single body has the difficulty that unlike a single rigid body, the two external contact points can move with respect to each other.

According to this approach, rigid bodies B and C are replaced by a new rigid body D whose mass, moment of inertia and position of its center of gravity are defined as $m^*, J^*$ and $r^*$ as shown in Figure 20. Contact E transmits impulsive forces to object D, which is equivalent to objects B and C together, and therefore the expression of impulsive forces for both system at D can be specified by using the results in Sections 4.3.1 and 4.3.2. By equating these two expression, one can determine the parameters $m^*, J^*$ and $r^*$ for rigid body D. If these quantities are realizable, i.e. $m^*$ and $J^*$ are greater than zero, then the original three rigid bodies can be transformed into two rigid bodies where an inelastic contact is always consistent.

According to (4.18), the impulsive force at hinge E for the combined rigid body is expressed as

$$\begin{bmatrix}
\frac{1}{m^*} + \frac{r_{E}^2}{J^*} & -\frac{r_{E}^2}{J^*} \\
-\frac{r_{E}^2}{J^*} & \frac{1}{m^*} + \frac{r_{E}^2}{J^*}
\end{bmatrix}
\begin{bmatrix}
F_{IE_x}^* \\
F_{IE_y}^*
\end{bmatrix}
= \begin{bmatrix}
\Delta u_{E_x}^* \\
\Delta u_{E_y}^*
\end{bmatrix}.
$$

(4.42)
Figure 20: A 2-D example of a single contact in three rigid bodies where object B and C are considered as a single rigid body D.
The superscript * represents an effective quantity. On the other hand, based on (4.38) the impulsive force of rigid bodies B and C at the same point can be written as

\[
\begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\begin{bmatrix}
F_{IE_x} \\
F_{IE_y}
\end{bmatrix}
= 
\begin{bmatrix}
\Delta v_{E_x} \\
\Delta v_{E_y}
\end{bmatrix}
\]  

where the matrix on the left hand side is symmetric and positive definite. This implies that \(a\) and \(c\) are greater than zero. Because this combined object is supposed to provide the same impulsive effect as the original rigid bodies, the matrices in (4.42) and (4.43) should be equal,

\[
\begin{bmatrix}
\frac{1}{m^*} + \frac{r_y^2}{J_x} & -\frac{r_x r_y}{J^*} \\
-\frac{r_x r_y}{J^*} & \frac{1}{m^*} + \frac{r_x^2}{J_x}
\end{bmatrix}
= 
\begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\]  

or

\[
\frac{1}{m^*} + \frac{r_y^2}{J_x} = a,
\]

\[
-\frac{r_x r_y}{J^*} = b,
\]

\[
\frac{1}{m^*} + \frac{r_x^2}{J_x} = c.
\]

In this case, there are four unknowns and three equations. Therefore, one can pick any arbitrary positive number for \(J^*\) and redefine \(r_y'\) as \(\frac{r_y}{\sqrt{J_x}}\) and \(r_y'\) as \(\frac{r_y}{\sqrt{J_x}}\). Based on this new variables, (4.44) is rewritten as

\[
\begin{bmatrix}
\frac{1}{m^*} & 0 \\
0 & \frac{1}{m^*}
\end{bmatrix}
+ 
\begin{bmatrix}
r_y^2 & -r_x r_y' \\
r_y' r_x & r_x^2
\end{bmatrix}
= 
\begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\]  

To specify the expression of \(m^*\), one can first subtract \(\begin{bmatrix}
\frac{1}{m^*} & 0 \\
0 & \frac{1}{m^*}
\end{bmatrix}\) from both sides of (4.48),

\[
\begin{bmatrix}
r_y^2 & -r_x r_y' \\
r_y' r_x & r_x^2
\end{bmatrix}
= 
\begin{bmatrix}
a - \frac{1}{m^*} & b \\
b & c - \frac{1}{m^*}
\end{bmatrix}
\]  

(4.49)
It is clear that the determinant of the left hand side matrix is zero. Therefore, one can solve for \( m^* \) by computing the determinant of the right hand side matrix,

\[
\left( \frac{1}{m^*} \right)^2 - (a + c)\left( \frac{1}{m^*} \right) + (ac - b^2) = 0. \quad (4.50)
\]

Because (4.50) is a quadratic equation, its solutions can be expressed as

\[
\frac{1}{m^*} = \frac{a + c}{2} + \frac{\sqrt{(a + c)^2 - 4(ac - b^2)}}{2} \quad (4.51)
\]

or

\[
\frac{1}{m^*} = \frac{a + c}{2} - \frac{\sqrt{(a + c)^2 - 4(ac - b^2)}}{2}. \quad (4.52)
\]

The quantity inside the square root can also be rewritten as

\[
\frac{\sqrt{(a + c)^2 - 4(ac - b^2)}}{2} = \frac{\sqrt{(a - c)^2 + 4b^2}}{2}. \quad (4.53)
\]

According to (4.53), the solutions in (4.51) and (4.52) are real numbers. Furthermore, since both \( a \) and \( c \) are greater than zero and \( \frac{\sqrt{(a + c)^2 - 4(ac - b^2)}}{2} \) is less than \( \frac{\sqrt{(a + c)^2}}{2} \), the combined mass (\( m^* \)) in (4.51) and (4.52) is always a positive real number. So both solutions are physically acceptable with regarding to producing positive mass.

To examine which value of \( m^* \) is appropriate for the combined object, the last constraints on \( r'_x \) and \( r'_y \) are used. Here both \( r'_x \) and \( r'_y \) must also be real numbers. From (4.49), the expressions for \( r'_x \) and \( r'_y \) are described as

\[
r'_x = \sqrt{a - \frac{1}{m^*}}, \quad (4.54)
\]

\[
r'_y = \sqrt{c - \frac{1}{m^*}}. \quad (4.55)
\]
Both of these quantities will represent real numbers if \( a - \frac{1}{m^*} \) and \( c - \frac{1}{m^*} \) are greater than zero. These conditions can be verified by replacing \( \frac{1}{m^*} \) in the above equations with the solution in (4.52). As a result, (4.54) and (4.55) become

\[
\begin{align*}
    a - \frac{1}{m^*} &= \frac{a - c}{2} + \frac{\sqrt{(a - c)^2 + 4b^2}}{2}, \\
    c - \frac{1}{m^*} &= \frac{c - a}{2} + \frac{\sqrt{(c - a)^2 + 4b^2}}{2}
\end{align*}
\]

Based on this value of \( \frac{1}{m^*} \), the quantities inside the square root in (4.54) and (4.55) are always positive and \( r_x' \) and \( r_y' \) are both real numbers. On the other hand, the solution in (4.51) would be unacceptable because it would lead to complex distances.

Since \( m^*, J^* \) and \( r^* \) are always realizable, the original three articulated bodies can be simplified into two rigid bodies with one contact point. It should be noted that the equivalent quantities \( m^*, J^* \) and \( r^* \) are instantaneous quantities. Based on this simplified system, the consistency between impulsive forces and contact velocities is then verified as described in Section 4.3.2. The same process can be recursively applied to the case of four, five and \( n \) articulated bodies. As a result, one is able to verify the consistency of an inelastic impact in a 2-D single-chain rigid bodies with one contact point.

### 4.3.4 A Single Rigid Body in Three dimensions

To illustrate that inelastic contact can be used for general motion in 3-D, one may start with a 3-D example of a single rigid body with one-point contact as shown in Figure 21. During non-impulsive impact, both forces and torques at the center of
Figure 21: A 3-D example of a single rigid body with one-point contact showing the direction of impulsive forces and contact velocities.

Gravity of this rigid body are described as

\[ \mathbf{F} = m_A \dot{\mathbf{v}}_A - m_A \mathbf{g}, \quad (4.58) \]
\[ \mathbf{\tau} = J_A \dot{\omega}_A + \omega_A \times (J_A \omega_A) \quad (4.59) \]

where \( \omega_A \) is the angular velocity of this rigid body. Here, the linear and angular impulses can be obtained by integrating both sides of (4.58) and (4.59) within a small time interval \( \Delta t \),

\[ \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} \mathbf{F} \, dt = \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} m_A \dot{\mathbf{v}}_A \, dt - \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} m_A \mathbf{g} \, dt, \]
\[ \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} \mathbf{\tau} \, dt = \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} J_A \dot{\omega}_A \, dt + \lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} \omega_A \times (J_A \omega_A) \, dt. \]
\[
\lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} \omega_A \times (J_A \omega_A) dt.
\] (4.61)

Since angular velocities and gravitational forces are finite, the second terms on the right of (4.60) and (4.61) vanish as \( \Delta t \) goes to zero [18]. Therefore, (4.60) and (4.61) become

\[
\lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} F dt = m_A(v_A(t_0^+) - v_A(t_0^-)),
\] (4.62)

\[
\lim_{\Delta t \to 0} \int_{t_0}^{t_0 + \Delta t} \tau dt = J_A(\omega_A(t_0^+) - \omega_A(t_0^-)).
\] (4.63)

By replacing both terms on the left-hand side with impulsive forces and torques, one obtains similar results as described in (4.9) and (4.10)

\[
F_I = m_A \Delta v_A, \quad (4.64)
\]

\[
\tau_I = J_A \Delta \omega_A. \quad (4.65)
\]

During impact, the change of contact velocity of this rigid body is defined as

\[
\Delta v_B = \Delta v_A + \Delta \omega_A \times r.
\] (4.66)

Based on (4.64), (4.65) and (4.66), the impulsive force now becomes

\[
F_I = m_A \Delta v_B - m_A(J_A^{-1} \tau_I) \times r.
\] (4.67)

Since \( \tau_I \) can be defined in terms of \( F_I \) as shown in (4.15), the impulsive force as a function of contact velocities is

\[
F_I + m_A(J_A^{-1}(r \times F_I)) \times r = m_A \Delta v_B,
\] (4.68)

or
\[
\frac{\mathbf{F}_I}{m_A} - \mathbf{r} \times (J_A^{-1}(\mathbf{r} \times \mathbf{F}_I)) = \Delta \mathbf{v}_B.
\] (4.69)

In three dimensions, a cross product \( \mathbf{r} \times \mathbf{a} \) can be replaced by a matrix-vector product \( R \) where
\[
R = \begin{bmatrix}
0 & -r_z & r_y \\
 r_z & 0 & -r_x \\
-r_y & r_x & 0 \\
\end{bmatrix}.
\] (4.70)

In terms of this operator, the impulsive forces in (4.69) are
\[
\frac{\mathbf{F}_I}{m_A} - R(J_A^{-1}(RF_I)) = \Delta \mathbf{v}_B,
\] (4.71)
\[
(\frac{I}{m_A} - RJ_A^{-1}R)\mathbf{F}_I = \Delta \mathbf{v}_B.
\] (4.72)

Here, the notation \( I \) represents a 2 \( \times \) 2 identity matrix. In order to show that an inelastic contact is still applied for three dimensions, the matrix \( \frac{I}{m_A} - RJ_A^{-1}R \) has to be positive definite. Since \( J_A \) is positive definite and \( R^T \) is equal to \(-R\), the matrix \( \frac{I}{m_A} - RJ_A^{-1}R \) can be rewritten as
\[
\frac{I}{m_A} - RJ_A^{-1}R = \frac{I}{m_A} + R^TJ_A^{-\frac{1}{2}}J_A^{\frac{1}{2}}R,
\] (4.73)
\[
= \frac{I}{m_A} + A
\] (4.74)

where \( A \) is a positive semidefinite matrix. The eigenvalues of \( \frac{I}{m_A} - RJ_A^{-1}R \) satisfy
\[
(\frac{I}{m_A} + A)x_i = \lambda_ix_i.
\] (4.75)

The terms \( x_i \) and \( \lambda_i \) are respectively the orthonormal eigenvectors and the eigenvalues of \( \frac{I}{m_A} - RJ_A^{-1}R \). By multiplying both sides of (4.75) with \( x_i^T \), one finds that
\[
x_i^T\frac{I}{m_A}x_i + x_i^TAx_i = \lambda_i.
\] (4.76)
Because \( \frac{1}{m_A} \) is positive definite, the first term on the left hand side is always greater than zero. Moreover, since \( A \) is positive semidefinite, the second term on the left of (4.76) is greater than or equal to zero. Therefore, \( \lambda_i \) is always greater than zero and the matrix \( \left( \frac{1}{m_A} - RJ_A^{-1}R \right) \) is positive definite. This result verifies that it is possible to determine an impulsive force which can prevent the contact point from moving after impact. Hence, an inelastic model can still be applied for single rigid body in three dimensions.

### 4.3.5 Two or More Rigid Bodies in Three Dimensions

Before going to the general case of 3-D articulated bodies, a simpler mechanism consisting of two rigid bodies are used to demonstrate how the analysis should be implemented in this problem. After the consistency of an inelastic impact is verified for this simple mechanism, the same procedure can be extended to the case of \( n \) articulated bodies in three dimensions. Because both kinematics and dynamics of rigid bodies become more complicated in 3-D, the direct analysis based on Newton-Euler formulation may not be suitable for this problem. Therefore, a more systematic approach based on the Lagrangian formulation will be utilized to verify the consistency of an inelastic impact in this case.

Figure 22 displays the picture of two rigid bodies where rigid body A makes contact with the ground at B. The positions of the contact point from the center of gravity of rigid body A is represented by \( r_3 \) whereas the joint position from the center of gravity of both objects are specified by \( r_1 \) and \( r_2 \). The mass and the moment of inertia of these two rigid bodies are defined as \( m_1, J_1 \) and \( m_2, J_2 \). Both rigid bodies
Figure 22: A 3-D example of two rigid bodies with single-point contact where both positions and velocities at B are represented by joint positions and velocities of a massless manipulator during impact.

are connected by a rotary joint 7 where the arrow displays its positive direction.

In order to apply the Lagrangian formulation, one needs to define the generalized coordinates for these rigid bodies. This procedure can be implemented by modeling the contact point B at the time of collision as being the attachment point to a massless manipulator as shown in Figure 22. This manipulator consists of three prismatic joints whose axes are in the same direction as the world coordinate $x_w, y_w$ and $z_w$. The manipulator also possesses three rotary joints whose angular velocities are in the same directions as the rotational velocities at B. In Figure 22, Joints 1, 2 and 3 are prismatic joints where the arrow shows positive direction of travel. Joints 4, 5 and
6 are rotary joints where the arrow displays a positive angle according to the right hand rule. Based on this assumption, the whole system can be considered as a single open-chain mechanism during impact. Therefore, the system generalized coordinates can be defined in terms of these joint variables.

The general form of Lagrangian equation can be described as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \tau_i \quad (4.77)$$

where

- $L$ is a Lagrangian function which is equal to the differences of the kinetic and potential energies in the system,
- $q_i$ are generalized coordinates of the system,
- $\dot{q}_i$ is the first time derivative of $q_i$,
- $\tau_i$ are generalized forces or torques applied to the system at the coordinate $q_i$.

For this particular problem, seven generalized coordinates can be formed according to the number of joint variables. The variables $q_1 \cdots q_6$ represent the displacement of three prismatic joints and the angular position of three rotary joints of the manipulator. The last coordinate $q_7$ is defined as angular positions of the actual joint between both rigid bodies.

Since the displacement of all prismatic joints is equivalent to the position of contact B during impact, therefore $q_1, q_2, q_3$ can also be described as
\[
\begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \begin{bmatrix}
x_{Bx} \\
x_{By} \\
x_{Bz}
\end{bmatrix} = x_B
\]

where \(x_B\) represents the positions of contact B during impact. Furthermore, suppose the angles of three rotary joints of the massless manipulator as well as the actual joint are redefined as \(\theta_M\) and \(\theta\). Therefore, one may rewrite the generalized coordinates \(q_4 \cdots q_7\) based on these variables as

\[
\begin{bmatrix}
q_4 \\
q_5 \\
q_6 \\
q_7
\end{bmatrix} = \theta_M,
\]

(4.79)

\[
q_7 = \theta.
\]

(4.80)

Based on (4.78) – (4.80), the generalized coordinate \(q_1 \cdots q_7\) is now a vector \([x_B^T \; \theta_M^T \; \theta]^T\). Therefore, their derivatives \(\dot{q}_1 \cdots \dot{q}_7\) also correspond to \([\dot{x}_B^T \; \dot{\theta}_M^T \; \dot{\theta}]^T\) where \(\dot{\theta}_M\) is considered as the angular velocities of contact B (\(\omega_B\)) during impact.

According to these generalized coordinates, the Lagrangian formulation for this system can be written as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x_B} \right) - \frac{\partial L}{\partial x_B} = F_B,
\]

(4.81)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \theta_M} \right) - \frac{\partial L}{\partial \theta_M} = \tau_M,
\]

(4.82)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \theta} \right) - \frac{\partial L}{\partial \theta} = \tau
\]

(4.83)

where \(F_B, \tau_M, \tau\) are generalized forces and torques at each joint.

The kinetic and potential energies of these rigid bodies can be expressed as follows:

\[
k_A = \frac{1}{2} m_1 v_A^T v_A + \frac{1}{2} \omega_A^T J_1 \omega_A,
\]

(4.84)

\[
k_B = \frac{1}{2} m_2 v_C^T v_C + \frac{1}{2} \omega_C^T J_2 \omega_C,
\]

(4.85)
\[ p_A = m_1 g^T (x_B - r_3), \] (4.86)
\[ p_B = m_1 g^T (x_B + r_1 + r_2 - r_3). \] (4.87)

The variables \( v_A, \omega_A, v_C, \omega_C \) are translational and rotational velocities of both rigid bodies at the center of gravity with respect to the world. In order to define \( k_A \) and \( k_B \) in terms of generalized coordinates, one can apply the following kinematic relations:

\[ \omega_C = \omega_A + \dot{\theta} \hat{u}, \] (4.88)
\[ v_A = v_B - \omega_A \times r_3, \] (4.89)
\[ v_C = v_A + \omega_A \times r_1 + \omega_C \times r_2, \] (4.90)
\[ \omega_A = \omega_B = \dot{\theta} M, \] (4.91)
\[ \bar{x}_B = v_B. \] (4.92)

The vector \( \hat{u} \) in (4.88) represents the direction of \( \dot{\theta} \) in this case. Using (4.84), (4.85), and (4.88)–(4.92), one can determine the expressions of the kinetic energy in terms of generalized coordinates as follows:

\[ k_A = \frac{1}{2} m_1 \dot{x}_B^T \dot{x}_B + \frac{1}{2} m_1 \dot{\theta}_M^T R_3^T \dot{x}_B + \frac{1}{2} m_1 \dot{\theta}_M^T R_3 \dot{\theta}_M \]
\[ + \frac{1}{2} m_1 \dot{\theta}_M^T R_3^T R_3 \dot{\theta}_M + \frac{1}{2} \dot{\theta}_M^T J_1 \dot{\theta}_M, \] (4.93)
\[ k_B = \frac{1}{2} m_2 \dot{x}_B^T \dot{x}_B + \frac{1}{2} m_2 \dot{\theta}_M^T R_{312}^T \dot{x}_B + \frac{1}{2} m_2 \dot{\theta}_M^T R_{312} \dot{\theta}_M \]
\[ - \frac{1}{2} m_2 \dot{\theta}_M^T R_2^T \dot{x}_B - \frac{1}{2} m_2 \dot{x}_B^T R_2 \dot{\theta}_M + \frac{1}{2} m_2 \dot{\theta}_M^T R_{312}^T R_{312} \dot{\theta}_M \]
\[ - \frac{1}{2} m_2 \dot{\theta}_M^T R_2^T R_{312} \dot{\theta}_M - \frac{1}{2} m_2 \dot{\theta}_M^T R_{312}^T R_2 \dot{\theta}_M + \frac{1}{2} m_2 \dot{\theta}_M^T R_2^T R_2 \dot{\theta}_M \]
\[ + \frac{1}{2} \dot{\theta}_M^T J_2 \dot{\theta}_M + \frac{1}{2} \dot{\theta}_M^T J_2 \dot{\theta}_M + \frac{1}{2} \dot{\theta}_M^T J_2 \dot{\theta}_M + \frac{1}{2} \dot{\theta}_M^T J_2 \dot{\theta}_M. \] (4.94)

The matrices \( R_1, R_2, R_{312} \) are linear operators as defined in (4.70). Each of this
matrix represent the cross-product of vectors \( r_1 \times r_2 \times \) and \( r_{312} \times \) where \( r_{312} \) is equal to \( (r_3 - r_1 - r_2) \).

Based on these kinetic and potential energies, the terms on the left hand side of (4.81)–(4.83) can be written as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_B} \right) - \frac{\partial L}{\partial x_B} = (m_1 + m_2) \ddot{x}_B + (m_1 R_3 + m_2 R_{312}) \ddot{\theta}_M - m_2 R_2 \dot{\theta} \dot{u} + (m_1 \dot{r}_3 + m_2 \dot{r}_{312}) \times \dot{\theta}_M - m_2 \dot{r}_2 \times \dot{\theta} \dot{u}
\]

\[
- (m_1 + m_2) g,
\]

(4.95)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_M} \right) - \frac{\partial L}{\partial \theta_M} = (m_1 R_3^T + m_2 R_{312}^T) \ddot{x}_B + (m_1 R_3^T R_3 + m_2 R_{312}^T R_{312} + J_1 + J_2) \ddot{\theta}_M - (m_1 \dot{r}_3 + m_2 \dot{r}_{312}) \times \dot{x}_B - m_1 [\dot{r}_3 \times (r_3 \times \dot{\theta}_M)]
\]

\[
+ r_3 \times (\dot{r}_3 \times \dot{\theta}_M) - m_2 [\dot{r}_{312} \times (r_{312} \times \dot{\theta}_M)] + r_{312} \times (\dot{r}_{312} \times \dot{\theta}_M)] + \dot{\theta}_M \times (J_1 \dot{\theta}_M) + \dot{\theta}_M \times (J_2 \dot{\theta}_M)
\]

\[
+ m_2 [\dot{r}_{312} \times (r_2 \times \dot{\theta} \dot{z}) + r_{312} \times (\dot{r}_2 \times \dot{\theta} \dot{z})] + \dot{\theta} \dot{u} \times (J_2, \dot{\theta} \dot{u})
\]

(4.96)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \theta} \right) - \frac{\partial L}{\partial \theta} = -m_2 R_2^T \ddot{x}_B + (J_2 - m_2 R_2^T R_{312}) \ddot{\theta}_M + (J_2 + m_2 \dot{r}_2 \times \dot{r}_2 + \ddot{r} \times (\dot{r}_{312} \times \dot{\theta}_M)]
\]

\[
+ \dot{\theta}_M \times (J_2 \dot{\theta}_M) - m_2 [\dot{r}_2 \times (r_2 \times \dot{\theta} \dot{u})]
\]

\[
+ \dot{r}_2 \times (\dot{r}_2 \times \dot{\theta} \dot{u}) + \dot{\theta} \dot{u} \times (J_2 \dot{\theta} \dot{u})
\]

(4.97)

According to (4.95)–(4.97), the Lagrangian formulation for this system can be expressed in a matrix form as

\[
\begin{bmatrix}
F_B \\
\tau_M \\
\tau_D
\end{bmatrix} = D \begin{bmatrix}
\ddot{x}_B \\
\dot{\theta}_M \\
\dot{\theta} \dot{u}
\end{bmatrix} + h + c
\]

(4.98)
where

\[
\begin{align*}
\tau_D &= \text{the joint torque between two rigid bodies} \\
D &= \text{a } 9 \times 9 \text{ inertia matrix of these two rigid bodies,} \\
h &= \text{a } 9 \times 1 \text{ vector consists of all nonlinear terms which are the function of } \dot{x}_B, \dot{\theta}_M, \text{ and } \dot{\theta}\hat{u}, \\
c &= \text{the gravitational term which is equal to } [(m_1 + m_2)g^T \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}_6^T, \\
\end{align*}
\]

To determine the total impulse for this system, one may integrate both sides of (4.98) within a small period of time \( \Delta t \),

\[
\int_{t_0}^{t_0+\Delta t} \begin{bmatrix} F_B \\ \tau_M \\ \tau_D \end{bmatrix} dt = \int_{t_0}^{t_0+\Delta t} D \begin{bmatrix} \dot{x}_B \\ \dot{\theta}_M \\ \dot{\theta}\hat{u} \end{bmatrix} dt + \int_{t_0}^{t_0+\Delta t} [h + c] dt. \tag{4.99}
\]

Because the joint velocities of the manipulator and rigid body are finite, the second term on the right hand side of (4.99) becomes zero as \( \Delta t \to 0 \). Both \( \tau_M \) and \( \tau_D \) are assumed to be finite for a very small \( \Delta t \). The impulse caused by \( F_B \), however, becomes an impulsive force as \( \Delta t \to 0 \). Using this assumption, one can rewrite (4.99) as

\[
\begin{bmatrix} F_{IB} \\ 0 \\ 0 \end{bmatrix} = D \begin{bmatrix} \Delta \dot{x}_B \\ \Delta \theta_M \\ \Delta \dot{\theta}\hat{u} \end{bmatrix}. \tag{4.100}
\]

For an inelastic impact, the inner product between \( F_{IB} \) and \( \dot{x}_B(t_0^-) \) has to be negative. This condition can be verified by multiplying (4.100) with \( [\Delta \dot{x}_B^T \quad \Delta \theta_M^T \quad \Delta \dot{\theta}\hat{u}^T]^T \),

\[
\Delta \dot{x}_B^T F_{IB} = [\Delta \dot{x}_B \quad \Delta \theta_M \quad \Delta \dot{\theta}\hat{u}]^T D \begin{bmatrix} \Delta \dot{x}_B \\ \Delta \theta_M \\ \Delta \dot{\theta}\hat{u} \end{bmatrix}. \tag{4.101}
\]
Since $\Delta \dot{x}_B$ is equal to $-\dot{x}_B(t_0^-)$ for this impact model, the result in (4.101) will be consistent if $D$ is a positive definite matrix. According to the Lagrange-Euler formulation in (4.95) and (4.97), $D$ is equal to

$$
\begin{bmatrix}
(m_1 + m_2)I & m_1 R_3 + m_2 R_{312} & -m_2 R_2 \\
m_1 R_3^T + m_2 R_{312}^T & J_1 + J_2 + m_1 R_3^T R_3 + m_2 R_{312}^T R_{312} & J_2 - m_2 R_{312}^T R_2 \\
-m_2 R_2^T & J_2 - m_2 R_{312}^T R_{312} & J_2 + m_2 R_2^T R_2
\end{bmatrix}.
$$

To show that this matrix is positive definite, one needs to first redefine $D$ as

$$
D = E + F. \quad (4.102)
$$

where

$$
E = \begin{bmatrix}
m_1 I & m_1 R_3 \\
m_1 R_3^T & J_1 + J_2 + m_1 R_3^T R_3 & J_2 \\
0 & J_2 & J_2
\end{bmatrix},
$$

$$
F = \begin{bmatrix}
m_2 I & m_2 R_{312} & -m_2 R_2 \\
-m_2 R_{312}^T & m_2 R_{312}^T R_{312} & -m_2 R_{312}^T R_2 \\
-m_2 R_2^T & -m_2 R_2^T R_{312} & m_2 R_2^T R_2
\end{bmatrix}.
$$

In this form, both $E$ and $F$ can be factorized and (4.102) can be expressed in a product form as

$$
D = G^T G + H^T H \quad (4.103)
$$

where

$$
G^T = \begin{bmatrix}
0 & \sqrt{m_1 I} & 0 \\
\sqrt{J_1} & \sqrt{m_1 R_3^T} & \sqrt{J_2} \\
0 & 0 & \sqrt{J_2}
\end{bmatrix},
$$

$$
H^T = \begin{bmatrix}
-\sqrt{m_2 I} \\
\vdots \\
-\sqrt{m_2 R_{312}^T} \\
\vdots \\
\sqrt{m_2 R_2^T}
\end{bmatrix}.
Since $G$ is nonsingular, $E$ which is equal to $G^T G$ is a positive definite matrix [28]. The matrix $F$, however, is positive semidefinite. By applying the same procedure as shown in (4.75) and (4.76) to (4.102), one can verify that the eigenvalue of $D$ is always greater than zero. Thus, $D$ is a positive definite matrix. This result verifies the consistency of the impulsive forces which oppose the contact velocities of these rigid bodies. Therefore, an inelastic impact is also possible for two rigid bodies in three dimensions.

As previously mentioned, the same analysis can be extended to the general case of $n$ rigid bodies in three dimensions. Under this circumstance, the generalized coordinates can be represented by

$$\begin{bmatrix}
q_1 \\
\vdots \\
q_6 \\
q_7 \\
\vdots \\
q_{n+6}
\end{bmatrix} = \begin{bmatrix}
x_B \\
\theta_M \\
\dot{\theta}_1 \\
\vdots \\
\dot{\theta}_n
\end{bmatrix}$$

where $\theta_1 \cdots \theta_n$ represents the $n$ rotary joints of these rigid bodies. As a result, similar expression to (4.98) can be derived for this system,

$$\begin{bmatrix}
F_B \\
\tau_M \\
\tau_1 \\
\vdots \\
\tau_n
\end{bmatrix} = D \begin{bmatrix}
\ddot{x}_B \\
\ddot{\theta}_M \\
\ddot{\theta}_1 \dot{u}_1 \\
\vdots \\
\ddot{\theta}_n \dot{u}_n
\end{bmatrix} + h + c.$$  (4.105)

The matrix $D$ in (4.105) can be considered as the inertia matrix for this system. According to [23], the inertia matrix for a single open-chain mechanism is always symmetric and positive definite. Therefore, the assumption of impulsive forces which
oppose the contact velocities is still consistent in the case of general 3-D articulated bodies. Based on all of these examples, one can conclude that an inelastic impact is always possible for the problem of both 2-D and 3-D articulated bodies with a single-point contact.

4.4 The Impulsive Formulation for the Inelastic Impact with Multiple contacts

The collision which occurs in legged vehicles and multifingered robots will be studied in this section. One of the difficulties in this problem is how to justify the motion of each contact after the collision. This factor needs to be specified in order to determine the correct impulsive forces at each contact. When there was a single collision point, the previous sections showed that setting the contact point velocity to zero yields a consistent impulsive force. However, when there are multiple contact points, not all contact velocities will be zero after impact.

To demonstrate how this situation can occur, one may consider a 2-D example of a single rigid body with two-contact points as shown in Figure 23. Based on the Newton-Euler equations, the impulsive forces and the change of velocity at each contact can be described as

\[ F_{IA} + F_{IB} = m\Delta v_C, \]  
\[ r_A \times F_{IA} + r_B \times F_{IB} = J(\Delta \dot{\theta})\hat{z}, \]  
\[ \Delta v_A = \Delta v_C + (\Delta \dot{\theta})\hat{z} \times r_A, \]  
\[ \Delta v_B = \Delta v_C + (\Delta \dot{\theta})\hat{z} \times r_B. \]
Figure 23: A 2-D example of a single rigid body with two contact points showing the direction of impulsive forces and contact velocities.

From (4.106) – (4.109), one can express \( \mathbf{F}_I^A \) and \( \mathbf{F}_I^B \) in terms of \( \Delta \mathbf{v}^A \) and \( \Delta \mathbf{v}^B \) as follows:

\[
\begin{bmatrix}
\frac{l}{m} + \frac{r_{Ax}^T A}{J} & \frac{l}{m} + \frac{r_{Ax}^T B}{J} \\
\frac{l}{m} + \frac{r_{Bx}^T A}{J} & \frac{l}{m} + \frac{r_{Bx}^T B}{J}
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}_I^A \\
\mathbf{F}_I^B
\end{bmatrix}
= 
\begin{bmatrix}
\Delta \mathbf{v}^A \\
\Delta \mathbf{v}^B
\end{bmatrix}
\tag{4.110}
\]

where \( r_A \) and \( r_B \) are defined in (4.17). The coefficient matrix on the left hand side of (4.110) can be expanded as

\[
\begin{bmatrix}
\frac{1}{m} + \frac{r_{Ax}^T A}{J} & -\frac{r_{Ax}^T A}{J} & \frac{1}{m} + \frac{r_{Ax}^T B}{J} & -\frac{r_{Ax}^T B}{J} \\
-\frac{r_{Ax}^T A}{J} & \frac{1}{m} + \frac{r_{Bx}^T A}{J} & -\frac{r_{Ax}^T B}{J} & \frac{1}{m} + \frac{r_{Bx}^T B}{J} \\
\frac{1}{m} + \frac{r_{Bx}^T A}{J} & -\frac{r_{Bx}^T A}{J} & \frac{1}{m} + \frac{r_{Bx}^T B}{J} & -\frac{r_{Bx}^T B}{J} \\
-\frac{r_{Bx}^T A}{J} & \frac{1}{m} + \frac{r_{Ax}^T B}{J} & -\frac{r_{Bx}^T B}{J} & \frac{1}{m} + \frac{r_{Ax}^T B}{J}
\end{bmatrix}
\]

For this example, \( r_{Ay} \) is equal to \( r_{By} \). Therefore, rows 1 and 3 as well as columns 1 and 3 of this coefficient matrix are identical. This result illustrates the rank deficiency of the system equation in (4.110). Furthermore, it also implies that \( \Delta v_{Ax} = \Delta v_{Bx} \). Therefore, without loss of generality, we can set \( F_{IAx} \) and \( F_{IBx} \) equal and treat as a
single variable. The homogeneous solutions for this problem can be simply expressed as the impulsive forces which are equal and opposite along the line between A and B. Based on these properties, the original 4 x 4 system equation in (4.110) is simply reduced to the following 3 x 3 system,

\[
\begin{bmatrix}
\frac{1}{m} + \frac{r_A^2}{J} & -\frac{r_A r_B}{J} & -\frac{r_A r_B}{J} \\
-\frac{r_A r_B}{J} & \frac{1}{m} + \frac{r_B^2}{J} & -\frac{r_A r_B}{J} \\
-\frac{r_A r_B}{J} & -\frac{r_A r_B}{J} & \frac{1}{m} + \frac{r_B^2}{J}
\end{bmatrix}
\begin{bmatrix}
F_{I Ax} \\
F_{I Ay} \\
F_{IBy}
\end{bmatrix} =
\begin{bmatrix}
\Delta u_{Ax} \\
\Delta u_{Ay} \\
\Delta u_{By}
\end{bmatrix}.
\]

One might suppose that this two contact problem can be solved very similarly to the one contact problem. Before we assumed that there was an impulsive force of a size and direction to stop the contact point and we were able to show that this impulsive force opposed the original contact velocity. This assumption, however, does not generalize to the two contact case. In order for (4.111) to always be consistent, with regard to all physical choices of \(m, \dot{r}_A, \dot{r}_B, J, v_A(t_0^-), \) and \(v_B(t_0^-), \) both \(F_{IAx} v_A(t_0^-) \) and \(F_{IBy} v_B(t_0^-) \) must be negative. The following numerical example will demonstrate a contradiction.

