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Mechanics of bending, flanging, and deep drawing and a computer-aided modeling system for predictions of strain, fracture, wrinkling and springback in sheet metal forming

Wang, Chuantao, Ph.D.
The Ohio State University, 1993
MECHANICS OF BENDING, FLANGING, AND DEEP DRAWING
AND A COMPUTER-AIDED MODELING SYSTEM FOR
PREDICTIONS OF STRAIN, FRACTURE, WRINKLING AND
SPRINGBACK IN SHEET METAL FORMING

DISsertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy
in the Graduate School of The Ohio State University

by

Chuantao Wang, B.S., M.S.

*****

The Ohio State University

1993

Dissertation Committee:

Dr. Taylan Altan
Dr. Gary L. Kinzel
Dr. Rajiv Shivpuri

Approved by

Taylan Altan
Adviser, Professor
Department of Industrial and Systems Engineering
To my parents, wife and daughter.
ABSTRACT

This research establishes the failure criteria of localized necking, fracturing, and wrinkling in sheet metal forming and the fundamentals of deformation mechanics in plane-strain bending (bending around a straight line), contour flanging (bending around a curve), and stretch/draw forming operations, which are primarily used in forming the box-shaped and structural sheet components. The mechanics and the associated computer programs are able to predict the deformation (strains, stresses, and loads), the failures (necking, tearing, and wrinkling), and the springback in the forming operations.

Based on the advancements in continuum mechanics, plasticity, and the modern concepts in the understanding of sheet metal formability, the mechanics of plane-strain bending and contour flanging was established. A number of commonly as well as the newly developed bending and flanging processes were analyzed. A computer code BEND was developed to simulate air bending, rotary bending, and die bending (curved-die, tractrix-die, wiping-die, U-die, and V-die). A computer program FLANGE was developed to simulate the shrink and stretch flanging operations.

The bending effects were introduced to the membrane finite element program SECTIONFORM for analyzing stretch/draw forming processes. In order to maintain the computational efficiency and numerical stability, a decoupled method was proposed for step-by-step bending corrections for membrane solutions. This method is able to consider both the local and the global bending effects, as well as unbending and sliding. Extra strain hardening and thinning due to bending are also included in the formulation. The algorithm
and subroutines were developed and implemented into SECTIONFORM program. The modified version of SECTIONFORM was tested by a number of examples. The simulations showed that the step-wise bending correction causes neither the numerical instability nor appreciable increase of computation time (CPU). The simulations of the plane-strain stretch forming and deep drawing using a flat bottom punch were compared with measurements. Good agreements were achieved for three punch radii (3.18, 7.14, 9.53 mm).

A number of failure criteria were developed for bending, flanging, and stretch/draw forming operations. New bendability criteria were proposed to determine the minimum bend ratio based on both localized necking and fracture modes and anisotropic material properties. A localized necking criterion was established for the stretch flangability analysis based on the modification of Hill's instability criterion and incorporating the strain hardening and the plastic anisotropy of sheet materials subjected to prestrain. With a bifurcation analysis of a double curved and anisotropic shell subjected to the forming stresses, the wrinkling criteria, incorporating sheet anisotropy, strain hardening, and deformation geometry, were developed to predict the local wrinkling phenomena in the unsupported region of sheet in deep drawing operations and to determine wrinkling at the flange edge in shrink flanging operation.

Experiments were conducted to verify the proposed process models for bending and flanging operations and the wrinkling criteria. Simulation results were compared with measurements. The springback and the relation between bending angle vs. punch stroke in various bending operations were successfully predicted with good accuracy. The strains and wrinkles in shrink flanging tests were also well predicted.

The practical aspect of this research is to provide a scientific approach to analyze the
formability of complex sheet parts formed in multiple operations (bending, flanging, stretching and deep drawing). The mechanics models and the associated computer-aided analysis system are able to provide information necessary for engineers to design sheet parts, processes, and dies by a more efficient and optimum strategy which reduces and finally eliminates costly try-outs. This computer-aided analysis system can also be adopted to other CAD systems for formability analysis.
ACKNOWLEDGMENTS

I would like to thank prof. Taylan Altan and Prof. Gary L. Kinzel for their advice, guidance, encouragement, and criticism throughout the course of the research. I especially appreciate the opportunity they gave me to participate in selecting and solving engineering problems in practical sense. I also would like to thank Prof. Rajiv Shivpuri for his metal forming courses and constructive criticism as a member of the dissertation committee. Also, thanks go to Dr. Michael Wenner in General Motors Research Laboratory, and Dr. Norman Wang in Advanced Stamping Manufacturing Eng. in Chrysler Motors Corp. for their many discussions and suggestions.

I wish to thank my research project team members: M. Bhutpathiraji, S. Tufekci, and R. Mantripragada, for their cooperation and help in conducting the experiments and reading the manuscript. Thanks also go to all the members of the Sheet Forming Research Group at The Ohio State University for their information and suggestions.

Finally, I sincerely thank my wife, Zhongyi Qiu, my daughter, Yun, and my parents and parents-in-law for their love, patience, understanding, encouragement, and prayers in my journey towards the Ph. D.
VITA

July 1, 1953 Born - Chongqing, Sichuan, The People's Republic of China

1978 - 1982 B.S. 
Department of Mechanical Engineering 
Chongqing University, Chongqing, P.R. China

1982 - 1983 Research Engineer, Metal Forming Division 
Chongqing Technology Research Institute of China Northern Industries Corp. 
Chongqing, P.R. China

1983 - 1986 M.S. 
Department of Mechanical Engineering 
Chongqing University 
Chongqing, P.R. China

1986 - 1987 Research Engineer, Metal Forming Division 
Chongqing Technology Research Institute of China Northern Industries Corp.

1987 - 1988 Visiting Scholar 
Department of Materials Science and Engineering 
Lehigh University 
Bethlehem, PA, U.S.A.

1988 - 1990 M.S. 
Department of Materials Science and Engineering 
Engineering Research Center for Net Shape Manufacturing 
The Ohio State University 
Columbus, Ohio, U.S.A.

1990-present Graduate Research Associate 
Department of Industrial and Systems Engineering 
Engineering Research Center for Net Shape Manufacturing 
The Ohio State University 
Columbus, Ohio, U.S.A.
FIELDS OF STUDY

MAJORS FIELD: Industrial Engineering
Studies in:

COMPUTER AIDED ANALYSIS, DESIGN, & MANUFACTURING
- Elastic Finite Element Methods
- Elastic-Plastic Finite Element Methods
- Finite Element Method in Engineering Science.
- Computer Aided Design & Manufacturing.
- CAD/CAM Laboratory for Rapid Design Prototyping.
- Survey of Artificial Intelligence I: Basic Techniques.
- Knowledge-Based Expert System.
- Manufacturing Processes and Simulations.
- Advanced Metal Forming.
- Optimal Design Theory in Mechanical Engineering.

MATERIAL SCIENCE AND METALLURGY
- Elastic Fields of Defects in Solid (Dislocation Theory).
- Crystal Deformation Mechanisms at Elevated Temperature.
- Point Defects in Crystalline Materials.
- Chemical Metallurgy I: Thermodynamics and Metallurgy Processes.
- Chemical Metallurgy II: Kinetics and Statistical Thermodynamics.
- Solidification of Metals and Alloys.
- Processing Metallurgy and Metal Refining.
- Advanced Corrosion.
- Ceramics Science: Mechanical Properties.
- Electron Microscopy and Interface Structures.

MECHANICS
- Elasticity and Plasticity
- Continuum Mechanics
- Fracture Mechanics & Failure Modes
- Metal Forming Mechanics
- Mechanics of Sheet Forming
- Composite Mechanics
- Structure Composites - Theory of Beam, Plate, & Shell.
- Stress Analysis of Machinery.
- Experimental Strain and Stress Analysis

MATHEMATICS
- Advanced Algebra.
- Matrix and Tensor Analysis.
- Calculus and Numerical Methods.
- Probability and Statistics.
- Partial Differential Equations.


C.T. Wang: Finite Element Analysis and Experimental Investigation of Square Punch Stretching and Deep Drawing of AKDQ Steel Sheets, Master Thesis, the Ohio State University, 1990


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H  Flange height.

k  Strength coefficient.

\( K_{ij} \)  Incremental bending strains.

Ld  Bending die length

lo, l  Fiber lengths before and after bending.

L  Length of straight portion of flange.

\( L_{ij\ell} \)  Instantaneous moduli.

\( \lambda_i \)  Wave numbers.

M  Anisotropic index in Hill's 1979 yield function.

Mb  Total internal bending moment.

Me  Elastic portion of bending moment.

Mp  Plastic portion of bending moment.

\( M_{ij} \)  Incremental stress couples.

\( \tilde{N}_{ij} \)  Incremental stress resultants.

N  Axial force per unit width.

n  Strain hardening exponent.

v  Poison's ratio.

\( \overline{R} \)  Normal anisotropy.

R0, R45, R90  Anisotropy indexes along the directions of 0, 45 and 90 degree from rolling direction.

R1  Radius of inner surface in bending.

Rn  Radius of neutral layer in bending.

Rc  Radius of outer (convex) layer in bending.

\( R_1, R_2 \)  Radial distance to center of die shoulder in flanging.

\( R_{T} \)  Radial distance to undeformed and deformed flange edge.

Rr  Radial distance to the tangent point at die radius in flanging.

r  Radius of arbitrary layer in bending;

\( r \)  Initial radial distance coordinate to a material point in flanging.

\( \rho \)  Instantaneous radial distance coordinate to a material point in flanging.

rd  Radius of die shoulder.

\( r_d \)  Radius of mid-axis of sheet around the die shoulder (= rd + t/2).

s  Arc coordinate in bending, Arc length coordinate along flange (measured from the clamped edge at die shoulder).

\( S_{1} \)  Bending arc length
to, t  
Initial and deformed sheet thickness

θ, θc  
Bend angle (blank included angle in flanging), punch/sheet contact angle

θs  
Springback angle

θ1, θ2  
Over bent angles under load, bending angle after unloading (desired angle)

σt  
Critical hoop stress at the onset of wrinkling.

σ  
Effective stress.

σi  
Principal stress along directions of length or longitudinal (i = x, or 1),
thickness or radial (i = y or 2), and width (i = z or 3) in bending;
Principal stresses (i = 1, 2) or stress along hoop (σo) and radial (σr)
directions in flanging.

Ui  
Incremental displacements (in xy-plane).

w  
Width of sheet.

Wy  
Incremental displacement normal to the shell surface.

ye  
Half height of elastic core.

y  
Radial distance from the origin.

x,y,z: Coordinates designed along sheet length, thickness and width.

X,Y,Z: Yield stresses for uniaxial tension along principal directions x or 1, y
or 2, and z or 3 in bending.

ZL  
Subtangent at the onset of the localized necking instability.
CHAPTER I

INTRODUCTION

Sheet forming is a significant net shape manufacturing process in material shaping industry. Sheet forming technology has been widely used to produce consumer products (beer cans, automatic washers, cabinets, electronic and computer boxes, etc.), automotive components (body panels, gasoline tanks, etc.), and aerospace parts (panels, wingtip, nose skins, etc.). After more than half a century of practice and development, sheet forming engineering has achieved much progress, and it has undergone a slow transition from a craft-based art to a science-based technology. However, sheet forming practice is still based primarily on trial-and-error procedures and the use of "rules of thumb" in the design and development of dies and processes.

During the last several decades, sheet forming engineering has been developed in two directions: the improvement of sheet materials and reliable process design and control. Much more effort has been devoted by the academic and research communities to material development than to process control. Concerning materials, research has aimed at an understanding the effect of material composition and behavior on the sheet formability. The major research interest has been directed from work hardening (1940's), to plastic anisotropy (1950's), forming limit diagram (1960's), strain rate sensitivity (1970's), and the sheet surface topography and coated materials (1980's).
The design and control of sheet forming processes consist of the design of tooling (die, punch, and drawbeads), process sequence, selection of the proper forming process parameters (blankholder force, press velocity, lubricant, etc.), and the evaluation and control of the process performance. Sheet forming operation is a complex system including various engineering aspects, and process control plays a key role in forming sound sheet parts. Today's highly competitive market mandates the sheet forming industry to make major changes in their traditional product development cycle from conceptual design to production. There is a strong need to reduce cost, shorten development cycle, and improve product quality. Therefore, the design and simulation of sheet forming processes to reduce and finally eliminate costly die try-outs is of significant economic interest.

A scientific approach combined with the practical know-how of experienced engineers should be used to develop a novel design methodology and process control strategy (the know-how and know-why) in sheet forming practice. Development and application of an integrated computer-aided engineering (CAE) system for sheet forming processes and tooling design could speed up this transition and be the bridge connecting the scientific fundamentals with the know-how of engineer's experience to reduce and finally eliminate expensive die try-outs. A practical and reliable CAE system must have a prediction capability to provide the evaluation of sheet material formability and the performance of forming operations under various forming conditions. Such a prediction capability can be provided through a computer-aided simulation tool that is able to assist die designers, stamping and product engineers, to evaluate the manufacturing feasibility of parts at the design stage, to explore alternative designs and evaluate the trade-offs to arrive at the optimized design with the right sheet material at the lowest cost and shortest lead time.
The objective of the current research is to develop the fundamentals of sheet forming mechanics and failure (fracture and wrinkling) criteria and to apply those mathematical models to develop a computer-aided analysis (CAA) system for deformation modeling, process optimization and tooling design in sheet forming. The CAA package will include two systems: an analytical-solution-based (ASB) CAA system, and a numerical-solution-based (NSB) CAA system (FEM). The ASB-CAA system consists of two modules: (1) BEND for the analysis of straight line bending of wide sheet (plane strain) and narrow sheet (uniaxial), and (2) FLANGE for the contour bending (bending around a curve) analysis. The NSB-CAA system is a finite element modeling program, SECTIONFORM, which is mainly used in two-dimensional section analysis of stretch/draw forming operations. This FEM program was developed earlier based on a membrane model which neglects the bending deformation. Under this study, SECTIONFORM was further developed to include bending effect so that the FEM program can provide a more accurate analysis for stretch/draw forming of sheets and plates which have a relative small R/t ratio (t is sheet thickness, and R is the tool radius at which the bending occurs). The ASB-CAA system will be used in the analysis of either individual forming operation (such as bending) or sequential and multi-operations as in forming box-typed sheet parts that usually require bending, flanging, and local stretching. It is expected that the CAA system should be able to: (a) provide the mechanics of deformation (strains, strain rates, stresses, deformed part profiles, punch load, die contact pressure); (b) predict the formability and failures (modes and locations); (c) evaluate the effects of material properties, tooling geometries, process parameters (blankholder force and drawbead geometry, lubrication, press velocity, etc.), (d) simulate the try-Outs and assist in process control, and (e) assist in establishing forming sequences. This CAA system is a part of the ERC/NSM project for developing an integrated CAE system (with rule-based knowledge and formability-based analysis) for process sequence design in
manufacturing box-typed sheet parts. In addition, the CAA system itself can stand along as a simulation package and can be adopted to other CAD/CAE systems in practice.

This dissertation consists of eleven chapters. Chapter I is the introduction. Chapter II provides a literature review of sheet forming operations and process classification, mathematical modeling of sheet forming, knowledge-based expert system. The development of sheet forming mechanics is reviewed as individual topics: bending, flanging, stretching, and deep drawing. Chapter III presents the fundamentals of the plane-strain bending: the elementary bending theory, the advanced bending theory, and the mechanics of bending under tension. Sheet anisotropy, strain hardening, the neutral axis, and the nonlinear strain description along thickness, etc., have been introduced into the bending theories to establish more accurate bending models. Applications of the bending mechanics are presented in Chapters IV. The conventional bending operations such as air bending and die bending operations with U-die, wiping-die, and V-die are analyzed. The process models for newly developed bending techniques such as rotary bending and tractrix-die bending are proposed. The proposed bending models and formulations are implemented into a computer code called BEND. In Chapter V, the contour bending (or flanging) processes are analyzed. The mathematical models for both shrink and stretch flanging operation are established with an assumption of axisymmetric deformation mode. A computer code called FLANGE is developed based on these models. The failure criteria for localized necking in stretch flanging and the fracture in plane-strain bending are developed in Chapter VI. Based on the fracture criterion and the refined bending theory, several new bendability criteria for the minimum bending ratio are proposed to account for the influences of the material properties (anisotropy and strain hardening), the strain/stress state, the shift of the neutral axis, and the applied axial tension on the minimum bending ratio. Based on Hill’s instability model, a localized necking criterion incorporating the strain hardening and the anisotropy is developed to evaluate the stretch flanging formability. In Chapter VI, wrinkling in the unsupported
region of a sheet in deep drawing and shrink flanging is studied. The wrinkling criteria for a general double curved shell is proposed based on DMV (Donnel-Mushtari-Vlasov) shell theory and an incremental constitutive relation for elastic isotropic and plastic anisotropic solids. A specific wrinkling model for shrink flanging is obtained from the simplification of the general wrinkling criteria. The further applications of the mechanics of bending under tension and plane-strain condition are presented in the following chapter. Chapter VIII deals with the bending effects in stretch/draw forming. The bending correction models for membrane FEM are established based on a proposed "decoupled method" which accounts both the local and global bending effects. Experimental studies of plane strain bending and flanging processes are presented in Chapter IX. Rotary bending is compared with the conventional wiping-die bending. The air bending operations using a tractrix die and a conventional die are compared each other. The springback angle, the strains, the relationships between the bending angles and the punch positions, and between the punch loads and the strokes are measured in the bending operations (air bending, wiping-die bending, rotary bending, and tractrix die bending). In flanging experiments, two types of specimens were designed. The full-sized circular blank was used to achieve the axisymmetric deformation mode, so that it can be directly compared with the axisymmetric solutions obtained from the proposed flanging mechanics. The angular specimens with different included angles and the rectangular blanks with varying ratios of width/length were used to investigate the application region of the axisymmetric models of flanging mechanics. The fractures in stretch flanging and the wrinkles in shrink flanging were measured and investigated. The simulations of various forming operations (bending, flanging, stretching, and deep drawing) are presented in Chapter X. The numerical results from computer codes BEND and FLANGE are compared with the measurements. The finite element simulations using the modified SECTIONFORM code are compared with strain measurements in plane-strain stretch and draw forming of strips. Finally, Chapter XI gives a general summary.
CHAPTER II

LITERATURE REVIEW

2.1 SHEET FORMING TECHNOLOGY AND PROCESS CLASSIFICATION

Sheet forming operations can be classified into two broad categories: separation or cutting processes (Fig. 2.1), and forming or stamping processes (Fig. 2.2). Cutting of sheet is a partial or complete separation process in which the sheet is worked above its ultimate strength. Therefore cutting is a highly localized plastic deformation around the cutter edges where the excess shear stresses are higher than the material shear resistance causing separation. Sheet stamping is a plastic deformation process originating from the dislocation motion in the material which is undergoing a permanent shape change at stresses above its elastic limit but below its ultimate strength. In the cutting of sheet, a highly concentrated plastic flow is desired. The more localized flow, the more precise is the cut. In contrast, in stamping processes, localized deformation should be avoided as much as possible. Many sheet metal parts are manufactured by the combination of a number of cutting and forming operations. The state-of-the-art of sheet forming technology and development in process control were reviewed [Wang, 1991]. According to the common features of deformation, the sheet forming processes are classified into four basic categories: straight line bending (Fig. 2.3), flanging or contour bending (Fig. 2.4), stretching, and deep drawing (Fig. 2.2). These four major operations represent the
Fig. 2.1 Cutting operations.
Fig. 2.2 Sheet forming operations [Lange, 1985]
Fig. 2.3  Plane-strain bending (bending around a straight line) operations
Fig. 2.4 Types of flanges
basic elements in sheet forming. Such a deformation-feature-based classification helps to
determine the least number of mathematical models necessary for analyzing sheet
forming processes. Indeed, many box-typed structural sheet components are formed by
the combination of these operations. Fig. 2.5 illustrate a typical sheet part formed by
multiple operations: cutting (to make holes), local stretching (to form the bead), shrink
flanging, and bending.

2.2 Mathematical Modeling of Sheet Forming Processes

2.1.1 Plane Strain Sheet Bending

Bending type of forming operations have been widely used in sheet forming
industries to produce the structural stamping components such as braces, brackets,
supports, hinges, angles, frames, channels, and non-symmetrical box-shaped sheet metal
parts. In these operations, plane-strain bending or straight line bending with single
curvature is the common deformation mode. The understanding of the bending
mechanics is aimed at obtaining three kinds of information important for industrial
applications: (a) the "Bendability" assessment for given bend radius, bending angle, and
sheet/plate materials, (b) the springback prediction for die design and shape/ dimension
control, and (c) an estimation of the bending force for selection of press capacity and for
strength analysis and design of dies.

Based on the elementary bending theory of elastic beams, Ludwick [Ludwick,
1903] established the engineering theory for plastic bending, based on four assumptions:
(a) the strain is linearly distributed throughout sheet thickness, therefore the cross section
plane remains a plane during bending,
(b) the transverse stress (normal to the sheet) across sheet thickness is ignored,
(c) the neutral layer (no strain along it) coincides with the mid-plane since the radius of
the bend is much greater than the sheet thickness, and
Fig. 2.5 A bracket formed by multiple forming operations: cutting holes, local stretching to form bead, shrink flanging, and bending.

[Eaton Corp., 1992]
(d) the removal of bending moment after bending is equivalent to the elastic response by superposition of a moment with equal magnitude, but opposite sign.

For plane strain bending, Hill [1950] proposed an elementary bending model with the same assumptions as Ludwick's. This model is applicable only for isotropic and elastic-perfect-plastic materials which may not exist in engineering applications. A numerical integration of bending moment incorporating strain hardening was proposed by Altan [1962]. It was apparent that most of bending analysis were based on elastic and ideal-plastic material law before 1970's. The strain hardening has profound influences in springback and the bendability of the sheets. It can be introduced to the bending theory through either Hollomon type of hardening equation (\( \bar{\sigma} = k\bar{\varepsilon}^n \)) [Johnson and Mellor, 1972; Hosford and Caddell, 1983; Wang, 1984] or Swift type of hardening law (\( \bar{\sigma} = k(\varepsilon_0 + \bar{\varepsilon})^n \)) [Nagpal et al, 1980]. In most of bending analyses, the anisotropic material behavior was not considered, and the sheet materials were assumed to be isotropic in order to use Tresca's or Mises' yield criterion. To be able to account for the anisotropic behavior of sheet materials, Hill's 1948 [1948] quadratic yield function for transverse (normal) anisotropic materials was used to analyze bending under plane-strain and plane-stress conditions [Nagpal, 1980; Wang, 1984; Levy, 1984]. Hill's new non-quadratic yield function for normal anisotropic materials [Hill, 1979] is used in the current study. This new yield function covers all most widely used yield criteria previously proposed by Tresca [1864], Von Mises [1913], and Hill [1948]. The planar anisotropic behaviors have not yet been considered for bending analysis because of the great mathematical complexity. A general anisotropic bending model in plane strain and 3-dimensional stress (including the transverse stress) was been introduced in the advanced bending theory by Wang et al [Wang, Kinzel, and Altan, 1992b]

Springback was addressed by Gardiner [1957] who proposed a mathematical formula for springback calculation based on the elementary bending theory with ideal
plasticity. Johnson and Yu further developed Gardiner's work to the linear hardening materials [1981], and to the bending with tension [Yu and Johnson, 1982]. Forming of the HSLA (high strength and low alloy) sheet steels in automobile industry has encountered more severe elastic distortion problems: springback and side-wall curl. A number of studies dealt with the springback in press forming where the deformation mode is the bending under tension [Duncan and Bird, 1978; Nakagawa and Abe, 1980; and Yuen, 1990]. To reduce the springback, the tension can be applied before, during or after the bending process. In press forming, the tension is applied during bending, hence it reduces the bending moment and increases the plastic yielding. A compression force can also be applied after bending, through pad pressing, bead bottoming, or restriking the sheet on the bottom, sides, or edges to cause the compression yielding. Careful analysis and quantitative study of the relationship between restriking pressure and springback do not yet appear to have been addressed.

The error in the elementary bending theory depends on the ratio of the sheet thickness to the radius of the curvature, $t/R$, [Hill, 1950]. In general, the elementary bending theory gives acceptable accuracy for the bending operations with the ratio of the radius of curvature to the sheet thickness ($R/t$) is greater than 4, in which neglecting the transverse stress induced by the curvature can be justified. However, when the ratio of $R/t < 4$, the normal stresses (shear stress and radial stress), increases and may reaches to the same order as that of the axial stress. The ignorance of the normal stress is no longer justified. The normal stress cause the shift of the neutral plane and increase in thinning of sheet. The advanced plastic bending theory accounting for the normal stress was proposed by Lubahn and Sachs [1950] and Hill [1950] through the force equilibrium equations and the Tresca yield function. In their studies, the elastic response was neglected and an isotropic and ideal-plastic material law were assumed. The shift of neutral layer and the thickness change were analyzed by their model. Drucker investigated the influence of the shear force on the plastic bending of a centrally loaded
cantilever rectangular beam. His study indicates that the interaction between the shear and the bending moment is not significant - a 10% reduction in moment for a shear force equal to half the shear force in fully plastic yielding case. Thus, the approximate solution for pure moment bending, neglecting the shear stress, can also be applied to bending with transverse load. Proksa extended Hill's work to a rigid-plastic material with a linear strain hardening [1959] in pure moment bending. Martin and Tsang [1966] examined the validity of pure bending theory in bending with transverse load and friction at freely supported ends, i.e. die shoulders. In the simple bending of beam under a central load, they compared the bending moment predicted from Proksa's model with their experimental results, and concluded that Proksa's analysis of plane strain pure bending can be applied in bending of a transversely loaded beam with a span larger than 5 inches and well lubricated ends. They also found that friction loss at die shoulders expends about 10% of applied work. But the friction does not affect the fracture strains. Very few studies dealing with the real strain hardening have been introduced into the general bending theory. Dadras and Majlessi [1982] extended Proksa's analysis for linear hardening materials to work hardening materials which obey Ludwick's hardening equation ($\sigma = \sigma_0 + k\epsilon^n$). They suggested two models to account for the strain hardening and the Bauschinger effect when the reverse loading occurs because of the shift of the neutral axis. No attempt has been made before to account for material anisotropic behavior in the general bending theory due to a great deal of mathematical difficulty involved in the formulation. In the current study, Hill's 1979 general anisotropic yield criterion is used to incorporate the planar anisotropy with the bending mechanics. An explicit expression of Hill's general anisotropic yield function under plane strain and 3-D stress conditions is derived and used to establish an advanced bending model with planar anisotropy [Wang, Kinzel, and Altan, 1992b].

The bending under tension (BUT) is a useful technique used to reduce the springback. This deformation mode also exists in stretch forming and deep drawing
operations, in which the sheet wraps around the radii of the shoulders of die and punch. When the drawbead is used in stretch/draw forming, the metal flow through the bead is also characterized by this deformation mode. For the BUT process, the effects of the applied axial tensile force on the deformation are (a) to enlarge the plastic yielding zone through the sheet thickness, hence, to reduce the elastic core height and to reduce the springback significantly; and (b) to cause the neutral axis to shift toward the inner concave surface, which results in a great increase of the tensile strain and stress, therefore, the fracture tendency is also increased. The accurate prediction of the axial tension is essential for the careful control both of the springback and the fracture. The influences of the axial tension on elastic-plastic bending of a beam were studied by Yu and Johnson [1982] by assuming (a) a linear strain distribution, (b) the ideal-plasticity, and (c) an isotropic material under uniaxial stress state. El-Domiaty and Shabaik [1984] extended the work of Yu and Johnson to derive the expressions of the axial force and bending moment for strain hardening materials. Following this work, El-Megharbel et al [1990] analyzed the springback and residual stresses in stretch-bending of 7075 Aluminum sheet. However, in their study, the assumptions (a) and (c) that exist in Yu and Johnson's work, still remain. The assumption of a linear strain distribution through the sheet thickness may not be justified at a higher strain level resulting from the applied tensile force and the shift of the neutral axis. The true nonlinear strain distribution must be used. A plane strain bending model is more appropriate than the assumed uniaxial stress state in bending of wide sheets. The consideration of the plastic anisotropy of sheet is also necessary to establish a more accurate model for sheet bending analysis. Therefore, in our present study, we adopt the following realistic models (a) a nonlinear strain distribution for plastic bending, and a linear strain distribution for elastic bending, (b) plane strain deformation, (c) strain hardening, and(d) Hill's new non-quadratic yield theory for normal anisotropic sheet materials.
2.2.2 Contour Bending - Flanging Operations

Flanging operations are basically bending processes of contour flanged parts with compound curvatures. Two differences between bending and flanging are that the bent down metal, during flanging, is shorter compared to overall part size, and the flanges and bends have distinctly different functions. But there is no well-defined bent-over length that distinguishes bending from flanging. Flanging deformation also occurs in many deep drawing and stretching processes of cups, boxes, and panels. Flanges are used for appearance, rigidity, hidden joints, and strengthening of the edge of sheet parts such as automobile front fender and complex panels formed by drawing or stretch forming. Three basic flange types are the straight flange, stretch flange, and shrink flange as shown in Fig. 2.4. The jogged flange is a combination of all three major tapes. The reverse flange is a combination of the stretch and shrink flanges. The hole flange is a special case of the stretch flange.

**Straight Flange** is a single curvature bend without longitudinal stresses imposed on the sheet except the bend radius. This operation is essentially the same as the bending.

**Shrink Flange** exists when the flange curvature is convex and the metal in the flange is in compression along the hoop direction, Fig. 2.6a.

**Stretch Flange** exists when a flange has a concave curvature (looking from outside the flanged surface) and the edge of blank gains an increase in length or the metal in the flange is stretched along hoop direction, Fig 2.6b.

**Hole Flange or Hole Expansion** is a special case of the stretch flange in which the bend line is a complete circle, Fig. 2.6c.

**Reverse Flange** is a combination of the stretch and shrink flanges, Fig. 2.4.
**Jogged Flange** is a combination of all three major types of flanging (straight, stretch, and shrink flanges). This process is used to make an offset in a flat plane by two parallel bends in opposite directions at the same angle, Fig. 2.4.

In shrink type of flanges, the greatest compression in the hoop direction occurs at the edge of the flange and diminishes to zero at the bend radius, Fig. 2.6a. In stretch type of flanges, the maximum elongation along hoop direction occurs at the edge of the flange and vanishes to zero at the bend radius, Fig. 2.6b. The most common defects in flanges are fracture in a stretch flange, wrinkling in a shrink flange, and springback in both. The flange width, \( W_f = R_2 - R_1 \) (where \( R_1 \) and \( R_2 \) are flange edge radii before and after forming, respectively) is a controlling factor in the stretch flange limit (edge splitting) and the shrink flange limit (wrinkling). To reduce the tensile stress and splitting, the flange width must be limited, or the edge notches should be provided. The wrinkling tendency in shrink flanging increases with increasing in flange width, bending angle, and sheet yield strength, and decreasing in contour radius, sheet thickness, strain hardening, and elastic modulus. Wrinkles can be prevented or reduced by decreasing the flange width, or increasing the sheet thickness, or providing offsets in the flange to take up excess metal, or reducing the clearance between the wiper die and the male die to iron out wrinkles.

Flanging operations have not been studied extensively, especially, the shrink flanging. Both stretch and shrink flange can be approximated as an axisymmetric deformation of an initial flat annular shaped piece of sheet metal clamped along one edge. Reverse flanging is a combination of stretch and shrink flanges. For parts formed by several (same or different) types of flanging operations, the forming of one flanged portion usually does not interfere with the forming of other flanged portions of the part. For design purpose, hence, each flanged portion should be evaluated in the same manner as if it were a separate part [Sachs, 1951]. Therefore, the establishment of mathematical models for shrink and stretch flanging and straight flanging (or plane strain bending) will
Fig. 2.6 (a) Shrink flange, (b) stretch Flange, and (c) hole flange.
provide necessary evaluation tools for deformation and formability analysis. So far, the empirical formulae for strain calculations in shrink and stretch flanges have been used in sheet forming practice [Sachs, 1951, Smith, 1990]. These simple equations are based on the assumptions of (a) the flange length does not change after flanging, and (b) the radius of die profile and sheet thickness approach to zero. These unrealistic assumptions yield much lower strain than the measured ones.

Stretch flanging was studied by Wang and Wenner [1974] using two approaches: the total strain theory of rigid-plasticity and an approximate theory with the assumption of uniaxial stress state which was justified by the experiment. Both methods employed a plane stress plasticity incorporating the nonlinear strain hardening and the normal anisotropy. With the approximate theory, a closed-form expression of the maximum tensile strain in the flange edge was established in terms of flange geometrical parameters. Wang further studied the stretch flange with a notch cutout [1983] and flange with an initially curved in a “V” shape [1984]. In both cases, the finite element simulations were performed to construct the formability diagrams. Wenner [1991] studied the two-step stretch flanging of prestrained sheet preformed by primary forming operations. The prestrain may be caused by previous sheet forming processes with a balanced biaxial stretching mode. It was found that any reasonable preform may increase the flange length by 10-15% due to the change of strain path from biaxial to uniaxial tension. Hole flanging as a special case of stretch flanging was examined by several investigators [Yamada, 1968; Dange, 1972; Johnson, 1980; and Tang, 1981]. In this study, Wang and Wenner’s model will be further developed.

Very few detailed analyses of shrink flanging exist in the sheet forming literature [Wood, 1965, and Nagpal et al., 1979]. The major difficulty is the modeling of wrinkling phenomenon during flanging operation. In shrink flanging, the wrinkle occur along the unsupported free edge, which is similar to the wall wrinkling in deep drawing
process. The strategy to tackle shrink flange may involve three issues of: (a) establishing the strain model in terms of the flange dimensions (the flange length, the flange angle, the radii of the die opening and the die profile, and the initial blank thickness and radius), (b) developing a wrinkling criterion in terms of the critical compressive strain which is related to material properties and flange dimensions, and (c) correlating the two strain expressions in (a) and (b) to establish the formability of shrink flange. With Wang and Wenner’s approximate theory, the expression of the maximum strain based on the shrink flange dimensions will be derived in this study. However, it would be impossible to establish the appropriate wrinkling criterion for shrink flange without careful examination and understanding of the buckling instability theory and its applications in sheet forming. An extensive study of the wrinkling for a double curved shell is conducted in this study, and a number of wrinkling criteria are proposed for deep drawing and shrink flanging.

2.4. Finite Element Modeling of Sheet Forming Processes

In metal forming analysis, either elasto-(visco)plastic or rigid-(visco)plastic constitutive relations can be adopted regarding the appropriate applications. For most metal forming problems, the plastic deformation far dominates the elastic response by several orders. The plastic strain can reach to 100% for billet forming, and 30% for sheet forming operations. While, the elastic strain is usually in the region of 0.3% for most structure metals, and the elastic deformation is several orders of magnitude less than the plastic deformation. Therefore, it is justified to ignore the elastic response in analysis of most metal forming processes, especially, the billet forming operations such as forging, extrusion, rolling, etc. For stretch-type of sheet forming operations, the springback is less apparent, hence, the rigid-(visco)plastic model is also appropriate for the deformation analysis. However, for bending processes and draw-type sheet forming operations (like deep drawing), especially when the restraining force (generated by blankholder and/or drawbead) is low, the elastic response, thereby, the springback are significant. Forming
of HSLA (high strength and low alloying) steel sheets and aluminum alloy sheets has being encountered a serious springback problem. An elastic-(visco)plastic analysis is more accurate than the rigid-(visco)plastic approximation for modeling such sheet forming operations. Residual stress evaluation is another application of elasto-plastic analysis. After removal from the die, the formed part is elastically unloaded, and the residual stress may remain in the core and on the surfaces of the part. Such residual stresses can initiate cracks, and enhance stress corrosion.

In finite element modeling of sheet forming processes, the membrane model is commonly used because the sheet thickness is usually small relative to the tool radii and the bending effect is not significant [Wang, 1984a]. In membrane theory, the strain is assumed uniform throughout the thickness. This yields efficiency and accuracy in computation for most stretch type of operations with fairly large ratio of tool radius over sheet thickness (R/t). However, as R/t ratio falls bellow 10, the bending effect is significant, and the membrane solutions may give poor accuracy, especially for the deep drawing with low restraining force [Wang, 1990]. To deal with bending effect, shell theory has been used [Tang, 1988, Lee 1989], in which the strain varies through sheet thickness, and the bending moment and stresses must be taken into account. This introduces additional degrees of freedom and significant increase in computational time and cost. In order to take advantage of the computational efficiency of membrane formulation and also be able to considering the bending effect, several bending correction models for membrane solution have been proposed. A more time-efficient quasi-membrane/bending formulation was proposed and tested [Stoughton, 1985], in which the local thickness and curvatures along the principal directions for each element. Wang [1991a] suggested a simple bending correction procedure for the local bending effect (the bending only affects the local region of bends) by adding the local bending strains to the membrane strains. The comparisons with measured maximum strains indicate that the local bending correction is valid for bending ratio R/t larger than six, but deteriorates as
the ratio falls below six. For small ratio of \( R/t \), the global bending effect (the bending affects entire strain/stress distributions because of extra hardening and thinning caused by bending) becomes important. The situation is further complicated when bending, sliding, and unbending occur as sheet passes the die radius during drawing process. In this study, a decoupled method is suggested for a stepwise bending correction of the membrane-based finite element modeling.

2.5 Artificial Intelligence and Design Expert Systems in Sheet Forming

Since 1980's, the artificial intelligence has been introduced into sheet forming practice through a so called Expert System, which is an intelligent computer program that uses the knowledge and inference the procedures to solve the problems which usually require human experts to solve. The knowledge-based expert system is emerging now as a potential tool to increase the efficiency and productivity, a permanent source to keep the expertise of the domain experts, and a teaching aid to train the beginners. The knowledge-based expert system consists of three components: a knowledge base, an inference engine, and a control strategy. Several expert systems have been developed for sheet forming process sequence design. Typical expert systems for process planning are BUCCS (Boeing Process Planning System) [Allen, 1980], GECAPP (General Electric) and GENPLAN (Lockheed-Georgia) for automatic generation of deep drawing process outlines [Eshel, 1985], MANICAP (a Computer-Aided Planning System for Modular Automated NC Integration) for sheet bending sequence generation [Geiger 1988]. At the ERC/NSM of The Ohio State University, a CAE system for process sequence design in forming of axisymmetric sheet parts [Sitaraman, Kinzel, and Altan, 1989] was developed, and an integrated CAE system for the design of part and process sequence in manufacturing box-shaped nonsymmetrical sheet components is under development [Yu, Wang, Miller, Kinzel, and Altan, 1993].
In earlier development of the expert system, only a so called 'coarse-grained-knowledge, i.e. the specific experience from the domain experts and the rules from the handbooks, was most commonly used to build up the knowledge-base. It has been found that the rule and experience based expert systems are efficient in solving the routine problems of KNOW-HOW, but seldom solve problems of KNOW-WHY. Such systems are still limited by (a) the brittleness of the coarse-grained knowledge which is too specific to solve broad range of problems, (b) the validity of past experience for new material and new process in a day-to-day changing world, and (c) the inherited incompleteness of the knowledge base when rules are not available. Therefore, the rule-based systems, in long run, still can not significantly reduce man power and arrive at an optimized design. To improve the knowledge base, a so called 'fine-grained-knowledge', i.e. the information provided from mathematical modeling of deformation process, should be introduced into the rule-based systems. The fine-grained knowledge from process simulation is more general, quantitative, and fresh. Therefore, the expert system is more flexible than the rule-based system. The integration of both the fine-grained and the coarse-grained knowledge, and the interaction between them, are still active topics in applications of AI and expert systems in sheet forming. Expert system may be possibly integrated with the simulation system, if the output results from prior simulations can be used as the input data to an induction algorithm which then deduce production rules. In addition, simulation module can be used (a) to generate the rules when they are not available from a current knowledge base, and (b) to verify and modify the "old" rules and to check the accuracy in new applications. The expert system can assist the simulation in (a) defining the initial processing conditions (generate process sequence and geometry of the part and the die, and the boundary conditions), and (b) diagnosing and recommending remedies to rectify the problems. A possible integration of knowledge-based system and simulation is shown in Fig. 2.7.
Fig. 2.7 An integrated CAE system with deformation analysis and knowledge-based expert system for process optimization and tooling design
CHAPTER III
FUNDAMENTALS OF PLANE STRAIN BENDING

3.1 ELEMENTARY BENDING THEORY

3.1.1 Assumptions

The elementary bending theory deals with the pure moment bending where the bending moment is assumed to be constant along the sheet length, Fig. 3.1. Seven assumptions are commonly adopted in various elementary bending models:

(i) The cross section plane (y-z plane) normal to the sheet surface and parallel to the bend axis remains a plane during and after bending, and therefore, the strains are linearly distributed throughout the sheet thickness, Figs. 3.1;

(ii) Radial stress normal to the sheet (y-axis or thickness) is neglected, and therefore, only stresses along the longitudinal (x-axis or fiber length) and width directions (z-axis) are considered, i.e. a plane state of stress exists;

(iii) Sheet thickness remains constant because the bending radius is assumed to be much larger than the sheet thickness;

(iv) The neutral layer coincides with the middle plane of the sheet;

(v) Sheet is bent under plane-strain conditions, and the strain along the sheet width is assumed to be zero;

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Fig. 3.1 Pure moment bending of a sheet ($L_o$ - length before bending).
(vi) Sheet material is a homogeneous continuum with isotropic properties, and the
Bauschinger effect is neglected;
(vii) The removal of the bending moment after bending is equivalent to the elastic
response by superposition of a moment with equal magnitude but opposite sign.

In a wide range of engineering applications, the bending mechanics based on
these assumptions provide reasonable approximations, although none of the assumptions
may truly exist. The first four assumptions are directly related to the ratio \((R_i/t)\) of the
bend radius, \(R_i\), to the sheet thickness, \(t\). When the sheet is sufficiently thin relative to the
bend radius, or \(R_i/t\) is large enough, these assumptions are justified. The first assumption
also indicates that the shear stresses are relatively small compared with the longitudinal
stresses, in which case the longitudinal dimensions are far greater than the dimensions of
sheet cross section. When the radius of the curvature is large compared with the sheet
thickness, the transverse or radial stress may be safely neglected. A sufficiently large
ratio of bend radius to sheet thickness is also the precondition for the assumptions of
constant sheet thickness and the co-plane of the neutral layer and the middle layer of the
sheet. Studies indicate that a shift of the neutral layer is unlikely if the maximum strain
on the convex outer surface is about 5\%, or the ratio of bend radius to sheet thickness is
greater than 8 [Lange, 1985]. If the pressure or the tension is applied, the neutral axis
also shifts toward the tool surface even for a fairly large bending radius. For bending
with small radius, the neutral layer always shifts. Evaluation of the shift of the neutral
axis is the key issue to calculate the actual strain/stress distributions. Unfortunately, the
bending models based on the above assumptions fail to account for this important
concept. The plane strain bending condition is maintained when the ratio \(w/t\) of the sheet
width to the thickness is greater than 8 [Sachs, 1951]. The material inhomogeneity
(voids, thickness variation, and surface quality) in sheet has more significant influences
on formability, compared to the inhomogeneity in bulk materials. For heavily rolled
sheet material, neglecting the anisotropy of the sheet may not be justified anymore.
In this study, only assumptions of (ii), (iii), (v), and (vii) are adopted. A nonlinear strain description, the shift of neutral axis, the strain hardening and the anisotropy are taken into consideration.

### 3.1.3. Strain Analysis

The sheet is imaginarily divided into a number of layers throughout the thickness. By its definition, the neutral layer, with the radius $R_n$, is strain free. Therefore, the length ($L_o$) of this layer is constant during bending. For a given radius $r$, the true strain or the longitudinal strain for any layer other than the neutral axis is defined by its lengths before ($L_o$) and after ($L$) bending, Fig. 3.1:

$$
\varepsilon_x = \ln \frac{L}{L_o} = \ln \frac{rR_n}{R_n} = \ln \frac{r}{R_n} = \ln(1 + \frac{y}{R_n})
$$

(3.1)

And the engineering strain is defined as

$$
\varepsilon = \frac{L - L_o}{L_o} = \frac{r}{R_n} - \frac{R_n\theta}{R_n} = \frac{(R_n + y)\theta - R_n\theta}{R_n} = \frac{y}{R_n}
$$

(3.2)

The radius of the neutral axis can be determined by volume constancy and is found as

$$
R_n = \sqrt{R_oR_i} = R_i\sqrt{(1 + t/R_i)}
$$

(3.3)

Where $R_i$ and $R_o$ ($=R_i+t$) are the radii of the inner concave and outer convex surfaces, and $t$ is the sheet thickness. When sheet contacts the tool, $R_i$ is the tool radius.

For the above descriptions of strains, in general, the true strain $\varepsilon_x(y)$ is not linearly distributed across the sheet thickness. Only if the sheet is thin, the radius of bending is relative large, or the ratio, $t/R_i$, is small, is the strain distribution through thickness approximately linear; and in such a case, the strain level is low, therefore, the
logarithm or the true strain is the same as the engineering strain. Using Taylor's expansion and ignoring the terms higher than two, Eq. (3.2) can be expressed as

\[ \varepsilon_x = \frac{y}{R_n} - \left( \frac{y}{R_n} \right)^2 = \frac{y}{R_n} \left( 1 - \frac{y}{R_n} \right) \]

For a relative small value of \( \frac{y}{R_n} \), the true strain \( \varepsilon_x (y) \) can be assumed to be linear to the radial distance \( y \):

\[ \varepsilon_x = \frac{y}{R_n} = \varepsilon_x \]

In such a case, the true strain \( \varepsilon_x \) \((= \ln \frac{L-x}{L_0})\) is the same as the engineering strain \( \varepsilon_x \) \((= \frac{L-L_0}{L_0})\).

The maximum tensile and compressive strains occur on the convex outer surface and the concave inner surface of sheet and have the values of

\[ \varepsilon_{\text{max}} = \ln \frac{R_{n_1}}{R_n} = \frac{1}{2} \ln(1 + t / R_i) \]  \hspace{1cm} (3.4)

\[ \varepsilon_{\text{min}} = \ln \frac{R_{n_2}}{R_n} = -\frac{1}{2} \ln(1 + t / R_i) \]  \hspace{1cm} (3.5)

where '+' is for tensile, and '-' for compressive.

The engineering strain distribution and the true strain distribution are illustrated in Fig. 3.2 (a). The relative difference (error) compared with the true strain is shown in Fig. 3.2(b). When the strain level is low (less than 10%), that is, when the radius of the neutral axis \( R_n \), or the bend radius \( R_i \) is large, these two strain descriptions yield similar results. However, the error induced by using linear engineering strain description increases with the strain level. At a strain level of 20%, the error reaches 10%. Therefore, the true strain description must be used in the following cases: (a) bending with small radius, and (b) the shift of the neutral axis resulting in a great strain increase.
Fig. 3.2 (a) Comparisons between true and engineering strain descriptions, and (b) error vs. relative distance ($y/R_n$ - the relative distance from the neutral axis)
For plane-strain bending, the deformation along the sheet width is prevented, therefore, the strain along the thickness is defined by the volume constancy:

$$\varepsilon_y = -\varepsilon_x \quad (3.6)$$

### 3.1.3. Strain Hardening Laws

Two types of flow curves are most commonly used in bending analyses. The Hollomon type of hardening curve is described as

$$\bar{\sigma} = k\bar{\varepsilon}^n \quad (3.7)$$

and the Swift's hardening curve is described as

$$\bar{\sigma} = k(\varepsilon_0+\bar{\varepsilon})^n \quad (3.8)$$

Hollomon's equation has the advantage of being a simple mathematical relation, and it is easily introduced into the bending moment calculation which requires an integration. However, this relation misrepresents the hardening behavior at both low and high strain levels. In some studies, a so-called 'two-part power law' was used, in which the material coefficient $k_1$ and $n_1$, and $k_2$ and $n_2$ are determined through the straight line fits (representing different strain levels and hardening behaviors) in a log-log plot of the hardening curves [Wang, 1983; Mickalich, 1988]. The prestrain, $\varepsilon_0$, or the history of successive forming operations, can be taken into account by Swift's equation which is a better representation of flow behavior at low strain level than Hollomon law. However, Swift's equation is not accurate at high strain levels. Also, there are some mathematical difficulties in the integration of bending moments using this hardening law. The Hollomon type of hardening relations is suitable, for low strength aluminum alloys and mild steel, but not for high strength aluminum alloys. Strain-rate hardening has not yet been addressed much in bending analyses. In this study, the Swift's hardening law will be
used, and if necessary, it can be simplified to Hollomon's equation by setting the initial strain to zero.

3.1.4 Plastic Yield Criteria

In most elementary bending models, anisotropy is still ignored. Von Mises' and Tresca' yield functions have been widely accepted in bulk forming. However, the isotropic yield criteria may only be applied at the initial stage of plastic deformation and non-prestrained materials. For materials which have undergone large deformation, especially, for heavily rolled metal sheets, the anisotropic behavior must be taken into account and has been accepted more and more in recent years. Using Hill's 1979 new yield theory for normal anisotropic materials and the equivalent plastic work, the relationship between the effective strain and the axial strain components, and between the effective stress and the axial stress components are (Appendix A):

\[ \bar{\varepsilon} = F \varepsilon_1 = F \varepsilon_x \]  
\[ \sigma_x = F\bar{\sigma} \]

Where \( F \) is an important index introduced here to explore the influences of the anisotropy and the strain/stress states in the metal flow behavior, and it has different values for different materials and stress/strain states:

\[ F = \frac{1+R}{\sqrt{1+2R}} \]

1. Isotropy and uniaxial stress
2. Isotropy and plane strain
3. Normal anisotropy and plane strain [Hill, 1948]
4. Normal anisotropy and plane strain [Hill, 1979]
where $R$ is the normal (transverse) anisotropy, and $M$ is the anisotropy index describing the shapes of the yield loci. Note that for isotropic materials ($R=1$, and $M=2$), $F$ has different values for uniaxial stress ($F = 1$) as in narrow sheet bending, and for plane-strain ($F = 1.155$) as in wide sheet bending.

3.1.5 Stress Analysis and Constitutive Relations

In bending of sheet with a large radius/thickness ratio, the radial stress along the sheet thickness can be ignored with small error because the contact pressure has much less effect on in-plane deformation [Drucker, 1956]. To maintain the plane strain condition, stress along the sheet width must be present. Therefore, both plane-strain and plane-stress states are assumed to exist.

In elasto-plastic bending, the elastic stress component of the total tensile stress can be obtained through Hooke's law:

$$\sigma_x = \frac{E}{(1-\nu^2)} \varepsilon_x \quad 0 \leq \varepsilon_x \leq \varepsilon_{e,0}$$

(3.12)

Using Swift's hardening law, Eq. (3.8), and Eq. (3.9), the longitudinal plastic stress is defined as

$$\sigma_x = k F^{n+1} \left( \frac{\varepsilon_0 - \varepsilon_{e,0}}{F} + \varepsilon_x \right)^n = k F^{n+1} \left[ \frac{\varepsilon_0 - \varepsilon_{e,0}}{F} + \ln(1 \pm \frac{y}{R_n}) \right]^n \quad \varepsilon_{e,0} < \varepsilon_x$$

(3.13)

The longitudinal stress distribution through the thickness can be determined by substitution of the strain distributions along the thickness, Eq. (3.2), into the above equations, i.e.:

$$\sigma_x = k F^{n+1} \left[ \varepsilon_0 - \varepsilon_{e,0} + \ln \left(1\pm\frac{y}{R_n}\right) \right]^n$$

(3.14)
In Eq. (3.14), the stress is indeterminable without knowing the radius of the neutral layer, \( R_n \), which will be discussed further later. For a simplified case in which the neutral layer is assumed midway through the deformed sheet thickness, the stress distribution is symmetric about the middle layer, Fig. 3.3.

### 3.1.6 Internal Bending Moment

A pure moment bending is considered in this section. Only a bending couple is applied to the sheet as shown in Fig. 3.1. This pure-moment bending model has been found useful in air bending [Lange, 1985] and U-die bending [Wang, 1984], in which the tool is mainly used to transmit the forces and moments to the sheet. The internal bending moment, \( M \), balancing the applied bending moment exerted by the tool, is defined as:

\[
M = \int_0^t \sigma y \, w dy = M_e + M_p \tag{3.15}
\]

In elastic-plastic bending, the moment includes two parts: the elastic-bending moment, \( M_e \), within the elastic core band, \( 2y_e \), and the plastic bending moment, \( M_p \), in the plastic deformation layers. The limiting height of the elastic core, \( 2y_e \), the elastic bending moment, and the plastic moment are given as follows (Appendix B).

