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Electromagnetic scattering analysis of a three dimensional inhomogeneous material body using the finite element method

Andre, Robert Gilman, Ph.D.
The Ohio State University, 1993
Electromagnetic Scattering Analysis of a Three Dimensional Inhomogeneous Material Body using the Finite Element Method

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Robert G. Andre

* * * * *

The Ohio State University
1993

Dissertation Committee:
Dr. Benedikt Munk
Dr. Edward Newman
Dr. Robert Lee

Approved by:
Advisor
Department of Electrical Engineering
DEDICATION

To Serendipity
ACKNOWLEDGEMENT

Were it not for a war and a recession, this dissertation would never have been attempted much less completed. Therefore, I gratefully acknowledge the following people for giving me the support which finally ended my academic sojourn.

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I have also depended on several other people over the last few years. Harry Shamansky patiently endured my ignorance of a new computer system. Nan Wang provided hours of interesting discussions and taught me that an intelligent and humble Ph.D. need not be an oxymoron. I especially thank Mimi Hsu and friends from the peanut gallery for lending me their wisdom and advice, providing much needed moral support and humor, and for laughing at all my bad jokes. Last, but by no means least, I thank my parents for giving me just the right amount of support without the pressure.
VITA

April 22, 1960 ......................... Born in Cincinnati, Ohio

June 1982 ............................... B.S. Mathematics, California Institute of Technology, Pasadena, California

June 1982 - December 1985 .......... Graduate Research Associate, The Ohio State University, Columbus, Ohio

January 1986 - December 1987 ...... Research Engineer, Lockheed Missiles & Space Co., Sunnyvale, California

January 1988 - Present ............... Graduate Research Associate, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Electrical Engineering

Electromagnetics ...................... B. A. Munk

Physics ................................. A. O. Fansome

 Electronics & Optics .................. B. D. Clymer
TABLE OF CONTENTS

DEDICATION ................................................................. ii

ACKNOWLEDGEMENT .................................................. iii

VITA ................................................................................ iv

LIST OF TABLES ........................................................... ix

LIST OF FIGURES .......................................................... x

CHAPTER

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>Introduction</td>
</tr>
<tr>
<td></td>
<td>1.1 Problem Description</td>
</tr>
<tr>
<td></td>
<td>1.2 Solution Methods</td>
</tr>
<tr>
<td></td>
<td>1.3 Previous Work</td>
</tr>
<tr>
<td></td>
<td>1.4 Proposed Solution Method</td>
</tr>
<tr>
<td>II.</td>
<td>The Interior Finite Element Method</td>
</tr>
<tr>
<td></td>
<td>2.1 Introduction</td>
</tr>
<tr>
<td></td>
<td>2.2 The Differential Equations and Boundary Conditions</td>
</tr>
<tr>
<td></td>
<td>2.3 The Functional in Theory</td>
</tr>
<tr>
<td></td>
<td>2.4 The Functional in Practice</td>
</tr>
</tbody>
</table>
E. Numerical Surface Integrations .................................................. 116
E.1 One Source Element and One Test Element ......................... 116
E.2 Singularities ................................................................. 119

BIBLIOGRAPHY ................................................................. 122
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mesh parameters and CPU run times for the MOM examples.</td>
</tr>
<tr>
<td>2</td>
<td>Mesh parameters and CPU run times for the FEM examples.</td>
</tr>
<tr>
<td>3</td>
<td>Mesh parameters and CPU run times for the MOM/FEM examples.</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Plane wave incident on an inhomogeneous body with embedded perfect electric conductors.</td>
</tr>
<tr>
<td>2</td>
<td>A first order brick element and a second order tetrahedral element.</td>
</tr>
<tr>
<td>3</td>
<td>Moment method currents $\vec{J}$, $\vec{M}$ are placed on the surface $S_0$ surrounding the FEM domain.</td>
</tr>
<tr>
<td>4</td>
<td>Bounded FEM region surrounding inhomogeneous dielectric and perfect conductors.</td>
</tr>
<tr>
<td>5</td>
<td>First order tetrahedral element.</td>
</tr>
<tr>
<td>6</td>
<td>Second order tetrahedral element.</td>
</tr>
<tr>
<td>7</td>
<td>Part of a finite element mesh using first order tetrahedral elements.</td>
</tr>
<tr>
<td>8</td>
<td>A plane wave is incident on a body surrounded by surface $S_0$.</td>
</tr>
<tr>
<td>9</td>
<td>The total matrix used in the iterative method is a conglomeration of the FEM and MOM matrices.</td>
</tr>
<tr>
<td>10</td>
<td>Parallel polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\epsilon_r = 1$, $\mu_r = 4$ and nodal density $N/\lambda_0^2 = 202$ with $\lambda_0 = 5$ meters. MOM solution is in solid. Eigenfunction solution is dashed.</td>
</tr>
<tr>
<td>11</td>
<td>Perpendicular polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\epsilon_r = 1$, $\mu_r = 4$ and nodal density $N/\lambda_0^2 = 202$ with $\lambda_0 = 5$ meters. MOM solution is in solid. Eigenfunction solution is dashed.</td>
</tr>
<tr>
<td>12</td>
<td>Parallel polarization for the bistatic scattering from a PEC sphere with $R = 1$ meter, nodal density $N/\lambda_0^2 = 440$ and with $\lambda_0 = 5$ meters. MOM solution is in solid. Eigenfunction solution is dashed.</td>
</tr>
</tbody>
</table>
13 Perpendicular polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter, nodal density \( \frac{N}{\lambda_0^2} = 440 \) and with \( \lambda_0 = 5 \) meters. MOM solution is in solid. Eigenfunction solution is dashed.

14 Perpendicular and parallel polarization for the bistatic scattering from a material sphere with free space material parameters. The sphere has a radius of one meter, a nodal density \( \frac{N}{\lambda_0^2} = 440 \) and the incident wavelength is \( \lambda_0 = 5 \) meters.

15 Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \epsilon_r = 2, \mu_r = (1.5, -5) \). The solution uses the MOM only method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \phi \theta \) (solid) and \( \phi \phi \) (dashed) and the two cross polarized echo areas \( \phi \theta \) (dashed) and \( \phi \phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 7 nodes along the side of the cube.

16 Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \epsilon_r = 2, \mu_r = (1.5, -5) \). The solution uses the MOM only method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \phi \theta \) (solid) and \( \phi \phi \) (dashed) and the two cross polarized echo areas \( \phi \theta \) (dashed) and \( \phi \phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 8 nodes along the side of the cube.

17 Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \epsilon_r = 2, \mu_r = (1.5, -5) \). The solution uses the MOM only method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \phi \theta \) (solid) and \( \phi \phi \) (dashed) and the two cross polarized echo areas \( \phi \theta \) (dashed) and \( \phi \phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 9 nodes along the side of the cube.

18 Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \epsilon_r = 2, \mu_r = (1.5, -5) \). The solution uses the MOM only method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \phi \theta \) (solid) and \( \phi \phi \) (dashed) and the two cross polarized echo areas \( \phi \theta \) (dashed) and \( \phi \phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 10 nodes along the side of the cube.
19 $E_z$ field along the $z$ axis. The region is a cube, 2 meters on a side. The FEM solves for the fields reflected and transmitted from a half space with dielectric constant $\varepsilon_r = 4$. The incident field is $\hat{\phi}$ polarized from the $\theta_i = 30$ degrees, $\phi_i = 15$ degrees direction relative to the normal to the plane. The incident frequency is 50 MHz. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. ............................................................. 52

20 $H_y$ field along the $z$ axis. The region is a cube, 2 meters on a side. The FEM solves for the fields reflected and transmitted from a half space with dielectric constant $\varepsilon_r = 4$. The incident field is $\hat{\phi}$ polarized from the $\theta_i = 30$ degrees, $\phi_i = 15$ degrees direction relative to the normal to the plane. The incident frequency is 50 MHz. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. ............................................................. 53

21 The average of the absolute value of the gauge field along the $z$ axis versus frequency for a second order mesh (solid) and a first order mesh (dashed). The incident field is $\hat{\phi}$ polarized. ................................. 54

22 The average of the absolute value of the gauge field along the $z$ axis versus frequency for a second order mesh (solid) and a first order mesh (dashed). The incident field is $\hat{\phi}$ polarized. ................................. 55

23 $E_x$ field along the $z$ axis using the large mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. ............................................................. 56

24 $H_y$ field along the $z$ axis using the large mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. ............................................................. 57

25 $G$ field along the $z$ axis using the large mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution are shown. ................................. 57

26 $E_x$ field along the $z$ axis using the medium mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. ............................................................. 58
27  \( H_y \) field along the \( z \) axis using the medium mesh. The incident field propagates along the \( z \) axis and is \( \hat{z} \) polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution. .................................................. 58

28  \( G \) field along the \( z \) axis using the medium mesh. The incident field propagates along the \( z \) axis and is \( \hat{z} \) polarized. The real (solid) and imaginary (dotted) parts of the FEM solution are shown. ................................. 59

29  Parallel polarization for the bistatic scattering from a material sphere with \( R = 1 \) meter, \( \varepsilon_r = 1 \), \( \mu_r = 4 \) and FEM nodal density \( \frac{N}{\lambda_0} = 145 \times 10^3 \), MOM nodal density \( \frac{N}{\lambda_0} = 783 \) with \( \lambda_0 = 10 \) meters. MOM/FEM solution is in solid. Eigenfunction solution is dashed. .......................................................... 63

30  Perpendicular polarization for the bistatic scattering from a material sphere with \( R = 1 \) meter, \( \varepsilon_r = 1 \), \( \mu_r = 4 \) and FEM nodal density \( \frac{N}{\lambda_0} = 145 \times 10^3 \), MOM nodal density \( \frac{N}{\lambda_0} = 783 \) with \( \lambda_0 = 10 \) meters. MOM/FEM solution is in solid. Eigenfunction solution is dashed. .......................................................... 64

31  Cross section of the material sphere geometry in the \( x,z \) plane. The FEM region includes the material sphere with radius one meter and a half meter shell of free space. The moment method fields are expanded on the \( S_0 \) boundary surface. .......................................................... 65

32  \( E_x \) field along the \( z \) axis for a sphere with \( R = 1 \) meter, \( \varepsilon_r = 1 \), \( \mu_r = 4 \) and \( \lambda_0 = 10 \) meters. The material sphere (shaded region) is surrounded by a \( \frac{1}{2} \) meter thick shell of free space. The incident field is a plane wave, \( \hat{x} \) polarized, electric field propagating along the \( z \) axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction's real (dashed) and imaginary (short dashed) parts. .......................................................... 67

33  \( H_y \) field along the \( z \) axis for a sphere with \( R = 1 \) meter, \( \varepsilon_r = 1 \), \( \mu_r = 4 \) and \( \lambda_0 = 10 \) meters. The material sphere (shaded region) is surrounded by a \( \frac{1}{2} \) meter thick shell of free space. The incident field is a plane wave, \( \hat{z} \) polarized, electric field propagating along the \( z \) axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction's real (dashed) and imaginary (short dashed) parts. .......................................................... 68
34 $G$ field along the $z$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{x}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dashed) parts.

35 $H_y$ field along the $y$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{x}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction's real (dashed) and imaginary (short dashed) parts.

36 Scalar potential along the $y$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{x}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dashed) parts.

37 Plot of the residual error of the biconjugate gradient method versus iteration count for a typical scattering run. The residual error is defined as $R = \|AX - Y\|^2/\|Y\|^2$. The solid line is for a scaled matrix while the dashed line is for an unscaled matrix.

38 Bistatic scattering from a dielectric slab with $\varepsilon_r = 2.6$ and dimensions 2 in. by 2 in. by $\frac{3}{4}$ in. at a frequency of 2 GHz for parallel polarization. The incident field is propagating along the $z$ axis. The MOM/FEM (solid) solution is plotted along with Gilmore's mfb (dashed) solution.

39 Bistatic scattering from a dielectric slab with $\varepsilon_r = 2.6$ and dimensions 2 in. by 2 in. by $\frac{3}{4}$ in. at a frequency of 2 GHz for parallel polarization. The incident field is propagating along the $z$ axis. The MOM/FEM (solid) solution is plotted along with Gilmore's mfb (dashed) solution.

40 Parallel polarization for the bistatic scattering from a PEC sphere with $R = 1$ meter and $\lambda_0 = 40$ meters. The incident field propagates along the $z$ axis. The MOM/FEM is solved using the small mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed.
Perpendicular polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter \( \lambda_0 = 40 \) meters. The incident field propagates along the \( z \) axis. The MOM/FEM is solved using the small mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed. ........................................ ............................... 76

Parallel polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter and \( \lambda_0 = 40 \) meters. The incident field propagates along the \( z \) axis. The MOM/FEM is solved using the medium mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed. ........................................ ............................... 77

Perpendicular polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter \( \lambda_0 = 40 \) meters. The incident field propagates along the \( z \) axis. The MOM/FEM is solved using the medium mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed. ........................................ ............................... 78

Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \varepsilon_r = 2, \mu_r = (1.5, -0.5) \). The solution uses the MOM/FEM method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \theta\theta \) (solid) and \( \phi\phi \) (dotted) and the two cross polarized echo areas \( \phi\theta \) (dashed) and \( \theta\phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 7 nodes along the side of the cube. ........................................ ............................... 79

Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \varepsilon_r = 2, \mu_r = (1.5, -0.5) \). The solution uses the MOM/FEM method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \theta\theta \) (solid) and \( \phi\phi \) (dotted) and the two cross polarized echo areas \( \phi\theta \) (dashed) and \( \theta\phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 8 nodes along the side of the cube. ........................................ ............................... 80

Bistatic scattering in the \( \phi = 0 \) plane from a material cube, one meter on a side, with parameters \( \varepsilon_r = 2, \mu_r = (1.5, -0.5) \). The solution uses the MOM/FEM method. The incident field has a wavelength of \( \lambda_0 = 4 \) meters with incident angles \( \theta_i = 90, \phi_i = 0 \) degrees. The two copolarized echo areas \( \theta\theta \) (solid) and \( \phi\phi \) (dotted) and the two cross polarized echo areas \( \phi\theta \) (dashed) and \( \theta\phi \) (short dashed) are plotted or are of negligible magnitude. This mesh uses 9 nodes along the side of the cube. ........................................ ............................... 81

Division of a triangular element into three subtriangles used to define the simplex coordinates. ........................................ ............................... 90

Local nodes corresponding to a first order triangular boundary element. The vertex numbers are enclosed in boxes. ........................................ ............................... 91
49 Local nodes corresponding to a second order triangular boundary element. The vertex numbers are enclosed in boxes.

50 A tetrahedral element. The perspective assumes vertices 1,3,4 lie in the plane of the paper and vertex 2 is closer to the viewer.

51 Local nodes corresponding to a first order tetrahedral element. The vertex numbers are enclosed in boxes.

52 Local nodes corresponding to a second order tetrahedral element. The vertex numbers are enclosed in boxes.

53 An example of a very warped isoparametric element.

54 The submatrix $R_{pl}^{mn}$ gives the FEM coupling between the potentials $w_0$, $\tilde{w}$ at node $m$ and the potentials $F_0, \tilde{F}$ at node $n$ in a single element.

55 The geometry for the surface integral equation.

56 Application of the surface divergence theorem over a single triangular patch.

57 The geometry for the surface integral defined on the source element $e'$. 

xvi
CHAPTER I
Introduction

1.1 Problem Description

Several methods may be used to analyze the electromagnetic scattering from a material body. The preferred method depends on the size, shape, and composition of the scatterer. An electrically small body may best be analyzed using a static field approximation while a body which is large compared to a wavelength will be more amenable to asymptotic methods. This report examines a numerical method appropriate to the transition region where the body’s size is comparable to a wavelength. The approach taken here combines a finite element method (FEM) which solves a vector differential equation in a bounded domain with a surface method of moments (MOM) which couples the bounded finite element solution to the exterior region. This three dimensional problem is computationally very demanding both in terms of computing time and storage requirements. The practicality of using a numerical approach has increased in recent years as the state of the art in computing machinery has advanced.

The geometry is shown in Figure 1 where a plane wave field is incident on a body composed of spatially varying complex dielectric and ferrite constants $\varepsilon, \mu$. There may exist embedded perfectly conducting surfaces or regions. The far field scattering and the total internal fields are desired.
Figure 1: Plane wave incident on an inhomogeneous body with embedded perfect electric conductors.

1.2 Solution Methods

When the body is homogeneous or composed of a few homogeneous regions, a surface MOM can be used to analyze the scattering [9]. This is attractive because the number of unknowns increases as the surface area of the scatterer. However, as the inhomogeneity increases, the complexity and numerical advantage of a surface MOM decreases. A traditional approach in this case is to model the inhomogeneity as a set of equivalent volumetric currents and solve for the currents using a MOM [14, 11, 8]. The resulting code can be very flexible in its ability to analyze an arbitrarily varying inhomogeneous body. The number of unknowns, $N$, is proportional to the volume of the scatterer. Because this is an integral equation approach, the impedance matrix for the MOM is dense and thus requires storage of $N^2$ impedances. Matrix fill time is proportional to $N^2$ and matrix inversion is typically of order $N^3$. The computational time and storage grows quickly with the size of the scatterer. Nevertheless, the
volumetric current approach is popular because of its ability to give good results even when using a coarse approximation for the equivalent currents.

Interest has grown recently in the use of finite elements for solving the scattering problem. The FEM will numerically solve a scalar or vector differential equation in a bounded domain based on the sources inside the domain and the boundary conditions. Because it is a bounded domain technique, the FEM was initially used to calculate resonant frequencies for cavities [28] and to map the mode structure in waveguides [27]. It is used extensively in solving the eddy current problem at power line frequencies [22] and for calculating the modes of optical dielectric waveguides when the field is known to decay exponentially away from the guiding structure [38]. In these applications, the FEM matrix is sparse. The matrix fill time is proportional to \( N \) and the inversion is of order \( N^2 \) where \( N \) is proportional to the volume of the solution region. Compared to the MOM volumetric current technique, the FEM matrix uses less storage and requires less computation time. This is counterbalanced somewhat by a higher density of unknowns being required for the FEM compared to a comparable MOM solution.

In a three dimensional finite element problem, the FEM domain is divided into elements, usually either brick or tetrahedral elements. These are shown in Figure 2. The unknown field is represented as a linear or higher order function across each element. In a nodal based FEM, the function is defined according to its value at the nodes of the elements. The order of the approximation is increased by increasing the number of nodes per element. Recently, edge based finite elements have become popular because they automatically enforce the zero divergence field condition required by Maxwell's equations (\( \nabla \cdot \vec{D} = 0 \) or \( \nabla \cdot \vec{B} = 0 \)) [25, 19]. As the name implies, edge elements define a linear approximation of the vector field according to the value along the edges of each element. They have an advantage of a
A first order brick element and a second order tetrahedral element.

A nodal based FEM with tetrahedral elements will be used here. The equations which govern the FEM solution will be developed in Chapter II.

To use the FEM on a scattering problem, the boundary of the domain must be extended to allow radiation of the scattered fields. This extension is most accurately done using an integral equation technique at the surface of the FEM domain as shown in Figure 3. The integral equation couples the incident field to the internal fields at the surface of the FEM domain. The numerical implementation of this coupling must be done in a manner which takes advantage of the sparseness of the FEM matrix.

Alternative methods exist for coupling the FEM to the unbounded domain such as absorbing boundary conditions [13] or spherical wave function techniques [12]. The surface MOM will be used in this report for its potential for greater accuracy. Unfortunately, the cost of accuracy is computation speed.
1.3 Previous Work

Various methods have been developed to couple a FEM to a surface MOM. The unimoment method developed by Mei [31] and the bymoment method of Cangellaris and Lee [30] decouple the interior problem for a set of basis functions defined on the boundary. These basis functions can take the form of equivalent surface currents $\vec{J}$, $\vec{M}$ or fields on the boundary surface. An integral equation at the surface solves for the unknown coefficients of the basis functions based on the interior solution and the incident field. Two matrix equations are involved with this technique: a sparse matrix associated with the volumetric domain and a smaller dense matrix which couples the unknown fields at the surface of the domain. This same method of decoupling was used by Wang, Dominek, Munk and Andre [37] by implementing a nodal based subsectional MOM on the boundary (boundary element method) that matched the finite element's nodes at the surface.
These applications have been for the 2D scattering from inhomogeneous cylinders. The same technique of decoupling the sparse interior FEM matrix from a dense surface MOM matrix will be used here to solve the 3D scattering problem. An iterative method which considers the surface MOM matrix and the FEM matrix as part of one combined matrix equation will also be examined. The form of the surface MOM matrix equations is examined in Chapter III. Chapter IV is devoted to the relationship of the surface integral equation to the interior FEM.

