INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
On the edge reconstruction of planar graphs

Zhao, Yue, Ph.D.

The Ohio State University, 1992
ON THE EDGE-RECONSTRUCTION OF PLANAR GRAPHS

DISSERTATION

Presented in Partial Fulfilment of the Requirement for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Yue Zhao, B.S., M.S.

*****

The Ohio State University
1992

Dissertation Committee:
Professor Neil Robertson, Advisor
Professor Dijen K. Ray-Chaudhuri
Professor Thomas Dowling

Approved by
Neil Robertson
Adviser
Department of Mathematics
ACKNOWLEDGEMENTS

I would like to take this opportunity to express my gratitude to my supervisor, Professor Neil Robertson, for his guidance and for the useful suggestions, assistance and encouragement given to me. I also like to take this opportunity to express my gratitude to Professor Thomas Dowling and Professor Ray-Chaudhuri for their helpful comments.
# VITA

**August 31, 1957**

Born - Beijing, China

**1978-1982**

B.S.

Beijing Institute of Technology, China

**1982-1984**

M.S.

Beijing Institute of Technology, China

**1984-1986**

Instructor

Beijing Institute of Technology, China

**1986-present**

Department of Mathematics

The Ohio State University

## Field of Study

Major Field: Mathematics
# TABLE OF CONTENTS

**ACKNOWLEDGEMENTS** .................................................................................................................. ii

**VITA** .................................................................................................................................................. iii

**LIST OF FIGURES** .......................................................................................................................... vi

**CHAPTER I INTRODUCTION** ........................................................................................................ 1

1.1. What is the Graph Reconstruction Conjecture ........................................................................... 1

1.2. Some Well Known Results .......................................................................................................... 2

1.3. Some Open Problems .................................................................................................................. 3

1.4. Results in This Dissertation ....................................................................................................... 4

**CHAPTER II ON THE EDGE RECONSTRUCTION OF 3-CONNECTED PLANAR GRAPHS WITH MINIMUM VALENCY 4** ........................................................................................................ 6

2.1. Introduction ................................................................................................................................... 6

2.2. Some Simple Facts about G ........................................................................................................ 8

2.3. Bridges, Cut Sets and Connectivity of G - e ............................................................................. 9

2.4. Some Special Cases .................................................................................................................... 16

2.5. The Case of $d_1(G) \geq 2$, $d_2(G) \geq 2$ ................................................................................ 21

2.6. The Case of $d_1(G) \geq 2$, $d_2(G) = 1$ .................................................................................... 22

**CHAPTER III THE EDGE-RECONSTRUCTION OF MINIMALLY 3-CONNECTED PLANAR GRAPHS** ................................................................................................................................. 32

3.1. Introduction .................................................................................................................................. 32

3.2. Some Lemmas ............................................................................................................................. 33
3.3 Proof of Theorem 3.1.1 .................................................................43

CHAPTER IV THE EDGE RECONSTRUCTION OF MAXIMAL BIPARTITE
PLANAR GRAPHS WITH MINIMUM VALENCY 3 .......................44
4.1. Introduction ...........................................................................44
4.2. The Case of Connectivity 3 ..................................................45
4.3. Maximal Bipartite Planar Graphs with Minimum Valency 3 .....52

CHAPTER V THE EDGE RECONSTRUCTION OF 3-CONNECTER BIPARTITE
PLANAR GRAPHS .........................................................................63
5.1. Introduction ...........................................................................63
5.2. Proof of the Theorem 5.1.1 ....................................................64

CHAPTER VI ON THE EDGE RECONSTRUCTION OF 3-CONNECTED
PLANAR GRAPHS .........................................................................79
6.1. Introduction ...........................................................................79
6.2. Proof of the Theorem 6.1.1 ....................................................80

REFERENCES ..................................................................................94
LIST OF FIGURES

FIGURE 2.1 .........................................................................................................10
FIGURE 2.2 .........................................................................................................12
FIGURE 2.3 .........................................................................................................13
FIGURE 2.4 .........................................................................................................14
FIGURE 2.5 .........................................................................................................15
FIGURE 2.6 .........................................................................................................16
FIGURE 2.7 .........................................................................................................25
FIGURE 2.8 .........................................................................................................26
FIGURE 3.1 ........................................................................................................36
FIGURE 3.2 .........................................................................................................38
FIGURE 3.3 .........................................................................................................40
FIGURE 3.4 .........................................................................................................41
FIGURE 4.1 .........................................................................................................47
FIGURE 4.2 .........................................................................................................51
FIGURE 4.3 .........................................................................................................52
FIGURE 4.4 .........................................................................................................55
FIGURE 4.5 .........................................................................................................56
FIGURE 4.6 .........................................................................................................60
FIGURE 4.7 .........................................................................................................61
FIGURE 5.1 .........................................................................................................66
FIGURE 5.2 .........................................................................................................70
CHAPTER I
INTRODUCTION

1.1. What is the Graph Reconstruction Conjecture?

The Graph Reconstruction Conjecture is generally regarded to be one of the foremost unsolved problems in graph theory. Indeed, Harary [12] has even classified it as a "graphical disease" because of its contagious nature. This conjecture was first proposed by Kelly and Ulam (see [3]) in 1941 in its vertex version, i.e. the vertex reconstruction conjecture. In 1964, Harary [13] formulated this conjecture in its edge version, i.e. the edge reconstruction conjecture.

A graph $G$ with vertex set $V(G)$ and edge set $E(G)$ considered here is finite and simple, i.e. $G$ contains no loops or multiple edges. We call $|V(G)|$ the order of $G$ and $|E(G)|$ the size of $G$. In the vertex reconstruction problem, we form the vertex deck of a graph $G$ by taking the multiset of unlabelled subgraphs of $G$

$$\mathcal{D}_V(G) = \{G- v; \ v \in \ V(G)\}.$$ 

We say that $G$ is vertex reconstructible if $\mathcal{D}_V(G) = \mathcal{D}_V(H)$ implies $G \cong H$, where $\cong$ means graph isomorphism. Note that the only known graphs which are not vertex reconstructible are $2K_1$ and $K_2$, where $2K_1$ is an edgeless graph with two vertices and $K_2$ is a graph with two vertices and a single edge. It is clear that $2K_1$ and $K_2$ have the same vertex deck, but they are not isomorphic. This suggests the following conjecture, due to Kelly and Ulam in 1941.
VERTEX RECONSTRUCTION CONJECTURE: If $|V(G)| \geq 3$, then $G$ is vertex reconstructible.

In the edge reconstruction problem, we form the edge deck of a graph $G$ by taking the multiset of unlabelled maximal subgraphs of $G$

$$\mathcal{D}_E(G) = \{G - e; e \in E(G)\}.$$ We say that $G$ is edge reconstructible if $\mathcal{D}_E(G) = \mathcal{D}_E(H)$ implies $G \cong H$. The only graphs which are not edge reconstructible occur in two pairs, $P_3 \cup K_1$, $2K_2$, and $K_{1,3}$ (the graph $K_{1,3}$ is also called a claw), $K_3 \cup K_1$ where $V(P_3 \cup K_1) = \{v_1, v_2, v_3, v_4\}$, $E(P_3 \cup K_1) = \{v_1v_2, v_2v_3\}$, $V(2K_2) = \{u_1, u_2, u_3, u_4\}$, $E(2K_2) = \{u_1u_2, u_3u_4\}$, $V(K_{1,3}) = \{x_1, x_2, x_3, x_4\}$, $E(K_{1,3}) = \{x_1x_2, x_1x_3, x_1x_4\}$ and $V(K_3 \cup K_1) = \{y_1, y_2, y_3, y_4\}$, $E(K_3 \cup K_1) = \{y_1y_2, y_2y_3, y_1y_3\}$. It is clear that for either pair $P_3 \cup K_1$, $2K_2$ or $K_{1,3}$, $K_3 \cup K_1$ the two graphs have the same edge deck, but they are not isomorphic. This suggests the following conjecture due to Harary in 1964 with which we will be concerned in this dissertation.

EDGE RECONSTRUCTION CONJECTURE: If $|E(G)| \geq 4$, then $G$ is edge reconstructible.

1.2. Some Well Known Results

Since 1941, various people have made many attempts to solve the reconstruction conjecture. A number of classes of graphs and graph parameters have been shown to be reconstructible, and conditions sufficient to guarantee reconstructibility have been discovered. These results should be considered by other researchers in their approaches to the reconstruction conjecture either trying to prove
other classes of graphs are reconstructible or to find some counter-examples of the reconstruction conjecture in other classes of graphs.

Classes of graphs which are known to satisfy the vertex reconstruction conjecture include regular graphs, trees, disconnected graphs (all due to Kelly—see [3]), graphs of order at most nine (Mckay [18]), maximal planar graphs (Fiorini and Lauri [6], Lauri [16]), separable graphs without pendant vertices (Bondy [2]) and 2-trees (Le Fever and Ray-Chaudhuri [17]).

In 1971, Greenwell [9] showed that, for a graph without isolated vertices, vertex reconstructibility implies edge reconstructibility. Therefore, regular graphs, disconnected graphs with at least two nontrivial components, graphs of order at most nine, maximal planar graphs, separable graphs without pendant vertices and 2-trees are all edge reconstructible as well as vertex reconstructible if their sizes are at least four. There are also classes of graphs which are known to satisfy the edge reconstruction conjecture, but which are not known to satisfy the vertex reconstruction conjecture. These include graphs with $|E(G)| > |V(G)|(|\log_2|V(G)| - 1)$ (Muller [19]), hamiltonian graphs of sufficiently large order (Pyber [21]), bidegreed graphs (Myrvold, Ellingham and Hoffman [20]), claw-free graphs (Ellingham, Pyber and Yu [5]), 4-connected planar graphs (Fiorini and Lauri [7]) and connected planar graphs with minimum valency 5 (Lauri [14]).

1.3. Some Open Problems

In 1977, More than thirty five years had passed since the reconstruction conjecture was proposed by Kelly and Ulam. During this period of time many classes of graphs were shown to be vertex or edge reconstructible. But the conjecture was far from being completely solved. The question then was what will be the next step in an
approach to the reconstruction conjecture? In 1977, in their well-known survey paper of the graph reconstruction, Bondy and Hemminger proposed the following problems, some of which (for example, problem 1 and problem 3) can be viewed as the next step following the reconstruction of regular graphs and separable graphs without pendant vertices and some of which (for example, problem 7 and problem 10) are interesting in their own right.

PROBLEM 1. Show that bidegreed graphs are vertex reconstructible.

PROBLEM 2. Show that bipartite graphs are vertex reconstructible.

PROBLEM 3. Show that separable graphs with pendant vertices are vertex reconstructible. (Even the reconstruction of separable graphs with two blocks, one of which is $K_2$, would be considered a worthwhile achievement).

PROBLEM 7. Show that planar graphs are vertex reconstructible.

PROBLEM 9. Show that bipartite graphs are edge reconstructible.

PROBLEM 10. Show that planar graphs are edge reconstructible.

However, because of this conjecture's elusive nature, the above proposed problems, even though apparently very special cases of the reconstruction conjecture, have not been completely solved for fifteen years.

1.4. Results in This Dissertation
Since Bondy and Hemminger's survey paper was published, many people have attempted to prove that planar graphs and especially 3-connected planar graphs are edge reconstructible. From 1977 to 1982, the edge reconstruction of maximal planar graphs, 4-connected planar graphs and connected planar graphs of minimum valency 5 was proved. In this dissertation, in an attempt to solve the edge reconstructibility of 2- and 3-connected planar graphs, we prove the following results on the edge reconstruction of planar graphs.

In chapter II, we prove that 3-connected planar graphs with minimum valency 4 are edge reconstructible if no 4-vertex is adjacent to a 5-vertex.

In chapter III, we prove that minimally 3-connected planar graphs are edge reconstructible.

In chapter IV, we prove that maximal bipartite planar graphs with minimum valency 3 are edge reconstructible.

In chapter V, we prove that 3-connected bipartite planar graphs are edge reconstructible.

In chapter VI, we prove that a 3-connected planar graph G with minimum valency 4 is edge reconstructible if for every 4-vertex v of G, the induced subgraph of G, by the vertex v and the neighbors of the vertex v in G, is a claw-free graph (i.e. a $K_{1,3}$ free graph).

The results of this dissertation listed above have shown the edge reconstructibility of certain subclasses of 2- and 3-connected planar graphs. To complete the whole conjecture for these classes of graphs, further research is planned in the future.
CHAPTER II
ON THE EDGE RECONSTRUCTION OF 3-CONNECTED PLANAR GRAPHS WITH MINIMUM
VALENCY 4

2.1. Introduction

In this chapter and also in the following chapters, all graphs \( G = (V(G), E(G)) \) considered will be finite and simple. A connected graph \( G \) is said to have connectivity \( k = k(G) \) if the deletion of some set of \( k \) vertices disconnects \( G \) and \( k \) is the least integer with this property. For any \( j \leq k \), \( G \) is said to be \( j \)-connected. If \( k(G) = 1 \), \( G \) is said to be separable. A set \( S \) of vertices is said to be a cut-set of \( G \) if the deletion of the vertices of \( S \) from \( G \) disconnects \( G \). A graph is planar if it can be embedded in the plane and such an embedded graph is called a plane graph. A well-known result states that a 3-connected planar graph has a unique embedding (up to preserving face boundaries) in the plane. Thus, when \( G \) is a 3-connected planar graph we may assume with no loss of generality that \( G \) is a plane graph. A subgraph \( C[a, b] \) is said to be a chain (or a path) from \( a \) to \( b \), or a \([a, b]\)-chain, if

\[
V(C[a, b]) = \{a = v_0, v_1, ..., v_t = b, v_i \neq v_j \text{ if } i \neq j\}
\]

and

\[
E(C[a, b]) = \{v_i v_{i+1}; i = 0, 1, ..., t - 1\}.
\]

Sometimes we also write \( C[a, b] \) as \( v_0v_1...v_t \) where \( a = v_0, b = v_t \). In the above definition of a chain \( C[a, b] \), if \( a = b \), then it is called a circuit. If a circuit consists of \( k \)
edges, we call it a \textit{k-circuit}. A \textit{k-vertex} of a graph is a vertex of valency \(k\). A \textit{k-face} of a plane graph is a face whose boundary is a \(k\)-circuit. We use \(d(v)\) and \(s(x, y)\) to denote the degree of a vertex \(v\) and the distance between vertices \(x\) and \(y\), respectively. We define

\[
\begin{align*}
    s_0(G) &= \min\{s(x, y); x, y \in V(G) \text{ with } d(x) = d(y) = 4\}, \\
    s_1(G) &= \min\{s(x, y); x, y \in V(G) \text{ with } d(x) = 4, d(y) = 5\}, \\
    s_2(G) &= \min\{s(x, y); x, y \in V(G) \text{ with } d(x) = d(y) = 5\}.
\end{align*}
\]

Let \(\Gamma\) be a circuit of a plane graph \(G\). A \(\Gamma\)-\textit{avoiding} chain is a chain \(C[a_0, a_t]\) in which neither its edges nor its vertices belong to \(\Gamma\), except possibly the endpoints \(a_0, a_t\). Two edges \(e_1, e_2\) are said to be connected outside of \(\Gamma\) if there exists a \(\Gamma\)-avoiding chain beginning in \(e_1\) and ending in \(e_2\). We say that \(e_1\) and \(e_2\) are \textit{bridge-equivalent} with respect to \(\Gamma\) if they are connected outside of \(\Gamma\). All edges which are bridge equivalent to an edge \(e\) induce a connected subgraph \(B = B(e)\) called a \textit{bridge} of \(\Gamma\) in \(G\). A bridge can only have vertices in common with \(\Gamma\). Such vertices will be called \textit{vertices of attachment} of the bridge of \(\Gamma\) in \(G\). The set of vertices of attachment of the bridge \(B\) in \(G\) is denoted by \(A(G, B)\). Since a closed Jordan curve \(\Gamma\) divides the rest of the plane into two connected open domains, the exterior of \(\Gamma\) and the interior of \(\Gamma\), the bridges of a circuit \(\Gamma\) of a plane graph \(G\) fall into two categories; \textit{outer bridges} which are in the exterior of \(\Gamma\), and \textit{inner bridges} which are in the interior of \(\Gamma\). The theory of bridges of planar graphs is dealt with in detail in Ore's book [22], to which the reader is referred.

The following definition can be found in [22]. A nonseparable graph \(G\) is called \textit{properly two-vertex separable} (or \(G\) has a \textit{proper two-vertex separation}) when \(G\) has a decomposition

\[
G = H_1 + H_2
\]

in which \(H_1\) and \(H_2\) are edge disjoint connected subgraphs of \(G\) which are not chains, and have exactly two common vertices \(a_1\) and \(a_2\).
A graph $G$ is said to be edge-reconstructible if it can be determined uniquely (up to isomorphism) from the family (the edge-deck) $D(G) = \{G_e; G_e = G - e, e \in E(G)\}$ of single-edge-deleted subgraphs of $G$. The edge form of the reconstruction conjecture states that every graph with at least four edges is edge reconstructible. A graph $H$ is called an edge-reconstruction of $G$ if $D(G) = D(H)$. Therefore, when we say that $G$ is edge-reconstructible we mean that every edge-reconstruction (or reconstruction, for short) of $G$ is isomorphic to $G$. It is shown, in [14], that a 3-connected planar graph with minimum valency 5 is edge-reconstructible. In this chapter, we show that a 3-connected planar graph with minimum valency 4 is edge-reconstructible if no 4-vertex is adjacent to a 5-vertex.