Suppose the mass and moment of inertia of this rigid body are specified as 1 and 8. The positions of contact points with respect to the center of gravity are \([-5 \ 1]^T \) and \([-1 \ 1]^T \). In other words, the center of gravity of the rigid body is located on the right hand side of both contact points A and B. Let's also assume that this rigid body simply falls down and strikes a planar surface at A and B with a velocity of \([0 \ 1]^T \) m/sec.

Based on these parameters, one can evaluate the system equation in (4.111) as follows:
For an inelastic contact at A and B, $\Delta v_A$ and $\Delta v_B$ must be equal to $-v_A(t_0^-)$ and $-v_B(t_0^-)$. In this case,

$$\Delta v_A = \Delta v_B = [0 \ -1]^T.$$  \hspace{1cm} (4.113)

Using (4.112) and (4.113), one can determine the impulsive force as

$$\begin{bmatrix} F_{IAx} \\ F_{IAY} \\ F_{IBy} \end{bmatrix} = \begin{bmatrix} 1.125 & 0.625 & 0.125 \\ 0.625 & 4.125 & 1.625 \\ 0.125 & 1.625 & 1.125 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \\ -1.25 \end{bmatrix}. \hspace{1cm} (4.114)$$

According to (4.114), the inner product $F_{IA}^T v_A(t_0^-)$ is equal to 0.25 which is positive. This result illustrates the contradiction of the previous assumption that $F_{IA}^T v_A(t_0^-)$ must be less than zero. Therefore, one cannot find a consistent impulsive force which opposes the motion of both contacts simultaneously in this case.

As a result of this example, the impact model proposed by Rehman [27], which assumes the existence of impulsive forces that stops all contact velocities, is not always true for multiple contact cases. Moreover, since $F_{IA}$ as computed in (4.114) indicates a pulling effect, the actual motion of A in this case should be off the ground after impact. This result also demonstrates the existence of impulsive forces at contacts which leave the ground. Therefore, the second assumption proposed by Rehman, which assumes that each contact may leave the ground only with a zero impulsive force, is not consistent.

In order to derive the impulsive formulation for multiple contact points, a consistent way to express both impulsive forces and contact velocities is necessary. One
possible approach to model an inelastic contact is by using a damper. According to this impact model, it is possible to derive the consistent non-impulsive relationship between contact forces and velocities for this problem. Therefore, by making the damper stiffer to the limit of infinity, one can determine the impulsive formulation based on these non-impulsive contact forces.

Although this approach is very straightforward, special attention needs to be made on how and when to include the damper into the system. For consistent results, the damper should be put at the contact point once the impact occurs and removed whenever the contact is separated. Therefore, one should assign the damper at the position where this criterion can be easily examined. One simple approach is to designate the damper in the opposite direction of contact velocities. This method, however, can lead to an ambiguous situation when the contact is broken and the damper must be removed from the system.

Another possible approach is to designate a damper in the direction normal to the contact surface. Since the changes in normal contact velocities happen a lot faster than in the horizontal velocities, special attention should be focused on the amount of impulsive force generated along the normal direction. Based on this impact model, a consistent criterion for including and removing dampers can be established with respect to the contact velocity along this direction. For instance, the contact is broken whenever its normal component of velocity is leaving the surface. However, because there are no dampers along the surface, each contact point is free to move in this direction.
In order to illustrate how this technique is used determine the impulsive formulation for multi-contact cases, first a 2-D example of a single rigid body with two contact points will be considered. The discussion is then extended to the general case of multiple contacts of a 3-D quadruped walking machine.

### 4.4.1 A 2-D Single Rigid Body with Two Contact Points

The impact of a single rigid body in Figure 23 is reconsidered in this section. The contact surface is assumed to be horizontal. Therefore, an impact model consisting of two vertical dampers is used in this case. For this rigid body, there are several impact cases. The first case is when both A and B strike the ground at the same time and the second one is when either A or B makes a contact while the other is resting on the ground.

**Case I. Both contacts strike the ground simultaneously**

When both points contact the ground simultaneously, the forces and torques at the center of gravity can be described as

\[
F_A + F_B = m \dot{v}_C, \quad (4.115)
\]

\[
r_A \times F_A + r_B \times F_B = J \ddot{\theta}. \quad (4.116)
\]

These forces are generated by the damper at each contact and are defined as

\[
F_A = \begin{bmatrix} 0 \\ -\beta v_{Ay} \end{bmatrix}, \quad (4.117)
\]

\[
F_B = \begin{bmatrix} 0 \\ -\beta v_{By} \end{bmatrix} \quad (4.118)
\]
where $\beta$ represents the damping constant at each contact. Based on the same assumption used in the single-contact case, the gravitational force of this rigid body can be neglected during the collision. Substituting (4.117) and (4.118) into (4.115) and (4.116), one obtains
\[
\begin{align*}
\dot{v}_{Cz} &= 0, \quad \text{(4.119)} \\
-\beta v_{Ay} - \beta v_{By} &= m\dot{v}_{Cy}, \quad \text{(4.120)} \\
-\beta v_{Ay} \tau_{Ax} - \beta v_{By} \tau_{Bx} &= J\ddot{\theta}. \quad \text{(4.121)}
\end{align*}
\]

The kinematic equations for this rigid body during impact are described as
\[
\begin{align*}
\dot{v}_A &= \dot{v}_C + (\ddot{\theta}z) \times r_A, \quad \text{(4.122)} \\
\dot{v}_B &= \dot{v}_C + (\ddot{\theta}z) \times r_B. \quad \text{(4.123)}
\end{align*}
\]
In this case, $(\ddot{\theta}z) \times r$ is specified as $[-r_y\ddot{\theta} \quad r_x\ddot{\theta}]^T$. Therefore, one can rewrite (4.122) and (4.123) in the scalar form as
\[
\begin{align*}
\dot{v}_{Ax} &= \dot{v}_{Cx} - r_{Ay}\ddot{\theta}, \quad \text{(4.124)} \\
\dot{v}_{Ay} &= \dot{v}_{Cy} + r_{Ax}\ddot{\theta}, \quad \text{(4.125)} \\
\dot{v}_{Bx} &= \dot{v}_{Cx} - r_{By}\ddot{\theta}, \quad \text{(4.126)} \\
\dot{v}_{By} &= \dot{v}_{Cy} + r_{Bx}\ddot{\theta}. \quad \text{(4.127)}
\end{align*}
\]
According to (4.119)-(4.121) and (4.124)-(4.127), the velocities at each contact during the collision can be specified as
\[
\begin{align*}
\dot{v}_z &= \beta Rv_y, \quad \text{(4.128)} \\
\dot{v}_y &= -\beta Pv_y \quad \text{(4.129)}
\end{align*}
\]
To obtain the expression for the force at each contact, one may multiply both sides of (4.129) with \(-\beta\),

$$\dot{F}_y = -\beta P F_y. \quad (4.130)$$

Here, \(F_y\) represents the contact forces \([F_{Ay} \quad F_{By}]^T\) caused by the dampers.

In order to simulate the effect of impulsive force, a very large value of damping ratio is used in this impact model. As \(\beta \to \infty\), the solutions obtained from (4.128), (4.129) and (4.130) will represent an impulsive quantity for this problem.

Since each contact point may leave the ground after impact, any transition which occurs in these solutions need to be clearly examined. This procedure, however, is very difficult to implement in the original time scale where all events occur instantaneously. Therefore, it is more appropriate to transform the original system equations in (4.128), (4.129) and (4.130) into a new time scale where the transitions of their solutions can be clearly identified.

One possible approach is to transform (4.128), (4.129) and (4.130) to a new time scale \(t' = \beta t\). According to this transformation, \(F_y, v_x\) and \(v_y\) calculated at a large value of \(t'\) will correspond to the value of these functions determined at
a very short time in \( t \). For an infinite value of \( \beta \), the steady-state value of \( \mathbf{F}_y, \mathbf{v}_x \) and \( \mathbf{v}_y \) in \( t' \), therefore, corresponds to an impulsive quantity of these functions in the original time \( t \). By using this new-time scale, one can clearly examine any transition which occurs in both contact velocities. Therefore, both impulsive force and contact velocity under different circumstances can consistently determined.

Another advantage of this transformation is that the original non-linear dynamic and kinematic equations are simply resolved into a linear system where both contact forces and velocities are described by a first-order system. Therefore, one is able to determine the close-form solution for the impulsive force and system velocity after impact. Although this property cannot be seen in the 2-D case, its effect becomes apparent in the case of 3-D articulated bodies.

To transform (4.128), (4.129) and (4.130) to the \( t' \) domain, one may apply the following properties:

\[
\begin{align*}
\mathbf{x}(t) & \rightarrow \mathbf{x}(t'), \\
\mathbf{\dot{x}}(t) & \rightarrow \frac{d\mathbf{x}(t')}{dt'} = \beta \frac{d\mathbf{x}(t')}{dt'}. 
\end{align*}
\]

The first relationship indicates a one-to-one mapping between a similar function in \( t \) and \( t' \) whereas the second equation displays the mapping of its rate computed in \( t \) and \( t' \). For simplicity, the notations \( \cdot(t) \) and \( \cdot(t') \) will be used to represent the first derivative of the function in \( t \) and \( t' \). According to (4.131) and (4.132), (4.128), (4.129) and (4.130) can be expressed in \( t' \) as

\[
\mathbf{v}_y(t') = R\mathbf{v}_y(t'),
\]
\[ \dot{v}_y(t') = -Pv_y(t'), \quad (4.134) \]
\[ \dot{F}_y(t') = -PF_y(t'). \quad (4.135) \]

Since the positions of both contacts remain unchanged during impact, both \( R \) and \( P \) can be considered as constant matrices. Therefore, (4.133) and (4.135) are simply a first-order system where their solutions correspond to

\[ v_y(t') = e^{-Pt'}v_y(0^-), \quad (4.136) \]
\[ F_y(t') = e^{-Pt'}F_y(0^-) \quad (4.137) \]

The expression for \( v_x(t') \), however, is determined by direct integration of (4.134),

\[ v_x(t') = \int_{0^-}^{t'} Rv_y(t')dt' + v_x(0^-). \quad (4.138) \]

To obtain the impulsive forces in \( t' \), one may start with the definition of these forces in the original time \( t \) as follows:

\[ F_{iy} = \lim_{\beta \to \infty} \int_{0^-}^{\frac{1}{\beta}} F_y dt. \quad (4.139) \]

For a very large value of \( \beta \), the upper limit of this integration is approaching \( 0^+ \) in the original time \( t \) and \( \infty \) in \( t' \). Therefore, one can transform the above equation into \( t' \) as

\[ F_{iy} = \lim_{\beta \to \infty} \int_{0^-}^{\infty} \frac{F_y(t')}{\beta} dt'. \quad (4.140) \]

where \( dt \) is equal to \( \frac{dt'}{\beta} \).

In order to compute the steady-state value of \( v_y \) in (4.136), one needs to specify the expression of \( e^{-Pt'} \) in this problem. This matrix can be defined according to the
eigenvalues of $P$. For this particular matrix $P$, the eigenvalues can be written as:

$$
\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},
$$

and

$$
\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
$$

where

$$
a = 1,
$$

$$
b = -((\frac{1}{m} + \frac{r_{Ae}^2}{J}) + (\frac{1}{m} + \frac{r_{Bx}^2}{J})),
$$

$$
c = (\frac{1}{m} + \frac{r_{Ae}^2}{J})(\frac{1}{m} + \frac{r_{Bx}^2}{J}) - (\frac{1}{m} + \frac{r_{Ae}r_{Bx}}{J})^2.
$$

Based on this assignment, the term $b^2 - 4ac$ in (4.141) can be written as

$$
b^2 - 4ac = (\frac{1}{m} + \frac{r_{Ae}^2}{J})^2 + 2(\frac{1}{m} + \frac{r_{Ae}^2}{J})(\frac{1}{m} + \frac{r_{Bx}^2}{J})
$$

$$
+ (\frac{1}{m} + \frac{r_{Bx}^2}{J})^2 - 4(\frac{1}{m} + \frac{r_{Ae}^2}{J})(\frac{1}{m} + \frac{r_{Bx}^2}{J})
$$

$$
+ 4(\frac{1}{m} + \frac{r_{Ae}r_{Bx}}{J})^2
$$

$$
= \left((\frac{1}{m} + \frac{r_{Ae}^2}{J}) - (\frac{1}{m} + \frac{r_{Bx}^2}{J})\right)^2
$$

$$
+ 4(\frac{1}{m} + \frac{r_{Ae}r_{Bx}}{J})^2.
$$

(4.142)

Since $b^2 - 4ac$ is greater than zero, the eigenvalues in (4.141) result in two different real numbers. Furthermore, the principal minors of $P$, which are $\frac{1}{m} + \frac{r_{Ae}^2}{J}$ and $\frac{(r_{Ae} - r_{Bx})^2}{mJ}$, are also greater than zero. Thus, $P$ is a positive definite matrix. Based on these results, the eigenvalues of $P$ in this case can be specified by two distinct positive real numbers. Therefore, (4.136) and (4.137) both represent an underdamped system.
In order to determine the consistent final state for $v_y(t')$, one must also consider the transition of this contact velocity in $t'$. This effect results from the fact that a contact may not always stay on the ground as $t' \to \infty$. According to this condition, two possibilities needs to be considered for each contact point.

The first possibility is when both contacts remain on the ground as $t' \to \infty$. In this case, one can simply use (4.140), (4.138) and (4.136) to calculate the impulsive force and the final state of $v_x(t'), v_y(t')$.

The second possibility is when a contact is broken after impact. This situation occurs when $v_y(t')$ crosses the $t'$ axis. For a consistent result, one needs to determine the time in $t'$ when this transition occurs and then removes the damper from that contact. Therefore, the impact model with only one damper will be used to find the expression for $F_Iy, v_x(t')$ and $v_y(t')$ after the transition. Unlike the first possibility, (4.140), (4.136) and (4.138) can only provide the initial conditions for $F_Iy, v_x(t')$ and $v_y(t')$ after the transition.

The impulsive formulation for $v_x(t'), v_y(t')$ and $F_Iy$ in each possibility can be derived as following: first, one may assume that both contacts are still on the ground after impact. In this case, both dampers are always included in the dynamic model. From (4.136), the steady-state value of $v_y(t')$ as $t' \to \infty$ is

$$v_y(t') \mid_{t'=\infty} = e^{-P_{oo}}v_y(0^-). \quad (4.143)$$

Since $P$ is a positive definite matrix, $e^{-P_{oo}}$ in this equation is asymptotically approaching zero. Therefore, the corresponding $v_y$ after impact in the original time $t$ is equal to
From (4.138), one can specify the steady-state value of $v_x(t')$ as

$$v_x(t') |_{t'=\infty} = \int_{0^-}^{\infty} Re^{-P't'}v_y(0^-)dt' + v_x(0^-). \quad (4.145)$$

Based on the description of $v_y(t')$ in (4.136), this equation can be simplified into

$$v_x(t') |_{t'=\infty} = \int_{0^-}^{\infty} Re^{-P't'}v_y(0^-)dt' + v_x(0^-)$$

$$= RP^{-1}v_y(0^-) + v_x(0^-). \quad (4.146)$$

This expression also corresponds to $v_x(t)$ after the collision as follows:

$$v_x(t) |_{t=0^+} = v_x(t') |_{t'=\infty} = RP^{-1}v_y(0^-) + v_x(0^-). \quad (4.147)$$

From (4.137) into (4.140), the impulsive forces at both contacts are equal to

$$F_{Iy} = \lim_{\beta \to -\infty} \int_{0^-}^{\infty} e^{-P't'} \frac{F_y(0^-)}{\beta} dt'.$$

Since $F_y(0^-)$ is equal to $-\beta v_y(0^-)$, the above equation can be simplified into

$$F_{Iy} = \int_{0^-}^{\infty} e^{-P't'} v_y(0^-) dt'$$

$$= P^{-1}[e^{-P\infty} - I]v_y(0^-)$$

$$= -P^{-1}v_y(0^-) \quad (4.149)$$

The minus sign in this equation represents the direction of impulsive force which opposes $v_y$.

The second possibility is when one contact leaves the ground after impact. This situation occurs when the trajectory of $v_{Ay}(t')$ or $v_{By}(t')$ crosses the $t'$ axis. In other
words, the signs of $v_{Ay}(t')$ or $v_{By}(t')$ become different from their original value as $t'$ increases.

Suppose, contact A starts to move away from the ground at $t' = t'_A$ or $v_{Ay}(t'_A) = 0$. According to (4.136), the contact velocity $v_y(t')$ can be written in the state equation form as

$$\dot{v}_y(t') = -Pv_y(t').$$

(4.150)

To evaluate $t'_A$ in this case, one may first transform $v_y(t')$ to a new state $z(t')$ by using the similarity transformation $Q$ where

$$Qz(t') = v_y(t').$$

(4.151)

Each column of $Q$ is composed of the eigenvectors of the matrix $-P$. Based on this transformation, (4.150) in terms of $z(t')$ becomes

$$\dot{z}(t') = Q^{-1}(-P)Qz(t'),$$

(4.152)

$$\ddot{z}(t') = \Lambda z(t')$$

(4.153)

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

$$\lambda_1, \lambda_2 = \text{the eigenvalues of the matrix } -P.$$  

From (4.153), one can easily write down its solution as

$$z_A(t') = e^{\lambda_1 t'}z_A(0^-),$$

(4.154)

$$z_B(t') = e^{\lambda_2 t'}z_B(0^-),$$

(4.155)

$$z(0^-) = Q^{-1}v_y(0^-).$$

(4.156)
If the inverse of $Q$ is represented by $\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$, the new state $z(t')$ at $t' = t_A$ will be

$$\begin{bmatrix} z_A(t'_A) \\ z_B(t'_A) \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} v_{Ay}(t'_A) \\ v_{By}(t'_A) \end{bmatrix}.$$  \hfill (4.157)

Substituting for $v_{Ay}(t'_A) = 0$ in (4.157), one obtains

$$z_A(t'_A) = q_{12}v_{By}(t'_A),$$ \hfill (4.158)

$$z_B(t'_A) = q_{22}v_{By}(t'_A).$$ \hfill (4.159)

These equations can be further simplified by replacing $z_A(t'_A), z_B(t'_A)$ with the expression in (4.154) and (4.155),

$$e^{\lambda_1 t'_A} z_A(0^-) = q_{12}v_{By}(t'_A),$$ \hfill (4.160)

$$e^{\lambda_2 t'_A} z_B(0^-) = q_{22}v_{By}(t'_A).$$ \hfill (4.161)

According to (4.160) and (4.161), the time when contact $A$ starts to leave the ground is equal to

$$t'_A = \frac{1}{(\lambda_1 - \lambda_2)} \ln \left( \frac{z_B(0^-)q_{12}}{z_A(0^-)q_{22}} \right).$$ \hfill (4.162)

The impulsive force at each contact within this period of time can be computed as follows:

$$F_{iy} \mid t' = t'_A = \lim_{\beta \to -\infty} \int_{0^-}^{t'_A} \frac{F_y(t')}{\beta} dt',$$

$$= \lim_{\beta \to -\infty} \int_{0^-}^{t'_A} \frac{-\beta v_y(t')}{\beta} dt',$$

$$= \int_{0^-}^{t'_A} -v_y(t') dt',$$

$$= \int_{0^-}^{t'_A} e^{-\mu t} v_y(0^-) dt',$$

$$= P^{-1}[e^{-\mu t_A} - I]v_y(0^-).$$ \hfill (4.163)
From (4.158), one may describe \( v_B y \) at \( t' \) as \( \frac{e^{-\lambda_1 t' A Z A (0^-)}}{q_1} \). Therefore, the normal component of both contact velocities evaluated at \( t'_A \) is equal to

\[
v_b(t'_A) = \begin{bmatrix} 0 \\ e^{-\lambda_1 t'_A Z A (0^-)} \end{bmatrix}.
\] (4.164)

According to (4.138), the corresponding \( v_x \) at \( t' = t'_A \) is specified as follows:

\[
v_x(t'_A) = \int_0^{t'_A} R v_y(t') dt' + v_x(0^-),
\]

\[
= -\int_0^{t'_A} R P^{-1} \dot{v}_y(t') dt' + v_x(0^-),
\]

\[
= -R P^{-1}[v_y(t'_A) - v_y(0^-)] + v_x(0^-).
\] (4.165)

Once point A leaves the ground, the damper at this point needs to be removed from the impact model. Hence, the forces applied to this rigid body after \( t' > t'_A \) is solely depending on the damper at B. The dynamic equations of this new system can be simply modified from (4.128) and (4.129). Because the first column of both matrices in these equations represents the contribution of contact forces at A, it can be set to zero once the damper is removed. Therefore, one may express the dynamic equations for this system in the original time \( t \) as follows:

\[
\dot{v}_{A_1}(t) = \beta \frac{T_{A_1 B} T_{B_1}}{J} v_{B_1}(t),
\] (4.166)

\[
\dot{v}_{B_1}(t) = \beta \frac{T_{B_1 B} T_{B_1}}{J} v_{B_1}(t),
\] (4.167)

\[
\dot{v}_{A_2}(t) = -\beta \left( \frac{1}{m} + \frac{T_{A_2 B_1}}{J} v_{B_1}(t) \right) v_{B_1}(t),
\] (4.168)

\[
\dot{v}_{B_2}(t) = -\beta \left( \frac{1}{m} + \frac{r_{A_2}}{J} \right) v_{B_1}(t).
\] (4.169)

These equations can be transformed into \( t' \) as

\[
\dot{v}_{A_1}(t') = \frac{T_{A_1 B} T_{B_1}}{J} v_{B_1}(t'),
\] (4.170)
\[ \dot{v}_{By}(t') = \frac{r_{Bx}r_{By}}{J} v_{By}(t'), \quad (4.171) \]
\[ \dot{v}_{Ay}(t') = -\left(\frac{1}{m} + \frac{r_{Az}r_{Bx}}{J}\right) v_{By}(t'), \quad (4.172) \]
\[ \dot{v}_{By}(t') = -\left(\frac{1}{m} + \frac{r_{Az}^2}{J}\right) v_{By}(t'). \quad (4.173) \]

The solution of (4.173) can be specified as
\[ v_{By}(t') = e^{-\sigma_A(t'-t_A)} v_{By}(t_A'), \quad t' \geq t_A' \quad (4.174) \]

where \( \sigma_A \) represents \( \frac{1}{m} + \frac{r_{Az}^2}{J} \). In this form, it is obvious that this velocity asymptotically approaches zero as \( t' \to \infty \). In other words,
\[ v_{By}(t') \mid_{t'=\infty} = 0. \quad (4.175) \]

The velocity \( v_{Ay}(t') \) for \( t' \geq t_A' \) is simply determined by integrating both sides of (4.172)
\[ \int_{t_A'}^{t'} \dot{v}_{Ay}(t')dt' = \int_{t_A'}^{t'} -\left(\frac{1}{m} + \frac{r_{Az}r_{Bx}}{J}\right) e^{-\sigma_A(t'-t_A')} v_{By}(t_A')dt', \]
\[ v_{Ay}(t') = \frac{1}{\sigma_A} \left(\frac{1}{m} + \frac{r_{Az}r_{Bx}}{J}\right)(e^{-\sigma_A(t'-t_A')} - 1) v_{By}(t_A'). \quad (4.176) \]

For every \( t' \) which is greater than \( t_A' \), the value of \( v_{Ay}(t') \) in (4.176) is always negative. This minus sign guarantees that point A will not strike the ground again.

From (4.176), the steady-state value of \( v_{Ay}(t') \) is
\[ v_{Ay}(t') \mid_{t'=\infty} = -\frac{1}{\sigma_A} \left(\frac{1}{m} + \frac{r_{Az}r_{Bx}}{J}\right) v_{By}(t_A'). \quad (4.177) \]

The steady-state value of others velocities can be obtained by replacing \( v_{By}(t') \) in (4.170)–(4.172) with \( -\frac{\dot{v}_{Bs}(t')}{r_A} \) and then integrating both sides of each equation from
Based on this steady-state value, the velocities of A and B after the collision in the original time $t$ is then defined.

From (4.140), the impulsive forces at B during this period are determined as follows:

$$F_{IBy} = \lim_{\beta \to \infty} \int_{t_A}^{\infty} \frac{F_{By}(t')}{\beta} dt', \quad (4.180)$$

The total impulsive force at both contacts can be specified as the summation between (4.163) and (4.180),

$$F_{Iy} = P^{-1}[e^{-Pt_A} - I]v_y(0^-) + \left[ \frac{0}{v_{By}(t_A')} - \frac{v_{By}(t_A')}{\sigma_A} \right]. \quad (4.181)$$

The same procedure can be implemented if point B is assumed to leave the ground first. In this case, the time in $t'$ when this contact starts to move away from the ground is determined as

$$t_B' = \frac{1}{(\lambda_1 - \lambda_2)} \ln \left( \frac{z_B(0^-)q_1'}{z_A(0^-)q_2'} \right). \quad (4.182)$$
The impulsive forces as well as the velocity at each contact at \( t_B' \) can be calculated by replacing \( t_A' \) with \( t_B' \) in (4.163), (4.164) and (4.165). Once a transition occurs at B, the damper at this contact point is removed from the impact model. This corresponds to setting the second column of both matrices in (4.128), (4.129) to zero. As a result, \( \dot{v}_{A_x}(t'), \dot{v}_{A_y}(t'), \dot{v}_{B_x}(t') \) and \( \dot{v}_{B_y}(t') \) for \( t' > t_B' \) can be described as follows:

\[
\begin{align*}
\dot{v}_{A_x}(t') &= \frac{Ta_Ar_{A_y}}{J} v_{A_y}(t') \quad (4.183) \\
\dot{v}_{B_x}(t') &= \frac{Ta_Ar_{B_y}}{J} v_{A_y}(t') \quad (4.184) \\
\dot{v}_{A_y}(t') &= -\left(\frac{1}{m} + \frac{r_{A_x}^2}{J}\right) v_{A_y}(t') \quad (4.185) \\
\dot{v}_{B_y}(t') &= -\left(\frac{1}{m} + \frac{r_{A_x}r_{B_x}}{J}\right) v_{A_y}(t') \quad (4.186)
\end{align*}
\]

From (4.185), the velocity \( v_{A_y}(t') \) for \( t' \geq t_B' \) corresponds to

\[
v_{A_y}(t') = e^{-\sigma_B(t' - t_B')} v_{A_y}(t_B')
\]

where \( \sigma_B \) is equal to \( \frac{1}{m} + \frac{r_{A_x}^2}{J} \) in this case. Since \( \sigma_B \) is positive, the value of \( v_{A_y} \) is asymptotically approaching to zero as \( t' \to \infty \). Therefore, the steady-state value of \( v_{A_y}(t') \) in this case is

\[
v_{A_y}(t') \mid_{t' = \infty} = 0.
\]

The expression for \( v_{B_y}(t') \) after \( t_B' \) can be derived in the same way as shown in (4.176),

\[
v_{B_y}(t') = \frac{1}{\sigma_B} \left(\frac{1}{m} + \frac{r_{A_x}r_{B_x}}{J}\right)(e^{-\sigma_B(t' - t_B')} - 1)v_{A_y}(t_B').
\]

According to (4.189), the sign of \( v_{B_y}(t') \) is always negative for any \( t' \) which is greater than \( t_B' \). This condition verifies that B will not strike the ground again once it leaves the contact surface. The final value of \( v_{B_y}(t') \) when \( t' \to \infty \) is
\[ v_{B_y}(t') \mid t' = \infty = -\frac{1}{\sigma_B} \left( \frac{1}{m} + \frac{r_{A_x} r_{B_x}}{J} \right) v_{A_y}(t'_B). \] (4.190)

The steady-state value of \( v_{A_x}(t') \) and \( v_{B_x}(t') \) can be determined by integrating both sides of (4.183) and (4.184) from \( t' = t'_B \) to \( t' = \infty \),

\[
\begin{align*}
\left. v_{A_x}(t') \right|_{t' = \infty} &= \frac{r_{A_x} r_{A_y}}{J \sigma_B} v_{A_y}(t'_B) + v_{A_x}(t'_B) \\
\left. v_{B_x}(t') \right|_{t' = \infty} &= \frac{r_{A_x} r_{B_y}}{J \sigma_B} v_{A_y}(t'_B) + v_{B_x}(t'_B).
\end{align*}
\] (4.191) (4.192)

Using the same procedure as described in (4.180) and (4.181), one can determine the amount of impulsive force in this case as follows:

\[
F_{I_y} = P^{-1} \left[ e^{-P t'_B} - I \right] v_{y}(0^-) + \left[ -\frac{v_{A_y}(t'_B)}{\sigma_B} \right].
\] (4.193)

Although either A or B may be broken after impact, it is not possible that transitions will occur at both contact points. This condition is necessary for the inelastic impact with multiple contact points. In other words, at least one contact point should remain on the ground regardless of the transition at the others. According to (4.174) and (4.187), it is clear that when either A or B is leaving the ground, the other always remains on the ground after impact. Therefore, the only situation which needs to be examined is the possibility of both A and B leaving the ground simultaneously in \( t' \).

To verify that A and B cannot leave the ground at the same time, one may first assume that there exists a finite time \( t'_{AB} \) in \( t' \) where both \( v_{A_y}(t') \) and \( v_{B_y}(t') \) become zero simultaneously. To determine \( t'_{AB} \), one may use (4.136) as follows:

\[
\begin{bmatrix}
  v_{A_y}(t'_{AB}) \\
  v_{B_y}(t'_{AB})
\end{bmatrix} = e^{-P t'_{AB}} \begin{bmatrix}
  v_{A_y}(0^-) \\
  v_{B_y}(0^-)
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\] (4.194)
Because \( P \) is positive definite, the only possible solution is \( t_{AB}' \to \infty \). This solution, however, is contradictory to the assumption which was previously made. Therefore, it is not possible for both A and B to leave the ground at the same time in \( t' \).

Based on this result, when both points contact the planar surface simultaneously, there are only two possibilities which need to be considered. First, both contacts are on the ground after impact. Second, either contacts start to leave the ground after collision while the other is still on the ground.

**Case II. One contact point is resting on the ground while the other striking the ground**

Another situation, which can occur in multiple contacts, is when one contact point is resting on the ground and the other strikes the ground. In order to determine a consistent impulsive force for this problem, first one must identify the new motion of the contact which is resting on the ground. Once the motion of this contact is identified, the situation becomes similar to Case I. Therefore, the analysis as well as the impulsive formulation obtained in Case I can be reapplied.