**The Elastic Bending moment \( M_e \).** \( M_e \) is calculated by substituting Eq. (3.12) into Eq. (3.15) and considering the linear strain description, Eq. (3.3):

\[
M_e = \int_0^{y_e} \sigma_{xy} \, w dy = \frac{2w}{R_n} \int_0^{y_e} \frac{E}{1 - \nu^2} y^2 dy
\]

\[
= \frac{2w}{3} \frac{E}{1 - \nu^2} \left( \frac{y_e}{R_n} \right)^3 R_n^2 = \frac{2wR_n^2}{3} \frac{E}{1 - \nu^2} \varepsilon_{e,o}^3 \tag{3.16}
\]

\[
\varepsilon_{e,o} = \frac{F}{E' \sigma} = \frac{F(1 - \nu^2)}{E} \frac{1}{\sigma} \tag{3.17}
\]
Fig. 3.3 Stress distribution through sheet thickness, t.
A good approximation of $\varepsilon_{e,o}$ is 0.2%, which corresponds to the initial yield stress, $\sigma_{0.2}$. Hence, Eq. (3.16) can be expressed as

$$M_e = \frac{2wR_n^2}{3} \left( \frac{1 - \nu^2}{E} \right)^2 \sigma_{0.2}^3$$  \hspace{1cm} (3.19)

For fully elastic bending ($y_e = t/2$), this moment changes to

$$M_e = \frac{wt^2}{6} \frac{\sigma_{0.2}}{1 - \nu^2}$$  \hspace{1cm} (3.20)

**Plastic Bending moment $M_p$.** The Plastic bending moment, $M_p$, is calculated based on Swift's hardening law and Hill's 1979 anisotropy yield function. The expression of $M_p$ is obtained by substituting Eqs. (3.14) into Eq. (3.15) and integrating it from the elastic core height, $y_e$, to the sheet surface:

$$M_p = 2w \int_{y_e}^{t^2} \sigma_{xy} wdy = 2w \int_{y_e}^{e_{max}} F^{n+1} k \left( \frac{\varepsilon_{e,o} - \varepsilon_{e,o}}{F} + \varepsilon_{x} \right)^n y dy$$  \hspace{1cm} (3.21)

After integration, the plastic bending moment is determined as:

$$M_p = 2wkF \left( e_{max}^n + \frac{e_{max}^{n+1}R_n^2}{2} e_{e,o}/F \right) \sum_{j=0}^{\infty} \left\{ \frac{2^j}{j!} \left( \varepsilon_{\max} + \frac{\varepsilon_{e,o}}{F} \right)^{j+1+n} \right\}$$  \hspace{1cm} (3.22)

where the maximum strain at the convex outer surface with radius $R_o$ is defined in Eq. (3.4). When the ratio of $R_o/t$ is greater than 5, the neutral axis is nearly at the middle axis, therefore, the moment can be expressed as

$$y_e = R_n\varepsilon_{e,o} = R_n \left( \frac{1 - \nu^2}{E} \right) F \bar{\sigma}$$  \hspace{1cm} (3.18)
For an accuracy of 99%, only the first four terms in Eq. (3.22) are required. The true strain description, Eq. (3.1), is used in the above equations. When the ratio of $R_i/t$ is large and the linear (engineering) strain is small, the first two terms in Eq. (3.22) can be used. This moment equation reduces to Nagpal’s formula [Nagpal, 1980] and Hosford’s moment equation [Hosford and Caddell, 1983] by setting the prestrain strain, $\varepsilon_0$, to zero and anisotropic index $F$ to 1.

For fully plastic bending (neglecting the strain at elastic limit, i.e. $\varepsilon_{e,o} = 0$), Eq. (3.22) reduces to

$$M = M_p = 2wkF^{n+1}R_i^2 \varepsilon^2 e^{2 \varepsilon/F} \sum_{j=0}^{\infty} \left\{ \frac{2j - \varepsilon/F}{j! (j + 1 + n)} \left[ (\varepsilon_{\max} + \varepsilon_o)^{j+1+n} - \left(\frac{\varepsilon_o}{F}\right)^{j+1+n} \right] \right\}$$

Fully plastic bending (3.24)

**Elasto-Plastic Bending Moment $M_{ep}$.** With Eqs. (3.16) and (3.22), the elasto-plastic bending moment is found to be

$$M_{ep} = M_e + M_p = \frac{2wR_i^2}{3} \frac{E}{1-v^2} \varepsilon_{e,o}^3 + + 2wkF^{n+1}R_i^2 \varepsilon^{2(\varepsilon_o \varepsilon_o)}$$

$$\sum_{j=0}^{\infty} \left\{ \frac{2j - \varepsilon(\varepsilon_o - \varepsilon_{e,o})}{j! (j + 1 + n)} \left[ (\varepsilon_{\max} + \varepsilon_o - \varepsilon_{e,o})^{j+1+n} - \varepsilon_{e,o}^{j+1+n} \right] \right\}$$

(3.25)

For the elastic-plastic bending with relative large $R_i/t$ ratio, the total bending moment is found as (neglecting the prestrain for convenience):

$$M_{ep} = \frac{C_2}{R_i^n} + \frac{D_1}{R_i^{n+1}} = \frac{1}{2} K F^{n+1} w t^2 \left( \frac{t}{2R_i} \right)^n \left( 1 + \frac{3}{2} \frac{n+2}{n+3} \frac{t}{2R_i} \right)$$

(3.26)
3.1.7 Springback and Residual Stress

The elastic recovery after unloading causes the redistribution of stresses and the springback phenomenon in which the radius of curvature, \( r \), of any fiber in bending increases to \( r' \) after the bending moment is removed. The dimensional accuracy of the bent sheet components is affected by the springback, and the service quality and life are affected by the residual stresses within the sheet. Researchers have struggled for many years to develop mathematical models to predict and control the springback. Springback was first addressed by Gardiner [Gardiner, 1957] who proposed a mathematical formula for the change in the curvature based on the ideal plasticity:

\[
\frac{R_n}{R'_n} = 4 \left( \frac{\sigma_o R_n}{E t} \right)^3 - 3 \frac{\sigma_o R_n}{E t} + 1
\]

For the strain hardening and anisotropic materials, the springback is found as (Appendix C):

\[
\frac{1}{R_n} - \frac{1}{R'_n} = \frac{M}{E' I_x} = \frac{12 (1 - v^2)}{w t^3 E} (M_e + M_p)
\]  \hspace{1cm} (3.27)

Where the unloading moment is equal to the bending moment defined in Eq. (3.25), and \( R_n \) and \( R'_n \) are the radii of the neutral axis before and after unloading, respectively.

Substituting the moment equation (3.25) into the above relations, it is seen that springback is controlled by several important factors: material properties, sheet thickness, bend radius, bending types, and stress/strain states. The conclusions are:

(a) Springback increases with the yield stress or the strength coefficient (\( k \)-value), work hardening (\( n \)-value), and anisotropy (\( R \)-value), since the higher these values are, the greater the resistance to plastic yielding.

(b) Springback decreases with the elastic modulus \( E \) because the resistance to elastic bending increases with Young's modulus.
(c) Bending of a thin sheet with larger bend radius \((R_n)\), or large relative bending radius \((R_n/t)\), enhances springback.

(d) The springback angle increases linearly with bending angle, but the bending angle does not affect the curvature springback.

(e) Wide sheet bending for the plane strain condition has a larger springback than narrow sheet bending under a uniaxial stress condition for isotropic materials. Note that this result contrasts with the conclusion from the springback model based on perfect-plasticity.

(f) The total springback is the sum of springback at individual section along the bending arc. The longer the bending arm is, the greater the total springback. As indicated in the following chapter, the bending arm increases in some bending operations (like air bending), while it decreases in other types of bending operations (like V-die bending). Therefore, the total springback is different for these two types of bending.

It should be noted that all of our equations related to strain, stress, moment, and springback, are related to every section of sheet. Therefore, they change along the bending span, and the total springback is a summation of all the incremental springbacks of the individual sections. The variation of the bending moment along the bending span depends on the types of bending operations. For most cases, a linear distribution may be justified. Further discussion will be presented in following Chapters.

The residual stress is derived in Appendix C and can be expressed as:

\[
\sigma_{x,\text{residual}} = -\sigma_{x,\text{unloading}} = \sigma_{x, \text{loading} \ - |\Delta \sigma_x|} = k F^{n+1} \ln(1 + \frac{y}{R_n}) \frac{E}{1 - \nu^2} \left[ \ln(1 + \frac{y}{R_n}) - \frac{y}{R_n} \right] \]

(3.28)

Some important observations can be drawn from Eq. (3.28):
(a) the residual stress increases with strength (k-value), strain hardening (n-value), and anisotropy (R-value, which raises the F-value),
(b) the residual stress decreases with increase in the elastic properties of the sheet, i.e. Young's modulus E and Poisson's ratio, and
(c) the residual stress in wide sheet bending under plane-strain conditions (F = 1.155) is greater than that in narrow sheet bending under uniaxial stress conditions (F = 1).

3.2 ADVANCED BENDING THEORY

The elementary bending theory has been found practical for use in the analysis of bending operations with gentle curvatures, and it provides acceptable accuracy for engineering applications if: (a) the bending radius ratio (R/t) of the bend radius, R, to the sheet thickness, t, is greater than 4; (b) pure moment bending can be approximated, i.e. the axial force, the shear forces and the radial loads do not exist or are negligible; (c) the neutral axis coincides with the middle axis of the sheet and remains in its original position during bending; and (d) the thickness change during bending is not significant. All of the above prerequirements for accurate applications of the elementary bending theory are violated as the radius of the curvature decreases, specifically, when the ratio R/t < 4 [Hill, 1950], and/or the loading conditions change, such as bending under tension, bending plus coining, and bending using an elastomer tool. The normal contact pressure is proportional to the curvature and may reach the order of the longitudinal stress at a small bending radius (or large curvature). Therefore, the general or the advanced bending theory becomes necessary for analyzing the bending processes with general loads and/or sharp bend radii.

Very few studies in the general bending theory considered strain hardening and no attempt has been made before to account for the material anisotropic behavior because of
the mathematical complexity in solving the differential equations of stresses. In the current study, the true (non-linear) strain distribution along the sheet thickness and the evaluation of the neutral axis will be taken into account. Both strain hardening and anisotropy will be considered through the uses of a power law hardening equation and Hill’s 1979 yield criterion for general (planar and normal) anisotropic materials. For a small radius of bending, the elastic deformation is negligible, and the rigid-plasticity is adopted.

3.2.1 Strains and Stresses in Plane Strain Bending

The assumption made here is that the principle axes of the strains and stresses coincide with the directions of the sheet length (longitudinal), width, and thickness (radial). In bending of a wide sheet, the strain along the width of sheet is zero, the strain along the fiber length is still defined by Eq. (3.1), and the thickness strain has the same value, but opposite sign as the fiber strain, i.e.:

\[
\varepsilon_1 = \varepsilon_x = \ln \frac{r}{R_n} = \ln \left(1 + \frac{y}{R_n}\right) \quad (3.29a)
\]

\[
\varepsilon_2 = 0 \quad (3.29b)
\]

\[
\varepsilon_3 = -\varepsilon_1 \quad (3.29c)
\]

where \(R_n\) is the radius of the neutral plane, and \(r\) is the radius of an arbitrary plane.

The stress state under plane strain condition is given by:

\[
\sigma_1 \neq 0 \text{ along fiber length}
\]

\[
\sigma_3 \neq 0 \text{ along thickness}
\]

\[
\sigma_2 = f(\sigma_1, \sigma_3) \text{ along width} \quad (3.30)
\]

For anisotropic materials, the stress along the sheet width can not be expressed simply, i.e. \(\sigma_2 = \nu (\sigma_1 + \sigma_3)\), where \(\nu\) is Poisson’s ratio which is 0.5 for plastic loading.
The expression of this stress depends on the yield theories used and it will be discussed in the next section. Note that the notations for the principal axes are changed now. The y-direction or direction 2 is now along the sheet width, and direction 3 is along the sheet thickness. These two directions are switched from Fig. 3.1.

3.2.3 General Anisotropic Yield Criterion

Hill's normal anisotropic theories in plane stress [Hill, 1948, 1979] are not applicable to the three dimensional stress state for our general bending case. An explicit expression for the general anisotropy yield criterion under plane strain and 3-D stress states will now be derived from Hill's new anisotropic yield theory in 3-D stress state. When the principle directions of stresses are coaxial with the principle axes of anisotropy (which are the intersections of the mutually orthogonal planes of symmetry), Hill's 1979 non-quadratic yield function is in a form of:

\[ 2f(\sigma_{ij}) = F|\sigma_2 - \sigma_3|^M + G|\sigma_3 - \sigma_1|^M + H|\sigma_1 - \sigma_2|^M = 1 \]  

(3.31)

where \( f \) is a plastic potential, \( F, G, \) and \( H \) are parameters describing the current state of the anisotropy, and \( M \) is a new index describing the shape of the yield surface. Note that \( F \) here is not the same as that used in Eq. (3.10) and defined by Eq. (3.11). Hill's 1979 non-quadratic yield function has been found more versatile for materials with different microstructures and anisotropic behavior [Kobayashi et al, 1985; Hosford, 1988], and it covers the traditional isotropic yield theories proposed by Tresca [Tresca, 1864] and Von Mises [Mises, 1913], and Hill's 1948 quadratic anisotropic yield criterion [Hill, 1948]. Eq. (3.31) returns to Tresca's yield criterion by setting \( M = 1, F = H = 0, \) and \( G = 1/(2k) \) (\( k \) is shear yield stress), to Mises yield theory by setting \( M = 2, F = G = H = 1/(6k^2), \) and to Hill's 1948 quadratic yield function by setting \( M = 2. \) It was found that the effects of the \( M \) value on the shape of the yield locus is opposite to that of the normal anisotropy, \( \bar{R} = \frac{R_0 + 2R_{45} + R_{90}}{4}, \) an average anisotropy, and the yield locus expands or the yield
stress increases, along the equal biaxial strain direction (the 45 degree line in the first quadrant of the yield locus), as $M$ decreases [Wang, 1990]. This yield criterion is of particular interest for those materials having normal anisotropy less than unity, for which, the quadratic yield criterion gives a significant underestimation of the flow stress in biaxial tension. For Aluminum alloys, $M$ is around $1.6 - 2.0$ for a strain range of $0.02 - 0.18$ [Wagoner, 1980], and $M$ is correlated well with the normal anisotropy for Steel, Brass, Aluminum, and Copper if $M = 1 + R$ for $R \leq 1$ [Ragab and Abas, 1986] and $M = 2$ for $R > 1$ [Bressan and Williams, 1983].

The explicit expression for the general yield theory for special plane strain with plane stress conditions was given by Hill [1979]. For the general plane strain case with three dimensional stress state, the mathematical expression of the plane strain yield function is derived by this author in Appendix D) and the results are presented below:

$$\sigma_1 - \sigma_3 = C X$$

(3.32)

and

$$C = \left( \frac{A}{B} \right)^{1/M}$$

(3.33a)

$$A = \frac{1 + R_90}{R_0}$$

(3.33b)

$$B = \left( \frac{R_90^{M-1}}{1 + R_90^{M-1}} \right)^{M} \left( \frac{1}{R_90} \right)^{M} \left( \frac{1}{1 + R_90^{M-1}} \right)^{M}$$

(3.33c)

where $X$ is the tensile yield stress in the first principal direction of anisotropy, i.e., the rolling direction, and $R_0$ and $R_90$ are the anisotropy values measured along the sheet rolling direction (0 degree) and the transverse direction (90 degree to the rolling direction). These two anisotropic indexes can be determined in a simple tensile test and they are defined as:
The yield criterion defined in Eq. (3.32) can be applied to several important cases in plane strain deformation as discussed in the following.

Normal Anisotropic and Planar Isotropic Materials:

For such materials, the anisotropic behavior in the sheet plane is negligible, and only anisotropy in the normal direction (perpendicular to sheet) is considerable (normal anisotropy). Therefore,

\[ R_0 = R_{90} = R \]

The yield criterion for normal anisotropic materials is obtained by using the normal anisotropy to replace the planar anisotropy value \( R_0 \) and \( R_{90} \) in Eqs. (3.33).

Isotropic Materials:

For such materials, the yield criterion defined by Eq. (3.32) returns to Tresca's by setting \( M=1, R=1 \), or Mises' yield criterion by setting \( M=2, R=1 \), that is

\[ M = 1 \text{ and } R = 1, \text{ so that } C = 1 \text{ for Tresca yield criterion.} \]

\[ M = 2 \text{ and } R = 1, \text{ so that } C = 2\sqrt{3} \text{ for Mises yield criterion.} \]

It should be noted that Eqs. (3.32) and (3.33) are referred to the case in which the bending axis is perpendicular to the rolling direction, i.e. the maximum stress \( \sigma_1 \), intermediate stress \( \sigma_2 \) and the minimum stress \( \sigma_3 \) are parallel to the rolling, transverse, and thickness directions, respectively. For bending axis parallel to the rolling direction, i.e.:
\( \sigma_1 = \sigma_2 \) along 90° from rolling direction
\( \sigma_2 = \sigma_1 \) along rolling direction
\( \sigma_3 = \sigma_3 \) along thickness direction
and
\( d \sigma_2 = 0 \) along rolling direction

If \( M = 2 \), the yield criterion (3-32) can be expressed as

\[
|\sigma_1 - \sigma_3| = C X = X \frac{1 + \frac{1}{R_0}}{\sqrt{\frac{1}{R_0} + \frac{1}{R_{90}} + \frac{1}{R_0 R_{90}}}}
\]  
(3.35)

3.2.4 Equilibrium Condition and Stresses in Bending

The force equilibrium along the thickness or radial direction can be written as (Fig. 3.4):

\[
\frac{d\sigma_3}{dr} - \frac{\sigma_1 - \sigma_3}{r} = 0
\]  
(3.36)

In order to solve for the stress, we must adopt the yield criterion and the hardening curve for flow stress. Using a Hollomon type of hardening law in uniaxial tension, the stress and strain relation along the longitudinal direction, or the first principal direction can be written as

\[
X = \sigma_\parallel = k e^p
\]  
(3.37)

After substitution of Eqs. (3.29a), (3.32), and (3.37) into Eq. (3.36), the integration of Eq. (3.36) gives the stress components as (Appendix E):

\[
\sigma_3 = p_0 - \frac{Ck}{n+1} [ (\ln \frac{R_0}{R_n})^{n+1} - (\ln \frac{r}{R_n})^{n+1} ] \quad R_n \leq r \leq R_0
\]  
(3.38a)

\[
\sigma_1 = \sigma_3 + CX = p_0 - \frac{Ck}{n+1} [ (\ln \frac{R_0}{R_n})^{n+1} - (\ln \frac{r}{R_n})^{n+1} ] + Ck (\ln \frac{r}{R_n})^n
\]  
(3.38b)

where \( p_0 \) is the pressure exerted on the outer surface as in bending with a back pad.
Fig. 3.4 Force equilibrium at a differential element
The stress distribution for the fibers below the neutral axis is:

\[ \sigma_3 = p_i - \frac{C_k}{n + 1} \left[ \ln \frac{R_i}{R_n} \right]^{n+1} - \ln \frac{R_i}{R_n} \]  
\[ R_n \leq r \leq R_o \]  
\[ (3.39a) \]

\[ \sigma_1 = \sigma_3 - C \sigma_x = p_i - \frac{C_k}{n + 1} \left[ \ln \frac{R_i}{R_n} \right]^{n+1} - \ln \frac{R_i}{R_n} \]  
\[ R_n \leq r \leq R_o \]  
\[ (3.39b) \]

where \( p_i \) is the tool (punch) pressure exerted on the inner surface of the sheet.

With the plane strain condition, the stress along the width for materials with planar anisotropy is found to be

\[ \sigma_2 = \frac{1}{1 + R_{90}^{M-1}} (R_{90} \sigma_1^{M-1} + \sigma_3) \]  
\[ (3.40) \]

### 3.2.5 Neutral Axis \( R_n \) and the Thickness Change

The stress must continue at the neutral axis. Therefore, equating the two stresses defined in Eqs. (3.38a) and (3.39a), we obtain the following relations among the neutral axis, the outer and inner radii, and the pressures:

\[ \left( \ln \frac{R_o}{R_n} \right)^{n+1} - \left( \ln \frac{R_i}{R_n} \right)^{n+1} = -\frac{P_i - P_o}{C_k} (n + 1) \]  
\[ (3.41) \]

or

\[ \left( \ln \frac{R_o}{R_n} \right)^{n+1} - \left( \ln \frac{R_i}{R_i} \right)^{n+1} = -\frac{P_i - P_o}{C_k} (n + 1) \]  
\[ (3.42) \]

There are two unknowns, the neutral axis radius, \( R_n \), and the outer surface radius, \( R_o \), which is a function of the current sheet thickness (\( R_o = R_i + t \)). We need another independent equation so that these two unknown can be determined. The additional relation can be found with the consideration of the volume constancy (Appendix F):

\[ R_n = R_m \frac{t}{t_o} = \frac{R_i + R_o}{2} \frac{t}{t_o} \]  
\[ (3.43) \]
For rigid ideal-plastic materials \( (n = 0, k = \sigma_0) \), an estimation of the position of the neutral axis is found to be

\[
R_n = \sqrt{R_i R_o} \sqrt{\exp\left(-\frac{p_i - p_o}{C \sigma_0}\right)}
\]  

(3.44)

Therefore, the position of the neutral axis is affected by the material properties, the exerted tool pressures, and the bending radii. The particular cases are:

(i) The neutral axis is a geometric average of the radii of the inner and the outer surfaces, if the pressures are equal or both zero, i.e.,

\[
R_n = \sqrt{R_i R_o} \quad p_o = p_i \neq 0, \quad \text{or} \quad p_o = p_i = 0
\]  

(3.45)

and for such a case, the material behavior does not affect the neutral axis.

(ii) The neutral axis moves towards the inner surface, if the pressure on the outer surface does not exists, but the pressure on the inner surface (contacting with punch) is present, i.e.,

\[
R_n = \sqrt{R_i R_o} \sqrt{\exp\left(-\frac{p_i}{C \sigma_0}\right)} < \sqrt{R_i R_o} \quad p_i \neq 0, \quad \text{and} \quad p_o = 0
\]  

(3.46)

(iii) The neutral axis shifts towards the outer surface, if the pressure on the inner surface does not exists, but the pressure on the outer surface presents, i.e.,

\[
R_n = \sqrt{R_i R_o} \sqrt{\exp\left(-\frac{p_o}{C \sigma_0}\right)} > \sqrt{R_i R_o} \quad p_o \neq 0, \quad \text{and} \quad p_i = 0
\]  

(3.47)

In the later two cases, the material properties will affect the neutral axis as well as bending radii and pressures.

Combining Eqs. (3.42) with (3.45), the instantaneous thickness is found to be

\[
t = \frac{R_n}{R_m} t_o = 2 \frac{\sqrt{R_i R_o}}{R_i + R_o} t_o
\]  

(3.48)
That is, the sheet thins during pure bending even for non-work-hardening materials, because the neutral axis is shifted away from the middle layer. The thinning effect described by Eq. (3.47) will not be significant for fairly large bend ratios, $R_i/t_o > 5$, but the thinning increases as the bend ratio decreases (6% for a ratio of 1).

3.2.6 Internal Bending Moment

The bending moment per unit width at any section of the sheet is defined as

$$M = \int_{R_i}^{R_n} \sigma_x r \, dr = \int_{R_i}^{R_n} \sigma_1 r \, dr + \int_{R_i}^{R_n} \sigma_{1} r \, dr$$  \hspace{1cm} (3.49)

Substituting the stresses defined in Eqs. (3.38b) and (3.39b) into the above equation, the bending moment is derived as (Appendix G):

$$M = \left( \rho_i - \frac{C}{n+1} \right) \frac{R_i^2}{2} \left( 1 - \varepsilon_i^2 \right) + Ck \frac{R_i^2}{2} \sum_{j=0}^{\infty} \frac{2^j}{(n+j+1) \, j!} \left[ \varepsilon_i^{n+1+j} - |\varepsilon_i|^{n+1+j} \right]$$

$$+ \left( \rho_o - \frac{C}{n+1} \right) \varepsilon_o \frac{R_i^2}{2} \left( e^{2 \varepsilon_o} - 1 \right) + Ck \frac{R_i^2}{n+1} \sum_{j=0}^{\infty} \frac{2^j}{(n+2+j) \, j!} \left[ \varepsilon_o^{n+2+j} - |\varepsilon_o|^{n+2+j} \right]$$

$$\hspace{1cm} (3.50)$$

Another form of the moment related to the bending radii can be shown as

$$M = \frac{1}{2} \left[ \rho_i - \frac{C}{n+1} \left( \frac{R_i}{R_n} \right)^{n+1} \right] \left( R_i^2 - R_n^2 \right) + \frac{1}{2} \left[ \rho_o - \frac{C}{n+1} \left( \frac{R_o}{R_n} \right)^{n+1} \right] \left( R_o^2 - R_n^2 \right)$$

$$+ Ck \frac{R_i^2}{n+1} \sum_{j=0}^{\infty} \frac{2^j}{(n+j+1) \, j!} \left( \frac{R_o}{R_n} \right)^{n+1+j} \left[ \ln \frac{R_i}{R_n} \right]^{n+1+j}$$

$$+ \frac{Ck}{n+1} \frac{R_i^2}{R_n^2} \sum_{j=0}^{\infty} \frac{2^j}{(n+2+j) \, j!} \left( \frac{R_o}{R_n} \right)^{n+2+j} \left[ \ln \frac{R_i}{R_n} \right]^{n+2+j}$$

$$\hspace{1cm} (3.51)$$
The strain terms with orders higher than five may be safely neglected for accuracy higher than 99%, because the strain is generally less than 40%.

Therefore, the bending moment is a function of anisotropy index, C, the strain hardening exponent, n, the strength coefficient, k, the bending radius, R_i, and the sheet thickness, t (R_o = R_i + t). To calculate the moment, the instantaneous thickness and the neutral axis should be known beforehand. These variables can be determined in a manner as described in the previous section. The measurement or calculation of the instantaneous thickness is not a simple problem. With Proksa's moment equation, Martin and Tsang calculated the bending moments by using the initial thickness and the true thickness at fracture, and they found that the moment difference is less than 3% for a span of 5 to 7 inches, and 6% for a span of 3 inches [Martin and Tsang, 1966]. Therefore, if the true thickness is not available, it may be justified to use the initial thickness to calculate the outer radius, R_o = R_i + t_o.

3.2.7 Springback and Residual Stresses

Springback and residual stresses can be calculated in the same manner as that used in the elementary bending theory, i.e. the unloading moment has the equal magnitude but opposite sign of the bending moment:

\[
\frac{1}{R_n} - \frac{1}{R_n} = \frac{M}{EI} = \frac{12(1-v^2)}{E I^3} M
\]

(3.52)

where the bending moment is defined in Eq. (3.51). The springback angle at each section of a sheet is

\[
\Delta \theta = (\frac{1}{R_n} - \frac{1}{R_n}) \Delta S
\]

(3.53)

The total springback of a sheet is an integration of the above equation over the entire span of the sheet.
The residual stress along the fiber length direction is found by using Eqs. (3.38b) and (3.39b) as the loading stress in Eq. (3.18). Then

\[ \sigma_{x, \text{residual}} = \sigma_1 - \left( \frac{1}{R_n} - \frac{1}{R_n'} \right) y = \sigma_1 - \frac{12}{t^3} M y = \sigma_1 - \frac{\pm 12}{t^3} M (r - R_n) \]

(3.54)

The residual stresses below and above the neutral axis can be computed using Eqs. (3.38b) and (3.39b).

### 3.3 MECHANICS OF BENDING UNDER TENSION

#### 3.3.1 Overview

Bending under tension (BUT), Fig. 3.5, is a useful technique used to reduce the springback. This deformation mode also exists in stretch forming and deep drawing operations in which the sheet wraps around the radii of the shoulders of die and punch. When the drawbead is used in stretch/draw forming, the metal flow through the bead is also characterized by this deformation mode. For the BUT process, the effects of the applied axial tensile force on the deformation are (a) to enlarge the plastic yielding zone through the sheet thickness, and hence, to reduce the elastic core height and to reduce the springback significantly, and (b) to cause the neutral axis to shift towards the inner concave surface (the neutral axis may be totally out of the sheet thickness) which results in a great increase of the tensile strain and stress. Therefore, the fracture tendency is also increased. An accurate prediction of the axial tension is essential for the careful control of both springback and fracture.
Fig. 3.5 Bending under axial tension
3.3.2. Strain and Stress Distributions

First, we consider the secondary plastic bending (that is the elasto-plastic bending), which is applied to sheet bending practice, Fig. 3.6. Then, the solutions for the secondary plastic bending can be extended to cover the fully elastic loading and the primary plastic loading cases simply by setting the appropriate values of the characteristic strains at the outer and the inner fibers.

At the elastic core, the strain is given by Eq. (3.1b), i.e.:

\[ \varepsilon_x = \frac{y}{R_n} \]  \hspace{1cm} (3.55a)

The strain in plastic region is defined by Eq. (2.1a), i.e.:

\[ \varepsilon_x = \ln \frac{R}{R_n} = \ln \left(1 + \frac{y}{R_n}\right) \]  \hspace{1cm} (3.55b)

where \( R_n \) is the radius of the neutral axis, and \( y \) is the distance from the neutral axis.

Using Hook's law, Eq. (3.12), for elastic deformation, and Swift's hardening law, Eq. (3.8), for plastic deformation, the longitudinal stresses are still defined by Eqs. (3.12) and (3.13), that is

\[ \sigma_x = \frac{E \varepsilon_x}{(1-\nu^2)} \quad 0 \leq \varepsilon_x \leq \varepsilon_{e,o} \quad \text{Elastic deformation.} \]  \hspace{1cm} (3.56a)

and

\[ \sigma_x = k F^{n+1} \left( \frac{\varepsilon_o - \varepsilon_{e,o}}{F} + \varepsilon_x \right)^n = k F^{n+1} \left[ \frac{\varepsilon_o - \varepsilon_{e,o}}{F} + \ln \left(1 \pm \frac{y}{R_n}\right) \right]^n \]  \hspace{1cm} (3.56b)

\[ \varepsilon_{e,o} < \varepsilon_x \quad \text{Plastic deformation.} \]

where \( \varepsilon_{e,o} \) is the yield strain, and \( F \) is a correlation index to consider the influences of material normal anisotropy and the strain/stress state in the flow stress. The expression of \( F \) is defined by Eq. (3.11).
Fig. 3.6 Strain and stress distributions for sheet bending under axial tension.
For sheet bending, only the elastic-plastic deformation in the secondary plastic bending needs to be considered. The characteristic strains, i.e. the yield strain, $\varepsilon_{e,o}$, the maximum strain at the outer convex surface, $\varepsilon_{\text{max}}$, and the strain at the inner concave surface, $\varepsilon_{i}$, are found to be:

$$\varepsilon_{e,o} = \frac{\sigma_o}{E'} = \frac{c}{R_n} \quad (3.57a)$$

$$\varepsilon_i = \ln\left(1 - \frac{t/2 - d}{R_a}\right) \quad (3.57b)$$

$$\varepsilon_{\text{max}} = \ln\left(1 + \frac{t/2 + d}{R_n}\right) \quad (3.57c)$$

where $\sigma_o$ is the initial yield stress, and $d$ is the shift of the neutral axis measured from the middle axis. The radius of the neutral axis is found as

$$R_n = R_i + t/2 - d \quad (3.58)$$

For a fully elastic loading, we only need to set $\varepsilon_{\text{max}} = \varepsilon_o$ and $\varepsilon_i < \varepsilon_o$, and $\varepsilon_i$ is defined by Eq. (3.57b). In the primary plastic loading case, it is common to set $\varepsilon_i = \varepsilon_o$ and $\varepsilon_{\text{max}} > \varepsilon_o$, where $\varepsilon_{\text{max}}$ is still defined by Eq. (3.57c).

### 3.3.3 Axial Force, $N$, and Bending Moment, $M$

The axial force per unit width, $N$, and the moment per unit width, $M$, are defined as:

$$N = \int_{-2}^{-2} \sigma_x \, dy \quad (3.59)$$

and

$$M = \int_{-2}^{-2} \sigma_x \, y \, dy \quad (3.60)$$
After integration (Appendix H), the axial force and bending moment are finally determined as:

\[
N = kF^{n+1}R_n e^{\frac{e_{n0}-e_0}{F}} \sum_{j=0}^{m} \frac{1}{(j+n+1)!} \left[ (e_{\text{max}} + \frac{e_0-e_{\text{c},0}}{F})^{n+1} + (e_i + \frac{e_0-e_{\text{c},0}}{F})^{n+1} \right]
\]

(3.61)

and the bending moment is

\[
M = M_c + M_p = \frac{2}{3} E' R^2 e^{2e_{\text{c},0}} + kF^{n+1}R_n e^{2e_{n0}-e_0} \sum_{j=0}^{m} (2^j - e^{j}) \frac{e_{\text{max}}-e_{\text{c},0}}{F} \left[ (e_i + \frac{e_0-e_{\text{c},0}}{F})^{n+1} + (e_{\text{max}} + \frac{e_0-e_{\text{c},0}}{F})^{n+1} \right] - 2(e_{\text{c},0} + \frac{e_0-e_{\text{c},0}}{F})^{n+1}
\]

(3.62)

where all of the characteristic strains are defined in Eq. (3.57), and they are the function of the shifting distance of the neutral axis, \(d\), the bending radius, \(R_i\), and the current sheet thickness, \(t\). Therefore, once the axial force \(N\) is given, then the distance \(d\) can be determined via Eq. (3.61). Then the characteristic strains in Eq. (3.57), from which the bending moment in Eq. (3.62) is related, can be defined. In general, the initial yield strain is around 0.001 to 0.005; therefore, the term \(e_0^3\), may be neglected.

3.3.4 Springback and Residual Stress

Unlike springback and residual stress in pure moment bending, the elastic recovery in bending under tension includes two parts: the elastic recovery due to the removal of the bending moment, and the recovery due to the removal of the axial load. Substituting the moment defined in Eq. (3.62) into Eq. (3.27), the springback is found to be:
\[
\frac{1}{R_n} - \frac{1}{R'_n} = \frac{M}{E' I} = \frac{12M}{E' t^3}
\]  

(3.63)

Note that the influence of the axial force in springback can be found by the characteristic strains which depend on the inward shift distance, \(d\), of the neutral axis away from the middle layer of sheet. This distance increases with the axial force.

The elastic strain due to unloading of the bending moment is still defined as:

\[
\sigma_{x,\text{unloading}} \ = \ E' \left( \frac{1}{R_n} - \frac{1}{R'_n} \right) y
\]  

(3.64)

The recovery stress due to removal of the axial force, \(N\), may be defined as

\[
\sigma_{x,\text{unloading}} N \ = \ \lambda \frac{N}{t}
\]  

(3.65)

where \(\lambda\) is a fraction factor (\(\leq 1\)) measuring the contribution of the stress caused by tension \(N\) to the total average stress \((N/t)\), and \(\lambda\) may be chosen as the unity for a tension \(N\) in the elastic region and zero for a tension \(N\) beyond plastic yielding. Therefore, the residual stress is expressed as:

\[
\sigma_{x,\text{residual}} \ = \ \sigma_{x,\text{loading}} - \sigma_{x,\text{unloading}} M - \sigma_{x,\text{unloading}} N
\]  

(3.66)

Substituting Eqs.(3.64) and (3.65) into Eq.(3.66), we obtain the expression of the residual stress:

\[
\sigma_{x,\text{residual}} \ = \ k \ F^{n+1} \left[ \ln \left( 1 + \frac{y}{R_n} \right) \right]^n - \frac{12M}{t^3} y - \lambda \frac{N}{t}
\]  

(3.67)

where the axial force, \(N\), and the bending moment, \(M\), are given by Eqs.(3.61) and (3.62).
3.3.5 Changes of Neutral Axis and Thickness

During bending under tension, the sheet thickness is reduced. In the calculation of the bending moment and axial force, we need to use the current or true thickness. If the origins of the radii of the neutral axis and the middle layer are assumed to be the same, then the current thickness is expressed as a function of the shift distance, \( d \), of the neutral axis away from the middle layer.

**Thinning of Sheet**

If the origins of the radii of the neutral axis and the middle layer are assumed to be the same, then the strain at the middle fiber can be simply determined through the condition that the angles corresponding to the neutral fiber lengths, \( L_n \), and the middle fiber length must be same, i.e.:

\[
\theta = \frac{L_n}{R_n} = \frac{L_m}{R_m}
\]

Hence, the fiber length at the middle layer is found to be:

\[
L_m = \frac{R_m}{R_n} L_n = L_n \frac{R_m}{R_m - d} \tag{3-69}
\]

The thickness change can be obtained from the volume constancy, i.e.:

\[
L_n t_o W_o = L_m t W_o \tag{3-70}
\]

Using the relation defined in Eq. (3-69), the current thickness is expressed as a function of the shift distance of the neutral axis away from the middle layer:

\[
t = (1 - \frac{d}{R_m}) t_o \tag{3-71}
\]

Therefore, the thickness decreases with the distance \( d \), or the increase of the axial force \( N \).
Neutral Axis Shift Distance (d)

The shift distance of the neutral axis can be determined with Eq. (3.61). If only one term (j=0) in the series of Eq. (3.61) is used, which corresponds to a linear strain distribution along thickness, i.e. $\varepsilon_x = y / R_o$, then the equation to define the shift distance is given as:

$$N = kF_0^e R_o e^{\frac{e_x - e_\infty}{F}} \frac{1}{n+1} \left[ (\varepsilon_{max} + \frac{\varepsilon_o - \varepsilon_{eoa}}{F})^n - (\varepsilon_i + \frac{\varepsilon_o - \varepsilon_{eoa}}{F})^n \right]$$

$$= kF_0^e R_o e^{\frac{e_x - e_\infty}{F}} \frac{1}{n+1} \left[ (\frac{t/2+d}{R_o} + a)^n - (\frac{t/2-d}{R_o} + a)^n \right]$$

$$= kF_0^e e^{\frac{e_x - e_\infty}{F}} (R_o - t/2) \frac{1}{n+1} \left[ (\frac{t/2+d}{R_o - t/2} + a)^n - (\frac{t/2-d}{R_o - t/2} + a)^n \right]$$

where $a = (e_o - e_{eoa}) / F$. This equation can be solved numerically for a given axial tension $N$. If $N$ is small, then the neutral axis stays at the position of the mid-axis, that is the shift distance $d = 0$ from Eq. (3-72).

### 3.3.6 Solving Procedure

As a summary, the solution process for plane strain sheet bending under axial tension is outlined as follows:

(a) Determine the maximum shifting distance for the neutral axis, $d_{max}$, by setting the maximum surface strain equal to a fraction ($f$ where $f \leq 1$) of the maximum uniform elongation (which equals the strain hardening exponent, n) so that no necking instability occurs:

$$\varepsilon_{max} = \ln \left[ 1 + \frac{t/2+d}{R_i + t/2-d} \right] = f \varepsilon_u = f n$$

Therefore

$$d_{max} = (1 - e^{-fn})R_i - (2e^{-fn} - 1)\frac{t}{2}$$ (B)
(b) Calculate the characteristic strains, $\varepsilon_1$ and $\varepsilon_0$ and from Eqs. (3.57);

(c) Determine the maximum axial force, $N_{\text{max}}$, needed for plastic yielding, from Eq. (3.61);

(d) Evaluate the bending moment $M$ via Eq. (3.62).

(e) Calculate the springback through Eq. (3.63).

(f) Evaluate the residual stress using Eq. (3.64).
CHAPTER IV

ANALYSIS OF BENDING PROCESSES

4.1 OVERVIEW

The elementary bending theory for pure moment bending has been applied in analysis of pressbrake bending operations such as air bending, V-die bending, U-die or channel die bending. In press brake bending, the sheet is placed over an open die, and it is deformed by a moving punch that contacts the sheet at its mid-span. The punch is actuated by the ram of the machine called a press brake. This process is most often used in sheet bending. In this chapter, air bending or "three-point" bending will be first analyzed in detail. Then the formulations are extended to analyze the conventional die bending processes with wiping-die and U-die, and to analyze the new bending processes such as rotary bending and tractrix-die bending.

4.2 ANALYSIS OF AIR BENDING

The air bending process considered here is a so-called 'three-point bending' in which the sheet is supported by the two shoulders of a stationary die and bent by a moving punch which contacts the sheet at its mid-span (Fig. 4.1). The 'three-point' bending model covers the mechanics for many bending operations such as U-die, V-die, and tractrix-die bending, and it can also be modified to 'two-point' bending such as wipe-die and rotary bending operations. In air bending, the die is designed to be deep enough so that the
Fig. 4.1 Brake bending
overbending by the punch stroke is allowed and bottoming or striking does not take place. The required bending angle is adjusted by the die opening width and the punch displacement. The major advantages of the air bending are (a) the variety of angles that can be obtained with a minimum number of punches and dies, (b) the small capacity (force) requirement, and (c) the ease of control and compensation for springback. Raghupathi et al [1983] analyzed this bending process based on (a) isotropic material behavior, (b) linear strain hardening, c) uniaxial stress state, and (d) ideal lubrication (frictionless). In the present study, the normal anisotropy of material, the nonlinear strain hardening under plane strain conditions, and friction are implemented in the current bending model. This model includes (i) the predictions of springback and bending angle under load, (ii) the punch displacement necessary for overbending to compensate for springback, and (iii) the influences of material properties, sheet thickness, and geometry of the punch and die, in the bending deformation and punch load. With some modifications, this model is also applicable to bending operations using a wiping-die, and U-die, provided the bottoming action is prevented. Bottoming or coining is employed to reduce the springback in bending. During the bottoming operation, the punch force increases rapidly (up to 3-5 times of the bending force), and it is difficult to control the punch stroke. Therefore it is limited to the thin gage sheets. For the on-line control of the bending process, the control of the punch stroke to reduce springback is found to be more flexible and suitable than the bottoming operation [Yang and Shima, 1990]. Therefore, the proposed model can serve as the process model in the control of a bending press.

4.2.1 Assumptions

Four assumptions adopted in air bending analysis are:

(i) The thinning of sheet in bending is negligible;

(ii) The pure moment bending exists for every section along the sheet length so that the bending of the entire sheet can be modeled, section-by-section, using elementary
bending theory;

(iii) The external bending moment (produced by the punch and balanced by the internal bending moment of the sheet) is linearly distributed from its maximum value underneath the punch tip to zero at the die shoulders, Fig. 4.2; and

(iv) Three deformation modes exist in the span of sheet as shown in Fig. 4.2:

(a) Fully plastic bending in the punch-sheet contact region,
(b) Elasto-plastic bending in a region of $0 \leq S \leq S_e$, and
(c) Fully elastic bending in a region of $S_e \leq S \leq S_1$.

4.2.2 Bending Moments and Distribution

The critical value for the bending moment at which the fully elastic bending is changed to elasto-plastic bending can be calculated via Eq. (B-4) by setting the elastic core height, $2y_e$, equal to the sheet thickness, $t$, i.e.

$$M_E = \frac{wt^2}{6} \frac{\sigma_y}{1 - \nu^2}$$  \hspace{1cm} (4.1)

Equation (4.1) illustrates that the fully elastic bending moment, $M_E$, is related to sheet thickness and width, $w$, the yield stress and the Poisson's ratio. The transition from elastic bending to elasto-plastic bending is defined at the distance $S = S_e$ at which the applied external bending moment reaches a certain magnitude of the internal bending moment to cause partial plastic yielding. The moment and plastic yielding increase with the span distance from the die shoulder, and the fully plastic bending is assumed to appear at the punch-sheet contact region in which the inner bend radius of the sheet is a minimum and takes the value of the punch radius, i.e. $R_i = R_p$. Therefore, the critical internal bending moment corresponding to fully plastic yielding should equal the external moment at the punch tip at $S = S_A = 0$, and the magnetite of this transition moment can be determined
Fig. 4.2 Geometric relations and coordinate system in air bending
using Eq. (3.26) by setting the elastic strain to be zero, or, the elastic core height, $2y_e = 0$, i.e.:

$$M_A = 2wkF^{n+1}R_i^2 e^{-2e/\mu} \sum_{j=0}^{\infty} \left( \frac{2j - e\phi/F}{j!} \right)^{j+1+n} \left[ (\varepsilon_{\text{max}} + \varepsilon_0/j+1 + n - (\varepsilon_0/j+1 + n)] \right)$$

(4.2a)

with

$$\varepsilon_{\text{max}} = \ln \frac{R_i}{R_a} = \ln \frac{R_i + t}{\sqrt{R_i(R_i + t)}} = \ln \frac{\sqrt{1+R_i/t}}{R_i/t} = \ln \frac{\sqrt{1+R_p/t}}{R_p/t}$$

And the bending moment for elasto-plastic bending is from Eq. (3.26):

$$M_{ep} = C_4R^2 + \frac{C_2}{R^n} + \frac{C_1}{R^{n+1}}$$

(4.2b)

with

$$C_1 = \frac{3}{4} n + 2 \quad C_2 = \frac{2wkF^{n+1}}{n+2} \quad C_4 = \frac{2}{3} \left( \frac{\sigma_0}{E_0} \right)^2 \sigma_0 w = \frac{2}{3} \varepsilon_{\text{e.o}}^2 \sigma_0 w$$

The linear moment distribution is described by Eq. (4.3):

$$M(S) = M_B \frac{S_1 - S}{S_1 - S_E}$$

(4.3a)

or

$$M(S) = M_A \frac{S_T - S}{S_1} = M_A \left( 1 - \frac{S}{S_1} \right)$$

(4.3b)

The coordinate system is shown in Fig. 4.2, and $S$ is the distance from the section of interest to the punch tip, $A$, which is the origin of the $S$-coordinate system. The bending moments at the three regions can be described as
\[ M(S) = M_A \quad \text{plastic bending in the punch-sheet contact region} \quad (4.4a) \]

\[ M(S) = M_A \frac{S_1 - S}{S_1} = M_{ep} \quad 0 \leq S \leq S_E \quad \text{(elasto-plastic bending)} \quad (4.4b) \]

\[ M(S) = M_E \frac{S_1 - S}{S_1 - S_E} \quad S_E \leq S \leq S_A \quad \text{(elastic bending)} \quad (4.4c) \]

The critical distance \( S_E \), at which partial plastic yielding starts can be determined through the moment continuity condition at location \( A \) (\( S = 0 \)):

\[ S_E = S_1 \left( 1 - \frac{M_E}{M_A} \right) = S_1 - \frac{1}{C_3} \quad \text{and} \quad C_3 = \frac{M_A}{M_E} \frac{1}{S_1} \quad (4.5) \]

From this relation, it is seen that the elastic bending portion \((S_1 - S_E)\) depends on (a) the tool shape and dimensions through \( S_1 \), (b) the elastic and plastic properties of the materials and sheet thickness, and (c) the bend radius which depends on the ratio of the elastic bending moment, \( M_E \), to the plastic bending moment, \( M_A \). The wider the die opening, \( C \), or the greater the gap, the larger the elastic portion of the sheet, hence the more severe the springback.

### 4.2.3 Curvature Distribution

The curvature distribution along the length of a sheet is a function of the bending moment. For sections with different bending deformation, the curvatures have to be solved based on the bending moment distributions defined in Eqs. (4.4, a,b,c). The curvature distributions for the punch-sheet contact region (fully plastic bending) are found to be (Appendix I):

\[ \frac{1}{R} = \frac{1}{R_p + t/2} = \frac{1}{R_p'} \quad (4.6a) \]

where \( R_p' = R_p + t/2 \) is the radius at the middle axis. Further for \( 0 \leq S \leq S_E \) (Elasto-plastic bending):
\[ S = S_1 - C_6 R^2 - C_7 \frac{R^n}{R^{n+1}} \]  

(4.6b)

and for \( S_E \leq S \leq S_1 \) (Elastic bending):

\[ \frac{1}{R} = \frac{M(S)}{E' I} = \frac{M_E}{E'I} \frac{S_1 - S}{S_1 - S_E} = \frac{1}{R_E} \frac{S_1 - S}{S_1 - S_E} = \frac{C_3}{R_E} (S_1 - S) = C_5 (S_1 - S) \]  

(4.6c)

where \( 1/R_E = M_E / (E' I) \) is the curvature at the transition point, \( E \). The constants in the above equations are defined as

\[
\begin{align*}
C_3 &= \frac{C_3}{R_E} = \frac{C_3 M_E}{E'I} \\
C_6 &= \frac{S_1}{M_A} C_4 \\
C_7 &= C_2 S_1 M_A \\
C_8 &= \frac{D'}{M_A} = \frac{3t}{4n+3} C_7 
\end{align*}
\]

The plane strain module \( E' \) is defined as

\[ E' = \frac{E}{1 - v^2} \]  

(4.7a)

and \( I \) is the moment of the area about the middle axis:

\[ I = \frac{w L^3}{12} \]  

(4.7b)

Equation (4.6b) must be solved using a numerical technique such as the Newton-Raphson iterative procedure. The deformed shape of the bent sheet can be plotted using the curvature distributions defined in Eqs. (4.6) and (4.7).

### 4.2.4 Bending Angle \( \theta_1 \) and Springback Angle \( \theta_S \)

The total rotation angle or bending angle is the sum of the individual rotation at each section. By definition, the rotation of incremental arc, \( dS \), can be expressed as

\[ d\theta = \frac{dS}{R} \]
and the springback at each section is

\[ \delta \theta_S = K_S \delta S = \frac{M(S)}{E'T} \delta S \]

Therefore, the total rotation \( \theta_1 \) is found to be (Appendix J):

\[ \theta_1 = \theta_c - 2C_s(R_E - R_p') + C_f\left[\left(\frac{1}{R_p}\right)^n - \left(\frac{1}{R_E}\right)^n\right] \]

\[ + C_s\left(\frac{1}{R_p}\right)^{n+1} - \left(\frac{1}{R_E}\right)^{n+1}\right] + \frac{C_s}{2} (S_1 - S_E)^2 \]

(4.8)

The contact angle, \( \theta_c \), will be determined later. The total springback is (Appendix J):

\[ \theta_S = \frac{1}{2} \frac{M_A}{E'T} S_1 = \frac{3}{2} \frac{K_F^{n+1}}{2 + n} \frac{1 - \nu^2}{E} \left(\frac{1}{R_p} - \frac{1}{R_e}\right) + C_f \left[\left(\frac{1}{R_p}\right)^n - \left(\frac{1}{R_E}\right)^n\right] \]

Thus, the springback is affected by material properties, sheet thickness, bending arc length, and tool shape and dimensions.

The bending angle after unloading, i.e. the desired bending angle, \( \theta_2 \), is the difference between the overbending angle under load, \( \theta_1 \), and the total springback angle:

\[ \theta_1 - \theta_2 = \theta_S = \frac{M_A}{2E'T} S_1 = \frac{C_s}{2} S_1^2 \]

(4.10a)

\[ \theta_2 = \theta_c - 2C_s(R_E - R_p') + C_f\left[\left(\frac{1}{R_p}\right)^n - \left(\frac{1}{R_E}\right)^n\right] \]

\[ + C_s\left(\frac{1}{R_p}\right)^{n+1} - \left(\frac{1}{R_E}\right)^{n+1}\right] - \frac{C_s}{2} (S_1 - S_E)^2 - \theta_s \]

(4.10b)

Therefore, the punch/sheet contact angle can be defined now as:

\[ \theta_c = \theta_1 + 2C_s(R_E - R_p') - C_f\left[\left(\frac{1}{R_p}\right)^n - \left(\frac{1}{R_E}\right)^n\right] \]

\[ - C_s\left(\frac{1}{R_p}\right)^{n+1} - \left(\frac{1}{R_E}\right)^{n+1}\right] - \frac{C_s}{2} (S_1 - S_E)^2 \]

(4.11)

The span of the bending arm, \( S_1 \), is expressed as the kinematic relation from Fig. 4.2
(Appendix J), and can be numerically integrated:

\[
S_1 = \int_{\theta_c}^{\theta_1} \frac{dX}{\cos \theta} \quad 0 \leq X \leq X(\theta_1) \tag{4.12a}
\]

and

\[
X(\theta_1) = L_d - (R_p + t/2) \cos \theta - (R_d + t/2) \cos \theta_i \tag{4.12b}
\]

where the half length of the die, \( L_d \), is defined by the clearance (die gap), the radii of the punch and the die:

\[
L_d = C + R_p + R_d \tag{4.13}
\]

### 4.2.5 Punch Displacement (d)

Referring to Fig. 4.2, the punch stroke, d, corresponding to the bending angle under load, \( \theta_1 \), can be expressed as

\[
d = \int_0^{\theta_1} \sin \theta \, dS + [R_d - (R_d + t/2) \cos \theta_i] + [R_p - (R_p + t/2) \cos \theta_i] + t \tag{4.14}
\]

where the integration term in Eq. (4.14) is the vertical measure of the span arm, \( S_1 \), and the angle, \( \theta \), is the swiping angle of arc length, S, which varies from the zero to the span, \( S_1 \). The derivation for the expression of this integration term is presented in Appendix K.