2.2. Some Simple Facts about $G$

LEMMA 2.2.1. Let $G$ be a 3-connected planar graph with minimum valency 4. If $s_0(G) \geq 2$, $s_1(G) \geq 2$ and $s_2(G) \geq 2$ then one of the following cases must occur:

(i) $G$ contains a 4-vertex only incident to 3-faces;

(ii) $G$ contains a 4-vertex incident to three 3-faces and one 4-face;

(iii) $G$ contains a 4-vertex incident to three 3-faces and one 5-face with two 4-vertices on the boundary of the 5-face;

(iv) $G$ contains a 5-vertex incident only to 3-faces.

Proof. Suppose that the claim of this lemma is not true, then each 4-vertex $v$ of $G$ is incident to either at least one $k$-face with $k \geq 5$ or at least two faces with face valencies not less than 4, and each 5-vertex $v$ of $G$ is incident to at least one $k$-face with $k \geq 4$. Let $u$ be a vertex of $G$ with valency less than 6. We want to construct a new planar graph $G'$ from $G$ which has the following properties: (1) $d_G(u) \geq 6$; (2) each 4-
vertex $v$ of $G'$ (if there exist 4-vertices) is incident to either at least one $k$-face with $k \geq 5$ or at least two faces with face valencies not less than 4, and each 5-vertex $v$ of $G'$ (if there exist 5-vertices) is incident to at least one $k$-face with $k \geq 4$; (3) $s_i(G') \geq 2$ for $i = 0, 1, 2$. This construction is possible since $s_i(G) \geq 2$ for $i = 0, 1, 2$. Clearly, the number of vertices of valencies less than 6 in $G'$ is less than the number of vertices of valencies less than 6 in $G$. We choose a vertex $w$ of $G'$ with valency less than 6 and construct another planar graph $H'$ from $G'$ which has the following properties: (1) $d_H(w) \geq 6$; (2) each 4-vertex $v$ of $H'$ (if there exist 4-vertices) is incident to either at least one $k$-face with $k \geq 5$ or at least two faces with face valencies not less than 4, and each 5-vertex $v$ of $H'$ (if there exist 5-vertices) is incident to at least one $k$-face with $k \geq 4$; (3) $s_i(H') \geq 2$ for $i = 0, 1, 2$. Continue in this way, we can increase the valencies of all 4- and 5-vertices in $G$ to at least 6 by adding edges and still keep the planarity of the graph. Therefore we get a new graph $G''$ which is a planar graph with minimum valency at least 6. This is a contradiction.

2.3. Bridges, Cut Sets and Connectivity of $G - e$

LEMMA 2.3.1. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.1 as a subgraph, where $d(v) = \alpha \geq 4$, $v$ is incident to $\alpha$ faces and let $C$ be the circuit with $V(C) = \{v_i; i = 0, 1, \ldots, t-1\}$, $E(C) = \{v_i v_{i+1}; i = 0, 1, \ldots, t-1, v_t = v_0\}$. If $G - vv_0$ is not 3-connected, then there exists an integer $j$ such that $j \in \{k, k + 1, \ldots, t - 1\}$ and $\{v_0, v_i, v_j\}$ is a cut set of $G$, $G - vv_i$ is 3-connected and there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_i, v_j, \ldots, v_{t-1}\}$. 

\[ \alpha \leq k \leq t - 1; \ v_i = v_2, \ v_k = v_{t-1} \] are allowed and if \( i \neq 2 \) and \( k \neq t - 1 \),

\( v \) is not adjacent to \( v_p \) with \( 2 \leq p < i \) and \( k < p \leq t - 1 \).

**Figure 2.1**

**Proof.** Since \( G - vv_0 \) is not 3-connected, there is a proper two-vertex separation of \( G - vv_0 \) such that

\[ G - vv_0 = H_1 + H_2 \]

in which \( E(H_1) \cap E(H_2) = \emptyset \) and \( V(H_1) \cap V(H_2) = \{x, y\} \). Clearly, \( v \notin \{x, y\} \). This is because if \( \{v, w\} \) were a cut set of \( G - vv_0 \), then it would also be a cut set of \( G \), which is impossible, since \( G \) is 3-connected. Hence, without loss of generality, we can assume that \( v \notin V(H_2) \). Therefore \( v \in V(H_1) \). We claim that \( v_0 \notin V(H_1) \). If our claim is not true, then \( v_0 \in V(H_1) \). But that \( v, v_0 \in V(H_1) \) implies that \( \{x, y\} \) is also a cut set of \( G \), which contradicts the fact that \( G \) is 3-connected. Hence our claim is true. Since \( v \in V(H_1) - V(H_2) \), \( v_0 \in V(H_2) - V(H_1) \) and \( v_1 \) is adjacent to both \( v \) and \( v_0 \), then \( v_1 \in V(H_1) \cap V(H_2) \). Hence \( v_1 \in \{x, y\} \). Let \( v_1 = x \). We claim that \( y \in \{v_k, \ldots, v_{t-1}\} \). Since \( v \in V(H_1) - V(H_2) \) and \( v \) is adjacent to \( v_k \), \( v_k \in V(H_1) \). If \( v_k \notin V(H_1) \cap V(H_2) \), then \( v_k \)
\( e \in V(H_1) - V(H_2) \). Since \( v_0 \in V(H_2) - V(H_1) \) and there is a path \( v_kv_{k+1}...v_{t-1}v_0 \) in \( G - vv_0 \) which joins \( v_k \) to \( v_0 \), it is clear that there exists \( j \in \{ k + 1, ..., t - 1 \} \) such that \( v_j \in V(H_1) \cap V(H_2) \). Hence \( y \in \{ v_k, ..., v_{t-1} \} \). Therefore \( \{ v_1, v_j \} \) is a cut set of \( G - vv_0 \).

Among all integers \( j' \) such that \( j' \in \{ k, ..., t - 1 \} \) and \( \{ v_1, v_{j'} \} \) is a cut set of \( G - vv_0 \), we choose \( j \) to be the largest one. By this choice, it is clear that there is an outer bridge \( B \) of the circuit \( C \) in \( G \) with \( A(G, B) = \{ v_0, v_1, v_j, ..., v_{t-1} \} \). Now we show that \( G - vv_1 \) is 3-connected. Assume that \( G - vv_1 \) is not 3-connected, by applying what we just proved to \( G - vv_1 \), we obtain that \( \{ v_0, v_1, v_p \} \) for some \( p \in \{ 2, ..., i \} \) is a cut set of \( G \) and there is an outer bridge \( D \) of \( C \) in \( G \) with \( A(G, D) = \{ v_0, v_1, ..., v_p \} \). Therefore \( A(G, D) \cap A(G, B) = \{ v_0, v_1 \} \), which contradicts the fact that \( G \) is a planar graph. Hence, \( G - vv_1 \) is 3-connected.

We call an edge \( e \in E(G) \) a good edge if \( G - e \) is still 3-connected. Otherwise, we call it a bad edge. Now we apply Lemma 2.3.1 to the graph \( G \) and we can obtain the following corollaries.

**COROLLARY 2.3.2.** Let \( G \) be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.2 as a subgraph, where \( v \) is a 4-vertex incident to three 3-faces and one 4-face and let \( C \) be the circuit with \( V(C) = \{ v_i; i = 0, 1, ..., 4 \} \), \( E(C) = \{ v_iv_{i+1}; i = 0, 1, ..., 4, v_5 = v_0 \} \). If \( G - vv_0 \) is not 3-connected, then \( \{ v_0, v_1, v_4 \} \) or \( \{ v_0, v_1, v_3 \} \) is a cut set of \( G \), \( G - vv_1 \) is 3-connected and there is an outer bridge \( B \) of the circuit \( C \) in \( G \) with \( A(G, B) = \{ v_0, v_1, v_4 \} \) or \( \{ v_0, v_1, v_3, v_4 \} \).
COROLLARY 2.3.3. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.2 as a subgraph. If there are bad edges among $vv_0$, $vv_1$, $vv_2$ and $vv_3$, then one of them must be either $vv_0$ or $vv_3$.

Proof. If this is not true, without loss of generality, we may assume that $vv_1$ is a bad edge. By Lemma 2.3.1, there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_2\}$. Clearly, in this case, $vv_3$ must be a bad edge of $G$. 

Therefore, if there are bad edges among $vv_0$, $vv_1$, $vv_2$ and $vv_3$, without loss of generality, we will assume that $vv_0$ is a bad edge.

COROLLARY 2.3.4. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.2 as a subgraph. Then one of $vv_2$ and $vv_3$ is a good edge of $G$. 

Figure 2.2
Proof. If $v v_3$ is a good edge of $G$, we are done. If $v v_3$ is not a good edge of $G$, by Lemma 2.3.1, $v v_2$ is a good edge of $G$. 

COROLLARY 2.3.5. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing a graph in Figure 2.3 as a subgraph, where the vertex $v$ is only incident to 3-faces. If $G - v v_0$ is not 3-connected, then $G - v v_1$ is 3-connected.

\[ \text{Figure 2.3} \]

COROLLARY 2.3.6. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.4 as a subgraph, where the 4-vertex $v$ is incident to three 3-faces and one 5-face and let $C$ be the circuit with $V(C) = \{v_i, i = 0, 1, \ldots, 5\}$, $E(C) = \{v_i v_{i+1}, i = 0, 1, \ldots, 5, v_6 = v_0\}$. If $G - v v_0$ is not 3-connected, then $\{v_0, v_1, v_5\}$ or $\{v_0, v_1, v_4\}$ or $\{v_0, v_1, v_3\}$ is a cut set of $G$, $G - v v_1$ is 3-connected.
and there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_5\}$ or $\{v_0, v_1, v_4, v_5\}$ or $\{v_0, v_1, v_3, v_4, v_5\}$.

**Figure 2.4**

**COROLLARY 2.3.7.** Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.4 as a subgraph. If there are bad edges among $vv_0$, $vv_1$, $vv_2$ and $vv_3$, then one of them must be either $vv_0$ or $vv_3$.

The proof is similar to the one of Corollary 2.3.3.

**COROLLARY 2.3.8.** Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.4 as a subgraph. Then one of $vv_2$ and $vv_3$ is a good edge of $G$.

The proof is similar to the one of Corollary 2.3.4.
COROLLARY 2.3.9. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.5 as a subgraph, where the 5-vertex $v$ is incident to four 3-faces and one 5-face and let $C$ be the circuit with $V(C) = \{v_i; i = 0, 1, \ldots, 6\}$, $E(C) = \{v_iv_{i+1}; i = 0, 1, \ldots, 6, v_7 = v_0\}$. If $G - vv_0$ is not 3-connected, then $\{v_0, v_1, v_6\}$ or $\{v_0, v_1, v_5\}$ or $\{v_0, v_1, v_4\}$ is a cut set of $G$, $G - vv_1$ is 3-connected and there is an outer bridge $B$ of the circuit $C$ in $G$ with $\Delta(G, B) = \{v_0, v_1, v_6\}$ or $\{v_0, v_1, v_5, v_6\}$ or $\{v_0, v_1, v_4, v_5, v_6\}$.

![Figure 2.5](image)

COROLLARY 2.3.10. Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.5 as a subgraph. If $G - vv_0$ is not 3-connected, then $G - vv_1$ and $G - vv_i$, $i = 2$ or 3, are 3-connected.

Proof. By Lemma 2.3.1, if $G - vv_0$ is not 3-connected, then $G - vv_1$ is 3-connected. Similarly, if $G - vv_2$ is not 3-connected, $G - vv_3$ must be 3-connected.
Therefore, if $G - vv_0$ is not 3-connected, then $G - vv_1$ and $G - vv_i$, $i = 2$ or $3$, are 3-connected.

**COROLLARY 2.3.11.** Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.6 as a subgraph, where the 5-vertex $v$ is incident to four 3-faces and one 4-face and let $C$ be the circuit with $V(C) = \{v_i; i = 0, 1, \ldots, 5\}$, $E(C) = \{v_i v_{i+1}; i = 0, 1, \ldots, 5, v_6 = v_0\}$. If $G - vv_0$ is not 3-connected, then $\{v_0, v_1, v_5\}$ or $\{v_0, v_1, v_4\}$ is a cut set of $G$, $G - vv_1$ is 3-connected and there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_5\}$ or $\{v_0, v_1, v_4, v_5\}$.

![Figure 2.6](image)

**COROLLARY 2.3.12.** Let $G$ be a 3-connected planar graph with minimum valency 4 and containing the graph in Figure 2.6 as a subgraph. If $G - vv_0$ is not 3-connected, then $G - vv_1$ and $G - vv_i$, $i = 2$ or $3$, are 3-connected.

**2.4. Some Special Cases**
Through the rest of this chapter, we will assume that all graphs considered will satisfy \(|V(G)| \geq 10\), since \(G\) is shown in [3] to be edge-reconstructible if \(|V(G)| < 10\). Under this hypothesis, if \(G\) is a 3-connected planar graph with minimum valency 4, the planarity of \(G\) is shown in [8] to be recognizable from its edge-deck. This means that all reconstructions of \(G\) considered in this chapter are planar graphs. We will often use this fact in our proofs without a further explanation. Since a graph \(G\) with minimum valency 4 with two adjacent 4-vertices is obviously edge reconstructible, we will assume without loss of generality that \(s_0(G) \geq 2\).

**Lemma 2.4.1.** If \(G\) is a 3-connected planar graph with minimum valency 4 which contains the graph in Figure 2.2 as a subgraph, then \(G\) is edge-reconstructible.

**Proof.** We will first prove the following claim: If \(G\) is not edge-reconstructible, then there is only one reconstruction of \(G\) which is not isomorphic to \(G\). Assume that \(H\) is a reconstruction of \(G\) that is not isomorphic to \(G\). By Corollary 2.3.2, there exists an edge \(w_i\), for \(i \in \{0, 1, 2, 3\}\), such that \(G - w_i\) is 3-connected. Assume \(i = 1\). Since \(G - w_i\) is a 3-connected planar graph, it has a unique planar representation. Since we can identify the vertex \(v\) in \(G - w_1\), we can obtain reconstructions from \(G - w_1\) in only two ways: as \(G\) or as \((G - v v_1) + v v_4\). Thus \(H\) is isomorphic to \((G - v v_1) + v v_4\). Since \(H\) is arbitrary, this argument does not depend on the choice of the good edge incident to \(v\). This proves the claim.

To prove Lemma 2.4.1, we assume that \(G\) is not edge-reconstructible.

Case 1. Suppose that for all \(vv_i\), \(i = 0, 1, 2, 3\), \(G - vv_i\) is 3-connected. We can construct reconstructions \(G_1, G_2\) of \(G\) as follows: \(G_1 = (G - vv_0) + vv_4\), \(G_2 = (G - vv_1) + vv_4\). By our above claim, we have \(G_1 \cong G_2\), but \(G_1\) is not isomorphic to \(G\). Let \(C\) be as defined in Corollary 2.3.2. Since we only changed the inner bridge of \(C\) in \(G\) to
obtain $G_1$, we can prove that $vv_1$ is still a good edge in $G_1$. This is because if $vv_1$ is not a good edge in $G_1$, by Corollary 2.3.2, we would have an outer bridge $B$ of the circuit $C$ in $G_1$ such that $A(G_1, B) = \{v_0, v_1, v_2\}$ or $\{v_0, v_1, v_2, v_4\}$. Clearly, $B$ is also an outer bridge of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_2\}$ or $\{v_0, v_1, v_2, v_4\}$. But $A(G, B) = \{v_0, v_1, v_2\}$ implies that $G - vv_1$ is not 3-connected, contrary to our assumption. Hence $A(G, B)$ must be $\{v_0, v_1, v_2, v_4\}$. In this case, there exists another outer bridge $D$ of the circuit $C$ in $G$ with $A(G, D) = \{v_2, v_3, v_4\}$. This implies that $G - vv_3$ is not 3-connected, we have another contradiction. Since $vv_1$ is a good edge in $G_1$, the graph $G_2$ can also be obtained from $G_1$ by deleting the edge $vv_1$ and adding the edge $vv_0$. Therefore $G_2$ is a reconstruction of $G_1$. Since $G$ is not edge-reconstructible, $G_1$ is not edge-reconstructible. By our above claim, $G_2$ is not isomorphic to $G_1$, which is a contradiction. Thus in this case, $G$ is edge reconstructible.

