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Development of an anisotropic yield function for porous materials

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The Ohio State University, 1994
DEVELOPMENT OF AN ANISOTROPIC YIELD FUNCTION
FOR POROUS MATERIALS

DISsertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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Department of Materials
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Dedicated to

Kousalya Krishnaswami, my mother

and

Rajagopalan Krishnaswami, my father
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CHAPTER I

VOID COALESCENCE AND DUCTILE FRACTURE

Ductile fracture by growth of holes has long been recognized as a mode of failure in ductile forming materials. These observations have been reported by Puttick (1959), Rogers (1960), and Beachem (1963). Such a mode of fracture is important during metalworking. The phenomena of centerbursts in extrusion, and alligatoring in rolling are examples where the phenomenon of ductile fracture initiated by void growth limits formability.

Ductile fracture is the microscopic failure mode associated with the macroscopically observed phenomenon of necking in simple tension. In a pure metal, it is possible for the specimen to fail completely by pure shear deformation, where slip continues until the cross section reduces to a point (Broek, 1982). However, most engineering materials are not pure. They do contain second phase particles as inclusions. Elements are often added to form second phase particles in order to improve strength and machinability. Alternately, there may be accidental additions or local concentrations due to improperly controlled processing techniques. For instance, Sulphur additions in the steel making process create MnS particles, which are more deformable than the steel itself. Consequently, the MnS inclusions may deform during rolling into elongated stringers,
leading to decreased ductility during rolling. High Carbon steels frequently contain grain boundary carbides which range in shape from spheroids to lamellae. These carbides, which are hard and often poorly bonded to the matrix, have been observed to limit ductility during rolling. It has been experimentally observed that in general, the reduction in area during rolling is inversely related to the volume fraction of second phase particles (Gladman \textit{et al.}, 1971).

An important and unanswered question is how voids quantitatively limit ductility. Voids act as concentrators of deformation, often stretching and rotating at a rate faster than the workpiece material. A qualitative sequence of void growth in tension is shown in Fig. 1.1. A spherical void has an elastic stress concentration factor of three, in simple tension. This amplification of the applied stress initiates a plastic zone at the void surface which grows into the matrix. Goods and Brown (1979) predict that when the void length equals their spacing, plastic flow localizes in the ligament between the voids, causing a necking of the ligament and coalescence of the voids (Fig. 1.1).

Once flow localizes, an instability has occurred in which continued straining is confined to the ligament between the voids. This material instability occurs because the localized strain causes the nucleation of smaller voids which in turn link and concentrate strain between themselves. Physical separation then occurs in regions of high void content (or high particle volume fraction) producing fracture surfaces in alloys with regions of high particle content and a 'dimpled' surface corresponding to void growth and link up processes. As an example, Fig.
1.2 shows the fracture surface of a low Carbon steel (Martin, 1980). The presence of sulphide particles has led to a ductile fracture in this case. Figure 1.3 shows the coalescence between voids nucleated around sulphide particles, leading to the formation of a ductile crack in a free cutting mild steel (Honeycombe, 1981).

Material failure involving the link up of voids during deformation is commonly referred to as ductile fracture. The word ductile emphasizes that large plastic deformation is involved, and fracture indicates that the deformation localizes to produce free surface area involved with fracture. From a metal forming viewpoint, any design process which suppresses ductile fracture improves formability.

The size and spacing of particles (the relevant length scale in the problem) are important microstructural features that control in part, the ductile fracture process. As discussed by Broek (1982) second phase particles can be broadly classified into three groups. Large particles range in size from 1 to 20 mm and cannot accommodate the plastic deformation of the surrounding matrix. They fracture or debond early, thereby creating voids. The failure of the particles or the particle matrix interface, induces local deformations that trigger the ductile fracture process. Intermediate particles range in size from 500 to 5000 Å. These particles may nucleate voids at substantially higher plastic strains than larger particles when they are present as isolated particles. Nucleation at these particles can occur at earlier stages if they are situated near large particles. Small particles range in size from 50 Å to 500 Å and are not observed to participate in the final fracture process.
Ductile failure does not occur at small macroscopic strains and hence cannot occur by the debonding of the large particles alone. However, the failure of intermediate particles creates new microscopic voids, which further serves to concentrate the deformation between them. Such a process seems to feed itself by creating a succession of void sizes. This continued nucleation of smaller voids may be the microscopic explanation for the macroscopically observed exponential growth rate of the void volume fraction near failure. An example of such a phenomenon is seen in Koplik and Needleman (1988), Figs. 4b and 5b.

Final fracture is due to a local necking down of the ligaments between neighboring voids. Though the void containing matrix may continue to harden, the decreasing ligament size between voids leads to a "geometric softening effect". Thus, the instability or concentration in deformation between larger voids, leads to a mechanical or load carrying instability. This is one microscopic mechanism for ductile fracture.

Metal forming processes may cause part failure during forming or later in service. If the phenomenon of void nucleation can be prevented, then ductile failure can be avoided. However, this approach is not practically feasible in engineering metals which always contain second phase particles from which voids nucleate. Hence, it becomes necessary to understand the behavior of metals with voids and incorporate the mechanical softening due to voids into stress-strain response during the metal forming process.

The mechanical response of the material is strongly dependant on the orientation of the voids with respect to the loading axes (Nagpal et al., 1972;
Becker et al., 1989). Oriented voids cause the stress-strain response of an otherwise isotropic matrix to be anisotropic. Such an anisotropy has been experimentally observed (Kuhn, 1978). Bend test specimens cut from different orientations in a rolled plate show different fracture strains (Fig. 1.4). Voids present in the rolled material are flattened so that a larger projected area exists normal to the thickness direction. A bend test, oriented to produce tensile stresses in the thickness direction, is expected to show a lower fracture strain than orientations which apply tensile stresses either in the rolling or transverse directions. This is shown in Fig. 1.4.

This anisotropy is different than that due to crystallographic texturing, which occurs due to the orientation of slip planes during deformation. In addition to the anisotropy due to the void orientation, the interaction of plastic zones between growing voids also causes anisotropy. Plastic zones that grow from the void surface concentrate the deformation in bands between voids. Thus, the direction along the band in the vicinity of the void has a different flow stress compared to the material outside the band, causing anisotropy.

Currently available yield conditions show the basic feature of mechanical softening due to the presence of voids, but do not account for void orientation. Hence, they lead to a prediction that rolled material, which has a strong internal void orientation parallel to the rolling direction, has the same formability regardless of the direction of subsequent loading. Hence, present yield conditions are unable to predict the observed behavior described in Fig. 1.4.
The aim of the present investigation is to account for the effect of the microstructure in the development of a macroscopic yield criterion and an associated incremental flow law. The evolution of void volume and shape will be emphasized. Experimental observations, along with finite element studies, will be used in developing a physically based, anisotropic, macroscopic yield criterion. Using this yield criterion, a flow law will be developed to give the incremental stress-strain response. The flow law can then be implemented in a finite element scheme and used to predict bulk formability limits in practical metal forming simulations. Some common failures such as centerbursts in extrusion, and alligatoring in rolling, may then be modelled by simulation.

Figure 1.4 can be viewed as a type of a Forming limit diagram. Forming limit diagrams provide strains to failure. These strains actually depend on the path to failure and are hence not unique failure strains. However, they serve to provide a rough estimate for the strains to fracture. Since ductile fracture is often caused by void growth and link up, the stress-strain relation to be developed for porous materials should, in principle, permit prediction of forming limit diagrams.

In order to obtain strains to fracture in simulations of actual forming problems, the flow law that is to be developed can be used as the material law in a conventional FEM simulation of the metal forming process. This is suitable as long as void properties may be modelled as a continuum on the scale of the forming problem. Such a law would offer an improvement over use of existing laws, which are unsuitable for formability predictions since they either ignore the effect of porosity or neglect the orientation of
damage by assuming isotropic behavior. The law to be developed would be amenable for future implementation in existing FEM codes such as DEFORM or ABAQUS.

The yield criterion to be developed may also be applicable as a model for ductile crack growth, in which the fracture process involves microvoid coalescence. In such cases, it might be possible to quantify resistance to crack growth in terms of the void growth and coalescence phenomena occurring near the crack tip.
Figure 1.1: Stages leading to ductile failure under nominally tensile deformation.

a) Nucleation and growth of voids at larger, isolated inclusions,
b) Onset of void-void interaction and void nucleation at intermediate particles,
c) Necking of ligaments in final stage of failure.
Figure 1.2: Ductile fracture surface of a low Carbon steel showing the presence of sulphide particles (Martin, 1980)
Figure 1.3: Ductile crack growth in a free-cutting mild steel containing sulphides (Honeycombe, 1982).
Figure 1.4: Fracture strain anisotropy in rolled specimens.
 CHAPTER II
LITERATURE REVIEW

The prediction of formability or ductility limits for metalworking has been of interest to many early investigators. The deleterious effect of a tensile state of stress on ductility was quantified in various ways in order to predict formability.

One such attempt by Cockcroft and Latham (1968) produced a simple failure criterion,

\[ \int_0^{\varepsilon_f} \sigma^+ \, d\varepsilon = C, \quad (2.1) \]

where \( \sigma^+ \) is the largest principal tensile stress which equals zero if all principal stresses are negative, \( C \) is a constant for a given material, temperature and strain rate, \( \varepsilon \) is the effective strain, and \( \varepsilon_f \) is the fracture strain. The Cockcroft and Latham criterion has been used with limited success, for example in prediction of critical reductions in extrusion for which arrowhead cracking occurs. Based on their failure criterion, Cockcroft and Latham justify experimental observations of cracking which occur only at intermediate reductions. More specifically, the stress distribution in extrusion was obtained from a slip-line solution. At small reductions the tensile stress along the extrusion axis is high, but the strains
are too small for eqn. (2.1) to be satisfied. At large reductions, the strains are high, but the tensile stress drops and eqn. (2.1) is not satisfied. At intermediate reductions where fracture is observed to occur through the formation of arrowhead cracks, eqn. (2.1) is satisfied due to intermediate values of both tensile stress and effective strain. Though the Cockcroft and Latham criterion is not founded on any micromechanical basis, its successful application to extrusion emphasizes that ductile fracture depends on a combination of adequate tensile stress and deformation.

Another ductile fracture criterion used by early investigators such as Puttick (1959) emphasizes the role of inclusions in nucleating voids. Due to the early established connection between ductile fracture and voids in metals, this approach relates the fraction of inclusions to the final fracture strain.

The study of void nucleation has received considerable attention (e.g., Goods and Brown, 1979), principally because any ability to suppress void nucleation should result in enhanced ductility. Void nucleation may be homogeneous or heterogeneous. Although experimental evidence suggests that voids may form in precipitate-free regions by vacancy condensation, Balluffi and Seigle (1957) showed that such a process usually does not occur. The reason is that the excess vacancy concentration cannot reach a high enough value for vacancy condensation. Thus, most void formation occurs at heterogeneous sites such as previously existing particles, precipitates, or grain boundaries. Void nucleation at grain boundaries is usually driven by local stresses rather than by vacancy supersaturation. At intermediate temperatures and low strain rates, when
grain boundary sliding contributes substantially to deformation, grain boundary cavitation has been found to occur in metals (Intrater et al., 1959). Other studies show that alloy additions that suppress grain boundary sliding do suppress grain boundary nucleation of voids.

Most heterogeneous void nucleation occurs at second phase particles in the matrix. The relaxation of strain energy provides a driving force for void nucleation at second phase particles. However, the kinetics of the process requires that the stress at the interface reach the interface fracture strength. This implies that a critical far field strain is needed for void nucleation at particles. This nucleation strain has been found to be a decreasing function of the tensile hydrostatic stress and the volume fraction of second phase particles. Also, the smaller the particle size, the higher the critical strain for nucleation.

Void nucleation appears to occur continuously during deformation, along with the growth of previously nucleated voids. Needleman and Rice (1978) have suggested a nucleation model to account for increase in void volume fraction due to nucleation. Their relation is of the form

\[
\dot{f}_{\text{nucleation}} = A\dot{\sigma}_M + B\dot{\sigma}_m
\]  

(2.2)

where \(\dot{\sigma}_M\) is the effective stress rate and \(\dot{\sigma}_m\) is the mean stress rate. The two terms in this model account for nucleation that is controlled by the plastic strain in the matrix and by the maximum normal stress at the particle-matrix interface, respectively. If the nucleation is controlled by the plastic strain in the matrix, Chu and Needleman (1979) have proposed that
\[ A = \left( \frac{1}{E_i} - \frac{1}{E} \right) \frac{f_n}{s \sqrt{2\pi}} \exp \left( -0.5 \left( \frac{\varepsilon^p - \varepsilon}{s} \right)^2 \right), B = 0 \] (2.3)

and when void nucleation is controlled by the maximum normal stress at the particle matrix interface, then

\[ A = B = \frac{f_n}{s \sqrt{2\pi}} \exp \left( -0.5 \left( \frac{\sigma_M + \sigma_m + \sigma_N}{s} \right)^2 \right). \] (2.4)

The volume fraction of void nucleating particles is \( f_N \). \( \varepsilon_M^P \) is the plastic strain in the matrix, and \( \varepsilon_N \) and \( \sigma_N \) are, respectively, the mean values of nucleation strain and stress consistent with a normal distribution of void volume fraction. \( S \) denotes the product of the reference stress with the standard deviation of the normal distribution.

This nucleation model has been used to successfully model the void volume fraction due to nucleation as a function of the deformation history. For example, Tvergaard (1982a) predicts the distribution of void volume fraction in a localized band between two interacting voids, based on the nucleation criterion discussed above. As flow localizes between the two larger interacting voids, microscopic voids with a range of sizes are nucleated and eventually coalesce to produce physical separation inside the band.

A micromechanical study of void nucleation due to interfacial debonding was done by Needleman (1987). Needleman (1987) proposed a void nucleation stress which depends on the mean stress as

\[ \sigma_n = \sigma_0 + c \sigma_m, \quad c \in [0,1] \] (2.5)
A value of 0.3 was prescribed as a good estimate by Needleman (1987). As $c$ ranges from 0 to 1, the void nucleation criterion shifts from a plastic strain controlled criterion to a stress controlled criterion. The rate of void nucleation was given in a manner similar to Chu and Needleman (1979). The void nucleation criterion of Needleman (1987) has been used by Xu and Needleman (1992) to examine the effect of void nucleation on flow localization in bands. The formulation is akin to the one employed earlier by Saje et al. (1982) and Pan et al. (1983). A band of imperfection of higher porosity is assumed to exist in a porous hardening solid. Conditions for the bifurcation of the velocity field are solved for in the solid for boundary conditions of plane strain tension and axi-symmetric tension. The dependence of the localization strain and the band orientation at localization were studied as a function of the parameter $c$ (eqn. 2.5). The localization strain was found to decrease monotonically as $c$ increased from zero to unity.

Deformation of an incompressible elastic sheet with an elliptical void was first considered by Berg (1962). The loading was represented by the two principal stress components. The elliptical void was mapped onto a circle in the complex plane by an appropriate conformal mapping. The Muskhelishvili solution for a plane elasticity problem with remote loading showed that the deformed void was also an ellipse. The analogy between the static deformation of an incompressible elastic material and the velocity field in quasistatic incompressible viscous flow permitted extension of the solution to the deformation of an elliptical void in a viscous material. By tracking the incremental deformation of the void as
function of time, it was found that the deformed void remains elliptical and tends to rotate such that the major axis of the void aligns with the direction of the largest principal stress. It was also observed that the radius and eccentricity of the void depend exponentially on the remote hydrostatic stress.

The extension of void growth from viscous to plastic materials was done by Rhee and McClintock (1962). The change in mean radius of an elliptical void with effective strain was found to depend exponentially on the ratio of mean stress to effective stress. The large shearing present in bands of localized deformation motivated McClintock, Kaplan and Berg (1966) to study shear and superposed hydrostatic loading of viscous voided materials. They assumed that the band of localized deformation rotated with respect to the remote loading, unlike the approach of Berg (1962). Their extensions to plastic materials show an exponential dependence of the fracture strain on the ratio of hydrostatic stress to shear stress. The effect is more pronounced in materials with lower strain hardening.

McClintock (1968) developed an analytical criterion for ductile fracture by growth of voids based on solutions for growth of circular cylindrical voids in a plastic, non-hardening matrix. These results show an exponential dependence of the fractional increase in hole radius on the radial stress, when strained parallel to the cylindrical axis. Since no analytical expression for elliptical holes in a plastic medium was available, McClintock adopted the counterpart viscous solutions from Berg (1962) for the plastic case. The solutions obtained again showed an exponential dependence of fractional increase of mean hole radius on the mean stress.
The effect was more pronounced for materials with a low strain hardening index, as observed by McClintock et al. (1962), implying that fracture occurs at lower strains for materials with lower strain hardening. These observations are supported by the Considere criterion which states that the strain at instability is proportional to the hardening coefficient. Such an effect has also been observed by Magnusen et al. (1988) in an experimental study of void distribution in a tensile specimen. In that study, the fracture strain was found to increase with hardening coefficient. McClintock (1968) also estimated a fracture strain and found that the fracture strain decreased exponentially with increasing hydrostatic stress. This dependence was stronger for materials with a low strain hardening index. The anisotropy of the ductile fracture process, which could arise from anisotropic shape and/or spacing, has also been noted.

Further investigations by Rice and Tracey (1969) confirmed that growth rates for an isolated void are noticeably increased by the superposition of a hydrostatic stress state on an incompressible velocity field near the void. The velocity field was chosen from a set of incompressible, kinematically admissible fields as to minimize a plastic work functional. For a spherical void in a remote tensile strain field, the void volume was found to depend exponentially on the mean remote stress. Similar results were obtained for a long cylindrical void as addressed by McClintock (1968). A recent investigation by Huang (1989) indicates that the dilatation rates of spherical voids as given by Rice and Tracey are underestimated. The more accurate results of Huang, however, retain the nature of the exponential
dependence. It can be seen that, in general, void growth rates are exponentially amplified by the application of a hydrostatic stress.

The investigations discussed thus far address the behavior of isolated voids and do not explicitly account for void-void interaction. Nagpal et al. (1972) addressed void-void interaction by determining the normal and shear tractions required to yield a band of voids in a rigid-plastic material. Using slip-line analysis, a one-dimensional periodic array of circular voids or cracks of various sizes and spacings was considered. The strains parallel to the band are assumed to be zero, while those perpendicular to the band were non-zero. Their results show localization of deformation within the band, due to necking of ligaments between the defects. More importantly, the orientation of the voids with respect to the loading causes large changes in stress-strain response. In particular, the tractions at yield are larger for circular holes than for cracks, whether horizontal or vertical.

Nagpal et al. (1972) also demonstrate the anisotropy of void growth by considering the growth of hexagonally shaped holes. At low porosities, the hole grows along the direction of the remote displacement increments, so that void volume is increased. At higher porosities, hole growth proceeds parallel to and inside the band, even though the macroscopic principal strain is normal to the band. This strong void interaction can cause substantial mechanical softening of the band, even though the material may continue to work-harden. Hence, the orientation of the void, along with interaction between voids, causes the stress-strain behavior to be anisotropic.
Needleman (1972) studied void growth by elastic-plastic finite element analysis of a doubly-periodic array of voids in uniaxial plane strain tension. The studies show the evolution of the plastic zone from the void surface and the void evolution with deformation. At lower strains the void and the cell of material surrounding it elongate parallel to the tensile loading direction, but both the void and the cell boundaries contract in the transverse direction. Upon further straining, the plastic zone grows toward the outer cell boundary transverse to the direction of principal straining. This interaction promotes necking of the ligament between voids. Though the material in the ligament continues to harden, the lesser volume available to carry the load drops. The rest of the cell unloads while the strain concentrates in the necked region. This signifies localization and the path to final fracture. These results are similar to those of Nagpal et al. (1972). Earlier models, like those of Rice and Tracey (1969), do not explicitly include this interaction effect, and hence tend to overestimate the strain to failure.

The work of Nagpal et al. (1972) suggests that a voided solid may be treated as a continuum with the microstructural features being represented as material parameters. Such a formulation can help the analytical study of the deformation and fracture of porous solids. A simple procedure is to construct a yield function that depends on the porosity, and develop from it an incremental stress-strain relation.

One of the earliest investigations of this type (Green, 1972) assumed that the yield function for a voided continuum also had a dependence on the mean stress. The deformation theory of plasticity was used to obtain a
yield function for a voided rigid, perfectly-plastic material. The void growth process was assumed isotropic. Green quantified the dependence of yield on mean stress the material by assuming a yield function of the form,

\[ J_2' + \alpha (J_1)^2 = \delta (\sigma_y)^2, \tag{2.6} \]

where, \( J_2' \) is the second invariant of the stress deviator tensor, \( J_1 \) is the first invariant of the stress tensor, \( \sigma_y \) is the yield stress of the matrix, and \( \alpha, \delta \) are material parameters which are functions of the void volume fraction. An incremental stress-strain law was developed from eqn. 2.6 and used to predict flow softening in tension for a voided material. The behavior in compression was shown to be different from that in tension, even though the matrix shows no Bauschinger effect. This effect is due to different void evolution in tension versus compression. However, the yield function is incapable of representing the anisotropic effects of void growth and coalescence which were observed in the slip line studies of Nagpal et al. (1972), and the FEM studies of Needleman (1972).

A popular yield criterion by Gurson (1977) uses a void cell model to represent the void matrix continuum. The matrix surrounding the voids is assumed to be an incompressible von Mises material. A yield locus is defined, based on the use of kinematically admissible velocity fields to obtain an upper bound to the macroscopic stress for yielding. Since separate velocity fields were chosen to represent volume and shape changes, the microscopic velocity fields chosen are similar to those of Rice and Tracey (1969). The macroscopic yield functions constructed are dependent on hydrostatic stress. The yield functions obey the principle of
maximum plastic work, drawing on the ideas of Bishop and Hill (1951). Gurson considered the void shapes of long circular cylinders (in axisymmetric and plane strain deformation) and spheres. By using a flow law for the Gurson yield locus for a spherical void, the onset of flow softening due to void growth can be shown to occur. The Gurson model has been used by many investigators to study the failure processes due to void growth. For example, Yamomoto (1977) has used an infinite band analysis in order to predict macroscopic strains to localization. Gurson's model has also been used in the finite element analysis of central bursts in extrusion due to void growth (Aravas, 1986), and in studies of void growth near the tip of a growing crack (Aoki et al., 1984).

Some of the recent efforts to predict the yield and dilatational behavior of spherical voids have focused on the construction of stress and strain rate potentials for voided solids. The state of stress and strain in the matrix surrounding a void is termed the microscopic state. The state of stress and strain remote from the void is termed the macroscopic state. The microscopic states of stress and strain are usually non-uniform even for uniform macroscopic loading. Let $\sigma$ and $\dot{e}$ represent the microscopic stress and strain rates and $\Sigma$ and $\dot{\Sigma}$ represent the macroscopic stress and strain rates. The macroscopic quantities are defined in terms of the corresponding microscopic quantities as follows. Let $V_v$ denote the voided volume, $V_M$ denote the simply connected matrix volume, and $V$ denote the union of $V_v$ and $V_M$. Further, let $\partial V$ denote the external surface bounding $V$, $\partial V_M$ the surface bounding $V_M$, and $\partial V_v$ the surface
bounding $V_v$. Let $h$ be a bounded function defined over the domain $V$. $h$ is defined as

$$< h > \equiv \frac{1}{V} \int_V h \, dv. \quad (2.7)$$

Thus,

$$\Sigma_{ij} \equiv < \sigma > = \frac{1}{V} \int_{\partial V} \sigma_{ik} n_k x_j \, dS \ , \text{ and} \quad (2.8)$$

$$\dot{E}_{ij} \equiv \frac{1}{2V} \int_{\partial V} \{ \dot{u}_i n_j + \dot{u}_j n_i \} \, dS. \quad (2.9)$$

The prescription of the microscopic state completely specifies the macroscopic state, but the converse is not true. However, prescription of uniform macroscopic boundary conditions can lead to the construction of useful macroscopic potentials whose bounding nature can be established [Hashin (1983), Willis (1989)].

Thus, the microscopic stress satisfies equilibrium, according to

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \ \forall x \ in \ V_M \ , \ and \ \sigma_{ij} n_j = 0 \ on \ \partial V_v \ . \quad (2.10)$$

Also, one of the following uniform macroscopic boundary conditions hold.

$$\sigma_{ij} n_j = \Sigma_{ij} n_j \ on \ \partial V_M , \ or \quad (2.11)$$

$$\dot{u}_i = \dot{E}_{ij} x_j \ on \ \partial V_M . \quad (2.12)$$
If $\sigma$ is a microscopic stress field satisfying equilibrium and $\dot{u}$ is a velocity field, and if one of the two macroscopic boundary conditions (eqns. 2.11 and 2.12) hold, then Hill's lemma [Hill (1963)], gives

$$<\sigma : \dot{\varepsilon} > = <\Sigma : \dot{\varepsilon} > .\tag{2.13}$$

The microscopic strain rate potential $\psi(\sigma)$ defines the strain rate $\dot{\varepsilon}$ by,

$$\frac{\partial \psi(\sigma)}{\partial \sigma_{ij}} = \dot{\varepsilon}_{ij} .\tag{2.14}$$

The conjugate microscopic stress potential $\phi(\dot{\varepsilon})$ is defined by the Legendre transform,

$$\phi(\dot{\varepsilon}) = \sigma : \dot{\varepsilon} - \psi(\sigma) .\tag{2.15}$$

Thus the Maxwell type reciprocal relationship yields,

$$\frac{\partial \phi(\sigma)}{\partial \dot{\varepsilon}_{ij}} = \sigma_{ij} .\tag{2.16}$$

The macroscopic potentials are defined by [Rice (1970)],

$$\Psi(\Sigma) = <\psi(\sigma) >, \quad \text{and} \quad \Phi(\dot{\varepsilon}) = <\phi(\dot{\varepsilon}) > .\tag{2.17}$$

The macroscopic potentials are also conjugate to each other [Duva and Hutchinson (1984)] and satisfy [Michel and Suquet (1992)],
Once the constitutive behavior of the matrix is assumed, usually to be a power law hardening non-linear viscous material, the microscopic stress potential \( \phi(\dot{\varepsilon}) \) is known. A functional form is assumed for the macroscopic stress potential \( \Phi(\dot{\varepsilon}) \). Once \( \Phi(\dot{\varepsilon}) \) is known, duality is invoked [Michel and Suquet (1992)] to calculate the macroscopic strain rate potential \( \Psi(\Sigma) \), which is subsequently assumed to be the yield function. For instance, Duva and Hutchinson (1984) use the dilatation rate results of Budiansky et al. (1982) to derive the macroscopic stress potential for a dilutely voided non-linear viscous material. Cocks (1989) has proposed a general approximate form for \( \Psi(\Sigma) \) for a non-linear power law hardening viscous material of void volume fraction \( f \) as,

\[
\Psi(\Sigma) = \frac{\sigma_0 \dot{\varepsilon}_0}{n+1} \left\{ \frac{9}{4} A(f) \left( \frac{\Sigma_{\text{mean}}}{\sigma_0} \right)^2 + B(f) \left( \frac{\Sigma_{\text{eq}}}{\sigma_0} \right)^2 \right\} \left( \frac{n+1}{2} \right). \tag{2.19}
\]

Michel and Suquet (1992) have proposed various functional forms for \( A(f) \) and \( B(f) \).

