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Application of FDTD in hybrid methods for EM scattering analysis of cavities with complex terminations

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The Ohio State University, 1994
APPLICATION OF FDTD IN HYBRID METHODS FOR EM SCATTERING ANALYSIS OF CAVITIES WITH COMPLEX TERMINATIONS

A DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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*****

The Ohio State University

1994

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Approved by

Adviser
Department of Electrical Engineering
To my wife Patricia

It is the glory of God to conceal a matter; to search out a matter is the glory of man.
Proverbs 25:2
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- “Implementing the hybrid ray-FDTD method for computing the RCS of 3D open-ended waveguide cavities,” co-author R. Lee, to be presented at the IEEE APS International Symposium and URSI Radio Science Meeting, June 20–26, 1994, Seattle, WA.


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CHAPTER I
INTRODUCTION

The analysis of electromagnetic (EM) scattering from the interior of jet engine inlet/exhaust cavities is an important part of modern aircraft radar cross section (RCS) reduction and radar target identification since the cavity scattered field contributes significantly to the overall RCS of an aircraft. However, this analysis is no easy task, particularly for an arbitrarily shaped inlet cavity with a complex termination (hub, blades, struts, etc.). In this dissertation, the finite-difference time-domain (FDTD) method is combined with three separate methods for the analysis of EM scattering from three-dimensional (3D) open-ended waveguide cavities with interior obstacles or terminations. The cavity structure under consideration can be divided into three regions corresponding roughly to the air intake, the engine and the exhaust sections of a jet engine inlet/exhaust as shown in Fig. 1. The hybrid methods are used specifically to find the scattering from the interior of the cavity only. All other scattering due to the exterior of the cavity, including the rim, are not considered in this analysis although their contributions can be accounted for using the geometry theory of diffraction [1] and its uniform version [2].

Traditionally, the method of moments [3, 4] and modal analysis [5, 6, 7] have been used to find the RCS of cavities. However, the use of these methods are restricted
to electrically small cavities or geometrically simple cavities like the rectangular and cylindrical cavities. Recently, ray methods have been developed to find the RCS of non-canonical and electrically large cavities. They are the Shooting and Bouncing Rays (SBR) method [8, 9], the Generalized Ray Expansion (GRE) method [10, 11] and the Gaussian Beam method [10]. These methods have been shown to produce results that are in good agreement with reference modal solutions for electrically large cavities with planar terminations [10]–[13]. They have also been used with some success for finding the RCS of S-bend cavities with planar terminations [9, 14]. However, realistic cavities, like jet engine inlets or exhaust cavities, have complex terminations. In the realistic case of a jet engine consisting of a hub, fan blades, compressor blades, etc. ray methods are not expected to produce reasonable results because the diffracted fields due to edges (eg. blade edges) and tips (eg. tip of the hub) cannot be accurately modeled by ray methods.

In recent times, physical optics (PO) has also been applied to find the scattering from cavity inlets. In [15, 16], PO was used to characterize the disc-blade termination.
in a semi-infinite cavity in terms of a modal reflection matrix. The approximate approach provided an estimate of the scattering and the modulation expected from a jet inlet engine. Less than two years ago, an iterative PO (IPO) method was proposed as a means of determining the scattering from open-ended waveguide cavities [17]. The method is based on the iterative approach presented in [18] for solving scattering problems of closed bodies. However, the IPO method had problems with convergence and sampling density requirements. These problems have been overcome in [19] where the shadowing effects due to obstacles are taken into account. Relatively good results have been obtained for cavities with a cylindrical hub or a flat termination. However, the IPO method is not expected to be as accurate for cavities with complex terminations.

To handle the complex termination in a cavity, the finite element method (FEM) [20] or the method of moments (MOM) [4] can be used. These methods have the capability to model arbitrary geometries. However, these methods are not without drawbacks. Both methods spend the majority of their computation time in the solution of a matrix equation. For a direct solver, the number of floating point operations (FLOPS) needed to solve the matrix equation in MOM and FEM is of the order of $N^3$ and $N^{5/3}$ respectively, where $N$ is the number of unknowns in the matrix equation. Another drawback of the two methods is their large memory storage requirement, especially for MOM. The memory storage required for MOM is proportional to $N^2$ compared to $N \log N$ for FEM. Therefore, for very large $N$, as in the case of an electrically large three-dimensional cavity problem, the memory storage and computation
time may become prohibitive for both FEM and MOM.

Currently, most of the solutions to cavity scattering problems are obtained via frequency domain methods, such as those mentioned above. In many situations, the transient response of a cavity due to a pulsed signal is desired. In order to find the transient response using frequency domain methods, the problem will have to be solved for a large number of frequencies (probably of the order of 200-1000). The desired transient solution is then obtained via an inverse Fourier transform of the frequency domain solutions. Therefore, it is not efficient to use frequency domain methods to get time domain solutions because of the many repetitious computations.

Time domain methods, on the other hand, are better suited for transient response analysis as they model the actual real-time behavior of the fields. Moreover, broad-band frequency information can be easily obtainable via a single fast Fourier transform (FFT). Using frequency domain methods to obtain the frequency spectrum/time variation will require solving the problem repeatedly for multiple frequencies over the entire frequency range of interest.

Time domain methods evaluate the solution directly in the time domain either explicitly or implicitly [21, 22]. Explicit methods are schemes in which the current field values are dependent only on their past values. Hence, the field values can be evaluated by stepping them in time. Implicit methods are schemes in which the current field values are dependent on both their past and current values so that a matrix equation has to be solved at each time step. An implicit scheme is, therefore, computationally more expensive than an explicit scheme. The attractive feature
of an implicit scheme, however, is that it may be unconditionally stable (although this does not guarantee accuracy) whereas an explicit scheme has to satisfy certain stability requirements. However, owing to the relative ease with which a solution can be obtained by an explicit scheme, explicit time domain methods like the finite difference time domain (FDTD) method and the time-domain finite element method (TDFEM) are generally preferable for transient/broadband analysis.

FDTD [23, 24] and TDFEM [25] overcome many of the difficulties and inefficiencies associated with the use of frequency domain methods in deriving time domain solutions. Both methods evaluate their solutions directly in the time domain by stepping their solutions in time. Although their solutions must be obtained for a multiple number of time steps, the two methods are still very efficient compared to solving matrix equations in frequency domain methods. Moreover, frequency domain solutions are easily obtainable from the Fourier transform of the time domain solutions. Like MOM and FEM, FDTD and TDFEM can also model complex geometries.

Table 1 compares the number of FLOPS and the storage requirements between time and frequency domain methods. The number of FLOPS and the storage requirements are both of the order of $N$ per iteration for FDTD and TDFEM. Therefore, if time domain solutions and broadband frequency information are desired, FDTD and TDFEM are more attractive options since they are substantially better than frequency domain methods in terms of both computation time and memory requirements. However, as in frequency domain methods, FDTD and TDFEM can also be computationally expensive for solving realistic three-dimensional cavity problems.
Table 1: Comparison of the order of computation time and storage requirement between time and frequency domain methods

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<th>Method of Solution</th>
<th>No. of FLOPS</th>
<th>Storage</th>
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<tr>
<td>FEM (banded matrix)</td>
<td>$N^{5/3}$</td>
<td>$N \log N$</td>
</tr>
<tr>
<td>MOM (full matrix)</td>
<td>$N^3$</td>
<td>$N^2$</td>
</tr>
<tr>
<td>FDTD/TDFEM</td>
<td>$N$/iteration</td>
<td>$N$</td>
</tr>
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For example, a rectangular cavity, with a cross-section of $30\lambda \times 40\lambda$ and a depth of $100\lambda$, requires 5.76 billion unknowns for a mesh density of 20 cells/$\lambda$. Moreover, the field equations have to be repeatedly evaluated for at least 8,000 time steps (or iterations, assuming two time steps for the field to traverse from one cell to another along the length of the cavity), before the interior irradiated field can be observed at the aperture of the cavity.

From the preceding paragraphs, it is clear that no single method is capable of and sufficiently efficient in analyzing the scattering from electrically large three-dimensional cavities, let alone cavities with complex terminations. As such, several hybrid methods which combine two or more methods, have been developed to solve the cavity scattering problem [26, 27, 28]. The idea behind hybrid methods is to combine into one method all the useful properties and advantages of the individual methods, so that the problem of interest can be solved more efficiently. In [26], a
hybrid combination of asymptotic high frequency and modal methods is used within the framework of the generalized scattering matrix method to find the EM scattering from open-ended cavities made up of piecewise separable waveguide sections. The hybrid asymptotic modal method has also been applied to find the RCS patterns of open-ended circular waveguide cavities with simple geometrical terminations like a disk-hemispherical hub termination and a hub-blade termination. In [27], modal analysis and MOM are combined to find the scattering from two-dimensional (2D), linearly tapered, open-ended waveguide cavities. In [28], a hybrid ray-mode formulation is employed to describe the initial beam-like propagation of the incident field and the gradual diffusion of the beam as it progresses down the waveguide through successive reflections.

In the above hybrid methods, a complex cavity is analyzed by combining different methods. However, a limitation still exists in all these hybrid methods in that they can only produce multiple frequency solutions by solving the same problem repeatedly at different frequencies. In this dissertation, the FDTD method is combined with three separate methods to analyze the scattering from cavities with complex terminations. The resultant hybrid methods are the modal-FDTD, the PO-FDTD and the GRE-FDTD methods. These hybrid methods are designed to efficiently determine the broadband EM scattering from the interior of open-ended three-dimensional (3D) waveguide cavities with complex terminations. The first method in each hybrid method is used to couple the externally applied field into the cavity where their field values are determined at a conveniently chosen sectional surface, $S_{T_1}$ (see Fig. 1).
The FDTD method is then used to find the interaction of these fields with the rest of the cavity interior including the complex termination. Finally, the desired scattered field is evaluated via the termination reciprocity integral [29].

The rationale for choosing these methods to combine with the FDTD method is dictated to a large extent by the cavity structure. A typical jet engine inlet/exhaust has an air intake (or front) section (Region 1 in Fig. 1) that may be long and arbitrarily shaped. In addition, the interior cross-section may not be uniform throughout the length of the intake section. However, the cross-sections of the engine (or termination) section and its vicinity are typically circular (Region 2 in Fig. 1). The exhaust section (Region 3 in Fig. 1), may or may not be circular. For a front section that can be approximated with one or more piecewise separable waveguide sections, modal analysis can provide very accurate field solutions [26]. Even though it is limited as such, modal analysis is still a useful method as it provides a valuable independent check on the accuracy of other more versatile but approximate methods.

When conventional waveguide modes cannot be defined for arbitrarily shaped or non-uniform cavities, other approximate but efficient methods like the IPO and the GRE methods are used. The IPO approach in [19] has been shown to be a useful approximate procedure for analyzing the scattering from moderately large open-ended cavities. It is efficient since only a small number of iterations is required to arrive at a convergent solution. The method also works well for smoothly varying cavities where the surface diffraction effects are minimal. However, the method is expected to require more iterations for deep cavities and steep angles of incidence. Furthermore,
the solution need not necessarily converge to the exact solution since high frequency asymptotic approximations of PO are used, unlike a more rigorous approach such as that used in [18].

Of the three methods, rays provide the most flexibility as it can be used in arbitrarily shaped cavities. As will be explained in the following chapter, the GRE method is more efficient and accurate than the SBR method in predicting the fields inside the cavity. Recently, its efficiency has been improved through the use of shaped beams as its ray basis [30, 31] instead of geometrical optics rays. Hence, the GRE method has been selected for analyzing the front section of an engine cavity. However, like the SBR and the IPO methods, it cannot account for surface diffraction effects and discontinuities due to edges and corners.

The rationale for choosing FDTD for analyzing the termination section is as follows. As alluded earlier, time domain methods are more efficient than frequency domain methods for obtaining broadband frequency information. Between FDTD and TDFEM, the FDTD method is chosen to handle the complex termination of the engine section for the following reasons. In FDTD, the fields are updated using their values from the previous time step. Moreover, the FDTD equations comprise only simple additions, divisions, and multiplications with no matrix manipulation. These equations can also be easily vectorized for running on the Cray supercomputer. However, in TDFEM, field values from two previous time steps are required in the updating process. In addition, a sparse matrix-vector multiplication has to be performed at each time step. Hence, TDFEM requires an order of magnitude more
CPU time and memory. The only advantage that TDFEM has over FDTD is that it can use conformal grids. Finally, FDTD has seen widespread applications while TDFEM has only recently been introduced and thus, has not been thoroughly proven.

There are two possible approaches to implementing the hybrid methods. In the first approach, all computations are carried out at a single frequency of interest. Modal analysis or PO or GRE is used to compute the fields incident at $S_{T1}$. For the latter two methods, their fields may be converted into modal fields to obtain a smooth and evenly spaced field distribution on $S_{T1}$. From the amplitude and phase information of the incident fields, a sinusoidal time variation of the field is generated. Using the FDTD method, the sinusoidal field is propagated to the termination where it is reflected back to $S_{T1}$. Once steady-state has been achieved, the reflected field on $S_{T1}$, together with the incident field on $S_{T1}$, are used in the modal form of the reciprocity integral to find the cavity scattered field. This approach is demonstrated for 2D cavities.

In the second approach, the pulsed-basis approach is used in the FDTD method to characterize the termination section in terms of modal reflection matrices at multiple frequencies. In this approach, the FDTD input is a time pulse modulated consecutively by each waveguide mode. From the reflected field, a single column of the modal reflection matrix is computed at each frequency of interest. By repeating the process for all the modes, the entire modal reflection matrix for each frequency is obtained. For each incident angle of the applied field, the field solution on $S_{T1}$ is expressed in terms of a vector of incident modal coefficients. This incident vector and the modal
reflection matrix are used in the reciprocity integral to find the cavity scattered fields. This approach is demonstrated for 3D cavities.

In contrast to the first approach, the computation in the front and termination section are decoupled in the pulsed-basis FDTD approach. Therefore, a change in the geometry of either sections does not require reworking the entire problem again. The second approach is also able to obtain broadband solutions of the termination more efficiently than the first approach. The termination reflection matrices obtained are not only useful for this present study but they can be a valuable database for other researchers in this area.

This dissertation is organized as follows. In Chapter II, the SBR and GRE methods are described in general mathematical terms. The accuracy of the interior cavity fields obtained by these two methods are compared. Chapter III describes the implementation of the hybrid GRE-FDTD method for 2D geometries. The 2D case serves two purposes. Firstly, it is used to validate the method against existing analytical and numerical solutions. Secondly, it serves as a test-bed for trying out new ideas since it is generally easier to modify a 2D code than a 3D code. Different versions of the hybrid method have been implemented by combining the GRE method with the sinusoidal FDTD or the pulsed-basis FDTD method. The latter was done to validate the pulsed-basis FDTD approach before implementation in 3D cavities. Results for various terminations are presented. The hybrid methods for 3D cavities are described in Chapter IV. Specifically, the absorbing boundary condition, the pulsed-basis FDTD algorithm for finding the reflection matrix, the computation of the scattered field and
the coupling between FDTD and the other methods are examined in details. Finally, a brief description of the implementation of the FDTD code on a parallel machine is given. Chapter V presents the results of the 3D hybrid methods. The results are presented in the form of RCS patterns for different cavity sizes and terminations. Most of the results shown are for the verification of the accuracy of the reflection matrices obtained by the pulsed-basis FDTD algorithm. The accuracies of the field solutions obtained by PO and GRE are also compared against modal solutions. Then, illustrative results are presented for the hybrid methods. A comparison of the CPU time for the serial and parallel code is given at the end of the chapter. The conclusions as well as recommendations for future work are given in the final chapter.
CHAPTER II

THE RAY METHODS

In the hybrid ray-FDTD method, rays are used to determine the fields at a chosen cross-section $S_T$ within the cavity due to an external field incident at the aperture of the cavity. For this purpose, the SBR and GRE methods are considered primarily due to their simplicity, ease of use, and versatility. As the ray solutions at $S_T$ are coupled to FDTD, the accuracy of the results obtained by the hybrid method will depend significantly on the accuracy of the ray solutions. Although SBR and GRE have produced RCS results that are in good agreement with reference solutions, the results do not necessarily guarantee the accuracy of the ray fields within the cavity due to the fact that the RCS is evaluated via integration of the cavity ray fields. The effect of the integration is to smooth out some of the errors in the ray fields. Thus, it is important to study and compare the accuracy of the interior fields generated by the two ray methods. In this regard, the accuracy of the SBR method has been studied by comparing the power flow inside a parallel-plate waveguide with reference modal solution [32]. Using the same approach in this chapter, the accuracy of the two ray methods are compared and their difference illustrated through power flow diagrams in a parallel-plate waveguide and an S-shaped waveguide. For the sake of completeness, a brief description of the SBR and GRE methods is given before the
comparison is made.

### 2.1 The Shooting and Bouncing Ray Method

As the detailed description of the SBR method can be found in [8, 9], only a brief summary is given here. In the SBR method, the incident plane wave field intercepted by the open end of the cavity is divided into a dense grid of parallel ray tubes as shown in Fig. 2. These ray tubes are launched into the cavity from an equi-phased plane orthogonal to the propagation direction of the plane wave. The density of the ray tubes is dependent on the geometry of the cavity. In general, a density of ten to twenty ray tubes per wavelength is used. The ray tubes are tracked within the cavity via the laws of geometrical optics as they undergo multiple reflections between the cavity walls. As each ray tube undergoes multiple reflections within the cavity, its field amplitude is modified by divergence factors and reflection coefficients. In particular, the electric field $E(r_i)$ of the ray immediately after reflection from the point $r_i$ is found recursively from the field $E(r_{i-1})$ immediately after the previous

![Diagram](image)

**Figure 2:** The Shooting and Bouncing Ray Method.
reflection point $\vec{r}_{i-1}$ through the following relation:

$$\vec{E}(\vec{r}_i) = \overline{G} \cdot \vec{E}(\vec{r}_{i-1}) \text{ (DF)}_{i-1} e^{-j k_0 s}$$  \hspace{1cm} (2.1)$$

where $k_0$ is the free-space propagation constant and $s = |\vec{r}_i - \vec{r}_{i-1}|$. $\overline{G}$ is the planar dyadic reflection matrix evaluated at $\vec{r}_i$ and (DF)$_{i-1}$ is the divergence factor governing the spreading of the ray tube associated with the ray after its reflection from the point at $\vec{r}_{i-1}$. The divergence factor is usually evaluated via the $Q$-matrix formulation of Deschamps [33]. A summary of the pertinent equations for this evaluation is given in Appendix A.

2.2 The Generalized Ray Expansion Method

Since the detailed formulation of the GRE method can be found in [10], only the working principles and pertinent equations of the method are presented here. In the GRE method, the externally applied field intercepted by the aperture of the cavity is replaced by equivalent electric ($\vec{J}_s^{eq}$) and magnetic ($\vec{M}_s^{eq}$) surface currents as shown in Fig. 3. These currents radiate the same field into the cavity as in the original problem. They are found via Kirchhoff’s approximation of the aperture fields and are given by

$$\vec{J}_s^{eq}(\vec{r}') = -\hat{n} \times \vec{H}^{i}(\vec{r}')$$  \hspace{1cm} (2.2)$$
$$\vec{M}_s^{eq}(\vec{r}') = -\vec{E}^{i}(\vec{r}') \times \hat{n}$$  \hspace{1cm} (2.3)$$

where $\vec{E}^{i}$ and $\vec{H}^{i}$ are the incident electric and magnetic fields in the absence of the cavity walls. $\vec{r}'$ is the vector from the origin $O$ to any point on the aperture and $\hat{n}$ is
Figure 3: Coupling of the external field into the cavity using equivalent surface currents on the aperture of the cavity. (a) original problem; (b) equivalent problem.

the outward-pointing unit vector normal to the aperture plane. The approximation is good for sufficiently large apertures and incident angles which are not close to grazing angles.

The infinitesimal current sources $\mathcal{J}_s^e(\mathbf{r'})\,ds'$ and $\mathcal{M}_s^e(\mathbf{r'})\,ds'$, where $ds'$ is an infinitesimal area on the aperture at $\mathbf{r'}$, radiates spherical waves which can be tracked as ordinary geometrical optics rays within the cavity. However, instead of dealing with a continuum of infinitesimal sources, the aperture is divided into an array of smaller subapertures. The spherical waves are then launched from the phase centers of the subapertures as shown in Fig. 4.

The shape and dimensions of the subapertures, and the number of subapertures depend on the shape and size of the original aperture, and on the overall length of the waveguide cavity. In most applications, rectangular subapertures are used as they are convenient although polygonal subapertures can be used to better match
the boundary of the aperture. Based on numerical experiments, the maximum linear dimension of any subaperture should in general be less than $\sqrt{L\lambda}$, where $L$ is the overall cavity length and $\lambda$ is the free-space wavelength [12].

The field within the cavity is now found by a superposition of the fields radiated by the equivalent currents over each of the subapertures. In particular, rays or ray tubes are launched in all directions in the forward half-space of the cavity from the phase centers of the subapertures. To determine the field at a point $P$ in space, the field contributions from rays whose paths cross $P$ are summed at $P$. In practice, however, it is not an easy task to find the rays that cross a point in space, particularly for a point in 3D space. An approach that circumvents this difficulty is presented in a later chapter.

For each subaperture, rays are generally launched within a half-cone angle of between $60^\circ$ to $80^\circ$ with respect to $-\hat{n}$. The initial field amplitude of a ray is computed

![Figure 4: Rays emanating from the phase center $O_i$ of the $l$th subaperture of area $S_i$ (shaded).](image-url)
from the far-field radiation pattern of the subaperture. Specifically, the electric field along the \( p \)th ray of the \( l \)th subaperture prior to any reflection is given by [12]

\[
E_{pl}(\vec{r}_{pl}) = \mathcal{C}_l(\vec{r}_{pl}) \frac{e^{-j \omega \tau_{pl}}}{r_{pl}} \tag{2.4}
\]

where

\[
\mathcal{C}_l(\vec{r}_{pl}) = \frac{j k_o Z_o}{4 \pi} \int \int_{S_l} \left[ \hat{\vec{r}}_{pl} \times \hat{\vec{r}}_{pl} \times \hat{J}_a (\vec{r}_{l}^{'}) + \hat{Y_o} \hat{r}_{pl} \times \hat{M}_a (\vec{r}_{l}^{'}) \right] e^{j k_o \vec{r}_{pl} \cdot \vec{r}_{l}'} ds_{l}' \tag{2.5}
\]

In (2.4) and (2.5), \( \vec{r}_{pl} \) is the position vector of a point along the \( p \)th ray with respect to the phase center \( O_l \) of the \( l \)th subaperture and \( \vec{r}' \) is the position vector of the equivalent sources with respect to \( O_l \). \( \mathcal{C}_l(\vec{r}_{pl}) \) is the far-field vector radiation pattern of the electric field evaluated in the direction \( \vec{r}_{pl} \) in the absence of the cavity walls. Equation (2.5) can be evaluated numerically or in closed form depending on the field excitation and the shape of the subaperture [12].

The use of (2.4) is governed by the far-zone criterion [34] given by

\[
|\vec{r}_{pl}| \geq \frac{2 D_l^2}{\lambda} \tag{2.6}
\]

where \( D_l \) is the maximum linear dimension of the subaperture \( S_l \). However, it has been shown that in the case of GRE, this criterion can be relaxed [12] so that

\[
|\vec{r}_{pl}| \geq \frac{D_l^2}{\lambda} \tag{2.7}
\]

Therefore, the use of subapertures permit the computation of the cavity fields within the near-zone of the original aperture.