Work on a combined 3D FEM and surface MOM has been done by previous authors. Wang & Ida [17] and Yuan [7] describe a tetrahedral edge element FEM which uses the triangular patch MOM of Rao, Wilton & Glisson [15] on the surface. The FEM solves for the internal electric field in the FEM domain. One method uses a first order FEM which couples the surface MOM matrix and the FEM matrix into one large partially dense matrix [17]. The other decouples the matrices by solving the FEM solution for each of the MOM basis functions [7].

Paulsen, Lynch and Strohbehn [10] combine a nodal based FEM with a surface MOM for use in a code to analyze hyperthermia cancer treatments. The source of the field is in the vicinity of the lossy dielectric body. Their FEM method solves for the internal electric field. Because the normal electric field is discontinuous at a material boundary, they establish a connection matrix for the field at an interface between different media. The connection matrix requires a unique vector normal to the interface. This can be done exactly for a planar interface. For an interface with curvature, they use conservation principles to approximately satisfy the boundary conditions. Their method is most appropriate for inhomogeneities with constant \( \epsilon \) or constant \( \mu \) or small curvature between different media.

Boyse, Lynch, Paulsen and Minerbo [1, 2] have since developed a method which uses scalar and vector potentials which obviates the need for a connection matrix.
between inhomogeneous regions. Potentials remain continuous in inhomogeneous media. The exterior boundary conditions, however, do require a connection matrix since the normal and tangential components of the vector potential on the boundary are treated differently. This is done so that the boundary conditions may remain general.

1.4 Proposed Solution Method

The method contained in this report will also use nodal based potentials for the FEM solution. However, the boundary conditions will be chosen specifically to match a nodal based surface MOM. This simplifies the treatment of the fields on the boundary. The boundary conditions must also be chosen to insure uniqueness of the solution. Instead of using equivalent currents which are most appropriate to an edge based expansion, the surface MOM will expand over a nodal basis of the boundary electric and magnetic fields.

The electric vector potential $\vec{F}$ and the magnetic scalar potential $F_0$ will be used. This allows the scalar potential to be ignored when the FEM region is only inhomogeneous in the dielectric permittivity $\varepsilon$ (constant $\mu$) with a corresponding reduction of unknowns and matrix size. Internal perfectly conducting bodies will be simulated as holes in the FEM mesh. The requirement of zero tangential electric field appears as a natural boundary condition of the FEM with this formulation. An FEM solution using the magnetic vector potential $\vec{A}$ and electric scalar potential $A_0$ would require an additional computation over the surface of the internal conducting bodies in order to enforce the proper boundary conditions. Tetrahedral elements will be used as the finite elements with the potentials expanded over the nodes of the mesh.
The differential equation satisfied by the scalar and vector potentials and the development of the FEM matrix equation is examined in the next chapter.
CHAPTER II
The Interior Finite Element Method

2.1 Introduction

The form of the finite element solution dictates the choice of elements and the type of surface MOM needed to couple the FEM solution to the unbounded domain. The derivation of the FEM matrix equation is done in this chapter.

The FEM is based on a solution of a partial differential equation. The differential equations and the independent variables are chosen according to the problem being solved. For an arbitrarily inhomogeneous body, a vector and scalar potential method will be used since the potentials are continuous across boundaries of different media. The next section in this chapter will develop the differential equations satisfied by the vector and scalar potentials. Also, boundary conditions will be derived which assure a unique solution for a given impressed tangential electric field applied to the boundary.

The solution of these differential equations will be established through a functional formulation. This will be examined from both the theoretical and the numerical point of view.

A simplification can be made when $\mu$ is constant. In this commonly occurring case, the magnetic field $\vec{H}$ is continuous and therefore can be used as the independent variable in a nodal based FEM. Use of $\vec{H}$ as the unknown in the FEM reduces the number of unknowns at a node from four for the vector and scalar potentials to three
for the $\vec{H}$ field. This special case will be treated as an extension of the potential method.

### 2.2 The Differential Equations and Boundary Conditions

The electric field $\vec{E}$ and magnetic field $\vec{H}$ in an isotropic, inhomogeneous region $V$ surrounded by a surface $S$ satisfy the following Maxwell’s equations for an $e^{j\omega t}$ time dependence,

\begin{align*}
\nabla \times \vec{E} &= -j\omega \vec{B} \\
\nabla \times \vec{H} &= j\omega \vec{D} \\
\nabla \cdot \vec{D} &= 0 \\
\nabla \cdot \vec{B} &= 0.
\end{align*}

The constitutive equations are given by

\begin{align*}
\vec{D} &= \epsilon \vec{E} \\
\vec{B} &= \mu \vec{H}
\end{align*}

where $\epsilon, \mu$ can be complex for lossy media and are functions of position. Equations (2.3) and (2.5) imply the existence of an electric vector potential $\vec{\Phi}$,

\[ \vec{E} = -\frac{1}{\omega \epsilon} \nabla \times \vec{\Phi}. \] (2.7)

This form is different than the one used in Harrington [40]. The notation used here is chosen for numerical convenience. Substitution of (2.7) and (2.5) into (2.2) gives

\[ \nabla \times (\vec{H} + j \vec{\Phi}) = 0 \] (2.8)

which implies the existence of a magnetic scalar potential $F_0$ such that

\[ \vec{H} = -j \vec{\Phi} - \nabla F_0. \] (2.9)
In order to restrict the divergence of the vector potential, a gauge field $G$ is introduced,

$$G = -\frac{1}{\omega e} \left( \nabla \cdot \vec{F} + j\alpha k^2 F_0 \right)$$  \hspace{1cm} (2.10)$$

where $\alpha$ is a parameter equal to zero or one as an aid to handling both the Coulomb and Lorentz gauge and $k = \omega \sqrt{\epsilon \mu}$ is the wave number in the dielectric medium.

The usual gauge condition is written as $\{G = 0, \alpha = 0\}$ for the Coulomb gauge or $\{G = 0, \alpha = 1\}$ for the Lorentz gauge. For the purpose of this analysis, the gauge field $G$ will be considered a scalar variable and not automatically set to zero. $G$ is functionally equivalent to $\chi$ given in [1] but is placed on equal status with $\vec{E}$ and $\vec{H}$ in the following equations.

With the addition of a gauge field, Maxwell's equations are modified to the form,

$$\nabla \times \vec{H} - j\omega e \vec{E} = 0$$  \hspace{1cm} (2.11)$$

$$\nabla \cdot (\omega e \vec{E}) = 0$$  \hspace{1cm} (2.12)$$

$$\nabla \times \vec{E} + j\omega \mu \vec{H} - \nabla G = 0$$  \hspace{1cm} (2.13)$$

$$\nabla \cdot (\omega \mu \vec{H}) - j\alpha k^2 G = 0$$  \hspace{1cm} (2.14)$$

in terms of the three fields $\vec{E}$, $\vec{H}$, $G$. As long as the gauge condition $G = 0$ can be enforced throughout the finite element region $V$, this modified set of equations is identical to the original Maxwell's equations. This form is specifically chosen because it is the result of a stationary functional formulation to be detailed later.

Equations (2.11) and (2.12) are automatically satisfied by the definitions of $\vec{E}$ and $\vec{H}$ in Equations (2.7) and (2.9). Substituting Equations (2.7), (2.9) and (2.10) into Equations (2.13) and (2.14) gives the differential equations satisfied by the potentials $\vec{F}$, $F_0$,

$$\nabla \times \left( \frac{1}{\omega e} \nabla \times \vec{F} \right) - \omega \mu \vec{F} + j\omega \mu \nabla F_0 - \nabla \left[ \frac{1}{\omega e} \left( \nabla \cdot \vec{F} + j\alpha k^2 F_0 \right) \right] = 0$$  \hspace{1cm} (2.15)$$
\[ \omega \nabla \cdot \left( -j \mu \vec{F} - \mu \nabla F_0 \right) + j \alpha k^2 \frac{1}{\omega \epsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) = 0 \quad (2.16) \]

which will be solved by the finite element method once the proper boundary conditions have been established.

The dependency between Equations (2.13) and (2.14) has been broken by adding the gauge field. By taking the divergence of Equation (2.13) and substituting Equation (2.14), the gauge field satisfies the Laplace (Coulomb gauge) or Helmholtz (Lorentz gauge) equation,

\[ \nabla^2 G + \alpha k^2 G = 0 \quad (2.17) \]

as previously shown in [1]. The gauge field must be forced to zero in order to have Equations (2.11) - (2.14) represent Maxwell's equations. By uniqueness of Equation (2.17), specifying either \( \{ G = 0 \} \) or \( \{ \frac{\partial G}{\partial n} = 0 \} \) on the boundary \( S \), where \( \hat{n} \) is the unit outward normal, will force a zero gauge field \( G \) throughout the region \( V \) except at internal resonances when using the Lorentz gauge. The boundary condition

\[ G|_S = G^B = 0 \quad (2.18) \]

will be chosen. With this boundary condition, Equations (2.11) - (2.14) will reduce to Maxwell's equations.

By specifying either \( \vec{E}_{tan} \) or \( \vec{H}_{tan} \) on the surface \( S \), \( \vec{E} \) and \( \vec{H} \) are uniquely determined throughout the region \( V \) [40] except at possible internal resonances. The boundary condition

\[ \vec{E}_{tan}|_S = \vec{E}_{tan}^B \quad (2.19) \]

will be used where \( \vec{E}_{tan}^B \) represents a tangential electric field on the boundary which is determined by the coupling to the external scattering region. As far as the finite element equations are concerned, \( \vec{E}_{tan}^B \) is a known quantity.

The boundary conditions given in Equations (2.18) and (2.19) represent three degrees of freedom on the surface \( S \) which uniquely determines the fields \( \vec{E}, \vec{H} \), and
\( G \) in the region \( V \). However, \( \vec{E} \) and \( \vec{H} \) in Equations (2.7) and (2.9) are invariant under the gauge transformation [42]

\[
\vec{F} \Rightarrow \vec{F} + j \nabla \psi \\
\vec{F}_0 \Rightarrow \vec{F}_0 + \psi
\]

(2.20) (2.21)

where \( \psi \) is an arbitrary scalar variable. Furthermore, any two potential solutions which represent the same electromagnetic fields must be related by this transformation. The gauge condition \( \{ G = 0 \} \) forces \( \psi \) to satisfy the equation,

\[
\nabla^2 \psi + \alpha k^2 \psi = 0.
\]

(2.22)

In order to restrict the potentials \( F_0, \vec{F} \) to one solution, the arbitrariness of \( \psi \) is restricted by once again exploiting the uniqueness properties of Equation (2.22). The boundary condition \( \psi|_S = 0 \) is enforced by specifying \( F_0|_S = 0 \) and the boundary condition \( \partial \psi / \partial n = 0 \) is enforced by specifying \( \hat{n} \cdot \vec{F} = 0 \). The boundary condition,

\[
F_0|_S = 0
\]

(2.23)

is chosen for this application.

Note a limitation to boundary condition (2.23). Since \( F_0 = 0 \) on the boundary \( S \) and \( F_0, \vec{F} \) are continuous, an interface between two media with different \( \mu \) cannot extend to the surface \( S \). This is because a discontinuity in the normal (to the interface) \( \vec{H} \) field cannot be represented unless \( F_0 \neq 0 \) on part of the surface. The discontinuity in \( \vec{H} \) requires a discontinuity in the first derivative of \( F_0 \). This is a minor limitation. The surface \( S \) need only be extended slightly away from any region of inhomogeneous \( \mu \).

Throughout the derivation, the specific gauge has been left optional through the parameter \( \alpha \). Only the Coulomb gauge \( (\alpha = 0) \) gives an unconditionally unique
solution to Equations (2.17) and (2.22). The Lorentz gauge ($\alpha = 1$) can suffer from internal resonances from the Helmholtz equation. From the theoretical point of view, the Coulomb gauge is more appealing. The Lorentz gauge, however, has properties which are beneficial when the FEM is numerically implemented. This will be discussed later in this chapter. The choice of gauge is left as a user option in the computer code.

By the above construction, the uniqueness of the potentials $\vec{F}$ and $F_0$ in the region $V$ has been established by using both Neumann and Dirichlet boundary conditions. The Dirichlet boundary condition, Equation (2.23), requires $F_0$ be forced to zero on $S$. The Neumann boundary conditions represented by Equations (2.18) and (2.19) require that all components of the vector potential $\vec{F}$ be left free on the boundary surface $S$ and the fields $\vec{E}_{\text{tan}}^B$ and $G^B = 0$ be specified. This becomes clearer by looking at the functional formulation in the next section.

When $\mu$ is constant and the boundary conditions given above have been enforced, then Equations (2.15) and (2.16) can be manipulated to show that the scalar potential satisfies the Helmholtz or Laplace equation,

$$\nabla^2 F_0 + \alpha \kappa^2 F_0 = 0. \quad (2.24)$$

With boundary condition (2.23), $F_0 = 0$ throughout the FEM region $V$. The Lorentz gauge ($\alpha = 1$) will be chosen so that Equation (2.16) is trivially satisfied for any $\vec{F}$ when $\mu$ is constant and $F_0$ is zero. From Equation (2.15), the vector potential $\vec{F}$ satisfies the differential equation,

$$\nabla \times \left( \frac{1}{\omega \epsilon} \nabla \times \vec{F} \right) - \omega \mu \vec{F} - \nabla \left[ \frac{1}{\omega \epsilon} \nabla \cdot \vec{F} \right] = 0 \quad (2.25)$$

and the scalar potential $F_0$ has been removed from the problem. Equation (2.9) shows that $\vec{F}$ is equal to $\vec{H}$ multiplied by a constant phase factor. Solving for $\vec{F}$.
in this case is equivalent to solving for $\vec{H}$. Both boundary conditions (2.18) and (2.19) are required for uniqueness. As before, all components of $\vec{F}$ are left free on the boundary.

For a region with inhomogeneous $\mu$, the scalar potential $F_0$ is in general nonzero and the interfaces between homogeneous subregions act as sources for the scalar potential.

2.3 The Functional in Theory

The geometry being considered is shown in Figure 4. The FEM region $V$ is bounded by a surface $S$ composed of an exterior surface $S_0$ and several interior surfaces $S_i$ ($i = 1 \ldots n$). The interior surfaces bound perfect electric conductors. The functional can be written in the form [41] [1],

$$\mathcal{F} [F_0, \vec{F}] = \int_V \frac{1}{2} (\epsilon \vec{E}^2 + \mu \vec{H}^2 + \epsilon G^2) \, dv + S \left[ F_0, \vec{F} \right]$$  \hspace{1cm} (2.26)

where $\vec{E}$, $\vec{H}$, $G$ are to be replaced by their representations in terms of the potentials $F_0, \vec{F}$ from Equations (2.7), (2.9), and (2.10). The surface functional $S$ on the
boundary surface $S$ is given by

$$S \left[ F_0, \vec{F} \right] = \frac{1}{\omega} \int_S - \left( \hat{n} \times \vec{E}^B \right) \cdot \vec{F} + G^B \hat{n} \cdot \vec{F} + \left( \omega \mu \vec{H}^B \cdot \hat{n} \right) F_0 \, ds \quad (2.27)$$

where $\vec{H}^B, \vec{E}^B, G^B$ are the respective field values imposed on the boundary $S_0$. This functional will be shown to be stationary for the exact fields.

While the functional $\mathcal{F}$ may look like the total energy, it is not. If a pure real electric field is imposed on the boundary of a lossless inhomogeneous medium, the true internal electric field will be pure real and the magnetic field will be pure imaginary. Ignoring the gauge field for the moment, the condition of a stationary functional in Equation (2.26) for the exact fields can be interpreted as follows. For an arbitrary small perturbation from the exact fields, the change in the electric field energy density at each point will equal the change in the magnetic field energy density at that point. The addition of the gauge field contributes to the electric energy density and acts exactly like a penalty function [27].

Stated mathematically, consider a small perturbation of the potentials $F_0, \vec{F}$,

$$\vec{F} \rightarrow \vec{F} + \eta \vec{w}$$

$$F_0 \rightarrow F_0 + \eta w_0$$

(2.28)

where $w_0, \vec{w}$ are arbitrary functions and $\eta$ is a small factor. The functional is perturbed into an equation quadratic in $\eta$,

$$\mathcal{F} \left[ F_0 + \eta w_0, \vec{F} + \eta \vec{w} \right] = \mathcal{F} \left[ F_0, \vec{F} \right] + \eta \delta \mathcal{F} \left[ F_0, \vec{F}; w_0, \vec{w} \right] + \eta^2 \varepsilon \left[ F_0, \vec{F}; w_0, \vec{w} \right].$$

(2.29)

A stationary functional results when $\delta \mathcal{F} = 0$. The proof that this occurs when the potentials satisfy Equations (2.15) and (2.16) for arbitrary test functions $w_0, \vec{w}$ and for boundary conditions (2.23), (2.18) and (2.19) is given in Appendix A.
With boundary condition (2.23), \( F_0 = 0 \) on \( S \) and the last term in the surface integral in Equation (2.27) is zero. On the interior PEC surfaces \( S_i, \vec{E}_{tan}^B = 0 \) while on all surfaces \( G^B = 0 \). Therefore the surface functional simplifies to

\[
S \left[ F_0, \vec{E} \right] = -\frac{1}{\omega} \int_{S_0} (\hat{n} \times \vec{E}^B) \cdot \vec{F} ds.
\]  

(2.30)

2.4 The Functional in Practice

For the numerical solution of the differential equations, the region \( V \) is meshed with tetrahedral elements. Either first order, Figure 5, or second order elements, Figure 6, are used in this implementation of the finite element method. Four nodes per element are required to represent a linear function throughout the finite element mesh using the first order elements while ten nodes per element are required to represent a quadratic function using the second order elements. Figure 7 shows a part of a mesh consisting of seven nodes and four first order elements. One of
the difficulties in working with a three dimensional finite element method is the construction of these meshes.

The potentials are expanded in terms of the nodal based finite element functions $u_n$ defined at each of the $N$ nodes in the FEM mesh [38],

$$F_l = \sum_{n=1}^{N} f_{l,n} u_n$$  \hspace{1cm} (2.31)

where $l = 0, x, y, z$ and $f_{l,n}$ are the unknown potential values at node $n$ for component $l$. The function $u_n$ has the property of being one at node $n$ and zero at every other node. It is continuous across the interfaces between elements and is nonzero only in the tetrahedral elements adjacent to the node. The expansion function $u_n$ is represented in each element by the shape functions described in Appendix B.

Substituting Equation (2.31) into the functional Equations (2.26) and (2.30) yields the matrix form of the functional relation,

$$\mathcal{F} [F_0, \vec{F}] = \frac{1}{2} f_{\mu} \tilde{M}^{\nu}_{\mu} f^{\mu} - f_{\nu} \tilde{\varepsilon}^{\nu} [\vec{E}^B]$$  \hspace{1cm} (2.32)
Figure 7: Part of a finite element mesh using first order tetrahedral elements.

where \( \mu = (l, n) \), \( \nu = (p, m) \) and the summation convention has been assumed. The boundary field \( \tilde{E}_{\text{tan}}^B \) is represented by the term \( \tilde{\varepsilon}^\nu \left[ \tilde{E}^B \right] \). Boundary condition (2.23) is used by forcing \( f_{0,k} = 0 \) for all nodes \( k \) which lie on the surface \( S \) of the FEM region. Note that \( \tilde{M}_\mu^\nu \) is a symmetric matrix.

Applying the stationary functional principle, \( \frac{\partial F}{\partial f^\nu} = 0 \), to Equation (2.32) yields the finite element matrix equation,

\[
\tilde{e}^\nu \left[ \tilde{E}^B \right] = \tilde{M}_\mu^\nu f^\mu \tag{2.33}
\]

in terms of the impressed boundary field and the unknown nodal values of the potentials. For numerical convenience, this equation is multiplied by the factor \( \omega^2 \varepsilon_0 \) to yield,

\[
e^\nu \left[ \tilde{E}^B \right] = M_\mu^\nu f^\mu \tag{2.34}
\]
where $e^\nu \left[ \vec{E}^B \right]$ and $M^\nu_\mu$ are defined by,

$$e^\nu \left[ \vec{E}^B \right] = \omega^2 \varepsilon_0 e^\nu \left[ \vec{E}^B \right]$$

(2.35)

$$M^\nu_\mu = \omega^2 \varepsilon_0 \tilde{M}^\nu_\mu.$$  

(2.36)

The method of calculating $M^\nu_\mu$ and $e^\nu \left[ \vec{E}^B \right]$ is described in Appendix C.

The advantage of using the finite element method comes from the structure of the matrix $M^\nu_\mu$. Because the FEM is a differential equation based method, there is only coupling between nodes which have a tetrahedral element in common. This means the matrix $M^\nu_\mu$ is sparse. The matrix fill time is proportional to the number of elements and not the square of the number of elements which a volumetric method of moments technique would require. A much larger number of unknowns can be used by taking advantage of the matrix sparsity. This advantage is tempered by a poorer matrix condition number and the requirement of a larger density of unknowns for a given wavelength compared to the method of moments.

