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The effects of guided discovery style teaching and graphing calculator use in differential calculus

Emese, George Laszlo, Ph.D.
The Ohio State University, 1993
THE EFFECTS OF GUIDED DISCOVERY STYLE TEACHING AND
GRAPHING CALCULATOR USE IN DIFFERENTIAL CALCULUS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

George L. Emese, B.S., M.A.

* * * * *

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CHAPTER I
INTRODUCTION

1.1 Rationale

The objective of this study was to examine the effects of graphing calculator use and
discovery style teaching in differential calculus. A number of studies have examined the
effects of either technology use or discovery style teaching. However, no studies that
examined the effects of both, and tried to isolate the effects of technology use and
discovery style teaching were found. This study was designed to do so and examine the
interaction between technology use and discovery style teaching.

The development of interactive computer and calculator graphing can give a new
momentum to the discovery movement since it is a very effective tool for student
experimentation and discovery. Students can examine a large number of examples in a
short time to form a base for discovery. This was not the case in the early days of pre­
technology. Thus, research was constrained away from a true test of discovery.

1.2 Introduction

Recent breakthroughs in computing technology, computer/calculator graphing and
computer/calculator algebra systems are becoming part of the mathematics instruction at the
secondary and college level. Research shows that these technologies can be used
successfully in mathematics education. Students using graphing calculators and/or
computer graphing utilities to graph functions in high school precalculus courses exhibited
higher competency in analytic geometry and functions, and more of them were ready for university calculus, when compared with the control groups (Demana, Foley, Osborne & Waits, 1989). Students achieved a higher level of graphical understanding and they were more likely to make connections between algebraic and graphical representations of a function (Browning, 1988/89, 1990). Beckmann (1990) found that "developing calculus concepts through the use of a graphical representation system, especially through computer graphics, can positively affect student understanding and interest without negatively influencing skill acquisition" (p. 107). Most studies using computer algebra systems (Dubinsky and Schwingendorf, 1990; Heid, 1988; Palmiter, 1986, 1991) reported positive findings on student achievement, primarily on conceptual understanding.

Dick and Shaughnessy (1988) found positive effects of symbolic/graphing calculator use on students' attitudes toward mathematics and calculators and on students' perceptions of mathematics. Farrell (1989/1990, 1991) found that students are more exploratory, experimental, and more active in classrooms where graphing calculators are used, than in traditional classrooms. Students were more likely to work together on mathematics problems in these settings. Teachers are more likely to exhibit the teaching behaviors recommended by the Curriculum and Evaluation Standards for School Mathematics by the National Council of Teachers of Mathematics (1989).

There is no doubt that technology use is a promising area in mathematics education. It can open new horizons and can revolutionize what and how we teach in mathematics (Fey, 1984; Hansen, 1984; Smith, Porter, Leinbach, and Wenger, 1988; Wilf, 1982). However, since these opportunities became available only recently, this is still a "new, unexpected, unstudied, and unpredictable" area in mathematics education. "Research will be important both to evaluate the impact of technology on mathematics instruction and to inform the development of software and applications" according to the Research Agenda Project of the National Council of Teachers of Mathematics (Sowder, 1989, p. 29).
Learning by discovery, on the other hand, has been one of the most studied and most controversial issues in mathematics education. Shulman (1970) described the origins of a theory of learning by discovery as a "melange of Piaget and Plato." A major factor in justifying the discovery approach was Piaget's teaching about the active interaction with the environment leading to the construction of knowledge and understanding by the student.

Discovery learning was the subject of a long, heated debate, and was in the center of mathematics education research in the 1960s (Begle, 1979; Fey, 1980). A great number of arguments were given both for and against. A great number of experiments were conducted. However, research results are still conflictive and inconclusive (Begle, 1979; Cronbach, 1977; Orton, 1987; Shulman and Keislar, 1966; Fey, 1980).

There were numerous calls for further research on the effectiveness of the discovery style teaching (e.g., Ausubel, 1978; Begle, 1979; Wittrock, 1966). Begle (1979) expressed his impression that most of the studies involved above average students. He raised the question: "Can all students discover? If so, what does the teacher do with his bright students, who discover quickly, while waiting for his slower students?" (p. 121). Bruner (1961) emphasized that his hypothesis had to be tested under regular classroom circumstances. Ausubel (1978) pointed out that "such research must also deal with large segments of instructional material and not merely with short-term problem-solving exercises in the laboratory" (p. 561).

The rationale for choosing a calculus course for the study lies in the importance of calculus in the college curriculum and in the growing consensus among mathematicians and math educators that a calculus reform is necessary. A calculus sequence is a prerequisite for all engineering and science programs, and many other majors require some calculus. Over one million students study calculus each year, but fewer than 25% survive to enter the science and engineering pipeline (Steen, 1987). But calculus should be the pump, not a filter (White, 1987; quoted in Steen, 1989); or the door, not a barrier, as others suggested.
There have been several calls for a calculus reform in the 1980s. Some of the more recent ones are the Sloan Conference/Workshop Toward a Lean and Lively Calculus at Tulane University (1986) and the A Pump, Not a Filter colloquium in Washington, DC (1987) organized by the National academy of Sciences. Several MAA CUPM (Mathematical Association of America Committee on Undergraduate Programs in Mathematics) projects examined the college teaching of calculus (Steen 1989).

1.3 Problem Statement

The objectives of the study were to

1. verify that students can discover a significant portion of differential calculus;

2. investigate the effects of the use/non use of graphing calculators and the instructional technique (lecture/discussion or discovery style teaching) (independent variables) on the following dependent variables: student achievement on the final exam (computational skills, conceptual understanding and ability to transfer what has been learned to a different, but related situation), short and long term retention (same three categories as above), students' time spent on course, and whether students worked with classmates outside of class or not. In the discovery sections the extent to which students actually discovered the new material (how successfully they proceeded on the worksheets; on the other hand, how much they had learned earlier, in previous high school or college courses, pre-reading the textbook, etc.) was also measured. The following background variables were examined: placement level and year, grade in the previous math class, Precalculus (Math 150) and the year in which it was taken, whether the student has already attempted Differential Calculus (Math 151) or not, and pretest score.

3. investigate the question whether the positive results of technology use (graphing calculators/computers) reported by many studies (e.g., Beckmann, 1990; Browning,
1990; Demana, Foley, Osborne & Waits, 1989; Dick & Shaughnessy, 1988; Farrell, 1989; Heid, 1988; Judson, 1988; Palmiter, 1986) are due to the technology use itself or to the increase in discovery activity on the part of the students which takes place as a result of technology use (W. Davis, personal communication, November 21, 1990; Dick, 1990).

1.4 Definition of Terms

Terms related to the independent variables are defined as follows.

There is no precise, consistent definition for discovery learning (Ausubel, 1978; Begle, 1979; Dunkin and Biddle, 1974; Good and Biddle, 1988; Shulman and Keislar, 1966; Fey, 1980). In this study Ausubel's (1978) definition was used. He contrasted discovery learning with reception learning:

In reception learning, the principal content of the learning task is merely presented to the learner; and he is only required to relate it actively and meaningfully to relevant aspects of his cognitive structure and to retain it for later recall or recognition, or as a basis for learning related new material. In discovery learning, on the other hand, the principal content of what is to be learned must be discovered independently before it can be assimilated into cognitive structure. (p. 4)

Ausubel added that discovery learning is not an absolute. It can be located on a reception-discovery continua. In other words, discovery methods can be classified according to the amount of guidance provided to the students (Biggs, 1972). In unguided discovery no external help is given to the students, only practice. In guided discovery there is some external help or guidance.

In this study, guidance was provided through the use of worksheets, hint-sheets and solution sheets. In the worksheets a chain of questions and problems led to the new concept, relationship, or technique. The solution-sheets contained complete solutions to the problems, the hint-sheets contained leading questions, some suggestions on to how to solve the problem or the first step of the solution. The use of worksheets, hint-sheets and
solution-sheets and the instructional techniques are described in 3.5 (Treatments).

A **graphing calculator** is a programmable, hand-held, scientific calculator with the capability of graphing functions. The use of graphing calculators in this study is described in 3.5 (Treatments).

**Calculator/computer graphing** is the use of a graphing calculator or computer software to graph functions.

**Computer algebra systems** (or **symbol manipulators**) are computer softwares that can perform operations on algebraic expressions.

A **supercalculator** is a graphing calculator with computer algebra capability.

The following definitions are related to the dependent variables.

Student achievement was measured in the following three categories: computational skills, conceptual understanding and transfer skills. Each problem on the tests was classified by a panel of doctoral students as an item measuring one of these variables. The **computational skills** problems required students to perform numerical and/or symbolic computations, to carry out algorithms. These computations, algorithms had been taught and practiced prior to the test, both in class and in homework assignments. The **conceptual understanding** problems required little or no computations, but they required the understanding and proper use of one or more mathematical concepts. The way these concepts had to be used were familiar to the students, problems of these types had been discussed prior to the tests. The **transfer** problems, however, presented students with an unfamiliar situation. In these problems they had to transfer what has been learned to a different, but related situation. In solving some these problems they needed to perform computations and use algorithms that had been taught prior to the test and use the appropriate concepts, so solving some of these problems also involved skills belonging to the previous two categories.
The retention is the extent to which acquired knowledge, skills, etc. were retained by the learner (Ellington and Harris, 1986).

The placement score is the score on the mathematics placement exam given to entering students at The Ohio State University. This score determines the mathematics courses for which the student can register.

The next chapter provides a review of literature. Chapter Three describes the methods and procedures. Chapter Four contains the results. Chapter Five gives a summary and the discussion, along with some recommendations for further research.
CHAPTER II
REVIEW OF RELATED LITERATURE

2.1 Discovery Learning

Research results on discovery learning are conflictive and inconclusive. A conference on discovery learning (Shulmann and Keislar, 1966) found that for each experiment showing the discovery approach superior over exposition there was another experiment with the opposite result. Begle (1979), Cronbach (1977), and Fey (1980) also spoke of contradictory, inconclusive findings. The issue is still under debate (Orton, 1987).

This section reviews the research results and the arguments given for and against the discovery approach. Since there were at least several hundred studies in this area, research summaries rather than individual reports were used as primary source. The section also investigates the reasons for the contradictory findings. Finally, a research-based rationale is given for the techniques used in this study to facilitate discovery.

As Ausubel (1978) pointed out, most articles on discovery learning "report no research findings whatsoever. They consist mainly of theoretical discussions, assertion, and conjecture; of descriptions of existing programs utilizing discovery methods; and of enthusiastic but wholly subjective testimonials regarding the efficacy of discovery approaches." (p. 554). This stands for most works cited in this section. When reference was made to research findings, it will be indicated.

A leading advocate of discovery learning, in the 1960s and 70s was Bruner. Bruner (1961) listed the major benefits resulting from discovery learning as follows: greater
intellectual potency, intrinsic motivation, memory processing, and the learning of the heuristics of discovery.

It is, if you will, a necessary condition for learning the variety of techniques of problem solving, of transforming information for better use, indeed for learning how to go about the very task of learning. Practice in discovering for oneself teaches one to acquire information in a way that makes that information more readily available in problem solving.

Piaget and Bruner agreed that conceptions that children self-discover are usually more meaningful than those provided by others. Bruner added that this way children not only develop problem solving skills, but also acquire confidence in their own learning abilities as well as a propensity to function later in life as problem solvers. "They learn how to learn as they learn" (Biehler and Snowman, 1982, p. 254).

Bruner (1966) also believed the following:

We teach a subject not to produce little living libraries on that subject, but rather to get a student to think . . . for himself, to consider matters as an historian does, to take part in the knowledge-getting. Knowing is a process, not a product. (p. 72)

Additional arguments for learning by discovery include:

- Knowledge can be more thorough and complete if obtained by discovery rather than by exposition (Orton, 1987).
- Students may understand a principle better if they themselves have extracted it from experience (Cronbach, 1977).
- There is some research indication that students may retain their knowledge longer and be able to apply them more widely if they discover it rather than absorb it from exposition (Begle, 1979).
- Discovery experience promotes autonomy (Cronbach, 1977).
- Discovery learning arouses students' curiosity, motivates them (Berlyne, 1965).
- Finding things out can be fun (Cronbach, 1977).
Let us close the series of arguments for the discovery approach with that of Biggs (1972):

I believe this method [discovery] is the best way to give our pupils real excitement in mathematics. I believe too, that it is only when we give our children a chance to think for themselves that they realize their full potential.

Skinner (1968) was one of the major critics of the discovery approach. He pointed out that:

- Teachers may choose to use discovery techniques to avoid the possibility of failure since they cannot fail if they do not even attempt to provide instruction.
- It is impossible for any group of inexperienced students to discover for themselves more than a tiny fraction of the accumulated knowledge of a culture.
- It is counterproductive to tell students that it is beneath their dignity to learn what others have already discovered.
- Genuine discovery is extremely rare. And "you cannot 'learn by discovery' if you fail to discover" (Cronbach, 1977, p. 548).
- Teachers may have to pretend that they do not know very much about a topic. This may cause their pupils to lose respect for them.
- The best students in any class are likely to make most of the discoveries. Less capable students may feel discouraged and lose interest.

(Summarized by Biehler and Snowman, 1982, p. 259-260.)

Additional arguments against the discovery approach include:

- The discovery approach is not appropriate for most learning at the abstract level (Ausubel, 1963).
- Students may not know what is important or relevant (Ausubel, 1978).
There may be some students who never come up with a discovery of their own. This can create jealousy, resentment, or feelings of inferiority (Biehler and Snowman, 1982).

Bruner (1961) was careful to present his teaching about the superiority of the discovery approach as a hypothesis, rather than an established fact: "So goes the hypothesis [the superiority of the discovery approach]. It is still in need of testing. But it is an hypothesis of such important human implications that we cannot afford not to test it" (p. 26).

There have been no shortage of discovery enthusiasts who claimed to have verified the hypothesis. Gagne and Brown (1961) claimed to have shown that guided discovery was the best method (of those used) for the learning of certain rules. Goldin (1989) wrote: "I think it is empirical fact . . . that methods based on 'transcription' and application of rules are less successful than those of mathematical discovery" (p. 22).

Ausubel (1963) argued that the discovery approach only seemed to be the best because of what it had been compared with--usually rote learning. After a careful analysis of the research literatureAusubel (1978) reached the following three conclusions:

1. Most of the articles . . . supportive of discovery techniques actually report no research findings whatsoever. They consist mainly of theoretical discussion, assertion, and conjecture; of descriptions of existing programs utilizing discovery methods; and of enthusiastic but wholly subjective testimonials regarding the efficacy of discovery approaches.

2. Most of the reasonably well-controlled studies report negative findings.

3. Most studies reporting positive findings either fail to control other significant variables or employ questionable techniques of statistical analysis. (p. 554)

He added: "This is not to say that the evidence is negative, but rather that there is just no evidence" (p. 555). Wittrock (1966) arrived at the same conclusion.
A major reason for the contradictory findings was the lack of a precise, consistent definition for the discovery teaching. A great variation in the nature of the experimental treatments was reported (Ausubel, 1978; Begle, 1979; Shulman and Keislar, 1966; Fey, 1980). The message of the discovery approach is vague and inconsistent and, therefore, apt to be misapplied (Dunkin and Biddle, 1974; Good and Biddle, 1988). But even the creation and general acceptance of a precise definition would not necessarily solve this controversy since "one cannot compare, say, 'discovery teaching' with 'non-discovery teaching'... one can only compare some specific attempts to do 'discovery' teaching, vs. some specific attempts to do 'non-discovery' teaching" (Davis, 1984).

A related problem is "the failure of most investigators to measure the extent to which their prescribed instructional treatments were accurately carried out by the experimental teachers" (Fey, 1980, p. 405). Another major reason is the lack of control for other variables (e.g., inductive-deductive, verbal-nonverbal approach, intramaterial organization of the subject matter) in many of the studies, as pointed out by Ausubel (1978), and Shulman and Keislar (1966). A significant portion of the studies employed questionable research design, data analysis techniques or the interpretation of the results was questionable (Ausubel, 1978; Shulman and Keislar, 1966; Fey, 1980). Bias of the experimenter was common, usually in favor of the experimental treatment (Fey, 1980).

In spite of the fact that the research results in general can be best described as contradictory and inconclusive, there are some regularities in the experimental findings (Begle, 1979). When discovery was superior to reception learning, it was usually on tests measuring problem solving ability (Ausubel, 1978; Begle, 1979), transfer (Babikan, 1971; Begle, 1979; Olander and Robertson, 1973; Worthen, 1967), or retention (delayed achievement test) (Begle, 1979; Olander and Robertson, 1973; Worthen, 1967). The discovery method is particularly appropriate for teaching the scientific method (how new knowledge is discovered) (Ausubel, 1978). Expository teaching, on the other hand, seems
more effective for immediate recall (Ausubel, 1978; Babikan, 1971; Begle, 1979; Olander and Robertson, 1973; Worthen, 1967), or for teaching computational skills (Begle, 1979; Olander and Robertson, 1973; Worthen, 1967). Begle (1979) added that "there is some indication that for mathematical concepts expository teaching is definitely better than discovery" (p. 121), but there is less agreement about this aspect.

A number of research studies compared the effectiveness of instructional styles on students of different ability, cognitive style, and personality (aptitude-treatment interaction (ATI) research). Discovery teaching seemed to work well for high ability learners, but was less effective or even detrimental for low ability learners (Babikan, 1971; Cronbach and Snow, 1977; Olander and Robertson, 1973; Snow, 1982; Worthen, 1967). The discovery approach tended to be more successful in the preschool and early elementary school years (Ausubel, 1978; Orton, 1987) when more concept formation than concept assimilation takes place, even though according to Biehler and Snowman (1982) "discovery techniques may not be appropriate for primary grade pupils" (p. 262). Ausubel (1978) stated that discovery teaching can work well with "adolescents and adults who are relatively unsophisticated in the basic concepts and terminology of a given principle" (p. 528). Biehler and Snowman (1982) disagreed: they suggested the use of discovery approach "primarily with older students, who have acquired a broad background of information" (p. 262). Another finding is that the discovery method does not seem to fit students with disadvantaged background (Biehler and Snowman, 1982). Finally, there is some evidence to the effect that students who are successful in discovery do better the next time and discover faster (Begle, 1979).

These results show that the issue of discovery vs. reception learning is much more complex that it may seem to be at the first look. There is no short answer. "An all-or-none position . . . is warranted by neither logic nor evidence" (Ausubel, 1978, p. 527). Discovery is certainly not an all-purpose instructional method (Cronbach, 1977). It is too
time-consuming and difficult (if not impossible) to use as a primary way of instruction to transmit the content of an academic discipline (Ausubel, 1978; Biehler and Snowman, 1982; Slavin, 1986), but it can be an important supplement to more traditional instructional methods (Slavin, 1986). The evidence suggests the use of discovery methods at selected places in the curriculum, but "research has not come to grips with the problem of combining inductive and expository modes in a curriculum sequence" (Cronbach, 1977, p. 547). (Cronbach uses the term inductive approach as a synonym for discovery approach.)

Several experimental studies compared discovery methods with different amounts of guidance provided for the students. Discovery learning without any guidance is a "terribly slow process. It is quite evident that adult human beings do not typically learn concepts by this method" (Gagne, 1966, p. 142). Guided discovery techniques have shown to be consistently more effective than unguided discovery (Ausubel, 1978; Cronbach, 1977; Woolfolk, 1987). Providing an "intermediate" amount of guidance seems to be superior over both providing very little guidance or providing complete guidance (Ausubel, 1978).

In this study guidance was provided in form of worksheets, hint-sheets and solution-sheets, in order to minimize the following difficulties with discovery learning.

- Students discover at different pace. Without individual or group work on worksheets or similar techniques only the fastest students would discover, others would be involved in reception learning, listening to the "discoverer", as a result they may also feel discouraged and lose interest (Cronbach, 1977). Or, if the teacher would wait for the slower students to discover it would be a waste of time for the faster students.
- Students can discover only a tiny fraction of the accumulated knowledge of a culture (Skinner, 1968). With the use of the hint-sheets this fraction can probably be significantly increased (but certainly it will still only be a fraction). Learning using the hint-sheets can still be considered primarily discovery rather than reception learning, only the amount of guidance is different, and, therefore, the position of the learning on
the reception-discovery continua (Ausubel, 1978). Different students may need
different amount of guidance. If the hint-sheets are used properly, line-by-line, several
levels of guidance can be provided to a given problem. This way the instruction can be
individualized to some extent.

- Research shows that discovery is very time-consuming (Ausubel, 1963, 1978; Begle,
  1979; Goldin, 1990; Orton, 1987). The teacher cannot wait forever for students to
discover (Bruner, 1960). The hint-sheets were designed to reduce this problem. If
students are unsuccessful they can get help from the hint-sheets, and they can get it
when they need it, instantly; so they can move on to the next problem (which is often the
next piece in a chain of problems leading to a major result).

- There may be some students who never come up with a discovery of their own. This
can create jealousy, resentment, or feelings of inferiority (Biehler and Snowman, 1982).
With the use of the hint-sheets all students may make at least some discovery of their
own.

- The rationale for the solution-sheets is obvious. It would be unrealistic to hope that all
students will discover everything we would like them to discover using the worksheets
(even with the hint-sheets). And "you cannot 'learn by discovery' if you fail to
discover" (Cronbach, 1977, p. 548)]. Even the most radical discovery enthusiasts
would agree that reception learning is better than no learning at all.

2.2 Calculator/Computer Graphing and Computer Algebra Systems

Due to the newness of this technology there are a relatively small number of studies in
this area. Most have investigated the impact of the use of calculator/computer graphics
and/or computer algebra systems on student achievement. The typical results are that
students in the experimental groups show a higher level of conceptual understanding on
written tests and in interviews than students in the control groups, and have about the same
achievement as control group students on the computational tests. Students' attitudes toward technology and their beliefs about mathematics and technology change as a result of technology use and these changes can generally be considered positive. Student and teacher roles and behaviors are different when technology is used. There appear to be sex differences in some aspects of attitudes, perceptions, confidence, problem solving approaches and performance.

2.2.1 Studies with Focus on Student Achievement

(1) Studies Using Graphics Only

Preece (1984) investigated students' concepts of gradient (e.g. slope) and the relationship between skills involved in sketching and interpreting graphs. Four 14 years old students were observed working with an interactive computer graphing program, SKETCH. Students watched a dynamic screen simulation (the mercury rising/falling in a thermometer) and then attempted to sketch a graph representing the process. Then they compared their graphs with that of the computer. Subjects were interviewed, their sketches and answers to a written test were analyzed.

The researcher used a modified version of Janvier's translation skills model (Janvier, 1978) to describe students' moves between different representations. This study and a survey conducted by the author earlier (Preece, 1983) showed that the problems associated with the two translations of the Janvier model [situation -> graph (sketching (modelling)) and graph -> situation (interpretation)] are different and not directly comparable in difficulty.

Goldenberg and Kliman (1988) interviewed 18 bright and articulate 8-, 11- and 12-graders with computer graphing experience to investigate students' understanding of graphs. They gave particular attention to the issue of scaling, since it is a central issue in computer graphing.
The main observations of the study were given as metaphors inferred from students' words:

1. "Computer as automatic paper and pencil: Computer graphs are treated like paper and pencil graphs that can be cropped, but not stretched or shrunk."

2. "Scaling is like using a magnifying glass: As one looks closer at a curve, one sees its true nature and composition better. . . . As with a physical object, magnification shows roughness that may not otherwise be visible."

3. "Mathematical curve as a bead necklace: Points in a curve, like beads in a bead necklace, line up 'next to each other.' A radical enough scale can magnify these points so that they can be seen, and certain scale changes can distort the appearance of these points."

Browning (1989, 1990) investigated the impact of the use of graphing calculators on graphical understanding. She developed a Graphing Levels Test. It was administered to more than 200 precalculus students in central Ohio high schools, most of whom participated in the C²PC project; the rest served as a control group.

Results showed the benefits of using graphing technology: over 68% of the control group remained at low levels of understanding while over 73% of the C²PC students were operating at higher levels of understanding. Posttest results also indicated substantial growth in graphical understanding for the C²PC students as compared to the control group.

Interviews suggested that students receiving the experimental treatment were making connections between algebraic and geometric representations of a function.

Demana, Foley, Osborne and Waits (1989) conducted a high school curriculum revision project using graphing calculators and computer graphing, and putting strong emphasis on applications. The 1665 field test students and 660 control class students took both the Ohio State University Calculus Readiness Test and the C²PC test (where most items are based on items of the Second International Mathematics Study test concerned with
analytic geometry and functions). Project students showed greater improvement both on the graphing and non-graphing items of the tests.

Beckmann (1990) compared four treatments in a college first-semester calculus course, investigating student understanding of selected calculus concepts through graphical representation. The four treatment conditions were: "Graphics (G), exposure to a computer-graphically developed conceptual course; Graphics Plus (G+), exposure to the same course as G subjects plus the provision of computer graphics software and related supplemental assignment; Standard 1 (S1), exposure to a graphically-developed, conceptual course; and Standard 2 (S2), exposure to a traditional skill-oriented course."

Graphics subjects scored significantly higher than Standard 2 subjects on nonroutine symbolic questions. There were no significant differences between students in the Graphics and Graphics Plus sections on any of the cognitive variables. Non-significant differences favored the Graphics Plus subjects. Students' attitudes toward the use of graphs, toward the course, and toward mathematics were generally positive. The author concluded that "developing calculus concepts through the use of a graphical representation system, especially as presented through computer graphics, can positively affect student understanding and interest without negatively influencing skill acquisition" (p. 107).

Dunham (1990, 1990/1991, 1991) investigated relationships between confidence and performance in a college precalculus course which fully integrated graphing calculators. She measured the confidence and performance of 213 students in the first and last weeks of the 10-week course and interviewed eight high and eight low confidence subjects after each of the four exams to obtain information about shifts in attitudes and technology use patterns.

She found significant positive correlations (from 0.17 to 0.30) between problem-specific confidence and performance more often for women than for men. Individual measures of confidence and performance showed significant interaction between sex and
problem context on both pre- and posttest. Low confidence females used graphing
calculators more and algebra less than any other group. Some students felt "algebraic guilt"
for "cheating"—choosing the "easy way" of solving problems by graphing calculator.

Ruthven (1990) compared the achievement of the high school students in the Graphic
Calculators in Mathematics project (England), who regularly used graphing calculators in
class, with that of the control group students. The project group showed superior
performance (with the use of the graphing calculator) on the symbolization items, calling
for an algebraic description of a graph; but not on the interpretation items, calling for the
extraction of information from some verbally contextualized graph. Female students
outperformed their male counterparts on the symbolization items in the project group,
revising the pattern in the comparison group. Males had superior performance on the
interpretation items in both groups.

Tufte (1990a, 1990b) examined the influence of computer programming and
computer graphics on conceptual understanding in college introductory calculus classes. In
Experiment I, 20 students enrolled in a three-semester engineering calculus sequence which
met for an additional period each week during the first semester to receive supplemental
instruction. In Experiment II, the 32 second semester calculus students had no additional
classtime for the supplemental instruction.

Both experimental groups had significantly higher scores on the conceptually-oriented
tests than did the control groups. In Experiment II, the experimental group performed no
worse than other calculus classes on the common final exam, testing primarily procedural
knowledge. Students in the experimental groups developed a geometric perspective of
derivatives and integrals that was not exhibited by members of the comparison group.