Once the rays are launched, their paths and field amplitudes are tracked in the same manner as the rays in SBR are tracked. In particular, the electric field along a
ray path is updated recursively through the relation

\[ \bar{E}(\bar{r}_i) = \bar{T} \cdot \bar{E}(\bar{r}_{i-1}) (DF)_{i-1} e^{-jka} \]  

(2.8)

where the various parameters are defined as before.

### 2.3 Comparison between the SBR & the GRE methods

An important difference between the SBR and the GRE methods is the manner in which the ray tubes are initially launched into the cavity. In SBR, the incident field illuminating the aperture of the cavity is a plane wave field. The portion of the incident plane wave field intercepted by the aperture of the cavity is divided into a dense grid of parallel ray tubes with the same initial amplitude and a relative linear phase variation. In GRE, no restriction is placed on the source producing the incident field so long as the equivalent currents \( \bar{J}_s^q \) and \( \bar{M}_s^q \) can be determined from the source field incident on the aperture. In contrast to SBR, it is these equivalent currents which radiate (in a cone of rays) the desired fields into the cavity interior. Moreover, the ray tubes of each subaperture are weighted by the far-field radiation pattern of the subaperture.

Although both methods launch ray tubes into the cavity, they differ in the number of ray tubes launched. For a single incident angle, SBR generally requires less ray tubes, which also means less ray tracing to perform, than GRE. However, a new set of rays has to be tracked in SBR, including the computation of their divergence factors and reflection coefficients, for every new incident angle. In GRE, the same set of rays is used regardless of the incident angle of the field as the initial amplitude of a
ray is the only variable with the incident field angle. This fact can be deduced from equations (2.4) and (2.5). The divergence factors and reflection coefficients for each ray remain unchanged because their paths are independent of the incident field angle.

As a result of the different coupling of the external field into the cavity, the interior cavity fields obtained by the two ray methods are not expected to be the same. In fact, the cavity field obtained by SBR only has the geometrical optics field intercepted by the aperture. On the other hand, GRE intrinsically includes the incident fields that are diffracted into the cavity by the aperture edges through the integration of the equivalent currents over the aperture of the cavity. In the next section, it is shown that these edge diffracted fields may contribute significantly to the total field inside the cavity.

2.4 Power flow diagrams of SBR & GRE

The difference in the accuracy of the SBR and GRE methods is illustrated here via power flow diagrams in a 2D parallel-plate waveguide cavity and an S-shaped waveguide cavity. Fig. 5 shows the power flow inside a $20\lambda$-wide parallel-plate waveguide for a vertically polarized plane wave field incident at $30^\circ$. Each arrow points in the direction of the Poynting vector $\vec{P}$, where $\vec{P} = \text{Re}(\vec{E} \times \vec{H}^*)$ and has a length proportional to the magnitude of $\vec{P}$. The SBR solution (in Fig. 5(a)) shows a physically intuitive collimated power flow inside the waveguide which is in good agreement with the modal solution (in Fig. 5(b)) for the initial $50\lambda$ of the waveguide. On the other hand, the GRE solution (in Fig. 5(c)) shows excellent agreement with the modal solution throughout the waveguide except for the initial $20\lambda$ of the waveguide. The
Figure 5: Power flow in a $20\lambda$ wide parallel-plate waveguide for a vertically polarized plane wave excitation incident at $30^\circ$ using (a) the SBR method, (b) modal analysis (reference solution) & (c) the GRE method (3 subapertures).
initial discrepancy is to be expected as GRE uses the far-field radiation pattern of each subaperture to find the fields radiated into the waveguide. For the case shown in Fig. 5(c), the far-field is \((a/3)^2/2\lambda\) [12] or \(22\lambda\), where \(a\) is the waveguide separation. Therefore, the waveguide cavity fields predicted by GRE is reliable only beyond the far-field of the subapertures. If necessary, this far-field distance can be reduced by increasing the number of subapertures.

The difference in the power flow or cavity fields predicted by SBR and GRE is attributed to the exclusion/inclusion of the fields diffracted into the cavity by the aperture edges [35]. For waveguides with electrically large apertures, the contribution from the edge diffracted fields to the cavity fields is negligible near the aperture compared to that from the incident G.O. field intercepted by the aperture. Therefore, the initial power flow is collimated. However, these edge diffracted fields become more significant further inside the waveguide where the transition regions of their reflection shadow boundaries increase (in width) with each successive reflection between the parallel plates. The wider transition regions result in the smearing of the collimated power flow. Therefore, the SBR solution is good only up to a certain depth from the aperture where the edge diffracted field is not dominant. On the other hand, the GRE solution which includes the edge diffracted fields, agrees very well with the modal solution (with the exception of the near-field of the subapertures).

The contribution of the edge diffracted fields to the cavity fields is even more significant for waveguides with small apertures. Fig. 6 shows the power flow inside a 3\(\lambda\)-wide parallel-plate waveguide. SBR again predicts a collimated beam behavior
Figure 6: Power flow in a 3λ wide parallel-plate waveguide for a vertically polarized plane wave excitation incident at 30° using (a) the SBR method, (b) modal analysis (reference solution) & (c) the GRE method (1 subaperture).
(see Fig. 6(a)) which is almost entirely incorrect compared to the modal power flow in Fig. 6(b). In fact, the modal solution shows a beam behavior that quickly becomes diffused after propagating $5\lambda$ into the waveguide. The beam behavior reappears momentarily further in the waveguide before becoming diffused again. The reason for the diffused power flow is as follows. As the aperture of the waveguide decreases, the incident G.O. field that is directly coupled into the cavity decreases. On the other hand, the edge diffracted fields is unchanged. Therefore, the diffracted fields quickly overwhelm the G.O. field as shown in Fig. 6(b). So, as the aperture of the waveguide decreases, the region (or depth) where the SBR solution is reliable decreases until it is no longer reliable even close to the aperture. Conversely, a comparison of Figs. 6(b) and 6(c) shows that GRE is able to predict the correct field even inside a waveguide with a small aperture. An enlargement of a section of this waveguide is given in Fig. 7 to show the excellent agreement between the GRE and modal solutions.

For the final example, we consider the power flow in an S-shaped waveguide with dimensions given in Fig. 8. Fig. 9 shows the power flow inside the S-shaped cavity for a plane wave incident at 30°. Once again, the SBR solution shows clearly demarcated lit and shadow regions. Compared to the modal solution, it is clear that the aperture intercepted G.O. field in SBR does not give the complete picture inside the waveguide. In fact, the edge diffracted fields in this case are significant right from the aperture of the waveguide as seen by the good agreement between the GRE solution and the modal solution. Therefore, the accuracy of the SBR method is also dependent on the shape of the waveguide.
Figure 7: Enlargement of a section of the $3\lambda$ wide parallel-plate waveguide for the modal analysis and the GRE method.
Figure 8: Geometry of an S-shaped cavity.
Figure 9: Power flow in a 9.9λ wide S-shaped cavity for a vertically polarized plane wave excitation incident at 30° using (a) the SBR method, (b) modal analysis (reference solution) & (c) the GRE method (5 subapertures).
2.5 Summary

The SBR and GRE methods were briefly described in order for comparisons between them to be made. It has been shown that the SBR cavity field solution is fairly accurate for a waveguide with a large aperture; the solution is, however, valid only up to a certain depth in the waveguide. This depth is a function of the angle of the incident field, the aperture size and the shape of the waveguide. The GRE cavity field solution, on the other hand, is more accurate than the SBR solution. The difference in the accuracy of the two ray methods is attributed to the presence (absence) of edge diffracted fields in GRE (SBR). Therefore, the choice between SBR and GRE for an application has to take into account, among other factors, the incidence angle of the incident field, the depth where the ray solution is required inside the waveguide as well as the shape and size of the waveguide. Based on these results and the fact that GRE is also computationally more efficient than SBR when RCS results are required at multiple angles, GRE has been chosen for use in the hybrid ray-FDTD method [36, 37].
CHAPTER III

THE TWO-DIMENSIONAL HYBRID RAY-FDTD METHOD

In this chapter, the hybrid GRE-FDTD method is validated by formulating and implementing it for 2D cavities. The reason for doing so is that analytical and numerical solutions are more widely available for 2D geometries than for 3D geometries. The GRE and the FDTD methods for 2D geometries are presented in the following two sections. The coupling between the two methods is then discussed followed by the treatment of the boundary conditions. The computation of the cavity scattered field is discussed before RCS results are presented for parallel-plate waveguide cavities with different terminations.

3.1 The 2D GRE method

The application of GRE in a 2D geometry is relatively easier than in a 3D geometry. In 2D, the externally applied field incident at the aperture of the cavity is replaced by equivalent line currents which radiated the desired fields (in the form of ray tubes) into the cavity as shown in Fig. 10. The propagation of the ray fields and their amplitude variations due to multiple reflections within the cavity walls are done by tracking the central rays of the ray tubes. To track the field amplitude of each ray
tube, its initial amplitude has to be found. With reference to Fig. 11, the initial field amplitude of the ray tube at \((\rho, \phi)\) is given by

\[
U(\rho, \phi) = 2 \sqrt{\frac{j k}{8 \pi}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} U^i(y') \cos \phi' \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \, dy'
\]  

(3.1)

where equivalent currents at the aperture have been applied in the 2D radiation integral. For an aperture that is subdivided into \(2N + 1\) equal subapertures of width \(\Delta\), it can be shown (as in Appendix B) that the above integral reduces to

\[
U(\rho, \phi) = 2 \sqrt{\frac{j k}{8 \pi}} \cos \phi \sum_{l=-L}^{L} \frac{e^{-jk\rho_l}}{\sqrt{\rho_l}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} U^i(y' + l\Delta) \, e^{jk|y'|\sin \phi_l} \, dy'.
\]  

(3.2)

Once the initial field amplitude of a ray is known, its variation along the ray path is monitored via (2.4).
3.2 The 2D Finite-Difference Time-Domain Method

The FDTD method is a direct solution of Maxwell's time-dependent curl equations [23]. It applies second-order accurate central difference approximations for both the space and time derivatives of the electric and magnetic fields directly to the differential operator of the curl equations. The derivation of the 2D finite-difference approximation of Maxwell's equations is given below.

In 2D, we can consider either the TE\(_z\) polarization or the TM\(_z\) polarization. We will derive the finite-difference equations only for the TE\(_z\) polarization and present the equations for the TM\(_z\) polarization as they can be obtained in a similar fashion.
Maxwell's equations in free space for the TE\textsubscript{z} polarization can be written as

\[ \varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} \]  
\[ \varepsilon_0 \frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x} \]  
\[ \mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \]  

where the dielectric parameters \( \mu_0 \) and \( \varepsilon_0 \) are free-space constants. Following Yee’s notation \[23\], we denote a point in space as

\[ (i,j) = (i\Delta x,j\Delta y) \]  

and any function of space and time as

\[ F^n(i,j) = F(i\Delta x,j\Delta y,n\Delta t) \]  

Applying second-order accurate central difference approximations for the space and time derivatives of \(3.7\), we get

\[ \frac{\partial F^n(i,j)}{\partial x} = \frac{F^n(i+\frac{1}{2},j) - F^n(i-\frac{1}{2},j)}{\Delta x} + O\left(\Delta x^2\right) \]  

\[ \frac{\partial F^n(i,j)}{\partial t} = \frac{F^{n+\frac{1}{2}}(i,j) - F^{n-\frac{1}{2}}(i,j)}{\Delta t} + O\left(\Delta t^2\right) \]  

respectively. To realize the space derivatives in \(3.3\) to \(3.5\) and achieve the accuracy of \(3.8\), the field components \( (E_x,E_y,H_z) \) are positioned on a spatial grid as shown in Fig. 12(a). To achieve the accuracy of \(3.9\), the electric and magnetic fields are evaluated at alternate half time steps. Based on these premises, equations \(3.3\) to
(3.5) become

\[ E_{x}^{n+\frac{1}{2}}(i, j + \frac{1}{2}) = E_{x}^{n-\frac{1}{2}}(i, j + \frac{1}{2}) + \frac{\Delta t}{\varepsilon_{0}\Delta y}[H_{z}^{n}(i, j + 1) - H_{z}^{n}(i, j)], \quad (3.10) \]

\[ E_{y}^{n+\frac{1}{2}}(i + \frac{1}{2}, j) = E_{y}^{n-\frac{1}{2}}(i + \frac{1}{2}, j) - \frac{\Delta t}{\varepsilon_{0}\Delta x}[H_{z}^{n}(i + 1, j) - H_{z}^{n}(i, j)], \quad (3.11) \]

\[ H_{z}^{n+1}(i, j) = H_{z}^{n}(i, j) + \frac{\Delta t}{\mu_{0}\Delta y}\left[E_{x}^{n+\frac{1}{2}}(i, j + \frac{1}{2}) - E_{x}^{n+\frac{1}{2}}(i, j - \frac{1}{2})\right] \]

\[ + \frac{\Delta t}{\mu_{0}\Delta x}\left[E_{y}^{n+\frac{1}{2}}(i + \frac{1}{2}, j) - E_{y}^{n+\frac{1}{2}}(i - \frac{1}{2}, j)\right]. \quad (3.12) \]

From (3.10) to (3.12), it is clear that the new value of a field component depends on its previous value and on the previous values of the other field vector components at half spatial steps away. The time variation of the electromagnetic field is thus obtained by updating the electric and magnetic field alternately in time.

The finite-difference equations for the TMz polarization can be similarly derived with the help of the spatial grid in Fig. 12(b). They are given by

\[ H_{z}^{n+\frac{1}{2}}(i, j + \frac{1}{2}) = H_{z}^{n-\frac{1}{2}}(i, j + \frac{1}{2}) - \frac{\Delta t}{\mu_{0}\Delta y}[E_{x}^{n}(i, j + 1) - E_{x}^{n}(i, j)], \quad (3.13) \]

\[ H_{y}^{n+\frac{1}{2}}(i + \frac{1}{2}, j) = H_{y}^{n-\frac{1}{2}}(i + \frac{1}{2}, j) + \frac{\Delta t}{\mu_{0}\Delta x}\left[E_{x}^{n}(i + 1, j) - E_{x}^{n}(i, j)\right], \quad (3.14) \]

\[ E_{x}^{n+1}(i, j) = E_{x}^{n}(i, j) - \frac{\Delta t}{\varepsilon_{0}\Delta y}\left[H_{z}^{n+\frac{1}{2}}(i, j + \frac{1}{2}) - H_{z}^{n+\frac{1}{2}}(i, j - \frac{1}{2})\right] \]

\[ + \frac{\Delta t}{\varepsilon_{0}\Delta x}\left[H_{y}^{n+\frac{1}{2}}(i + \frac{1}{2}, j) - H_{y}^{n+\frac{1}{2}}(i - \frac{1}{2}, j)\right]. \quad (3.15) \]

The size of the time step, \( \Delta t \), is governed by the Courant-Friedrichs-Lewy (or CFL) stability criterion. This criterion ensures the stability of the FDTD solution as the solution marches on in time. The CFL stability criterion is derived by decomposing Maxwell's time-dependent equations into a pair of time and space eigenvalue
Figure 12: Field distribution on a two-dimensional finite-difference grid.
problems [24]. For the 2D case, the CFL criteria works out to be

$$v_{\text{max}} \Delta t \leq \frac{1}{\sqrt{(\frac{1}{\Delta x})^2 + (\frac{1}{\Delta y})^2}}$$

(3.16)

where $v_{\text{max}}$ is the maximum phase velocity in the finite difference space. For our purpose, $v_{\text{max}}$ is equated to the speed of light in free space.

3.3 Coupling between ray & FDTD

As with most EM hybrid methods, the coupling between the individual methods has to be handled with care. Consider the cavity shown in Fig. 13. The cavity has been divided into three regions corresponding roughly to the air intake, the engine, and the exhaust sections of a jet engine inlet. The imaginary surface $S_{T1}$ separates Regions 1 and 2 while $S_{T2}$ separates Regions 2 and 3. In Region 1, the cavity is assumed to be smoothly varying for high frequency methods like ray methods to be valid. Ray tubes are traced from the aperture of the cavity to $S_{T1}$ where they are summed to form a high frequency solution of the cavity fields across $S_{T1}$. This solution on $S_{T1}$ is used as the excitation for the FDTD computation in Region 2.

---

**Figure 13:** Geometry of a cavity with a termination.
We will now describe two approaches to the coupling between the GRE and the FDTD methods [37]. Depending on the size of a ray tube when it reaches $S_{T1}$, it will either not intersect any of the FDTD grid points on $S_{T1}$ or it will intersect one or more of those grid points. Therefore, different ray tubes will contribute differently to the total field of a grid point. Moreover, it is possible that some grid points will not have any ray tubes intersecting them. As a result, the incident fields on $S_{T1}$ evaluated via rays will not be smoothly varying. In order to obtain a smoothly varying as well as an accurate field on $S_{T1}$, the size of the ray tube can be restricted so that it will only intersect one FDTD grid point. Alternatively, some form of interpolation can be applied to the ray tubes that intersect more than one grid point. In both of these approaches, a ray tube is launched and tracked via its central ray to $S_{T1}$. The projected ray tube width (for 2D) on $S_{T1}$ is then determined. If this width is greater than some specified width, $A_p$, the ray tube is subdivided and the process of tracking and determining the projected width of the smaller ray tubes is repeated.

In the first approach, $A_p$ is equal to the FDTD grid spacing (usually $\lambda/20$) so that if the projected ray tube width is less than $A_p$, that ray tube can intersect at most one FDTD grid point on $S_{T1}$. In such an event, the field specified by the central ray will be added to the intersected grid point. Ray tubes that do not intersect any grid point are ignored.

For the second approach, $A_p$ is larger, but no greater than $\lambda/2$ for reasons given in [10]. In our 2D implementation, we have used $A_p = \lambda/4$ for greater accuracy in the evaluation of the fields on $S_{T1}$. Since a larger ray tube may intersect more than
one FDTD grid point on \( S_{T1} \), its contribution to the field on \( S_{T1} \) is determined by converting its ray field into modal fields. Specifically, the contribution of the ray field to the modal coefficients are determined by integrating the ray field over the projected area of the ray tube assuming a linear phase variation in the field over the projected area with respect to the field of the central ray. This approach assumes that the fields on \( S_{T1} \) are expressible in terms of parallel-plate waveguide modes for 2D problems. This assumption can usually be satisfied by a suitable choice of \( S_{T1} \).

In realistic 3D problems, there is usually a narrow section in front of the termination which is cylindrical so that the fields in this narrow section can also be expressed in terms of modes. In any case, when the modal coefficients have been obtained by summing the contributions due to all the ray tubes, the desired field at each FDTD grid point on \( S_{T1} \) can be determined.

Comparing the two approaches, it is clear that the first approach requires more ray tracing (which means more computational time and storage) than the second approach because of the smaller \( A_p \). However, once the rays are traced, the first approach uses only a simple summation of the ray fields to obtain the desired fields at the FDTD grid points on \( S_{T1} \). The second approach is less efficient in this latter aspect as it has to compute the modal coefficients and then sum the modal fields to get the desired fields. However, in realistic 3D problems, the first approach may not be viable because the amount of ray tracing can become overwhelmingly excessive.

Another consideration in the coupling of the GRE and FDTD methods is the selection of a suitable time variation for the excitation since the former is a frequency-
domain method while the latter is a time-domain method. There are two possible schemes for the time variation: the sinusoidal steady-state time variation [38] and the pulsed (usually Gaussian or raised-cosine) time variation. For the steady-state FDTD, the ray solution on $S_{T1}$ is evaluated only at a single frequency of interest. Based on the complex ray field solution at $S_{T1}$, the excitation can be made to vary sinusoidally with time. For the pulsed FDTD, there are two possible alternatives. For the first alternative, the ray solution on $S_{T1}$ is computed over a range of frequencies corresponding to the frequency content of the pulse. The excitation can then be obtained by an inverse Fourier transform of the product of the ray solution and the Fourier transform of the pulse. This alternative is not attractive as the resultant inverse transform will have a wide time window with a number of significant pulses due to the different arrival times at $S_{T1}$ of the reflected, diffracted and reflected-diffraction fields. A better alternative is to use a basis set (e.g. modes) as excitation for the pulsed FDTD to characterize the termination section in terms of a termination scattering matrix. This scattering matrix, together with the ray solution on $S_{T1}$ (expressed also in terms of the basis set), can then be used to find the cavity scattered field. The pulsed time variation scheme is more efficient for problems which require multiple frequency solutions while the steady-state scheme is more efficient for problems which require only a single frequency solution.

Regardless of the time variation used, the excitation produces a wave which propagates toward the termination and interacts with it. For the geometry shown in Fig. 13, part of the wave may be transmitted to Region 3 through $S_{T2}$ while the remainder is
reflected back towards $S_{T1}$. If we assume that the waves leaving Region 2 through the imaginary surfaces $S_{T1}$ and $S_{T2}$ do not return, then an absorbing boundary condition (ABC) such as the ones introduced by Higdon [39, 40] or Mur [41] can be applied in the FDTD computations at each of the two surfaces. The above assumption is reasonable as most jet engine inlets are shaped in such a way that there is very little energy that returns to Region 2 upon its exit from there. Otherwise, we can convert the waves leaving Region 2 back into rays (using GRE) and track those rays that return to Region 2. These returning rays act as an additional excitation.

3.4 Absorbing boundary conditions at $S_{T1}$ & $S_{T2}$

As mentioned previously, absorbing boundary conditions (ABCs) are applied at the imaginary boundary surfaces $S_{T1}$ and $S_{T2}$ for the proper transmission of waves through these surfaces. However, the two ABCs cannot be implemented in the same way because of different field conditions at their respective boundary surfaces. With reference to Fig. 13, the absorbing boundary condition at $S_{T1}$ has to properly account for the waves which pass through $S_{T1}$ in both directions; it has to account for the *incident* excitation at $S_{T1}$ and the scattered field due to the termination. In contrast, the absorbing boundary at $S_{T2}$ only has waves transmitted through it from Region 2 to Region 3 assuming that there are no waves transmitted through $S_{T2}$ from Region 3 to Region 2 (recall discussion in previous section).