The FEM is notorious for the presence of spurious resonances and poorly conditioned matrices. Lynch and Paulsen [3, 5] performed a spectral analysis of the differential equations used in implementations of the finite element and finite difference methods. Their results showed that spurious modes can appear through the process of discretizing the differential equation. One way around this problem is to take advantage of the properties of scalar finite elements. The scalar Helmholtz equation is known to be resistant to spurious effects. Under the Lorentz gauge or when $F_0 = 0$ throughout $V$, the components of $F_0, \vec{F}$ separate into four decoupled scalar Helmholtz equations in homogeneous subregions of $V$. The current belief is that this property will aid the stability of the FEM matrix. The Coulomb gauge does not have this same property.
A spectral analysis can also yield information about the dispersive effect of the discretization process. By modelling the material by finite elements, the numerical solution can only approximate the physical solution. For a given periodic mesh in a homogeneous medium, an analytic solution can be derived. The derived numerical wavelength will be different from the physical wavelength and their ratio characterizes the dispersion induced by the numerical procedure. Lynch, Paulsen, and Strohbehn [4] performed this analysis on a two dimensional, first order mesh. Their results suggested that a mesh density of at least 20 elements per wavelength will produce negligible dispersion. In addition to dispersion, the mesh density must be chosen to accurately model the discontinuities in the material properties. Dispersion becomes of greater importance as the size of the body increases.

The decoupling of the vector differential equation into four scalar Helmholtz equations by the Lorentz gauge has another desirable effect. As shown in Appendix C, the matrix element \( M_{(p,m)}^{(l,n)} \) is zero when \( p \neq l \) and one of \( m \) or \( n \) is a node within a homogeneous subregion. By not including these terms while building the sparse FEM matrix, the FEM matrix size can be reduced to as little as one fourth its original size. This reduction is not available with the Coulomb gauge.

Equation (2.34) completes the goal of this chapter. It describes a matrix equation which solves for the unknown internal potentials given a known tangential electric boundary field. The internal potentials can be used to find the internal electric and magnetic fields. In particular, the potentials provide the magnetic field on the boundary. Therefore, this matrix equation represents one equation between the total tangential electric field on the boundary and the total magnetic field on the boundary. Solving a surface method of moments provides another matrix equation between the boundary electric and magnetic fields in terms of the incident field. Combination of the FEM and MOM matrix equations yields enough equations to
solve for the total internal potentials and external fields for a given incident field.

The next chapter develops the surface method of moments matrix equation.
CHAPTER III
The Exterior Method of Moments

3.1 Introduction

A surface integral equation, applied to the FEM boundary surface $S_0$, provides a relationship between the electric and magnetic fields on the boundary in terms of the incident field. This relationship accounts for radiation into the far field which a pure FEM solution can only approximate through absorbing boundary conditions. This chapter will examine the characteristics of the surface integral equation and the corresponding MOM matrix equation which will be coupled to the FEM. More detail may be found in Appendices D and E. The MOM may also be used by itself to analyze the scattering from a body composed of a homogeneous dielectric or a perfect electric conductor (PEC). This will also be described.

3.2 The Surface Integral Equation

Free space is assumed to exist outside a boundary surface $S_0$ as shown in Figure 8. The incident electric field $\vec{E}_{\text{inc}}(\hat{r})$ in the absence of the perturbing body is a plane wave with polarization $\hat{p}_{\text{inc}}$ incident from the $\hat{s}$ direction. For this exterior problem, the scattered far field $\vec{E}^{\text{scat}}(\hat{r})$ and radar cross section $\sigma(\hat{r})$ is desired in the direction $\hat{r}$ with polarization $\hat{p}$. The scattered field depends on the value of the total electric field $\vec{E}^B$ and magnetic field $\vec{H}^B$ on the surface. Once these boundary fields are known, the scattering problem is solved.
The interior solution provides one set of equations between $\mathbf{E}^B$ and $\mathbf{H}^B$. Another set of equations is derived from an exterior integral equation. One method of forming the surface integral equation is to represent the boundary fields in terms of equivalent currents. These equivalent currents, along with the incident field, form a null field within the bounding surface $S_0$ [40]. From the null field, a surface integral equation can be derived. This popular method is not easily coupled to an interior nodal based FEM because the fields of the equivalent currents involve only the tangential components on the boundary surface. The surface integral used here was derived in [39] using Green’s theorem. It has the form,

$$
\begin{align*}
\mathbf{E}_{\text{inc}}(\mathbf{r}) &= \mathbf{E}^B(\mathbf{r}) - \int_{S_0} - j \omega \mu_0 \left( \mathbf{n}' \times \mathbf{H}^B(\mathbf{r}') \right) G_0(R) \\
&+ \left( \mathbf{n}' \times \mathbf{E}^B(\mathbf{r}') \right) \times \nabla' G_0(R) \\
&+ \left( \mathbf{n}' \cdot \mathbf{E}^B(\mathbf{r}') \right) \nabla' G_0(R) \, ds'.
\end{align*}
$$

(3.1)
The free space scalar Green's function $G_0(R)$ is defined by,

$$G_0(R) = \frac{e^{-jk_0 R}}{4\pi R}$$  \hspace{1cm} (3.2)

where $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$ is the free space wave number. The boundary fields must be continuous on $S_0$. An integral equation based on equivalent currents would have this same form except the equivalent charge term $(\hat{n}' \cdot B^B (r'))$ in Equation (3.1) would be represented in terms of the surface divergence of the equivalent current $(\hat{n}' \times H^B (r'))$. Notice that Equation (3.1) uses all components of $E^B$. This is important since it allows a representation of $E^B$ in which each Cartesian component is free and independent and coupled through the integral equation. The fact that $H^B$ is only represented in terms of tangential components is immaterial since the magnetic boundary field is ultimately represented in terms of the electric boundary field through the interior matrix equation. The matrix equation will then reduce to an equation with $E^B$ as the only unknown. This is shown later in this chapter for the scattering from a homogeneous body.

### 3.3 The MOM Matrix Equation

The method of moments expands the boundary fields $E^B$, $H^B$ in terms of an independent set of expansion functions. Because the Cartesian components can be treated independently, the boundary fields are expanded as,

$$E^B = \sum_q \tilde{u}_q^B \tilde{E}^B_q$$

$$H^B = \sum_s \tilde{u}_s^B \tilde{H}^B_s$$

$$\tilde{u}_q^B = u_{k_i}^q \hat{x}_i$$

$$\tilde{u}_s^B = u_{k^m}^B \hat{x}_m$$  \hspace{1cm} (3.4)
in which \( q = (l', k') \), \( s = (l'', k'') \), \((l', l'' = x, y, z) \) and \( k', k'' \) are boundary nodes. The summation is over the \( N^B \) boundary nodes and three Cartesian coordinates. The scalar boundary function \( u^B_k \) has a value of one at boundary node \( k \) and is zero at every other boundary node. These boundary expansion functions are the restriction to the boundary \( S_0 \) of the FEM expansion functions \( u_n \) and thus provide a natural method of coupling to an interior FEM method. The expansion functions are described in detail in Appendix B. The field expansion in Equation (3.4) satisfies the requirement of a continuous field on \( S_0 \).

The same set of functions is used to test the surface integral equation. This yields the matrix equation,

\[
V^E = j\omega\mu_0 \left[ M^{0+}\right]^t_s H^B s + \left[ M^{\nabla+}\right]^t_q E^B q. \tag{3.5}
\]

The incident field is represented in the known vector \( V^{B t} \). For \( N^B \) boundary nodes, the matrices are \( 3N^B \times 3N^B \) square matrices. In contrast to the FEM matrix, these matrices are dense. The unknown electric and magnetic fields are represented by their values at the nodes of the surface mesh through the vectors \( E^B q, H^B s \). A simultaneous solution of this equation and the interior matrix equation provides a solution for the unknown vectors \( E^B q, H^B s \). The scattering cross section can be obtained for any direction \( \hat{r} \) from these vectors using the method in Appendix D.

Equation (3.5) is an integral equation of the second kind. This helps to stabilize a numerical solution of the matrix equation. Problems with internal resonances are less likely to affect the MOM matrix equation or the inversion of \( [M^{\nabla+}]^t_q \) since all components of \( \vec{E}^B \) are coupled into the matrix equation. A similar technique was used by Yaghjian [36] for controlling spurious resonances by using an augmented integral equation.
3.4 Pure MOM Solutions

The dual form of the matrix equation is given by,
\[ V^H t = -j\omega \epsilon_0 \left[ M^{0+} \right]_s^t E^B_s + \left[ M^{\nabla+} \right]_q^t H^B_q \]  
(3.6)
in terms of the incident magnetic field \( \vec{H}_{inc}(r) \). This form of the MOM matrix equation is useful for analyzing the scattering from a PEC. A PEC has the boundary condition \( \hat{n}' \times \vec{E}^B = 0 \). The matrix equation reduces in this case to,
\[ V^H t = \left[ M^{\nabla+} \right]_q^t H^B_q \]  
(3.7)
with the solution
\[ H^B_q = \left[ M^{\nabla+} \right]_t^{-1} q V^H t \]  
(3.8)
for the total boundary magnetic field \( \vec{H}^B \). The scattered field may then be computed from \( \vec{H}^B \) with \( \hat{n}' \times \vec{E}^B = 0 \).

When the interior volume consists of a homogeneous dielectric \( \epsilon_-, \mu_- \), an interior matrix equation can be derived,
\[ 0 = -j\omega \epsilon_- \left[ M^{0-} \right]_s^t E^{B-}_s + \left[ M^{\nabla-} \right]_q^t H^{B-}_q \]  
(3.9)
where \( \vec{E}^{B-} \) and \( \vec{H}^{B-} \) are internal fields adjacent to the boundary surface \( S_0 \). Equation (3.9) yields an admittance relation between the electric and magnetic boundary fields,
\[ H^{B-}_q = Y^q_s E^{B-}_s \]  
(3.10)
\[ Y^q_s = j\omega \epsilon_- \left[ M^{\nabla-} \right]_t^{-1} q \left[ M^{0-} \right]_s^t. \]  
(3.11)
Using the continuity of tangential components of electric and magnetic fields across the boundary surface \( S_0 \), a combined matrix equation can be written which couples the interior and exterior matrix equations using Equations (3.5) and (3.11),
\[ V^E t = \left[ M^{tot} \right]_q^t E^B_q \]  
(3.12)
\[ 27 \]
\[
[M_{\text{tot}}]^t_q = j\omega \mu_0 [M^0+]^t \nu_q^r + [M^\nabla+]^t_q
\] (3.13)

with the solution,
\[
E^{B_q} = [M_{\text{tot}}]^{-1} q \nu E^t.
\] (3.14)

This equation, with Equation (3.10), provides the scattering solution from a homogeneous body. It requires calculating four matrices, two internal and two external. A finite element solution replaces Equation (3.10) with the FEM equivalent. This is considered in the next chapter.
CHAPTER IV
Coupling the Exterior to the Interior

4.1 The Coupled Matrix Equations

Chapter III presented a matrix equation which related the boundary electric field $\vec{E}^B$, the boundary magnetic field $\vec{H}^B$, and the incident field $\vec{E}_{inc}$,

$$V^E = j\omega\mu_0 \left[ M^{0+}\right]^t s H^B s + \left[ M^{\nabla+}\right]^t q E^B q.$$  \hspace{1cm} (4.1)

The implied summation for this equation is over the boundary nodes and the three Cartesian components of the fields.

Chapter II described the development of a matrix equation which solved for the internal scalar potential $F_q$ and vector potential $\vec{F}$ given a known electric field $\vec{E}^B$ on the boundary surface $S_0$, 

$$e^\nu \left[ \vec{E}^B \right] = M^\nu_\mu f^\mu.$$  \hspace{1cm} (4.2)

The implied summation is over the nodes which form the tetrahedral mesh and each component of $F_0$, $\vec{F}$. The field $\vec{E}^B$ is represented on the boundary using the MOM expansion given in equation (3.4). This yields the FEM matrix equation,

$$e^\nu_q E^B q = M^\nu_\mu f^\mu.$$  \hspace{1cm} (4.3)

The matrix $e^\nu_q$ establishes a coupling of the exterior MOM into the interior FEM region. The calculation of $e^\nu_q$ is described in Appendix C.

To couple the interior FEM to the exterior MOM, the tangential magnetic field $\vec{H}^B$ is required from the FEM. The magnetic field is provided throughout $V$ and on
the boundary $S_0$ through the definition,

$$\vec{H} = -j \vec{F} - \nabla F_0.$$  \hspace{1cm} (4.4)

The scalar potential $F_0$ is constant on the surface $S_0$. Therefore, $\nabla F_0$ is normal to $S_0$ which implies the tangential magnetic field takes the form,

$$\hat{n}' \times \vec{H}^B = -j \left( \hat{n}' \times \vec{F}' \right)$$  \hspace{1cm} (4.5)

where $\vec{F}'$ is the value of the vector potential on the boundary. This is an important result. No derivatives of $F_0$ nor of $\vec{F}$ are involved in Equation (4.5). This means the order of the approximation of $\hat{n}' \times \vec{H}^B$ stays at the level solved for in the FEM. The ability of the FEM to model the tangential boundary magnetic field would be reduced if $\hat{n}' \times \nabla F_0$ were nonzero.

In coupling these two matrix equations, the tetrahedral mesh of the FEM forms the triangular boundary mesh for the MOM matrix equation. If a first order FEM approximation is made, each FEM node $n$ on the boundary $S_0$ will correspond to a boundary node $k$. This is formally written as a mapping $\Upsilon$ from boundary nodes to FEM mesh nodes,

$$n = \Upsilon(k).$$  \hspace{1cm} (4.6)

Equation (4.5) can therefore be written as

$$H^B s = \vec{H}^B (l'', k'') = -j f(l'', \Upsilon(k''))$$  \hspace{1cm} (4.7)

where $\vec{H}^B$ and $-j \vec{F}'$ are equated on a component by component basis since only the tangential part of $\vec{H}^B$ is used in the MOM matrix equation. This equation can be used to eliminate $H^B s$ from the MOM matrix equation and reduce Equations (4.1) and (4.3) to a coupled set of equations in the unknowns $E^B q$ and $f^\mu$. The next two sections consider two methods for solving this coupled set of equations.
Not all FEM mesh nodes on the surface $S_0$ need be part of the basis modes of the MOM. Usually a linear approximation is sufficient for a surface MOM while the internal FEM solution can benefit from a higher order finite element approximation. For example, if the FEM uses a quadratic interpolation, there is a node at the center of each edge in addition to the nodes at the vertices of the triangles which form the surface mesh. This is shown in Figure 6 of Chapter II for one tetrahedral element. Six nodes, three vertex and three edge nodes, are available for each triangular surface element of the FEM mesh. By using all the surface nodes of the mesh, a quadratic interpolation of the FEM surface fields will be made. However, the MOM assumes a linear field variation and uses only the three vertex nodes for each boundary element as the basis for this approximation. While the FEM and MOM field variation may not correspond exactly on the surface, Equation (4.7) is still used to establish a coupling with the idea of point matching the two fields at the vertex nodes. A good solution results from this approach even though information about the interior FEM field at the unused edge nodes is being lost. An averaging process which uses the value of the potential field at these edge nodes might help improve the dynamic range of the numerical solution but this will be left for future study.

4.2 The Admittance Matrix Method

For each of the MOM basis functions $\tilde{u}_q^B$, there is a solution of the FEM matrix equation $f^\mu \left[ \tilde{u}_q^B \right]$ with this given impressed boundary electric field. Equation (4.5) gives the boundary magnetic field. The $q^{th}$ column of the admittance matrix $Y_q^s$ is constructed from this solution,

$$Y_q^s = H^{B,s} \left[ \tilde{u}_q^B \right].$$  

(4.8)
This matrix is a $3N^B \times 3N^B$ square matrix. Construction of $Y^s_q$ requires solving the FEM matrix $3N^B$ times. For an arbitrary boundary electric field, the magnetic field has the solution,

$$H^B \{E^B\} = Y^s_q E^B q.$$  

(4.9)

Combining this equation with the MOM matrix equation yields,

$$V^E t = [M^{tot}]^t_q E^B q$$  

(4.10)

$$\left[ M^{tot} \right]^t_{q} = j \omega \mu_0 \left[ M^0 \right]^t_q Y^s_q + \left[ M^\nabla \right]^t_q$$  

(4.11)

with the solution,

$$E^B q = \left[ M^{tot} \right]^{-1}_q V^E t.$$  

(4.12)

The boundary magnetic field is found from Equation (4.9) and the internal potential fields from Equation (4.3).

The disadvantage of this method is the multiple FEM solutions required to build $Y^s_q$ and $\left[ M^{tot} \right]^t_q$, and the subsequent factoring of $\left[ M^{tot} \right]^t_q$. This requires a lot of computer time and memory. This method is best when multiple solutions are required as in a backscattering analysis over a sweep of incident angles.

The FEM matrix equation is solved using a direct matrix solver. Direct solvers are available which take advantage of the sparse and symmetric structure of the FEM matrix [49, 50, 51, 43]. This code uses the Sparse 1.3 package developed by Kenneth S. Kundert and Alberto Sangiovanni-Vincentelli at the University of California, Berkeley. It was originally written for use in circuit simulations.

4.3 The Iterative Method

An alternative solution method considers the unknowns $f^\mu$ and $E^B q$ as a single vector of unknowns in a larger matrix equation. The matrix combines the FEM
Figure 9: The total matrix used in the iterative method is a conglomeration of the FEM and MOM matrices.

Matrix $M_\mu^\nu$, the MOM matrices $[M^0]^t_s$, $[M^\nabla]^t_q$, the coupling matrix $e^\nu_q$, and it includes Equation (4.7) implicitly. This is shown schematically in Figure 9. This is a square matrix. The known vector is zero for the rows corresponding to $f^\mu$ and $V^E t$ for the rows corresponding to $E^B q$. As a reminder, the number of $\mu$ rows (columns) is much greater than the number of $q$ rows (columns). However, the matrices $M_\mu^\nu$ and $e^\nu_q$ are sparse whereas the matrices $[M^0]^t_s$ and $[M^\nabla]^t_q$ are dense.

The combined matrix is solved using an iterative matrix method. For each incident field, the unknown potentials and surface electric fields are iteratively improved until the solution to the matrix equation is found within an acceptable error. The advantage of this solution method is the much smaller memory requirements and
the time required to yield a single solution compared to the previous method. The disadvantage is the time required to find multiple solutions.

There are many iterative techniques for the solution of this combined matrix system [47]. The most popular methods are an extension of the conjugate gradient method developed by Hestenes and Steffel [44]. The original conjugate gradient method solves the matrix equation by minimizing a related functional. The convergence is therefore monotonic in the residual error. This technique chooses a set of orthogonal search directions which will converge to the solution in a maximum of N steps for a matrix equation of order N. Ideally, convergence to an acceptable error will be reached in much fewer than N steps. The conjugate gradient method usually requires the matrix to be hermitian and positive definite. This is not the case for the matrix developed in this report.

The biconjugate gradient method is a modification of the conjugate gradient algorithm which can be applied to complex and unsymmetric matrices. However, no functional is being minimized which often leads to an erratic path towards a converged solution. Furthermore, this method can break down though this has not been observed in the present application. The combined MOM/FEM matrix system is solved using a biconjugate gradient solution with scaling to force the entries in the main diagonal to have unit magnitude.

A quasi-minimal residual method (QMR) has been developed which corrects some of the shortcomings of the biconjugate gradient method [45]. This method will be implemented when a robust code is available.

The rate of convergence of an iterative matrix solver is dependent upon the spread of eigenvalues of the matrix [46, 48]. Iterative solvers can be forced to converge faster by using a matrix preconditioner which attempts to cluster these eigenvalues to improve convergence. Preconditioning can be very effective on the sparse matrices.
generated by finite element solutions of partial differential equations such as the FEM matrix. The typical preconditioner builds a pseudo or incomplete inverse of the original matrix. The iterative solver then solves a matrix equation with a matrix which is close to the identity matrix. The convergence is very fast. The combined matrix, however, includes the MOM matrices which are dense and which have strong coupling between all the surface elements. A preconditioner for these matrices would be harder and more costly to develop. Furthermore, parts of the matrix \( e_q^\nu \) are not stored but are calculated as needed. For simplicity, no preconditioning was used beyond scaling the matrix equation.

### 4.4 Method Comparisons

The preferred method depends on the size and type of problem and the available computing resources. This code was developed and executed on an SGI Indigo workstation. In order to analyze as large a problem as possible, the iterative method was the only practical matrix method with the available matrix solvers because of the more modest storage requirements. The number of FEM unknowns never exceeded 31,000 and the number of MOM unknowns were constrained to less than 1,500. With this restriction, the analysis was left to bistatic incidence in which only two solution vectors (for two polarizations) were required. Multiple far field observation angles were quickly generated from these solutions. The admittance matrix approach was used if the problem was small enough and a backscattering solution was desired.