(2) Studies Using Both Computer Algebra and Graphics

Heid (1988) used a computer algebra system, muMath, a function grapher program,
Graph Functions, Fit Functions to Data, Table of Values, and other demonstration
programs as tools in a concept-oriented introductory calculus course. Two experimental classes of an applied college calculus course studied calculus concepts using graphical and symbol-manipulation computer programs to perform routine manipulations during the first 12 weeks of the semester. The last 3 weeks were spent on skill development. The control class had the emphasis on skills in demonstrations, assignments, quizzes and examinations in all 15 weeks of the semester. The experimental classes showed a higher level of conceptual understanding and about equal level of skill development on the final exam and in tape-recorded interviews. The study showed that the concepts of calculus can be learned without concurrent or previous mastery of the usual algorithmic skills of computing derivatives and integrals, and sketching curves.

Dubinsky and Schwingendorf (1990) used the interactive programming language \textit{ISETL}, the computer algebra system \textit{MAPLE}, and computer graphing in their college introductory calculus curriculum development project. The project was built on the constructivist learning theory. They believed the primary role of the teacher was "not to lecture, explain or otherwise attempt to 'transfer' mathematical knowledge, but to create situations for students that will foster their making the necessary constructions" (p. 2).

Engineering, science and mathematics project students outperformed their counterparts on the common departmental multiple choice final exam testing primarily mechanical skills, even though the project does not emphasize mechanics. Management, social science, and life science project students performed at about the same level as non-project students on these exams.

(3) \textbf{Studies Using Computer Algebra Only}

Palmiter (1986, 1991) investigated the use of a computer algebra system \textit{(MACSYMA)} in a college introductory integral calculus course. The experimental group of 40 students completed the same conceptual material in half the time using \textit{MACSYMA} instead of performing the traditional paper and pencil techniques of integration. Both the
experimental and the control groups took conceptual and computational exams; the experimental group after five weeks, the control group after ten weeks of instruction. The experimental group scored significantly higher on the conceptual test, without the use of MACSYMA. On the computational exam the experimental group scored significantly better than the control group and completed the computations in half the time, with the use of the computer.

Palmiter concluded that, with the use of a computer algebra system, integral calculus could be taught in substantially less time, eliminating the teaching of techniques of integration, and yet produce equivalent or better conceptual understanding.

Judson (1988a, 1988b) conducted an experimental study with random assignment to investigate the effects of resequencing skills and applications in an introductory college business calculus course. Students in the two experimental classes were taught the same curriculum, but learned concepts and applications of differentiation before the rules of differentiation. A computer algebra system, MAPLE, was used to perform calculations. The two control classes followed the traditional pattern: differentiation rules and skill development before applications.

There were no significant differences in differentiation skills, conceptual understanding and in the ability to solve application problems between the experimental and the control groups. The author pointed out, however, that the sample size was small ($N = 39$), a larger sample would have been needed to detect differences, if the differences were small. On the other hand, lack of statistically significant differences does not mean lack of educationally significant differences. The experimental treatment had a positive effect on student interest, attitude, and motivation. Class participation and enthusiasm was notably higher in the MAPLE group. This study supports the claim that skills acquisition is not prerequisite to understanding.
2.2.2 Studies with Focus on Attitudes, Perceptions, Student and Teacher Behaviors

Dick & Shaughnessy (1988) provided each of two high school mathematics departments with a classroom set of HP 28 calculators. Five (self-selected) teachers participated in the 6 months long project. They used the calculators in their Intermediate Algebra, Trigonometry, and Calculus classes, providing a classroom environment where every student had individual access to one of these calculators on a regular basis. At the beginning and at the end of the project the 294 students in the experimental classes were given multiple Likert-type attitude scales to assess their attitudes and beliefs about mathematics and calculator usage. In addition, the researchers made classroom observations and conducted interviews with the teachers.

At the end of the project students indicated more agreement with the statements that calculators helped them learn more about difficult math topics, they made math more fun, and they were easy to use. There were surprisingly big differences between Algebra II and Calculus students' attitudes and perceptions, and significant differences between male and female students. Male students were more likely to view mathematics as a rule-dominated discipline, and female students had more anxiety toward calculator usage. Teachers were most impressed with the graphing capabilities of the calculator, they felt the symbolic capabilities were less useful. Teachers felt the calculators brought only minor changes in the dynamics of classroom interaction. This was contrary to the findings of other similar studies.

Farrell (1989/1990, 1991) examined teacher and student behaviors when graphing utilities (graphing calculators and/or computer graphing) were integrated into a precalculus course. She analyzed six videotaped lessons from each of six teachers using the Systematic Classroom Analysis Notation (SCAN).

Students were more active and exhibited a wider variety of roles (Manager, Task Setter, Explainer, Consultant, Fellow Investigator, and Resource) when technology was
used. They worked in groups, consulted each other, and were involved in problem solving more often with technology. Teachers remained primarily managers, but were observed as task setters and explainers less often and as consultants more often when technology was in use.
CHAPTER III
METHODS AND PROCEDURES

3.1 Research Design

This research was a three group experimental study.

**Group 1:** Use of graphing calculators + (guided) discovery approach with worksheets, hint-sheets and solution-sheets

**Group 2:** Use of graphing calculators without discovery

**Group 3:** No graphing calculators, no discovery (traditional instruction)

In Winter 92 Group 1 was taught by the investigator, Group 2 by James Ham, and Group 3 by Chun Lin Qian. All were teaching assistants of the Department of Mathematics and had comparable teaching experiences and student evaluations. The study was repeated the next quarter with a switch of instructors between Groups 1 and 2. Pooled results from the two quarters were used for the data analysis to minimize the instructor effect.

Every effort was made to insure that the instruction of the three groups only differed in the use/non-use of graphing calculators and discovery style teaching, and they were comparable otherwise; so that the study could be free of contaminations as much as possible. The selection of the instructors was done with this objective in mind, as described in the paragraph above. The three groups followed the same syllabus, the same problems were planned for discussion in the groups, the instructors used the same lesson plans, except for the differences due to graphing calculator use and discovery style.
teaching. The instructors met before each class to discuss the lesson plan for that day and the extent to which they followed the previous lesson plan. When it turned out that there were differences (e.g., there was no time to do all the problems planned, students' questions led to different discussions in the groups, et cetera) corrective actions were planned for the next class to keep the groups as close as possible. The differences were recorded and kept in mind for test problem selection to avoid selecting a test problem that would favor one of the groups. The entire class period was spent on the new section planned for that day, except for the cases when there was no time to cover the entire previous lesson plan. The instructors discussed homework problems, answered students' questions on the previous sections in office hours and help sessions before/after class, rather than in class, in order to minimize the differences in what was covered in the three groups.

The following steps were taken to insure that the instruction in the study's sections follow the design of the study and to document this. Other mathematics education doctoral students were asked to observe the instruction to check its consistency with the prescribed treatment. Daily teachers' notes were kept and the lessons were audiotaped to document the instruction as carefully as possible. The researcher listened to a random selection of these tapes to discover possible unplanned differences in the instruction. If such differences were discovered, corrective action was taken, whenever possible.

3.2 Pilot Study

A pilot study was conducted in Fall 91 to test the instruments and instructional materials. James Ham taught a graphing calculators without discovery section, and the investigator taught a graphing calculators + discovery section. These were night classes, met twice a week for 2 hours 15 minutes, similar to the sections in the actual study. Thus, it was possible to test the instructional techniques under the same circumstances. The pilot
study provided valuable information on how students used the worksheets, hint-sheets and solution-sheets and how graphing calculators can be used in differential calculus.

3.3 Population

College students enrolled in introductory calculus classes in the U.S. comprise the intended target population for this study. The accessible population for the investigation consisted of students registered for Differential Calculus (Math 151) for a given time slot (night) at The Ohio State University in Winter 92/Spring 92. There was no sampling done for the study. All three Math 151 night classes were used in the study both in Winter 92 and in Spring 92.

Students registered for the sections without knowing about the experiment and without knowing who their instructors would be. The schedule of classes only gave the time and place of the classes for each call number. All three classes met at the same time (Tuesdays and Thursdays 7-9:15 P.M.), in the same building. If different number of students attempted to register for sections offered at the same time the registration computer evened out the differences by random reassignment. After the initial registration students were free to switch sections provided the section they tried to change to was not closed (i.e., the number of students in that section did not exceed a predetermined maximum number).

Students who entered any of the study's sections after the first day of classes were interviewed to determine whether they chose that particular section by random or they self-selected the instructional technique or the instructor. In the latter cases their data were not included in the data analysis. For Group 1 (graphing calculators + discovery), 3 out of the 8 add students were excluded, for Group 2 (graphing calculators, no discovery), 4 out of 10, and for Group 3 (control), 1 out of the 4 add students were excluded for this reason. Data of audit students and of a student who had a learning disability were excluded.
Students who dropped out of any of the experimental sections were interviewed to determine whether they dropped the course because of the experimental treatment or for some other reason. Nine students dropped out of Group 1 (graphing calculators + discovery), 8 of them said the experimental treatment had not played a role in their decision, one student could not be contacted. For Group 2 (graphing calculators, no discovery), 15 out of 18 dropped out for other reasons, 2 in part because of the graphing calculator use, and one student could not be contacted.

3.4 Hypotheses

The following hypotheses were tested at the $\alpha = 0.05$ level:

For computational skills:

H 1: There will be no significant difference between the three groups on computational skills on the final exam and on the two retention tests.

For conceptual understanding:

H 2a: Both graphing calculator groups will exhibit significantly higher conceptual understanding on the final exam and on the retention tests than the control group will.

H 2b: There will be no significant difference between the two graphing calculator groups (i.e. with/without discovery) on conceptual understanding.

For transfer skills:

H 3a: Students in the graphing calculator + discovery section will exhibit significantly higher transfer skills on the final exam and on the retention tests than students in either of the other two groups will.

H 3b: Among the other two groups, the graphing calculator group will score significantly higher.
H 3c: These differences (H 5a, 5b) will be the greatest on the long term retention test.

For the other dependent variables:

H 4: There will be no significant difference between the groups on the amount of time students spent on the course.

H 5: Students in the discovery group will work with classmates outside of class to a significantly greater extent than students in the other two groups.

3.5 Treatments

All groups covered the same curriculum as other sections of Math 151: Chapters 2 through 4 of Calculus by Finney and Thomas (the required text for Math 151 at The Ohio State University).

In Groups 1 and 2 each student who did not have a graphing calculator was issued a TI 81 graphing calculator. They could take the calculator home and keep it until the end of the quarter. Calculator use was an integral part of the course, with a portion of the in class and homework problems requiring their use. The graphing calculators were most often used to obtain a graphical representation of a problem. They were also often used for numerical computations, at times using short programs provided and briefly explained by the instructors. The graphing calculators made it possible to take advantage of multiple (algebraic, numerical and graphical) representations. They also served as time savers by doing a large number of computations, graphing several functions in a short amount of time. They made it possible to examine, explore a large number of examples.

In Group 3 graphing calculators were not required but allowed, by department policy. As was expected most of the students in Winter and in Spring did have a Casio or TI 81 graphing calculator since it was required in the prerequisite course, Math 150. Only about half of the students in the Fall pilot study had graphing calculators since for many this was their first OSU math course. A few had a supercalculator (HP 28S, 48S, or 48SX). These
experiences were in line with the past experiences of the Mathematics Department.

Groups 2 and 3 followed traditional discussion format. In Group 1 part of the new material was covered in guided discovery style using worksheets, hint-sheets and solution-sheets. In the worksheets a chain of questions/problems led to the new concept, relationship or technique. Students tried to do the problems on their own or in groups. Students were free to choose between individual and group work at any time, and they were also free to choose/change their groupmates at any time. If students needed help they could look at the hint-sheets and solution-sheets, but only one line at a time, and then they were to continue on their own. They could ask for help from their classmates and/or the instructor. They were to check their answers after each problem using the solution-sheets and correct them if necessary. A typical student was able to do some of the problems without the hint-sheets, he or she was able to do some other problems with the hint-sheet, and he or she needed to use the solution-sheets for the rest. At times there was a fourth group of problems where some students needed some further help from a classmate or the instructor. Data were collected via questionnaire on how and to what extent students used the worksheets, hint-sheets and solution-sheets.

Obviously, students cannot discover everything mankind has discovered in 2000 years (especially not in 50 x 48 minutes); thus, part of the material was covered in the traditional discussion format in the discovery sections, too. Section 2.2 provides for research evidence to support the statement that the discovery approach is not a viable option as a primary way of instruction, but can be successfully used as a supplement to exposition. On the other hand, there was some student discovery activity in the discussion section, too. It would have been artificial and not desirable to keep a 100% lecture style, especially in small classes (20-30 students).

Therefore, the study compared not the pure discovery and pure exposition types of instruction. The study compared an instruction where an attempt was made to include as
much discovery activity as possible in the given time and curriculum frame, using worksheets, hint-sheets, and solution-sheets with individual and group work against the "traditional discussion type instruction", with or without graphing calculators where there were no worksheets, no individual or group work in class, the material was presented in a lecture + teacher question, student answer type discussion with the class. The instruction in Group 1 was an attempt to combine exposition and discovery style teaching into an effective instructional sequence as called for by Cronbach (1977). Furthermore, the study did not compare these kinds of instructions in general, only specific attempts to implement these instructions for a given curriculum, since the comparison in general is not possible (Davis, 1984).

Appendix D contains the worksheets, hint-sheets and solution-sheets. They were used to teach about 20% of the material. The topics that were taught in part or entirely using discovery approach are the following (section numbers refer to the textbook Calculus by Finney and Thomas): slopes and derivatives (3.1), derivatives of trigonometric functions (3.4), chain rule (3.5), implicit differentiation (3.6), Newton's method (3.8), maximum, minimum and Mean Value Theorem (4.2) and curve sketching with \( y \) and \( y' \) (4.3).

Other mathematics education doctoral students observed each section three times in each quarter to verify the validity and distinctness of the treatments. They and the researcher had a meeting each quarter to discuss the observations. The observers confirmed that the instruction was in accordance with the research proposal. One of them was unable to continue the observations in the second quarter of the study, he was replaced by a science faculty member who had sufficient background in mathematics for this job.
3.6 **Follow-up**

In order to measure long-term retention, a delayed achievement test was administered toward the end of the quarter following the instruction. In order to determine how students of the study performed in the follow-up math course, Integral Calculus (Math 152), their grades in the follow-up course were analyzed for those students who took the follow-up course in the quarter after the experiment. In order to complete the study in Summer 92, these follow-ups were only done for students who participated in the experiment in Winter 92.

3.7 **Instrumentation**

The major instruments of the study were the pretest, the final exam and the two retention tests. They can be found in the Appendix B. Each problem on these tests was classified as an item measuring computational skills, conceptual understanding or transfer skills (ability to transfer what has been learned to a different, but related situation). This classification was done by a panel of doctoral students in mathematics education.

The pretest was administered in the second week of the quarter; by that time most late registrations were completed. Most of the items were chosen from precalculus and algebra, with a few calculus items to measure how much calculus students already knew from their prior calculus classes in high school or in college. It required 48 minutes of classtime. Scores of this test were not used for the purpose of determining grades.

The short-term retention of knowledge was measured by a 30-minute, unannounced test given toward the end of the quarter, containing problems about the material covered at the beginning of the quarter (Chapter 2: Limits and Continuity). The test items were selected from those areas that were not used directly after Chapter 2. Scores of this test were used for the purpose of determining grades only if it improved the grade of a given student.
The final exam was comprehensive. Students had 120 minutes to complete the exam. The same final exam was used in the two quarters of the study so that pooled results from the two quarters could be used in the data analysis. The exams were not returned to the students and students did not know that the same exam would be used so there is no reason to believe that students in the second quarter had prior knowledge of the test problems.

The 48 minute long-term retention test contained selected problems from the final exam. It was administered toward the end of the quarter following the instruction. A copy of the test was mailed at the beginning of the next quarter to those students who did not show up at any of the times when the test was given. Students were paid $10 for their time to take this test.

To insure consistency in grading, each problem on the final exam was graded by the same instructor for all students, rather than each instructor doing the grading for his class; and the grading was done one problem at a time rather than one student at a time, following a predetermined grading guide. To insure unbiased grading, teaching assistants of the Mathematics Department not involved in the study graded the pretests and retention tests using a grading guide constructed by the researcher. For consistency, each problem was graded by the same teaching assistant for all students and the grading was done one problem at a time rather than one student at a time.

Partial credit was awarded on all tests in problems where the solution consisted of more than one step. For most problems it was done by dividing the solution into small parts and assigning a point value for each part. Students who did a particular part correctly received the corresponding credit, others did not. If an incorrect result was carried over from one part to the other and it was handled correctly there, then full credit was given in this second part, even though all numerical values may have been wrong. In other words, students were not penalized twice for the same error. For some other problems the partial credit was determined by subtracting a certain number of points for each error.
The placement scores and the grades in the previous OSU math course were obtained from university records. The other variables were measured by a questionnaire completed after the final exam (see Appendix B).

3.8 Data Analysis Procedures

The SPSSX statistical program package was used on a mainframe computer to analyze the data. When a procedure was not available in SPSSX, the SAS program package was used instead.

The major part of the data analysis was to test if there were statistically significant differences between the three groups on several background and dependent variables. The a priori level of significance was 0.05 for all the comparisons.

The major comparisons of this study were the comparisons of students' scores on the computational, conceptual and transfer parts of the final exam, and the short and long term retention tests. Analysis of covariance was implemented for these comparisons, using the computational pretest subscore as covariate for computational skills, the conceptual pretest subscore as covariate for conceptual skills, the transfer pretest subscore as covariate for transfer skills, and the overall pretest score as covariate for the overall performance. The computations were carried out using the General Linear Methods procedure of the SAS program package. The analysis of covariance requires a linear relationship between the covariate and the dependent variable. The existence of this linear relationship was tested at the $\alpha = 0.05$ level in each case. The analysis of covariance requires the homogeneity of variance within the treatment groups. Univariate homogeneity of variance tests (Bartlett) were implemented at the $\alpha = 0.05$ level to test if the data satisfied this requirement. The analysis of covariance requires the homogeneity of regression within the groups. Therefore, the slopes of the regression lines were compared in each case at the $\alpha = 0.05$ level.
The analysis described in the previous paragraph was repeated for the transfer skills items of the final exam that belonged to topics taught by the discovery technique. These items were 8, 9, 10 and 14 (see Appendix B). Since there was only one item on the final exam measuring conceptual understanding in the topics taught by discovery (13) and one such item measuring computational skills (7a), this comparison was not implemented for conceptual understanding and computational skills.

Analysis of variance in SPSS$\text{X}$ was used to compare the three groups on the other continuous variables: students’ computational skills, conceptual understanding, transfer skills and their overall performance on the pretest, and students' time spent on the course. These comparisons were also done at the $\alpha = 0.05$ level. Analysis of variance requires the homogeneity of variance among the groups. Univariate homogeneity of variance tests (Bartlett) were used to test if the data satisfies this assumption at the $\alpha = 0.05$ level.

The chi-square test of association in SPSS$\text{X}$ was implemented to determine if there were significant differences between the groups in the nominal and ordinal variables. The chi-square test was implemented in two different ways:

(a). Traditionally it has been required that the expected frequency is at least 5 in each cell, or at least there are not too many cells with expected frequency less than 5 (e.g., Hays, 1973; Marasculo, 1971; Norusis/SPSS-X Inc., 1988; Siegel, 1956). There is some disagreement on what not too many exactly means, not more than 20% is a typical interpretation. Most of the nominal and ordinal data did not meet this requirement; thus it was necessary to collapse the values of the variables to reduce the number of cells. For example, the + and - signs were omitted from the grades.

(b) Several studies (Camilli & Hopkins, 1978, 1979; Conover, 1974; Everitt, 1977; Roscoe & Byars, 1971) indicate that the chi-square test can be used without these requirements, even when the average expected frequency is as low as 2. Therefore the data was analyzed without recoding, too.
In the comparison of the grades of the follow-up course, Integral Calculus, even the average expected frequency was less than 2 because of the small sample size \((n = 17, \text{ and } 15 \text{ cells after collapsing})\). Fisher's exact test was used in SAS for this variable since it has no assumptions on the expected frequencies.
CHAPTER IV
RESULTS

4.1 Comparison of Groups at the Beginning of the Study

A pretest was administered in all groups in the second week of the quarter. Pretest scores were used as covariates in the data analysis and to test whether there were significant differences between the groups at the beginning of the quarter. In addition to the pretest scores; students' precalculus grades and the year in which they took precalculus (if it was at this university), students' prior differential calculus experience at this university, mathematics placement levels and the year in which students took the placement test, and students' sex were analyzed. Significant differences were found in the distribution of male and female students in the three groups ($p = 0.007$). For the other variables listed above the groups were not significantly different at the $\alpha = 0.05$ level. This section gives the details of these comparisons.

Analysis of variance was used to compare students' computational skills, conceptual understanding, transfer skills and overall performance on the pretest. Table 1 shows that the procedure did not find statistically significant differences between the groups at the $\alpha = 0.05$ level. Analysis of variance requires the homogeneity of variance among the groups. Univariate homogeneity of variance tests (Bartlett) indicated that homogeneity of variance could be assumed at the $\alpha = 0.05$ level for computational skills, conceptual understanding and overall performance, but not for transfer skills (Table 2). Several data transformations (logarithmic, exponential, radical root, square) were attempted to obtain a data set with
homogeneous variance, but none of them was successful. However, given the
approximately equal sample sizes (38, 45, 40), given how close the transfer skills means
are (7.97, 7.62, 7.70) and given the high $F$ probability (0.913) it seemed reasonable to
assume that there were no significant differences between the groups on transfer skills.
Thus, the groups were judged to have comparable computational skills, conceptual
understanding and transfer skills at the beginning of the study.

Table 3 shows the precalculus (Math 150) grades of those students who took the
course at this university. The course has departmental syllabus, departmental midterms and
final examinations that are considered to be consistent in content and difficulty level from
quarter to quarter thus, the grades can be considered comparable even though they were
given in different quarters. The chi-square test of association did not show significant
differences between the groups in the grade distribution (Pearson chi-square $p = 0.92$).
Since 76% of the cells had expected frequency less than 5, the chi-square test was also
implemented after collapsing the values of this variable, omitting the + and - signs from the
grades (Table 4). No significant differences were found between the groups (Pearson chi-
square $p = 0.65$). The precalculus grade point averages of the groups were calculated and
found to be very close (Table 5).

It was tested if the groups differed on the year in which students took Math 150
(Table 6). The chi-square test of association did not show significant differences (Pearson
chi-square $p = 0.46$). Since 75% of the cells had expected frequency less than 5, the
procedure was repeated after collapsing the values of the variable to avoid cells with very
small expected frequencies (Table 7). No significant differences were found (Pearson chi-
square $p = 0.87$).

Table 8 shows how many students have already attempted Math 151, Differential
Calculus prior to the study. The information was obtained from the students' grade
records. A course appears in a student's grade record if he or she stays in that course
beyond the third week of the quarter. The chi-square test of association did not show significant differences at the $\alpha = 0.05$ level (Pearson chi-square $p = 0.12$).

Students entering The Ohio State University (except those who transfer from another university) are required to take a mathematics placement test. Table 9 shows the mathematics placement levels of the students in the three groups. The chi-square test of association did not show significant differences at the $\alpha = 0.05$ level (Pearson chi-square $p = 0.27$). However, since 48% of the cells had expected frequency less than 5, the data was also analyzed after collapsing some categories (Table 10, Pearson chi-square $p = 0.43$).

Students took the placement test in different years, ranging from 1983 to 1992 (Table 11). The differences were not found to be significant at the $\alpha = 0.05$ level (Pearson chi-square $p = 0.25$). Since 78% of the cells had expected frequency less than 5, the data was also analyzed after collapsing some categories (Table 12, Pearson chi-square $p = 0.71$).

Table 13 shows the distribution of male and female students in the three groups. This time the chi-square test of association did show significant differences at the $\alpha = 0.05$ level (Pearson chi-square $p = 0.007$). In Group 1 (graphing calculators + discovery) the ratio of female students is less (12.5%), in Group 2 (graphing calculators without discovery) the ratio of female students is more (43.5%) than in the entire sample (30.1%). The data analysis was done after excluding the data from those few students who self-selected either the treatment or the instructor. Therefore, these differences cannot be interpreted as a result of a negative attitude on part of the female students towards discovery style teaching. There is no reason to believe that these differences are a result of anything but chance.

The data analyzed indicated the three groups had comparable mathematical backgrounds.
4.2 Comparison of Groups on the Final Exam, Short And Long Term Retention Tests

Each examination consisted of computational, conceptual and transfer problems. Students' performance on the exams were first analyzed in these categories, then their overall performance were compared giving equal weight to each of the three categories. Analyses of covariance were implemented for the comparisons, using the computational pretest subscore as covariate for computational skills, the conceptual pretest subscore as covariate for conceptual skills, the transfer pretest subscore as covariate for transfer skills, and the overall pretest score as covariate for the overall performance.

This analysis was repeated for those transfer skills items on the final exam that belonged to topics taught by the discovery technique. These items were 8, 9, 10 and 14 (see Appendix B). Since there was only one item on the final exam measuring conceptual understanding in the topics taught by discovery (13) and one such item measuring computational skills (7a), this comparison was not implemented for conceptual understanding and computational skills.

The existence of a linear relationship between the covariate and the dependent variable was tested at the $\alpha = 0.05$ level in each case. With one exception, the existence of this linear relationship was confirmed at the $\alpha = 0.05$ level. (It should be noted that in over half of the cases the $p$ values were below 0.001.) For the conceptual part of the long term retention test $p = 0.57$ was found.

Univariate homogeneity of variance tests (Bartlett) indicated that the homogeneity of variance within the groups could be assumed at the $\alpha = 0.05$ level for 10 out of the 13 comparisons. For the conceptual part of the final exam Bartlett $p = 0.050$, for the transfer part of the short term retention test Bartlett $p = 0.021$, and for the overall performance on the short term retention test Bartlett $p = 0.045$. Several data transformations (logarithmic, exponential, radical root) were attempted to obtain a data set with homogeneous variance, but no single transformation was successful for all three variables. It did not seem
desirable to use different transformations for the different variables. Given the approximately equal sample sizes (34, 33, 27 for the final exam and 26, 30, 23 for the short term retention test) the unequal variances are not likely to have a significant effect on the results.

The analysis of covariance requires the homogeneity of regression within the treatment groups. Therefore the slopes of the regression lines were compared in each case. No significant differences were found at the $\alpha = 0.05$ level. In other words, no statistically significant pretest/treatment interaction was found at the $\alpha = 0.05$ level.

Table 14 gives a summary of the comparisons. It includes the $F$ probability of the treatment main effect only when this probability is below the preset level of significance, 0.05. (The $F$ probability of the treatment main effect is the lowest significance level at which the null hypothesis that the different treatments result in identical test scores would be rejected.) In case of statistically significant treatment main effect the $p$ value of the pairwise comparisons were computed and included in the table when they were below 0.05.

Tables 15-27 provide the details, including tests for pretest/treatment interaction and homogeneity of variance. For the adjusted mean scores, the tables give the $F$ probability of the treatment (or group) main effect. This $p$ value can be found in the ‘Group’ line. The line above confirms the linear relationship between the pretest score (covariate) and the posttest score (dependent variable). The $p$ value in this line is the lowest level of significance at which the existence of this linear relationship would be accepted.