The ABC at $S_{T1}$ has to transmit (or absorb) the waves that are *scattered* by the termination toward $S_{T1}$ without destroying or affecting the *incident* excitation at $S_{T1}$. We will demonstrate how this function can be accomplished with the second-order
ABC of Mur [41] in a 2D problem. Applying the ABC given by equation (17) of [41] to the scattered electric field $E_{z,s}$ (for the TM$_z$ case) at $S_{T1}$, we have

$$E_{z,s}^{n+1}(0, j) = E_{z,s}^n(1, j) + a_1 \left[ E_{z,s}^{n+1}(1, j) - E_{z,s}^n(0, j) \right] - a_2 \left[ H_{z,s}^{n+\frac{1}{2}}(0, j + \frac{1}{2}) - 

H_{z,s}^{n+\frac{1}{2}}(0, j - \frac{1}{2}) + H_{z,s}^{n+\frac{1}{2}}(1, j + \frac{1}{2}) - H_{z,s}^{n+\frac{1}{2}}(1, j - \frac{1}{2}) \right]$$

(3.17)

(following Mur's FDTD indexing; see Fig. 14 also for indexing) where $E_{z,s}$ and $H_{z,s}$ are the fields scattered by (or reflected from) the termination. The constants $a_1$ and $a_2$ are given by

$$a_1 = \frac{c_0 \Delta t - \Delta x}{c_0 \Delta t + \Delta x}, \quad (3.18)$$

Figure 14: A section of the FDTD grid next to the absorbing boundary $S_{T1}$ for the TE$_z$ case.
Replacing the scattered field components \((E_{z,s}, H_{z,s})\) with

\[
(E_{z,s}, H_{z,s}) = (E_{z}, H_{z}) - (E_{z,i}, H_{z,i})
\]

where the subscript \(i\) in \(E_{z,i}(H_{z,i})\) denotes the incident component of the total \(E_z\) (\(H_z\)) field, (3.17) becomes

\[
\begin{align*}
E_{z,i}^{n+1}(0,j) & = E_z^n(1,j) - E_{z,i}^n(1,j) + a_1 \left[ E_z^{n+1}(1,j) - E_{z,i}^{n+1}(1,j) - E_{z,i}^n(0,j) \right] - \\
& \quad a_2 \left[ H_z^{n+\frac{1}{2}}(0,j + \frac{1}{2}) - H_{z,i}^{n+\frac{1}{2}}(0,j + \frac{1}{2}) - \\
& \quad \quad H_z^{n+\frac{1}{2}}(0,j - \frac{1}{2}) + H_{z,i}^{n+\frac{1}{2}}(0,j - \frac{1}{2}) + \\
& \quad \quad H_z^{n+\frac{1}{2}}(1,j + \frac{1}{2}) - H_{z,i}^{n+\frac{1}{2}}(1,j + \frac{1}{2}) - \\
& \quad \quad H_z^{n+\frac{1}{2}}(1,j - \frac{1}{2}) + H_{z,i}^{n+\frac{1}{2}}(1,j - \frac{1}{2}) \right].
\end{align*}
\]

Except for \(E_z^{n+1}(1,j)\) and \(E_{z,i}^{n+1}(1,j)\), all the other field components in (3.21) are computed values for the previous (one or half) time step. \(E_z^{n+1}(1,j)\) can be calculated for the current time step from the regular (total-field) difference equation. \(E_{z,i}^{n+1}(1,j)\) can be obtained via the ray method (as was done for the excitation at the absorbing boundary) or from the propagation of the incident excitation. After \(E_{z,s}\) on \(S_{T1}\) has been found, the total field components of \(H_z\) and \(H_y\) in Region 2 can be updated via the regular FDTD equations.

The procedure described above for obtaining the scattered fields at \(S_{T1}\) due to the reflection of the fields from the termination has to be used whenever the steady-state sinusoidal time variation is chosen for the FDTD computation in Region 2. However,
this procedure may or may not be necessary when the pulsed time variation is chosen, depending on the time window of the excitation and the closest distance between $S_{T_1}$ and the termination.

The ABC at $S_{T_2}$ does not require any special treatment like the one at $S_{T_1}$ if we assume that waves are transmitted through $S_{T_2}$ only from Region 2 to Region 3 and not vice versa. In this case, the transmitted field is also the total field. Therefore, any suitable ABC for the absorption of the total field at $S_{T_2}$ can be applied. Note that the ABC (3.17) can also be applied at $S_{T_2}$ with the scattered field variables replaced appropriately by the total field variables (with the proper spatial indices).

3.5 Scattered field computation

To determine the cavity scattered fields, the appropriate field solutions in Regions 1 and 2 have to be used. One possible way of finding the cavity scattered field is to launch GRE rays into Region 1 using the FDTD solution at $S_{T_1}$. These rays are tracked to the front aperture of the cavity where aperture integration can be applied to find the scattered field. Unfortunately, this method requires that rays be traced both into the cavity for the incident excitation at $S_{T_1}$ and out of the cavity for the cavity scattered field computation.

A more suitable and efficient way for finding the cavity scattered field is based on the termination reciprocity integral developed by Pathak and Burkholder [29]. This integral is given by

$$E_c^s(P) \cdot \bar{P}_t \approx \int_{S_{T_1}} \left( E_c^s \times H_t - E_t \times H_c^s \right) \cdot \hat{n} \, ds \quad (3.22)$$
where $\overline{E}_c^t(P)$ is the desired cavity scattered field at the observation point $P$ and $\overline{P}_t$ is the strength of an electric current point (test) source. $(\overline{E}_c^t, \overline{H}_c^t)$ are the fields scattered by the termination in the cavity while $(\overline{E}_t, \overline{H}_t)$ are the fields radiated by the test source in the presence of the cavity structure without the termination. $\hat{n}$ is the unit vector normal to the surface $S_{T1}$ as shown in Fig. 13. The approximation in (3.22) assumes that the source and observer are in direct view of the open front end so that the contribution to $\overline{E}_c^t(P)$ from the scattered field exiting through $S_E$ is negligible compared to the fields exiting through $S_{T1}$.

To see the usefulness of (3.22), consider how the cavity scattered fields are obtained using a purely ray-based approach. For example, in the original implementation of SBR [8, 9], two-way (in and out of the cavity) ray tracing has to be performed before the scattered field can be determined via aperture integration. With the reciprocity integral, the rays are only traced from the open end of the cavity to the termination and back to $S_{T1}$. Therefore, the amount of ray tracing is basically reduced by about half. This reduction is even more significant when the reciprocity integral is used in the hybrid method because the rays are only traced from the open end to $S_{T1}$.

From (3.22), we see that the integral is independent of the method(s) used to find the two sets of tangential fields on $S_{T1}$. It is particularly easy to apply the integral to find the backscattered field in our hybrid method since both sets of fields at $S_{T1}$ are readily available. Specifically, for backscatter computation, $(\overline{E}_t, \overline{H}_t)$ is the initial excitation on $S_{T1}$ evaluated by the GRE method for the FDTD computation in Region 2; its evaluation has already been described earlier. $(\overline{E}_c^t, \overline{H}_c^t)$ is the termination
scattered field that is transmitted through $S_{T1}$ from Region 2. It has to be evaluated on $S_{T1}$ from the FDTD algorithm. However, the tangential $\bar{E}$ and $\bar{H}$ fields in a Yee cell lie on different planes. In particular, the tangential $\bar{E}$ and $\bar{H}$ planes are half a cell width apart in the direction normal to both of these planes. Consider for example, the 2D TE case shown in Fig. 14, where the tangential magnetic field $H_y(0,j)$ lies on $S_{T1}$ and the tangential electric field $E_y(\frac{1}{2},j)$ lies on a plane parallel to $S_{T1}$ at $\Delta x/2$ away. To obtain $E_{y,s}(0,j)$ (which is symbolically, the termination scattered field $E_{y,s}$ on $S_{T1}$), an extrapolation of the neighboring values of $E_{y,s}$ is used. Specifically, we assumed that the gradient of the scattered field $E_{y,s}$ in the normal direction of $S_{T1}$, $dE_{y,s}/dx$, at $x = \Delta x/4$ is equal to that at $x = \Delta x$, so that

$$E_{y,s}(0,j) = \frac{3}{2}E_{y,s}(\frac{1}{2},j) - \frac{1}{2}E_{y,s}(\frac{3}{2},j).$$ \hspace{1cm} (3.23)

For the steady-state FDTD, the scattered field components ($E_{y,s}$) on the right-hand side of (3.23) are replaced by the difference between their respective total ($E_y$) and incident ($E_{y,i}$) field components. This extrapolation scheme can be similarly applied to the tangential electric or magnetic field in the 3D case. When all the tangential fields within the integral of (3.22) have been appropriately manipulated into the frequency domain, the cavity scattered field $E^s(P)$ can be computed.

In summary, the hybrid ray-FDTD method uses high frequency ray solutions as input excitation for the evaluation of the termination scattered field via the low frequency FDTD method; the scattering of the cavity is then obtained via the termination reciprocity integral. The hybrid ray-FDTD method combines the efficiency of the ray method (and the reciprocity integral) with the modeling flexibility of the FDTD
method. At the same time, it overcomes some of the limitations of the individual methods in analyzing the scattering from a cavity.

3.6 Results

The hybrid GRE-FDTD method has been implemented in three different programs (see Table 2) using various combinations of ray tracing and time variations to determine the radar cross-section (RCS) of parallel-plate cavities with different PEC plug terminations. Program 1 implements the GRE method such that the projected areas of the ray tubes are less than $\lambda/20$ to find the incident field on $S_{T1}$ while the other two programs implement the GRE method such that the ray fields are converted into modes to find the same fields. Programs 1 and 2 implement the FDTD algorithm using the steady-state sinusoidal time variation while Program 3 implements the FDTD algorithm using the basis-pulsed time variation approach.

In all three programs, the aperture of the cavity is divided into three subapertures from which rays are launched within an angle of $\pm 75^\circ$ of the cavity axis to compute the excitation at $S_{T1}$. $S_{T1}$ is fixed at $4\lambda$ from the closed end of the cavity. For the FDTD algorithm, a $\lambda/20 \times \lambda/20$ spatial grid is used (unless it is specified otherwise) with a time step of $\Delta t/(2c_0)$; Mur's [41] absorbing boundary condition is applied at $S_{T1}$ to absorb the fields scattered toward $S_{T1}$ by the termination. No absorbing boundary condition is implemented at $S_{T2}$ since Region 3 (see Fig. 13) is non-existent for our terminated cavity. The scattered field is obtained via the reciprocity integral (3.22) which is now exact since $S_E$ is perfectly conducting for the terminated (PEC) parallel-plate waveguide.
Table 2: Different implementations of the hybrid GRE-FDTD method.

<table>
<thead>
<tr>
<th>Program</th>
<th>GRE</th>
<th>FDTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A_p = \frac{\lambda}{20} )</td>
<td>sinusoidal</td>
</tr>
<tr>
<td>2</td>
<td>( A_p = \frac{\lambda}{4} )</td>
<td>sinusoidal</td>
</tr>
<tr>
<td>3</td>
<td>( A_p = \frac{\lambda}{4} )</td>
<td>pulsed-basis</td>
</tr>
</tbody>
</table>

In the following examples, we will only present results obtained via Program 2. These results are compared with reference solutions obtained via the hybrid asymptotic modal-method of moments (Modal-MM) [27]. The results from the other two programs are not presented here as there is little difference between their results and that of Program 2; In fact, the purpose of Programs 1 and 3 is to check the accuracy of the approach employed in Program 2. Moreover, Program 3 validates the time-domain (pulsed-basis) approach which has great potential in (2D and 3D) scattering problems where the cavity RCS is required at multiple frequencies. For the purpose of this dissertation, the results from Program 2 are sufficient to illustrate the utility and accuracy of the hybrid method.

Figure 15 shows the RCS patterns of a \( 9.6\lambda \)-wide and \( 30\lambda \)-long parallel-plate waveguide cavity with a \( 2\lambda \times 4\lambda \) rectangular PEC plug termination for both the TM\(_z\) and TE\(_z\) cases. The GRE-FDTD patterns in both cases show excellent agreement with the corresponding Modal-MM patterns. The slight differences between the GRE-FDTD and the Modal-MM solutions can be attributed to two causes. One of the
Figure 15: RCS patterns of parallel-plate cavity with a rectangular-shaped PEC plug termination using the hybrid Modal-MM, GRE-FDTD & GRE methods. $a = 9.6\lambda$, $l = 30\lambda$, $d = 4\lambda$, $h = 2\lambda$, $w = 4\lambda$; — Modal-MM; + GRE-FDTD; - - - GRE (3 subapertures).
causes is the inability of the absorbing boundary condition to absorb the higher order modes excited by the rectangular plug termination. These higher order modes result from the reflection and edge diffraction of the fields from the termination. They are particularly significant at larger incident angles since the termination reflected fields (which are, the incident fields after undergoing multiple reflections in the termination section) are incident at large angles from the normal on the absorbing boundary $S_{T1}$. These higher order modes are not transmitted (or absorbed) as well as the lower order modes by the second-order Mur absorbing boundary condition (see Table 3 [42]). In addition, their multiple reflections between the absorbing boundary and the termination result in appreciable errors in the scattered field. Another source of error is the ray solution on $S_{T1}$. These solutions are less accurate at large incident angle $\theta$ than at small $\theta$. This difference in accuracy is a result of limiting the ray solution to those rays whose launch angles are within $\pm 75^\circ$ of the cavity axis. At large $\theta$, this (artificial) angular limit may not be adequate. For example, for an incident angle of $60^\circ$, the magnitude of the radiation power pattern of each subaperture at $75^\circ$ is -8dB (40%) with respect to the beam maximum while the magnitude of the first and second sidelobes are -9dB (35%) and -13dB (22%) respectively. Therefore, significant contribution by rays whose launch angles are greater than $75^\circ$ are omitted from the final ray solution on $S_{T1}$.

Also shown in Fig. 15 are the respective patterns obtained using the GRE method alone. In this purely GRE approach, the rays are launched from the aperture and traced to $S_{T1}$ where they form the incident field. The rays are then traced beyond
Table 3: Reflection coefficients for the Mur absorbing boundary condition at the modal angles of the higher order modes in a 9.6λ-wide parallel plate waveguide cavity.

<table>
<thead>
<tr>
<th>approx. modal angle of higher order modes (deg)</th>
<th>approx. reflection coefficient (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>3</td>
</tr>
<tr>
<td>51</td>
<td>5</td>
</tr>
<tr>
<td>56</td>
<td>8</td>
</tr>
<tr>
<td>62</td>
<td>10</td>
</tr>
<tr>
<td>70</td>
<td>25</td>
</tr>
<tr>
<td>82</td>
<td>50</td>
</tr>
</tbody>
</table>

$S_{T1}$ to the termination where they are scattered back to $S_{T1}$ to form the termination scattered field. Both the incident and scattered fields are converted into modal fields before they are used in the reciprocity integral to obtain the cavity scattered fields. It is clear from Fig. 15 that the GRE solutions agree very well with the reference and hybrid solutions for $\theta$ up to about 15°. This agreement suggests that the diffracted fields from the termination are negligible for small $\theta$. Beyond that, the GRE solutions show significant deviations from the reference solutions at certain angles. From these results, it is envisaged that for a complex termination, the hybrid GRE-FDTD method will produce far more accurate results than the purely GRE approach which does not account for the termination diffracted fields.

Fig. 16 shows the resultant RCS patterns when the rectangular plug termination is replaced by a wedge-shaped plug termination. The height and base width of the
Figure 16: RCS patterns of parallel-plate cavity with a wedge-shaped PEC plug termination using the hybrid Modal-MM & GRE-FDTD methods. $a = 9.6\lambda, l = 30\lambda, d = 4\lambda, h = 2\lambda, w = 4\lambda$; — Modal-MM; + GRE-FDTD.
wedge-shaped plug are $2\lambda$ and $4\lambda$, respectively. For the TM$_z$ case, the wedge is modeled in the FDTD code with the electric field $E_z$ tangent to the wedge surface. For the TE$_z$ case, the wedge is modeled with the magnetic field $H_z$ tangent to the wedge surface. In contrast to the TM$_z$ case where the tangential electric field $E_z$ vanishes on the wedge surface, the tangential magnetic field $H_z$ for the TE$_z$ case does not vanish. Instead, it has to be specially treated. Specifically, we apply Stoke’s Theorem to one of Maxwell’s curl equations to obtain a difference equation for updating the tangential $H_z$ correctly. Applying Stoke’s Theorem to Maxwell’s curl-$E$ equation results in

$$\oint_{\partial} \mathbf{E} \cdot d\mathbf{l} = -\mu_0 \int_{\partial} \frac{\partial \mathbf{H}}{\partial t} \cdot d\mathbf{s}$$

(3.24)

where the line and surface integrals are taken around the closed loop $ABC$ and the triangle $ABC$ respectively as shown in Fig. 17. After integration and rearrangement, (3.24) becomes

$$H_z^{n+1}(i_m,j_n) = H_z^n(i_m,j_n) + \frac{\Delta t}{\mu_0 \Delta y} \left[ E_x^{n+\frac{1}{2}}(i_m,j_n + \frac{1}{2}) - E_x^{n+\frac{1}{2}}(i_m,j_n - \frac{1}{2}) \right] - \frac{\Delta t}{\mu_0 \Delta x} \left[ E_y^{n+\frac{1}{2}}(i_m + \frac{1}{2},j_n) - E_y^{n+\frac{1}{2}}(i_m - \frac{1}{2},j_n) \right] + CF$$

(3.25)

where the correction factor $CF$ is given by

$$CF = \frac{\Delta t}{\mu_0 \Delta y} \left[ E_x^{n+\frac{1}{2}}(i_m,j_n + \frac{1}{2}) - E_x^{n+\frac{1}{2}}(i_m,j_n - \frac{1}{2}) \right] - \frac{\Delta t}{\mu_0 \Delta x} \left[ E_y^{n+\frac{1}{2}}(i_m + \frac{1}{2},j_n) - E_y^{n+\frac{1}{2}}(i_m - \frac{1}{2},j_n) \right].$$

(3.26)

Notice that without $CF$, (3.25) is just the regular FDTD equation for updating $H_z$. Therefore, $CF$ adjusts for the $H_z$’s that are tangent to the wedge surface. The GRE-FDTD patterns in Fig. 16 show excellent agreement with the Modal-MM pattern except for large $\theta$. 
Figure 17: Geometry of wedge with respect to FDTD grid for TE$_z$ case.

For the final example, a PEC semi circular plug of radius $2\lambda$ is substituted for the wedge-shaped plug. The curved surface of the semi circular plug is modeled using its stair-stepped approximation as shown in Fig. 18 for the TM$_z$ case. A similar approximation of the semi circular plug is used for the TE$_z$ case. The RCS patterns of the new cavity are shown in Fig. 19 for both the TM$_z$ and TE$_z$ cases. In this figure, we have also plotted the respective patterns obtained with the smaller $\lambda/40$ FDTD grid. The patterns obtained with the $\lambda/20$ grid show some semblance to the
Figure 18: Geometry of semi circular plug of radius $2\lambda$ for the $TM_z$ case; • FDTD node where $E_z$ is on or within the plug and thus set to zero; — outline of surface of plug.

reference patterns while those obtained with the finer grid show better agreement with the reference patterns, particularly for the $TE_z$ case. The significant improvement in the latter case is attributed to the fact that in two dimensional geometries, $TE_z$ modes suffer dispersion due to the stair-stepped approximation while $TM_z$ modes does not [43]. This dispersion (or numerical error) reduces when a finer FDTD grid is used. Therefore, the pattern for the $TE_z$ case shows marked improvement while that of the $TM_z$ case is basically unchanged.
Figure 19: RCS patterns of parallel-plate cavity with a semi circular-shaped PEC plug termination using the hybrid Modal-MM & GRE-FDTD methods. $a = 9.6\lambda$, $l = 30\lambda$, $d = 4\lambda$, $r = 2\lambda$; — Modal-MM; + GRE-FDTD ($\frac{A}{20}$); o GRE-FDTD ($\frac{A}{40}$).
CHAPTER IV

HYBRID METHODS FOR 3-D CAVITY SCATTERING PROBLEMS

In this chapter, three hybrid methods for analyzing the scattering from 3-D cavities are presented. They are the modal-FDTD (MO-FDTD) method, the GRE-FDTD method and the physical optics-FDTD (PO-FDTD) method. The common denominator among these methods is the use of the FDTD method to characterize the termination in terms of a reflection matrix.

The pertinent equations of the 3-D FDTD method and the Higdon's absorbing boundary condition are developed in the following two sections. The determination of the termination reflection matrix by the FDTD method and the computation of the cavity scattered field are then described. The coupling between FDTD and GRE and between FDTD and PO are presented in the next two sections. Finally, a brief description of the parallel implementation of the FDTD method is given.
4.1 The 3-D Finite-Difference Time-Domain Method

The extension of the finite-difference approximation of Maxwell’s equation to three-dimension is fairly straightforward. Maxwell’s equations in the rectangular coordinate system can be written as the following system of equations:

\[
\begin{align*}
\frac{\partial H_x}{\partial t} &= \frac{1}{\mu} \left( \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} \right) \quad (4.1) \\
\frac{\partial H_y}{\partial t} &= \frac{1}{\mu} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \quad (4.2) \\
\frac{\partial H_z}{\partial t} &= \frac{1}{\mu} \left( \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) \quad (4.3) \\
\frac{\partial E_x}{\partial t} &= \frac{1}{\epsilon} \left( \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right) \quad (4.4) \\
\frac{\partial E_y}{\partial t} &= \frac{1}{\epsilon} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \quad (4.5) \\
\frac{\partial E_z}{\partial t} &= \frac{1}{\epsilon} \left( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} \right). \quad (4.6)
\end{align*}
\]

Following Yee's notation, a point in space is denoted as

\[(i,j,k) = (i\Delta x, j\Delta y, k\Delta z) \quad (4.7)\]

and any function of space and time is denoted as

\[F^n(i,j,k) = F(i\Delta x, j\Delta y, k\Delta z, n\Delta t). \quad (4.8)\]

As in the two-dimensional case, applying second-order accurate central difference approximations for the space and time derivatives of (4.8) result in:

\[\frac{\partial F^n(i,j,k)}{\partial x} = \frac{F^n(i + \frac{1}{2}, j, k) - F^n(i - \frac{1}{2}, j, k)}{\Delta x} + O\left(\Delta x^2\right) \quad (4.9)\]
and
\[
\frac{\partial F^n(i,j,k)}{\partial t} = \frac{F^{n+\frac{1}{2}}(i,j,k) - F^{n-\frac{1}{2}}(i,j,k)}{\Delta t} + O\left(\Delta t^2\right) \tag{4.10}
\]
respectively. To realize the space derivatives of (4.1) - (4.6) and achieve the accuracy of (4.9), the components of \( E \) and \( H \) are positioned on a special lattice as shown in Fig. 20. To achieve the accuracy of (4.10), the \( E \) and \( H \) fields are evaluated at alternate half time steps. Based on these premises, the system of equations (4.1)-(4.6) becomes:

\[
H_x^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2}) = H_x^{n-\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2}) + \frac{\Delta t}{\mu(i,j+\frac{1}{2},k+\frac{1}{2})} \cdot \left[ \frac{E_x^n(i,j+\frac{1}{2},k+1) - E_x^n(i,j+\frac{1}{2},k)}{\Delta z} - \frac{E_x^n(i,j+1,k+\frac{1}{2}) - E_x^n(i,j,k+\frac{1}{2})}{\Delta y} \right] \tag{4.11}
\]

\[
H_y^{n+\frac{1}{2}}(i+\frac{1}{2},j,k+\frac{1}{2}) = H_y^{n-\frac{1}{2}}(i+\frac{1}{2},j,k+\frac{1}{2}) + \frac{\Delta t}{\mu(i+\frac{1}{2},j,k+\frac{1}{2})} \cdot \left[ \frac{E_x^n(i+1,j,k+\frac{1}{2}) - E_x^n(i,j,k+\frac{1}{2})}{\Delta x} - \frac{E_x^n(i+\frac{1}{2},j,k+1) - E_x^n(i+\frac{1}{2},j,k)}{\Delta z} \right] \tag{4.12}
\]

\[
H_z^{n+\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k) = H_z^{n-\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k) + \frac{\Delta t}{\mu(i+\frac{1}{2},j+\frac{1}{2},k)} \cdot \left[ \frac{E_x^n(i+\frac{1}{2},j+1,k) - E_x^n(i+\frac{1}{2},j,k)}{\Delta y} - \frac{E_x^n(i+1,j+\frac{1}{2},k) - E_x^n(i,j+\frac{1}{2},k)}{\Delta x} \right] \tag{4.13}
\]

\[
E_x^{n+1}(i+\frac{1}{2},j,k) = E_x^n(i+\frac{1}{2},j,k) + \frac{\Delta t}{\varepsilon(i+\frac{1}{2},j,k)} \cdot \left[ \frac{\partial F^n(i,j,k)}{\partial x} \right]
\]
\[
\left[ \begin{array}{c}
\frac{H_z^{n+\frac{1}{2}}(i + \frac{1}{2}, j + \frac{1}{2}, k) - H_z^{n+\frac{1}{2}}(i + \frac{1}{2}, j - \frac{1}{2}, k)}{\Delta y} \\
\frac{H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j, k + \frac{1}{2}) - H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j, k - \frac{1}{2})}{\Delta z}
\end{array} \right]
\] (4.14)

\[E_y^{n+1}(i, j + \frac{1}{2}, k) = E_y^n(i, j + \frac{1}{2}, k) + \frac{\Delta t}{\epsilon(i, j + \frac{1}{2}, k)} \cdot \left[ \begin{array}{c}
\frac{H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) - H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}, k - \frac{1}{2})}{\Delta z} \\
\frac{H_z^{n+\frac{1}{2}}(i + \frac{1}{2}, j + \frac{1}{2}, k) - H_z^{n+\frac{1}{2}}(i - \frac{1}{2}, j + \frac{1}{2}, k)}{\Delta x}
\end{array} \right].\] (4.15)

\[E_x^{n+1}(i, j, k + \frac{1}{2}) = E_x^n(i, j, k + \frac{1}{2}) + \frac{\Delta t}{\epsilon(i, j, k + \frac{1}{2})} \cdot \left[ \begin{array}{c}
\frac{H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j, k + \frac{1}{2}) - H_y^{n+\frac{1}{2}}(i - \frac{1}{2}, j, k + \frac{1}{2})}{\Delta x} \\
\frac{H_z^{n+\frac{1}{2}}(i, j + \frac{1}{2}, k + \frac{1}{2}) - H_z^{n+\frac{1}{2}}(i, j - \frac{1}{2}, k + \frac{1}{2})}{\Delta y}
\end{array} \right].\] (4.16)