Case 2. There exists an edge $vv_i$, for $i \in \{0, 1, 2, 3\}$, such that $G - vv_i$ is not 3-connected. By Corollary 2.3.3, we can assume $i = 0$. Therefore, there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_4\}$ or $\{v_0, v_1, v_3, v_4\}$. Since $vv_0$ is not a good edge of $G$, then $vv_1$ is a good edge of $G$. By Corollary 2.3.4, one of $vv_2$ and $vv_3$ is a good edge of $G$. First, we assume that $vv_2$ is a good edge of $G$. Let $G_1 = (G - vv_1) + vv_4$ and $G_1' = (G - vv_2) + vv_4$. By our claim, $G_1$ and $G_2$ are reconstructions of $G$ such that $G_1 \cong G_1'$ and $G_1$ is not isomorphic to $G$. Since we only changed the inner bridge of the circuit $C$ in $G$ to get $G_1$, we can show that $vv_2$ is also a good edge in $G_1$. This is because if $vv_2$ is not a good edge in $G_1$, we would have an outer bridge $B'$ of the circuit $C$ in $G_1$ with $A(G_1, B') = \{v_1, v_2, v_3\}$ or $\{v_0, v_1, v_2, v_3\}$. Clearly, $B'$ is also an outer bridge of the circuit $C$ in $G$. Since $G$ is a planar graph and $A(G, B) = \{v_0, v_1, v_4\}$, $A(G, B') \neq \{v_0, v_1, v_2, v_3\}$. Hence $A(G, B') = A(G_1, B') = \{v_1, v_2, v_3\}$. But in this case, $vv_2$ is a bad edge of $G$. This is a contradiction and it follows that $vv_2$ is a good edge of $G_1$. Let $G_2 = (G_1 - vv_2) + vv_1$. Then $G_2$ is a
reconstruction of $G_1$ that is not isomorphic to $G_1$. But $G_2 \cong G_1'$, which is a contradiction and it follows that $vv_2$ can not be a good edge of $G$. Therefore $vv_3$ must be a good edge of $G$.

Now, we have that both $vv_1$ and $vv_3$ are good edges of $G$. Let $G_1 = (G - vv_1) + vv_4$ and $G_1' = (G - vv_3) + vv_4$. Then $G_1$ and $G_2$ are reconstructions of $G$ with $G_1 \cong G_1'$, but $G_1$ is not isomorphic to $G$. Since $vv_3$ is also a good edge in $G_1$, the graph $G_2 = (G_1 - vv_3) + vv_1$ is a reconstruction of $G_1$ not isomorphic to $G_1$. But $G_2 \cong G_1'$, this is a contradiction, which implies that $G$ is edge reconstructible.

Lemma 2.4.2. If $G$ is a 3-connected planar graph with minimum valency 4 and contains the graph (a) in Figure 2.3 as a subgraph, then $G$ is edge-reconstructible.

Proof. By Corollary 2.3.5, there exists an edge $e$ incident to $v$ such that $G - e$ is 3-connected. Since a 3-connected planar graph has a unique planar representation, we can reconstruct $G$ from $G - e$ by joining the unique 3-vertex $v$ of $G - e$ and the unique vertex on the 4-face to which $v$ is not adjacent. Therefore $G$ is edge-reconstructible. #

In the previous lemmas of this section, we did not make any assumptions about $s_1(G)$ and $s_2(G)$, but in the next lemma, we will assume $s_1(G) \geq 2$.

Lemma 2.4.3. Assume $s_1(G) \geq 2$. If $G$ is a 3-connected planar graph with minimum valency 4 which contains the graph in Figure 2.4 as a subgraph with $d(v_4) = 4$ or $d(v_5) = 4$, then $G$ is edge-reconstructible.

Proof. Without loss of generality, we assume that $d(v_4) = 4$ then $d(v_5) > 4$. Assume that $G$ is not edge-reconstructible. By Corollary 2.3.6, we know that there
exists an edge vw such that $G - vw$ is 3-connected. Since a 3-connected planar graph has a unique planar representation and $\{d(v), d(w)\}$ is edge-reconstructible by [10], we cannot obtain a reconstruction of $G$ by adding an edge which joins the 3-vertex $v$ in $G - vw$ to $v_4$. Therefore, there is only one reconstruction $H$ of $G$ which is not isomorphic to $G$.

If for all $vv_i$, $i = 0, 1, 2, 3$, $G - vv_i$ is 3-connected, we can use a similar proof to the one of Lemma 2.4.1 to show that $G$ is edge-reconstructible.

If there is an edge $vv_i$ such that $G - vv_i$ is not 3-connected, by Corollary 2.3.7, we know that we can assume $i = 0$. Then $vv_1$ is a good edge of $G$ and by Corollary 2.3.8, one of $vv_2$ and $vv_3$ is a good edge of $G$. First, we consider the case that $vv_2$ is a good edge of $G$. As we did in the proof of Lemma 2.4.1, we construct reconstructions $G_1$ and $G_1'$ of $G$ as follows: $G_1 = (G - vv_1) + vv_5$ and $G_1' = (G - vv_2) + vv_5$. Then $G_1 \cong G_1'$ and $G_1$ is not isomorphic to $G$. Clearly, $vv_2$ is also a good edge in $G_1$, otherwise, we would have an outer bridge $B'$ of the circuit $C$ in $G_1$ with $A(G_1, B') = \{v_1, v_2, v_3\}$, where the definition of the circuit $C$ is the same as one in Corollary 2.3.6. Therefore there is an outer bridge $B'$ of the circuit $C$ in $G$ with $A(G, B') = \{v_1, v_2, v_3\}$, which implies that $vv_2$ is not a good edge in $G$. This is a contradiction. Therefore, we can construct a reconstruction $G_2$ of $G_1$ by $G_2 = (G_1 - vv_2) + vv_1$. Then $G_2$ is not isomorphic to $G_1$. But $G_2 \cong G_1'$, this is a contradiction. Hence $vv_2$ cannot be a good edge of $G$. Thus $vv_3$ is a good edge of $G$. For the case that $vv_3$ is a good edge in $G$, by using a similar proof as we used above, we can also get a contradiction. Therefore $G$ is edge-reconstructible under the conditions of our lemma. #

**Lemma 2.4.4.** If $G$ is a 3-connected planar graph with minimum valency 4 and contains the graph (b) in Figure 2.3 as a subgraph, then $G$ is edge-reconstructible.
Proof. If $G$ contains the graph in Figure 2.2 as a subgraph, by Lemma 2.4.1, $G$ is edge-reconstructible. Therefore we assume that $G$ does not contain such a graph as a subgraph. By Corollary 2.3.5, there exists an edge $e$ incident to $v$ such that $G - e$ is still 3-connected. Since a 3-connected planar graph has a unique planar representation and $G$ does not contain a 4-vertex incident to three 3-faces and one 4-face, we can reconstruct $G$ from $G - e$ by joining the unique 4-vertex, which is incident to one 4-face and three 3-faces, and the unique vertex on the 4-face to which $v$ is not adjacent. Hence $G$ is edge reconstructible.

2.5. The Case of $s_1(G) \geq 2$, $s_2(G) \geq 2$

**Theorem 2.5.1.** If $G$ is a 3-connected planar graph with minimum valency 4 and $s_1(G) \geq 2$, $s_2(G) \geq 2$, then $G$ is edge-reconstructible.

Proof. Since $s_1(G) \geq 2$ and $s_2(G) \geq 2$, $G$ contains neither two adjacent vertices with valencies 4 and 5 each, nor two adjacent vertices of valency 5. Hence by Lemma 2.2.1, we have the following cases:

(i) $G$ contains a 4-vertex only incident to 3-faces;
(ii) $G$ contains a 4-vertex incident to three 3-faces and one 4-face;
(iii) $G$ contains a 4-vertex incident to three 3-faces and one 5-face with two 4-vertices on the boundary of the 5-face;
(iv) $G$ contains a 5-vertex incident only to 3-faces.

If $G$ contains a 4-vertex incident only to 3-faces, $G$ contains the graph (a) in Figure 2.3 as a subgraph. By Lemma 2.4.2, $G$ is edge-reconstructible. If $G$ contains a 4-vertex incident to three 3-faces and one 4-face, $G$ contains the graph in Figure 2.2 as a subgraph. By Lemma 2.4.1, $G$ is edge-reconstructible. If $G$ contains a 4-vertex incident
to three 3-faces and one 5-face with two 4-vertices on the boundary of the 5-face, then
G contains the graph in Figure 2.4 as a subgraph with \( d(v_4) = 4 \) or \( d(v_5) = 4 \). By
Lemma 2.4.3, G is edge-reconstructible. If G contains a 5-vertex only incident to 3-
faces, by Lemma 2.4.4, G is edge-reconstructible.

### 2.6. The Case of \( s_1(G) \geq 2, s_2(G) = 1 \)

In this section, we assume that G is a 3-connected planar graph with minimum
valency 4 and \( s_1(G) \geq 2, s_2(G) = 1 \).

**Lemma 2.6.1.** If \( H \) is a reconstruction of \( G \), then \( s_1(H) \geq 2 \).

**Proof.** By [10], if \( e = \{v_1, v_2\} \), then \( \{d(v_1), d(v_2)\} \) is edge-reconstructible.
Therefore, \( s_1(G) \geq 2 \) implies \( s_1(H) \geq 2 \).

We give one more definition here. For \( t \geq 0 \), we define a chain \( P_t \) with
\( V(P_t) = \{v_i; i = 0, 1, \ldots, t\} \), \( E(P_t) = \{v_i v_{i+1}; i = 0, 1, \ldots, t - 1\} \), \( d(v_i) = 5, i = 0, 1, \ldots, t \), and
when \( t \geq 3, v_t = v_0 \) is allowed and when \( t = 0, E(P_0) = \emptyset \).

**Lemma 2.6.2.** If G contains a chain \( P_t \) with \( t \geq 3 \), then G is edge-
reconstructible.

**Proof.** Consider \( G - e, e = \{v_1, v_2\} \). Clearly we have \( s_1(G - e) = 1 \). By Lemma
2.6.1, if \( H \) is a reconstruction of \( G \), then \( s_1(H) \geq 2 \). Therefore, the only way to add an
edge in \( G - e \) is to join \( v_1 \) and \( v_2 \). Hence G is edge-reconstructible.
LEMMA 2.6.3. Let $G$ be a 3-connected planar graph with minimum valency 4 and with $s_1(G) \geq 2$, $s_2(G) = 1$ and let $C$ be a boundary circuit of a face of $G$ which contains more than two non-adjacent vertices with valencies less than 6. Suppose that $G$ is not edge-reconstructible. Then we can increase the valencies of 4- and 5-vertices on $C$ to at least 6 by adding some edges in the interior of $C$ and still keep the planarity of the graph obtained by adding edges to $G$.

Proof. Let $C$ be a boundary circuit of a face of $G$ which contains more than two non-adjacent vertices with valencies less than 6. Assume $V(C) = \{v_i; i = 0, 1, ..., t - 1\}$, $E(C) = \{v_iv_{i+1}; i = 0, 1, ..., t - 1, v_t = v_0\}$ with $d(v_0) = 4$ or 5 and $d(v_{t-1}) \geq 6$. Moreover, we assume that $C$ contains $t_1$ chains of $P_0$, $t_2$ chains of $P_1$, $t_3$ chains of $P_2$ and $t_4$ 4-vertices. Since $G$ contains neither two adjacent vertices with valencies 4 and 5 each, nor $P_t$ for $t \geq 3$, $G$ contains at least $t_1 + t_2 + t_3 + t_4$ vertices which have degrees at least 6 and separate all chains $P_0, P_1, P_2$ and 4-vertices on $C$. We add edges from $v_0$.

Case 1. $d(v_0) = 4$.

Since $d(v_0) = 4$, then $d(v_1) \geq 6$ and $d(v_{t-1}) \geq 6$. Let $a = \min\{j; 1 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$ and $b = \max\{j; 1 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. By our assumptions, clearly $a \neq b$.

1.1. If $d(v_{a+1})$ and $d(v_{b-1})$ are greater than 5, we add edges $v_0v_a$ and $v_0v_b$, then consider $v_k$ with $k = \min\{j; a \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$ (k can be equal to $a$ if $d(v_a) = 4$).

1.2. If at least one of $d(v_{a+1})$ and $d(v_{b-1})$ is a 5-vertex, without loss of generality, we may assume that $d(v_{a+1}) = 5$. If $d(v_{a+2}) > 5$, we add edges $v_0v_a$ and $v_0v_{a+1}$, then consider $v_k$ with $k = \min\{j; a + 2 < j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+2}) = 5$, we add edges $v_0v_a, v_0v_{a+1}$ and $v_0v_{a+2}$, then consider $v_k$ with $k = \min\{j; a + 2 < j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$.
Case 2. $d(v_0) = 5$, $d(v_1) > 5$.

Let $a = \min\{j; 1 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) > 5$, we just add the edge $v_0v_a$, then consider $v_k$ with $k = \min\{j; a \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) = 5$, $d(v_{a+2}) > 5$, we add edges $v_0v_a$ and $v_0v_{a+1}$, then consider $v_k$ with $k = \min\{j; a + 2 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) = d(v_{a+2}) = 5$, we add edges $v_0v_a$, $v_0v_{a+1}$ and $v_0v_{a+2}$, then consider $v_k$ with $k = \min\{j; a + 3 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$.

Case 3. $d(v_0) = d(v_1) = 5$, $d(v_2) \geq 6$.

Let $a = \min\{j; 2 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) > 5$, we add edges $v_0v_a$ and $v_1v_a$, then consider $v_k$ with $k = \min\{j; a + 1 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) = 5$, $d(v_{a+2}) > 5$, we add edges $v_0v_{a+1}$ and $v_1v_a$, then consider $v_k$ with $k = \min\{j; a + 2 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$. If $d(v_{a+1}) = d(v_{a+2}) = 5$, we add edges $v_0v_{a+2}$, $v_1v_a$ and $v_1v_{a+1}$, then consider $v_k$ with $k = \min\{j; a + 3 \leq j \leq t - 1, d(v_j) = 4 \text{ or } 5\}$.

Case 4. $d(v_0) = d(v_1) = d(v_2) = 5$.

We can similarly deal with this case as we did above.

Continue in this way, we can increase the valencies of all 4- and 5- vertices on $C$ to at least 6 and still keep the planarity of the graph obtained by adding edges to $G$. Our lemma is proved. 

From Lemma 2.6.3, we have the following corollary.

**COROLLARY 2.6.4.** If $G$ does not contain any graph in Figure 2.3 as a subgraph, then $G$ must contain at least one of the graphs in Figure 2.7 as a face boundary.
where in (a), d(v_0) = 4, or d(v_0) = d(v_1) = 5 or d(v_0) = d(v_1) = d(v_2) = 5;

in (b), d(v_0) = d(v_1) = d(v_2) = 5 or d(v_1) = d(v_3) = 4.

**Figure 2.7**

**LEMMA 2.6.5.** Let $G$ be a 3-connected planar graph with minimum valency 4 and with $s_1(G) \geq 2$, $s_2(G) = 1$. Suppose that $G$ is not edge-reconstructible. Then we can increase the valency of all 4-vertices of $G$ to at least 6 and still keep the planarity of the graph obtained by adding edges to $G$.

**Proof.** By Lemma 2.4.1, Lemma 2.4.2, Lemma 2.4.3 and Lemma 2.4.4, we know that we can assume that $G$ does not contain any graphs in Figure 2.2, Figure 2.3 and Figure 2.4 as a subgraph, where the graph in Figure 2.4 with $d(v_4) = 4$ or $d(v_5) = 4$. From this assumption, we know that there are only two possibilities for any 4-vertex $v$ in $G$:

(i) $v$ is incident to at least one face with face valency $\geq 5$;

(ii) $v$ is incident to at least two faces with face valencies $\geq 4$. 
It is clear that in these cases we can increase the valency of all 4-vertices of \( G \) to at least 6 and still keep the planarity of the graph obtained by adding edges to \( G \). #

**Observation.** If a 5-vertex and a 4-vertex are incident to the same \( k \)-face with \( k \geq 4 \) in \( G \), we can increase the valency of the 5-vertex to at least 6 at the same time when we increase the valency of the 4-vertex and still keep the planarity of the graph obtained by adding edges to \( G \). Therefore we just need to consider those 5-vertices which are not on a boundary of a \( k \)-face with any 4-vertices. #

![Diagram](a)

where in (a): \( v \) is incident to four 3-faces and one 4-face, \( d(v) = d(v_0) = d(v_5) = 5 \) or \( d(v) = d(v_0) = 5, d(v_i) \geq 6, i = 1, ... , 5 \); in (b): \( v \) and \( v_6 \) are incident to four 3-faces and one 5-face, \( d(v) = d(v_0) = d(v_6) = 5 \). In both (a) and (b), \( C_{V_0} \) does not contain any 4-vertex.

Figure 2.8
For each vertex $v$ of the graph $G$, we denote $C_v$ to be the symmetric difference of the boundaries of the faces incident to $v$. Clearly, $C_v$ is the circuit through all the neighbors of $v$.

**Lemma 2.6.6.** Let $G$ be a 3-connected planar graph with minimum valency 4 and with $s_1(G) \geq 2$, $s_2(G) = 1$. Suppose that $G$ is not edge-reconstructible. Then $G$ must contain one of the graphs in Figure 2.8 as a subgraph.

**Proof.** First we consider that $G$ contains the graph (a) in Figure 2.7 as a face boundary. From Lemma 2.6.5 and above observation, we may assume $d(v_0) = d(v_1) = 5$, $d(v_2) \geq 6$, $d(v_3) \geq 6$ or $d(v_0) = d(v_1) = d(v_2) = 5$.