As shown in Chapter V, comparisons to FEM simulations show that the Gurson model and the models of Cocks (1989) and Michel and Suquet (1992) are reasonably accurate in characterizing a material with spherical voids when loaded under both uniaxial and hydrostatic stress states. The
works of Cocks (1989) and Michel and Suquet (1992) will be revisited in Chapter V.

An important drawback of the Gurson model and the models of Cocks (1989) and Michel and Suquet (1992) is that the stress-strain behavior is isotropic. The microstructure is represented only by the volume fraction, so that void shape and orientation are not adequately modeled. The final coalescence process is highly anisotropic and this process cannot be modeled accurately by an isotropic model.

The softening rate predicted by the Gurson model after the peak in "true stress" is not consistent with the actual rate as given by finite element cell studies of void coalescence (see Fig. 9a, Koplik and Needleman, 1988). According to the Gurson model, the complete loss of all stress carrying capacity occurs at a void volume fraction of unity, a value that is too high. There have been many attempts to modify the Gurson model. Tvergaard (1982b) introduced three parameters \( q_1, q_2 \) and \( q_3 \) in eqn. 2.20 to improve the agreement between finite element cell results and the Gurson predictions, especially at low void volume fractions,

\[
\Phi = \frac{\sigma_{\text{eff}}}{\sigma_M} + 2q_1 \sigma_{\text{eff}} \cosh\left(\frac{3q_2 \sigma_{\text{mean}}}{2} \frac{1 + q_3 \sigma_{\text{eff}}^2}{\sigma_M}\right) = 0 \tag{2.20}
\]

The values of the parameters were also prescribed by Tvergaard based on finite element studies of a voided material. This modified model was used by Tvergaard (1982a), to study void coalescence. Here the rate of increase in void volume fraction was assumed to be due to both void nucleation and growth. In addition, Tvergaard proposed that when the
void volume fraction reaches a critical value, an extra term should be added to the void volume increase rate. This was done to better model the rapid growth in void volume fraction near failure.

Tvergaard and Needleman (1984) propose the use of an 'effective void volume fraction' to accurately predict the critical failure strain. Spitzig et al. (1988) experimentally studied the growth of voids in compacted iron specimens with different initial porosities. Based on uniaxial tension tests, it was proposed that the void volume fraction should be replaced by an effective void volume fraction \( f_{\text{eff}} \) equal to \( f^m \), where \( m = \left(\frac{n+2}{3}\right) \) and is a function of the strain hardening co-efficient, \( n \). The work of Bourcier et al. (1986) shows that the area fraction of porosity on the fracture surface of specimens is eight to ten-fold higher than the initial porosity. This increase cannot be ascribed to void growth alone. Instead it indicates that ductile fracture proceeds along paths of high initial void volume fraction and that local fluctuations or void clustering are important homogeneities on which the ductile failure process strongly depends. The effect of local void concentrations on the ductile process is also demonstrated by the work of Magnusen et al. (1988), who studied the effect of random arrays of voids on ductility. Their results show the effect of a local concentration of voids by demonstrating that a random distribution of voids lowers ductility in tensile specimens by 20 to 75%, as compared to specimens with a regular distribution of voids.

Koplik and Needleman (1988) suggested improvements to the Gurson model, based on finite element analysis of a circular cell with a spherical void. An effective void volume fraction due to Tvergaard and Needleman
(1984) was employed. They reported different values for the \( q_1, q_2 \) and \( q_3 \) parameters as compared to Tvergaard (1982b), in order to better represent the final loss of stress carrying capacity. The factors \( q_1, q_2 \) and \( q_3 \) are simply empirical quantities and do not have any microstructural basis. At present, they are constants and do not evolve with deformation.

The Gurson model has been derived for the ideal case of symmetric deformation of a spherical void. It is therefore not suitable to model general void shapes and loading. An inspection of the Gurson model reveals that a pure shear loading does not affect the void growth rate. Hence, under pure shear, there would be a very small reduction in flow stress of order \( f \) due to existing voids, but no void growth. By the consistency criterion, there would be no softening. But the recent work of Anderson et al. (1990) shows a case in point, where a material with a periodic array of cracks is subjected to pure shear. The cell model is employed and is studied using the finite element method. The shear stress-strain curve shows a peak due to material softening caused by rotation and stretching of the cracks. Hence, the predictions of the Gurson model under macroscopic pure shear need to be improved.

Fleck and Hutchinson (1986) extended the earlier studies of elliptical voids in linear viscous materials by Berg (1962) and McClintock et al. (1966) to non-linear viscous materials under superposed shear and hydrostatic tension. The void shape and size were obtained as a function of the deformation imposed and the ratio of \( (\sigma_m/\sigma) \). Circular voids evolved into ellipses with the major axis rotating so as to coincide with the direction of the largest principal strain. However, for non-linear viscous materials, the
void was found to dilate more rapidly in a direction normal to the
direction of the largest principal strain. This effect is similar to the FEM
results of Needleman (1972).

The work of Budiansky et al. (1982) shows that under various ratios of
\((\sigma_m/\tau)\), the void shape evolves differently from that of a sphere, in a non-
linear viscous material subjected to axisymmetric loading along with a
uniaxial stress, especially at low stress triaxialities. They also observed that
voids grow in a direction transverse to the direction of principal loading.
Such features are not accounted for by the predictions of the Gurson model
or its modifications.

The dilation normal to the direction of principal straining observed by
Fleck and Hutchinson (1986) and Budiansky et al. (1982) was for the case of
an isolated void. The reason for this transverse dilation is the non-linear
material behavior (Budiansky et al., 1982) and not void-void interaction.
However, the studies of Needleman (1972) showed the transverse growth
of a void due to void-void interaction. Both these studies emphasize that
the void shape can change with deformation, though the reason for the
growth may be different.

The recent work of Becker et al. (1989) also demonstrates the effect of
void shape on ductile failure. The results show that void growth rates in
simple tension are governed by the void dimension perpendicular to the
tensile axis. However, at higher triaxialities, the effect diminishes, so that
void growth rate depends principally on void volume fraction. Further,
the failure strains are proportional to the projected area of voids onto a
plane with the tensile axis as the normal, and not to the actual void volume fraction.

Thus, the slip line studies by Nagpal (1972), void shape studies by Budiansky et al. (1982) and Fleck and Hutchinson (1986), and finite element studies by Becker et al. (1989) show the effect of void shape in determining ductility. Experimental work by Spitzig et al. (1988), and Bourcier et al. (1986), and finite element studies by Tvergaard (1982) show that the void volume fraction alone cannot adequately represent the effect of damage. In summary, the review of the literature pertaining to the ductile failure process shows the importance of oriented damage. The inadequacy of some of the existing models in characterizing this effect emphasizes the need for improvement.
A. Void geometry

In order to model the effect of void size and shape on the mechanical behavior of voided materials, it is necessary to develop a yield function. This yield function would give the relation that needs to be satisfied by the macroscopic stresses, in order for the void-containing material to yield. The existence of such a function is being tacitly assumed, in light of the work done by earlier investigators such as Gurson (1977).

Previous efforts at developing a yield function for porous materials have considered the voids to be spherical. The resulting yield functions have thus been isotropic, for example Green (1972) and Gurson (1977). However, voids in materials are seldom spherical, their shapes being distorted due to previous mechanical working. Neglecting void shape leads to erroneous predictions of yield behavior, and this error increases with increasing departure from the spherical shape. This fact has been pointed out by Krishnaswami and Anderson (1991), and also substantiated by their two-dimensional finite element studies. Furthermore, it has been
seen experimentally that fracture strains in rolling stock can be anisotropic [Kuhn (1978)]. The strain to fracture is predominantly dependent on the final stages to fracture. Thus, the anisotropy is a reflection of the anisotropy of the ductile rupture mode due to void coalescence. We postulate that such an anisotropy is predominantly due to the effect of the texturing of elongated voids. Hence, we focus our present effort on developing a yield function for porous materials including the effects of void shape.

A typical engineering metal workpiece contains many second phase particles which nucleate voids when the workpiece is stressed. The effect of a random initial distribution of particles on the distribution of voids is approximated by a locally periodic distribution of voids within the

![Figure 3.1: Finite element cell for analysis](image)

Figure 3.1: Finite element cell for analysis
workpiece. The orientation of the particles varies, depending upon the local state of stress. This allows the assumption that regions of the workpiece can be described by a periodic distribution of oriented voids. Thus, the initial state of the workpiece can be classified as an incompressible matrix with many regions, each containing periodically distributed, identically oriented voids. This periodicity can be gainfully exploited in reducing the size of the domain to be analyzed, as shown in Fig. 3.1.

Figure 3.2: Spheroidal void in a cubical cell.
Thus, instead of studying the entire domain under consideration, we choose a representative cell. One of the most important criteria for choosing the shape of the cell is that it be space filling. The simplest space filling shape is the cube and this has been adopted for study. To study the effect of voids, we consider that a single ellipsoidal void exists at the center of each cell.

Fig. 3.2 shows an octant of a cell under consideration. This view shows the void geometry inside the cell. The void has been assumed to be a spheroid, such that any (001) plane section through the spheroid yields a circle (Fig. 3.2). The volume fraction of the void, $f$, is given by,

$$f = \frac{4\pi}{3} \left( \frac{a}{2} \right)^2 \left( \frac{c}{2} \right).$$

(3.1)

The yielding of the void cell aggregate is defined as follows. The void will serve to amplify the local stresses under a given applied load. This means that yield of the matrix material inside the cell will occur first at the void surface. Upon further loading, the plastic zones grow out toward the edges of the cell. If the matrix is a perfectly plastic material, the existence of a fully yielded ligament of matrix material that extends outward from the void surface to an edge of the cell implies a limit load for the material. This limit load is assumed to characterize the yield stress of the void cell aggregate. Thus, the yield stress of the void cell aggregate is larger than the stress required to cause initiation of yielding in the matrix, but smaller than the yield stress of a cell of unvoided pure matrix material.
B. Requirements for the yield function

The yield function of Gurson (1977) predicts that a voided aggregate can yield under a purely hydrostatic state of stress. Hence, a line in stress space that is equally inclined to the three principal directions must intersect the yield function at a finite distance. The Gurson model assumes that voids remain spheres at all stages during deformation. Hence, the stress vector and the strain increment vector are collinear during hydrostatic loading of a material containing spherical voids.

Fig. 3.3: Two dimensional hydrostatic loading of a square cell with an elliptical void.
Let us now consider a case of a material with a non-spherical void under a two dimensional hydrostatic loading (Fig. 3.3). From the study of Becker et al. (1989), it is known that the normal void growth rate is greater at point B than at point A. We can thus postulate that \(\text{d}E_{22} > \text{d}E_{11}\), where \(\text{d}E_{ij}\) denotes the \(ij^{th}\) component of the macroscopic strain increment tensor \(\text{d}E\).

Thus, a non-spherical void can lead to an anisotropic strain response under a hydrostatic applied stress state. Assuming normality for the yield function, we conclude that the [111] direction in stress space is not necessarily a normal to the yield surface at the point of intersection of the [111] direction and the yield function.

C. Form for the proposed yield function

We now examine the form for the yield function for a porous material. We seek to show the effect of void shape along with the effect of void size. The basic requirements for the yield function defined in stress space are that it be a convex function that is quadratic in the stress components. We also assume that the principle of normality [Drucker (1952)] holds for such a yield function. This is a consequence of the maximum dissipation of plastic work. In fact, the two are equivalent and one may be used to obtain the other. As has been pointed out earlier [Gurson (1977)], the assumption of normality for a porous material can be justified using arguments similar to those used by Bishop and Hill (1952) for the plastic response of a polycrystalline aggregate.
The yield surface in this work is assumed to be a hyper-ellipsoid in six-dimensional stress vector space. Such a choice automatically ensures the convexity of the yield surface. Since we use the Cauchy measure of stress, the symmetry of such a stress tensor allows us to conclude that the stress vector space is indeed six dimensional. We assume the existence of an orthogonal basis for such a space and label them as $m^{(i)} \ (i = 1, 2, ..., 6)$. Let $\Sigma$ denote the macroscopic stress tensor in such a stress space. We then represent the hyper-ellipsoidal yield function in stress space as,

$$\Phi = \left[ \Sigma \cdot m^{(1)} \right]^2 + \left[ \Sigma \cdot m^{(2)} \right]^2 + \left[ \Sigma \cdot m^{(3)} \right]^2 + \left[ \Sigma \cdot m^{(4)} \right]^2 + \left[ \Sigma \cdot m^{(5)} \right]^2 + \left[ \Sigma \cdot m^{(6)} \right]^2 - \sigma_m^2 = 0. \quad (3.2)$$

Thus, $\sigma_m$ denotes the equivalent stress of the material. It can be seen that the above form enforces the condition that the normal to the yield surface lies along the directions $m^{(i)}$. Thus these vectors conveniently form the axes of the hyper-ellipsoid. The lengths of vectors $m^{(i)}$ are seen to be the inverse of the lengths of the axes of the hyper-ellipsoid. The vectors $m^{(i)}$ are assumed to depend on void size and shape. For ease of manipulation, we need to refer the $m^{(i)}$ vectors to some convenient basis. For simplicity, we choose the same set of axes as those along which the components of the macroscopic stress vector are expressed. We further assume that this basis is a cartesian orthogonal basis, thereby obliterating any distinction between covariant and contravariant components.

Let $m^{(i)}_k$ be the $k$th component of $m^{(i)}$. We denote,
This allows us to recast the yield function (Equation 3.2) into the form,

\[
\Phi = Q_{ik} A_{ij} Q_{jk} - 1 = 0, \quad (i, j, k = 1, \ldots, 6),
\]

where,

\[
A_{ij} = \frac{\Sigma_i}{\bar{\sigma}_m} \frac{\Sigma_j}{\bar{\sigma}_m}.
\]  

It is to be noted that \(Q_{ij}\) are in actuality the components of a fourth order quantity. The precise representation of the hyper-ellipsoid in a \(3^4\) dimensional space is,

\[
\Phi = Q_{ipmn} A_{iprs} Q_{rstn} - 1 = 0
\]

where each of the indices run from 1 to 3, and

\[
A_{ijmn} = \frac{\Sigma_{ij}}{\bar{\sigma}_m} \frac{\Sigma_{mn}}{\bar{\sigma}_m}.
\]

We note from Equation 3.6 that \(Q_{ijkl} = Q_{jikl}\). The use of the form given in Equation 3.4 is justified as follows. We represent the voided material to be composed of a lattice of points spread continuously in space. The motif at each such lattice point is then the 'cell' as mentioned above. The symmetry of the structure is then given by symmetry of the motif alone, due to the assumption that the lattice points are spread continuously in three dimensional space. It will be shown that this symmetry yields less than 36 independent terms in the fourth order quantity \(Q_{ijmn}\) thereby justifying the use of a second order quantity \(Q_{kl}\).
Let us now examine the symmetries of the cell. While two-fold axes of rotational symmetry exist about the [100] and [010] directions, a higher four fold axis of rotational symmetry exists along the [001] direction due to the fact that it is the spheroidal axis. This symmetry is represented as \(4[001]^2[100]^2[110]\) or simply as 422 in the regular crystallographic notation [Burns and Glazer (1990)]. But, as can be observed, a higher symmetry exists for the cell. There are mirror planes of symmetry along the three axes of the void. This is usually represented as \(4[001]m[001], 2[100]m[100], 2[110]m[110]\) or just as 4/mmm [Burns and Glazer (1990)]. Any operation consistent with the symmetry of the cell leaves the cell identically disposed to an external impulse. We then impose the 4/mmm symmetry for the cell and examine the \(Q_{ijkl}\). Consider the symmetry group 4/mmm, to examine the individual symmetries in the group. The stereogram for this group is shown in Fig. 3.4[Burns and Glazer (1990)].

Figure 3.4: Stereogram for the symmetry group 4/mmm.
The sixteen symmetry elements in the group 4/mmm are then derived to be,
\[
\{ 1, \overline{4}, 4[001], 4^2[001], 4^3[001], 2[100], 2[\overline{1}40], 2[010], 2[110], m[001], \\
m[\overline{1}40], m[110], m[010], m[100], 4[001], 4^3[001] \}.
\]

Each of these sixteen symmetry elements corresponds to a point on the stereogram for the group 4/mmm, as can be seen from Figure 3.4. However we need not apply all the sixteen elements to observe the symmetry in $Q_{ijkl}$. It can be seen that one can just use three operations to obtain the necessary symmetry, as shown in Figure 3.5.

\[\begin{align*}
4[001] & \rightarrow 2[100] \\
m[001] & \rightarrow m[001]
\end{align*}\]

Figure 3.5: The operations $4[001]$, $2[100]$ and $m[001]$ form a basis for the group 4/mmm.
So, we use the operations $4_{[001]}$, $2_{[100]}$ and $m_{[001]}$ on $Q_{ijkl}$. We assume $Q_{ijkl}$ to be components of a fourth order tensor, and thus invoke the transformation,

$$Q_{ijkl} = R_{ai} R_{bj} R_{ck} R_{dl} Q_{abcd}$$

(3.8)

where $R_{ij}$ are the components of the rotation matrix.

Applying the above mentioned symmetries, we obtain seven independent, non-zero terms in $Q_{ijkl}$.

$$Q_{1111} = Q_{2222}$$

(3.9a)

$$Q_{1122} = Q_{2211}$$

(3.9b)

$$Q_{1133} = Q_{2233}$$

(3.9c)

$$Q_{3311} = Q_{3322}$$

(3.9d)

$$Q_{1212} = Q_{2121} = Q_{1221} = Q_{2112}$$

(3.9e)

$$Q_{1313} = Q_{2323} = Q_{1331} = Q_{2332} = Q_{3113} = Q_{3223} = Q_{3131} = Q_{3232}$$

(3.9f)

$$Q_{3333}$$

(3.9g)

We now need to consider the relation between the fourth order quantity $Q$ represented by its components $Q_{ijkl}$ and the second order quantity $Q_{mn}$. The only condition that we could use for this purpose is that the yield functions represented by Equations 3.4 and 3.6 are identically the same. We thus use the equivalence of yield in the two forms. We
consider pairs of loadings due to the fact that the yield function is quadratic in stresses. For each such pair, the forms in Equations 3.4 and 3.6 may be expanded and compared to obtain a relation between the second and fourth order representation for the material property matrix $Q$.

First, let us examine the nature of the yield function. Using equations 3.9a - 3.9g in Equation 3.6, there are no interaction terms between tension and shear stress loadings. Further, individual shear stresses themselves have no interaction terms with other shear stresses. However, normal stresses do have interaction terms with other normal stresses. Thus,

for $i \neq j$,

$$Q_{ijn} Q_{klmn} = 0, \text{ unless } (k = i \text{ and } l = j) \text{ or } (k = j \text{ and } l = i). \quad (3.10)$$

It must be noted that the product in Equation 3.10 does not equal $(\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li})$. Similarly, for $i = j$,

$$Q_{ijn} Q_{klmn} = 0, \text{ for } k \neq l. \quad (3.11)$$

The product in Equation 3.11 does not equal $\delta_{kl}$. The use of Equations 3.10 and 3.11 helps to relieve the tedium of expanding Equation 3.6.

Such a nature of non-interaction of shear stresses with other stresses and the presence of interaction terms between normal stresses is a common characteristic of other yield functions, for instance the Von Mises $J2$ yield criterion.
Enforcing the equality of yield in Equations 3.4 and 3.6 under pairs of stress loadings, we obtain the following after a straightforward but rather tedious calculation:

\[ Q_{11} = Q_{22} = Q_{1111} \]  \hspace{1cm} (3.12a)

\[ Q_{12} = Q_{21} = Q_{1122} \]  \hspace{1cm} (3.12b)

\[ Q_{13} = Q_{23} = Q_{1133} = Q_{2233} \]  \hspace{1cm} (3.12c)

\[ Q_{31} = Q_{32} = Q_{3311} = Q_{3322} \]  \hspace{1cm} (3.12d)

\[ Q_{33} = Q_{3333} \]  \hspace{1cm} (3.12e)

\[ Q_{44} = Q_{55} = 2 \sqrt{2} Q_{2323} \]  \hspace{1cm} (3.12f)

\[ Q_{66} = 2 \sqrt{2} Q_{1212} \]  \hspace{1cm} (3.12g)

These are the seven independent constants in the second order representation for \( Q \). Thus \( Q \) is represented as,

\[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\
Q_{12} & Q_{11} & Q_{13} & 0 & 0 & 0 \\
Q_{31} & Q_{31} & Q_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{66}
\end{bmatrix}
\]

The relations 3.12a-g parallel those in Equations 3.9a-g. Henceforth, all references to the fourth order quantity and its components will be eliminated and \( Q \) will be referred to purely by its second order components.

In conjunction with Equation 3.3, we now observe that,

\[
\begin{align*}
\{m^{(4)}\}^T &= \begin{bmatrix} Q_{11} & Q_{12} & Q_{31} & 0 & 0 & 0 \end{bmatrix}^T \\
\{m^{(5)}\}^T &= \begin{bmatrix} Q_{12} & Q_{11} & Q_{31} & 0 & 0 & 0 \end{bmatrix}^T \\
\{m^{(6)}\}^T &= \begin{bmatrix} Q_{13} & Q_{13} & Q_{33} & 0 & 0 & 0 \end{bmatrix}^T \\
\{m^{(7)}\}^T &= \begin{bmatrix} 0 & 0 & 0 & Q_{44} & 0 & 0 \end{bmatrix}^T \\
\{m^{(8)}\}^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & Q_{44} & 0 \end{bmatrix}^T \\
\{m^{(9)}\}^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix}^T
\end{align*}
\]

Using elementary row operations, the row reduced echelon form for the matrix \( Q \) exists, thereby proving that the \( m^{(i)} \) vectors are independent. We assume that the \( m^{(i)} \) vectors are orthogonal, but not orthonormal. Thus,

\[
m^{(i)} \cdot m^{(j)} = 0 \text{ for } i \neq j.
\]

The vectors \( m^{(4)} \), \( m^{(5)} \), and \( m^{(6)} \) are each already orthogonal to all other vectors, as can be observed from Equations 3.13 - 3.18.

We have proposed a functional form for a yield function for porous materials in terms of the \( m^{(i)} \) vectors. It is assumed that these vectors are functions of void shape and size alone. By construction, the vectors are
consistent with the periodicity and the 4/mmm symmetry of the void cell arrangement.

D. Resolution of the \( m^{(i)} \) vectors

Let us now examine the nature of the \( m^{(i)} \) vectors. The \( m^{(i)} \) vectors have the physical significance that along such directions, the normal to the yield surface (and thus the strain increment vector) is collinear with the corresponding \( m^{(i)} \) vector. For spherical voids, the \([1 1 1 0 0 0]^T\) direction in stress space is one such direction. We assume that the \( m^{(3)} \) vector lies along the \([1 1 1 0 0 0]^T\) direction in stress space, for spherical voids. However, the magnitude of this vector is as yet, unknown.

Let us now assume that for spherical voids,

\[
[m^{(3)}]^T = K(f) [1 1 1 0 0 0]^T
\]  

(3.20)

where \( K(f) \) is a function of the void volume fraction \( f \). Let us consider the loading of spheres in hydrostatic tension. Then,

\[
\Sigma_{11} = \Sigma_{22} = \Sigma_{33} = \Sigma_m \quad \text{where} \quad \Sigma_m \quad \text{denotes the mean stress.} \quad (3.21)
\]

Using orthogonality of the \( m^{(i)} \) vectors and Equation 3.20, we obtain for spheres,

\[
[m^{(1)}]^T = C^*(f) \begin{bmatrix} 2 \sqrt{3} & -1 & (\sqrt{3} - 1) & 0 & 0 & 0 \end{bmatrix}^T
\]

(3.22)

\[
[m^{(2)}]^T = C^*(f) \begin{bmatrix} -1 & (2 \sqrt{3}) & (\sqrt{3} - 1) & 0 & 0 & 0 \end{bmatrix}^T
\]

(3.23)

where \( C^*(f) \) is an arbitrary function of \( f \), which is independent of \( K(f) \).
Using the condition for yield (Equation 3.4) under the hydrostatic loading shown in Equation 3.21 we obtain,

\[ \Sigma_m = \frac{1}{3 K(f)} \]  

(3.24)

and is thus independent of \( C^*(f) \). For an elastic plastic spherical shell under internal pressure, Hill's solution [Hill (1949)] can be used to obtain the critical mean hydrostatic tension required to completely yield the spherical shell as

\[ \Sigma_m^{\text{shell}} = -\frac{2}{3} \ln(f) \]  

(3.25)

The Gurson yield function [Gurson (1977)] also tends to this classical limit under hydrostatic loading. Equation 3.25 shows that the distance in stress space from the origin to the point of intersection of the [111] direction with the yield surface increases in a logarithmic fashion. Thus, as \( f \) tends to zero, the spherical shell can never yield under hydrostatic tension.