### 4.2.6. Punch Force and Die Friction Effect

The maximum external bending moment generally depends on the type of bending and the bending tool geometry. Consider the air bending as shown in Fig. 4.3, Friction around the punch is neglected since the relative sliding in the punch/sheet contact regions is negligible. Only the friction around the die shoulders is considered and discussed here.

Introducing the geometry used by Altan [Altan, 1962], the geometric relations for beam bending are found from Fig. 4.3 as
Fig. 4.3 Bending with varying span
\[
\begin{align*}
a &= c \sin \beta = (L_d - R_d \sin \theta) (\sin \theta - \cos \theta \tan \alpha) \\
b &= c \cos \beta = (L_d - R_d \sin \theta) (\cos \theta + \sin \theta \tan \alpha)
\end{align*}
\]

where \( R_d \) is the radius of the die shoulder, \( L_d \) is the half die length or \( 2L_d \) is the original length between the two contact points of the flat sheet, \( L'_d \) is the reduced span due to punch travel, \( d, d' \) is the midspan deflection of the sheet referred to the reduced span, and \( \theta \) is the bending angle or the deflection angle at midspan.

The force balance for the beam gives the normal force, \( N \), and the friction (tangential) force, \( F \), at the contact points around the die shoulders as shown in Fig. 4.3:

\[
\begin{align*}
N &= \frac{P}{2 \cos \theta (1 + \mu \tan \theta)} \\
F &= \mu N = \frac{P \mu}{2 \cos \theta (1 + \mu \tan \theta)}
\end{align*}
\]

The moment balance at point A in the midspan is

\[ M = N b + F a \]

Using the geometric relations defined and the forces in Eq. (4.15), the external moment is found to be:

\[
M_{\text{external}} = \frac{PL_d}{2} \left[ 1 - \frac{R_d \sin \theta}{L_d} \right] + \frac{PL_d}{2} \frac{\tan \theta - \mu}{1 + \mu \tan \theta} \left[ \frac{d}{L_d} - \frac{R_d}{L_d} (1 - \cos \theta) \right]
\]

\[ = M_{\text{bending}} + M_{\text{friction}} \]

Therefore, the maximum external bending moment at the midspan consists of two parts: the bending moment, \( M_{\text{bending}} \), due to the punch force, \( P \), and punch displacement, \( d \), and the additional work, \( M_{\text{friction}} \), to overcome the friction around the die shoulders. \( M_{\text{bending}} \) decreases with the punch travel because the bending angle increases and the span decreases. The friction loss will be higher without lubrication, and this loss increases as
The bending angle increases or the punch travels increases. As indicated in the Eq. (4.16), a large ratio of the die radius, $R_d$, to the half die length (span) $L_d$, will reduce the total moment by reducing both $M_{bending}$ and $M_{friction}$.

The punch force can be calculated by equating the internal bending moment, $M$, with the external moment expression, Eq. (4.16), i.e.:

$$P = \frac{2M}{L_d \left\{ \left[ 1 - \frac{R_d}{L_d} \sin \theta \right] + \frac{\tan \theta - \mu}{1 + \mu \tan \theta} \left[ \frac{d}{L_d} - \frac{R_d}{L_d} (1 - \cos \theta) \right] \right\}}$$ (4.17a)

The moment increases with punch stroke and reaches the maximum value, i.e. the fully plastic bending moment, $M_A$, at punch tip. For this bending moment, the punch force is:

$$P = \frac{K F^{n+1}}{2+n} \frac{w t^2 (\frac{1}{2})^n (1 + \frac{3 n+2}{2R_p} - \frac{1}{2R_p})}{L_d \left\{ \left[ 1 - \frac{R_d}{L_d} \sin \theta \right] + \frac{\tan \theta - \mu}{1 + \mu \tan \theta} \left[ \frac{d}{L_d} - \frac{R_d}{L_d} (1 - \cos \theta) \right] \right\}}$$ (4.17b)

Hence, the proposed punch load is related to (a) sheet material properties (thickness $t$, strain hardening exponent $n$-value, strength coefficient $k$-value, and the plastic anisotropy index $F$-value), (b) the tool parameters (the radii of punch $R_p$ and die $R_d$, and the die opening $2L_d$ which is related to the sheet thickness, die gap, die radius and punch radius), and (c) the process parameters (the bending angle, the punch displacement $d$, and the friction coefficient).

### 4.2.7 Solving Procedures

Up to this stage, all necessary governing equations have been established, and the solution procedures are as follows:

(i) Determine an initial guess for the bending span, $S_1$, from Eq. (E.15) by assuming

$$\theta_C = \theta_1 = \theta_2$$
and
\[ S_1 = \frac{L_d}{\cos \theta_2} - (R_p + R_d + t) \tan \theta_2 \]

(ii) Evaluate the springback angle, \( \theta_s \), via Eq. (4.9),

(iii) Calculate the new value of \( \theta_1 \) from Eq. (4.10a), and the desired bending angle, \( \theta_2 \),

(iv) Compute the contact angle, \( \theta_c \) using Eq. (4.11),

(v) Repeat the calculation of the bending arc length, \( S_1 \), from Eq. (4.12), using \( \theta_c \) and \( \theta_1 \),

(vi) Repeat steps (ii) to (v) until the relative variation of the span length, \( S_1 \), between two adjacent \((j+1)-th\) and \(j-th\) iterations is within an acceptable error limit (say, 0.001):

\[
\left| \frac{S_1^{(j+1)} - S_1^{(j)}}{S_1^{(j+1)}} \right| \leq 0.001
\]

(vi) Integrate Eqs. (F-5) and (F-6) by Simpson's numerical procedure,

(vii) Calculate the punch displacement, \( d \), via Eq. (4.14) or Eq. (F-3). This punch stroke ensures the desired bending angle, \( \theta_2 \), by over bending with deeper stroke to overcome the springback.
4.3 ANALYSIS OF WIPING-DIE BENDING

In this bending operation, a pressure pad, either a spring loader or a hydraulic cylinder, is used to clamp the sheet to the die, before the punch contacts the workpiece. The punch wipes one side of the sheet as the punch travels, Fig. 4.4a. For simple hold-down action, the pad force is about one half the bending force or punch force [Eary, 1974]. Cracks often occur in bending with sharp radius because the sharp edge of the punch or die will shear the sheet instead of bending it. Overbending is usually used to compensate the springback. If the part angle is acute (≤90°) or the bending angle is greater than 90°, then the die has to be undersized and the clearance must be reduced to less than sheet thickness, Fig. 4.4b. While, if the bending angle is less than 90°, then overbending by deeper punch travel can be used, as in air bending case.

The analysis of the wiping-die bending is a close analogy to that of air bending. Due to the geometrical symmetry in air bending, only half of the system needs to be modeled, as shown in Fig. 4.2. The center section of the workpiece beneath the punch is fixed in the horizontal direction. In such a model, the air bending process can also be modeled by assuming that the punch is stationary with one end of the sheet clamped at the punch, and the die moves to bend the sheet. This is the same as the wiping-die bending. In other word, if we look at the wiping-die bending (Fig. 4.4a) in the up-side-down direction, switch the names of the punch and die, and compare it with the air bending model shown in Fig. 4.2, we will find a close analogy between them. The only difference between them is the clearance, C, which is much smaller in wiping-die (also in U-die) bending than that in air bending. Therefore, the formulation and solution procedure derived for air bending can be directly adopted to analyze wiping-die bending. For desired bending angles greater than 90°, the springback angle for a undersized die can be evaluated using Eq. (4.9). The punch force in wiping-die bending can be determined by Eq. (4.17) in air bending, with a factor of 0.5 based on the cantilever beam analysis.
Fig. 4.4 Wiping-die bending: (a) dimensions, and (b) oversized die to overcome springback.
4.4 ANALYSIS OF U-DIE BENDING WITH BACK PAD

In most cases of U-die bending processes, the backing pad or an ejector is often used to force the sheet web to keep contact with the punch bottom, Fig. 4.5. It was found that a backing pad of 30% of the bending force is required to keep the sheet in contact with punch face. Without the pad, three times the bending force is needed for the coining action [Lange, 1985]. The pad pressure and die gap are two important process variables in controlling springback. An increase in the backing pressure and decrease in the die gap reduces springback. At certain pressure levels, the springback is independent to the pressure [Liu, 1984].

U-die bending can be analyzed by the same manner as that used in wiping-die bending. For the simulation, only one half of the sheet needs to be considered because of geometrical symmetry, Fig. 4.6a. The center section laying at the center line of the punch is the neutral section which can not move horizontally. Therefore, the half of the U-die is just the same as the wiping-die, Fig. 4.6b. Based on the analogy among wiping-die, U-die, and air bending, the mathematical models developed in air bending can be used for analyzing U-die bending.
Fig. 4.5 U-die bending with back pad
Fig. 4.6 Analysis models for U-die bending: (a) normal view, and (b) inverted view (similar to wiping-die)
4.5 ANALYSIS OF ROTARY BENDING

Rotary bending [Ready Bender, 1992] is a bending process using a rocker to simultaneously hold, bend, and overbend the sheet (Fig. 4.7). The advantage of rotary bending comparing to wiping-die bending are (a) eliminating the blankholder, (b) compensating springback of the sheet by overbending, and (c) requiring less tonnage.

The strategies to analyze the rotary bending are (a) to define the relationships between the ram stroke and the rocker rotation, and between the rocker rotation and the sheet bending angle, (b) to determine the bending arm length $S_1$ in terms of the current position of rocker, (c) to calculate the maximum internal bending moment which appears at the anvil radius using Eq. (4.12a), and (d) to calculate the springback angle using Eq. (4.9) in air bending.

4.5.1 Rocker Rotation Angle

Refer to Fig. 4.7, the angles $\theta_a$ and $\theta_b$ are defined as the angles between the clamping (left) leg and the rocker center line, and between the bending (right) leg and the rocker center line, respectively, and they are subjected the design constrain by

$$\theta_R = \theta_a + \theta_b$$  

(4.18)

where $\theta_R$ is the included angle of the rocker. For a symmetrical arrangement, $\theta_a$ and $\theta_b$ are equal and take the half inclined angle $\theta_R$, i.e.

$$\theta_a = \theta_b = \theta_R / 2$$

(4.19)

At any instantaneous position of ram (or punch), the rotation angle of rocker, $\theta$, is the sum of the angle changes, $d\theta_a$ and $d\theta_b$ which are equal, that is

$$\theta = d\theta_a + d\theta_b = 2d\theta_a = 2d\theta_b$$

(4.20)
Fig. 4.7 Geometrical relations of rotary bending
When the bending leg moves from position b to position b', the new angle, \( \theta_b' \), between the rocker center and the bending leg due to the rotation is

\[
\theta_b' = \theta_b - d\theta_b = \theta_b - \theta / 2 \tag{4.21a}
\]

Similarly for the clamping leg which moves from position a to position a', the new angle, \( \theta_a' \), between the rocker center and the clamping leg due to the rotation is

\[
\theta_a' = \theta_a + d\theta_a = \theta_a + \theta / 2 \tag{4.21b}
\]

4.5.2. Bending Length

The bending length \( S_1 \) is defined as the sheet length between the contact point, e, at anvil (or die) radius and the contact point, b', at the left leg (Fig. 4.7). The bending lengths are derived for three possible cases in which the rocker rotation angle, \( \theta \), is less than (Fig. 4.7), or equal to (Fig. 4.8), or greater than 90° (Fig. 4.9). These correspond to a bending angle of workpiece, \( \theta_p \) (= 180° - \( \theta \)), greater than, or equal to, or less than 90° degree.

(a) \( \theta < 90^\circ \)

With the geometry shown in Fig. 4.7, the bending length \( S_1 \) can be calculated as follows. Length of arc along die radius:

\[
S_{el} = (R_d + t / 2)\theta \tag{4.22}
\]

Length of line fg:

\[
S_{fg} = [K - (R_d + t / 2)\sin\theta] / \cos\theta \tag{4.23}
\]

and \( K = (R_d + t) / \tan(\theta_a / 2) \tag{4.24} \)
Length of line gb':

\[ S_{gb'} = B' / \cos \theta = \overrightarrow{o'b'} \sin \theta_b / \cos \theta = \frac{C}{\cos(\theta_R / 2)} \frac{\sin \theta_b}{\cos \theta} \]  (4.25)

and \[ B' = o'b' \sin \theta_b \] and \[ o'b' = ob = C / \cos(\theta_R / 2) \]  (4.26)

The total bending length \( S_t \) is

\[ S_t = qS_{ef} + S_{tb} + S_{gb'} \]

or

\[ S_t = q(R_d + t / 2) \theta + [K - (R_d + t / 2) \sin \theta] / \cos \theta + \frac{C}{\cos(\theta_R / 2)} \frac{\sin \theta_b}{\cos \theta} \theta < 90^\circ \]  (4.27)

where \( q \) \((0 \leq q \leq 1)\) is a factor to consider partial elastic recovery of arc segment \( S_{ef} \) at the die radius. \( q = 1 \) for small bending angles or large \( R_d/t \) ratio of the die radius \( (R_d) \) over the sheet thickness \( t \) at which the elastic deformation is significant, and \( q = 0 \) for large bending angle or small \( R_d/t \) ratio at which plastic deformation dominates the elastic response.

(b) \( \theta = 90^\circ \)

Referring to Fig. 4.8, for bending angle equal to 90 degrees, the bending arc length is simply defined as

\[ S_t = q(R_d + t / 2) \frac{\pi}{2} + [\overrightarrow{o'b'} -(R_d + t / 2)] = (R_d + t / 2)(q \frac{\pi}{2} - 1) + \frac{C}{\cos(\theta_R / 2)} \]

\[ \theta = 90^\circ \]  (4.28)

(c) \( \theta > 90^\circ \)

Referring to Fig. 4.9, for bending angle greater than 90 degrees, the bending arc length is found as follows.
Fig. 4.8 Geometrical relations for bending angle = 90°.
Fig. 4.9 Geometrical relations for bending angle $> 90^\circ$. 
\[ S'_{ef} = q(R_d + t/2) \theta \]  
\[ S_{fb'} = \overline{b'} - S_{o_f} = \overline{b} - S_{o_f} \]

and

\[ S_{o_f} = \frac{((R_d + t/2)[(1 + \cos(180^\circ - \theta))] / |\cos \theta - 90^\circ|}{(1 + \cos(180^\circ - \theta) \}} \]

Therefore, the bending length is

\[ S_i = S_{ef} + S_{fb'} \]

or

\[ S_i = q(R_d + \frac{t}{2}) \theta + \frac{C}{\cos(\theta_R / 2)} + \frac{((R_d + t/2)[(1 + \cos(180^\circ - \theta))] / |\cos \theta - 90^\circ|}{(1 + \cos(180^\circ - \theta) \}} \]

where \( \theta > 90^\circ \)  

4.5.3. Springback Angle \( \theta_s \)

The springback angle is defined as the angular difference between the angle unload, \( \theta' \), and the angle under load, \( \theta \). The formula to calculate springback angle is the same as that defined in Eq. (4.9), i.e.

\[ \theta_s = |\theta' - \theta| = \frac{1}{2} \frac{M}{E' I} S_i \]  

where \( M \) is the maximum bending moment at the die (or anvil) radius, \( E' \) is the plane-strain modulus, and \( I \) is the inertia moment of area. These are defined in Eqs. (4.2) and (4.7).

4.5.4. Punch or Ram Displacement

The rotation of rocker is controlled by the ram stroke. The geometrical relationship between the ram displacement, \( d \), and the rotation angle of the rocker, \( \theta \), is found as (Fig. 4.7)

\[ d = C - H = C[1 - \frac{\cos(\theta_s + \theta / 2)}{\cos \theta_s}] \]
where
\[ H = \frac{\bar{a}}{\cos \theta_a} \cos (\theta'_a) = \frac{\bar{a}}{\cos \theta_a} \cos (\theta_a + \theta / 2) \]

In order to compensate the springback, a overbending is necessary. Overbending can be achieved by an additional rotation with amount of springback angle. In adapter control of pressbrake bending, a deeper stroke necessary for this overbending can be determined from Eq. (4.34) by using the over-rotation angle under load, \( \theta' \), i.e.
\[ \theta' = \theta + \theta_1 = \theta + \frac{M}{2 E_1} S_1 \]  (4.35)
and
\[ d = C - H = C[1 - \frac{\cos (\theta_a + \theta'/2)}{\cos \theta_a}] \]  (4.36)

4.5.5 Punch (Ram) Force

The total force (P) supplied by the press includes two parts: \( P_C \) to clamp the sheet and \( P_B \) to bend the sheet, as illustrated in Fig. 4.7. The energy conservation requires that the work supplied by the punch, \( W_P \), balances the bending energy \( W_B \) and friction dissipation energy \( W_F \), i.e.:
\[ W_P = W_B + W_F \]  (4.37)

The work done by the punch force P is
\[ W_P = P \cdot d \]  (4.37a)

The work needed for plastic deformation in bending of the sheet is the product of the bending moment and rotation angle, that is:
\[ W_M = \int_0^\theta \theta dM = \int_0^\theta \theta P_B ds = \theta P_B S_1 = M_{\max} \cdot \theta \]  (4.37b)

The work spent in friction losses due to sliding between the claming (left) leg and
sheet and between the bending (right) leg of the rocker and sheet is:

\[ W_F = \mu P_C \cdot \ddot{a} \cdot \dddot{a} + \mu P_b |L| \] (4.37c)

where \( L \) is the sliding distance of the bending leg of the rocker, and it is defined by the difference between the original length (unbent) \( eb \) and the current bending length \( S_i \), i.e.

\[ L = eb - Si = K + C \tan \frac{\theta_a}{2} - S_i \] (4.38a)

And the sliding distance of the clamping (left) leg of the rocker is

\[ \frac{\ddot{a} \cdot \dddot{a}}{\cos \theta_a} = \frac{C}{\cos \frac{\theta_a}{2}} \left( \frac{\sin \frac{\theta_a}{2} + \theta}{2} - \frac{\sin \frac{\theta_a}{2}}{2} \right) \] (4.38b)

Therefore, the energy balance gives

\[ P \cdot d = M_{\max} \theta + \mu P_C \ddot{a} \dddot{a} + \mu P_b |L| \] (4.39)

Considering the force equilibrium along vertical and horizontal directions, and the moment balance about the contact point (b') between the rocker and the sheet, the following equations are obtained:

\[ P = P_b \cos \theta + I \mu P_b \sin \theta + P_c \] (4.40a)

or

\[ P_c = P - P_b (\cos \theta + I \mu \sin \theta) \] (4.40b)

where

\[ I = -1 \text{ for } L < 0 \text{ or } S_i < K + C \tan \frac{\theta_a}{2}, \text{ that is, the bending leg of rocker slides outward.} \]

\[ I = 1 \text{ for } L > 0 \text{ or } S_i < K + C \tan \frac{\theta_a}{2}, \text{ that is, the right leg of rocker slides inward.} \]

Substituting Eqs. (4.38) and (4.40) into Eq. (4.39), the punch force is found to be:
The bending force $P_B$ can be expressed by the maximum bending moment and the bending span $S_1$ as

$$P_B = \frac{M_{\text{max}} \theta - \mu P_g a (\cos \theta + \mu \sin \theta) + \mu P_l L}{H - \mu a^2}$$

(4.41a)

The clamping force $P_c$ is then determined by substituting Eqs. (4.41b) and (4.42) into Eq. (3.40b).

4.5.6 Modeling of Special Rotary Bender (CB3)

There is another special rotary die which can overbend to 120 degrees. This bender has a nonsymmetric arrangement of the bending leg and the clamping leg (Fig. 4.11). The equations to calculate the loads (bending force, clamping force, and total force $P$) are the same as those in Eqs. (4.40b) - (4.42). Following the similar procedures as in the previous sections, the formulation to calculate the bending length ($S_1$) is obtained and
Fig. 4.10 A special overbender
summarized as follows.

\[ S_1 = q(R_d + \frac{t}{2})\theta + \overline{fb} \]  
\[ \overline{fb} = \overline{jb} \sin(90^\circ + \theta_b) / \sin \theta \]  
\[ \overline{jb} = \overline{b'm'} - \overline{m'h} = \overline{bm} - \overline{m'h} / \cos \theta_b \]  
\[ \overline{m'h} = \overline{a'm'} \cos \theta_a + (R_d + \frac{t}{2})[1 + \sin(\theta - 90^\circ)] \]

where the lengths of the bending leg (\( \overline{bm} = \overline{b'm'} \)) and the clamping leg (\( \overline{am} = \overline{a'm'} \)) can be determined from the specifications of rocker.

### 4.6 TRACTRIX DIE BENDING

Based on the proposed mathematical model for springback calculation (\( \theta_s = \frac{1}{2} \frac{M_{\text{max}}}{EI} S_1 \)), a constant springback angle, \( \theta_s \), for various bending angles can be achieved by keeping the bending arc length \( S_1 \) constant during the punch stroke via a proper design of die profiles. Such a die design suggested here is the one using the tractrix curve in which the length of any tangent between the curve (die profile) and a fixed line (punch center line) is a constant, Fig. 4.11. The technological significance of this optimal die design are that (a) the angular tolerance of the components for variety of bending angles can be controlled using a single value of the springback angle which could be controlled within the tolerance requirement by a careful selection of the dimensions of tractrix profile, and (b) if necessary, the overbending of the parts for a variety of bending angles are easy to achieve and maintain by overbending using a single value of the springback angle. Also, the sensitivity of the bending angle to variation in the punch stroke and press setup is greatly reduced. As a result, the time necessary to adjust the press stroke and for try-outs
Fig. 4.11 Tractrix Die Bending
in production should be reduced.

4.6.1 Geometrical Relations in Tractrix Die Profile

1. Mathematical Equation for Tractrix Curve

The mathematical description of the tractrix curve is given by

\[ y = a \ln \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2} \]  

(4.44)

where the x-y coordinate system is shown in Fig. 4.11.

2. Instantaneous Radius of the Tangent Point on Die Profile

The instantaneous radius of the tangent point on the die profile is a function of x and y coordinates at that point (Fig. 4.12), and it can be determined by the bending angle (under load) and the length of tangent which is approximately given by the initial die opening \( L_d \), that is

\[ R_d = L_d \tan \theta \]  

(4.45)

The die radius is needed in calculations of punch and die forces.

3. Bending Arc Length \( S_1 \)

The bending arc length is defined as the length between the two tangent points contacting with die and punch, and it can be numerically integrated as

\[ S_1 = \int_{\theta}^{\pi/2} \frac{x}{x \cos \theta} \, dx = \int_{\theta}^{\pi/2} \frac{x}{x \cos \theta} \, dx \]  

(4.46)

and the relative coordinates (refer to the tangent point, \( A_1 \), at punch) for these two tangent points are
Fig. 4.12 Geometrical relations in tractrix die
\[ x_A' = y_A' = 0 \]  

and

\[ x_B' = L_{d\theta} \cos \theta \]
\[ y_B' = L_{d\theta} \sin \theta \]  

4. Punch Displacement

The punch stroke measured from the origin of the global coordinate (x-y) system to the current position of punch tip is determined as

\[ d = y_B + y_B' + y_A \]
\[ d = \text{a} \ln \frac{a + \sqrt{a^2 - x_B^2}}{x_B} - \sqrt{a^2 - x_B^2} + L_{d\theta} \sin \theta + (R_p + t / 2)(1 - \cos \theta) \]  

4.6.2 Punch and Die Loads

In tractrix die bending, the bending arm is nearly constant (=L_d). As the bending moment (M) increases from zero at initial stage to the maximum at the punch tip, the normal force of die (N) increases in a similar manner. The punch and die forces were derived before for air bending. For tractrix die bending, the reduced span (horizontal) \( L_{d'} \) is defined as

\[ L_{d'} = x_B = L_{d\theta} \cos \theta + R_p \sin \theta \]  

Using this span, other geometrical relations for force calculation can be obtained with the same equations derived for air bending. Therefore, the punch and die forces can be determined by the same equations used in air bending, i.e. Eqs. (4.15) and (4.17).
4.7 ANALYSIS OF V-DIE AND CHANNEL-DIE BENDING

There are some differences between bending operations with V-dies and channel-dies and bending operations with U-dies, wiping-dies, and air bending. The major difference is the die contact. In air bending or U-die bending, the sheet wraps around the punch radius and there is only line contact along the sheet width between the sheet and die radius throughout the bending process. In wiping-die bending, the sheet wraps around the die radius and touches the punch radius with only a line contact. The bending arm increases and the punch load decreases as bending proceeds. In V-die or channel-die bending, the sheet not only wraps the punch radius, but also part of the sheet conforms to the die wall with area contact, and this part will take the die angle as the bending angle and its curvature becomes zero. The conformation process proceeds until the end of the bending process, Figs. 4.13 and 4.14. The bending span decreases and the punch force increases as the conformation increases during bending. At the coining stage, the punch force increases significantly. Besides these differences, the die angle or inclination becomes an important variable in V-die or channel-die bending (a U-die with inclination). In the following, the V-die bending will be modeled first, and the formulation is valid for channel-die bending because these two process are very similar.

4.7.1. Assumptions

The assumptions, in defining the strain, stress, and internal bending moment, are similar to those in the analysis of air bending, i.e.:

(i) the thinning of sheet is negligible in bending, hence the neutral axis coincides with the middle layer of sheet;
Fig. 4.13 Analysis model for V-die bending (half side)
Fig. 4.14 Analysis model for channel-die bending (half side)
(ii) the bending moment is linearly distributed from its maximum value underneath the punch tip to zero at the first contact point B, between the sheet and die wall, Fig. 4.13;

(iii) two deformation modes exist in the span of sheet as shown in Fig. 4.13:

(a) elasto-plastic bending in the punch contact region and unsupported region (span of $0 \leq S \leq S_B$)

(b) Rigid-body motion in the die contact region ($S_B \leq S$);

(iv) Sheet materials obeys Hill's transverse (normal) anisotropic yield criterion and Swift's strain hardening law.

4.7.2 Internal Bending Moment

The descriptions of bending strains/stresses and hardening behavior are the same as those discussed in Chapter III (see Eqs. (3.1) to (3.14). For any section of the sheet with an arbitrary radius of the neutral axis, $R_n$, the internal bending moment, $M_i$, can be calculated using Eq. (3.25), i.e.

$$M_i = M_e + M_p$$

(4-51a)

where the elastic part of bending moment, $M_e$, is

$$M_e = \frac{2wR_n^2}{3} \left(\frac{1-v^2}{E}\right)^2 \sigma_y^3$$

(4-51b)

And the plastic part of the bending moment, $M_p$, is

$$M_p = 2wkF \sum_{j=0}^{\infty} \left(\frac{\varepsilon_{e,o} - \varepsilon_{e,o}}{j! (j+1+n)} \right) [ (\varepsilon_{max} + \varepsilon_0 - \varepsilon_{e,o})^{j+1+n} - \varepsilon_0^{j+1+n} ]$$

(4-51c)

where $k$ is the strength coefficient, $n$ is the strain hardening exponent, $F$ is the anisotropy index defined in Eq. (3.11), and $\varepsilon_{e,o}$ and $\varepsilon_0$ are the elastic limit strain and prestrain, $w$ is
the sheet width, \( R_n \) is the radius of the neutral axis. For an accuracy of 99\%, only the first three terms in Eq. (4-51c) are required.

Equation (4-51) defines the moment-curvature relation or curvature-moment relation (curvature \( k = 1/R_n \)),

\[
k = k(M) \quad (4.52a)
\]
or

\[
M = M(k) \quad (4.52b)
\]

The internal moment balances the external bending moment applied by the tool. This moment varies along the span and can be assumed to be linear with the x-coordinate.

**4.7.3 Kinematics and Moment Distributions**

Here we introduce and modify Wang's kinematic relations [Wang, 1984]. Introducing a variable, \( \varphi \), which measures the inclination angle with respect to the x-axis, the shape of the bent sheet can be defined by the following differential equations:

\[
k = \frac{\partial \varphi}{\partial S} \quad (4.53a)
\]

\[
\cos \varphi = \frac{\partial X}{\partial S} \quad (4.53b)
\]

and

\[
\sin \varphi = \frac{\partial Y}{\partial S} \quad (4.53c)
\]

where the X-Y coordinate system is defined for the middle layer or the neutral axis. The geometrical relations for different regions of the sheet are described as follows.
(a) **Punch Contact Region** \((0 \leq S \leq S_A, \text{ or } 0 \leq X \leq X_A)\)

In this region, the sheet takes the shape of the punch, and the curvature of neutral axis is:

\[
k_o = 1/R_{np} \quad \text{and} \quad R_{np} = R_p + t/2
\]  

(4.54)

With an integration of Eqs.(4-53a,b,c), the inclination angle and \(X, Y\) coordinates at any section in this region are found to be:

\[
\varphi(S) = k_o S \quad \text{(4.55a)}
\]

\[
X = \sin \varphi / k_o = R_{np} \sin \varphi \quad \text{(4.55b)}
\]

\[
Y = \cos \varphi / k_o = R_{np} \cos \varphi \quad \text{(4.55c)}
\]

The inclination angle and the coordinates at separation point, \(A\), are designated as \(\varphi_A, X_A\) and \(Y_A\).

Because of the constant curvature, the bending moment is also a constant in this region from the moment-curvature relation, and can be shown to have the value \(M_A\), by setting \(R_n = R_{np}\) in Eqs. (4.51b, c).

(b) **Unsupported Region** \((S_A \leq S \leq S_B, \text{ or } X_A \leq X \leq X_B)\)

The bending moment is assumed to be a linear function of the span or the \(X\)-coordinate:

\[
M(X) = M_A \frac{X_B - X}{X_B - X_A}
\]  

(4.56)

By relating Eq. (4.56) and Eq. (4.51) and using Eqs. (4.52) and (4.53a), the following relations can be found:
\[
\cos \phi \ d\phi = k[M(X)] \ dX
\]

The integration within the region of \( X_A \leq X \leq X_B \), gives the relation between the inclination angle and \( X \) coordinate at any section:

\[
\phi(X) = \sin^{-1} \{ \sin \phi_A + \int_{X_A}^{X} k[M(X)] \ dX \} \tag{4.57}
\]

The arc length, \( S \), and \( Y \)-coordinate at any section in this region are defined by the integration of Eqs.(4.55b) and (4.55c):

\[
S(X) = S_A + \int_{X_A}^{X} \sec \phi(X) \ dX \tag{4.58a}
\]

\[
Y = Y_A + \int_{X_A}^{X} \tan \phi(X) \ dX \tag{4.58b}
\]

(c) **Die Contact Region (\( S \geq S_B \) or \( X \geq X_B \))**

The curvatures and bending moments are all nil because the sheet conforms to the die wall. The inclination angle of the sheet takes the value of the die angle, \( \alpha \), i.e. \( \phi_B = \alpha \)

4.7.4 **Springback**

Following the same treatment in as that given in Section 4.2.4, the springback angle at any section, after removal of load, is defined as

\[
\delta \theta_s = K_S \delta S = \frac{M(S)}{E' I} \delta S
\]

The total springback is the sum or integration of the springback for all of the individual sections:
4.7.5 Punch Force and Coining Action

To ensure further the shape accuracy, the coining action is needed. Thus, the punch strikes the sheet and forces it to conform the tool shape. Additional yielding of the sheet occurs in coining stage, which helps to reduce the springback. During coining, the bending arm reduces rapidly, and, the punch load rises rapidly. Therefore, careful control of the punch stroke is very important and also very difficult.

The maximum punch force during coining can be formulated using a similar procedure to that in Section 4.2.6. The normal force, $N$, and the frictional (tangential) force, $F$, are the same as those defined in Eq. (4.65), i.e.:

$$
N = \frac{P}{2 \cos\theta (1 + \mu \tan\theta)} \quad \text{and} \quad F = \mu N = \frac{P}{2 \cos\theta (1 + \mu \tan\theta)}
$$

(4.60)

And these forces are assumed to appear at the tangent (or separation) point B, Fig. 4.13. The moment balance at the origin in the midspan gives

$$
M_{\text{external}} = \frac{P}{2} (X_B + Y_B \frac{\tan\alpha - \mu}{1 + \mu \tan\alpha})
$$

(4.61)

The punch force can be calculated by equating this moment with the maximum internal bending moment, $M_A$, defined by Eq. (4.51) with $R_n = R_p + \nu/2$, i.e.:

$$
P = \frac{M_A}{X_B + Y_B \frac{\tan\alpha - \mu}{1 + \mu \tan\alpha}}
$$

(4.62a)
Therefore, in the proposed model, the punch load is related to (a) the sheet material properties: width, w, thickness, t, strain hardening exponent, n-value, strength coefficient, k-value, and the plastic anisotropy index, F-value defined in Eq. (3.11); (b) the tool parameters: punch radius R_p and the die gap which is included in the coordinates X_B and Y_B; and (c) the process parameters: the die angle, θ, and the friction coefficient. The punch force increases with the width and thickness of the sheet and with the die friction, and it decreases with the increases in the die gap, punch radius, and die angle. A higher punch load is required for sheets with higher strength, strain hardening, and normal anisotropy.

It is seen that the punch force for V-die bending is not related to the die radius which affects the load in air bending as discussed in Section 4.2.6. The reasons for this is that sheet contacts die radius only at the beginning stages of punch stroke. Once the contact point passes the last tangential point at the die radius, the sheet contacts the flat die wall. Therefore, the bending arm is determined by only the tangential point at the die wall and the punch tip. On the other hand, the bending arm in air bending is determined by the contact points on die radius and on punch tip; therefore, the bending moment and the punch force are related to the die radius. Another important observation from Eq. (4.62) is that at the coining action, the bending arm, which can be decomposed into coordinates X_B and Y_B, reduces rapidly; therefore, the force required for coining increases rapidly. However, the reduction of the bending arm in V-die bending has the advantage of less springback than air bending, because the total springback is an integration of the individual ones at each section of the sheet along the bending arm. In air bending, the bending arm increases, hence, the final springback is greater than that in V-die bending. Based on these comparisons, it is possible to combine the advantages in air bending and V-die bending to develop a die in which the bending span reduces during bending, as in the V-die bending, but the coining is eliminated and the springback can be compensated by deeper punch travel, as in the air bending. One such die design is proposed by Yang and Shima [Yang and Shima, 1990], in which the flat die wall in the V-die is replaced by a curved die wall, Fig. 4.15.
Fig. 4.15 Curved bending die
4.7.6 Geometrical Constraints and Punch Displacement

Equations (4.51) to (4.59) formulate all necessary governing relations to model V-die and channel-die bending. The only variables in the governing equations (4.51-4.59) are \( \Phi_A \) (the inclination angle or punch contact angle) and the X-coordinate, \( X_B \), at the tangent point, B, on the die. These two variables are subjected to the geometrical constraint (Appendix L) in the solutions by:

\[
\Phi(X_B) = \Phi_B = \alpha \tag{4.63a}
\]

and

\[
X_B \sin \alpha - Y_B \cos \alpha = C + R_p \left( 1 - \cos \alpha \right) - \frac{1}{2} \left( 1 + \cos \alpha \right) \tag{4.63b}
\]

because \( \Phi_B \) is related to \( \Phi_A \) and \( X_B \) by Eq. (4.57). In Eq. (4.63b), \( C \), is the clearance or die gap, \( R_p \) is the punch radius, \( t \) is the sheet thickness, and \( \alpha \) is the die angle. Both the tangent point, B, on the die, and the die gap, \( C \), are variables which decrease with the punch displacement. The relation between the punch travel, \( d \), and the die gap are found to be (Appendix L):

\[
d = L_d \tan \alpha - X_G \tan \alpha + \frac{1}{2} \\
= L_d \tan \alpha - R_p \left( \frac{1 - \cos \alpha}{\cos \alpha} \right) - \frac{C}{\cos \alpha} + t \\
= L_d \tan \alpha + Y_B - X_B \tan \alpha + \frac{1}{2} \left( 1 - \frac{1}{\cos \alpha} \right) \tag{4.64a}
\]

or

\[
C = t \cos \alpha - d \cos \alpha + L_d \sin \alpha - R_p \left( 1 - \cos \alpha \right) \tag{4.64b}
\]

Eq. (4-64b) indicates that the die gap, \( C \), decreases as the punch displacement, \( d \), increases. Once the tool parameters are known, the die gap can be defined for a given punch displacement. The geometrical constrain condition in Eq. (4-64b) is also determined
at the punch depth.

4.7.7 Solution Procedures

There are two control variables, namely, the punch contact angle, $\phi_A$, and the $X$-coordinate of the separation point, $B$, at the die wall, $X_B$. The solution algorithm is described as follows:

(i) Specify the punch depth, $d$, $(0 < d \leq H_d - t)$,

(ii) Define the die gap via the die gap - punch displacement relation in Eq. (4-64b),

(iii) Assume an initial guess solution for the $Y$-coordinate at point $B$, i.e. $Y_B^*$,

(iv) Calculate the $X$-coordinate, $X_B^*$, at the point $B$ using the geometrical constrain in

Eq. (4-63b),

(v) Solve the punch contact angle, $\phi_A$, using the $\phi$-$X$ relationship in Eq. (4.57) and the constraint condition in Eq. (4.63a):

$$\phi(X_B) = \alpha = \sin^{-1} \left\{ \sin \phi_A + \int_{X_A}^{X_B} k[M(X)] \, dX \right\}$$

with

$$X_A = \sin \phi_A / k_o$$

(vi) Calculate $Y_B$ using Eq. (4.8b):

$$Y_B = Y_A + \int_{X_A}^{X_B} \tan \phi(X) \, dX$$

with
\[ Y_A = \cos \Phi_A / K_0 \]

(vii) Check the relative error:

If

\[ \delta = \left| \frac{Y_B - Y_B^*}{Y_B} \right| \leq 0.0001 \]  \hspace{1cm} (C)

then convergence is achieved. Otherwise, update the values of \( Y_B^* \), and repeat steps (iv) to (vii) until the error is within an accepted region.

In solving Eq. (A), we must explicitly find the curvature-moment relation, \( k = k[M(x)] \). This may be extremely difficult or impossible for our current analysis in which the true strain description, strain hardening, and anisotropy are included. The procedure to relate the moment distribution to the curvature distribution follows:

(a) Evaluate the maximum internal bending moment, \( M_A \), in the punch contact region, using Eqs.(4.51) and with punch radius \( R_p \);

(b) Define the internal bending moment, \( M(X) \), for any arbitrary section with the curvature, \( k (= 1/R_n) \) using Eqs.(4.51), to obtain the \( M = M(k) \) equation;

(c) Calculate the external moment at that section using the moment distribution equation

(4.56) and the \( X \) coordinate \( X_A^* \) defined in Eq. (A):

\[ M(X) = M_A \frac{X_B - X}{X_B - X_A^*} \]

(D)

(d) Equate the internal and the external moments found in Steps (b) and (c) to obtain the moment-curvature relations explicitly.

\[ M(k) = M(X) = M_A \frac{X_B - X}{X_B - X_A^*} \]

(E)

(e) Calculate the data table or construct the graph of the moment-curvature, then find the curvature-moment relation numerically.

(f) Integrate the second term in Eq. (A) numerically.
4.8 COMPUTER PROGRAM: BEND

Based on the proposed formulations, a computer program called 'BEND,' has been developed. The main flow chart of the program is shown in Fig. 4.15. The input and output information are listed as follows.

4.8.1 Input Data

To run BEND, following data are required:

**MATERIAL DATA**

- $n$ - Strain hardening coefficient.
- $K$ - Strength coefficient (MPa).
- $a_0$ - Prestrain (default = 0.0001).

(These three parameters are used in flow curve $\sigma = K (a_0 + \varepsilon)^n$.

- $\sigma_0$ - Initial yield stress (MPa).
- $E$ - Young's module (MPa).
- $\nu$ - Poisson's ratio.
- $M$ - Anisotropic index in Hill's new (1979) yield function. (default = 2)
- $\overline{R}$ - Normal anisotropy (default = 1, isotropic).
- $R_0$ - Planar anisotropy along sheet rolling direction (default = 1, isotropic).
- $R_{45}$ - Planar anisotropy in 45 degree with rolling direction (default = 1, isotropic).
- $R_{90}$ - Planar anisotropy in 90 degree with rolling direction (default = 1, isotropic).
- $\varepsilon_f$ - Fracture strain (to evaluate the minimum bending ratio)
- $t$ - Sheet thickness (mm).
- $w$ - Sheet width (mm).

**TOOL DATA**

- $R_p$ - Punch radius (mm).
Fig. 4.15 The main flow chart of program BEND
Rd - Die radius (mm).
C - Die gap (mm).
H - Die height (mm).
Da - Die inclination angle (degree).

**PROCESS PARAMETERS**

μ - Friction coefficient.
θ2 - Desired bending angle (degree).

4.8.2 Output from BEND

Three kinds of information are provided from the simulation:

**BENDABILITY INFORMATION**:

The minimum bending ratio of \(R_p/t\) or \(R_d/t\).
The maximum tensile strain and stress (MPa).

**GEOMETRY INFORMATION**:

Bending angle under load.
Springback angle.
Strain distribution along bending arc length.
Punch displacement to ensure the desired bending angle.
Curvature distribution along bending arm at each punch height.

**LOAD INFORMATION**:

The maximum punch force (N).
Punch force vs. punch stroke.
Bending moment distribution along bending arm at each punch height.
Maximum forces on die shoulders.
CHAPTER V

MATHEMATICAL MODELING OF STRETCH AND SHRINK FLANGING OPERATIONS

5.1 ASSUMPTIONS

1. Axisymmetric Deformation under Uniaxial Stress State.

An axisymmetric flanging deformation is considered here (Fig. 5.1), in which an angular flat blank with a initial radius $R_1$ at its (right) edge is clamped along its outer boundary with a radius of $R_0$, and the portion of the blank near the clamped boundary is forced to wrap around a circular die through an angle, $\alpha$, to form a flange with a new radius, $R_2$, at its edge. It was found that the axisymmetric model of stretch and shrink flanges approximates the maximum hoop strains at the flange edges quite accurately in practical flanging operations [Wang and Wenner, 1974, Wang 1983, 1984, Sachs 1951]. Fig. 5.1 illustrates a portion of a circular flange formed by the shrink flanging operation.

In stretch and shrink flanging operations, the mixed deformation modes exist along the flange height (or the length $L$ as shown in Fig. 5.1). Around the die shoulder ($0 \leq s \leq r_\theta \alpha$, Fig. 5.1) there are compound curvatures resulting from the radii of the flange and the die shoulder. Therefore, in this region, a biaxial stretch state exists along both the hoop and the radial directions in a stretch flange, and a state of hoop compression and
Fig. 5.1 Shrink Flanging
radial tension exists in shrink flange. The radial tension, resulting from bending along die profile radius and friction, decreases rapidly and vanishes at the end of this region \( (s = r'_d \alpha) \). In the region of the flange wall \( (r'_d \alpha \leq s \leq r'_d \alpha + L) \), the sheet takes a single curvature of the flange radius. Hence, the deformation in this region is uniaxial, that is, a tension along the hoop direction in a stretch flange, and a compression along the hoop flange. Wang and Wenner [1974] compared the approximation solution of a stretch flange based on the assumption of the uniaxial stress state with the experimental measurements. They found that there was an excellent agreement between the theory and the experiment for the hoop strains up to 100%. Many engineering examples [Sachs, 1951] also reveals the similar results.

2. Membrane Approximation.

The dominant deformation in flanging is the hoop tension or compression in the flange. The radius of flange usually is much greater than the sheet thickness, thereby, the bending moment is negligible along the hoop direction. A local bending occurring at the die radius may be important for die design consideration of the minimum bending ratio, however, it does not significantly affect the overall flanging deformation. Furthermore, the failures of necking, tearing, and wrinkling are mainly caused by the excess in-plane membrane strain/stress at the flange edge. Therefore, a membrane analysis is justified.

5.2 STRAIN AND STRESS ANALYSES

In both stretch and shrink flanging (Fig. 5.1), for a material point with initial radial distance, \( r \), and new distance, \( \rho \), the true (hoop) strain is defined as:

\[
\varepsilon_\theta = \ln \frac{\rho}{r} \quad \text{(5.1a)}
\]
If a differential segment along flange has an initial length, \( dr \), and a final length, \( ds \), then the radial strain can be expressed as:

\[
\varepsilon_r = \ln \frac{ds}{dr}
\]  

(5.1b)

The thickness strain is found by volume constancy, i.e.:

\[
\varepsilon_t = -(\varepsilon_\theta + \varepsilon_r)
\]  

(5.1c)

Using Hill's 1979 yield function for normal anisotropic materials [1979], the radial strain component is related to the hoop strain component by:

\[
\varepsilon_r = -\frac{R}{1+R}\varepsilon_\theta = -fe_\theta
\]  

(5.2)

where \( f \) is the strain ratio and \( R \) is the normal anisotropy, and \( f \) is defined as:

\[
f = \frac{\varepsilon_r}{\varepsilon_\theta} = \frac{R}{1+R}
\]  

(5.3)

Therefore, using Eqs. (5.2) and (5.1a) and (5.1b), the following kinematic relationship is obtained:

\[
\frac{ds}{dr} = \left(\frac{r}{\rho(s)}\right) r^e \quad \text{or} \quad [p(s)] r^e ds = r^e dr
\]  

(5.4)

The integration of this equation with the appropriate geometrical boundary conditions will provide a relationship among the initial blank sizes, tool dimensions, and the flange geometry. Then the maximum strain at the flange edge can be evaluated. The explicit integration of Eq. (5.4) depends on the types of flange: shrink or stretch. These will be discussed in the following two sections.

The stress-strain relationship is described by Swift's hardening equation:

\[
\bar{\sigma} = k(\varepsilon_\theta + \varepsilon)^n
\]  

(5.5a)
Under the uniaxial stress state ($\sigma = \sigma_0$ and $\varepsilon = \varepsilon_0$), this equation is identical to

$$\sigma_0 = k(\varepsilon_0 + \varepsilon_0)^n \quad (5.5b)$$

### 5.3 ANALYSIS OF SHRINK FLANGING PROCESS

Following the analysis of stretch flange by Wang and Wenner (1974), the flange is divided into two regions (Fig. 5.1): a curved portion around die shoulder and flange wall. These two regions are jointed at tangent point T. The radial distance, $\rho(s)$, of an arbitrary material point in these two regions is found to be:

**In Region I:** $0 \leq s \leq r'_d \alpha$ and $R_o \leq r \leq R_T$:

$$\rho(s) = R_o + r'_d \sin(s / r'_d) \quad (5.6)$$

**In Region II:** $(r'_d \alpha) \leq s \leq (r'_d \alpha + L)$ and $R_T \leq r \leq R_i$:

$$\rho(s) = R_o + r'_d \sin \alpha + (s - r'_d \alpha) \cos \alpha = R_o [a_1 + \left(\frac{s}{R_o} - a_2 \alpha \cos \alpha\right)] \quad \alpha \neq 90^\circ \quad (5.7a)$$

and for a 90 degree flange:

$$\rho(s) = R_o + r'_d \quad \alpha = 90^\circ \quad (5.7b)$$

where $a_1 = 1 + \frac{r'_d}{R_o} \sin \alpha$ and $a_2 = \frac{r'_d}{R_o} \quad (5.8)$

Using Eqs. (5.6) - (5.8), the integration of Eq. (5.4) in these two regions provides a relationship among the initial blank sizes (the radius $R_1$, the thickness, $t$, and the radial distance, $R_o$, at the clamped edge), the flange sizes (the flange length, $L$, and the flange angle $\alpha$), the tool dimension (die radius, $r_d$), and the sheet anisotropy $\bar{R}$ (Appendix M) as follows.
Inclined Flange (flange angle $\alpha \neq 90^\circ$):

$$
\frac{R_1}{R_o} = \left[1 + \frac{1 + 2R}{1 + R} \frac{r_d + t/2}{R_o} \left[\alpha + \frac{R}{1 + 2R} \frac{r_d + t/2}{R_o} (1 - \cos \alpha)\right] + \frac{1}{\cos \alpha} \left[(1 + \frac{r_d + t/2}{R_o} \sin \alpha + \frac{L}{R_o} \frac{1 + 2R}{1 + R} \frac{r_d + t/2}{R_o} \right] - (1 + \frac{r_d + t/2}{R_o} \sin \alpha)^{1 + \frac{1}{2R}} \right]^{1/2} \right]^{1/2}
$$

(5.9a)

Vertical Flange (flange angle $\alpha = 90^\circ$):

$$
\frac{R_1}{R_o} = \left[1 + \frac{1 + 2R}{1 + R} \frac{r_d + t/2}{R_o} \left(\frac{\pi}{2} + \frac{R}{1 + 2R} \frac{r_d + t/2}{R_o}\right) + (1 + \frac{r_d + t/2}{R_o} \frac{L}{R_o} \right]^{1 + \frac{1}{2R}} \right]^{1/2}
$$

(5.9b)

From the geometry shown in Fig. 5.1, the flange radius $R_2$ at the flange edge is found as a function of flange angle, $\alpha$, flange length, $L$, and die radius, $R_d$:

$$
\frac{R_2}{R_o} = \frac{a_1 + \frac{L}{R_o} \cos \alpha = (1 + \frac{r_d + t/2}{R_o} \sin \alpha) + \frac{L}{R_o} \cos \alpha \neq 90^\circ (5.10a)}{}
$$

and

$$
\frac{R_2}{R_o} = a_1 = 1 + \frac{r_d + t/2}{R_o} \quad \alpha = 90^\circ (5.10b)
$$

Using the relationships defined in Eqs. (5.9) and (5.10), the maximum compressive strain at the flange edge can be determined as

$$
e_{\text{max}} = \frac{R_2 - R_1}{R_1} \quad \text{(engineering strain)} \quad (5.11a)
$$

or

$$
e_{\text{max}} = \ln \frac{R_2}{R_1} \quad \text{(true strain)} \quad (5.11b)
$$

This strain is a function of flange dimensions (flange angle, $\alpha$, length, $L$, and thickness, $t$, and the radial distance of the clamped edge, $R_o$), die dimension (die radius, $R_d$), and the sheet normal anisotropy, that is...
This maximum compressive strain must not exceed the limit strain at which buckling occurs. A wrinkling criterion will be derived and presented in Chapter VII and compared to the maximum strain at flange edge to establish a workable condition for shrink flange operation.