As the number of unknowns increases, the matrix condition number becomes worse. The iterative matrix solvers (and the direct solvers to a certain extent) are at the mercy of the condition number for their convergence. Furthermore, the true error in the solution for an iterative method is related to the calculated residual.
error through the matrix condition number [48]. For large problems, an iterative technique would need to use some type of preconditioning in order to control the condition number. The problems associated with preconditioning the combined matrix might make the admittance matrix method a more attractive option. The admittance matrix has the added benefit of allowing the direct matrix solvers to work separately with each of the strengths of the FEM (large and sparse) and MOM (small and dense) matrices.
CHAPTER V
Numerical Results

5.1 A Word About The Code

The hybrid combination of an internal finite element method using potentials and an external surface method of moments has been implemented as a computer code called PEON (Potential Elements Over Nodes). It was designed to calculate the scattering from a material body due to a plane wave field. The bistatic or backscattered far field echo area is available as is the internal $\vec{E}$, $\vec{H}$, $G$ fields at a set of user supplied points when using the FEM. Three different solution methods are possible with this program. A MOM only solution is available for a homogeneous dielectric body and for a conducting body. An FEM only internal solution may be found for a given impressed boundary field. This is useful for looking at eigenfunction results without the MOM affecting the solution. Lastly, the MOM and FEM can be coupled together as outlined in Chapter IV.

The FEM matrix is calculated in double precision because the matrix can often exhibit a poor condition number and because of the large number of unknowns involved with an FEM solution. The choice in gauge is left as a user option. Both the Lorentz gauge and the Coulomb gauge produce nearly identical numerical results. When no scalar potential is used or if the Lorentz gauge is used, the matrix size is reduced due to the large number of zeros as described in Chapter II. All the examples in this report which use the FEM and which require a scalar potential are computed using the Lorentz gauge in order to take advantage of the reduced storage.
requirements. Since the FEM matrix is symmetric, only the lower triangular part is stored.

The MOM matrix is calculated in single precision for speed. When an iterative solution with the FEM is required, the MOM matrices are first converted to double precision. Two surface integrals are required between each MOM matrix element: one over the source triangle, the other over the test triangle. Both integrals use a Gaussian quadrature method appropriate for triangles. The source integration is chosen for convergence at each field point on the test triangle. The test integration can be set for a one, four or seven point quadrature. The four point quadrature was found to give much better dynamic range compared to the one point (point matching) quadrature.

The mesh is built using either first or second order tetrahedrons. The MOM assumes a linear behaviour on the triangles which form the boundary. FEM matrix elements for first order or second order tetrahedrons with linear geometry use precalculated parameters resulting in a very fast FEM matrix fill time. The isoparametric elements require an approximate numerical quadrature in which the order of the quadrature is set according to the degree of curvature of the isoparametric elements.

The iterative method uses the biconjugate gradient method described by Sarkar et al. [20] with a modification which scales the matrix equation so that the main diagonal has elements with unit magnitude. The scaling compensates for the different dimensions of the unknown potentials and surface fields. The scaling is actually a simple form of preconditioning using a diagonal preconditioning matrix with real elements.

The computations were performed on an SGI Indigo workstation with an R3000 MIPS processor and 80 megabytes of memory.
One of the problems associated with a MOM or FEM implementation is generating the meshes. The cubical and spherical meshes used as examples come from a simple mesh generator. The mesh is built out of bricks and each of those bricks is divided into five tetrahedrons. The rectangular mesh is warped into a sphere if desired. This is a convenient method of generating a spherical mesh. Unfortunately, the isoparametric elements near the origin can become very warped. The large curvature necessitates a high quadrature order for the calculation of the FEM matrix elements. In addition, the corners can form singular points on some of the tetrahedrons which also hinders accuracy. Nevertheless, useful results can be generated from these meshes.

This chapter presents various numerical results generated by PEON. The sections are divided according to the three solution methods provided by this code. The first section examines the nodal based MOM for the scattering from homogeneous dielectric and conducting bodies. The second section compares the FEM by itself with known eigenfunction solutions. Finally, the hybrid combination of the MOM and FEM is used to generate both scattering results and internal fields which are compared with known solutions.

All units are assumed to be MKSA. The scattered field magnitude is given in terms of the scattering cross section in decibels relative to a square meter. When material parameters are required, the values are specified as $\varepsilon_r$, $\mu_r$ and are relative to free space ($\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$, $\mu_r = \frac{\mu}{\mu_0}$).
Table 1: Mesh parameters and CPU run times for the MOM examples.

<table>
<thead>
<tr>
<th>Description</th>
<th>MOM Mesh</th>
<th>Run Time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>nodes</td>
<td>elements</td>
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<tr>
<td>Material Sphere</td>
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<td>192</td>
</tr>
<tr>
<td>PEC Sphere</td>
<td>218</td>
<td>432</td>
</tr>
<tr>
<td>Free Space Sphere (Noise)</td>
<td>218</td>
<td>432</td>
</tr>
<tr>
<td>Cube 7 nodes/side</td>
<td>218</td>
<td>432</td>
</tr>
<tr>
<td>Cube 8 nodes/side</td>
<td>296</td>
<td>588</td>
</tr>
<tr>
<td>Cube 9 nodes/side</td>
<td>386</td>
<td>768</td>
</tr>
<tr>
<td>Cube 10 nodes/side</td>
<td>488</td>
<td>972</td>
</tr>
</tbody>
</table>

5.2 MOM

This section presents results using only the moment method part of the code. Table 1 lists the mesh information and CPU run times for the examples in this section.

5.2.1 Material Sphere

Figures 10 and 11 show the bistatic scattering from a material sphere with relative material parameters \( \varepsilon_r = 1, \mu_r = 4 \) and with radius one meter \( (R = 1) \). Figure 10 is for parallel polarization and Figure 11 is for perpendicular polarization relative to the plane of scattering. The incident field approaches the sphere along the \( z \) axis. The incident wavelength is five meters in free space. The mesh requires 294 boundary unknowns. At this frequency, there are about 200 nodes per free space wavelength squared.
Figure 10: Parallel polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and nodal density $N_{\lambda_0^2} = 202$ with $\lambda_0 = 5$ meters. MOM solution is in solid. Eigenfunction solution is dashed.

The solution is compared against the eigenfunction solution for a material sphere with an equivalent radius $R_e = .976$ in order to equate the volumes of the mesh sphere and the eigenfunction sphere. The agreement is very good.

5.2.2 PEC Sphere

Figures 12 and 13 show the bistatic scattering from a perfectly conducting sphere with radius one meter ($R = 1$). Figure 12 is for parallel polarization and Figure 13 is for perpendicular polarization. As before, the incident field approaches the sphere along the $z$ axis. The incident wavelength is five meters. The mesh in this example requires 654 boundary unknowns. At this frequency, there are 440 nodes per free space wavelength squared.
Figure 11: Perpendicular polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and nodal density $\frac{N}{\lambda_0} = 202$ with $\lambda_0 = 5$ meters. MOM solution is in solid. Eigenfunction solution is dashed.

The solution is compared against the eigenfunction solution for a sphere with an equivalent radius $R_e = 0.989$. The agreement is good for this case. However, when the mesh in the previous example is used to compute the scattering from a perfectly conducting sphere, the result does not match the eigenfunction result. The MOM solution has not converged. Increasing the wavelength does not help. There are not enough nodes to compensate for the field variation on the surface. The number of unknowns for this convergent case is about one fifth the number of unknowns used by Gilmore [8] for a comparable scattering problem.

5.2.3 Noise

An interesting experiment with a material body code is examining the scattering from a material sphere with material parameters of free space. The bistatic scattering
Figure 12: Parallel polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter, nodal density \( \frac{N}{\lambda_0^2} = 440 \) and with \( \lambda_0 = 5 \) meters. MOM solution is in solid. Eigenfunction solution is dashed.

Figure 13: Perpendicular polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter, nodal density \( \frac{N}{\lambda_0^2} = 440 \) and with \( \lambda_0 = 5 \) meters. MOM solution is in solid. Eigenfunction solution is dashed.
using the mesh of Section 5.2.2 is shown in Figure 14 for both polarizations at a wavelength of five meters. The wiggles in this figure are due to the numerical noise of the computations.

The dynamic range between the PEC scattering and the free space sphere is over 80 decibels. The dynamic range for the mesh in Section 5.2.1 has been found to be 70 decibels. Ironically, the exact fields, when specified at the nodes of the mesh, produce a scattered field 30 decibels higher than the one shown in Figure 14. The error results from the linear approximation across the elements of the mesh. The MOM solution produces an average solution with a smaller echo area.

Part of the dynamic range is limited by the mesh density and part is limited by the quadrature methods of the code. The current configuration of the code uses four points to test the field on the boundary triangles for the MOM. When one point is used for a point matching solution, the dynamic range decreases by 35 decibels. Using seven points does not significantly alter the result compared to four point testing. This type of test was used along with other considerations for choosing the quadrature methods and orders for the surface MOM.

Point matching is often advocated for a triangular mesh method of moments [15] because it is simple to implement. Those applications are usually for an electric field integral equation over conducting bodies in contrast to the augmented magnetic field integral equation used by the MOM code in PEON. The improvement seen here by increasing the integration order may or may not have the same effect on those codes.

5.2.4 Convergence

The scattering from a material cube, one meter on a side, is considered in this section. The material consists of \( \varepsilon_r = 2, \mu_r = (1.5, -0.5) \). The incident field has a
Figure 14: Perpendicular and parallel polarization for the bistatic scattering from a material sphere with free space material parameters. The sphere has a radius of one meter, a nodal density $N = 440$ and the incident wavelength is $\lambda_0 = 5$ meters.

wavelength of 4 meters and is incident from the $\theta_i = 90, \phi_i = 0$ degrees direction. The bistatic scattering in the $\phi = 0$ plane is plotted as $\theta$ sweeps from 0 to 360 degrees in Figures 15, 16, 17 and 18 for four different meshes. The meshes range from 7 to 10 nodes on a side. The scattered field converges as the density of nodes increases.

The cross polarized fields are especially sensitive to the symmetry properties of the mesh. This can be seen in Figure 16 and to a lesser degree in Figure 18 for the even meshes which have a different symmetry than the odd meshes.

A corresponding analysis of spheres shows a much faster rate of convergence. This would appear to indicate that this particular nodal based MOM has difficulties analyzing bodies with sharp edges and corners.
Figure 15: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM only method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90, \phi_i = 0$ degrees. The two copolarized echo areas $\theta\theta$ (solid) and $\phi\phi$ (dotted) and the two cross polarized echo areas $\phi\theta$ (dashed) and $\theta\phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 7 nodes along the side of the cube.
Figure 16: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM only method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90, \phi_i = 0$ degrees. The two copolarized echo areas $\theta\theta$ (solid) and $\phi\phi$ (dotted) and the two cross polarized echo areas $\phi\phi$ (dashed) and $\theta\phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 8 nodes along the side of the cube.
Figure 17: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM only method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90^\circ$, $\phi_i = 0^\circ$ degrees. The two copolarized echo areas $\theta\theta$ (solid) and $\phi\phi$ (dotted) and the two cross polarized echo areas $\phi\theta$ (dashed) and $\theta\phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 9 nodes along the side of the cube.
Figure 18: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM only method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90, \phi_i = 0$ degrees. The two copolarized echo areas $E' \theta$ (solid) and $\phi \phi$ (dotted) and the two cross polarized echo areas $\phi \theta$ (dashed) and $\theta \phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 10 nodes along the side of the cube.
Table 2: Mesh parameters and CPU run times for the FEM examples.

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<th>Description</th>
<th>FEM Mesh</th>
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<td>nodes</td>
<td>elements</td>
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<tr>
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<td></td>
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<td>6,030</td>
</tr>
</tbody>
</table>

5.3 FEM

A known electric field is applied to the surface of the FEM region. The internal potentials and fields are derived through the FEM. Table 2 lists the mesh information and CPU run times for the examples in this section.

5.3.1 Half Space

The total fields are known for the plane wave reflection and transmission from a material half space. This example applies the FEM to a cubical volume surrounding the interface between free space and the material half space. The cubical FEM region is $2 \text{ m} \times 2 \text{ m} \times 2\text{m}$. The material half has material constants $\varepsilon_r = 4$, $\mu_r = 1$. 

50
The total electric field is specified on the boundary using the known solution. The FEM solves for the internal fields which can then be compared with the known plane wave fields.

The mesh is composed of second order tetrahedral elements. It requires 1995 unknowns. This mesh is a regular mesh with a density of 4 second order elements per side. The scalar potential is not needed for this constant $\mu$ example. The incident field arrives from the direction $\theta_i = 30$ degrees, $\phi_i = 15$ degrees where the $z$ axis is normal to the interface and the cube's sides are aligned with the Cartesian directions.

The $z$ component of the electric field along the $z$ axis is shown in Figure 19 for an incident frequency of 50 MHz ($\lambda_0 = 6$ meters) and polarization $\hat{\phi}$. The $y$ component of the magnetic field is shown in Figure 20. The magnetic field is continuous since it is proportional to the vector potential. The electric field, however, can be discontinuous because it involves the curl of the vector potential.

This example provides an opportunity for comparing first and second order tetrahedral elements in the FEM. A first order mesh with 8 first order elements per side is constructed. This mesh requires 2187 unknowns. The nodal density for the first and second order meshes is about the same.

A first order approximation of the fields will necessarily be poorer than a second order approximation because of the different orders of interpolation across the elements. Therefore, the gauge field $G$ will be compared. The correct solution satisfies the gauge condition $G = 0$. The average of the absolute value of the gauge field $|G|$ along the $z$ axis will be used as the basis of comparison. The smaller $|G|$, the better the FEM solution.

The incident field arrives from the $\theta_i = 30$ degrees, $\phi_i = 15$ degrees direction. The average of the absolute value of the gauge field for both the first and second order meshes is plotted versus frequency. Figure 21 corresponds to a $\hat{\phi}$ polarized
Figure 19: $E_z$ field along the $z$ axis. The region is a cube, 2 meters on a side. The FEM solves for the fields reflected and transmitted from a half space with dielectric constant $\varepsilon_r = 4$. The incident field is $\hat{\theta}$ polarized from the $\theta_i = 30$ degrees, $\phi_i = 15$ degrees direction relative to the normal to the plane. The incident frequency is 50 MHz. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution.
Figure 20: $H_y$ field along the $z$ axis. The region is a cube, 2 meters on a side. The FEM solves for the fields reflected and transmitted from a half space with dielectric constant $\varepsilon_r = 4$. The incident field is $\hat{\theta}$ polarized from the $\theta_i = 30$ degrees, $\phi_i = 15$ degrees direction relative to the normal to the plane. The incident frequency is 50 MHz. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution.
Figure 21: The average of the absolute value of the gauge field along the z axis versus frequency for a second order mesh (solid) and a first order mesh (dashed). The incident field is \( \phi \) polarized.

The incident field and Figure 22 corresponds to a \( \theta \) polarized incident field. These plots demonstrate the better performance of the second order mesh compared to the first order mesh for the same nodal density. Only when the nodal density becomes unreasonably large (such as a frequency larger than 75 MHz) do the two meshes give similar errors.

Notice the spikes around 52 MHz. These are spurious responses (spurs) of the FEM and do not correspond to a natural resonance of the structure. This spur is attributed to the small number of FEM elements used along the normal to the interface. When the the density of nodes along the z direction is doubled for the second order mesh, no spurs are observed for a frequency sweep from 4 MHz to 80 MHz with a 1 MHz step. The spurs are very narrow in frequency. As in the moment method, their appearance acts as an annoyance, not as a limitation, in practical applications.
5.3.2 PEC Sphere

The scattering from a perfectly electrically conducting (PEC) sphere due to a plane wave incident field can be solved exactly in terms of an eigenfunction expansion [40]. When these known fields are applied to the outer boundary of an FEM region with an inner surface which models the PEC sphere, the near fields around the conductor can be derived using the FEM. This FEM solution is examined in this section. The PEC sphere has a one meter radius and the incident field has a free space wavelength of 40 meters, is $\hat{z}$ polarized and propagates along the $\hat{z}$ axis.

The main difficult in analyzing a PEC body is modelling the rapidly varying fields which can surround a PEC. This requires a mesh with a large number of nodes. The mesh is expanded in the spherical shell which separates the PEC sphere from the outer boundary surface. Two meshes are used. The medium size mesh uses two
layers of second order elements and requires 12,366 unknowns. The outer surface is at a radius of \( R_{S0} = 2 \) meters. The larger mesh uses three layers of elements with 30,696 unknowns and has an outer radius of \( R_{S0} = 2.2 \) meters.

The figures plot the three fields \( E_x, H_y \) and \( G \) along the \( z \) axis from the PEC surface outward in the forward scattering direction. The real and imaginary parts of the FEM solution for \( E_x \) and \( H_y \) are compared with the eigenfunction solution. The real and imaginary parts of the gauge field should satisfy the gauge condition \( (G = 0) \). Figures 23, 24 and 25 are for the large mesh and Figures 26, 27 and 28 are for the medium mesh.

The field variations along other contours are comparable to the plots shown here. The gauge field \( G \) for the large mesh in Figure 25 is unusually good along the \( z \) axis. Along the \( y \) axis, for example, the gauge field for this mesh rises from zero at
Figure 24: $H_y$ field along the $z$ axis using the large mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution.

Figure 25: $G$ field along the $z$ axis using the large mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution are shown.
Figure 26: $E_x$ field along the $z$ axis using the medium mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution.

Figure 27: $H_y$ field along the $z$ axis using the medium mesh. The incident field propagates along the $z$ axis and is $\hat{x}$ polarized. The real (solid) and imaginary (dotted) parts of the FEM solution is compared to the real (dashed) and imaginary (short dashed) parts of the exact solution.
the PEC surface to about .3 (V/m) then quickly decreases to zero. This is similar to the behaviour of the gauge field near a dielectric interface.

These plots show that the electric field and the gauge field improve with increasing mesh density. However, the magnetic field deviates from the expected value next to the PEC sphere. To interpret this behaviour, the results should be compared to the functional solution on which they are based. For this constant $\mu$ example, the scalar potential is zero so that Equation (A.15) in Appendix A represents the FEM functional being used for this example.

This functional minimizes the electric and gauge field variations at the boundary surfaces which the numerical results support. It does not explicitly enforce any boundary conditions on the magnetic fields. The magnetic field boundary condition must result from the correct solution of the differential equation. In order to solve the differential equation, the functional attempts to extremize the differences
between the electric and gauge field energies versus the magnetic field energy. The difference between this example and examples without a PEC surface is the large field variations around a PEC. This tends to enhance the importance of the electric and gauge fields, which depend on the derivatives of $\vec{F}$, compared to the magnetic field which is proportional to $\vec{F}$ and is scaled by the frequency squared.

The numerical results show good agreement between the FEM solution and the eigenfunction solution for the electric and gauge fields. However, the magnetic field boundary condition is not satisfied because there is too much error in the numerical solution of the differential equation. Increasing the mesh density has improved the gauge and electric field solutions but has not shown a corresponding improvement in the magnetic field boundary values. Further increases in the mesh density may improve the magnetic field solution at the boundary. Unfortunately, the densities being used now are already beyond the range of practical application.

These problems only occur near the PEC. Away from the conductor, the magnetic field approaches the correct value. As a result, the far field scattering compares favorably with the eigenfunction result. This is demonstrated in Section 5.4.4.

This example shows the weakness of a nodal based FEM when applied to a PEC using a natural (Neumann) boundary condition. The boundary conditions near the PEC might be improved by using an alternative set of boundary conditions on the PEC surface. A set of boundary conditions which explicitly enforces (Dirichlet) zero tangential electric and normal magnetic fields could help to improve the magnetic field behaviour near the conducting surface. This would require forcing parts of the vector potential to be zero on a curved surface and parts to be left free. This can only be done in an average sense and would necessitate making approximations [1].
5.4 MOM/FEM

The numerical results using the coupled MOM and FEM are examined in this section. The coupling uses the iterative method in which the MOM and FEM matrix equations are combined and solved using a scaled biconjugate gradient method. Table 3 lists the mesh information and CPU run times for the examples in this section.

5.4.1 Material Sphere

The bistatic scattering from a material sphere with $\varepsilon_r = 1, \mu_r = 4$ and radius one meter is shown in Figure 29 for parallel polarization and Figure 30 for perpendicular polarization. The free space wavelength is ten meters for this case. The agreement with the eigenfunction solution is acceptable.

The FEM is composed of second order isoparametric tetrahedrons with 7058 unknowns. The boundary requires 654 unknowns. The mesh is composed of two concentric spheres. The inner sphere is the material sphere. The outer spherical surface forms the MOM boundary surface and has a radius of $1\frac{1}{2}$ meters. The half meter shell is composed of free space. A cross section of the geometry is shown in Figure 31.

The gauge condition $(G = 0)$ is part of the solution in this nodal based FEM. The units of the gauge field $G$ were chosen to be the same as the electric field $\vec{E}$, volts per meter. The degree that $G$ deviates from zero gives an indication of the quality of the solution. For this example, the absolute value of the gauge field is approximately $|G| = 0.07 \pm 0.04$ with a unit incident electric field.