In equations (4.11)-(4.13), a magnetic field vector component is updated using its previous value and the previous values of the other vector components of the electric field at half spatial steps away. Similarly, in equations (4.14)-(4.16), an electric field vector component is advanced in time using its previous values and the previous values of the other vector components of the magnetic field at half spatial steps away. The time variation of the electromagnetic field is thus obtained by updating the $\mathbf{E}$ and $\mathbf{H}$ field alternately in time.
The CFL stability criteria for the three-dimensional case is given by

\[ v_{\text{max}} \Delta t \leq \frac{1}{\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta z}\right)^2}}. \]  

(4.17)

For \( \Delta x = \Delta y = \Delta z \), (4.17) reduces to

\[ v_{\text{max}} \Delta t \leq \frac{\Delta x}{\sqrt{3}}. \]  

(4.18)

### 4.2 Absorbing boundary condition

For the proper transmission of the reflected fields through \( S_{T1} \) (or the input plane), an absorbing boundary condition (ABC) has to be applied at \( S_{T1} \). For 3D cavity problems, Mur's second order ABC [41] does not provide adequate absorption at large
angles. Therefore we have chosen to use Higdon's ABC because of the flexibility allowed in the design of the absorption properties. Specifically, the angles of best absorption can be easily adjusted and the order increased to suit a given problem. Another reason for choosing Higdon's ABC is that the implementation near corners of rectangular FDTD cells is the same as that at other points on the absorbing boundary. The rest of this section presents the development and implementation of Higdon's ABC.

Higdon [39, 40] and Keys [44] independently developed ABCs through the discretization of analytical boundary conditions given by:

\[ BE(x, y, z = 0) = \left[ \prod_{m=1}^{M} B_m(z, t) \right] E(x, y, z = 0) = 0 \] (4.19)

where

\[ B_m(z, t) = \cos \alpha_m \frac{\partial}{\partial t} - c \frac{\partial}{\partial z} \] (4.20)

with \(|\alpha_m| < \pi/2\) for all \(m\). \(E(x, y, z = 0)\) is a tangential field component at the absorption plane, \(z = 0\). Equation (4.19) is satisfied by any linear combination of plane waves that are incident at \(z = 0\) from the \(z > 0\) half-space at angles of incidence \(\pm \alpha_1, \ldots, \pm \alpha_M\) (measured with respect to \(\hat{z}\)). For such a linear combination, there is no reflection. Therefore, the angles \(\alpha_m\) can be distributed to optimize the absorption properties of (4.19).

The discretization of (4.19) can be reduced to the discretization of the individual boundary operator \(B_m\) in (4.20) since the former is a composite of boundary operators. Let \(E_{i,j,k}^n\) denote \(E(i\Delta x, j\Delta y, k\Delta z, n\Delta t)\). Following Higdon's formulation, let \(Z\) and
$T$ denote the forward shift operation with respect to $z$ and $t$ respectively, such that

\begin{align}
ZE_{i,j,k}^n &= E_{i,j,k+1}^n \\
TE_{i,j,k}^n &= E_{i,j,k}^{n+1}
\end{align}  \quad (4.21)  \quad (4.22)

Equation (4.20) is now approximated by

\[ B_m(Z,T^{-1}) = \cos \alpha_m \left( \frac{I - T^{-1}}{\Delta t} \right) \left( \frac{I + Z}{2} \right) - c \left( \frac{Z - I}{\Delta z} \right) \left( \frac{I + T^{-1}}{2} \right) \]  \quad (4.23)

using space and time differences and their respective averages. After some algebraic manipulation,

\[ B_m(Z,T^{-1}) = C \left[ \left( I - Z T^{-1} \right) + \frac{\cos \alpha_m - \delta}{\cos \alpha_m + \delta} \left( Z - T^{-1} \right) \right] \]  \quad (4.24)

where $C$ is an inconsequential constant and $\delta = c \Delta t / \Delta z$. Equation (4.19) is now discretized by a composition of the form

\[ BE_{i,j,0}^{n+1} = \left[ \prod_{m=1}^{M} B_m(Z,T^{-1}) \right] E_{i,j,0}^{n+1} = 0 . \]  \quad (4.25)

The direct implementation of (4.25) for large $M$ involves high order derivatives in both space and time. In fact, the algebra becomes cumbersome and unwieldy for $M > 2$. To circumvent this problem, Fang's multi-absorbing boundary condition approach [42] is used to implement (4.25). The idea behind this approach is to apply each boundary operator $B_m$ sequentially with the result of each operator being used as the operand for the next operator. In so doing, the derivatives involved are of the same order as those in $B_m$ but smaller than those in $B$. Using this approach for
\( M = 4 \) in (4.25), a system of equations is defined such that
\[
B_1 E^{n+1}_{i,j,k} = h^{n+1}_{1,i,j,k} \tag{4.26}
\]
\[
B_2 h^{n+1}_{1,i,j,k} = h^{n+1}_{2,i,j,k} \tag{4.27}
\]
\[
B_3 h^{n+1}_{2,i,j,k} = h^{n+1}_{3,i,j,k} \tag{4.28}
\]
\[
B_4 h^{n+1}_{3,i,j,k} = 0 \tag{4.29}
\]

Note that the \( h \)'s are introduced in (4.26)-(4.29) so that the higher order derivatives (up to eighth) in \( B \) are avoided and replaced by lower order derivatives (first and second). The evaluation of \( E^{n+1}_{i,j,k} \) can now be obtained through a process of forward and backward substitution as summarized in Table 4. In the forward substitution, the \( h \)'s are evaluated on planes that are parallel and adjacent to the absorbing plane. In the backward substitution, the \( h \)'s are evaluated on the absorbing plane so that \( E^{n+1}_{i,j,k} \) can be finally evaluated.

The reflection coefficient for the composite operator, \( B \), in (4.25) is given by
\[
R_B = - \prod_{m=1}^{M} [-R_m] \tag{4.30}
\]
where [42]
\[
R_m = - \frac{\sin \left[ \frac{1}{2} k \Delta z \left( \delta - \cos \theta \right) \right] + \cos \alpha \sin \left[ \frac{1}{2} k \Delta z \left( \delta + \cos \theta \right) \right]}{\sin \left[ \frac{1}{2} k \Delta z \left( \delta + \cos \theta \right) \right] + \cos \alpha \sin \left[ \frac{1}{2} k \Delta z \left( \delta - \cos \theta \right) \right]} e^{-jk \Delta z \cos \theta}. \tag{4.31}
\]

For \( k \Delta z \ll \pi \), (4.31) reduces to the expression
\[
R_m \approx - \frac{\cos \alpha_m - \cos \theta}{\cos \alpha_m + \cos \theta} \tag{4.32}
\]
given in [39]. A plot of the magnitude of \( R_B \) is given in Fig. 21 for \( M = 4, \Delta z = \lambda/20, \delta = 0.5, \alpha_1 = 11.4^\circ, \alpha_2 = 35.4^\circ, \alpha_3 = 51^\circ \) and \( \alpha_4 = 58.4^\circ \). The \( \alpha_m \)'s are chosen...
Table 4: The forward & backward substitution process in Fang's multi-absorbing boundary condition approach.

Forward substitution:

<table>
<thead>
<tr>
<th>Step</th>
<th>From</th>
<th>calculate</th>
<th>for k =</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4.26)</td>
<td>$h_{1,i,j,k}^{n+1}$</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>2</td>
<td>(4.27)</td>
<td>$h_{2,i,j,k}^{n+1}$</td>
<td>1, 2</td>
</tr>
<tr>
<td>3</td>
<td>(4.28)</td>
<td>$h_{3,i,j,k}^{n+1}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Backward substitution:

<table>
<thead>
<tr>
<th>Step</th>
<th>From</th>
<th>calculate</th>
<th>for k =</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>(4.29)</td>
<td>$h_{3,i,j,k}^{n+1}$</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(4.28)</td>
<td>$h_{2,i,j,k}^{n+1}$</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>(4.27)</td>
<td>$h_{1,i,j,k}^{n+1}$</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>(4.26)</td>
<td>$E_{i,j,k}^{n+1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

through trial and error so that the reflection coefficient is less than $10^{-4}$ for incident angles less than 60°. From Fig. 21, it is clear that (4.32) is a good approximation for (4.31) for the given parameters.

Recently, it was shown that computer round-off error is a cause of numerical instability in ABCs [45, 46]. The higher the order of the ABC, the greater the instability. To reduce the instability, it was suggested that damping factors be added to the boundary operators. In our program, we have added damping factors, $c\alpha$, to
Figure 21: Reflection coefficient for a 4th-order Higdon's absorbing boundary condition with $\Delta z = 0.15 \text{ cm}$, $\delta = 0.5$, $f = 10 \text{ GHz}$, $\alpha_1 = 11.4^\circ$, $\alpha_2 = 35.4^\circ$, $\alpha_3 = 51^\circ$ and $\alpha_4 = 58.4^\circ$

all the boundary operators except $B_1$ so that

$$B_m(z, t) = \cos \alpha_m \frac{\partial}{\partial t} - c \frac{\partial}{\partial z} + \alpha, \quad \text{for } m = 2, 3, 4$$

(4.33)

with $\alpha = 0.08/\Delta z$.

4.3 Determination of $[S_T]$ via FDTD

In the sinusoidal-FDTD used in the 2-D GRE-FDTD hybrid method, ray solutions at $S_{T1}$ are required as input excitation for the FDTD code. Therefore, the computation time of the FDTD code is dependent on the number of observation angles where the
cavity scattered field is desired. To overcome this dependency, the FDTD algorithm uses a basis set to characterize the termination section in terms of a scattering or reflection matrix. The use of a basis set (e.g., waveguide modes or plane waves [47]) allows us to decouple the FDTD computation from the field computation in the front section of the cavity. In particular, the FDTD computation in the GRE-FDTD method can now be carried out independently of the ray solution at $S_{T1}$. Moreover, the FDTD computation time is determined by the size of the basis set which is not expected to exceed the number of observation angles generally required in a 3D problem. Another advantage of the decoupling in the computation is that a change in the front section of the cavity does not require the re-evaluation of the termination (of FDTD) section and vice versa. This is particularly useful for analyzing different combinations of the termination and front sections.

Critical to the pulsed-basis FDTD algorithm is the availability of a suitable basis set. The basis set is obtained by choosing $S_{T1}$ at an appropriate cross-section of the cavity. For our problems, we have chosen waveguide modes to form the basis set. Therefore, $S_{T1}$ is chosen close to the front of the termination section in a realistic 3D engine cavity where it is usually circular in shape. In cases where waveguide modes cannot be used, a set of plane waves may be suitable as a basis set [47].

The basis set has to be combined with a suitable time variation for use in the FDTD method. The two commonly used time variations are the sinusoidal and the pulsed time variations. We have selected to use the latter because it is more efficient than the sinusoidal time variation for generating broadband frequency solutions. By
combining the pulsed time variation with a basis set in the FDTD (hence, the pulsed-basis FDTD), we can obtain multiple reflection matrices for the termination section at various frequencies in a single run.

The reflection matrix is obtained via the concept of the generalized scattering matrix of a multi-port junction in microwave circuits [48]. To find the modal reflection matrix \([S_T]\) of the termination via FDTD, the field distribution of each waveguide mode at \(S_{T1}\) is used to amplitude modulate the FDTD input time waveform. When this field interacts with the termination, part or all of it (depending on the termination) is reflected back towards \(S_{T1}\). From the reflected field at \(S_{T1}\), the coupling (or reflection coefficient) between the applied mode and each of the waveguide modes is determined. These reflection coefficients constitutes a column of the reflection matrix. The entire matrix, \([S_T]\), can therefore be found by modulating the input time waveform consecutively with each of the waveguide modes.

The modulation by a waveguide mode of the FDTD time waveform is performed differently in the sinusoidal and the pulsed FDTD. Consider for example, the tangential \(\vec{H}\)-field of the \(TM_{pq}\) mode, \(\hat{h}_{pq}\), of a circular waveguide of radius \(a\), given by (see Appendix C)

\[
\hat{h}_{pq} = \frac{kN_{pq}\kappa_{pq}}{Z_0} \left( \hat{\rho} \frac{pJ_p(\kappa_{pq}\rho)}{\kappa_{pq}\rho} \left\{ \sin p\phi \cos p\phi \right\} + \hat{\phi} J_p(\kappa_{pq}\rho) \left\{ \frac{\cos p\phi}{-\sin p\phi} \right\} \right)
\]

(4.34)

where \(N_{pq}\) is a frequency-dependent normalization constant for unit modal power and \(\kappa_{pq}\) is a solution of \(J_p(\kappa_{pq}a) = 0\). When \(\hat{h}_{pq}\) is used to modulate the time waveform...
in the sinusoidal FDTD, the reflected field at $S_{T1}$ can, in general, be expressed as

$$
\mathbf{H}^r = \sum_{n',m'} \left[ A_{n',m'}' \left(-\tilde{h}_{n'm't} + h_{n'm'tz}^r\right) \right] + \sum_{n,m} A_{nm} \left(-\tilde{h}_{nmt}\right) \quad (4.35)
$$

where $\tilde{h}_{n'm't}$ and $h_{n'm'tz}^r$ are the respective tangential and $z$ components of the $\mathbf{H}$-field for the $\text{TE}_{n'm'}$ mode. $A_{n'm'}'$ and $A_{nm}$ are the respective $\text{TE}_{n'm'}$ and $\text{TM}_{nm}$ coefficients of the reflected field $\mathbf{H}^r$ due to the $\text{TM}_{pq}$ input excitation. They are given by

$$
A_{n'm'}' = \int_{S_{T1}} \mathbf{H}^r \times \mathbf{\hat{e}}_{n'm't} \cdot \mathbf{\hat{z}} \, dS \quad (4.36)
$$

$$
A_{nm} = \int_{S_{T1}} \mathbf{H}^r \times \mathbf{\hat{e}}_{nmt} \cdot \mathbf{\hat{z}} \, dS \quad (4.37)
$$

where $\mathbf{\hat{e}}_{n'm't}$ and $\mathbf{\hat{e}}_{nmt}$ are the tangential $\mathbf{E}$-field of the $\text{TE}_{n'm'}$ and $\text{TM}_{nm}$ modes respectively (see Appendix C).

To apply the waveguide mode modulation in the pulsed FDTD, the frequency-independent form of the field distribution of the waveguide mode has to be used. Specifically, the $\text{TM}_{pq}$ mode in (4.34) is scaled by $kN_{pq}k_{pq}/Z_0$, so that the coefficients of the reflected field are now given by

$$
A_{n'm'}' = \frac{kN_{pq}k_{pq}}{Z_0} \int_{S_{T1}} \mathbf{H}^{rs} \times \mathbf{\hat{e}}_{n'm't} \cdot \mathbf{\hat{z}} \, dS \quad (4.38)
$$

$$
A_{nm} = \frac{kN_{pq}k_{pq}}{Z_0} \int_{S_{T1}} \mathbf{H}^{rs} \times \mathbf{\hat{e}}_{nmt} \cdot \mathbf{\hat{z}} \, dS \quad (4.39)
$$

where $\mathbf{H}^{rs}$ is the reflected field due to the scaled input excitation. The reflection coefficients due to TE modes can also be similarly obtained.

As in the 2D FDTD method, the tangential $\mathbf{E}$ and $\mathbf{H}$ planes in the 3D FDTD space are also half a cell width apart. An extrapolation scheme was used in the 2D case to find the reflected fields at $S_{T1}$. For the 3D case, a simple analytical expression
is used to account for the different planes of excitation used for the TE and TM modes. If the TE modes are applied at \( S_{T1} \) where \( z = 0 \) and the TM modes are applied at \( z = \Delta z/2 \) (where \( z \) increases towards the termination), then each of the reflection coefficients in (4.38) and (4.39) for the TM input excitation is modified by the phase factor \( e^{-j(\beta^i+\beta^r)\Delta z/2} \), where \( \beta^i \) and \( \beta^r \) are the propagation constants of the applied (TM) and the reflected (TE or TM) modes respectively.

### 4.4 Scattered field computation

In the 2D GRE-FDTD method, the termination reciprocity integral was used to find the cavity scattered field. Specifically, the desired scattered field, \( \mathbf{E}_c^s(P) \), at the observation point \( P \) is given by [29]

\[
\mathbf{E}_c^s(P) \cdot \mathbf{P}_i \approx \int_{S_{T1}} (\mathbf{E}_c^s \times \mathbf{H}_t - \mathbf{E}_t \times \mathbf{H}_c^s) \cdot \mathbf{n} \, ds \quad (4.40)
\]

where \( \mathbf{P}_i \) is the strength of an electric current point (test) source. \((\mathbf{E}_c^s, \mathbf{H}_c^s)\) are the fields scattered by the termination in the cavity while \((\mathbf{E}_t, \mathbf{H}_t)\) are the fields radiated by the test source in the presence of the cavity structure without the termination. When the termination is characterized by a modal reflection matrix \([S_T]\) as in our case, the cavity scattered field can be conveniently obtained via the modal form of (4.40). For backscatter computation, the modal form of (4.40) is given by [10]

\[
\mathbf{E}_c^s(P) \cdot \mathbf{P}_i = -2 \sum_n A_n^- A_n^+ \quad (4.41)
\]

(4.41) is obtained by substituting the modal expansion of \((\mathbf{E}_c^s, \mathbf{H}_c^s)\) and \((\mathbf{E}_t, \mathbf{H}_t)\) into (4.40) and using the orthogonality property of the modes to reduce the resultant
integral (see Appendix D). Specifically,

\[
(E_x, H_x) = \sum_n A_n^+ (\hat{e}_n^+, \hat{h}_n^+) e^{-j\beta_n a^+} \tag{4.42}
\]

\[
(E^a_x, H^a_x) = \sum_n A_n^- (\hat{e}_n^-, \hat{h}_n^-) e^{j\beta_n a^-} \tag{4.43}
\]

where

\[
(\hat{e}_n^\pm, \hat{h}_n^\pm) = n^{th} electric and magnetic modal fields propagating in the \\
\pm \hat{z} directions respectively, with \int_{S_T} \hat{e}_n^\pm \times \hat{h}_n^\pm \cdot \hat{z} \ dS = \delta_{lm},
\]

\[
\beta_n = \text{propagation constant of the } n^{th} \text{ modal field},
\]

\[
\hat{z} = \text{unit vector parallel to theaxis of the termination section}
\]

and pointing into the termination section.

Assuming \(A_{n}^+\) is known, then for backscatter, \(E^a_x\) can be computed via (4.41) with \(A_{n}^-\) given by

\[
[A_{n}^-] = [S_T] [A_{n}^+] \tag{4.44}
\]

Up to this point, no mention has been made as to the methods used to find the coefficients \(A_{n}^+\) in (4.42). For the hybrid methods of interest, \(A_{n}^+\) will be determined by modal analysis, the IPO method and the GRE method. The first method assumes that the front section can be analyzed in terms of waveguide modes. The GRE and IPO methods are more versatile in that they can be used for quite arbitrary front sections. The derivation of \(A_{n}^+\) for circular waveguides via modal analysis will not be discussed since it can be found in [13, 49]. The derivation of \(A_{n}^+\) by the remaining two methods are discussed in the following two sections.
4.5 Coupling between GRE & FDTD

In the 2-D hybrid ray-FDTD method [37], two different schemes were used for the coupling between the GRE and the FDTD methods. In the first scheme, the ray tube size was limited to the FDTD grid spacing of $\lambda/20$ so that the field of a ray tube was added to the field of the FDTD grid point intersected by that ray tube. In the second scheme, the ray tube size was limited to $\lambda/4$ and its field was expressed as a sum of waveguide modes. Although it is possible to apply these two schemes in the 3-D case, there are serious limitations. For the first scheme, an enormous number of ray tubes has to be tracked as the ray tube size is limited to $(\lambda/20)^2$ in 3-D problems. For the second scheme, the number of ray tubes required is less than that for the first scheme since the ray tube size can be as large as $(\lambda/2)^2$ [10]. However, the conversion of the ray fields into waveguide modes can still be time consuming since there may be many waveguide modes (and even more ray tubes) in a large 3-D cavity. Therefore, the application of either of the two schemes in their present form in a 3-D problem is not computationally efficient.

Besides efficiency, the coupling scheme has to be compatible with the excitation of the FDTD algorithm. Since waveguide modes are used to characterize the termination section in terms of reflection matrices and the cavity scattered field can be computed efficiently via (4.41), the ray solution at $S_{T1}$ should therefore be expressed in terms of the same set of waveguide modes. For this purpose, a new coupling scheme, combining the previous two coupling schemes, is used. Beginning with the second scheme, the
ray solution at \( S_{T1} \) is described by a set of modal coefficients, \([A_n^+]\), given by

\[
A_n^+ = \sum_{p=1}^{P} \int_{S_p} \frac{E_p}{M_n} \cdot \hat{z} \, ds \quad \text{for } n = 1, \ldots, N \quad (4.45)
\]

where

- \( A_n^+ \) = modal coefficient of the \( n^{th} \) waveguide mode,
- \( M_n \) = \( n^{th} \) waveguide mode,
- \( E_p^i \) = electric field of the \( p^{th} \) ray tube incident at \( S_{T1} \),
- \( S_p \) = footprint of the \( p^{th} \) ray tube on \( S_{T1} \).