Suppose $d(v_0) = d(v_1) = 5$, $d(v_2) \geq 6$, $d(v_3) \geq 6$. Since $G$ does not contain $P_3$, there is only one 5-vertex among the neighbors of $v_0$ or $v_1$. Without loss of generality, we may assume that $v_1$ is the only 5-vertex neighbor of $v_0$. By the above observation, it is clear that if either $v_0$ is incident to more than one $k$-faces with $k \geq 4$ or $C_{v_1}$ contains a 4-vertex, then we can increase the valency of both $v_0$ and $v_1$ to at least 6 and still keep the planarity of the graph obtained by adding edges to $G$.

Suppose $d(v_0) = d(v_1) = d(v_2) = 5$, then each of $v_0$ and $v_2$ has only one 5-vertex neighbor. If either $v_0$, $v_2$ are incident to more than one $k$-faces with $k \geq 4$ or $C_{v_1}$ contains a 4-vertex, then we can increase the valency of $v_0$, $v_1$ and $v_2$ to at least 6 and still keep the planarity of the graph obtained by adding edges to $G$.

Similarly, we can consider the case that $G$ contains the graph (b) in Figure 2.7 as a face boundary and prove that if either (1) one of the vertices $v_0$ and $v_2$ is incident to more than one $k$-faces with $k \geq 4$ or (2) $C_{v_1}$ contains a 4-vertex, then we can increase the valency of $v_0$, $v_1$ and $v_2$ to at least 6 and still keep the planarity of the graph. Therefore $G$ must contain one of the graphs in Figure 2.8 as a subgraph. 

#
LEMMA 2.6.7. If $G$ contains any one of the graphs in Figure 2.8 as a subgraph, then for any good edge $vv_i$, $i \neq 0$, $v$ is recognizable in $G - vv_i$.

Proof. Without loss of generally, we assume $i = 1$. In $G - vv_1$, the 5-vertex $v_0$ is adjacent to the 4-vertex $v$. Since $s_1(G) \geq 2$, $s_1(G - vv_1) = 1$ and $\{d(v), d(v_1)\}$ is edge-reconstructible and since $G - vv_1$ is a 3-connected planar graph, it follows that we can construct reconstructions of $G$ from $G - vv_1$ only in the following ways: as $(G - vv_1) + vv_i$, $i \in \{1, 5\}$ (or $\{1, 5, 6\}$) or else as $(G - vv_1) + uv_0$, where $u \neq v$ is a 4-vertex in $G$. Since $G - vv_1$ is a 3-connected planar graph, we could add the edge $uv_0$ only when vertices $u$ and $v_0$ are incident to the same face. But that $u$ and $v_0$ are incident to the same face means that $C_{v_0}$ contains a 4-vertex in $G$. This contradicts Lemma 2.6.6. Hence we can only construct reconstructions of $G$ in the way as $(G - vv_1) + vv_i$. This means that $v$ is recognizable in $G - vv_1$. #

THEOREM 2.6.8. If $G$ contains any one of the graphs in Figure 2.8 as a subgraph, then $G$ is edge-reconstructible.

Proof. First we assume that $G$ contains the graph (a) in Figure 2.8 as a subgraph. If $d(v) = d(v_0) = d(v_5) = 5$, then $d(v_i) \geq 6$, $i = 1, 2, 3, 4$.

Case 1. $G - vv_0$ is not 3-connected.

First by Lemma 2.6.7, we know that if $vv_i$ is a good edge of $G$, then in $G - vv_i$, we can identify $v$. Therefore, by using a similar argument as we used before, we can show that if $G$ is not edge-reconstructible, there is only one reconstruction of $G$ which is not isomorphic to $G$. Then we use a similar argument as we did in the proof of Lemma 2.4.1, we can show that $G$ is edge-reconstructible in this case.

Case 2. $G - vv_0$ is 3-connected.
In this case, first we assume that $G - vv_1$ is 3-connected. Since one of $G - vv_3$ and $G - vv_4$ is also 3-connected, it is clear that $G$ is edge-reconstructible. If $G - vv_1$ is not 3-connected, then $G - vv_2$ and one of $G - vv_i$, $i = 3, 4$, are 3-connected. Clearly, in this case, $G$ is also edge-reconstructible.

If $d(v) = d(v_0) = 5$, $d(v_5) > 6$, by Lemma 2.6.6, we know that we also have $d(v_i) > 6$, $i = 1, 2, 3, 4$. If $G - vv_0$ is 3-connected, we can identify $v$ and $v_0$. This is because that we already assume that $G$ does not contain the graph in Figure 2.4 as a subgraph with two nonadjacent 4-vertices on the boundary of the 5-face and $C_{v_0}$ does not contain any 4-vertex in $G$ by Lemma 2.6.6. Therefore, we can reconstruct $G$ uniquely. $G$ is edge-reconstructible.

If $G - vv_0$ is not 3-connected, then $G - vv_1$ and one of $G - vv_2$, $G - vv_3$ will be 3-connected. From the above discussion, we know that $G$ is also edge-reconstructible.

Now, we assume that $G$ contains the graph (b) in Figure 2.8 as a subgraph. Since $G$ does not contain $P_3$, $d(v_i) \geq 6$, $i = 1, ..., 5$. Since one of $G - vv_1$ and $G - vv_2$ is 3-connected (say $G - vv_1$), by Lemma 2.6.7, we can identify the 4-vertex $v$ in $G - vv_1$. Therefore, if $G$ is not edge-reconstructible, there is only one reconstruction $H$ of $G$ which is not isomorphic to $G$, where $H = (G - vv_1) + vv_5$ if $d(v_1) = d(v_5) + 1$ or $H = (G - vv_1) + vv_6$ if $d(v_1) = 6$.

Case 1. $G - vv_0$ is not 3-connected.

Since $G - vv_0$ is not 3-connected, then $G - vv_1$ and one of $G - vv_i$, $i = 2, 3$, are 3-connected.

(1.1) $G - vv_3$ is 3-connected.

There are three cases. First we consider the case in which $d(v_1) = d(v_3) = 6$. Since $G - vv_1$ and $G - vv_3$ are 3-connected, we can construct reconstructions $G_1$, $G_1'$ of $G$ as follows: $G_1 = (G - vv_1) + vv_6$ and $G_1' = (G - vv_3) + vv_6$. Clearly, $G_1 \cong G_1'$.
and $G_1$ is not isomorphic to $G$. Since $vv_3$ is also a good edge in $G_1$, we can also construct $G_1'$ from $G_1$ by $G_1' = (G_1 - vv_3) + vv_1$. Since $G$ is not edge-reconstructible, $G_1$ is not edge-reconstructible. Therefore, $G_1'$ is not isomorphic to $G_1$. This is a contradiction.

For the case $d(v_1) = d(v_3) = d(v_5) + 1$, we can use a similar proof to show that $G$ is edge-reconstructible. If at least one of $d(v_1)$, $d(v_3)$ is not equal to 6 or $d(v_5) + 1$, it is easy to show that $G$ is edge-reconstructible. Therefore, we only need to consider the case $d(v_1) = 6$, $d(v_3) = d(v_5) + 1$ or $d(v_1) = d(v_5) + 1$, $d(v_3) = 6$.

Assume that $d(v_1) = 6$, $d(v_3) = d(v_5) + 1$. We define that $f_3(G)$ is the number of 3-faces of $G$. Since $G - vv_1$ and $G - vv_3$ are 3-connected, we can construct reconstructions $G_1$, $G_1'$ of $G$ as follows: $G_1 = (G - vv_1) + vv_6$ and $G_1' = (G - vv_3) + vv_5$. Clearly, $f_3(G_1) = f_3(G_1') = f_3(G) - 1$, $G_1 \cong G_1'$, and $G_1$ is not isomorphic to $G$. Since $vv_3$ is also a good edge in $G_1$, we can construct a reconstruction $G_2$ of $G_1$ by $G_2 = (G_1 - vv_3) + vv_5$. Clearly, we have $f_3(G_2) = f_3(G_1) = f_3(G) - 1$ and $G_2 \cong G$. Since $G$ is a 3-connected planar graph, it has a unique planar representation. Now, $G$ has two different planar representations, one is the planar representation of $G$ and the other is the planar representation of $G_2$. This is a contradiction.

Similarly, we can consider the case $d(v_1) = d(v_5) + 1, d(v_3) = 6$. (1.2) $G - vv_3$ is not 3-connected.

In this case, $vv_1$, $vv_2$ and $vv_4$ are good edges of $G$. If at least two of $d(v_1)$, $d(v_2)$ and $d(v_4)$ are equal to each other, by the argument similar to the above one, we can show that $G$ is edge-reconstructible. If none of $d(v_1)$, $d(v_2)$ and $d(v_4)$ are equal to each other, then at least one of them is not equal to $d(v_5) + 1$ and 6, say $d(v_1)$. Since $G - vv_1$ is 3-connected and $d(v_1) \neq d(v_5) + 1$, $d(v_1) \neq 6$. The only way to construct reconstruction of $G$ from $G - vv_1$ is to add back the edge $vv_1$. $G$ is edge-reconstructible.
Case 2. $G - vv_0$ is 3-connected.

In this case, if $G - vv_1$ is also 3-connected, from above, we know that we are done. Hence we assume that $G - vv_1$ is not 3-connected. Then $G - vv_2$ is 3-connected and one of $G - vv_i$, $i = 3, 4$, is 3-connected.

(2.1) $G - vv_2$ is 3-connected.

If $d(v_2) = d(v_3)$, we can easily prove that $G$ is edge-reconstructible. Hence we consider the case $d(v_2) \neq d(v_3)$. If one of $d(v_2)$, $d(v_3)$ is not equal to 6 or $d(v_5) + 1$, then we can also easily prove that $G$ is edge-reconstructible. Therefore we only need to consider the cases $d(v_2) = 6$, $d(v_3) = d(v_5) + 1$ or $d(v_3) = 6$, $d(v_2) = d(v_5) + 1$. Assume $d(v_3) = d(v_5) + 1$. If $G$ is not edge-reconstructible, we construct a reconstruction $H$ of $G$ from $G - vv_3$ by $H = (G - vv_3) + vv_5$. $H$ is not isomorphic to $G$. From Lemma 2.6.6, we know that $v_6$ is incident to four 3-faces and one 5-face in $G$. Hence $H$ contains the graph (a) in Figure 2.8. By the first part of our proof, $H$ is edge-reconstructible. Hence $G$ is edge-reconstructible. A similar proof can be applied to the case $d(v_2) = d(v_5) + 1$ to show that $G$ is edge-reconstructible.

(2.2) $G - vv_3$ is not 3-connected.

Since $G - vv_3$ is not 3-connected, $G - vv_4$ is 3-connected. By the discussion in (2.1), we know that $G$ is edge-reconstructible.

COROLLARY 2.6.9. If $G$ is a 3-connected planar graph with minimum valency 4 and $s_1(G) \geq 2$, $s_2(G) \geq 1$, then $G$ is edge-reconstructible.
CHAPTER III
EDGE-RECONSTRUCTION OF MINIMALLY 3-CONNECTED PLANAR GRAPHS

3.1. Introduction

In this chapter, we consider the problem of the edge reconstruction of minimally 3-connected planar graphs. We call a graph G minimally 3-connected if G is 3-connected but for every e \( \in \) \( E(G) \), \( G_e \) is not 3-connected. Let \( d \) and \( \delta \) be the average degree of \( G \) and the minimum degree of \( G \), respectively. Let \( n = |V(G)| \), \( m = |E(G)| \) and \( d(v) \) be the degree of a vertex \( v \) in \( G \). Sometimes in order to avoid confusion, we use \( d_G(v) \) to denote the degree of \( v \) in \( G \). We use \( n_k \) to denote the number of \( k \)-vertices of \( G \) and we use \( f, f_k \) to denote the number of faces and the number of \( k \)-faces of a plane graph, respectively. We are going to prove the following theorem in this chapter.

THEOREM 3.1.1 If G is a minimally 3-connected planar graph, then G is edge-reconstructible.

Throughout the rest of this chapter, \( G \) is always a minimally 3-connected planar graph. Since \( G \) is trivially edge reconstructible from \( G_{uv} \) for any two adjacent 3-vertices \( u \) and \( v \) in \( G \), we will assume that there do not exist any two adjacent 3-vertices in \( G \). By
[3], G is edge-reconstructible if \( |V(G)| < 10 \). We will assume that \( |V(G)| \geq 10 \). Hence, the planarity of G, by [8], is recognizable from its edge-deck. This means that all reconstructions of G are planar graphs. When we construct reconstructions of G, we will often use this fact without a further explanation.

### 3.2. Some Lemmas

The first two lemmas are due to Hoffman (see [4] for a proof).

**Lemma 3.2.1.** A graph of minimum degree \( \delta \) and average degree \( d \) is edge-reconstructible if

\[
d < \delta + 1 - \frac{1}{\delta + 1}.
\]  

**Lemma 3.2.2.** Let \( H \) be a graph of minimum degree \( \delta \). Suppose that, for some \( k \geq 0 \), there is a vertex in G of degree \( \delta + k \) adjacent to \( k + 1 \) or more vertices of degree \( \delta \). Then G is edge-reconstructible.

The next lemma is due to Halin [11].

**Lemma 3.2.3.** Every circuit of a minimally 3-connected graph contains at least two vertices of valency 3.

By this lemma, each face of G will contain at least two 3-vertices, and each 4-face of G will contain just two 3-vertices. We will use these facts without a further explanation.

The following lemma can be found in Ore's book [22].
LEMMA 3.2.4. Let $H$ be a nonseparable planar graph. A necessary and sufficient condition that its planar representation is unique (up to preserving face boundaries) is that it has no proper two-vertex separation.

LEMMA 3.2.5. If $f_4 < \frac{5}{8} n$, then $G$ is edge-reconstructible.

Proof. By [11], $G$ contains at least one 3-vertex. Hence $\delta = 3$. Since $G$ contains no two adjacent 3-vertices, by [11], $G$ does not have triangles. Therefore,

$$4f_4 + 5(f - f_4) \leq 2m \text{ and } f \leq \frac{2}{3}m + \frac{1}{3}f_4.$$

By Euler's formula

$$n - m + f = 2,$$  \hspace{1cm} (2)

we have

$$2 = n - m + f \leq n - m + \frac{2}{3}m + \frac{1}{3}f_4 = n - \frac{3}{5}m + \frac{1}{5}f_4.$$

Hence,

$$m < \frac{5}{3}n + \frac{1}{3}f_4.$$  \hspace{1cm} (3)

Since

$$d = \frac{\sum d(x_i)}{n} = \frac{2m}{n}$$  \hspace{1cm} (4)

where $d(x)$ denotes the degree of the vertex $x$ of $G$, we have

$$d = \frac{2m}{n} < \frac{\frac{10}{3}n + \frac{2}{3}f_4}{n} = \frac{10}{3} + \frac{2f_4}{3n}.$$

By Lemma 3.2.1, if $d < 3 + 1 - \frac{1}{4} = \frac{15}{4}$, $G$ is edge-reconstructible. From the condition $f_4 < \frac{5}{8} n$, we have

$$d < \frac{15}{4}.$$
\[ d < \frac{10}{3} + \frac{2f_4}{3n} < \frac{10}{3} + \frac{2}{3n} \left( \frac{5}{8} n \right) = \frac{10}{3} + \frac{5}{12} = \frac{15}{4}. \]

Therefore, G is edge-reconstructible.

By Lemma 3.2.5, to prove Theorem 3.1.1, we only need to consider the case \( f_4 \geq \frac{5}{8} n \). We will assume \( f_4 \geq \frac{5}{8} n \) throughout the rest of this chapter.

**Lemma 3.2.6.** \( n_3 < \frac{2}{3} n \).

**Proof.** Since there do not exist two adjacent 3-vertices, \( 3n_3 \leq m \leq 2n - 4 \). Therefore, \( n_3 < \frac{2}{3} n \). 

If two faces have a common edge, we say that they are adjacent by the edge (or for short, adjacent). If two faces have a common vertex, but they do not have a common edge, we say that they are adjacent by the vertex.

**Lemma 3.2.7.** \( G \) contains two adjacent 4-faces.

**Proof.** If there were no two adjacent 4-faces, we would have

\[ m > 4f_4 \geq 4 \left( \frac{5}{8} n \right) = \frac{5}{2} n. \]

This contradicts the fact that \( m \leq 2n - 4 \).
Where in Figure 3.1, $d(v_1) = d(v_3) = 3$.

Let $C$ be a circuit of the graph in Figure 3.1 with

\[ V(C) = \{v_0, v_1, \ldots, v_{t-1}\} \]
\[ E(C) = \{v_i v_{i+1}; \ i = 0, 1, \ldots, t-1, v_t = v_0\}. \]

Consider an exterior bridge of $C$ in $G$. Since $G$ is a 3-connected planar graph, there is no exterior bridge $B$ with $A(G, B) \subseteq \{v_0, v_1, v_2, v_3, v_4\}$ or $A(G, B) \subseteq \{v_4, \ldots, v_{t-1}, v_0\}$.