In 12 out of the 13 comparisons no statistically significant differences were found at the $\alpha = 0.05$ level. For the computational subtest of the long term retention test Groups 1 and 3 performed significantly better than Group 2 (graphing calculators, no discovery) ($p = 0.047$ overall, $p = 0.041$ for the pairwise comparisons Group 1 vs. Group 2 and Group 2 vs. Group 3). It should be noted that only 6 students from Group 3 (control) and only 7
students from Group 1 (graphing calculators + discovery) took the long term retention test. On no other component of the long term retention test did Groups 1 and 3 outperform Group 2, nor did this happen on the computational (or, in fact, any other) component of the final exam or the short term retention test. For the computational subtest of the long term retention test $p = 0.047$, just slightly below the preset level of significance, $\alpha = 0.05$. Considering all these factors, it seems very likely that these differences are the results of chance.

There are substantial differences on the adjusted means of the transfer part of the short term retention test. They are not statistically significant at the $\alpha = 0.05$ level ($p = 0.13$). The transfer component of both other tests resulted in a different order of adjusted means among the three groups. On the computational component of the short term retention test the order is the same, but there the difference between the lowest and highest adjusted means is less than 14% of the lowest adjusted mean (or less than 6% of the maximum possible score for that variable). On the conceptual component of the short term retention test the order is different. Thus, it seems very likely that these differences are the results of chance.

Table 14 includes the sums of the computational, conceptual and transfer scores on the three tests. For computational skills Group 1 (graphing calculators + discovery) has the highest sum. However, Group 1 was best on the computational part of the final exam only, and not on the computational part of the short and long term retention tests. For conceptual understanding Group 3 (control) has the highest sum. However, this is not the case on the conceptual part of all three exams. For transfer skills Group 2 (graphing calculators, no discovery) has the highest sum. This is again not the case on the transfer part of all three exams.

Group 3 (control) scored the highest for the largest number of subtests. However, Group 3 was not the best on any part of the final exam, which was the longest of the three
tests and was taken by the largest number of students. (In fact, Group 3 was the weakest on two out of three components of the final exam.) For overall performance there are three different orders among the groups on the three tests.

In summary, there is no reason to believe that the differences in the adjusted mean scores are caused by anything but chance. Thus, Hypothesis 1 ("There will be no significant difference between the three groups on computational skills on the final exam and on the two retention tests.") should be accepted. For conceptual understanding, Hypothesis 2a ("Both graphing calculator groups will exhibit significantly higher conceptual understanding on the final exam and on the retention tests than the control group will.") should be rejected. Hypothesis 2b ("There will be no significant difference between the two graphing calculator groups (i.e. with/without discovery) on conceptual understanding.") should be accepted. For transfer skills, Hypothesis 3a ("Students in the graphing calculator + discovery section will exhibit significantly higher transfer skills on the final exam and on the retention tests than students in either of the other two groups will.") should be rejected. Hypothesis 3b ("Among the other two groups, the graphing calculator group will score significantly higher.") should be rejected. Hypothesis 3c ("These differences (H 5a, 5b) will be the greatest on the long term retention test.") should be rejected.

4.3 Comparison of Groups on the Other Dependent Variables

After the final exam students completed a questionnaire about the course (Appendix B). One question asked them to estimate the time they had spent on the course in a typical week. Students' answers were compared using analysis of variance to test if there were statistically significant differences between the groups at the $\alpha = 0.05$ level. No significant differences were found (Table 28, $p = 0.29$, 10.35 hours/week, 12.25 hours/week, 11.73 hours/week, respectively). Thus, Hypothesis 4 ("There will be no significant difference
between the groups on the amount of time students spent on the course.' should be accepted.

Students were asked if they worked together with their classmates outside of class. Table 29 shows that the chi-square test of association did not find statistically significant differences between the groups at the $\alpha = 0.05$ level (Pearson chi-square $p = 0.67$). Thus, Hypothesis 5 ('Students in the discovery group will work with classmates outside of class to a significantly greater extent than students in the other two groups.') should be rejected.

Attendance was taken at 15 (out of the 40) classes. The following results were found: Group 1 (graphing calculators + discovery): 66.7%, Group 2 (graphing calculators without discovery): 79.7%, Group 3 (control): 80.4%. A possible explanation was that when students used the worksheets in class they thought they could do the same work at home at their convenience. Students' comments on the course evaluation and conversations with students confirmed this to be the main reason, along with some students having a negative attitude towards discovery style teaching.

4.4 Student Evaluation of Discovery Activities

In the discovery groups, the questionnaire students completed after the final exam included several questions on the discovery activities. Thirty-five of the 36 students completed the questionnaire, most of them answered all the questions. Students were asked how successfully they used the worksheets. Table 30 shows students' responses to the item 'For the questions on the worksheets where the answer was not previously known I/we found the answer on my/our own to about ___% of the questions.' (mean = 47%, median = mode = 50%). They found the answer with hint (but not solution) from other classmates or their instructor or from the hint-sheets in average to about 22% of the questions (Table 31). Table 32 gives the distribution of the sum of these two variables, i.e. the percentage of questions on the worksheets that students were able to answer on their
own or with hint (mean = 69%, median = 75%, mode = 60%). Interestingly, the answer was previously known to the students 47% of the time: from previous coursework (25%), from pre-reading the textbook (14%), or from other sources (8%) (Tables 33 - 36).

Students were free to choose between individual, pair and group work in class. Most of them preferred individual work, in average 67% of the time (Table 37). Working in pairs was the second choice, in average 25% of the time (Table 38). Working in groups of 3 or more was the least favorite, students chose this only 8% of the time (Table 39). Sixty percent of the students never worked in groups.

Most students liked using the worksheets (Table 40). Half of them found that the hint-sheets and solution-sheets helped most of the time, and only a few said it was of little help (Tables 41 and 42).

Students suggested that 30% of the classtime be spent on individual/group work using worksheets, hint-sheets and solution-sheets (Table 43). Twelve percent of them thought no classtime should be spent on discovery style teaching and another 12% suggested more than half of the classtime for discovery activities. Students were asked a long question about when they think hint-sheets and solution-sheets should be given out.

Some say they both should be given to students when the individual/group work starts so students can get the hints and/or the solutions instantly when they feel they need them. Others argue that if students have these cheat-sheets from the beginning then there is too much temptation just to read them rather than working through the problems--and then students will not experience the advantages of discovery learning. A possible compromise is to display these cheat-sheets in class, but distribute them at the end of the individual/group work only--then they are in fact available anytime, but the temptation is less. Considering all the pros and cons what do you think is the best?

A. For the hint-sheet:
   - 76% Distribute at the beginning
   - 21% Display at the beginning, distribute at end
   - 3% Neither distribute nor display before the end

B. For the solution-sheet:
   - 13%
   - 63%
   - 23%
CHAPTER V
SUMMARY AND DISCUSSION

This chapter gives a summary of the study, followed by a discussion of the conclusions and recommendations for further research based on the results of the study.

5.1 Summary

This was an experimental study examining the effects of graphing calculator use and guided discovery style teaching with the following research design:

**Group 1:** Use of graphing calculators + (guided) discovery approach with worksheets, hint-sheets and solution-sheets

**Group 2:** Use of graphing calculators without discovery

**Group 3:** No graphing calculators, no discovery (traditional instruction)

One objective of the study was to verify that students can discover a significant portion of differential calculus using worksheets, hint-sheets and solution-sheets. This objective has been accomplished. According to the questionnaire students completed after the final exam, they found the answer on their own to 47% of those questions on the worksheets where the answer was not previously known to them (Table 29). They found the answer to an additional 22% of the questions with hint (but not solution) from the hint-sheets, from classmates or the instructor (Table 30). Over 75% of the students found the answer to the majority of such questions with or without hint (Table 31). This shows that discovery
style teaching is a viable alternative to traditional teaching for at least part of the new material.

Another objective of the study was to compare the three groups on student achievement on the final exam and on the short and long term retention tests, and on other dependent variables. To insure a fair comparison, students' mathematical background at the beginning of the study was compared. The major instrument for this comparison was the pretest students took at the beginning of the quarter. The pretest, the final exam and the short and long term retention tests were divided into three parts: items measuring computational skills, conceptual understanding and transfer skills. No statistically significant differences were found on any of these three parts of the pretest and the groups' mean scores were remarkably close (Table 1). No statistically significant differences were found on the following background variables: placement level, the year in which students took the placement test, their precalculus grade and the year in which they took precalculus.

To increase the statistical power and the precision of the student achievement comparisons, analyses of covariance were used for this part of the data analysis. The scores on the corresponding subtest of the pretest served as covariates. No instructional method proved superior to the others on comparison.

The other two dependent variables of the study were students' time spent on the course and whether students worked with their classmates outside of class or not. Data for these variables were collected by the questionnaire students completed after the final exam. Statistically significant differences were not found between the groups on these variables.

The third objective of the study was to investigate the question whether the positive results of technology use (graphing calculators/computers) reported by many studies (e.g., Beckmann, 1990; Browning, 1990; Demana, Foley, Osborne & Waits, 1989; Dick & Shaughnessy, 1988; Farrell, 1989; Heid, 1988; Judson, 1988; Palmiter, 1986) were due to the technology use itself or to the increase in discovery activity on the part of the students
which takes place as a result of technology use. The study failed to reproduce these results.

5.2 Discussion

What can be the reasons for the findings of no significant differences in this study, contradicting the findings of some related studies? One possible reason was that the differences between the treatments were not as great as in many other studies. In part, this was done on purpose as part of the research design. The instructors used the same lesson plans, discussed the same problems in class, except for the differences due to graphing calculator use and discovery style teaching for part of the new material. Other steps were taken to insure that the instruction in the three groups would be as close to one another as possible, except for the planned differences. A very strong effort was made to control for the contaminating variables to a maximum possible extent. This was not the case in many related studies (Ausubel, 1978; Fey, 1980; Shulman and Keislar, 1966).

For the other part, graphing calculator use and especially discovery style teaching was not as extensive as it could have been and probably should have been. The ratio of the new material covered using discovery style teaching was only about 20%. One reason for this was that the instructors of the experimental groups did not want to depart too radically from the traditional instructional practice of the department. They observed a strong negative attitude towards discovery style teaching on the part of several students. Some students walked out of the classroom as soon as the discovery style teaching started, without even giving it a chance. Some had the view that if the instructor is not lecturing then there is in fact no instruction, there is no reason to stay. It seemed “like a waste of time (i.e. time that I could do the same work and grasp the same concepts at home)”--as one of the students wrote in the evaluation at the end of the quarter. This attitude caused several problems. For the instruction, these students did not get the benefit of continuous feedback and help
from their classmates and from the instructor, as did those students who stayed for the
discovery style teaching. For the research, these students did not receive the experimental
treatment. Many of them learned the new material from the textbook or from the solution
sheets, rather than from the worksheets and hint-sheets, thus, they were not involved in
discovery learning. ("I did not like using worksheet in fact, I never used them."--wrote
another student.) Since absenteeism was close to 50% at times--about 33% in average, this
was a significant contamination to the study.

Many students view mathematics as a collection of algorithms to solve certain classes
of problems. There is no room for discovery style teaching in this view of mathematics.
The role of the instructor is to explain these algorithms and to demonstrate them on as many
examples as time permits. The students' role is to memorize these algorithms and practice
them on a large number of examples so they can repeat them correctly on tests. This view
of mathematics is probably developed during previous coursework, both at the high school
and college level. The investigator had the opportunity to participate in a 4-year long,
Grades 9-12 curriculum development project on discovery style teaching (National
Pedagogical Institute, Hungary, 1982, 1983, 1984, 1985). This view of mathematics and
this categorical rejection of the discovery style teaching was almost nonexistent there.

There were other students with a very positive view of the discovery method,
claiming that it "gave a better understanding of the material." "I definitely liked the
worksheets. Puzzling through them made things make a lot more sense." It gave the
instructors great pleasure to observe that on several occasions, while using the worksheets,
students did not realize that it was time for them to leave, they all stayed over 15 minutes
beyond the official class time. The instructors have never experienced this with traditional
instruction: then any attempt to keep their students in the classroom more than 15 seconds
beyond the official classtime was 'dead on arrival.' The following statement from another
student also summarizes the experiences and beliefs of the two 'discovery instructors': "I
think that both ways have their good and bad points and using them together would work better than either alone.” This balanced view seems to be reflected in that students in average suggested 30% of the classtime be spent on discovery style teaching and that only 12% of the students had the view that the discovery method should be completely eliminated, while also only 12% suggested to spend more than half of the classtime on discovery style teaching (Table 42).

These modest views are in agreement with the literature. Ausubel (1978, p. 527) argues against “an all-or-none position”, so does Cronbach (1977). Discovery style teaching can be a successful part of an instruction (Slavin, 1986), but not the major part of it (Ausubel, 1978; Biehler and Snowman, 1982; Slavin, 1986). Cronbach (1977) calls for research to combine discovery and expository style instruction into an effective curriculum sequence. This study attempted to address this issue, but, unfortunately, did not get closer to the solution.

Since students had very different views on the usefulness of the discovery method it seems likely that the approximately equal mean scores on the posttests do not mean that each individual student would score approximately the same with discovery and non-discovery treatments, but that the individual differences evened out when the group adjusted means were computed. The literature supports this assumption: discovery techniques seem to be effective for high ability learners, but less effective for low ability learners (Babikan, 1971; Cronbach and Snow, 1977; Olander and Robertson, 1973; Snow, 1982; Worthen, 1967). This study did not find statistically significant pretest/treatment interaction, but this does not contradict the above studies, since the pretest was an achievement test, it can not be considered an instrument measuring mathematical ability.

Accepting the results of the above studies there is implication for instruction: it seems reasonable to offer both a (partially) discovery and a non-discovery instruction to students and let them choose—perhaps make a suggestion to them based on the results of some
ability test. The second part of this suggestion is quite controversial: classifying a someone as 'low ability student' (whether these words are used or not) can have a negative effect on him or her, especially if the student has math anxiety.

The graphing calculator use was well received the study. Most students had a positive attitude towards graphing calculators and most of them were proficient in using them since graphing calculators were required in the precalculus classes at this university and in many high schools. Those few students who did not know how to use a graphing calculator learned it fast at the beginning of the quarter using the handouts the instructors gave out and some help of their classmates.

Since most students in the control group owned a graphing calculator and they were proficient in using them for the reasons mentioned above and used them on tests, the differences between the three groups on graphing calculator use were not as great as it would have been desirable for an experimental study to detect differences in student achievement. This is a likely explanation of why this study found no statistically significant differences between the graphing calculator and control groups on student achievement, unlike most related studies. In most other studies graphing calculator use in the control group was minimal, if any.

5.3 Recommendations for Further Research

The following studies are recommended for further research:

1. A replication of this study with the following differences:
   (a) Increase the amount of discovery activity in Group 1 to maximize the difference between treatments.
   (b) Choose a population were there is no extensive graphing calculator use in the control group, in order to maximize the differences between treatments.
(c) Conduct the study for a period longer than 10 weeks. Ten weeks may not be long enough for some students to overcome their negative perceptions of discovery learning. Students need time to adjust to the discovery approach and to experience the benefits of discovery learning.

(d) Include a guided discovery approach without graphing calculators, for comparison. That design would serve the objectives of this study better, it would give more information on the effects of graphing calculator use and discovery style teaching and on their potential interaction. Also, it could give valuable information to address the third item in the Problem Statement section: investigate the question whether the positive results of technology use reported by many studies are due to the technology use itself or to the increase in discovery activity on the part of the students which takes place as a result of technology use.

It was not possible for the investigator to get four sections of the course; so one of the four treatments had to be omitted from the study. It is recommended that a replication of this study should include this fourth group (discovery without graphing calculators).

(e) Administer a test at the beginning of the study to measure mathematical ability. This would make it possible to confirm the results of related studies that discovery is successful with high ability students but less successful with low ability students.

2. Choosing a younger (high school) population would probably reduce the problem of some students having a negative attitude towards discovery learning. At a younger age these negative attitudes probably either have not developed yet, or at least they are not as strong as at some older students. Choosing a high school population would reduce another problem with discovery learning, too: the curricula in high school mathematics courses are less crowded; thus, the fact
that discovery takes more time would cause less problem there.

3. Investigate which types of students have a strong negative attitude towards discovery learning and how this can be changed.

4. Examine students' success in follow-up mathematics courses.

5. This study used engineering and science oriented students who generally have a relatively strong background in and positive attitude towards mathematics. Would the results be different if more generally oriented students are used?
LIST OF REFERENCES


Dick, T. P. (1990, April). *Equal access to representations--research questions about mathematical toolkits in the classroom*. A paper presented at the research presession of the annual meeting of The National Council of Teachers of Mathematics, Salt Lake City, UT.


APPENDIX A

DATA RELATIVE TO CHAPTER IV
Table 1

Comparison of Groups on the Pretest

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<td>(control)</td>
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<td>2</td>
<td>21.08</td>
<td>10.54</td>
<td>0.471</td>
</tr>
<tr>
<td>Within groups</td>
<td>120</td>
<td>2682.76</td>
<td>22.36</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>122</td>
<td>2703.85</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. PCOMP = computational skills, PCONC = conceptual understanding
Table 1 (continued)

Comparison of Groups on the Pretest

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>38</td>
<td>45</td>
<td>40</td>
<td>123</td>
</tr>
<tr>
<td>PTRANSF Mean</td>
<td>7.97</td>
<td>7.62</td>
<td>7.70</td>
<td>7.76</td>
</tr>
<tr>
<td>SD</td>
<td>2.81</td>
<td>3.86</td>
<td>4.69</td>
<td>3.85</td>
</tr>
<tr>
<td>Source</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Between Groups</td>
<td>2</td>
<td>2.73</td>
<td>1.37</td>
<td>0.091</td>
</tr>
<tr>
<td>Within Groups</td>
<td>120</td>
<td>1807.95</td>
<td>15.07</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>122</td>
<td>1810.68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTOT(%) Mean</td>
<td>35.44</td>
<td>34.28</td>
<td>34.86</td>
<td>34.83</td>
</tr>
<tr>
<td>SD</td>
<td>12.06</td>
<td>12.47</td>
<td>12.35</td>
<td>12.21</td>
</tr>
<tr>
<td>Source</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Between Groups</td>
<td>2</td>
<td>27.39</td>
<td>13.70</td>
<td>0.090</td>
</tr>
<tr>
<td>Within Groups</td>
<td>120</td>
<td>18170.26</td>
<td>151.42</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>122</td>
<td>18197.65</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. PTRANSF = transfer skills, PTOT(%) = overall performance (in %)
Table 2

Univariate Homogeneity of Variance Tests (Bartlett) for the Pretest Scores

<table>
<thead>
<tr>
<th></th>
<th>F(2, 32019)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCOMP</td>
<td>1.28</td>
<td>0.27</td>
</tr>
<tr>
<td>PCONC</td>
<td>0.032</td>
<td>0.96</td>
</tr>
<tr>
<td>PTRANSF</td>
<td>4.67</td>
<td>0.009</td>
</tr>
<tr>
<td>PTOT(%)</td>
<td>0.023</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Note. PCOMP = computational skills, PCONC = conceptual understanding, PTRANSF = transfer skills, PTOT(%) = overall performance (in %)
Table 3
Comparison of Groups on Precalculus Grades

<table>
<thead>
<tr>
<th>Precalculus grade</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>Row</th>
<th>Column</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>grade</td>
<td>A</td>
<td>A-</td>
<td>B</td>
<td>B+</td>
</tr>
<tr>
<td>gr.c.+ discov.</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>8.7</td>
<td>3.9</td>
<td>.9</td>
<td>5.7</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>12</td>
<td>3</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>4.5</td>
<td>1.0</td>
<td>6.6</td>
</tr>
<tr>
<td>control</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>10.2</td>
<td>4.6</td>
<td>1.1</td>
<td>6.7</td>
</tr>
</tbody>
</table>

Column Total: 29 | 13 | 3 | 19 | 6 | 12 | 15 | 4 | 14 | 8 | 10 | 133

Total: 21.8% | 9.8% | 2.3% | 14.3% | 4.5% | 9.0% | 11.3% | 3.0% | 10.5% | 6.0% | 7.5% | 100.0%

Chi-Square Value | DF | Significance
--- | --- | ---
Pearson | 11.95009 | 20 | .91778
Minimum Expected Frequency | .902
Cells with Expected Frequency < 5 | 25 OF | 33 ( 75.8%)
Table 4

Comparison of Groups on Precalculus Grades After Recoding

<table>
<thead>
<tr>
<th>Precalculus grade</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>Row</th>
<th>Discov.</th>
<th>Control</th>
<th>Column</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gr. C. + Discov.</td>
<td>7</td>
<td>6</td>
<td>13</td>
<td>10</td>
<td>4</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8.7</td>
<td>4.8</td>
<td>11.1</td>
<td>9.9</td>
<td>5.4</td>
<td>30.1%</td>
<td></td>
</tr>
<tr>
<td>Gr. C., No Discov.</td>
<td>12</td>
<td>3</td>
<td>15</td>
<td>9</td>
<td>7</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>5.5</td>
<td>12.8</td>
<td>11.4</td>
<td>6.2</td>
<td>34.6%</td>
<td></td>
</tr>
<tr>
<td>Control</td>
<td>10</td>
<td>7</td>
<td>9</td>
<td>14</td>
<td>7</td>
<td>47</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.2</td>
<td>5.7</td>
<td>13.1</td>
<td>11.7</td>
<td>6.4</td>
<td>35.3%</td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td>29</td>
<td>16</td>
<td>37</td>
<td>33</td>
<td>18</td>
<td>133</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>21.8%</td>
<td>12.0%</td>
<td>27.8%</td>
<td>24.8%</td>
<td>13.5%</td>
<td>100.0%</td>
<td></td>
</tr>
</tbody>
</table>

Chi-Square

<table>
<thead>
<tr>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.98045</td>
<td>8</td>
<td>.64942</td>
</tr>
</tbody>
</table>

Minimum Expected Frequency: 4.812

Cells with Expected Frequency < 5: 1 OF 15 (6.7%)
Table 5

Precalculus Grade Point Averages

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>33</td>
<td>34</td>
<td>37</td>
<td>104</td>
</tr>
<tr>
<td>GPA</td>
<td>2.60</td>
<td>2.41</td>
<td>2.39</td>
<td>2.46</td>
</tr>
<tr>
<td>SD</td>
<td>0.92</td>
<td>0.89</td>
<td>0.97</td>
<td>0.92</td>
</tr>
</tbody>
</table>
Table 6

Comparison of Groups on the Year Students Took Precalculus

<table>
<thead>
<tr>
<th>Precalculus year</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>Row %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>83</td>
<td>86</td>
<td>87</td>
</tr>
<tr>
<td>GROUP</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>gr.c. + discov.</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>.6</td>
<td>.3</td>
<td>.3</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.7</td>
<td>.3</td>
<td>.3</td>
</tr>
<tr>
<td>control</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.7</td>
<td>.4</td>
<td>.4</td>
</tr>
<tr>
<td>Column</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>1.9%</td>
<td>1.0%</td>
<td>1.0%</td>
</tr>
</tbody>
</table>

Chi-Square

<table>
<thead>
<tr>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.84004</td>
<td>14</td>
<td>.46169</td>
</tr>
</tbody>
</table>

Minimum Expected Frequency .317

Cells with Expected Frequency < 5 18 OF 24 (75.0%)

Did not take precalculus: 29
Table 7

Comparison of Groups on the Year Students Took Precalculus, After Recoding

<table>
<thead>
<tr>
<th>Precalculus year</th>
<th>Count</th>
<th>Exp. Val.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Did not take</td>
<td>&lt;91/92</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GROUP</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>gr.c. + discov.</td>
<td>7</td>
<td>7</td>
<td>26</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>8.7</td>
<td>7.5</td>
<td>23.8</td>
<td>30.1%</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>12</td>
<td>8</td>
<td>26</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>8.6</td>
<td>27.3</td>
<td>34.6%</td>
</tr>
<tr>
<td>control</td>
<td>10</td>
<td>10</td>
<td>27</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>10.2</td>
<td>8.8</td>
<td>27.9</td>
<td>35.3%</td>
</tr>
<tr>
<td>Column</td>
<td>29</td>
<td>25</td>
<td>79</td>
<td>133</td>
</tr>
<tr>
<td>Total</td>
<td>21.8%</td>
<td>18.8%</td>
<td>59.4%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chi-Square</th>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>1.27623</td>
<td>4</td>
<td>.86540</td>
</tr>
<tr>
<td>Minimum Expected Frequency</td>
<td>7.519</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 8
Comparison of Groups on Prior Differential Calculus (Math 151) Experience

<table>
<thead>
<tr>
<th>GROUP</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>Row</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>N</td>
</tr>
<tr>
<td>gr.c.+ discov.</td>
<td>32</td>
<td>28.61</td>
<td>321</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>28</td>
<td>32.91</td>
<td>281</td>
</tr>
<tr>
<td>control</td>
<td>35</td>
<td>33.61</td>
<td>351</td>
</tr>
<tr>
<td>Column</td>
<td>95</td>
<td>11.41</td>
<td>11.4%</td>
</tr>
<tr>
<td>Total</td>
<td>133</td>
<td>30.1%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

Chi-Square

<table>
<thead>
<tr>
<th>Chi-Square</th>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>4.16580</td>
<td>2</td>
<td>.12457</td>
</tr>
<tr>
<td>Minimum Expected Frequency</td>
<td>11.429</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 9

Comparison of Groups on Mathematics Placement Level

<table>
<thead>
<tr>
<th>Count (Exp. Val.)</th>
<th>No placm. row</th>
</tr>
</thead>
<tbody>
<tr>
<td>GROUP</td>
<td>L</td>
</tr>
<tr>
<td>-------------------</td>
<td>----</td>
</tr>
<tr>
<td>gr.c. + discov.</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>12.3</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>14.2</td>
</tr>
<tr>
<td>control</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>14.5</td>
</tr>
<tr>
<td>Column Total</td>
<td>41</td>
</tr>
<tr>
<td>Total</td>
<td>30.8%</td>
</tr>
</tbody>
</table>

Chi-Square Value | DF | Significance
-----------------|----|------------
Pearson          | 14.52631 | 12 | 0.26837
Minimum Expected Frequency = 1.805
Cells with Expected Frequency < 5 = 10 of 21 (47.6%)%

Note. L = Calculus, M = Precalculus II., N = Precalculus I., R = Intermediate Algebra, S = Remedial Algebra II., T = Remedial Algebra I.
Table 10  

Comparison of Groups on Mathematics Placement Level, After Recoding

| Placement level | \( |L\ OR\ M|N\ |R\ |S\ OR\ T|\) | Row Total |
|-----------------|-----------------|------------------|-----------------|------------------|------------------|-----------------|------------------|------------------|------------------|------------------|------------------|
| GROUP           |                 | \( |L\ OR\ M|N\ |R\ |S\ OR\ T|\) |                 |
| gr.c.+ discov.  | 11 7 16 3 3 | 40 | 12.3 6.6 11.1 6.0 3.9 | 30.1% |
| gr.c., no discov.| 13 10 10 8 5 | 46 | 14.2 7.6 12.8 6.9 4.5 | 34.6% |
| control         | 17 5 11 9 5 | 47 | 14.5 7.8 13.1 7.1 4.6 | 35.3% |
| Column          |               | 41 22 37 20 13 133 | 30.8% 16.5% 27.8% 15.0% 9.8% 100.0% |