Equation 3.25 cannot be used to obtain the value of \( K(f) \) because the Hill solution has been derived for isolated spherical shells. The problem at hand is for a periodic array of spheroidal voids. The actual method used to obtain the values of \( K(f) \) as a function of void volume fraction is explained in Chapter IV. The following form for \( K(f) \) is suggested based on Equation 3.25 and is justified by results reported in Chapter IV,

\[ K(f) = \frac{1}{3 \Sigma_{\text{mean}}^{\text{FEM}}} = \left[ \alpha_1 - \alpha_2 \ln(f) \right]^{-1}, \]  

where,  

(3.26)

\[ K(0) = 0 \]  

(3.27)
It is also possible to estimate the value of $C^*(0)$. It is reasonable to assume that as the void volume fraction reduces to zero, the yield function reduces to the Von Mises $J_2$ yield function. For this we expand the yield function (Equation 3.4), and compare it termwise to the terms in the $J_2$ yield function in the limit of $f$ tending to zero. This gives,

\[ Q_{1k}Q_{1k} = 1 = Q_{3k}Q_{3k} \tag{3.28} \]
\[ Q_{1k}Q_{2k} = -\frac{1}{2} = Q_{1k}Q_{3k} \tag{3.29} \]
\[ Q_{44} = \sqrt{3} = Q_{66} \tag{3.30} \]

Using Equations 3.22 and 3.23, and Equation 3.27, we obtain,

\[ C^*(0) = \frac{\sqrt{2 + \sqrt{3}}}{2} \tag{3.31} \]

![Diagrams of voids](image)  

a: Prolate void  
b: Oblate void  

Figure 3.6: Voids under hydrostatic tension.
We have thus far been concerned with the behavior of the $m^{(i)}$ axes as a function of void size. We shall now consider the effect of shape. First let us consider the axis $m^{(3)}$. The relevant components are $Q_{13}$ and $Q_{33}$ as given by Equation 3.15. Let us examine the case of an oblate versus a prolate void. The $X_3$ axis is the axis normal to which all sections of the ellipsoid yield circles.

Based on Figure 3.6a, there is greater area to carry load normal to the $X_3$ axis than the area normal to the $X_1$ and $X_2$ axes. Thus, we expect the compliance to be the largest when measured along the $X_3$ axis, for the prolate void case. Similarly, we expect the compliance along the $X_3$ axis to be the smallest for the oblate void case as shown in Figure 3.6b.

We therefore expect that for loading in hydrostatic tension,

$$dE_{11} = dE_{22} > dE_{33} \quad \text{for a prolate void}$$

$$dE_{11} = dE_{22} < dE_{33} \quad \text{for an oblate void}$$

where $dE$ denotes the macroscopic strain tensor.

Assuming that normality holds for the yield function, at the point of intersection of the $[1 \ 1 \ 1 \ 0 \ 0 \ 0]$ direction with the yield surface, the normal rotates toward the $\Sigma_{33}$ axis for an oblate void and away from the $\Sigma_{33}$ axis for a prolate void. This implies that the $m^{(3)}$ vector rotates toward the $X_1$-$X_2$ plane for an oblate void, and toward the $X_3$ axis for a prolate void. Using Equation 3.15, we conclude that,

$$Q_{13} - Q_{33} < 0 \quad \text{for a prolate void},$$

$$Q_{13} - Q_{33} > 0 \quad \text{for an oblate void}.$$
and

\[ Q_{13} - Q_{33} > 0 \quad \text{for an oblate void} \quad (3.35) \]

In the case of a spherical void, Equations 3.34 and 3.35 are consistent with Equation 3.20. Though the direction of rotation of the \( m^{(3)} \) axis is given by Equations 3.34 and 3.35, the extent of rotation is as yet undetermined.

In order to completely characterize the yield surface, the values of \( Q_{ij} \) need to be found for arbitrary void shapes and sizes. This aspect is dealt with in the next section.

E. Calculation of the \( m^{(i)} \) vectors

In this section, we explain the method of calculating the values of the seven independent non-zero terms in the material property matrix \( Q \). The matrix \( Q \) consists of the \( m^{(i)} \) vectors as columns. The components of these vectors are not functions of stress. Instead they depend solely on the void size and shape.

Assume now that \( \Sigma_{1}^y \) represents the yield stress under uniaxial tension along the \( X_1 \) direction (Figure 3.2) and \( \Sigma_{3}^y \) represents the yield stress under uniaxial tension along the \( X_3 \) direction, for the voided material. Further, let \( \Sigma_{1}^E, \Sigma_{2}^E, \Sigma_{3}^E \) represent the stresses required to yield the voided material subject to the remote boundary condition of equal normal deformation rates in the \( X_1, X_2, \) and \( X_3 \) directions. In each of these cases, the void is assumed to be in an elastic perfectly plastic matrix.
Considering a uniaxial stress along the $X_1$ direction, the condition of yield as given by Equation (3.4) is,

$$Q_{1k} Q_{1k} = Q_{11}^2 + Q_{12}^2 + Q_{13}^2 = [\Sigma_1 y]^2.$$  \hfill (3.36)

Yield under uniaxial stress along the $X_3$ direction is given by,

$$Q_{3k} Q_{3k} = 2Q_{31}^2 + Q_{33}^2 = [\Sigma_3 y]^2.$$  \hfill (3.37)

The $m^{(i)}$ vectors are also orthogonal. Examining Equations (3.13) to (3.18), we see that there remain only two non-trivial relations. The orthogonality of the $m^{(1)}$ and $m^{(2)}$ vectors yields

$$2 Q_{11} Q_{12} + Q_{31}^2 = 0.$$  \hfill (3.38)

The orthogonality of $m^{(1)}$ and $m^{(3)}$ yields,

$$(Q_{11} + Q_{12}) Q_{13} + Q_{31} Q_{33} = 0.$$  \hfill (3.39)

Let us now consider the case of loading a periodic array of spheroidal voids under the condition of equal macroscopic normal deformation rates along the $X_1$, $X_2$, and $X_3$ directions. Using normality,

$$D_{ij}^p = \Lambda \frac{\partial \Phi}{\partial \Sigma_{ij}} \quad \{i,j = 1,2,3\}.$$  \hfill (3.40)

Equivalently we write,

$$D_m^p = \Lambda \frac{\partial \Phi}{\partial \Sigma_m} \quad \{m = 1..6\}.$$  \hfill (3.41)

Using Equation (3.4),
\[ D_m^P = \Lambda \frac{\partial [Q_{ik} A_{ij} Q_{jk} - 1]}{\partial \Sigma_m} , \]  
(3.42)

\[ = \Lambda Q_{ik} \frac{\partial A_{ij}}{\partial \Sigma_m} Q_{jk} \quad \{ \text{All indices run from 1 to 6}. \} \]  
(3.43)

Using the definition of \( A_{ij} \) given by Equation (3.5), we arrive at,

\[ D_m^P = 2 \Lambda Q_{ik} \Sigma_i Q_{mk} \]  
(3.44)

The condition that \( D_1^P = D_3^P \) yields,

\[ \Sigma_3^E = \gamma \Sigma_1^E , \]  
(3.45)

where,

\[ \gamma = \frac{[\Sigma_1^y]^2 + Q_{1k} Q_{2k} - 2 Q_{1k} Q_{3k}}{[\Sigma_3^y]^2 - Q_{1k} Q_{3k}} . \]  
(3.46)

We then use the yield condition to obtain,

\[ [\Sigma_1^E]^2 = \left( 2 [\Sigma_1^y]^2 + \gamma [\Sigma_3^y]^2 + 2 \gamma Q_{1k} Q_{3k} + 2 Q_{1k} Q_{2k} \right)^{-1} , \]  
(3.47)

where the contractions used in Equation (3.46) and (3.47) are given by,

\[ Q_{1k} Q_{2k} = 2 Q_{11} Q_{12} + Q_{13}^2 , \]  
(3.48)

\[ Q_{1k} Q_{3k} = (Q_{11} + Q_{12}) Q_{31} + Q_{13} Q_{33} . \]  
(3.49)

The values of \( Q_{11} , Q_{12} , Q_{13} , Q_{31} \) and \( Q_{33} \) can now be easily calculated for a given void shape and size using Equations (3.36), (3.37), (3.38), (3.39) and a
combination of Equations (3.45) and (3.47). We note that for this purpose, we need the values of \( \Sigma_1^y \), \( \Sigma_3^y \), \( \Sigma_1^E \) and \( \Sigma_3^E \). The calculation is carried out as follows. From Equations (3.38) and (3.39) we obtain,

\[
Q_{11} = \frac{-Q_{31} Q_{33} + Q_{31} \sqrt{Q_{33}^2 + 2Q_{13}^2}}{2Q_{13}} \quad (3.50)
\]

\[
Q_{12} = \frac{-Q_{31} Q_{33} + Q_{31} \sqrt{Q_{33}^2 + 2Q_{13}^2}}{2Q_{13}} \quad (3.51)
\]

Using Equations (3.50) and (3.51) in Equation (3.36), we obtain,

\[
Q_{31} = \frac{\left( \left( \Sigma_3^y \right)^2 + Q_{13}^2 \right) + \sqrt{\left( \left( \Sigma_3^y \right)^2 + Q_{13}^2 \right)^2 - 8Q_{13}^2 \left( \left( \Sigma_1^y \right)^2 - Q_{13}^2 \right)}}{4} \quad (3.52)
\]

Using Equation (3.52) in Equation (3.38) yields \( Q_{33} \). We have then the need to evaluate \( Q_{13} \). Equation (3.15) shows that \( Q_{13} \) is the component of the \( m^{(3)} \) vector along the \( X_1 \) (or \( X_2 \)) direction. For spherical voids, \( m^{(3)} \) is chosen to lie along the hydrostatic direction [See Equation (3.20)]. Now, let \( \Sigma^E \) represent the length of the line joining the origin to the point of intersection of the line \( \{1\ 1\ 1\ 0\ 0\ 0\} \) line in stress space with the yield surface. Thus,

\[
\Sigma^E = \sqrt{[\Sigma_1^E]^2 + [\Sigma_2^E]^2 + [\Sigma_3^E]^2} \quad (3.53)
\]

We stipulate further that the value of \( Q_{13} \) is such that the deviation of the value of \( \Sigma^E \) calculated using the \( Q_{ij} \) values shows the minimum
percentage deviation from the actual value given by Equation (3.53). To this end, Equations (3.45), (3.46) and (3.47) are used.

The calculation of $Q_{44}$ and $Q_{66}$ are simpler in comparison. Using the condition of normality given by Equation (3.44), let us consider the case of the loading of a periodic array of voids under a pure remote shear deformation rate $D_{23}^P$ or $D_4^P$. Then,

$$D_4^P = 2 \Lambda Q_{ik} Q_{4k} \Sigma_i$$  \hfill (3.54)

But, Equations 3.13 to 3.18 give,

$$Q_{ik} Q_{4k} = Q_{44}^2 \delta_{i4}$$  \hfill (3.55)

Here $\delta_{ij} = 0$ for $i \neq j$ and equals one for $i = j$, where the indices run from 1 to 6. Thus, Equation 3.54 yields,

$$D_4^P = 2 \Lambda Q_{44}^2 \Sigma_4.$$  \hfill (3.56)

Equation 3.56 proves that $D_4^P$ is affected only by $\Sigma_4$ even though other stresses may exist.

Next, consider the case of remote loading by a pure shear stress of $\Sigma_{23}$ or $\Sigma_4$. Normality (Equation 3.44) yields,

$$D_m^P = 2 \Lambda Q_{mk} Q_{4k} \Sigma_4.$$  \hfill (3.57)

Using Equation 3.55 in 3.57, we obtain,

$$D_m^P = 2 \Lambda Q_{44}^2 \Sigma_4 \delta_{m4}.$$  \hfill (3.58)
Thus, under a remote shear stress $\Sigma_4$, the only deformation rate component is $D_4^p$.

Thus, we may conclude that a state of pure remote shear deformation rate $D_4^p$, is identical to a state of a uniaxial remote shear stress $\Sigma_4$, with respect to yielding. Similarly, a state of pure remote shear deformation rate $D_6^p$, is identical to a state of a uniaxial remote shear stress $\Sigma_6$, for purposes of calculating yielding stresses.

Thus, we may use limit loads obtained from remote shear loadings of deformation rate as the limit loads obtained from remote shear loadings of stress.

Using this idea, we employ the yield criterion to conclude that,

$$Q_{4k} Q_{4k} = Q_{44}^2 = \left( \Sigma_4^y \right)^2,$$

where $\Sigma_4^y$ is the yield stress for yielding the void cell aggregate in remote shear loading. Similarly,

$$Q_{6k} Q_{6k} = Q_{66}^2 = \left( \Sigma_6^y \right)^2,$$

where $\Sigma_6^y$ is the yield stress for yielding the void cell aggregate in remote shear loading. Equations (3.59) and (3.60) are used to obtain the values of $Q_{44}$ and $Q_{66}$.

We have thus enumerated the procedure to calculate the values of the components of the $Q$ matrix. These components are the components of $Q$ represented in a stress space. The stress space axes are the stress
values when the stresses are referred to a set of axes aligned to the principal void axes. Thus, these components of $Q$ are referred to the material axes and reflect the symmetry and texturing of the periodic distribution of voids in the matrix.

All that remains is to obtain the yield values under the uniaxial and the equal deformation rate mode loading conditions, for various void shapes and sizes. We have chosen to obtain these values using the finite element method. The advantages of this method are its ability to handle complex geometries and the possibility of applying complex boundary conditions. Experimental methods may not be easy and may prove to be too time intensive. The next chapter will discuss the finite element methodology and the results obtained.

Using the FEM results of Chapter IV, the values of the $Q$ matrix are calculated in Appendix A. Appendix A also gives the functional forms for the variation of $Q$ with void volume fraction and aspect ratio. Using the functional forms for the $Q$ matrix, the yield function can be constructed from eqns. (3.4) and (3.5) for a porous material of given void volume fraction of periodically distributed ellipsoidal voids.
CHAPTER IV

FEM STUDY TO CALIBRATE YIELD FUNCTION

A. Introduction

Finite element analysis is a powerful numerical tool for solving boundary and initial value problems on arbitrarily shaped domains. The solution space and the geometric space are discretized, and the governing differential equations are converted into algebraic equations. The boundary conditions on the problem are then applied. The algebraic equations are solved to obtain the solution variables at fixed points (either material or spatial) called 'nodes'. These nodal values are used to obtain the variation of the solution variable(s) within the domain. Finite element analysis can be applied to domains with moving boundaries and for constrained problems. Thus, it is well suited for simulation of metal forming problems, where boundaries deform with a constraint of zero dilatation.

To model the periodicity in the field variables of stress and strain, we need to impose periodic boundary conditions. The finite element program ABAQUS (1991) has been used for this purpose. Twenty-noded hybrid "brick" elements with independent interpolation functions for the displacement and pressure were used. These elements are especially well suited to study the deformation behavior of incompressible materials. The
matrix is treated as elastic-perfectly plastic. Further, we take advantage of the symmetry of the loading and geometry to restrict the size of the representative finite element cell. This part of the cell is then discretized into elements.

We now examine the meshing scheme used for dividing an octant of the cell. The mesh was generated using the pre-processor of ABAQUS (1991).

Figure 4.1: Mesh generation in octant by subdivision into prisms.
Generation of the mesh is facilitated by the use of the following idea. An octant with an eighth of the void in the cube is a composite figure with three plane rectangular faces, three more non-rectangular plane faces, and the surface of the void. Paving a full cube with brick elements is easy because there exist exactly six rectangular faces with a one to one correspondence between points on opposite faces. For the problem at hand, the octant is thought to be made of three prisms. The void surface is divided into three sub-areas. Each sub-area is a base of one of the three prisms. Each of the three plane rectangular external faces constitutes the other base. This is shown in Fig. 4.1. The three sub-areas on the void surface for each prism are \(A_y-D_y-C_y-B_y\), \(D_y-H_y-G_y-C_y\) and \(C_y-G_y-F_y-B_y\). The corresponding external faces of the octant for each prism are \(A-D-C-B\), \(D-H-G-C\) and \(C-G-F-B\). Thus, the octant is now divided into three prisms \(A_y-D_y-C_y-B_y-A-D-C-B\), \(D_y-H_y-G_y-C_y-D-H-G-C\) and \(C_y-G_y-F_y-B_y-C-G-F-B\). Nodes are then generated in each prism such that there is a one to one correspondence between nodes on each void sub area and its corresponding external rectangular face. Such a method enables us to create the required mesh of brick elements in the octant of the cell.

To determine the smallest mesh size that would give accurate yield stress values, finite element test runs of uniaxial and hydrostatic tension of spherical voids were employed. It was concluded that a mesh of 81 twenty noded brick elements in an octant yielded values as accurately as more dense meshes of 192 and 375 elements. Thus, a mesh of 81 twenty noded elements in an octant of the cell was used for finite element simulations to determine yield values. However, to obtain contour plots
of the Von Mises effective stress and the effective plastic strain, a more dense mesh with 192 twenty noded brick elements (e.g., see Fig. 4.4) was used.

B. Uniaxial tension loadings

The finite element cell was first subjected to a series of uniaxial tension simulations using FEM. The loading direction was assumed to be along one of the void axis directions. For instance, Fig. 4.2 shows the case of a macroscopic stress $\Sigma_{33}$ along $X_3$.

![Figure 4.2: Loading of the cubical cell of dimension $s$ along the $X_3$ direction.](image-url)
For uniaxial tension, we need to impose periodic boundary conditions. For example, we write the boundary conditions for loading along $X_3$ as follows.

\[
\Delta \left( u_3(X_1, X_2, -\frac{s}{2}) - u_3(X_1, X_2, \frac{s}{2}) \right) = s \Delta E_{33} ;
\]

\[
t_3(X_1, X_2, -\frac{s}{2}) + t_3(X_1, X_2, \frac{s}{2}) = 0 ;
\]

\[
\frac{1}{s} \int_{-\frac{s}{2}}^{\frac{s}{2}} t_2 \, dX_3 = 0, \quad X_2 = \pm \frac{s}{2} ;
\]

\[
\frac{1}{s} \int_{-\frac{s}{2}}^{\frac{s}{2}} t_1 \, dX_3 = 0, \quad X_1 = \pm \frac{s}{2} ;
\]  

\[
t_1 = 0 = t_2 \text{ on } X_3 = \pm \frac{s}{2} ;
\]

\[
t_2 = 0 = t_3 \text{ on } X_1 = \pm \frac{s}{2} ;
\]

\[
t_3 = 0 = t_1 \text{ on } X_2 = \pm \frac{s}{2} .
\]  

Here, $\Delta$ denotes a time increment associated with deformation, $s$ is the dimension of the cubical cell, $u$ denotes the displacement vector, and $t$ denotes the traction vector. Due to the symmetry of the cell and loading, we need to consider only an eighth of the cell for the finite element simulation of uniaxial tension.

A typical finite element macroscopic stress-strain plot for uniaxial $X_3$ loading is shown in Figure 4.3. The void was initially an oblate void with an aspect ratio of $\frac{a}{c} = 5$ and of 1% volume fraction. The matrix was assumed to be an isotropic-elastic, perfectly-plastic material with a yield stress of $10^4$ units. The Young's modulus was assumed to be $10^7$ units and the Poisson's ratio was assumed to be 0.30. The assumption of perfect plasticity
enables a clear definition of a limit load. The macroscopic stress-strain plot shown in Figure 4.3 is typical of other uniaxial finite element simulations.

Upon loading, the stress concentration due to the oblate void causes the Von Mises effective stress to be highest around the equatorial $X_1$-$X_2$ plane, as shown in Fig. 4.4. The region of the matrix adjoining the void along the $X_3$ direction is observed to remain rigid. The cell remains fully elastic until a macroscopic strain of about $0.27\varepsilon_y$. At this strain plasticity begins to develop at the equator of the void along the $X_1$-$X_2$ plane. Upon further loading, this plastic zone grows until the entire $X_1$-$X_2$ plane at the void section yields fully. Figure 4.5 shows the contours of the Von Mises effective stress at a macroscopic strain of $E_{33} = 2.5\varepsilon_y$. It is seen that the entire $X_1$-$X_2$ plane at the void section has effective stress equal to or exceeding the matrix yield stress. Though the matrix is elastic perfectly plastic, the mesh artificially induces hardening resulting in regions with effective stress higher than the yield stress of the matrix. This indicates that there are some insufficiencies in implementation of the ABAQUS hybrid elements. The effect of this artificial hardening on the limit load is negligible, as can be observed from Fig. 4.3. The rigid region along the $X_3$ direction is also found to have shrunk as the strain is increased to $E_{33} = 2.5\varepsilon_y$. Figure 4.6 shows the contours of the effective plastic strain at a macroscopic strain of $E_{33} = 2.5\varepsilon_y$. The entire $X_1$-$X_2$ plane at the void section has yielded. It is also seen that the magnitude of the effective plastic strain on the equatorial $X_1$-$X_2$ plane at the void decreases with
increasing distance from the void. This is due to the fact that the plastic zone grew outward from the void surface.

The contour plots of Von Mises effective stress for spherical voids is shown in fig. 4.7 for simple tension along the $X_3$ direction at a strain of $E_{33} = 2.5\varepsilon_y$. The entire equatorial $X_1-X_2$ plane at the void section has yielded and the low strain region along the $X_3$ axis remains. The plasticity which initiated at the void surface on the equatorial $X_1-X_2$ plane, was observed to grow with loading until the entire plane yielded, as shown in fig. 4.8.

The Von Mises stress distribution for prolate voids of aspect ratio $\frac{a}{c} = \frac{1}{2}$ is shown in fig. 4.9 for simple tension along the $X_3$ direction at a strain of $E_{33} = 2.5\varepsilon_y$. The entire void section is observed to have yielded. The effective plastic strain contours are shown in fig. 4.10 at the same strain. A lower strain region is still observed along the $X_3$ axis.

From figs. 4.6, 4.8 and 4.6, the maximum plastic strain at the void surface under uniaxial $X_3$ loading increases as the void changes from prolate to spherical to oblate at a fixed void volume fraction. This trend scales with the void projected area normal to the loading direction $X_3$. As the void becomes more oblate, the elastic stress concentration due to the void increases. This can be seen by considering the limiting case of maximum stress concentration which is due to a crack in the $X_1-X_2$ plane, loaded along the $X_3$ direction. The increased stress concentration due to the oblate shaped void results in the earlier evolution of plasticity in oblate voided materials as compared to prolate voided materials of the
same void volume fraction. This also results in a higher maximum plastic
strain at the void surface for oblate voids, at a given macroscopic strain
$E_{33}$.

Finite element simulations of uniaxial tension along $X_3$ were
performed for various void volume fractions and aspect ratios. These
results are summarized in Table 4.1.
Figure 4.3: FEM macroscopic stress-strain plot for uniaxial loading along $X_3$. Initial void volume fraction $= 1\%$. Initial aspect ratio $= 5$. 
Figure 4.4: FEM Von Mises effective stress contours at a strain of $E_{33} = 0.047\sigma_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. $\sigma_y = 10^4$ units.
Figure 4.5: FEM Von Mises effective stress contours at a strain of $E_{33} = 2.5 \varepsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c}=5$. $\sigma_y = 10^4$ units.
Figure 4.6: FEM effective plastic strain contours at a strain of $E_{33} = 2.5\varepsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. $\varepsilon_y = 10^{-3}$. 
Figure 4.7: FEM Von Mises effective stress contours at a strain of $E_{33} = 2.5\varepsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $a_c = 1$. $\sigma_y = 10^4$ units.
Figure 4.8: FEM effective plastic strain contours at a strain of $E_{33} = 2.5\epsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 1$. $\epsilon_y = 10^{-3}$. 
Figure 4.9: FEM Von Mises effective stress contours at a strain of $E_{33} = 2.5\varepsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$. $\sigma_y = 10^4$ units.
Figure 4.10: FEM effective plastic strain contours at a strain of $\varepsilon_{33} = 2.5\varepsilon_y$ for uniaxial loading along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$. $\varepsilon_y = 10^{-3}$. 
Table 4.1: FEM results of uniaxial simple tension along $X_3$.

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{33}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.102</td>
<td>0.8748</td>
</tr>
<tr>
<td>4.0</td>
<td>0.9594</td>
</tr>
<tr>
<td>2.0</td>
<td>0.9757</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9835</td>
</tr>
<tr>
<td>2/3</td>
<td>0.9860</td>
</tr>
<tr>
<td>1/2</td>
<td>0.9874</td>
</tr>
<tr>
<td>1/4</td>
<td>0.9892</td>
</tr>
<tr>
<td>0.209135</td>
<td>0.9895</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{33}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.9161</td>
</tr>
<tr>
<td>2.0</td>
<td>0.9486</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9569</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9663</td>
</tr>
<tr>
<td>2/3</td>
<td>0.9710</td>
</tr>
<tr>
<td>1/2</td>
<td>0.9738</td>
</tr>
<tr>
<td>1/4</td>
<td>0.9782</td>
</tr>
</tbody>
</table>

<table>
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<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{33}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.8410</td>
</tr>
<tr>
<td>2.0</td>
<td>0.8764</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8952</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9164</td>
</tr>
<tr>
<td>2/3</td>
<td>0.9286</td>
</tr>
<tr>
<td>1/2</td>
<td>0.9360</td>
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<tr>
<td>1/3</td>
<td>0.9446</td>
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</table>
Table 4.1 (continued): FEM results of uniaxial simple tension along $X_3$.

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\Sigma_{33}$</th>
<th>$\sigma_{\text{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f=0.10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td></td>
<td>0.7750</td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td>0.8065</td>
</tr>
<tr>
<td>1.25</td>
<td></td>
<td>0.8229</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>0.8420</td>
</tr>
<tr>
<td>4/5</td>
<td></td>
<td>0.8557</td>
</tr>
<tr>
<td>2/3</td>
<td></td>
<td>0.8667</td>
</tr>
<tr>
<td>1/2</td>
<td></td>
<td>0.8820</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>$\Sigma_{33}$</th>
<th>$\sigma_{\text{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f=0.20$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td>0.6601</td>
</tr>
<tr>
<td>1.25</td>
<td></td>
<td>0.6898</td>
</tr>
<tr>
<td>1.1</td>
<td></td>
<td>0.7074</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>0.7206</td>
</tr>
<tr>
<td>1/1.1</td>
<td></td>
<td>0.7309</td>
</tr>
<tr>
<td>4/5</td>
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<td>0.7456</td>
</tr>
<tr>
<td>2/3</td>
<td></td>
<td>0.7653</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\Sigma_{33}$</th>
<th>$\sigma_{\text{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f=0.30$</td>
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<td></td>
</tr>
<tr>
<td>1.25</td>
<td></td>
<td>0.5572</td>
</tr>
<tr>
<td>1.10</td>
<td></td>
<td>0.5928</td>
</tr>
<tr>
<td>1.05</td>
<td></td>
<td>0.6030</td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td>0.6147</td>
</tr>
<tr>
<td>1/1.05</td>
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</tr>
<tr>
<td>4/5</td>
<td></td>
<td>0.6509</td>
</tr>
</tbody>
</table>
From Fig. 4.11, we see that the macroscopic stress $\Sigma_{33}$ decreases as the void becomes increasingly oblate. This is again seen to correlate with the fact the area normal to the $X_3$ axis decreases as the void becomes more oblate.