We call an edge $e$ a good edge if $G - e$ has a unique planar embedding.

**Lemma 3.2.8.** The common edge of any pair of 4-faces of $G$ is a good edge.

**Proof.** By the above definition of a good edge, we just need to prove that $G - vv_2$ has a unique planar embedding. By Lemma 3.2.4, to show that $G - vv_2$ has a unique planar embedding, it suffices to show that $G - vv_2$ has no proper two-vertex separation. We assume that $G - vv_2$ has a proper two-vertex separation. Then $G - vv_2$ has a decomposition.
\[ G - vv_2 = H_1 + H_2 \]
in which the edge disjoint subgraphs \( H_1 \) and \( H_2 \) are not graph chains and they are only connected at the two common vertices \( x \) and \( y \). Clearly, \( v \notin \{x, y\} \). This is because if \( \{v, w\} \) were a cut set of \( G - vv_2 \), then it would also be a cut set of \( G \), which is impossible, since \( G \) is 3-connected. It is also clear that \( \{x, y\} \subseteq V(C) \). Since if one of \( x \) and \( y \) does not belong to \( V(C) \), then \( V(C) \) and \( v \) belong to the same \( H_i \), say \( i = 1 \). This implies that for any vertex \( z \) in \( H_2 \), where \( z \notin \{x, y\} \), a chain from \( z \) to \( w \), where \( w \in V(C) - \{x, y\} \), must pass through \( x \) or \( y \). Therefore in \( G \), any chain from \( z \) to \( w \) must pass through \( x \) or \( y \). This is impossible, since \( G \) is 3-connected.

Now, \( \{x, y\} \subseteq V(C) \). Clearly \( \{x, y\} \neq \{v_0, v_4\} \). This is because that (1) either one of \( H_1 \) and \( H_2 \) would be a graph chain \( C[v_0, v_4] \) with
\[
V(C[v_0, v_4]) = \{v_0, v, v_4\},
\]
\[
E(C[v_0, v_4]) = \{v_0v, vv_4\},
\]
which is not a proper two-vertex separation; or (2) there is an exterior bridge \( B \) of \( C \) in \( G \) with \( A(G, B) \subseteq \{v_4, ..., v_{t-1}, v_0\} \), which contradicts the fact that \( G \) is 3-connected. Therefore, \( v_0, v \) and \( v_4 \) belong to the same \( H_i \), say \( i = 1 \). Hence \( \{x, y\} \subseteq \{v_0, v_1, v_2, v_3\} \) or \( \{x, y\} \subseteq \{v_1, v_2, v_3, v_4\} \). But this is impossible. Because in either case, it implies that there is an exterior bridge \( B \) of \( C \) in \( G \) with \( A(G, B) \subseteq \{v_0, v_4, ..., v_{t-1}\} \). Therefore, \( vv_2 \) is a good edge. 

**Lemma 3.2.9.** If \( H \) is a reconstruction of \( G \), then \( H \) is a minimally 3-connected planar graph.

**Proof.** Clearly, we only need to show that \( H \) is 3-connected. If \( H \cong G \), there is nothing to prove. Hence we assume that \( H \) is not isomorphic to \( G \). By Lemma 3.2.7, \( G \) contains the graph in Figure 3.1 as a subgraph. By Lemma 3.2.8, the edge \( vv_2 \) is a good
edge. By Lemma 3.2.2, $d(v^2) \geq 6$. Since $v$ can be recognized in $G - vv_2$ and \{d(v), d(v_2)\} is edge reconstructible, $H$ can be constructed from $G - vv_2$ by adding an edge $vv_i$ with $i \in \{5, ..., t - 1\}$. By the construction of $H$ and the proof of Lemma 3.2.8, $H$ is 3-connected. Therefore, $H$ is a minimally 3-connected planar graph.

**Lemma 3.2.10.** If $H$ is a reconstruction of $G$, then $f_4(H) = f_4(G)$.

**Proof.** Since $G$ contains the graph in Figure 3.1 as a subgraph and $vv_2$ is a good edge, we have $f_4(H) \leq f_4(G - vv_2) + 2 = f_4(G)$. On the other hand, $G$ is also a reconstruction of $H$. Since $H$ is also a minimally 3-connected planar graph, there exists a good edge $e$ in $H$ such that $f_4(G) \leq f_4(H - e) + 2 = f_4(H)$. Hence $f_4(H) = f_4(G)$. 

![Figure 3.2](image)

Where in Figure 3.2, $d(v_1) = d(v_3) = d(v_5) = 3$. 

---

**LEMMA 3.2.10.** If $H$ is a reconstruction of $G$, then $f_4(H) = f_4(G)$.

**Proof.** Since $G$ contains the graph in Figure 3.1 as a subgraph and $vv_2$ is a good edge, we have $f_4(H) \leq f_4(G - vv_2) + 2 = f_4(G)$. On the other hand, $G$ is also a reconstruction of $H$. Since $H$ is also a minimally 3-connected planar graph, there exists a good edge $e$ in $H$ such that $f_4(G) \leq f_4(H - e) + 2 = f_4(H)$. Hence $f_4(H) = f_4(G)$. 

![Figure 3.2](image)

Where in Figure 3.2, $d(v_1) = d(v_3) = d(v_5) = 3$. 

---
LEMMA 3.2.11. If $G$ contains the graph in Figure 3.2 as a subgraph, where $v$ is incident to three 4-faces, then $G$ is edge-reconstructible.

Proof. By Lemma 3.2.8, $vv_0$ is a good edge of $G$. Since $f_4(G)$ is edge-reconstructible and $v$ can be recognized in $G - vv_0$, there is only one way to add an edge to $G - vv_0$. Therefore, $G$ is edge-reconstructible. 

By Lemma 3.2.11, we can assume that $G$ does not contain the graph in Figure 3.2 as a subgraph.

LEMMA 3.2.12. For each pair of 4-faces of $G$, if the 3-vertex on the common edge of that pair of 4-faces is incident to two 4-faces and one $k$-face with $k \neq 6$, then $G$ is edge-reconstructible.

Proof. Let $v$ be the 3-vertex and $e$ be the common edge. By Lemma 3.2.8, $e$ is a good edge of $G$. Since $f_4(G - e) = f_4(G) - 2$ and $f_4(G)$ is edge-reconstructible, to obtain a reconstruction of $G$, we must construct two 4-faces by adding an edge in $G - e$. Since we can recognize $v$ in $G - e$ and the other face incident to $v$ with face valency $\neq 6$, there is only one way to add back the edge $e$ in $G - e$. Therefore, $G$ is edge-reconstructible.

By Lemma 3.2.12, we can assume that for each pair of 4-faces of $G$, there is a 6-face of $G$ which is incident to the 3-vertex on the common edge of that pair of 4-faces and we call that 6-face the face corresponding to that pair of 4-faces. Therefore, for each pair of 4-faces of $G$, we have the graph in Figure 3.3 as a subgraph of $G$, where $v$ is incident to two 4-faces and one 6-face.
Figure 3.3

Where in Figure 3.3, \( d(v_1) = d(v_3) = d(v_5) = d(v_7) = 3 \) and \( d(v_2) = d(v_6) + 1 \).

We claim that if \( G \) is not edge-reconstructible, there is only one reconstruction \( H \) of \( G \) which is not isomorphic to \( G \). This is because that \( G \) contains the graph in Figure 3.3 as a subgraph and \( v_2 \) is a good edge. Therefore, the only way to construct \( H \) from \( G - vv_2 \), which is not isomorphic to \( G \), is to add the edge \( vv_6 \). Clearly, the list of face valencies of \( H \) is the same as the one of \( G \).

**Lemma 3.2.13.** If \( G \) contains the graph in Figure 3.4 as a subgraph where \( v \) and \( u \) are incident to 4-faces and one 6-face, then \( G \) is edge-reconstructible.
Figure 3.4

Where in Figure 3.4, \( d(v_0) = d(v_2) = d(v_4) = d(v_6) = d(v_8) = 3 \).

**Proof.** If \( G \) contains the graph in Figure 3.4 as a subgraph, then the reconstruction \( H \) of \( G \), which can be constructed from \( G - v_3 \) by adding the edge \( vv_9 \), will contain the graph in Figure 3.2 as a subgraph. Therefore, \( H \) is edge-reconstructible. Hence, \( G \) is edge-reconstructible.

From Lemma 3.2.12, we can see that for each pair of 4-faces of \( G \), there is a 6-face of \( G \) corresponding to it (see Figure 3.3). By Lemma 3.2.13, we can make a further assumption about \( G \). We can assume that for different pairs of 4-faces of \( G \), there are different 6-faces of \( G \) corresponding to them. Therefore, if \( G \) has \( k \) different pairs of 4-faces, there will be \( k \) different 6-faces in \( G \) corresponding to them.

**Lemma 3.2.14.** \( n_3 \geq f_4 \).
Proof. Since each 4-face contains two 3-vertices and each 3-vertex is incident to at most two 4-faces, the conclusion follows.

COROLLARY 3.2.15. \( \frac{5}{6} n \leq f_4 < \frac{2}{3} n \).

COROLLARY 3.2.16. \( m < \frac{17}{9} n \).

Proof. Since \( m < \frac{5}{3} n + \frac{1}{3} f_4 < \frac{5}{3} n + \frac{2}{9} n = \frac{17}{9} n \).

We define \( E_{3,i} = \{ e \in E(G); \text{one end of } e \text{ is incident to a 3-vertex and the other end of } e \text{ is incident to an } i \text{-vertex} \} \) and \( t_{3,i} = |E_{3,i}| \) for \( i \geq 4 \). Then \( 3n_3 = t_{3,4} + ... + t_{3,\Delta} \), where \( \Delta \) is the maximum degree of \( G \). By Lemma 5.2.2, we know \( \Delta \geq 6 \). Also by Lemma 5.2.2, if \( G \) is not edge reconstructible, we will have \( t_{3,i} \leq (i - 3)n_i \) for \( i \geq 4 \).

LEMMA 3.2.17. If \( G \) is not edge reconstructible, then \( n_3 \leq \frac{13}{36} n \).

Proof. Since \( 2m = \sum_{i=3}^{\Delta} in_i \), \( n = \sum_{i=3}^{\Delta} n_i \) and \( 2m = dn \), we have

\[
n_3 = 4n - 2m + n_5 + ... + (\Delta - 4)n_\Delta.
\]

By \( m \leq \frac{17}{9} n \) and \( d \geq \frac{15}{4} n \), we obtain

\[
\frac{2}{9} n \leq 4n - 2m \leq n_3 \leq \frac{1}{4} n + n_5 + ... + (\Delta - 4)n_\Delta.
\]

Since \( n_3 \geq \frac{2}{9} n \) and \( n_3 + n_4 \leq n \), we have \( n_4 \leq \frac{7}{9} n \). Hence, we have

\[
n_4 \geq t_{3,4} = 3n_3 - t_{3,5} - ... - t_{3,\Delta}
\]

\[
= 3n_5 + ... + 3(\Delta - 4)n_\Delta + 3(4n - 2m) - t_{3,5} - ... - t_{3,\Delta}
\]

\[
= (2n_5 - t_{3,5}) + ... + [(\Delta - 3)n_\Delta - t_{3,\Delta}] + n_5 + ... + (2\Delta - 9)n_\Delta + 3(4n - 2m)
\]
\[ \geq n_5 + \ldots + (\Delta - 4)n_\Delta + 6(2n - m) \]
\[ \geq n_5 + \ldots + (\Delta - 4)n_\Delta + 6(2n - \frac{17}{9}n) \]
\[ \geq n_5 + \ldots + (\Delta - 4)n_\Delta + \frac{6}{9}n. \]

It follows that \( n_5 + \ldots + (\Delta - 4)n_\Delta \leq \frac{1}{9}n \). Hence, \( n_3 \leq \frac{1}{4}n + \frac{1}{9}n = \frac{13}{36}n \). #

### 3.3. Proof of Theorem 3.1.1

Using the above lemmas, we are going to prove Theorem 3.1.1 by showing that there does not exist a graph \( G \) that satisfies the properties which we got above. Therefore we finish proving our theorem 3.1.1.

**Proof of Theorem 3.1.1.** We call a 4-face of \( G \) isolated if it is not adjacent to any other 4-face of \( G \). Let \( \beta \) be the number of isolated 4-faces of \( G \). By Lemma 3.2.17 and the fact that the boundary of each 4-face has two 3-vertices, it is clear that \( \beta \leq \frac{13}{72}n \). Therefore, there are at least \( f_4 - \beta \geq \frac{5}{8}n - \frac{13}{72}n = \frac{4}{9}n \) 4-faces in \( G \) which are not isolated. By Lemmas 3.2.12 and 3.2.13, \( G \) contains at least \( \frac{1}{2} \cdot \frac{4}{9}n = \frac{2}{9}n \) 6-faces.

Since \( 2m \geq 4f_4 + 6f_6 \geq 4 \cdot \frac{5}{8}n + 6 \cdot \frac{2}{9}n = \frac{23}{6}n \), \( m \geq \frac{23}{12}n \) which is greater than \( \frac{17}{9}n \). It contradicts \( m \leq \frac{17}{9}n \). Hence we finished our proof. #
CHAPTER IV
THE EDGE-RECONSTRUCTION OF MAXIMAL BIPARTITE PLANAR GRAPHS WITH MINIMUM VALENCY 3

4.1. Introduction

In this chapter, we consider maximal bipartite planar graphs with minimum valency 3. We call an edge whose contraction results in a 3-connected graph a contractible edge and we call an edge whose deletion results in a 3-connected graph a deletible edge. Let $E_c(G)$ and $E_d(G)$ be the set of contractible edges of $G$ and the set of deletible edges of $G$, respectively.

Let $H$ be an induced subgraph of $G$. A vertex of attachment of $H$ in $G$ is a vertex of $H$ that is incident in $G$ to an edge not belonging to $E(H)$. The set of vertices of attachment of $H$ in $G$ is denoted by $A(G, H)$.

Let $k(G) = k$ and $S$ a cut set of $G$ containing $k$ vertices. Let the connected components of $G - S$ be $H_1, \ldots, H_r$. We then write

$$G - S = \bigcup_{i=1}^{r} H_i,$$

and we have that $H_i \cap H_j = \emptyset$ for $i \neq j$. $\overline{H_i}$ shall denote that subgraph of $G$ induced by $V(H_i) \cup S$. Then $\overline{H_i} \cap \overline{H_j} = S$ for $i \neq j$.

In the case of $k(G) = 2$, if $G$ is a planar graph and $S = \{a, b\}$ is a cut set for $G$ such that $G - S = \bigcup_{i=1}^{r} H_i$. Then $\overline{H_i}$, $i = 1, \ldots, r$ will be called lobules of $G$. 
In this chapter, our main purpose is to prove the following theorem.

THEOREM 4.1.1. *Maximal bipartite planar graphs with minimum valency 3 are edge-reconstructible.*

We prove Theorem 4.1.1 by two steps. In section 4.2, we consider the case in which maximal bipartite planar graphs are 3-connected. In section 4.3, we drop off the condition on the connectivity, and prove that maximal bipartite planar graphs with minimum valency 3 are edge-reconstructible.

4.2. The Case of Connectivity 3

In this section, we are going to prove the following theorem which is a special case of Theorem 4.1.1.

THEOREM 4.2.1. *Let G be a maximal bipartite planar graph. If G is 3-connected, then G is edge-reconstructible.*

Throughout the rest of this section, G is always a 3-connected maximal bipartite planar graph. A well-known result states that a 3-connected planar graph has a unique embedding on the plane. Since in this section, we only consider the case that G is a 3-connected planar graph, without loss of generality, we can assume that G is a plane graph. Since G is trivially edge-reconstructible from $G_{uv}$ for any two adjacent $\delta$-vertices $u$ and $v$ in G, we will assume that there do not exist any two adjacent $\delta$-vertices in G. By [3], G is edge-reconstructible if $|V(G)| < 10$. We will assume that $|V(G)| \geq 10$. Hence by [8], the planarity of G is recognizable from its edge-deck. This means that all
reconstructions of $G$ are planar graphs. We will often use this fact in our proofs without a further explanation.

Before we prove our theorem 4.2.1, we need several lemmas. The first lemma can be found in [22].

**Lemma 4.2.2.** Let $H$ be a nonseparable planar graph. A necessary and sufficient condition that its planar representation is unique is that it has no proper two-vertex separation.

The second lemma is due to Hoffman (see [4] for a proof).

**Lemma 4.2.3.** Let $G$ be a graph of minimum degree $\delta$. Suppose that, for some $k \geq 0$, there is a vertex in $G$ of degree $\delta + k$ adjacent to $k + 1$ or more vertices of degree $\delta$. Then $G$ is edge-reconstructible.

**Lemma 4.2.4.** If $H$ is a reconstruction of $G$, then $H$ is also a maximal bipartite planar graph.

**Proof.** First we show that $H$ is a bipartite graph. Suppose that $H$ is not a bipartite graph, then $H$ contains an odd circuit. Clearly, $H$ itself is not a circuit. Therefore there exists $e \in E(H)$ such that $H - e$ contains an odd circuit. On the other hand, $\mathcal{D}(H) = \mathcal{D}(G)$ and $G$ is a bipartite graph. So $H - e$ cannot contain an odd circuit for every $e \in E(H)$. This is a contradiction. Hence $H$ is a bipartite graph. Since $m(H) = m(G) = 2n - 4$ and $H$ is a bipartite planar graph, $H$ is a maximal bipartite planar graph.