Chi-Square Value DF Significance
Pearson 8.02780 8 .43076
Minimum Expected Frequency 3.910
Cells with Expected Frequency < 5 3 OF 15 (20.0%)

Note. \( L = \) Calculus, \( M = \) Precalculus II., \( N = \) Precalculus I., \( R = \) Intermediate Algebra, \( S = \) Remedial Algebra II., \( T = \) Remedial Algebra I.
Table 11

Comparison of Groups on the Year Students Took the Placement Test

<table>
<thead>
<tr>
<th>Placement year</th>
<th>83</th>
<th>85</th>
<th>86</th>
<th>87</th>
<th>88</th>
<th>89</th>
<th>90</th>
<th>91</th>
<th>92</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GROUP</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>gr.c. + discov.</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>14</td>
<td>0</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>.3</td>
<td>.6</td>
<td>.9</td>
<td>.9</td>
<td>1.6</td>
<td>3.5</td>
<td>7.9</td>
<td>12.6</td>
<td>.6</td>
<td>31.5%</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>14</td>
<td>1</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>.4</td>
<td>.7</td>
<td>1.1</td>
<td>1.1</td>
<td>1.8</td>
<td>3.9</td>
<td>9.0</td>
<td>14.3</td>
<td>.7</td>
<td>35.9%</td>
</tr>
<tr>
<td>control</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>1</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td></td>
<td>.3</td>
<td>.7</td>
<td>1.0</td>
<td>1.0</td>
<td>1.6</td>
<td>3.6</td>
<td>8.2</td>
<td>13.0</td>
<td>.7</td>
<td>32.6%</td>
</tr>
<tr>
<td>Column</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>25</td>
<td>40</td>
<td>2</td>
<td>92</td>
</tr>
<tr>
<td>Total</td>
<td>83</td>
<td>85</td>
<td>86</td>
<td>87</td>
<td>88</td>
<td>89</td>
<td>90</td>
<td>91</td>
<td>92</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chi-Square Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>19.42875</td>
<td>16</td>
</tr>
</tbody>
</table>
| Minimum Expected Frequency | .315 | 21 OF | 27 (77.8%) | Did not take placement test: 41
Table 12

Comparison of Groups on the Year Students Took the Placement Test After Recoding

<table>
<thead>
<tr>
<th>Placement year</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>No. placm.</th>
<th>Row Test</th>
<th>83-89</th>
<th>90-91 or 92</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>GROUP</td>
<td>------</td>
<td>-----------</td>
<td>------------</td>
<td>---------</td>
<td>-------</td>
<td>-------------</td>
<td>-------</td>
</tr>
<tr>
<td>gr.c. + discov.</td>
<td>11</td>
<td>5</td>
<td>10</td>
<td>14</td>
<td>40</td>
<td>12.3</td>
<td>7.5</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>13</td>
<td>11</td>
<td>7</td>
<td>15</td>
<td>46</td>
<td>14.2</td>
<td>8.6</td>
</tr>
<tr>
<td>control</td>
<td>17</td>
<td>9</td>
<td>8</td>
<td>13</td>
<td>47</td>
<td>14.5</td>
<td>8.8</td>
</tr>
<tr>
<td>Column</td>
<td>41</td>
<td>25</td>
<td>25</td>
<td>42</td>
<td>133</td>
<td>30.8%</td>
<td>18.8%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chi-Square</th>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>3.76813</td>
<td>6</td>
<td>.70802</td>
</tr>
<tr>
<td>Minimum Expected Frequency</td>
<td>7.519</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 13

Comparison of Groups on the Distribution of Male and Female Students

<table>
<thead>
<tr>
<th>Sex</th>
<th>Count</th>
<th>Exp. Val.</th>
<th>Row</th>
<th></th>
<th></th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F</td>
<td>M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GROUP</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>gr.c.+ discov.</td>
<td>5</td>
<td>35</td>
<td>40</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>12.0</td>
<td>28.0</td>
<td>30.1%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>20</td>
<td>26</td>
<td>46</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>13.8</td>
<td>32.2</td>
<td>34.6%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>control</td>
<td>15</td>
<td>32</td>
<td>47</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>14.1</td>
<td>32.9</td>
<td>35.3%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Column</td>
<td>40</td>
<td>93</td>
<td>133</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>30.1%</td>
<td>69.9%</td>
<td>100.0%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chi-Square

<table>
<thead>
<tr>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>9.88022</td>
<td>2</td>
</tr>
<tr>
<td>Minimum Expected Frequency</td>
<td>12.030</td>
<td></td>
</tr>
</tbody>
</table>
Table 14
Overview of Test Scores: Adjusted Means

<table>
<thead>
<tr>
<th></th>
<th>Max. possible (gr. c. + discov.)</th>
<th>Group 1</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
<th>Significance (if &lt; .05)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>FINAL</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N = 94</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCOMP</td>
<td>76</td>
<td>52.28</td>
<td>50.36</td>
<td>45.47</td>
<td></td>
</tr>
<tr>
<td>FCONC</td>
<td>53</td>
<td>38.69</td>
<td>42.11</td>
<td>38.81</td>
<td></td>
</tr>
<tr>
<td>FTRANSF</td>
<td>85</td>
<td>18.92</td>
<td>24.05</td>
<td>17.82</td>
<td></td>
</tr>
<tr>
<td>FTRDISCOV</td>
<td>32</td>
<td>9.58</td>
<td>14.13</td>
<td>11.44</td>
<td></td>
</tr>
<tr>
<td>FTOT(%)</td>
<td>52.54</td>
<td>55.91</td>
<td>49.63</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>34</td>
<td>33</td>
<td>27</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>SHORT RET.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N = 79</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SCOMP</td>
<td>32</td>
<td>13.83</td>
<td>14.36</td>
<td>15.73</td>
<td></td>
</tr>
<tr>
<td>SCONC</td>
<td>33</td>
<td>23.62</td>
<td>23.28</td>
<td>24.10</td>
<td></td>
</tr>
<tr>
<td>STRANSF</td>
<td>30</td>
<td>3.57</td>
<td>4.79</td>
<td>6.59</td>
<td></td>
</tr>
<tr>
<td>STOT(%)</td>
<td>41.56</td>
<td>44.42</td>
<td>47.98</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>26</td>
<td>30</td>
<td>23</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>LONG RET.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N = 25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LCOMP</td>
<td>28</td>
<td>18.65</td>
<td>9.60</td>
<td>19.03 p = .047 overal, .041 for Gr. 1, 2 and 2, 3</td>
<td></td>
</tr>
<tr>
<td>LCONC</td>
<td>57</td>
<td>31.56</td>
<td>36.15</td>
<td>40.72</td>
<td></td>
</tr>
<tr>
<td>LTRANSF</td>
<td>35</td>
<td>5.70</td>
<td>9.54</td>
<td>8.60</td>
<td></td>
</tr>
<tr>
<td>LTOT(%)</td>
<td>48.28</td>
<td>44.77</td>
<td>56.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n</td>
<td>7</td>
<td>12</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>COMPsum</strong></td>
<td>136</td>
<td>84.76</td>
<td>74.32</td>
<td>80.23</td>
<td></td>
</tr>
<tr>
<td><strong>CONCsum</strong></td>
<td>143</td>
<td>93.87</td>
<td>101.54</td>
<td>103.63</td>
<td></td>
</tr>
<tr>
<td><strong>TRANSFsum</strong></td>
<td>150</td>
<td>28.19</td>
<td>38.38</td>
<td>33.01</td>
<td></td>
</tr>
</tbody>
</table>

**Note.** COMP = computational skills, CONC = conceptual understanding, TRANSF = transfer skills, TOT(%) = overall performance (in %)
COMPsum = sum of computational skills scores, CONCsum = sum of conceptual understanding scores, TRANSFsum = sum of transfer skills scores
FTRDISCOV = transfer skills problems from topics taught by discovery approach
Table 15

Comparison of Computational Skills on the Final Exam

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>34</td>
<td>33</td>
<td>27</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>52.28</td>
<td>50.36</td>
<td>45.47</td>
</tr>
<tr>
<td>Standard error</td>
<td>2.58</td>
<td>2.62</td>
<td>2.89</td>
</tr>
</tbody>
</table>

Source

<table>
<thead>
<tr>
<th>Type III ss</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCOMP</td>
<td>1</td>
<td>4251</td>
<td>4251</td>
<td>18.80</td>
<td>0.0001</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>723</td>
<td>362</td>
<td>1.60</td>
<td>0.208</td>
</tr>
</tbody>
</table>

PCOMP/group interaction

<table>
<thead>
<tr>
<th>Homogeneity of variance</th>
<th>F(2, 18169)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bartlett</td>
<td>1.53</td>
<td>0.217</td>
</tr>
</tbody>
</table>

Note. PCOMP = computational skills pretest score
Table 16

Comparison of Conceptual Understanding on the Final Exam

<table>
<thead>
<tr>
<th>Group</th>
<th>n</th>
<th>Adjusted mean</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1 (gr. c. + discov.)</td>
<td>34</td>
<td>38.69</td>
<td>1.38</td>
</tr>
<tr>
<td>Group 2 (gr. c. no discov.)</td>
<td>33</td>
<td>42.11</td>
<td>1.39</td>
</tr>
<tr>
<td>Group 3 (control)</td>
<td>27</td>
<td>38.81</td>
<td>1.54</td>
</tr>
</tbody>
</table>

Source                  df | ss  | ms  | F ratio | F prob. |
Type III ss              |
PCONC                    1  | 794 | 794 | 12.43   | 0.0007  |
Group                    2  | 243 | 121 | 1.90    | 0.156   |
PCONC/group interaction  |    |     | 0.74   | 0.480   |

Homogeneity of variance  F(2, 18169) | p  |
(Bartlett)               3.00         | 0.050 |

Note. PCONC = conceptual understanding pretest score
### Table 17

**Comparison of Transfer Skills on the Final Exam**

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTRANSF</td>
<td>1</td>
<td>4680</td>
<td>4680</td>
<td>21.19</td>
<td>0.0001</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>693</td>
<td>347</td>
<td>1.57</td>
<td>0.214</td>
</tr>
<tr>
<td>PTRANSF/group interaction</td>
<td></td>
<td></td>
<td></td>
<td>0.56</td>
<td>0.573</td>
</tr>
</tbody>
</table>

**Homogeneity of variance**

<table>
<thead>
<tr>
<th>F(2, 18169)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.234</td>
<td>0.791</td>
</tr>
</tbody>
</table>

**Note.** PTRANSF = transfer skills pretest score
Table 18

Comparison of Overall Performance on the Final Exam

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(gr.c. + discov.)</td>
<td>(gr. c., no discov.)</td>
<td>(control)</td>
</tr>
<tr>
<td>n</td>
<td>34</td>
<td>33</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>52.54</td>
<td>55.91</td>
</tr>
<tr>
<td>Standard error</td>
<td>2.27</td>
<td>2.30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTOT(%)</td>
<td>1</td>
<td>6162</td>
<td>6162</td>
<td>35.19</td>
<td>0.0001</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>591</td>
<td>295</td>
<td>1.69</td>
<td>0.191</td>
</tr>
</tbody>
</table>

| PTOT(%)/group interaction |  |     |     | 1.19    | 0.310   |

<table>
<thead>
<tr>
<th>Homogeneity of variance</th>
<th>F(2, 18169)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Bartlett)</td>
<td>1.057</td>
<td>0.348</td>
</tr>
</tbody>
</table>

Note. PTOT(%) = overall performance on the pretest (in %)
Table 19

**Comparison of Computational Skills on the Short Term Retention Test**

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>26</td>
<td>30</td>
<td>23</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>13.83</td>
<td>14.36</td>
<td>15.73</td>
</tr>
<tr>
<td>Standard error</td>
<td>1.62</td>
<td>1.51</td>
<td>1.73</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCOMP</td>
<td>1</td>
<td>1026</td>
<td>1026</td>
<td>14.99</td>
<td>0.0002</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>47</td>
<td>23</td>
<td>0.34</td>
<td>0.712</td>
</tr>
</tbody>
</table>

PCOMP/group interaction

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogeneity of variance</td>
<td>F(2, 12630)</td>
<td>p</td>
</tr>
<tr>
<td>(Bartlett)</td>
<td>1.650</td>
<td>0.192</td>
</tr>
</tbody>
</table>

**Note.** PCOMP = computational skills pretest score
Table 20

Comparison of Conceptual Understanding on the Short Term Retention Test

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c. no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>26</td>
<td>30</td>
<td>23</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>23.62</td>
<td>23.30</td>
<td>24.10</td>
</tr>
<tr>
<td>Standard error</td>
<td>0.75</td>
<td>0.69</td>
<td>0.78</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCONC</td>
<td>1</td>
<td>152</td>
<td>152</td>
<td>10.75</td>
<td>0.0016</td>
</tr>
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<td>Group</td>
<td>2</td>
<td>8.8</td>
<td>4.4</td>
<td>0.31</td>
<td>0.7325</td>
</tr>
<tr>
<td>PCONC/group interaction</td>
<td></td>
<td></td>
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<td>0.49</td>
<td>0.6149</td>
</tr>
</tbody>
</table>

Homogeneity of variance

<table>
<thead>
<tr>
<th>F(2, 12630)</th>
<th>p</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.467</td>
<td>0.627</td>
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<td></td>
</tr>
</tbody>
</table>

Note. PCONC = conceptual understanding pretest score
Table 21

Comparison of Transfer Skills on the Short Term Retention Test

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c. no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>26</td>
<td>30</td>
<td>23</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>3.57</td>
<td>4.79</td>
<td>6.59</td>
</tr>
<tr>
<td>Standard error</td>
<td>1.03</td>
<td>0.96</td>
<td>1.09</td>
</tr>
</tbody>
</table>

Source

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTRANSF</td>
<td>1</td>
<td>169</td>
<td>169</td>
<td>6.26</td>
<td>0.0145</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>112</td>
<td>56</td>
<td>2.07</td>
<td>.1333*</td>
</tr>
</tbody>
</table>

PTRANSF/group interaction

<table>
<thead>
<tr>
<th>Homogeneity of variance</th>
<th>F(2, 12630)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Bartlett)</td>
<td>3.853</td>
<td>0.021</td>
</tr>
</tbody>
</table>

Note. PTRANSF = transfer skills pretest score
Table 22

Comparison of Overall Performance on the Short Term Retention Test

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>26</td>
<td>30</td>
<td>23</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>41.56</td>
<td>44.42</td>
<td>47.98</td>
</tr>
<tr>
<td>Standard error</td>
<td>2.32</td>
<td>2.16</td>
<td>2.45</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTOT(%)</td>
<td>1</td>
<td>4751</td>
<td>4751</td>
<td>34.31</td>
<td>0.0001</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>503</td>
<td>251</td>
<td>1.82</td>
<td>0.1698</td>
</tr>
<tr>
<td>PTOT(%)/group interaction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Homogeneity of variance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Bartlett)</td>
<td></td>
<td></td>
<td></td>
<td>3.096</td>
<td>0.045</td>
</tr>
</tbody>
</table>

**Note.** PTOT(%) = overall performance on the pretest
Table 23

Comparison of Computational Skills on the Long Term Retention Test

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>7</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>18.65</td>
<td>9.60</td>
<td>19.03</td>
</tr>
<tr>
<td>Standard error</td>
<td>3.30</td>
<td>2.50</td>
<td>3.54</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCOMP</td>
<td>1</td>
<td>428</td>
<td>428</td>
<td>5.73</td>
<td>0.0260</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>529</td>
<td>264</td>
<td>3.54</td>
<td>0.0472*</td>
</tr>
</tbody>
</table>

PCOMP/group interaction

<table>
<thead>
<tr>
<th>Homogeneity of variance</th>
<th>F(2, 848)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bartlett</td>
<td>0.265</td>
<td>0.767</td>
</tr>
</tbody>
</table>

Note. PCOMP = computational skills pretest score

* p (1, 2) = p (2, 3) = 0.0407
### Table 24

**Comparison of Conceptual Understanding on the Long Term Retention Test**

<table>
<thead>
<tr>
<th>Group</th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>7</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>31.56</td>
<td>36.15</td>
<td>40.72</td>
</tr>
<tr>
<td>Standard error</td>
<td>3.25</td>
<td>2.53</td>
<td>3.58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type III ss</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCONC</td>
<td>1</td>
<td>24</td>
<td>24</td>
<td>0.33</td>
<td>0.5705</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>270</td>
<td>135</td>
<td>1.84</td>
<td>0.1833</td>
</tr>
</tbody>
</table>

| PCONC/group interaction       |    |     |     | 0.43    | 0.6559  |

| Homogeneity of variance       | F(2, 848) | p    |     |         |         |
| (Bartlett)                    | 0.559     | 0.572|     |         |         |

**Note.** PCONC = conceptual understanding pretest score
Table 25  
Comparison of Transfer Skills on the Long Term Retention Test

<table>
<thead>
<tr>
<th></th>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(gr.c. + discov.)</td>
<td>(gr. c., no discov.)</td>
<td>(control)</td>
</tr>
<tr>
<td>n</td>
<td>7</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>Adjusted mean</td>
<td>5.70</td>
<td>9.54</td>
<td>8.60</td>
</tr>
<tr>
<td>Standard error</td>
<td>2.72</td>
<td>2.19</td>
<td>3.15</td>
</tr>
</tbody>
</table>

Source | df | ss  | ms  | F ratio | F prob.  
Type III ss |     |     |     |         |          |
PTRANSF | 1  | 447 | 447 | 8.68    | 0.0077   |
Group | 2  | 63.65 | 31.82 | 0.62    | 0.5486   |

PTRANSF/group interaction | 2.81 | 0.0854 |

Homogeneity of variance | F(2, 848) | p     |  
(Bartlett)  | 0.423    | 0.655 |

Note. PTRANSF = transfer skills pretest score
### Table 26

**Comparison of Overall Performance on the Long Term Retention Test**

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>F Ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type III ss</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PTOT(%)</td>
<td>1</td>
<td>2421</td>
<td>2421</td>
<td>7.67</td>
<td>0.0115</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>472</td>
<td>235</td>
<td>0.75</td>
<td>0.4858</td>
</tr>
<tr>
<td>PTOT(%)/group interaction</td>
<td></td>
<td></td>
<td></td>
<td>0.26</td>
<td>0.7750</td>
</tr>
<tr>
<td>Homogeneity of variance</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Bartlett)</td>
<td>0.544</td>
<td>0.581</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. PTOT(%) = overall performance on the pretest
Table 27

Comparison of Transfer Skills on the Final Exam on Topics Taught by Discovery

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>ss</th>
<th>ms</th>
<th>F ratio</th>
<th>F prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>PTRANSF</td>
<td>1</td>
<td>1162</td>
<td>1162</td>
<td>14.48</td>
<td>0.0003</td>
</tr>
<tr>
<td>Group</td>
<td>2</td>
<td>350</td>
<td>175</td>
<td>2.18</td>
<td>0.119</td>
</tr>
<tr>
<td>PTRANSF/group interaction</td>
<td></td>
<td></td>
<td></td>
<td>0.06</td>
<td>0.94</td>
</tr>
</tbody>
</table>

Homogeneity of variance

F(2, 18169) = 0.826, p = 0.438

Note. PTRANSF = transfer skills pretest score
Table 28

Comparison of Groups on Time Spent on Course (hr/wk)

<table>
<thead>
<tr>
<th></th>
<th>Group 1 (gr.c. + discov.)</th>
<th>Group 2 (gr. c., no discov.)</th>
<th>Group 3 (control)</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>33</td>
<td>31</td>
<td>30</td>
<td>94</td>
</tr>
<tr>
<td>Mean</td>
<td>10.35</td>
<td>12.25</td>
<td>11.73</td>
<td>11.42</td>
</tr>
<tr>
<td>SD</td>
<td>5.23</td>
<td>4.70</td>
<td>5.02</td>
<td>5.01</td>
</tr>
<tr>
<td>Source</td>
<td>df</td>
<td>ss</td>
<td>ms</td>
<td>F ratio</td>
</tr>
<tr>
<td>Between groups</td>
<td>2</td>
<td>62</td>
<td>31.16</td>
<td>1.25</td>
</tr>
<tr>
<td>Within groups</td>
<td>91</td>
<td>2268</td>
<td>24.92</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>93</td>
<td>2330</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Homogeneity of variance</td>
<td>F(2, 18563)</td>
<td>p</td>
<td>0.175</td>
<td>0.839</td>
</tr>
</tbody>
</table>
Table 29

Comparison of Groups on the Number of Students Who Worked With Their Classmates Outside of Class

<table>
<thead>
<tr>
<th>Work with classmates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

GROUP

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Y</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>gr.c. + discov.</td>
<td>21</td>
<td>14</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>20.8</td>
<td>14.2</td>
<td>36.5%</td>
</tr>
<tr>
<td>gr.c., no discov.</td>
<td>20</td>
<td>11</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>18.4</td>
<td>12.6</td>
<td>32.3%</td>
</tr>
<tr>
<td>control</td>
<td>16</td>
<td>14</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>17.8</td>
<td>12.2</td>
<td>31.3%</td>
</tr>
</tbody>
</table>

Column Total

<table>
<thead>
<tr>
<th>N</th>
<th>Y</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>57</td>
<td>39</td>
<td>96</td>
</tr>
</tbody>
</table>

Total 59.4% 40.6% 100.0%

Chi-Square Value DF Significance

<table>
<thead>
<tr>
<th>Chi-Square</th>
<th>Value</th>
<th>DF</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pearson</td>
<td>.79934</td>
<td>2</td>
<td>.67054</td>
</tr>
<tr>
<td>Minimum Expected Frequency</td>
<td>12.188</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 30

The Percentage of Questions on the Worksheets Students Were Able To Answer on Their Own

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
<th>Percent</th>
<th>Cum. Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2.9</td>
<td>2.9</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.9</td>
<td>5.9</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2.9</td>
<td>8.8</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>5.9</td>
<td>14.7</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>5.9</td>
<td>20.6</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>2.9</td>
<td>23.5</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
<td>2.9</td>
<td>26.5</td>
</tr>
<tr>
<td>35</td>
<td>1</td>
<td>2.9</td>
<td>29.4</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>5.9</td>
<td>35.3</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>35.3</td>
<td>70.6</td>
</tr>
<tr>
<td>60</td>
<td>2</td>
<td>5.9</td>
<td>76.5</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
<td>2.9</td>
<td>79.4</td>
</tr>
<tr>
<td>70</td>
<td>1</td>
<td>2.9</td>
<td>82.4</td>
</tr>
<tr>
<td>75</td>
<td>2</td>
<td>5.9</td>
<td>88.2</td>
</tr>
<tr>
<td>85</td>
<td>1</td>
<td>2.9</td>
<td>91.2</td>
</tr>
<tr>
<td>95</td>
<td>1</td>
<td>2.9</td>
<td>94.1</td>
</tr>
<tr>
<td>99</td>
<td>2</td>
<td>5.9</td>
<td>100.0</td>
</tr>
<tr>
<td>Total</td>
<td>34</td>
<td>100.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Mean 47.029  Std err 4.658  Median  50.000
Mode 50.000  Std dev 27.162  Variance 737.787
Table 31

The Percentage of Questions on the Worksheets Students Were Able To Answer with Hint

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
<th>Percent</th>
<th>Cum. Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>11.8</td>
<td>11.8</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5.9</td>
<td>17.6</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>17.6</td>
<td>35.3</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>2.9</td>
<td>38.2</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>11.8</td>
<td>50.0</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>14.7</td>
<td>64.7</td>
</tr>
<tr>
<td>30</td>
<td>4</td>
<td>11.8</td>
<td>76.5</td>
</tr>
<tr>
<td>35</td>
<td>1</td>
<td>2.9</td>
<td>79.4</td>
</tr>
<tr>
<td>40</td>
<td>3</td>
<td>8.8</td>
<td>88.2</td>
</tr>
<tr>
<td>45</td>
<td>1</td>
<td>2.9</td>
<td>91.2</td>
</tr>
<tr>
<td>50</td>
<td>3</td>
<td>8.8</td>
<td>100.0</td>
</tr>
<tr>
<td>Total</td>
<td>34</td>
<td>100.0</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Mean 22.353  Std err 2.630  Median 22.500
Mode 10.000  Std dev 15.336  Variance 235.205
Table 32
The Percentage of Questions on the Worksheets Students Were Able To Answer on Their Own or with Hint

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
<th>Percent</th>
<th>Percent Cum.</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>2.9</td>
<td>2.9</td>
</tr>
<tr>
<td>24</td>
<td>1</td>
<td>2.9</td>
<td>5.9</td>
</tr>
<tr>
<td>25</td>
<td>2</td>
<td>5.9</td>
<td>11.8</td>
</tr>
<tr>
<td>30</td>
<td>1</td>
<td>2.9</td>
<td>14.7</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>5.9</td>
<td>20.6</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>2.9</td>
<td>23.5</td>
</tr>
<tr>
<td>60</td>
<td>4</td>
<td>11.8</td>
<td>35.3</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
<td>2.9</td>
<td>38.2</td>
</tr>
<tr>
<td>70</td>
<td>2</td>
<td>5.9</td>
<td>44.1</td>
</tr>
<tr>
<td>75</td>
<td>4</td>
<td>11.8</td>
<td>55.9</td>
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<tr>
<td>80</td>
<td>4</td>
<td>11.8</td>
<td>67.6</td>
</tr>
<tr>
<td>85</td>
<td>1</td>
<td>2.9</td>
<td>70.6</td>
</tr>
<tr>
<td>90</td>
<td>2</td>
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<td>76.5</td>
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<td>95</td>
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<td>5.9</td>
<td>82.4</td>
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<tr>
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<td>2</td>
<td>5.9</td>
<td>88.2</td>
</tr>
<tr>
<td>100</td>
<td>4</td>
<td>11.8</td>
<td>100.0</td>
</tr>
</tbody>
</table>

Total: 34 100.0 100.0

Mean: 69.382  Std err: 4.469  Median: 75.000  
Mode: 60.000  Std dev: 26.056  Variance: 678.910  

Table 33

Percentage of the Questions on the Worksheets Students Already Knew the Answer to from Previous Coursework

<table>
<thead>
<tr>
<th>Q8A</th>
<th>Value</th>
<th>Frequency</th>
<th>Percent</th>
<th>Valid Percent</th>
<th>Cum. Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>6</td>
<td>17.1</td>
<td>18.2</td>
<td>18.2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>2.9</td>
<td>3.0</td>
<td>21.2</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2</td>
<td>5.7</td>
<td>6.1</td>
<td>27.3</td>
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<td>8</td>
<td>1</td>
<td>2.9</td>
<td>3.0</td>
<td>30.3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>7</td>
<td>20.0</td>
<td>21.2</td>
<td>51.5</td>
</tr>
<tr>
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<td>15</td>
<td>1</td>
<td>2.9</td>
<td>3.0</td>
<td>54.5</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>2</td>
<td>5.7</td>
<td>6.1</td>
<td>60.6</td>
</tr>
<tr>
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<td>3</td>
<td>8.6</td>
<td>9.1</td>
<td>69.7</td>
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<td>2</td>
<td>5.7</td>
<td>6.1</td>
<td>75.8</td>
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<td>50</td>
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<td>5.7</td>
<td>6.1</td>
<td>81.8</td>
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<td>60</td>
<td>3</td>
<td>8.6</td>
<td>9.1</td>
<td>90.9</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>1</td>
<td>2.9</td>
<td>3.0</td>
<td>93.9</td>
</tr>
<tr>
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Mean: 25.152  Std err: 4.490  Median: 10.000
Mode: 10.000  Std dev: 25.795  Variance: 665.383
Table 34

Percentage of the Questions on the Worksheets Students Already Knew the Answer to from Pre-reading the Textbook

<table>
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<tr>
<th>Value</th>
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<th>Cum. Percent</th>
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Total 35 100.0 100.0

Mean 13.758 Std err 2.874 Median 10.000

Mode .000 Std dev 16.511 Variance 272.627
Table 35

Percentage of the Questions on the Worksheets Students Already Knew the Answer to from Some Other Source

<table>
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Mean 8.030  Std err 3.321  Median .000
Mode .000  Std dev 19.078  Variance 363.968
Table 36

Percentage of the Questions on the Worksheets Students Did Not Know the Answer to Previously

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<th>Percent</th>
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</table>

Mean 52.697  Std err 5.245  Median 50.000
Mode 40.000  Std dev 30.128  Variance 907.718

Note: To simplify data entry, 100% was entered as 99%
Table 37

Percentage of Time Students Chose To Work on Their Own

<table>
<thead>
<tr>
<th>Value</th>
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<th>Percent</th>
<th>Cum. Percent</th>
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Total 35 100.0

Mean 66.800 Std err 6.299 Median 90.000
Mode 99.000 Std dev 37.266 Variance 1388.753

Note: To simplify data entry, 100% was entered as 99%
Table 38

Percentage Of Time Students Chose To Work in a Pair

<table>
<thead>
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<th>Percent</th>
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</tr>
<tr>
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<td>40.0</td>
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</tr>
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<td>5.7</td>
<td>65.7</td>
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</tr>
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<td>2.9</td>
<td>68.6</td>
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</tr>
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Total 35 100.0

Mean 24.571 Std err 5.425 Median 5.000
Mode .000 Std dev 32.092 Variance 1029.899

Note: To simplify data entry, 100% was entered as 99%
Table 39

Percentage Of Time Students Chose To Work in a Group of 3 or More Students

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<td>94.3</td>
<td></td>
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Total 35 100.0

Mean 8.286 Std err 2.887 Median .000
Mode .000 Std dev 17.082 Variance 291.798
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<th>Percent</th>
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Table 41

Students' Opinion on the Hint-sheets

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Table 42

Students' Opinion on the Solution-sheets

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### Table 43

Percentage of the Class time Students Think Should be Spent on Individual/Group Work Using Worksheets, Hint-sheets and Solution-sheets

<table>
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<th>Frequency</th>
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<th>Cum. Percent</th>
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**Note:** To simplify data entry, 100% was entered as 99%
APPENDIX B

INSTRUMENTS AND GRADING GUIDES TO THE TESTS
MATH 151 FINAL EXAM (120min)

Important: You need to give complete solutions (w/reasons), not only answers. Answers without reasons earn no credit. If your answer is based on graphing calc., say so. Approx. answers (eg., by graphing calc.) 50% credit, except problems where only approx. answer possible.