5.4 ANALYSIS OF STRETCH FLANGLING PROCESS

Following the same procedure in shrink flange analysis, the radial distance, \( p(s) \), of an arbitrary material point in these two regions are found to be (Fig. 5.2):

In Region I: \( 0 \leq s \leq r_d \alpha \) and \( \varepsilon_{\text{max}} = f(\alpha, L, t, R, r_d, \overline{R}) \):

\[
p(s) = R_0 - r'_d \sin (s / r'_d)
\]

(5.12)

In Region II: \( (r_d \alpha) \leq s \leq (r_d \alpha + L) \) and \( R_1 \leq r \leq R_7 \):

\[
p(s) = R_o - r'_d \sin \alpha - (s - r'_d \alpha) \cos \alpha = R_o [a_1 - \left( \frac{s}{R_o} - a_2 \cos \alpha \right)] \quad \alpha \neq 90^\circ
\]

(5.13a)

And

\[
p(s) = R_o - r'_d \quad \alpha = 90^\circ
\]

(5.13b)

where \( a_1 = 1 - \frac{r'_d}{R_o} \sin \alpha \) and \( a_2 = \frac{r'_d}{R_o} \)

(5.14)

Using Eqs. (5.12) - (5.14) and following the same integration procedure in shrink flange analysis, the integration of Eq. (5.4) in these two regions gives following relationships.
Fig. 5.2 Stretch Flange
Inclined Flange (flange angle $\alpha \neq 90^\circ$):

$$R_o = \frac{1}{1 + \frac{2R_o}{R_d} \frac{t/2}{\cos \alpha} \left[ \frac{1}{1 + \frac{R_d}{R_o}} \frac{r_d + t/2}{(1 - \cos \alpha)} - \frac{1}{\cos \alpha} \left[ \left( 1 - \frac{r_d + t/2}{R_o} \sin \alpha \right) \left( 1 - \frac{L}{R_o} \cos \alpha \right) \right] \right]^{1+2R_o} (5.15a)$$

Vertical Flange (flange angle $\alpha = 90^\circ$):

$$R_o = \frac{1}{1 + \frac{2R_o}{R_d} \frac{t/2}{\cos \alpha} \left[ \frac{1}{1 + \frac{R_d}{R_o}} \frac{r_d + t/2}{\pi} + (1 - \frac{r_d + t/2}{R_o}) \frac{L}{R_o} \right]^{1+2R_o} (5.15b)$$

From the geometry shown in Fig. 5.2, the flange radius $R_2$ at the flange edge is found as a function of flange angle, flange length, and die radius:

$$\frac{R_2}{R_o} = \left( 1 - \frac{r_d + t/2}{R_o} \sin \alpha \right) - \frac{L}{R_o} \cos \alpha \quad \alpha \neq 90^\circ (5.16a)$$

and

$$\frac{R_2}{R_o} = 1 - \frac{r_d}{R_o} - \frac{r_d + t/2}{R_o} \quad \alpha = 90^\circ (5.16b)$$

Using the relationships defined in Eqs. (5.15) and (5.16), the maximum tensile strain at the flange edge can be determined using Eq. (5.11). This maximum tensile strain must not exceed the limit strain at which fracture occurs. The limit strain will be derived and presented in Chapter VI, based on the localized necking instability which follows by fracture.

5.5 COMPUTER PROGRAM: FLANGE

Based on the proposed formulations, a computer program called FLANGE has been developed. The main flow chart of program FLANGE is shown in Fig. 5.3. The input and output information are listed as follows.
Input
Material
Tool
Process

Calculate
Max. bending strain
Min. bending ratio (R/t)
at die shoulder

calculate
Radius at flange edge: R2
Flange length: L

Calculate
Max. strain / stress
at flange edge

Calculate
Failure strain

Calculate
Strain distribution
Deformed profile
(x-y coordinate)

Output

Fig. 5.3 Main flow chart of program FLANGE
5.5.1 Input Data

To run FLANGE, the following input data are required:

**MATERIAL DATA**

- \( n \) - Strain hardening coefficient.
- \( K \) - Strength coefficient (MPa).
- \( a_0 \) - Prestrain (default = 0.0001).

(These three parameters are used in flow curve: \( \sigma = K (a_0 + \varepsilon)^n \).)

- \( \sigma_0 \) - Initial yield stress (MPa).
- \( E \) - Young's module (MPa).
- \( \nu \) - Poisson's ratio.
- \( M \) - Anisotropic index in Hill's new (1979) yield function. (default = 2)
- \( \bar{R} \) - Normal anisotropy (default = 1, isotropic).

**BLANK GEOMETRY**

- \( t \) - Sheet thickness (mm).
- \( \theta \) - Included angle of blank.
- \( R_o \) - Radial distance from the center of blank to the center of die shoulder.
- \( R_1 \) - Radial distances from the center of the blank to the undeformed free edge.
- \( w \) - Sheet width (mm). It is calculated by \( \theta \times R_o \) for circular blanks.

**TOOL DATA**

- \( r_d \) - Radius of die shoulder.

**PROCESS PARAMETERS**

- \( \alpha \) - Flange angle.
5.5.2 Output from FLANGE

The information provided from the simulation are:

**FORMABILITY INFORMATION:**

The minimum bending ratio of \( r_d/t \).

The critical tensile strain and stress for fracture.

The critical tensile strain and stress for wrinkling.

**DEFORMATION INFORMATION:**

Strain distribution along flanging length.

The maximum strains and stresses at flange edge.

Springback angle.
6.1 Bendability: The Minimum Bending Ratio (MBR)

The initial consideration in the design of bending tools and the selection of sheet materials is to determine the minimum bending ratio (MBR) which is defined as the ratio of the bending tool radius to the sheet thickness. According to the failure theory of the maximum tensile strain, cracks will occur on the convex surface of bent sheet if the maximum surface bending strain exceeds the tensile fracture strain, or the bend ratio is less than the minimum value allowed. For designs with a good margin of safety, the failure criterion may be established based on plastic necking instability. Both fracture and necking models for the minimum bend ratios in various types of bending modes will be derived and discussed here.

6.1.1 The Minimum Bend Ratio for an Anisotropic Sheet

Under Plane Strain Condition

There has been a criterion of the minimum bend ratio based on the area reduction for the isotropic materials under a uniaxial tension. It was reported [Barlow, 1956] that
the width of sheet specimen affects the minimum bend ratio by changing the strain states. The plane strain bending of a wide sheet requires a larger bend ratios than that in bending of a narrow sheet. It was noticed that the orientation of the bend axis with respect to sheet rolling direction also has significant influence in the maximum strain [Lange, 1985]. The minimum bend ratio has to be increased to avoid cracking if the bending axis is perpendicular to the rolling direction. However, in most handbooks and standards, the recommended values of the minimum bend ratios are still based on the results obtained in uniaxial tension of the isotropic materials. The new bendability models are proposed here (Appendix N) to account for the effects of both material anisotropy and stress/strain states on the minimum bend ratio.

Introducing the parameter $F$ (which indicates the effect of the sheet anisotropy and stress/strain state on the flow stress and effective strain in plane strain deformation) defined in Eq. (3.11), the ratio of minimum bend ratio for a constant neutral axis location can be expressed as a function of $F$ value and the area reduction ratio $A_r$ ($A_r = (A_o - A_f) / A_o$, where $A_o$ is the initial section area of a specimen before tensile test, and $A_f$ is the section area at the fracture, Appendix N):

$$\left( \frac{R_t}{t} \right)_{\text{min}} = 0.5 \left\{ \frac{1}{(1 - A_o^{-1/F})^{-1} - 1} \right\} \quad \text{Neutral axis coincides with mid-axis} \quad (6.1)$$

Setting $F = 1$, Eq. (6.1) returns to the commonly used model for isotropic sheet under the uniaxial stress state as defined by Eq. (6.2):

$$\left( \frac{R_t}{t} \right)_{\text{min}} = \frac{1}{2A_r} - 1 \quad (6.2)$$

Eq. (6.1) implies that materials with more ductility have a higher bendability, i.e. a smaller bend ratio, than the less ductile sheets. The influence of the sheet anisotropy on the minimum bending ratio is illustrated in Fig. 6.1 The values of $F$ increases with the normal anisotropy. Therefore, as shown in Fig. 6.1, the bendability decreases, that is the
Fig. 6.1 Influence of area reduction ratio \( \left[ A_r = (A_o - A_f)/A_o \right] \) and transverse anisotropy (R-value) upon the minimum bending ratio \( (R/t) \)
minimum bend ratio has to be increased for materials with more anisotropic behavior. The width of the sheet determines the strain/stress states. In bending of a narrow sheet (the W/t ratio of the width, W, to thickness, t, is less than 8), a uniaxial tension on the convex outer fiber may exist. The plane strain condition occurs in bending of a wide sheet with a W/t ratio greater than eight. The F-values for an isotropic sheet in these two forming conditions are unity and 1.16 (R-value = 1), respectively. Therefore, the bending of a wide sheet has less (about 20%) bendability for 20% area reduction, and requires a larger bend ratio than that in bending of a narrow sheet. This may be the explanation for a number of experimental results reported early by Sachs [1951, Barlow [1956], Turno [1980], and Lange [1985], stating that the minimum bend ratio increases with the ratio of the sheet width to the sheet thickness. The effect of the sheet anisotropy on the minimum bend ratio is demonstrated in Fig. 6.1 and by the following examples. For a 20% area reduction, the minimum bend ratios are 1.68 and 2.06 respectively for anisotropic sheets (anisotropy = 0.5 and 1.5) under the plane strain condition, and the minimum bend ratio is 1.5 for an isotropic sheet under the uniaxial tension. The relative difference between the predictions using Eq. (6.1) and Eq. (6.2) is about 10% for an anisotropy R-value of 0.5, and 57% for R-value of 1.5, which are significant. The discrepancy increases for higher anisotropic and less ductile materials and decreases for more ductile sheets.

A structural weakening of bent parts may occur when the tensile strain of the outer fiber exceeds the maximum uniform elongation, \( e_u \), of the sheet materials. At the point of plastic instability (diffusive necking), the maximum uniform elongation is equal to the strain hardening exponent. If the minimum bending ratio is expressed in terms of the instability strain at the diffusive necking, then the MBR is a function of the strain hardening exponent, \( n \), and anisotropy index, \( F \), and is given by (Appendix N):

\[
\left( \frac{R_1}{t} \right)_{\text{min}} = \frac{1}{2} \frac{1}{e^{n/F} - 1}
\]  

(6.3)
Fig. 6.2 Influence of strain hardening exponent n-value and plastic anisotropy r-value on the minimum bend ratio $R_j/t$. 
Eq. (6.3) returns to the isotropic sheet under the uniaxial stress state:

\[
\left(\frac{R}{t}\right)_{min} = \frac{1}{2} \left(1 - \frac{1}{e^n + 1}\right)
\]  

(6.4)

The effect of work hardening on the minimum ratio of the bend radius to the sheet thickness is clearly shown in Fig. 6.2. The curve is very similar to the curves in Fig. 6.1, because the n-value is also a measure of ductility and the extent in which the material can be elongated without necking or significant thinning. The higher work hardening a sheet has, the more ductile the material is. Therefore, the minimum bend radius decreases as the strain hardening exponent increases. The bendability reduces (the MBR value increases) as the plastic anisotropy increases. The stress/strain state also affects the bending limit. The minimum bend ratio required in plane strain bending is greater than that in the bending of a narrow sheet under the uniaxial stress state.

6.1.2 Influence of the Neutral Axis Shift on the Minimum Bend Ratio

Considering the shift of the neutral axis, the minimum bending ratio is found to be (Appendix N):

\[
\left(\frac{R}{t}\right)_{min} = \frac{1}{1 - A_0^2 + A_0^2F - 1} = \frac{(1 - A_0)^{2/F}}{1 - (1 - A_0)^{2/F}}
\]  

(6.5)

Eq. (6.5) is plotted in Fig. 6.3 to compare the difference between the minimum bend ratios in plane strain bending of an anisotropic sheet with R-value of 2, and that in bending of an isotropic sheet under the uniaxial stress state. The relative difference is significant and increases as the area reduction raises. For a 30% area reduction, the error by using the model which assumes a constant neutral axis coinciding with the mid-axis of sheet is 50%. This model is also used to compare the experimental results measured by Datsko and Yang [Datsko and Yang, 1960]. In their study, the shift of the neutral axis
Fig. 6.3 Comparisons between the minimum bend ratios in uniaxial bending of isotropic sheet with mid-axis as the neutral axis and in plane strain bending of anisotropic sheet (R-value =2) with neutral axis position: (a) $R_l/t$ vs. area reduction $A_r$, and (b) relative difference between these two models.
Table 6.1 Comparisons between the measured minimum bend ratio with the predicted ratios using anisotropic model and isotropic model (normal anisotropy R = 5)

<table>
<thead>
<tr>
<th>Alloys</th>
<th>Minimum Bend Ratio (R/t)\text{\textsubscript{min}}</th>
<th>A\textsubscript{r}</th>
<th>Area Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Measured</td>
<td>Current Model</td>
<td>Isotropic Model</td>
</tr>
<tr>
<td>0.050 AZ31 BH24 Mg</td>
<td>3.70</td>
<td>3.81</td>
<td>2.0</td>
</tr>
<tr>
<td>0.032 HK31 X1 H24 Mg</td>
<td>3.0</td>
<td>2.67</td>
<td>1.65</td>
</tr>
<tr>
<td>130B Ti</td>
<td>2.0</td>
<td>1.88</td>
<td>1.0</td>
</tr>
</tbody>
</table>

was considered in their theoretical model of the minimum bending ratio, but the strain/stress state and the anisotropy were not included in their model. They found that there were good agreements between the predicted bending ratios and the measured bending ratios for a number of the sheets in the twenty materials tested. However, for high anisotropic materials such as 130B Ti, 0.050 AZ31 BH 24 Mg alloy, and 0.032 Hk 31 X1 H24 Mg alloy, their predictions were far away from the measurements. There were no anisotropy data reported for these Mg-alloy in their study. The normal anisotropy for the hexagonal close-packed alloys is usually around 3 to 6 [Backofen, 1972]. The minimum bend ratios for these three alloys were calculated using the present model in Eq. (6.5), and assuming an anisotropy value of 5, i.e. R = 5. The comparisons are excellent as shown in Table 6.1.

If the necking criterion is used and the shift of the neutral axis is considered, the minimum bending ratio is found to be (Appendix N):

$$\left(\frac{R_i}{t}\right)_{\text{\textsubscript{min}}} = \frac{1}{\exp(4n/F) - 1} \quad (6.6)$$

For materials with low strain hardening (n-value less than 10%), the difference between R\textsubscript{i}/t ratios with and without the consideration of the neutral axis shift is not
significant. However, this difference is considerable (10 to 20% for commonly used sheet materials with the n-value ranging from 0.1 to 0.3). The difference increases with the strain hardening exponent, but decreases as plastic anisotropy reduces.

6.1.3 Influence of the Applied Axial Tension on the Minimum Bend Ratio

When an axial tension is applied, the minimum bending ratio must increase to avoid cracking. The minimum bend ratio can be defined either through the fracture or the necking criteria as described in previous sections.

1. The Minimum Bending Ratio \( (R_l/t)_{min} \) Based on Fracture Limit

Following the derivation of Eq. (6.1), see Appendix N:

\[
\bar{\varepsilon} = \ln \frac{1}{1 - A_t} = F \ln \left[ 1 + \frac{t/2 + d}{R_i + t/2 - t} \right] = F \ln \left[ 1 + \frac{1 + \delta}{2 R_i/t + (1 - \delta)} \right]
\]

where

\[\delta = \frac{d}{t/2}\]

and d is the shifting distance of the neutral axis away from the middle axis, and it can be evaluated by the formula and procedures described in Chapter III under the Section ('Mechanics of Bending under Tension').

Therefore, the ratio \( (R_l/t)_{min} \) is found to be:

\[
\left( \frac{R_l}{t} \right)_{min} = \frac{1}{2} \left[ \frac{1 + \delta (1 - A_t)^{1/F}}{(1 - A_t)^{1/F} - 1} - 1 \right]
\]

Eq. (6.8) returns to Eq. (6.1) when \( d = 0 \), i.e. no axial force is applied and the neutral axis coincides with the middle axis. The shift distance, \( d \), increases with the axial force, \( N \), and the neutral axis may shift outside of the sheet thickness. Therefore, the minimum bending ratio must increase to avoid fracture when the axial load is present.
2. The Minimum Bending Ratio \((R/t)_{min}\) Based on Necking Instability

Following the same procedures in derivation of Eq. (6.3), the minimum bending ratio is expressed as:

\[
\left( \frac{R_l}{t} \right)_{min} = \frac{1}{2} \left[ 1 + \frac{\delta e^{\gamma \sigma}}{e^{\gamma \sigma} - 1} - 1 \right]
\]

(6.8)

Eq. (6.8) returns to Eq. (6.3) when \(d=0\), i.e. no axial force is applied and the neutral axis coincides with the middle axis. As the axial load increases, the shift of the neutral axis, \(d\), also rises. Therefore, the minimum bending ratio must increase to avoid the necking instability.

6.2 STRETCH FLANGE FORMABILITY

The fracture along the flange edge is the most common failure in stretch flanging. The formability is mainly limited by the localized necking instability which causes significant thinning and tearing at the edge. Hill [1952] proposed a criterion for the localized necking instability for a specimen deformed in two principal directions under the plane-stress state, that is

\[
\frac{d\sigma}{d\varepsilon} = \frac{\sigma}{Z_L}
\]

(6.9)

and the subtangent \(Z_L\) at the onset of the localized necking instability is

\[
\frac{1}{Z_L} = \left( \frac{\partial f}{\partial \sigma_1} + \frac{\partial f}{\partial \sigma_2} \right) / \frac{\partial f}{\partial \bar{\sigma}}
\]

(6.10)

where \(f\) is the loading function or plastic potential \(f(\sigma)\) which can be expressed by the yield locus \(\sigma(\sigma)\) if the flow (or incremental deformation) theory is applied.
In order to consider the effect of the plastic anisotropy on the instability, we introduce Hill’s nonquadratic yield criterion for normal anisotropic materials [Hill, 1979]. When the principal directions of the stresses are coaxial with the principal axes of anisotropy (which are the intersections of the mutually orthogonal planes of symmetry), Hill’s 1979 non-quadratic yield function is given by:

\[
f(\sigma_{ij}) = \tilde{\sigma} = C_1 (C_2 |\sigma_1 - \sigma_2|^M + |\sigma_1 + \sigma_2|^M)^{1/M}
\]

and

\[
C_1 = \frac{1}{2(1 + \bar{R})} \quad \text{and} \quad C_2 = (1 + 2\bar{R})
\]

where \( \bar{R} \) is the normal anisotropy and \( M \) is a new index describing the shape of the yield locus.

With a stress ratio, \( \alpha_\sigma \), this yield function can be further expressed as

\[
f(\sigma_{ij}) = \tilde{\sigma} = C_1 (C_2 |\sigma_1 - \sigma_2|^M + |\sigma_1 + \sigma_2|^M)^{1/M} \\
= C_1 (C_2 |1 - \alpha_\sigma|^M + |1 + \alpha_\sigma|^M)^{1/M} \sigma_1 \\
= C_1 B_1 \sigma_1
\]

where

\[
B_1 = (C_2 |1 - \alpha_\sigma|^M + |1 + \alpha_\sigma|^M)^{1/M}
\]

\[
\alpha_\sigma = \frac{\sigma_2}{\sigma_1}
\]

The derivatives of the effective stress with respect to the stress components are as follows:
and

\[
A_1 = C_1 \{ C_2 \{ 1 - \alpha \}^{M-1} + [1 + \alpha]^{M-1} \}^{1/M} \\
A_1 = C_1 \{ -C_2 \{ 1 - \alpha \}^{M-1} + [1 + \alpha]^{M-1} \}^{1/M} \tag{6.17}
\]

Substitute these derivatives into Eq. (6.10), we obtain the expression of the instability index \((1/Z_L)\) as

\[
\frac{1}{Z_L} = \frac{1}{C_1B_1} (A_1 + A_2) = 2C_1\{1 + \alpha\}^{M-1} / B_1^{M-1} \tag{6.18}
\]

or

\[
\frac{1}{Z_L} = \frac{2\{1 - \frac{1}{2(1 + R)}\}^{M-1} + [1 + \alpha]\}^{M-1}}{(1 + 2R)\{1 - \alpha\}^{M} + [1 + \alpha]^{M-1} \}^{1/M} \tag{6.19}
\]

Under uniaxial stress state \((\sigma_2 = 0 \text{ and } \alpha = \sigma_2 / \sigma_1 = 0)\), therefore the localized necking instability index is

\[
\frac{1}{Z_L} = \frac{1}{1 + R} \tag{6.20}
\]

Using the flow stress and strain curve described by Swift's equation, \(\bar{\sigma} = K(\varepsilon_o + \bar{\varepsilon})^n\), the instantaneous slope (tangent) of the flow curve can be determined as

\[
\frac{d\bar{\sigma}}{d\varepsilon} = nK(\varepsilon_o + \bar{\varepsilon})^{n-1} = \frac{n\bar{\sigma}}{\varepsilon_o + \bar{\varepsilon}} \tag{6.21}
\]

With Eqs. (6.9) and (6.21), the critical strain at the localized instability point is

\[
\bar{\varepsilon}_{cr} = (1 + R)n - \varepsilon_o \tag{6.22}
\]
In stretch flanging under the uniaxial stress state, the effective strain is given by the hoop strain component. Therefore, the limit strain for the onset of the localized necking is

\[ \varepsilon^* = (1 + \bar{R})n - \varepsilon_o \]  

(6.23)

This criterion indicates that the limit strain is proportional to the strain hardening exponent \( n \), the normal anisotropy \( \bar{R} \), and the prestrain \( \varepsilon_o \). The stretch flanging formability is enhanced by an increase of strain hardening and normal anisotropy, because both of these two material properties are the indices to resist the thinning. The flangability is poor for those materials with anisotropy values less than unity (such as aluminum alloys). A compressive prestrain \( \varepsilon_o < 0 \) is beneficial for flangability, while a tensile prestrain \( \varepsilon_o > 0 \) deteriorates the formability. Relating the limit strain to the maximum strain in stretch flanging, defined in Eqs. (5.11) and (5.16), a set of working limits on tool dimensions and process parameters can be established.
CHAPTER VII

WRINKLING CRITERIA

IN DEEP DRAWING AND SHRINK FLANKING

Wrinkling at the edge of a shrink flange is the major defect in shrink flanging operation. In deep drawing operations, the body wrinkling in the unsupported region of the sheet in the die wall is a common defect in forming of deep drawn sheet components and automotive body panels. However, most previous wrinkling studies have been limited to the wrinkles on the flat flange under the blankholder in deep drawing. In this study, a body wrinkling criterion is proposed for an elastic isotropic and plastic anisotropic shell with compound curvatures in noncontact region of sheet subjected to internal forming stresses. A quasi-shallow shell is modeled by Donnell-Mushtari-Vlasov (DMV) shell theory. A bifurcation functional from Hill's general theory of uniqueness and bifurcation in elastic-plastic solids is used to model the local wrinkling phenomenon. Both strain hardening and transverse anisotropy are taken into consideration. A wrinkling criterion for the shrink flanging is obtained by the simplification of the general wrinkling criterion. This wrinkling criterion is especially useful as a failure criterion in 3-D finite element modeling (FEM) of sheet forming. Given the principal stresses or strains, and geometry provided at each incremental deformation step, the criterion can be used to predict wrinkles in the elements in the unsupported region. When wrinkles are detected, an adjustment of the restraining force generated by the blankholder force and
the drawbead is necessary to suppress the wrinkles. The predicted curve of the restraining force versus punch stroke can then be used as a process controller when monitoring press forming.

7.1. OVERVIEW

In deep drawn sheet components, wrinkling and tearing are two major failure modes. Excessive compressive stresses cause wrinkling, while excessive tensile stresses lead to fracturing. Careful control of the tensile restraining force generated by blankholders and drawbeads can avoid wrinkles and cracks. A number of approaches to control the blankholder force (BHF) and drawbead force (DBF) have been reported elsewhere [Wang, Kinzel, and Altan, 1992]. For prediction and control of fracture, a number of failure criteria such as the forming limit diagram and localized necking instability may be used in conjunction with finite element modeling. However, there has been a lack of reliable wrinkling criteria for prediction and control of wrinkling, especially, body wrinkling.

There are three types of compressive instability problems that may occur in sheet forming operations. These instability problems are: flange wrinkling, wall wrinkling, and elastic buckling in the undeformed area due to residual elastic compressive stress. All of these can be observed in the forming of a common pie pan made from aluminum sheet. The flange wrinkling and wall wrinkling are basically the same, and caused by the compressive circumferential stress. However, wall wrinkling occurs far more easily than the wrinkling on the flat flange since the wall is relatively unsupported by the tool. The suppression of wall wrinkles by control of the blank holder force (BHF) which affects the radial tensile stress component is more difficult than the suppression of flange wrinkles. It is known that wall wrinkling (or body wrinkling) is more of a problem than flange wrinkling in most press forming, and the blankholder force is far greater than that necessary to suppress flange wrinkling. The drawbead is then particularly designed to
add an additional restraining force to prevent metal flowing into the die too freely. Therefore wall wrinkling is the problem of most industrial importance and interest. However, very few studies have been done in understanding and controlling wall wrinkling.

The studies of wrinkling phenomenon in sheet forming are mostly limited to flange wrinkling. Based on an energy approach, that is the energy expended in the flange by the circumferential stress is equal to the energy dissipated in buckling the flange, Senior [1956] proposed a one-dimension beam buckling model to determine the critical stress and dimensions, and number of wrinkles in flange wrinkling of isotropic sheet. In his model, the flange was approximated as a number of linked struts, and the blankholder pressure is assumed to be distributed only at the inner edge of flange. His study indicated that (a) proper control of the BHF can reduce the amplitude and increase the number of wrinkles, which is desired in practice, (b) the BHF should be sufficiently high to prevent the lifting of the blankholder (more than twice the wrinkle amplitude) when a constant BHF is used, and (c) in the drawing of non-circular cups (made of two semi-circular ends jointed by straight walls), the number of wrinkles formed in the semi-circular ends corresponded to those formed in circular cups, while the number of wrinkles per unit length in the straight walls were fewer due to the reduction of hoop compressive stresses in this region. The later observation is important and it points out the possibility of applying the flange wrinkling models of circular cups to flange wrinkling in non-circular sheet parts where the worst wrinkles always occur in the curved regions. A similar study of flange wrinkling was carried out by Kawai [1961] based on the moment equilibrium of a half-wave segment of wrinkled flange and a spring model of the blankholder. A model of the critical blankholder pressure to prevent the growth of wrinkles was determined and verified. His experimental results showed that a wrinkle-free part could be produced by increasing the blankholder force to such an extent that the sheet thickness and the local clearance which appears during drawing are reduced. It was also found that excess
lubricant should be removed since it causes an excessive blankholder force on the blank before the drawing operation. Naziri and Pearce [1968] experimentally investigated the influences of material properties, especially, planar and normal anisotropy of sheet, on wrinkling and on the necessary blankholder pressure (BHP) in deep drawing of four metal alloys (steel, aluminum, zinc, and titanium). Their experimental results illustrated that (a) the BHP necessary to suppress flange wrinkling decreases with increasing normal anisotropy, and the BHP increases with increasing planar anisotropy; and (b) the wrinkles occur at the location with lower planar anisotropy (in a direction of 45° from rolling direction). Triantafyllidis and Needleman studied the wrinkling problem in Swift cup test, and they considered the onset of flange wrinkling as a plastic bifurcation problem [1980]. They proposed a two-dimension model (numerical) for flange wrinkling based on the variational principle which incorporates both flow theory and total strain (or deformation) theory of plasticity under plane stress. In this model, the flange was modeled as an annular flat plate with orthotropic elastic-plastic properties (normal anisotropy), and the blankholder was modeled as a simple linear elastic foundation on which the flange rests. Their simulations showed that the limiting drawing ratio (LDR), the critical drawing stress, and the punch displacement at the onset of flange wrinkling increase with the normal anisotropy. Yu and Johnson [1982] established the analytical models for wrinkling of a two-dimensional elastic-(perfect)-plastic annular plate, using well-known energy methods. The neglecting of strain hardening and anisotropy is the major drawback of this model, although it improves upon the predictions of Senior's model.

Modeling wall wrinkling occurring in the unsupported and curved (conical for example) wall presents a more challenging task for sheet forming researchers. The wall is curved and it is not appropriate to treat it as a flat plate. Moreover, for a tapered wall, the buckling solution for a conical shell must be used instead of the cylindrical shell solution. One approach in modeling wall wrinkling is to approximate the curved wall as
a ‘flat plate’ if the radius of wall (or die opening), \( R \), is much greater than the sheet thickness, \( t \), or \( L^2 < 4Rt \) where \( L \) is the unsupported length. Logan approximated the tapered wall of the cup as a rectangular flat plate subjected to circumferential compression and radial tension. The edges of the plate (corresponding to the die and punch radii) were considered as “clamped” due to the constraints resulting from bending under tension around the respective radii. By these approximations, he was able to apply Bulson’s flat-plate wrinkling model (critical hoop stress at onset of wrinkling) [Bulson, 1969] for deep drawing of conical cup [Logan, 1985]. However, an addition adjustable parameter accounting for appropriate dimensions of ‘rectangular plate” had to be introduced to Bulson’s model to fit the experimental data. This model was then used with a finite element simulation of deep drawing. The stresses from a FEM prediction provides necessary information to use the buckling criterion. For an appropriate choice of the adjustable parameters, a fairly good comparison was achieved between the analysis and experiment. Again, it was found that planar and normal anisotropy play the most important role in controlling wall wrinkling, although sheet thickness is also important. Havranek [1975,1977] conducted a systemic experimental study of the influences of sheet mechanical properties, blankholder force, tool radii, and friction on wall wrinkling and fracture in the deep drawing of conical cups. His study indicated that (a) sheet materials with a high normal anisotropy (r-value) are more wrinkle-resistant, and the first wrinkle appears in the lowest r-value directions; (b) the high strain hardening (n-value) increases the compressive hoop stress and thereby increases the tendency for wall wrinkling. On the other hand, a high n-value is more resistant to strain-localization and allows a higher BHF resulting in more stretch and less draw-in along the radial direction and reduces the tendency to wrinkling. The overall effect is that a higher n-value material has a greater drawing depth or limiting drawing ratio (LDR); (c) high yield stress or low ratio of elastic module to yield stress promotes wall wrinkling; (d) there exists an optimum blankholder force which produces the deepest cup, and at this load, both wall wrinkling and fracture
over the punch nose occur simultaneously; and (e) the changes of die profile radius, blank diameter or lubrication under the blankholder have little effect on LDR and workable BHF range. Donoghue et al. [1989] conducted experiments and used finite element modeling to investigate wall wrinkling in a hemispherical cup subjected to a non proportional loading path. The prebifurcation functional is based on the virtual work principle and incorporates a membrane approximation and flow theory of plasticity. The experimental and simulation results indicated that the membrane bifurcation solution was able to achieve good agreement with experiment, and the flow theory provided a better comparison than deformation theory for the non proportional loading path. The later conclusion is contradictory to what is observed in the vast majority of plastic buckling experiments in which proportional or near proportional loading conditions exist.

In modeling wall wrinkling, a curved shell model should be used to achieve better accuracy than a flat plate model. In the elastic buckling literature, DMV (Donnel-Mushtari-Vlasov) shell theory has been applied to study the wrinkling phenomenon of a three-dimensional shell with compound curvatures. The DMV model is based on shallow shell theory and is able to determine the critical condition (stress or strain) for a short wavelength wrinkling mode [Brush, 1975]. Hutchinson [1974] applied Hill’s bifurcation criterion [Hill, 1958, 1961] for a class of three-dimension solids to the DMV theory and proposed a general elastic-plastic bifurcation criterion for the DMV theory of plates and shells. The body wrinkling can be considered as the bifurcation of the plane stress (in-plane) deformation to the out-of-plane bending type of deformation. In this wrinkling model, both the total strain (or deformation) theory and the incremental (or flow) theory are considered. With the context of this model, a specific instability criterion was suggested to predict the conditions for the onset of wall wrinkling in doubly-curved and isotropic sheet metal undergoing a forming operation [Hutchinson and Neale, 1985]. However, the importance of plastic anisotropy was not considered in this model, and the enhancement of this model for anisotropic sheet materials is necessary.
In this study, wrinkling criteria for an elastic isotropic and plastic anisotropic shell with compound curvature is proposed based on DMV shell theory, bifurcation analysis, and an incremental constitutive law incorporating with Hill’s new yield criterion for transverse anisotropic materials. The critical stresses and strains for the onset of wall wrinkling during sheet forming are related to the material properties (elastic modulus and Poisson’s ratio, flow stress, strain hardening, and plastic anisotropy), geometrical dimensions (wall inclination and die clearance, sheet thickness and principle curvatures determined), and loading conditions (ratios of radial tensile stress or strain to hoop compressive stress or strain). The effects of blankholder force and drawbead force, friction, punch and die profile radii around which bending, unbending and sliding occur, are reflected by the magnitude of the radial tensile stress or strain. In general, this wall wrinkling model can be applied to both shrink flanging and deep drawing. In shrink flanging operations, the only failure mode is wrinkling at the flange edge which undergoes maximum compression. The wrinkling criterion which is related to the flange height and sheet thickness and properties can be used for approximate part and tool design and selection of sheet materials. In deep drawing applications, this criterion is a dynamic one, i.e. the critical stress or strain at the onset of wrinkling changes for each incremental deformation or punch height because of the changes in the stress ratio, deformation geometry, and incremental moduli which are related to current stress state as the punch travels. At each incremental deformation or punch height, the information on stresses (or strains) and geometrical configuration provided by numerical (FEM) simulation can be used to calculate the critical wrinkling stress or strain corresponding to the current stress or strain level and deformation geometry. This critical value is then compared with the maximum stress in each element of the unsupported wall. The wrinkling tendency is predicted if the maximum stress or strain reaches the critical value. To suppress the wrinkles, an increase in the BHF or drawbead force (DBF) which is related to bead penetration is necessary. Through finite element simulations and using
both wrinkling and tearing criteria, a relationship or a curve of BHF or DBF vs. punch stroke could be generated and used as an adaptive controller of deep drawing operations to form defect-free sheet metal parts.

7.2. INCREMENTAL STRESS-STRAIN CONSTITUTIVE EQUATIONS FOR ELASTIC ISOTROPIC AND PLASTIC ANISOTROPIC SOLIDS

For solids undergone a large or finite elasto-plastic deformation, the Jaumann rates or increments of stresses must be used to ensure that the rigid-body motion and rotation will not produce any stress change in the solid, or the measure of the stress and strain increments must be independent of the current rate of rigid body motion. Therefore, a rate or incremental form of the stress-strain constitutive relationship is necessary for the study of large elasto-plastic deformations.

Most constitutive equations for elasto-plastic deformation of a solid are derived based on the generalized Hook’s law for elastic response and Von Mises’s yield theory for isotropic solids undergoing plastic deformation [Lee, 1962; Needleman and Rice, 1978; Hutchinson and Neale, 1978]. A few studies deal with the anisotropic plastic response but none of these provide an explicit expression of the incremental moduli [Triantafyllidis and Needleman, 1980; Wang and Lee, 1989]. In this section, a general expression of a constitutive equation will be given first for solids with the deformation behavior of elastic isotropy and plastic anisotropy. Then, an explicit formulation of the incremental stress-strain relation will be derived under plane stress condition which is a primary deformation mode in most sheet forming operations. Both strain hardening and plastic anisotropy are incorporated in this constitutive relation.
7.2.1 Constitutive Equation for Elasto-Plastic Solids

For a solid statically stressed beyond its elastic limit, a linear decomposition of strain increments into elastic and plastic parts is valid because the recoverable elastic strains are small. The tensor expression for the strain increments is

\[ \Delta e_{ij} = \Delta e^e_{ij} + \Delta e^p_{ij} \quad (i, j = 1, 2, 3) \]  \hspace{1cm} (7.1)

where \( \Delta e_{ij} \) is the total increment strain, and the superscripts 'e' and 'p' stand for elastic and plastic, respectively.

The generalized Hook’s law relates the elastic strain increments to the stress increments (\( \Delta \sigma_{ij} \)) as follows

\[ \Delta e^e_{ij} = \frac{1}{E} [(1 + v)\Delta \sigma_{ij} - v\Delta \sigma_{kk}\delta_{ij}] \]  \hspace{1cm} (7.2)

where \( E \) and \( v \) are Young’s modulus and Poisson’s ratio, \( \Delta \sigma_{kk} \) (\( k = 1, 2, \) and \( 3 \)) is a sum of the three normal stresses according to the tensor summation convention, and \( \delta_{ij} \) is the Kronecker delta defined by

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

The plastic strain increments may be defined in terms of a plastic potential \( f(\sigma_{ij}) \) or yield locus \( \overline{\sigma}(\sigma_{ij}) \) if the flow (or incremental deformation) theory is applied. According to Hill [1950], the plastic strain increments are related to the stress increments by

\[ \Delta e^p_{ij} = \alpha H \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \Delta \sigma = \alpha H \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \frac{\partial f(\sigma_{kl})}{\partial \sigma_{kl}} \Delta \sigma_{kl} \]  \hspace{1cm} (7.3)

where \( \alpha \) is a parameter to identify the plastic loading (\( \alpha=1 \)) and elastic loading and unloading (\( \alpha=0 \)), that is
\[ \alpha = \begin{cases} 1 & \text{if } df \geq 0 \\ 0 & \text{if } df < 0 \end{cases} \]  

(7.4)

and \( H \) is a scalar factor related to strain history, current stress state, and strain hardening, and it is found from

\[ H = \frac{1}{E_T} - \frac{1}{E} \]  

(7.5)

where \( E_T \) is the instantaneous slope of the flow curve, i.e. the effective stress - effective strain curve, or

\[ E_T = \frac{d\bar{\sigma}}{d\varepsilon} \]  

(7.6)

Substituting Eqs. (7.2) - (7.6) into Eq. (7.1), the constitutive relationship between the strain increments and stress increments is given by

\[ d\varepsilon_{ij} = \frac{1}{E} \left[ (1 + v) d\sigma_{ij} - v d\sigma_{kk} \delta_{ij} \right] + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \frac{\partial f(\sigma_{kl})}{\partial \sigma_{kl}} d\sigma_{kl} \]  

(7.7)

7.2.2. Explicit Formulae of a Constitutive Equation for Elastic Isotropic and Plastic Anisotropic Solids

The plastic potential, \( f(\sigma_{ij}) \), may take the form of the yield locus or effective stress, \( \bar{\sigma}(\sigma_{ij}) \). In order to consider the plastic anisotropy, we introduce Hill’s nonquadratic yield criterion for normal anisotropic materials [Hill, 1979]. When the principle directions of the stresses are coaxial with the principle axes of anisotropy (which are the intersections of the mutually orthogonal planes of symmetry), Hill’s 1979 non-quadratic yield function is given by:

\[ \bar{\sigma} = C_1 \left| C_2 \right| 2 \delta_{ii} \delta_{jj} \sigma_{ij} - \delta_{ij} \delta_{kk} \sigma_{kk} \sigma_{ij}^{M/2} + \left| \delta_{ij} \sigma_{ij}^{M} \right|^{1/M} \]  

(7.8)
with

\[ C_1 = \left[ \frac{1}{2(1 + \bar{R})} \right]^{1/M} \text{ and } C_2 = (1 + 2\bar{R}) \quad (7.9) \]

where \( \bar{R} \) is the normal anisotropy and \( M \) is a new index describing the shape of the yield locus. It was found that the effect of the \( M \) value on the shape of the yield locus is opposite to that of normal anisotropy, and the yield locus expands or the yield stress increases along equal biaxial strain directions (45 degree line in first quadrant of yield locus) as \( M \) decreases. Hill's 1979 non-quadratic yield function has been found more versatile for materials with different microstructures and anisotropic behavior [Kobayashi et al, 1985; Hosford, 1988], and it covers the traditional isotropic yield theories proposed by Tresca if \( M=1 \) and \( \bar{R}=1 \), and Von Mises if \( M=2 \) and \( \bar{R}=1 \), and Hill's 1948 quadratic anisotropic yield criterion [Hill, 1948] if \( M=2 \). This yield criterion is of particular interest for those materials having a normal anisotropy less than unity for which the quadratic yield criterion gives a significant underestimation of the flow stress in biaxial tension. For aluminum alloys, \( M \) is around 1.6 ~ 2.0 for a strain range of 0.02 ~ 0.18 [Wagoner, 1980]. \( M \) is correlated well with the normal anisotropy for steel, brass, aluminum, and copper using the formula \( M = 1 + \bar{R} \) for \( \bar{R} \leq 1 \) [Ragab and Abas, 1986] and \( M = 2 \) for \( \bar{R} > 1 \) [Bressan and Williams, 1983].

Under plane stress conditions, Hill's nonquadratic yield function is simplified when principal stresses are of interest as in many sheet forming operations. The effective stress is related to the two in-plane principal stresses as follows

\[ \bar{\sigma} = C_1 \left[ C_2 [\sigma_{11} - \sigma_{22}]^M + [\sigma_{11} + \sigma_{22}]^M \right]^{1/M} \quad (7.10) \]

An explicit expression for the constitutive equation (7.7) can then be derived under the 2-D principal stress state as follows.
\[
\begin{align*}
\varepsilon_{11} &= \frac{1}{E} (d\sigma_{11} - v d\sigma_{22}) + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \frac{\partial \sigma}{\partial \sigma_{11}} \left( \frac{\partial \sigma}{\partial \sigma_{11}} d\sigma_{11} + \frac{\partial \sigma}{\partial \sigma_{22}} d\sigma_{22} \right) \\
&= \left[ \frac{1}{E} + \alpha \left( \frac{1}{E} - \frac{1}{E_T} \right) \left( \frac{\partial \sigma}{\partial \sigma_{11}} \right)^2 \right] d\sigma_{11} + \left[ \frac{-v}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \frac{\partial \sigma}{\partial \sigma_{11}} \frac{\partial \sigma}{\partial \sigma_{22}} \right] d\sigma_{22} \tag{11a}
\end{align*}
\]

\[
\begin{align*}
\varepsilon_{22} &= \frac{1}{E} (d\sigma_{22} - v d\sigma_{11}) + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \frac{\partial \sigma}{\partial \sigma_{22}} \left( \frac{\partial \sigma}{\partial \sigma_{11}} d\sigma_{11} + \frac{\partial \sigma}{\partial \sigma_{22}} d\sigma_{22} \right) \\
&= \left[ \frac{-v}{E} + \alpha \left( \frac{1}{E} - \frac{1}{E_T} \right) \frac{\partial \sigma}{\partial \sigma_{11}} \frac{\partial \sigma}{\partial \sigma_{22}} \right] d\sigma_{11} + \left[ \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \left( \frac{\partial \sigma}{\partial \sigma_{22}} \right)^2 \right] d\sigma_{22} \tag{11b}
\end{align*}
\]

\[
\begin{align*}
\varepsilon_{33} &= \frac{1}{E} [d\sigma_{33} - v (d\sigma_{11} + d\sigma_{22})] + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \frac{\partial \sigma}{\partial \sigma_{33}} \left( \frac{\partial \sigma}{\partial \sigma_{11}} d\sigma_{11} + \frac{\partial \sigma}{\partial \sigma_{22}} d\sigma_{22} \right) \\
&= \frac{-v}{E} (d\sigma_{11} + d\sigma_{22}) \tag{11c}
\end{align*}
\]

or

\[
\begin{align*}
\varepsilon_{11} &= C_{11} d\sigma_{11} + C_{12} d\sigma_{22} \\
\varepsilon_{22} &= C_{12} d\sigma_{11} + C_{22} d\sigma_{22} \\
\varepsilon_{33} &= \frac{-v}{E} (d\sigma_{11} + d\sigma_{22}) \tag{7.12}
\end{align*}
\]

where the compliance moduli \( C \) are defined as

\[
\begin{align*}
C_{11} &= \frac{1}{E} + \alpha \left( \frac{1}{E} - \frac{1}{E_T} \right) \left( \frac{\partial \sigma}{\partial \sigma_{11}} \right)^2 \\
C_{12} &= \frac{-v}{E} + \alpha \left( \frac{1}{E} - \frac{1}{E_T} \right) \frac{\partial \sigma}{\partial \sigma_{11}} \frac{\partial \sigma}{\partial \sigma_{22}} \\
C_{22} &= \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \left( \frac{\partial \sigma}{\partial \sigma_{22}} \right)^2 \tag{7.13}
\end{align*}
\]

An inverse of the first two terms in Eq. (7.12) is found to be

\[
\begin{align*}
\sigma_{11} &= L_{11} \varepsilon_{11} + L_{12} \varepsilon_{22} \\
\sigma_{22} &= L_{12} \varepsilon_{11} + L_{22} \varepsilon_{22} \tag{7.14}
\end{align*}
\]

and the instantaneous stiffness moduli \( L \) are
With the yield criterion in Eq. (7.10), the derivatives are found to be
\[
\frac{\partial \sigma}{\partial \sigma_{11}} = \frac{C_{11}^M}{\sigma_{\text{y}}^{M-1}} \left[ C_2 |\sigma_{11} - \sigma_{22}|^{M-1} + |\sigma_{11} + \sigma_{22}|^{M-1} \right]
\]
(7.16)
\[
\frac{\partial \sigma}{\partial \sigma_{22}} = \frac{C_{12}^M}{\sigma_{\text{y}}^{M-1}} \left[ -C_2 |\sigma_{11} - \sigma_{22}|^{M-1} + |\sigma_{11} + \sigma_{22}|^{M-1} \right]
\]
(7.17)

If the strain hardening is described by Swift's equation, i.e.:
\[
\sigma = k(\varepsilon_0 + \bar{\varepsilon})^n,
\]
(7.18)
then the instantaneous slope \( E_T \) of the flow curve can be defined as
\[
E_T = \frac{\partial \sigma}{\partial \varepsilon} = nk(\varepsilon_0 + \bar{\varepsilon})^{n-1} = \frac{n}{(\varepsilon_0 + \varepsilon)} \overline{\sigma}
\]
(7.19)

where \( k \) is a strength coefficient, \( n \) is strain hardening exponent, \( \varepsilon_0 \) is the prestrain, and \( \bar{\varepsilon} \) is the effective strain. For a proportional loading path, the effective strain is given by
\[
\bar{\varepsilon} = \frac{1}{D_1}[D_2(\varepsilon_{11} - \varepsilon_{22})^{\frac{M}{M-1}} + (\varepsilon_{11} + \varepsilon_{22})^{\frac{M}{M-1}}]^{\frac{M-1}{M}}
\]
(7.20)

with
\[
D_1 = [2(1 + R)]^{\frac{M}{M-1}} / 2 = \frac{1}{2C_1}
\]
(7.21)
\[
D_2 = (1 + 2R)^{\frac{1}{M-1}} = C_2^{\frac{1}{M-1}}
\]
Note that all the instantaneous moduli $C$ (in Eq. (7.13)) or $L$ (in Eq. (7.15)) are stress dependent in the elasto-plastic deformation, and they change as the current stress state changes for every incremental deformation step.

7.3. DONNELL-MUSHTARI-VLASOV (DMV) THEORY FOR QUASI-SHALLOW SHELLS OF GENERAL SHAPES

A shell (Fig. 7.1) is defined as a body in which the distance from any point within the body to some reference surface (usually the shell middle surface) is relatively small compared with any typical dimension of the reference surface (such as the radius of the curvature). Because of this smallness in the dimension normal to the surface, a three-dimensional deformation problem in a shell may be simplified to a two-dimensional one. For the shell stability analysis, the simplest possible form of the shell equations is the Donnell-Mushtari-Vlasov (DMV) equation for shells with general shape [Brush and Almroth, 1975]. A quasi-shallow shell is the one that is relative flat before deformation, and its displacement components in the deformed configuration are rapidly varying functions of the shell coordinates. For such quasi-shallow shells, the rotations about the normal are functions of the displacement normal to the shell, but independent of the in-plane (tangential) displacements.

7.3.1. Assumptions

(i) The shell is thin relative to its curvature, i.e., $t/R$ is much less than unity (where $R$ is the smallest principal radius of curvature of the undeformed middle surface),

(ii) A plane stress state exists at each point through the thickness, i.e., the contributions of the transverse (normal and shear) stresses to the strain energy are ignored,
Fig. 7.1 A shell element (a) force and moment equilibrium, and (b) deformation and geometrical relations.
(iii) The strains are small compared with unity, and hence the strains on the middle surface are linear function of the middle surface displacements,

(iv) The characteristic wavelength of deformation is large compared with the shell thickness but small compared to the radii of curvature of its middle surface.

3.2. Incremental Strains

The geometry of a shell can be completely defined by the shape of its middle surface and its thickness at all points. Therefore, the displacement of any point inside the shell can be expressed in terms of the displacement of a corresponding point on the middle surface.

Using a convective or co-moving coordinate system $x_i$ (i=1,2,3), the initial and deformed configurations of a shell are referred to the same coordinates. That is, when the coordinate system is set in the middle surface of the unformed (prebuckling) shell, the material points in a shell are identified by coordinates $x_i$ (i=1,2) lying in the middle surface of the undeformed body and coordinate $x_3$ normal to the undeformed middle surface. Let the shell coordinates $x_1$ and $x_2$ coincide with lines of principal curvatures ($1/r_x$ and $1/r_y$) of the shell, $dU_i$ or $\hat{U}_i$ (i = 1, 2) be the in-plane incremental displacements in the $x_1$ and $x_2$ (or $x$ and $y$) directions, $\hat{W}$ be the incremental (buckling) displacement normal to the middle surface of the shell, and $b_{ij}$ be the curvature tensor of the middle surface in the undeformed (prebuckling) shell. The incremental stretch strains $dE_{ij}$ (or $\hat{E}_{ij}$) and the incremental bending strains (or the changes of the curvatures) $dK_{ij}$ (or $\hat{K}_{ij}$) in the middle surface are defined as

$$\hat{E}_{ij} = \frac{1}{2}(\hat{U}_{i,j} + \hat{U}_{j,i}) + b_{ij}\hat{W} + \frac{1}{2}\hat{\beta}_i\hat{\beta}_j \quad (i, j = 1, 2)$$  \hspace{1cm} (21a)
\[ \dot{K}_{ij} = -\dot{W}_{ij} \]  
\[(21b)\]

where the incremental rotations \( \dot{\beta}_i \) are defined as

\[ \dot{\beta}_i = -\dot{W}_{.i} = \frac{\partial \dot{W}}{\partial x_i} \quad (i=1, 2) \]  
\[(7.22)\]

In the above equations, the commas (, ) denote the covariant differentiations with respect to the general surface coordinates \( x_1 \) and \( x_2 \), i.e., \( \dot{U}_{i,j} = \frac{\partial \dot{U}_i}{\partial x_j} \) and \( \dot{W}_{.ij} = \frac{\partial^2 \dot{W}_i}{\partial x_j^2} \).

The nonlinear terms in Eq. (21a) come from the rotations about the normals (or the slopes along the \( x_i \) directions). These nonlinear terms may be omitted in comparison with unity when the rotations are small. In such a case, a linear theory is recovered.

The incremental Lagrangian strains for any point inside the shell with distance \( x_3 \) or \( z \) can then be defined as

\[ \dot{e}_{ij} = \dot{E}_{ij} + x_3 \dot{K}_{ij} \]
\[ \dot{K}_{ij} = -\dot{W}_{ij} \]  
\[(7.23)\]

### 7.3.3. Incremental Stress Resultants and Stress Couples

The incremental stress resultants and stress couples (per unit width) are defined as

\[ \dot{N}_i = \int_{-t/2}^{t/2} \dot{\sigma}_i dx_3 \]
\[ \dot{\sigma}_{ij} = \int_{-t/2}^{t/2} \dot{\sigma}_{ij} x_3 dx_3 \]  
\[(7.24)\]

Substitute the incremental strains in (7.23) into the constitutive equations in Eq. (7.14), and then substitute the results into Eq. (7.24). The incremental stress resultants and stress couples are then found to be
When the rotations are small, the nonlinear terms can be neglected. Then Eq. (7.24) reduces to

\[ \dot{N}_{ij} = t L_{ijkl} \dot{E}_{kl} \]
\[ \dot{M}_{ij} = \frac{t^3}{12} L_{ijkl} \dot{K}_{kl} \]  

(7.26)

where the \( M_{ij} \) are bending moments when \( i = j \), and twisting moments when \( i \neq j \).

Note that the instantaneous moduli \( L \) is rewritten as the 4th-order tensor \( (L_{ijkl}) \) for convenience of the general expression of the constitutive equations, and they will return to two index forms as \( L_{1111} = L_{11} \), \( L_{2222} = L_{22} \), and \( L_{1122} = L_{12} \), etc. for explicit expression of the stress-strain relationship.