In order to examine the motion of a contact which is resting on the ground, one may use the first-order approximation of a Taylor series for the normal velocity of that contact. For instance, suppose contact \( i \) simply rests on the ground. The normal velocity of this contact point at \( t_0' + \Delta t' \) can be written as

\[
 v_{iy}(t_0' + \Delta t') = v_{iy}(t_0') + \dot{v}_{iy}(t_0')\Delta t'.
\]  

(4.195)

Since the initial velocity \( v_{iy}(t_0') \) is equal to zero, (4.195) can be simplified to

\[
 v_{iy}(t_0' + \Delta t') = \dot{v}_{iy}(t_0')\Delta t'.
\]  

(4.196)
In other words, the direction of normal velocities for this contact at \( t'_0 + \Delta t' \) can be evaluated based on its acceleration \( \dot{v}_{iy}(t'_0) \). Since the positive y direction is toward the ground, \( \dot{v}_{iy}(t'_0) > 0 \) will indicate a motion which is pressing on the ground whereas \( \dot{v}_{iy}(t'_0) < 0 \) will correspond to a motion which is away from the ground.

For a 2-D single rigid body, each contact which is resting on the ground can possibly stay on the ground or leave the ground once the collision occurs. This generates two possible impact models: one with a single vertical damper and the other with two vertical dampers. Suppose contact A is simply resting on the ground while B is striking the ground. First, let's assume that A is moving toward the ground and the impact model consisting of two vertical dampers is used. From (4.129) and (4.150), one can write the state equation for \( v_{Ay}(t') \) and \( v_{By}(t') \) as follows:

\[
\begin{bmatrix}
\dot{v}_{Ay}(t') \\
\dot{v}_{By}(t')
\end{bmatrix} = - \begin{bmatrix}
\frac{1}{m} + \frac{\tau_x}{J} & \frac{1}{m} + \frac{\tau_{Ax} + \tau_{Bx}}{J} \\
\frac{1}{m} + \frac{\tau_{Bx}}{J} & \frac{1}{m} + \frac{\tau_x}{J}
\end{bmatrix} \begin{bmatrix}
v_{Ay}(t') \\
v_{By}(t')
\end{bmatrix}.
\]

(4.197)

On the other hand, if A is moving away from the ground and the impact model with a single damper at B is used, the state equation for \( v_{Ay}(t') \) and \( v_{By}(t') \) will be described based on (4.172) and (4.173) as follows:

\[
\begin{bmatrix}
\dot{v}_{Ay}(t') \\
\dot{v}_{By}(t')
\end{bmatrix} = - \begin{bmatrix}
0 & \frac{1}{m} + \frac{\tau_{Ax} + \tau_{Bx}}{J} \\
0 & \frac{1}{m} + \frac{\tau_x}{J}
\end{bmatrix} \begin{bmatrix}
v_{Ay}(t') \\
v_{By}(t')
\end{bmatrix}.
\]

(4.198)

To decide which impact model should be used, one needs to evaluate \( \dot{v}_{Ay}(0^-) \) in (4.197) and (4.198). Since \( v_{Ay}(0^-) \) is initially equal to zero, \( \dot{v}_{Ay}(0^-) \) obtained from (4.197) and (4.198) is simply \( \left( \frac{1}{m} + \frac{\tau_{Ax} + \tau_{Bx}}{J} \right) v_{By}(0^-) \). Since the value of \( \dot{v}_{Ay}(0^-) \) is identical in both equations, one can always select a consistent impact model for this rigid body according to the sign of \( \left( \frac{1}{m} + \frac{\tau_{Ax} + \tau_{Bx}}{J} \right) v_{By}(0^-) \).
If \((\frac{1}{m} + \frac{\mathbf{F}_{\text{imp}}}{m})v_{B_y}(0^-)\) is less than zero, contact A will leave the ground once the collision occurs at B. Therefore, the formulation in (4.175), (4.178), (4.179) and (4.180) with \(t_A' = 0^-\) can be used to determine the final value of \(v_x(t')\) and \(v_y(t')\) as well as the total impulsive force after impact. However, if \((\frac{1}{m} + \frac{\mathbf{F}_{\text{imp}}}{m})v_{B_y}(0^-)\) is greater than zero, the acceleration of A at \(t' = 0^-\) will be toward the ground and therefore, the same procedure as previously described in (4.143)–(4.193) can be applied. The same procedure can be implemented for the case where contact B is resting on the ground.

This approach can also be applied to the general case of multiple contacts in 3-D articulated bodies. Based on the first-order approximation of the normal velocity, one can identify the motion of a contact which is resting on the ground. As a result, there always exists only one impact model which is consistent with this estimation. The proof of this principle will be presented for a general 3-D articulated bodies in Section 4.4.5.

4.4.2 Numerical Examples of a 2-D Single Rigid Body with Two Contact Points

The numerical simulation of each possibility described in Section 4.4.1 are illustrated in this section. Three different examples are used to demonstrate how each situation can occur. The first example represents the case where both contacts stay on the ground after the collision. The second example illustrates the situation where only one contact remains on the ground after impact. Finally, the third example demonstrates the case where one contact is resting on the ground while the other striking the
ground.

**Example I. Both A and B strike the ground simultaneously and remain on the ground after impact**

Figure 24 illustrates an example where both contacts stay on the ground after impact. In this case, the center of gravity of the object is located between both contact points. The mass and the moment of inertia of this rigid body are arbitrarily assigned as 1 and 8 units. The coordinates of each contact with respect to the world are also displayed in the figure. The initial velocities of A and B used in this example are equal to $[2 \ 1]^T$ and $[2 \ 2]^T$, respectively.

According to (4.128) and (4.129), matrices $P$ and $R$ in this case are determined as 

$$
\begin{bmatrix}
2.125 & -0.5 \\
-0.5 & 3.0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1.125 & -1.5 \\
1.125 & -1.5
\end{bmatrix}
$$

Based on the similarity transformation in (4.151), $Q$ in this example can be described as 

$$
\begin{bmatrix}
-0.9106 & 0.4132 \\
-0.4132 & -0.9106
\end{bmatrix}
$$

The initial condition of the new state $z(t')$ in (4.151) is therefore equal to

$$
\begin{bmatrix}
z_A(0^-) \\
z_B(0^-)
\end{bmatrix} = Q^{-1} \begin{bmatrix}
v_{A_0}(0^-) \\
v_{B_0}(0^-)
\end{bmatrix} = \begin{bmatrix}
-1.7371 \\
-1.408
\end{bmatrix}.
$$

The eigenvalues of the matrix $-P$, $\lambda_1$ and $\lambda_2$, are also determined as -1.8981 and -3.2269.

To examine whether A or B may leave the ground after impact, one may apply (4.162) and (4.182),

$$
t'_A = \frac{1}{(\lambda_1 - \lambda_2)} \ln \left( \frac{z_B(0^-)q_{12}}{z_A(0^-)q_{22}} \right) = -0.7527,
$$

$$
t'_B = \frac{1}{(\lambda_1 - \lambda_2)} \ln \left( \frac{z_B(0^-)q_{11}}{z_A(0^-)q_{21}} \right) = 0.4366 + 2.3643i.
$$
Figure 24: A 2-D example of a single rigid body illustrates the situation where both contact points are still on the ground after impact. The picture also displays the coordinates of two contact points and the center of gravity as well as the direction of contact velocities.
Because neither $t'_A$ and $t'_B$ are realizable, i.e., not positive real numbers, both contacts will remain on the ground after the collision. In this case, the impact model with two vertical dampers will be used.

Figures 25 and 26 show the trajectories of $v_A(t')$ and $v_B(t')$ as obtained from (4.133) and (4.134). Both $v_{Ay}(t')$ and $v_{By}(t')$ asymptotically approach zero as $t'$ increases. The tangential velocities $v_{Ax}(t')$ and $v_{Bx}(t')$, on the other hand, start to
Figure 26: The trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ based on the collision in Figure 24.
decrease at the beginning and finally reach their steady states. The impulsive forces \( F_{Iy} \) as obtained from (4.149) are equal to \([-0.6531, -0.7755]^T\) and the final value of \( v_{Ax}(t') \) and \( v_{Bx}(t') \) according to (4.147) is equal to 1.5714 m/sec. After the collision, both A and B will have the velocity of 1.5714 m/sec in the direction as shown in Figure 24.

**Example II. Both A and B strike the ground simultaneously and B leaves the ground after impact**

Figure 27 illustrates a situation where one of the contact points leaves the ground after impact. Unlike the previous example, the center of gravity of this rigid body is located on the right of contact A as shown in the figure. The mass and the moment of inertia are assumed to be the same as before. The initial velocity of both contacts are set to \([2 \ 1]^T\) in this case. According to (4.128), (4.129) and (4.151), one can specify the matrices \( P, R \) and \( Q \) as

\[
\begin{bmatrix}
2.125 & 3.25 \\
3.25 & 5.5
\end{bmatrix},
\begin{bmatrix}
-1.125 & -2.25 \\
-1.125 & -2.25
\end{bmatrix},
\begin{bmatrix}
-0.8546 & -0.5192 \\
0.5192 & -0.8546
\end{bmatrix}
\]

respectively. The corresponding \( z(0^-) \) in (4.151) is described as

\[
\begin{bmatrix}
z_A(0^-) \\
z_B(0^-)
\end{bmatrix} = Q^{-1}
\begin{bmatrix}
v_{Ay}(0^-) \\
v_{Bx}(0^-)
\end{bmatrix} = 
\begin{bmatrix}
-0.3354 \\
-1.3739
\end{bmatrix}.
\]

(4.202)

The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the matrix \(-P\) are also determined as -0.1505 and -7.4745.

To examine whether A or B will leave the ground after impact, one can apply the results in (4.162) and (4.182) as follows:

\[
t'_A = \frac{1}{\lambda_1 - \lambda_2} \ln \left( \frac{z_B(0^-)g_{12}}{z_A(0^-)g_{22}} \right) = 0.1245 + 0.4289i, \quad (4.203)
\]

\[
t'_B = \frac{1}{\lambda_1 - \lambda_2} \ln \left( \frac{z_B(0^-)g_{11}}{z_A(0^-)g_{21}} \right) = 0.2606. \quad (4.204)
\]
Figure 27: A 2-D example of a single rigid body demonstrates the case where one contact point leaves the ground after impact. The figure also shows the coordinates of two contact points and the center of gravity plus the direction of contact velocities.
Since $t'_B$ is realizable in this case, contact B will leave the ground at $t' = 0.2606$ sec. The contact A, on the other hand, will remain on the ground after the collision. According to this transition, the damper at B will be removed from the impact model at $t' = 0.2606$ sec.

Based on (4.133) and (4.134), the trajectory of $v_x(t')$ and $v_y(t')$ which starts from $t' = 0$ to $t' = 0.2606$ can be described as

$$
\begin{bmatrix}
\dot{v}_{Ax}(t') \\
\dot{v}_{Bx}(t') \\
\dot{v}_{Ay}(t') \\
\dot{v}_{By}(t')
\end{bmatrix} =
\begin{bmatrix}
-1.125 & -2.25 \\
-1.125 & -2.25 \\
2.125 & 3.25 \\
3.25 & 5.5
\end{bmatrix}
\begin{bmatrix}
v_{Ax}(t') \\
v_{Ay}(t') \\
v_{Bx}(t') \\
v_{By}(t')
\end{bmatrix},
$$

(4.205)

After $t' \geq 0.2606$, the trajectories of these velocities will be specified according to (4.183)–(4.186),

$$
\begin{align*}
\dot{v}_{Ax}(t') &= \dot{v}_{Bx}(t') = -1.125 v_{Ay}(t'), \\
\dot{v}_{Ay}(t') &= -2.125 v_{Ay}(t'), \\
\dot{v}_{By}(t') &= -3.25 v_{By}(t').
\end{align*}
$$

(4.207) \quad (4.208) \quad (4.209)

Figures 28 and 29 show the trajectories of contact velocities $v_x(t')$ and $v_y(t')$ as obtained from (4.205)–(4.209). The velocity $v_{Ay}$ is asymptotically approaching zero as $t' \to \infty$. The velocity $v_{By}$, on the other hand, starts to decrease and crosses zero at $t' = 0.2606$ sec as shown in Figure 29. After that, it begins to increase and reaches the steady-state value of -0.5772 m/sec. Based on the trajectory of $v_{By}(t')$, B will not strike the ground again after it takes off.

Although the model is changed discontinuously at $t' = 0.2606$, the contact velocities appear continuous and smooth at this point as shown in Figure 29. Since
Figure 28: The trajectories of $v_{Ax}(t')$ and $v_{Bx}(t')$ based on the collision in Figure 27.
Figure 29: The trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ based on the collision in Figure 27.
the damper is removed when the velocity is zero, force is continuous and therefore velocity should be smooth which is consistent with simulation results.

The trajectories of $v_{A}(t')$ and $v_{B}(t')$ are still the same in this case. Their final value as computed by (4.191) and (4.192) are equal to 1.4228 m/sec. Therefore, after the collisions, both A and B will have the velocities of $[1.4228\; 0]^T$ m/sec and $[1.4228\; -0.5772]^T$ m/sec as shown in Figure 27. The impulsive force at each contact as obtained from (4.193) are -0.3327 and -0.0902.

**Example III. Contact A is resting on the ground while the collision occurs at B**

Figure 30 illustrates an example where one contact is resting on the ground and the other makes a contact with the ground. In this case, A is resting on the ground while the collision happens at B. The mass and the moment of inertia for this rigid body are the same as in the two previous examples. However, the center of gravity is assumed to be on the left of B in this case. The velocity of B before the collision is specified as $[0\; 1]^T$ m/sec.

According to (4.129), the matrix $P$ for this rigid body can be determined as

\[
\begin{bmatrix}
4.125 & 2.25 \\
2.25 & 1.5
\end{bmatrix}
\]

To examine whether A will leave the ground after impact, one needs to determine $\dot{v}_{Ag}(0^-)$ which is equal to $(\frac{1}{m} + \frac{\sum_{j=1}^{n} \alpha_j I_j}{I}) v_{Bg}(0^-)$. For this particular case, $\dot{v}_{Ag}(0^-)$ is equal to -2.25. Since $\dot{v}_{Ag}(0^-)$ is less than zero, A will leave the ground immediately after the collision. Therefore, the impact model with one vertical damper will be used in this example.

From (4.170) to (4.173), one can describe the accelerations of A and B in this case.
Figure 30: A 2-D example of a single rigid body demonstrates the situation where one point is simply resting on the ground while the other is making contact. The coordinates of both contact points and the center of gravity as well as the direction of contact velocities are also shown in this figure.
Figure 31: The trajectories of $v_{Ax}(t')$ and $v_{By}(t')$ based on the collision shown in Figure 30.

as follows:

\[ \dot{v}_{Ax}(t') = \dot{v}_{Bx}(t') = 0.75v_{By}(t'), \quad (4.210) \]
\[ \dot{v}_{Ay}(t') = -2.25v_{By}(t'), \quad (4.211) \]
\[ \dot{v}_{By}(t') = -1.5v_{By}(t'). \quad (4.212) \]
Figure 32: The trajectories of $v_{Ax}(t')$ and $v_{Bx}(t')$ based on the collision shown in Figure 30.
Figure 31 and 32 show the trajectories of \( u_{Ay}(t'), u_{By}(t'), u_{Ax}(t') \) and \( u_{Bx}(t') \) as obtained from (4.210)-(4.212). From the trajectory of \( u_{Ay}(t') \), it is obvious that contact A separates from the ground immediately once the impact occurs. This velocity also increases at the beginning and reaches its steady state as \( t' \to \infty \). The final value of \( u_{Ay} \) as obtained from (4.176) is equal to -1.5 m/sec. The contact B, on the other hand, will remain on the ground after the collision. This result is indicated by the trajectory of \( u_{By}(t') \) which is approaching zero as \( t' \to \infty \). According to Figure 32, A and B will also have the velocity in the x direction due to the collision at B. This result can be seen from the plots of \( u_{Ax}(t') \) and \( u_{Bx}(t') \) which start to increase and finally reach their steady states as \( t' \to \infty \). The final value of these velocities as obtained from (4.191) and (4.192) corresponds to 0.5 m/sec. Therefore, the velocities of A and B after impact can be written as \([0.5 - 1.5]^T\) and \([0.5 0]^T\) respectively. Since A starts to leave the ground at \( t' = 0 \), the impulsive force exerted on the system will come from contact B. The amount of impulsive force from (4.180) is equal to -0.6667.

As a result of these three examples, a few observations can be made for a 2-D single rigid body with two contact points. After a contact separates from the ground, it will stay in the air. Based on examples II and III, it is clear that only one transition can occur for a 2-D single rigid body with two contact points. Moreover, when one contact is resting on the ground, it is possible to specify the motion of this contact after the other strikes the ground. Although this point may be trivial for the case of a 2-D single rigid body where the number of possibilities is very limited, it is very
important factor for the case of a 3-D quadruped with many possible next states.

4.4.3 Sensitivity of the Impact Model in a 2-D Single Rigid Body

In the technique discussed in the last two sections, a linear damper has been used to simulate the effect of impulsive forces at each contact. As a result, one is able to find a closed-form solution for both impulsive forces and contact velocities after impact. Nevertheless, some assumptions have been made in order to obtain these solutions. For instance, the damping ratio at each contact are assumed to be the same. Moreover, where there are impacts at two points, the collisions are assumed to be happen simultaneously.

In this section, the effects of these assumptions on the impact model will be presented. The discussion will focus on how these conditions can affect the model in terms of the impulsive force.

The effect due to different values of damping coefficient

First, one may consider the effect due to different value of damping coefficient. Suppose both A and B contact the ground simultaneously. Let $\beta_A$ and $\beta_B$ be the damping coefficients at contact point A and B. According to (4.117) and (4.118), the contact forces during impact can be described as

$$F_A = \begin{bmatrix} 0 \\ -\beta_A v_{Ay} \end{bmatrix}, \quad F_B = \begin{bmatrix} 0 \\ -\beta_B v_{By} \end{bmatrix}. \quad (4.213)$$

(4.214)
Substituting these contact forces into (4.115) and (4.116), one obtains

\[ \dot{v}_{Cz} = 0, \]

(4.215)

\[ -\beta_A v_{Ay} - \beta_B v_{By} = m \ddot{v}_{Cy}, \]

(4.216)

\[ -\beta_A v_{Ay} r_{Ax} - \beta_B v_{By} r_{Bx} = J \ddot{\theta}. \]

(4.217)

Using these equations and the results in (4.124)-(4.127), one can solve for \( \dot{v}_{Ay} \) and \( \dot{v}_{By} \) as follows:

\[
\begin{bmatrix}
\dot{v}_{Ay} \\
\dot{v}_{By}
\end{bmatrix} =
\begin{bmatrix}
-\beta_A \left( \frac{1}{m} + \frac{r_{Ax}}{J} \right) & -\beta_B \left( \frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} \right) \\
-\beta_A \left( \frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} \right) & -\beta_B \left( \frac{1}{m} + \frac{r_{Bx}}{J} \right)
\end{bmatrix}
\begin{bmatrix}
v_{Ay} \\
v_{By}
\end{bmatrix}.
\]

(4.218)

This equation is further simplified to

\[
\begin{bmatrix}
\dot{v}_{Ay} \\
\dot{v}_{By}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{m} + \frac{r_{Ax}}{J} & \frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} \\
\frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} & \frac{1}{m} + \frac{r_{Bx}}{J}
\end{bmatrix}
\begin{bmatrix}
-\beta_A v_{Ay} \\
-\beta_B v_{By}
\end{bmatrix}.
\]

(4.219)

To rewrite (4.219) in terms of \( F_{Ay} \) and \( F_{By} \), one simply multiplies both sides of this equation with \(- \begin{bmatrix} \beta_A & 0 \\ 0 & \beta_B \end{bmatrix}\) as follows:

\[
\begin{bmatrix}
\dot{F}_{Ay} \\
\dot{F}_{By}
\end{bmatrix} =
\begin{bmatrix}
-\beta_A & 0 \\
0 & -\beta_B
\end{bmatrix}
\begin{bmatrix}
\frac{1}{m} + \frac{r_{Ax}}{J} & \frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} \\
\frac{1}{m} + \frac{r_{Ax}r_{Bx}}{J} & \frac{1}{m} + \frac{r_{Bx}}{J}
\end{bmatrix}
\begin{bmatrix}
F_{Ay} \\
F_{By}
\end{bmatrix},
\]

(4.220)

\[ \dot{F}_y = -\gamma \mathcal{P} F_y. \]

(4.221)

Suppose \( \beta_A \) and \( \beta_B \) are generally specified as \( m\beta \) and \( n\beta \) where \( m \) and \( n \) represent two arbitrary positive numbers. In order to find the impulsive force, one may transform (4.221) to \( t' \) where \( t' = \beta t \) and \( \beta \to \infty \). Based on (4.131) and (4.132), (4.221) in \( t' \) can be written as follows:

\[
\dot{F}_y(t') = -\frac{1}{\beta} \gamma \mathcal{P} F_y(t'),
\]

\[ = -\gamma' PF_y(t') \]

(4.222)
where $T'$ is now a matrix $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$.

The state equation for $v_Ay(t')$ and $v_By(t')$ in this case can be obtained from (4.219) as

$$\begin{bmatrix} \dot{v}_{Ay} \\ \dot{v}_{By} \end{bmatrix} = -\beta \begin{bmatrix} \frac{1}{m} + \frac{r_A^2}{f} & \frac{1}{m} + \frac{r_AfR_B}{f^2} \\ \frac{1}{m} + \frac{r_AfR_B}{f^2} & \frac{1}{m} + \frac{r_B^2}{f} \end{bmatrix} \begin{bmatrix} -\beta_A v_{Ay} \\ -\beta_B v_{By} \end{bmatrix},$$

$$= -\beta \begin{bmatrix} \frac{1}{m} + \frac{r_A^2}{f} & \frac{1}{m} + \frac{r_AfR_B}{f^2} \\ \frac{1}{m} + \frac{r_AfR_B}{f^2} & \frac{1}{m} + \frac{r_B^2}{f} \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} v_{Ay} \\ v_{By} \end{bmatrix}.$$  \hspace{1cm} (4.223)

Likewise, one can rewrite this equation in $t'$ as

$$\begin{bmatrix} \dot{v}_{Ay}(t') \\ \dot{v}_{By}(t') \end{bmatrix} = -\beta \begin{bmatrix} \frac{1}{m} + \frac{r_A^2}{f} & \frac{1}{m} + \frac{r_AfR_B}{f^2} \\ \frac{1}{m} + \frac{r_AfR_B}{f^2} & \frac{1}{m} + \frac{r_B^2}{f} \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} v_{Ay} \\ v_{By} \end{bmatrix},$$

$$\dot{v}_y(t') = -P'T'v_y(t').$$  \hspace{1cm} (4.224)

Since (4.222) represents a linear system, its closed-form solution can be written as

$$F_y(t') = e^{-T'P't'}F_y(0^-).$$ \hspace{1cm} (4.225)

One may express this equation in terms of $v_y(0^-)$ as

$$F_y(t') = -\beta e^{-T'P't'}T'v_y(0^-).$$ \hspace{1cm} (4.226)

From the previous two sections, both contacts can either remain on the ground or one of them leaves the ground after impact. When both contacts remain on the ground, their velocities in $t'$ are asymptotically approaching zero as $t' \to \infty$. Under this circumstance, one can determine the impulsive force using (4.140) and (4.226) as follows:

$$F_{Iy} = \lim_{\beta \to \infty} \int_{0^-}^{\infty} \frac{F_y(t')}{\beta} dt'.$$
\[ \begin{align*}
&= - \int_{0^-}^{\infty} e^{-T'P't'} Y'v_y(0^-) dt', \\
&= (Y'P')^{-1} [e^{-T'P't'} |_{t' \to \infty} - I] Y'v_y(0^-). \quad (4.227)
\end{align*} \]

Since \( P \) and \( T' \) are both positive definite, the matrix \( e^{-T'P't'} \) is then asymptotically approaching zero as \( t' \to \infty \). Therefore, (4.227) is simply resolved to

\[ F_{iy} = -P^{-1}v_y(0^-). \quad (4.228) \]

According to (4.228) and (4.149), it is clear that the same amounts of impulsive force are obtained regardless of different value of \( \beta_A \) and \( \beta_B \). In other words, the impulsive force, which causes both contacts to remain on the ground, is not a function of the damping coefficient.

Now, suppose one contact leaves the ground after impact. For a 2-D single rigid body, this corresponds to the situation where the center of gravity of the object is not between both contact points. To examine the effect of the damping coefficient under this circumstance, one may consider an example of a rigid body shown in Figure 30. The coordinates of contact A and B are changed to \([8,0]\) and \([1,0]\). The center of gravity of this object has been relocated to \([0,-1]\) instead of \([0,-3]\) as shown in Figure 30. The initial velocity at both points before collisions are both set to \([0 \ 1]^T\) and \( \beta_A \) and \( \beta_B \) are assigned as \( \frac{3}{2}\beta \) and \( 3\beta \), respectively.

Figure 33 displays the plots of contact velocities \( v_{Ay}(t') \) and \( v_{By}(t') \) as obtained from (4.224). Contact A starts to leave the ground approximately at \( t' = 0.075 \) sec. Therefore, the damper at A is removed from the impact model after \( t' = 0.075 \) sec. Contact B, on the other hand, is still on the ground since \( v_{By}(t') \) asymptotically approaches zero as \( t' \to \infty \).
Figure 33: The trajectories of $v_{A}(t')$ and $v_{B}(t')$ where the damping coefficients at A and B are set to $\frac{3}{2}\beta$ and $3\beta$, respectively. The dashed line represents contact velocities at B.
The impulsive force at each contact can be determined using (4.181). For this particular example, \( t'_A \) is equal to 0.075 and the matrix \( P \) is now \( PT' \) where \( \gamma' \) is equal to \[
\begin{bmatrix}
\frac{3}{2} & 0 \\
0 & 3
\end{bmatrix}.
\]
As a result, the impulsive force at the contacts is calculated as \([-0.0515 - 0.7959]^T\).

In order to show the effect of the damping coefficient, a new damping coefficient of 4.5\( \beta \) is used at contact B instead. Therefore, \( \gamma' \) in this case corresponds to \[
\begin{bmatrix}
\frac{3}{2} & 0 \\
0 & 4.5
\end{bmatrix}.
\]
The trajectories of \( v_{Ay}(t') \) and \( v_{By}(t') \) as obtained from (4.224) are displayed in Figure 34. In this case, the normal contact velocities \( v_{Ay}(t') \) starts to leave the ground approximately at \( t' = 0.06 \) sec instead of 0.075 sec as before. The impulsive force as determined from (4.181) also corresponds to \([-0.0426 - 0.8131]^T\) instead of \([-0.0515 - 0.7959]^T\).

As a result of this example, it is obvious that different values of damping coefficient may lead to different amounts of impulsive force once a transition occurs at the contact. This result can be considered as the uncertainty of the final velocity of a contact which leaves the ground. Despite different amount of impulsive force found at the contacts, the state of each contact point after the collision is still the same in both cases, i.e., contact A still remains on the ground while B leaves the ground. The same result, however, may not be obtained if there are more than two contact points during impact. Under this circumstance, different amount of impulsive force as well as unexpected transitions may be found due to different values of damping coefficient. Therefore, the multiple contact problem is not generally well posed in terms of the damping coefficient.
Normal contact velocities

Figure 34: The trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ where the damping coefficients at A and B are set to $\beta$ and $4.5\beta$, respectively. The dashed line represents contact velocities at B.
The effect due to non-simultaneous collisions

For the problem of multiple contacts, it is possible that each collision may not happen simultaneously but in a random sequence. Even though each collision may occur almost instantaneously in the original time $t$, the effect due to non-simultaneous collisions may lead to a finite amount of delay in the $\beta$ scaled time $t'$. To examine whether this factor can affect the model in terms of impulsive forces, one may consider these following two impact cases.

Like the previous studies on damping coefficients, one may first consider the situation where both contact points remain on the ground after impact. In other words, the final state of these contact points is uniquely defined. The 2-D example of a single rigid body shown in Figure 24 can be used to simulate this impact case. All of the parameters and initial conditions are assumed to be the same. However, instead of assuming that both A and B strike the ground simultaneously, a finite amount of delay due to the non-simultaneous collision is included at contact B, i.e., the impact at B does not occur until $t' = 2$ sec.

Figure 35 displays the trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ from $t' = 0$ to $t' = 8$ sec. Since B does not make contact until $t' = 2$, the impact model containing only one damper is used from $t' = 0$ to $t' = 2$. After the collision occurs at B, the impact model with two vertical dampers will be used.

According to Figure 35, both $v_{Ay}(t')$ and $v_{By}(t')$ asymptotically approach zero as $t' \to \infty$. This shows that both contact points still remain on the ground after impact. Moreover, each of these graphs also demonstrates the transition due to a delayed
Figure 35: The trajectories of $v_{A_y}(t')$ and $v_{B_y}(t')$ when a delayed collision occurs at B. The dashed line in this graph represents the contact velocity $v_{B_y}(t')$. 
collision at B. From (4.140), the impulsive force at the contacts can be determined to be $[-0.6531\ -\ 0.7755]^T$.

Figure 36 displays the trajectories of $\frac{F_{A\beta}(t')}{\beta}$, $\frac{F_{B\beta}(t')}{\beta}$ showing the effect of the delay at point B. According to Figure 36b, the delay primarily causes a finite delay of scaled contact forces $\frac{F_{B\beta}(t')}{\beta}$. On the other hand, different trajectories are found in the scaled contact force $\frac{F_{A\beta}(t')}{\beta}$ due to the delay as shown in Figure 36a.

Since both contact points still remain on the ground regardless of the delay, the final states of $v_{Ay}(t')$ and $v_{By}(t')$ in this problem are uniquely determined. Therefore, identical amounts of impulsive force should be obtained despite a delay at B. According to (4.140), the impulsive forces determined based on the dashed curves in Figure 36 are also equal to $[-0.6531\ -\ 0.7755]^T$. Thus, non-simultaneous collisions will not affect the model in terms of impulsive forces if the final state of both contact points are uniquely determined.

Now suppose only one contact remains on the ground after the collisions. In other words, the final velocity of a contact which leaves the ground is not uniquely defined. In this case, one may reconsider the impact of a single rigid body shown in Figure 27. All parameters are kept the same except that the position of B is changed to $[-4, 0]$. The initial velocities at A and B are specified as $[2\ 1]^T$ m/sec. Like the previous example, the collision also starts with A and then B instead of A and B at the same time. The delay due to the non-simultaneous collision in $t'$ is specified as 0.2 sec.

Figure 37 displays the trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ from $t' = 0$ to 3. In this case, contact B starts to leave the ground approximately at $t' = 0.63$ sec. Contact
Figure 36: The trajectories of scaled contact forces $\frac{F_{AB}(t')}{\beta}$ and $\frac{F_{BA}(t')}{\beta}$ in Figure 35b as compared to the results when simultaneous collisions are assumed. The dashed line in this figure represents $\frac{F_{AB}(t')}{\beta}$ and $\frac{F_{BA}(t')}{\beta}$ with simultaneous collisions at A and B.
A, on the other hand, remains on the ground after the collision. Since B does not touch the ground until \( t' = 0.2 \) sec, the impact model with only one damper is used from \( t' = 0 \) to \( t' = 0.2 \) sec. From 0.2 to 0.63 sec, the impact model which consists of two linear dampers is applied. After contact B is broken at \( t' = 0.63 \) sec, the impact model with a single damper will be used in the simulation. According to (4.193), the impulsive forces at the contacts can be determined as \([-0.3747 - 0.0812]^T\).

Figure 38 shows the comparison of scaled contact forces in Figure 37b and the results obtained when there are no delayed collisions at B. Based on (4.193), one can calculate the impulsive force when a simultaneous collision occurs at A and B as \([-0.2856 - 0.1569]^T\). Since this impulsive force differs from the previous results when a delayed collision occurs at B, non-simultaneous collisions do affect the impact model once a transition happens after impact.

Although this factor does not affect the model when both contacts remain on the ground, different amounts of impulsive force are found due to non-simultaneous collisions with a transition at the contact. However, the smaller the amount of the delay in \( t' \), the less effect it generates in terms of the impulsive force at each contact point. Therefore, with a small amount of delay in the \( \beta \) scaled time \( t' \), a consistent amount of impulsive force can be obtained from the model and the problem of non-simultaneous collisions is well posed for a 2-D single rigid body with two contact points.

Similar discussion on these two issues will be reexamined in the general case of 3-D articulated bodies in Section 4.4.7. The studies will focus on the multiple contacts in
Figure 37: The trajectories of $v_{Ay}(t')$ and $v_{By}(t')$ when a delayed collision occurs at B. The dashed line represents the contact velocity $v_{By}(t')$ which crosses the $t'$ axis at 0.63 sec.
Figure 38: The trajectories of scaled contact forces $\frac{F_{dx}(t')}{\beta}$ and $\frac{F_{dy}(t')}{\beta}$ in Figure 37 as compared to the results when simultaneous collisions are assumed. The dashed lines in this figure represent the quantities obtained with simultaneous collisions at A and B.
4.4.4 The Impulsive Formulation for a 3-D Single Rigid Body with Two Contact Points

The problem of multiple contacts for a 3-D articulated bodies is presented in this section. According to the results in Section 4.4.1, one is able to determine the impulsive forces for multiple contacts in a 2-D single object based on the steady-state value of contact forces and velocities obtained in $\beta$ scaled time, $t'$. This procedure is quite straightforward in 2-D since both contact forces and velocities are described by first-order systems. For the general motion in 3-D, the dynamic and kinematic equations are both described by non-linear functions. Therefore, the contact forces and velocities in this case are represented by non-linear systems.