Good Luck & Enjoy Your Final! (I'm sure you will.)

1. BACK BY POPULAR DEMAND (in an improved form)
   (a) Your first name: _________________ (1p)
   (b) Last: ________________________ (1p)
   (c) Middle: ______________________ (1p)

   Circle your TA: Chunlin George Jim

2. (1p X credit) "Mathematics is where we never know what we are talking about, nor if it is true or not."

   This is a quote of (circle one)
   A. A. Einstein
   B. B. Russell
   C. Your TA
   D. My Mom
   E. Bart Simpson
   F. Leave me alone
   G. None of the above. Please specify: ______________________

3. (a) (COMP*) WARM-UP PROBLEM: Differentiate $f(x) = \frac{1}{2x}$. (4 points)

   (b) (COMP*) Just for the fun of it (+ for the credit) find the derivative by the DEFINITION, too. (10 points)

* COMP=computational skills item, CONC=conceptual understanding item, TRANSF=transfer skills item
(c) After comparing your two answers: (4 points)

\[ \text{A. } \] \hspace{2cm} \[ \text{B. } \]

(d) (CONC) On the drawing your mouth is concave ______ward. (4 points)

4. (COMP) \[ \lim_{x \to 2} \frac{2 - x}{x^2 + 5x - 14} = (8 \text{ points}) \]

5. (COMP) \[ \lim_{x \to -\infty} \frac{5x^7 - 2x^6 + x^2 - 1}{3 - x^7} = \]

Show your work carefully. (8 points)

6. (CONC)

(a) List all real numbers where \( f(x) \) is discontinuous: ______________________

\[ \text{Diagram of } f'(x) \text{ against } x \]
(b) \[ \lim_{{x \to -2}} f(x) = \quad \lim_{{x \to -1}} f(x) = \]

\[ \lim_{{x \to 1}} f(x) = \quad \lim_{{x \to 2}} f(x) = \]

(c) \[ \lim_{{x \to 2^-}} f(x) = \quad \lim_{{x \to 2^+}} f(x) = \]

\[ \lim_{{x \to \_}} f(x) = 2 \]

(List all solutions if there is more than one.)

(d) \[ f'(-1) = \quad f'(0) = \]

\[ f'(1) = \quad f'(1.5) = \quad f'(4) = \]

(e) Determine all \( x \) values where \( f'(x) = 0 \)

(f) Sketch \( f'(x) \) on \([-2,2]\).
(g) Does $f(x)$ have a maximum on $(-2, 6)$? Y / N

If yes: max = ___ at $x = ___$. (43 points)

7. (COMP) Differentiate the following functions. Need not simplify

(unless you can't resist the temptation).

(a) $f(x) = \sin \sin 3x$  $f'(x) =$ ________________ (6 points)
(b) $g(x) = \frac{x \cos x}{\sin 1992 \cdot x}$  $g'(x) =$ ________________ (10 points)

8. (TRANSF) Given the graph of the derivative of $f(x)$, $f'(x)$ what can you say about $f(x)$ at $x = a$? (5 points)

A. $f$ has a local max. at $x = a$.  
B. $f$ has a local min. at $x = a$.  
C. $f$ has an inflection point at $x = a$.  
D. $f$ cannot have any of the above at $x = a$.  
E. Cannot be determined which of the above is true.
9. (TRANSF) Given the graph of the derivative of f(x), f'(x) what can you say about f(x) at x = a? (5 points)

   A. f has a local max. at x = a.
   B. f has a local min. at x = a.
   C. f has an inflection point at x = a.
   D. f cannot have any of the above at x = a.
   E. Cannot be determined which of the above is true.

10. (TRANSF) Below is the graph of the derivative of f(x), f'(x). (12 points)

   (a) f(x) is increasing on the following interval(s): ________
   (b) f(x) is concave upwards on the following interval(s): ________
   (c) f(x) is has inflection point(s) at the following x value(s): ________
11. **(COMP)** The velocity function of an object moving along a line is 
\[ v(t) = 2t^2 + t + 2, \] initial position is 5.

(a) Find the position function. (10 points)

(b) Find the acceleration function. (5 points)

12. **(COMP)** (a) Find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \) if 
\[ 4xy^2 = y^3 - 4x - 9. \] (10 points)

(b) Find an equation of the tangent line at \( P(-1, 1) \). (5 points)

13. **(CONC)** Describe what linear approximation is good for and how it works. Make sure to include the geometry of the linear approximation. (6 points)

14. **(TRANSF)** You want to solve \( f(x) = 0 \) by Newton's method and start with \( x_0 = a \).

Describe what will happen in both algebraic and geometric terms. (\( f \) has a local max. at \( a \).)

\[ f(x) \]

(a) Algebraically: (5 points)
(b) Geometrically: (5 points)

15. (TRANSF) A farmer has 600 yards of fencing to build a rectangular corral having two internal dividers parallel to two sides of the corral. What is the maximum total area of such a corral? (Give careful reasoning.) (18 points)

16. (TRANSF) A rocket is launched vertically and is tracked by a station on the ground 5 mi from the launch pad. The angle of elevation \( \theta \) of the line of sight to the rocket is increasing at \( 3^\circ/\text{s} \) when \( \theta = 60^\circ \). What is the velocity of the rocket at this instant? (15 points)
BONUS: (TRANSF) Find the length of the longest rod that can be carried horizontally from a 4 m wide hallway to a 2 m wide, perpendicular hallway. (20 points)

Have a Terrific Break!

(w/ or w/o derivatives - your choice)
GRADING GUIDE TO THE FINAL EXAM

Important: You need to give complete solutions (w/reasons), not only answers. Answers without reasons earn no credit. If your answer is based on graphing calc., say so. Approx. answers (eg., by graphing calc.) 50% credit, except problems where only approx. answer possible.

Good Luck & Enjoy Your Final! (I'm sure you will.)

1. BACK BY POPULAR DEMAND (in an improved form)
   (a) Your first name: ________________ (1p)
   (b) Last: ________________________ (1p)
   (c) Middle: ______________________ (1p)

   Circle your TA: Chunlin George Jim

2. (1p X credit) "Mathematics is where we never know what we are talking about, nor if it is true or not."

   This is a quote of (circle one)
   A. A. Einstein                   B. B. Russell
   C. Your TA                       D. My Mom
   E. Bart Simpson                  F. Leave me alone
   G. None of the above. Please specify: ___________________

   B. (1 point extra credit)

3. (a) WARM-UP PROBLEM: Differentiate \( f(x) = \frac{1}{2x} \).
   \[ f'(x) = -\frac{1}{2} x^{-2} = -\frac{1}{2x^2} \] (4 points)

   (b) Just for the fun of it (+ for the credit) find the derivative by the DEFINITION, too.
   \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{2(x+h)} - \frac{1}{2x} = \] (3 + 2 points)
\[
\lim_{h \to 0} \frac{x - (x+h)}{2h(x+h)x} = \lim_{h \to 0} \frac{-1}{2h(x+h)x} = \frac{-1}{2x} \quad (2 + 1 + 2 \text{ points})
\]

(c) After comparing your two answers:

A. 

B.

(d) On the drawing your mouth is concave \underline{up}_ward. (4 points)

4. \[
\lim_{x \to 2} \frac{2 - x}{x^2 + 5x - 14} = \lim_{x \to 2} \frac{2 - x}{(x-2)(x+7)} \quad (4 \text{ points})
\]

\[
\lim_{x \to 2} \frac{1}{x+7} = \frac{1}{9} \quad (2 + 2 \text{ points})
\]

5. \[
\lim_{x \to -\infty} \frac{5x^7 - 2x^6 + x^2 - 1}{3 - x^7} = \\text{Show your work carefully.}
\]

\[
= \lim_{x \to -\infty} \frac{1}{x^7} \left( \frac{5}{x} - \frac{2}{x^5} + \frac{1}{x^7} \right) = -5 \quad (4 \times 2 \text{ points})
\]
(a) List all real numbers where \( f(x) \) is discontinuous:
-2, -1, 2  

(1 + 1 + 1 point)

(b) \( \lim_{{x \to -2}} f(x) = -1 \) \( \lim_{{x \to -1}} f(x) = \text{DNE} \)  

(2 + 2 points)

\( \lim_{{x \to 1}} f(x) = 2 \) \( \lim_{{x \to 2}} f(x) = 0 \)

(2 + 2 points)

(c) \( \lim_{{x \to 2^-}} f(x) = 0 \) \( \lim_{{x \to 2^+}} f(x) = 0 \)

(2 + 2 points)

\( \lim_{{x \to -1.4}} f(x) = 2 \)

(2 + 2 points)

(List all solutions if there is more than one.)
(d) \( f'(1) = \text{DNE} \quad f'(0) = 0 \quad (2 + 2 \text{ points}) \)
\( f'(1.5) = -2 \quad f'(4) = 0 \quad (2 + 2 + 2 \text{ points}) \)

(e) Determine all \( x \) values where \( f'(x) = 0 \)
\((-\infty, -2), (-2, -1), (-1, 0], 4, (6, \infty) \) \( (5 \times 1 \text{ point}) \)

(f) Sketch \( f'(x) \) on \([-2, 2]\).

![Graph of f'(x) on [-2,2]](attachment:image.png)

(2 + 2 + 2 points)

(g) Does \( f(x) \) have a maximum on \((-2, 6)\)? \( \text{Y} / \text{N} \)
If yes: \( \text{max} = 2.5 \) at \( x = -2 \). \( (1 + 1 + 1 \text{ point}) \)

7. Differentiate the following functions. Need not simplify
(unless you can't resist the temptation).

(a) \( f(x) = \sin(3x) \quad f'(x) = (\cos(3x))3 \cos x \quad (2 + 2 + 2 \text{ points}) \)

(b) \( g(x) = \frac{x \cos x}{\sin 1992x} \quad g'(x) = \frac{(1 \cos x - x \sin x) \sin 1992x - x \cos x 1992 \sin 1991x \cos x}{\sin 3984x} \)

8. Given the graph of the derivative of $f(x)$, $f'(x)$, what can you say about $f(x)$ at $x = a$?

- A. $f$ has a local max. at $x = a$.
- B. $f$ has a local min. at $x = a$.
- C. $f$ has an inflection point at $x = a$.
- D. $f$ cannot have any of the above at $x = a$.
- E. Cannot be determined which of the above is true.

C. (5 points)

9. Given the graph of the derivative of $f(x)$, $f'(x)$, what can you say about $f(x)$ at $x = a$?

- A. $f$ has a local max. at $x = a$.
- B. $f$ has a local min. at $x = a$.
- C. $f$ has an inflection point at $x = a$.
- D. $f$ cannot have any of the above at $x = a$.
- E. Cannot be determined which of the above is true.
11. The velocity function of an object moving along a line is $v(t) = 2t^2 + t + 2$, initial position is 5.

(a) Find the position function.
\[ s(t) = \frac{2}{3} t^3 + \frac{1}{2} t^2 + 2t + C \] (6 points)

At \( t = 0 \) \( C = 5 \).

\[ s(t) = \frac{2}{3} t^3 + \frac{1}{2} t^2 + 2t + 5 \] (4 points)

(b) Find the acceleration function.

\[ a(t) = v'(t) = 4t + 1 \] (5 points)

12. (a) Find \( dy/dx \) in terms of \( x \) and \( y \) if \( 4xy^2 = y^3 - 4x - 9 \).

\[ 4y^2 + 4x \frac{dy}{dx} = 3y^2 \frac{dy}{dx} - 4 \]

\[ \frac{dy}{dx} \left( 8xy - 3y^2 \right) = -4y^2 - 4 \]

\[ \frac{dy}{dx} = \frac{-4y^2 + 4}{8xy - 3y^2} \] (10 points, 2 points off for each error)

(b) Find an equation of the tangent line at \( P(-1, 1) \).

\[ y - y_0 = m(x - x_0) \]

\[ m = \frac{dy}{dx}(-1, 1) = -\frac{8}{8 - 3} = \frac{8}{5} \]

\[ y - 1 = \frac{8}{11}(x + 1) \] (5 points, 2 points off for each error)

13. Describe what linear approximation is good for and how it works. Make sure to
include the geometry of the linear approximation.
2 points for word description of what linear approximation is good for
2 points for word description of how linear approximation works
2 points for geometric interpretation of linear approximation

14. You want to solve \( f(x) = 0 \) by Newton's method and start with \( x_0 = a \). Describe what will happen in both algebraic and geometric terms. (\( f \) has a local max. at \( a \)).

(a) Algebraically:
\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\]
\( f'(x_0) = 0 \), ratio does not exist, thus Newton's method does not work.
(5 points total, 2 points for \( f'(x_0) = 0 \))

(b) Geometrically:
Tangent line horizontal, hence does not intercept \( x \)-axis, thus Newton's method does not work. (5 points total, 2 points if horizontal tangent line sketched)

15. A farmer has 600 yards of fencing to build a rectangular corral having two internal dividers parallel to two sides of the corral. What is the maximum total area of such a corral? (Give careful reasoning.)
A = x(300 - 2x) = 300x - 2x^2, should find maximum on [0, 150] (7 points)

A'(x) = 300 - 4x (2 points)

Possibilities:
A'(x) = 0  x = 75  A = 11250. (2 points)
A'(x) DNE, there is no such point. (2 points)
Endpoints: x = 0 and x = 150, A = 0 at both. (2 + 2 points)
Hence maximal area: 11250 yrd^2. (1 point)

16. A rocket is launched vertically and is tracked by a station on the ground 5 mi from the launch pad. The angle of elevation α of the line of sight to the rocket is increasing at 3°/s when α = 60°. What is the velocity of the rocket at this instant?

\[
\frac{d\alpha}{dt} = 3^\circ /s = 3 \frac{\pi}{180} /s = \frac{\pi}{60} \frac{\text{rad}}{s}
\]

\[
\frac{dy}{dt} = ?
\]

(4 points)
\[
\tan \alpha = \frac{y}{x} \quad \text{(2 points)}
\]

\[
\frac{d}{dt} \sec \alpha \frac{d\alpha}{dt} = \frac{1}{5} \frac{dy}{dt} \quad \text{(4 points)}
\]

\[
\frac{dy}{dt} = 5 \sec^2 \alpha \frac{d\alpha}{dt} = 5 \frac{\pi}{3} \frac{60}{\pi} = \frac{\pi}{9} \text{ mi/s} \quad \text{(5 points)}
\]

**BONUS:** Find the length of the longest rod that can be carried horizontally from a 4 m wide hallway to a 2 m wide, perpendicular hallway.

![Diagram of hallway and rod](image)

\[
L = \frac{4}{\cos \theta} + \frac{2}{\sin \theta} \quad \text{should find minimum on \([0, \frac{\pi}{2}]\).} \quad \text{(7 points)}
\]

\[
L(\theta) = 4 \sec \theta + 2 \csc \theta 
\]

\[
L'(\theta) = 4 \sec \theta \tan \theta - 2 \csc \theta \cot \theta = 0 \quad \text{(5 points)}
\]

\[
\tan^3 \theta = \frac{1}{2} \quad \text{(3 points)}
\]
\[ \theta = 0.8999 \quad \text{(2 points)} \]
\[ L(\theta) = 8.987 \text{ m} \quad \text{(3 points)} \]

*Have a Terrific Break!*

*(w/ or w/o derivatives - your choice)*
CALCULUS PRETEST (48 min)

Important: Please give complete solutions (w/reasons), not only answers.

If your answer is based on graphing calc., say so.

Give exact answers (not only approximations) whenever possible.

Thank you for your cooperation.

Your name: _________________________________

Your TA: Chunlin George Jim (Circle one.)

1. (COMP*) Determine the value of $x$ (see figure).

2. (TRANSF) DO THE FOLLOWING PROBLEM WITHOUT GRAPHING ON YOUR GRAPHING CALCULATOR (HOWEVER, YOU CAN USE IT FOR COMPUTATION).

   $\sin \sqrt{x} > 0$

   (a) Solve this inequality over the interval $[0, 2\pi)$ (i.e. find all solutions that are between 0 and $2\pi$). ($x$ is measured in radians.)

*COMP=computational skills item, CONC=conceptual understanding item, TRANSF=transfer skills item
(b) Solve the inequality over the interval [0, 50) (i.e. find all solutions that are between 0 and 50). (x is measured in radians.)

3. (COMP.) Differentiate the following functions (i.e. compute the derivative). Do not simplify (unless you can't resist the temptation).

(a) \( f(x) = 3x^5 - 7x^4 + 2 \) 
\( f'(x) = \) 

(b) \( f(x) = \sqrt{x} \) 
\( f'(x) = \) 

(c) \( f(x) = \cos x \) 
\( f'(x) = \) 

(d) \( f(x) = \sin 5x \) 
\( f'(x) = \) 

(e) \( f(x) = x^2 \cos x \) 
\( f'(x) = \) 

(f) \( f(x) = \frac{x^2 \cos x}{\sin 5x} \) 
\( f'(x) = \) 

4. (COMP) Simplify the following expression: 
\( \frac{1}{2(x+h)} - \frac{1}{2x} \)

5. (COMP) Solve for \( x \): 
\( 4(2abx + b^2) = 3b^2x - 4 \)
6. (CONC)

Give approximate answers to the following questions based on the graph above.

(a) The maximum value of \( f(x) \) is ________, achieved at the following \( x \) value(s): ___________.

(b) The minimum value of \( f(x) \) is ________, achieved at the following \( x \) value(s): ___________.

(c) \( f(x) \) is decreasing on the following interval(s): ___________

(d) The range of \( f(x) \) is ___________________________.

(e) \( f'(-3) = ________ \quad f'(-1) = ________ \quad f'(0) = ________ \quad f'(1) = ________ \quad f'(1.5) = ________ \quad f'(4) = ________

7. (TRANSF) A commuter train carries 600 passengers. It costs $1.50 per person to ride the train. It is found that 40 fewer people will ride the train for each 5c increase in the fare.

(a) How many people will ride the train if the fare is increased to $1.65?

(b) How much will be the total revenue in this case?

(c) Now let us assume that (after another fare increase) 360 people ride the train. How much is this new fare?

(d) How much would be the fare if the total revenue would be $832?
8. **TRANSF** A bookseller sells a book for $52.44. His profit in % is the same number as the price in $ for which he gets the book from the publisher. Write an equation from which you could determine this price (but don't need to solve the equation).

9. Find all x values (approximately from the graph) for which

(a) \( f(x) = 0 \)  
(b) \( f(x) = g(x) \)  
(c) \( f(x) < g(x) \)  
(d) \( f(x) - g(x) = 0 \)
10. (CONC) Given the graph of \( f(x) \) sketch the graph of its derivative, \( f'(x) \).
1. Determine the value of $x$ (see figure).

\[
\sin 20^\circ = \frac{2}{x} \quad (2 \text{ points})
\]

\[
x = \frac{2}{\sin 20^\circ} = 5.8476 \quad (2 + 2 \text{ points})
\]

Full credit was given for approximate answer.

2. DO THE FOLLOWING PROBLEM WITHOUT GRAPHING ON YOUR GRAPHING CALCULATOR

(HOWEVER, YOU CAN USE IT FOR COMPUTATION).
\[
\sin \sqrt{x} > 0
\]
(a) Solve this inequality over the interval \([0, 2\pi]\) (i.e. find all solutions that are between 0 and \(2\pi\)). (\(x\) is measured in radians.)
\[
0 < \sqrt{x} < \pi \quad 0 < x < 2\pi^2 \quad \text{(2 + 2 points)}
\]
\[
2\pi < \sqrt{x} < 3\pi \quad x \text{ is too big} \quad \text{(no points are taken off if this is missing)}
\]
(b) Solve the inequality over the interval \([0, 50]\) (i.e. find all solutions that are between 0 and 50).
(\(x\) is measured in radians.)
\[
0 < \sqrt{x} < \pi \quad 0 < x < 2\pi^2 \quad \text{(1 point)}
\]
\[
2\pi < \sqrt{x} < 3\pi \quad 4\pi^2 < x < 9\pi^2 \quad \text{(1 + 1 point)}
\]
\[
4\pi^2 < x < 50 \quad \text{(2 points)}
\]
\[
4\pi < \sqrt{x} < 5\pi \quad x \text{ is too big} \quad \text{(no points are taken off if this is missing)}
\]

3. Differentiate the following functions (i.e. compute the derivative). Do not simplify (unless you can't resist the temptation).

(a) \(f(x) = 3x^5 - 7x^4 + 2\)
\[
f'(x) = 15x^2 - 28x^3 \quad \text{(2 points)}
\]
(b) \(f(x) = \sqrt{x}\)
\[
f'(x) = \frac{1}{2\sqrt{x}} \quad \text{(2 points)}
\]
(c) \(f(x) = \cos x\)
\[
f'(x) = -\sin x \quad \text{(2 points)}
\]
(d) \(f(x) = \sin 5x\)
\[
f'(x) = 5\cos 5x \quad \text{(4 points)}
\]
(e) \(f(x) = x^2 \cos x\)
\[
f'(x) = 2x \cos x - x^2 \sin x \quad \text{(2 + 2 points)}
\]
(f) \(f(x) = \frac{x^2 \cos x}{\sin 5x}\)
\[
f'(x) = \frac{(2x \cos x - x^2 \sin x) \sin 5x - x^2 \cos x (5 \cos 5x)}{\sin^2 5x} \quad \text{(2 + 2 points)}
\]

1 point was taken off for each minor error (e.g. sign, coefficient, ...)

No credit was taken off if wrong result from (d) or (e) is carried over, but otherwise
4. Simplify the following expression: \[
\frac{1}{2(x+h)} - \frac{1}{2x} =
\]

\[
\frac{x - (x+h)}{2x(x+h)h} =
\]

(2 points)

\[
= \frac{x - x - h}{2x(x+h)h} =
\]

(1 point numerator, 2 points for making numerator and denominator simple)

\[
= \frac{-h}{2x(x+h)h} =
\]

(1 point)

\[
= -\frac{1}{2x(x+h)}
\]

(2 points)

5. Solve for x: \[
4(2abx + b^2) = 3b^2x - 4
\]

\[
8abx + 4b^2 = 3b^2x - 4
\]

(2 points)

\[
8abx - 3b^2x = -4b^2 - 4
\]

(2 points)

\[
(8ab - 3b^2)x = -4b^2 - 4
\]

(2 points)

\[
x = \frac{-4b^2 - 4}{8ab - 3b^2} = 4 \frac{b^2 + 1}{3b^2 - 8ab}
\]

(2 points for either form)

if \(8ab - 3b^2 \neq 0\)  (no points taken off if this is missing)
Give approximate answers to the following questions based on the graph above.

(a) The maximum value of $f(x)$ is 2, achieved at the following $x$ value(s):
   \[ 1, 4 \]  
   (1 + 1 + 1 point)

(b) The minimum value of $f(x)$ is -1, achieved at the following $x$ value(s):
   \[ (-\infty, -2) \cup (-2, -1) \]  
   (1 + 1 + 1 point)

(c) $f(x)$ is decreasing on the following interval(s):
   \[ [1.2] \cup [4.6] \]  
   (1 + 1 point)

(d) The range of $f(x)$ is $\{-1\} \cup [0.2]$.
   (1 + 2 points)

(e) $f'(-3) = 0$ \hspace{1cm} $f'(-1) = \text{DNE}$ \hspace{1cm} $f'(0) = 0$ \hspace{1cm} $f'(4) = 0$ \hspace{1cm} $f'(1.5) = -2$
   (1 + 1 + 1 point)

7. A commuter train carries 600 passengers. It costs $1.50 per person to ride the train. It is found that 40 fewer people will ride the train for each 5c increase in the fare.

(a) How many people will ride the train if the fare is increased to $1.65$?
600 - 3.40 = 480 (2 points)

(b) How much will be the total revenue in this case?

