Rational period functions for the modular group and related discrete groups

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RATIONAL PERIOD FUNCTIONS FOR THE MODULAR GROUP AND RELATED DISCRETE GROUPS

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CHAPTER I

Introduction

1.1 Some Preliminaries

Felix Klein viewed geometry as the study of those properties of a space which remain invariant under the action of a group. Therefore, different geometries may be identified by their respective spaces, groups, and group actions [Le1, p. 11]. Perhaps the most familiar geometry is Euclidean geometry, where the space is the Euclidean plane, and the group is the group of isometries of the Euclidean plane. Abstractly this notion of geometry, or alternatively, of invariance, provides a means by which we may view a mathematical object (a space) from a variety of viewpoints. That is, we have a 'before picture' and an 'after picture' of a space, where the 'after picture' is arrived at by applying a group element to the 'before picture.' Though these pictures may look fundamentally different, they are, in reality, the same, and therein lies a basis for comparison.

The topic of this thesis is rational period functions for the modular group and related discrete groups. We will return often to Klein's simple yet powerful vision of geometry, in the context of vector spaces of meromorphic functions which are invariant under an action of the modular group, in order to motivate and clarify some of the
fundamental ideas that occur naturally in this subject.

To understand fully the answers to the questions which make up a large part of this thesis, we must first motivate the questions themselves. Therefore, in this chapter, we begin with a brief overview of the theory of modular functions and forms, and then more to the matter at hand, a history of the development of the theory of rational period functions.

In Chapter 2 we address the main purpose of this thesis, which is to answer the following question [Kn3]: 'Are there rational period functions (defined on the modular group) with irrational poles which are eigenfunctions of the induced Hecke operators $\hat{T}_{2k}(n)$?' Answering this question will enable us, in Chapter 3, to answer an analogous question about rational period functions defined on the two Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$. Lastly, in Chapter 4, we will shift emphasis by beginning an investigation into the location of zeros of rational period functions.

1.2 Modular Functions and Forms

We wish to motivate the definitions of modular functions and forms from first principles. Therefore, before giving these definitions, we begin by considering the notion of periodicity in the context of the so-called elliptic functions. More precisely, suppose $\omega_1, \omega_2 \in \mathbb{C}$ are non-zero such that $\frac{\omega_1}{\omega_2}$ is not real. If $f$ is meromorphic in $\mathbb{C}$ and satisfies $f(z + \omega_1) = f(z)$ and $f(z + \omega_2) = f(z)$ for all $z \in \mathbb{C}$, then $f$ is an elliptic function. The condition that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ implies that $\omega_1$ and $\omega_2$, considered as vectors in the complex plane, are independent. That is, $f$ has two 'different' periods, and therefore may legitimately be thought of as a doubly periodic function. An example
of such a function is the famous Weierstrass \( g \) function, which is defined as follows: with \( \omega_1, \omega_2 \in \mathbb{C} \) as before, let \( \Omega = \{m\omega_1 + n\omega_2 : (m, n) \in \mathbb{Z} \times \mathbb{Z}\} \), where \( \Omega \) is called a period lattice. Then

\[
\vartheta(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \mathbb{C} \\
\omega \neq (0,0)}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).
\]

(1.1)

The group \( G_\Omega = \{T_\omega(z) = z + \omega : \omega \in \Omega\} \) is generated by \( T_{\omega_1}(z) = z + \omega_1 \) and \( T_{\omega_2}(z) = z + \omega_2 \). All questions of convergence aside, a straightforward computation shows that \( \vartheta(z) \) is invariant under the action of \( G_\Omega \), where by action we mean composition of the function \( \vartheta \) with the function \( T_\omega \in G_\Omega \). That is, \( \vartheta(T_\omega z) = \vartheta(z) \) for every \( T_\omega \in G_\Omega \) and for all \( z \in \mathbb{C} \). The convergence of \( \vartheta(z) \) is settled in the usual way by showing that \( \vartheta(z) \) converges uniformly on compact subsets of \( \mathbb{C} \) (excluding the obvious places where \( \vartheta(z) \) has singularities, namely at points in \( \Omega \)). See, for example, [Sc1, p. 156] and [Ap, p. 10].

In the language of Felix Klein, we have 'identified' the following geometry: the vector space of all elliptic functions with period lattice \( \Omega \). Such a space is invariant under the group \( G_\Omega \), where the group action is given by composition of functions.

One nice property of the above geometry is that we can understand completely the behavior of any function with period lattice \( \Omega \), for example \( \vartheta(z) \), by restricting the domain to the parallelogram, \( \mathcal{R} \), determined by the two vectors \( \omega_1 \) and \( \omega_2 \). That is, \( \mathcal{R} = \{z : z = t_1\omega_1 + t_2\omega_2 : 0 < t_1, t_2 < 1\} \). See Figure 1. Note that all translates of \( \bar{\mathcal{R}} \), the closure of \( \mathcal{R} \), by elements of \( G_\Omega \) cover the complex plane, with overlap only at the boundaries of translates of \( \mathcal{R} \). See Figure 2. In fact, by virtue of invariance under the action of \( G_\Omega \), any elliptic function with period lattice \( \Omega \) behaves the same way.
Figure 1: The parallelogram $\mathcal{R}$ determined by $\omega_1$ and $\omega_2$.

on each translate of $\mathcal{R}$. In some sense, $\mathcal{R}$ is a 'smallest' such region and therefore is fundamental in determining the overall behavior of $\varphi(z)$. For further details, see also [Fo, Section 61]. In order to define modular function as an analogue of elliptic function, we shift to a slightly different point of view. Consider those functions meromorphic in $\mathcal{H}$, the upper half-plane (rather than all of $\mathbb{C}$), whose associated group of invariance is the modular group. Next we define the modular group.

**Definition 1.2.1** The modular group, $\Gamma(1)$, is defined to be the set of linear fractional transformations $Mz = \frac{az + b}{cz + d}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

The modular group is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the translation, and $T =$
Figure 2: A covering of $C$ by all translates of $\mathcal{H}$ by elements of $G_{\Omega}$

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ the inversion. We refer to } S \text{ and } T \text{ as the standard generators of } \Gamma(1).
\]

The use of matrix notation is convenient because matrix multiplication corresponds to composition of linear fractional transformations. Also, since $-M$ and $M$ are identical linear fractional transformations, we have $\Gamma(1) \cong SL(2, \mathbb{Z})/\pm I$, and consequently, $(ST)^3 = T^2 = I$ (whereas in $SL(2, \mathbb{Z})$, $(ST)^6 = T^4 = I$).

We now give

**Definition 1.2.2** A function $f$ meromorphic in $\mathcal{H}$ is said to be a modular function if

(a) $f(Mz) = f(z)$ for all $M \in \Gamma(1)$, and

(b) there exists a positive real number $y_0$ such that for all $\text{Im}(z) > y_0$, $f$ has an
expansion of the form $\sum_{n=-m}^{\infty} a_n e^{2\pi i n z}$.

The first condition describes the invariance of modular functions under the action of $\Gamma(1)$. The second condition describes the behavior of $f$ at the point $z = i\infty$. That is, if we let $q = e^{2\pi i z}$, the expansion in (b) may be rewritten as $f(q) = \sum_{n=-m}^{\infty} a_n q^n$. Since $z \mapsto e^{2\pi i z}$ maps $\mathcal{H}$ onto the punctured unit disc $\{z: 0 < |z| < 1\}$, the expansion for $f(q)$ is simply the Laurent expansion about the point $q = 0$, which in turn corresponds to the point $z = i\infty$. In other words, condition (b) says that $f(q)$ cannot have an essential singularity at $q = 0$, and quite possibly may be analytic there (if $m < 0$). Therefore, in the worst case, $f(z)$ has a pole of order $m$ at $i\infty$. In general, the Laurent expansion in (b) is called the Fourier expansion of $f$ and the coefficients of the expansion are called the Fourier coefficients of $f$.

It is straightforward to check that the family of modular functions is a vector space over $\mathbb{C}$. Therefore, with the group given by $\Gamma(1)$, and the group action given by composition of modular functions with linear fractional transformations in $\Gamma(1)$, the space of modular functions is a geometry in the spirit of Felix Klein.

To complete the analogy of modular functions with that of the elliptic functions, we return to the idea of giving a suitable 'small' connected subset of $\mathcal{H}$ on which to study the overall behavior of modular functions. To this end, let $\mathcal{F} = \{z: z + \bar{z} < 1 \text{ and } |z| > 1\}$, which we show in Figure 3. Applying all linear fractional transformations in $\Gamma(1)$ to $\mathcal{F}$, the closure of $\mathcal{F}$, produces a cover of $\mathcal{H}$ with overlapping only at the boundaries of copies of $\mathcal{F}$. See Figure 4. Further, by virtue of Definition 1.2.2 (a), a modular function behaves the same way on each copy of $\mathcal{F}$. 
Figure 3: The region $\mathcal{F}$

Figure 4: A covering of $\mathcal{H}$ by 'translates' of $\mathcal{T}$ by all elements of $\Gamma(1)$
Remark. In Figure 2, the copies of \( \mathcal{R} \) under translates by elements of \( G_0 \) are all congruent with respect to the usual Euclidean metric. In Figure 4, with a different metric, the same can be said of \( \mathcal{F} \) and all of its copies. That is, with the right metric, the 'translates' of \( \mathcal{F} \) by elements of \( \Gamma(1) \) are all congruent. This, in fact, is Klein's upper half-plane model of hyperbolic geometry. See, for example, [Ma, Chapter 1], [Be, Section 9.4], or [Le1, p. 29].

Perhaps the best known modular function is \( J(z) \), Klein's famous \textit{J-invariant}, which we will define shortly. See [Ap, p. 15]. At present, we merely offer \( J(z) \) as collateral for the existence of modular functions, which, in turn, provides a springboard from which we will motivate the definition of a modular form. If \( f \) is a modular function, then for every \( M \in \Gamma(1) \), we have \( f(Mz) = f(z) \). Formally, we may take the derivative of \( f \) as follows: \( f'(Mz) \frac{d(Mz)}{dz} = f'(z) \). Then \( (cz + d)^{-2} f'(Mz) = f'(z) \). We see that \( f' \) is not quite a modular function because it is not quite invariant under the group action given by composition of functions. In particular, the extra factor \( (cz + d)^{-2} \) prevents \( f' \) from satisfying condition (a) in Definition 1.2.2. We turn this possible obstruction to our advantage by generalizing the notion of composition of functions in the following definition.

**Definition 1.2.3** The slash operator is given by

\[
(F|_r M)(z) = (cz + d)^{-r} F(Mz),
\]

where \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \) and \( r \in \mathbb{R} \).

Therefore, if \( f \) is a modular function, then we have seen that \( (f'|_2 M)(z) = f'(z) \) for every \( M \in \Gamma(1) \). This idea of not quite composition of functions gives new energy
to Klein's notion of the geometry of a space of functions. In particular we will use
the slash operator to describe a group action for \( \Gamma(1) \), other than composition of
functions, to give the definition of entire modular form. Before doing so, it is useful
to note that \( (F|_{r}M_{1})|_{r}M_{2})(z) = (F|_{r}M_{1}M_{2})(z) \).

**Definition 1.2.4** A function \( f \) analytic in \( \mathcal{H} \) is said to be an entire modular form
of weight \( 2k \), for \( k \in \mathbb{Z} \) if

(a) \( (f|_{2k}M)(z) = f(z) \) for all \( M \in \Gamma(1) \), and

(b) \( f \) has a Fourier expansion given by \( \sum_{n=0}^{\infty} a_{n}e^{2\pi inz} \), valid for all \( z \in \mathcal{H} \).

If \( a_{0}=0 \), then \( f \) is said to be a cusp form of weight \( 2k \).

One can show that the existence of nontrivial entire modular forms requires that
\( k \geq 2 \). See for example [Ap, p. 116]. Condition (a) in Definition 1.2.4 describes
the invariance of modular forms under the action of \( \Gamma(1) \), and condition (b) requires
that \( f \) be analytic at \( \infty \). It is not hard to see that the family of entire modular
forms of a given weight forms a vector space over \( \mathbb{C} \). Finally, we note that quite
often the Fourier coefficients of entire modular forms turn out to be number theoretic
functions, and thus these coefficients provide a link between complex function theory
and multiplicative number theory. We will say more about this beautiful connection
presently.
1.3 Existence of Entire Modular Forms

As evidence for the existence of entire modular forms of weight $2k$, we offer the Eisenstein series $G_{2k}(z)$, defined by

$$G_{2k}(z) = \sum_{(m,n) \neq (0,0)}^{\infty} \frac{1}{(m+nz)^{2k}},$$

where, for $k > 1$, the series $G_{2k}(z)$ converges uniformly on compact subsets of $\mathcal{H}$, and therefore represents an analytic function on $\mathcal{H}$. In fact, for $k > 1$, $G_{2k}(z)$ is an entire modular form of weight $2k$, and moreover, every entire modular form is a polynomial in $G_4(z)$ and $G_6(z)$. For details, see [Ap, p.12]. When $k = 1$, $G_2(z)$ does not satisfy part (a) of Definition 1.2.4 because $G_2(z)$ converges only conditionally in $\mathcal{H}$. This 'failure' will be one of the motivating forces behind the definition of Modular Integral with associated Rational Period Function in Section 1.5 of this chapter.

Finally, we note that if $f_1$ and $f_2$ are modular forms of the same weight, then $\frac{f_1}{f_2}$ and $\frac{f_2}{f_1}$ are both modular functions. In fact, the modular function $J(z)$ given in section 1.2 of this chapter, is the quotient of $60G_4(z)^3$ and $G_4(z)^3 - 27G_6(z)^2$, which, incidentally, are both modular forms of weight 12, and are both polynomials in $G_4(z)$ and $G_6(z)$.

1.4 The Hecke Operators

We mentioned earlier that the Fourier coefficients of entire modular forms often turn out to be number theoretic functions. More precisely, a function $g$ is said to be
multiplicative if, for \( m \) and \( n \) positive integers, \( g \) satisfies

\[
g(m)g(n) = \sum_{d \mid \gcd(m,n)} d^\alpha g\left(\frac{mn}{d^2}\right),
\]

where \( \alpha \in \mathbb{Z} \). In particular, if \( \gcd(m, n) = 1 \), then \( g(m)g(n) = g(mn) \). In [He2], Erich Hecke was able to find all entire modular forms with multiplicative Fourier coefficients by creating a family of linear operators \( \{T_{2k}(n)\} \) which are defined as follows.

**Definition 1.4.1**

\[
T_{2k}(n)f = n^{2k-1} \sum_{ad = n} \sum_{d > 0} f\left(\frac{az + b}{d}\right)
\]

With the right tools from algebra, it is not hard to see that \( T_{2k}(n) \) maps the space of entire modular forms of weight \( 2k \) to itself, and in fact is independent of the choice of representatives of \( b(\text{mod } d) \). See, for example [Gu, p. 58] or [Ap, pp. 122-124]. Furthermore, Hecke showed that these operators commute, that is, \( T_{2k}(n)T_{2k}(m) = T_{2k}(m)T_{2k}(n) \), and using this, that the only entire modular forms with multiplicative Fourier coefficients must be eigenfunctions of *all* of the operators \( T_{2k}(n), n > 1 \).

In addition, Hecke took \( f \), an entire modular form of weight \( 2k \), with Fourier expansion \( f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i nz} \) and created the Dirichlet series \( \varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \). Because of certain growth conditions on the sequence \( \{a_n\} \), \( \varphi(s) \) converges in a right
half-plane, the location of which depends on $2k$, the weight of $f$, and on whether or not $f$ is a cusp form.

Next, Hecke showed that $\varphi$ has an Euler product, which means that $\varphi(s)$ has an infinite product expansion of the form

$$\prod_{p \text{ prime}} (1 - a_p p^{-s} - p^{2k-1-2s})^{-1}, \quad (1.7)$$

if and only if the Fourier coefficients are multiplicative. Moreover, the Euler product converges in the same half-plane with $\varphi(s)$. Altogether, Hecke proved

**Theorem 1.4.2** *If f is an entire modular form of weight 2k, then the following are equivalent:

(a) $f$ is an eigenfunction of $T_{2k}(n)$ for all $n > 1$,

(b) the Fourier coefficients of $f$ are multiplicative,

(c) the Dirichlet series for $f$ has an Euler product expansion of the form $\varphi(s) = \prod_{p \text{ prime}} (1 - a_p p^{-s} - p^{2k-1-2s})^{-1}$.

Condition (c) provides evidence that the behavior of a multiplicative function (namely, the Fourier coefficients of $f$) is completely determined by its values at prime numbers. For details of Hecke's work, see [He1, p. 668] or [He2, p. 37].

Both $G_4(z)$ and $G_6(z)$ are examples of entire modular forms which are eigenfunctions of all of the Hecke operators $T_4(n)$ and $T_6(n)$ respectively, and hence by Theorem 1.4.2 have multiplicative Fourier coefficients. In particular,

$$G_4(z) = \frac{\pi^4}{15} \left( 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi inz} \right), \quad (1.8)$$
and

\[
G_6(z) = \frac{\pi^6}{945} \left( 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i nz} \right),
\]

(1.9)

where \( \sigma_i(n) = \sum_{d|n} d^i \).

Dirichlet series can also be created from certain generalizations of modular forms, in particular from modular integrals, which will be defined in the next section.

1.5 Modular Integrals and Rational Period Functions

The theory of modular forms lends itself naturally to a variety of generalizations, and we will focus on one such for the remainder of this chapter.

In section 1.3 we claimed that the Eisenstein series \( G_2(z) \) was not an entire modular form. Specifically, \( G_2(z) \) fails to satisfy the functional equation in Definition 1.2.4 (a). To understand the nature of this failure, since \( (G_2|_2 M_2)(z) = (G_2|_2 M_1 M_2)(z) \), it suffices to examine \( G_2|_2 S \) and \( G_2|_2 T \), where \( S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) are the standard generators of \( \Gamma(1) \). In particular, \( (G_2|_2 S)(z) = G_2(z) \), but \( (G_2|_2 T)(z) = G_2(z) - \frac{2\pi i}{z} \). See for example [Hu].

In [Kn1], Marvin Knopp used the example provided by \( G_2(z) \) to motivate the following generalization of modular form:

**Definition 1.5.1** Suppose \( f \) is meromorphic in \( \mathcal{H} \) and satisfies

\[
(f|_{2k} S)(z) = f(z)
\]

(1.10)

and

\[
(f|_{2k} T)(z) = f(z) + q(z),
\]

(1.11)
where $k$ is an integer and $q(z)$ is a rational function. If, in addition, $f$ is meromorphic at $i\infty$, then $f$ is a modular integral of weight $2k$ with associated rational period function (abbreviated as RPF) $q(z)$.

Under such circumstances, we say that $q(z)$ is an RPF of weight $2k$. Note that when $q(z) \equiv 0$, $f$ is a (not necessarily entire) modular form.

Remark. This definition is also a generalization of an Eichler integral, where $q(z)$, a rational function, is replaced by $p(z)$, a polynomial. Eichler integrals are $(2k-1)$-fold integrals of cusp forms and entire modular forms of positive weight $2k$. For further details, see [Ei] and [Kn4].

For many reasons, it is worthwhile to note that the Hecke operators $\{T_{2k}(n)\}$ defined in Section 1.4 also act as operators on the space of modular integrals of weight $2k$. In other words, if $f$ is a modular integral of weight $2k$, then so is $T_{2k}(n)f$. Moreover, in [Kn1], Knopp defined the induced Hecke operator, $\hat{T}_{2k}(n)$, an operator on the space of RPFs of weight $2k$, as follows.

**Definition 1.5.2** If $f$ is a modular integral of weight $2k$ with associated RPF $q(z)$ then $\hat{T}_{2k}(n)q = (T_{2k}(n)f)|_{2kT} - T_{2k}(n)f$.

The induced Hecke operator, $\hat{T}_{2k}(n)$, inherits the multiplicative property of $T_{2k}(n)$. Moreover, it is of particular interest to note that if $f$ is an eigenfunction of $T_{2k}(n)$, then $q$ is an eigenfunction of $\hat{T}_{2k}(n)$. To see why, suppose that $T_{2k}(n)f = sf$ for some $s \neq 0$ in $\mathbb{C}$. Then

$$\hat{T}_{2k}(n)q = (T_{2k}(n)f)|_{2kT} - T_{2k}(n)f$$

$$= (sf)|_{2kT} - sf$$
In other words, if \( f \) is an eigenfunction of \( T_{2k}(n) \), then \( q \) is an eigenfunction of \( \hat{T}_{2k}(n) \), and in that case, \( f \) and \( q \) share the same eigenvalue. Equivalently, if \( q \) fails to be an eigenfunction of \( \hat{T}_{2k}(n) \), then \( f \) is not an eigenfunction of \( T_{2k}(n) \).

Following the lead of Hecke in \([\text{He3}]\), Goldstein and Razar in \([\text{GR}]\) gave a method with which to associate a Dirichlet series to the Fourier expansion of functions that are more general than the entire modular forms defined in Section 1.3. In \([\text{Kn1}]\), Knopp used the methods of Goldstein and Razar to associate a Dirichlet series to the Fourier expansions of modular integrals, and these results have recently been applied further in \([\text{Kn5}]\) and \([\text{HK}]\).