In order to find the impulsive formulation for multiple contacts in 3-D, first one needs to show that the contact forces and velocities for this system can also be described by the first-order systems in $\beta$ scaled time, $t'$. Once this criterion is satisfied, a similar approach to the 2-D case can be used to determine the impulsive forces for this system.

To examine whether this condition can be satisfied, one may first consider an example of a 3-D single rigid body which is similar to that shown in Figure 23. The expression of non-impulsive forces and torques at the center of gravity of this rigid body can be written as

\[
\begin{align*}
F_A + F_B - mg &= m\dot{v}_C, \\
\tau_A \times F_A + \tau_B \times F_B &= J\dot{\omega} + \omega \times (J\omega).
\end{align*}
\]
The kinematic equations which describe the contact velocities of this rigid body are

\[ \begin{align*}
\dot{v}_A &= \dot{v}_C + \omega \times r_A + \omega \times (\omega \times r_A), \\
\dot{v}_B &= \dot{v}_C + \omega \times r_B + \omega \times (\omega \times r_B).
\end{align*} \tag{4.231} \tag{4.232} \]

Unlike the 2-D case, these equations contain nonlinear terms as a result of \( \omega \times \dot{r}_A, \omega \times \dot{r}_B \). From (4.229) – (4.232), one can solve for \( \dot{v}_A \) and \( \dot{v}_B \) in terms of contact forces as follows:

\[ \begin{align*}
\dot{v}_A &= \left( \frac{F_A}{m} + (r_A \times F_A) \times r_A \right) + \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_A \right] + \\
&\quad [\omega \times (\omega \times r_A) - g - (\omega \times (J\omega)) \times r_A], \tag{4.233} \\
\dot{v}_B &= \left( \frac{F_A}{m} + (r_A \times F_A) \times r_B \right) + \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_B \right] + \\
&\quad [\omega \times (\omega \times r_B) - g - (\omega \times (J\omega)) \times r_B]. \tag{4.234}
\end{align*} \]

The contact velocities for this rigid body at time \( t \) can be specified by integrating (4.233) and (4.234),

\[ \begin{align*}
v_A(t) - v_A(0^-) &= \int_{0^-}^{t} \left( \frac{F_A}{m} + (r_A \times F_A) \times r_A \right) dt + \\
&\quad \int_{0^-}^{t} \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_A \right] dt + \\
&\quad \int_{0^-}^{t} [\omega \times (\omega \times r_A) - g - (\omega \times (J\omega)) \times r_A] dt, \tag{4.235} \\
v_B(t) - v_B(0^-) &= \int_{0^-}^{t} \left( \frac{F_A}{m} + (r_A \times F_A) \times r_B \right) dt + \\
&\quad \int_{0^-}^{t} \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_B \right] dt + \\
&\quad \int_{0^-}^{t} [\omega \times (\omega \times r_B) - g - (\omega \times (J\omega)) \times r_B] dt. \tag{4.236}
\end{align*} \]

As illustrated in the 2-D case, the velocity of each contact after the collision can be specified through its steady-state value in \( t' \) where \( t' = \beta t \) and \( \beta \rightarrow \infty \). From (4.235)
and (4.236), one can simply write the expression of \( v_A \) and \( v_B \) in \( t' \) as follows:

\[
\begin{align*}
v_A(t') - v_A(0^-) &= \int_{0^-}^{t'} \frac{1}{\beta} \left[ \frac{F_A}{m} + (r_A \times F_A) \times r_A \right] dt' + \\
&\quad \int_{0^-}^{t'} \frac{1}{\beta} \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_B \right] dt' + \\
&\quad \int_{0^-}^{t'} \left[ \omega \times (\omega \times r_A) - g - (\omega \times (J\omega)) \times r_A \right] dt', \quad (4.237)
\end{align*}
\]

\[
\begin{align*}
v_B(t') - v_B(0^-) &= \int_{0^-}^{t'} \frac{1}{\beta} \left[ \frac{F_A}{m} + (r_A \times F_A) \times r_B \right] dt' + \\
&\quad \int_{0^-}^{t'} \frac{1}{\beta} \left[ \frac{F_B}{m} + (r_B \times F_B) \times r_B \right] dt' + \\
&\quad \int_{0^-}^{t'} \left[ \omega \times (\omega \times r_B) - g - (\omega \times (J\omega)) \times r_B \right] dt'. \quad (4.238)
\end{align*}
\]

For the planar surface, the contact forces \( F_A \) and \( F_B \) which are normal to the surface can be defined as

\[
\begin{align*}
F_A &= -\beta v_A \mathbf{z} = -\beta B v_A, \quad (4.239) \\
F_B &= -\beta v_B \mathbf{z} = -\beta B v_B \\
\end{align*}
\]

where

\[
B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Substituting (4.239), (4.240) into (4.237), (4.238), one obtains

\[
\begin{align*}
v_A(t') - v_A(0^-) &= -\int_{0^-}^{t'} \frac{B v_A}{m} + (r_A \times B v_A) \times r_A] dt' - \\
&\quad \int_{0^-}^{t'} \left[ \omega \times (\omega \times r_A) - g - (\omega \times (J\omega)) \times r_A \right] dt', \quad (4.241)
\end{align*}
\]

\[
\begin{align*}
v_B(t') - v_B(0^-) &= -\int_{0^-}^{t'} \frac{B v_A}{m} + (r_A \times B v_A) \times r_B] dt' - \\
&\quad \int_{0^-}^{t'} \left[ \omega \times (\omega \times r_B) - g - (\omega \times (J\omega)) \times r_B \right] dt'.
\end{align*}
\]
For a large value of $\beta$, the last terms in (4.241) and (4.242) become insignificant. Therefore, these equations are simplified to

$$\begin{align*}
\vec{v}_A(t') - \vec{v}_A(0^-) &= -\int_{0^-}^{t'} \left[ \frac{\vec{B}\vec{v}_B}{m} + (\vec{r}_B \times \vec{B}\vec{v}_B) \times \vec{r}_B \right] dt' - \\
&\quad - \int_{0^-}^{t'} \frac{\vec{B}\vec{v}_B}{m} + (\vec{r}_B \times \vec{B}\vec{v}_B) \times \vec{r}_B dt', \\
\vec{v}_B(t') - \vec{v}_B(0^-) &= -\int_{0^-}^{t'} \left[ \frac{\vec{B}\vec{v}_A}{m} + (\vec{r}_A \times \vec{B}\vec{v}_A) \times \vec{r}_A \right] dt' - \\
&\quad - \int_{0^-}^{t'} \frac{\vec{B}\vec{v}_A}{m} + (\vec{r}_A \times \vec{B}\vec{v}_A) \times \vec{r}_A dt'.
\end{align*}$$

(4.243)

(4.244)

The corresponding differential equations of (4.243) and (4.244) can be described as

$$\begin{align*}
\ddot{\vec{v}}_A(t') &= -\left[ \frac{\vec{B}\vec{v}_A}{m} - \vec{r}_A \times (\vec{r}_A \times \vec{B}\vec{v}_A) \right] - \left[ \frac{\vec{B}\vec{v}_B}{m} - \vec{r}_A \times (\vec{r}_B \times \vec{B}\vec{v}_B) \right], \\
\ddot{\vec{v}}_B(t') &= -\left[ \frac{\vec{B}\vec{v}_A}{m} - \vec{r}_B \times (\vec{r}_A \times \vec{B}\vec{v}_A) \right] - \left[ \frac{\vec{B}\vec{v}_B}{m} - \vec{r}_B \times (\vec{r}_B \times \vec{B}\vec{v}_B) \right].
\end{align*}$$

(4.245)

(4.246)

If one replaces $\vec{r}_A \times $ and $\vec{r}_B \times $ with the matrix operators $R_A$ and $R_B$ as defined in (4.70), (4.245) and (4.246) will resolve into

$$\begin{bmatrix}
\ddot{\vec{v}}_A(t') \\
\ddot{\vec{v}}_B(t')
\end{bmatrix} = -\begin{bmatrix}
\frac{\vec{B}}{m} - R_A R_A B & \frac{\vec{B}}{m} - R_A R_B B \\
\frac{\vec{B}}{m} - R_B R_A B & \frac{\vec{B}}{m} - R_B R_B B
\end{bmatrix}
\begin{bmatrix}
\vec{v}_A(t') \\
\vec{v}_B(t')
\end{bmatrix}.$$  

(4.247)

Since the positions of both contacts do not change during the collision, the coefficient matrix in (4.247) is constant and (4.247) now represents a first-order linear system. Based on this result, it is clear that non-linear terms in the 3-D kinematics and dynamics do not affect the system in the $\beta$ scaled time $t'$. Therefore, one can use the same approach as implemented in the 2-D case to find the impulsive forces for this rigid body.
4.4.5 The Impulsive Formulation of a Quadruped on a Planar Surface

In order to show that the same approach as described in Section 4.4.1 can be used for multiple contacts of 3-D articulated bodies 3-D, the impact of a 3-D quadruped on a planar surface will be presented in this section. Like a 3-D single rigid body, first one needs to verify that both contact forces and velocities in this case are simplified to first-order linear systems in $t'$. Once this condition is satisfied, one can derive the impulsive formulation for different cases of impact in this problem.

According to the recursive Newton-Euler formulation shown in Figure 4, the joint torques as well as forces and torques at each foot during impact can be written as

$$\begin{bmatrix}
\tau_k \\
n_{4k} \\
f_{4k}
\end{bmatrix} = C_k \mathbf{x}_b + D_k x_k + \eta_k, \quad k = 1, 4 \quad (4.248)$$

The variables $\mathbf{x}_b$ and $x_k$ as defined in (2.13) represent $[\mathbf{v}_b^T \quad \mathbf{\omega}_b^T]^T$ and $[\mathbf{f}_{1k}^T \quad \mathbf{n}_{1k}^T \quad \mathbf{\hat{\theta}}_k^T]^T$ where

- $\mathbf{v}_b, \mathbf{\omega}_b$ = the translational and rotational accelerations at the body.
- $\mathbf{f}_{1k}$ = the force at the hip of leg $k$,
- $\mathbf{n}_{1k}$ = the torque at the hip of leg $k$,
- $\mathbf{\hat{\theta}}_k$ = the joint accelerations at leg $k$.

The variable $\eta_k$, on the other hand, represents the summation of the non-linear and gravitational terms for leg $k$. The corresponding forces and torques at the body are also specified as follows:

$$\begin{bmatrix}
m_b g \\
(J_b \mathbf{\omega}_b) \times \mathbf{\omega}_b
\end{bmatrix} = A \mathbf{x}_b + \sum_{k=1}^{4} B_k x_k,$$
\[
\begin{bmatrix}
  m_b g \\
  0 
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  (J_b \omega_b) \times \omega_b 
\end{bmatrix}
= Ax_b + \sum_{k=1}^{4} B_k x_k. 
\] (4.249)

If \([\tau_k^T, n_{4k}^T, f_{4k}^T]^T, [mg^T, 0]^T\) and \([0, (J_b \omega_b) \times \omega_b]^T\) are defined as \(F_k, F_b\) and \(\eta_b\), one can rewrite the system equation according to (4.248) and (4.249) as

\[
\begin{bmatrix}
  A & B_1 & B_2 & B_3 & B_4 \\
  C_1 & D_1 & 0 & 0 & 0 \\
  C_2 & 0 & D_2 & 0 & 0 \\
  C_3 & 0 & 0 & D_3 & 0 \\
  C_4 & 0 & 0 & 0 & D_4 
\end{bmatrix}
\begin{bmatrix}
  x_b \\
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 
\end{bmatrix}
= \begin{bmatrix}
  F_b \\
  F_1 \\
  F_2 \\
  F_3 \\
  F_4 
\end{bmatrix}
- \begin{bmatrix}
  \eta_b \\
  \eta_1 \\
  \eta_2 \\
  \eta_3 \\
  \eta_4 
\end{bmatrix}. 
\] (4.250)

In order to derive the impulsive formulation in this case, one also needs to rearrange both columns and rows of the matrix in (4.250) as well as the corresponding right-hand side so that

\[
\begin{bmatrix}
  x_b^T, x_1^T, \ldots, x_4^T 
\end{bmatrix}^T 
\rightarrow
\begin{bmatrix}
  x_b^T, \dot{\theta}_1^T, \ldots, \dot{\theta}_4^T, f_{11}^T, n_{11}^T, \ldots, f_{44}^T, n_{44}^T 
\end{bmatrix}^T
\]

and

\[
\begin{bmatrix}
  F_b^T, F_1^T, \ldots, F_4^T 
\end{bmatrix}^T 
\rightarrow
\begin{bmatrix}
  F_b^T, \tau_1^T, \tau_2^T, \ldots, \tau_4^T, f_{11}^T, n_{11}^T, \ldots, f_{44}^T, n_{44}^T 
\end{bmatrix}^T.
\]

Based on this modification, (4.250) is changed to

\[
\begin{bmatrix}
  A & B \\
  C & D 
\end{bmatrix}
\begin{bmatrix}
  \dot{x} \\
  y 
\end{bmatrix}
= \begin{bmatrix}
  \tau \\
  \ldots \\
  \ldots \\
  F_F \\
  \eta_t 
\end{bmatrix}. 
\] (4.251)

where

\[
\begin{align*}
x & = [\dot{v}_b^T, \omega_b^T, \dot{\theta}_1^T, \ldots, \dot{\theta}_4^T]^T, \\
y & = [f_{11}^T, n_{11}^T, \ldots, f_{44}^T, n_{44}^T]^T, \\
\tau & = [F_b^T, \tau_1^T, \ldots, \tau_4^T]^T, \\
F_F & = [f_{41}^T, n_{41}^T, \ldots, f_{44}^T, n_{44}^T]^T,
\end{align*}
\]
\[ \eta_u, \eta_l = \text{the partitions of } [\eta_1^T \eta_2^T \cdots \eta_4^T]^T \]

after rearranging rows.

The dimensions of \( A, B, C, D \) in this equation are equal to \( 18 \times 18, 18 \times 24, 24 \times 18 \) and \( 24 \times 24 \), respectively. The length of \( \tau \) and \( \eta_u \) are 18 and the length of \( \mathbf{F}_F \) and \( \eta_l \) are 24. One may also rewrite (4.251) as

\[
A\ddot{x} + By = \tau - \eta_u, \quad (4.252) \\
C\ddot{x} + Dy = \mathbf{F}_F - \eta_l, \quad (4.253)
\]

provided \( D \) is non-singular, one can determine \( y \) from (4.253) as

\[
y = D^{-1}[\mathbf{F}_F - \eta_l] - D^{-1}C\ddot{x}. \quad (4.254)
\]

The expression for \( \dot{x} \) in this case is also specified by substituting (4.254) into (4.252),

\[
[A - BD^{-1}C]\ddot{x} = [\tau - BD^{-1}\mathbf{F}_F] - [\eta_u + BD^{-1}\eta_l], \\
D\ddot{x} = \tau - BD^{-1}\mathbf{F}_F - [\eta_u + BD^{-1}\eta_l]. \quad (4.255)
\]

The joint velocities and body's velocity during impact can be determined by integrating both sides of (4.255) as follows:

\[
\int_0^t D\dot{x}dt = \int_0^t \tau dt - \int_0^t BD^{-1}\mathbf{F}_F dt - \int_0^t [\eta_u + BD^{-1}\eta_l]dt. \quad (4.256)
\]

Since the position of the vehicle's body and its links do not change during impact, \( D \), which is the function of these variables, can be regarded as constant. Thus, (4.256)
is resolved to

$$\mathcal{D}[\dot{x}(t) - \dot{x}(0^-)] = \int_{0^-}^{t} \tau dt - \int_{0^-}^{t} BD^{-1}F_p dt - \int_{0^-}^{t} [\eta_u + BD^{-1}\eta_l] dt.$$  \quad (4.257)

As described in the 2-D case, the joint velocities and body’s velocity after the collision can be specified according to the steady-state value of $\dot{x}(t')$ where $t' = \beta t$ and $\beta \to \infty$.

In this case, the expression of $x(t')$ can be obtained from (4.257) as follows:

$$\mathcal{D}[\dot{x}(t') - \dot{x}(0^-)] = \frac{1}{\beta} \int_{0^-}^{t'} \tau(t') dt' - \frac{1}{\beta} \int_{0^-}^{t'} BD^{-1}F_p(t') dt' - \frac{1}{\beta} \int_{0^-}^{t'} [\eta_u(t') + BD^{-1}\eta_l(t')] dt'.$$  \quad (4.258)

For this impact model, forces exerted by the ground at each contact are modeled by the damper which is in the $z$ direction. The torques at each contact, however, are assumed to be zero. Based on this impact model, $f_{4k}$ and $n_{4k}$ can be specified as

$$f_{4k} = \beta B v_{3k},$$  \quad (4.259)

$$n_{4k} = 0.$$  \quad (4.260)

where $v_{3k}$ represents the foot velocity and $B$ is defined as shown in (4.240). According to these constraints, $F_p(t')$ in (4.258) can be described as

$$F_p(t') = \beta F_p'(t')$$  \quad (4.261)

where

$$F_p'(t') = [(Bv_{31})^T 0 (Bv_{32})^T 0 (Bv_{33})^T 0 (Bv_{34})^T 0]^T.$$
Therefore, one can rewrite (4.258) in terms of $F'_F(t')$ as follows:

$$D[\dot{x}(t') - \dot{x}(0^-)] = \frac{1}{\beta} \int_{0^-}^{t'} \tau(t') dt' - \int_{0^-}^{t'} BD^{-1} F'_F(t') dt' - \frac{1}{\beta} \int_{0^-}^{t'} [\eta_u(t') + BD^{-1} \eta_l(t')] dt'. \quad (4.262)$$

For a large value of $\beta$, all the terms on the right-hand side of (4.262) except $F'_F(t')$ become insignificant and can be ignored. Thus, (4.262) is simplified to

$$D[\dot{x}(t') - \dot{x}(0^-)] = - \int_{0^-}^{t'} BD^{-1} F'_F(t') dt'. \quad (4.263)$$

One may also express (4.263) as

$$\mathcal{D}\ddot{x}(t') = - BD^{-1} F'_F(t')$$

or

$$\ddot{x}(t') = - D^{-1} BD^{-1} F'_F(t'). \quad (4.264)$$

From the recursive kinematic equations, the translational acceleration at each foot can be described in terms of body's and joint accelerations as

$$\ddot{v}_{3k} = C'_k \begin{bmatrix} \ddot{v}_b \\ \dot{\omega}_b \end{bmatrix} + D'_k \ddot{\theta}_k \eta'_k, \quad k = 1, 4 \quad (4.265)$$

where $\eta'_k$ represents the summation of all non-linear terms at leg $k$. According to (4.265), one can formulate the system equation for $\dot{v}_{31} \cdots \dot{v}_{34}$ as follows:

$$\begin{bmatrix} C'_1 & D'_1 & 0 & 0 & 0 \\ C'_2 & 0 & D'_2 & 0 & 0 \\ C'_3 & 0 & 0 & D'_3 & 0 \\ C'_4 & 0 & 0 & 0 & D'_4 \end{bmatrix} \begin{bmatrix} \ddot{v}_b \\ \dot{\omega}_b \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} \ddot{v}_{31} \\ \ddot{v}_{32} \\ \ddot{v}_{33} \\ \ddot{v}_{34} \end{bmatrix} - \begin{bmatrix} \eta'_1 \\ \eta'_2 \\ \eta'_3 \\ \eta'_4 \end{bmatrix},$$
\[ \Psi \dot{x} = \dot{v}_F + \eta'. \]  

(4.266)

Since the body’s position and orientation as well as joint positions do not change during impact, \( \Psi \) can be considered as constant in this case. According to (4.266), both body and joint velocities during impact can be written in terms of foot velocities as follows:

\[ \Psi [\dot{x}(t) - \dot{x}(0^-)] = [v_F(t) - v_F(0^-)] + \int_{0^-}^{t} \eta' dt \]  

(4.267)

One may express (4.267) in \( t' \) as

\[ \Psi [\dot{x}(t') - \dot{x}(0^-)] = [v_F(t') - v_F(0^-)] + \frac{1}{\beta} \int_{0^-}^{t'} \eta'(t') dt'. \]  

(4.268)

For a large value of \( \beta \), the non-linear term \( \eta'(t') \) also becomes insignificant. Therefore, (4.266) during impact can be linearized in \( t' \) as

\[ \Psi \dot{x}(t') = \dot{v}_F(t') \]  

(4.269)

Based on (4.269), one is able to describe (4.264) in terms of \( v_F(t') \) by multiplying both sides of this equation with \( \Psi \) as follows:

\[ \Psi \dot{x}(t') = -\Psi D^{-1}BD^{-1} F_F'(t'), \]  

(4.270)

\[ \dot{v}_F(t') = -\Psi D^{-1}BD^{-1} F'_F(t'). \]  

(4.271)

Based on the value of \( F_F'(t') \) in (4.261), the right-hand side of (4.271) can be simplified to

\[ \dot{v}_F(t') = -ASv_F(t') \]  

(4.272)
where $\mathbf{A}$ is a $12 \times 12$ matrix whose columns are columns $1 - 3, 7 - 9, 13 - 15$ and $19 - 21$ of $\Psi D^{-1}BD^{-1}$ in (4.271). The matrix $\mathbf{S}$, on the other hand, is defined as follows:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & 0 & 0 & 0 \\ 0 & \mathbf{B} & 0 & 0 \\ 0 & 0 & \mathbf{B} & 0 \\ 0 & 0 & 0 & \mathbf{B} \end{bmatrix}.$$ 

If the leg's configuration is not singular, $\Psi$ will have full-row rank and the rank of $\Psi D^{-1}BD^{-1}$ will be equal to 12. In this case, $\mathbf{A}$ will represent a non-singular matrix.

Based on the state equation in (4.272), one can apply the same technique as shown in the 2-D example to calculate the steady-state value of $v_F(t')$ and the impulsive force at each foot. This procedure, however, will be impossible unless the eigenvalues of $-\mathbf{A}$ are always negative. In other words, $\mathbf{A}$ has to be a symmetric and positive definite matrix.

In order to verify this condition, a systematic approach based on the Lagrangian formulation will be used. To apply this technique, one needs to firstly define the generalized coordinates for this system. Due to the fact that the body’s position and orientation do not change during impact and the fact that its velocities are also finite, it is possible to consider these parameters during the collision as the joint variables of a massless manipulator which is connected to the vehicle’s body as shown in Figure 39. This manipulator is composed of three prismatic joints 1, 2 and 3 whose axes coincide with the world coordinate system. Therefore, the displacement of these joints will represent the positions of the body with respect to the world. Their velocities, on the other hand, will correspond to body translational velocities.
Figure 39: A schematic diagram of a quadruped during impact where body translational and rotational velocities as well as body positions and orientations are represented by joint velocities and positions of a massless manipulator.

The manipulator also possesses three rotary joints 4, 5 and 6 whose velocities are in the same direction as body angular velocities. Based on this assumption, the original closed-chain mechanism can be considered as an open-chain system where the body positions and velocities during impact are described by the joint positions and velocities of this massless manipulator.

The generalized coordinates in this case can be chosen as the joint variables of
both vehicle and manipulator. From the Lagrange-Euler formulation, the generalized forces in this system can be described as

\[
\tau(t) = \mathcal{H}(q)\ddot{q}(t) + C(q, \dot{q}) + G + J^T F_R(t)
\]  

(4.273)

where

- \(\tau\) = generalized forces at each joint,
- \(q\) = the joint variables of both vehicle and manipulators,
- \(\mathcal{H}(q)\) = the inertial matrix for this system,
- \(C(q, \dot{q})\) = the coriolis and centripetal terms in this system,
- \(G\) = the gravitational terms in this system,
- \(J\) = the Jacobian matrix for this system.

The last term in (4.273) represents the contribution of contact forces and torques expressed in the generalized coordinates. According to (4.273), one can solve for the joint velocities by integrating both sides of this equation as follows:

\[
\int_{0^-}^{t} \tau(t) dt = \int_{0^-}^{t} \mathcal{H}(q)\ddot{q}(t) dt + \int_{0^-}^{t} [C(q, \dot{q}) + G] dt + \int_{0^-}^{t} J^T F_R(t) dt.
\]  

(4.274)

Since the joint position does not change during impact, the inertia matrix in this equation can be considered as constant. Therefore, (4.274) is simplified to

\[
\mathcal{H}(q)[\ddot{q}(t) - \dot{q}(0^-)] = \int_{0^-}^{t} \tau(t) dt - \int_{0^-}^{t} [C(q, \dot{q}) + G] dt - \int_{0^-}^{t} J^T F_R(t) dt.
\]  

(4.275)
To express (4.275) in $t'$, one may follow the same procedure as shown in (4.258) - (4.263). As a result, (4.275) in $t'$ can be written as

$$
\mathcal{H}(q)[\dot{q}(t') - \dot{q}(0^-)] = -\int_{0^-}^{t'} \mathcal{J}^T \mathbf{F}'_{F}(t') dt',
$$

$$
\dot{q}(t') - \dot{q}(0^-) = -\mathcal{H}^{-1}(q) \int_{0^-}^{t'} \mathcal{J}^T \mathbf{F}'_{F}(t') dt'
$$

$$
= -\int_{0^-}^{t'} \mathcal{H}^{-1}(q) \mathcal{J}^T \mathbf{F}'_{F}(t') dt'
$$

(4.276)

where $\mathbf{F}'_{F}(t')$ is defined according to (4.261). To express (4.276) in terms of foot’s velocities, one may multiply both sides of this equation with $\mathcal{J}$,

$$
\mathcal{J}[\dot{q}(t') - \dot{q}(0^-)] = -\int_{0^-}^{t'} \mathcal{J}^T \mathcal{H}^{-1}(q) \mathcal{J}^T \mathbf{F}'_{F}(t') dt',
$$

$$
\begin{bmatrix}
\mathbf{v}_{31}(t') \\
\omega_{31}(t') \\
\vdots \\
\mathbf{v}_{34}(t') \\
\omega_{34}(t')
\end{bmatrix}
- \begin{bmatrix}
\mathbf{v}_{31}(0^-) \\
\omega_{31}(0^-) \\
\vdots \\
\mathbf{v}_{34}(0^-) \\
\omega_{34}(0^-)
\end{bmatrix}
= -\int_{0^-}^{t'} \mathcal{J}^T \mathcal{H}^{-1}(q) \mathcal{J}^T \mathbf{F}'_{F}(t') dt'.
$$

(4.277)

The differential equation based on (4.277) can also be expressed as follows:

$$
\begin{bmatrix}
\dot{\mathbf{v}}_{31}(t') \\
\dot{\omega}_{31}(t') \\
\vdots \\
\dot{\mathbf{v}}_{34}(t') \\
\dot{\omega}_{34}(t')
\end{bmatrix}
= -\mathcal{J}^T \mathcal{H}^{-1}(q) \mathcal{J}^T \mathbf{F}'_{F}(t') = -\Phi \mathbf{F}'_{F}(t').
$$

(4.278)

Based on the value of $\mathbf{F}'_{F}(t')$ defined in (4.261), $\mathcal{J}$ in (4.272) corresponds to the rows and columns 1-3, 7-9, 13-15, 19-21 of $\Phi$ in (4.278). In other words, $\mathcal{J}$ in this case is one of the principle minors of $\Phi$ in (4.278). Therefore, if $\Phi$ is a symmetric and positive semi-definite matrix, $\mathcal{J}$ will be symmetric and either positive semi-definite or positive definite. However, since $\mathcal{J}$ has full rank for non-singular leg’s configurations, it can represent a symmetric and positive definite matrix under this condition. In order to
show that $\Phi$ is indeed symmetric and positive semi-definite, one has to verify that $H$
or the inertia matrix for this system is positive definite.

Usually, the inertia matrix is defined according to the torques and accelerations
at each joint. Without loss of generality, one may assume that all the joints are
initially resting in space and the effect due to gravitational and contact forces are not
considered when determining the expression of this matrix. Suppose the generalized
forces input $\tau$ are applied to each joint of the system in Figure 39. The translational
and rotational accelerations at each link, which are generated by these forces, are also
specified as $\ddot{v}_i$ and $\ddot{\omega}_i$. Both are measured in world coordinates.

If $\Gamma_i(q)$ represents the Jacobian from the base of the massless manipulator to the
center of gravity of this link, $v_i$ and $\omega_i$ can be expressed in the joint space as

$$
\begin{bmatrix}
  v_i \\
  \omega_i
\end{bmatrix} = \Gamma_i(q)\dot{q}.
$$

(4.279)

From (4.279), one may obtain $\dot{v}_i$ and $\dot{\omega}_i$ by taking the time derivative of (4.279),

$$
\begin{bmatrix}
  \dot{v}_i \\
  \dot{\omega}_i
\end{bmatrix} = \Gamma_i(q)\ddot{q} + \dot{\Gamma}_i(q, \dot{q})\dot{q}.
$$

(4.280)

Since $\dot{q}$ is equal to zero, (4.280) is simply resolved to

$$
\begin{bmatrix}
  \dot{v}_i \\
  \dot{\omega}_i
\end{bmatrix} = \Gamma_i(q)\ddot{q}.
$$

(4.281)

From the Newton-Euler formulation, the forces and torques at the center of gravity
of this link without gravitation can be described as

$$
\begin{bmatrix}
  f_i \\
  n_i
\end{bmatrix} = \begin{bmatrix}
  m_i I & 0 \\
  0 & J_i
\end{bmatrix} \begin{bmatrix}
  \ddot{v}_i \\
  \ddot{\omega}_i
\end{bmatrix} + \begin{bmatrix}
  0 \\
  \omega_i \times (J_i \omega_i)
\end{bmatrix}
$$

(4.282)
where \( m_i \) and \( J_i \) are the mass and the moment of inertia of this link at its center of gravity. Because the joint velocity is set to zero, the value of \( \omega_i \) in (4.282) is also equal to zero. Therefore, (4.282) can be simplified to

\[
\begin{bmatrix}
\mathbf{f}_i \\
\mathbf{n}_i
\end{bmatrix} =
\begin{bmatrix}
m_i I & 0 \\
0 & J_i
\end{bmatrix}
\begin{bmatrix}
\dot{\psi}_i \\
\dot{\omega}_i
\end{bmatrix}
\]

(4.283)

Substituting (4.281) into (4.283), one obtains

\[
\begin{bmatrix}
\mathbf{f}_i \\
\mathbf{n}_i
\end{bmatrix} = M_i \Gamma_i \ddot{q}
\]

(4.284)

where \( M \) represents the inertia matrix of this link which is defined as \( \begin{bmatrix} m_i I & 0 \\ 0 & J_i \end{bmatrix} \).

Both \( \mathbf{f}_i \) and \( \mathbf{n}_i \) in (4.284) can be expressed in terms of the generalized coordinates or the joint space as

\[
\tau_i = \Gamma_i^T \begin{bmatrix}
\mathbf{f}_i \\
\mathbf{n}_i
\end{bmatrix}
\]

(4.285)

Multiplying both sides of (4.284) with \( \Gamma_i^T \), one is able to determine the joint acceleration due to generalized forces \( \tau_i \) as follows:

\[
\tau_i = \Gamma_i^T M_i \Gamma_i \ddot{q}
\]

(4.286)

The same procedure can be applied to the others mechanisms as well. Therefore, the total generalized force input \( \tau \) which drives the whole system is determined by taking the summation of the result in (4.286) as follows:

\[
\tau = \sum_{i=1}^{k} \tau_i = \Gamma^T M \Gamma \ddot{q}
\]

(4.287)
where

\[ k = \text{the number of links in the system,} \]

\[ \Gamma^T = [\Gamma_1^T \Gamma_2^T \cdots \Gamma_k^T]^T, \]

\[ M = \begin{bmatrix}
    M_1 & 0 & 0 & \cdots & 0 \\
    0 & M_2 & 0 & \cdots & 0 \\
    0 & 0 & M_3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & M_k
\end{bmatrix}. \]

Since each \( M_i \) is positive definite and \( \Gamma \) is linearly independent, \( \Gamma^T M \Gamma \) in (4.287) will represent a positive definite matrix. This result shows that the inertia matrix \( \mathcal{H} \) for this system is positive definite. Therefore, \( \Phi \) in (4.278) can be factorized as

\[ \Phi = \mathcal{J} \mathcal{H}^{\frac{1}{2}} \mathcal{H}^{\frac{1}{2}T} \mathcal{J}^T \]

(4.288)

which is a symmetric and positive semi-definite matrix. Hence for a non-singular leg configurations, \( A \) is also a symmetric and positive definite matrix. As a result, the steady-state solution for \( \mathbf{v}_F(t') \) in (4.272) can be obtained in closed form and the approach presented in Section 4.4.1 can still be applied.