We now consider finite element simulations of uniaxial tension along the $X_1$ direction. Figure 4.12 shows the contour plot of Von Mises
effective stress for oblate voids of aspect ratio $\frac{a}{c}=5$ at a strain of $E_{11} = 0.05\varepsilon_y$. The elastic stress concentration due to the void is observed. The cell is fully elastic at this strain. A rigid region is observed along the $X_1$ direction adjacent to the void. As loading proceeds, a region with high effective stress grows out from the void surface predominantly in the $X_2$-$X_3$ plane. Simultaneously, the rigid region shrinks in extent. The $X_2$-$X_3$ plane at the void yields completely at a macroscopic strain of $E_{11} = 1.09\varepsilon_y$. At this strain, the $X_3$-$X_1$ and $X_1$-$X_2$ planes at the void section are yet to yield completely due to a more rigid region along the $X_1$ direction adjacent to the void. At $E_{11} = 2.5\varepsilon_y$, the rigid region shrinks almost totally, and all faces on the void section yield completely as shown in fig. 4.13. Figure 4.14 shows the variation of effective plastic strain in the cell at the strain of $E_{11} = 2.5\varepsilon_y$. The effective plastic strain is maximum at the void surface on the $X_2$ plane because plasticity initiated in that region.

Figure 4.15 depicts the variation of the Von Mises effective stress in the cell for uniaxial loading along $X_1$ direction at a strain of $E_{11} = 2.5\varepsilon_y$. The void is prolate with an aspect ratio of $\frac{a}{c} = \frac{1}{2}$. The rigid region is seen to be larger in extent for the case of the prolate void as compared to the oblate void shown in fig. 4.13. Figure 4.16 portrays the variation of the effective plastic strain with position at a strain of $E_{11} = 2.5\varepsilon_y$. The prolate voided case exhibits a larger projected void area normal to the loading direction, $X_1$. This results in a more marked concentration of effective plastic strain along the $X_2$-$X_3$ plane at the void section, when compared to the oblate voided case shown in fig. 4.14, or the spherical void shown in fig. 4.8.
Finite element simulations of uniaxial loading along the $X_1$ direction were performed for various void sizes and aspect ratios. These results are summarized in Table 4.2.
Figure 4.12: FEM Von Mises effective stress contours at a strain of $E_{11} = 0.05 \varepsilon_y$ for uniaxial loading along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. $\sigma_y = 10^4$ units.
Figure 4.13: FEM Von Mises effective stress contours at a strain of $E_{11} = 2.5E_y$ for uniaxial loading along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. $\sigma_y = 10^4$ units.
Figure 4.14: FEM effective plastic strain contours at a strain of $E_{11} = 2.5\varepsilon_y$ for uniaxial loading along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. $\varepsilon_y = 10^{-3}$. 
Figure 4.15: FEM Von Mises effective stress contours at a strain of $E_{11} = 2.5\varepsilon_y$ for uniaxial loading along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$, $\sigma_y = 10^4$ units.
Figure 4.16: FEM effective plastic strain contours at a strain of $E_{11} = 2.5\varepsilon_y$ for uniaxial loading along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 2$. $\varepsilon_y = 10^{-3}$. 
Table 4.2: FEM results of uniaxial tension along $X_1$.

\[
\begin{array}{l|c}
\text{f=0.01} & \frac{\Sigma_{11}}{\sigma_{yield}} \\
\hline
\text{Aspect Ratio } \frac{a}{c} & \\
14.102 & 0.9898 \\
4.0 & 0.9878 \\
2.0 & 0.9857 \\
1.0 & 0.9835 \\
2/3 & 0.9812 \\
1/2 & 0.9802 \\
1/4 & 0.9782 \\
0.209135 & 0.9778 \\
\end{array}
\]

\[
\begin{array}{l|c}
\text{f=0.02} & \frac{\Sigma_{11}}{\sigma_{yield}} \\
\hline
\text{Aspect Ratio } \frac{a}{c} & \\
4.0 & 0.9753 \\
2.0 & 0.9710 \\
1.5 & 0.9687 \\
1.0 & 0.9663 \\
2/3 & 0.9620 \\
1/2 & 0.9599 \\
1/4 & 0.9559 \\
\end{array}
\]

\[
\begin{array}{l|c}
\text{f=0.05} & \frac{\Sigma_{11}}{\sigma_{yield}} \\
\hline
\text{Aspect Ratio } \frac{a}{c} & \\
3.0 & 0.9357 \\
2.0 & 0.9288 \\
1.5 & 0.9231 \\
1.0 & 0.9164 \\
2/3 & 0.9070 \\
1/2 & 0.9022 \\
1/3 & 0.8951 \\
\end{array}
\]
Table 4.2 (Continued): FEM results of uniaxial tension along $X_1$.

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\Sigma_{11}$</th>
<th>$\sigma_{\text{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f=0.10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.8667</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.8559</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>0.8488</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.8420</td>
<td></td>
</tr>
<tr>
<td>4/5</td>
<td>0.8326</td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>0.8264</td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>0.8155</td>
<td></td>
</tr>
<tr>
<td>$f=0.20$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.7450</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>0.7333</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>0.7257</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.7206</td>
<td></td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.7134</td>
<td></td>
</tr>
<tr>
<td>4/5</td>
<td>0.7048</td>
<td></td>
</tr>
<tr>
<td>2/3</td>
<td>0.6888</td>
<td></td>
</tr>
<tr>
<td>$f=0.30$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>0.6322</td>
<td></td>
</tr>
<tr>
<td>1.1</td>
<td>0.6215</td>
<td></td>
</tr>
<tr>
<td>1.05</td>
<td>0.6174</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.6147</td>
<td></td>
</tr>
<tr>
<td>1/1.05</td>
<td>0.6079</td>
<td></td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.6027</td>
<td></td>
</tr>
<tr>
<td>4/5</td>
<td>0.5860</td>
<td></td>
</tr>
</tbody>
</table>
The macroscopic yield stress plotted in Fig. 4.17 is obtained by summing the tractions on the face with $X_1$ as normal and dividing it by the area of that face. This stress is then normalized by the yield stress in tension of the elastic-perfectly plastic matrix. It can be seen from Figure 4.17 that as the void becomes more oblate, the yield stress increases. This behavior correlates with the fact that the projected area at the minimum
section normal to the loading axis $X_1$ increases as the void becomes more oblate.

It is interesting to compare the results for uniaxial tension to the results of Gurson (1977). Let $\Sigma_{uni}^{Gur}$ denote the uniaxial stress to yield a porous material of void volume fraction $f$. Let $\sigma_y$ denote the yield stress of the matrix. The Gurson yield function (eqn. 2.20) gives

$$\Phi = \left(\frac{\Sigma_{uni}^{Gur}}{\sigma_y}\right)^2 + 2f \cosh\left(\frac{3\sigma_{\text{mean}}}{2\sigma_y}\right) - \{1 + f^2\} = 0 . \quad (4.2)$$

Since $\cosh(x) \geq 1$ for $x > 0$, we obtain,

$$\frac{\sigma}{\sigma_y} < (1-f) . \quad (4.3)$$

Examining the FEM values for the uniaxial yield stresses reported in Tables 4.1 and 4.2, it is seen that the inequality (4.3) holds for the FEM values also. The ratio approaches unity as the voids elongate along the direction of uniaxial stressing.

Table 4.3 lists the values of the yield stress calculated by using the Gurson yield function for the case of uniaxial tension. Figure 4.18 compares the Gurson yield predictions to the FEM results for spherical voids under uniaxial tension.
Table 4.3: The Gurson yield values under uniaxial tension.

<table>
<thead>
<tr>
<th>Void volume fraction f</th>
<th>$\Sigma_{uni}^{Gur}$</th>
<th>$\Sigma_{uni}^{Gur}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.9772</td>
<td>$\sigma_{yield}$</td>
</tr>
<tr>
<td>0.02</td>
<td>0.9550</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.8936</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>0.8038</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>0.6706</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>0.5849</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.18: Comparison of the Gurson yield predictions to the FEM yield values for spherical voids.
It is seen that the Gurson yield stress predicts a lower stress for uniaxial yield than shown by the FEM results. However, it must be noted that the Gurson model predicts bounds for a spherical void in a spherical shell. Such a shell model does not satisfy compatibility of displacements if a periodic array of such shells is assumed to compose the entire material. In fact, the spherical shell cell model is not space filling. The conclusion that can be drawn from Fig. 4.18 is that the Gurson model is reasonably accurate in predicting the yield values under uniaxial yield for spherical voids. In other words, if the void volume fraction is an appropriate parameter to characterize the yield value of a voided material in uniaxial tension, then the Gurson model can be employed.

However, it is clear from Fig. 4.11 and Fig. 4.17 that for a given void volume fraction, a change in the aspect ratio changes the yield value. This change of yield value with aspect ratio increases with void volume fraction. Thus, the void volume fraction alone does not adequately represent the yield behavior of a voided material.
C. Pure Shear Loadings

The finite element cell shown was subjected to loadings of pure macroscopic shear deformation rates. Once again, to model the periodicity in loading and of the geometry, we impose periodic boundary conditions. Referring to fig. 4.2, we write the boundary conditions for a macroscopic shear deformation rate $D_{23}$:

$$\Delta [ u_2(X_1, X_2, s/2) - u_2(X_1, X_2, -s/2) ] = s \Delta E_{23},$$

$$t_2 (X_1, X_2, s/2) + t_2 (X_1, X_2, -s/2) = 0,$$

$$u_3 (X_1, X_2, s/2) - u_3 (X_1, X_2, -s/2) = 0,$$

$$t_3 (X_1, X_2, s/2) + t_3 (X_1, X_2, -s/2) = 0,$$

$$\Delta [ u_3(X_1, s/2, X_3) - u_3(X_1, -s/2, X_3) ] = s \Delta E_{23},$$

$$t_3 (X_1, s/2, X_3) + t_3 (X_1, -s/2, X_3) = 0,$$

$$\Delta [ u_2(X_1, s/2, X_3) - u_2(X_1, -s/2, X_3) ] = 0,$$

$$t_2 (X_1, s/2, X_3) + t_2 (X_1, -s/2, X_3) = 0.$$

The above conditions do not assume any conditions about the symmetry of the cell itself. Imposing the symmetry of the cell and loading, we can see that only a quarter of the cell needs to be used in order to complete the finite element simulations of pure shear loadings. Table 4.4 lists the values of the yield stress for shear loading deformation rate $D_{12}$, while table 4.5 lists the yield stress values for loading under $D_{23}$. The yield stresses reported in these tables are normalized with respect to the uniaxial yield stress of the unvoided elastic-perfectly plastic matrix material.
Table 4.4: FEM results of pure shear \( \frac{\Sigma_{12}}{\sigma_{yield}} \).

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{12}}{\sigma_{yield}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.5604</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5572</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5537</td>
</tr>
<tr>
<td>2/3</td>
<td>0.5508</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5486</td>
</tr>
<tr>
<td>1/4</td>
<td>0.5443</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{12}}{\sigma_{yield}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.5489</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5427</td>
</tr>
<tr>
<td>1.5</td>
<td>0.5400</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5356</td>
</tr>
<tr>
<td>2/3</td>
<td>0.5307</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5276</td>
</tr>
<tr>
<td>1/4</td>
<td>0.5194</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{12}}{\sigma_{yield}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.5164</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5084</td>
</tr>
<tr>
<td>1.5</td>
<td>0.5021</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4924</td>
</tr>
<tr>
<td>2/3</td>
<td>0.4823</td>
</tr>
<tr>
<td>1/2</td>
<td>0.4752</td>
</tr>
<tr>
<td>1/3</td>
<td>0.4651</td>
</tr>
</tbody>
</table>
Table 4.4 (continued): FEM results of pure shear $\frac{\Sigma_{12}}{\sigma_{yield}}$.

<table>
<thead>
<tr>
<th>$f=0.10$</th>
<th>$\frac{\Sigma_{12}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect Ratio $\frac{a}{c}$</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.4636</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4528</td>
</tr>
<tr>
<td>1.25</td>
<td>0.4452</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4364</td>
</tr>
<tr>
<td>4/5</td>
<td>0.4267</td>
</tr>
<tr>
<td>2/3</td>
<td>0.4188</td>
</tr>
<tr>
<td>1/2</td>
<td>0.4059</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f=0.20$</th>
<th>$\frac{\Sigma_{12}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect Ratio $\frac{a}{c}$</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>0.3744</td>
</tr>
<tr>
<td>1.25</td>
<td>0.3622</td>
</tr>
<tr>
<td>1.1</td>
<td>0.3535</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3467</td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.3400</td>
</tr>
<tr>
<td>4/5</td>
<td>0.3305</td>
</tr>
<tr>
<td>2/3</td>
<td>0.3166</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f=0.30$</th>
<th>$\frac{\Sigma_{12}}{\sigma_{yield}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect Ratio $\frac{a}{c}$</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>0.2926</td>
</tr>
<tr>
<td>1.10</td>
<td>0.2803</td>
</tr>
<tr>
<td>1.05</td>
<td>0.2758</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2709</td>
</tr>
<tr>
<td>1/1.05</td>
<td>0.2664</td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.2619</td>
</tr>
<tr>
<td>4/5</td>
<td>0.2488</td>
</tr>
</tbody>
</table>
Figure 4.19: The variation of $\frac{\Sigma_{12}}{\sigma_{\text{yield}}}$ with aspect ratio.

Figure 4.19 shows the variation of the yield stress $\frac{\Sigma_{12}}{\sigma_{\text{yield}}}$ as a function of aspect ratio for various void volume fractions. As the voids become more oblate the yield stress is found to increase at fixed void volume fraction. This behavior is consistent with the fact that the matrix cross sectional areas with normals along $X_1$ and $X_2$ increase, as the void becomes more oblate at fixed void volume fraction.
Table 4.5: FEM results of pure shear $\frac{\Sigma_{23}}{\sigma_{\text{yield}}}$.  

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{23}}{\sigma_{\text{yield}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.5240</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5438</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5537</td>
</tr>
<tr>
<td>2/3</td>
<td>0.5514</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5501</td>
</tr>
<tr>
<td>1/4</td>
<td>0.5438</td>
</tr>
</tbody>
</table>

f=0.02

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{23}}{\sigma_{\text{yield}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>0.4842</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5179</td>
</tr>
<tr>
<td>1.5</td>
<td>0.5273</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5356</td>
</tr>
<tr>
<td>2/3</td>
<td>0.5324</td>
</tr>
<tr>
<td>1/2</td>
<td>0.5289</td>
</tr>
<tr>
<td>1/4</td>
<td>0.5203</td>
</tr>
</tbody>
</table>

f=0.05

<table>
<thead>
<tr>
<th>Aspect Ratio $\frac{a}{c}$</th>
<th>$\frac{\Sigma_{23}}{\sigma_{\text{yield}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.4227</td>
</tr>
<tr>
<td>2.0</td>
<td>0.4557</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4743</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4924</td>
</tr>
<tr>
<td>2/3</td>
<td>0.4858</td>
</tr>
<tr>
<td>1/2</td>
<td>0.4785</td>
</tr>
<tr>
<td>1/3</td>
<td>0.4700</td>
</tr>
</tbody>
</table>
Table 4.5 (continued): FEM results of pure shear \( \frac{\Sigma_{23}}{\sigma_{\text{yield}}} \).

\( f = 0.10 \)

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{23}}{\sigma_{\text{yield}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.3749</td>
</tr>
<tr>
<td>1.5</td>
<td>0.4047</td>
</tr>
<tr>
<td>1.25</td>
<td>0.4208</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4364</td>
</tr>
<tr>
<td>4/5</td>
<td>0.4305</td>
</tr>
<tr>
<td>2/3</td>
<td>0.4237</td>
</tr>
<tr>
<td>1/2</td>
<td>0.4130</td>
</tr>
</tbody>
</table>

\( f = 0.20 \)

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{23}}{\sigma_{\text{yield}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.2924</td>
</tr>
<tr>
<td>1.25</td>
<td>0.3192</td>
</tr>
<tr>
<td>1.1</td>
<td>0.3365</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3467</td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.3433</td>
</tr>
<tr>
<td>4/5</td>
<td>0.3354</td>
</tr>
<tr>
<td>2/3</td>
<td>0.3254</td>
</tr>
</tbody>
</table>

\( f = 0.30 \)

<table>
<thead>
<tr>
<th>Aspect Ratio ( \frac{a}{c} )</th>
<th>( \frac{\Sigma_{23}}{\sigma_{\text{yield}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>0.2325</td>
</tr>
<tr>
<td>1.10</td>
<td>0.2559</td>
</tr>
<tr>
<td>1.05</td>
<td>0.2642</td>
</tr>
<tr>
<td>1.0</td>
<td>0.2709</td>
</tr>
<tr>
<td>1/1.05</td>
<td>0.2693</td>
</tr>
<tr>
<td>1/1.1</td>
<td>0.2655</td>
</tr>
<tr>
<td>4/5</td>
<td>0.2574</td>
</tr>
</tbody>
</table>
Figure 4.20 plots the variation of \( \frac{\Sigma_{23}}{\sigma_{y}} \) as a function of aspect ratio for various void volume fractions. Once again, the effect of aspect ratio on yield is seen clearly.

To compare the FEM shear results for spheres versus the predictions of the Gurson model [Gurson (1977)], we solve the Gurson yield function for the case of pure shear. Such a stress state has a vanishing trace, and
hence the exponential behavior given by the hyperbolic cosine term in the Gurson yield function vanishes. Thus, for pure shear the Gurson model predicts,

\[ \Sigma_{\text{Gur}} = \frac{1-f}{\sqrt{3}}. \]  

(4.5)

Fig. 4.21 plots the Gurson predictions versus the FEM yield values in shear. All the yield values have been normalized with respect to the uniaxial tensile yield stress of the unvoided elastic-perfectly plastic material.

Figure 4.21: Comparison of FEM and Gurson yield values in pure shear for spherical voids.
It is obvious from Fig. 4.21 that the Gurson yield function is unable to capture the effects of periodicity in a material with spherical voids loaded in shear. The divergence of the Gurson predictions from the FEM values decreases with decreasing void volume fraction because the effects of periodicity are smaller at lower void volume fractions.

Thus, even in those cases where the void volume fraction is a sufficient factor in characterizing the effects of voids, the Gurson model is incapable of properly characterizing the yield in shear. However, it is clear from Fig. 4.19 and Fig. 4.20 that the void volume fraction does not fully characterize the yield behavior of a voided material. The effect of void shape is important in shear and the effect of shape is more predominant at higher void volume fractions.

D. Equal deformation rate loadings

We now examine the yield behavior of a voided material with spherical voids under the macroscopic boundary condition of equal normal deformation rates. The boundary conditions used are similar to those used in uniaxial tension, except that normal deformation rates are applied equally along all the three void axis directions. The normal macroscopic stresses needed to impose the boundary conditions are calculated by summing up the reaction forces on the necessary faces. Again, for purposes of the simulation, we invoke symmetry to justify the use of just an octant of the cell with a void at the center.
A typical macroscopic stress-strain plot along one of the three equivalent loading directions is shown in fig. 4.22 for the equal deformation rate loading of spherical voids with an initial void volume fraction of 1%. It is observed that the stress-strain curve shows softening, even though the matrix is perfectly plastic. The reason for the drop in macroscopic stress is due to logarithmic variation of the hydrostatic mean yield stress with void volume fraction. To examine this further, we use volume conservation to obtain,

\[ \dot{f} = (1-f) D_{kk} . \]  

(4.6)

\( f \) denotes the void volume fraction, \( \dot{f} \) denotes the rate of change of void volume fraction and \( D_{kk} \) denotes the trace of the macroscopic rate of deformation tensor. We assume that the macroscopic deformation rate tensor may be approximated by the macroscopic strain rate \( \dot{E}_{kk} \). We then integrate eqn. (4.6) to obtain

\[ f = 1 - (1-f_o)\exp(-E_{kk}) . \]  

(4.7)

Thus, at a macroscopic strain of \( 3\varepsilon_y \) an initial void volume fraction of \( f_o = 0.01 \) becomes 0.0189, a value almost twice the initial value. As explained later in this section, eqn. (3.25) is a reasonable approximation for the rate of variation of the mean stress with void volume fraction.

\[ \frac{\Sigma_{\text{mean}}}{\sigma_y} = -\frac{2}{3} \ln(f) . \]  

(3.25)
Using equation (3.25) we calculate that \( \frac{\Sigma_{\text{mean}}}{\sigma_y} = 3.07 \) for \( f = 0.01 \), but falls to \( \frac{\Sigma_{\text{mean}}}{\sigma_y} = 2.64 \) when \( f = 0.0189 \) at a macro strain of \( 3\varepsilon_y \) along each of the three loading directions. The rate of change of \( \frac{\Sigma_{\text{mean}}}{\sigma_y} \) is highest at low void volume fraction because of the nature of the logarithmic function. This explains why the macroscopic stress-strain curve in fig. 4.22 levels off at higher macroscopic strains, for which void volume fractions reach larger values. Thus, the yield stress for a voided material under equal deformation rate loadings is taken to be the peak stress in the stress-strain curve.

As observed in the uniaxial loading cases, the void acts to concentrate the stresses at the void cross-sections normal to the loading direction. Figure 4.23 shows the effective stress distribution in the cell for a macroscopic strain of \( 0.55\varepsilon_y \) along \( X_1, X_2 \) and \( X_3 \). A region of high effective stress is found to grow out concentrically from the void. Yielding of the void sections is observed at a macroscopic strain of \( 3.03\varepsilon_y \) along the three loading directions, as shown by the effective stress distribution in fig. 4.24. A rigid exists in the region of the cell diagonally opposite the void. Figure 4.25 shows the variation of the effective plastic strain in the cell at the macroscopic strain of \( 3.03\varepsilon_y \). Upon further loading, the subsequent plasticity is seen to develop strongly along the three loading directions reflecting the strong interaction between adjacent voids under hydrostatic stress states. This aspect is clear from the contours of effective plastic strain at a macroscopic strain of \( 4.55\varepsilon_y \) shown in fig. 4.26. The magnitude of the maximum plastic strain at \( E_1 = E_2 = E_3 = 3.03\varepsilon_y \) is two orders of magnitude higher than the corresponding values under uniaxial tension at the same
strain. This observation supports previous findings that predict an exponential void growth rate under hydrostatic conditions, e.g. Rice and Tracey (1969).

Table 4.6 lists the mean stress values required to yield spheres of various void volume fractions under the boundary conditions of equal deformation rate along the three loading directions. For spherical voids, the equal macroscopic deformation rate boundary condition ensures that the voids remain spherical throughout the deformation. Table 4.6 also lists the mean stress values required to yield an isolated spherical void in a spherical shell of material [Hill (1950)], which is given by eqn. (3.25).
Figure 4.22: FEM macroscopic stress-strain plot for equal deformation loading of spherical voids. Initial void volume fraction= 1%. $\Sigma_{11} = \Sigma_{22} = \Sigma_{33} = \Sigma$. $E_{11} = E_{22} = E_{33} = E$. 
Figure 4.23: FEM effective stress contours for equal deformation rate loading of spherical voids. Initial void volume fraction = 1%. \( E_{11} = E_{22} = E_{33} = E = 0.55\varepsilon_y \). \( \sigma_y = 10^4 \) units.
Figure 4.24: FEM effective stress contours for equal deformation rate loading of spherical voids. Initial void volume fraction = 1%. $E_{11} = E_{22} = E_{33} = E = 3.03\varepsilon_y$. $\sigma_y = 10^4$ units.
Figure 4.25: FEM effective plastic strain contours for equal
deformation rate loading of spherical voids.
Initial void volume fraction = 1%. $E_{11} = E_{22} = E_{33} = E = 3.03 \varepsilon_y$, $\varepsilon_y = 10^{-3}$. 
Figure 4.26: FEM effective plastic strain contours for equal deformation rate loading of spherical voids. Initial void volume fraction = 1%. $E_{11} = E_{22} = E_{33} = E = 4.55\varepsilon_y$, $\varepsilon_y = 10^{-3}$. 
Table 4.6 lists the mean stress values required to yield spheres of various void volume fractions under the boundary conditions of equal deformation rate along the three loading directions. For spherical voids, the equal macroscopic deformation rate boundary condition ensures that the voids remain spherical throughout the deformation. Table 4.6 also lists the mean stress values required to yield an isolated spherical void in a spherical shell of material [Hill (1950)], which is given by eqn. (3.25).

Table 4.6: FEM mean stress required to yield spherical void in cell under hydrostatic tension.

<table>
<thead>
<tr>
<th>Void volume fraction, ( f )</th>
<th>( \sigma_{\text{mean}} )</th>
<th>( \sigma_{\text{mean}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2.89</td>
<td>3.070</td>
</tr>
<tr>
<td>0.02</td>
<td>2.46</td>
<td>2.610</td>
</tr>
<tr>
<td>0.05</td>
<td>1.88</td>
<td>1.997</td>
</tr>
<tr>
<td>0.10</td>
<td>1.43</td>
<td>1.540</td>
</tr>
<tr>
<td>0.20</td>
<td>0.969</td>
<td>1.070</td>
</tr>
<tr>
<td>0.30</td>
<td>0.698</td>
<td>0.803</td>
</tr>
</tbody>
</table>

Figure 4.27 shows that the yield values for the periodic array of spherical voids are lower than the corresponding analytical values. Further, the percentage deviation from the FEM value is found to increase with increasing void volume fraction. This deviation is a result of the periodic
boundary conditions imposed during the finite element simulation. However, it turns out that the mean hydrostatic stress is still linear in \( \ln(f) \). This justifies the use of Equation 3.26.