Note that only the tangential components of \( E_p^i \) are required in (4.45). The field of each of the \( P \) ray tubes at \( S_{T1} \) has to be expanded into a sum of \( N \) waveguide modes to determine its contribution to \( A_n^+ \). Depending on the shape and size of \( S_p \), the integral in (4.45) may have to be evaluated numerically. The integral may, however, be evaluated approximately if \( S_p \) is sufficiently small [10]. In any case, the determination of \( A_n^+ \) via (4.45) can be very time consuming when there is a large number of modes and an even larger number of ray tubes.

Now, in the first scheme, the desired field is obtained essentially by sampling the ray fields at \( S_{T1} \) with a fine grid. Applying the same principle but using a coarser grid, (4.45) can be rewritten as

\[
A_n^+ = \sum_{l} \sum_{m} E_{lm}^i \cdot \hat{z} \Delta x \Delta y \quad (4.46)
\]

where \( \Delta x \) and \( \Delta y \) are the grid spacing in the \( x \) and \( y \) directions in the plane of \( S_{T1} \) as shown in Fig. 22. \( E_{lm}^i \) is the sum of the incident electric fields of the ray tubes.
hitting $S_{T1}$ in the vicinity of the grid point $(l\Delta x, m\Delta y)$, given by

$$\overline{E}_{lm}^i = \sum_{p=1}^{p_{lm}} \overline{E}_{pim}^i$$

(4.47)

where $p_{lm}$ is the index of the ray tube whose field is assigned to the grid point $(l\Delta x, m\Delta y)$. $\overline{E}_{pim}^i$ is given by

$$\overline{E}_{pim}^i = \overline{E}_p(\overline{r}_p) e^{-jk_p(\overline{r}_{lm} - \overline{r}_p)} \frac{S_p}{\Delta x \Delta y}$$

(4.48)

where

$\overline{E}_p(\overline{r}_p) = \text{electric field of } p^{th} \text{ ray tube at } \overline{r}_p,$

$\overline{r}_p = \text{position vector of point where the axis of the } p^{th} \text{ ray tube hits } S_{T1},$

$k = \text{free space propagation constant},$

$\hat{k}_p = \text{unit vector of propagation direction of } p^{th} \text{ ray tube},$

$\overline{r}_{lm} = \text{position vector of grid point } (l\Delta x, m\Delta y),$
\[ S_p = \frac{\Delta A_p}{|\vec{k}_p \cdot \hat{z}|}, \]

\[ \Delta A_p = \text{cross-sectional area of the } p^{th} \text{ ray tube.} \]

The purpose of the term \( S_p/(\Delta x \Delta y) \) in (4.48) is for the conservation of the power of the \( p^{th} \) ray tube when it is eventually integrated in (4.46).

The critical factors in the new scheme are the sizes of the ray tubes and the grid. It is preferable that the ray tubes be sufficiently large to reduce the number of ray tubes required. However, to evaluate the ray field in closed form, the size of a ray tube cannot be greater than \((\lambda/2)^2\) [10]. For this limit, we have found via numerical experiments that a grid size of \( \lambda/8 \) is required for a convergent field solution on \( S_{T1} \).

In summary, the new scheme traces each ray tube to \( S_{T1} \) and assigns its field to the nearest grid point according to (4.47) and (4.48). When all the ray fields have been assigned, the resultant grid fields in (4.47) are substituted into (4.46) to find \( A_h^+ \). Compared to the direct ray-to-mode conversion scheme which requires \( PN \) integrations (see (4.45)), the new scheme requires only \( N \) integrations (in the form of a double summation) to obtain \( A_h^+ \). Moreover, (4.46) can be evaluated very efficiently via fast transform techniques [50].

### 4.6 Coupling between PO and FDTD

In the hybrid PO-FDTD method, the iterative physical optics (IPO) method [19] is used to find the fields at the cross-section \( S_{T1} \). These cross-sectional fields are then expressed in terms of the basis set (waveguide modes) for use in computing the cavity scattered field. A summary of the pertinent equations used in the IPO
A detailed description of the IPO method can be found in [19].

The IPO method approximately solves the magnetic field integral equation (MFIE) for an applied field incident on a PEC scatterer. For our purpose, the sources of the applied field are the equivalent currents at the aperture of the cavity and the PEC scatterer is the interior cavity wall between the aperture and $S_{TI}$ as shown in Fig. 23. Specifically, the MFIE for field points on the PEC walls of the cavity is given by [18, 19, 51]

$$\mathbf{J}(\mathbf{r}_c) = 2\hat{n} \times \mathbf{H}_a(\mathbf{r}_c) + 2\hat{n} \times \oint_{S_c} \mathbf{J}(\mathbf{r}_c') \times \nabla G_0(\mathbf{r}_c - \mathbf{r}_c') \, dS'_c$$

(4.49)

where $\mathbf{J}$ denotes the principal value of the integral and

- $\mathbf{J}(\mathbf{r}_c)$ = equivalent electric current density,
- $\mathbf{r}_c$ = position vector of a point on the interior cavity wall,
- $\hat{n}$ = unit normal surface vector pointing into cavity,
- $G_o$ = free space Green’s function.

Figure 23: Equivalent currents of the magnetic field integral equation.
$\overline{H}_a$ is the magnetic field radiated into the cavity via the aperture due to the external field $(\overline{E}^i, \overline{H}^i)$ incident at the aperture ($S_a$) of the cavity. It is given by [19]

$$\overline{H}_a(\overline{r}_c) = \int_{S_a} \left[ \hat{n} \times \overline{H}_a' \right] \times \nabla G_o(\overline{r}_c - \overline{r}_a') \, dS_a' + \frac{1}{jkZ_o} \nabla \times \int_{S_a} \left[ \overline{E}_a' \times \hat{n} \right] \times \nabla G_o(\overline{r}_c - \overline{r}_a') \, dS_a' \quad (4.50)$$

Equation (4.49) is solved via the iterative physical optics algorithm, taking into account shadowing effects [19]. The IPO algorithm uses as its initial guess the PO currents on the cavity walls excited by $\overline{H}_a^i$. This initial guess is used in the integral of (4.49) to find a better estimate of $\overline{J}_c$. This process is repeated until a convergent solution is obtained. Once $\overline{J}(\overline{r}_c)$ is found, the desired fields $\overline{H}_{ST1}(\overline{r}_s)$ on $S_{T1}$ can be obtained by integrating $\overline{J}(\overline{r}_c)$ as

$$\overline{H}_{ST1}(\overline{r}_s) = \int_{S_{T1}} \overline{J}(\overline{r}_c') \times \nabla G_o(\overline{r}_s - \overline{r}_c') \, dS_c' \quad (4.51)$$

The IPO method is implemented by modeling the cavity and the aperture end with flat facets [19]. The integrals in (4.49) and (4.50) are evaluated numerically via summations by assuming the fields and currents are constant over each facet. Facets that are not within line-of-sight of each other do not contribute mutually. A $\lambda/8 \times \lambda/8$ grid similar to the one used in the GRE method is set up at $S_{T1}$ to sum the fields due to the (convergent solutions of the) currents of the faceted cavity walls. Once the grid fields have been summed, they are expressed in terms of waveguide modes as in (4.42).
4.7 Implementation of a parallel FDTD code

As the size of the cavity increases, the memory required in the FDTD code increases in a disproportionate fashion. This disproportionality is due to the greater number of field variables, the Fourier transformed field variables, and the field variables for the absorbing boundary condition. Therefore, the size of a cavity that can be analyzed is limited by the available memory in a given computer system. A larger cavity also requires more computation time because of the greater number of field variables and propagating waveguide modes.

To overcome these constraints, the pulsed-basis FDTD algorithm has also been implemented on the Touchstone DELTA system for analyzing larger cavities. The DELTA system is a high-speed concurrent multi-computer consisting of 576 processors connected in a 16 x 36 two-dimensional mesh [52, 53]. There are four different types of processors:

- numeric processors, for numerically intensive tasks,
- mass storage processors, providing input/output services to the numeric processors,
- gateway processors, which connect the system to an Ethernet network, and
- service processors, for remote access to the system.

The numeric processors are based on Intel’s i860™ processing board whose processor operates at 40 MHz and is rated at 33 MIPS, 80 (peak) single-precision MFLOPS
and 60 (peak) double-precision MFLOPS. Each numeric processor has 16 Mbytes of memory, of which 500 Kbytes is used for the NX/M operating system and 3 Mbytes is used for message buffering, leaving 12.5 Mbytes per processor for user application.

A parallel computer reduces the run time of a program by distributing its computation load among a set of independent processors. In addition, by combining the memory resources of a set of processors through proper partitioning of the problem geometry, a larger geometry can be solved. In our parallel FDTD code, the 1-way domain decomposition method [54] is used to divide the FDTD space into rectangular boxes (or computational domains) which are assigned to different processors as shown in Fig. 24. Note that the widths of the boxes (along the y axis) can be different although the program has to be written so as to run on all the processors.

At every half time step, each processor updates the electric or magnetic fields within its domain independently of the other processors except for field values which are on the boundary plane(s) shared with neighboring processor(s). An internal processor shares boundaries with two other processors, a left processor and a right processor, while an end processor shares a single boundary with one other processor. When the electric (magnetic) field on a shared boundary is computed, it requires the magnetic (electric) field values from the appropriate neighboring processor. To accomplish this task, the original domain is divided such that each processor shares a buffer region with its neighboring processor as illustrated by the shaded volume in Fig. 24. Each buffer region consists of two planes, one $E_x - H_y - E_z$ plane and one $H_x - E_y - H_z$ plane. The field values on these planes are shared by two adjacent
Figure 24: 1-way domain decomposition and buffering used in the parallel FDTD code.
We now consider how the electric and magnetic fields on the boundaries of an internal processor are computed. Assume that there are $J_M + 1/2$ Yee cells (corresponding to $(J_M + 1) E_x - H_y - E_z$ planes and $(J_M + 1) H_z - E_y - H_x$ planes) in the middle processor along the y-axis. Then, the fields are updated systematically as summarized in Table 5 where the indices $M$, $L$ and $R$ refers to the middle, left and right processors respectively. For simplicity, the time index and the space indices in the $x$ and $z$ directions are not shown. To update $E_x^{M}(J_M + 1)$, the previous time step values of $H_z^{M}(J_M + 1)$ (and other magnetic field components) are required. These values have to be obtained from $H_z^{R}(1)$ of the right processor as they cannot be computed by the middle processor. In the same way, the computation of $E_x$, $H_x$ and $H_z$ at the boundaries of the middle processor requires the appropriate data to be transferred from the left or right processors. Note that the computation of $E_y$ and $H_y$ do not require any data from the neighboring processors. The data communication between all the processors can be summarized as depicted in Fig. 25. Arrows pointing into a processor indicate incoming data while outward pointing arrows indicate outgoing data. The side from which an arrow enters (exits) a processor is the boundary where the data is transferred to (from).

![Figure 25: Data communication between processors.](#)
Table 5: Data transfer to compute the fields in an internal processor

<table>
<thead>
<tr>
<th>update</th>
<th>$j$, $y$-axis index</th>
<th>data transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x^M(j)$</td>
<td>2 to $J_M + 1$</td>
<td>$H_x^M(J_M + 1) \leftarrow H_x^R(1)$</td>
</tr>
<tr>
<td>$E_y^M(j)$</td>
<td>1 to $J_M$</td>
<td>$-$</td>
</tr>
<tr>
<td>$E_z^M(j)$</td>
<td>2 to $J_M + 1$</td>
<td>$H_x^M(J_M + 1) \leftarrow H_x^R(1)$</td>
</tr>
<tr>
<td>$H_x^M(j)$</td>
<td>1 to $J_M$</td>
<td>$E_x^L(J_L + 1) \rightarrow E_x^M(1)$</td>
</tr>
<tr>
<td>$H_y^M(j)$</td>
<td>2 to $J_M + 1$</td>
<td>$-$</td>
</tr>
<tr>
<td>$H_z^M(j)$</td>
<td>1 to $J_M$</td>
<td>$E_x^L(J_L + 1) \rightarrow E_x^M(1)$</td>
</tr>
</tbody>
</table>

Data transfer between processors is accomplished by using the asynchronous send and receive commands provided for in the NX/M operating system. These commands are non-blocking calls in that the program continues without waiting for their completion. In contrast, synchronous commands are blocking calls as the program can only continue after these commands have completed their execution. The advantage of asynchronous calls over synchronous calls is that they allow the overlapping of communication and computation in a program. Therefore, the performance of a program can generally be improved by using asynchronous calls.

Some of the results presented in the next chapter are obtained by the parallel FDTD code. A comparison of the computation time between the parallel FDTD code running on the DELTA machine and the serial vectorized FDTD code running on the Cray will also be given in the next chapter.
CHAPTER V

NUMERICAL RESULTS

This chapter presents the results obtained by the three hybrid methods discussed in the previous chapter. Important to these hybrid methods is the accuracy of the termination reflection matrix obtained by the FDTD method. Therefore, most of the results shown here, in the form of RCS patterns, are for the purpose of evaluating the accuracy of the reflection matrices obtained by the FDTD method. These results are obtained for two different terminations hubs, the cylindrical and conical hubs. Fan blades are also attached to these hubs to observe the effects of the blades on the cavity scattering. Besides its accuracy, we will also examine the properties of the termination reflection matrix and their implications in terms of storage requirements. The accuracy of the fields obtained by the IPO and the GRE methods are also evaluated by comparing their field solutions to modal field solutions. The results obtained by the hybrid methods are then presented. Finally, a comparison of the computation times between a parallel FDTD code and a serial FDTD code is given.
5.1 Accuracy of the FDTD method

The accuracy of any hybrid method is determined by the combined accuracies of the individual methods. Therefore, we will first examine the accuracy of the reflection matrices obtained by the pulsed-basis FDTD algorithm. To do so, the radar cross-section (RCS) of the termination section of the cavity is computed via (4.41) and (4.44) where \( A_n^+ \) are the modal coefficients of a plane wave field incident at the open end of the termination section. Whenever possible, the FDTD solutions are compared to solutions obtained by the method of moments for bodies of revolution (BOR-MM) [55]. Note, however, that the cavity used in the BOR-MM code has a hemispherical cap added to the external back end of the cavity for reducing the scattering from the back rim of the cavity.

5.1.1 Cylindrical hub termination

The first set of results is for a circular cavity with a cylindrical hub termination as depicted in Fig. 26. The hub is attached to the flat end of the cavity. Since cubic cells are used in the FDTD code to model the cavity, a stair-stepped approximation of the cavity is produced. An example of a such an approximation is shown in Fig. 27. The cylindrical hub termination is similarly modeled by its stair-stepped approximation.

Fig. 28 shows the co-polarized RCS patterns at 10 GHz for the cavity obtained by the FDTD method and the BOR-MM. The diameters of the cavity and the hub are 4\( \lambda \) and 2\( \lambda \), and their lengths are 1.5\( \lambda \) and 0.5\( \lambda \), respectively. The circular waveguide section of this cavity can support a total of 77 (even and odd) propagating modes.
Figure 26: A circular cavity with a cylindrical hub termination.

However, because of symmetry, only the 42 (even) modes are used in the FDTD algorithm to find the $77 \times 77$ modal reflection matrix. Although there is overall agreement between the FDTD and the reference BOR-MM solutions, there are still some differences, particularly in the $\phi - \phi$ polarization. These differences are due mainly to the modeling differences in the two methods. For the FDTD method, some errors are introduced by the use of the stair-stepped approximations of the cavity and the hub. However, as will be shown later, these errors are small. The major cause of the differences is the external scattering effects in the BOR-MM characterization of the cavity. These effects, which are not included in the FDTD solutions, are the edge-diffracted fields from the front and back rims of the cavity, and the surface-diffracted fields from the hemispherical cap of the cavity. These fields are also more significant in the $\phi -$ (or the horizontal-) polarization than in the $\theta$-polarization since creeping waves are most strongly excited on the hemispherical surface and rim-diffracted fields are greater for the horizontal polarization. The surface-diffracted fields are also more
Figure 27: Stair-stepped approximation of a circular cavity. The dashed line indicates the circumferential wall of the actual cavity.
Figure 28: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. 

$dc = 12$ cm, $Lc = 4.5$ cm, $dh = 6$ cm, $Lh = 1.5$ cm.
significant at wider angles of incidence when the path around the hemispherical cap is shorter. Therefore, the differences in the RCS patterns are most likely due to these diffraction effects. These effects are, however, expected to become less significant as the electrical size of the cavity increases.

For the next example, the diameters of the cavity and the hub are increased from $4\lambda$ and $2\lambda$ to $6\lambda$ and $3\lambda$, respectively, while their lengths are both increased by $\lambda/2$ to $2\lambda$ and $1\lambda$. Of the 175 circular waveguide modes, 93 are used to determine the reflection matrix. The RCS patterns for this larger cavity are shown in Fig. 29. The FDTD solutions in both polarizations are now in better agreement with the corresponding BOR-MM solutions. The improvement is due to the better FDTD stair-stepped model and the reduction of the diffraction effects in the BOR-MM solutions.

The diameters of the cavity and the hub are further increased to $8\lambda$ and $4\lambda$ while their lengths remain unchanged at $2\lambda$ and $1\lambda$ respectively. For this cavity, 164 of the 313 modes are used to generate the modal reflection matrix. Fig. 30 shows the RCS patterns for this cavity. The FDTD solutions show excellent agreement with the reference solutions. These results are expected as the external effects in the reference solutions are considerably less significant in a larger cavity. Moreover, the errors due to the stair-stepped modeling of the cavity are also minimized.

From the above examples, we see that the pulsed-basis FDTD algorithm with stair-stepped modeling is a viable method for obtaining the reflection matrix of the termination section. It is also a sufficiently accurate method even for stair-stepped models of cavities as small as $4\lambda$ in diameter.
Figure 29: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. $d_e = 18$ cm, $L_e = 6$ cm, $d_h = 9$ cm, $L_h = 3$ cm.
Figure 30: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. 
\( d_c = 24 \, \text{cm}, \, L_c = 6 \, \text{cm}, \, d_h = 12 \, \text{cm}, \, L_h = 3 \, \text{cm} \).
5.1.2 Conical hub termination

In the next example, the effects of stair-stepped modeling in the FDTD algorithm is further investigated. For this purpose, an axially symmetric cone is used. The base of the cone is attached to the flat end of a circular cavity as shown in Fig. 31. The cone is modeled in the FDTD code as a series of stair-stepped circles (as in Fig. 27(b)) with increasing radius in the axial direction. The cone has a base diameter of 6 cm and a height of 6 cm. The cavity is 12 cm in diameter and 9 cm long. Reflection matrices at 8, 9 and 10 GHz are obtained for cell sizes of 0.10 cm and 0.15 cm (corresponding to 30 and 20 cells per wavelength respectively at 10 GHz).

Figs. 32 to 34 show the RCS patterns for this cavity at the 3 respective frequencies. The agreement between the FDTD and the BOR-MM solutions are rather good considering the fact that a stair-stepped model of the cone is used in the FDTD code. The edges and corners of the stair-stepped model give rise to diffraction and interaction effects that are not present in the real cone except for the diffraction due to the

Figure 31: A circular cavity with a conical hub termination.
tip of the cone. However, these effects do not seem to have affected the accuracy of the reflection matrix seriously. As before, the reference solutions include the external scattering effects.

The solutions obtained with the $\lambda/30$ cells show some, albeit not spectacular, improvement over those obtained with the $\lambda/20$ cells. In Figs. 32 to 34, the solutions obtained with the smaller cells inch toward the reference solutions. The improvement is particularly evident in the nulls of the RCS patterns.

The above results have shown that reasonable accuracy is still obtainable even with the approximate cone model. These results also improve with frequency. Therefore, we can expect accurate solutions for realistic engine models which are tens of wavelengths across. The RCS results of the last two subsections have shown that the pulsed-basis FDTD algorithm with stair-stepped modeling can produce sufficiently accurate termination reflection matrices. In the next two subsections, the simple yet versatile modeling capabilities of the stair-stepped approximation is used to model more realistic jet engine terminations.
Figure 32: Co-polarized RCS patterns at 8 GHz for a circular cavity with a conical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. $d_c = 12$ cm, $L_c = 9$ cm, $d_h = 6$ cm, $L_h = 6$ cm.
Figure 33: Co-polarized RCS patterns at 9 GHz for a circular cavity with a conical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. $d_c = 12$ cm, $L_c = 9$ cm, $d_h = 6$ cm, $L_h = 6$ cm.
Figure 34: Co-polarized RCS patterns at 10 GHz for a circular cavity with a conical hub termination obtained by the pulsed-basis FDTD method and the BOR-MM. $d_c = 12\text{ cm}$, $L_c = 9\text{ cm}$, $d_h = 6\text{ cm}$, $L_h = 6\text{ cm}$. 
5.1.3 Cylindrical hub and blades termination

To simulate the front frame of a jet engine, fan blades are attached to the hub. We will use both the cylindrical hub and the conical hub. The blades are either *sectorial* or *straight*. The *sectorial* blades are defined by their angular widths while the *straight* blades are defined by their constant thickness. For our purpose, the shape and orientation of these blades do not change with its length along the axis of the cavity.

Fig. 35 shows an example of a circular cavity with a termination comprising a cylindrical hub and some blades. The subscripts $c$, $h$ and $b$ in the figure refer to the cavity, blades, and hub respectively. The parameter $w_b$ denotes the width of the blades in meters for *straight* blades and in degrees for *sectorial* blades. In all the examples shown here, the hub and blades are attached to the flat end of the cavity. In addition, the tips of the blades touch the circumferential walls of the cavity although

![Figure 35: A circular cavity with a termination comprising a cylindrical hub and blades.](image-url)
this condition can be easily altered in the FDTD program.

Fig. 36 shows a cross-sectional view of a cavity with 4 sectorial blades attached to a cylindrical hub termination. These 45°-wide blades are equally spaced and oriented such that one of the edges of each blade is aligned with either the vertical $x$-axis or the horizontal $y$-axis. As a result, the remaining edges of the blades are approximated via stair-stepping in the FDTD code as shown in Fig. 36. The RCS patterns of this cavity are shown in Fig. 37. For this small cavity with a diameter of $1.35\lambda$ and length of $1.325\lambda$, 40 cells per wavelength is used to model the cavity and its termination. All ten propagating modes are used in the determination of the reflection matrix since there is no symmetry about the axis of the cavity.

Figure 36: Stair-stepped approximation of 4 sectorial blades with $w_b = 45°$. 
Figure 37: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 4 sectorial blades obtained by the pulsed-basis FDTD method and the mode-matching method with $\phi = 0^\circ$. $r_c = 2.025$ cm, $L_c = 3.975$ cm, $r_h = 0.9$ cm, $L_h = 0.975$ cm, $r_b = 1.125$ cm, $L_b = 0.975$ cm, $w_b = 45^\circ$. The dash-dot lines indicate the modal RCS patterns of the same cavity with a PEC disc placed in front of the hub-blades termination.
For comparison, we have included in Fig. 37 the solutions obtained via the mode-matching method [56, 57]. For the $\theta$-polarization, there is good agreement between the RCS patterns of the two methods for $|\theta| \leq 30^\circ$. For the $\phi$-polarization, the patterns are in agreement for $|\theta| \leq 40^\circ$. Since both methods treat the cavity as an internal problem (as opposed to the external problem in the BOR-MM), the disagreement at wider angles of incidence between the two methods has to be due to the computational differences of the two methods. The disagreement is most likely due to the higher order modes that are not effectively transmitted through the absorbing boundary in the FDTD method. In particular, half of the propagating modes have angles greater than $60^\circ$. Recall from Fig. 21 that the magnitude of the reflection coefficient of the absorbing boundary increases rapidly from about $10^{-4}$ at $60^\circ$ to 1 at $90^\circ$. Therefore, the higher order modes are not effectively transmitted through the absorbing boundary and hence they undergo artificial multiple reflections between the absorbing boundary and the termination. These artificial reflections contaminate the fields at the absorbing boundary from which the coefficients of the reflection matrix are derived.