The gauge field $G$ is a useful indicator. A moment method solution will give a reasonable result with a poor approximation of the basis field since the integral
Table 3: Mesh parameters and CPU run times for the MOM/FEM examples.

<table>
<thead>
<tr>
<th>Description</th>
<th>FEM (MOM) Mesh</th>
<th>Run Time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>nodes</td>
<td>elements</td>
</tr>
<tr>
<td>Material Sphere, Second Order</td>
<td>1,981</td>
<td>1,080</td>
</tr>
<tr>
<td>Dielectric Slab, Second Order</td>
<td>1,831</td>
<td>1,960</td>
</tr>
<tr>
<td>Cube 7 nodes/side, Second Order</td>
<td>1,981</td>
<td>1,080</td>
</tr>
<tr>
<td>Cube 8 nodes/side, Second Order</td>
<td>3,032</td>
<td>1,715</td>
</tr>
<tr>
<td>Cube 9 nodes/side, Second Order</td>
<td>4,401</td>
<td>2,560</td>
</tr>
<tr>
<td>PEC Sphere, small mesh, Second Order</td>
<td>1,702</td>
<td>760</td>
</tr>
<tr>
<td>PEC Sphere, medium mesh, Second Order</td>
<td>4,122</td>
<td>2,240</td>
</tr>
</tbody>
</table>
Figure 29: Parallel polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and FEM nodal density $\frac{N}{\lambda_0^3} = 145 \times 10^3$, MOM nodal density $\frac{N}{\lambda_0^3} = 783$ with $\lambda_0 = 10$ meters. MOM/FEM solution is in solid. Eigenfunction solution is dashed.
Figure 30: Perpendicular polarization for the bistatic scattering from a material sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and FEM nodal density $N/\lambda_0^3 = 145 \times 10^3$, MOM nodal density $N/\lambda_0^3 = 783$ with $\lambda_0 = 10$ meters. MOM/FEM solution is in solid. Eigenfunction solution is dashed.
Figure 31: Cross section of the material sphere geometry in the $x,z$ plane. The FEM region includes the material sphere with radius one meter and a half meter shell of free space. The moment method fields are expanded on the $S_0$ boundary surface.
equation helps to smooth out any errors. A finite element solution, however, is solving a differential equation which is less forgiving of a poor basis set. When a finite element solution fails, it can be spectacular. By evaluating $G$, the user has an indication of when the code has failed or when the code has succeeded in the absence of an eigenfunction or other reference solution.

The internal fields are calculated for an incident plane wave field propagating along the $z$ axis and $\hat{z}$ polarized in the electric field. The incident magnetic field is $\hat{y}$ directed. Figure 32 shows the electric field $E_z$ along the $z$ axis from one end of the boundary surface to the other. The real and imaginary parts are compared to the eigenfunction solution. Figure 33 shows the magnetic field $H_y$. Figure 34 plots the real and imaginary parts of the gauge field $G$. Notice how the gauge field has the largest error at the interface of the sphere. Some of the roughness of these plots results from the fields being calculated along the boundary between elements. The fields require the derivatives of the potentials.

Figure 35 plots the $H_y$ field along the $y$ axis. The jump in normal component of the magnetic field results from the gradient of the scalar potential. The scalar potential is plotted in figure 36. The interfaces between material regions with different permeability $\mu$ act as sources for the scalar potential. If $\mu$ were constant, the scalar potential would be zero.

This example attempts to use the smallest wavelength for the given one meter sphere and still give useful results. When a wavelength of five meters is used, the average gauge field is $|G| = .15 \pm .05$. The internal field accuracy is comparable to the present case. The far field echo area, however, deviates from the eigenfunction solution by zero decibels at forward scattering to two decibels at backscattering. For the material, isoparametric spheres, an FEM node density around 40 nodes
Figure 32: $E_z$ field along the $z$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{z}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction's real (dashed) and imaginary (short dashed) parts.
Figure 33: $H_y$ field along the $z$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{z}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction's real (dashed) and imaginary (short dashed) parts.
Figure 34: $G$ field along the $z$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{z}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dashed) parts.
Figure 35: $H_y$ field along the $y$ axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{x}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dotted) parts against the eigenfunction’s real (dashed) and imaginary (short dashed) parts.

per wavelength gives acceptable results. This may be different for different meshes, geometries, elements, or material inhomogeneities.

This solution was found using an iterative method on the combined matrix. If the matrix equation is written in the form $AX = Y$, than an error estimate can be defined as, $R = \|AX-Y\|^2/\|Y\|^2$. Figure 37 plots this error estimate versus iteration count for a typical scattering run using the scaled biconjugate gradient method and the biconjugate gradient method without scaling. The oscillatory convergence is one of the “features” of the biconjugate gradient method. The stopping criterion of $R = 10^{-11}$ was used in order to show the behaviour of the error over a large iteration count. A much more modest value of residual error can be used in practice. Because
Figure 36: Scalar potential along the y axis for a sphere with $R = 1$ meter, $\varepsilon_r = 1$, $\mu_r = 4$ and $\lambda_0 = 10$ meters. The material sphere (shaded region) is surrounded by a $\frac{1}{2}$ meter thick shell of free space. The incident field is a plane wave, $\hat{x}$ polarized, electric field propagating along the $z$ axis. The MOM/FEM solution is plotted as the real (solid) and imaginary (dashed) parts.
of the jaggedness of this method, the search continues for a better method for the iterative solution of matrices.

5.4.2 Dielectric Slab

This example is taken from Gilmore [8]. The scatterer is a 2 inch (5.04 cm) by 2 inch (5.04 cm) by \(\frac{3}{4}\) inch (1.905 cm) dielectric slab with relative dielectric constant \(\varepsilon_r = 2.6\). The incident plane wave field has a frequency of 2 GHz (\(\lambda_0 = 15\) cm). The bistatic scattering is plotted in Figure 38 for parallel polarization and Figure 39 for perpendicular polarization along with a similar calculation performed using the “mfb” code written by Gilmore. This code uses a volumetric MOM for the scattering calculations. While the results are within one decibel of each other, the average gauge field at the centroid of the FEM elements is \(|G| = .28 \pm .27\) for the FEM solution. This would not normally indicate a very good solution. Most of the
Figure 38: Bistatic scattering from a dielectric slab with $\varepsilon_r = 2.6$ and dimensions 2 in. by 2 in. by $\frac{3}{4}$ in. at a frequency of 2 GHz for parallel polarization. The incident field is propagating along the $z$ axis. The MOM/FEM (solid) solution is plotted along with Gilmore's mfb (dashed) solution.

error probably comes from the edges and corners of the slab where there is a large field variation. The far field is calculated by integrating over the entire surface of the slab and is therefore less affected by localized errors. A higher node density would improve the internal fields at the cost of a longer run time.

This example provides an opportunity for comparing the MOM/FEM method with the volumetric MOM of Gilmore. Gilmore's matrix requires 898 unknowns with a matrix size of 806,404 matrix elements. The FEM mesh uses second order tetrahedrons with 1,831 nodes for 5,493 unknowns. The FEM nodal density is $125 \times 10^3 \frac{N}{\lambda_0^2}$. The average number of nonzero matrix elements per row for this matrix is 19 to give an approximate size of 104,367 matrix elements. Added onto this, is 226 boundary nodes with 678 unknowns. The surface MOM matrices require 919,368 matrix elements. The MOM/FEM combined matrix requires 1,023,735 matrix elements.
Figure 39: Bistatic scattering from a dielectric slab with $\epsilon_r = 2.6$ and dimensions 2 in. by 2 in. by $\frac{3}{4}$ in. at a frequency of 2 GHz for parallel polarization. The incident field is propagating along the $z$ axis. The MOM/FEM (solid) solution is plotted along with Gilmore's mfb (dashed) solution.

The size of the solution methods is comparable for this example because of the size and shape of the body. As the body increases in size, the memory requirements will grow quadratically for the volumetric MOM, linearly for the FEM and to the $\frac{4}{3}$ power (area squared) for the surface MOM. The MOM/FEM will require fewer resources as the problem grows in size. Unfortunately, the surface MOM is dominating the memory requirements of the combined MOM/FEM method. This is a problem which needs to be addressed as the code develops.

5.4.3 Noise

The spherical body of sections 5.2.3 and 5.2.2 is now analyzed with the interior FEM and exterior surface MOM. The sphere encloses free space and has a radius of one meter.
For an incident wavelength of 5 meters, the free space scattering is \( \approx -40 \) dBsm (decibels relative to a square meter) which gives a dynamic range of 50 dB relative to the scattering from a perfectly conducting sphere. The FEM nodal density is \( 61 \times 10^3 \frac{N}{\lambda_0^3} \) or about 40 nodes per wavelength. The mesh is made out of second order tetrahedrons with linear geometry. Using isoparametric elements, the dynamic range is reduced by 5 dB. This is to be compared with the pure surface MOM result which showed 85 dB of dynamic range.

When the incident wavelength is increased to 10 meters, the dynamic range of the MOM/FEM scattering solution increases to 65 dB. The FEM nodal density is now \( 489 \times 10^3 \frac{N}{\lambda_0^3} \) or about 80 nodes per wavelength. These results and the scattering experiments from material bodies indicate that the interior FEM benefits greatly by using a nodal density far beyond that required by the surface MOM.

5.4.4 PEC Sphere

While the near fields for a PEC body are difficult to obtain with this nodal based FEM, the far fields involve an averaging over the MOM surface and are more forgiving of the near field errors. The scattering from a PEC sphere with a radius of one meter is examined in this section for two different mesh densities. A small mesh with 5,106 FEM unknowns and a medium one which requires 12,366 FEM unknowns. The surface MOM requires 654 and 1,158 unknowns respectively. The incident field is a plane wave propagating along the \( z \) axis.

Figures 40 and 41 compare the scattered field calculated using the small mesh with the eigenfunction solution for both polarizations. The maximum error is about 2 dB with a 10 degree error in the null for the parallel polarization echo area.

Figures 42 and 43 compare the scattered field calculated using the medium mesh with the eigenfunction solution. The maximum error is less than a decibel and the
Figure 40: Parallel polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter and \( \lambda_0 = 40 \) meters. The incident field propagates along the \( z \) axis. The MOM/FEM is solved using the small mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed.

Figure 41: Perpendicular polarization for the bistatic scattering from a PEC sphere with \( R = 1 \) meter \( \lambda_0 = 40 \) meters. The incident field propagates along the \( z \) axis. The MOM/FEM is solved using the small mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed.
null in the parallel polarization echo area is in the correct spot. The near fields for the scattering from the PEC sphere are comparable with the results in Section 5.3.2. The far field scattering converges much faster than the near fields.

5.4.5 Convergence

Section 5.2.4 presented the bistatic scattering from a material cube using the MOM alone. The results for four different mesh densities were compared for convergence. This section analyzes this same body and incident field using three of these meshes with the hybrid MOM/FEM method. The results are shown in Figures 44, 45 and 46 for meshes with 7, 8 and 9 nodes per side respectively. The convergence with the hybrid method is comparable to the use of the MOM by itself.
Figure 43: Perpendicular polarization for the bistatic scattering from a PEC sphere with $R = 1$ meter $\lambda_0 = 40$ meters. The incident field propagates along the $z$ axis. The MOM/FEM is solved using the medium mesh. MOM/FEM solution is in solid. Eigenfunction solution is dashed.
Figure 44: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM/FEM method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90^\circ, \phi_i = 0$ degrees. The two copolarized echo areas $\theta\theta$ (solid) and $\phi\phi$ (dotted) and the two cross polarized echo areas $\phi\theta$ (dashed) and $\theta\phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 7 nodes along the side of the cube.
Figure 45: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2, \mu_r = (1.5, -0.5)$. The solution uses the MOM/FEM method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90, \phi_i = 0$ degrees. The two copolarized echo areas $\theta\theta$ (solid) and $\phi\phi$ (dotted) and the two cross polarized echo areas $\phi\theta$ (dashed) and $\theta\phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 8 nodes along the side of the cube.
Figure 46: Bistatic scattering in the $\phi = 0$ plane from a material cube, one meter on a side, with parameters $\varepsilon_r = 2$, $\mu_r = (1.5, -0.5)$. The solution uses the MOM/FEM method. The incident field has a wavelength of $\lambda_0 = 4$ meters with incident angles $\theta_i = 90, \phi_i = 0$ degrees. The two copolarized echo areas $\theta \theta$ (solid) and $\phi \phi$ (dotted) and the two cross polarized echo areas $\phi \theta$ (dashed) and $\theta \phi$ (short dashed) are plotted or are of negligible magnitude. This mesh uses 9 nodes along the side of the cube.
CHAPTER VI

Conclusions

The vector finite element method is very efficient with respect to memory storage requirements compared to a volumetric method of moments. Even though the FEM requires a higher density of unknowns than a corresponding volumetric method of moments, the storage increases only linearly with the size of the volume instead of quadratically. A problem with thousands of unknowns is considered common and can be run on a small computer. The FEM, however, must be coupled to the far field for a scattering analysis. The method presented in this report uses a subsectional surface method of moments. The storage requirements scale as the volume to the \( \frac{4}{3} \) power. The MOM ultimately determines the maximum size of a body which can be solved.

Ideally, the storage requirements would be equally divided between the MOM and the FEM. Part of the problem comes from the different scales of application of the MOM versus the FEM for the same level of accuracy. The FEM requires a much higher node density than the MOM. The use of second order FEM elements with first order MOM elements was an attempt to alleviate this discrepancy. This was not enough. The use of a third or fourth order FEM element would help to increase the FEM node density for a given MOM node density and MOM matrix size. A more desirable alternative would be to have the capability of independently scaling the densities of unknowns for the FEM and the MOM. The use of an entire domain basis function such as spherical wave functions instead of the subsectional basis functions

82
would definitely reduce the MOM matrix size and give some user control over the level of MOM accuracy required. This would be a worthwhile experiment to make with this code.

The FEM was presented in terms of the scalar and vector potentials. The potentials effectively account for the field discontinuities at interfaces between dielectric regions. A modest amount of additional storage is required with the addition of a fourth unknown per node. The one disadvantage of using potentials is the need to calculate derivatives in order to recover the field quantities. This has been addressed in [2]. Fortunately, the method of coupling to the surface MOM does not require derivatives of the potentials. This helps the far field scattering results.

The performance of this code with perfect conductors is disappointing. The Neumann boundary conditions used for the vector potential on the surface of the PEC was sufficient to enforce the electric and gauge field boundary conditions but not the magnetic field boundary condition. As indicated in Chapter V, the solution may be improved by using a Dirichlet type of boundary condition for the tangential components of the vector potential at the surface of the PEC. This was initially avoided because it can only be enforced in an approximate sense. However, this approximation may prove to be less severe then the error introduced by the Neumann boundary conditions.

The gauge field has proven to be a valuable indicator of the quality of the finite element method. This is especially true when working with meshes which require a very high node density and the correct node density is not known a priori. The gauge field can also provide an indication of problems with spurious resonances. Spurious resonances have only been seen at low FEM node densities and on pathological meshes (with pancake elements).
A weakness of this computer code is the iterative solution method. The iterations over the combined matrix require the majority of the computer run time. While the biconjugate gradient method works, the oscillations in the residual error can vary by over four orders of magnitude. This makes it difficult to set a good stopping point for the iterations. Other iterative methods will be tried in the future.

One possible extension of this code may prove interesting. Currently, the interior region is solved using potentials. The potentials are then converted to fields at the surface for the exterior surface MOM. Another possibility is to eschew the use of fields entirely and solve the exterior region in terms of potentials. Instead of three unknown fields at each boundary node, there would be four potentials. This seems like a disadvantage. However, the potentials separate into four identical scalar wave equations in free space under the Lorentz gauge. The two exterior vector matrices could then be reduced to two scalar matrices. The scalar matrices would relate the boundary potentials to their normal derivatives and the incident potential field. The reduction in storage would amount to a factor of nine from the current configuration. The functional equation in Appendix C was derived with this extension in mind.
APPENDIX A

The Stationary Functional

The finite element equations can be derived through a functional formulation [18] or by minimizing the residuals of the differential equation [22]. The former technique is used here. An excellent discussion of functionals of Maxwell's equations can be found in Berk [28]. Berk uses the energy density as the integrand for the functional to be minimized. Accompanying Berk's discussion is a letter by Rumsey which describes an equivalent derivation using the reaction principle.

The functional from Chapter II, written as a function of the potentials alone, is given by,

\[
\mathcal{F} [F_0, \bar{F}] = \frac{1}{2\omega} \int_V \frac{1}{\omega \varepsilon} (\nabla \times \bar{F})^2 + \omega \mu (j \bar{F} + \nabla F_0)^2 \\
+ \frac{1}{\omega \varepsilon} (\nabla \cdot \bar{F} + j \alpha k^2 F_0)^2 \, dv + S [F_0, \bar{F}]. \tag{A.1}
\]

The surface functional \( S \) is,

\[
S [F_0, \bar{F}] = \frac{1}{\omega} \int_{S_0} - (\hat{n} \times \bar{E}^B) \cdot \bar{F} + G^B \hat{n} \cdot \bar{F} + (\omega \mu \bar{H}^B \cdot \hat{n}) F_0 \, ds \tag{A.2}
\]

where \( \bar{H}^B, \bar{E}^B, G^B \) are the respective field values imposed on the boundary \( S_0 \).

The Euler equations are found by considering a small perturbation in the potentials,

\[
\bar{F} \rightarrow \bar{F} + \eta \bar{u} \\
F_0 \rightarrow F_0 + \eta w_0 \tag{A.3}
\]
where \( w_0, \bar{w} \) are arbitrary functions and \( \eta \) is a small factor. Substituting Equations (A.3) into (A.1) and retaining terms which are first order gives the variation of the functional \( \mathcal{F} \),

\[
\delta \mathcal{F} = \frac{\eta}{\omega} \int_V \frac{1}{\omega} \left[ \left( \nabla \times \vec{F} \right) \cdot \left( \nabla \times \bar{w} \right) \right] + \omega \mu \left[ -\vec{F} \cdot \bar{w} + j \left( \vec{F} \cdot \nabla w_0 + \bar{w} \cdot \nabla F_0 \right) + \nabla w_0 \cdot \nabla F_0 \right] + \frac{1}{\omega} \left[ (\nabla \cdot \vec{F}) (\nabla \cdot \bar{w}) + j \alpha k^2 \left( (\nabla \cdot \vec{F}) w_0 + (\nabla \cdot \bar{w}) F_0 \right) \right] - \alpha^2 k^4 F_0 w_0 \right) \, dv + \delta S. \tag{A.4}
\]

The following identities are needed,

\[
\frac{1}{\omega} \left( \nabla \times \vec{F} \right) \cdot \left( \nabla \times \bar{w} \right) = \nabla \cdot \left[ \bar{w} \times \frac{1}{\omega} \left( \nabla \times \vec{F} \right) \right] + \bar{w} \cdot \nabla \times \left[ \frac{1}{\omega} \left( \nabla \times \vec{F} \right) \right] \tag{A.5}
\]

\[
\omega \mu \left[ j \vec{F} + \nabla F_0 \right] \cdot \nabla w_0 = \nabla \cdot \left[ \omega \mu w_0 \left( j \vec{F} + \nabla F_0 \right) \right] - w_0 \nabla \cdot \left[ j \omega \mu \vec{F} + \omega \mu \nabla F_0 \right] \tag{A.6}
\]

\[
\frac{1}{\omega} \left( \nabla \cdot \vec{F} \right) (\nabla \cdot \bar{w}) + \frac{j \alpha k^2}{\omega} F_0 (\nabla \cdot \bar{w}) = \nabla \cdot \left[ \frac{1}{\omega} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \bar{w} \right] - \bar{w} \cdot \nabla \left[ \frac{1}{\omega} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right] \tag{A.7}
\]

Substituting these equations into Equation (A.4) and using Stoke’s theorem on the divergence terms gives,

\[
\delta \mathcal{F} = \frac{\eta}{\omega} \int_V \bar{w} \cdot \nabla \times \left[ \frac{1}{\omega} \left( \nabla \times \vec{F} \right) \right] + \omega \mu \left[ -\vec{F} \cdot \bar{w} + j \bar{w} \cdot \nabla F_0 \right] - w_0 \nabla \cdot \left[ j \omega \mu \vec{F} + \omega \mu \nabla F_0 \right] - \bar{w} \cdot \nabla \left[ \frac{1}{\omega} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right] + \frac{j \alpha k^2}{\omega \mu} w_0 \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \, dv + \frac{1}{\omega} \eta \int_S \left[ \bar{w} \times \frac{1}{\omega} \left( \nabla \times \vec{F} \right) \right] \cdot \hat{n} + \omega \mu w_0 \left( j \vec{F} + \nabla F_0 \right) \cdot \hat{n} + \frac{1}{\omega} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \bar{w} \cdot \hat{n} ds + \delta S. \tag{A.8}
\]
When the first variation of the surface functional,

$$\delta S = \frac{\eta}{\omega} \int_S - (\hat{n} \times \vec{E}^B) \cdot \vec{w} + G^B \hat{n} \cdot \vec{w} + (\omega \mu \vec{H}^B \cdot \hat{n}) w_0 ds$$  \hspace{1cm} (A.9)$$

is substituted into this equation and terms are rearranged, the first variation of the total functional becomes,

$$\delta \mathcal{F} = \frac{\eta}{\omega} \int_V \vec{w} \cdot \left\{ \nabla \times \left( \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right) - \omega \mu \vec{F} + j \omega \mu \nabla F_0 \right\}$$

$$- \nabla \left[ \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right]$$

$$+ w_0 \left\{ \omega \nabla \cdot (-j \mu \vec{F} - \mu \nabla F_0) + j \alpha k^2 \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right\} dV$$

$$+ \frac{\eta}{\omega} \int_S (\hat{n} \times \vec{w}) \cdot \left[ \vec{E}^B + \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right] + \left[ G^B + \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right] (\hat{n} \cdot \vec{w})$$

$$+ \left[ \omega \mu \vec{H}^B - \omega \mu (-j \vec{F} - \nabla F_0) \right] \cdot \hat{n} w_0 ds.$$  \hspace{1cm} (A.10)$$

A stationary functional implies $\delta \mathcal{F} = 0$ for arbitrary testing functions $\vec{w}, w_0$. This requires the terms of Equation (A.10) in $\{ \}$ be zero,

$$\nabla \times \left( \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right) - \omega \mu \vec{F} + j \omega \mu \nabla F_0 - \nabla \left[ \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) \right] = 0$$  \hspace{1cm} (A.11)$$

$$\omega \nabla \cdot (-j \mu \vec{F} - \mu \nabla F_0) + j \alpha k^2 \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right) = 0.$$  \hspace{1cm} (A.12)$$

These are the volumetric Euler equations and the partial differential equations derived in Chapter II.