Figure 4.27: Plot of the mean stress in hydrostatic tension for spherical voids comparing FEM values with the shell model.
Next, we consider the effect of void shape upon deformation under conditions of equal macroscopic deformation rates. Specifically, let us examine the combination of macroscopic stresses $\Sigma_1^E, \Sigma_2^E$ and $\Sigma_3^E$ needed to yield the material with the above boundary condition. These stresses are normalized with respect to the yield stress of the unvoided matrix material. Define,

$$\Sigma^E = \sqrt{[\Sigma_1^E]^2 + [\Sigma_2^E]^2 + [\Sigma_3^E]^2}, \text{ and }$$ (3.53)

$$\zeta = \arctan \left( \frac{\Sigma_3^E}{\Sigma_1^E} \right)$$ (4.8)

Table 4.7 reports the values of $\Sigma_1^E, \Sigma_2^E, \Sigma_3^E, \Sigma^E$ and $\zeta$ for various void shapes as a function of void volume fraction.

For both the void volume fractions tested, $\Sigma_1^E > \Sigma_3^E$ for oblate voids while $\Sigma_1^E < \Sigma_3^E$ for prolate voids. This behavior follows the same trend as that followed by the projected area normal to the three loading directions as the void shape changes. For oblate voids, there is minimum area normal to the direction $X_3$ while for prolate voids the area is minimum normal to the $X_1$ or $X_2$ axis. Table 4.7 also shows that $\Sigma^E$ is maximum for spherical voids, for fixed void volume fraction.
Table 4.7: Effect of void shape on normalized stresses for yield under equal macroscopic deformation rate.

### $f=0.01$

<table>
<thead>
<tr>
<th>$\frac{a}{c}$</th>
<th>$\Sigma_1 E$ ($= \Sigma_2 E$)</th>
<th>$\Sigma_3 E$</th>
<th>$\Sigma E$</th>
<th>$\zeta$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.102</td>
<td>2.030</td>
<td>1.986</td>
<td>3.491</td>
<td>44.37</td>
</tr>
<tr>
<td>1.0</td>
<td>2.890</td>
<td>2.890</td>
<td>5.006</td>
<td>45.00</td>
</tr>
<tr>
<td>0.209135</td>
<td>2.636</td>
<td>2.832</td>
<td>4.682</td>
<td>47.05</td>
</tr>
</tbody>
</table>

### $f=0.05$

<table>
<thead>
<tr>
<th>$\frac{a}{c}$</th>
<th>$\Sigma_1 E$ ($= \Sigma_2 E$)</th>
<th>$\Sigma_3 E$</th>
<th>$\Sigma E$</th>
<th>$\zeta$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.823</td>
<td>1.759</td>
<td>3.128</td>
<td>43.98</td>
</tr>
<tr>
<td>1.5</td>
<td>1.869</td>
<td>1.824</td>
<td>3.211</td>
<td>44.30</td>
</tr>
<tr>
<td>1.0</td>
<td>1.880</td>
<td>1.880</td>
<td>3.256</td>
<td>45.00</td>
</tr>
<tr>
<td>2/3</td>
<td>1.833</td>
<td>1.906</td>
<td>3.218</td>
<td>46.12</td>
</tr>
<tr>
<td>1/2</td>
<td>1.774</td>
<td>1.920</td>
<td>3.159</td>
<td>47.27</td>
</tr>
</tbody>
</table>

### $f=0.10$

<table>
<thead>
<tr>
<th>$\frac{a}{c}$</th>
<th>$\Sigma_1 E$ ($= \Sigma_2 E$)</th>
<th>$\Sigma_3 E$</th>
<th>$\Sigma E$</th>
<th>$\zeta$ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.409</td>
<td>1.220</td>
<td>2.336</td>
<td>40.89</td>
</tr>
<tr>
<td>1.5</td>
<td>1.440</td>
<td>1.330</td>
<td>2.432</td>
<td>42.73</td>
</tr>
<tr>
<td>1.0</td>
<td>1.430</td>
<td>1.430</td>
<td>2.477</td>
<td>45.00</td>
</tr>
<tr>
<td>2/3</td>
<td>1.359</td>
<td>1.494</td>
<td>2.434</td>
<td>47.70</td>
</tr>
<tr>
<td>1/2</td>
<td>1.282</td>
<td>1.533</td>
<td>2.372</td>
<td>50.10</td>
</tr>
</tbody>
</table>
A plot of the variation of $\Sigma^E$ is shown in Fig. 4.28, for two void volume fractions. The distance from the origin in stress space to the point on the yield surface where $D_1 = D_2 = D_3$, is a maximum for spherical voids. Also, as expected the distance from the origin increases with increasing void volume fraction.
The yield values calculated by FEM can be used to find the $Q_{ij}$ values. The procedure for such a calculation has already been outlined in Chapter III. Appendix A gives the calculated values of $Q_{ij}$ and the subsequent interpolation to obtain expressions for the variation of the $Q_{ij}$ values with void volume fraction and aspect ratio.
CHAPTER V

PREDICTIONS OF THE PROPOSED YIELD FUNCTION
AND COMPARISONS TO EXISTING MODELS AND FEM

A. Introduction

This section describes the predictions of the yield function proposed in Chapter III. Yield values for various types of loadings are predicted. Comparisons are made to existing models and FEM for the case of spherical voids. For textured voids, comparisons are made to finite element results. The results indicate the accuracy of the proposed model for both spherical and non-spherical voids.

B. Predictions for spherical voids and comparison to existing models under simple loadings

The predictions of the proposed model for the uniaxial yield stress of a material with spherical voids is shown in Fig. 5.1. The variation of the uniaxial yield stress with void volume fraction $f$, can be reasonably approximated by a linear function, for void volume fractions not exceeding 5%. Comparisons are drawn to three other existing models for
spherical voids: Gurson (1977), Cocks (1989), and Michel-Suquet (1992). These models also show a linear variation of the uniaxial yield stress with void volume fraction in the dilute limit. Denoting the Michel-Suquet model by M-S and the Cocks model by C, we represent the functional forms for the variation of the uniaxial yield stress with void volume fraction as follows:

\[
\sigma_{\text{M-S}} = \frac{2\sqrt{3}}{\sqrt{12 + 11f}} (1-f) = \left( 1 - \frac{11}{24} f + \frac{121}{384} f^2 + O(f^3) \right) (1-f)
\]

(5.1)

\[
\sigma_C = \sqrt{\frac{6(1+f)}{4f^2 + 13f + 6}} (1-f) = \left( 1 - \frac{7}{12} f + \frac{73}{96} f^2 + O(f^3) \right) (1-f)
\]

(5.2)

It is clear from (5.1) and (5.2) that in the dilute limit, the uniaxial stress is linear in void volume fraction. The expression for the uniaxial yield stress for the Gurson model cannot be obtained in an analytical form. It is given by the solution to eqn. (4.2). Figure 5.1 shows that the existing models for spherical voids agree with the predictions of the proposed model for the uniaxial yield behavior of spherically voided materials. For \( f < 10\% \) the other models are within 6\% of the predicted value. The predictions of the uniaxial yield stress of the proposed model are lower than the other models and hence will provide a conservative estimate.

The mean stress required to yield a voided material with spherical voids under a purely hydrostatic loading, is shown in Fig. 5.2. Hill (1950) calculates the mean stress required to yield an elastic perfectly-plastic spherical shell of void volume fractions \( f \) as:
\[
\frac{\sigma_{\text{mean}}}{\sigma_{\text{yield}}} = \frac{2}{3} \ln(f). \tag{5.3}
\]

Fig. 5.2 shows that both the predicted yield function and the Hill limit provide a good representation of the yield of spheres in hydrostatic tension when compared with the corresponding FEM values. The good agreement with FEM supports the dependence of \( \frac{\sigma_{\text{mean}}}{\sigma_{\text{yield}}} \) on \( \ln(f) \) as adopted in equation (3.26). The good agreement of the Hill limit and the FEM results suggests that the mean stress is not sensitive to whether the packing geometry is space filling.

The Michel-Suquet (1992) and Cocks (1989) models do not show a logarithmic variation of mean stress with void volume fraction:

\[
\sigma_{\text{mean}}^{\text{MS}} = \sqrt{\frac{2(1 + f)}{9f}} (1 - f) = \frac{\sqrt{2}}{3} f^{\frac{1}{2}} - \frac{\sqrt{2}}{6} f^{\frac{1}{2}} + O(f^{\frac{3}{2}}) \tag{5.4}
\]

\[
\sigma_{\text{mean}}^{C} = \sqrt{\frac{4}{9f}} (1 - f) = \frac{2}{3} f^{\frac{1}{2}} - \frac{2}{3} f^{\frac{1}{2}} \tag{5.5}
\]

According to equations (5.4) and (5.5), the Cocks (1989) and Michel-Suquet (1992) models suggest a \( f^{\frac{1}{2}} \) variation for the mean yield stress as \( f \) tends to zero. Thus, at low void volume fractions (\( f < 1\% \)) both models do not accurately predict the yield condition under a hydrostatic state of stress. The models become more accurate at non dilute void volume fractions, which will be considered later. Cocks (1989) has developed alternate expressions for the dilute limit of small \( f \) which have a logarithmic dependence of the mean stress on void volume fraction.
C. Predictions of yield surfaces for spherical voids and comparison to existing models under axi-symmetric loading

Having evaluated the predictions of the model under simple loadings for spherical shapes, we seek to show the accuracy of the proposed model in describing yield under mixed loadings. The axi-symmetric (S-T) loading of spherical voids will be considered, where $\sigma_{11} = \sigma_{22} = T$, and $\sigma_{33} = S$. Fig. 5.3 shows the S-T yield loci for spherical voids of 1% void volume fraction. The FEM results for yield under uniaxial tension and yield under hydrostatic tension are also included for comparison. As observed earlier in Fig. 5.2, the Gurson and the proposed yield functions are more accurate under both these stress states. The excursion of the Michel-Suquet and Cocks yield functions from the FEM value at the hydrostatic yield point can be attributed to the $f^{-1/2}$ variation for the mean yield stress in the dilute limit in these models. Also, as observed in Fig. 5.1 for small $f$, all models are in better agreement in describing yield in the uniaxial yield case. Fig. 5.3 also indicates that under purely bi-axial in-plane loading, all yield functions predict close to identical yield values. At a higher void volume fraction of 10%, the predictions under hydrostatic yield for all the models compare well with FEM as shown in Fig. 5.4. Under uniaxial tension and bi-axial in-plane tension, all models predict almost the same values.

Fig. 5.5 shows the predictions for the proposed model for S-T yield surfaces for spherical voids for void volume fractions ranging from 1% to 20%. The yield functions progressively become smaller with an increase in
void volume fraction. The contraction of the yield surface with void volume fraction along the hydrostatic axis of $S = T$ is more rapid than the contraction of the yield surfaces along the uniaxial or the biaxial loadings. The Gurson yield function predicts a similar trend with void volume fraction as seen in Fig. 5.6.

D. Curvature of proposed yield function and comparison to the Gurson yield function

The rate of change of the normal vector with arc length is an important quantity in strain localization. If the normal vector around the hydrostatic yield point rotates at a rapid rate, normality implies that the strain rate vector changes direction rapidly in stress space. Thus, small changes in the stress state can cause different modes of deformation that lead to instability. In an attempt to quantify this, we can consider the rate of change of the unit normal vector with arc length.

Let us denote the yield function in stress space by $\Phi$ and the vector gradient operator by $\nabla$. We thus obtain the unit normal vector $n$ to the yield function in stress space as

$$n = \frac{\nabla \Phi}{|\nabla \Phi|}.$$  \hspace{1cm} (5.5)

The tangent vectors $t^{(i)}$ span the tangent sub-space. The curvature tensor is usually defined as the rate of change of the tangent vector with arc
length. We are interested in the rate of change of \( n \) and we denote this tensor by \( C \). Denoting the gradient operator on a vector by \( \nabla \)

\[
C = \nabla n \tag{5.6}
\]

Thus, the rate of change of the normal vector with arc length along a tangent direction \( t(k) \) is given by

\[
c = t(k) \cdot C t(k) . \tag{5.7}
\]

We evaluate the tensor \( C \) in the Cartesian basis in stress space. The yield function \( \Phi \) as given in eqn. (3.4) is

\[
\Phi = Q_{ik} A_{ij} Q_{jk} - 1 = 0 .
\]

The seven independent non-trivial functions in stress space are known as functions of void volume fraction and aspect ratio (Appendix A). In the Cartesian basis, equation (5.5) becomes

\[
n_i = \frac{\partial \Phi}{\partial \Sigma_i} \frac{1}{\left( \frac{\partial \Phi}{\partial \Sigma_k} \frac{\partial \Phi}{\partial \Sigma_k} \right)^{1/2}} . \tag{5.8}
\]

The Cartesian components of \( C \) are given as

\[
[C]_{ij} = \frac{\partial n_i}{\partial \Sigma_j} . \tag{5.9}
\]
Equation (5.7) can then be used to evaluate the rate of change of the normal vector along any direction in the tangent sub-space of $ \Phi $ at any given point on the yield function in stress space.

Table 5.1: The value of $ c $ given by eqn. (5.7) as a function of void volume fraction for spherical voids. Values are calculated using both the Gurson yield function and the proposed yield function.

<table>
<thead>
<tr>
<th>Void volume fraction</th>
<th>$ c_{\text{Gurson}} $</th>
<th>$ c_{\text{Proposed}} $</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.E-05</td>
<td>3.46410</td>
<td>19.89787</td>
</tr>
<tr>
<td>1.E-04</td>
<td>3.46410</td>
<td>15.92643</td>
</tr>
<tr>
<td>1.E-03</td>
<td>3.46410</td>
<td>11.97377</td>
</tr>
<tr>
<td>1.E-02</td>
<td>3.46445</td>
<td>8.15122</td>
</tr>
<tr>
<td>2.E-02</td>
<td>3.46549</td>
<td>7.09792</td>
</tr>
<tr>
<td>5.E-02</td>
<td>3.47278</td>
<td>5.92608</td>
</tr>
<tr>
<td>1.E-01</td>
<td>3.49909</td>
<td>5.45458</td>
</tr>
<tr>
<td>2.E-01</td>
<td>3.60844</td>
<td>6.00318</td>
</tr>
<tr>
<td>3.E-01</td>
<td>3.80671</td>
<td>7.48817</td>
</tr>
</tbody>
</table>
For the (S-T) loading considered, we calculate the components of \( C \) by setting \( \Sigma_{11} = \Sigma_{22} = T, \Sigma_{33} = S \). The components of \( C \) were calculated using MapleV(1991). To compare the Gurson model with the proposed model, we evaluate \( C \) for spherical voids of various void volume fractions.

For spherical voids, the S-T yield surfaces have the \( S=T \) line as an axis. The \( m^{(i)} \) vectors have been constructed such that the \( m^{(3)} \) direction coincides with the \( S=T \) direction for spherical voids. Since there are no shears along the directions aligned with the void axes, normality and orthogonality of the \( m^{(i)} \) vectors dictate that the tangent space at the point \( S=T \) on the yield surface is spanned by \( m^{(1)} \) and \( m^{(2)} \). Hence, the required projection of \( C \) in the tangent space is given by

\[
c = m^{(1)} \cdot C m^{(1)} = m^{(2)} \cdot C m^{(2)}.
\] (5.10)

Fig. 5.7 shows the variation of \( c \) with void volume fraction, for both the Gurson and the proposed yield functions. The values of \( c \) are tabulated in Table 5.1. It can be seen that the curvature of the proposed yield function at the point \( S=T \) on the yield surface, is higher than the corresponding Gurson value. For the Gurson yield function, \( c \) reaches a limiting value of \( 2\sqrt{3} \) as \( f \) tends to zero. However, the proposed yield function predicts that the value of \( c \) grows unbounded as \( f \) tends to zero. This difference is due to the assumed quadratic approximation for the proposed yield function. Fig. 5.8 shows the variation of \( c \) in S-T space for \( f=0.01 \) and \( f=0.10 \). It can be seen that the value of \( c \) grows rapidly in the vicinity of \( \tan^{-1}(\frac{S}{T}) = 45 \) degrees. Normality of \( \Phi \) implies that the normal vector rapidly changes direction around \( S = T \). This in turn reflects the physical phenomenon of rapidly
changing deformation modes for small perturbations in stresses about the hydrostatic loading. It can be observed from Fig. 5.8 that the value of $c$ around the $S=T$ point, is higher for the lower void volume fraction of 1%. The rate of change of $c$ with $f$ at the point $S=T$, is an indirect reflection of the difference in the rates of change of uniaxial versus the hydrostatic yield stress of the voided material. The yield stress along the hydrostatic direction decreases logarithmically. Fig. 5.1 suggests that a linear variation for the uniaxial yield stress is a reasonable approximation. Thus, the hydrostatic yield stress decreases much more rapidly than the uniaxial yield stress leading to a decrease in the value of $c$ with increasing void volume fraction, especially in the range of dilute void volume fractions.

Thus far, we have contrasted the predictions of the proposed yield function for spherical voids with predictions from previously available yield functions. It is seen that the available yield functions are quite adequate in describing the behavior of voided materials with spherical voids. The following conclusions can be made about the yield functions for spherical voids. Under uniaxial loadings, the proposed model and other available models considered, predict accurate uniaxial yield values. Under hydrostatic states of stress, the proposed model and the Gurson model show good agreement with FEM values at both dilute and non-dilute void volume fractions up to 20%. Under hydrostatic tension, the predictions of the Cocks (1989) and Michel-Suquet (1992) models are more accurate at non-dilute void volume fractions.

E. Predictions of the model for non-spherically voided materials and comparison to FEM under simple loadings
The proposed yield function models the effects of void shape on yield, and we now seek to show these features of the proposed model. The predictions are compared to finite element results wherever possible.

Fig. 5.9 shows the predictions for the uniaxial yield stress $\Sigma_{11}$ as a function of void shape. Void shape is expressed in terms of the variable $u$.

$$u = \frac{a}{S} - \frac{c}{S}.$$  \hspace{1cm} (5.11)

Thus, $u=0$ denotes a sphere, and $u>0$ and $u<0$ denote an oblate and a prolate void respectively. The choice of the variable $u$ is explained in Appendix A. Fig. 3.2 shows the geometry of the cell and the direction of stressing $X_1$. The predictions are compared to corresponding FEM predictions at various void volume fractions. It is seen that at a fixed void volume fraction, the uniaxial yield stress $\Sigma_{11}$ varies with void shape. For instance, at a void volume fraction of 0.1, the difference between the $\Sigma_{11}$ values for the most oblate and most prolate cases is about 10% of the corresponding spherical value. The variation of $\Sigma_{11}$ with void shape is in conformity with the remaining projected ligament area normal to the loading axis. As the void becomes more oblate at a fixed void volume fraction, it is clear from Fig. 3.2 that the available area left to carry load increases. Thus, for a perfectly plastic material, the value of $\Sigma_{11}$ will increase from the spherical value as the void becomes more oblate.

The predictions for the variation of the value of $\Sigma_{33}$ with void shape are consistent with the corresponding FEM values for void volume fractions up to 20%. Fig. 5.10 shows the variation of $\Sigma_{33}$ with void shape. Once again, it is seen that the variation of $\Sigma_{33}$ with shape is quite large. As
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the voids become more oblate, Fig. 3.2 indicates that the void area normal to the loading axis $X_3$ increases, causing the value of $\Sigma_{33}$ to decrease. The predictions for the variation of the value of $\Sigma_{33}$ with void shape are consistent with the corresponding FEM values for void volume fractions up to 20%.

Fig. 5.11 shows the variation of the pure shear stress $\Sigma_{12}$ with void shape for void volume fractions up to 30%. As the voids become oblate, the matrix cross sectional area normal to the $X_1$ and $X_2$ directions increases. This leads to an increase in $\Sigma_{12}$ as the voids become more oblate. Fig. 5.12 shows the variation of the pure shear stress $\Sigma_{23}$ with void shape at void volume fractions not exceeding 30%. The predictions indicate that the spherical void is stronger than any ellipsoidal void of the same void volume fraction under the shear loading $\Sigma_{23}$.

It is clear from Figs. 5.9, 5.10, 5.11 and 5.12 that the proposed model predicts that the variation of the yield stress with void shape decreases with decreasing void volume fraction. This is due to the fact that all the $Q_{ij}$ values were assumed to equal their Von Mises limits in the limit of $f \to 0$.

F. Predictions of yield surfaces for non-spherically voided materials and comparison to FEM under axi-symmetric loadings

Fig. 5.13 depicts the S-T yield surface for the S-T loading of a voided material of 1% void volume fraction. The rotation of the yield surfaces with aspect ratio is a striking feature. Fig. 5.14 illustrates the S-T yield surface for the S-T loading of a voided material of 10% void volume
fraction. It is seen that the rotation of the yield surface increases with increasing void volume fraction. As the void becomes more oblate, the yield function rotates away from the S direction. Invoking normality, the rotation implies that a purely hydrostatic loading of $S=\tau$ will cause a greater component of the incremental strain vector along the $X_3$ direction. Alternatively, in order to obtain equal macroscopic strain increments in the three directions, the applied loading must satisfy $S<\tau$. The opposite is true for prolate voids. As the void becomes more prolate, the yield function rotates toward the S direction. This implies that for a prolate voided material, a purely hydrostatic loading produces a larger incremental strain along the $X_1$ or $X_2$ directions than the $X_3$ direction. Alternatively, in order to obtain equal macroscopic strain increments along the three directions, the applied loading is characterized by $S>\tau$.

The hydrostatic stress to yield a voided material under a state of a purely hydrostatic stress is seen to be highest for a sphere. The length of the $m^{(3)}$ axis is maximum for a sphere and decreases with evolving void shape at constant void volume fraction. This behavior is also seen in the FEM results as shown in Fig. 4.11.

One of the main effects of the rotation of the yield surface with void shape is to cause anisotropy in the uniaxial yield stresses along the $X_1$ and $X_3$ directions. The increase in rotation with void volume fraction implies that the anisotropy increases with increasing void volume fraction.

To compare the yield surface predictions for non-spherical voids under more mixed loadings, FEM simulations were done at a void volume
fraction of 10% for an oblate void characterized by an aspect ratio of $\frac{a}{c} = 2$. Fig. 5.15 compares the predictions of the yield surface for this case versus the FEM results. The proposed yield surface is reasonably close values to the FEM results.

Fig. 5.16 shows the ratio of the components of the rate of deformation vector $\mathbf{D}$ as predicted using normality for the proposed yield function. The predicted values of $\frac{D_3}{D_1}$ are seen to be close to the FEM values. The close agreement of the predicted ratio and the observed FEM value is an indication that the normal vectors predicted by the model are close to being parallel to the actual values as observed through FEM. It is also a validation of the assumption of normality for the porous material model.

It can also be observed that as $\frac{S}{T}$ tends to unity, the value of $\frac{D_3}{D_1}$ increases very rapidly. This reflects the anisotropy in the components of the strain increment vector due to the oblate shape of the void.

G. Conclusions

The proposed model for the yield function for porous materials has been found to be in close agreement with the existing models for spherically voided materials under both simple uniaxial and axisymmetric loadings. Good agreement is also observed with FEM results under both loadings.

For non spherical voids, the model is seen to show close agreement with finite element values of yield under simple uniaxial and axisymmetric loadings for void volume fractions up to 20%. The uniaxial
yield stresses predicted are the lowest among the models examined. Further, the normal vectors predicted by the proposed yield function are close to parallel to the actual normal vectors as predicted from FEM.

Using the yield function as a plastic potential, it is possible to derive the incremental stress-strain curve (flow law) for a voided solid. This aspect will be explored in Chapter VIII.
Figure 5.1: Comparison of predicted uniaxial yield stress for spherical voids with the Gurson, Cocks, and Michel-Suquet model predictions.
Figure 5.2: Comparison of the predicted mean stress for spherical voids with the Hill limit.
Figure 5.3: Comparison of predicted $S$-$T$ yield surfaces for spherical voids with Gurson, Cocks, and Michel-Suquet yield surfaces. Uniaxial and hydrostatic FEM values are also shown. Void volume fraction = 1%.
Figure 5.4: Comparison of predicted $S$-$T$ yield surfaces for spherical voids with Gurson, Cocks, and Michel-Suquet yield surfaces. Uniaxial and hydrostatic FEM values are also shown. Void volume fraction $= 10\%$. 
Figure 5.5: Predictions of $S-T$ yield surfaces for spherical voids as a function of void volume fraction, $f$. 

\[ \frac{T}{\sigma_y} \quad \frac{S}{\sigma_y} \]
Figure 5.6: Predictions of $S-T$ yield surfaces for the Gurson model as a function of void volume fraction, $f$. 
Figure 5.7: Comparison of the predicted variation of the rate of change of normal vector along arc length at the hydrostatic yield point with the Gurson model, as a function of void volume fraction, \( f \). Voids are assumed to be spherical.
Figure 5.8: Predictions of the rate of change of normal vector with arc length as a function of the angle $\tan^{-1}(\frac{S}{T})$ in $S$-$T$ stress space. $f$ denotes the void volume fraction.
Figure 5.9: Prediction of variation of the uniaxial yield stress $\Sigma_{11}$ along the $X_1$ direction as a function of void volume fraction $f$, and void shape $u$. $u > 0$ for oblate voids, $u = 0$ for spherical voids and $u < 0$ for prolate voids. FEM points are included for comparison. The solid bounding lines at the left and right indicate the limits of feasible void shapes before coalescence.
Figure 5.10: Prediction of variation of the uniaxial yield stress $\Sigma_{33}$ along the $X_3$ direction as a function of void volume fraction $f$, and void shape $u$. FEM points are included for comparison.
Figure 5.11: Prediction of variation of the uniaxial yield stress $\Sigma_{12}$ as a function of void volume fraction $f$, and void shape $u$. FEM points are included for comparison.
Figure 5.12: Prediction of variation of the uniaxial yield stress $\Sigma_{23}$ as a function of void volume fraction $f$, and void shape $u$. FEM points are included for comparison.
Figure 5.13: Variation of $S$-$T$ yield surfaces with void aspect ratio, $\frac{a}{c} > 1$ denotes oblate voids, $\frac{a}{c} = 1$ denotes spherical voids, and $\frac{a}{c} < 1$ denotes prolate voids. Void volume fraction = 1%. 
Figure 5.14: Variation of $S$-$T$ yield surfaces with void aspect ratio, $\frac{a}{c}$. $\frac{a}{c} > 1$ denotes oblate voids, $\frac{a}{c} = 1$ denotes spherical voids, and $\frac{a}{c} < 1$ denotes prolate voids. Void volume fraction = 10%.
Figure 5.15: Predictions $S-T$ yield surface for an oblate void of aspect ratio $\frac{a}{c} = 2$. Void volume fraction = 10%. FEM values are included for comparison.
Figure 5.16: Comparison of the predicted ratio of deformation rates $\frac{D_3}{D_1}$ as a function of $\frac{S}{T}$ with FEM. The void is oblate with aspect ratio $\frac{a}{c} = 2$. Void volume fraction = 10%.
A. Introduction

This chapter deals with an axi-symmetric model for fiber reinforced composites. The aim is to examine the limiting case of re-inforcements of zero strength in order to model cylindrical linked up voids in the matrix. The extension to voided materials with continuous cylindrical voids is made in the limit of setting the elastic modulus of the fiber to zero. The yield functions obtained by this approach will be compared to the predictions of the yield function proposed in Chapter III.