7.4. WRINKLING CRITERIA

7.4.1 Bifurcation Functional for a General Shell with Compound Curvatures

The body wrinkling can be considered as the bifurcation of the in-plane deformation (prebuckling) under the plane (membrane) stress condition to the out-of-plane bending type of deformation (buckling). Consider \( x_1 \) and \( x_2 \) to coincide with the principal axes; let \( \sigma_1 = \sigma_{11} \) and \( \sigma_2 = \sigma_{22} \) be the principal in-plane (membrane) stresses along these axes; and let \( R_1 \) and \( R_2 \) be the radii of the principal curvatures. A bifurcation functional was proposed by Hutchinson [1974] based on Hill’s general theory of uniqueness and bifurcation in elastic-plastic solids [Hill, 1958,1959, 1961]. This bifurcation functional is given by
\[ F(\hat{U}_i, \hat{W}) = \int_S \left\{ \hat{M}_{ij} \hat{K}_{ij} + \hat{N}_{ij} \hat{E}_{ij} + \hat{N}_{ij} \hat{\dot{\theta}}_{ij} \right\} dS \]
\[ = \int_S \left\{ \frac{t^2}{12} \hat{L}_{ijkl} \hat{K}_{kl} \hat{K}_{ij} + t \hat{L}_{ijkl} \hat{E}_{kl} \hat{E}_{ij} + \hat{N}_{ij} \hat{W}_i \hat{W}_j \right\} dS \]  

(7.27)

where \( S \) denotes the region of the shell middle surface over which the wrinkles appear.

This bifurcation functional is the total energy for wrinkling occurrence. In the right hand side of Eq. (7.27), the first term represents the bending energy \((i=j)\) and twisting energy \((i \neq j)\), the second term is the strain energy due to the membrane stresses, and the third term may be interpreted as the potential energy of the edge stress or the work done by the applied in-plane stresses in the middle surface. For all admissible incremental displacement fields \( \hat{U}_i \) and \( \hat{W} \), if \( F > 0 \), then the incremental solutions of the deformation are unique and bifurcation is not possible because to create wrinkles requires the energy supply, i.e. the total potential of the system to increase which is not a natural or spontaneous process from a thermodynamics point of view. Then \( F=0 \) corresponds the critical conditions for wrinkles to occur for some non-zero incremental displacement fields. That is, the wrinkling criterion may be written as

\[ F(\hat{U}_i, \hat{W}) = 0 \quad (\text{not all } \hat{U}_i \text{ and } \hat{W} = 0) \]  

(7.28)

### 7.4.2 Incremental Displacements in Wrinkling

The incremental displacement fields given by Hutchinson and Neale [1985] are employed for wrinkling in short-wavelength and shallow modes. In these fields, wrinkling is assumed to occur over a certain local region \( S \) of the sheet which spans many wavelengths of the wrinkling mode. Therefore, in a local wrinkling problem, the continuity conditions or boundary conditions along the boundaries of region \( S \) are relatively unimportant. The fields are given as follows
\[ \dot{W} = A \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]
\[ \dot{U}_1 = B \sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]
\[ \dot{U}_2 = C \cos(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi) \]

(7.29)

with

\[ \xi = \sqrt{R t} \]

(7.30)

where

\( t \) = the thickness of the sheet.
\( R \) = the radius of the principal curvature (\( R_1 \) or \( R_2 \)).
\( A, B, \) and \( C \) = constants representing the relative displacement amplitudes of the mode shape.
\( \lambda_1 \) and \( \lambda_2 \) = dimensionless wave numbers.

7.4.3. Wrinkling Criterion for a Double Curved Shell

The membrane forces due to the in-plane stresses can be defined as

\[ N_{11} = -\sigma_{11} t \]
\[ N_{22} = -\eta \sigma_{22} t \]

(7.31)

and

\[ \eta = \begin{cases} 1 & \text{\( \sigma_{22} \) is tensile stress} \\ -1 & \text{\( \sigma_{22} \) is compressive stress} \end{cases} \]

(7.32)

Finally, the bifurcation functional is obtained (Appendix O) by substituting Eqs. (7.31) and the strain increments [defined by the kinematic relationships in Eqs. (7.21), (7.25), and (7.29)] into Eq. (7.27), i.e.
\[ F = \dot{U}_b + \dot{U}_m + \dot{U}_N = \frac{t^3}{12} \zeta S (\frac{c_t}{\xi})^4 \left( L_{12} \lambda_1^4 + L_{22} \lambda_2^4 + (L_{12} + L_{44}) (\lambda_1 \lambda_2)^2 \right) \\
+ t^3 \zeta S \left( L_{11} \left( \frac{B \lambda_1}{\xi} + Ab_1 \right)^2 + L_{22} \left( \frac{C \lambda_2}{\xi} + Ab_2 \right)^2 + L_{12} \left( \frac{B \lambda_1}{\xi} + Ab_1 \right) \left( \frac{C \lambda_2}{\xi} + Ab_2 \right) \right) \\
+ L_{44} \left( \frac{1}{4} \frac{B \lambda_1}{\xi} + \frac{C \lambda_2}{\xi} + A (b_1 + b_2)^2 \right) - At^3 \zeta S \left[ \sigma_{11} (\frac{\lambda_1}{\xi})^2 + \eta \sigma_{22} (\frac{\lambda_2}{\xi})^2 \right] \]

(7.33)

or in the matrix form as

\[
F = \zeta S t \left( \frac{c_t}{\xi} \right)^2 \begin{bmatrix} A \\ B \\ C \end{bmatrix} [M] \{A B C\} = \zeta S t \left( \frac{c_t}{\xi} \right)^2 \begin{bmatrix} A \\ B \\ C \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \{A B C\} \quad (7.34)
\]

where the matrix \([M]\) is given by

\[
M_{11} = \frac{t^3}{12} \left( \frac{c_t}{\xi} \right)^2 \left[ L_{11} \lambda_1^4 + L_{22} \lambda_2^4 + 2 (L_{12} + 2 L_{44}) (\lambda_1 \lambda_2)^2 \right] \\
+ \left[ L_{11} \left( \frac{c_t}{R_1} \right)^2 + L_{22} \left( \frac{c_t}{R_2} \right)^2 + 2 L_{12} \left( \frac{c_t}{R_1} \right) \left( \frac{c_t}{R_2} \right) \right] - \left[ \sigma_{11} \lambda_1^2 + \eta \sigma_{22} \lambda_2^2 \right] \\
M_{22} = L_{11} \lambda_1^4 + L_{44} \lambda_2^4 \\
M_{33} = L_{22} \lambda_2^4 + L_{44} \lambda_1^4 \\
M_{12} = M_{21} = L_{11} \lambda_1 \left( \frac{c_t}{R_1} \right) + L_{12} \lambda_1 \left( \frac{c_t}{R_2} \right)
\]

and

\[
M_{13} = M_{31} = L_{22} \lambda_2 \left( \frac{c_t}{R_1} \right) + L_{12} \lambda_2 \left( \frac{c_t}{R_2} \right) \\
M_{23} = M_{32} = (L_{12} + L_{44}) \lambda_1 \lambda_2
\]

where
\[ \zeta S = \int_S [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 dS \]
\[ = \int_S [\sin(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi)]^2 dS \]
\[ = \int_S [\sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 dS \]  
(7.36)

and

\[ \zeta = \begin{cases} 
1/4 & \text{if both } \lambda_1 \text{ and } \lambda_2 \neq 0 \\
1/2 & \text{if either } \lambda_1 \text{ or } \lambda_2 = 0 
\end{cases} \]  
(7.37)

### 7.4.4 Criterion for Wrinkling along Two Principal Axes:

The wrinkling criterion \( F = 0 \), in Eq. (7.28), requires the determinant of the matrix \([M] \) in Eq. (7.34) to vanish since the constants \( A, B, \) and \( C \) are not all zero; that is

\[ \det[M] = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix} = 0 \text{ for wrinkling to occur} \]  
(7.38)

The critical stress values of \( \sigma_{11} \) and \( \sigma_{22} \) can be obtained by minimizing this wrinkling equation with respect to the wave numbers \( \lambda_1 \) and \( \lambda_2 \), i.e.

\[ \frac{\partial[M]}{\partial \lambda_i} = 0 \quad (i = 1, 2) \]  
(7.39)

### 7.4.5 Criterion for Wrinkling along One Principal Axis:

In most sheet forming operations, wrinkling takes place along one of the principal axes, that is, the wrinkles are aligned with one of the principal curvature. If wrinkles are assumed to be perpendicular to the \( x_1 \) direction, then the wave number \( \lambda_2 = 0 \), \( \xi = \sqrt{R_2 t} \), and \( M_{13} = M_{31} = M_{23} = M_{32} = 0 \). Let the principal curvatures be
\[ b_{11} = 1/R_1, \quad b_{22} = 1/R_2 \quad \text{and other } b_{ij} = 0 \quad (7.40) \]

where \( R_1 \) and \( R_2 \) are the radii along the principal axes. Also let the instantaneous moduli be

\[
L_{11} = L_{1111}, \quad L_{22} = L_{2222}, \quad L_{12} = L_{1122}, \quad \text{and} \quad L_{44} = L_{1212} \quad (7.41)
\]

Then, the wrinkling criterion in Eq. (7.38) is simplified to

\[ \det[M] = M_{33}(M_{11}M_{22} - M_{12}^2) = 0 \]

or

\[ \psi = M_{11}M_{22} - M_{12}^2 = 0 \quad (7.42) \]

and

\[
M_{11} = \frac{3}{12} \left( \frac{t}{\xi} \right)^2 L_{11} \lambda_1^4 + \left[ L_{11} \left( \frac{\xi_1}{R_1} \right)^2 + L_{22} \left( \frac{\xi_2}{R_2} \right)^2 + 2L_{12} \left( \frac{\xi_1}{R_1} \right) \left( \frac{\xi_2}{R_2} \right) \right] - \sigma_{11} \lambda_1^2, \\
M_{22} = L_{11} \lambda_1^4, \\
M_{33} = L_{44} \lambda_1^4, \\
M_{12} = M_{21} = L_{11} \lambda_1 \left( \frac{\xi_1}{R_1} \right) + L_{12} \lambda_1 \left( \frac{\xi_2}{R_2} \right) \quad (7.43)
\]

From Eq. (7.43), the wrinkling equation (7.42) depends explicitly on the stress component the \( \sigma_{11} \). The effect of the stress component \( \sigma_{22} \) on wrinkling along \( \sigma_{11} \) direction appears implicitly in the incremental moduli \( L \), defined by Eqs. (7.13), (7.15), and (7.16). With Eqs. (7.42) and (7.43), a relationship between the wrinkling wave number and the stress-can be established, i.e.

\[ \sigma_{11} = \frac{1}{12} \left( \frac{t}{\xi} \right)^2 L_{11} \lambda_1^2 + \frac{1}{\lambda_1^2} \left( L_{22} - \frac{L_{12}^2}{L_{11}} \right) \quad (7.44) \]
Minimizing the stress with respect to the wave number \( \frac{\partial \sigma_{11}}{\partial \lambda_1} = 0 \), we obtain the critical wave number or wavelength \( \lambda_1^c \):

\[
\lambda_1^c = \sqrt{2/3} \left( \frac{L_{11}L_{22} - L_{12}^2}{L_{11}^2} \right)^{1/4} = 1.8612 \left( \frac{L_{11}L_{22} - L_{12}^2}{L_{11}^2} \right)^{1/4} \tag{7.45}
\]

The critical wrinkling stress is then obtained by substituting \( \lambda_1^c \) into Eq. (7.44), i.e.

\[
\sigma_1^c = \frac{1}{\sqrt{3}} \frac{t}{R_2} \sqrt{L_{11}L_{22} - L_{12}^2} = 0.5774 \frac{t}{R_2} \sqrt{L_{11}L_{22} - L_{12}^2} \tag{7.46}
\]

Similarly, for wrinkling perpendicular to the \( x_2 \) direction (\( \xi = \sqrt{R_1} t \) and \( \lambda_1 = 0 \)), the critical wavelength and stress are

\[
\lambda_2^c = \sqrt{2/3} \left( \frac{L_{11}L_{22} - L_{12}^2}{L_{22}^2} \right)^{1/4} = 1.8612 \left( \frac{L_{11}L_{22} - L_{12}^2}{L_{22}^2} \right)^{1/4} \tag{7.47}
\]

and

\[
\sigma_2^c = \frac{1}{\sqrt{3}} \frac{t}{R_1} \sqrt{L_{11}L_{22} - L_{12}^2} = 0.5774 \frac{t}{R_2} \sqrt{L_{11}L_{22} - L_{12}^2} \tag{7.48}
\]

A procedures using those criteria are summarized as follows. The deformation geometry (the current thickness and principal radii), and the stress and strain components are assumed to be known or provided by the finite element simulations. Use these components to calculate the effective stress by Eq. (7.10) and the effective strain by Eq. (7.19). With the effective stress and strain, and the strain hardening equation (7.17), the moduli \( C \) and \( L \) are-defined. Wrinkling conditions are predicted by substituting the moduli \( L \) and the geometrical dimensions (the thickness and the principal radii) into Eqs. (7.45) - (7.48).
7.5 Wrinkling Criterion for Shrink Flanging

In shrink flanging operations (Fig. 7.2), a segmental or whole circular blank is firmly clamped by a blankholder at the radial distance \( R_0 \), and the blank is bent down, by a moving punch, around the curve line of the die shoulder (with a profile radius of \( r_d \)). The blank with the initial radius of \( R_1 \) is formed to a flange with a radius of \( R_2 \). The stress state at the flange edge with a principle curvature of \( \sin \alpha / R_2 \) is uniaxial compression along the hoop direction. Wrinkles occur at the edge where the maximum compressive hoop stress or strain appears. If the hoop stress is assumed to be \( \sigma_{11} \), then the effective stress \( \bar{\sigma} \) is identical to stress \( \sigma_{11} \), i.e. \( \bar{\sigma} = \sigma_{11} \). Therefore, the moduli \( C \) in Eq. (7.13) are

\[
C_{11} = \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right), \quad C_{22} = \frac{1}{E}, \quad \text{and} \quad C_{12} = \frac{-v}{E} \tag{7.49}
\]

From Eq. (7.15), the incremental moduli \( L \) can be defined using Eq. (7.49), that is

\[
L_{11} = \frac{1}{\left[ \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \right] - \frac{v^2}{E}} \left(1 + \alpha \left( \frac{E}{E_T} - 1 \right) \right)
\]

\[
L_{22} = \frac{1}{\left[ \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \right] - \frac{v^2}{E}} \left(1 + \alpha \left( \frac{E}{E_T} - 1 \right) \right)
\]

\[
L_{12} = \frac{v}{\left[ \frac{1}{E} + \alpha \left( \frac{1}{E_T} - \frac{1}{E} \right) \right] - \frac{v^2}{E}}
\]
Fig. 7.2 Shrink flanging: the maximum compression appears at flange edge which has initial radius of $R_1$ and final radius of $R_2$. 

$$R_0$$
$$R_1$$
$$R_2$$
$$R_T$$
$$R_O$$

principal radius $\frac{R_2}{\sin \alpha}$
7.5.1 Elastic Wrinkling

The elastic moduli are obtained by setting the loading index \( \alpha = 0 \) in Eq. (7.50), i.e.

\[
L_{11} = \frac{E}{1 - \nu^2}, \quad L_{22} = \frac{E}{1 - \nu^2}, \quad L_{12} = \frac{\nu E}{1 - \nu^2}
\] (7.51)

Therefore, the elastic wrinkling criterion is given by substituting these moduli into the wrinkling criterion in Eq. (7.46), that is

\[
\sigma_1^c = \frac{1}{\sqrt{3}} \frac{t}{R_2} \sin \alpha \frac{E}{\sqrt{1 - \nu^2}}
\] (7.52)

Or in terms of strain

\[
e_1^c = \sigma_1^c / E = \frac{1}{\sqrt{3}} \frac{t}{R_2} \sin \alpha \frac{1}{\sqrt{1 - \nu^2}}
\] (7.53)

where \( \alpha \) is the flange angle.

7.5.2 Plastic Wrinkling

For the plastic loading case (the loading index \( \alpha = 1 \)), the incremental moduli are

\[
L_{11} = \frac{EE_T}{E - E_T \nu^2}, \quad L_{22} = \frac{E_T^2}{E - E_T \nu^2}, \quad L_{12} = \frac{\nu EE_T}{E - E_T \nu^2}
\] (7.54)

Substituting these moduli into Eq. (7.46), the critical stress for wrinkling is

\[
\sigma_1^c = \frac{1}{\sqrt{3}} \frac{t}{R_2} \sin \alpha \frac{E_T / E}{\sqrt{1 - \nu^2}} \frac{E_T}{E}
\] (7.55)
For a flow curve described by Swift's hardening relation, Eq. (7.17), the slope of the flow curve is defined by Eq. (7.18), that is

\[ E_T = \frac{d\sigma_i}{d\varepsilon_i} = nK(\varepsilon_o + \varepsilon_i)^{n-1} = n\sigma_i / (\varepsilon_o + \varepsilon_i) \]

Finally, the critical strain for wrinkling to occur is found by substituting \( E_T \) into Eq. (7.46),

\[ \varepsilon_i^{\sigma} = \left[ \frac{1}{3} \left( \frac{t}{R_2} \sin \alpha \right)^2 \frac{E}{1 - \nu^2 K} \right]^{\frac{1}{n+1}} - \varepsilon_o \tag{7.56} \]
CHAPTER VIII
BENDING EFFECTS IN STRETCH/DRAW FORMING
AND BENDING CORRECTION FOR
MEMBRANE FINITE ELEMENT MODELING

8.1 Overview

During deep drawing, the sheet is drawn into the die cavity by the punch. As metal flows, the sheet, originally located in blankholder, will undergo three different types of deformation: (a) bending around the die radius, (b) sliding over the die radius, and (c) unbending after sheet passing through the die profile. These three deformation processes proceed under the tension generated by blankholders and drawbeads. This tension generates membrane strains (uniform if no friction at the die and punch) in the sheet. The portion of the sheet under the punch is also subjected to these three deformation processes when symmetry is not present and draw-in proceeds along all directions. For a symmetric tool, the sheet located at this symmetric center will not have in-plane motion; therefore the sliding over the punch radius is relatively small, and unbending is unlikely. In such case, the bending around the punch radius is significant.

The importance of the bending/unbending in stretch/draw forming depends mainly on two conditions: (a) the ratio (R/t) of tool radius, R, to the sheet thickness, t; and (b) the restraining force generated by the blankholder force (BHF) and drawbead force (DBF).
The R/t ratio determines the nominal maximum bending strain on the convex sheet surfaces. This strain increases asymptotically with the decrease of the R/t ratio, as illustrated by the relation below

$$\varepsilon_b = \ln \left( 1 + \frac{t}{2R} \right) = \pm \frac{t}{2R} = \pm \frac{1}{2R} \frac{t}{t}$$

where '+' is for convex surface, and '-' is for concave surface, and the neutral plane is assumed to be the same as the mid-plane (a more accurate formulation considering the true position of the neutral axis will be given later). From this relation, the bending strain is less than 5% when the R/t ratio is greater than 10 (Fig. 8.1). However, as the ratio falls below 5, the bending strain increases rapidly. It is seen that the bending strain increases asymptotically with the decrease in the tool radius and increase linearly with the sheet thickness. A 20% strain can be generated by bending alone for a R/t ratio less than 3. The bending moment is also a function of the bending ratio R/t (see Eq. (2.7)) as shown in Fig. 8.2. It is seen that the moment increases rapidly (asymptotically) as the R/t ratio falls below 5.

When the sheet is subjected to bending, sliding, and unbending as it passes through the die radius, there will be a non-local bending effect (the overall strain level is affected by bending, sliding and unbending which cause the increase of the in-plane tension force) in addition to the local bending effect (the bending affects mainly the local region of bend). The global bending effect increases rapidly as the R/t ratio decreases below 5. Wang demonstrated this phenomenon by pulling a sheet strip over a frictionless radius and found that the increases in drawing force is substantial for R/t ratio less than 5. To keep the global bending effect less than 5%, the R/t ratio must be greater than 8 [Wang, 1983]. The experimental work and FEM simulations in GM also
Fig. 8.1 Bending strain (on convex surface) as a function of R/t ratio in plane strain bending
Fig. 8.2 Bending moment vs. bending ratio R/t in plane strain bending
Fig. 8.3 Increase in draw force $T_s$ for sheet bending, sliding and unbending over a radius $R$ as a function of $R/t$ ($t$ - sheet thickness) for AKDQ steel with no friction. The back tension $T_b$ is taken to be the value for initial yielding at a strain of 0.002.
show that non-local bending is observed for R/t ratio below about 6 [Stoughton, 1985]. Fig. 8.3 shows an increase of the tensile force vs. the bending ratio as sheet passes a bend by bending, sliding, and unbending [Wang, Kinzel, Altan, 1992]. The tension increases rapidly as the bending ratio falls below 5.

The BHF and DBF determine the level of overall membrane strain in the sheet. Higher restraining forces generate higher membrane strains, and the stretching may dominate the draw-in deformation. For stretching dominated sheet forming, sheet thinning is significant, therefore, the bending effect decreases as the thickness decreases. However, for a small tool radius, the bending strain may still be important. For instance, let the tool radius be 3.2 mm (1/8"), the original sheet thickness be 1.0 mm, and thinning strain be 20% (the final thickness be 0.82 mm), then the bending strain is about 13% which is comparable with the membrane strain. In addition, as thinning starts, the neutral axis shifts towards the concave surface and may be totally out of the sheet. This shift causes even higher bending strains because the effective radius decreases with the shift in the neutral axis. When the BHF and DRF are low, the draw-in of the sheet metal may dominate the stretch deformation as in many deep drawing operations for automotive panels. It is difficult to generate membrane strains higher than 5% (which is desired for rigidity and avoiding surface distortion) in the large flat area of panels. Therefore, the importance of bending is more obvious. Neglecting the bending effect can cause significant loss in the accuracy of the process simulation even for relatively large R/t ratios. As an example, for R/t = 10, the bending strain is 5% which is comparable with the membrane strain. Moreover, the lack of bending stiffness in membrane finite elements creates a numerical instability in the simulation of wrinkling dominated deep drawing processes because the wrinkles are surface distortions with small radii, and the wrinkling can be considered as a bifurcation from an in-plane deformation to an out-of-plane bending deformation.
There are two major methods for the analysis of sheet forming: the membrane (no bending) approach and the shell bending theory. In the membrane model, the strain across the sheet thickness is assumed to be constant and bending effects (bending moment and bending stresses) are ignored. The membrane approach is often used in most stretch types of forming with a relative large ratio of tool radius to sheet thickness, R/t. This usually gives adequate accuracy, significant simplification in the analysis, and great efficiencies in computation. The bending effects in stretch/draw forming become important as the ratio of tool radius to sheet thickness, R/t, decreases below 20 [Stoughton, 1985]. In such cases, the membrane analysis of sheet forming is not accurate in general. To some extent, the sole membrane results are not acceptable, especially for draw types of forming with low restraining force [Wang, 1990, 1991]. To deal with this problem, the shell bending model can be used in which the strain varies throughout the sheet thickness, and the bending moment and bending stresses are taken into account. These additional degrees of freedom cause a significant increase in computational time and complexity in the formulation of the analysis using the bending shell model especially in a finite element analysis.

In order to take advantage of the computational efficiency of the membrane formulation and also to consider the bending effects, several bending correction models have been suggested. Stoughton proposed a modified membrane-bending formulation in which bending effects are incorporated into the membrane formulation by considering the local sheet thickness and curvatures along the principal directions for each finite element [Stoughton, 1985]. The bending around the radius gives two types of contributions to the total strains in stretch/draw forming: (a) local variation of strains due to pure bending strain relative to the middle plane (\(\varepsilon_b = \ln (1+y/R_m)\)), and (b) global variation of strains due to additional stress and work which are required to stretch or compress the fibers to different curvatures so that the sheet is moved into or out of a bend. This effect may be interpreted as a local reduction in the stiffness of sheet across
a sharp bend [Wang, 1982; Wenner, 1983]. The additional stress is a function of material properties, friction, current stress level, and the R/t ratio (increases as the ratio of R/t decreases). The local thicknesses are calculated through membrane theory and the local curvatures of each element are obtained through the tool surface description which gives the first and the second derivatives of the surface.

The simple bending correction based on geometric consideration was used to evaluate the maximum surface strains by simply adding the local bending strain to the membrane strain [Wang and Wagoner 1991]. Using elementary bending theory, the local bending strain is calculated based on the contact tool radius and the current sheet thickness provided by membrane analysis. It was found that this simple bending correction gave much better results than the membrane analysis alone for the ratio, R/t, greater than six. However, the simple bending correction is not good enough for two cases: (a) stretch forming with small tool radii (R/t = 3.2), and (b) deep drawing with a low blankholder force. In case (a), the neutral axis was assumed to be the same as the middle axis, and global bending effects were not considered. In case (b), the strain hardening history during unbending and sliding were not considered. The neutral axis shift toward the concave side of sheet during bending because of (a) the increase in the local contact pressure in bending with a small radius, as indicated in Section 2.2 ("Advanced Bending Theory"), and (b) the applied axial tension, as studied in Section 3.3 ("Bending under Tension"). The determination of the actual position of the neutral axis is considered to be a key to evaluate the bending effects accurately. In this chapter, more advanced models of bending correction for membrane analysis are proposed to account for (a) the shift of the neutral axis induced by tension (membrane force) and sheet thinning, (b) local and non-local bending effects, (c) the strain hardening history in bending, unbending, and sliding for a sheet to pass through a bend, and (d) a decoupled method for bending correction for membrane finite element modeling of stretch/draw forming processes.
8.2 Evaluation of Curvatures of Sheet Elements

8.2.1 Radii of the Mid-Plane, Convex and Concave Surfaces

For the membrane finite elements, the curvature of the mid-plane may be calculated as the average value of the radii at each node of an element [Stoughton, 1985]. For a three-node triangular element whose nodes contact with the tool, the principal curvature is described as (Fig. 8.4)

\[
K = \frac{1}{R_m} = \frac{\frac{\partial^2 z}{\partial x^2} \cos^2 \theta + 2 \frac{\partial^2 z}{\partial x \partial y} \cos \theta \sin \theta + \frac{\partial^2 z}{\partial y^2} \sin^2 \theta}{[1 + (\frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta)^2] \sqrt{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}}}
\]

(8.1)

where \( \theta \) is the principal direction. This description requires a knowledge of the first and the second derivatives of the tool surface equation:

\[
z = z(x, y)
\]

(8.2)

For the element partially in contact with tool radius, if one of the nodes has coordinates \( X_n \), and a vector pointing in the principal direction and dissecting the line connecting the other two nodes at a point \( X_i \), then the curvature may better be evaluated as:

\[
K = \frac{1}{R_m} = \frac{2[z_i - z_n + \frac{\partial z}{\partial x} (x_i - x_n) + \frac{\partial z}{\partial y} (y_i - y_n)]}{|X_i - X_n| \sqrt{1 + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}}}
\]

(8.3)
Fig. 8.4 Geometrical construction used to determine the curvature of an element in the direction of a principal strain axis [Stoughton, 1985]
For a fine mesh, the difference between Eqs. (8.1) and (8.3) may be negligible.

For a section analysis using a 'line' element with two nodes (Fig. 8.4), as in the SECTIONFORM finite element program, there are two approaches in evaluating the curvatures of elements. One method is to use Eq. (8.3) based on the first and second derivatives of nodal coordinates. In this study, we proposed another approach to define the curvatures of elements using three consecutive nodal points which construct two adjacent elements (Fig. 8.5).

In this method, a curve (with a radius R) passes the three nodal points which correspond to the two adjacent elements. The radius of this curve can be evaluated using the three nodal coordinates. The mathematical equations to define the curvatures are given as follows.

The lengths of elements I and J are defined by their corresponding nodes as

\[
L_{ij} = \sqrt{(x_j - x_i)^2 - (y_j - y_i)^2} \quad (8.4a)
\]

\[
L_{jk} = \sqrt{(x_k - x_j)^2 - (y_k - y_j)^2} \quad (8.4b)
\]

And the distance between node K and node I is defined as

\[
L_{ki} = \sqrt{(x_k - x_i)^2 - (y_k - y_i)^2} \quad (8.5)
\]

The angles A and B can be calculated using these quantities by the cosine law, i.e.

\[
\cos A = \frac{L_{ki}^2 + L_{jk}^2 - L_{ij}^2}{2L_{jk}L_{ki}} \quad (8.6a)
\]

\[
\cos B = \frac{L_{ij}^2 + L_{ki}^2 - L_{jk}^2}{2L_{jk}L_{ki}} \quad (8.6b)
\]

The height of the triangle, h, is then determined by
Fig. 8.5 Evaluation of the curvatures of elements by three nodal coordinates
\[ h = L_{ij} \sin B = L_{jk} \sin A \] (8.7)

The height \( h \) is used as the criterion for a flat sheet element if the relative value \( (h/L_{kl}) \) is very small (say, < 0.0001).

Using sine rule, the radius of the curve passing the three nodes is defined as

\[ \frac{L_{ij}}{\sin A} = \frac{L_{jk}}{\sin B} = 2R \]

or

\[ R = \frac{L_{ij}}{2 \sin A} = \frac{L_{jk}}{2 \sin B} \] (8.8)

This radius can be considered as the radius of the mid-plane, \( R_m \). Then the radii of the outer convex surface (\( R_o \)) and inner concave surface (\( R_i \)) can be calculated by the mid-plane radius and current sheet thickness of the elements. That is

\[ R_o = R_m + \frac{t}{2} \] (8.9)

\[ R_i = R_m - \frac{t}{2} \] (8.10)

### 8.2.2 Radius of Neutral Plane

For bending with large curvature or bending with tension, the neutral axis moves towards the tool surface or the concave side of the sheet because of the local contact pressure build-up and the thinning of sheet. Finding the actual position of the neutral axis is essential for the evaluation of bending effects. For a rigorous analysis, the shift of the neutral axis can be evaluated using both the theory of bending under tension and the advanced bending theory. The shift of the neutral layer due to the local pressure increase for bending around a small radius can be estimated using Eq. (3.42), as discussed in Section 3.2 (‘Advanced Bending Theory’) in Chapters three. The shift of
the neutral axis due to applied tension (here it is the membrane force) can be evaluated using Eq. (3.72) as discussed in Section 3.3 ('Mechanics of Bending under Tension'). In this chapter, a simple model, based on the plastic volume conservation, can also be derived to approximately account for the shift of the neutral axis (Appendix F). The radius of the neutral axis can be expressed in terms of the membrane tensile strain $\varepsilon_m$, tool radius $R_t$, and sheet thickness $t_o$:

$$R_n = R_m - \frac{t_o}{t_o} = (R_t + \frac{t_o}{2} e^{-\varepsilon_m}) e^{-\varepsilon_m}$$  \hspace{1cm} (8.11)

This relation implies that the neutral axis shifts towards the concave surface of the sheet if thinning starts, and $R_n < R_m$. This is the actual case for stretch/draw forming where bending and tension coexist. The shift distance ($d$) of the neutral axis away from the middle axis is determined by

$$d = R_m - R_n = R_m (1 - \frac{t_o}{t_o})$$  \hspace{1cm} (8.12)

The instantaneous thickness, $t$, can be approximated by the membrane strain:

$$t = t_o e^{-\varepsilon_m}$$  \hspace{1cm} (8.13)

where $\varepsilon_m$ is the membrane strain along the longitudinal direction, and $t_o$ is the initial thickness.

8.3 Bending Strains

In plane strain stretch and draw forming, the tension is generated along the length direction, and the sheet is undergone bending, sliding, and unbending in die shoulder and punch shoulder. Therefore, the sheet can be assumed to consist of a straight line in unsupported region (die wall) and flange under the blankholder, and curves around the die and the punch shoulders and drawbead. Between the curved and
straighten regions, there are some transition curvatures for the elements partially contacting with tool surfaces. In finite element modeling, for all nodes with radii, the bending strain can be calculated based on Eq. (8.14), i.e.

$$\varepsilon_x = \ln \left( \frac{L_o}{L_n} \right) = \ln \left( \frac{r_n}{r_n} \right) = \ln \left( \frac{y + R_n}{R_n} \right) = \ln (1 + y/R_n)$$  

(8.14)

where $R_n$ is the radius of the neutral axis, and $y$ is the distance from the neutral axis ($R_m - 0.5t \leq y \leq R_m + 0.5t$). This strain is added to the membrane strain to obtain the total strain.

The maximum bending strain on the convex outer surface (with radius $R_o$) is then:

$$\varepsilon_{bend, \text{max}} = \ln \left( \frac{R_o}{R_n} \right)$$  

(8.15a)

And the minimum bending strain on the concave inner surface (with radius $R_i$) is determined by

$$\varepsilon_{bend, \text{min}} = \ln \left( \frac{R_i}{R_n} \right)$$  

(8.15b)

8.4 A Decoupled Method of Bending Correction for Membrane FE Analysis

8.4.1 Concepts

Based on the superposition principle, the membrane (stretching) strain and the bending strain can be decoupled (note that, the bending strain along thickness can be approximated as a linear function of the distance from the neutral plane as indicated by equation (2.1)). Therefore, a decoupled method is proposed for the bending correction in the membrane finite element analysis. In this method, the concept of 'decoupled' has two meanings:

(i) the bending strains and the stretching membrane strains are decoupled and subjected to the superposition principle.
(ii) the membrane analysis and bending correction are decomposed in each incremental
deformation step, but the stresses are not subjected to the superposition principle
because they are not linear through the thickness. That is, for the i-th incremental
deformation step, the bending correction procedure is carried out only after the
solution for the membrane analysis of that step is converged. Then, the new
solution (modified thickness and effective strains) for the i-th incremental step after
bending correction is used in the next (i+1-th) incremental membrane analysis.

The advantages of this decoupled method for the bending correction are
described as follows:
(a) The global bending effect is taken into account because the corrected effective
strain and thickness determined by the bending modification in the i-th step will be
accumulated and used in next membrane force calculation. The effective stress and
the membrane stresses in the i+1-th step will increase due to the accumulated
effective strain including bending/unbending history.
(b) This method causes the minimum disturbance in the finite element system equations.
Therefore, it may not cause numerical instability and converging problems
because the bending correction is not included in the Newton-Raphson iteration loop
for equilibrium.
(c) The computational efficiency will not suffer because the CPU time used for the
bending calculation is very short.

8.4.1 Procedures

The procedures to evaluate the bending effect for each incremental step of
membrane finite element analysis are:
(a) Calculate the radii of the mid-plane, the convex and the concave surfaces of all
elements using the three consecutive nodal coordinates (these three nodes
corresponding to two adjacent elements), and thicknesses, using Eqs. (8.4) - (8.10),
(b) Evaluate the radii of the neutral axis for those elements, using Eqs. (8.11) and (8.13),

(c) Calculate the bending strains on both inner concave surface and the outer convex surfaces by Eqs. (8.15a) and (8.15b),

(d) Calculate the total strains (including membrane and bending strains) on the convex and concave surfaces by Eq. (8.16), Fig. 8.6:

\[ \varepsilon_{\text{total,outer}} = \varepsilon_{\text{membrane}} + \varepsilon_{\text{bend,max}} \]
\[ \varepsilon_{\text{total,inner}} = \varepsilon_{\text{membrane}} - |\varepsilon_{\text{bend,min}}| \]

(e) Average the strains on the convex (outer) and concave (inner) surfaces to get the 'corrected strain' (major principal strain along the length direction) through thickness using Eqs. (8.17) and (8.18), :

\[ \varepsilon_{\text{avg}} = \xi_1 \varepsilon_{\text{total,outer}} + \xi_2 \varepsilon_{\text{total,inner}} \]

And the weight factors considering the neutral axis position (Fig. 8.6) are defined as

\[ \xi_1 = \frac{t/2 + d}{t} = 0.5 + \frac{d}{t} \]
\[ \xi_2 = \frac{t/2 - d}{t} = 0.5 - \frac{d}{t} \]

(f) Modify the effective strain using the 'corrected strain' and the relation between the strain components and the effective strain defined by Eqs. (8.19):

\[ \bar{\varepsilon}_{\text{avg}} = F \varepsilon_{\text{avg}} \]

where \( F \) is defined in Eq. (3.11).

(g) Modify the thickness using the 'corrected strain' via Eqs. (8.20):
\[ t = t_0 e^{-\varepsilon_{sm}} \]  \hfill (8.20)

where \( t_0 \) is the initial thickness of the sheet.

Fig. 8.6 Strain distribution and the averaged tensile strain through thickness.
(i) Use the modified thickness and effective strain to calculate the effective stress and the internal membrane force in the next incremental step of the membrane analysis.

(j) Repeat (a) to (g) for every incremental step until the total deformation is completed.

8.5. Finite Element Formulation

SECTIONFORM, a plane-strain section analysis FEM program, employs a membrane approximation and an implicit, incremental type, updated Lagrangian formulation which introduces a minimum plastic work path assumption and handles material, geometric and contact nonlinearities. The material constitutive relations include a rigid-viscoplastic material hardening law, and Hill's non-quadratic yield criterion for normal anisotropic materials [Hill, 1979]. The contact and friction in sheet/tool interface are treated by a modified Coulomb friction model with a smooth transition from the sticking to sliding condition, a unilateral contact law (contact force must be towards the tool), and a "CFS algorithm" (Consistent Full Set) [Saran and Wagoner, 1990]. In CFS algorithm, the complete set of the governing relations, which include kinematic, constitutive, force equilibrium, and the interfacial equations, are linearized to obtain a consistent stiffness matrix which improves the numerical stability. The contact and friction problems are treated by introducing an additional degree of freedom at each node (the contact pressure) so that the contact conditions are directly included in the system equations without the need for arbitrary projection schemes used previously [Wang, 1990]. This idea is similar to the Lagrangian Multiplier Method which introduces the Lagrangian multipliers (represent the contact pressure) into the governing system to enforce the contact conditions.

8.5.1. Incremental Virtual Work Theorem

At each incremental deformation step with an incremental displacement $du$, the incremental virtual work theorem states that the internal work for plastic deformation
equals the external work (supplied by the forming tools) for every kinematically admissible virtual incremental displacement $\delta u = \delta(du)$ during the time interval $dt$, i.e.

$$\delta W_i = \delta W_e$$  \hspace{1cm} (8.21)

where the incremental internal work $dW_i$ and the external work $dW_e$ are given by

$$dW_i = \int_{A_o} \left[ \int_{\tilde{e}_e} \sigma(\tilde{e}) d\tilde{e} \right] d\tilde{A}$$

$$= \int_{A_o} \left[ \int_{\tilde{e}_e} \sigma(\tilde{e}) \frac{\partial(\tilde{e})}{\partial(du)} d\tilde{A} \right] d\tilde{A}$$  \hspace{1cm} (8.22)

$$dW_e = F_e du$$  \hspace{1cm} (8.23)

where the external force vector $F_e$ is defined by the normal and tangent (frictional) components as

$$F_e(du) = F_n + F_t = (n - \mu \phi(du) t) P$$  \hspace{1cm} (8.24)

where $n$ and $t$ are the normal and tangent vectors at the contact nodes, $\phi$ is regulation function used to make a smooth transition from sticking to sliding condition. Therefore, the internal virtual work can be derived as

$$\delta W_i = \delta(dW_i) = \int_{A_o} \overline{\sigma}(\tilde{e}) \frac{\partial(\tilde{e})}{\partial(du)} \delta(du) d\tilde{A} = F_i \delta(du)$$  \hspace{1cm} (8.25)

where the internal force is defined as

$$F_i = \int_{A_o} \overline{\sigma}(\tilde{e}) \frac{\partial(\tilde{e})}{\partial(du)} d\tilde{A}$$  \hspace{1cm} (8.26)

The external virtual work is

$$\delta W_e = F_e \delta(du)$$  \hspace{1cm} (8.27)

Substituting Eqs. (8.25) and (8.27) into Eq. (8.21), we obtain the force equilibrium:
\[ F_j(du) = F_e(du) \]  \hspace{1cm} (8.28)

### 8.5.2 Geometrical Constrain at Contact

The geometrical constrain for the contact is that the sheet surface \( S(x,y) \) mates the tool surface \( S_T(x,y) \) for any contacting node with the coordinates \((x,y)\), that is

\[ S(x,y) = S_T(x,y) \] \hspace{1cm} (8.29)

### 8.5.3 Governing Equations of System

Equations (8.28) and (8.29) are the governing relations. Linearizing these system equations with the unknowns (the incremental displacement vector \( du \) and the contact pressure \( P \)), we finally obtain the matrix form of the formulation

\[
\begin{bmatrix}
    K_i - K_e \\
    C
\end{bmatrix}
\begin{bmatrix}
    \delta u \\
    \delta P
\end{bmatrix}
= \begin{bmatrix}
    F_e - F_i \\
    G
\end{bmatrix}
\] \hspace{1cm} (8.30)

where \( K = K_i - K_e \) is the tangent stiffness matrix, and

\[ K_i = \frac{\partial F_e(du)}{\partial (du)} \] \hspace{1cm} (8.31)

\[ K_e = \frac{\partial F_e(du)}{\partial (du)} \] \hspace{1cm} (8.32)

\[ N = n - \mu \phi(du_i) t \] \hspace{1cm} (8.33)

\[ C = \begin{bmatrix}
    \frac{\partial S(x,y)}{\partial x} & \frac{\partial S(x,y)}{\partial y} & -1
\end{bmatrix} \] \hspace{1cm} (8.34)

The displacement perturbation \( \delta u \) (= \text{\( du^{j+1} \)} - \text{\( du^j \)}, and \( j \) and \( j+1 \) are the iteration numbers for any two consecutive iterations) is the perturbation of the increment displacement vector, and \( \delta p \) is the perturbation of the contact pressure (magnitude), \((F_e - F_i)\) is the residual force vector, and \( G \) is the contact error (the measure of the gap or the penetration between the finite element node and the tool surface). The contact error \( G \)
(measured from the finite element normal (n) direction to the tool surface) indicates the contact situation and is associated to the pressure perturbation \( \delta p \).

The Newton-Raphson iteration scheme is used to solve the system equation (8.30) at every incremental step \( n \). The perturbation of the unknowns are updated as

\[
n_{du}^{n+1} = n_{du} + n_{du}^{n+1}
\]  

(8.35)

The incremental effective strain, the total effective strain are updated accordingly

\[
^\text{inc} e_{i+1} = f(du_{i+1}^{n})
\]  

(8.36)

\[
^\text{inc} e_{i+1} = \varepsilon + ^\text{inc} e_{i} + d e^{i+1}
\]  

(8.37)

The internal force \( F_i \) depends on the total effective strain and the strain hardening, and it is given by

\[
^\text{inc} F_i^{i+1} = F_i(\varepsilon_i^{i+1}) = F_i(\varepsilon_i + ^\text{inc} e_i + d e_{i+1})
\]  

(8.38)

When the norms of the displacement perturbation \( \delta u \) and the residual force \( (F_e - F_i) \) satisfy the specified converging criteria, say 0.00001, the total (membrane) effective strain at step \( n \) is updated by the summation of the effective strain at the previous step (step \( n-1 \)) and the incremental effective strain at the current step (step \( n \)), that is

\[
^\text{inc} \varepsilon = \varepsilon_i^{n-1} + d \varepsilon
\]  

(8.39)

The bending correction for the membrane solution at step \( n \) then follows as described in the previous sections. The effective strain and thickness are corrected by the bending effects. The new effective strain includes the contributions from the bending and the in-plane stretching and it is greater than the membrane effective strain, i.e.
The sheet thickness in the contact area is also reduced due to bending, that is

\[ t_{\text{corrected}} = f(t_{\text{membrane}}, d_{\text{bending}}) \]  

(8.41)

Therefore, the global bending effect is introduced to next incremental step (step n+1) by using the corrected effective strain and thickness at step-n in which the extra strain hardening and the thickness reduction are included as indicated by Eqs. (8.40) and (8.41). These corrected quantities affect the total effective strain and the internal force at the step n+1, as indicated by Eqs. (8.36) - (8.37).
CHAPTER IX

EXPERIMENTAL INVESTIGATION OF BENDING AND FLANGING OPERATIONS

9.1 Plane-Strain Bending Experiments

Plane-strain condition (no deformation along sheet width which is parallel to the bending axis) can be achieved when the ratio \((w/t)\) of the width, \(w\), to the thickness, \(t\), is greater than 8. In most sheet bending operations, this ratio is far greater than eight. Therefore, only the plane-strain bending has industrial interest. In this experimental study, the conventional air bending, and two newly developed bending operations, rotary bending and tractrix bending, are investigated.

The rectangular specimens were cut from HS (high strength) steel sheets to the dimensions of length of 101.6 mm (4 inches) and width of 76.2 mm (3 inches). The material properties of HS steel are given in Table 9.1. At each ram position, the bending angles before and after unloading were measured with a digital protractor which has precision of \(\pm 0.1^\circ\). Fig. 9.1 illustrate the tooling setups.

Table 9.1 Material properties of HS (high strength) steel sheets.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(K)</th>
<th>(\varepsilon_0)</th>
<th>(\sigma_0)</th>
<th>(\bar{R})</th>
<th>(M)</th>
<th>(E)</th>
<th>(v)</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.143</td>
<td>603.8</td>
<td>0.0003</td>
<td>289</td>
<td>1.4</td>
<td>2</td>
<td>206.8</td>
<td>0.3</td>
<td>0.91</td>
</tr>
</tbody>
</table>
Fig. 9.1 Experiment set up
9.1.1 Rotary Bending of HS Steel Sheets

A die set provided by Ready Tools Inc. was used in the experiment. This is a special bender (CB3) which can bend the sheet to 110° through the rocker rotation. The die configuration and dimensions are shown in Fig. 9.2. The rocker is attached to a saddle which is connected to the ram of pressbrake. When the ram moves downwards, the rocker rotates about its geometrical center and bends the sheet.

At each ram position, the bending angles before and after unloading were measured. The ram loads vs. strokes were also recorded using a load cell with a capacity of 50 tons. The measured angles and ram positions are listed in Table 9.2. The springback angles can be calculated by the differences of the bending angles before and after unloading. The load-stroke data is not reported here because the maximum load is about 600 Newtons or 0.06 tons which is too low to be measured with a reasonable accuracy (the maximum load is within the error region of the 50-ton load cell).

<table>
<thead>
<tr>
<th>Stroke (mm)</th>
<th>Angle under load (degree)</th>
<th>Angle after load Sample A (degree)</th>
<th>Angle after load Sample B (degree)</th>
<th>Angle after load Sample C (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5000</td>
<td>6.2</td>
<td>8.0</td>
<td>17.5</td>
<td>18.0</td>
</tr>
<tr>
<td>1.0500</td>
<td>15.0</td>
<td>17.0</td>
<td>32.5</td>
<td>32.5</td>
</tr>
<tr>
<td>1.9000</td>
<td>30.0</td>
<td>61.8</td>
<td>97.2</td>
<td>97.4</td>
</tr>
<tr>
<td>2.6520</td>
<td>44.5</td>
<td>46.5</td>
<td>88.0</td>
<td>86.8</td>
</tr>
<tr>
<td>3.4250</td>
<td>60.0</td>
<td>77.2</td>
<td>92.1</td>
<td>92.6</td>
</tr>
<tr>
<td>4.2100</td>
<td>75.0</td>
<td>87.3</td>
<td>98.0</td>
<td>97.6</td>
</tr>
<tr>
<td>4.8400</td>
<td>85.0</td>
<td>92.1</td>
<td>98.3</td>
<td>97.1</td>
</tr>
<tr>
<td>5.1500</td>
<td>90.0</td>
<td>92.9</td>
<td>103.0</td>
<td>103.1</td>
</tr>
<tr>
<td>5.5760</td>
<td>95.0</td>
<td>92.9</td>
<td>103.0</td>
<td>103.1</td>
</tr>
<tr>
<td>6.0750</td>
<td>100.0</td>
<td>103.1</td>
<td>103.0</td>
<td>103.1</td>
</tr>
</tbody>
</table>
Fig. 9.2 A special rotary bending die set for 110° overbending.

Included angles $\theta_R = 70^\circ$, $\theta_s = 44^\circ$ and $\theta_b = 26^\circ$,
Rocker radius $R = 12.7$ mm,
Anvil radius $R_d = 1.52$ mm
$K = 0.24$ mm
9.2 Tractrix Die Bending of HS Steel Sheets

Based on the proposed mathematical model for springback calculation \( \theta_s = \frac{1}{2} \frac{M_{\text{max}}}{E} S_1 \), a constant springback angle, \( \theta_s \), for various bending angles can be achieved by keeping the bending arc length \( S_1 \) constant during the punch stroke via a proper design of die profiles. Such a die design suggested here is the one using the tractrix curve in which the length of any tangent between the curve (die profile) and a fixed line (punch center line) is a constant, Fig. 9.3. The technological significance of this optimal die design are that (a) the angular tolerance of the components for variety of bending angles can be controlled using a single value of the springback angle which could be controlled within the tolerance requirement by a careful selection of the dimensions of tractrix profile, and (b) if necessary, the overbending of the parts for a variety of bending angles are easy to achieve and maintain by overbending using a single value of the springback angle. Also, the sensitivity of the bending angle to variation in the punch stroke and press setup is greatly reduced. As a result, the time necessary to adjust the press stroke and for try-outs could be shortened.

The mathematical equation for the tractrix curve is given by Eq. (4.44) which is restated as follows:

\[
y = a \ln \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2}
\]  

(4.44)

where \( a \) is the distance \( L_d \) between the punch center to the die shoulder, and the \( x-y \) coordinate system is shown in Fig. 9.3. For this experiment, the dimension \( a \) is chosen as 7.5 mm which corresponds to the half die opening in air bending die. With this dimension and the calculated coordinates, the tractrix die profiles were machined with the wire EDM (electrical discharge machining). The instantaneous die radius is a function of
Fig. 9.3 Tractrix bending die

- $R_p =$ punch radius $= 1.52 \text{ mm}$,
- $L_d =$ half die opening $= 7.5 \text{ mm}$
- $R_d =$ the instantaneous die radius (depends on the die opening and bending angle).
the bending angle and the coordinates of the tangent points on the die profile. The punch is the same as that used in the air bedding.

The experiment procedure is identical to that in rotary bending. The relationship between the bending angle under load and the punch stroke was measured first. The bending angles before and after unloading were measured at each punch position. Tables 9.3 presents the measured data for angle-stroke relationship. Table 9.4 summarizes the measurements in bending angles, springback angles, and the strokes.

The measurements indicated that the springback angles are nearly constant and insensitive to the bending angles and strokes. The technological significance of this die design is that the part tolerance and quality for a variety of bending angles are easy to control and maintain by overbending using a single value of the springback angle.

Table 9.3 Measurement of bending angle (under load) and stroke in the tractrix die bending of HS steel sheets

<table>
<thead>
<tr>
<th>Stroke (mm)</th>
<th>1.01</th>
<th>2.027</th>
<th>3.029</th>
<th>4.027</th>
<th>5.026</th>
<th>6.030</th>
<th>7.028</th>
<th>8.038</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle (deg.)</td>
<td>12.2</td>
<td>28.4</td>
<td>42.2</td>
<td>58.2</td>
<td>73.2</td>
<td>84.4</td>
<td>99.</td>
<td>108.4</td>
</tr>
</tbody>
</table>
Table 9.4  Measurements of stroke, bending angles before and after unloading in tractrix die bending of HS steel sheets.

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>Stroke (mm)</th>
<th>Angle under load (degree)</th>
<th>Angle after load (degree)</th>
<th>Springback Angle (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>1.0120</td>
<td>13.0</td>
<td>9.0</td>
<td>4.0</td>
</tr>
<tr>
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<td>2.0770</td>
<td>29.4</td>
<td>24.6</td>
<td>4.8</td>
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<td>39.6</td>
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</tr>
<tr>
<td>A4</td>
<td>4.0580</td>
<td>59.2</td>
<td>55.4</td>
<td>3.8</td>
</tr>
<tr>
<td>A5</td>
<td>5.0490</td>
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<td>68.4</td>
<td>4.2</td>
</tr>
<tr>
<td>A6</td>
<td>6.0880</td>
<td>84.4</td>
<td>84.0</td>
<td>4.4</td>
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<td>93.2</td>
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<td>106.6</td>
<td>102.4</td>
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<td>24.8</td>
<td>3.6</td>
</tr>
<tr>
<td>B2</td>
<td>3.0540</td>
<td>44.0</td>
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<td>8.0520</td>
<td>108.6</td>
<td>105.0</td>
<td>3.6</td>
</tr>
</tbody>
</table>
9.3 Air Bending of HS Steel Sheets

Air bending experiments were conducted to compare with the tractrix die bending. The radii of punch and die are 1.52 mm. The die opening is 15 mm. The high strength (HS) steel sheets of 0.91 (0.036") thickness were used in the tests. The same testing and measuring procedures in the tractrix die bending were carried out in the air bending. The measurements of the strokes, and bending angles before and after unloading are presented in Tables 9.5 and 9.6.