**Lemma 4.2.5.** $\delta(G) = 3$. 
Proof. Since \( d = \frac{2m}{n} = \frac{4n-8}{n} < 4 \) and \( G \) is 3-connected, we have \( 4 > d \geq \delta(G) \geq 3 \). Therefore \( \delta(G) = 3 \).

By Lemma 4.2.5, \( G \) contains the graph in Figure 4.1 as a subgraph, where \( v \) is incident to three 4-faces in \( G \).

![Figure 4.1](image)

**LEMMA 4.2.6.** \( G \) contains a 3-vertex which is adjacent to a 4-vertex.

**Proof.** Let \( n_i \) be the number of \( i \)-vertices in \( G \), \( \Delta \) be the maximum degree of \( G \) and \( f \) be the number of faces of \( G \). Since bidegreed graphs are edge-reconstructible [20], we can assume \( \Delta \geq 5 \). Since \( 2m = \sum_{i=3}^{\Delta} in_i, 2m = 4(n - 2), n = \sum_{i=3}^{\Delta} n_i \) and \( f = n - 2 \), and since by Euler's Formula
\[ n - m + f = 2, \]

we have

\[ \Delta \sum (4 - i) n_i = 8. \]

Therefore, \( n_3 = 8 + n_5 + \ldots + (\Delta - 4)n_\Delta. \)

If \( G \) does not contain a 3-vertex which is adjacent to any 4-vertex, by Lemma 4.2.3, we would have

\[ 2n_5 + \ldots + (\Delta - 3)n_\Delta \geq 3n_3 \]

Hence

\[ 2n_5 + \ldots + (\Delta - 3)n_\Delta \geq 24 + 3n_5 + \ldots + 3(\Delta - 4)n_\Delta. \]

This is a contradiction.

Since \( G \) is a 3-connected planar graph, there is no outer bridge \( B \) of \( C \) in \( G \) with

\( A(G, B) = \{v_2, v_3, v_4\} \) or \( \{v_0, v_1, v_2, v_4, v_5\} \), where \( V(C) = \{v_i; i = 0, 1, \ldots, 5\} \), \( E(C) = \{v_i v_{i+1}; i = 0, 1, \ldots, 5, v_6 = v_0\} \) (see Figure 4.1).

**Lemma 4.2.7.** If there is a proper two-vertex separation of \( G - vv_0 \) such that

\( G - vv_0 = H_1 + H_2 \)

and \( E(H_1) \cap (H_2) = \emptyset, V(H_1) \cap V(H_2) = \{x, y\} \), then \( \{x, y\} = \{v_1, v_3\} \) or \( \{v_2, v_3\} \) or \( \{v_1, v_4\}. \)

**Proof.** Clearly \( v \not\in \{x, y\} \). This is because if \( \{v, w\} \) is a cut set of \( G - vv_0 \), then it would also be a cut set of \( G \), which is impossible since \( G \) is 3-connected. It is also clear that \( V(C) \supset \{x, y\} \). Since if one of \( x \) and \( y \) does not belong to \( V(C) \), then \( V(C) \) and \( v \) belong to the same \( H_i \), say \( i = 1 \). This implies that for any vertex \( z \) in \( H_2 \), where \( z \not\in \{x, y\} \), a chain from \( z \) to \( w \), where \( w \in V(C) - \{x, y\} \) must pass through \( x \) or \( y \).
Therefore in $G$, any chain from $z$ to $w$ must pass through $x$ or $y$. This contradicts the fact that $G$ is 3-connected. Hence $\{x, y\} \subset V(C)$. Clearly $\{x, y\} \neq \{v_2, v_4\}$. This is because if $\{x, y\} = \{v_2, v_4\}$, then either (1) one of $H_1$ and $H_2$ would be a graph chain $C[v_2, v_4]$ with 

$$V(C[v_2, v_4]) = \{v_2, v, v_4\}$$
$$E(C[v_2, v_4]) = \{vv_2, v_4\}$$

which is not a proper two-vertex separation by our definition, or (2) there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_2, v_3, v_4\}$ which contradicts the fact that $G$ is 3-connected. Therefore, $\{v_0, v_1, v_2, v_5\} \supseteq \{x, y\}$ or $\{v_0, v_1, v_4, v_5\} \supseteq \{x, y\}$. There are three possibilities for $\{x, y\}$, $\{x, y\} = \{v_2, v_5\}$ or $\{x, y\} = \{v_1, v_4\}$ or $\{x, y\} = \{v_1, v_5\}$.

**Lemma 4.2.8.** If there is a proper two-vertex separation of $G - vv_0$ such that 

$$G - vv_0 = H_1 + H_2$$

and $E(H_1) \cap E(H_2) = \emptyset$, $V(H_1) \cap V(H_2) = \{v_2, v_5\}$ (or $\{v_1, v_4\}$), then $G$ is edge-reconstructible.

**Proof.** Since there is a proper two-vertex separation of $G - vv_0$ such that 

$$G - vv_0 = H_1 + H_2$$

and $E(H_1) \cap E(H_2) = \emptyset$, $V(H_1) \cap V(H_2) = \{v_2, v_5\}$, there is an outer bridge $B$ of $C$ in $G$ such that $A(G, B) = \{v_0, v_1, v_2, v_5\}$. If $G - vv_2$ also has a proper two-vertex separation such that 

$$G - vv_2 = H_3 + H_4$$

and $E(H_3) \cap E(H_4) = \emptyset$, $V(H_3) \cap V(H_4) = \{x, y\}$, by Lemma 4.2.7, $\{x, y\} = \{v_1, v_3\}$ or $\{v_0, v_3\}$ or $\{v_1, v_4\}$. Therefore, there is an outer bridge $D$ of $C$ in $G$ with $A(G, D) \supseteq \{v_1, v_2, v_3\}$. But this contradicts the fact that $G$ is a planar graph. Hence, $G - vv_2$ has a
unique planar representation. Since in $G - vv_2$, we can recognize $v$ which has degree 2 and is incident to the unique 6-face; and since any reconstruction of $G$ is also a maximal bipartite planar graph, there is only one way to add an edge $e$ to $G - vv_2$ to obtain a reconstruction of $G$. Therefore, $G$ is edge-reconstructible.

In the case of $V(H_1) \cap V(H_2) = \{v_1, v_4\}$, the proof is similar. 

It is clear from the proof of Lemma 4.2.8 that if $G$ contains a 3-vertex which is incident to an edge $e$ such that $G - e$ has a unique planar representation, then $G$ is edge-reconstructible. Hence, by Lemma 4.2.8, for any 3-vertex $v$ of $G$ (see Figure 4.1), we can assume that there is a proper two-vertex separation of $G - vv_1$ ($i = 0, 2, 4$) such that

$$G - vv_i = H_i + H_{i+1}$$

and $E(H_i) \cap E(H_{i+1}) = \emptyset$, $V(H_i) \cap V(H_{i+1}) = \{v_{i-1}, v_{i+1}\}$ (where $v_0 = v_5$). Therefore, there are three outer bridges $B_1, B_2, B_3$ of the circuit $C$ in $G$ with $A(G, B_1) = \{v_0, v_1, v_5\}$, $A(G, B_2) = \{v_1, v_2, v_3\}$ and $A(G, B_3) = \{v_3, v_4, v_5\}$.

Let $G^*$ be a dual graph of $G$, then $G^*$ is a 4-regular 3-connected planar graph. Like in [1], we define

$$U_i = \{x \in V(G^*); d_{G^*}(x) = 3, |E_G(x) \cap E_c(G^*)| = i\}, i = 0, 1, 2, 3,$$

and

$$W_i = \{x \in V(G^*); d_{G^*}(x) \geq 4, |E_G(x) \cap E_c(G^*)| = i\}, i \geq 0,$$

where $E_G(x)$ is the set of edges which are incident with $x$.

Clearly, $U_i = \emptyset$ ($i = 0, 1, 2, 3$). By Theorem 5 of [1], $|U_2| \geq 2|W_1| + 3|W_0|$. Hence we have $|W_0| = |W_1| = 0$. Therefore, each vertex $v^*$ of $G^*$ is incident to at least two edges belonging to $E_c(G^*)$. This implies that each 4-face of $G$ is incident to at least two edges belonging to $E_d(G)$. Therefore, in Figure 4.1, $v_0v_1$ belongs to $E_d(G)$. By Lemma 4.2.6, we can assume $d(v_0) = 4$ in Figure 4.1.
Proof of Theorem 4.2.1. By our assumption, we know that there is an outer bridge $B_1$ of $C$ in $G$ with $A(G, B_1) = \{v_0, v_1, v_5\}$. Therefore, the graph in Figure 4.2 is contained by the union of the bridge $B_1$ and the graph in Figure 4.1, where $v_0$ is incident to four 4-faces.

Since $v_0v_1 \in E_d(G)$, we can delete $v_0v_1$ such that $G - v_0v_1$ is still 3-connected. In $G - v_0v_1$, we have two adjacent 3-vertices which are incident to the unique 6-face. Therefore, if $G$ is not edge-reconstructible, there is only one way to obtain a reconstruction $H$ of $G$ from $G - v_0v_1$ such that $H$ is not isomorphic to $G$. This can be done by adding the edge $vv_8$. Hence $H$ contains the graph in Figure 4.3 as a subgraph.
Consider the outer bridge of the circuit \( C' \) of \( H \), where \( V(C') = \{v, v_4, v_5, v_6, v_7, v_g\} \), \( E(C') = \{v v_8, v v_4, v_4 v_5, v_5 v_6, v_6 v_7, v_7 v_g\} \). Clearly, \( v_8, v_1, v_2, v, v_3, v_4 \) belong to the same outer bridge of \( C' \) in \( H \), say \( B' \). We are going to show \( v_5 \in V(B') \). By our assumption, \( v_5, v_4, v_3 \) belong to the same outer bridge \( B_3 \) of the circuit \( C \) in \( G \). Therefore, there are \( C \)-avoiding chains \( C[v_5, v_4] \) and \( C[v_5, v_3] \) in \( B_3 \). By the construction of \( H \), this implies that there are \( C' \)-avoiding chains from \( v_5 \) to \( v_4 \) and \( v_3 \) in \( H \). Therefore \( v_5 \in V(B') \). By Lemma 4.2.8, \( H \) is edge-reconstructible. Hence, \( G \) is edge-reconstructible.

4.3. Maximal Bipartite Planar Graphs with Minimum Valency

#
Throughout the rest of this section, $G$ always stands for a maximal bipartite planar graph with minimum valency 3. Since $G$ is trivially edge reconstructible from $G_{uv}$ for any two adjacent 3-vertices $u$ and $v$ in $G$, we will assume that there do not exist any two adjacent 3-vertices in $G$. By [3], $G$ is edge-reconstructible if $|V(G)| < 10$. We will assume that $|V(G)| \geq 10$. Hence by [8], the planarity of $G$ is recognizable from its edge-deck. This means that all reconstructions of $G$ are planar graphs. We will often use this fact without a further explanation.

Before we prove our theorem 4.1.1, we need several lemmas. The first lemma can be found in [22]. To state this lemma, we need to define the decomposition of the vertex set of a bipartite planar graph. For a bipartite planar graph $G'$, we let $V(G') = V_1 \cup V_2$ be the corresponding decomposition of the vertex set of $G'$ such that $V_1 \cap V_2 = \emptyset$ and each edge $e$ of $G'$ is of the form $e = v_1v_2$, $v_1 \in V_1$, $v_2 \in V_2$.

**Lemma 4.3.1.** A maximal bipartite planar graph $G'$ without multiple edges is two-vertex connected. It is three-vertex connected when in addition $G'$ has no pair of faces with just two boundary vertices in common, both belonging to $V_1$ or both to $V_2$.

Before we state the second lemma, we need a definition.

Let $G'$ be a planar graph of connectivity 2, and

$L = \{H; H$ is a lobule of $G'$, $A(G', H) = \{a, b\}$, for all cut sets $\{a, b\}$ of $G'\}$.

By a minimal lobule of $G'$ we mean a lobule of minimal order, where minimality is taken over all $L$.

The second lemma is due to Lauri [14].

**Lemma 4.3.2.** Let $H$ be a minimal lobule of a graph $G'$ of minimum valency 3 and connectivity 2, and let $A(G', H) = \{u, v\}$. Then
(i) $H$ is 2-connected.

(ii) In $H$, degrees of $u$ and $v \geq 2$.

(iii) If $S = \{a, b\}$ is a cut set for $H$, then $S$ cannot be a cut set for $G$.

By section 4.2, to prove Theorem 4.1.1, we only need to consider the case in which the connectivity of maximal bipartite graphs is 2. Hence we can assume that $G$ has a cut set $S$ with $|S| = 2$. Also by section 4.2, we know that all reconstructions of $G$ are also maximal bipartite planar graphs.

Let $v \in V(G)$, we define $N(v) = \{u \in V(G); u$ is adjacent to $v\}$.

**Lemma 4.3.3.** Let $H$ be a minimal lobule of $G$ and let $A(G, H) = \{a, b\}$. Then

(i) $H$ is also a maximal bipartite planar graph.

(ii) $H$ has at least four 3-vertices, say $u_1, u_2, u_3, u_4$.

(iii) $H$ has at least one 3-vertex $u$ with $\{a, b\} \not\subseteq N(u)$.

**Proof.** (i) By the definition of $H$ and Lemma 4.3.1, it is clear that $H$ is a maximal bipartite planar graph.

(ii) Let $n_i$ be the number of i-vertices of $H$, $\Delta$ be the maximum degree of $H$ and $f$ be the number of faces of a planar embedding of $H$. Let $m = |E(H)|$, $n = |V(H)|$. Since $2m = \sum_{i=2}^{\Delta} n_i$, $2m = 4(n - 2)$, $n = \sum_{i=2}^{\Delta} n_i$ and $f = n - 2$, and since by Euler's Formula

$$n - m + f = 2,$$

we have

$$\sum_{i=2}^{\Delta} (4 - i)n_i = 8.$$
Therefore, \(2n_2 + n_3 = 8 + n_5 + \ldots + (\Delta - 4)n_\Delta\). Since \(H\) has at most two 2-vertices, we have

\[n_3 \geq 4 + n_5 + \ldots + (\Delta - 4)n_\Delta.\]

(iii) By Lemma 4.3.2, we know that \(\{a, b\}\) is not a cut set of \(H\). If there is one 3-vertex \(u\) of \(H\) with \(\{a, b\} \not\subset N(u)\), we are done. Hence we assume that all 3-vertices of \(H\) are adjacent to both \(a\) and \(b\). Since \(\{a, b\}\) is not a cut set of \(H\), there is a path \(P\) in \(H\) which passes neither \(a\) nor \(b\) and connects \(u_1\) and \(u_2\). Hence \(H\) contains the graph in Figure 4.4 as a subgraph.

![Figure 4.4](image)

Since \(G\) is a bipartite graph, \(P\) can not be a single edge. Hence we assume that \(P\) consists of more than one edge. Since \(H\) is a planar graph, it contains no subdivision of \(K_{3,3}\). Therefore there does not exist such a path that passes neither \(a\) nor \(b\) and connects \(u_3\) and any vertex \(v\) on \(P\). This implies that \(\{a, b\}\) is a cut set of \(H\). This is a contradiction.

Throughout the rest of this section, we let \(H\) be a minimal lobule of \(G\) and let \(A(G, H) = \{a, b\}\). By Lemma 4.3.3, we can also assume that there is a 3-vertex \(v\) in \(H\)
such that $v \notin \{a, b\}$ and $(a, b) \notin N(v)$. Hence $H$ contains the graph in Figure 4.5 as a subgraph, where $v$ is incident to three 4-faces.

Let $C$ be a circuit of $G$ with $V(C) = \{v_1, ..., v_6\}$ and $E(C) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$.

**Lemma 4.3.4.** Let $S$ be a cut set of $G$ with $|S| = 2$. Then $S \neq \{v_1, v_3\}, S \neq \{v_3, v_5\}$ and $S \neq \{v_1, v_5\}$.

**Proof.** Suppose $S = \{v_{2i-1}, v_{2i+1}\}$ ($v_7 = v_1$) for some $i$ (say $i = 1$). Since $N(v)$ does not contain $(a, b)$, $(v_1, v_3) \neq (a, b)$. Therefore $(v_1, v_3) \subseteq V(H)$ and $(v_1, v_3) \not\subseteq V(G) - V(H).$ By the minimality of $H$, $s = \{v_1, v_3\}$ can not be a cut set of $G.$ This is a contradiction. 
By Lemma 4.3.4, there is no outer bridge $B$ of $C$ in $G$ such that $A(G, B) = \{v_1, v_2, v_3\}$ or $\{v_3, v_4, v_5\}$ or $\{v_5, v_6, v_1\}$. This means that in any planar embedding of $G$, $v$ is always incident to these 4-faces $vv_1v_2v_3v$, $v_3v_4v_5v_3$, $v_4v_6v_1v$.