As shown in Chapter II, $F_0 = 0$ on the boundary so that $w_0 = 0$ on $S$ also. The last term in Equation (A.10) therefore goes to zero. With an arbitrary variation of $\vec{w}$ on the boundary, Equation (A.10) implies the boundary fields satisfy the equations

$$G^B = \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} + j \alpha k^2 F_0 \right)$$  \hspace{1cm} (A.13)$$

$$\vec{E}^B_{tan} = \left[ -\frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right]_{tan}$$  \hspace{1cm} (A.14)$$

which are the required Neumann boundary conditions.
When \( \mu \) is constant, \( F_0 = 0 \) and \( w_0 = 0 \). The functional in Equation (A.1) reduces to

\[
\mathcal{F} \left[ \vec{F} \right] = \frac{1}{2\omega} \int_V \frac{1}{\omega \varepsilon} \left( \nabla \times \vec{F} \right)^2 - \omega \mu \vec{F}^2 + \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} \right)^2 \, dv \\
+ \mathcal{S} \left[ \vec{F} \right].
\] (A.15)

with the surface functional \( \mathcal{S} \),

\[
\mathcal{S} \left[ \vec{F} \right] = \frac{1}{\omega} \int_S - \left( \vec{n} \times \vec{E}^B \right) \cdot \vec{F} + G^B \vec{n} \cdot \vec{F} \, ds.
\] (A.16)

A small perturbation of the vector potential \( \left( \vec{F} + \eta \vec{w} \right) \) yields the first variation of the total functional,

\[
\delta \mathcal{F} = \frac{\eta}{\omega} \int_V \vec{w} \cdot \left\{ \nabla \times \left( \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right) - \omega \mu \vec{F} - \nabla \left[ \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} \right) \right] \} \, dv \\
+ \frac{\eta}{\omega} \int_S \left( \vec{n} \times \vec{w} \right) \cdot \left[ \vec{E}^B + \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right] \\
+ \left[ G^B + \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} \right) \right] \left( \vec{n} \cdot \vec{w} \right) \, ds. 
\] (A.17)

The volumetric Euler equation results from applying the stationary functional principle \( \delta \mathcal{F} = 0 \) for arbitrary testing functions \( \vec{w} \),

\[
\nabla \times \left( \frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right) - \omega \mu \vec{F} - \nabla \left[ \frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} \right) \right] = 0. 
\] (A.18)

This is the partial differential equation described in Chapter II for the special case of a constant \( \mu \). As before, the boundary fields must satisfy the Neumann boundary conditions,

\[
G^B = -\frac{1}{\omega \varepsilon} \left( \nabla \cdot \vec{F} \right) 
\] (A.19)

\[
\vec{E}^B_{tan} = \left[ -\frac{1}{\omega \varepsilon} \nabla \times \vec{F} \right]_{tan} 
\] (A.20)

since \( \vec{w} \) has an arbitrary variation on the boundary.
APPENDIX B

Element Shape Functions

This appendix describes the geometry and shape functions associated with tetrahedral volume elements and triangular boundary elements. The shape functions are local functions used to expand the fields in each element. They are described in terms of a simplex coordinate system. A thorough discussion of simplex coordinates and shape functions as used here may be found in [38].

B.1 Triangular Elements

A triangular element $T^B$ consists of three vertices located at coordinates $\vec{r}_i$. The side numbered $i$ is defined to be the line segment opposite vertex $i$. The area $A$ is given by,

\[ A = \frac{1}{2} |(\vec{r}_2 - \vec{r}_1) \times (\vec{r}_3 - \vec{r}_1)|. \]  

(B.1)

The three simplex coordinates $\zeta_i$ ($i=1,2,3$) are defined for each point $\vec{r}$ in the plane of the triangle and within the triangle's boundary by considering the three subtriangles $T^B_i$ formed by the vertices $\vec{r}_i$ and $\vec{r}$ shown in Figure 47. With $A_i$ equal to the area of subtriangle $i$, the simplex coordinates are defined as,

\[ \zeta_i = A_i/A. \]  

(B.2)

Coordinate $\zeta_i$ has the property of being one at vertex $i$ and zero at side $i$. This equation represents a transformation to the local simplex coordinates from the Cartesian
coordinate system. They satisfy the equation,

\[ 1 = \zeta_1 + \zeta_2 + \zeta_3 \]  \hspace{1cm} (B.3)

and so are not independent.

The triangular elements are used to expand fields on surfaces. For a first order (linear) approximation, the nodes correspond to the vertices as shown in Figure 48 and the shape functions \( U_j^B \) are defined as,

\[ U_1^B = \zeta_1 \]
\[ U_2^B = \zeta_2 \]
\[ U_3^B = \zeta_3. \]  \hspace{1cm} (B.4)

For a second order (quadratic) approximation, the six nodes shown in Figure 49 are required with shape functions,

\[ U_1^B = \zeta_1 (2\zeta_1 - 1) \]

90
Figure 48: Local nodes corresponding to a first order triangular boundary element. The vertex numbers are enclosed in boxes.

Figure 49: Local nodes corresponding to a second order triangular boundary element. The vertex numbers are enclosed in boxes.
\[
U_2^B = 4\zeta_1\zeta_2 \\
U_3^B = 4\zeta_1\zeta_3 \\
U_4^B = \zeta_2(2\zeta_2 - 1) \\
U_5^B = 4\zeta_2\zeta_3 \\
U_6^B = \zeta_3(2\zeta_3 - 1).
\] (B.5)

The shape function \(U_j^B\) has the property of being one at local node \(j\) and zero at every other node.

The expansion functions \(u_k^B\), corresponding to global boundary node \(k\) and used to expand an arbitrary boundary scalar field, is defined in each triangular element \(e\) by the local shape function \(U_j^B\). Global node \(k\) corresponds to local node \(j\) in element \(e\). This is usually defined in the program by establishing a connection array \(\Phi^B\) such that

\[ k = \Phi^B(j, e). \] (B.6)

### B.2 Tetrahedral Elements

A tetrahedral element \(T\) consists of four vertices located at coordinates \(\vec{r}_i\) \((i = 1, 2, 3, 4)\) as in Figure 50. The side or face labeled \(i\) is defined to be the face opposite vertex \(i\). The volume \(V\) of the tetrahedron is given by,

\[ V = \frac{1}{6} (\vec{r}_1 - \vec{r}_2) \cdot [(\vec{r}_2 - \vec{r}_3) \times (\vec{r}_4 - \vec{r}_3)]. \] (B.7)

The four simplex coordinates are defined for each point in the tetrahedron by evaluating the volume \(V_i\) of the four subtetrahedrons formed by the vertices \(\vec{r}_i\) and \(\vec{r}\). The simplex coordinates are given by,

\[ \zeta_i = V_i/V \] (B.8)
and they satisfy the equation,

$$1 = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4.$$  

(A.9)

A first order approximation of the fields inside a tetrahedral element requires a node at each vertex as shown in Figure 51. The shape functions for a first order approximation of the fields are,

$$U_i^B = \zeta_i$$

(A.10)

A second order approximation of the fields requires ten local nodes defined in Figure 52. The shape functions for a second order approximation are,

$$U_i^B = \zeta_1 (2\zeta_1 - 1)$$
Figure 51: Local nodes corresponding to a first order tetrahedral element. The vertex numbers are enclosed in boxes.

Figure 52: Local nodes corresponding to a second order tetrahedral element. The vertex numbers are enclosed in boxes.
\begin{align*}
U_2^B &= 4\zeta_1\zeta_2 \\
U_3^B &= 4\zeta_1\zeta_3 \\
U_4^B &= 4\zeta_1\zeta_4 \\
U_6^B &= \zeta_2(2\zeta_2 - 1) \\
U_6^B &= 4\zeta_2\zeta_3 \\
U_7^B &= 4\zeta_2\zeta_4 \\
U_8^B &= \zeta_3(2\zeta_3 - 1) \\
U_9^B &= 4\zeta_3\zeta_4 \\
U_{10}^B &= \zeta_4(2\zeta_4 - 1).
\end{align*}
(B.11)

By using the simplex coordinates, the definition of the shape functions becomes independent of the geometry of the element. The geometry enters through the definition of the simplex coordinates.

So far, the geometry has been linear as defined by Equation (B.8). A curved geometry may also be represented by using the quadratic shape functions \( U_j \) and the coordinates \( \bar{x}_j \) of the ten local nodes,

\[ \bar{r} = \sum_{j=1}^{10} \bar{x}_j U_j^B. \]  
(B.12)

This is shown in Figure 53. Using the property of the shape functions, \( \bar{r} = \bar{x}_j \) when the simplex coordinates \( \zeta_i \) correspond to local node \( j \). When the local nodes are chosen as the vertices and edge centers of a linear tetrahedron, Equations (B.12) and (B.8) represent identical coordinate transformations. Use of Equation (B.12) as the coordinate transformation results in an isoparametric element with curved surfaces passing through the local nodes. While the transformation \( \bar{r}(\zeta_i) \) is quadratic in \( \zeta_i \), the inverse transformation \( \zeta_i(\bar{r}) \) is not quadratic. Therefore, if the six nodes on face \( s \) of a tetrahedron lie on the surface of a sphere, the transformation (B.12)
Figure 53: An example of a very warped isoparametric element.

will not give a surface coincident with the sphere since that surface of the element corresponds to \( \zeta_s (\vec{r}) = 0 \). Nevertheless, an isoparametric tetrahedral element will more accurately represent a curved geometry than a tetrahedral element with linear geometry.

As in the triangular case, the expansion functions \( u_n \), corresponding to global node \( n \) and used to expand an arbitrary interior scalar field, is defined in each tetrahedral element \( e \) by the local shape function \( U_j \). Global node \( n \) corresponds to local node \( j \) in element \( e \). This is defined in the program by establishing a connection array \( \Phi \) such that

\[
n = \Phi (j,e).
\]  

(B.13)

The connection array relating the boundary nodes to the interior nodes is described in Chapter IV.
C.1 Matrix Equation

The FEM matrix equation from Chapter II is given by,

\[ e^\nu \left[ e^B \right] + g^\nu \left[ G^B \right] + h^\nu \left[ H^B \right] = M^\nu_{\mu} f^\mu \]  

(C.1)

where all external boundary terms have been retained. These terms can be written in the form,

\[ e^\nu \left[ e^B \right] = \left( \frac{k_0}{\eta_0} \right) \int_S (\hat{n} \times \vec{E}^B) \cdot \vec{w} ds \]

\[ g^\nu \left[ G^B \right] = -\left( \frac{k_0}{\eta_0} \right) \int_S G^B (\hat{n} \cdot \vec{w}) ds \]

\[ h^\nu \left[ H^B \right] = -\left( \frac{k_0}{\eta_0} \right) \int_S (\hat{n} \cdot \omega \mu H^B) w_0 ds \]

\[ M^\nu_{\mu} = \int_V \frac{1}{\epsilon_r} \left\{ (\nabla \times \vec{w}) \cdot (\nabla \times \vec{F}) + (\nabla \cdot \vec{w}) (\nabla \cdot \vec{F}) - k^2 \vec{w} \cdot \vec{F} \right\} \]

\[ + \frac{jk^2}{\epsilon_r} \left\{ (\vec{F} \cdot \nabla w_0 + \alpha w_0 (\nabla \cdot \vec{F})) + (\vec{w} \cdot \nabla F_0 + \alpha F_0 (\nabla \cdot \vec{w})) \right\} \]

\[ + \frac{k^2}{\epsilon_r} \left\{ \nabla F_0 \cdot \nabla w_0 - \alpha^2 k^2 F_0 w_0 \right\} dv \]  

(C.2)

with \( \mu = (l,n) \), \( \nu = (p,m) \) and

\[ \vec{F} = u_n \hat{x}_l \ (l = x,y,z) \]

\[ F_0 = u_n \ (l = 0) \]  

(C.3)

\[ \vec{w} = u_m \hat{x}_p \ (p = x,y,z) \]

\[ w_0 = u_m \ (p = 0). \]  

(C.4)
The free space impedance $\eta_0$ is defined as,

$$\eta_0 = \sqrt{\frac{\mu}{\epsilon}}$$

and $\epsilon_r = \frac{\epsilon}{\epsilon_0}, \mu_r = \frac{\mu}{\mu_0}$ are the relative dielectric parameters. The test functions are represented by $w_0, \bar{w}$ with index $\nu$ and the expansion functions are represented by $F_0, \bar{F}$ with index $\mu$.

The dielectric parameters $\epsilon, \mu$ are assumed to be constant in each tetrahedral element. The expansion functions $\phi_n$ take on the values of the shape functions described in Appendix B in terms of each element's simplex coordinates. The integral is thus most easily performed by integrating over each element separately in terms of the simplex coordinates and collecting the results to form the matrix equation. The integral over simplex coordinates is performed numerically using a standard Gaussian quadrature method. The surface and volume integrals will be treated separately.

### C.2 Volume Integrations

Consider a single tetrahedral element $e$ with simplex coordinates $\zeta_i$. Also consider a single test node $m$ and expansion node $n$. The volume integral over element $e$, represented by the term $R_{pl}^{mn}$, corresponds to the coupling between the four components of the potentials at node $m$ and the four components at node $n$. $R_{pl}^{mn}$, for fixed $m$ and $n$, forms the four by four submatrix shown in Figure 54. Once calculated, this matrix is added into the FEM matrix $M_\mu'$. This is done for each pair of local nodes in element $e$ and each tetrahedral element of the mesh. Four cases will be considered in calculating $R_{pl}^{mn}$ for the four combinations between $F_0, \bar{F}$ and $w_0, \bar{w}$. 


98
Figure 54: The submatrix $\mathcal{R}_{pl}$ gives the FEM coupling between the potentials $w_0$, $\bar{w}$ at node $m$ and the potentials $F_0, \bar{F}$ at node $n$ in a single element.

The following identities will be needed using Equations (C.3) and (C.4),

\[
\begin{align*}
\nabla \times \bar{F} & = \nabla u_n \times \hat{x}_l \\
\nabla \cdot \bar{F} & = \nabla u_n \cdot \hat{x}_l \\
\nabla F_0 & = \nabla u_n \\
\nabla \times \bar{w} & = \nabla u_m \times \hat{x}_p \\
\nabla \cdot \bar{w} & = \nabla u_m \cdot \hat{x}_p \\
\n\nabla w_0 & = \nabla u_m.
\end{align*}
\]

(C.6)

The Jacobian matrix of the transformation between simplex and Cartesian coordinates is represented by,

\[
J_{ki} = \frac{\partial x_k}{\partial \zeta_i}
\]

(C.7)

with determinant $J$. Only the independent set $\zeta_1, \zeta_2, \zeta_3$ of simplex coordinates is used in this equation. Note that the inverse transformation gives the inverse Jacobian matrix,

\[
J^{-1}_{ik} = \frac{\partial \zeta_i}{\partial x_k} = \nabla \zeta_i.
\]

(C.8)
The dependency between the simplex coordinates gives the relation,

\[ \nabla \zeta_4 = -\nabla \zeta_1 - \nabla \zeta_2 - \nabla \zeta_3. \]  

(C.9)

The gradients of the scalar functions \( u_m, u_n \) can be written in the form,

\[ \nabla u_m = \nabla \zeta_i \frac{\partial u_m}{\partial \zeta_i} \]

\[ \nabla u_n = \nabla \zeta_j \frac{\partial u_n}{\partial \zeta_j} \]  

(C.10)

where an implied summation \((i, j = 1, 2, 3, 4)\) has been assumed and \( \zeta_4 \) is considered an independent variable in the terms \( \frac{\partial u_m}{\partial \zeta_i} \) and \( \frac{\partial u_n}{\partial \zeta_j} \).

C.2.1 Vector - Vector

For the coupling between \( \bar{w} \) and \( \bar{F} \), the volume integration reduces to,

\[ R_{pl}^{mn} = \frac{1}{\epsilon_r} \int_{V_e} \left\{ (\nabla \times \bar{w}) \cdot (\nabla \times \bar{F}) + (\nabla \cdot \bar{w}) (\nabla \cdot \bar{F}) - k^2 \bar{w} \cdot \bar{F} \right\} dv \]  

(C.11)

for \((p = x, y, z)\) and \((l = x, y, z)\). Substituting Equations (C.6) into (C.11) and simplifying gives,

\[ R_{pl}^{mn} = \frac{1}{\epsilon_r} \int_{\zeta_i} \left\{ \delta_{pl} \left[ (\nabla u_m \cdot \nabla u_n) - k^2 u_m u_n \right] 
+ \left[ (\hat{x}_p \cdot \nabla u_m) (\hat{x}_l \cdot \nabla u_n) - (\hat{x}_l \cdot \nabla u_m) (\hat{x}_p \cdot \nabla u_n) \right] \right\} J d\zeta_i \]  

(C.12)

where \( \delta_{pl} = 0 \) for \( p \neq l \) and \( \delta_{ll} = 1 \). This equation has four independent components.

The first \( [ ] \) contains the diagonal terms and the second \( [ ] \) contains the off diagonal terms of the \( R_{pl}^{mn} \) submatrix.

For a tetrahedron with linear geometry, the Jacobian terms are constants giving the simplification,

\[ R_{pl}^{mn} = \frac{J}{\epsilon_r} \left\{ \delta_{pl} T^{ij} + (\hat{x}_p \cdot \mathbf{K}^{ij} \cdot \hat{x}_l + \hat{x}_l \cdot \mathbf{K}^{ij} \cdot \hat{x}_p) \right\} Q_{ij}^{mn} 
- \left( \frac{J k^2}{\epsilon_r} \right) \delta_{pl} T^{mn} \]  

(C.13)
\[ Q_{ij}^{mn} = \int_{\zeta_i} \frac{\partial u_m}{\partial \zeta_i} \frac{\partial u_n}{\partial \zeta_j} d \zeta_i \]  
(C.14)
\[ T_{ij}^{mn} = \int_{\zeta_i} u_m u_n d \zeta_i \]  
(C.15)
\[ \mathcal{K}^{ij} = \nabla \zeta_i \nabla \zeta_j \]  
(C.16)
\[ T^{ij} = \nabla \zeta_i \cdot \nabla \zeta_j \]  
(C.17)

All the geometry dependent terms are contained in the dyad \( \mathcal{K}^{ij} \) and the scalar \( T^{ij} \). The integrals \( Q_{ij}^{mn}, T_{ij}^{mn} \) can be done once for all elements since the same shape functions are used for all the elements of a given order.