Fiber composites have yield properties which clearly differ from the typical von Mises behavior used to describe yield of homogeneous plastic materials. Earlier models based on yield of a cylindrical composite element, in which an elastic fiber is surrounded by an annulus of non hardening elastic-plastic material, predict a bilinear stress-strain behavior during monotonic uniaxial loading along the fiber direction [Hill (1964)], and qualitatively predict experimental measurements of hysteretic behavior during cyclic
uniaxial loading [Mulhern et al. (1967)]. Unlike pure or solid solution
materials, two-phase materials are susceptible to yield under pure hydrostatic
loading [Ashby et al. (1969), Radcliffe et al. (1964)]. In particular, TEM studies
by Ashby et al. (1969) have documented high dislocation densities around
SiO2 and Al2O3 particles in a Cu matrix, when loaded to 2.5 GPa pressure.
Corresponding continuum analyses have quantified the deviatoric stresses
produced near particles [Ashby et al. (1969)] and near cylindrical fibers
[Dvorak et al. (1973)] under hydrostatic loading, when the bulk moduli of the
two phases are different. Other departures from von Mises behavior are the
marked anisotropy of yielding under various multiaxial stress states, as well
as yielding due to heating or cooling. For example, finite element studies of a
hexagonal array of aligned elastic fibers in an elastic-perfectly plastic matrix,
with material properties chosen to simulate a B-Al composite, were used to
determine the loading and uniformly distributed temperatures at which
yielding first occurs [Dvorak et al. (1973), Dvorak et al. (1974)]. Although the
von Mises yield surface used to describe matrix yield is a circular cylinder extend ing indefinitely along the hydrostatic axis, the composite yield surface
is truncated along the hydrostatic axis, and resembles an irregular ellipsoid
when plotted as a function of the direct stresses acting on the composite
[Dvorak et al. (1974)]. The same finite element studies have shown that
heating or cooling the composite by an amount $\Delta T$ effectively shifts the surface
for initial yield along the hydrostatic axis by an amount, $3\Delta T (\alpha_m - \alpha_f)/(1/K_f - 1/K_m)$, where $\alpha$ and $K$ are the coefficient of thermal expansion and bulk
modulus of the matrix ($m$) and fiber ($f$), respectively. More recent work has
examined the critical $\Delta T$ to yield the composite, as a function of elastic
properties and volume fractions of the components [Daehn et al. (1991), Zhang et al. (1993a)].

This chapter documents the critical axisymmetric stress states to yield continuous fiber composites, as a function of the elastic properties and volume fractions of the components. In the results presented here, the surfaces correspond to yield of the entire matrix. Some other analyses have based surfaces on the initial matrix yield [Dvorak et al. (1973), Dvorak et al. (1974)]. The yield surfaces produced here are nearly elliptical and can be characterized by three quantities: the lengths of the major and minor axes and the orientation angle of the ellipse. Although yield surfaces of specific fiber composites have been reported previously, this study addresses axisymmetric yield behavior over a wide range of component properties and volume fractions, in an effort to provide useful yield design criteria. Where appropriate, comparison will be made to estimates of yield of fiber composites based on finite element [Dvorak et al. (1973), Brockenborough et al. (1991), Nakamura et al. (1993)] and self-consistent approaches [Dvorak et al. (1979)], as well as particulate composite models based on upper bound approaches in the limit of rigid spherical particles [Chu and Hashin (1971)]. As a further application of the composite model, the strength of the fiber will be set to zero to model a cylindrical void in a matrix. The ensuing results will be compared to predictions of yield for spherical cavities Hill (1950), Oyane et al. (1989), and Gurson (1977). The predictions will also be compared to the predictions from the model proposed in Chapter III. These results may also be found in Zhang et. al (1993b).
B. Formulation

A more detailed formulation of the axisymmetric analysis is in Zhang et al. (1993a), which considers the stress state and eventual yield in the composite when heated or cooled, due to different coefficients of thermal expansion in each component. In comparison, the present chapter addresses the isothermal response of initially stress-free metal matrix composites loaded by an axisymmetric (S-T) stress state, as shown in Fig. 6.1. The fibers are assumed to be parallel, infinitely long compared to their spacing, and perfectly bonded to the surrounding metal matrix.

![Fig. 6.1. Axisymmetric cell used to simulate continuous fiber composites.](image)

Although the cell shown in Fig. 6.1 is not space filling, periodic conditions are applied to the outer wall at \( r = r_m \). The axisymmetric geometry, S-T loading, and infinite fiber length greatly simplify the analysis so that the three
direct components of stress and strain in the cylindrical coordinate system shown correspond to the principal stress and strain states, \((\sigma_r, \sigma_\theta, \sigma_z)\) and \((\varepsilon_r, \varepsilon_\theta, \varepsilon_z)\). The corresponding equilibrium equation can be written as (e.g., see Chakrabarty (1987), pp. 316-323),

\[
\frac{\partial \sigma_r}{\partial r} = \frac{\sigma_\theta - \sigma_r}{r} \quad (6.1)
\]

and the corresponding compatibility equations consistent with the displacement field \(u_r = u_r(r), u_\theta = 0, u_z = cz\) are

\[
\frac{\partial \varepsilon_\theta}{\partial r} = \frac{\varepsilon_r - \varepsilon_\theta}{r}, \quad \text{and} \quad \frac{\partial \varepsilon_z}{\partial r} = 0. \quad (6.2)
\]

These equations must be solved subject to boundary conditions that the stresses on the cell boundary match the applied stresses, \((S, T)\), and that the fiber-matrix interface at \(r = r_f\) is fully bonded,

\[
\sigma_r(r = r_m) = T, \quad \varepsilon_\theta^n|_{r=r_f} = \varepsilon_\theta^f|_{r=r_f}, \quad \sigma_r^n|_{r=r_f} = \sigma_r^f|_{r=r_f}, \quad \varepsilon_t^n|_{r=r_f} = \varepsilon_t^f|_{r=r_f}. \quad (6.3)
\]

The matrix region, \(r_f < r < r_m\), is assumed to be elastic-plastic, and the inner fiber region, \(r < r_f\), is assumed to be elastic only. The matrix is assumed to yield when a von Mises yield function defined by

\[
\Phi(\sigma_i, \sigma_m) = \frac{1}{2}[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_r - \sigma_z)^2] - \sigma_m^2 \quad (6.4)
\]

reaches zero. Here, \(\sigma_m^2\) is the current flow stress of the matrix in simple tension, and for isothermal conditions, it is assumed to depend on effective plastic strain, \(\varepsilon_m^p\). When the yield surface is used as a plastic potential, and
conditions of normality and consistency are imposed (e.g., see Chakrabarty (1987), pp. 77-84), the resulting incremental constitutive relation is

\[
d\epsilon^m_i = \begin{cases} \frac{1}{E_m}[(1 + \nu_m)d\sigma^m_i - \nu_m d\sigma^m_{kk}], & \Phi < 0 \\ \frac{1}{E_m}[(1 + \nu_m)d\sigma^m_i - \nu_m d\sigma^m_{kk}] + \frac{1}{H} \frac{\partial \Phi}{\partial \sigma^m_i} \left( \frac{1}{2\sigma_m} \frac{\partial \Phi}{\partial \sigma^m_j} d\sigma^m_j \right), & \Phi = 0 \end{cases} (6.5)
\]

The strain increment in (6.5) has an elastic contribution dependent on Young's modulus \( E_m \) and Poisson's ratio \( \nu_m \) of the matrix. Here, \( \sigma_{kk} \) denotes the trace of the stress tensor. During yielding, the strain increment has an additional contribution dependent on the plastic modulus, \( H \), defined by \( d\sigma_m / d\epsilon^p_m \). A power-law relation and corresponding plastic modulus of the form,

\[
\frac{\bar{\sigma}_m}{\sigma_0} = \left( \frac{\epsilon^p_m}{\epsilon_0} + 1 \right)^N, \quad H = E_m N \left( \frac{\bar{\sigma}_m}{\sigma_0} \right)^{N-1} \quad (6.6)
\]

are used, where \( \sigma_0, \epsilon_0 \) are the uniaxial yield stress and strain of the matrix.

The fiber is assumed to be elastic, with different isotropic elastic properties, \( E_f \) and \( \nu_f \) than the matrix. Under such conditions, the solution to the Lame problem dictates that the only admissible stress field in the fiber is a uniform one, described by \( (\sigma'_1 = \sigma'_\theta, \sigma'_2) \) (e.g., see Hill (1950), pp. 107-110). The elastic states in the fiber and matrix can be easily solved for. Denoting the volume fraction of fiber by \( f \), the elastic quantities in the fiber are given by,

\[
\epsilon^F_z = \epsilon^M_z = \epsilon_z. \quad (6.7)
\]

\[
\sigma^F_r = \sigma^F_\theta = T_F; \sigma^F_z = S_F = E_F \epsilon_z + 2 \nu_F T_F. \quad (6.8)
\]
The elastic state in the matrix is then given by,

\[ \sigma^M_r = C + \frac{D}{r^2}, \quad \sigma^M_\theta = C - \frac{D}{r^2}, \]

where,

\[ C = \frac{T - fT_F}{(1-f)}, \quad D = \frac{(T_F - T)r_F^2}{(1-f)}. \]

(6.10)

Using continuity of \( \varepsilon_\theta \) at \( r = r_F \), we obtain,

\[ T_F = \frac{(1 - \nu_M)T}{1 - 2\nu_F} + \frac{1 + f - 2f\nu_M}{2\mu_m(1-f)} \left( \frac{1}{2\mu_F} + \frac{1 + f - 2f\nu_M}{2\mu_m(1-f)} \right), \]

(6.11)

where \( \mu \) is defined by \( \mu = \frac{E}{2(1 + \nu)} \).

Using the equations (6.7)-(6.11) the elastic problem is completely solved. These equations can be used to obtain the analytical locus in S-T space for the inception of yield at the fiber matrix interface. Now, given an increment \( \text{dof } r = \text{dof } 0 \) in the fiber stress state, the corresponding increment in remote stresses \((S, T)\) are determined by first using the elastic constitutive relations to find \( \text{d}e^f \), then using the continuity conditions in (6.3) to find \( \text{d}e^m \), \( \text{d}e^m_\theta \), \( \text{d}e^m_\phi \) at \( r = r_F \), and finally integrating (6.1), (6.2), and (6.5) from \( r = r_F \) to \( r = r_M \). The resulting values of \( dS, dT \) are determined using the two relations on the left side of (6.3). The ratio of \( d\sigma^f = d\sigma^f_\theta \) to \( d\sigma^f_\phi \) is chosen so that a proportional S-T path is followed from an initially stress-free state to the point where the entire matrix yields. The resulting values of \((S, T)\) for various proportional paths are recorded to produce the entire S-T yield surface.
C. Results and Discussion

This section describes the anisotropic yield behavior of fiber composites in terms of a departure of the yield surface from a von Mises surface. The principal composite variables considered are the volume fraction, \( f \), of the fiber and the ratio, \( E_f/E_m \), of Young's moduli. Poisson's ratio for each component is usually set to 0.3 in the results shown, although in some specific cases, the values for each component are varied from 0.25 to 0.35. In all cases, the hardening exponent \( N = 0.05 \) to model an essentially perfectly plastic material with yield stress \( \sigma_0 \), and \( \sigma_0/E_m = 0.0004 \).

The important yield surface features for the case of stiffer fibers is shown in Fig. 6.2, where \( E_f/E_m = 10 \). The dashed line shows the corresponding von Mises surface for the matrix, or equivalently, the surface for \( f = 0 \), and the solid lines show a succession of approximately elliptical surfaces as \( f \) is increased. Over the large range of \( f \) shown, the surface becomes more equiaxed and rotates counterclockwise with increasing \( f \). The more equiaxed shape and rotation indicate that both hydrostatic and deviatoric loadings cause yield of these composites. For example, the \( f = 0.5 \) curve shows that the yield stress in hydrostatic tension is lower than that for pure axial (S) loading. This feature is qualitatively consistent with Chu and Hashin (1971), which reports that the hydrostatic stress required to yield a rigid particulate composite with \( f = 0.1 \) is approximately five times that for \( f = 0.5 \).
Fig. 6.2. S-T yield surfaces for composites with \( \frac{E_f}{E_m} = 10 \) and \( \nu_f = \nu_m = 0.3 \) for \( f = 0 \) (dashed line), 0.05, 0.25, and 0.5 (solid lines). \( \sigma_o \) is the yield stress of the matrix. The shaded region shows the variation in the case for \( f = 0.25 \) which occurs when \( \nu_f \) and \( \nu_m \) are independently varied from 0.25 to 0.35.

The observed rotation in Fig. 6.2 also introduces anisotropic yield behavior. In particular, the counterclockwise rotation causes the axial yield stress to become larger than transverse one. The rotation also defines a major axis direction, \( S \neq T \), along which the composite is most resistant to yield, and a perpendicular minor axis direction, \( S \neq -T \), along which the composite is easiest to yield.

Changes in Poisson's ratio of each component generally produces modest variation along the minor axis dimension and much larger variation along the
major axis dimension. In particular, the shaded region in Fig. 6.2 shows the region within which yield surfaces for $f = 0.25$ lie when $\nu_f$ and $\nu_m$ are varied independently from 0.25 to 0.35. Over this range, the length of the major axis increases in length by approximately 18 percent for the case $\nu_f = 0.25$ and $\nu_m = 0.35$, and decreases by approximately 27 percent when Poisson's ratios are reversed. In comparison, the change in the length of the minor axis is modest. These trends are consistent with Ashby (1969), which asserts that yield under hydrostatic loading depends on differences, $\Delta(1/K)$, in bulk moduli of the components. Clearly, in the limit of equal bulk moduli, an admissible solution for hydrostatic loading is that the stress state in each component is simply a uniform hydrostatic stress equal to the applied value. In this limit, the matrix will not yield, and the corresponding yield surface extends indefinitely along the hydrostatic axis. Over the limited range of Poisson's ratios shown for $f = 0.25$, the major axis increases in length and rotates clockwise as the bulk moduli become more similar. In this case, $K_f/K_m$ decreases from approximately 17 for $(\nu_f, \nu_m) = (0.35, 0.25)$ to 6 when $\nu_f, \nu_m$ are reversed. Although the case $f = 0.25$ is discussed in detail here, the trend over the range of $(\nu_f, \nu_m)$ reported is similar for other volume fractions.

The important yield surface features for the case of more compliant fibers are shown in Fig. 6.3, where $E_f/E_m = 0.1$ and again, the Poisson's ratio for each component is 0.3. The trend in rotation is similar to, although less pronounced than that in the stiffer fiber case. Consequently, anisotropy is less pronounced, in that the axial yield stress for a given $f$ is only modestly higher than the transverse value. The reduction in major axis length with $f$ is much larger here, and the minor axis dimension decreases rather than increases, so
that all yield surfaces lie within the von Mises ($f = 0$) surface. As for the stiffer fiber case, the shaded region in Fig. 6.3 shows the locus of yield surfaces for $f = 0.25$ when $v_f$ and $v_m$ are independently varied from 0.25 to 0.35. Again, the major axis increases and rotates clockwise as the bulk moduli of each component become more equal. Results for a hollow spherical shell of rigid perfectly plastic material show a similar reduction in major and minor axes, but no rotation occurs in that case [Oyane et al. (1989)].

![Fig. 6.3. S-T diagram for composites with $E_i/E_m = 0.1$ and $v_f = v_m = 0.3$ for $f = 0$ (dashed line), 0.05, 0.25, and 0.5 (solid lines). $\sigma_o$ is the yield stress of the matrix. The shaded region shows the variation in the case for $f = 0.25$ which occurs when $v_f$ and $v_m$ are independently varied from 0.25 to 0.35.](image)
Figures 6.4, 6.5 and 6.6 document, respectively, the yield surface orientation $\theta$, length $a$ of the major axis, and length $b$ of the minor axis over a wide range of fiber volume fraction and ratio of Young's moduli of the two components, with $v_f = v_m = 0.3$. These three quantities may be used to define elliptical yield surfaces that reasonably approximate the computed ones.

![Graph showing orientation angle $\theta$ as a function of fiber volume fraction and ratio of Young's moduli](image)

**Fig. 6.4.** Orientation angle $\theta$ of the major axis of $S-T$ yield surfaces as a function of fiber volume fraction and ratio of Young's moduli, for $v_f = v_m = 0.3$. (+) denotes finite element based estimates from [Dvorak et al. (1973)], and (o) denotes self-consistent based estimates from [Dvorak et al. (1979)]; both assume $E_f/E_m = 6$, $v_f = 0.21$, $v_m = 0.33$. 
Using an elliptical approximation for the yield surfaces, the yield values for pure axial loading \( (S_y) \), pure axisymmetric loading \( (T_y) \), and hydrostatic loading \( (P_y) \) are simply functions of \( a, b \), and \( \theta \) provided by these plots,

\[
S_y = \pm \left( \frac{a^2 b^2}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \right)^{1/2} = \pm \frac{b}{\cos \theta} \quad \text{for } f << 1 \quad (6.12)
\]

\[
T_y = \pm \left( \frac{a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \right)^{1/2} = \pm \frac{b}{\sin \theta} \quad \text{for } f << 1 \quad (6.13)
\]

\[
P_y = \pm \left( \frac{2a^2 b^2}{(a^2 + b^2) + (b^2 - a^2) \sin 2\theta} \right)^{1/2} \quad (6.14)
\]

The orientation angle \( \theta \) is a basic measure of anisotropy, in that the ratio, \( S_y/T_y = \tan \theta \) in the dilute (small \( f \)) limit, where \( a >> b \). Further, the loading paths \( S/T = \tan \theta \), \( \tan \theta \pm 90^\circ \) are the directions along which the composite is most resistant and least resistant to yield, respectively.

The plot of yield surface orientation in Fig. 6.4 shows that \( \theta = 45^\circ \) when the elastic properties of the fiber and matrix are equal, and that \( \theta \) increases when the fiber and matrix moduli become more different, regardless of whether the fibers are more or less stiff. In fact, \( \theta \) for the case of cylindrical voids, approximated by \( E_f/E_m = 10^{-4} \), is comparable to that for stiff fibers in the range \( E_f/E_m = 5 \) to \( 8 \). In all cases, the rotation appears to increase most rapidly with \( f \) in the dilute range.
Fig. 6.5. The length of the major axis of S-T yield surfaces, as a function of fiber volume fraction and ratio of Young's moduli, for \( \nu_f = \nu_m = 0.3 \). The dashed line shows the prediction for spherical voids based on Hill (1950). See Fig. 6.4 caption for meaning of (+), (o).

Figure 6.5 demonstrates that the major axis of the yield surface decreases in length as the fiber and matrix moduli become more different. Clearly, the effect is much more pronounced for less stiff fibers. In the limiting case of cylindrical voids, modeled by \( E_f/E_m = 10^{-4} \), the length \( a \) of the major axis is comparable to the external hydrostatic stress, \( P_y = -2\ln(f)/3 \) to yield a thick-walled spherical shell surrounding a spherical void of volume fraction \( f \) [Hill (1950)].
Fig. 6.6. The length of the minor axis of $S-T$ yield surfaces, as a function of fiber volume fraction $f$ and ratio of Young's moduli, for $v_f = v_m = 0.3$. The spherical void result is based on Gurson (1977). See Fig. 6.4 caption for meaning of (+), (o).

In comparison, Fig. 6.6 shows that the minor axis of the yield surface increases or decreases in length as the fiber is made more stiff or more compliant, respectively, than the matrix. In the more dilute limit, the major axis length $a$ is a measure of the composite resistance to hydrostatic loading, while the minor axis limit $b$ is a measure of the composite resistance to deviatoric and uniaxial loading. The comparison in Figs. 6.4, 6.5, 6.6 with results from finite element based results [Dvorak et al. (1973)] suggests that the non space-filling cell employed here reasonably approximates $\theta$ and $b$ for arrays of hexagonally packed fibers with $E_f/E_m = 6$ and $(v_f, v_m) = (0.21, 0.33)$, although it appears to overestimate $a$. However, the finite element work
[Dvorak et al. (1973)] is based on onset of yield in the matrix compared to full matrix yield in the present work. In general, packing geometry has little effect on axial yield, although it does have strong effects on transverse yield behavior [Brockenborough et al. (1991), Nakamura and Suresh (1993)].

Thus, the results of the numerical study on composites shows that although the matrix is assumed to obey a pressure-independent von Mises yield criterion, composites with different bulk moduli for each component yield under pure hydrostatic loading. The analysis also shows that the hydrostatic load to yield decreases with increasing difference in bulk modulus of the components, regardless of which component has the higher bulk modulus.

Continuous fiber composites are also shown to have preferred axisymmetric loadings for which they are most resistant and least resistant to yield. Although the matrix material is most resistant to yield when $|S| = |T|$, the composites tend to be most resistant to yield for specific loadings with the general feature $|S| > |T|$, regardless of whether the fiber is more or less stiff than the matrix. In a related manner, each composite has $(S-T)$ loading directions for which it is easiest to yield, which differ from the von Mises prediction that $|S| = -|T|$. These features are linked to an inherent anisotropy in fiber composites, the magnitude of which is quantified in terms of a rotation of the $(S-T)$ yield surface that is dependent on the relative elastic properties and volume fraction of the fiber. The large rotations observed over a wide range of elastic properties and volume fractions suggest important design criteria that are not modeled well by isotropic theories of yielding.
D. Application to continuously voided materials

Finally, we seek to show an application of the composite model to simulate a cylindrical void in an elastic-perfectly plastic matrix. For the composite model, the ratio of the elastic modulus of the fiber to the elastic modulus of the matrix is set to a low value ($<10^{-4}$).

For comparison to the predictions of the yield function proposed in Chapter III, the void aspect ratio is set to the limitingly prolate value in order to simulate a linked up array of circular cylindrical voids in a matrix.

Figs. 6.7 and 6.8 show a comparison between the void-composite yield surface, the yield surface from the proposed model in Chapter III, and the Gurson model for void volume fractions of 10% and 20% respectively. The anisotropy present in the cylindrical geometry is apparent when model predictions for $\frac{E_f}{E_m} \to 0$ are compared with the Gurson yield function. The decrease in the size of the yield surface, and rotation associated with a spherical to cylindrical change in void shape, contribute to a lower yield under both hydrostatic and transverse (T) loadings, but provide a comparable yield for axial (S) loading. The yield surface obtained from the void-composite model shows good agreement with the model proposed in Chapter III for both the void volume fractions considered.

Figs. 6.7 and 6.8 show that the predictions of the yield function proposed in Chapter III are accurate in the case of limitingly prolate (continuous) voids. In addition they also show the anisotropy induced in the S-T yield function due to void shape.
Figure 6.7: Comparison of void-composite model predictions with the yield function predictions for continuously linked up prolate voids. Void volume fraction = 10%. Gurson model predictions for a spherical void of 10% void volume fraction is included for comparison.
Figure 6.8: Comparison of void-composite model predictions with the yield function predictions for continuously linked up prolate voids. Void volume fraction = 20%. Gurson model predictions for a spherical void of 20% void volume fraction is included for comparison.
CHAPTER VII

KINEMATIC RELATIONS FOR VOID EVOLUTION

A. Introduction

In this chapter, relations are developed for the rate of change of the void axes lengths with macroscopic deformation. For the yield function model proposed in chapter III, the voids were assumed to be ellipsoidal in shape. We denote \( d \) to be the deformation rate of the void and \( D \) to be the macroscopic deformation rate. The void update relations link the components of \( d \) to \( D \). We assume that

\[
d_{ij} = N_{ijkl} D_{kl} .
\]

(7.1)

The components \( N_{ijkl} \) are assumed to depend only on the void and cell dimensions. This assumption is reasonable for linear materials [Budiansky et al. (1982)].

We assume that

\[
d_{ij} = 0 \quad i \neq j .
\]

(7.2)

Further, we assume that the macroscopic shear deformation rates cause no increase in void axes lengths. This yields,

\[
N_{ijkl} = 0 \quad i \neq j, \quad N_{ijkl} = 0 \quad k \neq l .
\]

(7.3)
This implies that there are only nine non-zero \( N_{ijkl} \). Since the void has
been assumed to be ellipsoidal in a cubical cell, the symmetries of the cell
reduce the number of independent \( N_{ijkl} \) to five.

\[
N_{1111} = N_{2222} \tag{7.4}
\]
\[
N_{1133} = N_{2233} \tag{7.5}
\]
\[
N_{3311} = N_{3322} \tag{7.6}
\]
\[
N_{1122} = N_{2211} \tag{7.7}
\]
\[
N_{3333} \tag{7.8}
\]

Since there are only five non-trivial unique \( N_{ijkl} \), we can use a two index
representation. Assuming that \( d_1 = d_{11} \), \( d_2 = d_{22} \), \( d_3 = d_{33} \), \( D_1 = D_{11} \),
\( D_2 = D_{22} \), and \( D_3 = D_{33} \), we assume

\[
d_i = N_{ij} D_j \tag{7.9}
\]

Out of the nine \( N_{ij} \) components, only 5 are independent.

\[
N_{11} = N_{22} = N_{1111} = N_{2222} \tag{7.10}
\]
\[
N_{13} = N_{23} = N_{1133} = N_{2233} \tag{7.11}
\]
\[
N_{31} = N_{32} = N_{3311} = N_{3322} \tag{7.12}
\]
\[
N_{12} = N_{21} = N_{1122} = N_{2211} \tag{7.13}
\]
\[
N_{33} \tag{7.14}
\]

We now develop expressions for the \( N_{ij} \) components.
B. Formulation

We are seeking to develop relations for the lengthening rate of the void axes of an ellipsoidal void in terms of the macroscopic deformation rate in a plastic material. Solving this problem exactly is infeasible. Frequently, numerical approaches are used, for instance Budiansky et al. (1982).