All of these developments naturally led to the question of an analogue to Hecke's work as described in Section 1.4. In particular, it becomes important to know which modular integrals are eigenfunctions of the operators \( \{T_{2k}(n)\} \) and to study properties of Dirichlet series associated with modular integrals. Alternatively, it becomes important to know when a modular integral \( f \) is not an eigenfunction of \( T_{2k}(n) \), which in turn, is equivalent to knowing when \( q \), the RPF associated with \( f \), is not an eigenfunction of the induced Hecke operator \( \hat{T}_{2k}(n) \). Since RPFs are actually rational functions, the problem of determining when \( q \), and hence \( f \), is not an eigenfunction is more accessible. In fact, the main result in this thesis, a proof of a conjecture of Knopp in \([\text{Kn3}]\), is the following theorem, which we will restate in Chapter 2.

**Theorem.** If \( q \) is a rational period function of weight \( 2k \) with at least one real
quadratic irrational pole, then \( q \) is not an eigenfunction of \( \hat{T}_{2k}(n) \) for any \( n > 1 \).

To better understand the generality and substance of this theorem, we next provide a history of the development of the theory of rational period functions. For a more detailed treatment of this history, see [Kn3].

### 1.6 History of Rational Period Functions

In [Kn1] and [Kn4], Knopp showed that a rational function \( q \) is an RPF of weight \( 2k \) if and only if the following two functional equations are satisfied:

\[
q|_{2k} T + q = 0 \quad (1.12)
\]

and

\[
q|_{2k} (ST)^2 + q|_{2k} (ST) + q = 0. \quad (1.13)
\]

Moreover, in [Kn1], Knopp added to the collection of known RPFs (those with a pole at zero, and the Eichler polynomials, which have poles at \( \infty \)) by giving the first example of RPFs with real quadratic irrational poles, namely for \( k \) a positive odd integer,

\[
q(z) = \frac{1}{(z^2 - z - 1)^k} + \frac{1}{(z^2 + z - 1)^k}, \quad (1.14)
\]

and proved that \( q \) is never an eigenfunction of \( \hat{T}_{2k}(n) \). Note that the poles of \( q \) are \( \frac{\pm(1\pm\sqrt{5})}{2} \), which are in \( \mathbb{Q}(\sqrt{5}) \).

In [Kn2], using functional equations 1.12 and 1.13, Knopp completely classified RPFs with poles only at 0 as follows. If \( q \) is an RPF of weight \( 2k > 0 \) with poles only in \( \mathbb{Q} \), then
\[ q(z) = \begin{cases} 
  b_0 \left( 1 - \frac{1}{2k} \right) & \text{if } k > 1 \\
  b_0 \left( 1 - \frac{1}{z^2} \right) + \frac{b_1}{z} & \text{if } k = 1 
\end{cases} \]  
(1.15)

for some \( b_0, b_1 \in \mathbb{C} \).

If \( k = 1, \ b_0 = 0, \) and \( b_1 \neq 0, \) then \( q(z) = \frac{b_1}{z} \) is the RPF associated with a suitable multiple of \( G_2, \) the Eisenstein series of weight 2. In contrast to the RPFs given in equation 1.14, Knopp proved that \( q \) is always an eigenfunction of \( \hat{T}_2(n) \) by showing that \( G_2 \) is always an eigenfunction of \( T_2(n) \).

In fact, we can show that any RPF with poles only at zero, given by 1.15, is always an eigenfunction of \( \hat{T}_{2k}(n) \) as follows. Since \( \frac{b_1}{z} \) is always an eigenfunction of \( \hat{T}_2(n) \), it suffices to show that \( b_0 \left( 1 - \frac{1}{2k} \right) \) is always an eigenfunction of \( \hat{T}_{2k}(n) \), for \( k \geq 1 \).

To this end, we first show that \( f(z) = -b_0 \) is a modular integral of weight \( 2k \) with associated RPF \( b_0 \left( 1 - \frac{1}{2k} \right) \). In particular,

\[(f|_{2k}S)(z) = -b_0|_{2k}S \]
\[= -b_0, \]

and

\[(f|_{2k}T)(z) = -b_0|_{2k}T \]
\[= -z^{-2k}b_0 \]
\[= -b_0 + b_0 \left( 1 - \frac{1}{z^{2k}} \right) \]
\[= f(z) + b_0 \left( 1 - \frac{1}{z^{2k}} \right). \]
Next, we show that $f(z) = -b_0$ is always an eigenfunction of $T_{2k}(n)$. Specifically,

$$T_{2k}(n)f = n^{2k-1} \sum_{\substack{ad = n \\ d > 0 \\ b \pmod{d}}} f|_{2k} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= n^{2k-1} \sum_{\substack{ad = n \\ d > 0 \\ b \pmod{d}}} (-b_0)|_{2k} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$= n^{2k-1} \sum_{\substack{ad = n \\ d > 0 \\ b \pmod{d}}} d^{-2k}(-b_0)$$

$$= C_n f(z),$$

where

$$C_n = n^{2k-1} \sum_{\substack{ad = n \\ b \pmod{d}}} d^{-2k}. \tag{1.16}$$

Thus, $f(z) = -b_0$ is always an eigenfunction of $T_{2k}(n)$, and hence $b_0(1 - \frac{1}{2^{2k}})$ is always an eigenfunction of $\hat{T}_{2k}(n)$.

What ultimately turned out to be of greater consequence and interest in [Kn2] was that Knopp proved that the finite non-zero poles of RPFs are necessarily real quadratic irrational numbers. To help understand why this is so, we give a definition.

**Definition 1.6.1** A linear fractional transformation $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$ is hyperbolic if $|\text{Trace}(M)| = |\alpha + \delta| > 2$.

In fact what Knopp proved in [Kn2] was that the finite non-zero poles of RPFs are necessarily fixed points of hyperbolic elements of $\Gamma(1)$. Therefore, if $z_0$ is such a
pole and is fixed by $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$ with $|\alpha + \delta| > 2$, then $Mz_0 = z_0$ implies that 

$$\frac{\alpha z_0 + \beta}{\gamma z_0 + \delta} = z_0. \quad (1.17)$$

If $\gamma = 0$, then since $\det(M) = 1$, we have $\alpha = \delta = \pm 1$, which means that $M$ is not hyperbolic, a contradiction. Therefore, we may assume without loss of generality that $\gamma \neq 0$. In that case, equation 1.17 yields $\gamma z_0^2 + (\delta - \alpha)z_0 - \beta = 0$, and thus 

$$z_0 = \frac{\alpha - \delta \pm \sqrt{(\delta - \alpha)^2 + 4\beta\gamma}}{2\gamma}. \quad (1.18)$$

Therefore, 

$$z_0 = \frac{\alpha - \delta \pm \sqrt{(\alpha + \delta)^2 - 4}}{2\gamma} \quad (1.19)$$

because $\alpha\delta - \beta\gamma = 1$. Further, $|\text{Trace}(M)| = |\alpha + \delta| > 2$ implies that $(\alpha + \delta)^2 - 4 > 0$. Moreover, we claim that $(\alpha + \delta)^2 - 4$ cannot be a square in $\mathbb{Z}$. If so, then $(\alpha + \delta)^2 - 4 = r^2$ for some integer $r$. In fact, since $|\alpha + \delta| > 2$, we must have $r > 0$. Then $(\alpha + \delta)^2 - r^2 = 4$. But this is impossible, since the difference between two distinct non-zero squares in $\mathbb{Z}$ cannot equal 4. Consequently, if $N := (\alpha + \delta)^2 - 4$, we have $z_0 \in \mathbb{Q}(\sqrt{N}) \setminus \mathbb{Q}$. Hence, if $z_0$ is a finite non-zero pole of an RPF, then $z_0$ is a real quadratic irrational number. Therefore, since Eichler integrals have RPFs which are polynomials, and the RPFs given by 1.15 have poles at zero, we see that RPFs may only have poles at $0, \infty$, and at real quadratic irrational numbers.

This elegant revelation about poles provoked a variety of papers on the subject of RPFs. For our purposes, it is those papers describing RPFs with quadratic irrational poles which are of primary interest.
In [PR], A. Parson and K. Rosen found examples of RPFs with poles in \( \mathbb{Q}(\sqrt{3}) \) and \( \mathbb{Q}(\sqrt{21}) \), and showed that these examples were never eigenfunctions of \( \hat{T}_{2k}(n) \).

In [Ha], J. Hawkins began a study of quadratic irrational poles of RPFs by introducing the notion of **irreducible systems of poles** which are minimal sets of quadratic irrational numbers, that are arrived at because of the nature of the functional equations 1.12 and 1.13. For example, if \( z_0 \neq 0 \) is a finite pole of an RPF \( q \), then by functional equation 1.12, \( Tz_0 = \frac{1}{z_0} \) must also be a pole of \( q \). Hawkins also showed that for \( k < 0 \), an RPF \( q \) of weight \( 2k \) must be a polynomial, and for \( k = 0 \), \( q \) must be identically 0. In other words, RPFs with finite non-zero poles must be of strictly positive weight.

In [Ch], Y-J Choie developed two methods by which to construct RPFs of positive weight with poles in \( \mathbb{Q}(\sqrt{N}) \) for any positive non-square integer \( N \).

In [CP1] and [CP2] Choie and Parson began a more detailed classification of RPFs of positive weight, and to do so, drew extensively from the theory of indefinite binary quadratic forms. In particular, it follows from [CP1] that

\[
\sum_{\substack{a \gg b > c \geq 0 \\ b^2 - 4ac = D > 0}} \frac{1}{(az^2 + bz + c)^k} \quad (1.20)
\]

is always an RPF of weight \( 2k \) when \( k > 0 \) is an odd integer.

In [As], A. Ash also explained the existence of RPFs from the point of view of cohomology of groups.

In [Pal], Parson gave a formula for certain modular integrals \( f \) corresponding to RPFs as classified in [CP1]. She then used the formula to describe the Fourier coefficients of these modular integrals, and in addition gave an explicit formula for
$T_{2k}(p)f$ for $p$ prime, which in turn gave a formula for $\hat{T}_{2k}(p)q$, where $q$ is the RPF associated with $f$.

Finally, in [CZ] and in [Pa2], Y-J Choie and D. Zagier, and A. Parson have independently completed the classification of RPFs by providing an explicit construction for all RPFs of a given positive weight.

Even though RPFs with quadratic irrational poles have been classified, it is difficult, in general, to find a formula for $\hat{T}_{2k}(n)q$. However, it is possible to show that those RPFs with at least one quadratic irrational pole are not eigenfunctions. This is the topic of Chapter 2.

### 1.7 The Effect of $\hat{T}(n)$ on Some Examples of RPFs

The purpose of this section is two-fold. First, we will use some of the examples to demonstrate the effect of the induced Hecke operators $\hat{T}(p)$ for $p$ prime. Then, by examining the effect of the induced Hecke operators on these examples, we will gain some insight into the foundation of the proof of Theorem 1.5. To this end, we draw from the families of RPFs given by 1.20, and for contrast, from 1.15 in Section 1.6.

To begin, we introduce some convenient notation. Let

$$q_{k,D} = \sum_{\substack{a \geq 0 \geq c \geq 0 \geq b^2-4ac=D>0}} \frac{1}{(az^2 + bz + c)^k}$$  \hspace{1cm} (1.21)

By [CP1], we know that when $a, b, c \in \mathbb{Z}$, and $k$ is a positive odd integer, then $q_{k,D}$ is an RPF of weight $2k$ with quadratic irrational poles. By means of a Basic program, the following is a sample of the effect of $\hat{T}(p)$ (see Definition 1.5.2), where $p$ is prime, on some explicit RPFs given by $q_{k,D}$. From now on, we will assume that
$k$ is a positive odd integer.

**Example 1.7.1** The family of RPFs given by $q_{k,5} = \frac{1}{(z^2 - z - 1)^k} + \frac{1}{(z^2 + z - 1)^k}$ is the original example of RPFs with quadratic irrational poles found by Knopp in [Kn1]. We have

$$
\hat{T}(2)q_{k,5} = \frac{1}{(z^2 + 2z - 4)^k} + \frac{1}{(z^2 - 2z - 4)^k} + \frac{1}{(4z^2 + 2z - 1)^k} \\
+ \frac{1}{(4z^2 - 2z - 1)^k} + \frac{1}{(5z^2 - 1)^k} + \frac{1}{(z^2 - 4z - 1)^k} \\
+ \frac{1}{(z^2 - 5)^k} \\
= \sum_{a>0, b>0, c=20} \frac{1}{(az^2 + bz + c)^k} \\
= q_{k,20}.
$$

The poles of $q_{k,5}$ are $\{ \frac{\pm 1 \pm \sqrt{5}}{2} \}$. Therefore, since none of the poles of $\hat{T}(2)q_{k,5}$ are poles of $q_{k,5}$, we have that $q_{k,5}$ cannot be an eigenfunction of $\hat{T}(2)$. In fact, in general, we need only find one pole of $\hat{T}(n)q$ that is not a pole of $q$ in order to prove that $q$ is not an eigenfunction of $\hat{T}(n)$.

**Example 1.7.2**

$$
\hat{T}(3)q_{k,5} = \frac{1}{(z^2 + 3z - 9)^k} + \frac{1}{(z^2 - 3z - 9)^k} + \frac{1}{(9z^2 + 3z - 1)^k} \\
+ \frac{1}{(9z^2 - 3z - 1)^k} + \frac{1}{(5z^2 + 5z - 1)^k} + \frac{1}{(11z^2 - z - 1)^k} \\
+ \frac{1}{(z^2 - 5z - 5)^k} + \frac{1}{(z^2 + z - 11)^k} + \frac{1}{(11z^2 + z - 1)^k} \\
+ \frac{1}{(5z^2 - 5z - 1)^k} + \frac{1}{(z^2 + 5z - 5)^k} + \frac{1}{(z^2 - z - 11)^k}
$$
\[ \sum_{\substack{a > 0 > c \\ b^2 - 4ac = 45}} \frac{1}{(az^2 + bz + c)^k} = q_{k,45}. \]

We see that \( \frac{3 + 3\sqrt{5}}{2} \) is a pole (arising from the first term) of \( T(3)q_{k,5} \) which is not a pole of \( q_{k,5} \), and therefore \( q_{k,5} \) is not an eigenfunction of \( T(3) \).

For the remaining two examples, \( D \) and \( p \) are large enough so that the functions \( q_{k,D} \) and \( T(p)q_{k,D} \) are cumbersome to write in their entirety, and so we simply give the result of the effect of \( T(p) \) on \( q_{k,D} \).

**Example 1.7.3** \( T(7)q_{k,8} = q_{k,392} + 2(7)^{-k}q_{k,8} \)

In the above example, although all of the poles of \( q_{k,8} \) are poles of \( T(7)q_{k,8} \), it is still the case that \( q_{k,8} \) is not an eigenfunction of \( T(7) \) because \( \sqrt{98} \) is a pole of \( T(7)q_{k,8} \) that is not a pole of \( q_{k,8} \). In particular, \( \sqrt{98} \) is a pole of \( \frac{1}{(z^2 - 98)^k} \) which is one of the terms of \( q_{k,392} \).

In our final example, we show the effect of an induced Hecke operator on an RPF with poles at zero as well as at real quadratic irrational numbers.

**Example 1.7.4**

\[
T(5)[q_{k,21} + b_0 \left(1 - \frac{1}{z^{2k}}\right)] = T(5)q_{k,21} + T(5)[b_0 \left(1 - \frac{1}{z^{2k}}\right)]
= q_{k,525} + 2(5)^{-k}q_{k,21} + C[b_0 \left(1 - \frac{1}{z^{2k}}\right)],
\]
where $C$ is computed by using equation 1.16 in Section 1.6. In particular,

$$C = 5^{2k-1} \sum_{\substack{a \equiv b \pmod{d} \neq 0}} d^{-2k}$$

$$= 5^{2k-1} \left[1 + \sum_{i=0}^{4} 5^{-2k}\right]$$

$$= 5^{2k-1} \left[1 + 5(5)^{-2k}\right]$$

$$= 5^{2k-1} + 1.$$

We note that $g_{k,21} + b_0(1 - \frac{1}{2x})$ is not an eigenfunction of $\hat{T}(x)$ because $\frac{1 + \sqrt{525}}{2}$ is a pole of $\hat{T}(x)[g_{k,21} + b_0(1 - \frac{1}{2x})]$ which is not a pole of $g_{k,21} + b_0(1 - \frac{1}{2x})$. In particular, $\frac{1 + \sqrt{525}}{2}$ is a pole of $\frac{1}{(x^2 - 131x)^k}$ which is one of the terms of $q_{k,525}$. 
CHAPTER II

The Main Result

2.1 About the Poles of Rational Period Functions

In this chapter we will prove the main result of this thesis, which will also appear in
A Tribute to Emil Grosswald: Number Theory and Related Analysis, Contemporary
Mathematics Series, AMS., and is stated as follows.

Theorem 2.3.1 If \( q \) is a rational period function with at least one quadratic irrational
pole, then \( q \) is not an eigenfunction of \( \hat{T}_{2k}(n) \) for any \( n > 1 \).

Before proving Theorem 2.3.1, it is useful here to note that RPFs can be de­
defined on the family of Hecke groups which are defined as follows: for \( n \geq 3 \) in \( \mathbb{Z} \),
\( G(\lambda_n) = \langle S_{\lambda_n}, T \rangle \), where \( S_{\lambda_n} = \begin{pmatrix} 1 & 2\cos\frac{\pi}{n} \\ 0 & 1 \end{pmatrix} \) and \( T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) are the stan­
dard generators, and satisfy \( (S_{\lambda_n}T)^n = T^2 = I \). Note that when \( n = 3 \), \( \lambda_n = 1 \), and
in that case \( G(\lambda_3) = G(1) \) which we will use as an alternative notation for \( \Gamma(1) \), the
modular group. Note also, in analogy to the functional equations given by 1.12 and
1.13, that a rational function \( q \) is an RPF of weight \( 2k \) on \( G(\lambda_n) \) if and only if the
following two functional equations are satisfied [Kn4]:

\[
q|_{2k}T + q = 0
\]  \hspace{1cm} (2.1)
and
\[ \sum_{i=1}^{n-1} q_{2k} (S_{n,k} T)^i + q = 0. \tag{2.2} \]

In Chapter 3 we will derive an analogue to Theorem 2.3.1 for RPFs defined on \( G(\lambda_4) = G(\sqrt{2}) \) and \( G(\lambda_6) = G(\sqrt{3}) \). Much of the machinery which we will develop for RPFs on \( \Gamma(1) \), particularly concerning poles, has direct analogues for RPFs on \( G(\sqrt{2}) \) and \( G(\sqrt{3}) \) which will help facilitate the proofs of the results given in Chapter 3. Therefore, we spend the remainder of this section deriving results about finite non-zero poles of RPFs defined on \( G(\lambda_n) \) for \( n = 3, 4, \) and \( 6 \). From now on, we will write \( \lambda \) in place of \( \lambda_n \), and will assume that \( \lambda = 1, \sqrt{2}, \) or \( \sqrt{3} \) unless otherwise specified.

We mentioned in the Section 1.6, that in [Kn2], Knopp proved that the finite non-zero poles of RPFs on \( \Gamma(1) \) are necessarily fixed points of hyperbolic elements of \( \Gamma(1) \). We begin by giving an analogue for RPFs defined on \( G(\sqrt{2}) \) and \( G(\sqrt{3}) \).

First, we give a definition, which, incidentally, is a broader form of Definition 1.6.1.

**Definition 2.1.1** Let \( \lambda = 1, \sqrt{2}, \) or \( \sqrt{3}, \) and suppose \( z_0 \) is fixed by \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda) \). Then

(a) \( M \) is hyperbolic if \(|\text{Trace}(M)| = |\alpha + \delta| > 2\), and

(b) \( M \) is parabolic if \(|\text{Trace}(M)| = |\alpha + \delta| = 2\).

**Remark 2.1.** If \( M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda) \) is hyperbolic, then \( \beta, \gamma \neq 0 \). Otherwise, if, say, \( \beta = 0 \), then \( \alpha \delta - \beta \gamma = 1 \) implies that \( \alpha = \delta = \pm 1 \), in which case \(|\text{Trace}(M)| = 2\), a contradiction. The same argument holds if \( \gamma = 0 \).
Theorem 2.1.2 If $z_0$ is a finite non-zero pole of an RPF $q_\lambda$ defined on $G(\lambda)$, for $\lambda = 1, \sqrt{2}$ or $\sqrt{3}$, then $z_0$ is fixed by a hyperbolic element of $G(\lambda)$.

The proof that we will give of Theorem 2.1.2 for $\lambda = \sqrt{2}$ and $\sqrt{3}$ is almost verbatim that of the proof that Knopp gave in [Kn2] for the case $\lambda = 1$. Also, in [Sc] T. Schmidt has an alternative proof of Theorem 2.1.2 using $\lambda$-continued fractions.