**Lemma 4.3.5.** Let $S$ be a cut set of $G$ with $|S| = 2$. Then $S \neq \{v_1, v_4\}$, $S \neq \{v_2, v_5\}$ and $S \neq \{v_3, v_6\}$.

*Proof.* Assume that our lemma is not true. Then one of $\{v_1, v_4\}$, $\{v_2, v_5\}$ and $\{v_3, v_6\}$ equals $S$. Without loss of generality, we assume $\{v_1, v_4\} = S$. Since $\{v_1, v_4\}$ is a cut set of $G$, by Lemma 4.3.1, there is a pair of 4-faces with $v_1, v_4$ as their common vertices. Hence there is a path of length two joining $v_1$ and $v_4$. Therefore $G$ contains a 5-circuit. This contradicts the fact that $G$ is a bipartite graph. 

**Lemma 4.3.6.** Let $\{u, w\} \subset V(C)$. If $\{u, w\}$ is a cut set of $G - vv_1$, but it is not a cut set of $G$, then $\{u, w\} = \{v_2, v_6\}$ or $\{v_3, v_6\}$ or $\{v_2, v_5\}$.

*Proof.* Since $\{u, w\}$ is not a cut set of $G$, $\{u, w\} \neq \{v_2i, v_2(i+1)\}$ for $i = 1, 2$. It is also clear that $u$ and $w$ are not adjacent on $C$. Therefore $\{u, w\} = \{v_2, v_6\}$ or $\{v_3, v_6\}$ or $\{v_2, v_5\}$. 

**Lemma 4.3.7.** If $\{v_2, v_5\}$ or $\{v_3, v_6\}$ is a cut set of $G - vv_1$, then $G$ is edge reconstructible.

*Proof.* Without loss of generality, we assume that $\{v_2, v_5\}$ is a cut set of $G - vv_1$. Since $\{v_2, v_5\}$ is not a cut set of $G$, we can write $(G - vv_1) - \{v_2, v_5\}$ in the following way, $(G - vv_1) - \{v_2, v_5\} = H_1 \cup H_2$, where $H_1, H_2$ are connected.
components of \((G - v_1) - \{v_2, v_5\}\). Without loss of generality, we assume \(v_1, v_6 \in V(H_1), v_3, v_4 \in V(H_2)\). Consider \(G - vv_5\). Clearly, \(G - vv_5\) has a planar embedding \(\psi\) such that \(v\) is incident to a 4-face \(F\) with the face boundary \(\Gamma = vv_1v_2v_3v\) and a 6-face \(F'\) with the face boundary \(\Gamma' = v_1v_3v_4v_5v_6v_1\). Since the 4-face \(F\) has only one bridge, \(\Gamma'\) is always a face boundary in any planar embedding of \(G - vv_5\). Therefore ambiguity in reconstructing \(G\) from \(G - vv_5\) can only arise if there exists a planar embedding \(\psi'\) of \(G - vv_5\) in which \(F'\) is not a face. But if \(F'\) is not a face in some plane embedding, then \(\Gamma'\) will have at least two bridges. Since \(\{v_3, v_5\}, \{v_1, v_5\}\) and \(\{v_1, v_3\}\) are not cut sets of \(G - vv_5\), clearly, there exist \(C\)-avoiding paths which join \(v_4\) to \(v_2\) and \(v_6\) to \(v_2\). Hence there is a bridge \(B\) of \(\Gamma'\) in \(G - vv_5\) with \(A'(G - vv_5, B) \supseteq \{v_1, v_3, v_4, v_6\}\). Since \(\{v_2, v_5\}\) is a cut set of \(G - vv_1\) and \(v_6 \in V(H_1), v_4 \in V(H_2)\), clearly, there is no \(C\)-avoiding path which joins \(v_6\) to \(v_4\). Hence there is no bridge \(B'\) of \(\Gamma'\) in \(G - vv_5\) with \(A'(G - vv_5, B') = \{v_4, v_5, v_6\}\). Clearly, we do not have a bridge \(B'\) of \(\Gamma'\) in \(G - vv_5\) with \(A'(G - vv_5, B') = \{v_4, v_5\}\) or \(\{v_5, v_6\}\). Hence it is clear that \(\Gamma'\) has only one bridge \(B\) with \(A'(G - vv_5, B) = \{v_1, v_3, v_4, v_5, v_6\}\). Therefore in any planar embedding of \(G - vv_5\), \(F'\) is always a face. This implies that there is only one way to reconstruct \(G\) from \(G - vv_5\). Therefore \(G\) is edge-reconstructible. 

Remark. From the proof of Lemma 4.3.7, we notice that what we need in the proof is that the 6-face boundary \(\Gamma'\) of \(G - e\) for some \(e \in E(G)\) has only one bridge. Therefore in any planar embedding of \(G - e\), \(\Gamma'\) is always a face boundary. We will often use this fact through the rest of this section.

Lemma 4.3.8. If none of \(\{v_2, v_6\}, \{v_3, v_6\}\) and \(\{v_2, v_5\}\) is a cut set of \(G - vv_1\), then \(G\) is edge-reconstructible.
**Proof.** It is clear that the boundary $\Gamma'$ of the 6-face of $G - vv_1$ has only one bridge. By our remark, it is easy to see that $G$ is edge-reconstructible. 

By Lemma 4.3.7 and Lemma 4.3.8, we can make the following assumption. We assume that $\{v_2, v_5\}$, $\{v_3, v_6\}$ are not cut sets of $G - vv_1$, $\{v_2, v_5\}$, $\{v_1, v_4\}$ are not cut sets of $G - vv_3$ and $\{v_1, v_4\}$, $\{v_3, v_6\}$ are not cut sets of $G - vv_5$. Therefore, if $G$ is not edge-reconstructible, $\{v_2, v_6\}$ is a cut set of $G - vv_1$, $\{v_2, v_4\}$ is a cut set of $G - vv_3$ and $\{v_4, v_6\}$ is a cut set of $G - vv_5$. Hence there are three outer bridges $B_1$, $B_2$, $B_3$ of $C$ in $G$ with $A(G, B_1) = \{v_1, v_2, v_6\}$, $A(G, B_2) = \{v_2, v_3, v_4\}$, $A(G, B_3) = \{v_4, v_5, v_6\}$. By this assumption, we have the following lemma.

**Lemma 4.3.9.** $d(v_i) \geq 5$, $i = 2, 4, 6$, where $d(v)$ denotes the degree of a vertex $v$ in $G$.

**Proof.** Clearly, $d(v_i) \geq 4$ for $i = 2, 4, 6$. If there is some $v_i$ ($i = 2, 4, 6$) with $d(v_i) = 4$, then there exists a planar embedding $\psi$ of $G$ such that the boundary of the infinite face of $\psi$ consists of at least six edges. This is a contradiction, since $G$ is a maximal bipartite planar graph. 

**Proof of Theorem 4.1.1.** By Theorem 4.2.1, we only need to consider the case in which the connectivity of $G$ is 2. First we notice that among $V(B_1)$, $V(B_2)$ and $V(B_3)$, there is at most one which could contain $\{a, b\}$. Without loss of generality, we assume that $V(B_1)$ does not contain $\{a, b\}$. Under this assumption, we know that the union of the bridge $B_1$ and the graph in Figure 4.5 is an induced subgraph of $H$. We consider the following cases.

**Case 1.** $d(v_1) = 4$. 


Figure 4.6

Since \( d(v_1) = 4 \), the graph in Figure 4.6 is contained by the union of the bridge \( B_1 \) and the graph in Figure 4.5, where \( v_1 \) is incident to four 4-faces. Clearly, \( \{v_2, v_8\} \) and \( \{v_6, v_8\} \) are not cut sets of \( H \). Otherwise they will be cut sets of \( G \). This contradicts Lemma 4.3.2. Consider \( G - v_1v_2 \). It is clear that \( G - v_1v_2 \) has a planar embedding \( \psi \) in which \( v_1 \) is incident to two 4-faces and one 6-face \( F \), bounded by the circuit \( \Gamma = v_1v_8v_9v_2v_3v \). It is also clear that \( \Gamma \) has only one bridge. Therefore, in any plane embedding of \( G - v_1v_2 \), \( \Gamma \) is always a face boundary of \( F \). Since \( G \) does not contain two adjacent 3-vertices and \( G \) is a maximal bipartite planar graph, we can construct a reconstruction \( G' \) of \( G \) from \( G - v_1v_2 \) in only two ways: as \( G \) or else as \((G - v_1v_2) + vv_9\). If \( G \) is not edge-reconstructible, then the reconstruction \( G' = (G - v_1v_2) + vv_9 \) of \( G \) is not isomorphic to \( G \) and \( G' \) is also not edge-reconstructible. Now we consider the circuit \( C' \) of \( G' \) with \( V(C') = \{v, v_5, v_6, v_7, v_8, v_9\} \) and \( E(C') = \{vv_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v\} \). Clearly, there is an outer bridge \( B \) of \( C' \) in \( G' \) with \( A(G', B) \supseteq \{v_9, \)}
v, v_5, v_6). Hence \{v_5, v_9\} is not a cut set of \(G' - vv_1\). By Lemma 4.3.7 and Lemma 4.3.8, \(G'\) is edge-reconstructible. This is a contradiction. Therefore \(G\) is edge-reconstructible.

**Case 2.** \(d(v_1) \geq 5\).

![Figure 4.7](image)

Figure 4.7

Clearly, the graph in Figure 4.7 is contained by the union of the bridge \(B_1\) and the graph in Figure 4.5. We already know that \(\{v_1, v_3\}, \{v_3, v_5\}\) and \(\{v_1, v_5\}\) are not cut sets of \(G\). It is also clear that \(\{u_{2i-1}, u_{2i+1}\}\) (\(i = 1, ..., t - 1\)) are not cut sets of \(H\). Otherwise, they will be cut sets of \(G\). This contradicts (iii) of Lemma 4.3.2. Consider \(G - v_1v_2\). \(G - v_1v_2\) has a planar embedding \(\psi\) in which all faces incident to \(v_1\) are 4-faces.
except one which is 6-face F and is bounded by the circuit \( \Gamma = v_2v_3v_1u_{t-3}u_{t-2}v_2 \). It is clear that \( \Gamma \) has only one bridge. Therefore, in any planar embedding of \( G - v_1v_2 \), \( \Gamma \) is always a face boundary of F. Since \( \{d(v_1), d(v_2)\} \) is edge-reconstructible, \( d(v_1) \geq 5 \), \( d(v_2) \geq 5 \), and \( G \) is a maximal bipartite planar graph, we can construct reconstruction \( G' \) of \( G \) from \( G - v_1v_2 \) in only two ways: as \( G \) or else as \( (G - v_1v_2) + v_3u_{t-3} \). If \( G \) is not edge-reconstructible, then the reconstruction \( G' = (G - v_1v_2) + v_3u_{t-3} \) of \( G \) is not isomorphic to \( G \) and \( G' \) is also not edge-reconstructible. Now consider \( C' \) of \( G' \) with \( V(C') = \{v_3, v_4, v_5, v_6, v_1, u_{t-3}\} \) and \( E(C') = \{v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1u_{t-3}, u_{t-3}v_3\} \). Since \( \{v_2, u_{t-3}\} \) is not a cut set of \( G \), there is a path in \( B_1 \) which joins \( u_{t-2} \) to \( v_1 \) and does not pass through \( u_{t-3} \). Hence, it is clear that there is an outer bridge \( B \) of \( C' \) in \( G' \) with \( A(G', B) \supseteq \{v_4, v_3, u_{t-3}, v_1\} \). Hence \( \{v_4, u_{t-3}\} \) is not a cut set of \( G' - vv_3 \). By Lemma 4.3.7 and Lemma 4.3.8, \( G' \) is edge-reconstructible. This is a contradiction. Therefore \( G \) is edge-reconstructible. 

#
CHAPTER V
THE EDGE-RECONSTRUCTION
OF 3-CONNECTED BIPARTITE PLANAR GRAPHS

5.1. Introduction

In this chapter, we focus our attention to 3-connected bipartite planar graphs. For convenience, we introduce the following new notation. For graphs F and G, the number of subgraphs of G isomorphic to F is denoted by $S(F, G)$. Here, we are going to prove the following theorem.

**THEOREM 5.1.1.** If $G$ is a 3-connected bipartite planar graph, then $G$ is edge-reconstructible.

Throughout the rest of this chapter, $G$ is always a 3-connected bipartite planar graph. Since $G$ is trivially edge reconstructible from $G_{uv}$ for any two adjacent $\delta$-vertices $u$ and $v$ in $G$, we will assume that there do not exist any two adjacent $\delta$-vertices in $G$. Since by [3], $G$ is edge-reconstructible if $|V(G)| < 10$, we will also assume that $|V(G)| \geq 10$. Hence by [8], the planarity of $G$ is recognizable from its edge-deck. This means that all reconstructions of $G$ are planar graphs. We will often use this fact without a further explanation.

63
5.2. Proof of the Theorem 5.1.1

Before we prove Theorem 5.1.1, we need several lemmas. The first lemma and the second lemma are due to Ore [22] and Hoffman (see [4] for a proof), respectively.

LEMMA 5.2.1. Let $H$ be a nonseparable planar graph. A necessary and sufficient condition that its planar representation is unique is that it has no proper two-vertex separation.

LEMMA 5.2.2. Let $H$ be a graph of minimum degree $\delta$. Suppose that, for some $k \geq 0$, there is a vertex in $G$ of degree $\delta + k$ adjacent to $k + 1$ or more vertices of degree $\delta$. Then $G$ is edge-reconstructible.

LEMMA 5.2.3. If $H$ is a reconstruction of $G$, then $H$ is also a bipartite planar graph.

Proof. Suppose that $H$ is not a bipartite graph, then $H$ contains an odd circuit. Clearly, $H$ itself is not a circuit. Therefore there exists an edge $e \in E(H)$ such that $H - e$ contains an odd circuit. On the other hand, since $\Delta(H) = \Delta(G)$ and $G$ is a bipartite graph, $H - e$ cannot contain an odd circuit for every $e \in E(H)$. This is a contradiction. Thus $H$ is a bipartite graph. Hence $H$ is a bipartite planar graph.

LEMMA 5.2.4. $\delta(G) = 3$.

Proof. Since $d = \frac{2m}{n} \leq \frac{4n - 8}{n} < 4$ and $G$ is 3-connected, we have $3 \leq \delta(G) \leq d < 4$. Therefore $\delta(G) = 3$. 

#
Let $G^*$ be a dual graph of $G$, then $G^*$ is a 3-connected planar graph. Like in Ando [1], we define

$$U_i = \{x \in V(G^*); d_{G^*}(x) = 3, |E_{G^*}(x) \cap E_c(G^*)| = i\}, \quad i = 0, 1, 2, 3,$$

and

$$W_i = \{x \in V(G^*); d_{G^*}(x) \geq 4, |E_{G^*}(x) \cap E_c(G^*)| = i\}, \quad i \geq 0,$$

where $E_{G^*}(x)$ is the set of edges which are incident with $x$.

Since $G$ is a bipartite planar graph, then $\delta(G^*) \geq 4$. Hence, it is clear that $U_i = \emptyset$ ($i = 0, 1, 2, 3$). By Theorem 5 in [1], $|W_2| \geq 2|W_1| + 3|W_0|$, we have that $|W_0| = |W_1| = 0$. Therefore, each vertex $v^*$ of $G^*$ is incident to at least two edges in $E_c(G^*)$. This implies that each $2k$-face of $G$ for $k \geq 2$ is incident to at least two edges in $E_d(G)$. Therefore each $2k$-face of $G$ is incident to at most $(k - 1)$ 3-vertices.

**Lemma 5.2.5.** There exists a 3-vertex $v \in G$ which is only incident to 4-faces.

**Proof.** Let $\Delta$ be the maximum degree of $G$ and $t$ be the maximum face valency of $G$. Since bidegreed graphs are edge-reconstructible [20], we may assume $\Delta \geq 5$. If $t = 4$, the lemma is obviously true. Therefore we assume $t \geq 6$. Since $2m = \sum_{i=3}^{\Delta} in_i$, $2m = \sum_{i=4}^{t} if_i$,

$$n = \sum_{i=3}^{\Delta} n_i \quad \text{and} \quad f = \sum_{i=4}^{t} f_i,$$

and since by Euler's Formula

$$n - m + f = 2,$$

we have

$$\sum_{i=3}^{\Delta} (4 - i)n_i + \sum_{i=4}^{t} (4 - i)f_i = 8.$$

Therefore, $n_3 = 8 + n_5 + \ldots + (\Delta - 4)n_\Delta + 2f_6 + \ldots + (t - 4)f_t$.

Since each $2k$-face is incident to at most $(k - 1)$ 3-vertices, there are at most
3-vertices on the boundaries of 2k-faces of G for k ≥ 3. Therefore the proof follows from

$$2f_6 + \ldots + \left( \frac{t}{2} - 1 \right)f_t \leq 2f_6 + \ldots + (t - 4)f_t < n_3. \quad \#$$

From the proof of Lemma 5.2.5, we can see that there are at least \([8 + n_5 + \ldots + (\Delta - 4)n_\Delta]\) 3-vertices in G which are only incident to 4-faces. Also from Lemma 5.2.5, G contains the graph in Figure 5.1 as a subgraph, where the 3-vertex v is incident to three 4-faces.