Figure 7.1: Octant of assumed model of a cuboidal void in a cubical cell.
Since we are interested in the rate of changes of the lengthening rates of the three orthogonal void axes, we employ a model of a cuboidal void inside a cubical cell (Fig. 7.1). The extension of these update relations to the ellipsoidal void arrangement is dealt with in Section C of this chapter.

The void has dimensions of $a$ along the $X_1$ and $X_2$ axes and a dimension of $c$ along $X_3$. The cell is a cube of side $S$.

i) Loading with $D_3$

We consider first loading by $D_3$, with $D_1 = D_2 = 0$. Fig. 7.2 shows the projection of the octant on the $X_1 - X_2$ plane.

![Figure 7.2: Projection of cube on the $X_1 - X_2$ plane.](image)

The $X_3$ axis points out of the plane of the paper.

Regions (1) and (2) are shown by their projections.
We consider the octant to be made of four regions, each of which is assumed to deform at a homogeneous deformation rate:

Region 1: \( X_1 \epsilon [a, \frac{S}{2}], X_2 \epsilon [a, \frac{S}{2}], X_3 \epsilon [0, \frac{S}{2}] \)

Region 2: \( X_1 \epsilon [a, \frac{S}{2}], X_2 \epsilon [0, a], X_3 \epsilon [0, \frac{S}{2}] \)

(or)

Region 2: \( X_1 \epsilon [0, a], X_2 \epsilon [a, \frac{S}{2}], X_3 \epsilon [0, \frac{S}{2}] \)

Region 3: \( X_1 \epsilon [0, a], X_2 \epsilon [0, a], X_3 \epsilon [c, \frac{S}{2}] \)

Region 4: \( X_1 \epsilon [0, a], X_2 \epsilon [0, a], X_3 \epsilon [0, c] \).

Region 4 is the void volume and Region 3 may be thought of as the cap region along the \( X_3 \) axis that lies above the void. Thus, the deformation rate vector in region 4 is the void deformation rate, \( d \):

\[
\begin{align*}
{d_1}^{(4)} &= {d_1} \\
{d_2}^{(4)} &= {d_2} \\
{d_3}^{(4)} &= {d_3}
\end{align*}
\]

We assume that

\[
D_3^{(1)} = D_3.
\]

Using volume conservation and symmetry, we obtain in region 1,

\[
D_1^{(1)} = D_2^{(1)} = -\frac{1}{2} D_3
\]
Averaging the deformation rate along $X_2$ axis on a path cutting the void, we obtain

$$d^{(2)}_2 = \frac{(S - 2a)}{4a} D_3 \quad (7.20)$$

Let us assume that

$$d^{(2)}_3 = D_3 \quad (7.21)$$

Using volume constancy in region 2, we get,

$$d^{(2)}_1 = -\frac{(S + 2a)}{4a} D_3 \quad (7.22)$$

Averaging the deformation rate along $X_1$ along a path cutting a void, and using eqn. (7.22), we obtain,

$$d_1 = -\frac{(S - 2a)(S + 2a)}{8a^2} D_3 \quad (7.23)$$

By symmetry of the geometry and the loading,

$$d_2 = d_1 \quad (7.24)$$

Averaging the deformation rate along $X_1$, along a path cutting region 3 and region 2, we get,

$$d^{(3)}_1 = -\frac{(S - 2a)}{2a} d^{(2)}_1 \quad (7.25)$$

By volume conservation of region 3,

$$d^{(3)}_3 = -2 d^{(3)}_1 = \frac{(S - 2a)}{a} d^{(2)}_1 \quad (7.26)$$
Averaging the deformation rate along a path cutting region 3 and the void and using (7.26) we get
\[ d_3 = \left( \frac{S}{2c} + \frac{(S - 2c)(S - 2a)(S + 2a)}{8a^2c} \right) D_3. \] (7.27)

Equation (7.23) can be used to calculate \( N_{13}^c \) and eqn. (7.27) can be used to give \( N_{33}^c \). The superscript \( c \) indicates that the update expression is for a cuboidal void.

ii) Loading with \( D_2 \)

We now consider loading by \( D_2 \). Fig. 7.3 shows the projection of the octant on the \( X_3 - X_1 \) plane. We consider the octant to consist of five regions including the void volume. Each of these regions is assumed to have a homogenous deformation rate.

Figure 7.3: Projection of cube on the \( X_3 - X_1 \) plane.

The \( X_2 \) axis points out of the plane of the paper.

Regions (1) and (2) and (4) are shown by their projections.
We consider the octant to be made of five regions, each of which is assumed to have a homogeneous deformation rate.

**Region 1:** \[ \begin{align*} &X_1 \in [a, \frac{S}{2}], \quad X_2 \in [0, \frac{S}{2}], \quad X_3 \in [c, \frac{S}{2}] \end{align*} \]

**Region 2:** \[ \begin{align*} &X_1 \in [0, a], \quad X_2 \in [0, \frac{S}{2}], \quad X_3 \in [c, \frac{S}{2}] \end{align*} \]

**Region 3:** \[ \begin{align*} &X_1 \in [0, a], \quad X_2 \in [a, \frac{S}{2}], \quad X_3 \in [0, c] \end{align*} \]

**Region 4:** \[ \begin{align*} &X_1 \in [a, \frac{S}{2}], \quad X_2 \in [0, \frac{S}{2}], \quad X_3 \in [0, c] \end{align*} \]

**Region 5:** \[ \begin{align*} &X_1 \in [0, a], \quad X_2 \in [0, a], \quad X_3 \in [0, c] \end{align*} \]

Region 4 is the cap region that is located along the \( X_2 \) axis over the void as we look at the void down the \( X_2 \) axis. Region 5 is the void volume.

Let us assume that

\[
\begin{align*}
d_2^{(1)} &= d_2^{(2)} = d_2^{(4)} = D_2. \tag{7.28}
\end{align*}
\]

To satisfy incompressibility in region 1, we assume

\[
\begin{align*}
d_1^{(1)} &= -\frac{(1-\alpha)}{2} D_2 \tag{7.29}
\end{align*}
\]

\[
\begin{align*}
d_3^{(1)} &= -\frac{(1+\alpha)}{2} D_2. \tag{7.30}
\end{align*}
\]

\( \alpha \) is an unknown factor that depends on void shape. By symmetry of the geometry, we can see from fig. 7.3 that \( \alpha = 0 \) for cubical voids.

Averaging the rate of deformation along \( X_3 \) along region 1 and region 4,
Using incompressibility in region 4, we get

\[ d^{(4)}_1 = -\left\{ \frac{1+\alpha}{2} \right\} \frac{(S-2c)}{2c} D_2 \].

(7.31)

Averaging the rate of deformation along \( X_1 \) along region 1 and region 2,

\[ d^{(2)}_1 = \frac{(1-\alpha)}{2} \frac{(S-2a)}{2a} D_2 \].

(7.33)

Using incompressibility in region 2, we get

\[ d^{(2)}_3 = -\left\{ \frac{1-\alpha}{2} \right\} \frac{(S-2a)}{2a} + 1 \} D_2 \] .

(7.34)

Averaging the rate of deformation along \( X_1 \) along region 4 and the void,

\[ d_1 = -\frac{(S-2a)}{2a} \left\{ \frac{1+\alpha}{2} \right\} \frac{(S-2c)}{2c} + 1 \} D_2 \].

(7.35)

Averaging the rate of deformation along \( X_3 \) along region 2 and the void,

\[ d_3 = \frac{(S-2c)}{2c} \left\{ \frac{1-\alpha}{2} \right\} \frac{(S-2a)}{2a} + 1 \} D_2 \].

(7.36)

Averaging the rates of deformation along \( X_1 \) and \( X_3 \) along region 3 and region 4,

\[ d^{(3)}_1 = d_1 \]

(7.37)

\[ d^{(3)}_3 = d_3 \]

(7.38)

Volume conservation of region 3 gives,
Averaging rates of deformation along $X_2$ along region 3 and void,

$$
\frac{d_2^{(3)}}{D_2} = \left\{ \frac{(S - 2a)}{2a} \right\} D_2 - \frac{2a}{(S - 2a)^2} d_2 .
$$

Equating eqn. (7.39) and eqn. (7.40), we obtain,

$$
d_2 = \left\{ \frac{S}{2a} + \frac{(S - 2a)}{2a} \left[ \frac{(S - 2a)}{2a} \right] \right\} D_2 .
$$

Equations (7.35), (7.36) and (7.41) can now be used to obtain the expressions for $N_{12}^c$, $N_{31}^c$ and $N_{11}^c$ respectively. The superscript $c$ indicates that the void is cuboidal in shape.

The relevant update relations are obtained as,

$$
N_{11}^c = \frac{S}{2a} + \frac{(S - 2a)}{2a} \left[ \frac{(S - 2a)}{2a} \right] \frac{(S - 2a)(S - 2c)}{2a}
$$

$$
N_{12}^c = -\frac{(S - 2a)}{2a} \left\{ \frac{2}{2} \frac{(S - 2a)}{2a} \right\} + 1
$$

$$
N_{13}^c = -\frac{(S - 2a)(S + 2a)}{8a^2}
$$

$$
N_{31}^c = \frac{(S - 2c)}{2c} \left\{ \frac{2}{2} \frac{(1 - \alpha)(S - 2a)}{2a} \right\} + 1
$$

$$
N_{33}^c = \frac{S}{2c} + \frac{(S - 2c)(S - 2a)(S + 2a)}{8a^2c}
$$

It can be seen that for cubes $N_{12}^c = N_{13}^c = N_{31}^c$, and $N_{11}^c = N_{33}^c$. 

The relations (7.42) - (7.46) can be used to obtain the void axes lengthening rates of a cuboidal void in a cubical cell, as a function of deformation. The extension of these results for the case of ellipsoidal voids is examined next.

C. Update relations for ellipsoidal voids

To extend the equations (7.41) - (7.45) to the case of ellipsoidal voids, we consider the loading of spherical voids under $D_1 = D_2 = D_3$. Let $N_{ij}^e$ represent the quantities required to update the ellipsoidal void axes lengths. We seek to relate $N_{ij}^e$ to $N_{ij}^c$.

Under the loading considered, we assume that the ellipsoidal void dimensions are the same as the cuboidal void dimensions. Let us denote the volume of the ellipsoid as calculated from the void axes lengths by $g$. Thus,

$$g = \frac{4\pi}{3} \left[ \frac{a}{s} \right]^2 \left[ \frac{c}{s} \right].$$

In an actual material, an initially ellipsoidal void will distort to other irregular shapes as deformation proceeds. $g$ can be calculated if the void axes lengths are known as a function of the loading. The actual void volume fraction $f$, can be calculated by using the fact that all macroscopic dilatation is due solely to the void volume change. Let us assume that,

$$\dot{g} = \dot{\nu}^c$$
where \( V^c \) denotes the volume of the corresponding cuboidal void. Let \( a^c \) denote the side of the cubical void and \( a^e \) denote the radius of the sphere. Then, by assumption,

\[
a^e = a^c.
\]

(7.49)

Equation (7.48) yields,

\[
\dot{a}^e = \frac{6}{\pi} \dot{a}^c.
\]

(7.50)

Using eqn. (7.50) with eqn. (7.9), we obtain,

\[
N_{ij}^e = \frac{6}{\pi} N_{ij}^c.
\]

(7.51)

Thus, the \( N_{ij}^c \) underestimate the actual \( N_{ij} \) values for a spherical void by a factor of \( \frac{6}{\pi} \), for a loading \( D_1 = D_2 = D_3 \).

We assume that this correction of \( \frac{6}{\pi} \) is valid for all other loadings also.

Thus, we obtain the following estimates for the \( N_{ij}^e \) values.

\[
N_{11}^e = \frac{6}{\pi} \left\{ \frac{S}{2a} + \frac{(S - 2a)}{2a} \left[ \frac{(S - 2a)}{2a} + \frac{(S - 2c)}{2c} + \frac{(S - 2a)(S - 2c)}{2a 2c} \right] \right\}
\]

(7.52)

\[
N_{12}^e = -\frac{6}{\pi} \left\{ \frac{(S - 2a)}{2a} \left[ \frac{(1 + \alpha)(S - 2c)}{2} + 1 \right] \right\}
\]

(7.53)

\[
N_{13}^e = -\frac{6}{\pi} \left\{ \frac{(S - 2a)(S + 2a)}{8a^2} \right\}
\]

(7.54)

\[
N_{31}^e = \frac{6}{\pi} \left\{ \frac{(S - 2c)}{2a} \left[ \frac{(1 - \alpha)(S - 2a)}{2} + 1 \right] \right\}
\]

(7.55)
The accuracy of these relations can be verified by implementing them in a flow law to obtain stress strain curves. Comparing the stress strain curves with FEM results will indicate the accuracy of these expressions. This exercise is postponed to Section B of chapter IX.

D. FEM observations of ellipsoidal void shape evolution

The purpose of this section is to illustrate the evolution of ellipsoidal voids measured from FEM, under uniaxial and hydrostatic loadings. Yield values have been shown earlier (Chapter IV) to scale with projected area normal to the loading axis. In addition, void growth rate depends mainly on the projected void area normal to the loading axis [Becker (1989)]. Thus, it is important to examine the evolution of the voids along directions normal to the loading directions. In this section, FEM results will be used to plot the initial and deformed void shapes on the $X_1-X_2$, $X_2-X_3$ and $X_3-X_1$ planes along the void. Figure 3.2 is reproduced here to illustrate the geometry. The void shape evolution plots along the three planes at the void section will also be used to examine if the projected void curves along these planes continue to remain elliptical.

\[
N_{33}^e = \frac{6}{\pi} \left( \frac{S}{2c} + \frac{(S - 2c) (S - 2a) (S + 2a)}{8a^2 c} \right) \quad (7.56)
\]
Figure 3.2: Spheroidal void in a cubical cell.

The plots shown in this section are for an initially spheroidal void with a void volume fraction of 1%. Oblate (a>c), spherical (a = c) and prolate (a<c) voids are loaded in uniaxial tension along the X₁ or X₃ directions to a macroscopic strain of 5£y. The evolution of the voids are observed both on planes normal to and on planes containing the direction of loading. The void evolution under hydrostatic conditions is shown by considering the equal deformation rate loading of spherical voids at a macroscopic strain of £2 = £3 = £1 = 6£y.
We first consider uniaxial loading of voids under uniaxial tension. We define the void axes strains by considering the change in length of imaginary lines aligned with the void axes. It is seen that the void axis extends along the loading direction for an initial void volume of 1% [See figures 7.4 to 7.8]. Figures 7.4, 7.5 and 7.6 show that as the projected void area normal to the \( X_3 \) loading direction increases, the void axis strain along \( X_3 \) increases. Figures 7.8, 7.5 and 7.7 show that as the projected void area normal to the \( X_1 \) loading direction increases, the void axis strain along \( X_1 \) increases. Thus, as the projected void area normal to the loading direction increases the void axis strain along the loading direction increases, in accord with the conclusions of Becker(1989). The void is observed to contract along both void axis directions in the plane normal to the loading direction [See figures 7.9 to 7.13].

The void axes strains under equal deformation rate loadings are found to be orders of magnitude higher than the void axis strains under uniaxial loading [fig. 7.14]. This can also be inferred from the high effective plastic strains in equal deformation rate loadings, as shown in fig. 4.26. Figure 7.14 shows that transverse stresses amplify the rate of void growth, as proven earlier by other investigators, e.g. Rice and Tracey (1969).
Figure 7.4: Void shape evolution measured from FEM. Uniaxial tension along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.5: Void shape evolution measured from FEM. Uniaxial tension along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c}$ = 1. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.6: Void shape evolution measured from FEM. Uniaxial tension along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.7: Void shape evolution measured from FEM. Uniaxial tension along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.8: Void shape evolution measured from FEM. Uniaxial tension along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.9: Void shape evolution measured from FEM. Uniaxial tension along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.10: Void shape evolution measured from FEM. Uniaxial tension along $X_3$. Initial void volume fraction $= 1\%$. Initial aspect ratio $\frac{a}{c} = 1$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.11: Void shape evolution measured from FEM.

Uniaxial tension along $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = id.$

Dashed line shows deformed void.
Figure 7.12: Void shape evolution measured from FEM.
Uniaxial tension along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 5$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.13: Void shape evolution measured from FEM. Uniaxial tension along $X_1$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = \frac{1}{2}$. Solid line shows underformed void. Dashed line shows deformed void.
Figure 7.14: Void shape evolution measured from FEM. Equal deformation rates along $X_1$, $X_2$ and $X_3$. Initial void volume fraction = 1%. Initial aspect ratio $\frac{a}{c} = 1$. Solid line shows underformed void. Dashed line shows deformed void.
CHAPTER VIII

DEVELOPMENT OF FLOW LAW

We seek to develop a relation between stress and strain increments, i.e., a flow law, for a void containing material subjected to arbitrary loading. The purpose of the flow law to be derived is to relate macroscopic stress and strain increments for the voided material in terms of microscopic material variables such as void orientation, flow stress of the matrix, etc. The flow law assumes that microscopic homogeneity exists over a length scale greater than the void spacing. That is, regions of material may be characterized by a given void volume fraction, spacing, orientation, and flow stress.

The proposed yield function has the form

$$\Phi = \Phi (\Sigma, \frac{a}{S}, \frac{c}{S}, \bar{\sigma}_M, R) = 0.$$  \hspace{1cm} (8.1)

where $\Sigma$ denotes the macroscopic stress vector, $f$ denotes the void volume fraction, $u$ denotes the value $\{ \frac{a}{S} - \frac{c}{S} \}$ (See Fig. 3.2), $\bar{\sigma}_M$ denotes the matrix flow strength, and $R$ denotes the rotation tensor giving the relative orientation of the void axes with respect to the global macroscopic loading axes. The proposed yield criterion is for a macroscopic continuum of voids. Each 'point' in this continuum has porosity in the form of voids with dimensions and orientations that change as the loading progresses.
The yield function is always referred to material axes that rotate with the void axes, while the global loading axes are assumed stationary. The flow law is expressed in a set of co-ordinates that rotate with the material axes. Thus the flow law is independent of $R$. However, if quantities of interest are expressed with respect to stationary macroscopic axes such as the loading axes, then the rotation matrix would come into play.

The consistency condition continues to describe the yield surface during and after plastic deformation. The consistency condition states that upon continued plastic loading, any allowable state of stress must lie on the yield surface. Alternatively, it states that the yield surface evolves with loading increments on the material such that the current state of stress either lies on or within the current yield surface. Thus $\Phi = 0$ for initial and continued yielding. The fact that $\Phi = 0$ is usually expressed as

$$\dot{\Phi} = 0. \quad (8.2)$$

Equation (8.2) is an approximate expression of consistency, provided that terms involving derivatives of $\Phi$ of order greater than 1 are negligible.

Equation (8.2) may be written more explicitly for the proposed yield function as

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \Sigma_{ij}} \dot{\Sigma}_{ij} + \frac{\partial \Phi}{\partial \sigma_{M}} \dot{\sigma}_{M} + \frac{\partial \Phi}{\partial \mu_{ij}} \dot{\mu}_{ij} + \frac{\partial \Phi}{\partial \phi_{ij}} \dot{\phi}_{ij} + \frac{\partial \Phi}{\partial R_{ij}} \dot{R}_{ij} = 0. \quad (8.3)$$

The stress increments $\dot{\Sigma}_{ij}$ are the increments of the Cauchy (true) stress tensor components at spatial points. In general, constitutive laws are
developed using objective stress rates for material points rather than spatial points. For example, pure rigid body rotation would produce a change in the Cauchy stress components if the components are not expressed in a co-ordinate system that rotates with the material coordinates. It is therefore desirable to use an objective stress rate that does not depend on rotation rate. One such objective stress rate, $\dot{\Sigma}_{ij}$, is used here and is defined by

$$\dot{\Sigma}_{ij} = \dot{\Sigma}_{ij} - W_{ik} \Sigma_{kj} - W_{jk} \Sigma_{ik}. \quad (8.4)$$

where $W_{ij}$ denotes the components of the rotation rate skew tensor.

An important property of the yield function is normality. Bishop and Hill (1951) prove that if the matrix material surrounding the voids in a porous aggregate obeys convexity and normality, then, the voided material also shows normality if the micro and macro plastic dissipations are equated. In other words, the direction of the macroscopic plastic strain increment at a given stress state (i.e., at a given point on the yield surface for the voided material) lies along the direction of the normal to the yield surface at that point. This is written as

$$D_{ij} = \Lambda \frac{\partial \Phi}{\partial \Sigma_{ij}} = \Lambda I_{ij}, \quad (8.5)$$

where

$$I_{ij} \equiv \frac{\partial \phi}{\partial \Sigma_{ij}} , \text{ and } I_{ij} = I_{ji} . \quad (8.6)$$
We now note that since $W$ is skew-symmetric, that

$$I_{ij} W_{ik} \Sigma_{kj} = I_{ji} W_{jk} \Sigma_{ki} = I_{ij} W_{jk} \Sigma_{ki} = I_{ij} W_{kj} \Sigma_{ik} \quad (8.7)$$

Thus,

$$I_{ij} W_{ik} \Sigma_{kj} + I_{ij} W_{kj} \Sigma_{ik} = 0. \quad (8.8)$$

This yields,

$$I_{ij} \Sigma_{ij} = I_{ij} (\dot{\Sigma}_{ij} - W_{ik} \Sigma_{kj} - W_{kj} \Sigma_{ik}) = I_{ij} \dot{\Sigma}_{ij} \quad (8.9)$$

The flow law can be derived from the yield function and update relations that describe how the following quantities evolve with deformation:

a) Matrix flow strength, $\sigma_M$: An increment in plastic deformation is associated with a corresponding increment in the matrix flow strength, through an effective stress-effective strain law that reflects the hardening behavior. An increment of macroscopic loading on a voided material must be dissipated as plastic work in the matrix surrounding the void. This is because the void is of zero strength. This concept is frequently described as the equivalence of plastic work (Yamamoto, 1978) and it permits an increment in the average flow stress $\sigma_M$ of the matrix to be expressed in terms of the macroscopic plastic work increment, $\sigma_{ij} D_{ij}^p$, void volume fraction $f$, and plastic modulus $h_M = d\sigma_M/d\dot{\varepsilon}_M^p$ of the matrix,

$$\frac{\dot{\sigma}_M}{\sigma_M} = \frac{h_M \Sigma_{ij} D_{ij}^p}{(1-f) \sigma_M}. \quad (8.10)$$

Using normality, we obtain
\[ \frac{\dot{\sigma}_M}{\Lambda} = \frac{h_M}{(1-f)} \Sigma_{ij} I_{ij} \]  

(8.11)

b) Void semi axes lengths, \( \frac{a}{S} \) and \( \frac{c}{S} \): Due to deformation, the void axes change length and orientation. The rate of change of length with deformation was developed in chapter VII from a model of a cuboidal void in a cubical cell. The results of this analysis yields relations between the rate of straining \( d_{ij}^p \) of the void axes in terms of the macroscopic rate of deformation \( D_{kl}^p \).

\[ d_{ij}^p = N_{ijkl} D_{kl}^p \]  

(8.12)

The expressions for \( N_{ijkl} \) were derived in chapter VII. We then obtain

\[ \frac{\dot{a}}{l_{S}} = \frac{a}{S} [ N_{11kl} \delta_{k1} \delta_{l1} ] D_{kl}^p , \text{ and} \]  

(8.13)

\[ \frac{\dot{c}}{l_{S}} = \frac{c}{S} [ N_{33kl} \delta_{k3} \delta_{l3} ] D_{kl}^p . \]  

(8.14)

Using normality, we obtain,

\[ \frac{1}{\Lambda} \frac{\dot{a}}{l_{S}} = \frac{a}{S} [ N_{11kl} - \delta_{k1} \delta_{l1} ] I_{kl} , \text{ and} \]  

(8.15)

\[ \frac{1}{\Lambda} \frac{\dot{c}}{l_{S}} = \frac{c}{S} [ N_{33kl} - \delta_{k3} \delta_{l3} ] I_{kl} . \]  

(8.16)
c) Void volume fraction, $f$: The rate of change of void volume fraction is obtained by using the fact that the observed macroscopic dilatation is solely due to the volume change of the void. Thus,

$$\dot{f} = (1 - f) D_{kk}^p.$$  \hspace{1cm} (8.17)

Upon examining the updates, it is seen that the updates for $\bar{\sigma}_M$, $\dot{a}$, and $\dot{c}$ are linear in the rate of deformation tensor $D$ and hence are linear in $\Lambda$. So, we use eqns. (8.3) and (8.9) to solve for $\Lambda$ as

$$\Lambda = \frac{1}{F} I_{ij} \Sigma_{ij}$$  \hspace{1cm} (8.18)

where,

$$F = - \left\{ \frac{\partial \Phi}{\partial \bar{\sigma}_M} \frac{\dot{\bar{\sigma}}_M}{\Lambda} + \frac{\partial \Phi}{\partial (\frac{a}{S})} \frac{\dot{a}}{\Lambda} + \frac{\partial \Phi}{\partial (\frac{c}{S})} \frac{\dot{c}}{\Lambda} \right\}$$  \hspace{1cm} (8.19)

Using eqn. (3.4),

$$\frac{\partial \Phi}{\partial \bar{\sigma}_M} = - \frac{2}{\bar{\sigma}_M}$$  \hspace{1cm} (8.20)

Using eqn. (8.15) and eqn. (8.5), we get,

$$D_{ij}^p \frac{1}{F} I_{kl} \Sigma_{kl} I_{ij}$$  \hspace{1cm} (8.21)

The elastic rate of deformation tensor is assumed to be the same as that for a void free material,
\[ D_{ij}^p = \frac{1}{2G} \Sigma_{ij} + \frac{1}{3} \left( \frac{1}{3K} - \frac{1}{2G} \right) \Sigma_{kk} \delta_{ij}. \]  
(8.22)

The total rate of deformation tensor is written as the sum of the elastic and plastic parts,

\[ D_{ij} = D_{ij}^e + D_{ij}^p = S_{ijkl} \Sigma_{kl}, \]  
(8.23)

where \( S \) is the fourth order compliance tensor. \( S \) evaluates to

\[ S_{ijkl} = \frac{1}{2G} \delta_{ik} \delta_{lj} + \frac{1}{3} \left( \frac{1}{3K} - \frac{1}{2G} \right) \delta_{ij} \delta_{kl} + \frac{1}{F} I_{ij} I_{kl}, \]  
(8.24)

with symmetry \( S_{ijkl} = S_{ijlk} \) and \( S_{ijkl} = S_{jikl} \).