Like a single rigid body in two dimensions, the 3-D multiple contact cases can be divided into two categories. The first category represents the situation when all feet strike the ground at the same time. The second category, on the other hand, is a collision when some feet are resting on the ground while the others make contact with the ground. Both cases can be further divided into two possibilities. The first possibility is when all feet are still on the ground after impact whereas the second possibility represents the case where some feet start to leave the ground after the
collision. As illustrated in the 2-D example, the closed-form solution for the feet velocity in each category will also be derived for this problem.

**Case I. All feet strike the ground simultaneously**

For the planar surface, the contact forces are assumed to be in the z direction. Therefore, all elements of $S$ in (4.272) are equal to zero except the diagonal terms $(3,3), (6,6), (9,9)$ and $(12,12)$ which is equal to 1. Based on this description, one can express each component of $v_F(t')$ in (4.272) as follows:

$$
\dot{v}_{3z}(t') = -Vv_{3z}(t'), \quad (4.289)
$$

$$
\dot{v}_{3h}(t') = -Wv_{3z}(t') \quad (4.290)
$$

where

$$
V = \begin{bmatrix}
A(3,3) & A(3,6) & A(3,9) & A(3,12) \\
A(6,3) & A(6,6) & A(6,9) & A(6,12) \\
A(9,3) & A(9,6) & A(9,9) & A(9,12) \\
A(12,3) & A(12,6) & A(12,9) & A(12,12)
\end{bmatrix},
$$

$$
W = \text{a 8} \times 4 \text{ coefficient matrix which corresponds to rows 1,2,4,5,7,8,10,11 and columns 3,6,9,12 of } A.
$$

The matrix $V$ consists of the z-component elements of the matrix $A$. It is a principle minor of $A$, although it would be necessary to renumber rows and columns to put it in the more usual block appearance. Because $A$ is symmetric and positive definite, so is $V$. 
The solution of (4.289) can be described as
\[ v_{3z}(t') = e^{-Vt'}v_{3z}(0^-). \] (4.291)

The expression of \( e^{-Vt'} \) in this equation may be calculated as
\[ e^{-Vt'} = Qe^{\Lambda t'}Q^{-1} \] (4.292)

where
\[ Q = \text{the matrix whose columns represent the eigenvectors of } V, \]
\[ \Lambda = \text{the diagonal matrix contains all eigenvalues of } -V. \]

Based on (4.292), the solution in (4.291) can be generally specified as
\[
\begin{bmatrix}
 v_{31z}(t') \\
 v_{32z}(t') \\
 v_{33z}(t') \\
 v_{34z}(t')
\end{bmatrix} = \begin{bmatrix}
 p_{11} & p_{12} & p_{13} & p_{14} \\
 p_{21} & p_{22} & p_{23} & p_{24} \\
 p_{31} & p_{32} & p_{33} & p_{34} \\
 p_{41} & p_{42} & p_{43} & p_{44}
\end{bmatrix} \begin{bmatrix}
 e^{\lambda_1 t'} \\
 e^{\lambda_2 t'} \\
 e^{\lambda_3 t'} \\
 e^{\lambda_4 t'}
\end{bmatrix},
\] (4.293)

\[
= [y_1(t') \quad y_2(t') \quad y_3(t') \quad y_4(t')]^T.
\] (4.294)

Suppose positive z component represents a unit vector directing toward the ground. Therefore, in order for the foot \( i \) to leave the ground after impact, its normal velocity \( v_{3izi}(t') \) must be less than zero. However, according to (4.293) \( v_{3izi}(t') \) will be less than zero if there is at least one \( p_{ij} \) in each row which is also negative. In other words, if \( p_{11} \cdots p_{44} \) are greater than zero, \( v_{3izi}(t') \) will be positive and that foot will remain on the ground after the collision. Thus, by examining the sign of \( p_{ij} \) in each row, one can predict whether that foot may leave the ground after impact.

First, let's assume that all the feet remain on the ground after impact. Since \( \lambda_1 \cdots \lambda_4 \) are less then zero, \( v_{31z}(t') \cdots v_{34z}(t') \) are asymptotically approaching zero as \( t' \to \infty \),
According to (4.289) and (4.290), both $x$ and $y$ components of the foot’s velocity can be described in terms of $v_{3z}(t')$ as

$$
\dot{v}_{3h}(t') = W V^{-1} v_{3z}(t').
$$

(4.296)

By integrating both sides of (4.296) from $0^-$ to $\infty$, one can specify the steady-state value of $v_{3h}(t')$ as follows:

$$
v_{3h}(t') \mid_{t' = \infty} = v_{3h}(0^-) = W V^{-1} v_{3z}(0^-),
$$

(4.297)

The last quantities which need to be specified are the impulsive forces at each contact.

For this impact model, the total impulse which occurs during the collision is equal to

$$
F_z(t) = -\beta \int_{0^-}^{t} v_{3z}(t) dt.
$$

(4.298)

This impulse in $t'$ can be specified as

$$
F_z(t') = -\beta \int_{0^-}^{t'} v_{3z}(t') \frac{dt'}{\beta},
$$

$$
= -\int_{0^-}^{t'} v_{3z}(t') dt'.
$$

(4.299)

As explained in the 2-D example, one may calculate the impulsive force in this case as

$$
F_{Iz} = -\int_{0^-}^{\infty} v_{3z}(t') dt',
$$

$$
= -\int_{0^-}^{\infty} e^{-\nu t'} v_{3z}(0^-) dt',
$$

$$
= V^{-1} \int_{0^-}^{\infty} -Ve^{-\nu t'} dt' v_{3z}(0^-),
$$

$$
= V^{-1}[e^{-\nu t'} \mid_{t'=\infty} - I] v_{3z}(0^-).
$$

(4.300)
Since $V$ is a positive definite matrix, $e^{-Vt'}$ is asymptotically approaching zero as $t' \to \infty$. Therefore, one can determine the impulsive force in this case as

$$F_{lz} = -V^{-1}v_{3z}(0^-). \quad (4.301)$$

Another possibility is when there is at least one foot leaving the ground after impact. Under this circumstance, first one needs to find the time in $t'$ when

$$v_{3iz}(t') = 0, \quad i = 1, 4. \quad (4.302)$$

According to (4.293), each one of these velocities are represented by the summation of four exponential functions. Therefore, it is very difficult to determine the solution of (4.302) in closed form. To specify the solution of (4.302), one may use an iterative method such as the Newton-Raphson’s method [15].

Suppose foot 1 starts to leave the ground at $t' = t'_1$, or

$$v_{3iz}(t') \big|_{t'=t_1} = 0. \quad (4.303)$$

From (4.294), the value of $v_{32z}(t'), v_{33z}(t'), v_{34z}(t')$ at $t' = t'_1$ can be written as

$$\begin{bmatrix} v_{32z}(t'_1) \\ v_{33z}(t'_1) \\ v_{34z}(t'_1) \end{bmatrix} = \begin{bmatrix} y_2(t'_1) \\ y_3(t'_1) \\ y_4(t'_1) \end{bmatrix}. \quad (4.304)$$

The corresponding $x$ and $y$ components of these velocities are described by

$$v_{3h}(t'_1) = WV^{-1}[v_{3z}(t'_1) - v_{3z}(0^-)] + v_{3h}(0^-). \quad (4.305)$$

The impulsive force once this foot starts to leave the ground is equal to

$$F_{lz} \big|_{t'=t'_1} = V^{-1}[e^{-Vt'_1} - I]v_{3z}(0^-). \quad (4.306)$$
At this point, the damper which is attached to foot 1 needs to be excluded from the impact model. Thus, the contact force contributed by this damper will be equal to zero for \( t' \geq t'_1 \). Based on this modification, the diagonal element of \( S \) which corresponds to \( v_{31z}(t') \) is set to zero. Therefore, both \( \dot{v}_3(t') \) and \( \dot{v}_3z(t') \) for \( t' \geq t'_1 \) will be expressed in terms of \( v_{32z}(t'), v_{33z}(t'), v_{34z}(t') \) instead.

According to this new \( S \) matrix, one can rewrite (4.289) and (4.290) as follows:

\[
\begin{align*}
\dot{v}_{3z}(t') &= - V_1 v_{3z}(t'), \\
\begin{bmatrix}
\dot{v}_{3h}(t') \\
\dot{v}_{31z}(t')
\end{bmatrix}
&= - \begin{bmatrix}
W_1 \\
W_2
\end{bmatrix} v_{3z}(t'), \quad t' \geq t'_1
\end{align*}
\]  

(4.307)  
(4.308)

where

\[
v_{3z}(t')^T = [ v_{32z}(t'), v_{33z}(t'), v_{34z}(t') ];
\]

\[
V_1 = \text{a principle minor of } A \text{ which is equal to }
\begin{bmatrix}
A(6,6) & A(6,9) & A(6,12) \\
A(9,6) & A(9,9) & A(9,12) \\
A(12,6) & A(12,9) & A(12,12)
\end{bmatrix},
\]

\[
W_1 = \text{a } 8 \times 3 \text{ coefficient matrix which corresponds to}
\]
\begin{enumerate}
\item rows 1, 2, 4, 5, 7, 8, 10, 11 and columns 6, 9, 12
\end{enumerate}

of \( A \),

\[
W_2 = [ A(3,6), A(3,9), A(3,12) ]
\]

Since \( V_1 \) is symmetric and positive definite, the solution of (4.307) is also specified as

\[
v_{3z}(t') = e^{-V_1(t'-t'_1)} v_{3z}(t'_1), \quad t' \geq t'_1.
\]  

(4.309)

To calculate \( e^{-V_1(t'-t'_1)} \), one may use the same approach as shown in (4.292). There-
fore, (4.308) can be simplified into the same form as (4.293),

\[
\begin{bmatrix}
v_{32}(t') \\
v_{33}(t') \\
v_{34}(t')
\end{bmatrix} = \begin{bmatrix}
\hat{p}_{11} & \hat{p}_{12} & \hat{p}_{13} \\
\hat{p}_{21} & \hat{p}_{22} & \hat{p}_{23} \\
\hat{p}_{31} & \hat{p}_{32} & \hat{p}_{33}
\end{bmatrix} \begin{bmatrix}
e^{\lambda_1(t'-t_1')} \\
e^{\lambda_2(t'-t_1')} \\
e^{\lambda_3(t'-t_1')}
\end{bmatrix},
\]

(4.310)

\[
\begin{bmatrix}
y_1'(t') \\
y_2'(t') \\
y_3'(t')
\end{bmatrix}^T
\]

(4.311)

where \(\lambda_1' \cdots \lambda_3'\) are the eigenvalues of \(-V_1\). According to (4.309), similar observations to (4.293) can be applied to each \(p'_{ij}\) to examine whether these feet can further leave the ground for any \(t' \geq t_1'\). However, it is also important to examine whether \(v_{31z}(t')\) will become zero again for \(t' \geq t_1'\). This condition represents the possibility of any repeating event at a contact foot. Unlike a 2-D single rigid body with two contact points, \(v_{31z}(t')\), which is a linear combination of \(v_{32z}(t'), v_{33z}(t')\) and \(v_{34z}(t')\), can have more than one transition in \(t'\). According to the studies on several random cases, a finite amount of transitions can be found for each contact velocity.

According to (4.307) and (4.308), \(v_{31z}(t')\) can be described in terms of \(v'_{3z}(t')\) as

\[
v_{31z}(t') = W_2 V_1^{-1} v'_{3z}(t').
\]

(4.312)

Therefore, one can determine \(v_{31z}(t')\) for \(t' > t_1'\) as follows:

\[
\int_{t_1'}^{t'} v_{31z}(t')dt' = \int_{t_1'}^{t'} W_2 V_1^{-1} v'_{3z}(t')dt',
\]

\[
v_{31z}(t') - v_{31z}(t_1') = W_2 V_1^{-1} [v'_{3z}(t') - v'_{3z}(t_1')],
\]

\[
v_{31z}(t') = W_2 V_1^{-1} [v'_{3z}(t') - v'_{3z}(t_1')] + v_{31z}(t_1').
\]

(4.313)

If no transitions occur at the other contact feet and \(v_{31z}(t')\) has only one transition, the steady-state value of \(v'_{3z}(t')\) in (4.310) will correspond to,

\[
\lim_{t' \to \infty} v'_{3z}(t') = 0.
\]

(4.314)
Based on (4.313), the steady-state value of \( v_{31z}(t') \) is determined as

\[
\lim_{t' \to \infty} v_{31z}(t') = -W_2V_1^{-1}v_{3z}(t_1') + v_{31z}(t_1').
\] (4.315)

The same procedure as shown in (4.313) can be applied to (4.308). Hence, \( v_{3h}(t') \) for \( t' > t_1' \) can be described as

\[
v_{3h}(t') = W_1V_1^{-1}[v_{3z}(t') - v_{3z}(t_1')]
\] (4.316)

where its steady-state value is described by

\[
\lim_{t' \to \infty} v_{3h}(t') = -W_1V_1^{-1}v_{3z}(t_1').
\] (4.317)

The impulsive force after \( v_{31z}(t') \) is equal to zero can be calculated as

\[
F_{Iz} \bigg|_{t_1'}^{\infty} = -\int_{t_1'}^{\infty} v_{3z}'(t')dt',
\]

\[
= -\int_{t_1'}^{\infty} e^{-V_1t'}v_{3z}'(t_1')dt',
\]

\[
= V_1^{-1} \int_{0^-}^{\infty} -Ve^{-V_1t'}dt'v_{3z}'(t_1'),
\]

\[
= V_1^{-1} [e^{-V_1t_1'}]v_{3z}'(t_1'),
\]

\[
= -V_1^{-1}v_{3z}'(t_1').
\] (4.318)

Therefore, the total impulsive force in this case is equal to the summation of (4.306) and (4.318),

\[
F_{Iz} = V^{-1}[e^{Vt_1'} - I]v_{3z}(0^-) + \left[ -V_1^{-1}v_{3z}'(t_1') \right].
\] (4.319)

On the other hand, if at least one element of \( v_{3z}'(t') \) becomes zero at \( t' > t_1' \), the same procedure as described in (4.303) – (4.313) will be reapplied and a similar form of state equation to (4.307) and (4.308) will be obtained. This procedure will be repeated until there are no transitions occur in \( \beta \) scaled time \( t' \).
Case II. Some feet are resting on the ground while the others make contact

The case where some feet are resting on the ground is very commonly found in a walking machine. For instance, when a quadruped is trotting, the feet across one diagonal of the body are always resting on the ground while the other diagonal pair strike the ground. As mentioned in the 2-D case, it is necessary to first identify the motion of these feet before an appropriate impact model can be selected.

A serious deficiency of the approach in [27] was that in some cases there was no impact model which was consistent with the direction of travel. It is important to know for this technique whether there always exists a consistent model. The following discussion will show that not only does a unique consistent assumption always exists, but that it can be determined easily.

**Theorem 1** For the impact model represented by normal dampers in which the contact velocities are linearized in the \( \beta \) scaled time \( t' \), there always exists a unique assumption for the model that is consistent with the motion of the contacts which are resting on the ground, i.e., if the motion is toward the ground, a normal damper will be included at that contact and if the motion is away from the ground, the damper must be removed.

To verify that this theorem is always true, one may consider a problem of multiple contacts consisting of four contact points. Suppose contacts 1 and 2 are simply resting on the ground while collisions occur at contacts 3 and 4. Since each of the resting contacts may remain or leave the ground as a result of these collisions, there are four
possibilities which need to be considered. First, both 1 and 2 might remain on the ground. Second, contact 1 could remain on the ground while contact 2 leaves the ground. Third, contact 1 could leave the ground and contact 2 might remain on the ground. Finally, both contacts 1 and 2 could leave the ground after impact. Based on these possibilities, one can construct four initial impact models consisting of different sets of dampers at contacts 1 and 2.

According to (4.289), the change of each contact velocity in z at $t' = 0^-$ can be described as

$$
\dot{v}_{3z}(0^-) = -V v_{3z}(0^-)
$$

(4.320)

where

$$
V = \begin{bmatrix}
A(3,3) & A(3,6) & A(3,9) & A(3,12) \\
A(6,3) & A(6,6) & A(6,9) & A(6,12) \\
A(9,3) & A(9,6) & A(9,9) & A(9,12) \\
A(12,3) & A(12,6) & A(12,9) & A(12,12)
\end{bmatrix}
$$

Based on (4.272), each column of $V$ represents the contribution of contact forces generated by the damper at that contact. Therefore, when a damper is not included at the contact, the column of $V$ corresponding to that contact point can be simply be set to zero. Based on this property, one can determine $\dot{v}_{31z}(0^-)$ and $\dot{v}_{32z}(0^-)$ subject to each possible impact model by setting columns 1 or 2 of $V$ in (4.320) to zero. However, since contacts 1 and 2 simply rest on the ground and $v_{31z}(0^-) = v_{32z}(0^-) = 0$, the contents of the first two columns has no effect on $\dot{v}_{31z}(0^-)$ and $\dot{v}_{32z}(0^-)$. In other words, $\dot{v}_{31z}(0^-)$ and $\dot{v}_{32z}(0^-)$ are independent of the impact model, and specifically are
Knowing this property, one may use the first-order approximation of the Taylor series to identify the motion of contacts 1 and 2 in this problem. Since the initial positions and velocities of both contacts are equal to zero, the sign of the velocities for small $t'$ depends on the sign of $\dot{v}_{31z}(0^-)$ and $\dot{v}_{32z}(0^-)$.

According to (4.321), there always exists only one consistent impact model based on the value of $\dot{v}_{31z}(0^-)$ and $\dot{v}_{32z}(0^-)$. If $\dot{v}_{31z}(0^-)$ is greater than zero, contact 1 will move toward the ground after impact and a vertical damper will be included at this contact. On the other hand, the negative value of $\dot{v}_{31z}(0^-)$ indicates the motion which is away from the ground and there will be no damper at this contact. Therefore, one can always find a consistent impact model with respect to the motion of each contact that rests on the ground.

This verifies the consistency of the impact model proposed in this work as compared to the approach in [27]. However, one may consider a higher order approximation of the normal velocities if the first order approximation is equal to zero. Under this circumstance, it is still possible to find an impact model which is consistent with the direction of travel. After the consistent impact model is obtained, this case will be similar to that of case I. Therefore, all previous impulsive formulation, which is described in case I, can be applied.

Once the impulsive forces are obtained, one can determine both joint velocity and body's velocity after impact from (4.263) as follows:

$$\mathcal{D}_{t' \to -\infty} [\dot{x}(t') - \dot{x}(0^-)] = -BD^{-1} \int_{0^-}^{t'} F'_F(t') dt', $$

\[ \begin{bmatrix} \dot{v}_{31z}(0^-) \\ \dot{v}_{32z}(0^-) \end{bmatrix} = \begin{bmatrix} A(3, 9) & A(3, 12) \\ A(6, 9) & A(6, 12) \end{bmatrix} \begin{bmatrix} v_{33z}(0^-) \\ v_{34z}(0^-) \end{bmatrix}. \]
\[
\lim_{t' \to \infty} \dot{x}(t') - \dot{x}(0^-) = -D^{-1}BD^{-1} \int_{0^-}^{t'} F_F(t') dt',
\]
\[
\dot{x}(t') |_{t' \to \infty} = x(0^-) - D^{-1}BD^{-1} \int_{0^-}^{t'} F_F(t') dt'.
\] (4.322)

Based on the definition of \( F_F(t') \) in (4.261), the integration on the right-hand side of (4.322) can be described as
\[
\int_{0^-}^{t'} F_F(t') dt' = \begin{bmatrix}
\int_{0^-}^{t'} (Bv_{31}(t'))^T dt' \
\int_{0^-}^{t'} (Bv_{32}(t'))^T dt' \
\int_{0^-}^{t'} (Bv_{33}(t'))^T dt'
\end{bmatrix}. \] (4.323)

For a planar surface, each \( \int_{0^-}^{t'} Bv_{3i}(t') dt' \) is simplified to
\[
\begin{bmatrix} 0 \\ 0 \\ \int_{0^-}^{t'} v_{3iz}(t') dt' \end{bmatrix}, \quad i = 1, 4. \] (4.324)

Once \( t' \to \infty \), this expression then resolves to the impulsive force at that foot,
\[
\lim_{t' \to \infty} \int_{0^-}^{t'} Bv_{3i}(t') dt' = \begin{bmatrix} 0 \\ 0 \\ \lim_{t' \to \infty} \int_{0^-}^{t'} v_{3iz}(t') dt' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F_{iz} \end{bmatrix}, \quad i = 1, 4. \] (4.325)

Therefore, (4.322) can be described in terms of impulsive forces as
\[
\dot{x}(t') |_{t'=\infty} = \dot{x}(0^-) - D^{-1}BD^{-1} F_{i},
\] (4.326)

The variable \( F_{i} \) in this formulation represents a 24 x 1 vector consisting of the impulsive force at each contact foot. Since \( \dot{x} |_{t'=\infty} \) is equal to \( \dot{x} |_{t=0^+} \), (4.326) can be used to calculate the joint velocities and body velocities of this quadruped after impact.
4.4.6 Numerical Examples of Multiple Contacts in a 3-D Quadruped

Some examples of the impact in a quadruped are illustrated in this section. These situations demonstrate the application of the impulsive formulation presented in Section 4.4.5. The origin of the world coordinate system is placed underneath the center of gravity of the vehicle and its orientation is also the same as the body coordinate system.

Example I. All feet strike the ground simultaneously

The first example illustrates the situation when all four feet strike the ground at the same time. The contact surface is assumed to be horizontal in this case. The mass of vehicle’s body used in this example is equal to 93 lbs. Its moment of inertia about the center of mass are described by \[
\begin{bmatrix}
155 & 0 & 0 \\
0 & 806 & 0 \\
0 & 0 & 899
\end{bmatrix}.
\]
The center of gravity of the body is located at 5.1 ft above the ground.

The weight of each link is specified as 0.6141, 1.8422 and 1.8422 lbs, respectively. The moment of inertia of these links about their coordinates are shown in Table 6. Table 7 displays the positions of center of gravity for each link as well as the joint angles. The foot positions, on the other hand, are specified as follows:

\[
[-0.2 \ 3.0 \ 0.0]^T, [-0.2 \ 3.0 \ 0.0]^T, [-7.0 \ -3.0 \ 0.0]^T, [-8.0 \ 3.0 \ 0.0]^T.
\]

Based on these foot positions, the center of gravity of this quadruped will be outside of the feet when they contact with the ground. As a result, some feet may not remain on the ground after the collision. The velocities at the body, joints and each contact foot before impact are also listed in in Table 8.
Table 6: The moment of inertia matrix about the link coordinate system for example I.

<table>
<thead>
<tr>
<th>Leg</th>
<th>link 1</th>
<th>link 2</th>
<th>link 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.061 0 0</td>
<td>0.358 0 0.562</td>
<td>0.777 0 0.675</td>
</tr>
<tr>
<td></td>
<td>0 0.061 0</td>
<td>0 1.411 0</td>
<td>0 1.411 0</td>
</tr>
<tr>
<td></td>
<td>0 0 0.019</td>
<td>0.562 0 1.110</td>
<td>0.675 0 0.692</td>
</tr>
<tr>
<td>2</td>
<td>0.061 0 0</td>
<td>0.778 0 0.675</td>
<td>0.357 0 0.562</td>
</tr>
<tr>
<td></td>
<td>0 0.061 0</td>
<td>0 1.411 0</td>
<td>0 1.411 0</td>
</tr>
<tr>
<td></td>
<td>0 0 0.019</td>
<td>0.675 0 0.691</td>
<td>0.562 0 1.111</td>
</tr>
<tr>
<td>3</td>
<td>0.061 0 0</td>
<td>1.382 0 -0.193</td>
<td>0.118 0 0.278</td>
</tr>
<tr>
<td></td>
<td>0 0.061 0</td>
<td>0 1.411 0</td>
<td>0 1.411 0</td>
</tr>
<tr>
<td></td>
<td>0 0 0.019</td>
<td>-0.193 0 0.086</td>
<td>0.278 0 1.351</td>
</tr>
<tr>
<td>4</td>
<td>0.061 0 0</td>
<td>1.361 0 0.253</td>
<td>0.122 0 0.288</td>
</tr>
<tr>
<td></td>
<td>0 0.061 0</td>
<td>0 1.411 0</td>
<td>0 1.411 0</td>
</tr>
<tr>
<td></td>
<td>0 0 0.019</td>
<td>0.253 0 0.107</td>
<td>0.288 0 1.346</td>
</tr>
</tbody>
</table>

Table 7: The center of gravity for each link and its joint angle measured in the link coordinate system for example I.

<table>
<thead>
<tr>
<th>Leg</th>
<th>Link</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ₁</th>
<th>θ₂</th>
<th>θ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4.5</td>
<td>-3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>-61.899°</td>
<td>18.699°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.177</td>
<td>-3.0</td>
<td>-2.893</td>
<td>0.0°</td>
<td>-61.899°</td>
<td>18.699°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.826</td>
<td>-3.0</td>
<td>-1.093</td>
<td>0.0°</td>
<td>-61.899°</td>
<td>18.699°</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>-43.156°</td>
<td>-18.786°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.474</td>
<td>3.0</td>
<td>-2.506</td>
<td>0.0°</td>
<td>-43.156°</td>
<td>-18.786°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.124</td>
<td>3.0</td>
<td>-0.706</td>
<td>0.0°</td>
<td>-43.156°</td>
<td>-18.786°</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-4.5</td>
<td>-3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>8.295°</td>
<td>-86.146°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.284</td>
<td>-3.0</td>
<td>-2.116</td>
<td>0.0°</td>
<td>8.295°</td>
<td>-86.146°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-5.534</td>
<td>-3.0</td>
<td>-0.316</td>
<td>0.0°</td>
<td>8.295°</td>
<td>-86.146°</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>-10.989°</td>
<td>-66.403°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.786</td>
<td>3.0</td>
<td>-2.128</td>
<td>0.0°</td>
<td>-10.989°</td>
<td>-66.403°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-6.536</td>
<td>3.0</td>
<td>-0.328</td>
<td>0.0°</td>
<td>-10.989°</td>
<td>-66.403°</td>
</tr>
</tbody>
</table>
Table 8: Initial velocities at the body, joint and each contact foot before impact for example I.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>Foot</th>
<th>$v_{3x}$</th>
<th>$v_{3y}$</th>
<th>$v_{3z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1109</td>
<td>4.3336</td>
<td>-7.6002</td>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>5.5</td>
</tr>
<tr>
<td>2</td>
<td>-0.1109</td>
<td>-2.9569</td>
<td>6.8202</td>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>-0.1109</td>
<td>-0.1925</td>
<td>-0.5178</td>
<td>3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>-0.1109</td>
<td>-0.1398</td>
<td>-0.7893</td>
<td>4</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>$\mathbf{v}_b$</td>
<td>[0 0 1]$^T$</td>
<td>$\omega_b$</td>
<td>[0.1 0.2 0]$^T$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Based on the parameters in Table 6 and 7, one can write the state equation in (4.289) as follows:

$$
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix} =
\begin{bmatrix}
0.7896 & 0.0001 & -0.0028 & 0.0091 \\
0.0001 & 1.5215 & 0.0037 & -0.0021 \\
-0.0028 & 0.0037 & 1.5868 & -0.0018 \\
0.0091 & -0.0021 & -0.0018 & 1.6229
\end{bmatrix}
\begin{bmatrix}
v_{31z}(t') \\
v_{32z}(t') \\
v_{33z}(t') \\
v_{34z}(t')
\end{bmatrix}
$$

(4.327)

The coefficient matrix $V$ in this equation can be obtained from the results in (4.250), (4.251), (4.271) and (4.272).

Figure 40 illustrates the trajectories of $v_{31z}(t') \cdots v_{34z}(t')$ as calculated from (4.327). Each plot simply displays the first 10 sec of the foot velocities $v_{31z}(t') \cdots v_{34z}(t')$. From this figure, $v_{31z}(t'), v_{32z}(t')$ and $v_{33z}(t')$ are all asymptotically approaching zero as $t' \to \infty$. Therefore, these feet will remain on the ground after the collision. Foot 4, on the other hand, starts to leave the ground after impact.

In order to determine the time where $v_{34z}(t')$ become zero, one may apply the method of Newton and Raphson to (4.327). In this case, the crossing time for $v_{34z}(t')$ is calculated as 1.2858 sec. Because $v_{34z}(t')$ starts to change the sign after $t' = 1.2858$
Figure 40: The trajectories of vertical foot velocities $v_{31z}(t') \cdots v_{34z}(t')$ when all feet strike the ground simultaneously.
sec, the damper at this foot will not be included in the impact model for \( t' \geq 1.2858 \) sec and the new expression for \( v_{31z}(t') \ldots v_{34z}(t') \) can be formed by simply setting the last column of \( V \) in (4.327) to zero,

\[
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix} = -
\begin{bmatrix}
0.7896 & 0.0001 & -0.0028 \\
0.0001 & 1.5215 & 0.0037 \\
-0.0028 & 0.0037 & 1.5868 \\
0.0991 & -0.0021 & -0.0018
\end{bmatrix}
\begin{bmatrix}
\nu_{31z}(t') \\
\nu_{32z}(t') \\
\nu_{33z}(t') \\
\nu_{34z}(t')
\end{bmatrix}, \quad t' \geq 1.2858.
\] (4.328)

From (4.328), one can specify the state equation for \( v_{31z}(t'), v_{32z}(t') \) and \( v_{33z}(t') \) as follows:

\[
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix} = -
\begin{bmatrix}
0.7896 & 0.0001 & -0.0028 \\
0.0001 & 1.5215 & 0.0037 \\
-0.0028 & 0.0037 & 1.5868 \\
0.0991 & -0.0021 & -0.0018
\end{bmatrix}
\begin{bmatrix}
\nu_{31z}(t') \\
\nu_{32z}(t') \\
\nu_{33z}(t') \\
\nu_{34z}(t')
\end{bmatrix}, \quad t' \geq 1.2858.
\] (4.329)

For this particular example, all of these velocities are asymptotically approaching zero as \( t' \to \infty \). Therefore, the final states of \( v_{31z}(t'), v_{32z}(t'), v_{33z}(t') \) in this case are equal to 0. To determine the final state of \( v_{34z}(t') \), one may use the result in (4.315) as follows:

\[
v_{34z}(t') = W_2 V_1^{-1}[v_{34z}(t') - v_{34z}(t')] + v_{34z}(t'),
\]

\[
v_{34z}(t') \mid t' = \infty = W_2 V_1^{-1}[v_{34z}(t') \mid t' = \infty - v_{34z}(t')] + v_{34z}(t').
\] (4.330)

The term \( W_2 \) in (4.330) represents the last row of \( V \) in (4.328). Since \( v_{31z}(t'), v_{32z}(t') \) and \( v_{33z}(t') \) are asymptotically approaching zero as \( t' \to \infty \), \( v_{34z}(t') \mid t' = \infty \) in (4.330) is then equal to zero. Based on the value of these parameters, \( v_{34z}(t') \mid t' = \infty \) in (4.330) can be calculated as -0.0220 m/sec. This result illustrates a non-repeating transition at foot 4 in this example.
The final quantity which needs to be determined is the impulsive force at each foot. In this case, one may consider the impulsive force at two different period of time. The first period starts from \( t' = 0 \) to \( t' = 1.2858 \) whereas the second one continues from \( t' = 1.2858 \) till \( t' \to \infty \). During the first period, all the dampers are included in the impact model. Therefore, one can determine corresponding impulsive forces using (4.306) and (4.327),

\[
F_{Iz} \mid_{t'=0}^{1.2858} = V^{-1}[e^{-1.2858V} - I]v_{3z}(0^-),
\]

\[
= [-4.4418 - 3.1037 - 0.0549 - 0.0406]^T. \tag{4.331}
\]

The impulsive forces for the second period, on the other hand, are obtained from (4.318) and (4.329),

\[
F_{Iz} \mid_{t'=1.2858}^{\infty} = -[(V^{-1}v_{3z}(t'))^T : 0]^T,\]

\[
= [-2.5236 - 0.5108 - 0.0120 0]^T. \tag{4.332}
\]

The term \( v_{3z}(t') \mid_{t'=1.2858} \) in this equation is determined from the closed-form solution of (4.328). Since foot 4 starts to leave the ground at \( t' = 1.2858 \) sec, the impulsive force at this foot within this period is equal to zero.