Table 9.5 Measurement of bending angle (under load) and stroke in air bending of HS steel sheets

<table>
<thead>
<tr>
<th>Stroke (mm)</th>
<th>Angle (deg.)</th>
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<tbody>
<tr>
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<td>4.4000</td>
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<td>14.937</td>
<td>135.2</td>
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</table>
Table 9.6 Measurements of stroke, bending angles before and after unloading in air bending of HS steel sheets.

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>Stroke (mm)</th>
<th>Angle under load (degree)</th>
<th>Angle after load (degree)</th>
<th>Springback Angle (degree)</th>
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</thead>
<tbody>
<tr>
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</table>
9.2. Shrink and Stretch Flanging Experiments

9.2.1 Shrink Flanging of HS Steel Sheets

The axisymmetric shrink flanging experiments were conducted to verify the proposed maximum strain model and the wrinkling criteria. The circular blanks were cut from HS sheets (thickness = 0.91 mm) to the initial diameters of 104.9, 114.5, and 126.7 mm. Two specimens for each diameter, and total of six specimens were prepared.

The ERC multi-action press was used in the experiments. The tooling used in deep drawing was modified for flanging experiment. Fig. 9.4 illustrates the geometry and dimensions of the tooling. The die corner radius and the punch corner radius are 1.52 mm (0.06"), and the clearance between the punch and the die is 1.02 mm (0.04"). The circular blank is centered and clamped to the flat bottom of a cylindrical punch by a circular blankholder using three bolts. The punch moves down and flanges the sheet over the concave edge of the die.

In order to detect the onset of the wrinkles (via visual observation), the incremental displacement of the punch was controlled by 0.25 mm (0.01"). The test stopped as soon as the wrinkles appeared. For blanks with initial diameter of 104.9 mm, no wrinkles were observed because the maximum compressive strain at the edge was only 4%. While, for blanks with larger initial diameters (114.5 mm), wrinkles appeared. For blanks with even larger initial diameters (126.7 mm), wrinkles occurred more earlier. Under the load, the pictures of the specimens and the wrinkled shapes were taken. After the diameter of the specimens were measured, the punch was slowly left up with the same incremental displacement in order to observe the elastic recovery of the wrinkles. No appreciable elastic recovery of the wrinkles were detected either by visual observation or by measuring the diameters after unloading. Therefore, there were no elastic wrinkles observed in all six specimens made from the HS steel sheets. The
Fig. 9.4 Tooling set for axisymmetric shrink flanging (the circular sheet is clamped to punch face by 3 bolts, and flanged around the punch shoulder when the punch moves down).
diameters and thickness at the edges of the initial blanks and the flanged parts, as well
as the flange height (a vertical measure from the flat top to the flange edge) were
measured and presented in Table 9.7. Using the initial and final diameters, the
maximum hoop strains at the flange edge can be calculated.

Table 9.7 Measurements in axisymmetric shrink flanging HS steel sheets

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>D₁ (mm)</th>
<th>t₀ (mm)</th>
<th>D₂ (mm)</th>
<th>tᵣ (mm)</th>
<th>H (mm)</th>
<th>Hoop Strain</th>
<th>Wrinkle</th>
<th>D₂/D₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>104.90</td>
<td>0.91</td>
<td>100.89</td>
<td>0.94</td>
<td>3.00</td>
<td>-3.9</td>
<td>no</td>
<td>0.962</td>
</tr>
<tr>
<td>A2</td>
<td>104.90</td>
<td>0.91</td>
<td>100.89</td>
<td>0.94</td>
<td>3.42</td>
<td>-3.9</td>
<td>no</td>
<td>0.962</td>
</tr>
<tr>
<td>B1</td>
<td>114.50</td>
<td>0.91</td>
<td>106.68</td>
<td>0.94</td>
<td>8.00</td>
<td>-7.1</td>
<td>yes</td>
<td>0.932</td>
</tr>
<tr>
<td>B2</td>
<td>114.50</td>
<td>0.91</td>
<td>106.96</td>
<td>0.94</td>
<td>7.90</td>
<td>-6.8</td>
<td>yes</td>
<td>0.932</td>
</tr>
<tr>
<td>C1</td>
<td>126.7</td>
<td>0.91</td>
<td>124.90</td>
<td>0.92</td>
<td>4.65</td>
<td>-1.4</td>
<td>yes</td>
<td>0.987</td>
</tr>
<tr>
<td>C2</td>
<td>126.64</td>
<td>0.91</td>
<td>124.86</td>
<td>0.92</td>
<td>4.50</td>
<td>-1.4</td>
<td>yes</td>
<td>0.986</td>
</tr>
</tbody>
</table>

D₁ = initial blank diameter,
D₂ = final diameter at the flange edge,
t₀ = initial thickness of blank,
tᵣ = final thickness at the flange edge,
H = flange height
9.2.2 Stretch Flanging of Non-axisymmetric Blanks

The axisymmetric stretch flanging was investigated by Wang and Wenner [1973] using hole flanging experiment in which a hemispherical punch was forced through the hole in a circular blank. Their measurements will be compared with simulations by program FLANGE in the next chapter. In this investigation, the segmental specimens with different included angles (90, 135, 180 degrees) and various radii (25.4, 31.8, and 38.1, mm), as shown in Fig. 9.5, were tested to study the application range of the axisymmetric models proposed. The rectangular specimens (Fig. 9.5b) with various W/L ratio (width to length) were also tested. The influences of the free boundary (two sides of flange) and the included angle, as well as the W/L ratio to the maximum flanging strain and flange length, have not been quantitatively investigated so far. It is expected that the free boundary will provide a relief of the strains at its adjacent areas. The smaller is the included angle or the smaller the W/L ratio, the greater is the relief effect on the strains by the free boundary is. It is also expected that the strain relief effect will be least in the center line of the specimen.

The tooling used in deep drawing was slightly modified for stretch flanging experiments (Fig. 9.6). The blanks were firmly clamped by the blankholder and were flanged around the concave edge of the die. The strains at the flange edge can be calculated via either the circle grid measurement or the measured flange radius (if the axisymmetric condition can be obtained). In this study, both methods were used. The circle grids with the diameter of 2.85 mm were electrochemically etched on the blanks. After flanging, the circles deformed to the ellipses. By measuring the lengths of the long axis and the short axis of the ellipse, the hoop strain (along the long axis) and the radial strain (along the short axis) can be determined. The thickness at the edge was measured using a dial gage with a accuracy of 0.01 mm (0.0005").
Fig. 9.5 Blank geometries (ABCD).
Fig. 9.6 Tooling configurations for stretch flanging.
It was observed in the experiments that the two sides of blanks sucked in towards the symmetry line (0°-line) along the hoop direction of the specimen (Fig. 9.7). This 'suck-in' has two effects on the deformation: (a) the total strain level everywhere in the flange were reduced, and (b) the strain relief is greater in the regions close to the these two free sides. In a truly axisymmetric stretch flanging of the full circular blanks, this 'suck-in' and its strain relief are prohibited. Therefore, the strains in axisymmetric flanging will be greater than those in angular flanges with free sides. Fig. 9.8 presents the hoop strain distributions along the flange edge. The maximum strains appear at the point M (Fig. 9.7) which is located at the center line (0°-line) along the flange edge. It should be pointed out that some of the strains are not exactly corresponding to the strains at the edge because there were no full circles in some locations at the edges. Whenever this happened, the circles close to edge, about 1 to 2 mm, were chosen and measured. Therefore, the strains reported here are lower boundaries of the edge strains. The strain relief effects are well demonstrated in Fig. 9.8. For a blank with 90° included angle, the strain relief effect is strong. The strain rapidly decreases and approaches to zero as the circles close to the free sides. For a blank with 180° included angle, the strains are nearly constant in a range of -35° to 35°, from the center line. Beyond this range, the strain decreases as the circles close to the free sides. In Fig. 9.8, the maximum strains (the open circle and the open square) at the intersection of the center line and flange edge were calculated using the axisymmetric formula, \( \varepsilon = \ln\left(\frac{R_2}{R_1}\right) \), and the measured radii (the initial blank radius \( R_1 \), and the flange radius \( R_2 \)) at that point. The open circle and the open square correspond to the maximum strains for specimens with 90° and 180° included angles, respectively. It is seen that the axisymmetric results for these two included angles do not have significant difference. However, the difference between the maximum strains (appearing at the 0°-line at the flange edge) from the circle grid measurement (actual) and the axisymmetric results are quite obvious. For a 90° angular specimen, the strain is about 26% from
Fig. 9.7 Angular specimen for stretch flanging.
Fig. 9.8 Measured strain distribution. Circle grids measurement gives actual strains along the edge. Axisymmetric results are calculated from the measured flange radius $R_2$ and the initial blank radius $R_1$ at the symmetric center line (0°-line) of blank.
the circle grid measurement, and the strain is about 58% from axisymmetric result. This indicates that the axisymmetric formula can not be applicable to flanges with the included angles ≤ 90° in which the free sides of blank have strong strain relief effect on the flanging deformation. The strain relief by the free sides of flange reduces as the blank angle increases. For the angular specimen with 180° included angle, the difference between the strain measurement (about 48%) and axisymmetric results (about 60%) decreases.

Table 9.8 presents the measurements at the point M which is located at the flange edge along the symmetry line (0°-line). This point is farest from two sides, hence it experienced a maximum deformation.
Table 9.8  Measurements along the edges of non-axisymmetric specimens in the stretch flanging of HS steel sheets (initial thickness = 0.91 mm, \( R_1 = \) initial blank radius, \( R_2 = \) final radius at the flange edge, \( t_f = \) final thickness at the flange edge, \( H = \) flange height)

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>Blank Angle ( \theta ) (°)</th>
<th>Range ( R_1 ) (mm)</th>
<th>Flange Angle ( \theta^0 ) (°)</th>
<th>Range ( R_2 ) (mm)</th>
<th>( t_f ) (mm)</th>
<th>( H ) (mm)</th>
<th>Hoop Failure Strain (^a) ( % )</th>
<th>Hoop Strain (^b) ( % )</th>
<th>Flange Failure Ratio ( R_2/R_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3A</td>
<td>135</td>
<td>25.6</td>
<td>80.0</td>
<td>46.02</td>
<td>0.71</td>
<td>24.03</td>
<td>59.4</td>
<td>48.5</td>
<td>neckin</td>
</tr>
<tr>
<td>3B</td>
<td>135</td>
<td>25.4</td>
<td>81.0</td>
<td>46.02</td>
<td>0.64</td>
<td>23.04</td>
<td>61.7</td>
<td>50.7</td>
<td>fracture</td>
</tr>
<tr>
<td>4A</td>
<td>180</td>
<td>25.6</td>
<td>81.8</td>
<td>46.56</td>
<td>0.78</td>
<td>22.26</td>
<td>60.6</td>
<td>48.0</td>
<td>neckin</td>
</tr>
<tr>
<td>4B</td>
<td>180</td>
<td>25.6</td>
<td>81.0</td>
<td>46.5</td>
<td>0.70</td>
<td>22.4</td>
<td>60.2</td>
<td>48.4</td>
<td>neckin</td>
</tr>
<tr>
<td>5A</td>
<td>90</td>
<td>38.1</td>
<td>72.8</td>
<td>46.31</td>
<td>0.864</td>
<td>12.22</td>
<td>19.9</td>
<td>19.5</td>
<td>no</td>
</tr>
<tr>
<td>5B</td>
<td>90</td>
<td>38.1</td>
<td>72.6</td>
<td>46.73</td>
<td>0.864</td>
<td>12.14</td>
<td>20.4</td>
<td>24.2</td>
<td>no</td>
</tr>
<tr>
<td>6A</td>
<td>90</td>
<td>25.4</td>
<td>72.8</td>
<td>45.47</td>
<td>0.826</td>
<td>23.47</td>
<td>58.2</td>
<td>26.2</td>
<td>no</td>
</tr>
<tr>
<td>6B</td>
<td>90</td>
<td>25.4</td>
<td>72.8</td>
<td>46.31</td>
<td>0.826</td>
<td>24.4</td>
<td>58.0</td>
<td>20.4</td>
<td>no</td>
</tr>
<tr>
<td>7A(^c)</td>
<td>90</td>
<td>19.1</td>
<td>86.3</td>
<td>48.5</td>
<td>0.838</td>
<td>31.65</td>
<td>93.4</td>
<td>22.4</td>
<td>no</td>
</tr>
<tr>
<td>7B(^c)</td>
<td>90</td>
<td>19.1</td>
<td>86.3</td>
<td>48.5</td>
<td>0.813</td>
<td>31.10</td>
<td>93.4</td>
<td>24.0</td>
<td>no</td>
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<tr>
<td>8A</td>
<td>180</td>
<td>31.8</td>
<td>79.8</td>
<td>46.33</td>
<td>0.775</td>
<td>18.10</td>
<td>37.6</td>
<td>32.9</td>
<td>no</td>
</tr>
<tr>
<td>8B</td>
<td>180</td>
<td>31.8</td>
<td>81.0</td>
<td>46.9</td>
<td>0.825</td>
<td>18.19</td>
<td>39.0</td>
<td>32.8</td>
<td>no</td>
</tr>
</tbody>
</table>

\(^a\) - from the measured radii at the flange edges by the axisymmetric formula: \( \ln(R_2/R_1) \).

\(^b\) - from circle grid measurement.

\(^c\) - blanks were not firmly clamped and draw-in occurred during punch stroke.
10.1. SIMULATIONS OF PLANE-STRAIN SHEET BENDING

10.1.1 Air Bending of 2024-O Aluminum Alloy Sheets

Using the computer code BEND, the air bending tested at Battelle [Nagpal, Subramania, and Altan, 1979], was simulated. The material data for Aluminum alloy 2024-O are listed in Table 10.1. The tooling configuration is shown in Fig. 10.1, and the dimensions are listed in Table 10.2. The simulation results with different radii of punch and die, and die opening are compared with experimental measurements by others [Nagpal et al, 1979; Raghupathi et al, 1983]. As shown in Table 10.2, springback angle (the difference between the bending angle under load, $\theta_1$ and the desired angle, $\theta_2$) increases with the die opening, $W_d$. With the same punch radii ($R_p = 2.54$ mm) and die radii ($R_d = 1.52$ mm), the predicted springback angle, for $W_d = 15.9$ mm is $3.81^\circ$ which is more than twice of the springback ($1.94^\circ$) when the die opening reduces to $9.53$ mm. The simulated curves of springback vs. punch stroke (Fig. 10.2a), and desired bending angle vs. stroke (Fig. 10.2b) are compared with the measured ones by Nagpal et al [Nagpal, Subramanian, and Altan, 1979]. A consistent agreement was again achieved. Notice that the bending angle $\theta_2$ is plotted against the stroke because of the symmetry. The potential application of the curve of bending angle vs. stroke is to pre-set the press.
Fig. 10.1 Air bending operation.

\[ 2\theta_1 = \text{bending angle under load (before springback)} \]
\[ 2\theta_2 = \text{bending angle after unloading (after springback)} \]
\[ 2\theta_s = \text{springback angle} \]
\[ 2\theta_p = \text{bending angle of the bent part (}\theta_p = 90^\circ - \theta_2\) \]
\[ d = \text{punch stroke} \]
\[ R_p = \text{punch radius} \]
\[ R_d = \text{die radius} \]
\[ W_d = \text{die opening} \]
Table 10.1 Material properties of 2024-O Aluminum Alloy [Nagpal, et al, 1979]

<table>
<thead>
<tr>
<th>n</th>
<th>K</th>
<th>a₀</th>
<th>σ₀</th>
<th>R₀</th>
<th>R₄₅</th>
<th>R₉₀</th>
<th>R₀ + 2R₄₅ + R₉₀</th>
<th>M</th>
<th>εₑ</th>
<th>εᵣₐₐₜ</th>
<th>εᵣₙₜ</th>
</tr>
</thead>
<tbody>
<tr>
<td>.242</td>
<td>339</td>
<td>0.0</td>
<td>90.0</td>
<td>0.74</td>
<td>0.77</td>
<td>0.64</td>
<td>0.73</td>
<td>1.73</td>
<td>73.08</td>
<td>0.3</td>
<td>0.62</td>
</tr>
<tr>
<td>.134</td>
<td>266</td>
<td>-0.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- a - for strain ≤ 0.05,  b - for strain > 0.05,  c - calculated by \( \overline{R} = \frac{R₀ + 2R₄₅ + R₉₀}{4} \).
- d - not reported but calculated by \( M = 1 + \overline{R} \).

Table 10.2 Comparisons between Simulations and Measurements from Nagpal et al [1979] and Raghupathi et al [1983]. (sheet thickness t = 1.27 mm, sheet width = 51 mm, and friction coefficient = 0.1).

<table>
<thead>
<tr>
<th>Rp</th>
<th>R_d</th>
<th>W_d</th>
<th>2θ₂</th>
<th>2θ₁</th>
<th>2θ₄</th>
<th>d</th>
<th>d</th>
<th>2θ₄</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(mm)</td>
<td>(mm)</td>
<td>(mm)</td>
<td>degree</td>
<td>degree</td>
<td>degree</td>
<td>(mm)</td>
<td>(mm)</td>
<td>degree</td>
<td></td>
</tr>
<tr>
<td>2.286</td>
<td>1.83</td>
<td>9.53</td>
<td>90</td>
<td>92.54</td>
<td>92.13</td>
<td>4.394</td>
<td>4.47</td>
<td>2.13</td>
<td>[A]</td>
</tr>
<tr>
<td>2.286</td>
<td>1.52</td>
<td>9.50</td>
<td>97.42</td>
<td>100.3</td>
<td>99.55</td>
<td>4.39</td>
<td>4.63</td>
<td>2.13</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.286</td>
<td>4.76</td>
<td>11.13</td>
<td>90</td>
<td>*</td>
<td>93.11</td>
<td>6.65</td>
<td>7.10</td>
<td>3.11</td>
<td>&quot;</td>
</tr>
<tr>
<td>3.302</td>
<td>4.76</td>
<td>11.13</td>
<td>96</td>
<td>*</td>
<td>98.48</td>
<td>6.95</td>
<td>7.02</td>
<td>2.48</td>
<td>&quot;</td>
</tr>
<tr>
<td>1.524**</td>
<td>1.52</td>
<td>6.35</td>
<td>90</td>
<td>*</td>
<td>91.88</td>
<td>3.07</td>
<td>3.08</td>
<td>1.88</td>
<td>&quot;</td>
</tr>
<tr>
<td>2.54</td>
<td>1.52</td>
<td>9.53</td>
<td>90</td>
<td>92.12</td>
<td>91.94</td>
<td>4.14</td>
<td>4.16</td>
<td>1.94</td>
<td>[B]</td>
</tr>
<tr>
<td>2.54</td>
<td>1.52</td>
<td>15.90</td>
<td>90</td>
<td>94</td>
<td>93.81</td>
<td>7.10</td>
<td>7.63</td>
<td>3.81</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

- * Not reported.  ** Sheet thickness t = 1.02 mm.


d = punch stroke, 2θ₁ = angle under load, 2θ₂ = angle after load, 2θ₄ = springback angle
Fig. 10.2 Process information: (a) springback vs. punch stroke, and (b) bending angle vs. stroke. Measurement by Nagpal et al [1979]
displacement in order to achieve the desired bending angle. Springback angle vs. bending angle could also be obtained from the simulation (not plotted here), and this curve could be used in the design of bending dies. The predicted punch load-stroke curves is compared with measured one, Fig. 10.3a. The maximum punch force is in excellent agreement with experiment; however, the stroke corresponding to the force is less accurate. The predicted moment-curvature curve (Fig. 10.3b) reflects the flow stress and strain relationship, and it is consistent with the characteristics of elasto-plastic deformation. In practical situations, sheet parts usually have holes before bending operation. The bending deformation may cause the distortion of these holes located along the bent arc. In order to assess such a distortion, a strain distribution along the bent arc is needed (Fig. 10.4). Finally, Fig. 10.5 illustrates the deformed shapes for three punch heights. It is seen that the bent arc is neither a portion of single circle nor a straight line.

10.1.2 Air Bending of 2024-T3 Aluminum Sheets

Additional simulation was conducted at the request for an assistance to the research in Department of Engineering Mechanics at the Ohio State University [Cho, 1993]. The air bending tool set (having the same configuration as that shown in Fig. 10.1) with a large die opening (76.2 mm), large die radius (12.7 mm), and large punch radius (6 mm) was used in the bending of thick (thickness = 2.5 mm) and high strength (tempered) aluminum alloy sheet (2024-T3). The material properties are listed in Table 10.3.

| Table 10.3 Material properties of 2024-T3 Aluminum Alloy [Cho, 1993] |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| n   | K   | a_o  | σ_o  | R   | M   | E   | ν   | t   |
|     | (MPa) | (MPa) | (GPa) |     |     |     |     | mm  |
| 0.2 | 879.1 | 0.0001 | 305   | 0.73 | 1.73 | 87  | 0.3 | 2.5 |
Fig. 10.3 Load information: (a) punch force vs. stroke, and (b) moment vs. curvature.
Strain Distribution (predicted)

2024-O Aluminum
Punch radius: 2.54
Die radius: 1.52
Die opening: 9.53
Sheet thickness: 1.27
Units: mm
Angle: 90 degree

Fig. 10.4 Strain distribution along bent arc

Deformed Profiles (predicted)

2024-O Aluminum
Punch radius: 2.54
Die radius: 1.52
Die opening: 9.53
Sheet thickness: 1.27
Units: mm

Fig. 10.5 Deformed shape
For this bending operation, the elastic-plastic finite element program SHEET-B encountered numerical converging difficulty. This may be attributed to the facts that (a) the large die opening causes the deformation mainly in the elastic region, and most of the elements had undergone unloading which is a problem for numerical simulations of the large plastic deformation, and (b) the large free draw-in of sheet to the die cavity can occur in addition to the rotation during punch stroke, and this large rigid-body motion presents the most difficult problem in finite element simulation. The material and tool data were given to this author. The simulation was conducted and the results were sent to the requester. The simulation was compared with their measurements and good agreements were reported [Cho, 1993]. Their comparisons are reproduced in Figs. 10.6 - 10.8.

Fig. 10.6 compares the predicted bending angle before unloading vs. punch stroke by the program BEND with the predictions by the FEM code SHEET-B and the measurements by Cho [1993]. The agreement is very good except at the large angles at which the bending deformation companies by a free draw-in with rigid-body motions. The same good agreements are also achieved in the predictions of bending angle after load vs. punch stroke (Fig. 10.7), and the springback angles vs. the desired bending angles (Fig. 10.8). The finite element code SHEET-B succeeded only at the beginning of the bending operations for bending angle about 40 degrees which corresponding to a part angle of 120 degrees.
Fig. 10.6 Comparisons between the measured bending angle under load and the predictions by program BEND and finite element code SHEET-B.
Fig. 10.7 Comparisons between the measured bending angle after unloading and the predictions by program BEND and finite element code SHEET-B.

AIR BENDING
2024-t3 Al-alloy
thickness = 2.5 mm
Punch radius = 6 mm
Die radius = 12.7 mm
Die opening = 76.2 mm
AIR BENDING
2024-t3 Al-alloy
thickness = 2.5 mm
Punch radius = 6 mm
Die radius = 12.7 mm
Die opening = 76.2 mm

Fig. 10.8 Comparison between the measured springback angle vs. bending angle after unloading and the predictions made by the program BEND and the finite element code SHEET-B.
10.1.3 Air Bending of HS Steel Sheets

The air bending experiments conducted at the ERC/NSM were also simulated using program \textit{BEND}. The material properties is given in Table 9.1. The tooling configuration is similar to the one shown in Fig. 10.1. The radii of the punch and the die are 1.52 mm (0.06\"), and the die opening is 15 mm.

The simulation results are compared with the measurements reported in previous chapter (see Chapter 9). Fig. 10.9 shows the comparison between the predicted and measured angle-stroke curves. It is seen that a nearly linear relationship between the bending angle and the stroke exists for bending angle up to 90 degrees. Beyond 90 degree, a nonlinear relationship appears because the draw-in of sheet into the die cavity increases rapidly for large bending angles. This large draw-in of sheet increases the bending arc length, hence the springback also increases rapidly when bending angles are greater than 90 degree (Fig. 10.10).
Fig. 10.9 Comparison between the simulated and the measured angle-stroke curves
Fig. 10.10 Comparison between the predicted and the measured springback angles vs. bending angles.
10.1.4 Tractrix Die Bending of HS Steel Sheets

Based on Eq. (4.9), the springback angle is proportional to the bending arc length. A constant springback angle, \( \theta_s \), for various bending angles may be achieved by keeping the bending arc length \( S_1 \) constant during the punch stroke via a proper design of die profiles. Such a die design suggested here is the one using the tractrix curve in which the length of any tangent between the curve and a fixed line is a constant, Fig. 10.11.

The simulations were compared with the measurements which were reported in Chapter 9. A nearly linear relationship between the bending angles and the punch strokes exists in a wide range of the bending angles (Fig. 10.12). Both the measurements and simulation results indicated that the springback angles are nearly constant and not very sensitive to the bending angles (Fig. 10.13) and strokes (Fig. 10.14). These results are quite different from the air bending in which the linear relationship exists for the bending angles less than 90 degrees (Fig. 10.9) and the springback angle increases with the bending angles and strokes (Fig. 10.10). The technological significance of the tractrix die design is that the part tolerance and quality for a variety of bending angles are easy to control and maintain by overbending using a single value of the springback angle (4 degrees for this example). Also, the sensitivity of the bending angle to variation in the punch stroke and press setup is greatly reduced. As a result, the time necessary to adjust the press stroke and for try-outs in production should be reduced.
Fig. 10.11 Tractrix bending die

\[ R_p = \text{punch radius} = 2.54 \text{ mm}, \]
\[ L_d = \text{half die opening} = 4.74 \text{ mm} \]
\[ R_d = \text{the instantaneous die radius (depends on the die opening and bending angle)}. \]
Fig. 10.12 Comparison between the predicted and the measured bending angles after unloading vs. strokes in tractrix die bending.
Fig. 10.13 Comparison between the predicted and the measured springback angles vs. bending angles after unloading in tractrix die bending.
Fig. 10.14 Comparison between the predicted and the measured springback angles vs. strokes in tractrix die bending.
10.1.5 Rotary Bending

The computer program ‘BEND’ is also used to perform the simulations of rotary bending of high strength (HS) sheet steels. The tooling configuration is shown in Fig. 10.15. This is a special rotary bending die for overbending up to 110 degrees [Ready Tools, Inc., 1993]. The clamping leg (left leg) and bending leg (right leg) are not symmetric with the rocker center. The dimensions of the rotary bending die set and the mechanical properties of HS steel sheets are given in Table 10.4.

<table>
<thead>
<tr>
<th>n</th>
<th>K</th>
<th>ε₀</th>
<th>σ₀</th>
<th>R</th>
<th>M</th>
<th>E</th>
<th>V</th>
<th>Thickness</th>
<th>Rocker radius</th>
<th>Die radius</th>
<th>K-size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(MPa)</td>
<td>(MPa)</td>
<td>(GPa)</td>
<td>(mm)</td>
<td>(mm)</td>
<td>(mm)</td>
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<td>0.0003</td>
<td>289</td>
<td>1.4</td>
<td>2</td>
<td>206.8</td>
<td>0.3</td>
<td>0.91</td>
<td>12.7</td>
<td>1.52</td>
<td>0.24</td>
</tr>
</tbody>
</table>

The simulated relationship between the springback angles and the desired bending angles is compared with the measured one, Fig. 10.16. A good agreement is achieved although some simplifications were made in the models to deal with the difficulty in mathematical modeling of the rocker rotation and ram displacement. The difference between the simulations and measurements are generally within 0.5 degree. The measurements of the bending angles before and after unloading were referred to the same reference plane on the sheet and not on the tool surface and are independent of the initial set up of the ram position and the die reference plane. However, the measurement of ram positions depends on the initial position to the reference plane on the die surface. The predicted curve of bending angle vs. ram position is compared with the experimental
Fig. 10.15 A special rotary bending die set for 110° overbending.
(Overbender CB3 with included angles \( \theta_R = 70^\circ \),
\( \theta_a = 44^\circ \) and \( \theta_b = 26^\circ \) ) [Ready Tools, Inc., 1993]
Fig. 10.16 Comparison between predicted and measured springback angle vs. bending angle.
measurements (Fig. 10.17). A fairly good agreement appears. The measured ram displacements are lower than the predicted ones. The differences may attribute to two possible causes: (a) there may be a slack in the ram motion, which underestimates the stroke, and (b) the specimen may be initially tilt due to the inclination of anvil surface. The ram force vs. stroke was recorded during rotary bending tests. However, the measurement was not accurate (the maximum ram force is only about 600 Newton, or 0.06 tons) because the bending load is too low and is within the error region of the load cell (with capacity is 50 tons). Also the die set had spring damps on the other side for bending thicker plate, which further reduces the force. It was reported by Ready Tools, Inc. that many of its customers of rotary benders failed to measure the loads in pressbrakes which usually have capacity greater than 30 tons. A low bending load is one of the major advantages in using the rotary bender. As discussed previously in Chapter 4, the total ram force along the vertical motion axis consists of two components: the clamping force and the bending force. According to the request of Ready Tools, the forming loads are predicted by program BEND. Fig. 10.18 illustrates that variations of the clamping, bending, and total ram forces during bending operations. The clamping force increases rapidly as the ram descends, and reaches the maximum at the end of the stroke. The bending force, however, increases rapidly only at the elastic deformation in the initial stage of the bending. After the plastic yielding, the bending force increases slowly because the plastically deformed sheet in contacting with the die radius serves as a 'plastic hinge' which can withstand the bending moment. The total ram force (the vertical component) increases with stroke, and it reflects the sum effect of the clamping and the bending forces. It is observed in Fig. 10.18, the clamping force contributes the major component to the total ram force, and the maximum clamping force at the end of stroke is almost twice that of the bending force. This large clamping load can generate a high contact pressure on the sheet surface, and may cause undesired marks or surface damage. This is especially important to the relative soft metals such as aluminum, to the
Fig. 10.17 Comparison between the predicted and the measured angle-stroke relation.
Fig. 10.18 Predicted forming load vs. ram stroke.
prepainted sheet steels used in appliance, and to the coated sheets used in automotive applications. The predicted clamping force can be used to evaluate the degree of the indentation, friction, and tearing of surface coating or paint. When the danger is detected, the techniques for surface protection must be adopted such as using protection pads, designing the rocker with appropriate profile and included angle, and adjusting the relative position of the die (anvil) and rocker.

The influences of material properties in springback were investigated by the simulations of two sheet materials: AKDQ steel and 2024-O aluminum alloy. The mechanical properties of these two materials are given in Table 10.5. In those cases, a regular rotary rocker (which has less overbending) with symmetric legs \((\theta_a = \theta_b = \theta_r / 2 = 44^\circ)\) was used. The dimensions of tool is also shown in Table 10.5.

The simulation results are shown in Figs. 10.19 and 10.20. A nearly linear relationship between the desired bending angle and ram displacement is observed from Fig. 10.19. For given bending angle, a necessary ram stroke or the position of the center of rocker can be obtained from the predicted curve. With such a relation of bending angle vs. stroke, multiple bending angles \((\theta \leq (180^\circ - \theta_r))\) can be achieved by control of ram positions and using only a single rotary bending die set. Springback angle vs. bending angles are illustrated in Fig. 10.20. The springback are larger for small bending angle because of significant elastic deformation at these angles. As bending angle increases, the springback decreases and remain nearly constant because the deformation at the die radius approaches to fully plastic bending and the bending length decreases as rocker rotates. These phenomena were observed in wiping-die bending experiments by Davies [1984] who plotted the springback angles vs. punch displacements which are related to bending angles. Another surprising result found in Fig. 10.20 is that common sense, the springback increases with yield strength and strain hardening. The yield
Table 10.5 Material properties of AKDQ steel and 2024-O aluminum alloy, and bending tool data.

<table>
<thead>
<tr>
<th>Material</th>
<th>n</th>
<th>K</th>
<th>ε₀</th>
<th>σ₀</th>
<th>̅R</th>
<th>M</th>
<th>E</th>
<th>v</th>
<th>Thickness</th>
<th>Rocker radius</th>
<th>Die radius</th>
<th>K-size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MPa</td>
<td>MPa</td>
<td>GPa</td>
<td>mm</td>
<td>mm</td>
<td>mm</td>
<td>mm</td>
<td>mm</td>
<td></td>
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<td></td>
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<tr>
<td>AKDQ</td>
<td>0.22</td>
<td>537</td>
<td>0.0</td>
<td>159</td>
<td>1.7</td>
<td>2</td>
<td>206.8</td>
<td>0.3</td>
<td>1.52</td>
<td>12.7</td>
<td>1.52</td>
<td>3.16</td>
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<tr>
<td></td>
<td>.242a</td>
<td>339a</td>
<td>0.0 a</td>
<td>90.0</td>
<td>0.73</td>
<td>1.73</td>
<td>73.1</td>
<td>0.3</td>
<td>1.52</td>
<td>12.7</td>
<td>1.52</td>
<td>3.16</td>
</tr>
<tr>
<td>2024-O</td>
<td>.134b</td>
<td>266b</td>
<td>-.023b</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

a - for strains ≤ 0.05,  b - for strain > 0.05

The stress of 2024-O aluminum is 90 MPa which is about 57% of the yield stress of AKDQ steel (159 MPa). The strain hardening of 2024-O aluminum is also lower than the springback for 2024-O aluminum alloy is larger than that of AKDQ steel. By that of AKDQ steel. It seems that the springback for aluminum alloy should be lower than that for AKDQ steel in conventional thoughts. However, another very important material property, i.e. the elastic modulus E, affects springback significantly. Equations (3.27) and (4.9) indicate that springback is proportional to 1/E. The greater the Young's modulus, the lower is the springback. A physical interpretation for this is that the resistance to elastic deformation (which is the source for springback) increases with Young's modulus. The elastic modulus for AKDQ steel (206.8 Gpa) is about three times greater than the modulus for 2024-O aluminum (73.1 Gpa). The ratio (σ₀/E) of yield stress to Young's modulus for 2024-O aluminum is 0.00123 which is 63% higher than that of AKDQ steel (0.00077). It is this ratio difference that may help to explain why the springback for aluminum is greater than that of steel. Therefore, the common belief in 'springback increases with yield strength' should be corrected to 'springback increases with the ratio of yield strength to elastic modulus'.
Fig. 10.19 Predicted relationship between bending angle after unloading vs. ram displacement in rotary bending.
Fig. 10.20 Predicted relationship between springback angle vs. bending angle after unloading in rotary bending.
10.2. FLANGING OPERATIONS

10.2.1 Shrink Flanging of High Strength Steel Sheets

The axisymmetric shrink flanging of HS (high strength) sheet steels was simulated with the program FLANGE. The material properties are given in Table 9.1, The tooling setup and dimensions is shown in Fig. 10.21. Three initial blank radii (R₁ = 52.45, 57.25, and 63.35 mm) were used in the experiments and simulations. The simulation results are compared with the measurements. In shrink flange, the maximum hoop strains increase as the flange ratio decreases or the initial blank radius (R₁) increases. Fig. 10.22 shows the comparison between the predicted and the measured maximum hoop strains at the flange edge (Note that the strains are compressive and have negative values, but are plotted using the absolute values). The agreement is very good and validates the proposed models for the maximum strains. The relationship between the flange height (the vertical measurement from the flat bottom to the flange edge) and the flange ratio is illustrated in Fig. 10.23. Again, a good match is attained. The predictions underestimate the flange height by 0.5 mm for the wrinkled specimens (about a half of the sheet thickness) and have a relative error less than 15%.

The proposed wrinkling criteria, Eqs. (7.53) and (7.56), were investigated by the visual observation of the onset of the wrinkles during shrink flanging experiments. No elastic wrinkles were observed because the unloading does not remove the wrinkles. Both the measurements and the predictions via the proposed wrinkling criterion proved that the critical strains to onset the wrinkles are greater than the initial yield strain (= yield stress / Young's modulus = 0.0014). Therefore, only plastic wrinkles occur in shrink flanging of the HS (high strength) steel sheets. The measured maximum strains in the wrinkled specimens (in these cases, they may also be the critical strains to onset the visible wrinkles at flange edge) are compared with the predicted wrinkling strains, Fig. 10.24. Wrinkles did not appear when a circular blank with a radius of 52.45 mm, is
Fig. 10.21 Tooling set for axisymmetric shrink flanging (the circular sheet is clamped to punch face by 3 bolts, and flanged around the punch shoulder when the punch moves down).
Fig. 10.22 The maximum hoop strains at flange edge vs. flange ratio

(\(R_1 = \) initial blank radius, \(R_2 = \) final radius at the flange edge)
Fig. 10.23 Flange height (see Fig. 5.1) vs. flange ratio ($R_1$ = initial blank radius, $R_2$ = final radius at the flange edge)
Fig. 10.24 Critical strains for wrinkling in shrink flanging of HS steel sheets
(t = sheet thickness, R2 = final radius at the flange edge,
alpha = flange angle)
formed to a flange of radius $R_2 = 50.43$ mm (the flange ratio is 0.96). The maximum strains measured and predicted are about 4% which is less than the critical strain (about 6%) predicted by the proposed wrinkling model. Wrinkles did appear for flanges with radii of $R_2 = 53.34$ and 62.49 mm, which are formed from the circular blanks with the initial radii of $R_1 = 57.25$, and 63.35 mm respectively. The wrinkling limits decreases as the flange ratio increases or the blank size increases. For a flange with initial blank radius of 57.25 mm (flange ratio = 0.932), the predicted critical strain for wrinkling is 4.5% which is less than the measured and predicted maximum strains (7%) at the flange edge. Therefore, wrinkles did appear in the experiment. For a flange with an even larger initial radius ($R_1 = 63.35$), the flange ratio has to be increased to 0.987. And the critical strain limit further reduces to 0.6%. At this flange ratio, the maximum strains measured and predicted are 1.4% which are greater than the limit. Therefore, for those specimens, wrinkles did occur earlier during shrink flanging tests.

Based on these comparisons, the following conclusions may be drawn:

(a) the mathematical models are accurate in determining the maximum strains in a shrink flange under axisymmetric condition.

(b) the proposed wrinkling criterion is applicable to establish the flanging limit in shrink flanging operations.
10.2.2 Stretch Flanging of AKDQ Steel Sheets

The axisymmetric stretch flanging was investigated by Wang and Wenner [1973] with the hole flanging experiments. The material used was AKDQ steel sheet with a thickness of 0.89 mm, a strain-hardening exponent (n-value) of 0.218, and the normal anisotropy (R-value) of 1.5. A circular blank having a central hole with an initial radius of $R_1$ was firmly clamped by the die and blankholder. A right angle flange was made by forcing a cylindrical punch with a hemispherical head through the hole in the specimen. The hoop strains at the flange edge and the flanging lengths were measured after tests.

Fig. 10.25 compares the flange length predicted by FLANGE and measured by Wang and Wenner. The agreement is very good and the maximum difference is less than 8%. Fig. 10.26 shows the comparison of the maximum hoop strains (logarithm strain) at the flange edge. Good agreements are indicated except at high strain level (> 80%) at which the necking and strain localization were observed at the edges of specimens in the experiments. At the high strain level, the strain path might be changed from the axisymmetric deformation to plane-strain condition which is the favorable condition for the occurrence of the necking instability.

To demonstrate the capability of program FLANGE, a flange with arbitrary dimensions (the initial radius $R_1 = 25$ mm, the die opening radius $R_o = 50$ mm, and die shoulder radius $R_d = 1.59$ mm) was chosen and simulated. The predicted strain distributions in a $90^\circ$ flange are shown in Fig. 10.27. The hoop tensile strain causes the thinning of sheet and compression along the radial or length direction. It should be pointed out that the flange length (which is defined as the length along the straight portion of the flange or the length between the tangent point on die profile radius to the flange edge) will be shortened in stretch flange because the radial strain is compressive. In conventional method to calculate the flange length, the radial strain component is
Fig. 10.25 Comparison of flange length L (see Fig. 5.2) in stretch flanges predicted by FLANGE and measured by Wang and Wenner [1973]
Fig. 10.26 Comparison of the maximum strains at stretched flange edge predicted by FLANGE and measured by Wang and Wenner [1973].
assumed to be zero, that is the sheet wraps around the die shoulder without deformation. The differences in flange lengths calculated by the current model and the traditional empirical method are shown in Table 10.6. The conventional method overestimates the flange length by 10-15%. This overestimation may cause the troubles in assembly operations.

The hoop strain distributions for three flange angles (45, 90, and 135 degrees) are illustrated in Fig. 10.28. As flange angle increases, the strains increase. The maximum strains appear at the flange edge with a initial radius of 25 mm. Fig. 10.29 shows the deformed profiles for three flange angles during flanging operations.

Table 10.6 Comparisons of flange lengths predicted by program FLANGE and by conventional method (initial blank radius $R_1 = 25$ mm)

<table>
<thead>
<tr>
<th>Flange angle (degree)</th>
<th>Flange radius $R_2$ (mm)</th>
<th>Flange length $L$ predicted (mm)</th>
<th>Flange length $L$ empirical (mm)</th>
<th>$\frac{L_{predicted} - L_{empirical}}{L_{predicted}}$ (%)</th>
</tr>
</thead>
<tbody>
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<td>80</td>
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<td>19.40</td>
<td>22.08</td>
<td>14.8</td>
</tr>
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<td>100</td>
<td>51.20</td>
<td>18.70</td>
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<td>14.2</td>
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<td>18.85</td>
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<td>61.40</td>
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<td>150</td>
<td>64.40</td>
<td>17.80</td>
<td>19.53</td>
<td>9.7</td>
</tr>
<tr>
<td>180</td>
<td>66.50</td>
<td>16.50</td>
<td>18.43</td>
<td>11.7</td>
</tr>
</tbody>
</table>
Fig. 10.27 Predicted strain distributions in a 90° stretch flange.
Fig. 10.28 Predicted hoop (major) strain distributions for three flange angles in a stretch flange.
DEFORMED PROFILES

Fig. 10.29 Predicted deformed profiles in a stretch flange.
10.3 PLANE STRAIN STRETCH AND DEEP DRAWING OPERATIONS

The modified FEM program SECTIONFORM-B (B stands for the bending correction included in the original FEM program SECTIONFORM) worked quite well in stretch and deep draw forming simulations. The bending correction does not cause any numerical instability and converging problems during the stepwise modifications of effective strain, stress, and the internal membrane forces. Also there was no appreciable increase of CPU time.

SECTIONFORM-B was tested in plane-strain stretch and draw forming of strips made from AKDQ sheet steels. The experiments and measurements were conducted previously in the Department of Materials Science and Engineering at the Ohio State University [Wang, 1990]. The material data is listed in Table 10.7. The forming tools consist of a square punch with a flat bottom, a die set with a rectangular cavity, and blankholders, Fig. 10.30. In stretch forming, the sheet was firmly clamped at the edge of the die with a sufficient blankholding force (50 KN). For an appropriate blankholding force, the draw-in condition was attainable.

Table 10.7 Material properties of AKDQ sheet
National Steel Corp., 1989

<table>
<thead>
<tr>
<th>n</th>
<th>K</th>
<th>a₀</th>
<th>m</th>
<th>R</th>
<th>M</th>
<th>E</th>
<th>w</th>
<th>t</th>
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<tbody>
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<td>0.23</td>
<td>598</td>
<td>0.0001</td>
<td>0.013</td>
<td>1.70</td>
<td>2.0</td>
<td>206.8</td>
<td>44.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

where

- n: strain hardening exponent
- K: strength coefficient
- a₀: prestrain
- m: strain-rate sensitivity
- R: normal anisotropy
- M: anisotropy index in Hill's 1970 yield criterion
- w: width of sheet
- t: thickness of sheet
Fig. 10.30 Tooling configurations (side view) of square punch and die
10.3.1 Simulations and Comparisons in Plane-Strain Stretch Forming

In stretch forming, the sheet is firmly clamped at the edge contacting with die and blankholder. Total 76 elements are used in the simulations. The finite element meshes are designed in such a way that the fine meshes with the element length of 0.5 mm were generated in the contact regions of radii of punch and die. Therefore, the bending effects can be well captured. The standard value of the friction coefficient in stretch/draw forming of steels (uncoated) with no lubrications is about 0.18 to 0.22 [Nine, 1978]. In the simulation, the friction coefficient was taken as 0.18.

The modified FEM program SECTIONFORM-B worked quite well in stretch forming simulations. The bending correction does not cause any numerical instability and converging problems during the stepwise modifications of the thickness, the effective strain/stress, and the internal membrane forces. Also there was no appreciable increase in CPU time. The simulation results for two punch radii (3.18 and 9.13) were compared with the measurements as shown in Figures 10.31 - 10.33. In general, the agreements are good. The simulated strains, especially the strains on the convex outer surface and the concave inner surface are well matched with the measured ones. The bending effects are well captured by the bending correction. It should be noted that there was an underestimation in the measured peak strains on the curved surfaces which are in contacting with the tool radii. This is because the curves of the deformed grid circles at the radii appeared as straight lines under the lens of the measuring microscopy. This underestimation is more serious for a sharp punch radius of 3.18 mm. At the same punch height of 10.7 mm, the bending effect is much obvious for smaller punch radius (3.18 mm) than that in larger punch radius (9.53 mm). That is, when the punch radius increases three times from 3.18 mm to 9.53 mm, the maximum surface strain on the convex surface reduces by a half from 19% to 9.5% (Figs. 31 - 32). The strain distributions for different punch heights (10.7 and 14.5 mm) are compared in Figs.
STRETCHING (AKDQ Steel, Friction coeff.=0.18)

Punch radius = 3.18, Die radius = 6.35 mm

Fig. 10.31 Comparison of surface strain distributions predicted by SECTIONFORM-B and measurements in stretch forming (Fig. 10.30) with punch radius of 3.18 mm, and punch height of 10.7 mm.
STRETCHING (AKDQ Steel, Friction coeff. = 0.18)
Punch radius =9.53 mm, Die radius = 6.35 mm

Fig. 10.32 Comparison of surface strain distributions predicted by SECTIONFORM-B and measurements in stretch forming with punch radius of 9.53 mm, and punch height of 10.7 mm.
STRETCHING (AKDQ steel, Friction coeff. = 0.18)
Punch radius = 9.53, Die radius = 6.35 mm.

Punch height = 14.5 mm

Fig. 10.33 Comparison of surface strain distributions predicted by SECTIONFORM-B and measurements in stretch forming with punch radius of 9.53 mm, and punch height of 14.5 mm.
10.32 and 10.33. The stretch (membrane) strains increase with punch travel and causes thinning which reduces the bending effect (the R/t ratio increases). While, the bending strains increases with punch height to some extant and then remain constant when the sheet elements take the curvatures of punch and die. Therefore, the total strains including bending and membrane strains on the sheet surface increase with punch travel.

The curvature influence on strains is illustrated in Fig. 10.34 for three punch radii (3.18, 7.14, and 9.53 mm). As expected, the peak strains increase with the decrease of bending ratio R/t. The global bending effect (that is, the bending deformation affects the entire strain distributions throughout the sheet length) can also be explored from this plot. For punch radii of 9.53 and 7.14 mm, the in-plane stretch strains in the region contacting with the punch flat bottom (the first plateau) and the unsupported region (the second plateau) are not different very much. However, when the radius reduces to 3.18 mm, the difference appears. To illustrate the global bending effect more clearly, the in-plane average strains from the surface strains on the convex and concave surfaces are plotted in Fig. 10.35. The relative difference of strains in the unsupported region is up to 36% between the punch radii of 3.18 and 9.53 mm. The increase of the in-plane strains in the unsupported region is important to the forming limit because the necking and fracture usually occur at the region just off the tool radius. This global bending effect explains (a) why the fracture appears at the region just off the bend, and (b) why the forming limit reduces as the tool radii decrease.
10.34 Surface strain distributions (on bottom surface of sheet) for three different punch radii, \( R_p \), in stretch forming.
Fig. 10.35 Average strain distributions for three punch radii in stretch forming
10.3.2 Simulations and Comparisons in Plane-Strain Deep Drawing

Bending plays a more important role in deep drawing operations. Experimental results in strip drawing tests showed that the strain increments are mainly produced by bending, unbending, and sliding once draw-in starts [Wang, 1990]. In order to simulate these processes, a mesh with total length of 85 mm and 102 elements were used in the simulations. The friction coefficient is 0.2 and the restraining force is 280 N/mm.

The comparisons between the FEM results and the measurements are presented in Figs. 10.36 to 10.39. The predicted membrane solutions in mid-plane match the measurements well only at the locations of punch flat bottom, unsupported region, and the blankholder. However, the membrane results are not comparable with the surface strains predicted and measured. The sole membrane solutions are not acceptable for all cases of the punch radii of 3.18, 7.14, and 9.53 mm. The FEM results including bending correction agree very well with the measured surface strains. Again, the peak strains increase with the decrease of R/t ratios. The bending, unending and sliding processes are shown in Fig. 10.39 for two punch heights (15 and 30 mm). Due to the symmetry of the tooling, the unbending and sliding at the punch radius are less likely, and the bending strains for these two punch height are limited to the region of the punch radius. While, the bending, unbending, and sliding did occur in the region of the die shoulder which is located in a region from distance of 55 to 60 mm measured from the punch center to the center of the die radius (Fig. 10.30). When punch height increased to 30 mm, the elements initially under the flat surfaces of the blankholder (with initial distances of 68 to 75 mm from the punch center) were drawn to the radius of die shoulder and undergone bending, sliding, and unbending.

Based the simulations and comparisons, it may conclude that the proposed decoupled method for the stepwise bending correction is efficient and accurate in predicting the surface strains in stretch/draw forming operations.
Fig. 10.36 Comparison of strain distributions predicted by SECTIONFORM-B and measured in strip drawing (Punch radius = 3.18 mm, Punch height = 30 mm, and Restraining force = 280 N/mm)
DRA WING (AKDQ Steel, Friction coeff. = 0.2)
Punch radius = 7.14, Die radius = 6.35 mm
Restraining force = 280 N/mm

Fig. 10.37 Comparison of strain distributions predicted by SECTIONFORM-B and measured in strip drawing
(Punch radius = 7.14 mm, Punch height = 30 mm, and Restraining force = 280 N/mm)
Fig. 10.38 Comparison of strain distributions predicted by SECTIONFORM-B and measured in strip drawing (Punch radius = 9.53 mm, Punch height = 30 mm, and Restraining force = 280 N/mm)
Fig. 10.39 Strain distributions during bending sliding, and unbending predicted by SECTIONFORM-B in strip drawing (Punch radius = 3.18 mm, Punch heights = 15, and 30 mm, and Restraining force = 280 N/mm)
CHAPTER XI

SUMMARY

Plane strain bending and contour flanging are two common sheet forming processes to produce structural sheet parts and non-symmetric box type components. Understanding the fundamentals of metal flow in these forming operations, and developing the mechanics and process simulation models will help to assess the sheet formability and to control failures (fracture and wrinkles) and dimension tolerance (springback), and to select the best process parameters for controlling the presses. This research establishes the failure criteria for localized necking, fracture, and wrinkling in sheet forming and the fundamentals of deformation mechanics in plane-strain bending (bending around a straight line), contour flanging (bending around a curve), and stretch/draw forming operations which are primarily used in forming box-shaped and structural sheet components.

Based on this study, the following conclusions can be drawn.

(a) Sheet forming mechanics and failure criteria in bending and flanging operations have been developed. The mechanics models incorporate the new concepts and understanding of sheet formability, the recent development in forming process control, and the advancement in computational mechanics and numerical methods.

(b) Based on these fundamentals of bending, the reliable process models have been developed for a number of the commonly practiced and newly developed bending
operations. A computer code, *BEND*, was developed to simulate the air bending, U-die bending, V-die bending, wiping-die bending, rotary bending, and tractrix die bending. This simulation program is able to provide reasonable results for the deformation information (strain, stress, and fracture), geometrical information (springback, deformed shapes), and the process information (bending angle vs. punch stroke) in control of pressbrakes.