### C.2.2 Scalar - Vector

For the coupling between \( u_0 \) and \( \vec{F} \), the volume integration reduces to,

\[ \mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} \int_{V_0} \{ \nabla u_0 \cdot \vec{F} + \alpha w_0 (\nabla \cdot \vec{F}) \} dV \]  
(C.18)

for \( (p = 0) \) and \( (l = x, y, z) \). Substituting Equations (C.6) into (C.18) and simplifying gives,

\[ \mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} \int_{\zeta_i} \{ (\hat{x}_l \cdot \nabla u_m) u_n + \alpha (\hat{x}_l \cdot \nabla u_n) u_m \} J d \zeta_i. \]  
(C.19)

For a tetrahedron with linear geometry,

\[ \mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} \int (\hat{x}_l \cdot \nabla \zeta_i) S_i^{mn} + \alpha (\hat{x}_l \cdot \nabla \zeta_j) S_j^{nm} \]  
(C.20)
\[ S_i^{mn} = \int_{\zeta_i} \frac{\partial u_m}{\partial \zeta_i} u_n d \zeta_i \]  
(C.21)

where \( S_i^{mn} \) is geometry independent.
C.2.3 Vector - Scalar

For the coupling between $\bar{w}$ and $F_0$, the volume integration reduces to,
\[
\mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} \int_{V_e} \left\{ \bar{w} \cdot \nabla F_0 + \alpha \left( \nabla \cdot \bar{w} \right) F_0 \right\} dv \quad \text{(C.22)}
\]
for $(p = x, y, z)$ and $(l = 0)$. Substituting Equations (C.6) into (C.22) and simplifying gives,
\[
\mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} \int_{\zeta_i} \left\{ \left( \hat{x}_p \cdot \nabla u_n \right) u_m + \alpha \left( \hat{x}_p \cdot \nabla u_m \right) u_n \right\} J d\zeta_i. \quad \text{(C.23)}
\]
This equation could also be found by exploiting the FEM matrix symmetry and Equation (C.18). For a tetrahedron with linear geometry,
\[
\mathcal{R}_{pl}^{mn} = \frac{j k^2}{\varepsilon_r} J \left\{ \left( \hat{x}_p \cdot \nabla \zeta_i \right) s_j^{mn} + \alpha \left( \hat{x}_p \cdot \nabla \zeta_i \right) s_{i}^{mn} \right\} \quad \text{(C.24)}
\]
with $s_{i}^{mn}$ defined in Equation (C.21).

C.2.4 Scalar - Scalar

For the coupling between $w_0$ and $F_0$, the volume integration reduces to,
\[
\mathcal{R}_{pl}^{mn} = \frac{k^2}{\varepsilon_r} \int_{V_e} \left\{ \nabla w_0 \cdot \nabla F_0 - \alpha^2 k^2 w_0 F_0 \right\} dv \quad \text{(C.25)}
\]
for $(p = 0)$ and $(l = 0)$. Substituting Equations (C.6) into (C.25) and simplifying gives,
\[
\mathcal{R}_{pl}^{mn} = \frac{k^2}{\varepsilon_r} \int_{\zeta_i} \left\{ \nabla u_m \cdot \nabla u_n - \alpha^2 k^2 u_m u_n \right\} J d\zeta_i. \quad \text{(C.26)}
\]
For a tetrahedron with linear geometry,
\[
\mathcal{R}_{pl}^{mn} = \frac{k^2}{\varepsilon_r} J \left\{ \tau_{ij} Q_{ij}^{mn} - \alpha^2 k^2 T_{mn} \right\} \quad \text{(C.27)}
\]
using the definitions in Equations (C.14), (C.15) and (C.17).
C.3 Surface Integrations

The surface integrations assume the boundary fields $\vec{E}^B$, $G^B$, and $\vec{H}^B$ are known and can take on any value on the surface $S$. For the present application, the boundary fields will be expanded in terms of the boundary functions $u_k^B$ defined in Appendix B,

$$
\begin{align*}
\vec{E}^B &= \sum_q \hat{x}_l u_k^B E^B q \\
G^B &= \sum_k u_k^B G^B k \\
\omega \mu \vec{H}^B &= \sum_q \hat{x}_l u_k^B \Delta^B q
\end{align*}
$$

(C.28)

where $q = (l, k)$, $l = x, y, z$ and $k$ is a node on the boundary. Notice that $\hat{n} \cdot \omega \mu \vec{H}^B$ is continuous across the boundary surface $S_0$ unlike $\hat{n} \cdot \vec{H}^B$. The sum is taken over the $N_B$ boundary nodes. $E^B q$, $G^B k$ and $\Delta^B q$ are the respective field values at node $k$. Substituting Equations (C.28) into Equations (C.2) gives the known part of the FEM matrix equation,

$$
\begin{align*}
e^\nu \begin{bmatrix} \vec{E}^B \end{bmatrix} &= e_q^\nu E^B q \\
g^\nu \begin{bmatrix} G^B \end{bmatrix} &= g_k^\nu G^B k \\
h^\nu \begin{bmatrix} \vec{H}^B \end{bmatrix} &= h_q^\nu \Delta^B q
\end{align*}
$$

(C.29)

with an implied summation over components and nodes. The integrals over the surface expansion functions are given by,

$$
\begin{align*}
e_q^\nu &= \left( \frac{k_0}{\eta_0} \right) \int_S (\hat{n} \times \hat{x}_l) u_k^B \cdot \vec{w} ds \\
g_k^\nu &= - \left( \frac{k_0}{\eta_0} \right) \int_S u_k^B (\hat{n} \cdot \vec{w}) ds \\
h_q^\nu &= - \left( \frac{k_0}{\eta_0} \right) \int_S (\hat{n} \cdot \hat{x}_l) u_k^B \omega_0 ds
\end{align*}
$$

(C.30)

for $p = x, y, z$ and

$$
\begin{align*}
h_q^\nu &= - \left( \frac{k_0}{\eta_0} \right) \int_S (\hat{n} \cdot \hat{x}_l) u_k^B \omega_0 ds
\end{align*}
$$

(C.31)
for \( p = 0 \). These equations form a link between the surface nodal fields and the FEM matrix equation. Only \( e_q \) is needed for the implementation of the FEM matrix equation detailed in Chapter II though all three integrals will be considered below.

The boundary elements correspond to the faces of the tetrahedrons forming the mesh. The representation of \( w_0, w \) in terms of tetrahedral simplex coordinates reduces to the triangular simplex coordinates for each boundary element. The above surface integrals are therefore most easily performed by integrating over each boundary element separately and collecting the results as was done for the FEM matrix.

Consider a single triangular surface element \( e \) with simplex coordinates \( \xi_1, \xi_2, \xi_3 \). Also consider a single test node \( m \) and boundary expansion node \( k \). The surface integrals in Equations (C.30) and (C.31), restricted to the one surface element \( e \), are represented by the terms \( \mathcal{E}_{pl}^{mk}, \mathcal{G}_{pl}^{mk}, \mathcal{H}_{pl}^{mk} \). They are computed for each pair of nodes in element \( e \). This pair of nodes consists of one FEM node and one boundary node. The boundary elements will be chosen to be first order so that the geometry is linear. This assumption helps simplify the MOM analysis. The surface integrals over one element become

\[
\mathcal{E}_{pl}^{mk} = \left( \frac{k_0}{\eta_0} \right) \hat{n} \cdot (\hat{x}_l \times \hat{x}_p) (2A) V^{mk}
\]

\[
\mathcal{G}_{pl}^{mk} = -\left( \frac{k_0}{\eta_0} \right) (\hat{n} \cdot \hat{x}_p) (2A) V^{mk}
\]

\[
\mathcal{H}_{pl}^{mk} = -\left( \frac{k_0}{\eta_0} \right) (\hat{n} \cdot \hat{x}_l) (2A) V^{mk}
\]

for \( p = x, y, z \) and

\[
V^{mk} = \int_{\xi_i} u_k^B u_m d\xi_i
\]

for \( p = 0 \) where,

is a geometry independent term which can be calculated once for all boundary elements. The area of the triangular element is \( A \) and the unit outward normal is \( \hat{n} \).
C.4 Zero Matrix Elements

This section will prove that \( M^{(p,m)}_{(l,n)} = 0 \) when \( l \neq p \) and one of \( u_n \) or \( u_m \) is contained within a homogeneous subregion. This is only true for the Lorentz gauge. While this property is true theoretically, it is only approximated by the numerical quadrature between isoparametric elements.

First consider the case \( p, l = x, y, z \) and assume \( u_m \) is nonzero only in a homogeneous subregion.

\[
M^{(p,m)}_{(l,n)} = \frac{1}{\varepsilon_r} \int_V \left\{ (\nabla \times \overrightarrow{w}) \cdot (\nabla \times \overrightarrow{F}) + (\nabla \cdot \overrightarrow{w}) (\nabla \cdot \overrightarrow{F}) - k^2 \overrightarrow{w} \cdot \overrightarrow{F} \right\} \, dv
\]

\[
= \frac{1}{\varepsilon_r} \int_V \left\{ (\nabla u_m \times \hat{x}_p) \cdot (\nabla u_n \times \hat{x}_l) + (\hat{x}_p \cdot \nabla u_m)(\hat{x}_l \cdot \nabla u_n)
- k^2 (\hat{x}_p \cdot \hat{x}_l) u_m u_n \right\} \, dv
\]

\[
= \frac{1}{\varepsilon_r} \int_V \left\{ \delta_{pl} [\nabla u_m \cdot \nabla u_n - k^2 u_m u_n] + [(\hat{x}_p \cdot \nabla u_m)(\hat{x}_l \cdot \nabla u_n) - (\hat{x}_l \cdot \nabla u_m)(\hat{x}_p \cdot \nabla u_n)] \right\} \, dv. \quad (C.36)
\]

The first term in the integrand corresponds to \( p = l \) and the second to \( p \neq l \). The second term reduces to

\[
(\hat{x}_p \cdot \nabla u_m)(\hat{x}_l \cdot \nabla u_n) - (\hat{x}_l \cdot \nabla u_m)(\hat{x}_p \cdot \nabla u_n)
= (\hat{x}_p \cdot \hat{x}_l) \cdot (\nabla u_m \times \nabla u_n)
= (\hat{x}_p \times \hat{x}_l) \cdot \nabla \times (u_m \nabla u_n)
= \nabla \cdot (u_m \nabla u_n \times (\hat{x}_p \times \hat{x}_l)). \quad (C.37)
\]

The matrix term becomes,

\[
M^{(p,m)}_{(l,n)} = \frac{1}{\varepsilon_r} \int_V \delta_{pl} [\nabla u_m \cdot \nabla u_n - k^2 u_m u_n] \, dv
+ \sum_e \frac{1}{\varepsilon_r} \int_{S_e} u_m \nabla u_n \cdot [(\hat{x}_p \times \hat{x}_l) \times \hat{n}] \, ds \quad (C.38)
\]

105
where the last integral is over the faces of all elements in which \( u_m \) is nonzero. While \( \hat{n} \cdot \nabla u_n \) may be discontinuous between elements, \( \nabla_{\text{tan}} u_n \) is continuous so that the integrals over opposing faces cancel. Therefore,

\[
M_{(i,n)}^{(p,m)} = \frac{1}{\varepsilon_r} \int_V \delta_{pl} \left[ \nabla u_m \cdot \nabla u_n - k^2 u_m u_n \right] dv
\]  

(C.39)

and the assertion is proved for this case. The procedure is identical when \( u_n \) is contained within a homogeneous subregion.

For \( p = 0, l = x, y, z \), choose the Lorentz gauge (\( \alpha = 0 \)). The matrix term becomes,

\[
M_{(i,n)}^{(p,m)} = \frac{jk^2}{\varepsilon_r} \int_V \left( \hat{F} \cdot \nabla w_0 + w_0 \left( \nabla \cdot \hat{F} \right) \right) dv
\]

\[
= \frac{jk^2}{\varepsilon_r} \int_V \hat{x}_l \cdot \nabla u_m u_n + u_m \hat{x}_l \cdot \nabla u_n dv
\]

\[
= \frac{jk^2}{\varepsilon_r} \int_V \hat{x}_l \cdot \nabla (u_m u_n) dv
\]

\[
= \frac{jk^2}{\varepsilon_r} \int_V \nabla \cdot (\hat{x}_l u_m u_n) dv
\]

\[
= \frac{jk^2}{\varepsilon_r} \sum_c \int_{S_c} u_m u_n \hat{x}_l \cdot \hat{n} ds.
\]  

(C.40)

Since \( u_m, u_n \) are continuous across element boundaries, this term goes to zero when either \( u_m \) or \( u_n \) are contained entirely within a homogeneous subregion. The same procedure applies for \( l = 0, p = x, y, z \). This completes the proof.
APPENDIX D

Surface Integral Equation and Matrix Equation

This appendix derives the basic MOM matrix equation from the exterior surface integral equation and the boundary shape functions described in Appendix B.

D.1 The Surface Integral Equation

The geometry for the integral equation is shown in Figure 55. A volume $V$ is

![Figure 55: The geometry for the surface integral equation.](image-url)
surrounded by a boundary $S_0$ with unit outward normal $\hat{n}'$. Electric and magnetic fields adjacent and exterior to the boundary will be denoted by $\vec{E}^B$, $\vec{H}^B$ while fields adjacent and interior to the boundary will be denoted by $\vec{E}^B_-$, $\vec{H}^B_-$. An incident electric field $\vec{E}_{\text{inc}}$ would exist in the absence of any perturbing effects from the volumetric region $V$. Free space is assumed outside of $S_0$. The total electric field satisfies the following integral equation in terms of the incident field and external boundary fields,

$$
\vec{E}(\vec{r}) \chi(\vec{r}) = \vec{E}_{\text{inc}}(\vec{r}) + \int_{S_0} -j\omega\mu_0 \left( \hat{n}' \times \vec{H}^B(\vec{r}') \right) G_0(R) \\
+ \left( \hat{n}' \times \vec{E}^B(\vec{r}') \right) \times \nabla' G_0(R) \\
+ \left( \hat{n}' \cdot \vec{E}^B(\vec{r}') \right) \nabla' G_0(R) \, ds'
$$

(D.1)

where $\chi(\vec{r}) = 1$ for $\vec{r}$ outside $V$ and $\chi(\vec{r}) = 0$ for $\vec{r}$ inside $V$. The free space scalar Green’s function $G_0(R)$ is defined by,

$$
G_0(R) = \frac{e^{-jk_0 R}}{4\pi R}
$$

(D.2)

$$
R = ||\vec{r} - \vec{r}'||
$$

(D.3)

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free space wave number. The boundary fields must be continuous on $S_0$. Equation (D.1) is manipulated to form the integral equation,

$$
\vec{E}_{\text{inc}}(\vec{r}) = \vec{E}^B(\vec{r}) \chi(\vec{r}) - \int_{S_0} -j\omega\mu_0 \left( \hat{n}' \times \vec{H}^B(\vec{r}') \right) G_0(R) \\
+ \vec{E}^B(\vec{r}') \times \left( \nabla' G_0(R) \times \hat{n}' \right) \\
+ \left( \hat{n}' \cdot \nabla' G_0(R) \right) \vec{E}^B(\vec{r}') \, ds'
$$

(D.4)

by restricting $\vec{r}$ to the boundary surface $S_0$. This equation yields the exact same integral equation for $\vec{r}$ outside or inside $V$ due to the singularities of $\nabla' G_0(R)$ cancelling the effect of $\chi(\vec{r})$. 

108
For a homogeneous volumetric region, a corresponding internal integral relation can be given for the internal electric field in terms of the internal boundary fields $E^B, H^B$. An internal integral equation is found by restricting $\vec{r}$ to the interior boundary surface,

$$0 = -\vec{E}^B(\vec{r})(1 - \chi(\vec{r})) - \int_{S_0} -j\omega\mu_- \left(\vec{n}' \times \vec{H}^B - (\vec{r}')\right) G_-(R)$$

$$+ \vec{E}^B(\vec{r}') \times \left(\nabla' G_-(R) \times \vec{n}'\right)$$

$$+ \left(\vec{n}' \cdot \nabla' G_-(R)\right) \vec{E}^B(\vec{r}') ds'$$

(D.5)

where $\varepsilon_-, \mu_-$ are the dielectric parameters for the homogeneous volume. The internal scalar Green's function is,

$$G_-(R) = e^{-jk_-R} \frac{4\pi R}{4\pi R}$$

(D.6)

with $k_- = \omega\sqrt{\mu_-\varepsilon_-}$.

D.2 The MOM Matrix Equation

The boundary fields are expanded in terms of a nodal based set of expansion functions in order to match with the FEM internal fields. The expansion functions are based on the scalar boundary functions $u^B_k$ introduced in Appendix B. The function $u^B_k$ has a value of one at node $k$ and zero at all other boundary nodes. It is represented in terms of simplex coordinates in each of the triangular boundary elements adjacent to node $k$. Only linear boundary elements are used for the MOM so as to reduce the number of unknowns. One unknown is used for each of the three coordinates of a field at each node. The boundary fields are therefore expanded as,

$$\vec{E}^B = \vec{u}^B q$$

$$\vec{H}^B = \vec{u}^B s$$

$$\vec{u}^B = u^B_k \hat{s}'q'$$

109
\( u_q^{B} = u_{k'}^{B} \hat{x}_{k''} \) \hspace{1cm} (D.7)

in which \( q = (l', k') \), \( s = (l'', k'') \), \((l', l'' = x, y, z)\) and \(k', k''\) are boundary nodes. There is an implicit summation over the \( N^B \) boundary nodes and Cartesian coordinates giving a total of \( 3N^B \) unknowns for the MOM. A set of test functions are also needed. The set of expansion functions \( \tilde{u}_t^B \) are used for the test functions with \( t = (l, k) \).

In order to derive the MOM matrix equation, Equations (D.7) are substituted into the integral Equation (D.4). The inner product of this equation with the test function \( \tilde{u}_t^B \) is taken to yield the matrix equation,

\[
V^E^t = j\omega \mu_0 \left[ M^{0+} \right]_s^t H^B \cdot s + \left[ M^{\nabla+} \right]_q^t E^B \cdot q.
\]

The matrices are defined by,

\[
\left[ M^{\nabla+} \right]_q^t = \int_{S_0} \tilde{u}_t^B(\vec{r}) \cdot \tilde{u}_q^B(\vec{r}) \, ds
- \int_{S_0} \int_{S_0} \tilde{u}_t^B(\vec{r}) \cdot \left[ \tilde{u}_q^B(\vec{r}') \times \left( \nabla' G_0(R) \times \hat{n}' \right) \right]
+ \left( \hat{n}' \cdot \nabla' G_0(R) \right) \left( \tilde{u}_t^B(\vec{r}) \cdot \tilde{u}_q^B(\vec{r}') \right) \, ds' \, ds
\]

\[
\left[ M^{0+} \right]_s^t = \int_{S_0} \int_{S_0} \tilde{u}_t^B(\vec{r}) \cdot \left( \hat{n}' \times \tilde{u}_s^B(\vec{r}') \right) G_0(R) \, ds' \, ds
\]

and the known voltage vector by,

\[
V^E^t = \int_{S_0} \tilde{u}_t^B(\vec{r}) \cdot \vec{E}_{inc}(\vec{r}) \, ds.
\]

An interior matrix equation can also be derived when the volume \( V \) is homogeneous,

\[
0 = j\omega \mu_0 \left[ M^{0-} \right]_s^t \mu^B \cdot s + \left[ M^{\nabla-} \right]_q^t E^B \cdot q
\]

(D.12)
with the internal matrices,

\[
[M^{\nabla^+}]_q^t = - \int_{S_0} \tilde{u}_q^B (\tilde{r}) \cdot \tilde{u}_q^B (\tilde{r}) \, ds
- \int_{S_0} \int_{S_0} \tilde{u}_q^B (\tilde{r}) \cdot \left[ \tilde{u}_q^B (\tilde{r}') \times \left( \nabla' G_- (R) \times \hat{n}' \right) \right]
+ \left( \hat{n}' \cdot \nabla' G_- (R) \right) \left( \tilde{u}_q^B (\tilde{r}) \cdot \tilde{u}_q^B (\tilde{r}') \right) \, ds' \, ds
\]  

(D.13)

\[
[M^{\nabla^+}]_s^t = \int_{S_0} \int_{S_0} \tilde{u}_q^B (\tilde{r}) \cdot \left( \hat{n}' \times \tilde{u}_q^B (\tilde{r}') \right) G_- (R) \, ds' \, ds.
\]  

(D.14)

The field point is taken to be adjacent to \( S_0 \) and internal to \( V \).

### D.3 Reduction of the Singularity

The scalar Green's function \( G_0 (R) \) has a singularity which must be accounted for in a numerical quadrature. Methods for handling this singularity are given in Appendix E. The gradient \( \nabla' G_0 (R) \) introduces more severe singularities. The methods in Appendix E can be used to find quadratures involving \( (\hat{n}' \cdot \nabla' G_0 (R)) \).

This section reduces the \([M^{\nabla^+}]_q^t \) matrix term to quadratures containing only the singularities of \( G_0 (R) \) and \( (\hat{n}' \cdot \nabla' G_0 (R)) \).