480 \cdot $1.65 = $792 (2 points, full credit if uses wrong result from (a))

(c) Now let us assume that (after another fare increase) 360 people ride the train. How much is this new fare?

\[
\frac{600 - (x - 1.50) \cdot 100}{40} = 360
\]

\[
600 - 800 (x - 1.50) = 360
\]

\[
600 - 800x + 1200 = 360
\]

\[
1440 = 800x
\]

\[
x = \$ 1.80
\]

(1 point for the last step if solved using a graphing calculator)

(d) How much would be the fare if the total revenue would be $832?

\[
(600 - \frac{(x - 1.50) \cdot 100}{5} \cdot 40) \cdot x = 832
\]

\[
x = \$1.62 \text{ or } \$0.65
\]

(1 point for the last step if solved using a graphing calculator)

8. A bookseller sells a book for $52.44. His profit in % is the same number as the price in $ for which he gets the book from the publisher. Write an equation from which you could determine this price (but don't need to solve the equation).
9. Find all x values (approximately from the graph) for which

\[ y = f(x) \]
\[ y = g(x) \]

(a) \( f(x) = 0 \) \hspace{1cm} \{-4, 2\} \hspace{1cm} (1 + 1 \text{ point})
(b) \( f(x) = g(x) \) \hspace{1cm} \{-3, 3\} \hspace{1cm} (1 + 1 \text{ point})
(c) \( f(x) < g(x) \) \hspace{1cm} \(-\infty, -3) \cup (3, \infty) \hspace{1cm} (2 + 2 \text{ points})
(d) \( f(x) - g(x) = 0 \) \hspace{1cm} \{-3, 3\} \hspace{1cm} (1 + 1 \text{ point})
10. Given the graph of $f(x)$ sketch the graph of its derivative, $f'(x)$.

(2 + 2 + 2 points)

a total of 1 point taken off if no holes indicated at $x = 0$ and/or 1.
1. (CONC*)

a. \( \lim_{x \to -2} f(x) = \) __________

b. \( \lim_{x \to -2} 2f(x) = \) __________

c. \( \lim_{x \to -2} -f(x) = \) __________

d. \( \lim_{x \to f(x)} = 2 \) __________

e. \( \lim_{x \to f(x)} = -1 \) __________

f. List the points of discontinuity __________

* COMP=computational skills item, CONC=conceptual understanding item, TRANSF=transfer skills item
2. (COMP) \[ \lim_{x \to 2} \frac{x - 2}{x^2 - 3x + 2} = \]

3. (COMP) \[ \lim_{x \to 0} \frac{\tan^2 7x}{13x^2} = \]

4. (CONC) Describe in your own words (but as precisely as you can) what you mean by \( \lim_{x \to -\infty} f(x) = \infty \).

5. (TRANSF) Let \( \lim_{x \to 1} f(x) = 5 \), \( \lim_{x \to 1} g(x) = \text{DNE} \). What can you say about \( \lim_{x \to 1} (f(x) + g(x)) \)? Give reasons, try to prove your claim.

6. (TRANSF) Let \( \lim_{x \to 3} f(x) = \text{DNE} \), \( \lim_{x \to 3} g(x) = \text{DNE} \). What can you say about \( \lim_{x \to 3} (f(x) + g(x)) \)? Give reasons, try to be as complete as you can.
GRADING GUIDE TO THE SHORT TERM RETENTION TEST

1. 

\[ \lim_{x \to -2} f(x) = \] 
(3 points)

\[ \lim_{x \to -2} f(x) = \text{DNE} \] 
(3 points)

\[ \lim_{x \to -2} f(x) = -\infty \] 
(3 points)

d. \[ \lim_{x \to -2} f(x) = 2 \] 
\[ -4, -2 \pm \] 
(3 + 3 points)

e. \[ \lim_{x \to -2} f(x) = -1 \] 
\[ +\infty \] 
(4 points)

f. List the points of discontinuity 
\[ -2, -1, 0 \] 
(2 + 2 + 2 points)
2. \[ \lim_{x \to 2} \frac{x - 2}{x^2 - 3x + 2} = \lim_{x \to 2} \frac{x-2}{(x-2)(x-1)} = \quad (4 \text{ points}) \]

\[ \lim_{x \to 2} \frac{1}{x-1} = \quad (4 \text{ points}) \]

\[ \frac{1}{2-1} = 1 \quad (4 \text{ points}) \]

A total of 6 points were given for approximate answers using graphing calculator.

3. \[ \lim_{x \to 0} \frac{\tan^2 7x}{13x^2} = \lim_{x \to 0} \frac{\sin^2 7x}{13x^2} \cdot \frac{1}{\cos^2 7x} = \quad (4 \text{ points}) \]

\[ \lim_{x \to 0} \left( \frac{\sin 7x}{7x} \right)^2 \cdot \frac{1}{13} \cdot \frac{1}{\cos^2 7x} = 1 \cdot \frac{1}{13} \cdot \frac{1}{13} = \frac{49}{13} \]

4 points for the form above, 4 points for the limit of \( \left( \frac{\sin 7x}{7x} \right)^2 \),

4 points for the limit of \( \frac{1}{\cos^2 7x} \) and 4 points for the final answer.

A total of 10 points were given for approximate answers using graphing calculator.

4. Describe in your own words (but as precisely as you can) what you mean by

\[ \lim_{x \to -\infty} f(x) = \infty. \]

8 points total. 4 points were given for answers showing some conceptual understanding of the limit concept, even if the answer was not precise. 1 point was given for examples by picture or formula.
5. Let \( \lim_{x \to 1} f(x) = 5 \), \( \lim_{x \to 1} g(x) = \text{DNE} \). What can you say about \( \lim_{x \to 1} (f(x) + g(x)) \)?

Give reasons, try to prove your claim.

DNE (4 points) 8 points were given for a precise proof. Partial credit was given for intuitive arguments. 1 point was given for an example by formula or graph.

6. Let \( \lim_{x \to 3} f(x) = \text{DNE} \), \( \lim_{x \to 3} g(x) = \text{DNE} \). What can you say about \( \lim_{x \to 3} (f(x) + g(x)) \)?

Give reasons, try to be as complete as you can.

a. DNE \( f(x) = g(x) = \frac{1}{x-3} \) (2 + 4 points)

b. finite \( f(x) = \frac{1}{x-3} \), \( g(x) = \frac{-1}{x-3} \) (2 + 4 points)

c. infinite \( f(x) = \frac{1}{x-3} \), \( g(x) = \frac{1}{(x-3)^2} - \frac{1}{x-3} \) (2 + 4 points)

The answer "may exist" without explanation was given 2 points.

The answer "may or may not exist" without explanation was given 4 points.
LONG TERM RETENTION TEST (48 min)

The long term retention test consisted of the following problems of the final exam:

**Computational skills items:** 3b, 4, 12a

**Conceptual understanding items:** 6, 13

**Transfer skills items:** 8, 9, 14, 16.
APPENDIX C

THE TWO LETTERS TO THE STUDENTS
ABOUT THE RETENTION TEST
May 29, 1992

Hi!

Thank you very much for participating in our MATH 151 experiment last quarter. As we talked about it last quarter we would like you to take a 48 min. long retention test at the end of this quarter, for which you will be paid $10. The test will be given at the times and locations listed below. Please give me a call if you cannot make it at any of these times so we can schedule a time that is convenient to you.

Tue., June 2 3 - 3:48 PM IV 221 (Ives Hall, Woodruff & Neil)
Tue., June 2 4 - 4:48 PM IV 221 (Ives Hall, Woodruff & Neil)
Wedn., June 3 9 - 9:48 AM Jones Tower Pool (see directions on other sheet)
Wedn., June 3 12 - 12:48 PM IV 219 (Ives Hall, Woodruff & Neil)
Wedn., June 3 1 - 1:48 PM IV 219 (Ives Hall, Woodruff & Neil)
Thur., June 4 12 - 12:48 PM JR 291 (Journalism Bldg., 18th & Neil)
Thur., June 4 1 - 1:48 PM JR 291 (Journalism Bldg., 18th & Neil)
Fri., June 5 9 - 9:48 AM Jones Tower Pool (see directions on other sheet)
Fri., June 5 11 - 11:48 AM JR 291 (Journalism Bldg., 18th & Neil)
Fri., June 5 12 - 12:48 PM JR 291 (Journalism Bldg., 18th & Neil)
Sat., June 6 5 - 5:48 PM Jones Tower Pool (see directions on other sheet)
Sun., June 7 5 - 5:48 PM Jones Tower Pool (see directions on other sheet)

Please have a picture ID with you (needed to receive the $10 payment).

Thank you very much for your cooperation.

George Emese
293-9717 (till 11 PM)
June 25, 1992

Hi!

I hope your Spring Quarter was successful and now you are enjoying your Summer.

I am sorry you could not make it to the MATH 151 retention test at the end of last quarter. Since I cannot complete my dissertation without these data I am sending you a copy of the test (along with a $10 check, as promised). I would greatly appreciate your completing it and mailing the test and the receipt in the enclosed stamped return envelope by Monday, July 6. (Make sure to return the receipt, otherwise the $10 will go from my own pocket.)

Please spend exactly 48 minutes on the test, without previously looking at the problems (i.e., start reading the test only when you are ready to spend 48 minutes continuously on it, without interruptions). You may use any kind of calculator, but please do not use your textbook or notes and do not consult anyone about the problems (i.e., same conditions as in class exams). Should you be prevented from having the same conditions (e.g., you are interrupted and continue later) please describe the exact circumstances on the attached questionnaire. Please do not prepare for the test in any way, what I am interested in is how much you remember without any review. Please let me know if you used a graphing calculator during the quarter but do not have access to one now, I can lend you one for the test.

Please call me (collect) if you have any questions.

Thank you very much for making it possible for me to finish my dissertation. Have a great Summer!

George Emese
(614) 293-9717 (home)

Hey Dude,

I wanna graduate this quarter & get out of this place, so please do return the test! Thx. 😊
APPENDIX D

WORKSHEETS, HINT-SHEETS AND SOLUTION-SHEETS
EX 1 Estimate the slopes of the following lines.

| Line | L1 | L2 | L3 | L4 | L5 | L6 | L7 |
| Slope |    |    |    |    |    |    |    |

EX 2 Sketch the tangent lines of the following curves at the indicated points.

Now estimate the slope of your tangent lines drawn at these points.

| Point | A | B | C | D | E | F | G | H |
| Slope |    |    |    |    |    |    |    |    |

Define tangent line (to a curve at a point):

Describe what is meant by “the slope of a curve at a point”:
Let $f(x) = x^2$. Sketch its graph.

Describe the features of this graph.

- **Domain:** ________________
- **Range:** __________________________
- **Symmetry:** _____________________________

Sketch the tangent line of the graph $y = x^2$ and estimate the slopes at the following $x$ - values.

<table>
<thead>
<tr>
<th>$x$ - value</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of Tangent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Look at the table of $x$ - values and slope values. Is there a function that would describe all of the slope values? This suggests that the function $y = x^2$ has a slope function $y = ________$
EX 4  Repeat EX 3 with the function \( y = |x| \). Graph it.

Compute the table of slope values as before.

\[
\begin{array}{cccccccc}
| \text{x} | & -3 & -2 & -1/2 & 1 & 2 & 1000 & 0 \\
| \text{Slope of } f(x) \text{ at } x | & & & & & & & \\
\end{array}
\]
EX 5  Repeat as before with the function \( y = \sin x \). Graph it.

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 0 & \pi/2 & \pi & 3\pi/2 & 2\pi \\
\hline
\text{Slope} & & & & & \\
\hline
\end{array}
\]

Let's determine the slope function. To do this consider the following statements.

1. \( \sin x \) has the period = ____; therefore it repeats itself every interval of length______;
   so its slope function ________________________________.

2. \( \sin x \) has a zero slope when \( x = \)____________________ so the slope function
   ____________________________.

3. \( \sin x \) has its largest slope of ___ when \( x = \)____________________, therefore the
   slope function ________________________________.

4. \( \sin x \) has its smallest slope of ___ when \( x = \)____________________,
   therefore the slope function ________________________________

5. What function has all these properties? ______

Thus, the slope function of \( y = \sin x \) is likely to be \( y = ______ \).

Proof will follow later.
3.1 Slopes, Tangent Lines and Derivatives

**HINT-SHEET**

**EX 1** Estimate the slopes of the following lines. *Hint: slope = rise / run*

<table>
<thead>
<tr>
<th>Line</th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L5</th>
<th>L6</th>
<th>L7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EX 2** Sketch the tangent lines of the following curves at the indicated points.

Now estimate the slope of your tangent lines drawn at these points.

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>2</td>
<td>-1.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Define tangent line (to a curve at a point): *a line through that point whose slope is* ....

Describe what is meant by “the slope of a curve at a point”: *Consider the slopes of the secant lines through that point* ....
EX 3  Let $f(x) = x^2$. Sketch its graph.

Describe the features of this graph.

Domain: possible $x$ values

Range: possible $y$ values

Symmetry: Circle all that apply: with respect to x axis / y axis / origin

Sketch the tangent line of the graph $y = x^2$ and estimate the slopes at the following $x$ - values.

<table>
<thead>
<tr>
<th>$x$-value</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of Tangent</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Look at the table of $x$ - values and slope values. Is there a function that would describe all of the slope values? This suggests that the function $y = x^2$ has a slope function $y = 2x$.
EX 4  Repeat EX 3 with the function $y = |x|$. Graph it.

*Hint:* $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Compute the table of slope values as before.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$-1$</th>
<th>$-2$</th>
<th>$-1/2$</th>
<th>$1$</th>
<th>$2$</th>
<th>$1000$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of $f(x)$ at $x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
EX 5 Repeat as before with the function $y = \sin x$. Graph it.

Let's determine the slope function. To do this consider the following statements.

1. $\sin x$ has the period $= \frac{2\pi}{2}$; therefore it repeats itself every interval of length $\frac{2\pi}{2}$; so its slope function also has the period $2\pi$.

2. $\sin x$ has a zero slope when $x = (2k + 1)\frac{\pi}{2}$ for $k = 0, 1, 2, 3, \ldots$, so the slope function

3. $\sin x$ has its largest slope of $1$ when $x = \ldots$, therefore the slope function

4. $\sin x$ has its smallest slope of $-1$ when $x = \ldots$, therefore the slope function

5. What function has all these properties? ____________

Thus, the slope function of $y = \sin x$ is likely to be $y = \ldots$

Proof will follow later.
3.1 Slopes, Tangent Lines and Derivatives

SOLUTION-SHEET

EX 1 Estimate the slopes of the following lines.

<table>
<thead>
<tr>
<th>Line</th>
<th>L1</th>
<th>L2</th>
<th>L3</th>
<th>L4</th>
<th>L5</th>
<th>L6</th>
<th>L7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>1</td>
<td>0</td>
<td>1.5</td>
<td>-8</td>
<td>-1</td>
<td>-1/2</td>
<td>DNE</td>
</tr>
</tbody>
</table>

EX 2 Sketch the tangent lines of the following curves at the indicated points.

Now estimate the slope of your tangent lines drawn at these points.

<table>
<thead>
<tr>
<th>Point</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>2</td>
<td>-1.4</td>
<td>1</td>
<td>0</td>
<td>-10</td>
<td>2.5</td>
<td>0.9</td>
<td>0</td>
</tr>
</tbody>
</table>

Define tangent line (to a curve at a point): *The line through the point \((x, f(x))\) with slope \(f'(x)\).*

Describe what is meant by “the slope of a curve at a point”: *It is the slope of the tangent to the curve at that point.*
EX 3. Let \( f(x) = x^2 \). Sketch its graph.

Describe the features of this graph.

Domain: all the real numbers
Range: the positive numbers and 0
Symmetry: to the y-axis

Sketch the tangent line of the graph \( y = x^2 \) and estimate the slopes at the following \( x \)-values.

<table>
<thead>
<tr>
<th>( x )-value</th>
<th>0</th>
<th>1</th>
<th>-1</th>
<th>2</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of Tangent</td>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>4</td>
<td>-4</td>
</tr>
</tbody>
</table>

Look at the table of \( x \)-values and slope values. Is there a function that would describe all of the slope values? This suggests that the function \( y = x^2 \) has a slope function \( y = 2x \).
EX 4  Repeat EX 3 with the function $y = |x|$. Graph it.

Compute the table of slope values as before.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1/2</th>
<th>1</th>
<th>2</th>
<th>1000</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of $f(x)$ at $x$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>DNE</td>
</tr>
</tbody>
</table>


EX 5  Repeat as before with the function \( y = \sin x \). Graph it.

\[ 
\begin{array}{cccccc}
 x & 0 & \pi/2 & \pi & 3\pi/2 & 2\pi \\
 \text{Slope} & 0 & 1 & 0 & -1 & 0 \\
\end{array}
\]

Let's determine the slope function. To do this consider the following statements.

1. \( \sin x \) has the period = \( 2\pi \); therefore it repeats itself every interval of length \( 2\pi \); so its slope function \( \text{also has the period } 2\pi \).

2. \( \sin x \) has a zero slope when \( x = (2k + 1/2)\pi \) for \( k = 0, 1, -1, 2, -2, ... \), so the slope function \( \text{is zero at these points} \).

3. \( \sin x \) has its largest slope of 1 when \( x = 2k\pi \) for \( k = 0, 1, -1, 2, 2, ... \), therefore the slope function \( \text{is never greater than 1 and is equal to 1 at } x = (2k)\pi \text{ for } k = 0, -1, 1, 2, 2, ... \).

4. \( \sin x \) has its smallest slope of -1 when \( x = (2k+1)\pi \) for \( k = 0, 1, -1, -2, 2, ... \), therefore the slope function \( \text{is never less then -1 and is equal to 1 at } x = (2k+1)\pi \text{ for } k = 0, -1, 1, -2, 2, ... \).

5. What function has all these properties? \( \cos(x) \)

Thus, the slope function of \( y = \sin x \) is likely to be \( y = \cos(x) \).

Proof will follow later.
EX 1  Discover the derivative of \( \cos x \). That is, \( \frac{d}{dx}(\cos x) = ? \)

(a) Sketch the graph of \( y = \cos x \) between \(-2\pi\) and \(2\pi\).

(b) For each \( x \)-value in the table below, find the slope of the tangent line of \( \cos x \) at the \( x \)-value. You may wish to estimate these slopes or you may want to use the SloPe VALue program of your calculator. To check your work, two of the slope values are given.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2\pi)</th>
<th>(-3\pi/2)</th>
<th>(-\pi)</th>
<th>(-\pi/2)</th>
<th>(0)</th>
<th>(\pi/2)</th>
<th>(3\pi/2)</th>
<th>(2\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tangent Line</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Plot the points of the slope function from the table in the graph below. The two points \((-\pi, 0)\) and \((\pi/2, -1)\) from the table are already plotted.
(d) Is there symmetry in this graph? Consider the slope values at \(-2\pi\) and \(2\pi\), 
\(-\pi\) and \(\pi\) etc., when answering. ______________________________

(e) Plot a few more points on the slope function from \(-2\pi\) to \(2\pi\). Does this 
function look familiar?
The slope function is __________________
Hence, it is likely that \( \frac{d}{dx}(\cos x) = \) ____________

EX 2 And the derivative of \(\sin x\) is?
Sketch the graph of \(y = \sin x\) and its slope function.
You may wish to use the SLoPe FUNCtion program of your graphing calculator.
Remember

\( y_1 = \sin x, \ y_2 = \text{NDERIV}(y_1, 0.001). \)

The slope function of \( \sin x \) looks like \_______________.
Hence, it is likely that \( \frac{d}{dx} (\sin x) = \_______________\). Proof will follow later.

**EX 3** Now let's use these two new derivatives in conjunction with our other rules (i.e. power rule, product rule, quotient rule, sum rule, constant multiple rule, difference rule) to compute the following derivatives.

(a) \( \frac{d}{dx} (2x^4 - 1992 \sin x) = \_______________ \)

Which rules of differentiation did you use? \_______________

(b) \( \frac{d}{dx} (x^3 \cos x) = \_______________ \)

Which rules? \_______________

(c) \( \frac{d}{dx} \frac{4}{\cos x} = \)

Which rules? \_______________
EX 1 Discover the derivative of $\cos x$. That is, $\frac{d}{dx}(\cos x) = ?$

(a) Sketch the graph of $y = \cos x$ between $-2\pi$ and $2\pi$.

(b) For each $x$-value in the table below, find the slope of the tangent line of $\cos x$ at the $x$-value. You may wish to estimate these slopes or you may want to use the SLoPe VALue program of your calculator. To check your work, two of the slope values are given.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-\pi$</th>
<th>$-\pi/2$</th>
<th>$0$</th>
<th>$\pi/2$</th>
<th>$\pi$</th>
<th>$3\pi/2$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Tangent Line</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Plot the points of the slope function from the table in the graph below. The two points $( - \pi, 0)$ and $( \pi/2, -1)$ from the table are already plotted.
(d) Is there symmetry in this graph? Consider the slope values at $-2\pi$ and $2\pi$, $-\pi$ and $\pi$ etc., when answering. ___________________________.

(e) Plot a few more points on the slope function from $-2\pi$ to $2\pi$. Does this function look familiar?

The slope function is ___________________.

Hence, it is likely that $\frac{d}{dx}(\cos x) =$ _________.

Proof will follow later.

EX 2 And the derivative of $\sin x$ is?

Sketch the graph of $y = \sin x$ and its slope function.
You may wish to use the SLoPe FUNCTION program of your graphing calculator.

Remember

\[ y_1 = \sin x, \quad y_2 = \text{NDERIV}(y_1, 0.001). \]

The slope function of \( \sin x \) looks like \[ \text{__________} \].

Hence, it is likely that \( \frac{d}{dx}(\sin x) = \text{__________} \). Proof will follow later.

**EX 3** Now let's use these two new derivatives in conjunction with our other rules (i.e. power rule, product rule, quotient rule, sum rule, constant multiple rule, difference rule) to compute the following derivatives.

(a) \( \frac{d}{dx}(2x^4 - 1992 \sin x) = \text{____________________}. \)

Which rules of differentiation did you use? *The difference, constant multiple, power rules.*

(b) \( \frac{d}{dx}(x^3 \cos x) = \text{____________________}. \)

Which rules? *The product and power rules.*

(c) \( \frac{d}{dx} \frac{4}{\cos x} = \)

Which rules? *The quotient rule.*
3.4 Derivatives of Trigonometric Functions

SOLUTION-SHEET

EX 1  Discover the derivative of $\cos x$. That is, $\frac{d}{dx}(\cos x) = ?$

(a) Sketch the graph of $y = \cos x$ between $-2\pi$ and $2\pi$.

(b) For each $x$-value in the table below, find the slope of the tangent line of $\cos x$ at the $x$-value. You may wish to estimate these slopes or you may want to use the SLoPe VALue program of your calculator. To check your work, two of the slope values are given.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2\pi$</th>
<th>$-3\pi/2$</th>
<th>$-\pi$</th>
<th>$-\pi/2$</th>
<th>$0$</th>
<th>$\pi/2$</th>
<th>$3\pi/2$</th>
<th>$2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope of</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Tangent Line</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Plot the points of the slope function from the table in the graph below. The two points $(-\pi, 0)$ and $(\pi/2, -1)$ from the table are already plotted.
(d) Is there symmetry in this graph? Consider the slope values at $-2\pi$ and $2\pi$, $-\pi$ and $\pi$ etc., when answering. *Symmetric with respect to the origin.*

(e) Plot a few more points on the slope function from $-2\pi$ to $2\pi$. Does this function look familiar?

The slope function is \( \frac{\sin(x)}{x} \).

Hence, it is likely that \( \frac{d}{dx}(\cos x) = -\sin(x) \).

Proof will follow later.

**EX 2** And the derivative of $\sin x$ is?

Sketch the graph of $y = \sin x$ and its slope function.
You may wish to use the SLoPe FUNCTION program of your graphing calculator. Remember

\[ y_1 = \sin x, \quad y_2 = \text{NDERIV}(y_1, 0.001). \]

The slope function of \( \sin x \) looks like \( \cos(x) \).

Hence, it is likely that \( \frac{d}{dx}(\sin x) = \cos(x) \). Proof will follow later.

**EX 3** Now let’s use these two new derivatives in conjunction with our other rules (i.e. power rule, product rule, quotient rule, sum rule, constant multiple rule, difference rule) to compute the following derivatives.

(a) \( \frac{d}{dx}(2x^4 - 1992 \sin x) = -8x^3 - 1992 \cos(x) \).

Which rules of differentiation did you use? **The difference, constant multiple, power rules.**

(b) \( \frac{d}{dx}(x^3 \cos x) = 3x^2 \cos(x) - x^3 \sin(x) \).

Which rules? **The product and power rules.**

(c) \( \frac{d}{dx} \frac{4 \sin(x)}{\cos^2(x)} = \frac{4 \sin(x)}{\cos^2(x)} \).

Which rules? **The quotient rule.**
3.5 The Chain Rule

WORK-SHEET

How do you compute the derivatives of the following functions?

a) \( y = (7x - 4)^2 \)

b) \( y = \sin(2x^6 - 1992) \)

c) \( y = \tan^4 x \)

Perhaps we could differentiate a) using a rule we have already discussed, but parts b) and c) can not be done without a new rule that we will discuss today - the chain rule.

Q1 What do the functions in a), b) and c) have in common?

Q2 Write the functions in a), b) and c) as compositions of two functions.

\[
\begin{align*}
\text{a)} & \quad y = (7x - 4)^2 \\
\text{b)} & \quad y = \sin(2x^6 - 1992) \\
\text{c)} & \quad y = \tan^4 x \\
& \quad y = \\
& \quad u = \\
& \quad y = \\
& \quad u = \\
\end{align*}
\]

Q3 How do we compute derivatives of such functions? Let’s consider example a).

EX 1 Compute the derivative of \( y = (7x - 4)^2 \).

Old way: \( y = (7x - 4)(7x - 4) = 49x^2 - 56x + 16 \), so \( y' = 98x - 56 \).

New way: \( \frac{dy}{du} = 2u \) because \( y = u^2 \); \( \frac{du}{dx} = 7 \) because \( u = 7x - 4 \).

In terms of \( x \), \( \frac{dy}{du} = 2(7x - 4) = 14x - 8 \).

Do you see a relationship between \( \frac{dy}{du} \), \( \frac{du}{dx} \) and \( \frac{dy}{du} \)?

The relationship is:

Not convinced? Try another example.
EX 2  A spider moves along the line $y = (1/2)x - 3$ in such a way that its $x$-coordinate at time $t$ (seconds) is given by $x = 4t$. Find the rate at which the $y$-coordinate is changing with respect to time (i.e. $\frac{dy}{dt}$).

Complete the following table showing the relationship between $t$, $x$ and $y$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As $t$ changes by 1 second, $x$ changes by ______ and $y$ changes by ______.

\[
\frac{dy}{dt} = \frac{dx}{dt} \quad \frac{dy}{dt} =
\]

Check the relationship found in EX 1: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$  (True / False)

The Chain Rule: If $y$ is a differentiable function of $u$ and $u$ is a differentiable function of $x$, then:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

We can alternately write the Chain Rule in another form: If $y = f(u)$ and $u = g(x)$, then:

\[
[f(g(x))]' = f'(u)g'(x) = f'(g(x))g'(x).
\]

Or, in words, the derivative of a composition of functions $f \circ g$ is equal to the derivative of the “outside” function $f$ evaluated at the “inside” function, times the derivative of the “inside” function.
EX 3  Pick out the “inside” and the “outside” functions in the compositions below and then compute their derivatives.

b) \( y = \sin(2x^6 - 1992) \)
Outside function: \( f(x) = \)
Inside function: \( g(x) = \)
\( y' = \)

c) \( y = \tan^4 x \)
Outside function: \( f(x) = \)
Inside function: \( g(x) = \)
\( y' = \)

EX 4  Practice a few more derivatives using the Chain Rule.

a) \( (d/dx)[(11x^{22} - 1963x)^{-14}] = \)
(Which date do you recognize in the problem ?)

b) \( (\sin \sin \sin x)' = \)

c) \( [\cos (x^{1/2})]' = \)

d) \( \frac{d}{dx}(u^n) = \)
where \( n \) is an integer and \( u \) is a function of \( x \). This is called the General Power Rule.
3.5 The Chain Rule

HINT-SHEET

How do you compute the derivatives of the following functions?

a) \( y = (7x - 4)^2 \)
b) \( y = \sin(2x^6 - 1992) \)
c) \( y = \tan^4x \)

Perhaps we could differentiate a) using a rule we have already discussed, but parts b) and c) cannot be done without a new rule that we will discuss today - the chain rule.