(c) The strain models for shrink and stretch flanging operations were established. Failure criteria for wrinkling in shrink flanging and fracture in stretch flanging were proposed. A computer program, *FLANGE*, was developed to simulate these two processes. The process models and computer program are able to predict the maximum strains at the flange edge, the strain distributions along the flange length, the deformed profiles, and the failures.

(c) A number of failure criteria were developed for bending, flanging, and stretch/draw forming operations. A new bendability criterion was proposed to determine the minimum bend ratio based on the fracture mode. A localized necking criterion was established for stretch flangability analysis based on the modification of Hill's instability criterion and incorporating the strain hardening and the plastic anisotropy of sheet materials subjected to prestrain. A wrinkling criteria for an elastic-isotropic and plastic-anisotropic shell with compound curvatures was developed to predict the body wrinkling in the unsupported region of sheet in deep drawing operations, and to determine the wrinkling at the flange edge in shrink flanging operation.

(d) Experiments were conducted to verify the proposed process models of bending and flanging operations, and the wrinkling criteria. Simulation results were compared with measurements. The springback and the relation between bending angle vs.
punch stroke in various bending operations were successfully predicted with a good accuracy. Strains and wrinkles in shrink flanging tests were well predicted.

(e) The bending effects were introduced to the membrane finite element program SECTIONFORM for the analyses of stretch/draw forming. A decoupled method was proposed for stepwise bending correction for the incremental membrane solutions. This method is able to consider both the local and the global bending effects as well as unbending and sliding. The extra strain hardening and thinning due to bending are also included in the formulation. The modified version of SECTIONFORM was tested by a number of examples. The simulations demonstrate that the step-wise bending correction causes neither numerical instability nor appreciable increase of computation time (CPU). The simulations of the plane-strain stretch forming and deep drawing using a flat bottom punch were compared with measurements. Good agreements were achieved for three punch radii (3.18, 7.14, 9.53 mm). These showed that the proposed decoupled method for stepwise bending correction is accurate and efficient in capturing the bending effects and predicting the surface strains in plane-strain stretch/draw forming operations.

(f) The good accuracy and the fast computation (less than 30 seconds on a VaxStation 3200) enable programs BEND and FLANGE to serve as the practical design aids. The applications of sheet metal forming mechanics and the computer-aided simulations will provide a scientific approach to analyze formability of complex sheet parts formed in multiple operations (bending, flanging, stretching and deep drawing). The mechanics models and the associated computer-aided analysis system are able to provide information necessary for engineers to design sheet parts, processes, and dies, by a more efficient and optimum strategy which reduces and finally eliminates costly try-outs. This computer-aided analysis system can be adopted to other CAD systems for a formability analysis.
Future Work

There are a number of issues needing to be addressed for further investigation. These are suggested as follows.

(a) Modeling of bending of shaped blanks such as U-channel section or tubes.

In sequential forming, a flat sheet will be bent or flanged into certain shape. The consequent bending operation will be carried on the preformed section made by the previous forming operations. The strains and springback in bending of U-channel are different from those in bending of flat blanks. Therefore, it is necessary to consider the shape factor in bending models and computer code. The key issue in the modeling work is to define the neutral axis of the section. Once the neutral axis is determined, the bending strain, bending moment, and springback can be calculated using the same equations developed for bending of the flat sheets.

(b) Investigation of the application region for the axisymmetric solutions in shrink and stretch flanging operations.

In industrial applications, the hoop strains in flanging operations are determined by a simple formula based on the assumption of axisymmetric deformation mode. That is, the maximum hoop strain $= \ln(R_2/R_1)$, where $R_1$ is initial blank radius, and $R_2$ is the final radius at the flange edge. However, this assumption may not always exist for all blank geometry and boundary conditions. For instance, the free sides in an open angular blank will provide the strain relief in the entire flange if the blank is not a full closed circle. Therefore, the calculated strain based on the axisymmetry assumption overestimates the deformation in the flange. This overestimation increases as the blank angle decreases and the flange length increases. In reverse flange which is a combination of the shrink and stretch flanges, the hoop compression in shrink flanging region may interact with the hoop tension in stretch flanging region. This interaction may result in no strain relief. These issues have not been well addressed
and investigated either by industrial practitioners or by research communities. The application region of the axisymmetric solution and the interaction between different type of flanges and different forming operations could be investigated either by experiments or three dimensional finite element modeling. The guidelines for applying the axisymmetric solution will be very valuable for sheet forming engineers and die designers.

(c) The computer programs BEND and FLANGE should be further tested in the industrial environment. The necessary modifications based on industrial user's response should be made.
BIBLIOGRAPHY


Backofen, W.A., 1972, Deformation Processing, Lecture notes, Department of Material Science and Engineering, Massachusetts Technology Institute, 1972, p.49.


Kergen, R., 1992, and Jodogne, Ph.: Computerized Control of the Blankholder Pressure on Deep Drawing Presses, SAE Paper No. 920433


Lascoe, O.D., 1988, Handbook of Fabrication Processes, ASM International, Metal Park, Ohio


Lee, L.H.N, 19662: Inelastic Buckling of Initially Imperfect Cylindrical Shells Subject to Axial Compression, J. Aerospace Sci. 29, pp. 87-95.


Ludwik, P., 1903: Technische Blatter, 1903, pp.133-159.

Lung, M., 1971: Ein Verfahren zur berechnung des Geschwindigkeits und Spannungsfeldes bei Stationaren Starr-plastischen Formandemugen mit Finite Element, Ph. Dissertation (in German), Univ. of Hanover, 1971


Mises, Von, 1913, Göttinger Nachrichten, Math - Phys., Klasse, 9, 1913, p.582.


Sevenler, K., 1986: Knowledge Based Systems Approach to Forming Sequence Design for Cold Forging, Ph.D Dissertation, The Department of Industrial and Systems Eng., The Ohio State University, Columbus, Ohio 43210, June, 1986


Shaffer, B.W., and House, R.N., 1955: The Elastic-Plastic Stress Distribution Within a


Wenner, M., 1991: Stretch-Flange Formability of High-Strength Steel, SAE paper No. 910513


APPENDIX A

RELATION BETWEEN EFFECTIVE STRESS AND AXIAL STRESS

The expressions for the effective stress and the axial stress and the effective strain and axial strain depend upon the yield theory used. Hill's normal anisotropic yield function for orthotropic materials [Hill, 1948] was used by several investigators in bending analysis and springback calculation [Nagpal, 1980; Wenner, 1983; Wang, 1984; Levy, 1984; and Makalich, 1988]. Hill's 1948 yield function has been found to be unsatisfactory for certain materials. By introducing a new anisotropic index, $M$, this new yield function has proven to be more promising in accounting for more anisotropic characteristics of various types of materials with different structures. Moreover, this yield criterion covers all commonly used yield criteria and easily reduces to Tresca's (anisotropy index $M = \infty$, and normal anisotropy $R = 1$), Von Mises' ($M = 2$, $R = 1$), and Hill's 1948 yield function ($M = 2$). In our study, Hill's new non-quadratic yield function [Hill, 1979] is used for the bending analysis.

Under the conditions of plane strain and plane stress, Hill's 1979 yield criterion is described as

$$\bar{\sigma} = \frac{1}{[2(1 + R)]^{1/M}} \left[ |\sigma_1 + \sigma_2|^M + (1 + 2R)|\sigma_1 - \sigma_2|^M \right]^{1/M}$$

(A-1)
or in terms of effective strain and longitudinal strain:

\[ \bar{\varepsilon} = F \varepsilon_1 = F \varepsilon_x \quad (A-2) \]

where \( F \) is an important index introduced here to explore the influences of anisotropy and strain/stress states in flow behavior, and it has different values for different materials and stress/strain states:

\[
\begin{align*}
F &= \frac{1+R}{\sqrt{1+2R}} & \text{Isotropy and uniaxial stress} \\
F &= \frac{1+R}{\sqrt{1+2R}} \left[ \frac{1+(1+2R)^{1/2}}{2} \right]^{M_{11}} & \text{Normal anisotropy and plane strain [Hill, 1948]} \\
F &= \frac{1+R}{\sqrt{1+2R}} \left[ \frac{1+(1+2R)^{1/2}}{2} \right]^{M_{12}} & \text{Normal anisotropy and plane strain [Hill, 1979]}
\end{align*}
\]

Note that for isotropic materials, \( F \) has different values for uniaxial stress (\( F = 1 \)) as in narrow sheet bending, and for plane-strain (\( F = 1.155 \)) as in wide sheet bending.

The plastic stress component of the total longitudinal stress can be derived through the plastic work equivalence and the hardening law as follows:

\[ \bar{\sigma} \bar{\varepsilon} = \sigma_x \varepsilon_x + \sigma_y \varepsilon_y \quad (A-4) \]

Because \( \varepsilon_y = 0 \) and \( \sigma_z = 0 \) for plane strain and plane stress and using the relation in Eq. (2.10), the plastic stress along the longitudinal direction is given by:

\[ \sigma_x = F \bar{\sigma} \quad (A-5) \]
In elastic-plastic bending, the moment includes two parts: the elastic bending moment, $M_e$, within the elastic core band, $2y_e$, and the plastic bending moment, $M_p$, in the plastic deformation layers. The limiting height of the elastic core, $2y_e$, the elastic bending moment, and the plastic moment are derived as follows.

**Elastic Core Height** The elastic limit strain, $\varepsilon_{e,o}$, determines the limiting height of the elastic band. The expression of $\varepsilon_{e,o}$ is obtained by substituting $\varepsilon_{e,o}$ into the elastic constitutive relation, Eq. (2.12), and the plastic relation, Eq. (2.13), and equating both equations:

$$\varepsilon_{e,o} = \frac{F}{E'} \frac{\sigma}{G} = \frac{F(1 - \nu^2)}{E} \frac{\sigma}{G} \quad \text{(B-1)}$$

$$y_e = R_\alpha \varepsilon_{e,o} = R_\alpha \left( \frac{1 - \nu^2}{E} \right) F \frac{\sigma}{G} \quad \text{(B-2)}$$

Therefore, the elastic core height increases proportionally to the flow stress, the factor $F$, and the radius of the neutral axis, but decreases as the Poisson's ratio and the elastic module increase. The factor $F$ is an index which describing the influences of stress/strain state and material anisotropy on flow stress. For AK (aluminum killed) steel sheet, $E = 200$ Gpa, initial yield stress = 200 Mpa, the normal anisotropy $R = 1.5$ (hence $F = 1.25$ calculated from Eq. (2.11), and Poisson's ratio = 0.3. The radius is chosen as
10 mm. The elastic core height $2y_e$ is found to be 0.01 mm for the 'isotropic' case (let $R = 1$, then $F = 1.155$). For an anisotropic sheet, like AK steel, the elastic core height is found to be 0.013 mm, which is 25% larger than the 'isotropic' case. The springback is directly related to the height of elastic core inside the sheet. The greater elastic core will generate larger springback.

For thin sheets used in automotive panels, the thicknesses are in a range of 0.5 to 1.0 mm; therefore, the elastic core is relatively small compared to the sheet thickness. Considering the shift in the neutral axis toward the concave surface for large curvature bending or bending under tension, the radius of the neutral axis decreases and this axis may totally move out of the sheet. In such cases, the elastic response may be safely neglected and the rigid-plasticity analysis is justified. However, in small curvature bending of high strength sheets, the elastic response is still important for the springback.

The Elastic Bending moment $M_e$. $M_e$ is calculated by substituting Eq. (2.12) into Eq. (2.15) and considering the linear strain description, Eq. (2.3):

$$M_e = \int_{y_e}^{y_e} \sigma_{xy} wdy = \frac{2w}{R_n} \int_{y_e}^{y_e} \frac{E}{1 - \nu^2} y^2 dy$$

$$= \frac{2w}{3} \frac{E}{1 - \nu^2} \left(\frac{y_e}{R_n}\right)^3 R_n^2 = \frac{2wR_n^2}{3} \frac{E}{1 - \nu^2} \varepsilon_{e,0}^3$$

(B-3)

A good approximation of $\varepsilon_{e,0}$ is 0.2 %, which corresponds to the initial yield stress, $\sigma_{0.2}$. Hence, Eq. (B-3) can be expressed as

$$M_e = \frac{2wR_n^2}{3} \left(\frac{1 - \nu^2}{E}\right)^2 \sigma_{0.2}^3$$

(B-4)

For fully elastic bending ($y_e = t/2$), this moment changes to
Plastic Bending Moment $M_p$

The Plastic bending moment, $M_p$, is calculated based on Swift's hardening law and Hill's 1979 anisotropy yield function. The expression of $M_p$ is obtained by substituting Eqs. (2.14) into Eq. (2.15) and integrating it from the elastic core height, $y_e$, to the sheet surface:

$$M_p = 2 \int_{y_e}^{y_*} \sigma_y y \, w \, dy = \frac{w}{2} \int_{\varepsilon_{\text{el},0}}^{\varepsilon_{\text{max}}} F^{n+1} k \left( \frac{\varepsilon_o - \varepsilon_{\text{el},0}}{F} + \varepsilon_x \right)^n \, y \, dy$$

(B-6)

For a more accurate and general expression of the plastic bending moment, a nonlinear or true strain distribution defined in Eq. (2.1) is used here. However, the direct use of this nonlinear relation causes mathematical difficulty in the integration. In order to avoid such a problem, we chose the integration variable as the strain instead of the radial distance, $y$.

Let

$$\varepsilon_x = \ln(1 + y / R_n)$$

(B-7)

then

$$y = R_n [\exp(\varepsilon_x) - 1], \quad \text{and} \quad dy = R_n \exp(\varepsilon_x) \, d\varepsilon_x$$

(B-8)

Therefore

$$M_p = 2w \int_{\varepsilon_{\text{el},0}}^{\varepsilon_{\text{max}}} k F^{n+1} R_n^2 \left( \varepsilon_x + \frac{\varepsilon_o - \varepsilon_{\text{el},0}}{F} \right)^n \left( e^{2\varepsilon_x/F} - e^{\varepsilon_x/F} \right) \, d\varepsilon_x$$

(B-9)

or
\[ M_p = 2 \dot{w} R_n^2 k F^{n+1} e^{2(e_{c.o} - \varepsilon_c)/F} \int_{e_{c,o}}^{e_{\text{max}}} (e'_x)^n \left( e^{2e'_x} - e^{(e_{c.o} - \varepsilon_c)/F} e^{e'_x} \right) de'_x \quad (B-10) \]

where

\[ e'_x = e_x + (e_{c.o} - \varepsilon_c)/F \quad (B-11) \]

Since \( e_x \) is generally less than unity for the application range ( \( R/t \geq 4 \) ) of elementary bending theory, the exponent terms, \( \exp(e_x) \) and \( \exp(2e_x) \), can be expanded in a Taylor's series:

\[ \exp(e_x) = 1 + e_x + \frac{1}{2} e_x^2 + \frac{1}{6} e_x^3 + ... + \frac{e_x^j}{j!} + ... = \sum_{j=0}^{\infty} \frac{(e_x^j)}{j!} \quad (B-12) \]

and

\[ \exp(2e_x) = \sum_{j=0}^{\infty} \frac{(2e_x^j)}{j!} \quad (B-8) \]

Therefore

\[ M_p = 2 \dot{w} R_n^2 k F^{n+1} e^{2(e_{c.o} - \varepsilon_c)/F} \int_{e_{c,o}}^{e_{\text{max}}} \left\{ \sum_{j=0}^{\infty} \frac{2^j - e^{(e_{c.o} - \varepsilon_c)/F}}{j!} (e'_x)^{j+n} \right\} de'_x \quad (B-13) \]

After integration, the plastic bending moment is determined as:

\[ M_p = 2\dot{w}kF^{n+1}R_n^2 e^{2(e_{c.o} - \varepsilon_c)/F} \sum_{j=0}^{\infty} \left\{ \frac{2^j - e^{(e_{c.o} - \varepsilon_c)/F}}{j!} (e_{\text{max}} + \varepsilon_c, j + 1 + n - \varepsilon_c, j + 1 + n) \right\} \quad (B-14) \]
where the maximum strain at the convex outer surface with radius $R_0$ is defined by Eq. (3.5). When the ratio of $R_i/t$ is greater than 5, the neutral axis is nearly at the middle axis, therefore, the moment can be expressed as

$$M_p = 2wKF^{n+1}R_0^2 e^{-2 \varepsilon/F} \sum_{j=0}^{\infty} \frac{2^j - e^{\varepsilon/F}}{j!(j+1+n)} \left[ (\ln(1 + \frac{t}{2R_n}) + \frac{\varepsilon_0}{F}j + 1 + n - \frac{\varepsilon_0}{F}j + 1 + n) \right]$$

(B-15)

For an accuracy of 99%, only the first four terms in Eq. (B-15) are required. The true strain description, Eq. (3.1), is used in the above equations. When the ratio of $R_i/t$ is large and the linear (engineering) strain is small, the first two terms in Eq. (B-15) can be used. This moment equation reduces to Nagpal's formula [Nagpal, 1980] and Hosford's moment equation [Hosford and Caddell, 1983] by setting the prestrain strain, $\varepsilon_0$, to zero and anisotropic index $F$ to 1.

For fully plastic bending (neglecting the strain at elastic limit, i.e. $\varepsilon_{e,o} = 0$), Eq. (B-15) reduces to

$$M = M_p = 2wKF^{n+1}R_0^2 e^{-2 \varepsilon/F} \sum_{j=0}^{\infty} \frac{2^j - e^{\varepsilon/F}}{j!(j+1+n)} \left[ (\varepsilon_{max} + \varepsilon_0)j + 1 + n - \frac{\varepsilon_0}{F}j + 1 + n \right]$$

Fully plastic bending (B-16)

With Eqs. (B-3) and (B-14), the elasto-plastic bending moment is

$$M_{ep} = Me + M_p = \frac{2wR_0^2}{3} E \varepsilon_{e,o}^3 +
+ 2wKF^{n+1}R_0^2 e^{2(\varepsilon_{e,o})} \sum_{j=0}^{\infty} \frac{2^j - e(\varepsilon_{e,o})}{j!(j+1+n)} \left[ (\varepsilon_{max} + \varepsilon_0 - \varepsilon_{e,o})j + 1 + n - \varepsilon_0j + 1 + n \right]$$

(B-17a)

For elastic-plastic bending with relative large $R_i/t$ ratio, the total bending moment is found as (neglecting the prestrain for convenience)

$$M_{ep} \approx \frac{C_2}{R_n} + \frac{D_1}{R_n+1} - \frac{K}{2} \left( \frac{t}{2R_n} \right)^{n+1} \left( 1 + \frac{3}{2} \frac{n+2}{2R_n} \right) t^2 (\varepsilon_{e,o})^n \left( 1 + \frac{3}{2} \frac{n+2}{2R_n} \right)$$

(B-17b)
APPENDIX C

RESIDUAL STRESS AND SPRINGBACK

The elastic recovery after unloading causes the redistribution of stresses and the springback phenomenon in which the radius of curvature, $r$, of any fiber in bending increases to $r'$ after the bending moment is removed. The dimensional accuracy of the bent sheet components is most affected by springback. And the service quality and life are affected by the residual stresses within the sheet.

1. RESIDUAL STRESS

   The recovered elastic strain can be defined as
   \[ \Delta \varepsilon_x = \varepsilon_{x, \text{unloading}} - \varepsilon_{x, \text{loading}} = \frac{y}{R_n} - \frac{y}{R_n} \]  \hspace{1cm} (C-1)

   The unloaded stress induced by this elastic recovery strain can be written as
   \[ \Delta \sigma_x = \sigma_{x, \text{unloading}} - \sigma_{x, \text{loading}} = E' \Delta \varepsilon_x = E' \frac{y}{R_n} \left( \frac{1}{R_n} - \frac{1}{R_n} \right) \]  \hspace{1cm} (C-2)

   Therefore, the residual stress after unloading is defined as
   \[ \sigma_{x, \text{residual}} = \sigma_{x, \text{unloading}} - \sigma_{x, \text{loading}} - |\Delta \sigma_x| \]
   \[ = kF_{n+1} \left( \frac{y}{R_n} \right)^n - \frac{E}{1 - v^2} \frac{y}{R_n} \left( \frac{1}{R_n} - \frac{1}{R_n} \right) \]  \hspace{1cm} (C-3)

   If the nonlinear strain description, Eq. (2.1), for plastic loading is used, then the
residual stress can be expressed as

\[
\sigma_{x,\text{residual}} = \sigma_{x,\text{unloading}} = \sigma_{x,\text{loading}} - |\Delta \sigma_x|
\]

\[
= kF^{n+1} \left[ \ln\left(1 + \frac{R}{y} \right)^n \right] \frac{E}{1 - v^2} \left[ \ln\left(1 + \frac{R}{R_n} \right) - \frac{y}{R_n} \right]
\]

(C-4)

Some important observations can be found in Eq. (C-4):

(a) the residual stress increases with the plastic properties of the sheet material: strength (k-value), strain hardening (n-value), anisotropy (R-value which raises F-value),

(b) the residual stress decreases with an increase in the elastic properties of the sheet: Young's module E, and Poisson's ratio, and

(c) the residual stress in wide sheet bending under plane-strain condition (F = 1.155) is greater than that in narrow sheet bending under a uniaxial stress state (F = 1).

The elastic recovery strain generates a recovery bending moment \( M' \):

\[
M' = 2w \int_0^{w^2} \Delta \sigma_x y \ dy = 2w \int_0^{w^2} E' \Delta \epsilon_{xy} y \ dy
\]

\[
= \frac{wE'}{12} E' \left( \frac{1}{R_n^3} - \frac{1}{R_n^3} \right) = \frac{I_x E'}{12} \left( \frac{1}{R_n^3} - \frac{1}{R_n^3} \right)
\]

where

\[
E' = \frac{E}{1 - v^2} \quad \text{and} \quad I_x = \frac{wt^3}{12}
\]

(C-5)

I\(_x\) is the secondary moment about the x-axis which passes the middle layer of sheet. \( I_x E' \) denotes the rigidity of the sheet, which increases with elastic properties (Young's module E, and Poisson's ratio), and sheet dimensions (thickness, t, and width, w).

According to the last basic assumption in elementary bending theory, the removal of bending moment after bending is equivalent to the elastic response by superposition of a moment with equal magnitude, but opposite sign. Therefore,
\[ M' = -M = -(M_e + M_p) \]  

(C-7)

The elastic bending moment \( M_e \) is given by Eq. (B-4), and the plastic bending moment \( M_p \) has forms defined in Eqs. (B-14), and (B-15). From above derivations, the springback is finally defined as the change of the curvature:

\[
\frac{1}{R_n} - \frac{1}{R'} = \frac{M}{E' I_x} = \frac{12 (1 - \nu^2)}{w t^3 E} (M_e + M_p)
\]

(C-8)

Assuming a linear (engineering) strain distribution across the sheet thickness, and ignoring the terms with order higher than 3 in Eq. (B-14), the springback is defined by substituting Eqs. (B-4) and (B-14) into above equation:

\[
\frac{1}{R_n} - \frac{1}{R'} = 8 \frac{R_n^2}{E' t^3} \left[ \frac{\sigma_0}{E'} \right]^3 + \frac{3}{n + 2} \frac{k E^{n+1}}{E'} \left[ \left( \frac{1}{2 R_n} \right)^{n+2} - \left( \frac{\sigma_0}{E'} \right)^{n+2} \right]
\]

(C-9)

If we set \( k = E' = E \) (narrow sheet), \( F=1 \) (isotropic), and \( n=0 \) (no work hardening) in Eq. (C-9), then Gardiner's springback model, Eq. (2.16), is recovered, which is applicable for bending of sheets with elastic and ideal-plastic stress-strain response.

The springback angle, \( \Delta \theta \), sometimes may be useful in the design of overbending to reduce springback. If the length of the bending arc segment is \( dS \), then the incremental springback angle is given by:

\[
\Delta \theta = \theta - \theta' = \Delta S \left( \frac{1}{R_n} - \frac{1}{R'} \right)
\]

(C-10)

Therefore, the springback angle increases linearly with bend angle, \( \theta \). The total springback is the sum of the individual springback angles in each cross-section of the sheet. For bending with a V-die, the total springback angle is expressed as [Wang, 1984]:

\[
\theta_t = \frac{12 (1 - \nu^2)}{E} \int_0^{\psi^*} M \csc \phi \, dx
\]

(C-11)
and

$$\varphi = \sin^{-1}\left[ \int_0^{L_d/2} \frac{1}{R_n} \, dx \right]$$

(C-12)

where $L_d$ is the length of the die (along x-axis), and $\varphi$ is the inclination angle with respect to the x-axis. This angle varies from zero underneath the punch tip to the bending angle, $\theta$, at the die shoulder. $\varphi$ is a function of the curvature, hence it depends on the bending moment variation through the die length, $L_d$.

Schwark proposed a springback model considering the shift of fibers of sheet after bending [Schwark, 1952], and it was given as

$$\frac{R_n}{R'_n} = 1 - \frac{12 R_n M}{E w t_0^2}$$

(C-13)

where the bending moment can be given by the summation of the elastic moment, Eq. (B-14), and plastic moment, Eq. (B-15).
Hill's non-quadratic yield function for general anisotropic materials with orthogonal symmetry is given in a formula [Hill, 1979]:

\[
2 f(\sigma_{ij}) = F|\sigma_y - \sigma_2|^M + G|\sigma_z - \sigma_3|^M + H|\sigma_x - \sigma_1|^M + \\
+ 2K \tau_{yz} + 2L \tau_{zx} + 2N \tau_{xy} = 1
\]  

(D-1)

and for principal stresses, it is simplified as

\[
2 f(\sigma_{ij}) = F|\sigma_2 - \sigma_3|^M + G|\sigma_3 - \sigma_1|^M + H|\sigma_1 - \sigma_2|^M = 1
\]  

(D-2)

where \( f \) is a plastic potential, \( F, G, ..., \) and \( N \) are parameters describing the current state of the anisotropy, and the \( M \) is a new index describing the shape of the yield surface. It was found that the effects of \( M \) value on the shape of the yield locus is opposite to that of the normal anisotropy, i.e., average values of planar anisotropy, and the yield locus expands or the yield stress increases, along equal biaxial strain direction (45 degree line in first quadrant), as \( M \) decreases [Wang, 1990]. For Aluminum alloys, \( M \) is around 1.6 ~ 2.0 for a strain range of 0.02 ~ 0.18 [Wagoner, 1980], and \( M \) is correlated well with the normal anisotropy for Steel, Brass, Aluminum, and Copper in the formula \( M = 1 + \bar{R} \) for \( \bar{R} \leq 1 \) [Ragab and Abas, 1986], and \( M = 2 \) for \( \bar{R} > 1 \) [Bressan and Williams, 1983]. The normal anisotropy is defined as
where $R_0$, $R_{45}$, and $R_{90}$ are the anisotropy values measured along the sheet rolling direction (0 degree to rolling direction), 45 degree axis to the rolling direction, and transverse direction (90 degree to the rolling direction). These two anisotropic indexes are defined as:

$$R_0 = \frac{de_2}{de_3}, \quad R_{90} = \frac{de_1}{de_3} \quad \text{and} \quad R_{45} = \frac{de_{45}}{de_3} \quad (D-4)$$

or

$$R_\alpha = \frac{de_\alpha}{de_3} = \frac{H + [2N - F - G - 4H] \sin^2 \alpha \cos^2 \alpha}{F \sin^2 \alpha + G \cos^2 \alpha} \quad (D-5)$$

Using uniaxial tension along the first principal direction (or $x$-axis), the second principal direction (or $y$-axis), and the third principal direction (or $z$-axis), then the parameters $F$, $G$, and $H$ can be found as:

$$2H = \frac{1}{X^M} + \frac{1}{Y^M} - \frac{1}{Z^M}$$

$$2G = \frac{1}{Z^M} + \frac{1}{X^M} - \frac{1}{Y^M}$$

$$2F = \frac{1}{Y^M} + \frac{1}{Z^M} - \frac{1}{X^M} \quad (D-6)$$

Where $X$, $Y$, and $Z$ are the current tensile yield stresses in the principal directions of the anisotropy.

With the associated flow law

$$de_{ij} = \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} d\lambda \quad (D-7)$$

or
\[ \begin{align*}
\text{de}_1 &= M \left[ H |\sigma_1 - \sigma_2|^{M-1} - G |\sigma_3 - \sigma_{||}|^{M-1} \right] d\lambda \\
\text{de}_2 &= M \left[ F |\sigma_2 - \sigma_3|^{M-1} - H |\sigma_1 - \sigma_{2||}|^{M-1} \right] d\lambda \\
\text{de}_3 &= M \left[ G |\sigma_3 - \sigma_{||}|^{M-1} - F |\sigma_2 - \sigma_3|^{M-1} \right] d\lambda 
\end{align*} \] (D-8)

Therefore, the parameters can be related to the anisotropy values:

\[ R_0 = \frac{\text{de}_2}{\text{de}_3} = \frac{F |\sigma_2 - \sigma_3|^{M-1} - H |\sigma_1 - \sigma_{2||}|^{M-1}}{G |\sigma_3 - \sigma_{||}|^{M-1} - F |\sigma_2 - \sigma_3|^{M-1}} \] (D-9a)

\[ R_{90} = \frac{\text{de}_1}{\text{de}_3} = \frac{H |\sigma_1 - \sigma_2|^{M-1} - G |\sigma_3 - \sigma_{||}|^{M-1}}{G |\sigma_3 - \sigma_{||}|^{M-1} - F |\sigma_2 - \sigma_3|^{M-1}} \] (D-9b)

Applying uniaxial tension along x-direction (\( \sigma_1 = X \), and \( \sigma_3 = \sigma_2 = 0 \)) to Eq. (D-9a), and uniaxial tension along y-direction (\( \sigma_2 = Y \), and \( \sigma_3 = \sigma_1 = 0 \)) to Eq. (D-9b), then we obtain

\[ H/G = R_0 \] (D-10a)

and

\[ H/F = R_{90} \] (D-10b)

Using the first and the second equations in Eq. (6), and the Eq. (10), the stress ratio is found to be

\[ \left( \frac{Z}{Y} \right)^M = \frac{F + H}{F + G} = \frac{(1 + R_{90}) R_0}{R_{90} + R_0} \] (D-11)

Similarly, using the second and third equations in Eq. (6), and the Eq. (10), another stress ratio is found to be

\[ \left( \frac{Z}{X} \right)^M = \frac{G + H}{F + G} = \frac{(1 + R_0) R_{90}}{R_{90} + R_0} \] (D-12)
Then, combining Eqs. (11) and (12), we obtain

\[
\begin{pmatrix} X \\ Y \end{pmatrix}^M = \frac{(1 + R_0) R_0}{(R_{90} + R_0) R_{90}}
\] \quad (D-13)

Substituting Eqs. (D-11), (D-12), and (D-13) into Eq. (D-2), the parameter \( H \) is determined as

\[
H = \frac{R_0}{1 + R_0} X^{-M}
\] \quad (D-14a)

Parameters \( G \) and \( F \) are decided by Eqs. (D-10) and (D-14a):

\[
G = \frac{1}{1 + R_0} X^{-M}
\] \quad (D-14b)

and

\[
F = \frac{R_0}{(1 + R_0) R_{90}} X^{-M}
\] \quad (D-14c)

Now we need to determine the second principal stress through the plane strain condition, i.e., the strain along the sheet width, or the transverse direction is zero. Setting the second equation in Eq. (D-8) to zero, and using Eqs. (D-13) and (D-14), the second principal stress is found to be

\[
\sigma_2 = \frac{1}{(1 + R_{90}^{1/(M-1)})} \left( R_{90}^{1/(M-1)} \sigma_1 + \sigma_3 \right)
\] \quad (D-15a)

For isotropic material, \( M = 1 \), and \( R_{90} = 1 \), Eq. (D-15a) reduced to common relation

\[
\sigma_2 = \frac{1}{2} (\sigma_1 + \sigma_3)
\] \quad (D-15b)

Finally substituting Eqs. (D-14) and (D-15a) into Eq. (D-2), the yield criterion under the plane strain condition with 3-D stress state is described by

\[
\sigma_1 - \sigma_3 = C X
\] \quad (D-16)
Let us examine several cases in which Eq. (D-16) can be applied.

**Planar Anisotropic Materials with** \( M = 2 \)

The correlation factor \( C \) is defined as

\[
C = \frac{(A/B)^{1/M}}{A} = \frac{1 + R_0}{R_0} \quad \text{(D-17)}
\]

\[
B = \frac{1}{R_0} + \left(1 + R_{90}^{1/(M-1)}\right)^{1-M} \quad \text{(D-19)}
\]

Normal Anisotropic and Planar Isotropic Materials

For such materials, the anisotropic behavior in the sheet plane is negligible, and only anisotropy in the normal direction (perpendicular to sheet) is considerable and it is described by the normal anisotropy. Therefore

\[
R_0 = R_{90} = R_\alpha = R \quad \text{(D-21)}
\]

The normal anisotropy is an average value of the anisotropy and defined in Eq. (D-3). The yield criterion for normal anisotropic materials is obtained by using the normal anisotropy to replace the planar anisotropy value \( R_0 \) and \( R_{90} \). If \( M = 2 \), then

\[
C = \frac{1 + R}{\sqrt{1 + 2R}} \quad \text{(D-22)}
\]

Isotropic Materials
For such materials, the yield criterion defined by Eq. (D-16) will return to Tresca's by setting $M=1$, $R=1$, or Mises' yield criterion by setting $M=2$, and $R=1$, that is

$$M = 1 \text{ and } R = 1,$$
so that $C = 1$ for Tresca's yield criterion and $M = 2$ and $R = 1$, so that $C = 2/\sqrt{3}$ for Mises's yield criterion.

If $M = 2$, the yield criterion (D-16) can be expressed as

$$|\sigma_1 - \sigma_3| = C X = \frac{1 + \frac{1}{R_0}}{\sqrt{\frac{1}{R_0} + \frac{1}{R_{90}} + \frac{1}{R_0 R_{90}}}}$$

(D-23)

It should be noted that the above equations (D-16) and (D-23) refer to the case in which the bending axis is perpendicular to the rolling direction, i.e. the maximum stress $\sigma_1$, intermediate stress $\sigma_2$, and the minimum stress $\sigma_3$ are parallel to the rolling, transverse, and thickness directions, respectively. For bending axis parallel to the rolling direction, these stresses are

$$\sigma_1' = \sigma_2 \text{ along } 90^\circ \text{ from rolling direction}$$
$$\sigma_2' = \sigma_1 \text{ along rolling direction}$$
$$\sigma_3' = \sigma_3 \text{ along thickness direction}$$

and

$$d\epsilon_2' = 0 \text{ along rolling direction}$$

And the flow stress $Y$ in the direction of $90$ degree from the rolling direction should be used in Eq. (D-16) or (D-23) instead of the flow stress $X$ along rolling direction.
APPENDIX E

EQUILIBRIUM CONDITION AND STRESSES IN BENDING

Force equilibrium along the thickness or radial direction can be written as (Fig. 3.4):

\[
2\sigma_1 \, w \, dr \sin\left(\frac{d\theta}{2}\right) + \sigma_3 \, w \, r \, d\theta - (\sigma_3 + d\sigma_3) (r + dr) \, d\theta \, w = 0
\]

or

\[
\frac{d\sigma_3}{dr} - \frac{\sigma_1 - \sigma_3}{r} = 0
\]

(E-1)

In order to solve the stress, we must adopt the yield criterion, Eq. (D-16), and the hardening curve for flow stress. Using a Hollomn type of hardening law in uniaxial tension, the stress and strain relation along the fiber length direction, or the first principal direction can be written as

\[
X = \sigma_x = k e^n_x
\]

(E-2)

Combining with Eq. (3.29a), the uniaxial flow stress in the fiber direction is defined

\[
\sigma_x = k \left[ \ln \left( \frac{R - r}{R_n} \right) \right]^n = k \left[ \ln \left( 1 + \frac{y}{R_n} \right) \right]^n
\]

(E-3)

Therefore, the yield condition, Eq. (D-16) can be rewritten as

\[
\sigma_1 - \sigma_3 = CX = Ck \left[ \ln \left( \frac{R - r}{R_n} \right) \right]^n \quad R_n \leq r \leq R_0
\]

(E-4a)
\[ \sigma_1 - \sigma_3 = -C[X] = -Ck \ln \frac{r}{R_i} \bigg|^n = -Ck \left[ \ln \frac{R_i}{r} \right]^n \quad R_i \leq r \leq R_o \quad (E-4b) \]

where \( R_i \) and \( R_o \) are the radii at the inner (concave) and the outer (convex) surfaces, respectively.

Substituting Eqs. (E-3) and (E-4) into the equilibrium condition (E-1), and solving the differential equation, we obtain the radial stress as

\[ \sigma_3 = p_o - \frac{Ck}{n + 1} \left[ (\ln \frac{R_o}{R_i})^{n+1} - (\ln \frac{r}{R_i})^{n+1} \right] \quad R_i \leq r \leq R_o \quad (E-5a) \]

With Eqs. (E-5a) and (E-4a), the stress along the fiber length direction can be determined as

\[ \sigma_1 = \sigma_3 + CX = p_o - \frac{Ck}{n + 1} \left[ (\ln \frac{R_o}{R_i})^{n+1} - (\ln \frac{r}{R_i})^{n+1} \right] + Ck \left[ \ln \frac{r}{R_i} \right]^n \quad R_i \leq r \leq R_o \quad (E-5b) \]

where \( p_o \) is the pressure exerted on the outer surface as in bending with a back pad.

The stress distribution for the fibers below the neutral axis can be determined in a similar manner and they are defined by Eq. (E-6):

\[ \sigma_3 = p_i - \frac{Ck}{n + 1} \left[ \left| \ln \frac{R_i}{R_o} \right|^{n+1} - \left| \ln \frac{r}{R_o} \right|^{n+1} \right] \quad R_i \leq r \leq R_o \quad (E-6a) \]

\[ \sigma_1 = \sigma_3 - C \sigma_i = p_i - \frac{Ck}{n + 1} \left[ \left| \ln \frac{R_i}{R_o} \right|^{n+1} - \left| \ln \frac{r}{R_o} \right|^{n+1} \right] - Ck \left[ \ln \frac{r}{R_o} \right]^n \quad R_i \leq r \leq R_o \quad (E-6b) \]

where \( p_i \) is the tool (punch) pressure exerted on the inner surface of the sheet.

With the plane strain condition, the stress along the width for a material with planar anisotropy is found to be:

\[ \sigma_2 = \frac{1}{(1 + R_{o0}^{M-1})} \left( R_{o0} \sigma_1^{M-1} + \sigma_3 \right) \quad (E-6c) \]
The simple models, based on the plastic volume conservation, can be derived to approximately account for the shift of the neutral axis. The models are derived and proposed below.

As shown in Fig. F-1, the radius of the neutral axis is defined as

\[ R_n = \frac{L_o}{\theta} \quad \text{or} \quad \theta = \frac{L_o}{R_n} \quad \text{(F-1)} \]

where \( L_o \) is the initial length of the fiber, and \( \theta \) is the bend angle. The initial volume, \( V_o \), and area, \( A_o \), of the sheet are

\[ V_o = L_o t_o w \]

and \( A_o = L_o t_o \) \quad \text{(F-2)}

The width of the sheet remains constant for plane strain deformation. Therefore, the volume constancy of plasticity can be written as area constancy. The area under the bending arc is:

\[ A = \frac{\theta}{2} (R_2^2 - R_1^2) = A_o = L_o t_o \]

\quad \text{(F-3)}
Fig. F-1  Bending of a sheet
Therefore, the bending angle can be written as

$$
\theta = 2 \frac{L_0 t_0}{(R_i^2 - R_e^2)} = 2 \frac{L_0 t_0}{(R_e - R_i)(R_e + R_i)} = \frac{L_0 t_0}{t R_m}
$$

where the current thickness, $t$, is defined as

$$
t = R_e - R_i
$$

and the radius of the middle layer is given by

$$
R_m = \frac{(R_e + R_i)}{2} = R_{tool} + \frac{t}{2}
$$

Equating the two relations in Eqs. (F-1) and (F-4), the radius of the neutral axis is modified by the thinning ratio of the sheet, $t / t_0$:

$$
R_n = R_m \left( \frac{t}{t_0} \right) = (R_i + \frac{t}{2}) \left( \frac{t}{t_0} \right)
$$

This relation implies that the neutral axis shifts towards the concave surface of sheet if the thinning starts, and $R_n < R_m$. This is the actual case for stretch/draw forming where bending and tension coexist. The current thickness, $t$, can be approximated by the membrane analysis:

$$
t = t_0 e^{-\varepsilon_m}
$$

where $\varepsilon_m$ is the membrane strain along the longitudinal direction, and $t_0$ is the initial thickness.

The neutral axis is finally expressed as the membrane tensile strain, $t_0$ radius, and sheet thickness:

$$
R_n = \left( R_i + \frac{t_0}{2} e^{-\varepsilon_m} \right) e^{-\varepsilon_m}
$$
APPENDIX G

INTERNAL BENDING MOMENT EVALUATION

The bending moment at any section of sheet is defined as

\[ M = \int_{R_i}^{R_o} \sigma_x r \, dr = \int_{R_i}^{R_n} \sigma_1 r \, dr + \int_{R_n}^{R_o} \sigma_1 r \, dr \]  
\[ (G-1) \]

In order to integrate the above equations conveniently, we transform the radius \( r \) at arbitrary fiber to the strain at that fiber. That is

\[ \varepsilon_1 = \ln \frac{r}{R_n}, \quad r = R_n e^{\varepsilon_1}, \quad \text{and} \quad dr = R_n e^{\varepsilon_1} \, d\varepsilon_1 \]  
\[ (G-2) \]

and

\[ \varepsilon_i = \ln \frac{R_i}{R_n} \quad r = R_i, \text{ at inner surface} \]
\[ \varepsilon_n = \ln \frac{R_n}{R_n} = 0 \quad r = R_n, \text{ at the neutral axis} \]
\[ \varepsilon_o = \ln \frac{R_o}{R_n} \quad r = R_o, \text{ at outer surface} \]
\[ (G-3) \]

Therefore, the stress along fiber length direction, Eqs. (3.38b) and (3.39b), can be rewritten as:

\[ \sigma_1 = p_0 - \frac{C_k}{n+1} \left[ \varepsilon_i^{n+1} - \varepsilon_i^{n+1} \right] + C_k \varepsilon_1^n \quad 0 \leq \varepsilon_1 \leq \varepsilon_o \]  
\[ (G-4) \]
\[ \sigma_1 = p_0 - \frac{C_k}{n+1} \left[ |\varepsilon_1|^{n+1} - |\varepsilon_1|^{n+1} \right] - C_k |\varepsilon_1|^n \quad \varepsilon_1 \leq \varepsilon_1 \leq 0 \]
Substituting the stresses defined in Eq. (G-4) and the relations defined by Eqs. (G-2) to (G-3) into Eq. (G-1), the bending moment is derived as

\[
M = \int_{\varepsilon^n}^{0} \left( [p_i - \frac{Ck}{n + 1} (e_{i}^{n+1} - e_{i}^{n})] - Ck |\varepsilon_i|^{n} \right) \, d\varepsilon_i + \\
+ \int_{0}^{0} \left\{ [p_0 - \frac{Ck}{n + 1} (e_{0}^{n+1} - e_{0}^{n})] + Ck e_{0}^{n} \right\} R^2 \, e^{2\varepsilon_1} \, d\varepsilon_1
\]  

(G-5)

Using Taylor's expansion for the exponent term:

\[
e^{2\varepsilon_1} = 1 + 2 \varepsilon_1 + 2 \varepsilon_1^2 + \frac{4}{3} \varepsilon_1^3 + \ldots = \sum_{j=0}^{\infty} \frac{(2 \varepsilon_1)^j}{j!}
\]  

(G-6)

Hence,

\[
M = \int_{\varepsilon^n}^{0} \left( [p_i + \frac{Ck}{n + 1} (e_{i}^{n+1} - e_{i}^{n})] - Ck |\varepsilon_i|^{n} \right) R^2 \sum_{j=0}^{\infty} \frac{(2 \varepsilon_1)^j}{j!} \, d\varepsilon_i + \\
+ \int_{0}^{0} \left\{ [p_0 + \frac{Ck}{n + 1} (e_{0}^{n+1} - e_{0}^{n})] + Ck e_{0}^{n} \right\} R^2 \sum_{j=0}^{\infty} \frac{(2 \varepsilon_1)^j}{j!} \, d\varepsilon_1
\]

Finally, the bending moment is found to be:

\[
M = (p_i - \frac{Ck}{n + 1} |\varepsilon_i|^{n+1}) \frac{R^2}{2} (1 - e^{2\varepsilon_i}) + Ck R^2 \sum_{j=0}^{\infty} \frac{2j}{(n+j+1)!} [e_{i}^{n+1+j} - |\varepsilon_i|^{n+1+j}] \\
+ (p_0 - \frac{Ck}{n + 1} e_{0}^{n+1}) \frac{R^2}{2} (e^{2\varepsilon_0} - 1) + Ck R^2 \sum_{j=0}^{\infty} \frac{2j}{(n+2+j)!} [e_{0}^{n+2+j} - |\varepsilon_0|^{n+2+j}]
\]  

(G-7)

Using the strain descriptions in Eq. (G-3), another form of the moment is
The strain terms with orders higher than five may be safely neglected for an accuracy higher than 99%, because the strain is generally less than 40%. Then other forms of the moment are

\[
M = 2 \left[ p_i - \frac{C k}{n+1} \ln \left( \frac{R_i}{R_n} \right)^{n+1} \right] \left( \frac{R_n^2 - R_i^2}{R_n^2} \right) + \frac{1}{2} \left[ p_o - \frac{C k}{n+1} \left( \ln \frac{R_o}{R_n} \right)^{n+1} \right] \left( \frac{R_o^2 - R_n^2}{R_o^2} \right) \\
+ \frac{C k}{n+1} R_n^2 \sum_{j=0}^{\infty} \frac{2^j}{(n+j+1)!} \left[ \ln \left( \frac{R_o}{R_n} \right)^{n+1+j} - \ln \left( \frac{R_i}{R_n} \right)^{n+1+j} \right] \\
+ \frac{C k}{n+1} R_n^2 \sum_{j=0}^{\infty} \frac{2^j}{(n+j+1)!} \left[ \ln \left( \frac{R_o}{R_n} \right)^{n+2+j} - \ln \left( \frac{R_i}{R_n} \right)^{n+2+j} \right]
\]

(G-8)

The strain terms with orders higher than five may be safely neglected for an accuracy higher than 99%, because the strain is generally less than 40%. Then other forms of the moment are

\[
M = \frac{p_i - \frac{C k}{n+1} \varepsilon_n^{n+1}}{2} \frac{R_n^2}{1 - e^{2\varepsilon_i}} + \frac{C k R_n^2}{n+1} \left\{ \frac{\varepsilon_{e_i}^{n+2}}{n+2} \right\} \\
+ \frac{4[(\varepsilon_{e_o}^{n+5}) \varepsilon_{e_i}^{n+5}]}{3(n+4)} \right\} \\
+ \frac{(p_0 + \frac{C k}{n+1} \varepsilon_o^{n+1})}{2} \frac{R_n^2}{(R_i/R_n^2) - 1} + \frac{C k R_n^2}{n+1} \left\{ \frac{\varepsilon_{e_i}^{n+1}}{n+1} \right\} \\
+ \frac{2[(\varepsilon_{e_o}^{n+3}) \varepsilon_{e_i}^{n+3}]}{3(n+4)} \right\} \\
+ \frac{4[(\varepsilon_{e_o}^{n+4}) \varepsilon_{e_i}^{n+4}]}{3(n+4)} \right\} \\
+ \frac{(p_0 + \frac{C k}{n+1} \varepsilon_o^{n+1})}{2} \frac{R_n^2}{(R_i/R_n^2) - 1} + \frac{C k R_n^2}{n+1} \left\{ \frac{\varepsilon_{e_i}^{n+1}}{n+1} \right\} \\
+ \frac{\varepsilon_{e_i}^{n+2} \varepsilon_{e_i}^{n+2}}{n+2} + \frac{2[(\varepsilon_{e_o}^{n+3}) \varepsilon_{e_i}^{n+3}]}{3(n+4)} \right\} \\
+ \frac{4[(\varepsilon_{e_o}^{n+4}) \varepsilon_{e_i}^{n+4}]}{3(n+4)} \right\}
\]

(G-9)
APPENDIX H

AXIAL FORCE AND BENDING MOMENT

The axial force per unit width, $N$, and the moment per unit width, $M$, are defined as:

$$N = \int_{-\frac{d}{2}}^{0} \sigma_x \, dy + \int_{0}^{c} \sigma_x \, dy + \int_{0}^{\frac{1}{2}+d} \sigma_x \, dy \quad (H-1a)$$

or

$$N = -\int_{\varepsilon_{e_0}}^{\varepsilon_{e_0}} \varepsilon_0 \, \sigma_x \, dy + \int_{0}^{e_{max}} \varepsilon_x \, \sigma_x \, dy + \int_{e_{max}}^{\varepsilon_{e_0}} \varepsilon_x \, \sigma_x \, dy$$

$$= -\int_{\varepsilon_{e_0}}^{\varepsilon_{e_0}} kF^{n+1} (\varepsilon_x + \frac{\varepsilon_0 - \varepsilon_{e_0}}{F}) \, dy - \int_{0}^{e_{max}} E' \varepsilon_x \, dy + \int_{0}^{e_{max}} E' \varepsilon_x \, dy + \int_{e_{max}}^{\varepsilon_{e_0}} kF^{n+1} (\varepsilon_x + \frac{\varepsilon_0 - \varepsilon_{e_0}}{F}) \, dy$$

$$= -\int_{\varepsilon_{e_0}}^{\varepsilon_{e_0}} kF^{n+1} (\varepsilon_x + \frac{\varepsilon_0 - \varepsilon_{e_0}}{F}) \, dy$$

$$= \int_{\varepsilon_{e_0}}^{e_{max}} kF^{n+1} (\varepsilon_x) \, dy$$

$$= \int_{\varepsilon_{e_0}}^{e_{max}} kF^{n+1} (\varepsilon_x)' \, dy$$

$$= \int_{\varepsilon_{e_0}}^{e_{max}} kF^{n+1} (\varepsilon_x)' \, dy$$

$$(H-1b)$$

and

$$M = \int_{-\frac{d}{2}}^{0} \sigma_x y \, dy + \int_{0}^{c} \sigma_x y \, dy + \int_{0}^{\frac{1}{2}+d} \sigma_x y \, dy \quad (H-2a)$$
or

\[ M = \int_{e_{\text{e}}}^{e_{\text{x}}} \sigma_x \, dy + \int_{0}^{e_{\text{x}}} \sigma_x \, dy + \int_{e_{\text{e}}}^{e_{\text{x}}} \sigma_x \, dy + \int_{e_{\text{e}}}^{e_{\text{x}}} \sigma_x \, dy \]

\[ = \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x + \frac{\varepsilon_x - \varepsilon_{\text{x},0}}{F}) \, dy + 2 \int_{0}^{e_{\text{x}}} E' \varepsilon_x \, dy + \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x + \frac{\varepsilon_x - \varepsilon_{\text{x},0}}{F}) \, dy \]

\[ = \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x + \frac{\varepsilon_x - \varepsilon_{\text{x},0}}{F}) \, dy + 2 \int_{0}^{e_{\text{x}}} E' \varepsilon_x \, dy + \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x + \frac{\varepsilon_x - \varepsilon_{\text{x},0}}{F}) \, dy \]

\[ = \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x) \, dy + 2 \int_{0}^{e_{\text{x}}} E' \varepsilon_x \, dy + \int_{e_{\text{e}}}^{e_{\text{x}}} kF^{n+1}(\varepsilon_x) \, dy \]

\[ \text{(H-2b)} \]

where

\[ \varepsilon'_x = \varepsilon_x + (\varepsilon_{\text{x},0} - \varepsilon_{\text{x},0}) / F \]

\[ \text{(H-3)} \]

The direct use of this nonlinear relation causes mathematical difficulty in the integrations. Following the same treatment in Appendix B, we chose the integration variable as the strain instead of the radial distance, \( y \). That is

\[ \varepsilon_x = \ln(1 + y / R_n) \]

\[ \text{(H-4)} \]

\[ y = R_n [\exp(\varepsilon_x) - 1], \text{ and } dy = R_n \exp(\varepsilon_x) \, d\varepsilon_x \]

\[ \text{(H-5)} \]

\[ d\varepsilon'_x = d\varepsilon_x \]

\[ \text{(H-6)} \]

and strains expressed by Taylor's series:

\[ \exp(\varepsilon_x) = 1 + \varepsilon_x + \frac{1}{2} \varepsilon_x^2 + \frac{1}{6} \varepsilon_x^3 + \ldots + \frac{\varepsilon_x^j}{j!} + \ldots = \sum_{j=0}^{\infty} \frac{(\varepsilon_x)^j}{j!} \]

\[ \text{(H-7)} \]

and

\[ \exp(2\varepsilon'_x) = \sum_{j=0}^{\infty} \frac{(2 \varepsilon'_x)^j}{j!} \]

\[ \text{(H-8)} \]
Therefore

\[ N = \int_{[\varepsilon_1]}^{\varepsilon_{\text{max}}} kF^{n+1}(\varepsilon'_x)^n R_n e^{e_x} \, d\varepsilon_x \]

\[ = \int_{[\varepsilon_1]}^{\varepsilon_{\text{max}}} kF^{n+1}R_n e^{e_x} e^{\frac{e_x - \varepsilon_{\text{e.o.}}}{F}} \, d\varepsilon_x \]

\[ = \int_{[\varepsilon_1]}^{\varepsilon_{\text{max}}} kF^{n+1}R_n e^{e_x} \frac{e_x - \varepsilon_{\text{e.o.}}}{F} \sum_{j=0}^{\infty} \frac{(\varepsilon'_x)^j}{j!} \, d\varepsilon_x \]

\[ = \int_{[\varepsilon_1]}^{\varepsilon_{\text{max}}} kF^{n+1}R_n e^{e_x} \sum_{j=0}^{\infty} \frac{(\varepsilon'_x)^j}{j!} \, d\varepsilon_x \]

and

\[ N = kF^{n+1}R_n e^{e_x} \frac{e_x - \varepsilon_{\text{e.o.}}}{F} \sum_{j=0}^{\infty} \frac{1}{(j + n + 1)j!} \left[ (\varepsilon_{\text{max}} + \frac{\varepsilon_0 - \varepsilon_{\text{e.o.}}}{F})^{e_{\text{e.o.}} + j} - (\varepsilon_1 + \frac{\varepsilon_0 - \varepsilon_{\text{e.o.}}}{F})^{e_{\text{e.o.}} + j} \right] \]

\[ (H-9) \]

The elastic moment within the elastic layer \( M_e \) is

\[ M_e = 2 \int_0^{e_{\text{e.o.}}} F' \varepsilon_x y \, dy = 2 \int_0^{e_{\text{e.o.}}} F' \varepsilon_x (\varepsilon_x R_n)(d\varepsilon_x R_n) = \frac{2}{3} F' R_n^2 e_x^3 \]

\[ (H-10) \]

The plastic moment \( M_p \) is
\[ M_p = \int_{e_o}^{e_{\text{max}}} kF^{n+1}(e')^n R_n \left( e'^\cdot e - \frac{\epsilon_o - \epsilon_{\text{max}}}{F} - 1 \right) R_n e'^\cdot e - \frac{\epsilon_o - \epsilon_{\text{max}}}{F} \, de' \]

\[ + \int_{e_o}^{e_{\text{max}}} kF^{n+1}(e')^n R_n \left( e'^\cdot e - \frac{\epsilon_o - \epsilon_{\text{max}}}{F} - 1 \right) R_n e'^\cdot e - \frac{\epsilon_o - \epsilon_{\text{max}}}{F} \, de' \]

\[ = \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} (e'^2 - e'^\cdot e) \, de' \]

\[ + \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} (e'^2 - e'^\cdot e) \, de' \]

\[ = \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} (e'^2 - e'^\cdot e) \left( \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \right) \, de' \]

\[ + \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} (e'^2 - e'^\cdot e) \left( \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \right) \, de' \]

\[ = \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} \left( \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \right) \, de' \]

\[ + \int_{e_o}^{e_{\text{max}}} kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} \left( \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \right) \, de' \]

or

\[ M_p = kF^{n+1} R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} \left( \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \right) \left( [\epsilon_o' + \frac{\epsilon_o - \epsilon_{\text{max}}}{F}]^{n+1+j} \right) \]

\[ + (\epsilon_{\text{max}} + \frac{\epsilon_o - \epsilon_{\text{max}}}{F})^{n+1+j} - 2(\epsilon_{e,0} + \frac{\epsilon_o - \epsilon_{\text{max}}}{F})^{n+1+j} \]  \( \text{(H-11)} \)

Combining Eqs. (H-10) and (H-11), the bending moment is

\[ M = M_c + M_p = \frac{2}{3} E' R_n^2 e^{\frac{\epsilon_o - \epsilon_{\text{max}}}{F}} \sum_{j=0}^{\infty} \left( \frac{2j}{j!} \right) \left( [\epsilon_o' + \frac{\epsilon_o - \epsilon_{\text{max}}}{F}]^{n+1+j} \right) \]

\[ + (\epsilon_{\text{max}} + \frac{\epsilon_o - \epsilon_{\text{max}}}{F})^{n+1+j} - 2(\epsilon_{e,0} + \frac{\epsilon_o - \epsilon_{\text{max}}}{F})^{n+1+j} \]  \( \text{(H-12)} \)
For an accuracy of 99%, only the first four terms in Eq. (1-15) are required. All characteristic strains are defined in Eq. (3.58), and they are the function of the shifting distance of the neutral axis, d, the bending radius, R_t, and the sheet thickness, t. Therefore, once the axial force N is given, then the shifting distance of the neutral axis, d, can be determined via Eq. (H-5). Then the characteristic strains in Eq. (3.58), from which the bending moment in Eq. (H-5) is related can be defined. The alternative procedure is described in Section 3.3.6. In general, the initial yield strain is around 0.001 to 0.005, therefore the term $e_0^3$, can be neglected.
APPENDIX I

CURVATURE DISTRIBUTION

There exists a one-to-one response in the curvature and bending moment. The curvature distribution along the length of a sheet is a function of the bending moment. For sections with different bending deformation, the curvatures have to be solved based on the bending moment distributions defined in Eqs. (3.5, a,b,c). In the punch contact region, the sheet wraps on the punch radius and takes the punch curvature. Therefore, in the punch-sheet contact region (fully plastic bending), the curvature distribution is a constant:

\[ \frac{1}{R} = \frac{1}{R_p + t/2} = \frac{1}{R_p'} \]  

(I-1)

where \( R_p' = R_p + t/2 \) is the radius at the middle axis.