Remark 2.2. It is convenient to note that for $\lambda = \sqrt{2}$ or $\sqrt{3}$, the elements of $G(\lambda)$ fall into two categories, the even elements \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] and the odd elements \[
\begin{pmatrix}
\alpha & b \\
c & d
\end{pmatrix}
\] where $a, b, c, d \in \mathbb{Z}$ and $ad - bc\lambda^2 = 1$ and $ad\lambda^2 - bc = 1$, respectively. In fact, this description also holds for the elements of $\Gamma(1)$ simply by observing that the two categories of elements actually coincide since $\lambda = 1$. For details, see [Hul] and [Yo].

Before proving Theorem 2.1.2, we give two lemmas.

Lemma 2.1.3 For $\lambda = 1, \sqrt{2}$, or $\sqrt{3}$, if $z_0$ is fixed by a hyperbolic or parabolic element of $G(\lambda)$, then for any $M \in G(\lambda)$, $Mz_0$ is fixed by a hyperbolic or parabolic element of $G(\lambda)$ respectively.

Proof: Suppose $z_0$ is fixed by $N \in G(\lambda)$. If $M \in G(\lambda)$, then $Mz_0$ is fixed by $MNM^{-1}$ because $MNM^{-1}(Mz_0) = MNz_0 = Mz_0$. Since $\text{Trace}(MNM^{-1}) = \text{Trace}(N)$ (see [Le2, p. 5]), if $N$ is hyperbolic or parabolic, then $MNM^{-1}$ is hyperbolic or parabolic respectively. □

Lemma 2.1.4 If $z_0 \not= \infty$ is fixed by a nontrivial parabolic element of $G(\lambda)$ for $\lambda = \sqrt{2}$ or $\sqrt{3}$, then $z_0 \in \lambda\mathbb{Q}$, where by $\lambda\mathbb{Q}$, we mean 'rational multiples of $\lambda$.'
Proof: By Remark 2.2, if \( z_0 \) is fixed by \( N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda) \), then since \(|\alpha + \delta| = 2 \in \mathbb{Z}\), \( N \) must be an even element. Therefore, \( \beta = \lambda \beta' \) and \( \gamma = \lambda \gamma' \), where \( \beta', \gamma' \in \mathbb{Z} \).

If \( \gamma' = 0 \), then \( \alpha = \delta = \pm 1 \), in which case, since \( z_0 \) is finite, we have \( \beta' = 0 \) and so \( N = I \), a contradiction. Therefore, \( Nz_0 = z_0 \) implies that

\[
\frac{\alpha z_0 + \lambda \beta'}{\lambda \gamma' z_0 + \delta} = z_0,
\]

and so \( \lambda \gamma'(z_0)^2 + (\delta - \alpha)z_0 - \lambda \beta' = 0 \). Therefore,

\[
z_0 = \frac{\alpha - \delta \pm \sqrt{(\alpha - \delta)^2 + 4\lambda^2 \beta' \gamma'}}{2 \lambda \gamma'}
\]

\[
= \frac{\alpha - \delta \pm \sqrt{(\alpha + \delta)^2 - 4}}{2 \lambda \gamma'},
\]

because \( \alpha \delta - \lambda^2 \beta' \gamma' = 1 \). Since \(|\alpha + \delta| = 2\), we have

\[
z_0 = \frac{\alpha - \delta}{2 \lambda \gamma'} = \frac{\alpha - \delta}{2 \lambda^2 \gamma'} \lambda,
\]

which is in \( \lambda \mathbb{Q} \), as desired. \( \square \)

Proof of Theorem 2.1.2

The proofs for \( \lambda = \sqrt{2} \) and \( \lambda = \sqrt{3} \) are essentially the same as the proof that Knopp gives in [Kn2] for \( \lambda = 1 \), and we demonstrate this by presenting the case \( \lambda = \sqrt{2} \).

Recall by functional equations 2.1 and 2.2 that if \( q_\lambda \) is an RPF on \( G(\sqrt{2}) \) of weight \( 2k \) then

\[
q_\lambda|_{2kT} = -q_\lambda \tag{2.3}
\]
and
\[ q_{\lambda}\mid_{2k}(S\sqrt{2}T)^3 + q_{\lambda}\mid_{2k}(S\sqrt{2}T)^2 + q_{\lambda}\mid_{2k}(S\sqrt{2}T) = -q_{\lambda}. \] (2.4)

For the remainder of the proof, we write \( S \) in place of \( S\sqrt{2} \) and \( \mid \) in place of \( \mid_{2k} \). Therefore,
\[ q_{\lambda}\mid T^2 = q_{\lambda}\mid(ST)^3T + q_{\lambda}\mid(ST)^2T + q_{\lambda}\mid(ST)T, \] (2.5)

and hence
\[ q_{\lambda} = q_{\lambda}\mid(ST)^3S + q_{\lambda}\mid(ST)S + q_{\lambda}\mid S. \] (2.6)

We proceed with the proof by rewriting equation 2.6, and in order to do so, we introduce some convenient notation. In particular, let
\[ M_1 = S = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}, M_2 = (ST)S = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \text{ and } M_3 = (ST)^3S = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}. \]
Note that the entries of \( M_i \) for \( i = 1, 2, 3 \) are all non-negative. We may now rewrite equation 2.6 as follows:
\[ q_{\lambda}(z) = (c_3z + d_3)^{-2k}q_{\lambda}(M_3z) + (c_2z + d_2)^{-2k}q_{\lambda}(M_2z) + (c_1z + d_1)^{-2k}q_{\lambda}(M_1z) \] (2.7)

The remainder of the proof is given in two cases:

**Case 1:** \( z_0 \) is a finite non-zero pole of \( q_{\lambda} \) such that \( z_0 \notin \sqrt{2}Q \)

Then \( z_0 \) must be a pole of \( (c_rz + d_r)^{-2k}q_{\lambda}(M_rz) \), one of the terms on the right-hand side of equation 2.7, where \( r \in \{1, 2, 3\} \). Since \( z_0 \notin \sqrt{2}Q \), we have \( c_rz_0 + d_r \neq 0 \), which means that the factor \( (c_rz + d_r)^{-2k} \) cannot account for a pole even when \( k < 0 \). Therefore, \( z_0 \) must be a pole of \( q_{\lambda}(M_rz) \), or alternatively, \( M_rz_0 \) is a pole of \( q_{\lambda}(z) \). By writing \( M_r \) as a word in \( S \) and \( T \), the standard generators of \( G(\sqrt{2}) \), it is easy to see that \( M_rz_0 \) is non-zero and finite, and \( M_rz_0 \notin \sqrt{2}Q \) because \( z_0 \) is non-zero and
finite, and \( z_0 \notin \sqrt{2}Q \). Therefore, we may invoke the above procedure again with \( z_0 \) replaced by \( M_r z_0 \). By doing so repeatedly, say \( s \) times, we generate a (partial) list of poles of \( q_\lambda \): \( \{ z_0, M_r z_0, M_r M_r z_0, \ldots, (M_r \ldots M_r) z_0 \} \), where \( r_i \in \{1, 2, 3\} \), and the \( r_i \)'s are not necessarily distinct. As a rational function, \( q_\lambda \) has only finitely many poles so that after invoking the procedure, say \( t \) times, an element in the list must be duplicated. That is, for some \( 0 < j < t \), we have \( M_r \ldots M_r z_0 = M_r \ldots M_r z_0 \) which means that \( z_j := M_r \ldots M_r z_0 \) is a fixed point of \( N := M_r \ldots M_{r+1} \). Also, note that since \( |\text{Trace}(M_i)| \geq 2 \) for \( i = 1, 2, 3 \), and since the entries of \( M_i \) are all non-negative, we have \( |\text{Trace}(N)| \geq 2 \). In other words, \( z_j \) is fixed by a parabolic or hyperbolic element of \( G(\sqrt{2}) \). Since, \( z_0 = (M_r \ldots M_r)^{-1} z_j \), Lemma 2.1.3 applies, and thus, \( z_0 \) is fixed by a parabolic or hyperbolic element of \( G(\sqrt{2}) \). But if \( z_0 \) is fixed by a parabolic element, then by Lemma 2.1.4, we have \( z_0 \in \sqrt{2}Q \), which is a contradiction. Thus, \( z_0 \) is fixed by a hyperbolic element of \( G(\sqrt{2}) \).

**Case 2:** \( z_0 \) is a finite non-zero pole of \( q_\lambda \) such that \( z_0 \in \sqrt{2}Q \)

First, by equation 2.1 both \( z_0 \) and \( T z_0 = \frac{z_0}{z_0} \) are poles of \( q_\lambda \) and hence by Lemma 2.1.3, we may assume without loss of generality that \( z_0 > 0 \). Therefore, the arguments given in Case 1 hold for \( z_0 \), except in this case, \( c_r z_0 + d_r \neq 0 \) because \( z_0 > 0 \). Thus, as before, \( z_j \) is fixed by \( N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) which is either an even or an odd element of \( G(\sqrt{2}) \), such that \( |\text{Trace}(N)| \geq 2 \). Recall that \( z_j = M_r \ldots M_r z_0 \). Since \( z_0 > 0 \) is finite, and \( z_0 \in \sqrt{2}Q \), the same can be said of \( z_j \). Because \( z_j \) is finite and is fixed by
\[ N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ then } \gamma \neq 0, \text{ and hence} \]

\[ z_j = \frac{\alpha - \delta \pm \sqrt{(\alpha + \delta)^2 - 4}}{2\gamma} \quad (2.8) \]

because \( \alpha\delta - \beta\gamma = 1 \). By Remark 2.2 if \( N \) is an odd element, then

\[ z_j = \frac{\alpha'\sqrt{2} - \delta'\sqrt{2} \pm \sqrt{2(\alpha' + \delta')^2 - 4}}{2\gamma}, \quad (2.9) \]

or else if \( N \) is an even element, then

\[ z_j = \frac{\alpha - \delta \pm \sqrt{(\alpha + \delta)^2 - 4}}{2\sqrt{2}\gamma'} \quad (2.10) \]

where \( \alpha, \alpha', \gamma, \gamma', \delta, \delta' \in \mathbb{Z} \).

If \( N \) is an odd element, then \( z_j \) is given by equation 2.9, and so

\[ z_j = \sqrt{2} \left( \frac{\alpha' - \delta'}{2\gamma} \right) \pm \sqrt{2} \left( \frac{\sqrt{(\alpha' + \delta')^2 - 2}}{2\gamma} \right). \]

Since \( z_j \in \sqrt{2}\mathbb{Q} \), we must have that \((\alpha' + \delta')^2 - 2\) is a square in \( \mathbb{Z} \), which is impossible, because the difference between a square and 2 cannot equal a square. On the other hand, if \( N \) is an even element, then by equation 2.10, we have

\[ z_j = \sqrt{2} \left( \frac{\alpha - \delta}{4\gamma'} \right) \pm \sqrt{2} \left( \frac{\sqrt{(\alpha + \delta)^2 - 4}}{4\gamma'} \right). \]

Since \( z_j \in \sqrt{2}\mathbb{Q} \), we must have \((\alpha + \delta)^2 - 4\) is a square in \( \mathbb{Z} \). Since the difference between two distinct non-zero squares can never equal 4, this means \((\alpha + \delta)^2 = 4\), so that \((\alpha + \delta)^2 - 4 = 0\). Since \( \alpha, \delta > 0 \), we must have \( \alpha = \delta = 1 \) and in this case implies that \( z_j = 0 \), which is a contradiction. Thus, \( z_0 \notin \sqrt{2}\mathbb{Q} \).
In total, then, if \( z_0 \) is a non-zero finite pole of \( q_\lambda \), then \( z_0 \) must be fixed by a hyperbolic element of \( G(\lambda) \).

In some sense, Theorem 2.1.2 describes what the finite non-zero poles of rational period functions are, namely, the fixed points of hyperbolic elements of \( G(\lambda) \). The following corollary of Theorem 2.1.2 sheds more light on the nature of these poles by describing where they are. But first, Theorem 2.1.2 inspires the following definition.

**Definition 2.1.5** If \( z_0 \) is a finite non-zero pole of an RPF \( q_\lambda \) defined on \( G(\lambda) \) for \( \lambda = 1, \sqrt{2}, \) or \( \sqrt{3} \), then \( z_0 \) is said to be a hyperbolic pole of \( q_\lambda \).

**Corollary 2.1.6** If \( z_0 \) is a hyperbolic pole of an RPF \( q_\lambda \) on \( G(\lambda) \) for \( \lambda = 1, \sqrt{2}, \) or \( \sqrt{3} \), then

(a) \( z_0 \) is a root of a quadratic polynomial of the form \( P(z) = \lambda az^2 + bz + \lambda c \), where \( a, b, c \in \mathbb{Z} \) such that \( a, c \neq 0 \), \( \gcd(a, b, c) = 1 \), and \( b^2 - 4\lambda^2 ac > 0 \). Consequently,

(b) \( z_0 \in \mathbb{Q}(\lambda, \sqrt{N}) \setminus \lambda \mathbb{Q} \), for some positive integer \( N \), where if \( N \) is a square or if \( N = \lambda^2 (N')^2 \) for some positive integer \( N' \), then \( z_0 \in \mathbb{Q}(\lambda) \setminus \lambda \mathbb{Q} \).

**Remark 2.3.**

(i) When \( \lambda = 1 \), Corollary 2.1.6(b) simply restates Knopp’s theorem regarding hyperbolic poles of RPFs defined on \( \Gamma(1) \). That is, such poles must belong to \( \mathbb{Q}(\sqrt{N}) \setminus \mathbb{Q} \), where, in this case, \( N \) is a positive non-square integer, and therefore poles are real quadratic irrational numbers.

(ii) It also follows from [MR, Theorem 2] that the finite poles of RPFs defined on \( G(\sqrt{2}) \) and \( G(\sqrt{3}) \) must be in \( \mathbb{Q}(\lambda, \sqrt{N}) \setminus \lambda \mathbb{Q} \), for some positive integer \( N \).
(iii) The condition that finite non-zero poles of RPFs are fixed points of hyperbolic elements of $G(\lambda)$ is necessary, but not sufficient.

**Proof of Corollary 2.1.6**

(a) Suppose $z_0$ is a hyperbolic pole of $q_\lambda$, an RPF on $G(\lambda)$. By Theorem 2.1.2, $z_0$ is fixed by $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\lambda)$ with $|\alpha + \delta| > 2$. By Remark 2.1, we have $\beta, \gamma \neq 0$, so that the standard computation following from $Mz_0 = z_0$ yields $\gamma z_0^2 + (\delta - \alpha)z_0 - \beta = 0$. If $M$ is an odd element, then by Remark 2.2, we have $\gamma' \lambda z_0^2 + (\delta - \alpha)z_0 - \beta' \lambda = 0$, where $\alpha, \beta', \gamma', \delta, \in \mathbb{Z}$. Then let

$$a = \frac{\gamma'}{\gcd(\gamma', \delta - \alpha, \beta')},$$

$$b = \frac{\delta - \alpha}{\gcd(\gamma', \delta - \alpha, \beta')},$$

and

$$c = \frac{\beta'}{\gcd(\gamma', \delta - \alpha, \beta')}.$$

Therefore, $z_0$ is a root of $P(z) = a\lambda z^2 + bz + c\lambda$ where, $\gcd(a, b, c) = 1$. Moreover,

$$b^2 - 4\lambda^2 ac = \frac{1}{(\gcd(\lambda', \delta - \alpha, \beta'))^2}((\delta - \alpha)^2 - 4\lambda^2 \gamma' \beta')$$

$$= \frac{1}{(\gcd(\lambda', \delta - \alpha, \beta'))^2}((\alpha + \delta)^2 - 4)$$

$$> 0,$$

because $|\text{Trace}(M)| = |\alpha + \delta| > 2$.

On the other hand, if $M$ is an even element, by Remark 2.2, we have $\gamma z_0^2 + \lambda(\delta' - \alpha')z_0 - \beta = 0$, where $\alpha', \beta, \gamma, \delta' \in \mathbb{Z}$. Multiplying through by $\lambda$ yields $\gamma \lambda z_0^2 + \lambda^2(\delta' - \alpha')z_0 - \beta \lambda = 0$. Then we simply proceed as in the case when $M$ is an odd element.
(b) Let \( N = b^2 - 4\lambda^2 ac \). Then from (a), since \( z_0 \) is a root of \( P(z) = a\lambda z^2 + bz + c\lambda \) with \( c \neq 0 \), we have

\[
z_0 = \frac{-b \pm \sqrt{N}}{2a\lambda},
\]

which means that \( z_0 \in Q(\lambda, N) \). Further, by Case 2 in the proof of Theorem 2.1.2, we have \( z_0 \notin \lambda Q \).

In total, if \( z_0 \) is a hyperbolic pole of \( q_\lambda \), then \( z_0 \in Q(\lambda, \sqrt{N}) \setminus \lambda Q \), for some positive integer \( N \).

Corollary 2.1.6 motivates the following definition.

**Definition 2.1.7** Suppose \( 0 \neq z_0 \in Q(\lambda, \sqrt{N}) \setminus \lambda Q \) for some positive integer \( N \). If \( z_0 \) is the root of a quadratic polynomial of the form \( P(z) = a\lambda z^2 + bz + c\lambda \) where \( a, b, c \in Z, ac \neq 0 \) and \( gcd(a, b, c) = 1 \), then \( P(z) \) is said to be an associated quadratic for \( z_0 \).

**Remarks.**

(i) Since \( z_0 \neq 0 \) and \( z_0 \notin \lambda Q \), a priori, an associated quadratic \( P(z) = a\lambda z^2 + bz + c\lambda \) must satisfy \( ac \neq 0 \). Moreover, since \( z_0 \) is real (and \( z_0 \notin \lambda Q \)) we must have \( b^2 - 4\lambda^2 ac > 0 \).

(ii) When \( \lambda = 1 \), an associated quadratic for \( z_0 \) is (up to multiplication by \(-1\)) the minimal polynomial for \( z_0 \), and hence is uniquely determined.

(iii) When \( \lambda = \sqrt{2} \) or \( \sqrt{3} \), associated quadratics are not necessarily minimal polynomials. For example, if \( z_0 = \frac{p}{q} \) is a rational number in lowest terms, then \( q^2\lambda z^2 - p^2\lambda \) is an associated quadratic for \( z_0 \), but is not the minimal polynomial for \( z_0 \). More
importantly, though, associated quadratics are unique up to multiplication by \(-1\), as will be verified in the following proposition.

**Proposition 2.1.8** Suppose \(0 \neq z_0 \in \mathbb{Q}(\lambda, \sqrt{N}) \setminus \lambda \mathbb{Q}\) for some positive integer \(N\). Then an associated quadratic for \(z_0\) (if it exists) is unique up to multiplication by \(-1\).

**Proof:** Since we wish to prove that an associated quadratic for \(z_0\) is unique up to multiplication by \(-1\), it suffices to show that if an associated quadratic for \(z_0\) exists, then there is exactly one associated quadratic for \(z_0\) with a positive lead coefficient.

To this end, by way of contradiction, suppose \(P_1(z) = a_1 \lambda z^2 + b_1 z + c_1 \lambda\) and \(P_2(z) = a_2 \lambda z^2 + b_2 z + c_2 \lambda\) are distinct associated quadratics for \(z_0\) with \(a_1, a_2 > 0\). Without loss of generality, assume that \(a_1\) is minimal with respect to all associated quadratics for \(z_0\) (with positive lead coefficient).

By the Euclidean Algorithm, there are positive integers \(r\) and \(s\) such that \(a_2 = a_1 s + r\) with \(0 \leq r < a_1\). Let \(P(z) = P_2(z) - sP_1(z)\). Then

\[
P(z) = (a_2 - sa_1) \lambda z^2 + (b_2 - sb_1) z + (c_2 - sc_1) \lambda
\]

Since \(z_0\) is a root of \(P_1(z)\) and \(P_2(z)\), \(z_0\) is a root of \(P(z)\). If \(r > 0\), then because \(z_0 \notin \lambda \mathbb{Q}\), the constant term of \(P(z)\) cannot equal 0. Therefore, by the method used in Corollary 2.1.6(a), \(P(z)\) can be normalized to be an associated quadratic for \(z_0\) with positive lead coefficient strictly less than \(a_1\), which contradicts the minimality of \(a_1\). Thus, \(r = 0\) and since \(z_0 \notin \lambda \mathbb{Q}\) and is a root of \(P(z) = P_2(z) - sP_1(z)\), we must have \(P_2(z) - sP_1(z) \equiv 0\), or alternatively, that \(P_2(z) \equiv sP_1(z)\). In that case we have
\[ a_2 = sa_1, \ b_2 = sb_1, \text{ and } c_2 = sc_1. \] Since \( \gcd(a_1, b_1, c_1) = \gcd(a_2, b_2, c_2) = 1 \), we have \( s = 1 \), and hence \( P_1(z) = P_2(z) \), which is a contradiction because \( P_1(z) \) and \( P_2(z) \) are distinct.