Q1 What do the functions in a), b) and c) have in common?

*They are compositions of two functions.*

Q2 Write the functions in a), b) and c) as compositions of two functions.

a) \( y = (7x - 4)^2 \)   \[ y = u^2 \]
   \[ u = 7x - 4 \]
b) \( y = \sin(2x^6 - 1992) \)   \[ y = \sin(u) \]
   \[ u = 2x^6 - 1992 \]
c) \( y = \tan^4x \)   \[ y = u^4 \]
   \[ u = \tan x \]

Q3 How do we compute derivatives of such functions? Let's consider example a).

EX 1 Compute the derivative of \( y = (7x - 4)^2 \).

Old way: \( y = (7x - 4)(7x - 4) = 49x^2 - 56x + 16 \), so \( y' = 98x - 56 = \frac{dy}{dx} \)

New way: \( \frac{dy}{du} = 2u \) because \( y = u^2 \);
\( \frac{du}{dx} = 7 \) because \( u = 7x - 4 \).

In terms of \( x \), \( \frac{dy}{du} = 2u = 2(7x - 4) = 14x - 8 \).

Do you see a relationship between \( \frac{dy}{dx} \), \( \frac{du}{dx} \) and \( \frac{dy}{du} \)?

The relationship is: \( \frac{dy}{dx} = \text{Express in terms of } \frac{du}{dx} \text{ and } \frac{dy}{du} \).

Not convinced? Try another example.
EX 2 A spider moves along the line \( y = (1/2)x - 3 \) in such a way that its \( x \)-coordinate at time \( t \) (seconds) is given by \( x = 4t \). Find the rate at which the \( y \)-coordinate is changing with respect to time (i.e. \( \frac{dy}{dt} \)).

Complete the following table showing the relationship between \( t, x \) and \( y \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>-3</td>
<td>-1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As \( t \) changes by 1 second, \( x \) changes by 4 and \( y \) changes by ________.

\[
\frac{dy}{dx} = \frac{1}{2} \quad \frac{dx}{dt} = \quad \frac{dy}{dt} =
\]

Check the relationship found in EX 1: \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \) (True / False)

**The Chain Rule:** If \( y \) is a differentiable function of \( u \) and \( u \) is a differentiable function of \( x \), then:

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We can alternately write the Chain Rule in another form: If \( y = f(u) \) and \( u = g(x) \), then:

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Or, in words, the derivative of a composition of functions \( f \circ g \) is equal to the derivative of the "outside" function \( f \) evaluated at the "inside" function, times the derivative of the "inside" function.
EX 3  Pick out the "inside" and the "outside" functions in the compositions below and then compute their derivatives.

b) \( y = \sin(2x^6 - 1992) \)
   Outside function: \( f(x) = \sin(x) \)
   Inside function: \( g(x) = 2x^6 - 1992 \)
   \( y' = \text{Use second form of chain rule.} \)

c) \( y = \tan^4 x \)
   Outside function: \( f(x) = x^4 \)
   Inside function: \( g(x) = \tan x \)
   \( y' = \)

EX 4  Practice a few more derivatives using the Chain Rule.

a) \( \frac{d}{dx}[(11x^{22} - 1963x)^{-14}] \) \( \text{outside function, } f(x) = x^{-14} \)
   (Which date do you recognize in the problem ?)

b) \( (\sin \sin \sin x)' \) \( \text{need chain rule twice, outside function, } f(x) = \sin x \)

c) \( [\cos (x^{1/2})]' \) \( \text{outside function, } f(x) = \cos x \)

d) \( \frac{d}{dx}(u^n) \) \( \text{outside function, } f(x) = x^n \)

where \( n \) is an integer and \( u \) is a function of \( x \). This is called the General Power Rule.
3.5 The Chain Rule

SOLUTION-SHEET

How do you compute the derivatives of the following functions?

a) \( y = (7x - 4)^2 \)

b) \( y = \sin(2x^6 - 1992) \)

c) \( y = \tan^4 x \)

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They are compositions of two functions.

Q2 Write the functions in a), b) and c) as compositions of two functions.

a) \( y = (7x - 4)^2 \)  
   \( y = u^2 \)  
   \( u = 7x - 4 \)

b) \( y = \sin(2x^6 - 1992) \)  
   \( y = \sin(u) \)  
   \( u = 2x^6 - 1992 \)

c) \( y = \tan^4 x \)  
   \( y = u^4 \)  
   \( u = \tan(x) \)

Q3 How do we compute derivatives of such functions? Let's consider example a).

EX 1 Compute the derivative of \( y = (7x - 4)^2 \).

Old way: \( y = (7x - 4)(7x - 4) = 49x^2 - 56x + 16 \), so \( y' = 98x - 56 = \frac{dy}{dx} \).

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In terms of \( x \), \( \frac{dy}{du} = 2u = 2(7x - 4) = 14x - 8 \).

Do you see a relationship between \( \frac{dy}{dx} \frac{du}{dx} \) and \( \frac{dy}{du} \)?

The relationship is: \( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \).

Not convinced? Try another example.
EX 2 A spider moves along the line \( y = \frac{1}{2}x - 3 \) in such a way that its \( x \)-coordinate at time \( t \) (seconds) is given by \( x = 4t \). Find the rate at which the \( y \)-coordinate is changing with respect to time (i.e. \( \frac{dy}{dt} \)).

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<table>
<thead>
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<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>( y )</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

As \( t \) changes by 1 second, \( x \) changes by ____ and \( y \) changes by ____.

\[
\frac{dy}{dx} = \frac{1}{2} \quad \frac{dx}{dt} = 4 \quad \frac{dy}{dt} = 2
\]

Check the relationship found in EX 1: \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \) (True / False)

The Chain Rule: If \( y \) is a differentiable function of \( u \) and \( u \) is a differentiable function of \( x \), then:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.
\]

We can alternately write the Chain Rule in another form: If \( y = f(u) \) and \( u = g(x) \), then:

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Or, in words, the derivative of a composition of functions \( f \circ g \) is equal to the derivative of the "outside" function \( f \) evaluated at the "inside" function, times the derivative of the "inside" function.
EX 3 Pick out the “inside” and the “outside” functions in the compositions below and then compute their derivatives.

b) \[ y = \sin(2x^6 - 1992) \]
   Outside function: \( f(x) = \sin(x) \)
   Inside function: \( g(x) = 2x^6 - 1992 \)
   \[ y' = \cos(2x^6 - 1992) \cdot 12x^5 = 12x^5 \cos(2x^6 - 1992) \]

c) \[ y = \tan^4 x \]
   Outside function: \( f(x) = x^4 \)
   Inside function: \( g(x) = \tan(x) \)
   \[ y' = 4 \tan^3 x \sec^2 x \]

EX 4 Practice a few more derivatives using the Chain Rule.

a) \( (dl/dx)((11x^{22} - 1963x)^{-14}) = -14 (11x^{22} - 1963x)^{-15} (22x^{21} - 1963) \)
   (Which date do you recognize in the problem?)

b) \( (\sin \sin \sin x)' = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x) \)

c) \( [\cos(x^{1/2})]' = -\sin(x^{1/2}) \cdot \left( \frac{1}{2}x^{-1/2} \right) = -\frac{1}{2}x^{-1/2} \sin(x^{1/2}) \)

d) \( \frac{d}{dx}(u^n) = n u^{n-1} \frac{d}{dx} u \)

where \( n \) is an integer and \( u \) is a function of \( x \). This is called the General Power Rule.
EX 1 Determine the "slope formula" for the following curve: \( x^2 + y^2 = 4 \).

(a) Sketch the graph of \( x^2 + y^2 = 4 \).

(b) Label a point on the graph where the slope is:
   (i) Large in absolute value and negative. Label the point A.
   (ii) Large in absolute value and positive. Label it B.
   (iii) Small in absolute value and negative. Label it C.
   (iv) Small in absolute value and positive. Label it D.

(c) What is the slope of the tangent line at the following points?
   (approximate the last one)

<table>
<thead>
<tr>
<th>point</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(0, 2)</th>
<th>(0, -2)</th>
<th>((\sqrt{3}, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(d) What is the equation of the tangent line to $x^2 + y^2 = 4$ at the following points? (Estimate the last one.)

<table>
<thead>
<tr>
<th>point</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(0, 2)</th>
<th>(0, -2)</th>
<th>$(\sqrt{3}, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>equation of tangent</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>line</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(e) To determine the slope formula, refer to Fig.1.

The point $(x, y)$ is on the circle. $R$ is the line containing the point $(x, y)$ and a radius of the circle.

What is the slope of the line $R$?
What is the relationship between $R$ and $T$? ________________
What is the slope of the line $T$?
So the slope formula for $x^2 + y^2 = 4$ is $\frac{x}{y}$. We usually call this $\frac{dy}{dx}$. 
What is the geometric meaning of \( \frac{dy}{dx} \)?

(f) Refine your answers from (c) and (d). Find the equation of the tangent line of \( x^2 + y^2 = 4 \) at the point \((\sqrt{3}, 1)\).

(g) Could you have computed the slope of \( x^2 + y^2 = 4 \) at any point using a technique we have already discussed? (Hint: Solve \( x^2 + y^2 = 4 \) for \( y \), then differentiate.) Verify the slopes that you computed in (c) using this "old" method.
Now we will discuss a technique that will allow us to compute \( dy/dx \) for any graph. This new technique is called \textbf{Implicit Differentiation}.

The graph \( G \) in Fig.2 is not a function. Why?

However \( G \) can be divided into 3 pieces, all of which are functions. Describe the pieces.

(1) 

(2) 

(3) 

What is the slope of \( G \) at \( x_0 \)? The curve is smooth at all points, so we can compute the slope of \( G \) at all points. However, for graphs like this one the slope also depends on the _____-coordinate.
The technique of implicit differentiation. An example:

**EX 2** Find \( \frac{dy}{dx} \) if \( x^2 + y^2 = 4 \).

**Step 1:** Differentiate both sides of the equation with respect to \( x \):

\[
\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = \frac{d}{dx} 4
\]

\[2x + 2y \frac{dy}{dx} = 0\]

**Step 2:** Solve algebraically for \( \frac{dy}{dx} \).

\[2y \frac{dy}{dx} = -2x\]

\[\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}\]

**EX 3** Find \( \frac{dy}{dx} \) if \( x + \sin y = xy \).

Can you solve this equation for \( y \)?

Like the graph \( G \) in Fig.2, \( x + \sin y = xy \) defines \( y \) implicitly as one or more differentiable function(s).

Find the tangent line of \( x + \sin y = xy \) at the point \( (0,0) \).

**Step 1:**

**Step 2:**
EX 1 Determine the "slope formula" for the following curve: \( x^2 + y^2 = 4 \).

(a) Sketch the graph of \( x^2 + y^2 = 4 \).

(b) Label a point on the graph where the slope is:
   (i) Large in absolute value and negative. Label the point A.
   (ii) Large in absolute value and positive. Label it B.
   (iii) Small in absolute value and negative. Label it C.
   (iv) Small in absolute value and positive. Label it D.

(c) What is the slope of the tangent line at the following points?
   (approximate the last one)

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<th>(0, 2)</th>
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<tbody>
<tr>
<td>slope</td>
<td>( \frac{dy}{dx} )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

slope = rise / run
(d) What is the equation of the tangent line to \( x^2 + y^2 = 4 \) at the following points? (Estimate the last one.)

<table>
<thead>
<tr>
<th>point</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(0, 2)</th>
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<td>( y = )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
<tr>
<td>line</td>
<td>( x = 2 )</td>
<td>( y = )</td>
<td>( )</td>
<td>( )</td>
<td>( )</td>
</tr>
</tbody>
</table>

(e) To determine the slope formula, refer to Fig. 1.

The point \((x, y)\) is on the circle. \( R \) is the line containing the point \((x, y)\) and a radius of the circle.

What is the slope of the line \( R \)? \( \text{slope} = \frac{\text{rise}}{\text{run}} \)

What is the relationship between \( R \) and \( T \)? 

What is the angle between them?

What is the slope of the line \( T \)? \( \text{For perpendicular lines } m_2 = -\frac{1}{m_1} \)

So the slope formula for \( x^2 + y^2 = 4 \) is \( \frac{x}{y} \). We usually call this \( \frac{dy}{dx} \).
What is the geometric meaning of \( \frac{dy}{dx} \)?

\( \frac{dy}{dx} \) **evaluated at a certain \((x,y)\) point gives the slope of the** _______.

(f) Refine your answers from (c) and (d). Find the equation of the tangent line of \( x^2 + y^2 = 4 \) at the point \((\sqrt{3}, 1)\). **The slope of the tangent at** \((\sqrt{3}, 1)\) is _______. thus the equation of the tangent at \((\sqrt{3}, 1)\) is _______________.

(g) Could you have computed the slope of \( x^2 + y^2 = 4 \) at any point using a technique we have already discussed? (Hint: Solve \( x^2 + y^2 = 4 \) for \( y \), then differentiate.) Verify the slopes that you computed in (c) using this "old" method.

Yes, for positive \( y \)'s we have:

\[ y = \sqrt{4-x^2} \quad \text{which gives } y' = \]

For negative \( y \)'s

\[ y = -\sqrt{4-x^2} \quad \text{which gives } y' = \]
Now we will discuss a technique that will allow us to compute \( \frac{dy}{dx} \) for any graph. This new technique is called **Implicit Differentiation**.

The graph \( G \) in Fig. 2 is not a function. Why?

*How many function values are there for \( x \)?*

However \( G \) can be divided into 3 pieces, all of which are functions. Describe the pieces.

1. **The piece between the points** \( a \) and \( b \)
2. ______________________________________________
3. ___________________________________

What is the slope of \( G \) at \( x_0 \)? The curve is smooth at all points, so we can compute the slope of \( G \) at all points. However, for graphs like this one the slope also depends on the __-coordinate.
The technique of implicit differentiation. An example:

EX 2 Find \( \frac{dy}{dx} \) if \( x^2 + y^2 = 4 \).

**Step 1:** Differentiate both sides of the equation with respect to \( x \):

\[
\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = \frac{d}{dx} 4
\]

\( 2x + 2y \frac{dy}{dx} = 0 \)

**Step 2:** Solve algebraically for \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}
\]

EX 3 Find \( \frac{dy}{dx} \) if \( x + \sin y = xy \).

Can you solve this equation for \( y \)?

Like the graph \( G \) in Fig.2, \( x + \sin y = xy \) defines \( y \) implicitly as one or more differentiable function(s).

Find the tangent line of \( x + \sin y = xy \) at the point \((0, 0)\).

**Step 1:**

\[
\frac{d}{dx} x + \frac{d}{dx} \sin y = \frac{d}{dx} (xy)
\]

**Step 2:**

\[
(\cos y - x) \frac{dy}{dx} = y - 1
\]
EX 1 Determine the "slope formula" for the following curve: $x^2 + y^2 = 4$.

(a) Sketch the graph of $x^2 + y^2 = 4$.

![Graph of $x^2 + y^2 = 4$]

This is the graph of the circle of radius 2, with center at the origin.

(b) Label a point on the graph where the slope is:

(i) Large in absolute value and negative. Label the point A.
(ii) Large in absolute value and positive. Label it B.
(iii) Small in absolute value and negative. Label it C.
(iv) Small in absolute value and positive. Label it D.

(c) What is the slope of the tangent line at the following points?
   (approximate the last one)

<table>
<thead>
<tr>
<th>point</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(0, 2)</th>
<th>(0, -2)</th>
<th>($\sqrt{3}$, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope</td>
<td>DNE</td>
<td>DNE</td>
<td>0</td>
<td>0</td>
<td>-1.7</td>
</tr>
</tbody>
</table>
(d) What is the equation of the tangent line to $x^2 + y^2 = 4$ at the following points? (Estimate the last one.)

<table>
<thead>
<tr>
<th>point</th>
<th>(2, 0)</th>
<th>(-2, 0)</th>
<th>(0, 2)</th>
<th>(0, -2)</th>
<th>(3, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>equation of tangent</td>
<td>$y = 2$</td>
<td>$y = -2$</td>
<td>$y = 2$</td>
<td>$y = -2$</td>
<td>$y = -1.7x + 4$</td>
</tr>
<tr>
<td>line</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(e) To determine the slope formula, refer to Fig. 1.

The point $(x, y)$ is on the circle. $R$ is the line containing the point $(x, y)$ and a radius of the circle.

What is the slope of the line $R$? \( \frac{y}{x} \)

What is the relationship between $R$ and $T$? They are perpendicular.

What is the slope of the line $T$? \( \frac{x}{y} \)

So the slope formula for $x^2 + y^2 = 4$ is $\frac{x}{y}$. We usually call this $\frac{dy}{dx}$. 
What is the geometric meaning of \( \frac{dy}{dx} \)?

\( \frac{dy}{dx} \) evaluated at a certain \((x,y)\) point gives the slope of the tangent to the curve at the \((x,y)\) point.

(f) Refine your answers from (c) and (d). Find the equation of the tangent line of \( x^2 + y^2 = 4 \) at the point \((\sqrt{3}, 1)\). The slope of the tangent at \((\sqrt{3}, 1)\) is \( -\sqrt{3} \), thus the equation of the tangent at \((\sqrt{3}, 1)\) is \( y = -\sqrt{3}x + 4 \).

(g) Could you have computed the slope of \( x^2 + y^2 = 4 \) at any point using a technique we have already discussed?

(Hint: Solve \( x^2 + y^2 = 4 \) for \( y \), then differentiate.) Verify the slopes that you computed in (c) using this "old" method.

Yes, for positive \( y \)'s we have:

\[
y = \sqrt{4-x^2} \quad \text{which gives} \quad y' = \frac{x}{\sqrt{4-x^2}} = \frac{x}{\sqrt{4x}} = \frac{x}{\sqrt{4x}},
\]

For negative \( y \)'s

\[
y = -\sqrt{4-x^2} \quad \text{which gives} \quad y' = -\frac{x}{\sqrt{4-x^2}} = -\frac{x}{\sqrt{4x}} = -\frac{x}{\sqrt{4x}}, \text{as well.}
\]

Thus we have \( y' = -\frac{x}{y} \), when \( y \) is not 0. When \( y = 0 \), \( x = \pm 2 \) or \(-2\), and the tangents at those points are vertical. Since \( y' \) is the same as before, we get the same slopes.
Now we will discuss a technique that will allow us to compute \( \frac{dy}{dx} \) for any graph. This new technique is called Implicit Differentiation.

The graph \( G \) in Fig. 2 is not a function. Why?

*Because there are 3 points on \( G(x,y) \) corresponding to \( x_0 \), i.e. \( G \) assigns more than 1 value to \( x_0 \).*

However \( G \) can be divided into 3 pieces, all of which are functions. Describe the pieces.

1. The piece between the points \( a, b \)
2. The piece between the points \( b, c \)
3. The piece between the points \( c, d \)

What is the slope of \( G \) at \( x_0 \)? The curve is smooth at all points, so we can compute the slope of \( G \) at all points. However, for graphs like this one the slope also depends on the \( y \) coordinate.
The technique of implicit differentiation. An example:

EX 2 Find $\frac{dy}{dx}$ if $x^2 + y^2 = 4$.

**Step 1:** Differentiate both sides of the equation with respect to $x$:
\[
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}4
\]
\[
2x + 2y \frac{dy}{dx} = 0
\]

**Step 2:** Solve algebraically for $\frac{dy}{dx}$.
\[
2y \frac{dy}{dx} = -2x
\]
\[
\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}
\]

EX 3 Find $\frac{dy}{dx}$ if $x + \sin y = xy$.

Can you solve this equation for $y$? No.

Like the graph $G$ in Fig.2, $x + \sin y = xy$ defines $y$ implicitly as one or more differentiable function(s).

Find the tangent line of $x + \sin y = xy$ at the point $(0,0)$.

1. **Step 1:**
\[
\frac{d}{d\zeta} x + \frac{d}{d\zeta} \sin y = \frac{d}{d\zeta} (xy)
\]
\[
1 + \cos y \frac{dy}{d\zeta} = y + x \frac{dy}{d\zeta}
\]

2. **Step 2:**
\[
(cos y \cdot x) \frac{dy}{d\zeta} = y \cdot 1
\]
\[
\frac{dy}{d\zeta} = \frac{y \cdot 1}{\cos y \cdot x}
\]
3.8 Newton's Method for Approximating Solutions of Equations

WORKSHEET

1. Find the $x$-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

   Solving this problem is equivalent to solving the equation _________________
   or _________________ = 0.

   The number of solutions and a rough estimate can be given by graphing calculator. In fact, a solution of any desired degree of accuracy (up to machine-precision, 8 - 10 significant digits for most) can be given by graphing calculator (zoom - zoom), but the graphing calculator alone is not the fastest way when high precision is required.

   This equation has ___ solution(s). Reasons (give a rough sketch + a few sentences):
Did you include your argument why you can be sure there are no additional solutions outside of your viewing rectangle (screen)? Make sure to check the cheat-sheet to see if your argument is complete.

Finding the solution of our equation is equivalent to finding where the graph of \( f(x) = \) \____________________. The graphing calculator shows that the solution is close to the integer \( x_0 = \) \_, this will be our first (or \( 0^{th} \)) approximation, and we will improve our approximation step by step.

Sketch the graph of our function \( f(x) = \) \______________ around its \( x \) intercept, on the interval \([1,2]\), and add everything we are talking about as we go along.
1.1 First, we can approximate \( f(x) \) around \( x_0 = 1 \) by _______________________.

Now, if our function can be approximated by this line, then its intercept (the solution of our equation) can be approximated by _______________________. The slope of this line is \( m = \) _____ = _____. Therefore, an equation of this line is ___________. The \( x \) intercept of this line is \( x_1 = _____. \) This will be our next (improved) approximation.

1.2 To improve our approximation, \( x_1 \), we can approximate our function around \( x_1 \) by ______________________, and proceed exactly the same way as in 1.1.

Use as many decimals as your calculator can give you.

Slope of tangent line, \( m = \) ____________

Equation of tangent line:

\( x \) intercept of tangent line: \( x = \) ____________

The improved approximation is \( x_2 = \) ____________.

1.3 You can keep going if you wish (but don't spend too much time on it) or just view the next four approximations. Be prepared to be impressed!
The results of applying Newton's method to \( x^3 - x - 1 = 0 \):

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1.5 \\
x_2 &= 1.347826087 \\
x_3 &= 1.325200399 \\
x_4 &= 1.324718174 \\
x_5 &= 1.324717957 \\
x_6 &= 1.324717957
\end{align*}
\]

How many correct decimals and how many significant digits do you think our last estimate, \( x_6 \), has?

Number of correct decimals: ______.

Number of significant digits: ______.

2. The Formula for Newton's Method

This method works nicely, but it would be too long to go through this process each and every time you need to solve an equation. Wouldn't it be nicer to derive a formula into which we can just plug in numbers to get the next approximation? (Then we can write a short, one line program that gives the next approximation using the formula. Or, better yet, write a short, few liner to give the sequence of approximations. A possible solution - in BASIC - is in your book, see pp 206 - 208, Ex 3, 4. The program can be much shorter if you want less features.)

So, now let's derive this formula. It will be easier than you think. We'll do exactly the same as in 1., but with parameters (letters), rather than numbers.
Let our equation be \( f(x) = 0 \), the starting estimate \( x_0 \). Compute \( x_1 \). Turn back to the previous page if needed.

\[
y - f(x_0) = f'(x_0) (x - x_0)
\]

\[x_1 = \text{________________________}.
\]

Guess what's next? Go for \( x_2 \). You might want to think a little if you can get this formula without any slave work (computation), just by taking a look at the previous formula \((x_1 = \ldots)\).

\[x_2 = \text{________________________}.
\]

In general, \( x_{n+1} = \text{________________________} \).

3. To see how this formula works, do our first problem with this formula and compare each step with our previous results (1.).
4. Can you summarize how you solve an equation of the form \( f(x) = 0 \) using Newton's method?

1.

2.

3.

4.
3.8 Newton's Method for Approximating Solutions of Equations

HINT-SHEET

1. Find the x-coordinate of the point where the curve \( y = x^3 - x \) crosses the horizontal line \( y = 1 \).

Solving this problem is equivalent to solving the equation \( x^3 - x = \) \underline{______} \text{ for the intersection point } \text{ or } \underline{____________} = 0.

The number of solutions and a rough estimate can be given by graphing calculator.
In fact, a solution of any desired degree of accuracy (up to machine-precision, 8-10 significant digits for most) can be given by graphing calculator (zoom - zoom), but the graphing calculator alone is not the fastest way when high precision is required.

This equation has \underline{______} solution(s). \textbf{Reasons:} (give a rough sketch + a few sentences):

\[ y = x^3 - x - 1. \]

\[ \text{If } x \text{ is a solution, then } x^3 - x - 1 = \underline{______} \text{ for that } x \text{ value.} \]

\[ \text{So that point is on the } \underline{__} \text{ axis.} \]
Did you include your argument why you can be sure there are no additional solutions outside of your viewing rectangle (screen)? Make sure to check the cheat-sheet to see if your argument is complete.

Finding the solution of our equation is equivalent to finding where the graph of \( f(x) = x^3 - x - 1 \) crosses the __________. The graphing calculator shows that the solution is close to the integer \( x_0 = 1 \), this will be our first (or 0th) approximation, and we will improve our approximation step by step.

Sketch the graph of our function \( f(x) = x^3 - x - 1 \) around its \( x \) intercept, on the interval \([1,2]\), and add everything we are talking about as we go along.
1.1 First, we can approximate \( f(x) \) around \( x_0 = 1 \) by \( (\text{See 3.7}) \). Now, if our function can be approximated by this line, then its intercept (the solution of our equation) can be approximated by \( \text{the x intercept of the tangent line} \). The slope of this line is \( m = f'(1) = \_ \). Therefore, an equation of this line is \( \_ \). The x intercept of this line is \( x_1 = \_ \) \( (\text{Use point-slope formula}) \). This will be our next (improved) approximation.

1.2 To improve our approximation, \( x_1 \), we can approximate our function around \( x_1 \) by its tangent line at \( x = \_ \), and proceed exactly the same way as in 1.1.

Use as many decimals as your calculator can give you.

\( \text{Slope of tangent line, } m = \_ \)

\( \text{Equation of tangent line:} \)

\( \text{x intercept of tangent line:} \)

The improved approximation is \( x_2 = \_ \).

1.3 You can keep going if you wish (but don't spend too much time on it) or just view the next four approximations. Be prepared to be impressed!
The results of applying Newton's method to \( x^3 - x - 1 = 0 \):

- \( x_0 = 1 \)
- \( x_1 = 1.5 \)
- \( x_2 = 1.347826087 \)
- \( x_3 = 1.325200399 \)
- \( x_4 = 1.324718174 \)
- \( x_5 = 1.324717957 \)
- \( x_6 = 1.324717957 \)

How many correct decimals and how many significant digits do you think our last estimate, \( x_6 \), has?