As a result of (4.331) and (4.332), the impulsive force for this collision is equal to

\[
F_{Iz} = [-6.9654 - 3.6145 - 0.0669 - 0.0406]^T \tag{4.333}
\]

Based on this impulsive force, one can determine the body and joint velocities after impact using (4.326). The result of this computation is displayed in Table 9.
Table 9: The velocities at the body and joint after the collision for example I.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\dot{\theta}_1$</th>
<th>$\dot{\theta}_2$</th>
<th>$\dot{\theta}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.086</td>
<td>4.1150</td>
<td>-4.4763</td>
</tr>
<tr>
<td>2</td>
<td>-0.0871</td>
<td>-3.4845</td>
<td>9.8541</td>
</tr>
<tr>
<td>3</td>
<td>-0.0606</td>
<td>-0.129</td>
<td>-0.5266</td>
</tr>
<tr>
<td>4</td>
<td>-0.0574</td>
<td>-0.1079</td>
<td>-0.7108</td>
</tr>
</tbody>
</table>

\[ \mathbf{v}_b = \begin{bmatrix} -0.0432 & 0 & 1.034 \end{bmatrix}^T \]
\[ \mathbf{\omega}_b = \begin{bmatrix} 0.0725 & 0.1786 & -0.0138 \end{bmatrix}^T \]

**Example II. Some feet are resting on the ground while the others make the contacts and no transitions occur after collisions**

The second example is to demonstrate the situation in Case II where some feet simply rest on the ground during impact. The mass and moment of inertia for both leg and body are assumed to be the same as previous example. The foot positions, however, are changed to $[6.2 \ -3.0 \ 0.0]^T$, $[6.0 \ 3.0 \ 0.0]^T$, $[-6.5 \ -3.0 \ 0.0]^T$ and $[-6.8 \ 3.0 \ 0.0]^T$ respectively. With these foot positions, the center of gravity of the quadruped will be inside these contact feet during impact. The position of the body center of gravity is located at 5.3 ft above the ground. Table 10 shows the center of gravity and the joint angle for each link with respect to its own coordinate system.

In this example, foot 1 and 2 are simply resting on the ground while foot 3 and 4 make the contacts. The initial velocities at the body, joints and the contact foot are listed in Table 11.

In order to find the state equation for $v_{31}(t') \cdots v_{34}(t')$, first one must examine
Table 10: The center of gravity for each link and its joint angle measured in the link coordinate system for example II.

<table>
<thead>
<tr>
<th>Leg</th>
<th>Link</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ₁</th>
<th>θ₂</th>
<th>θ₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4.5</td>
<td>-3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>70.171°</td>
<td>-92.137°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.177</td>
<td>-3.0</td>
<td>-2.893</td>
<td>0.0°</td>
<td>68.627°</td>
<td>-94.189°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.826</td>
<td>-3.0</td>
<td>-1.093</td>
<td>0.0°</td>
<td>16.541°</td>
<td>-88.604°</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>68.627°</td>
<td>-94.189°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.474</td>
<td>3.0</td>
<td>-2.506</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.124</td>
<td>3.0</td>
<td>-0.706</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>16.541°</td>
<td>-88.604°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.284</td>
<td>3.0</td>
<td>-2.116</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-5.534</td>
<td>3.0</td>
<td>-0.316</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>16.541°</td>
<td>-88.604°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.786</td>
<td>3.0</td>
<td>-2.128</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-6.536</td>
<td>3.0</td>
<td>-0.328</td>
<td>0.0°</td>
<td>11.057°</td>
<td>-84.485°</td>
</tr>
</tbody>
</table>

Table 11: Initial velocities at the body, joint and each contact foot before impact for example II.

<table>
<thead>
<tr>
<th>Leg</th>
<th>θ₁</th>
<th>θ₂</th>
<th>θ₃</th>
<th>Foot</th>
<th>v₃x</th>
<th>v₃y</th>
<th>v₃z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1104</td>
<td>-0.2993</td>
<td>0.0278</td>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>-0.1104</td>
<td>-0.1227</td>
<td>-0.2194</td>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>-0.1104</td>
<td>-0.2336</td>
<td>-0.1867</td>
<td>3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>-0.1104</td>
<td>-0.1817</td>
<td>-0.432</td>
<td>4</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>

vᵦᵀ = [0 0 1]  [ωᵦᵀ = [0.1 0.2 0]]
the motion of foot 1 and 2 once the collisions occur. As described in the previous section, the motion of these contact feet can be specified through the value of \( \dot{v}_{31z}(0^-) \) and \( \dot{v}_{32z}(0^-) \) which can be obtained as follows:

\[
\begin{bmatrix}
\dot{v}_{31z}(0^-) \\
\dot{v}_{32z}(0^-) \\
\dot{v}_{33z}(0^-) \\
\dot{v}_{34z}(0^-)
\end{bmatrix} = -V \begin{bmatrix}
v_{31z}(0^-) \\
v_{32z}(0^-) \\
v_{33z}(0^-) \\
v_{34z}(0^-)
\end{bmatrix},
\]

\[
= - \begin{bmatrix}
0.5732 & -0.0005 & 0.0028 & -0.0035 \\
-0.0005 & 0.6232 & -0.0038 & 0.0030 \\
0.0028 & -0.0038 & 1.5277 & -0.0016 \\
-0.0035 & 0.0030 & -0.0016 & 1.5471
\end{bmatrix} \begin{bmatrix}
v_{31z}(0^-) \\
v_{32z}(0^-) \\
v_{33z}(0^-) \\
v_{34z}(0^-)
\end{bmatrix}
\]

The coefficient matrix \( V \) in this equation is calculated based on (4.250), (4.251), (4.271) and (4.272). From the initial foot velocities in Table 11, both \( \dot{v}_{31z}(0^-) \) and \( \dot{v}_{32z}(0^-) \) can be determined as 0.0008 ft/sec\(^2\). Since \( \dot{v}_{31z}(0^-) \) and \( \dot{v}_{32z}(0^-) \) both direct toward the ground, the initial impact model consisting of four vertical damper can be applied. Therefore, one can write the state equation for \( v_{31z}(t') \cdots v_{34z}(t') \) based on this impact model as

\[
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix} = -V \begin{bmatrix}
v_{31z}(t') \\
v_{32z}(t') \\
v_{33z}(t') \\
v_{34z}(t')
\end{bmatrix}
\]

(4.335)

where \( V \) is determined as described in (4.334).

Figure 41 displays the trajectories of \( v_{31z}(t') \cdots v_{34z}(t') \) as obtained from (4.335). The figure shows the first 10 sec of each vertical foot velocity after impact. According to these trajectories, \( v_{33z}(t') \), \( v_{34z}(t') \) are both decreasing and asymptotically approaching zero as \( t' \to \infty \). The velocities \( v_{31z}(t') \) and \( v_{32z}(t') \), however, start to increase once the impact occurs and then asymptotically decrease to zero as \( t' \to \infty \).
Figure 41: The trajectories of vertical foot velocities $v_{31z}(t) \cdots v_{34z}(t)$ when foot 1 and 2 are simply resting on the ground and foot 3 and 4 make contacts.
Table 12: The velocities at the body and joint after foot 3 and 4 strike the ground in example II.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1106</td>
<td>-0.2936</td>
<td>0.0262</td>
</tr>
<tr>
<td>2</td>
<td>-0.1106</td>
<td>-0.1169</td>
<td>-0.2214</td>
</tr>
<tr>
<td>3</td>
<td>-0.1104</td>
<td>-0.2595</td>
<td>0.1901</td>
</tr>
<tr>
<td>4</td>
<td>-0.1104</td>
<td>-0.2079</td>
<td>-0.0549</td>
</tr>
<tr>
<td>$v_b$</td>
<td>[-0.0006 0 0.9945]$^T$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\omega_b$</td>
<td>[0.1 0.1972 -0.0006]$^T$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since $v_{31z}(t') \cdots v_{34z}(t')$ are all asymptotically approaching zero, all of the feet will remain on the ground after impact. One can also apply the Newton and Raphson method to solve for the crossing point of (4.335). After a finite amount of iterations, each normal velocity $v_{31z}(t') \cdots v_{34z}(t')$ is reaching its steady-state value without crossing the $t'$ axis.

To calculate the amount of impulsive force in this case, one may apply the result in (4.301) as follows:

$$F_{Iz} = -V^{-1}v_{3z}(0^-),$$

$$= [-0.0008 - 0.0009 - 0.6574 - 0.647]^T. \quad (4.336)$$

Based on these impulsive forces, both body and joint velocities after impact can be determined from (4.326). The result of this calculation is displayed in Table 12.
Table 13: The center of gravity for each link and its joint angle measured in the link coordinate system for example III.

<table>
<thead>
<tr>
<th>Leg</th>
<th>Link</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4.5</td>
<td>-3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>-68.3195°</td>
<td>31.1724°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.177</td>
<td>-3.0</td>
<td>-2.893</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.826</td>
<td>-3.0</td>
<td>-1.093</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>-40.194°</td>
<td>-24.8233°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3.474</td>
<td>3.0</td>
<td>-2.506</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.124</td>
<td>3.0</td>
<td>-0.706</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-4.5</td>
<td>-3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>54.3147°</td>
<td>-108.6293°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.284</td>
<td>-3.0</td>
<td>-2.116</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-5.534</td>
<td>-3.0</td>
<td>-0.316</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-4.5</td>
<td>3.0</td>
<td>-4.1</td>
<td>0.0°</td>
<td>49.2647°</td>
<td>-108.3274°</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-4.786</td>
<td>3.0</td>
<td>-2.128</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-6.536</td>
<td>3.0</td>
<td>-0.328</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example III. Some feet are resting on the ground while the others make a contact and a foot leaves the ground**

Another situation which can occur in Case II is when some feet start to leave the ground after impact. To simulate this effect, one may reconsider example II with the following foot positions: $[-0.1 -3.0 0.0]^T$, $[0.2 3.0 0.0]^T$, $[-4.5 -3.0 0.0]^T$, $[-4.8 3.0 0.0]^T$. The position of center of gravity at the body is relocated at 5.0 ft above the ground. Table 13 displays the center of gravity and the joint angle of each link used in this example. The initial velocities at the body, joints and each contact foot are listed in Table 14.

Like the previous example, foot 1 and 2 are initially resting on the ground once the impact occurs. Therefore, the procedure described in (4.334) can be used to
Table 14: Initial velocities at the body, joint and each contact foot before impact for example III.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>Foot</th>
<th>$v_{3x}$</th>
<th>$v_{3y}$</th>
<th>$v_{3z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1111</td>
<td>0.0193</td>
<td>-0.4464</td>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>-0.1111</td>
<td>-0.2867</td>
<td>-0.0027</td>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>-0.1111</td>
<td>-0.1626</td>
<td>-0.2462</td>
<td>3</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>-0.1111</td>
<td>-0.0737</td>
<td>-0.4811</td>
<td>4</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>

$v_{b}$ = [0 0 1]' $\omega_{b}$ = [0.1 0.2 0]'

determine the motion of these contact feet. According to (4.250), (4.251), (4.271) and (4.272), the coefficient matrix $V$ in this case corresponds to

$$V = \begin{bmatrix}
0.6249 & -0.0004 & 0.0042 & 0.0074 \\
-0.0004 & 1.6202 & 0.0006 & -0.0016 \\
0.0042 & 0.0006 & 1.1579 & 0.0005 \\
0.0074 & -0.0016 & 0.0005 & 1.2592
\end{bmatrix}.$$  \hspace{1cm} (4.337)

Based on this matrix $V$ and the initial foot velocity in Table 14, one can calculate $\dot{v}_{31z}(0^{-}), \dot{v}_{32z}(0^{-})$ as -0.0116 ft/sec² and 0.001 ft/sec². Unlike the previous example, the sign of $\dot{v}_{31z}(0^{-})$ indicates that foot 1 will leave the ground immediately after foot 3 and 4 making contacts. Under this circumstance, the impact model consisting of three vertical dampers will be used.

Based on this impact model, the state equation for $\dot{v}_{31z}(t') \cdots \dot{v}_{34z}(t')$ can be formed as

$$\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix} = -V' \begin{bmatrix}
v_{32z}(t') \\
v_{33z}(t') \\
v_{34z}(t')
\end{bmatrix}.$$  \hspace{1cm} (4.338)
where \( V' \) represents a \( 4 \times 3 \) matrix which corresponds to columns 2 to 4 of \( V \) in (4.337).

Figure 42 illustrates the trajectories of \( v_{31z}(t') \cdot v_{34z}(t') \) as obtained from (4.338). According to the plot of \( v_{31z}(t') \), foot 1 starts leaving the ground immediately after impact. Foot 2, on the other hand, still remains on the ground since \( v_{32z}(t') \) in this case is asymptotically approaching zero as \( t' \to \infty \). Similar results are also obtained from foot 3 and 4 based on the trajectories of \( v_{33z}(t') \) and \( v_{34z}(t') \) in Figure 42. The final value of \( v_{31z}(t') \) as \( t' \to \infty \) can be determined from (4.315) where

\[
t_1' = 0,
\]

\[
W_2 = \text{the first row of } V \text{ in (4.338),}
\]

\[
V_1 = \text{a } 3 \times 3 \text{ block matrix formed by rows 2 to 4 and columns 1 to 3 of the matrix } V' \text{ in (4.338),}
\]

\[
v_{3z}^*(t') = [v_{32z}(t') \ v_{33z}(t') \ v_{34z}(t')]^T.
\]

Based on these parameters, the steady-state value of \( v_{34z}(t') \) as obtained from (4.315) is equal to -0.0095 ft/sec. Like the first example, this final value of \( v_{34z}(t') \) shows the definite transition which occurs at foot 1.

To calculate the impulsive force at foot 2, 3 and 4, one may apply (4.301) as follows:

\[
\begin{bmatrix}
F_{1z2} \\
F_{1z3} \\
F_{1z4}
\end{bmatrix} = -
\begin{bmatrix}
1.6202 & 0.0006 & -0.0016 \\
0.0006 & 1.1579 & 0.0005 \\
-0.0016 & 0.0005 & 1.2592
\end{bmatrix}^{-1}
\begin{bmatrix}
v_{32z}(0^-) \\
v_{33z}(0^-) \\
v_{34z}(0^-)
\end{bmatrix},
\]

\[
= \begin{bmatrix}
-0.0005 \\
-0.8633 \\
-0.7938
\end{bmatrix}.
\]
Figure 42: The trajectories of vertical foot velocities $v_{31z}(t') \cdots v_{34z}(t')$ when foot 1 and 2 are resting on the ground before impact. The picture illustrates the transition of $v_{31z}(t')$ at $t' = 0$. 
Table 15: The velocities at the body and joint after foot 3 and 4 strike the ground for example III.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\dot{\theta}_1$</th>
<th>$\dot{\theta}_2$</th>
<th>$\dot{\theta}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1109</td>
<td>0.0188</td>
<td>-0.4368</td>
</tr>
<tr>
<td>2</td>
<td>-0.1109</td>
<td>-0.2812</td>
<td>-0.0042</td>
</tr>
<tr>
<td>3</td>
<td>-0.1108</td>
<td>-0.2148</td>
<td>0.1722</td>
</tr>
<tr>
<td>4</td>
<td>-0.1108</td>
<td>-0.1134</td>
<td>-0.0794</td>
</tr>
</tbody>
</table>

$v_b = \begin{bmatrix} 0.0048 & 0 & 0.9943 \end{bmatrix}^T$

$\omega_b = \begin{bmatrix} 0.0998 & 0.1968 & 0.0001 \end{bmatrix}^T$

Since the damper at foot 1 is not initially included in the model, the impulsive force at this foot is then equal to zero. Based on these impulsive forces, both body and joint velocities after impact can be determined from (4.326). Table 15 illustrates the result from this computation.

These three examples demonstrate the application of the impulsive formulation presented in Section 4.4.5. The last two cases are the closest representatives for the impact which normally occurs in the simulation. For instance, when a quadruped is trotting on the ground, two feet at one diagonal are simply resting on the ground while the others two making contact with the ground. In this situation, the same analysis as illustrated in these two examples can be use to determine impulsive forces and velocities of the vehicle after impact.

In case of a 2-D single rigid body, both contacts tend to remain on the ground after impact if the center of gravity is located between both contact points. This result, however is not applied to a 3-D quadruped because both body and legs are not strictly
connected. Therefore, a transition may occur at each contact foot regardless of the location of the center of gravity.

4.4.7 Sensitivity of the Impact Model in a 3-D Quadruped

The effects on the impact model due to different values of damping coefficient and non-simultaneous collisions for a 3-D quadruped will be discussed in this section. According to a 2-D single rigid body, different damping coefficients may lead to different amounts of impulsive forces once a contact is broken after impact. The same result is also obtained when non-simultaneous collisions are considered and there is a transition at the contact. In order to examine whether these results are still applied, the consistency of impulsive forces based on these factors will be reconsidered for a 3-D quadruped.

The effect due to different values of damping coefficients

First, one may examine the effect of the damping coefficient used in the model. This procedure can be divided into two cases. The first situation is when all contact feet remain on the ground after impact and the other one is when a foot is leaving the ground after the collision. To include the effect of this parameter in the model, one needs to rederive the state equation in (4.289) for different values of damping coefficients. This procedure can be implemented as follows: Suppose the damping coefficient at contact foot \( i \) is specified as \( \alpha_i \beta \) where \( m_i \) is an arbitrary positive number. Based on these damping coefficients, the matrix \( S \) in (4.272) can be described
as

\[ S = \begin{bmatrix}
B_1 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 \\
0 & 0 & B_3 & 0 \\
0 & 0 & 0 & B_4
\end{bmatrix} \]  

(4.340)

where each block matrix \( B_i \) is defined as

\[ B_i = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & a_i
\end{bmatrix} \]

The term \( \beta \) disappears due to the transformation from \( t \) to \( t' \). According to (4.340), the columns of \( AS \) in (4.272) are mostly equal to zero except columns 3, 6, 9 and 12. Therefore, by disregarding all zero columns in \( AS \), one can rewrite (4.272) as

\[ \dot{v}_F(t') = -A'v_{3z}(t') \]  

(4.341)

where

\[ A' = \begin{bmatrix}
\alpha_1 A(1,3) & \alpha_2 A(1,6) & \alpha_3 A(1,9) & \alpha_4 A(1,12) \\
\alpha_1 A(2,3) & \alpha_2 A(2,6) & \alpha_3 A(2,9) & \alpha_4 A(2,12) \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_1 A(12,3) & \alpha_2 A(12,6) & \alpha_3 A(12,9) & \alpha_4 A(12,12)
\end{bmatrix} \]

\[ A(i,j) = \text{the (i,j) element of matrix } A. \]

To specify the state equation for \( v_{3z}(t') \), one can simply select rows 3, 6, 9 and 12 of \( A' \) in (4.341). As a result,

\[ \dot{v}_{3z}(t') = -V'v_{3z}(t'). \]  

(4.342)

The matrix \( V' \) in this case represents the elements in rows 3, 6, 9, and 12 of \( A' \) in (4.341). When the same value of damping coefficient is assumed, the scales \( \alpha_1 \cdots \alpha_4 \) in (4.341) are all equal to 1. Therefore, the matrix \( V' \) in (4.342) becomes \( V \) as
previously described in (4.289). Comparing (4.289) with (4.342), one can also express $V'$ in terms of $V$ as

$$V' = VN,$$  \hspace{1cm} (4.343)

$$N = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 \\
0 & 0 & \alpha_3 & 0 \\
0 & 0 & 0 & \alpha_4 \end{bmatrix}.$$  \hspace{1cm} (4.344)

Since both $V$ and $N$ are positive definite and $N$ is a diagonal matrix, the product $VN$ or $V'$ in (4.342) is a positive definite matrix.

First, suppose all feet strike the ground simultaneously and all contact feet remain on the ground after impact. The total impulse at these contacts can be described as

$$F_z(t') = \int_{0^-}^{t'} -\beta N v_{3z}(t') \frac{dt'}{\beta},$$

$$= -\int_{0^-}^{t'} N v_{3z}(t') dt'.$$  \hspace{1cm} (4.345)

Using the same procedure as shown in (4.300), one also obtains the impulsive force in this case as follows:

$$F_{Iz} = -\int_{0^-}^{0} N v_{3z}(t') dt',$$

$$= -\int_{0^-}^{0} Ne^{-V N t'} v_{3z}(0^-) dt',$$

$$= -\int_{0^-}^{0} Ne^{-V N t'} v_{3z}(0^-) dt',$$

$$= V^{-1} \int_{0^-}^{0} -V N e^{-V N t'} dt' v_{3z}(0^-),$$

$$= V^{-1} [e^{-V N t'} \bigg|_{t' = 0^-} -Iv_{3z}(0^-)],$$

$$= -V^{-1} v_{3z}(0^-).$$  \hspace{1cm} (4.346)

Since the impulsive force in (4.346) is the same as previously obtained in (4.301), different values of damping coefficients do not affect the model in terms of impulsive
forces. Under this circumstance, there is only one set of impulsive forces which stop these feet.

For a 2-D single rigid body, this property is still applied to the case where one contact point is simply resting on the ground if there are no transitions occur after impact. This is because the motion of a contact, which is resting on the ground, does not depend on the value of damping coefficient used in the model. For a 3-D quadruped, the motion of a foot which is resting on the ground must be determined based on a linear combination of the columns of $V$ weighted by the initial velocities of the feet that strike the ground. According to (4.342), it is clear that different values of damping coefficients can lead to uncertainty motion of the foot which is resting on the ground. Therefore, the situation where some feet simply rest on the ground of a 3-D quadruped is not well posed in terms of the damping coefficient.

Next, one may examine the effect of this parameter when a foot is leaving the ground after impact. In this case, one may reconsider the first example illustrated in Section 4.4.6. All of the parameters and initial velocities are described by those listed in Table 6–8. The same damping coefficient is still applied at foot 1, 2 and 3. For foot 4, however, the damping coefficient has been set to $5\beta$ instead of $\beta$. According to (4.343), $N$ in this case can be described as

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \quad (4.347)$$

Based on the coefficient matrix $V$ in (4.327), one can determine the new coefficient matrix $V'$ using (4.343) and (4.347). As a result,
\[ V' = \begin{bmatrix}
0.7896 & 0.0001 & -0.0028 & 0.0457 \\
0.0001 & 1.5215 & 0.0037 & -0.0103 \\
-0.0028 & 0.0037 & 1.5868 & -0.0089 \\
0.0091 & -0.0021 & -0.0018 & 8.1144 \\
\end{bmatrix}. \quad (4.348) \]

From (4.342), one may write the state equation for \( v_{3i}(t') \) in this case as

\[
\begin{bmatrix}
\dot{v}_{31}(t') \\
\dot{v}_{32}(t') \\
\dot{v}_{33}(t') \\
\dot{v}_{34}(t') \\
\end{bmatrix} =
\begin{bmatrix}
0.7896 & 0.0001 & -0.0028 & 0.0457 \\
0.0001 & 1.5215 & 0.0037 & -0.0103 \\
-0.0028 & 0.0037 & 1.5868 & -0.0089 \\
0.0091 & -0.0021 & -0.0018 & 8.1144 \\
\end{bmatrix}
\begin{bmatrix}
v_{31}(t') \\
v_{32}(t') \\
v_{33}(t') \\
v_{34}(t') \\
\end{bmatrix}. \quad (4.349) \]

Figure 43 demonstrates the trajectories of \( v_{31}(t'), v_{32}(t') \) and \( v_{33}(t') \) are all asymptotically approaching zero as \( t' \to \infty \). Therefore, foot 1, 2 and 3 will remain on the ground after impact. Foot 4, on the other hand, leaves the ground after the collision. To determine the time when this transition occurs, one may apply the method of Newton and Raphson to (4.349). In this case, the time when \( v_{34}(t') \) becomes zero is changed from 1.2858 to 0.4013. This result is due to the differences of damping coefficients used in both cases.

After this transition occurs, the damper at foot 4 will be removed from the impact model and \( \dot{v}_{31}(t'), \ldots, \dot{v}_{34}(t') \) will be described by

\[
\begin{bmatrix}
\dot{v}_{31}(t') \\
\dot{v}_{32}(t') \\
\dot{v}_{33}(t') \\
\dot{v}_{34}(t') \\
\end{bmatrix} =
\begin{bmatrix}
0.7896 & 0.0001 & -0.0028 \\
0.0001 & 1.5215 & 0.0037 \\
-0.0028 & 0.0037 & 1.5868 \\
0.0091 & -0.0021 & -0.0018 \\
\end{bmatrix}
\begin{bmatrix}
v_{31}(t') \\
v_{32}(t') \\
v_{33}(t') \\
v_{34}(t') \\
\end{bmatrix}. \quad (4.350) \]

From (4.346), the impulsive force at each foot can be described as

\[
F_{iz} = - \int_{0}^{\infty} N v_{3z}(t') dt',
\]

\[
= - \int_{0}^{0.4013} N v_{3z}(t') dt' - \left[ \int_{0.4013}^{\infty} N v_{3z}(t') dt' \ldots \int_{0}^{0.4013} N v_{3z}(t') dt' \right]. \quad (4.351) \]
Figure 43: The trajectories of vertical foot velocities $v_{31z}(t') \cdots v_{34z}(t')$ when all feet strike the ground simultaneously. The damping coefficient at foot 4 in this case is five times greater than that used in Figure 40.
The terms \( N' \) and \( v'_{3z}(t') \) in this equation represent the first \( 3 \times 3 \) of matrix \( N \) in (4.347) and the contact velocity \( [v_{31z}(t') \ v_{32z}(t') \ v_{33z}(t')]^T \).

Based on the solutions of \( v_{3z}(t') \) and \( v'_{3z}(t') \) in (4.349) and (4.350), \( F_{tz} \) in (4.351) can be determined as \([-6.9654 - 3.6145 - 0.0669 - 0.0531]^T \). Because the impulsive force at foot 4 differs from the previous result in (4.333), the value of damping coefficient does affect the impact model in this case.

The result is similar to the case of a 2-D single rigid body. Because of the uncertainty of the foot velocity after the transition, different amounts of impulsive forces can be found by changing the value of damping coefficient. Therefore, the multiple contacts in a 3-D quadruped is also not well posed when a transition occurs at the contact foot.

**The effect due to non-simultaneous collisions**

Another issue concerning this impact model is its sensitivity toward the time of the collisions. Since each impact may not actually occur simultaneously but rather at slightly different time. Under this circumstance, a finite delay in the \( t' \) caused by this factor may lead to different amounts of impulsive forces found at each contact.

Like a 2-D case, one can first demonstrate the effect of non-simultaneous collisions when there are no transitions occur after impact. Suppose all four feet of a quadruped strike the ground. Instead of simultaneous collisions, foot 3 and 4 initially strike the ground at \( t' = 0 \) whereas the impact at foot 1 and 2 occur later at \( t' = 1 \) and \( t' = 2 \). All of the parameters used in this case are the same as the second experiment in Section 4.4.6. However, the initial velocities before foot 3 and 4 strike the ground are
Table 16: Initial body, joints and foot velocities before foot 3 and 4 strike the ground for the non-simultaneous-collision test.

<table>
<thead>
<tr>
<th>Leg</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>Foot</th>
<th>$v_{3x}$</th>
<th>$v_{3y}$</th>
<th>$v_{3z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.1104</td>
<td>-0.6086</td>
<td>0.4503</td>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>-0.1104</td>
<td>-0.4242</td>
<td>0.2039</td>
<td>2</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>-0.1104</td>
<td>-0.2336</td>
<td>-0.1867</td>
<td>3</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>4</td>
<td>-0.1104</td>
<td>-0.1817</td>
<td>-0.432</td>
<td>4</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

$v_b = [0 \ 0 \ 1]^T$, $\omega_b = [0.1 \ 0.2 \ 0]^T$

changed to those listed in Table 16.

Since all collisions do not occur at the same time, each damper needs to be properly included in the impact model. This situation is opposite to the case where a foot leaves the ground after impact. Therefore, similar procedure can be applied to obtain the state equation for $v_{31z}(t') \cdots v_{34z}(t')$ in this case. As previously described, a damper can be removed from the model by simply setting the column of $S$ in (4.272) to zero. Likewise, when a contact occurs and the damper is included in the model, the column of $S$ which corresponds to that foot will be set to one. Using this concept and the formulation in (4.250), (4.251), (4.271) and (4.272), the dynamic equation for $v_{31z}(t') \cdots v_{34z}(t')$ in this example can be determined as follows:

$$
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix}
= \begin{bmatrix}
0.0028 & -0.0035 \\
-0.0038 & 0.0030 \\
1.5277 & -0.0016 \\
0.0016 & 1.5471
\end{bmatrix}
\begin{bmatrix}
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix}
, \quad 0 \leq t' \leq 1, \quad (4.352)
$$

$$
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix}
= -\begin{bmatrix}
0.5732 & 0.0028 & -0.0035 \\
-0.0005 & -0.0038 & 0.0030 \\
0.0028 & 1.5277 & -0.0016 \\
-0.0035 & -0.0016 & 1.5471
\end{bmatrix}
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix},
$$

\[ v_{31z}(t') \cdots v_{34z}(t') \]
\[
\begin{bmatrix}
\dot{v}_{31z}(t') \\
\dot{v}_{32z}(t') \\
\dot{v}_{33z}(t') \\
\dot{v}_{34z}(t')
\end{bmatrix}
= -
\begin{bmatrix}
0.5732 & -0.0005 & 0.0028 & -0.0035 \\
-0.0005 & 0.6232 & -0.0038 & 0.0030 \\
0.0028 & -0.0038 & 1.5277 & -0.0016 \\
-0.0035 & 0.0030 & -0.0016 & 1.5471
\end{bmatrix}
\begin{bmatrix}
v_{31z}(t') \\
v_{32z}(t') \\
v_{33z}(t') \\
v_{34z}(t')
\end{bmatrix},
\]

\[1 \leq t' \leq 2,\]  \hspace{2cm} (4.353)

\[t' > 2.\]  \hspace{2cm} (4.354)

Figure 44 displays the trajectories of \(v_{31z}(t')\) \(\cdots\) \(v_{34z}(t')\) when simultaneous collisions are assumed as compared to when there are delays at the contacts. The dashed line in this picture represents the results when collisions do not occur at the same time. In both cases, \(v_{31z}(t')\) \(\cdots\) \(v_{34z}(t')\) are all asymptotically go to zero as \(t' \to \infty\). Therefore, all feet will remain on the ground after impact. According to these trajectories, the early collisions at foot 3 and 4 do not create any major effect on foot 1 and 2. The same results are also obtained when foot 1 and 2 make contacts with the ground at \(t' = 1\) and \(t' = 2\).

From (4.346), the impulsive force when non-simultaneous collisions occur at foot 1 and 2 can be determined as follows:

\[
\mathbf{F}_{Iz} = -\int_{0}^{\infty} N v_{3z}(t')dt',
\]

\[
= -N[\int_{0}^{1} v_{3z}(t')dt' + \int_{1}^{2} v_{3z}(t')dt' + \int_{2}^{\infty} v_{3z}(t')dt'].
\]  \hspace{2cm} (4.355)

Because all of the damping coefficients are assumed to be the same in this test, \(N\) in this case represents an identity matrix. From the expressions of \(\dot{v}_{3z}(t')\) in (4.352), (4.353) and (4.354), \(\mathbf{F}_{Iz}\) in (4.355) is calculated as \([-1.7469 \quad -1.6070 \quad -0.6582 \quad -0.6479]^{T}\).

To calculate the impulsive force when there are no delays, one may apply (4.346) to (4.354). As a result, the same amount of impulsive force is obtained at each contact foot. Like a 2-D single rigid body, when the final state of the normal foot velocity
Figure 44: The contact velocities $v_{31z}(t') \cdots v_{34z}(t')$ when simultaneous collisions are assumed as compared to when there are collision delays at foot 1 and 2. The dashed line represents the trajectories of these velocities with a delay of 1 and 2 sec at foot 1 and 2. In this case, all feet still remain on the ground after impact.
is uniquely defined in $t'$, i.e., $v_{3z}(t') \big|_{t \to \infty} = 0$, a finite delay due to non-simultaneous collisions should not affect the impulsive force. In other words, there is simply one set of impulsive force when all contact feet remain on the ground after impact.

According to a 2-D single object, if any transition occurs after impact, a finite delay in $t'$ due to non-simultaneous collisions will lead to inconsistent amount of impulsive force at each contact. By including the effect of non-simultaneous collisions to example I in Section 4.4.6, one is able to demonstrate the same result as found in the 2-D single rigid body. Moreover, the less amount of delay at each contact, the smaller effect on the impulsive force it generates. Therefore, the problem due to non-simultaneous collisions is still well posed for a 3-D quadruped.

Based on these results, the following observations can be made regarding the sensitivity of this impact model in a 3-D quadruped:

1. The value of damping coefficient and non-simultaneous collisions do not affect the impact model when all contact feet remain on the ground after impact. Because the final state of normal velocity is uniquely defined in $t'$ under this circumstance, the impulsive force obtained from the model is always consistent.

2. When at least one transition occurs after impact, different values of damping coefficients and non-simultaneous collisions can lead to inconsistent amount of impulsive force. However, the smaller the delay due to a non-simultaneous collision, the smaller is the effect generated in terms of the impulsive forces. With a small amount of delay in $\beta$ scaled time $t'$, a consistent amount of impulsive force can be obtained from the model and therefore the problem due to non-simultaneous collisions is still
well posed for a 3-D quadruped.