Thus, if an associated quadratic for \( z_0 \) exists, it must be unique up to multiplication by \(-1\). \( \square \)

By Corollary 2.1.6, all hyperbolic poles of RPFs defined on \( G(\lambda) \) have associated quadratics. To prove the main results in this chapter and the next, quite often we will study real numbers that are potential poles of RPFs because they are roots of polynomials of the form \[ Q(z) = a\lambda z^2 + bz + c\lambda. \] However, these quadratic polynomials will not necessarily be associated quadratics. That is, it may be the case that \( \gcd(a, b, c) \neq 1 \). These quadratic polynomials provide a means by which we can associate a potential pole of an RPF with a triple of integers, namely \([a, b, c]\). For convenience, then, we write \( Q(z) = [a, b, c] \) in place of \( Q(z) = a\lambda z^2 + bz + c\lambda \) wherever appropriate. Also, we say that \( a \) is the lead coefficient of \( Q(z) \) (as opposed to \( a\lambda \)), \( b \) is the second coefficient of \( Q(z) \), and \( c \) is the constant term (as opposed to \( c\lambda \)).

**Definition 2.1.9** Suppose \( z_0 \) is in \( \mathbf{Q}(\lambda, \sqrt{N})\backslash\mathbf{Q} \), for some positive integer \( N \), and has associated quadratic given by \( P(z) = [a, b, c] \). Define \( \text{disc}(z_0) \) to be the discriminant of the polynomial \( P(z) \). That is, \( \text{disc}(z_0) = b^2 - 4\lambda^2 ac \).

Note that \( \text{disc}(P(z)) = \text{disc}(-P(z)) \), so that by Proposition 2.1.8, \( \text{disc}(z_0) \) is well defined.
In order to make a distinction between the definition of $\text{disc}(z_0)$ and the discriminant of any quadratic polynomial of which $z_0$ is a root, we give the following definition.

**Definition 2.1.10** Suppose $P(z) = rz^2 + sz + t$ is in $\mathbb{R}[z]$. Then $D_{P(z)} = s^2 - 4rt$. In other words, $D_{P(z)}$ is the discriminant of the quadratic polynomial $P(z)$.

If in fact $P(z)$ is an associated quadratic for $z_0$, then $\text{disc}(z_0) = D_{P(z)}$.

**Lemma 2.1.11** Suppose $z_0$ is a root of the polynomial $P(z) = rz^2 + sz + t$ in $\mathbb{R}[z]$, and let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a linear fractional transformation such that $\det(M) = d \neq 0$. Then

(a) $Mz_0$ is a root of the (at most quadratic) polynomial $Q(z) = (P - 2M')(z)$, where $M' = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$, and

(b) $D_{Q(z)} = d^2 D_{P(z)}$.

**Proof:** First note that

\[
(P - 2M')(Mz_0) = (-\gamma(Mz_0) + \alpha)^2 P(M'(Mz_0))
\]

\[
= (-\gamma(Mz_0) + \alpha)^2 P\left( \begin{pmatrix} \alpha \delta - \beta \gamma & 0 \\ 0 & \alpha \delta - \beta \gamma \end{pmatrix} z_0 \right)
\]

\[
= (-\gamma(Mz_0) + \alpha)^2 P\left( \frac{d_0}{d} \right)
\]

\[
= (-\gamma(Mz_0) + \alpha)^2 P(z_0)
\]

\[
= 0,
\]

because $z_0$ is a root of $P(z)$. 

Moreover, \( Q(z) \) is a polynomial of at most degree 2 because

\[
Q(z) = (-\gamma z + \alpha)^2 \left( r \left( \frac{\delta z - \beta}{-\gamma z + \alpha} \right)^2 + s \left( \frac{\delta z - \beta}{-\gamma z + \alpha} \right) + t \right)
\]

\[
= r(\delta z - \beta)^2 + s(\delta z - \beta)(-\gamma z + \alpha) + t(-\gamma z + \alpha)^2
\]

\[
= (r\delta^2 - s\gamma \delta + t\gamma^2)z^2 + (s(\alpha \delta + \beta \gamma) - 2(r\beta \delta + t\alpha \gamma))z + (r\beta^2 - s\alpha \beta + t\alpha^2).
\]

Finally, the second statement follows from the first since

\[
D_{Q(z)} = (s(\alpha \delta + \beta \gamma) - 2(r\beta \delta + t\alpha \gamma))^2 - 4(r\beta^2 - s\beta \alpha + t\alpha^2)(r\delta^2 - s\gamma \delta + t\gamma^2)
\]

\[
= d^2(s^2 - 4rt)
\]

\[
= d^2 D_{P(z)},
\]

as desired. \( \square \)

**Corollary 2.1.12** Suppose \( z_0 \) is a hyperbolic pole of an RPF \( q_\lambda \) on \( G(\lambda) \) for \( \lambda = 1, \sqrt{2}, \) or \( \sqrt{3} \). If an associated quadratic for \( z_0 \) is \( P(z) = [r, s, t] \), then for any \( M \in G(\lambda) \),

(a) an associated quadratic for \( Mz_0 \) exists, and is given by \( Q(z) = (P|_z M^{-1})(z) \),

and consequently,

(b) \( \text{disc}(z_0) = \text{disc}(Mz_0) \).

**Proof:** (a) In the special case that \( M \in G(\lambda) \), we have \( M' = M^{-1} \) (\( M' \) as defined in Lemma 2.1.11). Furthermore, because the entries of \( M \) are in \( \mathbb{Z}[\lambda] \), it is clear that since \( z_0 \in \mathbb{Q}(\lambda, \sqrt{N}) \) then so is \( Mz_0 \). Also, since \( S_\lambda z_0 = z_0 + \lambda \) and \( Tz_0 = \frac{z_0}{z_0} \), we
have $Mz_0 \notin \lambda Q$, because $z_0 \notin \lambda Q$, and because $S\lambda$ and $T$ generate $G(\lambda)$. Therefore, $Mz_0$ satisfies the hypotheses of Definition 2.1.7, and hence may have an associated quadratic.

In fact by writing $M^{-1}$ as a word in $S\lambda$ and $T$, it is easy to show that $Q(z) = (P|_{-M^{-1}})(z)$ is a polynomial of the form $a\lambda z^2 + bz + c\lambda$, where $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and since $Mz_0 \notin \lambda Q$, we have $ac \neq 0$. Specifically, $(P|_{-S\lambda})(z) = [r, 2r\lambda^2 + s, r\lambda^2 + s+t]$ and $(P|_{-T})(z) = [t, -s, r]$, and therefore since the polynomial $P(z)$ when viewed as a triple of integers is relatively prime, then the same can be said of $(P|_{-S\lambda})(z)$ and $(P|_{-T})(z)$, and hence of $Q(z) = (P|_{-M^{-1}})(z)$. Thus, $Q(z)$ is an associated quadratic for $Mz_0$.

(b) By Lemma 2.1.11 (b), since $\det(M) = 1$, we have $D_p = D_Q$, and since both $P(z)$ and $Q(z)$ are associated quadratics, $\text{disc}(z_0) = \text{disc}(Mz_0)$. \hfill \Box

For $G(\lambda) = \Gamma(1)$, with the same hypotheses as in Corollary 2.1.12, statement (b) follows directly from the fact that we may view $P(z)$ and $Q(z)$ as binary quadratic forms. That is, if we let $Q_1(x, y) = ax^2 + bxy + cy^2$ and $Q_2(x, y)$ be the binary quadratic form with the same coefficients as $(P|_{-M^{-1}})(z)$ (so that $P(z) = Q_1(z, 1)$ and $(P|_{-M^{-1}})(z) = Q_2(z, 1)$), then since $Q_1(x, y)$ and $Q_2(x, y)$ are equivalent in the narrow sense, we have that the discriminant of $P(z)$ is the same as the discriminant of $(P|_{-M^{-1}})(z)$. In other words, $\text{disc}(Mz_0) = \text{disc}(z_0)$. (For more information on binary quadratic forms, see [Bu] and [Za].)

Lemma 2.1.13 Let $q$ be a rational period function on $G(\lambda)$ for $\lambda = 1, \sqrt{2}$ or $\sqrt{3}$. 

If \( z_0 \) is a hyperbolic pole of \( q \), then given a fixed prime \( p \), there is a hyperbolic pole \( z_1 \) of \( q \) satisfying \( \text{disc}(z_1) = \text{disc}(z_0) \) and with associated quadratic \([r, s, t]\) such that \( \gcd(r, p) = 1 \). In other words, \( q \) has a hyperbolic pole with associated quadratic whose lead coefficient is relatively prime to \( p \).

**Proof:** Because the proofs for \( \lambda = 1, \sqrt{2}, \) and \( \sqrt{3} \) are analogous, we present only the case \( \lambda = 1 \). To this end, suppose an associated quadratic for \( z_0 \) is \([a, b, c]\). Without loss of generality, assume \( \gcd(a, p) = p \). By functional equation 1.12, \( Tz_0 = \frac{-1}{z_0} \) is a pole of \( q(z) \) and by Corollary 2.1.12 (a) has associated quadratic \([c, -b, a]\). If \( \gcd(c, p) = 1 \), we are done. Otherwise, \( \gcd(c, p) = p \). Now, by functional equation 1.13 either \( (ST)^{-1}z_0 = (ST)^2z_0 \) or \( (ST)^{-2}z_0 = STz_0 \) is a pole of \( q(z) \) (or possibly both). An associated quadratic for \( (ST)^2z_0 \) is \([a + b + c, -2a - b, a]\), and an associated quadratic for \( STz_0 \) is \([c, -2c - b, a + b + c]\). In the first case, since \( \gcd(a, p) = \gcd(c, p) = p \) and \( \gcd(a, b, c) = 1 \), we must have \( \gcd(a + b + c, p) = 1 \). In the second case, consider \( \frac{1}{STz_0} \), which has associated quadratic \([a + b + c, 2c + b, c]\). As before, \( \gcd(a + b + c, p) = 1 \).

By Corollary 2.1.12, in either case, \( q(z) \) has a hyperbolic pole \( z_1 \) such that \( \text{disc}(z_1) = \text{disc}(z_0) \), and whose associated quadratic has lead coefficient relatively prime to \( p \). □

### 2.2 More About the Induced Hecke Operators

For the remainder of this chapter, we restrict our attention to RPFs defined on \( \Gamma(1) \), the modular group, which is the setting for the main result of this chapter.

We begin by taking a more detailed look at the induced Hecke operator \( \hat{T}_{2k}(n) \) defined on the space of RPFs on \( \Gamma(1) \). To this end, in Section 1.4 we saw that for
an entire modular form of weight $2k$, and $n$ a natural number the Hecke operator $T_{2k}(n)$ was defined as follows by

$$T_{2k}(n)f = n^{2k-1} \sum_{ad = n, \quad d > 0, \quad b(\text{mod} \ d)} f|_{2k} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

and in Section 1.5, that the induced Hecke operator $\hat{T}_{2k}(n)$ was given by $\hat{T}_{2k}(n)q = (T_{2k}(n)f)|_{2k}T - T_{2k}(n)f$, where $q$ is the RPF associated with $f$, a modular integral of weight $2k$. From now on, for convenience, we choose the set of representatives of $b(\text{mod} \ d)$ given by $\{0, 1, 2, ..., d-1\}$, and write $T(n)$ and $\hat{T}(n)$ in place of $T_{2k}(n)$ and $\hat{T}_{2k}(n)$ respectively.

By following [Ap, pp. 120-124], we can write the induced Hecke operator as follows. First, define $\Gamma_n = \{M_{b,d} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a, b, d \in \mathbb{Z}, a, b, d \geq 0, 0 \leq b < d, ad = n\}$. Then for each $M_{b,d}$ in $\Gamma_n$, there is a unique $V_{b,d}$ in $\Gamma(1)$ and a unique $M_{b',d'}$ in $\Gamma_n$ such that $M_{b,d}T = V_{b,d}M_{b',d'}$. For example, if $M_{b,d} = M_{0,n} = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, then since

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}T = T\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix},$$

we have $M_{b',d'} = M_{0,1}$ and $V_{b,d} = T$.

Next, examine one term of the sum representing $\hat{T}(n)q(z)$ corresponding to a fixed $M_{b,d}$. Let $V_{b,d}$ and $M_{b',d'}$ be the unique elements of $\Gamma(1)$ and $\Gamma_n$, respectively, satisfying $M_{b,d}T = V_{b,d}M_{b',d'}$. Since $V_{b,d}$ is in $\Gamma(1)$, and $\Gamma(1)$ is generated by $S$ and $T$, we can write $V_{b,d} = S^{\alpha_1}TS^{\alpha_2}...TS^{\alpha_r}$, where $\alpha_i \in \mathbb{Z}$. Then for the RPF $q(z)$, the induced Hecke operator can be written as follows:

$$\hat{T}(n)q = (T(n)f)|_{2k}T - T(n)f$$
\[ f_{2k}S = f \]

Since \( f_{2k}S = f \), we have

\[
\sum_{M \in \Gamma(n)} f_{2k} V_{b,d} M_{b,d'} - T(n) f
\]

\[ = n^{2k-1} \sum_{M \in \Gamma(n)} f_{2k} V_{b,d} M_{b,d'} - T(n) f. \]

where \( q_{b,d} \) is the sum of terms of the form \( q_{2k} M, M \in \Gamma(1) \). In total, then,

\[
\hat{T}(n)q = n^{2k-1} \sum_{M \in \Gamma(n)} (f + q_{b,d})_{2k} M_{b,d'} - T(n) f
\]
\[ = T(n)f + n^{2k-1} \sum_{M_{b,d} \in \Gamma_n} q_{b,d}|_{2k} M_{b',d'} - T(n)f \]

\[ = n^{2k-1} \sum_{M_{b,d} \in \Gamma_n} q_{b,d}|_{2k} M_{b',d'}. \]

**Remark 2.4.** For the proof of the main result, it is important to note that when \( z_0 \) is a pole of \( q(z) \), then \( nz_0 \) is a potential pole of \( \hat{T}(n)q(z) \). To see this, recall that when \( M_{b,d} = M_{0,1} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \) we must have \( V_{b,d} = V_{0,1} = T \) and \( M_{b',d'} = M_{0,n} = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \). In particular, the term \( q_{b,d}|_{2k} M_{b',d'} \) of \( \hat{T}(n)q \) corresponding to \( M_{0,1} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \) is \( n^{-2k} q(\frac{z}{n}) \) and is arrived at as follows:

\[
q_{b,d}|_{2k} M_{b',d'} = f|_{2k} V_{b,d} M_{b',d'} - f|_{2k} M_{b',d'}
\]

\[
= f|_{2k} T \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} - f|_{2k} \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}
\]

\[
= (f + q)|_{2k} \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} - f|_{2k} \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}
\]

\[
= q|_{2k} \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}
\]

\[
= n^{-2k} q(\frac{z}{n}).
\]

Therefore, \( n^{-2k} q(\frac{z}{n}) \) is one term in the sum which represents \( \hat{T}(n)q(z) \) and hence \( nz_0 \) is a potential pole of \( \hat{T}(n)q(z) \).

It is the above description of the induced Hecke operator \( \hat{T}(n) \), and Remark 2.4 which provide the foundation for the proof of the main result. For further details, see also [Kn1].
2.3 The Main Theorem

We are now ready to prove Theorem 2.3.1, which we restate using the terminology developed in Section 2.1.

**Theorem 2.3.1** Let \( q(z) \) be a rational period function with at least one hyperbolic pole. Then \( q(z) \) is not an eigenfunction of the induced Hecke operator \( \hat{T}(n) \) for any \( n > 1 \).

The idea of the proof is to show that the RPF \( \hat{T}(n)q(z) \) has a hyperbolic pole \( x_0 \) which is not a pole of \( q(z) \). In particular, for any hyperbolic pole \( z_p \) of \( q(z) \), \( x_0 \) will have the property that \( \text{disc}(x_0) > \text{disc}(z_p) \). The proof is given in three steps: for \( n = p^s \), where \( p \) is an odd prime and \( s \) is a positive integer; for \( n = 2^s \); and finally for any integer \( n > 1 \), where we use the multiplicative property of the operator \( \hat{T}(n) \).

In the proof of Theorem 2.3.1 we need two lemmas.

**Lemma 2.3.2** Suppose \( \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) \) is in \( \Gamma_{p^s} \), where \( p \) is prime, and \( s \) is a positive integer. Suppose \( z_0 \) is a hyperbolic pole of an RPF on \( \Gamma(1) \) with associated quadratic \( Q(z) = [a, b, c] \), satisfying \( \text{disc}(z_0) = D \). Then \( \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) p^s z_0 \) has an associated quadratic such that \( \text{disc} \left[ \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^s z_0) \right] \) is at most \( p^{4s} D \).

**Proof:** Let

\[
X = \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^s z_0)
\]

\[
= \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) \left( \begin{array}{cc} p^s & 0 \\ 0 & 1 \end{array} \right) (z_0)
\]
\[
\begin{pmatrix}
a'p^* & b' \\
0 & d'
\end{pmatrix} (z_0).
\]

First, since \(a', b', d', p^* \in \mathbb{Z}\) with \(a'd' \neq 0\), and since \(z_0\) is a real quadratic irrational number (see Remark 2.3(i)), we must have that \(X\) is a real quadratic irrational number. Thus, \(X\) satisfies the hypotheses of Definition 2.1.7, and hence has an associated quadratic, which, in this case, is actually the minimal polynomial.

Next, by Lemma 2.1.11 (a) and (b), since the determinant of \(\begin{pmatrix}
a'p^* & b' \\
0 & d'
\end{pmatrix}\) is \(p^{2s}\), we have that \(X\) is a root of \(P(z) = [a(d')^2, d'(-2ab' + a'p^*b), a(b')^2 - ba'b'p^* + (a')^2p^{2s}c]\) and \(D_P(z) = p^{4s}D_Q(z)\). That is, \(X\) is a root of a quadratic polynomial with integer coefficients that has discriminant \(p^{4s}D_Q(z)\). Note that since \(X \notin \mathbb{Q}\), by necessity, neither the lead coefficient nor the constant term of \(P(z)\) may be zero. Therefore, an associated quadratic for \(X\) is \(P(z)\) or else a normalization of \(P(z)\), and thus \(\text{disc}(X)\) is at most \(p^{4s}D_Q(z)\).

\[\square\]

**Lemma 2.3.3** Suppose \(z_0\) is a hyperbolic pole of an RPF defined on \(\Gamma(1)\) with associated quadratic \([a, b, c]\), such that \(\gcd(a, p) = 1\). Then \(p^*z_0\) has an associated quadratic which satisfies \(\text{disc}(p^*z_0) > \text{disc}(z_0)\).

**Proof:** It is straightforward to see that \(p^*z_0\) is a root of the quadratic polynomial \([a, bp^*, p^{2s}c]\). Moreover, since \(\gcd(a, p, c) = 1\) and \(ac \neq 0\), this quadratic polynomial is an associated quadratic for \(p^*z_0\). Thus, \(\text{disc}(p^*z_0) = p^{2s}(b^2 - 4ac) = p^{2s}\text{disc}(z_0)\), and hence, \(\text{disc}(p^{2s}z_0) > \text{disc}(z_0)\). \[\square\]
Proof of Theorem 2.3.1: Recall that we intend to show that $\hat{T}(n)q(z)$ has a pole which cannot be a pole of $q(z)$.

Step 1: $n = p^s$, $p$ an odd prime and $s$ a positive integer

Let $z_1$ be a hyperbolic pole of $q(z)$ such that $\text{disc}(z_1)$ is maximal with respect to all hyperbolic poles of $q(z)$. By Lemma 2.1.13 we can find a hyperbolic pole of $q(z)$, $z_0$, with associated quadratic $[a, b, c]$ satisfying $\gcd(a, p) = 1$, and such that $\text{disc}(z_1) = \text{disc}(z_0)$. For convenience, let $D = \text{disc}(z_0)$. In the alternative description of the induced Hecke operator $\hat{T}(p^s)$, we had

$$\hat{T}(p^s)q = (p^s)^{2k-1} \sum_{M' \in \Gamma_n} q_{b, d} |_{2k} M_{b', d'}$$

so that $\hat{T}(p^s)q$ is the sum of terms of the form $q|_{2k} M_{b', d'}$ with $M \in \Gamma(1)$ and $M_{b', d'} \in \Gamma_n$. In order to follow the initial reasoning in Knopp's original argument for the RPF $\left( \frac{1}{(x^2-x-1)^2} + \frac{1}{(x^2-x)(x-1)^2} \right)$ [Kn1], let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $M_{b', d'} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$. Then

$$q|_{2k} MM_{b', d'} = (\gamma a'z + \gamma b' + \delta d')^{-2k}q \left( \frac{\alpha a'z + \alpha b' + \gamma d'}{\gamma a'z + \gamma b' + \delta d'} \right).$$

Since the factor $(\gamma a'z + \gamma b' + \delta d')^{-2k}$ contributes only rational poles, a hyperbolic pole of $\hat{T}(p^s)q$ must be a pole of $q(MM_{b', d'}z) = q \left( \frac{\alpha a'z + \alpha b' + \gamma d'}{\gamma a'z + \gamma b' + \delta d'} \right)$. In other words, in order for $z_0$ to be a hyperbolic pole of $\hat{T}(p^s)q$, we must have that $MM_{b', d'}z_0 = z_2$, where $z_2$ is a hyperbolic pole of $q$. Specifically, by Remark 2.4, we know that $p^s z_0$ is a pole of the term of $\hat{T}(p^s)q(z)$ arising from $M_{b', d'} = \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix}$ and $V_{b, d} = T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is enough to show that no other term of $\hat{T}(p^s)q(z)$ has a pole at $p^s z_0$ so that cancellation does not occur. In particular, by the maximality of $\text{disc}(z_0)$ and by Lemma 2.3.3, we will have shown that $\hat{T}(n)q(z)$ has a pole, namely $p^s z_0$, which is not a pole of $q(z)$.  