Number of correct decimals: _______. (i.e. how many digits after the decimal point are correct)

Number of significant digits: _______. (i.e. how many digits altogether are correct)

2. The Formula for Newton's Method

This method works nicely, but it would be too long to go through this process each and every time you need to solve an equation. Wouldn't it be nicer to derive a formula into which we can just plug in numbers to get the next approximation? (Then we can write a short, one line program that gives the next approximation using the formula. Or, better yet, write a short, few liner to give the sequence of approximations. A possible solution - in BASIC - is in your book, see pp 206 - 208, Ex 3, 4. The program can be much shorter if you want less features.)

So, now let's derive this formula. It will be easier than you think. We'll do exactly the same as in 1., but with parameters (letters), rather than numbers.
Let our equation be \( f(x) = 0 \), the starting estimate \( x_0 \). Compute \( x_1 \). Turn back to the previous page if needed.

\[ y - f(x_0) = f'(x_0)(x - x_0) \]

\( x_1 = \) ________________.

Guess what's next? Go for \( x_2 \). You might want to think a little if you can get this formula without any slave work (computation), just by taking a look at the previous formula \((x_1 = \ldots)\). \( x_2 \) is obtained (computed) from \( x_1 \), the same way as \( x_1 \) is computed from \( x_0 \).

This should be reflected in the formula.

\( x_2 = \) ________________.

In general, \( x_{n+1} = \) ________________.

3. To see how this formula works, do our first problem with this formula and compare each step with our previous results (1.).

\( x_0 = 1 \) \hspace{1cm} \( f(x) = x^3 \cdot x \cdot 1 \) \hspace{1cm} \( f'(x) = \) __________

\( x_1 = x_0 \cdot \frac{f(x_0)}{f'(x_0)} = \) __________

\( x_2 = x_1 \cdot \frac{f(x_1)}{f'(x_1)} = \) __________

\( x_3 = \) ________________
4. Can you summarize how you solve an equation of the form \( f(x) = 0 \) using Newton's method?

1. Find initial approximation, \( x_0 \).

2. Use \( x_0 \) to get a 2\(^{nd} \) (better) approximation by the formula \( \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \).

3.

4.
3.8 Newton's Method for Approximating Solutions of Equations

SOLUTION-SHEET

1. Find the $x$-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solving this problem is equivalent to solving the equation $x^3 - x = 1$ for the intersection point or $x^3 - x - 1 = 0$.

The number of solutions and a rough estimate can be given by graphing calculator. In fact, a solution of any desired degree of accuracy (up to machine-precision, 8 - 10 significant digits for most) can be given by graphing calculator (zoom - zoom), but the graphing calculator alone is not the fastest way when high precision is required.

This equation has $1$ solution(s). Reasons (give a rough sketch + a few sentences):

Graph $y = x^3 - x - 1$.

If $x$ is a solution, then $x^3 - x - 1 = 0$ for the $x$ value.

So that point is on the $x$ axis. The solutions are the $x$ intercepts.
There is only one x intercept since the complete graph of a cubic is always one of four possibilities:

Therefore, the graph cannot 'turn back' and cross the x axis again.

Did you include your argument why you can be sure there are no additional solutions outside of your viewing rectangle (screen)? Make sure to check the cheat-sheet to see if your argument is complete.

Finding the solution of our equation is equivalent to finding where the graph of 
\[ f(x) = x^3 - x - 1 \] crosses the x axis. The graphing calculator shows that the solution is close to the integer \( x_0 = 1 \), this will be our first (or 0th) approximation, and we will improve our approximation step by step.

Sketch the graph of our function \( f(x) = x^3 - x - 1 \) around its x intercept, on the interval \([1,2]\), and add everything we are talking about as we go along.
1.1 First, we can approximate $f(x)$ around $x_0 = 1$ by its tangent line at $x_0 = 1$ (See 3.7).

Now, if our function can be approximated by this line, then its intercept (the solution of our equation) can be approximated by ______ the x intercept of the tangent line _______.

The slope of this line is $m = f'(1) = 2$. Therefore, an equation of this line is $y + 1 = 2(x - 1)$. The x intercept of this line is $x_1 = 1.5$. This will be our next (improved) approximation.

1.2 To improve our approximation, $x_1$, we can approximate our function around $x_1$ by its tangent line at $x = 1.5$, and proceed exactly the same way as in 1.1.

Use as many decimals as your calculator can give you.

\[ \text{Slope of tangent line, } m = 5.75 \quad f'(x) = 3x^2 - 1 \]

\[ \text{Equation of tangent line: } y \cdot .875 = 5.75(x - 1.5) \]

\[ \text{x intercept of tangent line: } x = 1.347826087 \]

The improved approximation is $x_2 = 1.347826087$.

1.3 You can keep going if you wish (but don't spend too much time on it) or just view the next four approximations. Be prepared to be impressed!
The results of applying Newton's method to $x^3 - x - 1 = 0$:

$x_0 = 1$
$x_1 = 1.5$
$x_2 = 1.347826087$
$x_3 = 1.325200399$
$x_4 = 1.324718174$
$x_5 = 1.324717957$
$x_6 = 1.324717957$

How many correct decimals and how many significant digits do you think our last estimate, $x_6$, has?  Probably the following numbers, but we cannot be absolutely sure.

Number of correct decimals: 8 or 9. (i.e. how many digits after the decimal point are correct)
Number of significant digits: 9 or 10. (i.e. how many digits altogether are correct)

2. The Formula for Newton's Method

This method works nicely, but it would be too long to go through this process each and every time you need to solve an equation. Wouldn't it be nicer to derive a formula into which we can just plug in numbers to get the next approximation? (Then we can write a short, one line program that gives the next approximation using the formula. Or, better yet, write a short, few liner to give the sequence of approximations. A possible solution - in BASIC - is in your book, see pp 206 - 208, Ex 3, 4. The program can be much shorter if you want less features.)

So, now let's derive this formula. It will be easier than you think. We'll do exactly the same as in 1., but with parameters (letters), rather than numbers.
Let our equation be \( f(x) = 0 \), the starting estimate \( x_0 \). Compute \( x_1 \). Turn back to the previous page if needed.

\[
y - f(x_0) = f'(x_0)(x - x_0) \quad \rightarrow \quad y = f(x_0) = x_0f'(x_0) - x_0f'(x_0) \quad \rightarrow \\
y = f(x_0) + x_0f'(x_0) - x_0f'(x_0) = 0 \quad \rightarrow \\
x_0f'(x_0) = x_0f'(x_0) \cdot f(x_0) \quad \rightarrow \quad x = \frac{x_0f'(x_0) \cdot f(x_0)}{f'(x_0)} = x_0 \cdot \frac{f(x_0)}{f'(x_0)}
\]

\[x_1 = -x_0 \cdot \frac{f(x_0)}{f'(x_0)}.
\]

Guess what's next? Go for \( x_2 \). You might want to think a little if you can get this formula without any slave work (computation), just by taking a look at the previous formula

\( x_1 = \ldots \). \( x_2 \) is obtained (computed) from \( x_1 \), the same way as \( x_1 \) is computed from \( x_0 \).

This should be reflected in the formula.

\[x_2 = -x_1 \cdot \frac{f(x_1)}{f'(x_1)}.
\]

In general, \( x_{n+1} = x_n \cdot \frac{f(x_n)}{f'(x_n)} \).

3. To see how this formula works, do our first problem with this formula and compare each step with our previous results (1.).

\[
x_0 = 1 \quad \quad f(x) = x^3 \cdot x \cdot 1 \quad \quad f'(x) = 3x^2 \cdot 1
\]

\[
x_1 = x_0 \cdot \frac{f(x_0)}{f'(x_0)} = 1 + .5 = 1.5
\]

\[
x_2 = x_1 \cdot \frac{f(x_1)}{f'(x_1)} = 1.5 \cdot \frac{.875}{5.75} = 1.347826087
\]

\[
x_3 = 1.325200399
\]
4. Can you summarize how you solve an equation of the form $f(x) = 0$ using Newton's method?

1. Find initial approximation, $x_0$. (e.g. by graphing calculator)

2. Use $x_0$ to get a 2nd (better) approximation by the formula $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

3. Use $x_1$ to get a 3rd (better) approximation by the formula $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

4. Continue until desired precision reached.
4.2 Maxima, Minima, and Mean Value Theorem

**WORKSHEET**

**Def. of local/absolute max/min:** textbook p. 226 highlighted. **Extreme value** = max. or min.

**EX 1**

The function below has local min(s) at $x = \underline{\phantom{0}}$.
The absolute min. is \underline{\phantom{0}} at $x = \underline{\phantom{0}}$.
The function below has local max(s) at $x = \underline{\phantom{0}}$.
The absolute max. is \underline{\phantom{0}} at $x = \underline{\phantom{0}}$.

![Graph of a function](image)

What can you say about $f'(x)$ at each of the local min/max. values?

**The First Derivative Theorem for Local Extreme Values:**

If $f$ has a local min. or max. at an interior point $c$ of an interval where it is defined, then $f'(x)$ is either \underline{\phantom{0}} or \underline{\phantom{0}}.

**EX 2**

Draw a few graphs that satisfy ALL of the following:

* $f(x)$ is continuous on the closed interval $[a, b]$
* $f(x)$ is differentiable on the open interval $(a, b)$
* $f(a) = f(b) = 0$.
These graphs all have a tangent line of what direction? ________________
What is the value of the derivative at that tangent line? ______
If this tangent line is at \( x = c \), then this observation can be put in the following form:

**Rolle's Theorem**

Suppose that

* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\)
* \( f(a) = f(b) = 0 \).

Then there is at least one number \( c \) between \( a \) and \( b \) at which \( f'(c) = 0 \).

The geometric meaning of Rolle's Thm. is that if a function \( f(x) \) satisfies the above three conditions, then its graph has at least one ___________ tangent line over the interval \((a, b)\).

**EX 3** Draw a few graphs that satisfy ALL of the following:

* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\)
* \( f(a) \) and \( f(b) \) have opposite signs
* \( f'(x) \neq 0 \) on \((a, b)\).

What can you say about the number of solutions of \( f(x) = 0 \) between \( a \) and \( b \)?

___________

Can you argue that this is always true?
**EX 4** Show that the equation $2x = \cos x$ has exactly one solution in $[-\pi, \pi]$.

**EX 5** Draw a few graphs that satisfy ALL of the following:

- $f(x)$ is continuous on the closed interval $[a, b]$
- $f(x)$ is differentiable on the open interval $(a, b)$

These graphs all have a tangent line of what direction? __________________

What is the value of the derivative at that tangent line? __________________.

If this tangent line is at $x = c$, then this observation can be put in the following form:

**Mean Value Theorem**

Suppose that

- $f(x)$ is continuous on the closed interval $[a, b]$
- $f(x)$ is differentiable on the open interval $(a, b)$

Then there is at least one number $c$ between $a$ and $b$ at which $f'(c) =$ __________________

The geometric meaning of the MVT is that if a function $f(x)$ satisfies the above two conditions, then its graph has at least one tangent line parallel to the secant line over $[a, b]$. 
Def. of increasing/decreasing: p. 232 highlighted

EX 6 Draw a graph that has both increasing and decreasing intervals.

Do you see some relationship between the increasing/decreasing nature of the function and the sign of the derivative?

The First Derivative Test for Increasing/Decreasing
If \( f'(x) > 0 \) then \( f(x) \) is ______________ .
If \( f'(x) < 0 \) then \( f(x) \) is ______________ .

More precisely:
Suppose that * \( f(x) \) is continuous on the closed interval \([a, b]\) * \( f(x) \) is differentiable on the open interval \((a, b)\).
Then - if \( f'(x) > 0 \) at each point of \((a,b)\), then \( f \) ______________
thruout \([a, b]\).
- if \( f'(x) < 0 \) at each point of \((a,b)\), then \( f \) ______________
thruout \([a, b]\).
In either case, \( f \) is one-to-one.
4.2 Maxima, Minima, and Mean Value Theorem

HINT-SHEET

Def. of local/absolute max/min: textbook p. 226 highlighted. Extreme value = max. or min.

EX 1 The function below has local min(s) at \( x = -2, 1, 5 \).
The absolute min. is \(-1\) at \( x = 1 \).
The function below has local max(s) at \( x = \) \______\.
The absolute max. is \______\ at \( x = \) \______\.

What can you say about \( f'(x) \) at each of the local min/max. values?

The First Derivative Theorem for Local Extreme Values:
If \( f \) has a local min. or max. at an interior point \( c \) of an interval where it is defined, then \( f'(x) \) is either \( 0 \) or \______\.

EX 2 Draw a few graphs that satisfy ALL of the following:
* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\)
* \( f(a) = f(b) = 0 \)
These graphs all have a tangent line of what direction? **horizontal**

What is the value of the derivative at that tangent line? ____

If this tangent line is at $x = c$, then this observation can be put in the following form:

**Rolle's Theorem**

Suppose that

* $f(x)$ is continuous on the closed interval $[a, b]$
* $f(x)$ is differentiable on the open interval $(a, b)$
* $f(a) = f(b) = 0$.

Then there is at least one number $c$ between $a$ and $b$ at which $f'(c) = 0$.

The geometric meaning of Rolle's Thm. is that if a function $f(x)$ satisfies the above three conditions, then its graph has at least one **horizontal** tangent line over the interval $(a, b)$.

**EX 3** Draw a few graphs that satisfy ALL of the following:

* $f(x)$ is continuous on the closed interval $[a, b]$
* $f(x)$ is differentiable on the open interval $(a, b)$
* $f(a)$ and $f(b)$ have opposite signs
* $f'(x) \neq 0$ on $(a, b)$.

What can you say about the number of solutions of $f(x) = 0$ between $a$ and $b$?

____

Can you argue that this is always true?
EX 4  Show that the equation \( 2x = \cos x \) has exactly one solution in \([-\pi, \pi]\).

Intermediate Value Theorem
Rolle's Theorem

EX 5  Draw a few graphs that satisfy ALL of the following:
* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\).

These graphs all have a tangent line of what direction? \( \text{parallel to the secant line over } [a, b]. \)
What is the value of the derivative at that tangent line? ______________
If this tangent line is at \( x = c \), then this observation can be put in the following form:

Mean Value Theorem
Suppose that  
* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\)
Then there is at least one number \( c \) between \( a \) and \( b \) at which \( f'(c) = \) ______________.

The geometric meaning of the MVT is that if a function \( f(x) \) satisfies the above two conditions, then its graph has at least one tangent line parallel to the secant line over \([a, b]\).
Def. of increasing/decreasing: p. 232 highlighted.

EX 6   Draw a graph that has both increasing and decreasing intervals.

Do you see some relationship between the increasing/decreasing nature of the function and the sign of the derivative?

The First Derivative Test for Increasing/Decreasing

If \( f'(x) > 0 \) then \( f(x) \) is increasing.
If \( f'(x) < 0 \) then \( f(x) \) is decreasing.

More precisely:
Suppose that *\( f(x) \) is continuous on the closed interval \([a, b]\)*
*\( f(x) \) is differentiable on the open interval \((a, b)\).*
Then  
- if \( f'(x) > 0 \) at each point of \((a,b)\), then \( f \) increases throughout \([a, b]\).
- if \( f'(x) < 0 \) at each point of \((a,b)\), then \( f \) decreases throughout \([a, b]\).

In either case, \( f \) is one-to-one.
4.2 Maxima, Minima, and Mean Value Theorem
SOLUTION-SHEET

Def. of local/absolute max/min.: textbook p. 226 highlighted.
**Extreme value** = max. or min.

**EX 1** The function below has local min(s) at \( x = -2, 1, 5 \).
The absolute min. is \(-1\) at \( x = 1 \).
The function below has local max(s) at \( x = -0.5, 2 \).
The absolute max. is \( 2 \) at \( x = -0.5 \).

What can you say about \( f'(x) \) at each of the local min/max. values?

**The First Derivative Theorem for Local Extreme Values:**
If \( f \) has a local min. or max. at an interior point \( c \) of an interval where it is defined, then \( f'(x) \) is either \( 0 \) or \( \text{DNE} \).

**EX 2** Draw a few graphs that satisfy ALL of the following:
* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\)
* \( f(a) = f(b) = 0 \)
These graphs all have a tangent line of what direction? **horizontal**

What is the value of the derivative at that tangent line? **0**

If this tangent line is at \( x = c \), then this observation can be put in the following form:

**Rolle's Theorem**

Suppose that
- \( f(x) \) is continuous on the closed interval \([a, b]\)
- \( f(x) \) is differentiable on the open interval \((a, b)\)
- \( f(a) = f(b) = 0 \).

Then there is at least one number \( c \) between \( a \) and \( b \) at which \( f'(c) = 0 \).

The geometric meaning of Rolle's Thm. is that if a function \( f(x) \) satisfies the above three conditions, then its graph has at least one **horizontal** tangent line over the interval \((a, b)\).

**EX 3** Draw a few graphs that satisfy ALL of the following:
- \( f(x) \) is continuous on the closed interval \([a, b]\)
- \( f(x) \) is differentiable on the open interval \((a, b)\)
- \( f(a) \) and \( f(b) \) have opposite signs
- \( f'(x) \neq 0 \) on \((a, b)\).

What can you say about the number of solutions of \( f(x) = 0 \) between \( a \) and \( b \)?

Can you argue that this is always true?
EX 4  
Show that the equation $2x = \cos x$ has exactly one solution in $[-\pi, \pi]$.

$f(x) = 2x \cdot \cos x = 0$ has at least one solution by the Intermediate Value Theorem.

Cannot have more than one, then $f'(c)$ would be 0 for some $c$ in $[\pi, \pi]$.

EX 5  
Draw a few graphs that satisfy ALL of the following:

* $f(x)$ is continuous on the closed interval $[a, b]$
* $f(x)$ is differentiable on the open interval $(a, b)$.

These graphs all have a tangent line of what direction? 
parallel to the secant line over $[a, b]$.

What is the value of the derivative at that tangent line? 
$f'(c) = \frac{f(b) - f(a)}{b - a}$

If this tangent line is at $x = c$, then this observation can be put in the following form:

**Mean Value Theorem**

Suppose that

* $f(x)$ is continuous on the closed interval $[a, b]$
* $f(x)$ is differentiable on the open interval $(a, b)$

Then there is at least one number $c$ between $a$ and $b$ at which $f'(c) = \frac{f(b) - f(a)}{b - a}$

The geometric meaning of the MVT is that if a function $f(x)$ satisfies the above two conditions, then its graph has at least one tangent line parallel to the secant line over $[a, b]$. 
Def. of increasing/decreasing: p. 232 highlighted.

EX 6  Draw a graph that has both increasing and decreasing intervals.

\[ \text{Do you see some relationship between the increasing/decreasing nature of the FUNCTION and the sign of the derivative?} \]

**The First Derivative Test for Increasing/Decreasing**

If \( f'(x) > 0 \) then \( f(x) \) is **increasing**.

If \( f'(x) < 0 \) then \( f(x) \) is **decreasing**.

More precisely:

Suppose that

* \( f(x) \) is continuous on the closed interval \([a, b]\)
* \( f(x) \) is differentiable on the open interval \((a, b)\).

Then
- if \( f'(x) > 0 \) at each point of \((a,b)\), then \( f \) **is increasing** throughout \([a, b]\).
- if \( f'(x) < 0 \) at each point of \((a,b)\), then \( f \) **is decreasing** throughout \([a, b]\).

In either case, \( f \) is one-to-one.
EX 1 Review: The First Derivative Test.

(a) \( f(x) \) is increasing on the following intervals: ____________________

(b) \( f(x) \) is decreasing on the following intervals: ____________________

(c) \( f'(b) = \) _____; \( f'(c) = \) ______

(d) If \( x \in (a, b) \), then \( f'(x) \) ___ 0.

(e) If \( x \in (b, c) \), then \( f'(x) \) ___ 0.

(Possible answers for (d) and (e) are <, >, =.)

Restate the First Derivative Test:
EX 2 Sketch a graph $f$ with the following property: $f'(c) = 0$ for some value $c$.

Now sketch another graph $g$ with $g'(c) = 0$ for some value $c$ and $g(c)$ is neither a max nor a min.

EX 3 If $f'(a)$ is not 0, can there exist a maximum or a minimum at $x = a$?
Concavity.

Concave up or concave down?

As you read this graph from left to right, the slopes of the tangent lines are ___________

As you read this graph from left to right, the slopes of the tangent are ___________

EX 4 Where is $f(x)$ concave up? ________________

Where is $f(x)$ concave down? ________________

Definition of concavity (refer to example 3, 4):

The graph of a differentiable function $y = f(x)$ is ________ on an interval where $y'$ is ________ and ________ on an interval where $y'$ is ________.
From the First Derivative Test we know that if \( y \) is increasing then \( y' > 0 \) and if \( y \) is decreasing then \( y' < 0 \). This is true for any function \( y = f(x) \). Consider the relationship between \( y' \) and \( y'' \). If \( y' \) is increasing then \( y'' ____ 0 \) and if \( y' \) is decreasing then \( y'' ____ 0 \). Now let's determine the relationship between \( y \) and \( y'' \).

1. If \( y'' > 0 \) then \( y' \) is ________________ Hence \( y \) is ________________
2. If \( y'' < 0 \) then \( y' \) is ________________ Hence \( y \) is ________________

Using this information state the Second Derivative Test for concavity:

Suppose \( y'' > 0 \) for \( a < x < b \). Then \( y = f(x) \) is concave ______ on \( (a,b) \).

If \( y'' < 0 \) for \( a < x < b \), then \( y = f(x) \) is concave ______ on \( (a,b) \).

**EX 5:** Determine where the following curves are concave up or concave down.

(a) \( f(x) = 2 + 27x + 1992x^2 \)

(b) \( f(x) = \cos x \) on \([0, 2\pi]\).
4.3 Curve Sketching with $y'$ and $y''$

HINT-SHEET

EX 1  Review: The First Derivative Test.

- (a) $f(x)$ is increasing on the following intervals: __________ [a, b] and __________.
- (b) $f(x)$ is decreasing on the following intervals: ________.
- (c) $f'(b) = _____$; $f'(c) = _____$.
- (d) If $x \in (a, b)$, then $f'(x) _____ 0$.
- (e) If $x \in (b, c)$, then $f'(x) _____ 0$.

(Possible answers for (d) and (e) are $<, >, =$.)

Restate the First Derivative Test:

\[ f \text{ increases when } f'(x) _____ 0 \text{ and decreases when } f'(x) _____ 0. \]

\[ \text{derivative = slope} \]
EX 2 Sketch a graph $f$ with the following property: $f'(c) = 0$ for some value $c$.

$derivative = slope$

Now sketch another graph $g$ with $g'(c) = 0$ for some value $c$ and $g(c)$ is neither a max nor a min.

EX 3 If $f'(a)$ is not 0, can there exist a maximum or a minimum at $x = a$?
Yes, if $f'$ is not defined at $x = a$, e.g. $f(x) = \ldots$ at $x = \ldots$
Concavity.

As you read this graph from left to right, the slopes of the tangent lines are **increasing or decreasing?**

EX 4 Where is $f(x)$ concave up? __________ $x < -2$ or __________

Where is $f(x)$ concave down? __________

Definition of concavity (refer to example 3, 4):

The graph of a differentiable function $y = f(x)$ is **concave up** on an interval where $y'$ is **increasing or decreasing?** and **concave down** on an interval where $y'$ is **_________.**
From the First Derivative Test we know that if \( y \) is increasing then \( y' > 0 \) and if \( y \) is decreasing then \( y' < 0 \). This is true for any function \( y = f(x) \). Consider the relationship between \( y' \) and \( y'' \). If \( y' \) is increasing then \( y'' > 0 \) and if \( y' \) is decreasing then \( y'' < 0 \). Now let's determine the relationship between \( y \) and \( y'' \).

1. If \( y'' > 0 \) then \( y' \) is \textit{increasing or decreasing?}. Hence \( y \) is \textit{(concave up or down?)}.
2. If \( y'' < 0 \) then \( y' \) is \underline{________} Hence \( y \) is \underline{________}.

Using this information state the Second Derivative Test for concavity:

Suppose \( y'' > 0 \) for \( a < x < b \). Then \( y = f(x) \) is concave \underline{_____} on \((a,b)\).

If \( y'' < 0 \) for \( a < x < b \), then \( y = f(x) \) is concave \underline{_____} on \((a,b)\).

\textbf{EX 5}: Determine where the following curves are concave up or concave down.

(a) \( f(x) = 2 + 27x + 1992x^2 \)

(b) \( f(x) = \cos x \) on \([0, 2\pi]\).

\[ f''(x) = ? \]
EX 1 Review: The First Derivative Test.

(a) \( f(x) \) is increasing on the following intervals: \([a, b]\) and \([c, d]\)

(b) \( f(x) \) is decreasing on the following intervals: \([b, c]\)

(c) \( f'(b) = 0 \); \( f'(c) = 0 \)

(d) If \( x \in (a, b) \), then \( f'(x) > 0 \).

(e) If \( x \in (b, c) \), then \( f'(x) < 0 \).

(Possible answers for (d) and (e) are \(<\), \(>\), \(=\).

Restate the First Derivative Test:

' \( f \) increases when \( f'(x) > 0 \) and decreases when \( f'(x) < 0 \).
**EX 2** Sketch a graph $f$ with the following property: $f'(c) = 0$ for some value $c$.

Now sketch another graph $g$ with $g'(c) = 0$ for some value $c$ and $g(c)$ is neither a max nor a min.

**EX 3** If $f'(a)$ is not 0, can there exist a maximum or a minimum at $x = a$?

Yes, if $f'$ is not defined at $x = a$. (e.g. $f(x) = x^{2/3}$ and $|x|$ have minimums at...
Concavity.

Concave up or concave down?

As you read this graph from left to right, the slopes of the tangent lines are increasing.

As you read this graph from left to right, the slopes of the tangent lines are decreasing.

**EX 4** Where is \( f(x) \) concave up? \(-2 < x < -1; \ -1 < x < 0\)

Where is \( f(x) \) concave down? \(-2 < x < -1; \ 0 < x < 2\)

Definition of **concavity** (refer to example 3, 4):

The graph of a differentiable function \( y = f(x) \) is **concave up** on an interval where \( y' \) is **increasing** and **concave down** on an interval where \( y' \) is **decreasing**.
From the First Derivative Test we know that if $y$ is increasing then $y' > 0$ and if $y$ is decreasing then $y' < 0$. This is true for any function $y = f(x)$. Consider the relationship between $y'$ and $y''$. If $y'$ is increasing then $y'' > 0$ and if $y'$ is decreasing then $y'' < 0$. Now let's determine the relationship between $y$ and $y''$.

1. If $y'' > 0$ then $y'$ is increasing. Hence $y$ is concave up.
2. If $y'' < 0$ then $y'$ is decreasing. Hence $y$ is concave down.

Using this information state the Second Derivative Test for concavity:

Suppose $y'' > 0$ for $a < x < b$. Then $y = f(x)$ is concave up on $(a,b)$.

If $y'' < 0$ for $a < x < b$, then $y = f(x)$ is concave down on $(a,b)$.

**EX 5:** Determine where the following curves are concave up or concave down.

(a) $f(x) = 2 + 27x + 1992x^2$

(b) $f(x) = \cos x$ on $[0, 2\pi]$.

a) $f'(x) = 27 + 3984x$

$f''(x) = 3984$, so $f''(x) > 0$ for every $x$, thus $f(x)$ is concave up on the whole real line.

b) $f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$, and $\cos(x) > 0$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$, hence $f(x)$ is concave up there.

$-\cos(x) < 0$ for $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, hence $f(x)$ is concave down on these intervals.