4.5 Summary

The impact model and the impulsive formulation which can be applied to single and multiple contacts are presented in this chapter. For the collision due to a single-point contact, an impact model with zero coefficient of restitution can always be applied. This assumption is verified through the consistency of impulsive forces at the contact. Based on the results obtained in Section 4.3, the impulsive force which generates the inelastic motion always exists for a single-point collision. Although this impact model is valid for a single contact, it cannot be used for multi-contact cases. This contradiction is caused by the inconsistency of impulsive forces which can simultaneously stop all contact velocities. Besides, it is also possible that some contacts may be broken after impact. Under these circumstances, the impact model based on the zero coefficient of restitution can lead to an inconsistent impulsive formulation.

In order to model an inelastic impact for multi-point contacts, the impact model consisting of a single damper is used. The damper is assigned in the normal direction of the contact surface so that it can be consistently removed from the system once the contact is broken. In order to model the effect of impulsive force at each contact, a very large value of damping coefficient is assumed in the model. Once the damping coefficient ($\beta$) is approaching infinity, the contact force generated by the damper will represent the impulsive force at the contact. Because the changes of contact velocities are instantaneous in the original time $t$, another way to analyze the problem based on the new timing scale $t'$ is implemented instead. By letting $t' = \beta t$, one can determine
the impulsive forces and contact velocities after impact based on the steady-state value of contact forces and velocities in $t'$. Because both contact forces and velocities are described by first-order systems, the impulsive formulation for different cases of impact are derived in closed forms.

The applications of this impact model to multiple contacts on a planar surface have been demonstrated for both a 2-D rigid body and a 3-D quadruped walking machine. The sensitivity of the model in terms of damping coefficient and non-simultaneous collisions has also been presented in this chapter. Based on the discussion in Sections 4.4.3 and 4.4.7, whether there is a foot contact transition after impact or not is the key factor on whether there is any sensitivity to damping coefficients or non-simultaneousness of impact.
CHAPTER V

Force Distribution for a Dynamically Stable Quadruped

5.1 Introduction

The last part of the dynamic simulation which has been developed in this work is the foot force allocation block. This issue is frequently found in both walking machines and multifingered robots. When the motion of the vehicle’s body is specified, the corresponding forces and torques which act on the body are also determined. These commanded body forces and torques are the results of reaction forces at the contacts that are generated by the input joint torques. Therefore, in order to determine the joint torque which moves the vehicle along the desired trajectories, the appropriate foot force solutions are necessary.

This problem has been first studied for the statically stable walking machine. In this case, the commanded forces and torques at the body are represented by an underdetermined system. Therefore, one is able to select the appropriate solution for the contact forces based on additional constraints. For example, all contact forces should be kept within a friction cone in order to prevent the feet from slipping.

Unlike statically stable legged vehicles, the commanded body forces and torques for the dynamically stable machine are described by an overdetermined system. As
a result, there are no exact solutions with respect to these commanded body forces and torques.

In this chapter, an optimization scheme, which can be used to solve the force distribution in the moderate and high-speed quadruped, will be presented. In these modes, the vehicle is always statically unstable and therefore, the commanded body forces and torques are described by an overdetermined system. In order to prevent the foot from slipping and assure the proper contact at the foot tip, both friction cone constraints and minimum vertical force are also considered in this problem.

The content of this chapter is organized as follows: the previous work on solving the force distribution is reviewed in Section 5.2. Section 5.3 illustrates the mathematical formulation of the foot force constraints which can be applied to the quadruped simulation. An example of the force distribution for a trotting quadruped is also discussed in this section. Section 5.4 describes an optimization scheme which can be used to determine the foot force solution for a dynamically stable quadruped. The application of this technique to the simulation of a trotting quadruped will be presented in Chapter 6.

5.2 Previous Work

Several mathematical techniques were previously used to determine the solution of the force distribution problem. The applications of these techniques was presented for both walking machine and multifingered robots.

One of the approaches which was widely used in the past is the linear programming method. McGhee and Orin [29] applied this technique to calculate the joint
torques which move a hexapod along its prescribed body trajectories with the minimum input energy. The frictional constraints as well as the maximum limitation on the joint torques was included in order to prevent the vehicle from slipping on the surface and overheating joint actuators. Since the force distribution in a hexapod is underdetermined, these joint torques also satisfy both commanded forces and torques at the body. A similar approach to solve the force distribution in closed-chain mechanisms was also presented by Orin and Oh [30]. The constraints on the vertical force component are added to the system equations to avoid penetration of the foot into a soft surface and improve the load balance among all supporting feet. Park [31] also applied the linear programming method to allocate the foot forces which are necessary to control a quadruped without exceeding its leg capacity. Since the quadruped can operate in statically stable and dynamically stable gaits, the linear programming was used to determine the foot force solutions for both overdetermined and underdetermined systems.

Due to the number of system variables, the original form of linear programming can lead to a time consuming algorithm and therefore may not be suitable for the simulation. In order to reduce the number of system variables, a dual formulation of linear programming was proposed by Cheng and Orin [32]. Based on the duality theory of linear programming, the original formulation can be transformed into a smaller system by exchanging between the system variables and its constraints. As a result, the algorithm requires less computation time than its original formulation. Even though this form of linear programming is quite practical, discontinuities can
occur in the solutions obtained from the simplex algorithm [33]. Experiments in [34] display this unwanted property when the linear programming was used to determine the foot force solutions for a hexapod.

Because of the disadvantages in the linear programming method, other mathematical techniques which provide better continuity and less computation time are therefore necessary. One such mathematical approach has been known as the pseudoinverse method. Klein, Olson and Pugh [35] applied this technique to determine the foot force solution for a hexapod while moving with a constant speed. Based on the assumption that there are no other commanded forces and torques at the body except the weight of the vehicle, the pseudoinverse solution in this case can be written in a very compact form which is suitable for the real-time force control system. Since the force distribution in a hexapod represents an underdetermined system, the homogeneous solutions can be used to minimize the discontinuities in the foot forces during the transfer and support phases. Klein and Chung [36] applied this concept to distribute the weight of the vehicle and reduce unexpected oscillations caused by the force feedback in the OSU hexapod.

Another view of the pseudoinverse method applied in legged locomotion was presented by Kumar and Waldron [37]. Based on the property that there are no interaction forces among the horizontal components along the line joining any two feet, the pseudoinverse solution can be simplified to a form which is moderate in its computational requirement. The vertical force components are also determined subject to the friction cone constraints in order to prevent the feet from slipping. Although this
formulation is quite practical, it can provide only a suboptimal solutions.

A method using both pseudoinverse and homogeneous solutions was developed by Klein and Kittivatcharapong [34]. Their technique has two different forms named the interior and exterior methods which illustrate two distinct views of analyzing the problem. According to basic properties of a pseudoinverse solution, an optimization scheme, such as Rosen's gradient projection method, can be used to determine the homogeneous force solution which provides a feasible solution subject to all necessary constraints. The force solutions obtained from these two techniques display better continuity than the linear programming method. Furthermore, with a suitable starting force solution, the exterior method can reach the solution point with fewer steps compared to the interior approach. Since the formulation is limited to an underdetermined system, further development is still required for solving an overdetermined system.

The application of the quadratic programming to the constrained force optimization in multifingered robots was introduced by Nahon and Angeles [33]. The algorithm was developed based on the method of Goldfarb and Idnani [38] which needs no initial guess to start the search unlike others gradient search methods. The force solution obtained from quadratic programming also provides better continuity than the linear programming method. Furthermore, its computational speed becomes apparent once the number of contacts increases. Another application of quadratic programming to force distribution in a trotting quadruped was also illustrated by Aebker [2].

Although the force distribution of statically unstable legged vehicles can be for-
mulated as a quadratic programming problem, different techniques can be applied to solve for the optimal solution. In this work, another view of the gradient projection method which can be applied to an overdetermined system will be presented. The optimal solution will be determined by using the null space of the least-square solution instead of the null space of the gradient of the objective function as implemented in the method of Goldfarb and Idnani [38].

5.3 Formulation of Foot Forces Constraints

The mathematical formulation of the force distribution in a trotting quadruped is presented in this section. In general, both forces and torques generated by each contact foot can be described as

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 1 & 0 \\
0 & 0 & 1 & \cdots & \cdots & 0 & 0 & 1 \\
0 & -z_1 & y_1 & \cdots & \cdots & 0 & -z_k & y_k \\
z_1 & 0 & -x_1 & \cdots & \cdots & z_k & 0 & -x_k \\
-y_1 & x_1 & 0 & \cdots & \cdots & -y_k & x_k & 0
\end{bmatrix}
\begin{bmatrix}
f_{ix} \\
f_{iy} \\
f_{iz} \\
f_{kx} \\
f_{ky} \\
f_{kz}
\end{bmatrix}
= \begin{bmatrix}
F_x \\
F_y \\
F_z \\
\tau_x \\
\tau_y \\
\tau_z
\end{bmatrix}
\tag{5.1}
\]

where

\[f_{ix}, f_{iy}, f_{iz} = \text{The foot forces for leg } i \text{ in the x, y and z directions,}\]

\[F_x, F_y, F_z = \text{The commanded force in the x, y and z directions,}\]

\[\tau_x, \tau_y, \tau_z = \text{The commanded torque acting on the vehicle body,}\]

\[x_i, y_i, z_i = \text{The foot position for leg } i,\]

\[k = \text{The number of feet on the ground.}\]
Partitioning by rows, this equation can be simplified to a more concise form as

\[
\begin{bmatrix}
J_F \\
\vdots \\
J_T
\end{bmatrix} = 
\begin{bmatrix}
F \\
\vdots \\
\tau
\end{bmatrix}
\]  

(5.2)

or

\[Jf = b.\]  

(5.3)

In order to prevent each contact foot from slipping, the foot forces need to be restricted within the friction cone generated at each contact point. Even though this constraint is originally non-linear, it can be linearized by using a square pyramid which is inscribed within the friction cone instead. Based on this approximation, the friction cone constraints can be described as

\[|f_{iz}| \leq \mu f_{iz}, \quad |f_{iy}| \leq \mu f_{iz}.\]  

(5.4)

where \(\mu\) represents the friction coefficient in this case.

The second set of constraints, which ensures that all supporting feet are always on the ground, is the minimum positive vertical force,

\[f_{iz} \geq f_{iz\text{min}} > 0.\]  

(5.5)

For a trotting quadruped, only two feet stay on the ground at one time. Therefore, \(J\) in (5.3) is simply specified as

\[
J = 
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & -z_1 & y_1 & 0 & -z_2 & y_2 \\
z_1 & 0 & -x_1 & z_2 & 0 & -x_2 \\
y_1 & x_1 & 0 & -y_2 & x_2 & 0
\end{bmatrix}
\]  

(5.6)
To solve for the foot forces based on (5.3)–(5.5), one needs to examine the rank of $J$ as defined in (5.7). Suppose the foot’s positions are randomly assigned and $x_i, y_i$ and $z_i$ are not equal to zero. For a very general case, let’s also assume that $x_1 \neq x_2, y_1 \neq y_2$ and $z_1 \neq z_2$. Based on these assumptions, the rank of $J$ in (5.7) can be examined by using the row operations as shown in Figure 45. In this form, it is obvious that last row of this matrix can be specified by multiplying the fourth row with $-\frac{(x_1-x_2)}{(z_1-z_2)}$. Therefore, the rank of $J$ in (5.3) is actually equal to 5. This situation corresponds to the one uncontrollable degree of freedom which is the torque about
the line connecting both feet. As a result, one cannot find the foot forces with respect to any arbitrary commanded forces and torques at the vehicle's body.

However, it is still possible to determine the foot forces which minimize $||Jf - b||$ based on the constraints in (5.4) and (5.5). Therefore, finding the solution of the force distribution in this case can be considered as solving a least-square problem subject to linear inequality constraints in (5.4) and (5.5).

### 5.4 Optimization Scheme

Since the technique proposed in this work is adapted from the Gradient Projection method, it may be useful to firstly review the basic concept of the Gradient Projection. In the Gradient Projection approach, one may begin with a feasible solution together with its constraints which are active or satisfy as equalities at that point. The next step is to determine another feasible solution which minimizes or maximizes the objective function. This procedure can be accomplished by using the null space solution subject to current active constraints. In this case, the null space solution is specified by orthogonally projecting the gradient of the objective function into the present active constraints. Since the new solution must be within all inequality constraints, only a specific amount of the null space solution is added to the current solution. This particular amount of the null space solution can be specified by performing a one-dimensional search in this direction against all inactive constraints. Once the next feasible solution is found, the new set of active constraints at that point is formed. The whole procedure is then repeated until the optimal solution is found according to the Kuhn-Tucker necessary condition [39].
The same concept can be used to determine the optimal force solution of an overdetermined system described by (5.3)-(5.5). Since the objective function in this case is non-linear, searching along the null space of the gradient of this function may require more steps before reaching the solution. In this approach, however, the optimal solution will be determined based on the null space of the least-square solution instead. Therefore, if the least-square solution subject to (5.3) does not violate the inequality constraints in (5.4) and (5.5), it can be considered as a solution. Using this approach, one may be able to find the least-square solution for an overdetermined system in (5.3) with fewer steps than the normal gradient projection method. The mathematical formulation of this algorithm can be described as following.

According to (5.3)-(5.5), the force distribution of a trotting quadruped can be rewritten in the form of a least-square problem as follows:

\[
\min \| Jf - b \| \quad \text{subject to}
\begin{bmatrix}
-1 & 0 & \mu & 0 & 0 & 0 \\
1 & 0 & \mu & 0 & 0 & 0 \\
0 & -1 & \mu & 0 & 0 & 0 \\
0 & 1 & \mu & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & \mu \\
0 & 0 & 0 & 1 & 0 & \mu \\
0 & 0 & 0 & 0 & -1 & \mu \\
0 & 0 & 0 & 0 & 1 & \mu \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
f_{1x} \\
f_{1y} \\
f_{1z} \\
f_{2x} \\
f_{2y} \\
f_{2z} \\
f_{x_{\text{min}}} \\
f_{x_{\text{min}}} \\
f_{x_{\text{min}}} \\
f_{y_{\text{min}}} \\
f_{y_{\text{min}}} \\
f_{y_{\text{min}}}
\end{bmatrix}
\leq
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(5.7)

To determine the least-square solution of (5.7), one may start with a feasible force constraint which satisfies the inequality constraints in (5.7). The force constraint which satisfies as equality at this stage can be described as

\[C_i f = a_i,\]

(5.8)
where $C_i$ represents an $n \times 6$ matrix and $n < 6$. Each row of $C_i$ represents the gradient of these active constraints with respect to $f$. Furthermore, these constraints are also linearly independent. According to (5.8), $f$ can be determined as follows:

$$f = C_i^+ a_i + (I - C_i^+ C_i)w_i.$$  \hfill (5.9)

The term $C_i^+ a_i$ corresponds to the minimum norm solution of (5.8) whereas $(I - C_i^+ C_i)w_i$ represents the null space solution of (5.8). As a result of (5.9), it is possible to use the null space solution to minimize $||Jf - b||$ by selecting the appropriate vector $w_i$.

To solve for $w_i$, one must formulate this least-square problem in terms of $w_i$ using (5.3) and (5.9),

$$J[C_i^+ a_i + (I - C_i^+ C_i)w_i] = b,$$

$$J(I - C_i^+ C_i)w_i = b - JC_i^+ a_i. \hfill (5.10)$$

In this form, the vector $w_i$ which provides the minimum norm solution of this least-square problem can be simply calculated as

$$w_i = [J(I - C_i^+ C_i)]^+(b - JC_i^+ a_i). \hfill (5.11)$$

If $f_i$ and $\Delta f_{Li}$ are defined as the minimum norm and null space solutions at this step, the force solution in (5.9) can be simplified to

$$f = f_i + \Delta f_{Li}. \hfill (5.12)$$

Because $f$ needs to be within the feasible region described by (5.7), only a specific amount of $\Delta f_{Li}$ is included. Therefore, the next feasible solution based on the null space of $w_i$ in (5.11) can be written as follows:
\[ f_{i+1} = f_i + \alpha_i \Delta f_i \]  

(5.13)

where \( \alpha_i \) represents the maximum length of the null space solution that can be added to \( f_i \) without violating the constraints in (5.7). If \( \alpha_i \geq 1 \), then the solution in (5.12) is feasible, and the value of \( f \) which provides the minimum \( ||Jf - b|| \) will be determined by setting \( \alpha \) in (5.13) to 1 and the procedure will be terminated at this point.

On the other hand, if \( \alpha_i < 1 \), then the solution in (5.12) is not feasible and the new feasible solution based on \( \alpha_i < 1 \), will not represent the minimum norm solution. In other words, the least-square solution is still outside the boundary of (5.7) and the procedure must continue.

Under this circumstance, the constraints satisfied as equalities at \( f_{i+1} \) are added to \( C_i \) and \( a_i \). Thus, \( C_{i+1} \) and \( a_{i+1} \) can be formulated as

\[
C_{i+1} = \begin{bmatrix} C_i & \cdots \end{bmatrix},
\]

(5.14)

\[
a_{i+1} = \begin{bmatrix} a_i & \cdots \end{bmatrix},
\]

(5.15)

where \( G, h \) are the coefficient of new active constraints and their corresponding right-hand sides. Normally, both \( G \) and \( h \) are simply obtained from the inequality constraints in (5.7). Based on the new \( f_{i+1} \) and \( C_{i+1} \), one can determine \( \Delta f_{i+1} \) by using (5.9) and (5.11). The corresponding \( \alpha_{i+1} \) is then specified according to (5.13) and finally, the new feasible solution \( f_{i+2} \) is determined. The same criterion as applied at \( f_{i+1} \) can be examined at \( f_{i+2} \) as well. This procedure will continue until the least-square solution is found or the null space solution is equal to zero. In the later case, the solution space is fully spanned by the gradient of active constraints and the
optimal criteria based on the Kuhn-Tucker necessary condition will be examined.

Suppose \( f_r \) represents the force solution where the null space solution is equal to zero. Since all of the constraints considered in this problem are linearly independent, each row of \( C_r \) plus the null space can be used as a basis in the force space \( f_r \). Thus, the gradient of the objective function in the force space can be generally described as follows:

\[
\nabla g(f_r) = C_r^T \nu_r + g_{\perp}
\]

where

\[
\nabla g(f_r) = \text{the gradient of the objective function at } f_r,
\]
\[
\nu_r = \text{a vector whose elements represent the Lagrange multipliers},
\]
\[
g_{\perp} = \text{a vector in the orthogonal subspace spanned by each row of } C_r.
\]

Since minimizing \( ||Jf - b|| \) is similar to minimizing \( \frac{1}{2}||Jf - b||^2 \), \( \nabla g(f_r) \) in (5.16) can be determined as

\[
\nabla g(f_r) = J^T(Jf_r - b).
\]

Multiplying both sides of (5.16) with \( C_r \), one obtains

\[
C_r \nabla g(f_r) = C_r C_r^T \nu_r.
\]

The last term of (5.16) disappears because \( g_{\perp} \) is in the orthogonal subspace spanned by the row vectors of \( C_r \). Based on (5.18), Lagrange multipliers can be determined as
\begin{equation}
\nu_r = (C_rC_r^T)^{-1}C_r\nabla g(f_r). \tag{5.19}
\end{equation}

Based on the Kuhn-Tucker necessary condition, \( f_r \) will represent the optimal solution if the force space is completely spanned by the rows of \( C_r \) and each element of \( \nu_r \) in (5.19) is greater than zero. However, if there is at least one element of \( \nu_r \) which is less than zero, \( f_r \) cannot be considered as the optimal solution. Under this circumstance, the most negative element of \( \nu_r \) is selected and the corresponding column of \( C_r^T \) is released from the active constraints. Based on this procedure, there exists one degree of freedom in the null space and therefore, the same process as previously explained in (5.8)-(5.15) can be used to calculate the new feasible solution.

Since the sign of the Lagrange multipliers indicates constraints which may be made inactive based on the gradient of the objective function, the same expression as (5.19) can be evaluated at each step in order to decide whether a constraint should be released from \( C_r \). However, because the null space solution is determined based on the least-square solution of \( ||Jf - b|| \) and not locally on its gradient, a constraint will be released if the sign of Lagrange multipliers is negative and the value of \( f_{i+1} \) obtained from (5.13) does not violate this constraint.

Using this hypothesis, one may first release a constraint based on the signs of the Lagrange multipliers and then examine whether \( f_{i+1} \) is still feasible. If the value of \( f_{i+1} \) does not violate this constraint, it can be considered as a new feasible solution and the new active constraint set \( C_{i+1} \) will be formed by releasing the constraint with the negative Lagrange multiplier. Otherwise the constraint will be kept in the active constraint set \( C_{i+1} \) regardless of the sign of the Lagrange multipliers. However,
when the dimension of the null space becomes zero, the optimal criterion of the force solution will be evaluated based on the Lagrange multipliers which are obtained from the gradient of the objective function.

One possible starting point is the pseudoinverse force solution. This solution is obtained from the force constraints,

$$ J_F f = F $$

where both $J_F$ and $F$ are defined according to (5.2). For a trotting quadruped, the pseudoinverse force solution is simply specified as

$$ f_i = J_F^+ F = \begin{bmatrix} \frac{F}{2} \\ \cdots \\ \frac{F}{2} \end{bmatrix}. $$  \tag{5.21}

If the value of $f_i$ does not violate any force constraints in (5.7), it can be considered as the starting point. Therefore, the initial active constraints $C_i$ in this case are equal to $J_F$.

Another starting point may be necessary if $f_i$ in (5.21) does not represent a feasible solution. Then a possibility is to use the pseudoinverse force solution for only the $z$-components. Equivalently, one can use $F = [0 \ 0 \ F_z]^T$ in (5.20).

Although there are no unique ways to determine the initial solution, a good starting guess can help minimizing the computation time. This result was illustrated in the method of Gradient Projection which was used in the hexapod walking machine [34].

5.5 Summary

A mathematical formulation of the force distribution in a trotting quadruped is presented in this chapter. Since the vehicle is statically unstable under this mode, both
commanded body forces and torques are resolved to an overdetermined system which provides no exact solutions. Under this circumstance, the force solution which represents a least-square fit to these commanded body forces and torques is used. In order to prevent a foot from slipping and to ensure a proper contact, both friction cone and minimum vertical force constraints are also included in this problem. Instead of using a quadratic programming approach, an optimization scheme which is modified from the gradient projection method is applied to determine the least-square solution with respect to these constraints.
CHAPTER VI

Simulation Results

6.1 Introduction

To demonstrate all the topics discussed earlier in this dissertation, the dynamic simulation of a trotting quadruped on a hard contact surface will be illustrated in this chapter. The direct dynamics are formulated based on the Newton-Euler formulation which is described in Chapter 3. The body and joint velocities during impact will be determined based on the impulsive formulation presented in Section 4.4.5. The control algorithm for this quadruped is based on the work developed in [2] and finally the force distribution routine is implemented using the modified gradient projection technique which is presented in Chapter 5.

6.2 Simulation Results based on the 3-D Impulsive Formulation in Section 4.4.5

The application of the impulsive formulation presented in Section 4.4.5 will be demonstrated in this section. Unlike the isolated experiments in Chapter 4, a complete dynamic simulation of a quadruped will be used.

In order to examine the effect of impulsive forces in the hard contact mode, both the trot and the walk gaits are evaluated in this simulation. The leg numbering
convention and leg sequencing diagram for these gaits are displayed in Figure 46. The solid lines in the diagram represents the time when a leg is on the ground. In the trot, the cycle begins with legs 1 and 4 in the supporting phase while legs 2 and 3 are in the air. After legs 2 and 3 are on the ground, the sequence then switches to legs 1 and 4. In this simulation, the transition from one pair to the next always results in a short period of all four feet being on the ground simultaneously. This implementation is to guarantee that a vehicle always has some feet on the ground to provide forces and torques at the body. Moreover, each pair also spends equal portions of the gait period on the ground.

Unlike the trot, the walk simply has one leg in the air while the other three are on the ground supporting the vehicle. The order of the leg which is lifted off the ground can be specified as 4, 2, 3 and 1. Each leg also spends similar portions of the gait period on the ground. A complete discussion of walking machine gaits can be found in [40].

6.2.1 An Experiment on a Quadruped with the Trot Gait

The first simulation presented in this section is an example of a trotting quadruped. The vehicle’s body is located at 6 ft above the ground. The body orientation during the simulation is described by row, pitch and yaw Euler angles where the values of these angles are initially set to zero. The vehicle is programmed to travel along the x direction with a constant speed of 5 ft/sec. The motion of the vehicle is generated through simulating the dynamics of the system but the update in state caused by impact is determined through the impulsive formulation. The total real time for the
Figure 46: The leg numbering convention and leg sequencing diagram for the trot and the walk gait used in the simulation. The solid line in the picture indicates the time when a leg is in the supporting phase.
simulation is equal to 5 sec and the step size used for the integration is 5 msec.

According to the impact model described in Section 4.4.5, only vertical impulsive forces are considered at the time of collision. This assumption is made because of the abrupt change in vertical velocities during impact as compared to the horizontal velocities. This situation occurs when a foot contacts with a hard surface. In other words, when a foot makes contact with hard ground and remains on the surface after impact, its vertical velocity immediately becomes zero. Based on this assumption, however, the impact model provides no impulsive forces in the horizontal direction. As a result, a foot may slide on the surface after the collision. Since these horizontal side effects due to the vertical impulses are expected to be relatively small, adding a small amount of horizontal damper in the continuous simulation should be sufficient to keep the foot from slipping.

Figure 47 displays the trajectories of vertical foot velocities during the simulation. The negative value of these vertical velocities indicates the time when a foot is lifted off the ground whereas the positive value of these velocities illustrates the time when a foot is placed on the ground. The trajectories of the velocity between these two transitions represent the transfer phase of each leg and the zero value of these velocities displays the time when a foot is in the supporting phase. Each vertical velocity also displays an overshoot due to the rolling of the vehicle’s body when a foot is lifted off the ground.

Tables 17-19 display the foot velocities, leg status and impulsive forces before and after impact. The data is collected at the end of each sampling time which is
Figure 47: The vertical foot velocities in ft/sec of a trotting quadruped with a constant speed of 5 ft/sec.
Table 17: The vertical foot velocities of a trotting quadruped with respect to the world coordinate before and after impact.

<table>
<thead>
<tr>
<th>Sampling Time</th>
<th>Before Impact</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>After Impact</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$v_{31x}$</td>
<td>$v_{32x}$</td>
<td>$v_{33x}$</td>
<td>$v_{34x}$</td>
<td></td>
<td>$v_{31x}$</td>
<td>$v_{32x}$</td>
<td>$v_{33x}$</td>
<td>$v_{34x}$</td>
</tr>
<tr>
<td>0.740</td>
<td>0.0</td>
<td>13.281</td>
<td>13.644</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.026</td>
<td>13.308</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.745</td>
<td>-0.026</td>
<td>10.978</td>
<td>-0.696</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.023</td>
<td>0.0</td>
<td>-0.674</td>
<td>-0.024</td>
</tr>
<tr>
<td>0.750</td>
<td>-0.023</td>
<td>-1.130</td>
<td>0.977</td>
<td>-0.024</td>
<td>0.0</td>
<td>-0.025</td>
<td>-1.128</td>
<td>0.0</td>
<td>-0.023</td>
</tr>
<tr>
<td>0.760</td>
<td>-11.592</td>
<td>1.807</td>
<td>0.0</td>
<td>-11.744</td>
<td>0.0</td>
<td>-11.591</td>
<td>0.0</td>
<td>0.0</td>
<td>-11.749</td>
</tr>
<tr>
<td>1.395</td>
<td>13.126</td>
<td>0.0</td>
<td>0.0</td>
<td>13.742</td>
<td>13.152</td>
<td>-0.036</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.400</td>
<td>10.840</td>
<td>-0.036</td>
<td>0.0</td>
<td>-0.697</td>
<td>0.0</td>
<td>-0.033</td>
<td>-0.023</td>
<td>-0.676</td>
<td>0.0</td>
</tr>
<tr>
<td>1.410</td>
<td>-0.164</td>
<td>-3.036</td>
<td>-3.073</td>
<td>2.285</td>
<td>-0.159</td>
<td>-3.042</td>
<td>-3.072</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.415</td>
<td>1.796</td>
<td>-8.9</td>
<td>-9.02</td>
<td>0.0</td>
<td>0.0</td>
<td>-8.899</td>
<td>-9.024</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2.055</td>
<td>0.0</td>
<td>13.136</td>
<td>13.751</td>
<td>0.0</td>
<td>-0.036</td>
<td>13.162</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2.060</td>
<td>-0.036</td>
<td>10.851</td>
<td>-0.698</td>
<td>0.0</td>
<td>-0.033</td>
<td>0.0</td>
<td>-0.677</td>
<td>-0.024</td>
<td>0.0</td>
</tr>
<tr>
<td>2.070</td>
<td>-7.574</td>
<td>1.829</td>
<td>0.0</td>
<td>-7.648</td>
<td>-7.573</td>
<td>0.0</td>
<td>0.0</td>
<td>-7.653</td>
<td>0.0</td>
</tr>
</tbody>
</table>

indicated in the first column of these tables. Since the collision occurs periodically, only events between the sampling time 0.74 and 2.07 are displayed. The first event which corresponds to the collisions of feet 2 and 3 with the ground occurs within 0.74 to 0.76 sec. The second event which demonstrates the impact of feet 1 and 4 occurs within 1.395 to 1.415 sec. The last event which starts from 2.055 to 2.07 sec represents the next cycle of the trotting in the simulation.

According to Tables 17 and 18, foot 3 first makes contact with the ground at 0.740 sec. Foot 2, on the other hand, is still in the air. Feet 1 and 4 are simply resting on the ground before the impact. The velocity of foot 3 before the collision is equal to 13.644 ft/sec.
Table 18: The leg status of a trotting quadruped before and after impact where 0 = in the air, 1 = on-the-ground-fixed and 2 = on-the-ground-rising. The transition 0/1 indicates an impact.

<table>
<thead>
<tr>
<th>Sampling Time</th>
<th>Before Impact/After Impact</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Leg 1</td>
</tr>
<tr>
<td>0.740</td>
<td>1/2</td>
</tr>
<tr>
<td>0.745</td>
<td>2/2</td>
</tr>
<tr>
<td>0.750</td>
<td>2/2</td>
</tr>
<tr>
<td>0.760</td>
<td>0/0</td>
</tr>
<tr>
<td>1.395</td>
<td>0/0</td>
</tr>
<tr>
<td>1.400</td>
<td>0/1</td>
</tr>
<tr>
<td>1.410</td>
<td>2/2</td>
</tr>
<tr>
<td>1.415</td>
<td>0/1</td>
</tr>
<tr>
<td>2.055</td>
<td>1/2</td>
</tr>
<tr>
<td>2.060</td>
<td>2/2</td>
</tr>
<tr>
<td>2.070</td>
<td>0/0</td>
</tr>
</tbody>
</table>

Table 19: The corresponding impulsive forces for a trotting speed of 5 ft/sec.

<table>
<thead>
<tr>
<th>Sampling Time</th>
<th>$F_{iz1}$</th>
<th>$F_{iz2}$</th>
<th>$F_{iz3}$</th>
<th>$F_{iz4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.740</td>
<td>-0.007</td>
<td>0.0</td>
<td>-34.693</td>
<td>-0.053</td>
</tr>
<tr>
<td>0.745</td>
<td>0.0</td>
<td>-28.27</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.750</td>
<td>0.0</td>
<td>0.0</td>
<td>-2.398</td>
<td>0.0</td>
</tr>
<tr>
<td>0.760</td>
<td>0.0</td>
<td>-4.413</td>
<td>-0.01</td>
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</tr>
<tr>
<td>1.395</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.013</td>
<td>-34.852</td>
</tr>
<tr>
<td>1.400</td>
<td>-28.25</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.410</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-5.566</td>
</tr>
<tr>
<td>1.415</td>
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<td>0.0</td>
<td>-0.009</td>
</tr>
<tr>
<td>2.055</td>
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<td>0.0</td>
<td>-34.863</td>
<td>-0.013</td>
</tr>
<tr>
<td>2.060</td>
<td>0.0</td>
<td>-28.181</td>
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<tr>
<td>2.070</td>
<td>0.0</td>
<td>-4.537</td>
<td>-0.01</td>
<td>0.0</td>
</tr>
</tbody>
</table>
As a result of the collision, the impulsive forces of -0.007, -34.693 and -0.053 are obtained at feet 1, 3 and 4. Since foot 2 is still in the air, there are no impulsive forces at this foot. The foot velocities after impact as determined from the impulsive formulation are equal to -0.026 ft/sec, 13.308 ft/sec, 0.0 ft/sec and 0.0 ft/sec. Since the velocity at foot 1 is away from the ground after impact, its leg status must be changed from on-the-ground-fixed to on-the-ground-rising. This indicates the situation where a foot absorbs an impulsive force and rise from the ground. According to the impact model proposed by Rehman [27], a foot may rise from the ground after impact only with no impulsive forces. This example, however, clearly shows the inadequacy of that assumption and justifies the more complete treatment used here.