To this end, suppose there exists \( M = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \Gamma(1) \) and \( M_{d',d''} = \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) \in \Gamma_{p^*} \) with
\[
\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) = z_2,
\]
where \( z_2 \) is a hyperbolic pole of \( q(z) \). We will show that that \( M_{d',d''} = \left( \begin{array}{cc} 1 & 0 \\ 0 & p^* \end{array} \right) \) which by Remark 2.4 implies that \( M = T V_{b,d} = T T = I \), and therefore that \( z_2 = z_0 \).

Since \( z_2 \) is a hyperbolic pole of \( q(z) \), Corollary 2.1.12 (b) applies, and hence
\[
\text{disc}(z_2) = \text{disc} \left[ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) \right] = \text{disc} \left[ \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) \right].
\]

Moreover, by Lemma 2.3.2 \( \text{disc}(z_2) = \text{disc} \left[ \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) \right] \) is at most \( p^*D \). Since \( D \) is maximal, we must have \( \text{disc} \left[ \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) \right] \leq D \). Recall that in the computation given in Lemma 2.3.2, we saw that \( X = \left( \begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right) (p^*z_0) \) was a root of \( P(z) = [a(d')^2, d'(-2ab' + a'p^*b), a(b')^2 - ba'b'p^* + (a')^2p^{2*}c] \), and \( D_{P(a)} = p^*D \). Therefore, in order for \( z_2 \) to be a pole of \( q(z) \), we must have \( P(z) = mQ(z) \), for some integer \( m \) with \( m \geq p^{2*} \), where \( Q(z) \) is an associated quadratic for \( z_2 \) and necessarily has discriminant less than or equal to \( D \). In fact, we show that \( m = p^{2*} \) as follows. First, we argue that the only prime number that can divide all of the coefficients of \( P(z) \) is \( p \). For, if there is a prime number \( r \) distinct from \( p \) such that \( r \) divides all of the coefficients of \( P(z) \), then in particular \( r \) must divide \( a(d')^2 \), the lead coefficient. But \( a'd' = p^* \), implies that \( d' \) and \( a' \) are both powers of \( p \), and hence \( r|a \). Now if \( r|d''(-2ab' + a'p^*b) \), the second coefficient of \( P(z) \), then since \( \gcd(r, d') = \gcd(r, a') = \gcd(r, p) = 1 \), and \( r|a \), we have \( r|b \). A similar argument for the constant term of \( P(z) \) shows that \( r|c \). Recall that \( a, b, \) and \( c \) are the coefficients
for an associated quadratic, so that \( \gcd(a, b, c) = 1 \). Therefore, we have shown that \( p \) is the only prime number that can divide each coefficient of \( P(z) \), and hence we have that \( P(z) = p^t Q(z) \), where \( p^t \geq p^{2*} \) for some positive integer \( t \). Thus, each coefficient of \( P(z) \) must be divisible by, at the least, \( p^{2*} \), or possibly a higher power of \( p \). But since \( \gcd(a, p) = 1 \) and, by the definition of \( \hat{T}(p^*) \), we have \( 0 \leq d' \leq p^{2*} \), the largest power of \( p \) which can divide \( a(d')^2 \) is \( p^{2*} \). That is, since the discriminant of \( P(z) \) is \( p^{4*}D \), we have \( p^t \leq p^{2*} \), and so \( P(z) = p^{2*} Q(z) \). Therefore, \( p^{2*} \) must divide each coefficient of \( P(z) \). In the case of the lead coefficient, because \( \gcd(a, p) = 1 \), we have \( p^{2*}(d')^2 \) which gives \( d' = p^* \) and hence \( a' = 1 \). Then the second coefficient of \( P(z) \), which is also divisible by \( p^{2*} \), becomes \(-2a'b'p^* + p^{2*}b' \), so that \( p^{2*} \mid (-2a'b'p^* + p^{2*}b') \), which implies \( p^* \mid b' \). Finally, since \( 0 \leq b' < p^* \), we have \( b' = 0 \).

The above computations show that the only term of \( \hat{T}(p^*)q(z) \) for which \( p^*z_0 \) is a pole arises from \( M_{b'd'} = \begin{pmatrix} 1 & 0 \\ 0 & p^* \end{pmatrix} \) and \( M = I \), as desired.

**Step 2**: \( n = 2^* \)

By the arguments presented in Step 1, we have \( a' = 1 \) and \( d' = 2^* \). It remains to show that \( b' = 0 \). It is still true that \( 2^{2*} \mid (-2a'b'2^* + 2^{2*}b) \) so that \( 2^{2*} \) divides \( b' \). This leaves only two possibilities, namely that \( b' = 0 \) or else \( b' = 2^{2*} \). In the first instance, we are done; in the second, examine the constant term of \( P(z) \) which is \( a2^{2*} - b2^{2*} + 2^{2*}c \). As before, \( 2^{2*} \) must divide this coefficient, but this is impossible since \( \gcd(a, 2) = 1 \). Thus, \( b' = 0 \), as desired.

Note that the proof of Steps 1 and 2 produces a hyperbolic pole \( x_0 \) of \( \hat{T}(p^*)q(z) \) with the property that for any hyperbolic pole \( z_p \) of \( q(z) \), \( \text{disc}(x_0) > \text{disc}(z_p) \).
Step 3: \( n > 1. \)

Let \( n = p_t^{s_t}p_{t-1}^{s_{t-1}}...p_1^{s_1}, \) where \( p_1, p_2, ..., p_t \) are distinct primes, and \( s_1, s_2, ..., s_t \) are positive integers. By the multiplicative property of operator \( \hat{T}(n) \), we have

\[
\hat{T}(n)q(z) = \hat{T}(p_t^{s_t})\hat{T}(p_{t-1}^{s_{t-1}})...\hat{T}(p_1^{s_1})q(z).
\]

Let \( h_i(z) \) be the RPF given by

\[
h_i(z) = \hat{T}(p_t^{s_t})\hat{T}(p_{t-1}^{s_{t-1}})...\hat{T}(p_1^{s_1})q(z).
\]

for \( i = 1, 2, ..., t. \) By the proof of Steps 1 and 2, for each \( i, h_i(z) \) has a hyperbolic pole \( x_i \) with the property that for any hyperbolic pole \( z_{p_i} \) of \( h_{i-1}(z), \text{disc}(x_i) > \text{disc}(z_{p_i}). \)

In particular, \( h_t(z) = \hat{T}(n)q(z) \) has a pole which is not a pole of \( q(z). \) Thus, \( q(z) \) is not an eigenfunction of \( \hat{T}(n) \) for any \( n > 1. \) This completes the proof of the theorem.

\( \square \)

**Corollary 2.3.4** Suppose \( q(z) \) is a rational period function with at least one hyperbolic pole. Then the RPF \( \hat{T}(n)q \) has a hyperbolic pole \( X_0 \) with the property that for every hyperbolic pole \( z_0 \) of \( q, \text{disc}(X_0) > \text{disc}(z_0). \)

**Proof:** This follows directly from Case 3 in the proof of Theorem 2.3.1.

\( \square \)

In the next chapter we will prove an analogous theorem for RPFs defined on \( G(\sqrt{2}) \) and \( G(\sqrt{3}). \)
3.1 The Hecke Groups $G(\lambda_n)$

In Chapter 2 we saw the modular group $\Gamma(1)$ as one of the infinite family of Hecke groups $G(\lambda_n) = \langle S_{\lambda_n}, T \rangle$, where $S_{\lambda_n} = \begin{pmatrix} 1 & 2\cos \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In particular, $\Gamma(1) = G(\lambda_3)$. The Hecke groups are interesting in a variety of contexts.

For example, one intriguing topic is the notion of generalizing on the idea of a simple (infinite) continued fraction expansion. That is, it is well known that a real number $\alpha$ has a periodic continued fraction expansion if and only if $\alpha$ is a real quadratic irrational number [Za, pp. 57-95]. The connection with real quadratic irrational numbers, continued fraction expansions, and the modular group is that any real quadratic irrational number is the fixed point of a hyperbolic $M$ element of $\Gamma(1)$, and the entries of $M$ together with the Euclidean Algorithm yield the periodic continued fraction expansion [Bu, pp. 35-48].

A generalization of continued fraction expansion, in the sense that $\Gamma(1)$ is replaced by $G(\lambda_n)$, is given by D. Rosen in [Ro1]. The classification of the periodic continued fraction expansions in the more general context is a much more difficult problem, and has applications in diophantine approximation. Using results of Rosen, J. Lehner in
[Le3] and [Le4], has addressed the generalized notion of diophantine approximation. For other results using Rosen's continued fraction expansions, see also [Ro2], [RS] and [SS].

Another well-researched topic is the study of Fourier coefficients of automorphic forms, and for our purposes, those forms defined on $G(\lambda_n)$. In analogy to results obtained for the Fourier coefficients of modular forms, there is a variety of papers on estimates of Fourier coefficients of automorphic forms defined on $G(\lambda_n)$. For a sampling of these results, see, for instance, [Ak], [Pa3], and [Ra].

Our main interest, at present, is that RPFs are defined on $G(\lambda_n)$, and this is the focus of the remainder of this chapter.

### 3.2 Automorphic Integrals and Rational Period Functions

In Section 2.1, we saw that rational period functions could be defined on the Hecke groups $G(\lambda_n) = \langle S_{\lambda_n}, T \rangle$ for $n \geq 3 \in \mathbb{Z}$. To make this more precise, we give a definition.

**Definition 3.2.1** Suppose $f$ is meromorphic in $\mathcal{H}$ for $n \geq 3$ in $\mathbb{Z}$, and satisfies

$$(f|_{2kS_{\lambda_n}})(z) = f(z)$$

and

$$(f|_{2kT})(z) = f(z) + q(z),$$

where $k$ is an integer and $q(z)$ is a rational function. If, in addition, $f$ is meromorphic at $i\infty$, then $f$ is an automorphic integral of weight $2k$ with associated rational period function (abbreviated as RPF) $q(z)$. 

Under such circumstances, we say that $q(z)$ is an RPF of weight $2k$ on $G(\lambda_n)$. If $q \equiv 0$, then $f$ is an automorphic form of weight $2k$. In the special case that $n = 3$, since $G(\lambda_3) = \Gamma(1)$, $f$ is a modular integral, and if, in addition, $q \equiv 0$, then $f$ is a modular form.

To give substance to the notion of RPFs defined on $G(\lambda_n)$, we note that Parson and Rosen in [PR] give an infinite family of (non-trivial) RPFs for each group $G(\lambda_n)$ as follows:

$$q_n(z) = \frac{1}{(z^2 - bz - 1)^k} + \frac{1}{(z^2 + bz - 1)^k},$$

where $k \geq 1$ is an odd integer, and $b = \frac{\lambda_n + \sqrt{\lambda_n^2 + 4}}{2} - \frac{\frac{2}{\lambda_n + \sqrt{\lambda_n^2 + 4}}}{\lambda_n + \sqrt{\lambda_n^2 + 4}}$.

In this chapter, we obtain an analogue to Theorem 2.3.1 for RPFs defined on $G(\lambda_4) = G(\sqrt{2})$ and $G(\lambda_6) = G(\sqrt{3})$.

We must restrict ourselves to the two groups $G(\sqrt{2})$ and $G(\sqrt{3})$ essentially because of the existence of Hecke operators on the space of automorphic integrals of a given weight, which, in turn, induce operators on the corresponding space of RPFs. To better understand why this is so, we give a definition.

**Definition 3.2.2** Suppose $G_1$ and $G_2$ are subgroups of a group $G$ such that for some $g, h \in G$, $[G_1 : g(G_1 \cap G_2)g^{-1}] < \infty$ and $[G_2 : h(G_1 \cap G_2)h^{-1}] < \infty$ (i.e., $g(G_1 \cap G_2)g^{-1}$ and $h(G_1 \cap G_2)h^{-1}$ are of finite index in $G_1$ and $G_2$ respectively). Then $G_1$ is said to be commensurable with $G_2$.

The Hecke groups, $G(\lambda_n)$, are subgroups of $SL(2, \mathbb{R})$. Leutbecher, in [Le], showed that the only Hecke groups which are pairwise commensurable are $\Gamma(1)$, $G(\sqrt{2})$, and...
and $G(\sqrt{3})$. In [BK], J. Bogo and W. Kuyk used the pairwise commensurability of $\Gamma(1)$, $G(\sqrt{2})$, and $G(\sqrt{3})$ to show the existence of, and subsequently define Hecke operators on the space of automorphic forms on $G(\sqrt{2})$ and $G(\sqrt{3})$. Implicit in their construction of Hecke operators was the use of map $\psi_\lambda$, defined by Hecke, which maps the space of automorphic forms on $G(\lambda)$, of weight $2k$ for $\lambda = \sqrt{2}$ or $\sqrt{3}$, to the space of modular forms of the same weight.

In [PR], A. Parson and K. Rosen applied results of [BK] to the space of automorphic integrals and the corresponding space of associated rational period functions. By doing so, they created new modular integrals and rational period functions defined on the modular group, from automorphic integrals and RPFs defined on $G(\sqrt{2})$ and $G(\sqrt{3})$.

Moreover, it is straightforward to see that the Hecke operators defined in [BK] also act as operators on the space of automorphic integrals of weight $2k$, and Parson and Rosen showed that these Hecke operators induce operators on the corresponding space of RPFs. Therefore, in view of Theorem 2.1.2, that the finite non-zero poles of RPFs on $G(\sqrt{2})$ and $G(\sqrt{3})$ are fixed points of hyperbolic elements of $G(\sqrt{2})$ and $G(\sqrt{3})$, and in light of the fact that induced Hecke operators exist on these spaces, we are irresistibly drawn to an analogue of Theorem 2.3.1.

To this end, the results mentioned above which are relevant to the results of this chapter are summarized in the following theorem and the next two definitions, which can be found, collectively, in [PR] and [BK].
Theorem 3.2.3 For $\lambda = \sqrt{2}$ or $\lambda = \sqrt{3}$,

(a) if $f_\lambda$ is an automorphic integral of weight $2k$ on $G(\lambda)$, then $\psi_\lambda(f_\lambda)$ is a modular integral of weight $2k$, where

$$\psi_\lambda(f_\lambda) = f_\lambda(\lambda z) + \lambda^{-2k} \sum_{t=0}^{\lambda^2-1} f_\lambda \left( \frac{z + t}{\lambda} \right),$$

and

(b) if $q_\lambda$ is the RPF associated with $f_\lambda$, then $\hat{\psi}(q_\lambda)$ is an RPF on $\Gamma(1)$, where

$$\hat{\psi}(q_\lambda) = (\psi_\lambda(f_\lambda))_{2k} T - \psi_\lambda(f_\lambda)$$

$$= q_\lambda(\sqrt{2}z) + (\sqrt{2})^{-2k} q_\lambda \left( \frac{z}{\sqrt{2}} \right)$$

$$+ (\sqrt{2})^{-2k} q_\lambda \left( \frac{z - 1}{\sqrt{2}} \right) + (1 - z)^{-2k} q_\lambda \left( \frac{\sqrt{2}z}{1 - z} \right)$$

if $\lambda = \sqrt{2}$, and

$$\hat{\psi}(q_\lambda) = q_\lambda(\sqrt{3}z) + (\sqrt{3})^{-2k} q_\lambda \left( \frac{z}{\sqrt{3}} \right) + (\sqrt{3})^{-2k} q_\lambda \left( \frac{z - 1}{\sqrt{3}} \right) + (\sqrt{3})^{-2k} q_\lambda \left( \frac{z + 1}{\sqrt{3}} \right)$$

$$+ (z + 1)^{-2k} q_\lambda \left( \frac{\sqrt{3}z}{z + 1} \right) + (z - 1)^{-2k} q_\lambda \left( \frac{\sqrt{3}z}{z - 1} \right),$$

if $\lambda = \sqrt{3}$.

Definition 3.2.4 For $\lambda = \sqrt{2}$ or $\sqrt{3}$ and $f_\lambda$ an automorphic integral of weight $2k$ on $G(\lambda)$, the Hecke operators $T_\lambda(n)$ are are defined as follows.

(a) If $\lambda^2 \nmid n$, then

$$T_\lambda(n)f_\lambda = n^{2k-1} \sum_{\substack{ad = n \\ d > 0 \\ b(\text{mod } d)}} d^{-2k} f_\lambda \left( \frac{az + b\lambda}{d} \right)$$
\[ \sum_{ad = n, \quad d > 0} f_\lambda|_{2k} \begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix}, \]

(b) if \( n = \lambda^2 \),

\[ T_\lambda(\lambda^2)f_\lambda = (\lambda^2)^{2k-1} \sum_{ad = \lambda^2, \quad d > 0} f_\lambda|_{2k} \begin{pmatrix} a & b\lambda \\ 0 & d \end{pmatrix} + \sum_{t=1}^{\lambda^2-1} f_\lambda|_{2k} \begin{pmatrix} \lambda t \\ \lambda \\ 0 \end{pmatrix}, \]

and

(c) if \( n = (\lambda^2)^r \) for some integer \( r > 1 \),

\[ T_\lambda((\lambda^2)^{r+1})f_\lambda = T_\lambda(\lambda^2)T_\lambda((\lambda^2)^r)f_\lambda - (\lambda^2)^k T_\lambda((\lambda^2)^r)f_\lambda - (\lambda^2)^{2k-1} T_\lambda((\lambda^2)^{r-1})f_\lambda. \]

As was the case for the spaces of modular forms and modular integrals, the Hecke operators defined on the space of automorphic integrals on \( G(\lambda) \) are multiplicative (see [PR] and [BK]), and are independent of the choice of representatives \( b(\text{mod } d) \).

Therefore, we again choose the set of representatives of \( b(\text{mod } d) \) given by \( \{0, 1, \ldots, d-1\} \). Moreover, in analogy to Definition 1.5.2, the Hecke operators in Definition 3.2.4 induce operators on the corresponding space of RPFs, which are given as follows.

**Definition 3.2.5** For \( \lambda = \sqrt{2} \) or \( \sqrt{3} \), if \( f_\lambda \) is an automorphic integral of weight \( 2k \) on \( G(\lambda) \) with associated RPF \( q_\lambda \), then \( \hat{T}_\lambda(n)q_\lambda = (T_\lambda(n)f_\lambda)|_{2k}T - T_\lambda(n)f_\lambda. \)

It is worth noting that the induced Hecke operators inherit the multiplicative property of \( T_\lambda(n) \), as well as the recursion formula in Definition 3.2.4 (c).
We now have enough information to state the main theorem of this chapter, which is a direct analogue of Theorem 2.3.1.

**Theorem 3.4.4** If $q_\lambda$ is a rational period function with at least one hyperbolic pole, then $q_\lambda$ is not an eigenfunction of $\hat{T}_\lambda(n)$ for any $n > 1$.

The proof of Theorem 3.4.4 will be a proof by contradiction, in which we will use Theorem 2.3.1 and the fact that 'essentially' $\hat{\psi}(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}(q_\lambda)$. However, in order to apply Theorem 2.3.1 in such a proof, we must guarantee that if $q_\lambda$ has a hyperbolic pole, then so does $\hat{\psi}(q_\lambda)$. This is entirely the purpose of the next section.

### 3.3 The Poles of $\hat{\psi}(q_\lambda)$

For the remainder of this chapter, assume that $\lambda = 1, \sqrt{2},$ or $\sqrt{3}$, and that all automorphic integrals and RPFs are of weight $2k$, $k$ is a positive integer, unless otherwise specified.

The goal of this section is to prove the following proposition.

**Proposition 3.3.1** For $\lambda = \sqrt{2}$ or $\sqrt{3}$, suppose $q_\lambda$ is an RPF defined on $G(\lambda)$ with a hyperbolic pole. If $\hat{\psi}_\lambda$ is the map defined in Theorem 3.2.3 (b), then $\hat{\psi}_\lambda(q_\lambda)$, an RPF defined on $\Gamma(1)$, has a hyperbolic pole.

Before proving Proposition 3.3.1 we give one lemma.

**Lemma 3.3.2** Suppose $q_\lambda$ is an RPF on $G(\sqrt{2})$ with a hyperbolic pole $z_0$ such that $\text{disc}(z_0) \equiv 1(\text{mod } 2)$. Then there is a hyperbolic pole $z_1$ of $q_\lambda$ satisfying $\text{disc}(z_1) =$
disc(z_0) with associated quadratic \([r, s, t]\) such that \(2|r\). In other words, \(q_\lambda\) has a hyperbolic pole with associated quadratic whose lead coefficient is divisible by 2.