To accomplish this task, the first term in the second integral of Equation (D.9) is reduced using the definitions in Equation (D.7),

\[
\tilde{u}_q^B (\tilde{r}) \cdot \left[ \tilde{u}_q^B (\tilde{r}') \times \left( \nabla' G_0 (R) \times \hat{n}' \right) \right] = \mathcal{M} \cdot \nabla' G_0 (R)
\]  

(D.15)

\[
\mathcal{M} = \hat{n}' \times \left( \tilde{u}_q^B (\tilde{r}) \times \tilde{u}_q^B (\tilde{r}') \right)
= \hat{n}' \times \left( \hat{\omega}_l \times \hat{\omega}_\mu \right) u_k^B (\tilde{r}) u_k^B (\tilde{r}') .
\]  

(D.16)

Note that \( \mathcal{M} \cdot \hat{n}' = 0 \). For a patch \( S_\Delta \) with boundary \( C_\Delta \) on the surface \( S_0 \),

\[
\int_{S_\Delta} \mathcal{M} \cdot \nabla' G_0 (R) \, ds' = \int_{S_\Delta} \mathcal{M} \cdot \nabla' sG_0 (R) \, ds'
\]  

(D.17)
where \( \nabla'_S \cdot \cdot \) is the surface divergence. The surface patch \( S_\Delta \) will correspond to a single triangular element in the present application as shown in Figure 56. The surface form of the divergence theorem gives,

\[
\int_{S_\Delta} \vec{M} \cdot \nabla' G_0(R) \, ds' = - \int_{S_\Delta} \left( \nabla'_S \cdot \vec{M} \right) G_0(R) \, ds' - \int_{C_\Delta} G_0(R) \vec{M} \cdot (\hat{n}' \times d\vec{\alpha}')
\]

and,

\[
\vec{M} \cdot (\hat{n}' \times d\vec{\alpha}') = \left\{ (\hat{x}_l \times \hat{x}_l') u^B_{k'} (\vec{\tau}) u^B_k (\vec{\tau}') \right\} \cdot d\vec{\alpha}'.
\]

Since the shape functions \( u^B_{k'} (\vec{\tau}') \) are continuous across the triangular boundary elements, the contour integral will disappear when the contributions from all the elements are added together. Therefore,

\[
\int_{S_\Delta} \vec{M} \cdot \nabla' G_0(R) \, ds' = - \int_{S_\Delta} \left( \nabla'_S \cdot \vec{M} \right) G_0(R) \, ds'.
\]

On each triangular boundary element on which \( \hat{n}' = \) constant,

\[
\nabla'_S \cdot \vec{M} = \hat{n}' \times (\hat{x}_l \times \hat{x}_l') \cdot \left( \nabla'_S u^B_{k'} (\vec{\tau}') \right) u^B_k (\vec{\tau}).
\]
The last term in equation (D.9) has the property,

\[
\int_{S_0} \left( \hat{n}' \cdot \nabla' G_0 (R) \right) \left( \bar{u}_t^B (\bar{r}) \cdot \bar{u}_q^B (\bar{r}') \right) \, ds' \\
= \frac{1}{2} \left( \bar{u}_t^B (\bar{r}) \cdot \bar{u}_q^B (\bar{r}) \right) \\
+ \mathcal{P} \nu \int_{S_0} \left( \hat{n}' \cdot \nabla' G_0 (R) \right) \left( \bar{u}_t^B (\bar{r}) \cdot \bar{u}_q^B (\bar{r}') \right) \, ds'
\] (D.22)

where the principle value of the integral is taken for the singular integrand. The principle value corresponds to integrating over a region which excludes a domain around the singular point and taking the limit as the domain shrinks to zero. The first integral in Equation (D.22) corresponds to the limiting operation of the field point \( \bar{r}' \) being brought down onto the integration surface \( S_0 \). The second integral corresponds to the limit as the excluded integration domain shrinks to zero around the singularity. Note that if the surface being integrated over is flat with the field point taken on this flat surface, then \( \left( \hat{n}' \cdot \nabla' G_0 (R) \right) \) is zero and the principle value contribution is zero. This is not the case for a curved surface.

Using the above results gives the simplified form for the matrix,

\[
\left[ M^{\nabla^+} \right]_{q} = \frac{1}{2} \int_{S_0} \left( \hat{x}_l \cdot \hat{x}_p \right) u_k^B (\bar{r}) u_k^B (\bar{r}) \, ds \\
- \int_{S_0} \mathcal{P} \nu \int_{S_0} \left( \hat{n}' \cdot \nabla' G_0 (R) \right) \left( \hat{x}_l \cdot \hat{x}_p \right) u_k^B (\bar{r}) u_k^B (\bar{r}') \, ds' \, ds \\
- \int_{S_0} \int_{S_0} \hat{n}' \times \left( \hat{x}_p \times \hat{x}_l \right) \left( \nabla' \cdot u_k^B (\bar{r}') \right) u_k^B (\bar{r}) \, G_0 (R) \, ds' \, ds.
\] (D.23)

Using Equations (D.7) in Equation (D.10) gives,

\[
\left[ M^{0^+} \right]_{q} = \int_{S_0} \int_{S_0} \hat{n}' \cdot \left( \hat{x}_p \times \hat{x}_l \right) u_k^B (\bar{r}) u_k^B (\bar{r}') \, G_0 (R) \, ds' \, ds.
\] (D.24)

For the interior matrices for a homogeneous volume,

\[
\left[ M^{\nabla^-} \right]_{q} = -\frac{1}{2} \int_{S_0} \left( \hat{x}_l \cdot \hat{x}_p \right) u_k^B (\bar{r}) u_k^B (\bar{r}) \, ds
\]
These matrix terms are now in a suitable form for a numerical quadrature using the methods in Appendix E.

D.4 The Scattered Field

The scattered far field may be found from Equation (D.1) by taking the limit as \( ||\vec{r}|| \to \infty \) and retaining only the leading order terms. With the definitions,

\[
r = ||\vec{r}||
\]

\[
\hat{r} = \frac{\vec{r}}{r}
\]

the scattered electric field in the direction \( \hat{r} \) is,

\[
\vec{E}^{\text{scat}}(\hat{r}) = \frac{e^{-jk_0r}}{4\pi r} \int_{S_0} \left\{ -j\omega \mu_0 \left( \hat{n}' \times \vec{H}^B \left( \vec{r}' \right) \right) \\
+ \left[ \left( \hat{n}' \times \vec{E}^B \left( \vec{r}' \right) \right) \times (j k_0 \hat{r}) + \left( \hat{n}' \cdot \vec{E}^B \left( \vec{r}' \right) \right) j k_0 \hat{r} \right] \right\} e^{jk_0\hat{r} \cdot \hat{r}' ds'}.
\]

This equation can be shown to satisfy \( \vec{E}^{\text{scat}}(\vec{r}) \cdot \hat{r} = 0 \) in the far field. Taking advantage of this fact, we choose a polarization vector \( \vec{p} \) such that \( \vec{p} \cdot \hat{r} = 0 \). The component of the scattered electric field in the \( \vec{p} \) direction is given by,

\[
\vec{p} \cdot \vec{E}^{\text{scat}}(\vec{r}) = \frac{e^{-jk_0r}}{4\pi r} \int_{S_0} \left\{ -j k_0 \eta_0 \left( \hat{n}' \times \vec{H}^B \left( \vec{r}' \right) \right) \cdot \vec{p} \\
- j k_0 (\vec{p} \times \hat{r}) \cdot \left( \hat{n}' \times \vec{E}^B \left( \vec{r}' \right) \right) \right\} e^{jk_0\hat{r} \cdot \hat{r}' ds'}.
\]
Using the field expansions in Equations (D.7) in this equation gives the simplification,

\[
\hat{p} \cdot \vec{E}^{\text{scat}}(\hat{r}) = -j k_0 \frac{e^{-jk_0 r}}{4\pi r} \left\{ \eta_0 \hat{p} \cdot \vec{S}_s(\hat{r}) H^B s + (\hat{p} \times \hat{r}) \cdot \vec{S}_q(\hat{r}) E^B q \right\} \quad (D.31)
\]

\[
\vec{S}_q(\hat{r}) = \int_{S_0} \left( \hat{n}' \times \vec{u}_q^B(\hat{r}') \right) e^{jk_0 \hat{r} \cdot \hat{r}'} d\hat{s}' . \quad (D.32)
\]

For a unit plane wave field with polarization \( \hat{p}_{\text{inc}} \) and incident from the direction \( \hat{s} \),

\[
\vec{E}_{\text{inc}}(\hat{r}) = \hat{p}_{\text{inc}} e^{jk_0 \hat{s} \cdot \hat{r}} . \quad (D.33)
\]

The voltage vector takes the form,

\[
\mathbf{V}^E = \int_{S_0} \hat{p}_{\text{inc}} \cdot \vec{u}_q^B(\hat{r}') e^{jk_0 \hat{s} \cdot \hat{r}'} d\hat{s}' . \quad (D.34)
\]

For this unit magnitude incident field, the far field scattering cross section becomes,

\[
\sigma(\hat{r}) = 4\pi r^2 |\hat{p} \cdot \vec{E}^{\text{scat}}(\hat{r})|^2 = \frac{k_0^2}{4\pi} \eta_0 \hat{p} \cdot \vec{S}_s(\hat{r}) H^B s + (\hat{p} \times \hat{r}) \cdot \vec{S}_q(\hat{r}) E^B q |^2 \quad (D.35)
\]

in terms of the nodal values of the boundary fields.
APPENDIX E
Numerical Surface Integrations

E.1 One Source Element and One Test Element

The matrix terms in Equations (D.23) and (D.24) involve surface integrals of the boundary functions and scalar Green's function. These integrals are most easily performed over each boundary element separately. This appendix will therefore consider the integrals over a single testing boundary element $e$ with surface integration variable $\vec{r}$ and a single source boundary element $e'$ with surface integration variable $\vec{r}'$. With this restriction, the normal vector $\hat{n}'$ becomes a constant.

In order to simplify the expressions, several terms need to be defined in terms of the geometry of the source triangle $e'$. The reader is referred to Figure 57. The nodes have position coordinates $\vec{r}_i$ for $i = 1, 2, 3$. The following definitions depend on these node coordinates,

$$\vec{r}_i^+ = \vec{r}_{i+2}$$  \hspace{1cm} (E.1)

$$\vec{r}_i^- = \vec{r}_{i+1}$$  \hspace{1cm} (E.2)

$$\vec{l}_i = \vec{r}_i^+ - \vec{r}_i^-$$  \hspace{1cm} (E.3)

$$\vec{l}_i = \frac{\vec{l}_i}{\|\vec{l}_i\|}$$  \hspace{1cm} (E.4)

$$\hat{n}_i = \vec{l}_i \times \hat{n}'$$  \hspace{1cm} (E.5)

All local node indices are calculated modulo three. For example, when $i = 2$, $i + 2 = 1$. In addition, there will be a need for the following definitions which
Figure 57: The geometry for the surface integral defined on the source element \( e' \).

depend on the observation point \( \vec{r}' \).

\[
\begin{align*}
I_i^\pm &= - (\vec{r} - \vec{r}_{i\pm}^\pm) \cdot \hat{l}_i \\
P_i^0 &= |(\vec{r} - \vec{r}_{i\pm}^\pm) \cdot \hat{u}_i| \\
\dot{P}_i^0 &= -(\vec{r} - \vec{r}_{i\pm}^\pm) \cdot \hat{u}\hat{u}/P_i^0 \\
d &= \hat{n}' \cdot (\vec{r} - \vec{r}_{i\pm}^\pm) \\
\bar{\rho} &= \vec{r} - \hat{n}' \left( \hat{n}' \cdot \vec{r} \right) \\
\bar{\rho}_i^\pm &= \vec{r}_i^\pm - \hat{n}' \left( \hat{n}' \cdot \vec{r}_i^\pm \right) \\
\bar{P}_i^\pm &= \bar{\rho} - \bar{\rho}_i^\pm \\
P_i^\pm &= \| \bar{P}_i^\pm \| \\
R_i^\pm &= \left( (P_i^\pm)^2 + d^2 \right)^{\frac{1}{2}}
\end{align*}
\]
The matrix terms reduce to,

\[
\left[ M^{\nabla+} \right]_q = \frac{1}{2} \left( \hat{x}_l \cdot \hat{x}_l \right) \int_e u_k^B (\vec{r}) u_k^B (\vec{r}) \, ds \\
- \left( \hat{x}_l \cdot \hat{x}_l \right) \int_e u_k^B (\vec{r}) \left\{ \mathcal{P}V \int_{e'} \left( \hat{n}' \cdot \nabla' G_0 (R) \right) u_k^B (\vec{r}') \, ds' \right\} \, ds \\
- \frac{\hat{c}' \cdot \hat{I}_{l(k')}^j}{2A} \int_e u_k^B (\vec{r}) \left\{ \int_{e'} G_0 (R) \, ds' \right\} \, ds
\]

\[
\left[ M^{0+} \right]_s = \hat{c}'' \cdot \hat{n}' \int_e u_k^B (\vec{r}) \left\{ \int_{e'} u_k^{B\eta} (\vec{r}') G_0 (R) \, ds' \right\} \, ds
\]

where

\[
\hat{c}' = \hat{x}_l \times \hat{x}_l
\]

\[
\hat{c}'' = \hat{x}_l \times \hat{x}_l.
\]

As before, \( q = (l', k') \), \( s = (l'', k'') \), \( t = (l, k) \) \((l', l'', l = x, y, z)\) and \( k', k'', k \) are boundary nodes. The area of the source element is \( A \). The local node \( i \) corresponding to boundary node \( k' \) is represented as \( i(k') \). The simplification of Equation (E.18) uses the following property of the local node coordinates \( \zeta_i \),

\[
\zeta_i = 1 + \left( \vec{r}' - \vec{r}_i \right) \cdot \frac{\hat{n}' \times \hat{I}_i}{2A}
\]

\[
\nabla^t \zeta_i = \frac{\hat{n}' \times \hat{I}_i}{2A}
\]

\[
\left( \hat{n}' \times \hat{c}' \right) \cdot \nabla^t \zeta_i = \frac{\hat{c}' \cdot \hat{I}_i}{2A}.
\]

In addition, the voltage and far field terms from Equations (D.34) and (D.32) reduce to,

\[
V^E t = \left( \hat{x}_l \cdot \hat{p}_{inc} \right) \int_{e'} u_k^B (\vec{r}') e^{jk_0 \hat{R}' \cdot \vec{r}'} \, ds'
\]
\[ \hat{p} \cdot \vec{S}_q (\vec{r}) = (\hat{p} \times \hat{n}^l) \cdot \hat{v}_l \int_{e} u_B^{k'} (\vec{r}'') e^{jk_0 \hat{p}' \cdot \vec{r}'} d\sigma. \] (E.25)

By assumption, the boundary functions \( u_B^k \) are chosen to be linear in the simplex coordinates. Therefore, the integrals over the source triangle can be reduced to the following forms which are dependent upon the field point \( \vec{r} \),

\[ I_0 (\vec{r}) = \int_{e} G_0 (R) \, ds' \] (E.26)
\[ I_i (\vec{r}) = \int_{e} \zeta_i G_0 (R) \, ds' \] (E.27)
\[ J_i (\vec{r}) = \mathcal{P} \mathcal{V} \int_{e} \zeta_i (\hat{n}' \cdot \nabla' G_0 (R)) \, ds' \] (E.28)

where the local node index can have the values \( i = 1, 2, 3 \). With these definitions, the matrix terms are given by,

\[ \left[ M^{\nabla^+} \right]^{t}_{q} = \frac{1}{2} \delta_{ll'} \int_{e} u_k^B (\vec{r}) u_{k'}^B (\vec{r}) \, ds \\
- \delta_{ll'} \int_{e} u_k^B (\vec{r}) J_i (k') (\vec{r}) \, ds \\
- \frac{\hat{c}' \cdot \hat{n}_{i(k')}}{2A} \int_{e} u_k^B (\vec{r}) I_0 (\vec{r}) \, ds \] (E.29)

\[ \left[ M^{0^+} \right]^{t}_{s} = \hat{c}'' \cdot \hat{n} \int_{e} u_k^B (\vec{r}) I_0 (k'') (\vec{r}) \, ds. \] (E.30)

The integrals in Equations (E.29) and (E.30) are well behaved since \( I_0 (\vec{r}) \), \( I_i (\vec{r}) \), \( J_i (\vec{r}) \) are bounded functions. This was achieved through a reduction of the singularity in the matrix term \( \left[ M^{\nabla^+} \right]^{t}_{q} \) as described in Appendix D.

### E.2 Singularities

When the source and test triangles are well separated, the integrals in Equations (E.26), (E.27) and (E.28) are well behaved and may be evaluated using a Gaussian quadrature method. However, when the source and test triangles are identical or
adjacent, the singularities in these equations must be taken into account. This is accomplished by removing the singularities from the Green's function and separating the integrals into a regular part and a wavelength independent singular part. The regular part is integrated by Gaussian quadratures and the singular part is reduced using the methods in [16].

The scalar Green's function and the normal derivative of the Green's function are separated into regular and singular parts as follows,

$$G_0(R) = \left( \frac{e^{-j k_0 R} - 1}{4\pi R} \right) + \frac{1}{4\pi R}$$  \hspace{1cm} (E.31)

$$\mathbf{n} \cdot \nabla' G_0(R) = \left[ \frac{(1 + j k_0 R) e^{-j k_0 R} - 1}{R^2} \right] \left( \frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}')} {4\pi R} \right) + \frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}')} {4\pi R^3}.$$  \hspace{1cm} (E.32)

The integrals in Equations (E.26), (E.27) and (E.28) reduce to the following forms by using Equations (E.21), (E.31) and (E.32),

$$I_0(\mathbf{r}) = \int_{e'} \left( \frac{e^{-j k_0 R} - 1}{4\pi R} \right) ds' + \frac{1}{4\pi} A_0(\mathbf{r})$$  \hspace{1cm} (E.33)

$$I_i(\mathbf{r}) = \int_{e'} \zeta_i \left( \frac{e^{-j k_0 R} - 1}{4\pi R} \right) ds'$$
$$+ \frac{1}{4\pi} \left[ 1 - \frac{l_i (\mathbf{r} - \mathbf{r}_i) \cdot \mathbf{u}_i}{2A} \right] A_0(\mathbf{r}) - \frac{l_i \mathbf{u}_i}{8\pi A} \cdot \mathbf{\bar{A}}_1(\mathbf{r}).$$  \hspace{1cm} (E.34)

$$J_i(\mathbf{r}) = d \int_{e'} \zeta_i \frac{(1 + j k_0 R) e^{-j k_0 R} - 1}{4\pi R^3} ds'$$
$$+ \frac{1}{4\pi} \left[ 1 - \frac{l_i (\mathbf{r} - \mathbf{r}_i) \cdot \mathbf{u}_i}{2A} \right] B_0(\mathbf{r}) - \frac{l_i \mathbf{u}_i}{8\pi A} \cdot \mathbf{\bar{B}}_1(\mathbf{r}).$$  \hspace{1cm} (E.35)

The functions $A_0(\mathbf{r})$ and $\mathbf{\bar{A}}(\mathbf{r})$ are defined in terms of the integrals evaluated in [16],

$$A_0(\mathbf{r}) = \int_{e'} \frac{1}{R} ds'$$
$$= \sum_{i=1}^{3} \mathbf{p}_i^0 \cdot \mathbf{u}_i \left[ P_i^0 \ln \frac{R_i^+ + t_i^+}{R_i^- + l_i^-} \right.$$  

$$- |d| \left( \tan^{-1} \frac{P_i^0 t_i^+}{(R_i^0)^2 + |d|R_i^0} - \tan^{-1} \frac{P_i^0 l_i^-}{(R_i^0)^2 + |d|R_i^0} \right).$$  \hspace{1cm} (E.36)
\[
\tilde{A}_1 (\vec{r}) = \int_{e'} \frac{\vec{\rho}'}{R} \cdot d\vec{s'}
= \frac{1}{2} \sum_{i=1}^{3} \hat{u}_i \left[ (R_i^0)^2 \ln \frac{R_i^+ + l_i^+}{R_i^- + l_i^-} + l_i^+ R_i^+ - l_i^- R_i^- \right]. \tag{E.37}
\]

The \( B_0 (\vec{r}) \) and \( \tilde{B} (\vec{r}) \) functions are evaluated using these same techniques,

\[
B_0 (\vec{r}) = d \int_{e'} \frac{1}{R^3} d\vec{s'}
= \frac{d}{|d|} \sum_{i=1}^{3} \hat{p}_i^0 \cdot \hat{u}_i \left[ \tan^{-1} \frac{P_i^0 l_i^+}{(R_i^0)^2 + |d| R_i^+}
- \tan^{-1} \frac{P_i^0 l_i^-}{(R_i^0)^2 + |d| R_i^-} \right]. \tag{E.38}
\]

\[
\tilde{B}_1 (\vec{r}) = d \int_{e'} \frac{(\vec{\rho}' - \vec{\rho})}{R^3} \cdot d\vec{s'}
= -d \sum_{i=1}^{3} \hat{u}_i \ln \frac{R_i^+ + l_i^+}{R_i^- + l_i^-}. \tag{E.39}
\]

The following property is often useful when implementing these formulas,

\[
\frac{R_i^+ + l_i^+}{R_i^- + l_i^-} = \frac{R_i^- - l_i^-}{R_i^+ - l_i^+}. \tag{E.40}
\]

The numerical accuracy of Equations (E.33), (E.34) and (E.35) decreases as the observation point \( \vec{r} \) increases in distance from the source triangle e.
BIBLIOGRAPHY


122