The disagreement at wide angles of incidence suggests that some of the higher order modes may have also been produced from the interaction of the incident lower order modes with the hub-blades termination. This situation indicates a potential problem in cases where there are as many higher order modes as there are lower order modes. If many of the higher order modes are somehow strongly excited at the same time, then depending on the absorbing boundary condition used, the reflection
coefficient can become contaminated with artificial reflections. In such an event, the
absorbing boundary condition should be replaced with one that has good absorption
properties at large angles of incidence.

Given the large area occupied by the blades and the hub, it is reasonable to
assume that most of the energy incident at the front of the termination is reflected
back towards the aperture, particularly at or close to boresight. The RCS patterns
of the same cavity with a short circuit placed in front of the hub-blades termination
are also plotted in Fig. 37. These patterns are obtained by modal analysis. These
patterns lend support to our expectation.

In the next example, the RCS patterns are obtained for a cavity with a cylindrical
hub and 4 blades of constant thickness. At 10 GHz, the cavity is 2λ wide and 1.5λ
long while the hub is 1λ wide and 0.5λ long. The λ/20-thick blades have equal radial
and axial lengths of 0.5λ. Two models of the same cavity are used to obtained the
reflection matrices from which the RCS patterns are derived. In one model, the planes
of the blades are parallel to either the x- or the y-axis, while in the other model, they
are offset by 45° with respect to the axes as shown in Fig. 38. In both cases, forty cells
per wavelength is used to model the cavity geometry. Seventeen propagating modes
are used to obtain the reflection matrices. The purpose of using the two models is
to evaluate the effect of the stair-stepped blades in model 2 on the accuracy of the
reflection matrix.

Fig. 39 shows the RCS patterns for the two cavity models. The RCS patterns are
computed at φ = 0° for model 1 and at φ = 45° for model 2. Thus, the RCS for the
two angles of $\phi$ applied to the two cavity models should be close if the stair-stepped approximation is accurate. The difference between the two $\sigma_{\phi\theta}$ plots is less than 2 dB while difference between the two $\sigma_{\phi\phi}$ plots is negligible. Fig. 40 compares the RCS patterns of the two models computed in a plane between the blades. In this case, the difference is seen only in the plots.

The differences in the solutions for the two models can only be due to the stair-stepped approximation of the blades used in model 2. However, in contrast to two-dimensional problems where the numerical dispersion due to stair-stepping is seen only in the TE case, the numerical dispersion in 3D problems is, in general, present in both polarizations [43]. From the two previous figures, the errors introduced by the stair-stepping are small and tolerable.
Figure 39: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 4 straight blades obtained by the pulsed-basis FDTD method. $r_c = 3$ cm, $L_c = 4.5$ cm, $r_h = 1.5$ cm, $L_h = 1.5$ cm, $r_b = 1.5$ cm, $L_b = 1.5$ cm, $w_b = \lambda/20$. — : model 1 ($\phi = 0^\circ$); — : model 2 ($\phi = 45^\circ$).
Figure 40: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 4 straight blades obtained by the pulsed-basis FDTD method. \( r_c = 3 \text{ cm}, \ L_c = 4.5 \text{ cm}, \ r_h = 1.5 \text{ cm}, \ L_h = 1.5 \text{ cm}, \ r_b = 1.5 \text{ cm}, \ L_b = 1.5 \text{ cm}, \ w_b = \lambda/20. \) — : model 1 (\( \phi = 45^\circ \)); — — : model 2 (\( \phi = 0^\circ \)).
In the next example, the RCS patterns are obtained for a bigger cavity with a termination comprising a cylindrical hub and 16 straight blades. The cavity has a diameter of 12 cm (4\(\lambda\)) and a length of 6 cm (2\(\lambda\)). The hub diameter and length are 6 cm (2\(\lambda\)) and 3 cm (1\(\lambda\)) respectively. The 0.3 cm (\(\lambda/10\)) thick blades have equal radial and axial lengths of 3 cm (1\(\lambda\)). The dimensions given within the brackets are with respect to 10 GHz. The blades are equally distributed in \(\phi\) so that they are approximately 22.5° apart. Fig. 41 shows a cross-section of the stair-stepped blades using 20 cells per wavelength.

The RCS patterns at 8, 9 and 10 GHz for this cavity are shown in Figs. 42 to 44 respectively. At these frequencies, the number of propagating modes are 51, 62 and 77 respectively. In the FDTD program, all 77 modes of the highest frequency are

![Figure 41: Stair-stepped approximation of 16 straight blades with \(w_b = 0.3\) cm.](image)
used although only the reflected fields corresponding to the appropriate propagating modes are used to determine the reflection matrices at the lower frequencies.

The complexity of the interactions between the open end and the termination and between the blades makes it difficult to analyze the results in any reasonable fashion. Moreover, because of the lack of comparable data in the open literature, we are unable to check the accuracy of the solutions. Therefore, we can only provide qualitative checks and observations. As a check, we have computed the RCS patterns along the planes of the blades and have found that the respective co-polarized RCS patterns are almost, if not exactly, identical. The RCS patterns along these planes have to be identical due to the symmetry of the termination along any plane formed by two opposite blades. For the same reason, the RCS at $\theta = 0^\circ$ has to be the same for the $\theta - \theta$ and $\phi - \phi$ polarization, which is evident from the plots in Figs. 42 to 44.

Finally, we have included the RCS patterns of the same cavity without the blades to evaluate the effects of the blades on the cavity backscatter. Note that these RCS patterns are obtained by the BOR-MM and thus include the external scattering effects inherent in the method. In general, the addition of the 16 blades have increased the on-axis RCS and reduced the levels of the sidelobes. The increase in the on-axis RCS is due to the strong reflection of the incident field from the pair of blades that are aligned along the polarization of the incident field. The changes in the sidelobe levels is due to a combination of the complex field interactions between the blades and between the termination and the aperture. Therefore, for a complex termination, the field scattered by the termination is a combination of the radiation from the different
Figure 42: Co-polarized RCS patterns at 8 GHz for a circular cavity with a termination comprising a cylindrical hub and 16 straight blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 43: Co-polarized RCS patterns at 9 GHz for a circular cavity with a termination comprising a cylindrical hub and 16 straight blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 44: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 16 straight blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
modes, and the phase differences between the modes which determine the shape of the RCS patterns. Since these modes interact differently with the termination, it is generally very difficult to predict the locations of the peaks and nulls in the scattered patterns.

An additional blade is inserted into the previous cavity so that angle between consecutive blades is about 21.2° apart. The new termination is also symmetrical only about the vertical x-axis as shown in Fig. 45. The RCS patterns at 8, 9 and 10 GHz for this new cavity are shown in Figs. 46 to 48, respectively for the two principal planes, \( x = 0 \) and \( y = 0 \). At 8 GHz, the additional blade did not cause any significant change in the RCS patterns of the cavity (compare Figs. 42 and 46). There is also little difference in the patterns of the two principal planes for the present

![Figure 45: Stair-stepped approximation of 17 straight blades with \( w_b = 0.3 \text{ cm} \).](image-url)
cavity. However, the differences between the patterns of the two cavities and between the patterns in the two principal planes increase with frequency. At 10 GHz, the peak RCS is more than 5 dB down from the value in the previous cavity and the patterns in the two principal planes also exhibit greater differences. The change or lack thereof in the patterns from one cavity to another could be due to the existence of a greater number of modes at the higher frequency in the *sectorial* space between two blades. It is likely that the changes observed in the patterns at 10 GHz are due to these additional modes.

The lack of change in the principal plane patterns can be reconciled. Although there is no symmetry about the \( z = 0 \) plane (where \( \phi = 90^\circ \)), this plane is very close to the plane defined by \( \phi = 84.7^\circ \) where there is symmetry. Therefore, we would not expect a drastic change in the RCS patterns, particularly for angles close to boresight, in going from the \( \phi = 90^\circ \) plane to the \( \phi = 84.7^\circ \) plane. Far from boresight, the difference between the patterns can be small or large depending on the greater field interactions at wider angles of incidence. These views are supported by the results in Figs. 46 to 48.

To illustrate the power of the pulsed-basis FDTD method, we generated in a single run the reflection matrices at frequencies between 9.5 GHz and 10.5 GHz in increments of 0.1 GHz for the same cavity. Composite plots of the co-polarized RCS patterns over this band of frequencies are shown in Figs. 49. These patterns are computed at the \( x = 0 \) plane. To obtain similar results with frequency domain methods, the same problem will have to be solved repeatedly for each of the frequencies.
Figure 46: Co-polarized RCS patterns at 8 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 straight blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 47: Co-polarized RCS patterns at 9 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 *straight* blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6 \text{ cm}, r_h = L_h = 3 \text{ cm}, r_b = L_b = 3 \text{ cm}, w_b = 0.3 \text{ cm}$. 
Figure 48: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 straight blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 49: Mesh plot of the co-polarized RCS patterns for 9.5 to 10.5 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 straight blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
The 17 straight blades in the previous example are replaced with 17 sectorial blades. The 3.6°-wide blades are chosen so that the total cross-sectional area of these blades in the $x - y$ plane, represented by the 658 PEC cells shown in Fig. 50, is about the same as the cross-sectional area of the 17 straight blades in the previous example. Therefore, the main difference between the cavity with the sectorial blades and the one with the straight blades is in the locations of some of the PEC cells used to model these blades.

The RCS patterns at 8, 9 and 10 GHz for the new cavity are again plotted in the two principal planes as shown in Figs. 51 to 53 respectively. The RCS patterns at 8 and 9 GHz for the cavity with the sectorial blades and the one with the straight blades did not show significant differences. However, at 10 GHz, the RCS patterns of the two cavities are very different. The patterns in the two principal planes are also quite dissimilar. Note, however, that at $\theta = 0°$,

$$\sigma_{\theta\theta}|_{\theta=0} \text{ plane } = \sigma_{\phi\phi}|_{\phi=0} \text{ plane}, \text{ and}$$

$$\sigma_{\theta\theta}|_{\phi=0} \text{ plane } = \sigma_{\phi\phi}|_{\phi=0} \text{ plane}$$

as required.

The last three sets of results have shown that the scattering patterns of cavities with complex terminations depend on the frequency of operation. Moreover, it can be quite sensitive to small changes in the termination section.
Figure 50: Stair-stepped approximation of 17 *sectorial* blades with $w_b = 3.6^\circ$. 
Figure 51: Co-polarized RCS patterns at 8 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 sectorial blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $\omega_b = 3.6^\circ$. 
Figure 52: Co-polarized RCS patterns at 9 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 sectorial blades obtained by the pulsed-basis FDTD method. \( r_c = L_c = 6 \text{ cm}, r_h = L_h = 3 \text{ cm}, r_b = L_b = 3 \text{ cm}, w_b = 3.6^\circ \).
Figure 53: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 sectorial blades obtained by the pulsed-basis FDTD method. $r_c = L_c = 6$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 3.6^\circ$. 
5.1.4 Conical hub and blades termination

The results in this subsection is for a cavity with a conical hub termination. Attached to the hub are several straight blades as depicted in Fig. 54. The cavity is 12 cm (4\(\lambda\)) wide and 9 cm (3\(\lambda\)) long. The 6 cm (2\(\lambda\)) long hub has a base diameter of 6 cm (2\(\lambda\)). The blade is 0.3 cm (\(\lambda/10\)) thick and 3 cm (1\(\lambda\)) long. In contrast to the previous cases where the blades are as long as the hub, the 3 cm long (1\(\lambda\)) blades in this case are shorter than the length of the conical hub.

Figs. 55 to 57 show the RCS patterns at 8, 9 and 10 GHz respectively, for two cavities. The first cavity has 4 blades and the second has 16 blades. Each of the 4 blades in the first cavity lies either in the \(x = 0\) plane or in the \(y = 0\) plane. The 16 blades in the second cavity are spaced at multiples of 22.5° apart, with one blade at \(\phi = 0°\). The BOR-MM solutions for a cavity with just the conical hub are also plotted in these figures. It is reasonable to assume that the reflection matrix has contributions

Figure 54: A circular cavity with a termination comprising a conical hub and some blades.
arising from the presence of the blades and their interactions among themselves. As would also be expected, the contributions from the interactions between the 4 blades and between the 16 blades are different resulting in the different RCS patterns.

In the final example, the RCS patterns are obtained for a larger cavity with a termination comprising a conical hub and 12 straight blades. This cavity has a diameter of 24 cm (8λ) and a length of 9 cm (3λ). The diameter and length of the hub are both 6 cm (2λ). The blades are 0.3 cm (λ/10) thick and 3 cm (1λ) long. The blades are equally distributed in φ so that they are about 30° apart. Fig. 58 shows a quadrant of the stair-stepped approximation of the 12 blades using 20 cells per wavelength.

The RCS patterns at 8, 9 and 10 GHz for this cavity are shown in Fig. 59. These patterns are computed at the $x = 0$ plane. It is interesting to note that the structure of the patterns for the respective polarizations are quite similar at the three different frequencies for angles close to the boresight. As the frequency increases, the main beam and the first two sidelobes become narrower. At the same time, more sidelobes appear within the same angular range.
Figure 55: Co-polarized RCS patterns at 8 GHz for a circular cavity with a termination comprising a conical hub with 4 or 16 straight fan blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_s = 6$ cm, $L_s = 9$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 56: Co-polarized RCS patterns at 9 GHz for a circular cavity with a termination comprising a conical hub with 4 or 16 \textit{straight} fan blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_c = 6 \text{ cm}$, $L_c = 9 \text{ cm}$, $r_h = 3 \text{ cm}$, $L_h = 6 \text{ cm}$, $r_b = L_b = 3 \text{ cm}$, $w_b = 0.3 \text{ cm}$.
Figure 57: Copolarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub with 4 or 16 straight fan blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_c = 6$ cm, $L_c = 9$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 58: A quadrant of the stair-stepped approximation of the 12 straight blades of thickness 0.3 cm. The dashed lines indicate the circumferential base of the conical hub.
Figure 59: Co-polarized RCS patterns for a circular cavity with a termination comprising a conical hub with 12 straight fan blades obtained by the pulsed-basis FDTD method with $\phi = 0^\circ$. $r_e = 12$ cm, $L_e = 9$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = 9$ cm, $L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 59: Continued.

(c) 10 GHz
5.2 Properties of $[S_T]$

As demonstrated in the previous section, the pulsed-basis FDTD method is capable of generating multiple reflection matrices for different frequencies in a single run. The storage of the reflection matrices becomes an important issue when reflection matrices are required at many frequencies for large termination sections. Therefore, we will examine the properties of $[S_T]$ that are useful from the viewpoint of minimizing the storage requirements. We will consider reflection matrices of rotationally symmetric terminations and of non-rotationally symmetric terminations.

Fig. 60 presents in a graphical form the reflection matrix for the termination section of Fig. 28. The area of each square at the $(i,j)$ location is equal to the power associated with the matrix element $S_{Tij}$. Since the power of each input mode has been normalized to 1, $\sum_j |S_{Tij}|^2 = 1$ and $|S_{Tij}|^2 \leq 1$ for a lossless termination section. Note that only elements whose powers are greater than or equal to a power threshold of $(0.05)^2$ are depicted in Fig. 60. In the original matrix, TM modes of increasing order are followed by TE modes of increasing order. The partitioning of the TM and the TE modes is indicated by the dotted lines in Fig. 60(a). In the reordered matrix, the modes are arranged in increasing order irrespective of the polarization, with the TM modes preceding the TE modes within each order.

From Fig. 60(b), we can make several observations about the reflection matrix. Each of the individual "blocks" along the diagonal of the matrix correspond to modes of the same order. This pattern suggests that modes of the same order (or with the same $\phi$ variation) are strongly coupled. At the same time, there is very little or
no coupling between modes of different orders. The difference in the coupling is a consequence of the rotational symmetry of the cylindrical hub termination. Now, the weak coupling between modes of different orders results in a very sparse matrix. As will be seen shortly, the sparsity of the matrix can be used to advantage from the storage point of view.

From the generalized scattering matrix concept [48], the termination reflection matrix for a reciprocal structure is symmetrical (i.e., $S_{ij} = S_{ji}$) when the powers of the input modes are appropriately normalized. However, due to numerical inaccuracy, the matrix in Fig. 60 does not exhibit perfect symmetry. Nevertheless, if we assume symmetry, then the minimum number of matrix elements to save ($N_s$) is given by

$$N_s = \frac{N_m + N}{2}, \quad (5.1)$$

where $N_m$ is the number of elements which satisfy $|S_{ij}| \geq 0.05$, and $N$ is the dimension of the matrix. For a full symmetric matrix, the minimum number ($N_f$) is

$$N_f = \frac{N^2 + N}{2}, \quad (5.2)$$

so that

$$R = \frac{N_s}{N_f} = \frac{N_m + N}{N^2 + N}. \quad (5.3)$$

For the matrix in Fig. 60, $N_m = 318$ and $N = 77$, so that $N_s = 198$ and $R = 6.6\%$. Therefore, we need to save only a very small percentage of the elements of $[S_t]$. As the size of the cavity increases, $N_s$ is expected to increase although $R$ may actually
decrease. For example, for the reflection matrix depicted in Fig. 61 for the termination section of Fig. 30, $N_m = 2075$ and $N = 313$ so that $N_s = 1194$ and $R = 2.4\%$.

In the two previous examples, the rotational symmetry of the terminations have resulted in sparse reflection matrices. However, realistic terminations in engine cavities are not rotationally symmetric. Fig. 62 depicts the reflection matrix for such a termination comprising a cylindrical hub and 4 blades. The matrix is symmetrical and sparse although there are significant coupling between modes of different orders. In this case, $N_m = 53$ and $N = 17$ so that $N_s = 35$ and $R = 22.9\%$. For another example, Fig. 63 depicts the matrix for the termination section of Fig. 59(c) which consist of a conical hub and 12 blades. In this case, $N_m = 4966$ and $R = 5.4\%$ compared to 2075 and 2.4\% respectively for the matrix depicted in Fig. 61. Except for the different terminations, the two cavities for the respective matrices in Figs. 61 and 63 have the same diameters.

As suggested in the preceding results, we only need to store a very small percentage of the matrix elements even for a complex termination consisting of a hub and several blades. To verify the accuracy of the reduced matrix which consists only of the stored elements, we computed the RCS patterns of the previous cavity with the original matrix and with the reduced matrix. The two sets of patterns are shown in Fig. 64. Notice that the differences between the two sets of patterns are very small. If necessary, the differences can be minimized by adjusting the power threshold used to select the elements of the reduced matrix.
In summary, we can expect to store more matrix elements for a non-rotationally symmetric termination than for a rotationally symmetric termination. Even so, the reflection matrix for the former is still expected to be sparse, resulting in tremendous savings in storage requirements. We have also shown that the RCS patterns obtained with the reduced matrix is close to the patterns obtained with the original reflection matrix for a power threshold of $(0.05)^2$. 
Figure 60: $[S_T]$ for the termination section of Fig. 28.
Figure 61: $[S_T]$ for the termination section of Fig. 30.
Figure 62: \([S_R]\) for the termination section of Fig. 39.
Figure 63: $[S_T]$ for the termination section of Fig. 59.
Figure 64: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub with 12 straight fan blades using the original matrix (obtained by the pulsed-basis FDTD method) and its reduced version. $\phi = 0^\circ$, $r_c = 12$ cm, $L_c = 9$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = 9$ cm, $L_b = 3$ cm, $w_b = 0.3$ cm.
5.3 Accuracy of the IPO & the GRE method

The accuracy of the IPO and the GRE methods are compared pictorially via the cross-sectional cavity fields obtained by these methods and modal analysis. The cross-sectional fields are computed with the modal coefficients, $A^+_n$, obtained by the respective methods. The fields are computed for two cavity sizes and at various depths for plane wave fields incident at the aperture at $\theta = 0^\circ$ and at $\theta = 30^\circ$ with $\phi = 0^\circ$ in all cases.

Figs. 65 and 66 show the cross-sectional fields obtained by modal analysis, the IPO and the GRE methods respectively, for a circular cavity with a diameter of $4\lambda$ and at a depth of $4\lambda$. Fig. 65 shows the $E_x$-field for a $\theta$-polarized field incident at the aperture at $\theta = 0^\circ$ while Fig. 66 shows $E_y$-field for a $\phi$-polarized field incident at the aperture at $\theta = 30^\circ$. Note that at $\theta = 0^\circ$, the magnitude of the field at a cross-section within the cavity is not constant as would have been predicted by the GO-based SBR method. In addition to the GO field intercepted by the aperture, there are the rim diffracted fields which can contribute significantly to the total field in a cavity. Both the IPO and the GRE methods inherently include these rim diffracted effects through the integration of the equivalent currents over the aperture of the cavity. Therefore, their field solutions are supposed to be comparable to the modal field solutions. Indeed, there is negligible difference between the IPO solutions and the modal solutions. However, this is not the case with the GRE solutions although they do show some resemblance to the modal solutions. There are several causes for this apparent inaccuracy. The most serious of these causes is ray caustic. Ray
caustics are particularly prevalent when rays are reflected off curved surfaces. The accuracy of the ray field is compromised when the caustic occurs before and close to the reflection point. Another factor in the accuracy of the GRE solutions is the approach used to find the cross-sectional field. In Section 4.5, a scheme was devised to expand the ray fields at the cross-section efficiently into a set of modal fields. The efficiency of this scheme comes at the expense of some loss in the accuracy of the field solutions. Finally, for the fields solutions shown in Figs. 65 and 66, rays were only launched within a cone angle of 65°. At θ = 30°, this cone angle may not be adequate. The combination of these factors result in a less accurate field solution for the GRE method. In contrast, the IPO method does not have caustic problems. The one factor that will however affect the accuracy of the IPO solutions is the size of the facet used since it is assumed that the field and current over each facet are constant. This approximation improves when smaller facets are used (at the expense of computation time). In Figs. 65 and 66, square facets of sides λ/3 were used. The good agreement between the IPO and the modal solutions means that the facets were not too large.

Figs. 67 and 68 show the same set of fields for a larger cavity with a diameter of 8λ and at a depth of 8λ. Both the IPO and the GRE solutions have the major features shown in the modal solutions although the former solutions are in better agreement with the modal solutions. However, compared to the 4λ-wide cavity, the accuracy of the GRE solutions in the larger cavity has improved as evident by the agreement between its peak field values and those of the modal solutions.
Figs. 69 and 70 show the fields obtained at a depth of 20\(\lambda\) in the same 8\(\lambda\)-wide cavity. At this depth, the IPO solutions are still reasonably good although \(E_x\) in Figs. 69(b) has a peak about the cavity axis when the modal solution shows a trough about the axis. Note that larger facets of sides \(\lambda/2\) were used to minimize the computation time. Therefore, it is probable that the larger facets cause the degradation in the accuracy of the field solutions. In addition, the IPO method used 5 iterations to obtain a "convergent" solution compared to only 2 iterations in the earlier examples. The increased in the number of iterations is related to the increased number of important reflections in a longer cavity [19]. It also means that more iterations might be required to get a more accurate solution. For the GRE solutions, they are somewhat worse off than before, which is to be expected when rays are traced deeper into a cavity with curved walls.