Lemma 3.3.2 is quite useful in proving Proposition 3.3.1 for the case \(\lambda = \sqrt{2}\). Unfortunately, the analogous lemma for \(\lambda = \sqrt{3}\) is false, as evidenced by the following example. For \(k\) a positive odd integer, let

\[
q(z) = \frac{1}{(\sqrt{3}z^2 - z - \sqrt{3})^k} + \frac{1}{(\sqrt{3}z^2 + z - \sqrt{3})^k}
\] (3.2)

First, we verify that \(q\) is an RPF of weight \(2k\) on \(G(\sqrt{3})\) by checking functional equations 2.1 and 2.2 from Section 2.1.

For convenience, we use the notation introduced in Section 2.1 after Proposition 2.1.8 to write \(q(z) = [1, -1, 1]^{-k} + [1, 1, -1]^{-k}\). In that case, we must show

\[
\sum_{i=1}^{5} [1, -1, -1]^{-k} |_{2k(ST)^i} + \sum_{i=1}^{5} [1, 1, -1]^{-k} |_{2k(ST)^i} = -([1, -1, 1]^{-k} + [1, 1, -1]^{-k}).
\] (3.3)

To this end, note that since \(k\) is odd, we have \(-[a, b, c]^{-k} = [-a, -b, -c]^{-k}\). Hence, it is straightforward to see that \(q\) satisfies 2.1, and therefore, we check only 2.2. To this end,

\[
\sum_{i=1}^{5} [1, -1, -1]^{-k} |_{2k(ST)^i} + \sum_{i=1}^{5} [1, 1, -1]^{-k} |_{2k(ST)^i} = \sum_{i=1}^{5} [1, -1, -1]^{-k} |_{2k(ST)^i}
\]

\[
= [1, -5, 1]^{-k} + [-1, -1, 1]^{-k} + [-3, 7, -1]^{-k} + [-3, 11, -3]^{-k} + [-1, 7, -3]^{-k}
\]

\[
+ [3, -7, 1]^{-k} + [3, -11, 3]^{-k} + [1, -7, 3]^{-k} + [-1, 1, 1]^{-k} + [-1, 5, -1]^{-k}
\]

\[
= -([1, -1, 1]^{-k} + [1, 1, -1]^{-k}).
\]
as desired.

Next, note that the poles of $q$ are $\frac{1 \pm \sqrt{3}}{2\sqrt{3}}$ and $\frac{-1 \pm \sqrt{13}}{2\sqrt{3}}$ which have associated quadratics $[1, -1, -1]$ and $[1, 1, -1]$, respectively, neither of which have lead coefficient divisible by 3, and both of which have discriminant equal to 13 ($\equiv 1 \pmod{3}$).

Therefore, we must resort to another technique, an algorithm, in the proof of Proposition 3.3.1 for the case $\lambda = \sqrt{3}$.

**Proof of Lemma 3.3.2** Suppose an associated quadratic for $z_0$ is $[a, b, c]$. Without loss of generality, assume $2 \nmid a$. By functional equation 2.1, $Tz_0 = \frac{-1}{z_0}$ is a pole of $q_\lambda$ which, by Corollary 2.1.12 (a) has associated quadratic $[c, -b, a]$. If $2 | c$, we are done. Otherwise, $gcd(c, 2) = 1$. Now by functional equation 2.2 and the fact that $z_0 \notin \sqrt{2} \mathbb{Q}$ (Corollary 2.1.6 (b)), at least one of $(S_{\sqrt{2}}T)^3z_0$, $(S_{\sqrt{2}}T)^2z_0$, or $(S_{\sqrt{2}}T)z_0$ is a pole of $q_\lambda$, and by Corollary 2.1.12 (a), these potential poles have associated quadratics $[2a + b + c, -4a - b, a]$, $[a + b + 2c, -4a - 3b - 4c, 2a + b + c]$, and $[c, -b - 4c, a + b + 2c]$, respectively. By hypothesis, $disc(z_0) = b^2 - 8ac$ is odd, so that $b$ is odd. Thus, since $a$ and $c$ are odd, the first two associated quadratics have even lead coefficients, and therefore if either $(S_{\sqrt{2}}T)^3z_0$ or $(S_{\sqrt{2}}T)^2z_0$ is a pole, we are done. Otherwise, $(S_{\sqrt{2}}T)z_0$ is a pole whose associated quadratic has even constant term, and in that case we need only use $\frac{-1}{(S_{\sqrt{2}}T)z_0}$ which has associated quadratic $[a + b + 2c, b + 4c, c]$ with even lead coefficient. □

The proofs of Proposition 3.3.1 for $\lambda = \sqrt{2}$ and $\lambda = \sqrt{3}$ are analogous only up to a point. Therefore, we present the proof of Proposition 3.3.1 for $\lambda = \sqrt{2}$ for as long
as the analogy holds, which will, in fact, completely take care of the case \( \lambda = \sqrt{2} \). To finish the proof, we then address what remains of the case \( \lambda = \sqrt{3} \).

**Proof of Proposition 3.3.1** We wish to show that if \( \lambda \), an RPF on \( G(\lambda) \) has a hyperbolic pole, then \( \hat{\psi}_\lambda(q_\lambda) \), an RPF on \( \Gamma(1) \), has a hyperbolic pole. To this end, recall from Theorem 3.2.3 that if \( \lambda = \sqrt{2} \),

\[
q_2(z) := \hat{\psi}_\lambda(q_\lambda(z)) = q_\lambda(\sqrt{2}z) + 2^{-k}q_\lambda\left(\frac{z}{\sqrt{2}}\right) + 2^{-k}q_\lambda\left(\frac{z - 1}{\sqrt{2}}\right) + (1 - z)^{-2k}q_\lambda\left(\frac{\sqrt{2}z}{1 - z}\right),
\]
and if \( \lambda = \sqrt{3} \),

\[
q_3(z) := \hat{\psi}_\lambda(q_\lambda(z)) = q_\lambda(\sqrt{3}z) + 3^{-k}q_\lambda\left(\frac{z}{\sqrt{3}}\right) + 3^{-k}q_\lambda\left(\frac{z - 1}{\sqrt{3}}\right) + 3^{-k}q_\lambda\left(\frac{z + 1}{\sqrt{3}}\right) + (z + 1)^{-2k}q_\lambda\left(\frac{\sqrt{3}z}{z + 1}\right) + (1 - z)^{-2k}q_\lambda\left(\frac{\sqrt{3}z}{1 - z}\right).
\]

For \( i = 2, 3 \) we wish to show that \( q_i \), a priori an RPF on \( \Gamma(1) \), has a hyperbolic pole. Since \( q_i \) is a sum of terms of the form \( q_\lambda(z)M \), where \( M \) is a linear fractional transformation of determinant \( \lambda \), we look to the poles of \( q_\lambda \) in order to search for potential poles of \( q_i \).

For example, if \( z_2 \) is a hyperbolic pole of \( q_\lambda \), for \( \lambda = \sqrt{2} \), then \( \sqrt{2}z_2 \) is potentially a pole of \( q_2 \) because the second term in equation 3.5 is \( 2^{-k}q_\lambda\left(\frac{z}{\sqrt{2}}\right) \). On the other hand, \( \sqrt{2}z_2 \) may be a removable singularity of \( q_2 \) if any of \( 2z_2, \frac{\sqrt{2}z_2 - 1}{\sqrt{2}}, \) and \( \frac{2z_2}{1 - \sqrt{2}z_2} \) are poles of \( q_\lambda \). Note here that \( (1 - z)^{-2k}q_\lambda\left(\frac{\sqrt{2}z_2}{1 - z}\right) \) is the fourth term in equation 3.5. By Corollary 2.1.6 (b), since \( z_2 \notin \sqrt{2}\mathbb{Q} \), we have \( 1 - \sqrt{2}z_2 \neq 0 \) and so \( (1 - z)^{-2k} \) cannot provide a zero-denominator when \( z = \sqrt{2}z_2 \).
Similarly, if $z_3$ is a hyperbolic pole of $q_\lambda$ for $\lambda = \sqrt{3}$, then by examining the second term in equation 3.6, we see that $\sqrt{3} z_3$ is potentially a pole of $q_\lambda$, but we have no guarantee that $\sqrt{3} z_3$ is not a removable singularity of $q_3$.

Therefore, we proceed as follows. First, for the sake of clarity, we temporarily restrict the discussion to the case $\lambda = \sqrt{2}$. We wish to find a hyperbolic pole $z_2$ of $q_\lambda$ such that $2z_2, \frac{\sqrt{2} z_2 - 1}{\sqrt{2}}$ and $\frac{2z_2}{1 - \sqrt{2} z_2}$ are not poles of $q_\lambda$, and in doing so, will guarantee that $\sqrt{2} z_2$ is a pole of $q_2$. This pole will necessarily be hyperbolic because since $z_2$ is non-zero and finite, then $\sqrt{2} z_2$ is also non-zero and finite. By Theorem 2.1.2, then, we will have that $\sqrt{2} z_2$ is a hyperbolic pole of $q_2$.

To this end, let $z_2$ be a hyperbolic pole of $q_\lambda$ with associated quadratic $[a_2, b_2, c_2]$ such that $D_2 = b_2^2 - 8a_2 c_2$ is maximal with respect to all hyperbolic poles of $q_\lambda$. For convenience, let $N_1 = \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right)$, $N_2 = \left( \begin{array}{cc} \sqrt{2} & -1 \\ 0 & \sqrt{2} \end{array} \right)$, and $N_3 = \left( \begin{array}{cc} 2 & 0 \\ -\sqrt{2} & 1 \end{array} \right)$, so that $N_1 z_2 = 2z_2$, $N_2 z_2 = \frac{\sqrt{2} z_2 - 1}{\sqrt{2}}$, and $N_3 z_2 = \frac{2z_2}{1 - \sqrt{2} z_2}$. Next, we apply Lemma 2.1.11 (a) to $N_1 z_2$, $N_2 z_2$, and $N_3 z_2$ to find that they satisfy the (not necessarily associated) quadratic polynomials $P_1(z) = a_2 \sqrt{2} z^2 + 2b_2 z + 4c_2 \sqrt{2}$, $P_2(z) = 2a_2 \sqrt{2} z^2 + (4a_2 + 2b_2)z + (a_2 + b_2 + 2c_2)\sqrt{2}$, and $P_3(z) = (a_2 + b_2 + 2c_2)\sqrt{2} z^2 + (2b_2 + 8c_2)z + 4c_2 \sqrt{2}$, respectively. First, note that we must show that $P_1(z)$, $P_2(z)$ and $P_3(z)$, or else normalizations of $P_1(z)$, $P_2(z)$ and $P_3(z)$, are of the necessary form with respect to Definition 2.1.7 so that $N_1 z_2$, $N_2 z_2$, and $N_3 z_2$ may legitimately be hyperbolic poles of $q_\lambda$. To this end, we need only show that the lead coefficients and constant terms of $P_1(z)$, $P_2(z)$ and $P_3(z)$ are all non-zero. This is clearly true of $P_1(z)$ because $a_2 c_2 \neq 0$ since $[a_2, b_2, c_2]$ is an associated quadratic for $z_2$. Therefore, it remains to
show that $a_2 + b_2 + 2c_2 \neq 0$. But, if $a_2 + b_2 + 2c_2 = 0$, then $N_2z_2 = \frac{\sqrt{2} b - 1}{\sqrt{2}}$ is a root of $P_2(z) = 2a_2\sqrt{2}z^2 + (4a_2 + 2b_2)z$. That is, $2a_2\sqrt{2}(N_2z_2)^2 + (4a_2 + 2b_2)N_2z_2 = 0$, which means that either $N_2z_2 = 0$ or else $2a_2\sqrt{2}N_2z_2 + (4a_2 + 2b_2) = 0$. This is a contradiction because in either case, we have $N_2z_2$ and hence $z_2$ is a rational multiple of $\sqrt{2}$. Thus, $N_1z_2, N_2z_2, \text{ and } N_3z_2$ all have associated quadratics, and hence, we return to the usual notation for the $P_j(z)$, namely $P_1(z) = [a_2, 2b_2, 4c_2], P_2(z) = [2a_2, 4a_2 + 2b_2, a_2 + b_2 + 2c_2], \text{ and } P_3(z) = [a_2 + b_2 + 2c_2, 2b_2 + 8c_2, 4c_2].$

Since for $j = 1, 2, 3$ the determinant of $N_j$ is 2, by Lemma 2.1.11 (b), we have $D_{P_j(z)} = 4D_P(z)$, or, alternatively, $D_{P_j(z)} = 4D_2$ because $P(z)$ is an associated quadratic for $z_2$. If it turns out that for $j = 1, 2, 3$, $P_j(z)$ is an associated quadratic for $N_jz_2$, then $N_jz_2$ will not be a pole of $q_\lambda$ by the maximality of $D_2$, and then we are done. That is, if $P_j(z)$, when viewed as a triple of integers, does not have 2 as a common factor, then $P_j(z)$ is an associated quadratic for $N_jz_2$, since the remaining condition is to check that the coefficients are relatively prime as a triple.

A similar discussion regarding potential hyperbolic poles of $q_3$ provides the following information. First, choose $z_3$ with associated quadratic $[a_3, b_3, c_3]$ so that $D_3 := \text{disc}(z_3)$ is maximal. Then in order to guarantee that $\sqrt{3}z_3$ is a (necessarily hyperbolic) pole of $q_3$, we must ensure that $3z_3, \frac{\sqrt{3}z_3 - 1}{\sqrt{3}}, \frac{\sqrt{3}z_3 + 1}{\sqrt{3}}, \frac{3z_3}{1-\sqrt{3}z_3}$ and $\frac{3z_3}{3z_3 - 1}$ are not poles of $q_\lambda$ for $\lambda = \sqrt{3}$. We apply Lemma 2.1.11 (a) to each of the above numbers, and discover that they satisfy, respectively, the quadratic polynomials $Q_1(z) = [a_3, 3b_3, 9c_3], Q_2(z) = [3a_3, 6a_3 + 3b_3, a_3 + b_3 + 3c_3], Q_3(z) = [3a_3, -6a_3 + 3b_3, a_3 - b_3 + 3c_3], Q_4(z) = [a_3 - b_3 + 3c_3, 3b_3 - 18c_3, 9c_3], \text{ and } Q_5(z) = \ldots$
\[ \lambda = \sqrt{2} \]

<table>
<thead>
<tr>
<th>Pole of ( q_\lambda )?</th>
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<td>2z_2</td>
<td>( P_1(z) = [a_2, 2b_2, 4c_2] )</td>
<td>3z_3</td>
<td>( Q_1(z) = [a_3, 3b_3, 9c_3] )</td>
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<td>( \sqrt{2z_2 - 1} \sqrt{2} )</td>
<td>( P_2(z) = [2a_2, 4a_2 + 2b_2, a_2 + b_2 + 2c_2] )</td>
<td>( \sqrt{3z_3 - 1} \sqrt{3} )</td>
<td>( Q_2(z) = [3a_3, 6a_3 + 3b_3, a_3 + b_3 + 3c_3] )</td>
</tr>
<tr>
<td>( \frac{2z_2}{1 - \sqrt{2z_2}} )</td>
<td>( P_3(z) = [a_2 + b_2 + 2c_2, 2b_2 + 8c_2, 4c_2] )</td>
<td>( \frac{\sqrt{3z_3 + 1}}{\sqrt{3}} )</td>
<td>( Q_3(z) = [3a_3, -6a_3 + 3b_3, a_3 - b_3 + 3c_3] )</td>
</tr>
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</table>

Figure 5: Potential poles of \( q_\lambda \) related to \( z_i \)

[\( a_3 + b_3 + 3c_3, 3b_3 + 18c_3, 9c_3 \)]. Note also that by Lemma 2.1.11 (b) that \( D_{Q_j(z)} = 9D_3 \) for \( j = 1, 2, 3, 4, 5 \); if it then turns out that each \( Q_j(z) \) is an associated quadratic, we are done by the maximality of \( D_3 \). That is, if each \( Q_j(z) \), when viewed as a triple of integers, does not have 3 as a common factor, then \( Q_j(z) \) is an associated quadratic.

We condense the information obtained thus far in Figure 5.

The remainder of the proof is given in two cases, the second of which includes an algorithm. Recall that \( D_2 \) and \( D_3 \) are maximal and fixed for the rest of the proof.

**Case 1: \( D_i \equiv 0 (mod \lambda^2) \)**

Since \( D_i = b_i^2 - 4\lambda^2 a_i c_i \), we have \( b_i \equiv 0 (mod \lambda^2) \). By Lemma 2.1.13, we may assume without loss of generality that \( gcd(a_i, \lambda^2) = 1 \). Then every potential pole of \( q_\lambda \) in Figure 5 is eliminated because each corresponding quadratic polynomial has either lead coefficient or constant term relatively prime to \( \lambda^2 \). In other words, all polynomials
in Figure 5 are associated quadratics. Therefore, \( \lambda z_i \) is a hyperbolic pole of \( q_i \).

**Case 2:** \( D_i \not\equiv 0 (mod \lambda^2) \)

Since \( D_i = b_i^2 - 4\lambda^2 a_i c_i \), we have \( b_i \not\equiv 0 (mod \lambda^2) \).

The remainder of Case 2 is given in two steps, the first of which gives a procedure for producing a hyperbolic pole of \( \hat{\psi}_\lambda(q_\lambda) \), although under somewhat restrictive circumstances (due, in part, to the failure of Lemma 3.3.2 for \( \lambda = \sqrt{3} \)). The purpose of the second step is to show that even in the worst case, we may always return to the first step.

**Step 1:**

If there exists a pole \( z_{m_i} \) with \( disc(z_{m_i}) = D_i \), and with associated quadratic \([r_i, s_i, t_i]\) such that \( \lambda^2 | r_i \), and if among all such poles, we choose \( z_{m_i} \) so that \( |z_{m_i}| \) is maximal, then all potential poles of \( q\lambda \), shown in Figure 6 are eliminated. That is, \( 2z_{m_2} \) and \( 3z_{m_3} \) are eliminated by virtue of the maximality of \( |z_{m_i}| \), and all of the remaining potential poles of \( q\lambda \) are eliminated because each has corresponding quadratic polynomial with either lead coefficient or constant term relatively prime to \( \lambda^2 \).

**Remark.** By Lemma 3.3.2, when \( \lambda = \sqrt{2} \), we can always find a hyperbolic pole of \( q\lambda \) satisfying the hypotheses given in Step 1. Therefore, when \( \lambda = \sqrt{2} \), if \( q\lambda \) has a hyperbolic pole, so does \( \hat{\psi}_\lambda(q_\lambda) \), and we are done. On the other hand, when \( \lambda = \sqrt{3} \), as promised, the situation is more complicated, and we deal with it in the next step.

**Step 2:** \( \lambda = \sqrt{3}, D_3 \not\equiv 0 (mod 3) \)
\[ \lambda = \sqrt{2} \]

<table>
<thead>
<tr>
<th>Pole of ( q_3 )?</th>
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</thead>
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<td>( 2z_{m_2} )</td>
<td>( [r_2, 2s_2, 4t_2] )</td>
<td>( 3z_{m_3} )</td>
<td>( [r_3, 3s_3, 9t_3] )</td>
</tr>
<tr>
<td>( \frac{\sqrt{2}z_{m_2} - 1}{\sqrt{2}} )</td>
<td>( [2r_2, 4r_2 + 2s_2, r_2 + s_2 + 2t_2] )</td>
<td>( \frac{\sqrt{3}z_{m_3} - 1}{\sqrt{3}} )</td>
<td>( [3r_3, 6r_3 + 3s_3, r_3 + s_3 + 3t_3] )</td>
</tr>
<tr>
<td>( \frac{2z_{m_2}}{1 - \sqrt{2}z_{m_2}} )</td>
<td>( [r_2 + s_2 + 2t_2, 2s_2 + 8t_2, 4t_2] )</td>
<td>( \frac{3z_{m_3}}{\sqrt{3}z_{m_3} + 1} )</td>
<td>( [r_3 - s_3 + 3t_3, 3s_3 - 18t_3, 9t_3] )</td>
</tr>
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</table>

Figure 6: Potential poles of \( q_3 \) related to \( z_m \).

Recall that \( z_3 \) is a hyperbolic pole of \( q_3 \) with associated quadratic \( [a_3, b_3, c_3] \) such that \( D_3 = disc(z_3) \) is maximal. Note that since \( D_3 \neq 0(mod\ 3) \), we must have \( gcd(b_3, 3) = 1 \).

If \( 3|a_3 \), then go to Step 1. Otherwise, \( gcd(a_3, 3) = 1 \). In that case, \( 3z_3 \) is eliminated as a possible pole of \( q_3 \) because the lead coefficient of \( Q_1 \) (see Figure 5) is not divisible by 3.