For the elasto-plastic portion, the total moment includes the elastic and the plastic bending moment defined in Eq. (B-17). The elastic moment, \( M_e \), and the plastic moment, \( M_p \), are found to be:

\[ M_e = \frac{2wR^2}{3} \left( \frac{\sigma_y}{E} \right)^2 \sigma_y \]  

(I-2)

and

\[ M_p = 2wkF^{n+1}R_n^2 e^{2(e_{e0}-e_0)/F} \sum_{j=0}^{\infty} \left\{ \frac{2^j}{j!} \left( \frac{e_{e0}}{F} \right)^j \left[ ( e_{\max} + \frac{e_0 - e_{e0}}{F} ) j + 1 + n - \left( \frac{e_0}{F} \right) j + 1 + n \right] \right\} \]  

(I-3)
Therefore, the total bending moment is

\[ M_{\text{ep}} = C_4 R^2 + \frac{C_2}{R^n} + \frac{C_1}{R^{n+1}} \]  

(I-4)

Note that only first three terms in Eq. (I-3) are chosen, and elastic limit strain \( \varepsilon_{e,o} \) and prestrain \( \varepsilon_o \) are neglected since they are much smaller than the maximum surface strain.

By equating Eqs. (I-4) and (3.4b), and using Eq. (3.3b) and the moment continuity condition at punch tip \((S=0, M(0) = M_A)\), we can derive the following relation:

\[ S_1 = \frac{M_A}{C_3 M_E} \]

Hence, the expression for the curvature distribution is found to be:

\[ S = S_1 - C_6 R^2 - \frac{C_7}{R^n} - \frac{C_8}{R^{n+1}} \quad 0 \leq S \leq S_E \quad (\text{Elasto-plastic bending}) \]  

(I-5)

For the elastic bending portion of the sheet, the curvature distribution is simply defined as

\[ \frac{1}{R} = \frac{M(S)}{E' I} = \frac{M_E}{E' I} \frac{S_1 - S}{S_1 - S_E} = \frac{1}{R_E} \frac{S_1 - S}{S_1 - S_E} = \frac{C_3}{R_E} (S_1 - S) = C_5 (S_1 - S) \]

\[ S_E \leq S \leq S_1 \quad (\text{Elastic bending}) \]  

(I-6)

where \( 1/R_E = M_E / (E'I) \) is the curvature at the transition point, E. The constants in the above equations are defined as

\[ C_1 = \frac{3t}{4} \frac{n+2}{n+3} C_2 \]  

(I-7a)

\[ C_2 = \frac{2wkF_{n+1}^{n+1}}{n + 2} \frac{1}{2} \]

(I-7b)

\[ C_3 = \frac{1}{S_1 - S_E} = \frac{M_A}{M_E} \frac{1}{S_1} \]

(I-7c)

\[ C_4 = \frac{2}{3} \frac{\sigma_y^2}{E'} \]

(I-7d)

\[ C_5 = \frac{C_3}{R_E} = \frac{C_3 M_E}{E' I} \]

(I-7e)
The plane strain module $E'$ is defined as

$$E' = \frac{E}{1 - v^2}$$  \hspace{1cm} (I-8)$$

and $I$ is the moment of the area about the middle axis:

$$I = \frac{W t^3}{12}$$  \hspace{1cm} (I-9)$$

The deformed shape of the bent sheet can be plotted using the curvature distributions defined in Eqs. (I-6) and (I-7).
APPENDIX J

BENDING ANGLE $\theta_1$ AND SPRINGBACK ANGLE $\theta_S$

1. Bending Angle

The total rotation angle or bending angle is the sum of the individual rotation at each section. By definition, the rotation of incremental arc, $ds$, can be expressed as

$$d\theta = \frac{ds}{R}$$

and the springback at each section is

$$\delta \theta_S = K_S \delta S = \frac{M(S)}{E'T} \delta S$$

Therefore, the total rotation $\theta_1$ is

$$\theta_1 = \int_0^{\theta_1} d\theta = \theta_C + \theta_{AE} + \theta_{EB} \quad \text{(J-1)}$$

where $\theta_C$ is the contact angle between the punch and sheet, $\theta_{AE}$ and $\theta_{EB}$ are the rotations in the regions of elasto-plastic bending ($0 \leq S \leq S_E$) and elastic bending ($S_E \leq S \leq S_I$), respectively.

From Eq. (3.6b), or (I-5), and (3.6c) or (I-6):

$$d\theta_{AE} = \frac{ds}{R} = [-2C_6 + nC_7 \left(\frac{1}{R}\right)^{2+n} + (1+n)C_8 \left(\frac{1}{R}\right)^{3+n}] dR \quad \text{(J-2a)}$$
and
\[ \frac{d\theta_{EB}}{dR} = C_5 (S_1 - S) dS \] (J-2b)

Therefore, the rotation angle in elasto-plastic region is found to be:

\[ \theta_{AE} = \int_{0}^{S_e} \frac{dS}{R} = \int_{R_p}^{R'_p} \left[ -2C_6 + nC_7 \left( \frac{1}{R} \right)^{n+1} + (n + 1)C_8 \left( \frac{1}{R} \right)^{n+2} \right] \]
\[ = -2C_6 (R'_p - R'_p) + C_7 \left[ \left( \frac{1}{R_p} \right)^n - \left( \frac{1}{R'_p} \right)^n \right] + C_8 \left[ \left( \frac{1}{R_p} \right)^{n+1} - \left( \frac{1}{R'_p} \right)^{n+1} \right] \] (J-3)

where \( R' \) is the radius at the neutral layer.

The rotation for the elastic portion \( (S_e \leq S \leq S_i) \) is found to be

\[ \theta_{EB} = \int_{S_e}^{S_i} \frac{dS}{R} = \int_{S_e}^{S_i} C_5 (S_1 - S) dS = \frac{C_5}{2} (S_1 - S_e)^2 \] (J-4)

Therefore, the total rotation or the bending angle under the load is

\[ \theta_i = \theta_e - 2C_6 (R'_p - R'_p) + C_7 \left[ \left( \frac{1}{R_p} \right)^n - \left( \frac{1}{R'_p} \right)^n \right] \]
\[ + C_8 \left[ \left( \frac{1}{R_p} \right)^{n+1} - \left( \frac{1}{R'_p} \right)^{n+1} \right] + \frac{C_5}{2} (S_1 - S_e)^2 \] (J-5)

The contact angle, \( \theta_C \), will be determined later.

2. Springback

At any section, the curvature change due to springback after removal of the load is defined as

\[ K_S = \frac{1}{R} - \frac{1}{R'} = \frac{M(S)}{E' I} \] (J-6)

The springback angle at each section is
\[ \theta_s = K_s \delta S = \frac{M(S)}{E'I} \delta S \quad (J-7) \]

The total springback is the sum or integration of the springback at individual section:

\[ \theta_s = \int_{0}^{s_1} \delta \theta_s = \int_{0}^{s_1} \frac{M(S)}{E'I} \delta S \]
\[ = \int_{0}^{s_1} \frac{C_3M_E}{E'I} (S_1 - S) \delta S \]
\[ = \frac{C_3M_E}{2E'I} S_1^2 = \frac{MA}{2E'I} S_1 \quad (J-8a) \]

Therefore, the springback is proportional to the bending arm length, \( S_1 \), and the bending moment. The influences of the material properties and tool shape and dimensions can be explored by

\[ \theta_s = \frac{1}{2} \frac{MA}{E'I} S_1 = 3 \frac{K_F^{m+1}}{2 + n} \frac{1}{E} \left( \frac{t}{2R_p} \right)^n \left( 1 + \frac{2}{n+2} \frac{t}{2R_p} \right) \frac{S_1}{t} \quad (J-8b) \]

The bending angle after unloading, i.e. the desired bending angle, \( \theta_2 \), is the difference between the overbending angle under load, \( \theta_1 \), and the total springback angle:

\[ \theta_2 = \theta_1 - \theta_s \quad (J-9a) \]

Combined with Eqs. (J-5) and (J-8b), Eq. (J-9a) is found to be

\[ \theta_2 = \theta_e - 2C_6(R_E - R_p') + C_7[(\frac{1}{R_p})^n - (\frac{1}{R_E})^n] \]
\[ + C_8(\frac{1}{R_p})^{n+1} - (\frac{1}{R_E})^{n+1} - \frac{C_5}{2} (S_1 - S_E)^2 - \theta_s \quad (J-9b) \]

Therefore, the punch/sheet contact angle can be defined now as:

\[ \theta_c = \theta_1 + 2C_6(R_E - R_p') - C_7[(\frac{1}{R_p})^n - (\frac{1}{R_E})^n] \]
\[ - C_8(\frac{1}{R_p})^{n+1} - (\frac{1}{R_E})^{n+1} - \frac{C_5}{2} (S_1 - S_E)^2 \quad (J-10) \]
3. Bending Span Length $S_1$

The span of bending arm, $S_1$, must be decided through the kinematic relation:

$$dX = \cos \theta \, dS \quad \text{(J-11)}$$

and

$$S_1 = \int_{\theta_c}^{\theta_1} dS = \int_{\theta_c}^{\theta_1} \frac{dX}{\cos \theta} \quad 0 \leq X \leq X(\theta_1) \quad \text{(J-12)}$$

with

$$X(\theta_1) = L_d - (R_p + t/2) \cos \theta_c - (R_d + t/2) \cos \theta_l \quad \text{(J-13)}$$

where the half length of the die, $L_d$, is defined by the clearance (die gap), the radii of the punch and the die:

$$L_d = C + R_p + R_d \quad \text{(J-14)}$$

Equations (J-12) and (J-13) must be solved by an iterative procedure. The initial guess of $S_1$ can be found by assuming that the punch contact angle reaches to the value of final bending angle under load, $\theta_1$, and the rotation angle in integration in Eq. (J-12) keeps constant. After integration, the initial guess of the span length is found to be:

$$S_1 = \frac{L_d}{\cos \theta_1} - (R_p + R_d + t) \tan \theta_l \quad \text{(J-15)}$$
From the kinematic relation
\[ \frac{dy}{dS} = \sin \theta \]  \tag{K-1}

then the vertical component of span Si is
\[ Y_1 = \int_0^{S_i} dy = \int_0^{S_i} \sin \theta \, dS \]  \tag{K-2}

Referring to Fig. 3.2, the punch stroke, d, corresponding to the bending angle under load, \( \theta_1 \), can be expressed as
\[ d = Y_1 + (R_d + \frac{1}{2}) (1 - \cos \theta_1) + (R_p + \frac{1}{2}) (1 - \cos \theta_c) \]
\[ = \int_0^{S_1} \sin \theta \, dS + [R_d - (R_d + \frac{1}{2}) \cos \theta_1] + [R_p - (R_p + \frac{1}{2}) \cos \theta_c] + t \]  \tag{K-3}

where the integration term is the vertical measure of the span arm, \( S_1 \), and the angle, \( \theta \), is the swiping angle of arc length, \( S \), which varies from zero to the span, \( S_1 \).

1. **The Rotation Angle \( \theta \)**

The rotation angle of an arc, \( ds \), is defined by (K-1), and the vertical measure, \( Y_1 \),
of the span arm, $S_1$, can be written as:

$$Y_1 = \int_0^{S_1} \sin \theta \, ds = u_1 + u_2$$  \hspace{1cm} (K-4)$$

where

$$u_1 = \int_0^{S_1} \sin \theta \, ds$$  \hspace{1cm} (K-5)$$

and

$$u_2 = \int_{S_1}^{S_E} \sin \theta \, ds$$  \hspace{1cm} (K-6)$$

Since

$$ds = R \, d\theta$$

hence

$$u_1 = \int_0^{S_E} R \sin \theta \frac{d\theta}{dR} \, dR$$  \hspace{1cm} (K-7)$$

The term of $d\theta /dR$ can be determined from rotation angle, $\theta_{AE}$, in the region of elastic and plastic bending through Eq. (J-3), by setting the upper limit of the integration to be the variable, $R$, i.e.:

$$\theta = \theta_c + \theta_{AE} = \int_{R_p}^{R} \left[ nC_7 \left( \frac{1}{R} \right)^{2+n} + (1+n)C_8 \left( \frac{1}{R} \right)^{3+n} \right] \, dR$$  \hspace{1cm} (K-8)$$

$$= \theta_c + \frac{n}{n+1} \frac{C_7}{R_p} \left[ \left( \frac{1}{R} \right)^{1+n} - \left( \frac{1}{R} \right)^{1+n} \right] + \frac{(1+n)C_8}{2+n} \left[ \left( \frac{1}{R} \right)^{2+n} - \left( \frac{1}{R} \right)^{2+n} \right]$$
At the given punch stroke, the contact angle, $\theta_c$, is a constant. Therefore, the derivative of the angle with respect to the radius is given by:

$$\frac{d\theta}{dR} = nC_7\left(\frac{1}{R}\right)^{2+n} + (1+n)C_8\left(\frac{1}{R}\right)^{3+n} \quad R_p \leq R \leq R_E \quad (K-9)$$

Substituting Eq. (K-9) into Eq. (K-7), we obtain

$$u_1 = \int_{S_E}^{S_I} \sin \theta \frac{d\theta}{dR} dR = \int_{R_p}^{R_E} \sin \theta \left[ nC_7\left(\frac{1}{R}\right)^{1+n} + (1+n)C_8\left(\frac{1}{R}\right)^{2+n} \right] dR \quad (K-10)$$

with the angle $\theta$ determined by Eq. (K-8). Noticed that the angle $\theta$ is a function of the radius $R$ which varies from $R_p' = R_p + t/2$ to $R_E' = R_E + t/2$.

In a same manner, the term $u_2$ is found to be:

$$u_2 = \int_{S_E}^{S_I} \sin \theta ds \quad (K-11)$$

with the angle $\theta$

$$\theta = \theta_c - \frac{C_5}{2} [(S_I - S)^2 - (S_I - S_E)^2] + \frac{n}{n+1}C_7\left(\frac{1}{R_p}\right)^{1+n} - \left(\frac{1}{R_E}\right)^{1+n} + \frac{n+1}{n+2}C_8\left(\frac{1}{R_p}\right)^{2+n} - \left(\frac{1}{R_E}\right)^{2+n} \quad (K-12)$$

Notice that the wiping angle depends on the arc length, $S$, which varies from $S_E$ to $S_I$. The punch contact angle is defined in Eq. (J-10).

The integration terms, $u_1$ and $u_2$, can be numerically integrated with the rotation angle defined in Eqs. (K-8) and (K-12), respectively.

2. **Numerical Integration of $u_1$ and $u_2$**

Let
\[ u_1 = \int_{R_p}^{R_E} \phi(R) \, dR \quad (K-13) \]

and

\[ \phi(R) = \sin[\theta(R)] \left[ nC_\alpha \left( \frac{1}{R} \right)^{1+n} + \frac{n+1}{n+2} C_\beta \left( \frac{1}{R} \right)^{3+n} \right] \quad (K-14) \]

Simpson's method for the numerical integration of Eq. (K-13) is then expressed as

\[ \int_{R_p}^{R_E} \phi(R) \, dR = \frac{\Delta R}{3} \left\{ \phi(R_p') + \phi(R_E') + 2 \sum_{k=1}^{N-2} \phi(R_{2k}) + 4 \sum_{k=1}^{N-1} \phi(R_{2k-1}) \right\} \quad (K-15) \]

where

\[ \Delta R = \frac{R_E - R_p}{N} \quad (K-16) \]

\[ R_{2k-1} = R_p' + (2k - 1) \Delta R \quad (K-17) \]

\[ R_{2k} = R_p' + 2k \Delta R \quad (K-18) \]

and \( 1 \leq k \leq N \)

The same integration scheme is applied for \( u_2 \), and the formulae are as follows:

\[ \phi(S) = \sin[\theta(S)] \quad (K-19) \]

\[ \Delta S = \frac{S_E - R_E}{N} \quad (K-20) \]

\[ S_{2k-1} = S_E + (2k - 1) \Delta S \quad (K-21) \]

and

\[ S_{2k} = S_E + 2k \Delta S \quad (K-22) \]

where the angle, \( \theta \), is defined by Eq. (K-12).
1. Geometrical Constraint

Using the local coordinate system $X'-Y'$ (Fig. L-1a), the $X'$ coordinate at the tangent point, $B$, on the die is found to be

$$X_B' = (C - t) + (R_p + \frac{t}{2})(1 - \cos \alpha)$$

$$= C + R_p (1 - \cos \alpha) - \frac{t}{2} (1 + \cos \alpha)$$

where $C$ is the clearance (die gap), $R_p$ is the punch radius, $t$ is the sheet thickness, and $\alpha$ is the die angle.

With the rotation transformation from the coordinate system $X'-Y'$ to the coordinate system $X-Y$,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix}$$

and the rotation angle $\theta$:

$$\theta = \frac{\pi}{2} - \alpha$$

Therefore
\[
\begin{pmatrix}
X \sin \alpha \\
Y \cos \alpha
\end{pmatrix} = \begin{pmatrix}
\sin^2 \alpha & \sin \alpha \cos \alpha \\
-\cos^2 \alpha & \sin \alpha \cos \alpha
\end{pmatrix} \begin{pmatrix}
X' \\
Y'
\end{pmatrix}
\] (L-4)

Finally, the geometrical constraint at the tangent point on die is found to be:

\[X_B \sin \alpha - Y_B \cos \alpha = X_B'\]

(L-5a)

or

\[X_B \sin \alpha - Y_B \cos \alpha = C + R_p (1 - \cos \alpha) - \frac{1}{2} (1 + \cos \alpha)\]

(L-5b)

In order to make sure that Eq. (L-5) is correct, we use another direct approach to find this geometrical constraint, Fig. L-1b. The coordinates at points D (D is the tangent point to join the radius and straight line on punch) and E on punch can be found as follows:

\[X_D = R_p \sin \alpha \quad \text{and} \quad Y_D = R_p (1 - \cos \alpha) + \frac{C}{2}\]

(L-6)

\[X_E = \frac{(Y_E - Y_D)}{\tan \alpha} + X_D\]

(L-7)

\[X_B = X_E + (C - \frac{1}{2}) \sin \alpha \quad \text{and} \quad Y_B = Y_E\]

(L-8)

The coordinate \(X_E\) can be defined by substituting Eq. (L-6) into (L-7), i.e.:

\[X_E = \frac{1}{\sin \alpha} \left[ Y_B \cos \alpha + R_p (1 - \cos \alpha) - \frac{1}{2} \cos \alpha \right]\]

(L-9)

Then the coordinate \(X_B\) is found using Eqs. (L-8) and (L-9):

\[X_B = \frac{1}{\sin \alpha} \left[ Y_B \cos \alpha + C + R_p (1 - \cos \alpha) - \frac{1}{2} (1 + \cos \alpha) \right]\]

(L-10a)

or

\[X_B \sin \alpha - Y_B \cos \alpha = C + R_p (1 - \cos \alpha) - \frac{1}{2} (1 + \cos \alpha)\]

(L-10b)
Therefore Eq. (L-10b) is the same as (L-5b).

2. Punch Displacement

Both the tangent point, B, on die wall, and the die gap, C, are variables that decrease with the punch displacement. We need to define a relation between punch travel, d, and the die gap, C. Referring to Fig. L-1b, the coordinates at point, G, on die wall, can be calculated as:

\[
X_G = X_F - Y_F \cot \alpha = \frac{1}{\sin \alpha} \left[ R_P (1 - \cos \alpha) + C - \frac{1}{2} \cos \alpha \right] \\
Y_G = 0
\]  

(L-11)

Another geometrical relation can be established by the punch travel and die length, \(L_d\), as follows:

\[
\frac{(d - \frac{t}{2}) - Y_G}{L_d - X_G} = \tan \alpha 
\]  

(L-12)

With these two relations, the punch displacement is found to be:

\[
d = L_d \tan \alpha - X_G \tan \alpha + \frac{1}{2} \\
= L_d \tan \alpha - R_P \left( \frac{1 - \cos \alpha}{\cos \alpha} \right) - \frac{C}{\cos \alpha} + t \\
= L_d \tan \alpha + Y_B - X_B \tan \alpha + \frac{1}{2} \left( 1 - \frac{1}{\cos \alpha} \right) \quad \text{for } 0 \leq d < H_d \\
\]  

(L-13)

or

\[
C = t \cos \alpha - d \cos \alpha + L_d \sin \alpha - R_P \left( 1 - \cos \alpha \right) 
\]  

(L-14)

Eq. (L-14) tells that the die gap, C, decreases as the punch displacement, d, increases.
Fig. L-1 Geometrical relations in V-die bending
**APPENDIX M**

**DERIVATIONS IN SHRINK FLANGING**

An integration of Eq. (2.4) can be written as:

$$ \int_0^{R_1} [p(s)]' \, ds = \int_{R_0}^{R_1} [r(s)]' \, dr $$

(M-1)

Using the boundary conditions in Eqs. (2.6) and (2.7), the integrations in the right-hand side (RHS) and left-hand side (LHS) are

$$ \text{RHS} = \int_{R_0}^{R_1} [r(s)]' \, dr = \frac{1}{f + 1} \left( \frac{R_1}{R_0} \right)^{f+1} \left( 1 - 1 \right) $$

(M-2)

and

$$ \text{LHS} = \int_0^{\alpha} [p(s)]' \, ds = \int_0^{\alpha} [p(s)]' \, ds + \int_{\alpha}^{\alpha + I} [p(s)]' \, ds = I_1 + I_2 $$

(M-3)

where

$$ I_1 = \int_0^{\alpha} [p(s)]' \, ds = \int_0^{\alpha} R_o' \left( 1 + \frac{r'_d \sin \frac{s}{r_d}}{r_d} \right) \, ds $$

$$ = \int_0^{\alpha} R_o' \left( 1 + f \frac{r'_d \sin \frac{s}{r_d}}{r_d} \right) ds = R_o' \left( 1 + f \frac{r'_d \sin \frac{s}{r_d}}{r_d} \right) $$

$$ \alpha \neq 90^\circ $$

(M-4a)

$$ I_1 = R_o' \left( 1 + f \frac{r'_d \sin \frac{s}{r_d}}{r_d} \right) \frac{r'_d - r_d}{R_o} $$

$$ \alpha = 90^\circ $$

(M-4b)

Note that in the integration of $I_1$, an approximation by Taylor’s expansion is used, i.e.

$$(1 + \frac{r'_d \sin \frac{s}{r_d}}{r_d})^f = (1 + f \frac{r'_d \sin \frac{s}{r_d}}{r_d}) \text{ because } R_o >> r'_d, \sin \frac{s}{r_d} < 1,$$

and $\frac{r'_d \sin \frac{s}{r_d}}{r_d} < 1$. 

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The second term in Eq. (M-3) is found to be

\[ I_2 = \int_{\alpha_d}^{\alpha_d+L} [p(s)]^f ds = \int_{\alpha_d}^{\alpha_d+L} R_o f [a_1 + \left( \frac{s}{R_o} - a_2 \alpha \right) \cos \alpha] ds \]

\[ = R_o \int_{f+1}^{f+1} \frac{1}{f+1} \cos \alpha \left[ -a_1^{f+1} + (a_1 + L \cos \alpha)^{f+1} \right] \]

where

\[ a_1 = 1 + \frac{r_o}{r_o} \sin \alpha \quad \text{and} \quad a_2 = \frac{r_o}{r_o} \]

\[ I_2 = \int_{\alpha_d}^{\alpha_d+L} R_o f (1 + \frac{r_o}{r_o}) ds = R_o f^{f+1}(1 + \frac{r_o}{r_o}) \frac{L}{r_o} \alpha = 90^\circ \quad (M-6b) \]

Substituting integrations in Eqs. (M-4) and (M-6) into Eq. (M-3), and equating Eqs. (M-2) and (M-3), the following relationship is defined:

\[ \frac{R_L}{R_o} = \left\{ 1 + \frac{1+2R}{1+R} \frac{r_d+t/2}{R_o} \left[ \alpha + \frac{R}{1+R} \frac{r_d+t/2}{R_o} (1-\cos \alpha) \right] + \right. \]

\[ \left. + \frac{1}{\cos \alpha} \left[ (1+\frac{r_d+t/2}{R_o} \sin \alpha) + \frac{L}{R_o} \cos \alpha \right]^{1+R} \right\}^{1+R} \]

\[ \alpha \neq 90^\circ \quad (M-8a) \]

and

\[ \frac{R_L}{R_o} = \left\{ \frac{1+2R}{1+R} \left[ \frac{r_d+t/2}{R_o} \left( \frac{R}{1+R} \frac{r_d+t/2}{R_o} \right) + (1+\frac{r_d+t/2}{R_o}) \frac{L}{R_o} \right] + 1 \right\}^{1+2R} \]

\[ \alpha = 90^\circ \quad (M-8b) \]
APPENDIX N

DERIVATION OF THE MINIMUM BENDING RATIO

If assume the neutral axis coincides with the middle axis, then the maximum tensile strain along the longitudinal direction is defined in Eq. (N—1)

\[ \varepsilon_{\text{max}} = \ln \left(1 + \frac{t}{2R_n}\right) = \ln \left(1 + \frac{t}{2R_i + t}\right) \]

\[ = \ln \left(1 + \frac{t}{2R_i + t}\right) = \ln \left(1 + \frac{1}{2R_i/t + 1}\right) \]

(N—1)

The neutral axis is assumed to be the same as the middle layer. With the relation in Eq. (N—2), the effective strain can be defined as

\[ \bar{\varepsilon} = F \varepsilon_{\text{max}} = F \ln \left(1 + \frac{1}{2R_i/t + 1}\right) \]

(N—2)

Minimum Bending Radius \((R_i/t)_{\text{min}}\) based on Fracture Limit

Introducing the maximum area reduction (%) at fracture, \(A_r\):

\[ A_r = \left(\frac{A_0 - A_{fr}}{A_0}\right) \]

(N—3)

where \(A_0\) and \(A_{fr}\) are the initial cross-section area and the final area at fracture, respectively. The effective strain then can be written in terms of these parameters:

\[ \bar{\varepsilon} = \ln \frac{A_0}{A_{fr}} = \ln \frac{1}{1 - A_r} \]

(N—4)
Equating the two relations in Eqs. (N–1) and (N–4):

\[
\frac{R_i}{t_{\text{min}}} = \frac{1}{2A_r} - 1
\]

This criteria is based on the area reduction of isotropic materials under uniaxial tension. This relation implies that materials with more ductility have high bendability, i.e. the smaller bend ratio, than the less ductile sheets. This criterion holds for \(A_r \leq 20\%\) or \(R/t > 2\). A new bendability model is proposed here to account for the effects of both material anisotropy and stress/strain states on the minimum bend ratio to sheet thickness. Introducing the parameter \(F\) defined in Eq. (3.11), and equating the two expressions for the effective strain defined in Eqs. (N–2) and (N–4), the ratio of the minimum bend radius to the sheet thickness can be expressed as:

\[
\frac{R_i}{t_{\text{min}}} = 0.5 \left\{ \frac{1}{(1 - A_r)^{-1/F} - 1} \right\}
\]

**Minimum Bending Radius \((R_i/t)_{\text{min}}\) Based on Localized Necking**

A structural weakening of bent parts occurs when the tensile strain of the outer fiber exceeds the critical strain of the sheet material. At this localized necking point of plastic instability, the critical strain is equal to the strain hardening exponent:

\[
\bar{\varepsilon} = 2n
\]

The relation in Eq. (N–2) must be used to correlate the deformation in different stress/strain states and anisotropic behaviors to the n-value determined in uniaxial test, i.e.:

\[
\bar{\varepsilon} = 2n = F_{\varepsilon_{\text{max}}}
\]

Substituting Eq. (N–1) into Eq. (N–8), the minimum ratio is found to relate to the strain hardening exponent, \(n\), as well as the anisotropy index \(F\):

\[
\frac{R_i}{t_{\text{min}}} = \frac{1}{2A_r} - 1
\]
This relation indicates that the higher work hardening a sheet has, the more ductile the material is. Therefore, the minimum bend radius decreases as the strain hardening exponent increases. The bendability reduces (the ratio increases) as the plastic anisotropy increases since the yield stress is raised as $R$-value increases to exceed the unity. The stress/strain state also affects the bending limit. The minimum bend ratio required in plane strain bending is greater than that in uniaxial stress state.

**Effect of Thinning and Shift of Neutral Axis on the Minimum Bend Ratio**

The minimum ratios of bend radius to sheet thickness, given in the above equations are only for pure bending cases in which the thinning of the sheet and the shift of neutral axis has not been considered yet. The sheet thins and the neutral axis shifts toward the concave surface in bending with a relatively small bend radius or bending under tension with a fairly large bend radius. This change in the position of the neutral axis raises the maximum tensile strains on the convex outer surface of the sheet and causes a drop in bendability. Therefore, the minimum bend ratio must increase to account for thinning due to either significant bending or the tension applied.

Based on the general bending theory, Eq. (3.45), the neutral axis is found to be

$$R_n = \sqrt{R_i R_c} = t \sqrt{\frac{(R_i)}{t}^2 + \frac{R_i}{t}}$$

This relation implies the neutral axis is always below the middle axis and within the sheet

$$R_i \leq R_n < R_m$$

The maximum bending strain on the convex surface, thereby, is
Combining Eqs. (N--10) and (N--12), the maximum tensile strain is expressed as a function of the bend ratio of $R_i/t$:

$$
\varepsilon_{\text{max}} = \ln \sqrt{\frac{R_d}{R_i}} = \ln \sqrt{\frac{R_i + t}{R_i}} = \ln \sqrt{1 + \frac{t}{R_i}}
$$

Using the relation in Eq. (N--2):

$$
\bar{\varepsilon} = F \varepsilon_{\text{max}} = \ln \frac{1}{1 - A_r}
$$

Substituting Eq. (N--13) into Eq. (N--14), the minimum bend ratio is finally expressed as

$$
\left(\frac{R_i}{t}\right)_{\text{min}} = \frac{1}{(1 - A_r)^{2F}} - 1 = \frac{(1 - A_r)^{2F}}{1 - (1 - A_r)^{2F}}
$$

If the failure criterion for localized necking is used, then the minimum bend ratio is found by substituting Eq. (N--13) into Eq. (N--7):

$$
\left(\frac{R_i}{t}\right)_{\text{min}} = \frac{1}{\text{Exp}(4n/F) - 1}
$$
APPENDIX O

WRINKLING CRITERION FOR A DOUBLE CURVED SHELL

1. Bifurcation Functional

Consider $x_1$ and $x_2$ to coincide with the principal axes, let $\sigma_1 = \sigma_{11}$ and $\sigma_2 = \sigma_{22}$ be the principal in-plane stresses along these axes, and let $R_1$ and $R_2$ be the radii of the principal curvatures. A bifurcation functional is given by

$$F(U, W) = \int_S \left\{ M_{ij} \hat{K}_{ij} + N_{ij} \hat{E}_{ij} + N_{ij} \hat{\delta}_{ij} \right\} dS$$

$$= \int_S \left\{ \frac{1}{12} L_{ijkl} \hat{K}_{ij} \hat{K}_{ij} + t L_{ijkl} \hat{E}_{ij} \hat{E}_{ij} + N_{ij} \hat{W}_{i,j} \hat{W}_{i,j} \right\} dS$$

where $S$ denotes the region of the shell middle surface over which the wrinkles appear. This bifurcation functional is the total energy for wrinkling occurrence. In the right hand side of Eq. (O-1), the first term represents the bending energy ($i=j$) and twisting energy ($i \neq j$), the second term is the strain energy due to the membrane stresses, and the third term may be interpreted as the potential energy of edge stress or the work done by applied stresses in the middle surface. For all admissible incremental displacement fields $\hat{U}_i$ and $\hat{W}$, if $F>0$, then the incremental solutions of the deformation are unique and bifurcation is not possible because to create wrinkles requires the energy supply, i.e. the total potential of the system increases which is not a natural or spontaneous process from a thermodynamics point of view. While $F=0$ corresponds the critical conditions for wrinkles to occur for some non-zero incremental displacement fields. That is, the wrinkling criterion may be written as
\[ F(\dot{U}, \dot{W}) = 0 \quad \text{(not all } \dot{U} \text{ and } \dot{W} = 0) \] (O-2)

2. Incremental Displacement Fields for Wrinkling

The incremental displacement fields are given as follows

\[ \dot{W} = A \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]
\[ \dot{U}_1 = B \sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]
\[ \dot{U}_2 = C \cos(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi) \] (O-3)

with

\[ \xi = \sqrt{Rt} \] (O-4)

where

\( t \) = the thickness of the sheet,
\( R \) = the radius of the principal curvature (\( R_1 \) or \( R_2 \))
\( A, B, \) and \( C \) = constants representing the relative displacement amplitudes of the mode shape
\( \lambda_1 \) and \( \lambda_2 \) = dimensionless wave numbers.

3. Wrinkling Criterion for a Double Curved Shell

Substitute the incremental displacement fields in Eq. (O-3) into the kinematic relationships in Eqs. (3.34) and (3.35), the incremental strains and rotations are found to be

\[ \dot{\varepsilon}_{11} = \frac{\partial \dot{U}_1}{\partial x_1} + b_{11} \dot{W} + \frac{1}{2} \left( \frac{\partial \dot{W}}{\partial x_1} \right)^2 \]
\[ = t \left[ \frac{B\lambda_1}{\xi} + A b_{11} \right] \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]
\[ + \left[ \frac{A\lambda_1}{2\xi} \right] \sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \] (O-5a)
\[ \dot{E}_{22} = \frac{\partial \dot{U}_2}{\partial x_2} + b_{22} \dot{W} + \frac{1}{2} \left( \frac{\partial \ddot{W}}{\partial x_2} \right)^2 \]

\[ = t \left[ \frac{C \lambda_2}{\xi} + Ab_{22} \right] \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) + \frac{1}{2} \left( \frac{\partial \ddot{W}}{\partial x_2} \right)^2 \]

\[ \dot{E}_{12} = \frac{1}{2} \left( \frac{\partial \dot{U}_1}{\partial x_2} + \frac{\partial \dot{U}_2}{\partial x_1} \right) + b_{12} \dot{W} + \frac{1}{2} \frac{\partial \dot{W}}{\partial x_1} \frac{\partial \dot{W}}{\partial x_2} \]

\[ = \frac{t}{2} \left[ \left( \frac{B \lambda_1}{\xi} + Ab_{11} \right) + \left( \frac{C \lambda_2}{\xi} + Ab_{22} \right) \right] \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) + \frac{1}{2} \left( \frac{\partial \ddot{W}}{\partial x_2} \right)^2 \]

where the rotations are

\[ \dot{\beta}_1 = -\frac{\partial \dot{W}}{\partial x_1} = At \cdot \frac{\lambda_1}{\xi} \sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]

\[ \dot{\beta}_2 = -\frac{\partial \dot{W}}{\partial x_2} = At \cdot \frac{\lambda_2}{\xi} \cos(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi) \]

For small rotations, the nonlinear terms in Eqs. (3.34) and (O-5) can be neglected, and the incremental strain fields simplified as

\[ \dot{E}_{11} = \frac{\partial \dot{U}_1}{\partial x_1} + b_{11} \dot{W} = t \left( \frac{B \lambda_1}{\xi} + Ab_{11} \right) \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]

\[ \dot{E}_{22} = \frac{\partial \dot{U}_2}{\partial x_2} + b_{22} \dot{W} = t \left( \frac{C \lambda_2}{\xi} + Ab_{22} \right) \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]

\[ \dot{E}_{12} = \frac{1}{2} \left( \frac{\partial \dot{U}_1}{\partial x_2} + \frac{\partial \dot{U}_2}{\partial x_1} \right) + b_{12} \dot{W} \]

\[ = \frac{t}{2} \left[ A(b_{11} + b_{22}) + \frac{B \lambda_1}{\xi} + \frac{C \lambda_2}{\xi} \right] \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]

The bending and twisting strains are
\[ \dot{K}_{11} = -\frac{\partial^2 \hat{W}}{\partial x_1^2} = \text{At} \left( \frac{\lambda_1}{\xi} \right)^2 \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]

\[ \dot{K}_{22} = -\frac{\partial^2 \hat{W}}{\partial x_2^2} = \text{At} \left( \frac{\lambda_2}{\xi} \right)^2 \cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi) \]  \hspace{1cm} \text{(O-8)}

\[ \dot{K}_{12} = -\frac{\partial^2 \hat{W}}{\partial x_1 \partial x_2} = -\text{At} \frac{\lambda_1 \lambda_2}{\xi^2} \sin(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi) \]

Let principal curvatures be

\[ b_{11} = 1/R_1, \quad b_{22} = 1/R_2 \quad \text{and other } b_{ij} = 0 \]  \hspace{1cm} \text{(O-9)}

where \( R_1 \) and \( R_2 \) are radii along the principal axes. And let the instantaneous moduli be

\[ L_{11} = L_{1111}, \quad L_{22} = L_{2222}, \quad L_{12} = L_{1122}, \quad \text{and } L_{44} = L_{1212} \]  \hspace{1cm} \text{(O-10)}

Substitute bending strain fields in Eq. (O-8) into the first term in the right hand side of Eq. (O-1), the incremental energy for bending and twisting is found to be

\[ \dot{U}_b = \frac{t^3}{12} \int_S L_{ijkl} \dot{K}_{ij} \dot{K}_{kl} dS \]

\[ = \frac{t^3}{12} \int_S \left[ L_{1111} \dot{K}_{11} \dot{K}_{11} + L_{2222} \dot{K}_{22} \dot{K}_{22} + L_{1122} \dot{K}_{11} \dot{K}_{22} + L_{1212} \dot{K}_{12} \dot{K}_{12} \right] dS \]

\[ = \frac{t^3}{12} \int_S \left[ L_{11} (\dot{K}_{11})^2 + L_{22} (\dot{K}_{22})^2 + L_{12} \dot{K}_{11} \dot{K}_{22} + L_{44} (\dot{K}_{12})^2 \right] dS \]

\[ = \frac{t^3}{12} \int_S \left[ (L_{11} + L_{22} (\dot{K}_{22})^2 + L_{12} \dot{K}_{11} \dot{K}_{22} + L_{44} (\dot{K}_{12})^2 \right] dS \]  \hspace{1cm} \text{(O-11a)}

\[ = \frac{t^3}{12} \int_S \left[ \frac{(\text{At})^2}{\xi^4} (L_{11} \lambda_1^4 + L_{22} \lambda_2^4 + L_{12} (\lambda_1 \lambda_2)^2 [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 \]

\[ + L_{44} (\lambda_1 \lambda_2)^2 [\sin(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi)]^2 \right] dS \]

or

\[ \dot{U}_b = \frac{t^3}{12} \zeta \int [-\frac{(\text{At})^2}{\xi^4} (L_{11} \lambda_1^4 + L_{22} \lambda_2^4 + (L_{12} + L_{44})(\lambda_1 \lambda_2)^2) \]  \hspace{1cm} \text{(O-11b)}
where
\[ \zeta S = \int_S [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 dS \]
\[ = \int_S [\sin(\lambda_1 x_1 / \xi) \sin(\lambda_2 x_2 / \xi)]^2 dS \]
\[ = \int_S [\sin(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 dS \] \hspace{1cm} (O-12)

and
\[ \zeta = \begin{cases} 
1/4 & \text{if both } \lambda_1 \text{ and } \lambda_2 \neq 0 \\
1/2 & \text{if either } \lambda_1 \text{ or } \lambda_2 = 0 
\end{cases} \] \hspace{1cm} (O-13)

Substitute the membrane strain increments in Eq. (O-7) into the second term in the right side of Eq. (O-1), the incremental strain energy due to the membrane stresses is
\[ \dot{U}_m = \int_S t L_{ij \alpha \beta} \dot{E}_{ij \alpha \beta} dS \]
\[ = \int_S t [L_{1111} \dot{E}_{11} + L_{2222} \dot{E}_{22} + L_{1112} \dot{E}_{11} \dot{E}_{22} + L_{1212} \dot{E}_{12} \dot{E}_{12}] dS \]
\[ = \int_S t [L_{11} (\dot{E}_{11})^2 + L_{22} (\dot{E}_{22})^2 + L_{12} \dot{E}_{11} \dot{E}_{22} + L_{44} (\dot{E}_{12})^2] dS \]
\[ = t^3 \int_S \left[ \frac{B \lambda_1}{\xi} + A \right] [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 \\
+ L_{22} \left( \frac{C \lambda_2}{\xi} + A \right) [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 \]
\[ + L_{12} \left( \frac{B \lambda_1}{\xi} + A \right) \left( \frac{C \lambda_2}{\xi} + A \right) [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 \]
\[ + L_{44} \left( \frac{B \lambda_1}{\xi} + A \right) [\cos(\lambda_1 x_1 / \xi) \cos(\lambda_2 x_2 / \xi)]^2 \] \hspace{1cm} (O-14a)
or
\[ \dot{U}_m = t^3 \zeta S \left[ \frac{B \lambda_1}{\xi} + A \right]^2 + L_{22} \left( \frac{C \lambda_2}{\xi} + A \right)^2 \]
\[ + L_{12} \left( \frac{B \lambda_1}{\xi} + A \right) \left( \frac{C \lambda_2}{\xi} + A \right) + L_{44} \left[ \frac{B \lambda_1}{\xi} + A \right] \left( \frac{C \lambda_2}{\xi} + A \right)^2 \] \hspace{1cm} (O-14b)

The membrane forces due to the applied membrane stresses can be defined as
\( N_{11} = -\sigma_{11}t \)  
\( N_{22} = -\eta \sigma_{22}t \) \hspace{1cm} (O-15) 

and

\[
\eta = \begin{cases} 
1 & \text{\( \sigma_{22} \) is tensile stress} \\
-1 & \text{\( \sigma_{22} \) is compressive stress} 
\end{cases} \hspace{1cm} (O-16)
\]

Substitute these forces and the incremental rotations in Eq. (O-6) into the second term in the right side of Eq. (O-1), the potential or work done by the applied stresses is found to be

\[
\dot{U}_N = \int_N \nu \eta \frac{\lambda_{11}}{\xi} dS = \int_{\partial N} \{N_{11}\dot{\theta}_1 + N_{22}\dot{\theta}_2\} dS
\]

\[
= -At^2 \int_N (\sigma_{11} \frac{\lambda_{11}}{\xi})^2 [\sin(\lambda_{11}x_1 / \xi) \cos(\lambda_{22}x_2 / \xi)]^2
\]

\[
+ \eta \sigma_{22} (\frac{\lambda_{22}}{\xi})^2 [\cos(\lambda_{11}x_1 / \xi) \sin(\lambda_{22}x_2 / \xi)]^2 \} dS
\]

or

\[
\dot{U}_N = -At^2 \xi S [\sigma_{11} (\frac{\lambda_{11}}{\xi})^2 + \eta \sigma_{22} (\frac{\lambda_{22}}{\xi})^2] \hspace{1cm} (O-17b)
\]

Finally, the bifurcation functional is obtained by substituting Eqs. (O-11b), (O-14b) and (O-17b) into Eq. (O-1), i.e.

\[
F = \dot{U}_b + \dot{U}_m + \dot{U}_N = \frac{t^3}{12} \xi S \{ (At)^2 \frac{(L_{11} \lambda_{11}^4 + L_{22} \lambda_{22}^4 + (L_{12} + L_{44})(\lambda_{11} \lambda_{22})}{\xi^4} 
\]

\[
+ t^3 \xi S \{L_{11}(\frac{B \lambda_{11}}{\xi} + Ab_{11})^2 + L_{22}(\frac{C \lambda_{22}}{\xi} + Ab_{22})^2 + L_{12}(\frac{B \lambda_{11}}{\xi} + Ab_{11})(\frac{C \lambda_{22}}{\xi} + Ab_{22}) 
\]

\[
+ L_{44} \frac{1}{4} (\frac{B \lambda_{11}}{\xi} + \frac{C \lambda_{22}}{\xi} + A(b_{11} + b_{22}))^2 \} - At^3 \xi S [\sigma_{11} (\frac{\lambda_{11}}{\xi})^2 + \eta \sigma_{22} (\frac{\lambda_{22}}{\xi})^2]
\]

\hspace{1cm} (O-18)

or in the matrix form as
\[
F = \zeta \text{St}(\frac{t^3}{\xi}) \begin{bmatrix} A \\ B \\ C \end{bmatrix} M \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \zeta \text{St}(\frac{t^3}{\xi}) \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \] (O-19)

where the matrix \([M]\) is given by

\[M_{11} = \frac{t^3}{12} \left( \frac{L_{11}}{\xi} \right)^2 [L_{11} \lambda_1^4 + L_{22} \lambda_2^4 + 2(L_{12} + 2L_{44})(\lambda_1 \lambda_2)^2]
\]
\[+ [L_{11} \left( \frac{R}{R_1} \right)^2 + L_{22} \left( \frac{R}{R_2} \right)^2 + 2L_{12} \left( \frac{R}{R_1} \right) (\frac{R}{R_2})] - [\sigma_{11} \lambda_1^2 + \eta \sigma_{22} \lambda_2^2]\]

\[M_{22} = L_{11} \lambda_1^4 + L_{44} \lambda_2^4\]

\[M_{33} = L_{22} \lambda_2^4 + L_{44} \lambda_1^4\]

\[M_{12} = M_{21} = L_{11} \lambda_1 \left( \frac{R}{R_1} \right) + L_{22} \lambda_2 \left( \frac{R}{R_2} \right)\] (O-20)

\[M_{13} = M_{31} = L_{22} \lambda_2 \left( \frac{R}{R_1} \right) + L_{12} \lambda_1 \left( \frac{R}{R_2} \right)\]

\[M_{23} = M_{32} = (L_{12} + L_{44}) \lambda_1 \lambda_2\]