Equation (8.21) can be inverted to give

\[ \Sigma_{ij} = K_{ijkl} D_{kl}, \]  
(8.25)

where

\[ \{K\} = \{S\}^{-1}. \]  
(8.26)

The relation (8.25) is written explicitly as

\[ \frac{1}{2G} \Sigma_{ij} = D_{ij} - \left( \frac{1}{3} - \frac{K}{2G} \right) D_{kk} \delta_{ij} - \frac{[I_{ij} - \left( \frac{1}{3} - \frac{K}{2G} \right) I_{kk} \delta_{ij}] [I_{ij} D_{ij} - \left( \frac{1}{3} - \frac{K}{2G} \right) I_{kk} D_{kk}]}{F \left( \frac{1}{2G} + \frac{I_{kl} I_{kl}}{F} - \frac{1}{F} \left( \frac{1}{3} - \frac{K}{2G} \right) (I_{kk})^2 \right)} \]  
(8.27)

Equation (8.23) or eqn. (8.27) is the required form of the flow law for the proposed yield function. Either forms of the flow law can be used in an
incremental approach to obtain the stress strain response of a voided material. For example, the use of the flow law in an FEM simulation of a metal forming problem can give the stress-strain distribution and the evolution of the void axes as a function of deformation. In a numerical procedure where the displacement rate increments are specified by the macroscopic boundary conditions, it may generally be easier to use eqn. (8.27). This approach avoids the problem in a softening material that two stress states exist for a given strain state.
A. Introduction

In this chapter, the flow law developed in Chapter VIII will be numerically integrated to provide the stress strain curve for some simple examples of representative loading. The loadings considered do not involve any rotation of the void axes. The change of length of the void axes with deformation is assumed to be given by the kinematic update relations obtained in Chapter VII. The matrix material is assumed to harden according to [Yamamoto (1978)]:

\[
\frac{\tau}{\tau_y} = \frac{\gamma}{\gamma_y} \quad \text{for } \tilde{\sigma}_M < \sigma_y , \quad \text{and} \quad \frac{\tau}{\tau_y} = \left(\frac{\gamma}{\gamma_y}\right)^N \quad \text{for } \tilde{\sigma}_M \geq \sigma_y . \quad (9.1)
\]

\(\tau\) denotes the true shear stress and \(\gamma\) denotes the equivalent shear strain.

The plastic hardening modulus is defined as

\[
h_M = \frac{d\tilde{\sigma}_M}{d\tilde{\varepsilon}_M} . \quad (9.2)
\]

Using (9.1), \(h_M\) evaluates to

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The flow law derived in Chapter VIII is now integrated for loadings such as simple tension and hydrostatic tension. The effect of void shape on the flow curve will be examined. Predictions for spherical voids will be compared against FEM results and against predictions for the Gurson model.

B. Modification to update relations

The update relations developed in chapter VII influence the flow curves directly. The proposed yield function shows good agreement with FEM values of yield of a perfectly plastic material. Continued accuracy of yield predictions for a hardening material require a closer scrutiny of the update relations (7.51)-(7.55). These update relations will now be contrasted to some FEM results.

A flow law based on the Gurson yield function [Yamamoto(1978)] for a loading of $D_1 = D_2 = D_3$ was implemented to obtain a stress strain curve.

Figures 9.7 and 9.8 show the resulting flow curves for the spherical voids of initial void volume fractions of 0.01 and 0.10 respectively. The flow law developed in Chapter VII was used to calculate the corresponding curve for the proposed yield function using the update relations developed in equations (7.51 - 7.55). It was observed that the both curves co-incided. It is thus ascertained that the update relations (7.51 - 7.55) are consistent with
the prescription that \( g = f \). We now contrast this by looking at FEM results for the flow curve for the same loading and hardening co-efficient.

FEM results indicate that under the loading of \( D_1 = D_2 = D_3 \), an initially spherical void does not remain spherical. Instead it deforms to an irregular shape. This fact can be checked by comparing the value of \( g \) (eqn. 7.46), and the actual void volume fraction \( f \). The actual void volume fraction \( f \) must satisfy

\[
\dot{f} = (1 - f) D_{kk}^P .
\]

Assuming \( D = D^p \), we obtain by integration for a proportional straining path that,

\[
f = 1 - (1 - f_0) \exp(-E_{kk}) .
\]

\( f_0 \) denotes the initial void volume fraction. By tracking the void axes length along the loading direction \( g \) is calculated according to eqn. (7.46). Figure 9.1 plots the ratio of \( \frac{g}{f} \) as a function of deformation. It is seen that \( \frac{g}{f} < 1 \) until the inception of plasticity, but exceeds 1 upon full development of plasticity. The FEM results therefore indicate that \( g > f \).

Thus, the update relations developed in equations (7.51 - 7.55) need to be modified further. Figure 9.1 was used to examine the amount by which the update relations must be amplified in order that \( g \) exceeds \( f \) as in FEM results. Thus, the following update relations are suggested:

\[
N_{11}^e = 2 \frac{\pi}{6} \left( \frac{S}{2a} + \frac{(S - 2a)}{2a} \left[ \frac{(S - 2a)}{2a} + \frac{(S - 2c)}{2c} + \frac{(S - 2a)(S - 2c)}{2a} \right] \right)
\]
These update relations are now used to obtain flow curves for various deformation paths.

C. Predictions for simple deformation paths

We first consider the simple unaxial loading of spherical voids along the $X_3$ direction. Figure 3.2 is reproduced here for ease of picturing the geometry and loading.
Sections of the ellipsoidal void normal to the $X_3$ axis always yield circles. Figure 9.2 shows the predicted flow curve for spherical voids of volume fractions of 1% and 10% for a hardening exponent of $N=0.1$. The solid line indicates the hardening behavior of the unvoided matrix, for $N = 0.1$. Initial yield stress decreases with increasing void volume fraction. Upon further loading, the stress strain curves for the voided material continue to show hardening. The rate of hardening decreases as the void volume fraction increases. The update relations in the flow law predict that as the loading along $X_3$ proceeds, the initially spherical voids...
elongate along the loading direction. For the 1% void volume fraction case, the initially spherical void continuously elongates along the loading direction until coalescence occurs along the loading direction at a macro strain of $3\varepsilon_y$. For the 10% void volume fraction case, the initially spherical void elongates to a prolate void of aspect ratio $\frac{a}{c} = \frac{1}{1.48}$ at a macroscopic strain of $4\varepsilon_y$.

Figure 9.3 shows the effect of void shape on the flow curves at a fixed initial void volume fraction of $f = 0.01$. Three cases were chosen for study, an oblate void of aspect ratio $\frac{a}{c} = 5$, a spherical void of aspect ratio $\frac{a}{c} = 1$ and a prolate void of aspect ratio $\frac{a}{c} = \frac{1}{2}$. Each case was run with a hardening exponent of $N = 0.1$, with the loading being along the $X_3$ direction. The matrix is seen to harden and predicts a higher flow stress than the voided material at all strains. The initial yield stress $\Sigma_{33}$ decreases as the void shape becomes oblate. The flow curve for the prolate void remains above the flow curve for the initially spherical void and the initially oblate void at all strains. The difference between the flow stresses of the three different initial void shapes decreases with increasing strain, because of the lengthening of all the voids along the loading direction. The initially prolate void of aspect ratio $\frac{a}{c} = \frac{1}{2}$ elongates further along the $X_3$ direction until coalescence at a strain of about $27\varepsilon_y$. The initially spherical void also elongates along the $X_3$ direction until coalescence at a strain of $31\varepsilon_y$. Thus, the macroscopic strain required for coalescence is lesser for the initially prolate void as compared to the initially spherical void. The initially
oblate void of initial aspect ratio of \( \frac{a}{c} = 5 \) evolves into a prolate void of
aspect ratio \( \frac{a}{c} = \frac{1}{6.28} \) at a strain of \( 42\varepsilon_y \).

Figure 9.4 shows the predictions of the variation of the mean stress
with strain required to yield spherically voided materials of void volume
fractions of 1% and 10% respectively. The value of \( N = 0.1 \). The unvoided
matrix cannot yield, since a \( J_2 \) flow law has been assumed for the matrix. It
is seen that the voided material softens with strain, exhibiting a negative
hardening modulus. As deformation proceeds, there are two competing
phenomena. The matrix yield stress increases due to hardening.
Simultaneously, the dilatation of the voids causes a geometric softening
effect. At large enough strains, the rate of strain hardening decreases to a
low enough value that the effect of increasing void volume fraction causes
the flow curve to show a negative slope.

The next section compares these results to those obtained from FEM
simulations and also to predictions from the Gurson model.

D. Comparison of flow law predictions to the Gurson model predictions
and to large strain FEM results

Figure 9.5 compares the flow law predictions for \( N=0.1 \) with the
corresponding Gurson flow law [Yamamoto (1978)] predictions for
spherical voids with an initial void volume fraction of 1%. Finite element
results using ABAQUS (1989) for \( N=0.1 \) are also included. It is seen that
the predictions of the flow law are in close agreement with the FEM
predictions. The Gurson flow law which assumes that the voids remain spherical as deformation proceeds, does manage to approximate the FEM results well until a strain of $6\varepsilon_y$.

Figure 9.6 compares the flow law predictions with FEM results and the Gurson flow law results for an initially spherical void of 10% volume fraction. The predicted flow law shows closer agreement to the FEM results than the Gurson flow law predictions.

Figures 9.7 and 9.8 compare the predictions of the flow law to FEM and Gurson flow law predictions for the case of $D_1 = D_2 = D_3$ for initial void volume fractions of 1% and 10% respectively. It is seen that the predictions are closer to the FEM values than the predictions of the Gurson flow law.
Figure 9.1: Ratio of the apparent void volume fraction, $g$, and the actual void volume fraction $f$ as a function of deformation. Values are from FEM runs with $D_1 = D_2 = D_3$. $f_0$ denotes the initial void volume fraction.
Figure 9.2: Flow law predictions for the uniaxial yield stress $\Sigma_{33}$ as a function of strain $E_{33}$. The hardening exponent is $N=0.1$. $f$ denotes void volume fraction. Voids are initially spherical.
Figure 9.3: Flow law predictions for the uniaxial yield stress \( \Sigma_{33} \) as a function of strain \( E_{33} \). The hardening exponent is \( N=0.1 \). \( \frac{a}{c} \) denotes void aspect ratio. Voids are initially of 1% volume fraction.
Figure 9.4: Flow law predictions for the mean stress as a function of strain. $E$ denotes the macroscopic strain along any of the three void axes directions. $f$ denotes the void volume fraction. The hardening exponent $N = 0.1$. 
Figure 9.5: Comparison of flow law predictions for uniaxial yield stress $\Sigma_{33}$ with Gurson flow law predictions and FEM. Initial void volume fraction = 1%. FEM values are shown as discrete points. The hardening exponent $N=0.1$. 
Figure 9.6: Comparison of flow law predictions for uniaxial yield stress $\Sigma_{33}$ with Gurson flow law predictions and FEM. Initial void volume fraction = 10%. FEM values are shown as discrete points. The hardening exponent $N = 0.1$. 
Figure 9.7: Comparison of the flow law predictions for the mean stress as a function of strain, with the Gurson predictions and FEM results. $E$ denotes the macroscopic strain along any of the three void axes directions. Initial void volume fraction = 1%. The hardening exponent $N = 0.1$. 
Figure 9.8: Comparison of the flow law predictions for the mean stress as a function of strain, with the Gurson predictions and FEM results. $E$ denotes the macroscopic strain along any of the three void axes directions. Initial void volume fraction = 1%. The hardening exponent $N = 0.1$. 

\[
\frac{\Sigma_{\text{mean}}}{\sigma_y} \quad \frac{E}{\varepsilon_y}
\]
CHAPTER X

CONCLUSIONS

This work has focused on developing an anisotropic yield function for porous materials including the effects of void shape and size. Previously existing yield functions for porous materials are isotropic, e.g. Gurson (1977). They are suitable for predicting the yield behavior of materials with spherical voids. But void shape evolves with deformation and leads to anisotropy in yield. Existing models cannot capture the anisotropy in yield due to void shape evolution during deformation.

The yield function developed can model the anisotropy in yield due to void shape in addition to modeling the effect of void size. The yield function has been calibrated to FEM simulations done using periodic boundary conditions. Thus, void-void interaction is also taken into account in the predictions of the proposed model.

The comparison of the predictions of existing yield functions with the predictions of the proposed model show that existing models are capable of adequately modeling the effect of spherical voids. The yield under hydrostatic tension continues to show a logarithmic behavior in void
volume fraction. It has also been concluded that this behavior is independent of packing geometry.

The proposed model has shown good agreement with FEM values of yield for non-spherical voids (Chapter V). The effect of a non-spherical void shape is to cause the yield function to rotate about the hydrostatic direction. FEM simulations confirm the rotation sense and magnitudes. Further confirmation of the rotation of the yield function has been possible through the development of an axi-symmetric void-composite model (Chapter VI).

A simple model for updating the length of the void axes during deformation has been proposed in Chapter VII. These update relations have been incorporated into a flow law framework and flow curves have been predicted in Chapter IX. Preliminary indications show reasonable agreement between the flow law and FEM results for low strains. There is scope for further work in developing more accurate update relations for void evolution. The update relations need to be developed from a more rigorous model which would include non-homogeneously deforming sub-regions with the possibility of slip between regions. Such a mode can also be extended to produce another upper bound estimate to the yield locus for the porous solid.

The flow law developed can be programmed into any numerical scheme to predict mechanical behavior of a porous solid. For instance, it is possible to use the flow law in an FEM code to study metal forming problems. Aravas (1986) has analyzed centerbursts in extrusion using a flow law derived from the Gurson model. The voids were assumed to be
spherical. However, it is clear that the voidage cannot be spherical at all points, since the process is predominantly compressive. An analysis of the centerburst problem through a finite element approach using the proposed flow law can show the effect of void shape on ductility.

An important application of the flow law would be an implementation to study localization of deformation in a band, e.g. Yamamoto(1978). A voided solid of homogeneous void volume fraction and aspect ratio is assumed to contain an initial imperfection band. This band has a perturbation of void size and or shape with respect to the rest of the solid. Upon deformation, it is possible to obtain conditions for the bifurcation of deformation wherein all deformation concentrates in the band. This leads to shear band formation. The criterion for this bifurcation is dependent on the initial orientation of the band with respect to the loading directions. Thus, differently oriented voids of a given void volume fraction could lead to varying predictions of the macroscopic strain to localization. Present predictions of localization strains are high compared to experimental values [Yamamoto (1978)]. The implementation of a flow law that includes void shape effects is a worthwhile future effort.

The actual comparison to experiment however involves identical initial material conditions in the simulation and experiment. The proposed model assumes that the voids are locally periodic. It would be experimentally tedious to characterize the initial material to obtain the void distribution to be input in a simulation. This would make an exact comparison of experimental and predicted macroscopic failure strains difficult.
The transition from void growth to actual fracture through the evolution of plasticity is not fully understood yet. Efforts on continuum analysis of this problem may not provide any fundamental insights. The transition from plasticity to fracture is best answered by looking at the void localization problem at a lower length scale. The plasticity must be modeled in terms of dislocations. If the dislocations are shown to redistribute to create a far field $r^{-2}$ field of stresses ahead of their location, this would constitute a transition to final fracture. This problem is worthy of further extensive study.
APPENDIX A

FUNCTIONAL FORMS FOR THE YIELD FUNCTION CO-EFFICIENTS

This appendix shows the calculated values of $Q_{ij}$ and the interpolation of the values to provide their variation with void volume fraction and aspect ratio. We let $f$ denote the void volume fraction. The void dimensions are $a$ along the $X_1$ (and $X_2$) axis and $c$ along the $X_3$ axis. The cubical cell has dimension $S$ (See Fig. 3.2). Let us define the variable $u$ as follows to denote the effect of void shape.

$$u = \frac{a}{S} - \frac{c}{S}$$

(A.1)

It must be noted that there can be many other choices of variables to represent the effect of shape, for example the variable $\{\frac{a}{c} - \frac{c}{a}\}$. However, the variable $u$ is chosen mainly because $|u| < 0.5$ for all values of $f$. We note that $u$ equals zero for spheres, is positive for oblate voids and is negative for prolate voids.

As mentioned in chapter III, the matrix $Q$ is assumed to be a material property matrix. Thus, given the void volume fraction and aspect ratio, it must be possible to completely determine the seven $Q_{ij}$ values.
Table A.1: $Q_{ij}$ values calculated using FEM results.

<table>
<thead>
<tr>
<th></th>
<th>$a/c = 0.2091$</th>
<th>$a/c = 1.0$</th>
<th>$a/c = 14.102$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{11}$</td>
<td>0.2558</td>
<td>0.2617</td>
<td>0.3459</td>
</tr>
<tr>
<td>$Q_{12}$</td>
<td>-0.9826</td>
<td>-0.9765</td>
<td>-0.9320</td>
</tr>
<tr>
<td>$Q_{13}$</td>
<td>0.1223</td>
<td>0.1086</td>
<td>0.1799</td>
</tr>
<tr>
<td>$Q_{31}$</td>
<td>0.7091</td>
<td>0.7149</td>
<td>0.8030</td>
</tr>
<tr>
<td>$Q_{33}$</td>
<td>0.1254</td>
<td>0.1086</td>
<td>0.1313</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$a/c = 0.5$</th>
<th>$a/c = 1.0$</th>
<th>$a/c = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{11}$</td>
<td>0.2958</td>
<td>0.2787</td>
<td>0.3148</td>
</tr>
<tr>
<td>$Q_{12}$</td>
<td>-1.0627</td>
<td>-1.0400</td>
<td>-1.0115</td>
</tr>
<tr>
<td>$Q_{13}$</td>
<td>0.1778</td>
<td>0.1773</td>
<td>0.1924</td>
</tr>
<tr>
<td>$Q_{31}$</td>
<td>0.7431</td>
<td>0.7614</td>
<td>0.7980</td>
</tr>
<tr>
<td>$Q_{33}$</td>
<td>0.1921</td>
<td>0.1773</td>
<td>0.1679</td>
</tr>
</tbody>
</table>
Using the ideas developed in Chapter III, we calculate the seven independent non-zero components of the $Q$ matrix for different void volume fractions and aspect ratios. The finite element yield values reported in Chapter IV are used for the purpose of calculation. Table A.1 reports the calculated $Q_{ij}$ values for non-shear loadings. Functional forms are then proposed for these five values.

In order to develop functional forms for the $Q_{ij}$ values, the behavior of these functions for spherical voids is first examined. The values of $Q_{13}$ and $Q_{33}$ are equal for spherical voids. In other words, the $m^{(3)}$ vector points along the [1 1 0 0 0] direction in stress space. This, when used with orthogonality (Eqn. 3.38 and Eqn. 3.39) yields specific directions for the $m^{(1)}$
and \( m^{(2)} \) vectors for spherical voids. Expressing these directions in terms of the components we obtained equations 3.22 and 3.23. For a material where \( f \to 0 \), the value of \( C^*(0) \) was given in Eqn. 3.31. Using Eqn. 3.31 and Eqn. 3.22, we conclude that, as \( f \to 0 \), for spherical voids,

\[
\lim_{f \to 0} Q_{31}(f,u=0) = (\sqrt{3} - 1) \frac{\sqrt{2 + \sqrt{3}}}{2} = \frac{1}{\sqrt{2}}. \tag{A.2}
\]

Based on this limit, we assume the following form for \( Q_{31} \),

\[ Q_{31}(f,u) = Q_{31}(f,u=0) + u \left[ a_3 + a_4 f \right] + uu \left[ a_5 + a_6 f \right] \tag{A.3} \]

where,

\[ Q_{31}(f,u=0) = (\sqrt{3} - 1) \left[ \frac{\sqrt{2 + \sqrt{3}}}{2} - a_1 f + a_2 f^2 \right]. \tag{A.4} \]

Using the calculated values of \( Q_{31} \) for \( f=0.01 \) and \( f=0.10 \) shown in Table A.1, we obtain

\[
\begin{align*}
  a_1 &= 1.003 \tag{A.5} \\
  a_2 &= 3.0 \tag{A.6} \\
  a_3 &= 0.0760 \tag{A.7} \\
  a_4 &= 2.3578 \tag{A.8} \\
  a_5 &= 0.0774 \tag{A.9} \\
  a_6 &= 0.2876. \tag{A.10}
\end{align*}
\]

Using equation 3.51 in the limit of \( f \to 0 \), we obtain for spherical voids,

\[
\lim_{f \to 0} Q_{12}(f,u=0) = \frac{(1+\sqrt{3})}{2} Q_{31}(f=0,u=0). \tag{A.11}
\]

Based on this limit, we assume the following functional form for \( Q_{12} \).
Using the calculated values of $Q_{12}$ for $f=0.01$ and $f=0.10$ shown in Table A.1, we obtain,

\begin{align*}
  b_1 &= 0.01632 \\
  b_2 &= 2.8377 \\
  b_3 &= 0.01704 \\
  b_4 &= 0.6862 .
\end{align*}

For spherical voids, $Q_{13} = Q_{33}$. By considering the hydrostatic loading of spherical voids and using the yield function (Eqn. 3.3), we obtain the following expression for the mean stress to yield a spherical voided material under a purely hydrostatic state of stress.

\[ \Sigma_m (f,u=0) = \frac{1}{3 Q_{13}} \]  

(A.17)

As discussed in Chapter IV and quantified by equation (3.26), \( \Sigma_m \) is linear in $f$. Thus, we obtain,

\[ Q_{13}(f,u=0) = Q_{33}(f,u=0) = (b_1 - b_2 \ln(f)) \]  

(A.18)

Using the calculated values of $Q_{13}$ from Table A.1, for spherical voids, we obtain,

\begin{align*}
  b_1 &= 0.029772157 \\
  b_2 &= 1.993049247 .
\end{align*}

(A.19)

(A.20)

Based on this limit, we assume the following functional form for $Q_{13}$. 

\[ Q_{12}(f,u) = \frac{(1+\sqrt{3})}{2} Q_{31}(f,u=0) + u[b_1 + b_2 f] + |u| [b_3 + b_4 f] \]  

(A.12)
\[ Q_{13}(f,u) = Q_{13}(f,0) + u \left[ c_1 + c_2 f \right] + u \left[ c_3 + c_4 f \right]. \]  
\tag{A.21}

Using the calculated values of \( Q_{13} \) for \( f = 0.01 \) and \( f = 0.10 \) shown in Table A.1, we obtain,
\[ c_1 = 0.012825 \]  
\tag{A.22}
\[ c_2 = 0.5535 \]  
\tag{A.23}
\[ c_3 = 0.0202 \]  
\tag{A.24}
\[ c_4 = 0.7877. \]  
\tag{A.25}

The values of \( Q_{22} \) and \( Q_{33} \) are obtained by using orthogonality, as follows:
\[
Q_{33} = \frac{-Q_{13} (2Q_{12}^2 - Q_{31}^2)}{2Q_{12}Q_{31}} \tag{A.26}
\]
\[
Q_{11} = \frac{-Q_{31}^2}{2Q_{12}^2}. \tag{A.27}
\]

We also need to obtain functional forms for the variation of the two remaining \( Q_{ij} \) values, \( Q_{44} \) and \( Q_{66} \). These are obtained using the calculated values of \( Q_{44} \) and \( Q_{66} \) from Table A.2.

Using Eqn. 3.30 for the limit for \( Q_{44} \) and \( Q_{66} \) for a void free material, we obtain,
\[
Q_{44}(f,u) = \sqrt{3} u_1 + u \left[ v_1 + v_2 f \right] + u \left[ v_3 + v_4 u \right] \tag{A.28}
\]

and,
\[
Q_{66}(f,u) = \sqrt{3} u_1 + u \left[ w_1 + w_2 f \right] + u \left[ v_1 \right]. \tag{A.29}
\]
We notice that both $Q_{44}$ and $Q_{66}$ have the same behavior in $f$ for spherical voids, as required by symmetry.

Table A.2: $Q_{ij}$ values calculated using FEM shear results.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$a/c$</th>
<th>$Q_{44}$</th>
<th>$Q_{66}$</th>
</tr>
</thead>
<tbody>
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<td>2.4213</td>
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</table>

Using the values of $Q_{44}$ and $Q_{66}$ from Table A.2, we obtain,
\[ u_1 = 5.5945 \quad \text{(A.30)} \]
\[ v_1 = 0.0594 \quad \text{(A.31)} \]
\[ v_2 = 6.9276 \quad \text{(A.32)} \]
\[ v_3 = 0.0979 \quad \text{(A.33)} \]
\[ v_4 = 12.2225 \quad \text{(A.34)} \]
\[ w_1 = -0.084225 \quad \text{(A.35)} \]
\[ w_2 = -6.761477 \quad \text{(A.36)} \]

We have thus developed functional forms to find the values of \( Q_{ij} \) for given values of void volume fraction and aspect ratio. These functional forms are found to be accurate in describing the yield of ellipsoidal void containing solids, as illustrated in Chapter VI.
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