It remains either to eliminate as potential poles of \( q_3 \), or else use to our advantage, the final four potential poles of \( q_3 \) in the second column of Figure 5. To this end, suppose \( X_4 = \frac{3z_3}{\sqrt{3}z_3 + 1} \) is a pole of \( q_3 \). By the maximality of \( D_3 \), and because \( gcd(3, b_3) = 1 \), we must have that \( \frac{3z_3 - b_3 + 3c_3}{3}, b_3 - 6c_3, 3c_3 \) is an associated quadratic for \( X_4 \). In that case, \( \frac{1}{X_4} \) is a pole of \( q_3 \) with associated quadratic whose lead coefficient is divisible by 3. Go to Step 1. Similarly, if \( X_5 = \frac{3z_3}{1 - \sqrt{3}z_3} \) is a pole of \( q_3 \), then \( \frac{1}{X_5} \) is a pole of \( q_3 \) with associated quadratic whose lead coefficient divisible by 3. Go to Step 1.

Now, without loss of generality, assume that neither \( X_4 \) nor \( X_5 \) are poles of \( q_3 \).
We will eliminate both \(X_2 = \frac{\sqrt{3}a_3 - 1}{\sqrt{3}}\) and \(X_3 = \frac{\sqrt{3}a_3 + 1}{\sqrt{3}}\) as potential poles of \(q_\lambda\) as follows. Observe that the constant terms of \(Q_2\) and \(Q_3\) (the corresponding quadratics for \(X_2\) and \(X_3\)) are \((a_3 + b_3 + 3c_3)\) and \((a_3 - b_3 + 3c_3)\), respectively. Therefore, we may eliminate one of \(X_2\) or \(X_3\) depending on whether or not \(a_3 \equiv b_3 (mod\ 3)\). By a solicitous choice of \(z_3\), we may eliminate \(X_2\) and \(X_3\) simultaneously.

In particular, if \(a_3 \equiv b_3 (mod\ 3)\), then among all such poles of \(q_\lambda\) with maximal discriminant and \(a_3 \equiv b_3 (mod\ 3)\), choose \(z_3\) to be the largest (furthest to the right on the real axis). Then \(X_2\) is eliminated because the constant term of \(Q_2\) is not divisible by \(3\), and \(X_3\) is eliminated because \(\frac{\sqrt{3}a_3 + 1}{\sqrt{3}} = z_3 + \frac{1}{\sqrt{3}} > z_3\).

Similarly, if \(a_3 \not\equiv b_3 (mod\ 3)\), then among all such poles of maximal discriminant, choose \(z_3\) to be the smallest. Then \(X_3\) is eliminated because the constant term of \(Q_3\) is not divisible by \(3\), and \(X_2\) is eliminated because \(\frac{\sqrt{3}a_3 - 1}{\sqrt{3}} = z_3 - \frac{1}{\sqrt{3}} < z_3\). Therefore, \(\frac{\sqrt{3}a_3}{\sqrt{3}}\) is a hyperbolic pole of \(q_3\). This concludes Step 2 and Case 2.

In all cases, we have shown that if \(q_\lambda\) is an RPF on \(G(\lambda)\) with a hyperbolic pole, then \(\hat{\psi}_\lambda(q_\lambda)\), an RPF on \(\Gamma(1)\), has a hyperbolic pole. □

### 3.4 Relationship Among \(\hat{T}(n)\), \(\hat{T}_\lambda(n)\), and \(\hat{\psi}_\lambda\)

In this section we find a formula for the relationship among \(\hat{\psi}_\lambda\), \(\hat{T}(n)\), and \(\hat{T}_\lambda(n)\), where \(\hat{T}(n)\) and \(\hat{T}_\lambda(n)\) are the induced Hecke operators on the space of RPFs on \(\Gamma(1)\) and \(G(\lambda)\) respectively. See Theorem 3.2.3 (b), Definition 1.5.2, and Definition 3.2.5.

In the next lemma, we begin by giving a formula for the relationship among \(T(n)\), \(T_\lambda(n)\), and \(\psi_\lambda\), the usual Hecke operators, and the map from the space of
modular integrals on $G(\lambda)$ to the space of modular integrals on $\Gamma(1)$, respectively. See Definitions 1.4.1, 3.2.4 and Theorem 3.2.3 (a). Then we will show that the corresponding formulas hold for the induced map and operators $T_\lambda$, $T(n)$, and $T_\lambda(n)$.

**Lemma 3.4.1** For $\lambda = \sqrt{2}$ or $\sqrt{3}$, if $f_\lambda$ is a modular integral of weight $2k$ on $G(\lambda)$, then

(a) $\psi(T_\lambda(n)f_\lambda) = T(n)\psi_\lambda(f_\lambda)$ if $\lambda^2 \not| n$

and

(b) $\psi(T_\lambda(n)f_\lambda) = T(n)\psi_\lambda(f_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\psi_\lambda(f_\lambda)$ if $n = \lambda^2$.

**Proof:** The proofs for $\lambda = \sqrt{2}$ and $\sqrt{3}$ are analogous, and so we present only the case $\lambda = \sqrt{2}$.

(a) First, for convenience we write $f$ and $\psi$ in place of $f_{\sqrt{2}}$ and $\psi_{\sqrt{2}}$ respectively. Next, by Definition 3.2.4 (a), since $2 \not| n$, we have

$$T_{\sqrt{2}}(n)f = n^{2k-1} \sum_{ad = n} f_{|2k} \begin{pmatrix} a & b\sqrt{2} \\ 0 & d \end{pmatrix}, \quad (3.7)$$

and by Theorem 3.2.3 (a),

$$\psi(f) = f_{|2k} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix} + f_{|2k} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} f_{|2k} \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{pmatrix}, \quad (3.8)$$

so that

$$\psi(T_{\sqrt{2}}(n)f) = \left( n^{2k-1} \sum_{ad = n} f_{|2k} \begin{pmatrix} a & b\sqrt{2} \\ 0 & d \end{pmatrix} \right)_{|2k} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$
\[
\begin{split}
&= n^{2k-1} \left[ \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} a & 2b \\ 0 & d \sqrt{2} \end{array} \right) + \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} a \sqrt{2} & b \sqrt{2} \\ 0 & d \end{array} \right) \\
&\quad + \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} a & a + 2b \\ 0 & d \sqrt{2} \end{array} \right) \right]. \tag{3.9}
\end{split}
\]

On the other hand, by Definition 1.4.1

\[
T(n) \psi(f) = n^{2k-1} \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{2} \end{array} \right) |_{2k} \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)
\]

\[
+ n^{2k-1} \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 0 \\ 0 & 1 \end{array} \right) |_{2k} \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right)
\]

\[
+ n^{2k-1} \sum_{ad = n} f|_{2k} \left( \begin{array}{cc} 1 & 1 \\ 0 & \sqrt{2} \end{array} \right) |_{2k} \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \tag{3.10}
\]
\[ n^{2k-1} \left[ \sum_{ad=n} \begin{array}{c} \sum_{0 \leq b < d} f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right) + \sum_{0 \leq b < d} f_{2k} \left( \begin{array}{cc} a\sqrt{2} & b\sqrt{2} \\ 0 & d \end{array} \right) \\ + \sum_{ad=n} f_{2k} \left( \begin{array}{cc} a & b+d \\ 0 & d\sqrt{2} \end{array} \right) \end{array} \right] \] (3.11)

In order to see that \( \psi(T_{\sqrt{n}}(n))f = T(n)\psi(f) \), it suffices to show that the summation of the first and last terms of equation 3.9 equals the summation of the first and last terms of equation 3.11, because the second terms in both equations are identical. Specifically, it suffices to show that

\[ \sum_{ad=n} \left[ f_{2k} \left( \begin{array}{cc} a & 2b \\ 0 & d\sqrt{2} \end{array} \right) + f_{2k} \left( \begin{array}{cc} a & a+2b \\ 0 & d\sqrt{2} \end{array} \right) \right] \] (3.12)

is the same as

\[ \sum_{ad=n} \left[ f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right) + f_{2k} \left( \begin{array}{cc} a & b+d \\ 0 & d\sqrt{2} \end{array} \right) \right] \] (3.13)

For convenience in computations to follow, we write \( \frac{n}{d} \) in place of \( a \).

First, observe that we need only examine the upper right-hand entries of the matrices in 3.12 and 3.13, since all other corresponding entries are identical. By doing so, we may rewrite 3.12 and 3.13 as follows. Recall that 2 \( \not| n \) so that \( d \) and \( \frac{n}{d} \) are odd. Let \( E1 = \{0, 2, \ldots, 2d-2\} \), \( O1 = \{ \frac{n}{d}, \frac{n}{d}+2, \ldots, \frac{n}{d}+2d-2\} \), \( O2 = \{1, 3, \ldots, 2d-1\} \).

In other words, \( E1 \) is the list of consecutive even integers from 0 to \( 2d-2 \), \( O1 \) is the
list of consecutive odd integers from \( \frac{n}{d} \) to \( \frac{n}{d} + 2d - 1 \), and \( O_2 \) is the list of consecutive odd integers from 1 to \( 2d - 1 \). In that case, 3.12 may be written as

\[
\sum_{a, d = n}^{b \in E_1} f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right) + \sum_{a, d = n}^{b \in O_1 \cap O_2} f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right)
\]

\[
+ \sum_{a, d = n}^{b \in O_1 \setminus (O_1 \cap O_2)} f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right), \tag{3.14}
\]

and 3.13 may be written as

\[
\sum_{a, d = n}^{\tilde{b} \in E_1} f_{2k} \left( \begin{array}{cc} a & \tilde{b} \\ 0 & d\sqrt{2} \end{array} \right) + \sum_{a, d = n}^{\tilde{b} \in O_1 \cap O_2} f_{2k} \left( \begin{array}{cc} a & \tilde{b} \\ 0 & d\sqrt{2} \end{array} \right)
\]

\[
+ \sum_{a, d = n}^{\tilde{b} \in O_2 \setminus (O_1 \cap O_2)} f_{2k} \left( \begin{array}{cc} a & \tilde{b} \\ 0 & d\sqrt{2} \end{array} \right). \tag{3.15}
\]

Therefore, it suffices to show that the last terms in 3.14 and 3.15 are the same. That is, we must show that

\[
\sum_{a, d = n}^{b \in O_1 \setminus (O_1 \cap O_2)} f_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right). \tag{3.16}
\]

is the same as

\[
\sum_{a, d = n}^{\tilde{b} \in O_2 \setminus (O_1 \cap O_2)} f_{2k} \left( \begin{array}{cc} a & \tilde{b} \\ 0 & d\sqrt{2} \end{array} \right). \tag{3.17}
\]

To this end, note that for \( d \neq n \), \( O_1 \setminus (O_1 \cap O_2) = \{2d + 1, 2d + 3, \ldots, \frac{n}{d} + 2d - 2\} \), and \( O_2 \setminus (O_1 \cap O_2) = \{1, 3, \ldots, \frac{n}{d} - 2\} \), and we see that the difference between every pair
of corresponding elements in the above two lists is \(2d\). When \(d = n\), \(O1 \setminus (O1 \cap O2)\) and \(O2 \setminus (O1 \cap O2)\) are empty. Therefore, we may rewrite 3.17 as

\[
\sum_{ad = n} \sum_{b \in O1 \setminus O1 \cap O2} f|_{2k} \left( \begin{array}{cc} a & b + 2d \\ 0 & d\sqrt{2} \end{array} \right),
\]

(3.18)

which means that 3.16 and 3.17 are identical because

\[
f|_{2k} \left( \begin{array}{cc} a & b + 2d \\ 0 & d\sqrt{2} \end{array} \right) = (d\sqrt{2})^{-2k}f \left( \frac{az + b + 2d}{d\sqrt{2}} \right)
\]

\[
= (d\sqrt{2})^{-2k}f \left( \frac{az + b}{d\sqrt{2}} + \sqrt{2} \right)
\]

\[
= f|_{2k} \left( \begin{array}{cc} a & b \\ 0 & d\sqrt{2} \end{array} \right)
\]

since \(f\) is periodic with period \(\sqrt{2}\). This proves (a).

(b) We will compute \(\psi(T_{\sqrt{2}}(2)f)\) and \(T(2)\psi(f)\), and then compare the results. To this end, by Definition 3.2.4 (b), we have

\[
T_{\sqrt{2}}(2)(f) = f|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 1 & \sqrt{2} \\ 0 & 2 \end{array} \right)
\]

\[+ f|_{2k} \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{array} \right),
\]

(3.19)

and by Definition 1.4.1

\[
T(2)\psi(f) = \psi(f)|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) + \psi(f)|_{2k} \left( \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right) + \psi(f)|_{2k} \left( \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right).
\]

(3.20)

Then combining 3.8 and 3.20 yields

\[
T(2)\psi(f) = f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 0 \\ 0 & 2 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 2\sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 1 & 2 \\ 0 & 2\sqrt{2} \end{array} \right).
\]
\[ f|_{2k} \left( \sqrt{2} \begin{array}{cc} 1 & 1 \\ 0 & 2 \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 1 & 3 \\ 0 & 2 \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 \sqrt{2} & 0 \\ 0 & 1 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & 0 \\ 0 & \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & 1 \\ 0 & \sqrt{2} \end{array} \right). \] (3.21)

On the other hand, combining 3.8 and 3.19 yields

\[
\psi(T_{\sqrt{2}}(2)f) = T_{\sqrt{2}}(2) f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 0 \\ 0 & 1 \end{array} \right) + T_{\sqrt{2}}(2) f|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{array} \right) + T_{\sqrt{2}}(2) f|_{2k} \left( \begin{array}{cc} 1 & 1 \\ 0 & \frac{\sqrt{2}}{2} \end{array} \right)
\]

\[ = f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 0 \\ 0 & 2 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & \sqrt{2} \\ 0 & 2 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 \sqrt{2} & 0 \\ 0 & 1 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & 1 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & \sqrt{2} \\ 0 & 1 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & \sqrt{2} \\ 0 & 2 \end{array} \right)
\]

\[ + f|_{2k} \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 1 & 2 \\ 0 & 2 \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & 0 \\ 0 & \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 2 \sqrt{2} \\ 0 & 2 \end{array} \right)
\]

\[ = T(2)\psi(f) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & \sqrt{2} \\ 0 & 2 \end{array} \right) + f|_{2k} \left( \begin{array}{cc} 2 & 0 \\ 0 & \sqrt{2} \end{array} \right) + f|_{2k} \left( \begin{array}{cc} \sqrt{2} & 2 \sqrt{2} \\ 0 & 2 \end{array} \right)
\]

\[ = T(2)\psi(f) + 2^{-k}f \left( \frac{\sqrt{2}z + \sqrt{2}}{2} \right) + (\sqrt{2})^{-k}f \left( \frac{2z + 2}{\sqrt{2}} \right) + 2^{-k}f \left( \frac{\sqrt{2}z + 2\sqrt{2}}{2} \right)
\]

\[ = T(2)\psi(f) + 2^{-k}f \left( \frac{z + 1}{\sqrt{2}} \right) + (\sqrt{2})^{-k}f(\sqrt{2}z + \sqrt{2}) + 2^{-k}f(\frac{z}{\sqrt{2}} + \sqrt{2}).
\]

Since \( f \) is periodic with period \( \sqrt{2} \), we conclude that

\[ \psi(T_{\sqrt{2}}(2)f) = T(2)\psi(f) + (\sqrt{2})^{-2k}\psi(f), \]
as desired.

The corresponding formulas hold for \( q \), the RPF associated with \( f \), as is shown in the next corollary.

**Corollary 3.4.2** For \( \lambda = \sqrt{2} \) or \( \sqrt{3} \), if \( q_\lambda \) is an RPF on \( G(\lambda) \), then

\[
\begin{align*}
(a) & \quad \hat{\psi}_\lambda(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}_\lambda(q_\lambda) \quad \text{if } \lambda^2 \nmid n \\
\text{and} \\
(b) & \quad \hat{\psi}_\lambda(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}_\lambda(q_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}_\lambda(q_\lambda) \quad \text{if } n = \lambda^2.
\end{align*}
\]

**Proof:** (a) First, for convenience we write \( q \) and \( \hat{\psi} \) in place of \( q_\lambda \) and \( \hat{\psi}_\lambda \) respectively. Recall by Theorem 3.2.3 that if \( f \) is an automorphic integral of weight \( 2k \) on \( G(\lambda) \) with corresponding RPF \( q \), then \( \hat{\psi}(q) \) is an RPF on \( \Gamma(1) \) with a corresponding modular integral \( \psi(f) \), and that \( \hat{\psi}(q) = (\psi(f))_{2k}T - \psi(f) \). Moreover, by Definition 1.5.2, we have \( \hat{T}(n)\hat{\psi}(q) = (T(n)\psi(f))_{2k}T - T(n)\psi(f) \). Finally, note that for any positive integer \( n \), \( T_\lambda(n)f \) is an automorphic integral on \( G(\lambda) \) of weight \( 2k \) with associated RPF \( \hat{T}_\lambda(n)q \). In that case,

(a) since \( \lambda^2 \nmid n \), Lemma 3.4.1 (a) applies, and so

\[
\hat{\psi}(\hat{T}_\lambda(n)q) = (\psi(T_\lambda(n)f))_{2k}T - \psi(T_\lambda(n)f) = (T(n)\psi(f))_{2k}T - T(n)\psi(f) = \hat{T}(n)\hat{\psi}(q),
\]

as desired.
(b) Since \( n = \lambda^2 \), Lemma 3.4.1 (b) applies, and hence

\[
\hat{\psi}(\hat{T}_\lambda(\lambda^2)q) = (\psi(T_\lambda(\lambda^2)f)|_{2kT} - \psi(T_\lambda(\lambda^2)f))
\]

\[
= (T(\lambda^2)\psi(f) + (\lambda^2 - 1)\lambda^{-2k}\psi(f))|_{2kT} - (T(\lambda^2)\psi(f) + (\lambda^2 - 1)\lambda^{-2k}\psi(f))
\]

\[
= [(T(\lambda^2)\psi(f))|_{2kT} - T(\lambda^2)\psi(f)] + (\lambda^2 - 1)\lambda^{-2k}[(\psi(f))|_{2kT} - \psi(f)]
\]

\[
= \hat{T}(\lambda^2)\hat{\psi}(q) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}(q).
\]

\(\Box\)

We need one final lemma in order to prove Theorem 3.4.4.

**Lemma 3.4.3** Suppose \( q_\lambda \) is an RPF of weight \( 2k \) on \( G(\lambda) \) for \( \lambda = \sqrt{2} \) or \( \sqrt{3} \), and suppose \( q_\lambda \) has a hyperbolic pole. If \( s \geq 1 \) is an integer, then the RPF \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda) \) (defined on \( \Gamma(1) \)) has a hyperbolic pole \( z_* \) with the following property: if \( z_0 \) is any hyperbolic pole of \( \hat{\psi}(q_\lambda) \), then \( \text{disc}(z_*) > \text{disc}(z_0) \).

**Proof:** We proceed by induction on \( r \). Specifically, we will show for all integers \( r \geq 1 \) that \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \) has a hyperbolic pole \( z_r \) such that for any hyperbolic pole \( z'_{r-1} \) of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda) \), we have \( \text{disc}(z_r) > \text{disc}(z'_{r-1}) \). To finish the proof, we apply this result to the case \( r = 1 \), or equivalently, to the hyperbolic pole of \( \hat{\psi}(q_\lambda) \).

Let \( r = 1 \). Then

\[
\hat{\psi}(\hat{T}_\lambda(\lambda^2)q_\lambda) = \hat{\psi}(\hat{T}_\lambda(\lambda^2)q_\lambda)
\]

\[
= \hat{T}(\lambda^2)\hat{\psi}(q_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}(q_\lambda)
\]
by Corollary 3.4.2 (b). Moreover, by Corollary 2.3.4, \( \hat{T}(\lambda^2)\hat{\psi}(q_\lambda) \) has a hyperbolic pole \( z_1 \) such that \( \text{disc}(z_1) > \text{disc}(z_0) \) for any hyperbolic pole \( z_0 \) of \( \hat{\psi}(q_\lambda) \), and therefore, by equation 3.22, so does \( \hat{\psi}(\hat{T}_\lambda(\lambda^2)q_\lambda) \).

Now suppose the induction hypothesis holds for all positive integers \( j \) such that \( 1 \leq j \leq r - 1 \). It remains to show that \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \) has a hyperbolic pole \( z_{r+1} \) such that \( \text{disc}(z_{r+1}) > \text{disc}(z_r) \) for any hyperbolic pole \( z_r \) of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \). To accomplish this, we use the recursive definition of \( \hat{T}_\lambda((\lambda^2)^{r+1}) \) as given by Definition 3.2.5 (c) in Section 3.2. Specifically,

\[
\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r+1})q_\lambda) = \hat{\psi}(\hat{T}_\lambda(\lambda^2)\hat{T}_\lambda((\lambda^2)^r)q_\lambda) - (\lambda^2)^k\hat{T}_\lambda((\lambda^2)^r)q_\lambda
\]

\[
- (\lambda^2)^{2k-1}\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda).
\]

(3.23)

Since \( \hat{\psi} \) is linear, and by applying Corollary 3.4.2 (b), we can rewrite equation 3.23 as

\[
\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) = \hat{T}(\lambda^2)\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) + (\lambda^2 - 1)\lambda^{-2k}\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)
\]

\[
- (\lambda^2)^k\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) - (\lambda^2)^{2k-1}\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda)
\]

\[
= \hat{T}(\lambda^2)\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)
\]

\[
+ [(\lambda^2 - 1)\lambda^{-2k} - (\lambda^2)^k]\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda)
\]

\[
- (\lambda^2)^{2k-1}\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda).
\]

In total,

\[
\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r+1})q_\lambda) = \hat{T}(\lambda^2)(\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda))
\]

\[
+ C_1\hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) + C_2\hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda),
\]

(3.24)
where \( C_1 = (\lambda^2 - 1)\lambda^{-2k} - (\lambda^2)^k \) and \( C_2 = -(\lambda^2)^{2k-1} \).