The results in this section have demonstrated that the IPO approach is more accurate than the GRE method when it comes to finding the cross-sectional fields inside a long straight duct. As the IPO method has only been introduced very recently, its accuracy in ducts with bends has yet to be investigated. On the other hand, the GRE method has been shown to provide sufficiently accurate solutions in ducts with bends and in very large cavities [10, 12]. Moreover, its accuracy improves with the size of the cavity. Therefore, it is envisaged that the IPO method will be used in small, straight ducts, while the GRE method will be used in larger, bent ducts.
Figure 65: $E_z$ at a depth of $4\lambda$ in a cavity of diameter $4\lambda$ for a $\theta$-polarized plane wave field incident at the aperture at $\theta = \phi = 0^\circ$. 
Figure 65: Continued.

(c) GRE method
Figure 66: $E_y$ at a depth of $4\lambda$ in a cavity of diameter $4\lambda$ for a $\phi$-polarized plane wave field incident at the aperture at $\theta = 30^\circ$ and $\phi = 0^\circ$. 
Figure 66: Continued.

(c) GRE method
Figure 67: $E_z$ at a depth of $8\lambda$ in a cavity of diameter $8\lambda$ for a $\theta$-polarized plane wave field incident at the aperture at $\theta = \phi = 0^\circ$.
Figure 67: Continued.

(c) GRE method
Figure 68: $E_y$ at a depth of $8\lambda$ in a cavity of diameter $8\lambda$ for a $\phi$-polarized plane wave field incident at the aperture at $\theta = 30^\circ$ and $\phi = 0^\circ$. 
Figure 68: Continued.

(c) GRE method
Figure 69: $E_z$ at a depth of $8\lambda$ in a cavity of diameter $20\lambda$ for a $\theta$-polarized plane wave field incident at the aperture at $\theta = \phi = 0^\circ$. 

(a) modal analysis

(b) IPO method
Figure 69: Continued.

(c) GRE method
Figure 70: $E_y$ at a depth of $8\lambda$ in a cavity of diameter $20\lambda$ for a $\phi$-polarized plane wave field incident at the aperture at $\theta = 30^\circ$ and $\phi = 0^\circ$. 

(a) modal analysis 

(b) IPO method
Figure 70: Continued.

(c) GRE method
5.4 Results from hybrid methods

In this section, we present the RCS patterns obtained by three hybrid methods for cavities made up of a front section and a termination section. These hybrid methods are the modal-FDTD, the PO-FDTD and the GRE-FDTD methods. The first method in each of these hybrid methods is used to determine the modal coefficients, $A_n^+$, of the field incident at the interface of the two cavity sections. The source of this field is an external plane wave field incident at the aperture of the front section. FDTD is used to derive the modal reflection matrix, $[S_R]$, for the termination section. The modal coefficients and the reflection matrix are substituted into equations (4.44) and (4.41) to compute the RCS for the entire cavity. Note that the reflection matrices used in the following examples are the same matrices obtained for the examples in the earlier sections. For cavities that are rotationally symmetric, the BOR-MM solutions are again used as reference solutions. Note, however, that the BOR-MM solutions include the external scattering effects as before. Owing to the lack of comparable data for the other cavities with hub-blades termination, the comparison for these cavities will be made among the hybrid methods themselves.

RCS patterns are presented for two cavity sizes, the $4\lambda$-wide and the $8\lambda$-wide circular cavities. Since the IPO and the GRE methods are frequency-domain methods, and $A_n^+$ are (currently) obtained for one frequency at a time, only the RCS patterns at 10 GHz are presented. Results for the smaller cavities will be present first. Each of the small cavities has a $4\lambda$-long ($L_f$) front section appended to some termination section. Fig. 71 shows the co-polarized RCS patterns for a cavity with a cylindrical
hub obtained by the three hybrid methods and the BOR-MM. The cavity is made up of a 4\(\lambda\)-long front section and the 1.5\(\lambda\)-long \((L_t)\) termination section of Fig. 28. The modal-FDTD and the PO-FDTD solutions show excellent agreement with the reference BOR-MM solutions. The differences at wide angles are probably the result of the external effects present in the reference solutions. Compared to the other two hybrid methods, the GRE-FDTD solutions are less accurate. From the discussion in the preceding section, the poorer accuracy is expected.

Fig. 72 presents the RCS patterns for a cavity with a conical hub termination. The 3\(\lambda\)-long termination section is similar to the one used in Fig. 34. Once again, the solutions of the modal-FDTD and PO-FDTD methods are in better agreement with the reference solutions. The good agreement suggests once again that the termination reflection matrix obtained with the stair-stepped cone model is adequately accurate.

Figs. 73 to 77 display the RCS patterns for cavities with various hub-blades terminations. In Figs. 73 to 75, the 2\(\lambda\)-long termination section consists of a 2\(\lambda\)-wide by 1\(\lambda\)-long cylindrical hub with a number of blades attached to the hub. For the cavities of Figs. 73 and 74, there are 16 and 17 \emph{straight} blades respectively. For the cavity of Fig. 75, there are 17 \emph{sectorial} blades. Note that these hub-blades terminations are the same terminations used in Figs. 44, 48 and 53 respectively. For the 3\(\lambda\)-long termination sections of Figs. 76 and 77, they have a conical hub with 4 and 16 \emph{straight} blades respectively. The hub is 2\(\lambda\) long and has a base diameter of equal length. Note that these termination sections are similar to the ones used in Figs. 57.
Figure 71: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the hybrid modal-FDTD, PO-FDTD, GRE-FDTD methods and the BOR-MM. $d_c = 12$ cm, $L_t = 4.5$ cm, $L_f = 12$ cm, $d_h = 6$ cm, $L_h = 1.5$ cm.
Figure 72: Co-polarized RCS patterns at 10 GHz for a circular cavity with a conical hub termination obtained by the hybrid modal-FDTD, PO-FDTD, GRE-FDTD methods and the BOR-MM. $d_c = 12$ cm, $L_t = 9$ cm, $L_f = 12$ cm, $d_h = 6$ cm, $L_h = 6$ cm.
Figure 73: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 16 straight blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods with $\phi = 0^\circ$. $r_c = L_t = 6$ cm, $L_f = 12$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 74: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 straight blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods with $\phi = 90^\circ$. $r_c = L_t = 6$ cm, $L_f = 12$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 75: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a cylindrical hub and 17 sectorial blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods with $\phi = 90^\circ$. $r_c = L_t = 6$ cm, $L_f = 12$ cm, $r_h = L_h = 3$ cm, $r_b = L_b = 3$ cm, $\omega_b = 3.6^\circ$. 
Figure 76: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub and 4 straight fan blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods with $\phi = 0^\circ$. $r_c = 6$ cm, $L_t = 9$ cm, $L_f = 12$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 77: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub and 16 straight fan blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods with $\phi = 0^\circ$. $r_e = 6$ cm, $L_t = 9$ cm, $L_f = 12$ cm, $r_h = 3$ cm, $L_h = 6$ cm, $r_b = L_b = 3$ cm, $w_b = 0.3$ cm.
In nearly all the cases, the PO-FDTD solutions track the modal-FDTD solutions. This close correspondence is expected since we have shown (in the preceding section) that the IPO method is able to obtain field solutions that are close to the modal solutions. The GRE-FDTD solutions, on the other hand, agree with the solutions of the other two hybrid methods for only a narrow band of angles, anywhere from \(|\theta| \leq 20^\circ\) to \(|\theta| \leq 40^\circ\). An interesting observation from the GRE-FDTD solutions is that the \(\sigma_{\phi}\) solutions always seem to agree better with the reference solutions than the \(\sigma_{\theta}\) solutions would. They also agree with the reference solutions over a wider range of incidence angles. This strange occurrence is also observed in the earlier figures. For some reason, the \(\phi\)-polarized field solutions are always more accurate than the \(\theta\)-polarized field solutions in the GRE method.

Figs. 78 and 79 present the RCS patterns for the larger (8\(\lambda\)-wide) cavity with a cylindrical hub termination. The termination section is 2\(\lambda\) long while the hub is 4\(\lambda\) wide and 1\(\lambda\) long (similar to the one used in Fig. 30). The front sections are 8\(\lambda\) and 20\(\lambda\) long in Figs. 78 and 79 respectively. As expected, the GRE-FDTD solutions are in better agreement with the solutions of the other two hybrid methods than they were in the smaller cavities. These figures also illustrate the change in the RCS patterns as a result of the redistribution of the energy of the various modes when the length of the cavity is altered.

Figs. 80 and 81 show the RCS patterns for the 8\(\lambda\)-wide cavity with a termination comprising a conical hub and 12 straight blades. The termination section is 3\(\lambda\) long and is similar to the one used in Fig. 59. The front sections are 8\(\lambda\) and 20\(\lambda\) long in
Figs. 80 and 81 respectively. Once again, the PO-FDTD solutions agree very well with the modal-FDTD solutions. On the other hand, the GRE-FDTD solutions agree with the modal-FDTD solutions only within ±15° to ±30° of boresight. As might be expected, the GRE-FDTD solutions for the longer cavity are inferior to those for the shorter cavity.

To illustrate the versatility of the PO method and the GRE method in analyzing non-canonical waveguide sections, we connect a super-elliptic front section to the termination section of Fig. 59. This super-elliptic cavity, shown in Fig. 82, has an elliptical front end and a circular back end. The length of the elliptical-to-circular transition \( (L_f) \) is 40 cm while the length of the circular termination section \( (L_t) \) is 9 cm. Fig. 83 shows the RCS patterns of this super-elliptical cavity computed along the horizontal plane. There is good agreement between the solutions of the PO-FDTD and the GRE-FDTD methods for \(|\theta| \leq 25°\). Beyond this range of angles, it is debatable as to which set of solutions is "correct". Further numerical analysis has to be carried out to ascertain the "correct" solution.

In summary, we have shown that the three hybrid methods, namely modal-FDTD, PO-FDTD and GRE-FDTD, are efficient for determining the scattering from long and relatively large cavities. Since the decoupling of the methods permits independent analysis of the front and termination sections, these hybrid methods are very useful for analyzing changes in either sections without having to solve the entire cavity problem again.
Figure 78: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods. $d_c = 24$ cm, $L_l = 6$ cm, $L_f = 24$ cm, $d_h = 12$ cm, $L_h = 3$ cm.
Figure 79: Co-polarized RCS patterns at 10 GHz for a circular cavity with a cylindrical hub termination obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods. $d_c = 24$ cm, $L_i = 6$ cm, $L_f = 60$ cm, $d_h = 12$ cm, $L_h = 3$ cm.
Figure 80: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub and 12 straight blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods. $d_c = 24$ cm, $L_t = 9$ cm, $L_f = 24$ cm, $d_h = 6$ cm, $L_h = 6$ cm, $r_b = 9$ cm, $L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 81: Co-polarized RCS patterns at 10 GHz for a circular cavity with a termination comprising a conical hub and 12 straight blades obtained by the hybrid modal-FDTD, PO-FDTD and GRE-FDTD methods. $d_c = 24$ cm, $L_t = 9$ cm, $L_f = 60$ cm, $d_h = 6$ cm, $L_h = 6$ cm, $r_b = 9$ cm, $L_b = 3$ cm, $w_b = 0.3$ cm.
Figure 82: Different views of a super-elliptic cavity.
Figure 83: Co-polarized RCS patterns at 10 GHz for a super-elliptic cavity with a termination comprising a conical hub and 12 straight blades obtained by the hybrid PO-FDTD and GRE-FDTD methods. $d_c = 24$ cm, $L_t = 9$ cm, $L_f = 40$ cm, $d_h = 6$ cm, $L_h = 6$ cm, $r_b = 9$ cm, $L_b = 3$ cm, $w_b = 0.3$ cm.
5.5 Comparison of CPU time on the Cray and the DELTA

This section presents a comparison of the run times achieved by the serial vectorized FDTD code running on the Cray Y-MP8/864 and the parallel FDTD code running on the Touchstone DELTA machine. Both codes are comparably vectorized. Since the bulk of the time is taken up by the updating of the field values and the determination of the reflection coefficients, the overheads associated with input and output have been excluded from the timing information obtained.

Tables 6 and 7 summarize the run times for both machines for various cavities. The dimensions of the cavities and the FDTD cell size are given in terms of the wavelength at 10 GHz. The number of \( \Delta t \)'s in both tables indicates the number of FDTD time steps used to obtain a single column of the reflection matrix corresponding to one input waveguide mode. The number of modes in Table 6 indicates the number of modes used to obtain the reflection matrix. For cavities B, C and E, only the even modes were used because of rotational symmetry in the cavity. The Cray run time is therefore the time taken to run the indicated number of modes.

The fourth column in Table 7 indicates the number of processors used on the DELTA machine to run the same number of waveguide modes indicated in Table 6. These processors are divided into groups of equal sizes (processors/group in Table 7). The group size is chosen so that the memory requirements of the program does not exceed the 12.5 Mbytes of memory available on each processor. Each group is assigned a certain number of waveguide modes to work on (no. of modes in Table 7). Different groups are assigned different waveguide modes. The processors within each group
work together (as in Fig. 24) to find the columns of reflection coefficients for their assigned modes. Therefore, all the groups are working at the same time but on different waveguide modes. Once all the groups are done, the reflection matrix is assembled with the columns of coefficients obtained by the various groups. Note that the number of modes indicated in the table is the maximum number of modes assigned to any one group. A group may have less than this number of modes since it is not always possible to assign an equal number of modes to each group. Therefore, there can be groups that will finish their assignments long before the rest are done. So, the CPU time shown in Table 7 indicates the average time of the processors in the “slowest” group. In most cases, more processors can be assigned to each group to reduce the CPU time. In addition, the program can be better optimized by dividing the computation load more evenly among the processors within each group.

The CPU times for cavity A are 350 seconds and 680 seconds for the Cray and the DELTA respectively. As noted earlier, the CPU time for DELTA can be reduced by increasing the number of processors per group and assigning only one mode (even or odd) to each group. For cavities B to D, their CPU times on the DELTA machine are between 2.5 to 8.9 times less than the corresponding runtimes on the Cray. Cavities E and F have been included in Table 7 to show the size of problem that can be run on the DELTA machine. Their CPU times on the DELTA machine are 2,900 and 10,000 seconds respectively. These times are about 22 times less than the corresponding CPU times on the Cray. Note that the Cray runtimes for both cavities are estimates based on the time taken to run the FDTD code for a fraction of the number of time
steps (between 100 to 200) for only one of the waveguides modes.

Table 6: CPU times of Cray code for various cavities.

<table>
<thead>
<tr>
<th>Cav.</th>
<th>Fig.</th>
<th>Cavity size (W x L)</th>
<th>Cell size</th>
<th>No. of field variables (millions)</th>
<th>No. of ( \Delta t )'s</th>
<th>No. of modes</th>
<th>Cray CPU (seconds)</th>
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<tr>
<td>A</td>
<td>40</td>
<td>( 1.350\lambda \times 1.325\lambda )</td>
<td>( \frac{1}{40} )</td>
<td>0.77</td>
<td>1,000</td>
<td>10</td>
<td>350</td>
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<td>B</td>
<td>31</td>
<td>( 4.0\lambda \times 1.5\lambda )</td>
<td>( \frac{1}{20} )</td>
<td>0.96</td>
<td>1,000</td>
<td>42</td>
<td>2,100</td>
</tr>
<tr>
<td>C</td>
<td>37</td>
<td>( 4\lambda \times 3\lambda )</td>
<td>( \frac{1}{20} )</td>
<td>1.89</td>
<td>2,000</td>
<td>42</td>
<td>6,930</td>
</tr>
<tr>
<td>D</td>
<td>55</td>
<td>( 4\lambda \times 2\lambda )</td>
<td>( \frac{1}{20} )</td>
<td>1.27</td>
<td>1,500</td>
<td>77</td>
<td>6,900</td>
</tr>
<tr>
<td>E</td>
<td>33</td>
<td>( 8\lambda \times 2\lambda )</td>
<td>( \frac{1}{20} )</td>
<td>5.01</td>
<td>1,000</td>
<td>164</td>
<td>64,000*</td>
</tr>
<tr>
<td>F</td>
<td>62</td>
<td>( 8\lambda \times 3\lambda )</td>
<td>( \frac{1}{20} )</td>
<td>7.45</td>
<td>1,440</td>
<td>313</td>
<td>220,000*</td>
</tr>
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* estimated.

Table 7: CPU times of DELTA code for various cavities.

<table>
<thead>
<tr>
<th>Cav.</th>
<th>Fig.</th>
<th>Cell size</th>
<th>No. of ( \Delta t )'s</th>
<th>No. of processors</th>
<th>Processors/group</th>
<th>Max. no. of modes/group</th>
<th>DELTA CPUs</th>
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<td>12</td>
<td>2</td>
<td>2</td>
<td>680</td>
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<tr>
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<td>42</td>
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<td>2</td>
<td>830</td>
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<tr>
<td>C1</td>
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<td>176</td>
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<td>2</td>
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<tr>
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* There are 6.23 million field variables for this case.
CHAPTER VI

CONCLUSIONS

In this dissertation, the use of the FDTD method in combination with three separate methods for analyzing the EM scattering for open-ended waveguide cavities has been described. The main contribution of this work is in the application of the FDTD method for determining the EM scattering from complex terminations of relatively large cavities. In this regard, the method has demonstrated its accuracy as well as its efficiency in generating broadband solutions. The hybridization of the FDTD method with modal analysis, PO and GRE were included to illustrate the utility of the method in solving general cavity problems that would have otherwise been difficult to solve.

The accuracy and the efficiency of two ray methods, namely SBR and GRE, were examined in Chapter II. GRE was shown to be more accurate and efficient than SBR. So, it was used in the evaluation of the hybrid ray-FDTD method as a suitable approach for finding the EM scattering from complex terminations in cavities.

In Chapter III, the 2D hybrid GRE-FDTD was shown to produce accurate solutions which were validated with solutions from established methods. The pulsed-basis approach to be adopted for the FDTD method in 3D was also tested in a 2D FDTD code and was shown to produce essentially the same results as the 2D sinusoidal FDTD approach.
The 3D pulsed-basis FDTD algorithm was described in Chapter IV. Although stair-stepped approximations were used to model the cavities and their various terminations, the results in Chapter V showed that the algorithm was able to produce sufficiently accurate reflection matrices. These matrices were also combined with the fields solutions of the separate methods of modal analysis, PO and GRE in the reciprocity integral to obtain the RCS patterns of longer cavities. For a given reflection matrix, the modal-FDTD method is the most accurate. This method is, however, limited to separable waveguide sections. The PO-FDTD and the GRE-FDTD methods are more useful for non-canonical waveguide cavities. For the examples shown in this dissertation, the PO-FDTD method was more accurate than the GRE-FDTD method. Finally, it was demonstrated that large cavities can be efficiently analyzed on the DELTA parallel machine by the pulsed-basis FDTD method.

More work can be done to improve the efficiency and accuracy of the pulsed-basis FDTD method. Since it was not the intent of this work to produce the fastest running code on either the Cray or the DELTA machines, there is obviously room for improving the efficiency of the code. Besides employing speed enhancement techniques, it might be worthwhile to consider the hybrid Yee-scalar wave approach of [58]. In this approach, divergence-free regions are computed with a set of equations that uses less arithmetics (additions and multiplications) than in the standard FDTD equations.

The efficiency and accuracy of the FDTD code also depend on the absorbing boundary condition (ABC) used. The fourth-order Higdon's ABC was designed for very low reflections (0.01%) between 0° and 60° to ensure that the ABC was not a
factor in the overall accuracy of the FDTD method. Nonetheless, it produced greater reflections at wider angles. Although higher order ABCs could be used, they would only increase the computation time of the code. Since Higdon’s ABC provides great flexibility in the design of an “optimum” ABC, it is proposed that wider angles of absorption be achieved with less stringent reflection/absorption requirements. For example, Fig. 84 shows that for the same order of ABC, 0.1% and 1% reflections can be achieved for angles up to 72° and 84° respectively. The former should be sufficient for many applications. If necessary, the superabsorption approach of [59] could be applied to Higdon’s ABC to further reduce the reflections.

![Graph showing reflection coefficient for the 4th-order Higdon’s absorbing boundary condition](image)

**Figure 84:** Reflection coefficient for the 4th-order Higdon’s absorbing boundary condition with $\Delta z = 0.15$ cm, $\delta = 0.5$, $f = 10$ GHz. $- -$ : $\alpha_1 = 16.8^\circ$, $\alpha_2 = 46.9^\circ$, $\alpha_3 = 64.4^\circ$ and $\alpha_4 = 72.0^\circ$; $- -$ : $\alpha_1 = 25.7^\circ$, $\alpha_2 = 61.9^\circ$, $\alpha_3 = 78.1^\circ$ and $\alpha_4 = 83.8^\circ$. 
The use of stair-stepped approximations in the FDTD method put a limit on the realistic modeling of curved boundaries/surfaces. These approximations also introduced some measure of errors, although these errors did not seem to have an adverse effect on the final solutions. Although these errors could be minimized with smaller computational cells, the accompanying increase in the computational load might be excessive. A compromise solution would be to incorporate the smaller cells only in the vicinity of curved surfaces [60]. Although these approximations may be adequate in some simple cases, it is doubtful whether they are adequate for complex structures. To model really complex terminations like the stator and/or rotor with angled blades, conformal FDTD (CFDTD) methods [61, 62, 63] could be used. As would be expected, CFDTD methods come with increase code complexity, memory requirements and computation time. Alternatively, time-domain finite element (TDFE) methods could be used. Many TDFE methods have been proposed [25, 64, 65] but few have seen wide applications, particularly in scattering analysis of open-ended waveguides. As the TDFE methods are based on the same principles of finite-element methods, they are also more flexible and less geometry dependent than some of the above CFDTD methods. Therefore, it might be preferably to look into TDFE methods.

As with any methods, it is vital that the solutions of the FDTD method be verified against the solutions of other methods. However, the latter solutions may not always be available or accessible. Therefore, the FDTD solutions should at least be compared with measured results.
As for the GRE method, its accuracy can perhaps be improved with the use of smaller subapertures to adequately sample the equivalent currents of small apertures. Alternatively, polygonal subapertures that conform to the aperture can be used. The former requires more ray tracing while the latter increases the complexity of the field computation. As a compromise, the subaperture gridding scheme of [66] could be used. To improve the efficiency of the GRE method, shaped beams [30, 31] can also be used.

The improved accuracies due to any of the above methods will be nullified if no improvement is made in the ray-to-mode conversion scheme. Although the scheme devised in Chapter 4 reduced the computation time of the conversion, it was at the expense of accuracy. Thus, for better accuracy, the slower but more accurate direct ray-to-mode conversion should be used albeit with a slight modification. The ray-to-mode conversion could be done selectively by using only modes whose angles are close to the propagation angle of a ray. This modification is similar to the selective modal scheme used in [6] to compute the RCS of large open-ended waveguides. With this approach, the ray tube size can still be as large as $(\lambda/2)^2$ for the regular GRE ray tubes.

As the IPO method is relatively new, its efficiency and accuracy in larger and curved cavities should be investigated. For deep or curved cavities, the method could be applied in succession to consecutive sections of the cavities to reduce the overall number of iterations and shadowing effects. This approach is similar in spirit to the division of the cavity into the front and termination sections for analysis, and to the
connection scheme of [67].