Once foot 3 is placed on the ground, the second collision then occurs at foot 2 \( t = 0.745 \). The status of legs 1 and 4 before the impact is similar to the previous sample. Leg 3, however, begins to rise from the ground with a velocity of -0.696 ft/sec. The velocity of foot 2 prior to the impact is equal to 10.978 ft/sec.

After impact, foot 2 remains fixed on the ground with zero vertical velocity. Feet 1 and 3 continue rising from the ground with vertical velocities of -0.023 ft/sec and -0.674 ft/sec, respectively. Foot 4, which is used to be resting on the ground, starts rising with a vertical velocity of -0.024 ft/sec after impact. The corresponding impulsive force which stops the vertical motion of foot 2 is equal to -28.27. Unlike the previous sample, foot 4 does not absorb any impulsive force while it starts rising from the ground. This event shows another possibility of a transition which can occur at a foot with zero impulsive force.
Since foot 3 is moving away from the ground, its vertical status starts changing at $t = 0.75$ from on-the-ground to in-the-air. However, this motion is actually the result caused by a small bump and is not intended lifting motion by the control system. Therefore, the simulation shows another small impact soon at foot 3 where its vertical velocity before the collision is equal to 0.977 ft/sec. The impulsive forces which stops the motion of this foot is equal to -2.398. The same circumstance also occurs at foot 2 when $t = 0.760$ sec. These two incidents demonstrate slightly hopping of feet 2 and 3 after their first collisions. At the end of this sampling period, feet 2 and 3 both remain fixed on the ground until the next locomotion cycle begins. Feet 1 and 4, on the other hand, are in the air with velocities of -11.591 ft/sec and -11.749 ft/sec.

Similar explanations can be applied to the next trotting cycle where feet 1 and 4 strike the ground and feet 2 and 3 are in the supporting phase. After the first collisions at 1.395 sec and 1.4 sec, both feet 1 and 4 also demonstrates slightly hopping before they remain fixed on the ground at $t = 1.415$ sec. Although it takes approximately two steps in the first two locomotion cycles for feet 2 and 3 as well as feet 1 and 2 to remain fixed on the ground, only one step of this effect is shown in the next trotting cycle.

When a foot strikes the ground, the impulsive force generated at the contact can cause abrupt changes to the body velocity at the hip. This result cannot be seen if the kinematic model, which assumes that a foot simply stops without any effect on the body, is used to update both joint and body velocities after impact. To demonstrate this effect, one may compare the vertical body velocities at the hip obtained from
the kinematic model and the impulsive formulation for a trotting quadruped. The
vehicle is programmed to travel with a constant speed of 5 ft/sec.

Figure 48 and 49 display the vertical body velocity at hips 1-4 obtained from
the kinematic model and the impulsive formulation. The dashed lines in the pic­
tures represent the quantities determined using the impulsive formulation. Before
the collisions occur at feet 2 and 3 (0 < t < 0.7), both the kinematic model and the
impulsive formulation demonstrate similar effects on the vertical body velocity at the
hips. However, after feet 2 and 3 make contact with the ground (0.74 < t < 0.76),
the results obtained from the kinematic model and impulsive formulation become
apparently different. The differences of the responses obtained from these two im­
 pact models are the result of different leg lifts and plants from the control system.
According to [2], triggering of leg lifts and plants in the simulation is based on the
positions of the stance feet with respect to their working ellipses. When a stance leg
moves to a point close to its working ellipse, it will be lifted off the ground. For the
simulation with a kinematic contact model, a foot simply stops after impact with zero
velocity in both horizontal and vertical directions. Therefore, no horizontal dampers
are required to prevent a foot from slipping in the continuous simulation. On the
other hand, a small amount of horizontal damper is added to the continuous simula­
tion to prevent a foot from slipping once the impulsive formulation is applied. Due
to the differences of this constraint, triggering of leg lifts and plants in the simulation
is also different between these two approaches. This result can be clearly seen from
the plots of vertical foot velocities shown in Figure 50.
Figure 48: The vertical body velocity in ft/sec obtained at hips 1 and 2. The solid line represents the quantity obtained from the kinematic model and the dashed line represents the quantity determined from impulsive formulation.
Figure 49: The vertical body velocity in ft/sec obtained at hips 3 and 4. The solid line represents the quantity obtained from the kinematic model and the dashed line represents the quantity determined from impulsive formulation.
Figure 50: The vertical foot velocities in ft/sec obtained from the kinematic model and the impulsive formulation. The solid line represents the quantity determined from the kinematic model and the dashed line is the quantity calculated using the impulsive formulation.
Based on this result, the impulsive force generated at each collision can provide an effect on the body velocity at the hip. Thus, the assumption that a foot simply stops after impact with no effect on the body velocity may provide inaccurate results. In other words, one cannot update the joint and body velocities by simply solving the kinematics from the body to the contact feet. Therefore, the hypotheses that a full impulsive formulation is worth performing has been justified.

6.2.2 An Experiment on a Quadruped with the Walk Gait

The second simulation presented in this section is an example of a walking quadruped. Unlike the trot, the walk has only one leg in the air at one time therefore provides another view of the relationship between the impulsive forces and foot velocities during impact.

In this experiment, a quadruped is programmed to walk in the x direction with a constant speed of 2 ft/sec. The vehicle's body is located at 5.5 ft above the ground. The sequence of each leg is also specified in the order as shown in Figure 46. The simulated time in this case is equal to 5 sec.

Figure 51 displays the trajectories of vertical foot velocities in the simulation. Like the previous result shown in Figure 47, the time when a foot is lifted off and placed on the ground can be seen from the negative and positive values of these foot velocities. The overshoot which occur in each curve are due to the rolling of the body caused by the motion of the leg in the air.

Tables 20-22 illustrate the vertical foot velocities, the leg status and the impulsive forces during the first locomotion cycle. In order to clearly examine the transition of
Figure 51: The vertical foot velocities in ft/sec of a walking quadruped with a constant speed of 2 ft/sec.
Table 20: The vertical foot velocities of a walking quadruped with respect to the world coordinate before and after impact.

| Sampling Time | Before Impact | | | | After Impact | | | |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
|               | \(v_{31z}\)  | \(v_{32z}\)  | \(v_{33z}\)  | \(v_{34z}\)  | \(v_{31z}\)  | \(v_{32z}\)  | \(v_{33z}\)  | \(v_{34z}\)  |
| 0.603         | 0.0           | 0.0           | 0.0           | 15.997        | 0.0           | -0.073        | -0.007        | 0.0           |
| 0.732         | 0.0           | -1.2          | 0.01          | 0.0           | 0.0           | -1.294        | 0.0           | 0.0           |
| 0.977         | 0.0           | 13.2          | 0.939         | -1.1E-6       | -0.004        | 13.183        | 0.0           | -5.9E-6       |
| 1.019         | -0.004        | 14.3          | 0.0           | 0.0           | 0.0           | 0.0           | 0.0           | -0.052        |
| 1.037         | 0.0           | 0.0           | -18.9         | 0.225         | 0.0           | -5.5E-4       | -18.9         | 0.0           |
| 1.354         | 0.0           | 0.04          | -0.039        | 0.0           | 0.0           | 0.0           | -0.039        | -1.5E-4       |
| 1.413         | 0.0           | 0.0           | 26.9          | 0.089         | 0.0           | -3.3E-4       | 26.9          | 0.0           |
| 1.426         | 0.0           | 0.14          | 22.69         | 0.0           | 0.0           | 0.0           | 22.69         | -7.6E-4       |
| 1.433         | 0.0           | 0.0           | 16.07         | -7.6E-4       | -0.073        | 0.0           | 0.0           | -7.4E-3       |
| 1.434         | -0.073        | 0.0           | 0.0           | 0.146         | -0.072        | -5.8E-4       | 0.0           | 0.0           |
| 1.459         | -24.0         | 0.22          | 2.2E-4        | 0.0           | -24.0         | 0.0           | 0.0           | -1.2E-3       |
| 1.471         | -25.2         | 0.0           | 0.0           | 0.013         | -25.2         | -4.3E-6       | -2.6E-6       | 0.0           |
| 1.874         | 18.35         | 0.12          | -1.0E-4       | 0.0           | 18.35         | 0.0           | 0.0           | -6.1E-4       |
| 1.910         | 13.9          | 0.0           | 0.0           | -6.1E-4       | 0.0           | 0.0           | -0.054        | 0.0           |
| 1.966         | 0.0           | 0.0           | 0.21          | -11.8         | -5.5E-4       | 0.0           | 0.0           | -11.803       |
Table 21: The legged status of a walking quadruped before and after impact where 0 = in the air, 1 = on-the-ground-fixed and 2 = on-the-ground-rising. The transition 0/1 indicates an impact.

<table>
<thead>
<tr>
<th>Sampling Time</th>
<th>Before Impact/After Impact</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Leg 1</td>
</tr>
<tr>
<td>0.603</td>
<td>1/1</td>
</tr>
<tr>
<td>0.732</td>
<td>1/1</td>
</tr>
<tr>
<td>0.977</td>
<td>1/2</td>
</tr>
<tr>
<td>1.019</td>
<td>2/1</td>
</tr>
<tr>
<td>1.037</td>
<td>1/1</td>
</tr>
<tr>
<td>1.354</td>
<td>1/1</td>
</tr>
<tr>
<td>1.413</td>
<td>1/1</td>
</tr>
<tr>
<td>1.426</td>
<td>1/1</td>
</tr>
<tr>
<td>1.433</td>
<td>1/2</td>
</tr>
<tr>
<td>1.434</td>
<td>2/2</td>
</tr>
<tr>
<td>1.459</td>
<td>0/0</td>
</tr>
<tr>
<td>1.471</td>
<td>0/0</td>
</tr>
<tr>
<td>1.874</td>
<td>0/0</td>
</tr>
<tr>
<td>1.910</td>
<td>0/1</td>
</tr>
<tr>
<td>1.966</td>
<td>1/2</td>
</tr>
</tbody>
</table>
Table 22: The corresponding impulsive forces for a walking quadruped with a speed of 2 ft/sec.

<table>
<thead>
<tr>
<th>Sampling Time</th>
<th>$F_{l1}$</th>
<th>$F_{l2}$</th>
<th>$F_{l3}$</th>
<th>$F_{l4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.603</td>
<td>-0.127</td>
<td>0.0</td>
<td>0.0</td>
<td>-37.213</td>
</tr>
<tr>
<td>0.732</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.116</td>
<td>-1.05E-4</td>
</tr>
<tr>
<td>0.977</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.981</td>
<td>0.0</td>
</tr>
<tr>
<td>1.019</td>
<td>-0.023</td>
<td>-33.485</td>
<td>-0.113</td>
<td>0.0</td>
</tr>
<tr>
<td>1.037</td>
<td>-0.0019</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.352</td>
</tr>
<tr>
<td>1.354</td>
<td>-9.9E-5</td>
<td>-0.075</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.413</td>
<td>-6.9E-4</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.11</td>
</tr>
<tr>
<td>1.426</td>
<td>-3.2E-4</td>
<td>-0.24</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>1.433</td>
<td>0.0</td>
<td>-0.13</td>
<td>-37.38</td>
<td>0.0</td>
</tr>
<tr>
<td>1.434</td>
<td>0.0</td>
<td>0.0</td>
<td>-1.3E-3</td>
<td>-0.18</td>
</tr>
<tr>
<td>1.452</td>
<td>0.0</td>
<td>-0.348</td>
<td>-0.002</td>
<td>0.0</td>
</tr>
<tr>
<td>1.459</td>
<td>0.0</td>
<td>-0.36</td>
<td>-1.9E-3</td>
<td>0.0</td>
</tr>
<tr>
<td>1.471</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-0.016</td>
</tr>
<tr>
<td>1.874</td>
<td>0.0</td>
<td>-0.15</td>
<td>-0.001</td>
<td>0.0</td>
</tr>
<tr>
<td>1.910</td>
<td>-32.25</td>
<td>-0.031</td>
<td>0.0</td>
<td>-0.11</td>
</tr>
<tr>
<td>1.966</td>
<td>0.0</td>
<td>-1.69E-3</td>
<td>-0.302</td>
<td>0.0</td>
</tr>
</tbody>
</table>
each foot during impact, an integration interval of 1 msec is used in the simulation. The locomotion cycle begins with foot 4 striking the ground at $t = 0.603$ sec. Feet 1, 2 and 3 simply rest on the ground before the collision. The vertical velocity of foot 4 before impact is equal to 15.997 ft/sec. After impact, feet 1 and 4 remain fixed on the ground with zero vertical velocity. Feet 2 and 3, on the other hand, change their status from on-the-ground-fixed to on-the-ground-rising. According to Table 22, the impulsive forces of -0.127 and -37.213 are obtained at feet 1 and 4. In this case, foot 1 simply absorbs impulsive forces without changing its status. Feet 2 and 3, however, absorb no impulsive forces despite the changes of their vertical velocities. Since foot 3 starts rising from the ground after impact, its status is further changed to in-the-air during the period of $t = 0.732$. As a result, this foot then slightly hops on the ground at $t = 0.732$. According to Table 20, foot 2 start lifting off the ground at $t = 0.733$. During continuous simulation, the status of foot 3 is changed from on-the-ground-fixed to on-the-ground-rising at $t = 0.966$ and further becomes in-the-air at $t = 0.967$. This also results in another hopping at this foot at $t = 0.977$.

The second collision of the locomotion cycle occurs at $t = 1.019$ when foot 2 makes contact with the ground. Feet 3 and 4 simply rest on the ground before impact. Foot 1, on the other hand, is rising from the ground as a result of the second hopping which previously occurs at foot 3. After the collision, feet 2 and 3 remain fixed on the ground. The status of foot 4 is changed from on-the-ground-fixed to on-the-ground-rising after impact. The status of foot 1, which used to be on-the-ground-rising, is also changed to on-the-ground-fixed after impact. The impulsive forces at feet 1 and
2 are found to be -0.023 and -33.485. Unlike most of previous transitions, this event demonstrates another possibility where a foot is switched from on-the-ground-rising to on-the-ground-fixed due to the impulsive force. From $t = 1.354$ to $1.413$, feet 2 and 4 both display the hopping effect after the collision at foot 2.

Similar situations also occur when feet 3 and 1 make contacts with the ground at $t = 1.433$ and 1.91. As a result of these collisions, feet 2, 3 and 4 also slightly hop on the surface. Based on these results, a collision at one foot can cause other feet to rise from the ground regardless of the number of supporting feet. Therefore, if the legs are in the configuration where these changes are possible, a foot can hop on the ground after impact. It should be noted that the hopping effect is theoretically entirely independent of the velocity at which a foot collides with the ground.

6.3 Simulation Results of the Force Distribution Algorithm

The foot force solution which is determined from the modified gradient projection technique will be illustrated in this section. The frictional coefficient and the minimum vertical force are specified as 1 and 30 in this simulation. Therefore, the foot force constraints in (5.4)-(5.5) can be described as

$$|f_{ix}| \leq f_{iz}, \quad (6.1)$$

$$|f_{iy}| \leq f_{iz}, \quad (6.2)$$

$$f_{iz} \geq 30 \quad (6.3)$$

where $f_{ix}$, $f_{iy}$ and $f_{iz}$ represent the foot force in x, y and z directions.

Due to the speed of the legs and their significant masses, the leg dynamics can
provide a substantial effect on the commanded forces and torques at the body. This, on the other hand, may lead to difficulties in studying the characteristics of foot force allocation solutions. Therefore, in order to examine the force responses provided by the force distribution algorithm, leg dynamics will be temporarily excluded from the simulation and only body dynamics are first considered. Each leg is assumed to follow its trajectories perfectly in the simulation. Furthermore, the pseudoinverse force solution as described in (5.21) will be used as a starting guess for the force distribution algorithm.

Figure 52 display the x and z components of the forces at feet 1 and 4 during the first and third trotting cycles. According to these curves, feet 1 and 4 are in the supporting phase approximately from 0 to 0.7 sec and 1.4 to 2 sec. Between $0 < t < 0.5$, all feet are in the supporting phase and the vehicle simply rests on the ground. This result can be seen from non-zero foot forces in the z direction in Figure 52.

The first trotting cycle begins when feet 2 and 3 start lifting off the ground at $t = 0.04$ sec. Thus, only feet 1 and 4 are on the ground supporting the vehicle from 0.05 to 0.7 sec. The switching of these leg status can be seen from the abrupt changes of the foot forces at $t = 0.04$ sec. The force constraints which are active at this moment are $f_{1z} = -f_{1z}$ and $f_{4z} = f_{4z}$. After feet 2 and 3 are in the air, there are no active force constraints. This situation corresponds to the case where the pseudoinverse force solution plus its null space solution minimize $\|Jf - b\|$ without violating the force constraints in (6.3). In other words, the scalar $\alpha_i$, which represents
Figure 52: The x and z components of the forces at feet 1 and 4 during their supporting phases using only body dynamics. The solid line is the x component and the dashed line is the z component of the foot forces.
the maximum length of the null space solution in (5.13), is greater than 1. Moreover, these force solutions also demonstrate continuous responses during this period of the locomotion cycle. The same result can be seen from the force response at feet 1 and 4 during the third period of the simulation ($1.4 < t < 2$).

Similar force responses are also obtained at feet 2 and 3 during the second and fourth locomotion cycle. When feet 1 and 4 start to lift off the ground at $t = 0.76$, the force constraints in (6.3), which is active, is $f_{2y} = -f_{2z}$. After feet 1 and 4 are in the air, the pseudoinverse force plus its null space solutions obtained at feet 2 and 3 represent a least-squares solution subject to the body commanded forces and torques as well as the force constraints in (6.3). As a result, the forces at feet 2 and 3 also demonstrate continuous responses during this second locomotion cycle. The same results can be seen during the fourth locomotion cycle of the simulation.

In order to illustrate the effects of leg dynamics, one may rerun the same simulation with both body and leg dynamics. Figure 53 displays the force responses at feet 1 and 4 during the first locomotion cycle. Unlike the results shown in Figure 52, the force responses in this case are not as smooth as the results obtained with only body dynamics. As described in [2], the motion of the legs which are in the air can provide a direct effect on the motion of the vehicle's body. In order to rectify these errors, the body controller must apply corrective forces and torques through the legs which are on the ground. This results in varying commanded body forces and torques used in each locomotion cycle.

Figure 54 displays the comparison of the x and z components of body velocities
with and without leg dynamics. From this figure, it is clear that the motion of the leg in the air can provide substantial changes in the motion of the vehicle’s body.

From Figure 53, the force constraints $f_{1x} = -f_{1z}$ and $f_{4x} = f_{4z}$ are both active when feet 2 and 3 start lifting off the ground ($t = 0.04$). From $t = 0.04$ to $t = 0.7$, there are no force constraints which are active. In other words, the least-squares solution is represented by the pseudoinverse force plus its homogeneous solutions during this period. From $t = 0.7$ to $t = 0.71$, the force constraints $f_{1x} = f_{1z}$, and $f_{4x} = f_{4z}$ become active. These results can be seen from the overlap in the foot forces in x and z directions in Figure 53.

Figure 55 displays the joint torques at leg 1 during the first two seconds of the simulation. These joint torques are determined by the controllers in order to provide the foot force during the supporting phase and the motion of the leg when it is in the air. According to the control implemented in [2], the ideal actuators are assumed and these joint torques are always realizable. However, one may add the limitation on these joint torques in the controllers in order to generate a more realistic simulation.

### 6.4 Summary

The application of the impulsive formulation presented in Section 4.4.5 is demonstrated in this chapter. Unlike the previous examples shown in Section 4.4.5, two complete simulations of the trot and the walk in a quadruped are used. The simulation is generated based on the recursive Newton-Euler formulation and the multischeme integration which consists of the fourth-order Runge-Kutta and the modified
Figure 53: The force allocation at feet 1 and 4 during the first locomotion cycle when leg dynamics are included in the simulation. The solid line is the x component and the dashed line is the z component of the foot forces.
Figure 54: The x and z components of body velocities with and without leg dynamics. The dashed line in this picture represents the quantities obtained with only body dynamics.
Figure 55: The commanded joint torques at leg 1 during the first two seconds of the simulation.
Euler.

According to the results obtained from each simulation, a foot which simply rests on the ground can change its status from on-the-ground-fixed to on-the-ground-rising after impact. In the trot, a contact foot can absorb an amount of impulsive force and rise from the ground. This result demonstrates a counterexample to the impact model in [27] which assumes that each contact foot may rise from the ground with only zero impulsive force. Moreover, since a foot does not possess any compliance, changes in the leg status due to impact can also make a foot hop on the ground. Because this event occurs right after the collision, a small step size has been used in both simulations to clearly demonstrate this effect.

The application of the force distribution algorithm to a trotting quadruped is also demonstrated in this chapter. The force solutions are determined based on the modified gradient projection technique described in Chapter 5. In order to clearly examine the force allocation provided by this algorithm, only the body dynamics are first considered in the simulation. As a result, the foot force obtained from this approach demonstrates the smoothness and continuity response during each supporting phase. These force allocations, however, becomes more complicated when the leg dynamics are included in the simulation. Furthermore, the force allocations also display the periods where there are no active force constraints. Therefore, the force solutions in these periods are simply composed of the pseudoinverse force and its null space solutions. Under this circumstance, a starting guess based on the pseudoinverse force solution can help the algorithm to reach the solution faster.
CHAPTER VII

Conclusions and Future Work

7.1 Summary

A complete dynamic simulation of a quadruped which moves on a hard contact surface is developed in this work. Unlike most of the previous work, here a contact model which is more appropriate than springs and dampers has been used to simulate the motion of the machine on a hard terrain.

The direct dynamics for the quadruped has been formulated based on the recursive Newton-Euler formulation where the vehicle's body is considered as a moving base in the simulation. Since the system is represented by multiple closed-kinematic chains, additional constraints must be assigned at the foot tips. In a hard contact mode, the status of a foot on the ground can either be on-the-ground-fixed or on-the-ground-rising. When a foot is fixed on the ground, there are no accelerations and torques at the contact. On the other hand, if a foot starts rising from the ground or in the air, the forces and torques at the tip are set to zero. These constraints are not necessary for the soft terrain where a contact is modeled by springs and dampers. Although the emphasis is on the hard contact mode, the formulation is developed with the flexibility where different contact models can be tested.
Once the appropriate foot constraints are specified, the system direct dynamics can be written as a \(186 \times 186\) matrix equation. Due to the recursive structure of the Newton-Euler formulation, this original matrix equation can be simplified into a \(42 \times 42\) system where its solution is determined by the Gaussian elimination of four \(9 \times 9\) plus one \(6 \times 6\) matrix equations.

In order to provide better accuracies, an adaptive integration scheme which consists of the fourth order Runge-Kutta and the modified Euler are used in the simulation. Moreover, another way to update the body orientation based on its angular velocity and not the Euler angles is also implemented in this work. Since the overall procedure is implemented in a matrix form, the formulation is very suitable for a supercomputer. With features such as scalar optimization, in-line expansions and vectorization providing on the Cray supercomputer, the computation time for solving direct dynamics can be increased to eight times faster than a primal version without these features.

Before the simulation can be fully utilized for various control algorithms, it is also necessary to verify its correctness. Since the simulation is composed of solving direct dynamics and motion integration, two testing schemes are developed to detect and identify the errors occurring in each part. The first testing scheme is derived based on the conservation of system kinetic and potential energies. For a quadruped, the system energy should be conserved before any feet strike the ground. Therefore, by computing the change of system energies with various step sizes before impact, one can determine the rate of convergence for the integration scheme used in the
For a correct simulation, this convergence rate must be consistent with the order of error provided by the integration. Since the energy involves both positions and velocities, this testing scheme can detect the errors in both direct dynamics and motion integration. However, one may not be able to identify the source of these errors based on the energy test alone.

The second testing scheme presented in this work is the power conservation test. Unlike the first testing scheme, the power conservation test deals with instantaneous quantities such as forces and accelerations. Therefore, it can be used to detect the errors in solving system direct dynamics. For a conservative system, the power flow into the system should be equal to its output. Therefore, by computing the power flow in and out of the system, one can verify the correctness of the solutions obtained from system direct dynamics. Since the power conservation can also be applied to each mechanism, one may use this technique to trace the errors in the dynamic formulation.

The most important part in the hard contact simulation is how to determine the system velocities after impact. When the vehicle operates in the soft contact mode, the contact forces are always specified through springs and dampers and therefore, no special treatment is required for the impact. For a hard contact model, the changes of body and joint velocities after impact must be obtained from the impulsive force at each contact.

Most of the previous impulsive formulations was derived based on the coefficient of restitution. However, according to an example of distributed masses shown in Chapter 4, inconsistent results among impulsive forces, contact velocities and the
value coefficient of restitution can be obtained using this approach. Therefore, a more consistent approach to derive the impulsive formulation is necessary. Furthermore, because collisions in leg vehicles are mostly inelastic, only the impulsive formulation for this type of impact is considered in this work.

Due to the complexity of the problem, preliminary studies for this problem were implemented on a single-point contact case. Based on the results presented in Sections 4.3.1-4.3.5, an impact model with zero coefficient of restitution can always be applied to a single and articulated rigid bodies with one contact point. In other words, one can always find an impulsive force which stops the motion at the contact. This principle, however, does not apply to the case of multiple contacts in leg vehicles since all contact points are not always stopped simultaneously.

Another technique which can be used to determine the impulsive force for multiple collisions is presented in this work. The idea is to first model each contact as a linear damper. Since the change of normal contact velocities is faster than its horizontal components, only the damper in the normal direction is considered. Based on this impact model, the non-impulsive contact forces and velocities are consistently defined. Therefore, by making the damper stiffer to the limit of infinity, one is able to determine the impulsive formulation according to these non-impulsive contact forces. This procedure can be implemented by transforming the contact forces in original time $t$ to a new time $t'$ where $t' = \beta t$ and $\beta$ represents the damping coefficient. Although the direct dynamics for a quadruped are non-linear in $t$, they can be linearized in the $\beta$ scaled time $t'$ once $\beta \to \infty$. As a result, the expression of contact forces
in $t'$ is simply resolved to a first-order system where its solution can be obtained in closed form. Based on this closed-form solution, one can determine the total impulse generated at each contact in $t'$ as $t' \to \infty$. This quantity, on the other hand, is equal to the impulsive force at that contact in the original time $t$. Furthermore, by transforming the system direct dynamics to a new time scale $t'$, one can clearly identify the transition which occurs at each contact as compared to the instantaneous event in the original time $t$.

Another important factor in multiple contacts is how to determine the motion of the contacts which simply rest on the ground. This situation is usually found in the case of multi-legged vehicles where some feet simply rest on the ground while the others make contacts. According to the impact model proposed in this work, one can always find a consistent motion of the contact point which rests on the ground. As a result, a consistent amount of impulsive force can still be obtained under this circumstance.

The application of this impact model and its impulsive formulation is also presented using both isolated examples and a complete simulation. According the isolated examples of a 2-D single rigid body and a 3-D quadruped, the impact model demonstrates the sensitivity toward the damping coefficient and the non-simultaneous collision once a transition occurs after impact. When a contact is broken, its velocity after impact is not uniquely defined. Therefore, different value of damping coefficient can lead to different amounts of impulsive forces. Similar results can be obtained when non-simultaneous collisions are assumed. However, the smaller the amount of
delay in the $t'$ scale, the less is the difference of impulsive forces obtained from the impact model.

In the complete simulation, a quadruped with the trot and the walk gaits are used to demonstrate the effect of impulsive forces on a hard surface. Both simulations represent the case where some feet simply rest on the ground before the impact. According to the results in the trot, a foot can change its status from on-the-ground-fixed to on-the-ground-rising after impact. Furthermore, a foot which starts rising from the ground does absorb a non-zero impulsive force. Therefore, Rehman's assumption [27] which assumes that a foot rises from the ground with no impulsive forces is not always consistent.

Because of the changes in the foot status, the legs which are on the ground will slightly hop on the surface after impact. The same results can also be seen in the walk and therefore, a single collision at one foot can make the others contacts to rise from the ground as well. Moreover, the impulsive forces at the contact also cause abrupt changes in body velocities at the hip. Thus, another assumption that a foot simply stops after the collision without no effect on the body may provide inconsistent results. This also implies that one cannot update both joint and body velocities by simply solving the kinematics from the body to the contact feet.

The last part of the dynamic simulation, which is developed in this work, is the foot force allocation. The function of this algorithm is to determine the desired body forces and torques to be provided by the supporting legs without violating foot force constraints. The friction cone and minimum vertical force constraints are used in
order to prevent a foot from slipping and to ensure a proper contact. Based on these desired body forces and torques, the control algorithm then determines corresponding joint torques input for the simulation.

For statically stable leg vehicles, the force distribution can be described by an underdetermined system. Therefore, it is always possible to find appropriate foot forces which meet the desired body forces and torques as well as the foot force constraints. Mathematical techniques such as linear programming, gradient projection and quadratic programming were previously applied to determine the force solution for this problem. Due to the continuity of response of the solution, the gradient projection method and the quadratic programming are preferable than linear programming.

Unlike statically stable legged vehicles, a quadruped with the moderate and high speed gaits is statically unstable. In this case, one cannot find the foot forces which always satisfy the commanded body forces and torques. In other words, the force distribution for this type of vehicle is described by an overdetermined system. The force distribution algorithm presented in this work is developed to determine the appropriate foot forces for this type of vehicle.

Since one cannot find a solution which satisfies the commanded body forces and torques, the solution which provides the best approximation for these body forces and torques should be considered. In other words, the force distribution in this case can be formulated as a least-square problem where the force solution minimizes the commanded body forces and torques in a least-squares sense. The optimization
scheme used in this work is modified from the method of gradient projection presented in [34]. However, instead of using along the projection of the gradient of the objective function, the solution is determined by searching along the null space of least-square solution itself. The optimal criteria of the solution is verified through the Kuhn-Tucker necessary condition. The algorithm is also developed to provide the flexibility in selecting the starting point. One of the effective starting guesses is the pseudoinverse force solution. Providing that it does not violate any foot force constraints, this solution is very easy to obtain.

The application of this technique to a trotting quadruped is also demonstrated in this work. Since only two feet are in the supporting phase, the force distribution in this case is described by an overdetermined system. In order to clearly examine the force response provided by the algorithm, the leg dynamics are temporarily excluded from the simulation. Based on the results in Section 6.3, the foot forces determined by the modified gradient projection demonstrate smooth and continuous responses. Furthermore, the force responses also illustrate no active force constraints and the force solution in this case is simply represented by the pseudoinverse force plus a portion of its null space solutions. This example demonstrates an advantage of a starting guess based on the pseudoinverse force solution. For this particular example, only one search along the null space of this solution is required and therefore the computation time can be reduced based on this starting guess.

Once the leg dynamics are included in the simulation, the foot forces at the supporting legs display non-smooth responses. This result demonstrates the effect of leg
dynamics on the body motion when the legs are moving in the air. As suggested in [2], these effects can be reduced with slower leg recovering speed, less massive legs and longer stride lengths.

7.2 Future Work

As previously mentioned, the impact model and the impulsive formulation are one important factor for the simulation on a hard terrain. In order to properly update the system velocity after impact, a consistent impact model as well as a correct impulsive formulation are both necessary. For multiple leg vehicles and multifingered robots, the problem becomes very complicated since the collisions involve more than one contact point. Despite their necessities, not many studies have been developed in this area. Therefore, the impact model and the impulsive formulation presented in this work can be regarded as a fundamental step for future development in this area.

Although the technique proposed in this work can be used to calculate the impulsive force for multiple contacts on a hard surface, the following studies can be further extended:

1) Since the formulation presented in this work is limited to a planar surface, a more general impulsive formulation, which works for multiple contacts on an uneven terrain, needs to be developed.

2) The application of this technique to multifingered robots can be demonstrated. Unlike leg vehicles, all fingers mostly come to contact with an object before grasping. Therefore, a situation where a contact simply rests on the surface is not quite applicable in this case. Due to the uneven shape of the object, the impulsive formulation
which works with non-planar contact is also required.

3) Since the changes of contact velocities in the normal direction occur faster than its horizontal components, the impact model consisted of a normal damper has been used in this work. Although only a normal impulsive force is generated from the model, it can provide changes in contact velocities along the horizontal direction. Since the collision is not always perpendicular, a contact may have initial horizontal velocities and therefore it may slide on the surface after impact. In this work, the horizontal dampers have been added during the continuous simulation to minimize this small effect. For a hard contact surface, this modification is reasonable because a foot normally stops instead of sliding on the surface. Based on this result, an extended impact model which takes into account the changes in horizontal velocities can be developed in the future. Although this may result in a more complicated impulsive formulation, the system can be used to design a better running machine in terms of both physical configuration and control software.
REFERENCES