By the induction hypothesis, \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \) has a hyperbolic pole, \( z_r \) such that for any hyperbolic pole \( z'_{r-1} \) of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r-1})q_\lambda) \), we have \( \text{disc}(z_r) > \text{disc}(z'_{r-1}) \). Moreover, by Corollary 2.3.4, \( \hat{T}(\lambda^2)^r \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \) has a hyperbolic pole \( z_{r+1} \) such that for any hyperbolic pole \( z'_{r} \) of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^r)q_\lambda) \), we have \( \text{disc}(z_{r+1}) > \text{disc}(z'_{r}) \). Therefore, by equation 3.24, the same can be said of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^{r+1})q_\lambda) \). This completes the induction on \( r \).

To finish the proof, simply note that for \( 1 < j < s \), we know that \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^j)q_\lambda) \) has a hyperbolic pole \( z_j \) such that \( \text{disc}(z_j) > \text{disc}(z'_{j-1}) \) for any hyperbolic pole \( z'_{j-1} \) of \( \hat{\psi}(\hat{T}_\lambda((\lambda^2)^{j-1})q_\lambda) \). In particular, if \( z_0 \) is any hyperbolic pole of \( \hat{\psi}(q_\lambda) \), then \( \text{disc}(z_s) > \text{disc}(z_0) \), as desired. \( \square \)

We are now ready to prove Theorem 3.4.4, which we restate.

**Theorem 3.4.4** For \( \lambda = \sqrt{2} \) or \( \sqrt{3} \), if \( q_\lambda \) is an RPF on \( G(\lambda) \) with at least one hyperbolic pole, then \( q_\lambda \) is not an eigenfunction of the induced Hecke operator \( \hat{T}_\lambda(n) \) for any \( n > 1 \).

**Proof:** We give a proof by contradiction, which is accomplished in two steps: for any integer \( n > 1 \) such that \( \lambda^2 \nmid n \), and for \( n = (\lambda^2)^sn' \), where \( s \geq 1 \), \( n' \geq 1 \) and \( \lambda^2 \nmid n' \).

**Step 1:** \( n > 1 \) is an integer with \( \lambda^2 \nmid n \).

By way of contradiction, suppose \( \hat{T}_\lambda(n)q_\lambda = Cq_\lambda \) for some \( C \neq 0 \) in \( \mathbb{C} \). Then by Corollary 3.4.2 (a),

\[
\hat{\psi}(\hat{T}_\lambda(n)q_\lambda) = \hat{T}(n)\hat{\psi}(q_\lambda), \tag{3.25}
\]
and by assumption,

$$\hat{\psi}(\hat{T}_\lambda(n)q_\lambda) = \hat{\psi}(Cq_\lambda) = C\hat{\psi}(q_\lambda)$$  \hfill (3.26)

so that

$$\hat{T}(n)\hat{\psi}(q_\lambda) = C\hat{\psi}(q_\lambda).$$  \hfill (3.27)

By Proposition 3.3.1, $\hat{\psi}(q_\lambda)$ has a hyperbolic pole, and hence Theorem 2.3.1 applies, and therefore equation 3.27 gives a contradiction. Specifically, by Theorem 2.3.1, $\hat{\psi}(q_\lambda)$ is not an eigenfunction of $\hat{T}(n)$.

**Step 2:** $n = n'(\lambda^2)^s$, where $s$ and $n'$ are positive integers, and $\lambda^2 \not| n'$.

By way of contradiction, suppose $\hat{T}_\lambda(n'(\lambda^2)^s)q_\lambda = Cq_\lambda$ for some $C \neq 0$ in $\mathbb{C}$, so that $\hat{\psi}(\hat{T}_\lambda(n'(\lambda^2)^s)q_\lambda) = C\hat{\psi}(q_\lambda)$. Since the induced Hecke operator is multiplicative and by Corollary 3.4.2 (a), we have

$$\hat{\psi}(\hat{T}_\lambda(n'(\lambda^2)^s)q_\lambda) = \hat{\psi}(\hat{T}_\lambda(n')\hat{T}_\lambda((\lambda^2)^s)q_\lambda)$$

$$= \hat{T}(n')\hat{\psi}(\hat{T}_\lambda(\lambda^2)^s)q_\lambda).$$

In total, with our original assumption, we have

$$\hat{T}(n')\hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda) = C\hat{\psi}(q_\lambda).$$  \hfill (3.28)

By Lemma 3.4.3, $\hat{\psi}(\hat{T}_\lambda(\lambda^2)^s)q_\lambda$ has a hyperbolic pole, $z_s$ such that $\text{disc}(z_s) > \text{disc}(z_0)$ for any hyperbolic pole $z_0$ of $\hat{\psi}(q_\lambda)$. Moreover, by Corollary 2.3.4, $\hat{T}(n')\hat{\psi}(\hat{T}_\lambda((\lambda^2)^s)q_\lambda)$ has a hyperbolic pole $Z_n$ such that $\text{disc}(Z_n) > \text{disc}(z_s)$. In other words, since $\text{disc}(Z_n) > \text{disc}(z_0)$, $\hat{T}(n')\hat{\psi}(\hat{T}_\lambda(\lambda^2)^s)q_\lambda)$ has a pole, $Z_n$, which cannot be a pole of $\hat{\psi}(q_\lambda)$, and this contradicts equation 3.28.
Therefore, $q_\lambda$ is not an eigenfunction of $\hat{T}_\lambda(n)$ for any integer $n > 1$. $\square$
CHAPTER IV

The Zeros of Rational Period Functions

4.1 An Explicit Description

In [PR], A. Parson and K. Rosen asked the following open-ended question: ‘What can be said about the zeros of a rational period function?’ The purpose of this chapter is to begin an investigation into the location of zeros of RPFs defined on $G(\lambda_n)$ for $n \geq 3$, and $\lambda_n = 2\cos\left(\frac{\pi}{n}\right)$. To this end, we will give an explicit description of the zeros of the following family of RPFs on $G(\lambda_n)$ given in [PR] by

$$
q_n(z) = \frac{1}{(z^2 - bz - 1)^k} + \frac{1}{(z^2 + bz - 1)^k},
$$

where $k \geq 1$ is an odd integer, and $b = \frac{\lambda_n + \sqrt{\lambda_n^2 + 4}}{2} - \frac{2}{\lambda_n + \sqrt{\lambda_n^2 + 4}}$. In addition, we will give an explicit description of the zeros of the family of RPFs defined on $G(\sqrt{3})$ given by

$$
q(z) = \frac{1}{(z^2 - \frac{1}{\sqrt{3}}z - 1)^k} + \frac{1}{(z^2 + \frac{1}{\sqrt{3}}z - 1)^k},
$$

where, again, $k \geq 1$ is an odd integer. Note that when $n = 3$ the family of RPFs given in equation 4.1 is the original family of RPFs with quadratic irrational poles found by Knopp in [Kn1] as described in equation 1.14 in Section 1.6. Moreover, the RPFs given in equation 4.2 are distinct from the RPFs in equation 4.1 for $n = 6$ because

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the poles of $q_8$ are $\pm \sqrt{3} \pm \sqrt{7}$, whereas the poles of $q$ are $\pm \sqrt{3} \pm \sqrt{39}$. Finally, recall that
the RPFs in equation 4.2 were given as example 3.2 in Section 3.3.

To help locate the zeros of the RPFs given above, we begin with a proposition.

**Proposition 4.1.1** Let $f(z) = \frac{1}{(x^2 - by^2 - 1)^k} + \frac{1}{(x^2 + by^2 - 1)^k}$ for any $b \neq 0$ in $\mathbb{R}$. If $f(x + iy) = 0$ then either $x = 0$ or else $x^2 + y^2 = 1$. In other words, the zeros of $f$ lie on
the unit circle or on the imaginary axis.

**Proof:** Suppose $f(x + iy) = 0$. Then

$$
\frac{1}{((x + iy)^2 - b(x + iy) - 1)^k} + \frac{1}{((x + iy)^2 + b(x + iy) - 1)^k} = 0 \quad (4.3)
$$

implies that

$$
\frac{1}{((x + iy)^2 - b(x + iy) - 1)^k} = -\frac{1}{((x + iy)^2 + b(x + iy) - 1)^k} \quad (4.4)
$$

which means

$$
\left( \frac{(x + iy)^2 - b(x + iy) - 1}{(x + iy)^2 + b(x + iy) - 1} \right)^k = -1, \quad (4.5)
$$

and therefore

$$
\left| \frac{(x + iy)^2 - b(x + iy) - 1}{(x + iy)^2 + b(x + iy) - 1} \right| = 1, \quad (4.6)
$$

or alternatively,

$$
\left| (x + iy)^2 - b(x + iy) - 1 \right|^2 = \left| (x + iy)^2 + b(x + iy) - 1 \right|^2. \quad (4.7)
$$

We simplify equation 4.7 and get

$$
\left| (x^2 - y^2 + bx - 1) + i(by + 2xy) \right|^2 = \left| (x^2 - y^2 - bx - 1) + i(-by + 2xy) \right|^2. \quad (4.8)
$$
Observe that most of the terms on the left-hand-side of equation 4.8 are the same as those on the right-hand side, so that when computing the norms, cancellation occurs, and we are left with

$$2b(x^2 - y^2 - 1) + 4bxy^2 = -2b(x^2 - y^2 - 1) - 4bxy^2,$$  \hspace{1cm} (4.9)

which gives

$$bx(x^2 + y^2 - 1) = 0.$$  \hspace{1cm} (4.10)

Therefore, $x = 0$ or $x^2 + y^2 = 1$ since $b \neq 0$. \hfill \Box

Once we know the general location of the zeros of functions of the form of $f$ as given in Proposition 4.1.1, we can explicitly compute the zeros of those functions as is shown in the next corollary.

**Corollary 4.1.2** With $k \geq 1$ odd and the same hypotheses on $f$ as in Proposition 4.1.1, the zeros of $f$ are given as follows.

(a) The purely imaginary zeros of $f$ are of the form $y_s i$, where

$$y_s = \frac{-btan\left(\frac{s\pi}{k}\right) \pm \sqrt{b^2tan^2\left(\frac{s\pi}{k}\right) - 4}}{2}$$

for all integers $s$ such that $1 \leq s \leq k - 1$ and $|tan\left(\frac{s\pi}{k}\right)| \geq \frac{2}{b}$.

(b) The zeros of modulus 1 of $f$ are of the form $z_s = \pm \sqrt{1 - y_s^2} + iy_s$, where

$$y_s = \frac{b}{2}tan\left(\frac{\pi s}{k}\right)$$

for all integers $s$ such that $0 \leq s \leq k - 1$ and $|tan\left(\frac{s\pi}{k}\right)| \leq \frac{2}{b}$.\hfill \Box
Proof: (a) By equation 4.5 with $k$ odd, we have

\[
\left( \frac{(iy)^2 - b(iy) - 1}{-(iy)^2 - b(iy) + 1} \right)^k = \left( \frac{-y^2 - b(iy) - 1}{y^2 - b(iy) + 1} \right)^k = 1,
\]

or alternatively, that \( \frac{-y^2 - b(iy) - 1}{y^2 - b(iy) + 1} \) is a $k$th root of unity. In that case, to find $y$, we must solve

\[
-\frac{y^2 - b(iy) - 1}{y^2 - b(iy) + 1} = \cos \left( \frac{2\pi s}{k} \right) + isin \left( \frac{2\pi s}{k} \right)
\]

for integers $s$ such that $0 \leq s \leq k - 1$. That is, we must determine which $k$th roots of unity provide a real solution for $y$ in equation 4.12. With this in mind, we rewrite equation 4.12 as

\[
y^2 \left( 1 + \cos \left( \frac{2\pi s}{k} \right) \right) + ybsin \left( \frac{2\pi s}{k} \right) + \left( 1 + \cos \left( \frac{2\pi s}{k} \right) \right) = 0
\]

Therefore,

\[
y^2 \left( 1 + \cos \left( \frac{2\pi s}{k} \right) \right) + ybsin \left( \frac{2\pi s}{k} \right) + \left( 1 + \cos \left( \frac{2\pi s}{k} \right) \right) = 0
\]

and

\[
y^2 sin \left( \frac{2\pi s}{k} \right) + yb \left( 1 - \cos \left( \frac{2\pi s}{k} \right) \right) + sin \left( \frac{2\pi s}{k} \right) = 0.
\]

When $s = 0$, equation 4.14 has no real solutions, and equation 4.15 provides no information about $y$. However, when $s \neq 0$, we can rewrite 4.14 as

\[
y^2 + b \left( \frac{sin \left( \frac{2\pi s}{k} \right)}{1 + \cos \left( \frac{2\pi s}{k} \right)} \right) y + 1 = 0
\]
since $1 + \cos \left( \frac{2\pi s}{k} \right) \neq 0$ for $k$ odd. Similarly, we can rewrite 4.15 as

$$y^2 + b \left( \frac{1 - \cos \left( \frac{2\pi s}{k} \right)}{\sin \left( \frac{2\pi s}{k} \right)} \right) y + 1 = 0$$

(4.17)

since $\sin \left( \frac{2\pi s}{k} \right) \neq 0$ for $k$ odd. Note that equation 4.16 is the same as equation 4.17 because

$$\frac{\sin \left( \frac{2\pi s}{k} \right)}{1 + \cos \left( \frac{2\pi s}{k} \right)} = \frac{1 - \cos \left( \frac{2\pi s}{k} \right)}{\sin \left( \frac{2\pi s}{k} \right)} = \tan \left( \frac{\pi s}{k} \right)$$

(4.18)

when $s \neq 0$. Therefore, we must determine for which $s$

$$y^2 + btan \left( \frac{\pi s}{k} \right) y + 1 = 0$$

(4.19)

gives a real solution for $y$. Specifically,

$$y = \frac{-btan \left( \frac{\pi s}{k} \right) \pm \sqrt{b^2tan^2 \left( \frac{\pi s}{k} \right) - 4}}{2}$$

(4.20)

and thus $y$ is real for those $s$ satisfying $|tan \left( \frac{\pi s}{k} \right)| \geq \frac{2}{b}$, as claimed.

(b) Since the proof is similar in spirit to that of a), we merely give a sketch. To this end, assume that $z = x + iy$ is a zero of $f$ such that $x^2 + y^2 = 1$. Then following the proof of Proposition 4.1.1 up through equation 4.5, we may conclude that

$$\left( \frac{(x + iy)^2 - b(x + iy) - 1}{-(x + iy)^2 - b(x + iy) + 1} \right)^k = 1,$$

(4.21)

and hence

$$\frac{(x + iy)^2 - b(x + iy) - 1}{-(x + iy)^2 - b(x + iy) + 1} = \cos \left( \frac{2\pi s}{k} \right) + isin \left( \frac{2\pi s}{k} \right).$$

(4.22)

Next, we simplify equations 4.21 and 4.22, and then equate the real and imaginary parts.

$$x^2 - y^2 - 1 + bx = (-x^2 + y^2 + 1 + bx)\cos \left( \frac{2\pi s}{k} \right) - y(-2x + b)\sin \left( \frac{2\pi s}{k} \right)$$

(4.23)
and

$$2xy + yb = (-2xy + yb)\cos\left(\frac{2\pi s}{k}\right) + (2y^2 + bx)\sin\left(\frac{2\pi s}{k}\right).$$ (4.24)

We can simplify equation 4.23 further by using $x^2 + y^2 = 1$. Hence,

$$-2y^2 + bx = (2y^2 + bx)\cos\left(\frac{2\pi s}{k}\right) + (2xy - by)\sin\left(\frac{2\pi s}{k}\right).$$ (4.25)

We combine 4.24 and 4.25 by multiplying 4.25 through by $x$, and multiplying 4.24 through by $y$, and then adding the resulting equations. By repeated use of the fact that $x^2 + y^2 = 1$, we get

$$b = b\cos\left(\frac{2\pi s}{k}\right) + 2ysin\left(\frac{2\pi s}{k}\right)$$ (4.26)

so that

$$\frac{b - b\cos\left(\frac{\pi s}{k}\right)}{2\sin\left(\frac{2\pi s}{k}\right)} = y,$$ (4.27)

or

$$y = \frac{b}{2} \tan\left(\frac{\pi s}{k}\right)$$ (4.28)

because $\frac{1 - \cos\left(\frac{2\pi s}{k}\right)}{\sin\left(\frac{2\pi s}{k}\right)} = \tan\left(\frac{\pi s}{k}\right)$. In that case, since $x$ is real and $x = \pm \sqrt{1 - y^2}$, we must have $1 - \left(\frac{b}{2} \tan\left(\frac{\pi s}{k}\right)\right)^2 \geq 0$, or alternatively, that $|\tan\left(\frac{\pi s}{k}\right)| \geq \frac{2}{b}$, as desired. □

Note that in b), $x = \pm 1$ are zeros (of modulus 1) of $f$, which arise when $s = 0$, and in fact correspond to 1, the trivial $k$th root of unity. In the case when $b = 1$, it is not hard to show that $|\tan\left(\frac{\pi s}{k}\right)| = 2$ cannot occur. Hence, there is no overlap in the solutions given by a) and b). Therefore, each $k$th root of unity provides two zeros, and thus in total, we have found $2k$ zeros, which, in fact, can easily shown to
be distinct. These account for all of the zeros of $f$ because solving equation 4.5 is equivalent to finding the roots of a polynomial of degree $2k$. On the other hand, if for some $b$ we can have $|\tan(\frac{\pi}{k})| = \frac{2}{b}$, then $\pm i$ occur as solutions in both a) and b). Since we don’t know about the multiplicity, we have listed only $2k - 2$ zeros. It seems quite likely that $|\tan(\frac{\pi}{k})| = \frac{2}{b}$ never occurs for the particular values of $b$ given in 4.1 and 4.2, but this still remains as an open question.

Note also that, except when $s = 0$, the zeros of modulus 1 obtained in b) come in groups of four, namely $\{\pm \sqrt{1 - y_s^2} \pm iy_s\}$. Similarly, the purely imaginary zeros can be grouped in pairs $\{y_s i, y_{k-s} i\} = \{\pm y_s i\}$, because $\tan(\frac{\pi}{k}) = -\tan(\frac{\pi(k-s)}{k})$. This accounts for some of the beautiful symmetry in the graphs of the zeros of functions like $f$. See, for example, Figure 7 and Figure 8 in Appendix A.

Since the families of RPFs given in 4.1 and 4.2 all satisfy the hypotheses of Proposition 4.1.1, Corollary 4.1.2 essentially provides an explicit description of their zeros.

In fact, the results of this section also apply to the families of RPFs in 4.1 and 4.2 with $k$ a negative odd integer. Under such circumstances, these RPFs are polynomials, but in the computations in Corollary 4.1.1 we replace $k$ with $-k$.

### 4.2 A Conjecture

In general, not much else is known about the zeros of RPFs, as most RPFs do not satisfy the hypotheses of Proposition 4.1.1. However, after extensive experimentation with the aid of Mathematica (see Appendix A), and with the results of this section, it appears that the graphs of the zeros of RPFs with at least one hyperbolic pole are highly symmetric. The following open-ended conjecture is suggested.
Conjecture. Suppose $q$ is an RPF with at least one hyperbolic pole. If \( z = x + iy \) is a zero of $q$, then either $x = 0$ or the ordered pair $(x, y)$ is a solution of $Ax^2 + By^2 + Cx + Dy + Eyz + F = 0$, where $A, B, C, D, E, F \in \mathbb{R}$. In other words, the zeros of $q$ are purely imaginary, or else lie on the graph of some conic section.
Appendix A

Graphs of Zeros of Rational Period Functions

In keeping with the notation of Chapter 4, we write $q_{k,D}$ in place of

$$\sum_{b^2-4ac=D>0} \frac{1}{(ax^2+bx+c)^k}.$$

Figure 7: The zeros of $q_{9,5}$
Figure 8: The zeros of $q_{19,5}$
Figure 9: The zeros of $q_{0.5} + 40(1 - \frac{1}{2\tau})$.

Figure 10: The zeros of $q_{21.5} + 40(1 - \frac{1}{2\tau})$.
Figure 11: The zeros of $q_{9,5} + i(1 - \frac{1}{2\pi})$

Figure 12: The zeros of $q_{21,5} + i(1 - \frac{1}{2\pi})$
Figure 13: The zeros of $q_{3, 8}$

Figure 14: The zeros of $q_{17, 8}$
Figure 15: The zeros of $q_{3,12}$

Figure 16: The zeros of $q_{5,12}$
Figure 17: The zeros of $q_{7,13}$

Figure 18: The zeros of $q_{19,13}$
Figure 19: The zeros of $q_{3.5} + q_{3.17}$

Figure 20: The zeros of $q_{9.5} + q_{9.8}$
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