Finally, it might be worthwhile to investigate the use of superhybrid methods where more than two methods are combined to solve for the EM scattering from accurate jet engine models. For example, UTD, PO and GRE could be combined to analyze the air-intake section while the FDTD and the TDFE methods could be combined to analyze the complex engine section. The results of the two sections could then be combined via the termination reciprocity integral. In any case, superhybrid methods could be the wave of the future for accurate EM analysis of jet engine inlets.
Appendix A

Determination of the ray divergence factor

In this appendix, the ray divergence factor is determined. This factor governs the spreading of a ray tube. Although this factor can be determined by various means, we will do so via the curvature matrix (or Q-matrix) formulation of Deschamps [33]. However, only the pertinent equations are presented as the detailed derivation are found in [33, 68, 69].

For a G.O. ray propagating in space from point $O$ to point $P$ as shown in Fig. 85, the electric field at the two points are related by

$$\vec{E}(P) = (DF) \vec{E}(O) e^{-jks}$$ (A.1)

where the divergence factor, $(DF)$, is given by

$$(DF) = \frac{1}{\sqrt{1 + s/R_1}} \frac{1}{\sqrt{1 + s/R_2}}$$ (A.2)

with the distance between $O$ and $P$ denoted by $s$. $R_1$ and $R_2$ are the principal radii of curvature of the wavefront at $O$. The square roots in (A.2) have the following sign convention:

$$\frac{1}{\sqrt{1 + s/R_{1,2}}} = \begin{cases} \frac{1}{\sqrt{1 + s/R_{1,2}}} & \text{for } 1 + s/R_{1,2} > 0 \\ \frac{1}{\sqrt{1 + s/R_{1,2}}} e^{j\pi/2} & \text{for } 1 + s/R_{1,2} < 0 \end{cases}$$ (A.3)
The principal radii of curvature \( (R_1, R_2) \) of a wavefront are given by

\[
\frac{1}{R_{1,2}} = \frac{1}{2} \text{tr } \overline{Q} \pm \frac{1}{2} \sqrt{\left[\text{tr } \overline{Q}\right]^2 - 4 \det \overline{Q}}
\]

where

- \( \overline{Q} = 2\times2 \) curvature matrix of the wavefront
- \( \text{tr } \overline{Q} = \text{trace of matrix } \overline{Q} \)
- \( \det \overline{Q} = \text{determinant of matrix } \overline{Q} \).

The curvature matrix of a wavefront or surface is defined as follows. Let the wavefront or surface be described parametrically by \( \overline{r}(u, v) \) where \( (u, v) \) are the parametric coordinates on the wavefront or surface. Let \( \hat{n}(u, v) \) be the unit vector normal to the wavefront or surface. Then, the curvature matrix \( \overline{Q} \) is defined by

\[
- \begin{bmatrix} \hat{n}_u \\ \hat{n}_v \end{bmatrix} = \overline{Q} \begin{bmatrix} \overline{r}_u \\ \overline{r}_v \end{bmatrix}
\]

where the subscript denotes differentiation with respect to the indicated parametric
coordinate. For example, the curvature matrix for a plane wavefront or surface is

\[
\overline{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]  
(A.6)

while that for a wavefront or surface with principal radii of curvature \((R_1, R_2)\) is

\[
\overline{Q} = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix}.
\]  
(A.7)

When a ray is reflected from a curved surface, its reflected wavefront will have, in general, a different curvature matrix from that of its incident wavefront. In particular, the curvature matrix for the reflected wavefront, \(\overline{Q}'\), can be related to the curvature matrices of the incident wavefront, \(\overline{Q}^i\), and the reflecting surface, \(\overline{Q}^D\), through phase matching [33, 68, 69] at the point of reflection such that

\[
\overline{P}^i_i^T \overline{Q}^i \overline{P}^i + 2 p^i_{33} \overline{Q}^D = \overline{P}^r_i^T \overline{Q}^r \overline{P}^r
\]  
(A.8)

where the superscript denotes matrix transpose and

\[
\overline{P}^{i,r} = \begin{bmatrix} \hat{x}_1^{i,r} \cdot \hat{x}_1 & \hat{x}_1^{i,r} \cdot \hat{x}_2 \\ \hat{x}_2^{i,r} \cdot \hat{x}_1 & \hat{x}_2^{i,r} \cdot \hat{x}_2 \end{bmatrix}
\]  
(A.9)

\[
p^i_{33} = \hat{z}^i \cdot \hat{n}.
\]  
(A.10)

The orthogonal unit vectors \((\hat{x}_1^{i,r}, \hat{x}_2^{i,r}, \hat{z}^{i,r})\), associated with the incident and reflected wavefronts, and the unit vectors \((\hat{x}_1, \hat{x}_2, \hat{n})\), associated with the reflecting curved surface are shown in Fig. 86. For ray tracing purposes, they are chosen as [9]

\[
\begin{align*}
\text{curved surface} & : \\
\hat{n} &= \text{unit normal to surface} \\
\hat{x}_1 &= \hat{r}_u \\
\hat{x}_2 &= \hat{n} \times \hat{x}_1 \\
\hat{z}^i &= \text{propagation direction of incident field} \\
\text{incident wavefront} & : \\
\hat{x}_1^i &= \hat{n} \times \hat{z}^i / \sin \theta^i \\
\hat{x}_2^i &= \hat{z}^i \times \hat{x}_1^i
\end{align*}
\]  
(A.11, A.12)
Figure 86: Coordinate systems of the curved surface, the incident ray and the reflected ray.

\[
\begin{align*}
\tilde{z}_r &= \tilde{z}_i - 2 (\tilde{z}_i \cdot \hat{n}) \hat{n} \\
\tilde{z}_1 &= \hat{n} \times \tilde{z}_r / \sin \theta^i \\
\tilde{z}_2 &= \tilde{z}_r \times \tilde{z}_1
\end{align*}
\]

(A.13)

The curvature matrix of the surface, \( \overline{Q}^\Sigma \), is given by

\[
\overline{Q}^\Sigma = \overline{V}^{-1} \tilde{Q}^\Sigma \overline{V}
\]

(A.14)

where

\[
\overline{V} = \begin{bmatrix}
\tilde{r}_u \cdot \tilde{x}_1 & \tilde{r}_u \cdot \tilde{x}_2 \\
\tilde{r}_v \cdot \tilde{x}_1 & \tilde{r}_v \cdot \tilde{x}_2
\end{bmatrix}
\]

(A.15)

\[
\tilde{Q}^\Sigma = \frac{1}{EG - F^2} \begin{bmatrix}
eG - fF & fE - eF \\
fG - gF & gE - fF
\end{bmatrix}
\]

(A.16)

\[
E = \tilde{r}_u \cdot \tilde{r}_u, \quad F = \tilde{r}_u \cdot \tilde{r}_v, \quad G = \tilde{r}_v \cdot \tilde{r}_v, \\
e = \tilde{r}_{uu} \cdot \hat{n}, \quad f = \tilde{r}_{uv} \cdot \hat{n}, \quad g = \tilde{r}_{vv} \cdot \hat{n}.
\]

(A.17)

The double subscripts denotes second order derivatives with respect to the indicated parametric coordinates. After \( \overline{Q}^r \) is determined from (A.8), the principal radii of
curvature \((R_1, R_2)\) of the reflected wavefront at the point of reflection can be obtained from (A.4), and the divergence factor at \(P\) can be found from (A.2). If \(P\) is the next reflection point on the curved surface, its curvature matrix will be required. This matrix can be obtained through the propagation relation given by

\[
\overline{Q}(P) = \left\{ \left[ Q^r(O) \right]^{-1} + s \overline{I} \right\}^{-1}
\]

(A.18)

where \(\overline{I}\) is the unitary dyad.
Appendix B

Derivation of the 2-D far-zone radiation field

In this appendix, the far-zone radiation field for an applied field incident at the aperture of a 2D cavity is derived in terms of the subaperture fields. The extension to the 3D case can be similarly derived although it is not shown in this appendix.

Consider an aperture of width $a$ at $x = 0$ with a known incident field as shown in Fig. 11 repeated here for convenience in Fig. 87. Using the equivalence principle and image theory [4], the field for $x > 0$ is given by

$$U(p, \phi) = 2 \sqrt{\frac{jk}{8\pi}} \cos \phi \int_{-\frac{a}{2}}^{\frac{a}{2}} U^i(y') \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \, dy'$$ (B.1)

where $k$ is the propagation constant and $\rho'$ is the distance from $(y',0)$ to $(\rho, \phi)$.

Dividing the aperture into $2L + 1$ equal subapertures of width $\Delta$, the integral in (B.1) becomes

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} U^i(y') \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \, dy' = \sum_{l=-L}^{L} \int_{(l-\frac{1}{2})\Delta}^{(l+\frac{1}{2})\Delta} U^i(y') \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \, dy'$$ (B.2)

where

$$\Delta = \frac{a}{2L + 1}.$$ (B.3)

The integration is simplified by defining a new variable of integration, $y'_l = y' - l\Delta$. 

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Figure 87: Subaperture field expansion in 2D. (a) Geometry of aperture with known incident field; (b) Aperture divided into subapertures.

Then, the distance $\rho'$ is given by

$$\rho' = \sqrt{\rho_l^2 + y_l'^2 - 2\rho_l y_l' \sin \phi_l}$$  \hspace{1cm} (B.4)$$

where $(\rho_l, \phi_l)$ are the polar coordinates of the field point with respect to the origin of the $l$th subaperture at $(l\Delta, 0)$. Equation (B.4) can be expanded in a Taylor series expansion so that

$$\rho' = \rho_l - y_l' \sin \phi_l + \frac{y_l'^2}{2\rho_l} \cos^2 \phi_l + O(x_l^3) .$$  \hspace{1cm} (B.5)$$

If the phase contribution due to the quadratic term is small enough, that is, if

$$k \frac{y_l'^2}{2\rho_l} \cos^2 \phi_l < \frac{\pi}{8} ,$$  \hspace{1cm} (B.6)$$
then

\[ \rho' \approx \rho_l - y'_l \sin \phi_l. \]  

(B.7)

Note that (B.6) is the standard far-field criterion used in defining a plane wave field [34]. When the maximum value of \( y'_l (= \Delta/2) \) is substituted into (B.6), the far-field distance for the \( l \)th subaperture is given by

\[ \rho_l > \frac{2\Delta^2}{\lambda} \cos^2 \phi_l. \]  

(B.8)

The inequality (B.8) defines the closest distance from the subaperture where the far-field form of the radiation integral is valid. Using (B.8), the far-field region for a subaperture of width \( \lambda \) is plotted in Fig. 88. In contrast, the far-field distance of the

![Figure 88: Far-zone region of an aperture of width \( \Delta = \lambda \).](image-url)
entire aperture is given by

$$\rho > 2 \frac{\Delta^2}{\lambda} N^2 \cos^2 \phi. \quad (B.9)$$

Therefore, the far-field distance where the aperture is divided into $N$ subapertures is reduced by a factor of about $N^2$.

From (B.2) and (B.7), (B.1) becomes

$$U(\rho, \phi) = 2 \frac{jk}{8\pi} \cos \phi \sum_{l=-L}^{L} \frac{e^{-jkl}}{\sqrt{\rho l}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} U^i(y'_l + l\Delta) e^{jky'_l \sin \phi_l} dy'_l. \quad (B.10)$$

Hence, the far-zone field radiated into the cavity is a sum of the fields radiated by the subapertures.
Appendix C

Waveguide modal fields

This appendix provides the equations of the waveguide modal fields used in this research. The normalized waveguide modal fields in a parallel-plate waveguide of width \( a \) (see Fig. 89(a)) and a circular waveguide of radius \( a \) (see Fig. 89(b)) are summarized in Tables 8 and 9 respectively.

![Diagram of parallel-plate waveguide](image1)

(a) parallel-plate waveguide

![Diagram of circular waveguide](image2)

(b) circular waveguide

Figure 89: Parallel-plate and circular waveguide geometries.
<table>
<thead>
<tr>
<th>Mode</th>
<th>TE&lt;sub&gt;n&lt;/sub&gt; modes</th>
<th>TM&lt;sub&gt;n&lt;/sub&gt; modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>e&lt;sub&gt;x&lt;/sub&gt;</td>
<td>( j N_n Z_0 \frac{n\pi}{ka} \sin \frac{n\pi y}{a} )</td>
<td>0</td>
</tr>
<tr>
<td>e&lt;sub&gt;y&lt;/sub&gt;</td>
<td>( N_n Z_0 \frac{\beta_n}{k} \cos \frac{n\pi y}{a} )</td>
<td>0</td>
</tr>
<tr>
<td>e&lt;sub&gt;z&lt;/sub&gt;</td>
<td>0</td>
<td>( N_n \sin \frac{n\pi y}{a} )</td>
</tr>
<tr>
<td>h&lt;sub&gt;x&lt;/sub&gt;</td>
<td>0</td>
<td>( j N_n Y_0 \frac{n\pi}{ka} \cos \frac{n\pi y}{a} )</td>
</tr>
<tr>
<td>h&lt;sub&gt;y&lt;/sub&gt;</td>
<td>0</td>
<td>( -N_n Y_0 \frac{\beta_n}{k} \sin \frac{n\pi y}{a} )</td>
</tr>
<tr>
<td>h&lt;sub&gt;z&lt;/sub&gt;</td>
<td>( N_n \cos \frac{n\pi y}{a} )</td>
<td>0</td>
</tr>
<tr>
<td>( N_n )</td>
<td>( \sqrt{\frac{2\varepsilon_0}{\alpha \beta_n \epsilon_{on}}} )</td>
<td>( \sqrt{\frac{2\varepsilon_0}{\alpha \beta_n \epsilon_{on}}} )</td>
</tr>
</tbody>
</table>

\[ \beta_n = \sqrt{k^2 - \left( \frac{n\pi}{a} \right)^2} \]

\[ \epsilon_{on} = \begin{cases} 
2, & n = 0 \\
1, & \text{otherwise}
\end{cases} \]

Table 8: Orthonormal waveguide modes of a parallel-plate waveguide of width <i>a</i>.
Table 9: Orthonormal waveguide modes of a circular waveguide of radius \( a \).
Appendix D

The Termination Reciprocity Integral

The termination reciprocity integral in (3.22) and its modal form in (4.41) are derived in this appendix.

D.1 Derivation of the termination reciprocity integral

Consider the open-ended waveguide cavity in Fig. 90. The cavity is illuminated by impressed electric and magnetic current sources, $\mathcal{J}^i(P')$ and $\mathcal{M}^i(P')$ respectively, at $P'$. Let $(E^i_c, H^i_c)$ denote the fields radiated by these time-harmonic current sources in the presence of the cavity without the termination. These fields satisfy the following Maxwell's curl equations:

\[ \nabla \times E^i_c = -j\omega\mu_0 H^i_c - \mathcal{M}^i(P') \]  
\[ \nabla \times H^i_c = j\omega\epsilon_0 E^i_c + \mathcal{J}^i(P') . \]  

However, in the presence of the cavity structure and the termination, these current sources radiate $(E, H)$ which satisfy the following Maxwell's curl equations:

\[ \nabla \times E = -j\omega\mu_0 H - \mathcal{M}(P') \]  
\[ \nabla \times H = j\omega\epsilon_0 E + \mathcal{J}(P') \]
Therefore, \( \overline{E} \) and \( \overline{H} \) denote the fields scattered by just the termination in the presence of the cavity structure. From (D.1) to (D.6), it can be shown that

\[
\nabla \times \overline{E} = -j \omega \mu_0 \overline{H}_c^s
\]

\[
\nabla \times \overline{H}_c^s = j \omega \epsilon_0 \overline{E}_c^s.
\]

To find \( \overline{E}_c^s, \overline{H}_c^s \) at some observation point \( P \), an electric current test source, \( J_t \), is placed at \( P \). \( J_t \) has the same frequency as the impressed sources, \( J'(P') \) and
$\mathcal{M}')$, and it radiates $(\mathcal{E}_t, \mathcal{H}_t)$ in the presence of the cavity structure without the termination. $(\mathcal{E}_t, \mathcal{H}_t)$ satisfy the following Maxwell’s curl equations:

\begin{align*}
\nabla \times \mathcal{E}_t &= -j\omega \mu_0 \mathcal{H}_t \quad \text{(D.9)} \\
\nabla \times \mathcal{H}_t &= j\omega \epsilon_0 \mathcal{E}_t + \mathcal{J}_t(P). \quad \text{(D.10)}
\end{align*}

From (D.8) and (D.10), we can get

\begin{equation}
\mathcal{E}_c^a \cdot \mathcal{J}_t(P) = -\mathcal{E}_c^a \cdot \nabla \times \mathcal{H}_t - \mathcal{E}_t \cdot \nabla \times \mathcal{H}_c. \quad \text{(D.11)}
\end{equation}

Using vector identities, it can be shown that

\begin{equation}
\nabla \cdot (\mathcal{E}_t \times \mathcal{H}_c^a) - \nabla \cdot (\mathcal{E}_c^a \times \mathcal{H}_t) = \mathcal{E}_c^a \cdot \nabla \times \mathcal{H}_t - \mathcal{E}_t \cdot \nabla \times \mathcal{H}_c. \quad \text{(D.12)}
\end{equation}

so that (D.11) becomes

\begin{equation}
\mathcal{E}_c^a \cdot \mathcal{J}_t(P) = \nabla \cdot (\mathcal{E}_t \times \mathcal{H}_c^a - \mathcal{E}_c^a \times \mathcal{H}_t). \quad \text{(D.13)}
\end{equation}

Applying the divergence theorem to (D.13) within the volume $V_\Sigma$ bounded by $\Sigma + S_{T1} + S_E + S_G$ (see Fig. 90), we get

\begin{equation}
\int_{V_\Sigma} \mathcal{E}_c^a \cdot \mathcal{J}_t(P) \, dv'' = \oint_{\Sigma + S_{T1} + S_E + S_G} (\mathcal{E}_t \times \mathcal{H}_c^a - \mathcal{E}_c^a \times \mathcal{H}_t) \cdot \mathbf{n} \, dS \quad \text{(D.14)}
\end{equation}

where $\mathbf{n}$ is the unit normal vector pointing into the volume $V_\Sigma$. Since the fields satisfy the radiation boundary condition on $\Sigma$ as $\Sigma \to \infty$, and both $\mathbf{n} \times \mathcal{E}_c^a$ and $\mathbf{n} \times \mathcal{E}_t$ vanish on the perfectly conducting walls of $S_G$, (D.14) reduces to

\begin{equation}
\int_{V_\Sigma} \mathcal{E}_c^a \cdot \mathcal{J}_t(P) \, dv'' = \int_{S_{T1} + S_E} (\mathcal{E}_c^a \times \mathcal{H}_t - \mathcal{E}_t \times \mathcal{H}_c^a) \cdot \mathbf{n} \, dS. \quad \text{(D.15)}
\end{equation}
Let the test source $\mathcal{J}_t(P)$ be a point source of strength $\bar{P}_t$ such that

$$\mathcal{J}_t(P) = \bar{P}_t \delta(\mathbf{r}_t - \mathbf{r})$$  \hspace{1cm} (D.16)

where $\mathbf{r}$ is the position vector of the observation point $P$ and $\mathbf{r}_t$ is the integration variable on the LHS of (D.15). When (D.16) is substituted into (D.15), we obtain

$$\mathcal{E}_c^s(P) \cdot \bar{P}_t = \int_{S_{T1} + S_E} \left( \mathcal{E}_c^s \times \mathcal{H}_t - \mathcal{E}_t \times \mathcal{H}_c^s \right) \cdot \hat{n} \, ds .$$  \hspace{1cm} (D.17)

When the source and observation points are located ahead of the open front end as shown in Fig. 90, then the contribution from the integration over $S_E$ to $\mathcal{E}_c^s(P)$ is generally negligible compared to that from the integration over $S_{T1}$ for a sufficiently large termination so that (D.17) reduces to

$$\mathcal{E}_c^s(P) \cdot \bar{P}_t \approx \int_{S_{T1}} \left( \mathcal{E}_c^s \times \mathcal{H}_t - \mathcal{E}_t \times \mathcal{H}_c^s \right) \cdot \hat{n} \, ds .$$  \hspace{1cm} (D.18)

### D.2 Modal form of the termination reciprocity integral

When the termination section is a uniform waveguide section, as in most realistic geometry, the termination section can be characterized by its modal reflection matrix, $[S_T]$. In such a case, the cavity scattered fields $(\mathcal{E}_c^s, \mathcal{H}_c^s)$ can be easily obtained via a modal form of (D.18). To do so, the incident field and scattered fields are expressed as a sum of orthonormal modes such that

$$(\mathcal{E}_t, \mathcal{H}_t) = \sum_n A_n^+ (\mathbf{e}_n^+, \mathbf{h}_n^+) e^{-j\beta_n z}$$  \hspace{1cm} (D.19)

$$(\mathcal{E}_c^i, \mathcal{H}_c^i) = \sum_n A_n^{+i} (\mathbf{e}_n^+, \mathbf{h}_n^+) e^{-j\beta_n z}$$  \hspace{1cm} (D.20)

$$(\mathcal{E}_c^s, \mathcal{H}_c^s) = \sum_n A_n^{-i} (\mathbf{e}_n^-, \mathbf{h}_n^-) e^{j\beta_n z}$$  \hspace{1cm} (D.21)
where

\((\hat{e}^\pm_n, \hat{h}^\pm_n) = \text{\textit{n}}^{\text{th}} \text{ electric and magnetic modal fields propagating in the} \ \pm \hat{z} \ \text{directions respectively} \)

\(\beta_n = \text{propagation constant of the} \ \text{\textit{n}}^{\text{th}} \text{ modal field} \)

\(\hat{z} = \text{unit vector along the axis of the termination section and}
\)

pointing into the termination.

The incident modal coefficients, \(A^+_n\), and the reflected modal coefficients, \(A^-_n\), are related via the reflection matrix \([S_r]\) through

\[
\begin{bmatrix}
A^-_n
\end{bmatrix} = [S_r] \begin{bmatrix}
A^+_n
\end{bmatrix} . \tag{D.22}
\]

The orthonormality of the modes, where each mode has unit power independent of the other modes, is expressed mathematically as

\[
\int \int_{S_{T1}} (\hat{e}_m \times \hat{h}_n) \cdot \hat{z} \ dS = \delta_{mn} \tag{D.23}
\]

where \(\delta_{mn}\) is the Kronecker delta. Substituting (D.19) to (D.21) into (D.18), and applying orthogonality of the modes, we get

\[
\bar{E}_c^s(P) \cdot \bar{P}_t = -\int_{S_{T1}} \left[ \left( \sum_m A^-_m \hat{e}^-_m \right) \times \left( \sum_n A^+_n \hat{h}^+_n \right) - \left( \sum_m A^+_m \hat{e}^+_m \right) \times \left( \sum_n A^-_n \hat{h}^-_n \right) \right] \cdot \hat{z} \ dS
\]

\[
= -\sum_m \sum_n A^-_m A^+_n \int_{S_{T1}} (\hat{e}^-_m \times \hat{h}^+_n) \cdot \hat{z} \ dS + \sum_m \sum_n A^+_m A^-_n \int_{S_{T1}} (\hat{e}^+_m \times \hat{h}^-_n) \cdot \hat{z} \ dS
\]

\[
= -2 \sum_n A^-_n A^+_n . \tag{D.24}
\]
BIBLIOGRAPHY