![Figure 5.1](image)

Since G is a 3-connected planar graph, there is no outer bridge B of the circuit C in G with \(A(G, B) = \{v_2, v_3, v_4\}\) or \(\{v_0, v_1, v_2, v_4, v_5\}\), where \(V(C) = \{v_i; i = 0, \ldots, 5\}\), \(E(C) = \{v_iv_{i+1}; i = 0, 1, \ldots, 5, v_6 = v_0\}\).
LEMMA 5.2.6. Let \( v \) be a 3-vertex of \( G \) which is only incident to 4-faces (see Figure 5.1). If there is a proper two-vertex separation of \( G - vv_0 \) such that

\[
G - vv_0 = H_1 + H_2,
\]

\[
E(H_1) \cap E(H_2) = \emptyset \text{ and } V(H_1) \cap V(H_2) = \{x, y\}, \text{ then } \{x, y\} = \{v_1, v_5\} \text{ or } \{v_2, v_5\} \text{ or } \{v_1, v_4\}.
\]

Proof. Clearly \( v \notin \{x, y\} \). This is because if \( \{v, w\} \) were a cut set of \( G - vv_0 \), then it would also be a cut set of \( G \), which is impossible, since \( G \) is 3-connected. It is also clear that \( V(C) \nsubseteq \{x, y\} \). Since if one of \( x \) and \( y \) does not belong to \( V(C) \), then \( V(C) \) and \( v \) belong to the same \( H_i \), say \( i = 1 \). This implies that for any vertex \( z \) in \( H_2 \), where \( z \notin \{x, y\} \), a chain from \( z \) to \( w \), where \( w \in V(C) - \{x, y\} \) must pass through \( x \) or \( y \). Therefore in \( G \), any chain from \( z \) to \( w \) must pass through \( x \) or \( y \). This contradicts the fact that \( G \) is 3-connected. Hence \( \{x, y\} \subseteq V(C) \). Clearly \( \{x, y\} \neq \{v_2, v_4\} \). This is because if \( \{x, y\} = \{v_2, v_4\} \), then either (1) one of \( H_1 \) and \( H_2 \) is a graph chain \( C[v_2, v_4] \) with

\[
V(C[v_2, v_4]) = \{v_2, v, v_4\}
\]

\[
E(C[v_2, v_4]) = \{vv_2, vv_4\}
\]

which is not a proper two-vertex separation, or (2) there is an outer bridge \( B \) of the circuit \( C \) in \( G \) with \( A(G, B) = \{v_2, v_3, v_4\} \) which contradicts the fact that \( G \) is 3-connected. Therefore, \( \{v_0, v_1, v_2, v_5\} \supset \{x, y\} \) or \( \{v_0, v_1, v_4, v_5\} \supset \{x, y\} \). There are three possibilities for \( \{x, y\}, \{x, y\} = \{v_2, v_5\} \) or \( \{x, y\} = \{v_1, v_4\} \) or \( \{x, y\} = \{v_1, v_5\} \). #

LEMMA 5.2.7. Let \( v \) be a 3-vertex of \( G \) which is only incident to 4-faces (see Figure 5.1). If there is a proper two-vertex separation of \( G - vv_0 \) such that

\[
G - vv_0 = H_1 + H_2,
\]
$E(H_1) \cap E(H_2) = \emptyset$ and $V(H_1) \cap V(H_2) = \{v_2, v_5\}$ (or $\{v_1, v_4\}$), then $G$ is edge-reconstructible.

**Proof.** Since there is a proper two-vertex separation of $G - vv_0$ such that

$$G - vv_0 = H_1 + H_2,$$

$E(H_1) \cap E(H_2) = \emptyset$ and $V(H_1) \cap V(H_2) = \{v_2, v_5\}$, there is an outer bridge $B$ of the circuit $C$ in $G$ such that $A(G, B) = \{v_0, v_1, v_2, v_5\}$. If $G - vv_2$ also has a proper two-vertex separation such that

$$G - vv_2 = H_3 + H_4,$$

$E(H_3) \cap E(H_4) = \emptyset$ and $V(H_3) \cap V(H_4) = \{x, y\}$, by Lemma 5.2.6, $\{x, y\} = \{v_1, v_3\}$ or $\{v_0, v_3\}$ or $\{v_1, v_4\}$. Therefore, there is an outer bridge $D$ of the circuit $C$ in $G$ with $A(G, D) \supseteq \{v_1, v_2, v_3\}$. This contradicts the fact that $G$ is a planar graph. Hence, $G - vv_2$ has a unique planar representation. Since in $G - vv_2$, we can recognize $v$ which has degree 2 and is incident to the unique 6-face, and since any reconstruction of $G$ is also a bipartite planar graph, there is only one way to add an edge $e$ in $G - vv_2$ to obtain a reconstruction of $G$. Therefore, $G$ is edge-reconstructible.

In the case of $V(H_1) \cap V(H_2) = \{v_1, v_4\}$, the proof is similar. 

It is clear from the proof of Lemma 5.2.7 that if $G$ contains a 3-vertex which is only incident to 4-faces and is also incident to an edge $e$ such that $G - e$ has a unique planar representation, then $G$ is edge-reconstructible. Hence, by Lemma 5.2.7, for any 3-vertex $v$ in $G$ which is only incident to 4-faces (see Figure 5.1), we can assume that there is a proper two-vertex separation of $G - vv_i$ ($i = 0, 2, 4$) such that

$$G - vv_i = H_i + H_{i+1},$$

$E(H_i) \cap E(H_{i+1}) = \emptyset$ and $V(H_i) \cap V(H_{i+1}) = \{v_{i-1}, v_{i+1}\}$ (where $v_{0-1} = v_5$). Therefore, there are three outer bridges $B_1, B_3, B_5$ of the circuit $C$ in $G$ with $A(G, B_1) =$.
\[ \{v_0, v_1, v_5\}, A(G, B_3) = \{v_1, v_2, v_3\} \text{ and } A(G, B_5) = \{v_3, v_4, v_5\}. \]

It is clear that if one of the three 4-faces which are incident to the 3-vertex \( v \) contains a 3-vertex distinct from \( v \), say \( v_i, i \in \{1, 3, 5\} \), then one of \( G - vv_{i-1} \) and \( G - vv_{i+1} (v_0 = v_6) \) has a unique planar representation. Therefore \( G \) is edge-reconstructible. Hence for any vertex \( v_i, i = 1, 3, 5 \), we may assume \( d(v_i) \geq 4 \).

We define \( E_{3,i} = \{ e \in E(G); \text{ one end of } e \text{ is incident to a 3-vertex and the other end of } e \text{ is incident to an } i-\text{vertex} \} \) and \( t_{3,i} = |E_{3,i}|, i \geq 4 \). Then \( 3n_3 = t_{3,4} + \ldots + t_{3,\Delta} \) and by Lemma 5.2.2, if \( G \) is not edge-reconstructible, then \( t_{3,i} \leq (i - 3)n_i \). We claim that if \( G \) is not edge-reconstructible, there exists at least one 3-vertex which is not only incident to three 4-faces but is also adjacent to a 4-vertex in \( G \). There are two cases.

Case 1. \( t = 4 \).

Since \( t = 4 \), then \( n_3 = 8 + n_5 + \ldots + (\Delta - 4)n_\Delta \). If \( G \) does not contain a 3-vertex which is adjacent to a 4-vertex, by Lemma 5.2.2, we will have

\[ 2n_5 + \ldots + (\Delta - 3)n_\Delta \geq 3n_3. \]

Hence

\[ 2n_5 + \ldots + (\Delta - 3)n_\Delta \geq 24 + 3n_5 + \ldots + 3(\Delta - 4)n_\Delta. \]

But this is impossible, since \( \Delta \geq 5 \).

Case 2. \( t \geq 6 \).

Since \( 3n_3 = t_{3,4} + \ldots + t_{3,\Delta} \) and \( n_3 = 8 + \sum_{i=5}^{\Delta} (i - 4)n_i + \sum_{i=6}^{t} (i - 4)f_i, \)

then

\[ t_{3,4} = (2n_5 - t_{3,5}) + \ldots + [(\Delta - 3)n_\Delta - t_{3,\Delta}] + 24 + n_5 + \ldots + (2\Delta - 9)n_\Delta + 6f_6 + \ldots + 3(t - 4)f_i. \]

Since there are at most \( \lceil 2f_6 + \ldots + (\frac{t}{2} - 1)f_i \rceil \) 3-vertices on the boundaries of 2k-faces of \( G \) with \( k \geq 3 \) and since \( 3(i - 4) - 3(\frac{i}{2} - 1) = \frac{3i}{2} - 9 \geq 0 \), with \( i \geq 6 \), we have \( 3\lceil 2f_6 + \ldots + (\frac{t}{2} - 1)f_i \rceil \leq [6f_6 + \ldots + 3(t - 4)f_i] \). Hence our claim follows. Actually from the
above discussion, we can see that there are at least $[24 + n_5 + ... + (2\Delta - 9)n_\Delta]$ edges which belong to $E_{3,4}$ and are incident to those 3-vertices which are only incident to 4-faces in $G$.

**Lemma 5.2.8.** Let $v$ be a 3-vertex of $G$ and be only incident to 4-faces (see Figure 5.1) and let $v_0$ be a neighbor of $v$ such that $d(v_0) = 4$. If $v_0$ is only incident to 4-faces in $G$, then $G$ is edge-reconstructible.

*Proof.* Since we know that there is an outer bridge $B_1$ of the circuit $C$ in $G$ with $A(G, B_1) = \{v_0, v_1, v_3\}$, the graph in Figure 5.2 is contained by the union of the bridge $B_1$ and the graph in Figure 5.1.

![Figure 5.2](image-url)
Since $v_0v_1 \in E_d(G)$, we can delete $v_0v_1$ such that $G - v_0v_1$ is still 3-connected. In $G - v_0v_1$, we have two adjacent 3-vertices which are incident to a 6-face. Therefore, if $G$ is not edge-reconstructible, there is only one way to construct a reconstruction $H$ of $G$ which is not isomorphic to $G$ and is constructed from $G - v_0v_1$ by adding the edge $vv_8$. Hence, $H$ contains the graph in Figure 5.3 as a subgraph.

Consider the outer bridge of the circuit $C'$ of $H$, where $V(C') = \{v, v_4, v_5, v_6, v_7, v_8\}$, $E(C') = \{vv_8, vv_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8\}$. Clearly, $v_8, v_1, v_2, v, v_3, v_4$ belong to the same outer bridge of the circuit $C'$ of $H$, say $B'$. We are going to show $v_5 \in V(B')$. Since $v_5, v_4, v_3$ belong to the same outer bridge $B_5$ of the circuit $C$ in $G$. Therefore, there are $C$-avoiding chains $C[v_5, v_4]$ and $C[v_5, v_3]$ in $B_5$. This implies that there are $C'$-avoiding chains from $v_5$ to $v_4$ and from $v_5$ to $v_3$. Therefore $v_5 \in V(B')$. By Lemma 5.2.7, $H$ is edge-reconstructible. Hence $G$ is edge-reconstructible. #

Apply this Lemma to 3-connected maximal bipartite planar graphs, we have the following corollary which was obtained as Theorem 4.2.1 in chapter 4.
COROLLARY 5.2.9. 3-connected maximal bipartite planar graphs are edge-reconstructible.

Proof of Theorem 5.1.1. By Lemma 5.2.8, we can make the following assumption that if \( v \) is a 3-vertex which is not only incident to three 4-faces, but it is also adjacent to a 4-vertex, then the 4-vertex is incident to at least one face with face valency \( \geq 6 \). By the assumption following the proof of Lemma 5.2.7, we know that if \( v \) is a 3-vertex which is only incident to 4-faces (see Figure 5.1), then \( G - vv_i \) \((i = 0, 2, 4)\) is not 3-connected and it has only one proper two-vertex separation

\[
G - vv_i = H_i + H_{i+1}
\]

where \( H_{i+1} \) consists of the chain graph \( C[v_{i-1}, v_{i+1}] \) together with the outer bridge \( B_{i+1} \), and \( H_i \) consists of the remaining part of \( G \), where \( V(C[v_{i-1}, v_{i+1}]) = \{v_{i-1}, v_i, v_{i+1}\} \), \( E(C[v_{i-1}, v_{i+1}]) = \{v_{i-1}v_i, v_iv_{i+1}\} \), and \( H_i, H_{i+1} \) are only connected at the two common vertices \( v_{i-1} \) and \( v_{i+1} \).

Clearly, \( G - vv_i \) has only two different planar embeddings and one of them which we have already seen is that \( B_{i+1} \) is an outer bridge of the circuit \( C \). The other is obtained by transferring the outer bridge \( B_{i+1} \) of the circuit \( C \) to an inner bridge \( B_{i+1}' \) of the circuit \( C \).

Let \( v \in G \) be a 3-vertex and only incident to 4-faces (see Figure 5.1). By our previous discussion, we can assume that one of the neighbors of \( v \) is a 4-vertex, say \( v_0 \). By our above discussion, \( G - vv_0 \) has only one proper two-vertex separation

\[
G - vv_0 = H_0 + H_1
\]

where \( H_1 \) consists of the chain graph \( C[v_5, v_1] \) together with the outer bridge \( B_1 \) and \( H_0 \) consists of the remaining part of \( G \), where \( V(C[v_5, v_1]) = \{v_0, v_1, v_5\} \) and \( E(C[v_5, v_1]) = \{v_0v_5, v_0v_1\} \). Let \( \Gamma \) be a path from \( v_5 \) to \( v_1 \) in \( H_1 \) such that \( C' = \Gamma \cup C[v_5, v_1] \) is a circuit of \( G \) and that any path from \( v_5 \) to \( v_1 \) in \( H_1 \) will be contained in the union of the
interior of the circuit $C'$ and its boundary $C'$. Let $V(\Gamma) = \{v_5 = u_0, u_1, ..., u_t = v_1\}$ and $E(\Gamma) = \{u_iu_{i+1}; i = 0, ..., t-1\}$. Then it is clear that if $G$ is not edge-reconstructible, all reconstructions of $G$ can be obtained from $G - vv_0$ by adding an edge joining $v$ and some vertex belonging to $V(C')$. There are two possibilities.

Case 1. $t = 2$. Since $t = 2$, there is only one reconstruction $H$ of $G$ which is not isomorphic to $G$ and is constructed from $G - vv_0$ by adding the edge $vu_1$. Clearly, the face list of $G$ is the same as one of $H$. This implies that the face list of $G$ is edge-reconstructible. If $u_1$ is only incident to 4-faces, then $H$ will contain the 3-vertex $v$ which is only incident to 4-faces, and further more this 3-vertex $v$ is adjacent to the 4-vertex $u_1$ which is only incident to 4-faces too. By Lemma 5.2.8, $H$ is edge-reconstructible. Hence $G$ is edge-reconstructible. Since $u_1$ is not only incident to 4-faces, it is clear that if we denote $A$ as a subgraph of $G$ such that $A$ consists of a 3-vertex and three 4-faces which are incident to that 3-vertex (see Figure 5.1) and if we call it a type A subgraph, then $S(G, A) = S(H, A)$. Hence $S(G, A)$ is also edge-reconstructible. Let $F_0, F_1, F_2$ and $F_3$ be the four faces to which $v$ is incident, where $V(F_0) = \{v_0, v_1, v_2, v\}, V(F_1) = \{v_0, v, v_4, v_5\}, V(F_2) = \{v_0, v_1, a_1, ..., a_k = p\}, V(F_3) = \{v_0, v_5, b_1, ..., b_i = p\}$ (see Figure 5.4). We assume $k \geq 6, k \geq i$. 
Since $v_0v_1 \in E_d(G)$, we can delete $v_0v_1$ such that $G - v_0v_1$ is still 3-connected. Then in $G - v_0v_1$, we have two adjacent 3-vertices. Since the face list of $G$ and $S(G, A)$ are edge-reconstructible and since $f_4(G - v_0v_1) = f_4(G) - 1$, $S(G - v_0v_1, A) = S(G, A) - 1$, we must construct one more 4-face and one more type A subgraph from $G - v_0v_1$ by adding an edge. There are three possibilities to add an edge in $G - v_0v_1$.

(a) We obtain a reconstruction $H'$ of $G$, which is isomorphic to $H$, from $G - v_0v_1$ by adding the edge $v_0a_{k-2}$, where $d(a_{k-1}) = 3$ and $a_{k-1}$ is only incident to 4-faces in $H'$. Besides $p$ and $a_{k-2}$, let $w$ be the third vertex adjacent to $a_{k-1}$. Clearly $w \neq a_{k-4}$, $w \neq b_{i-2}$ and $w$ is neither adjacent to $a_{k-3}$ nor $b_{i-1}$, this is because that $H'$ is 3-connected and $d(p) \geq 4$ (see Figure 5.5). Therefore $a_{k-2}, v_0, p$ belong to the same outer bridge $B$ of the circuit $C_1$, where $V(C_1) = \{v_0, p, w_1, w_2, w, a_{k-2}\}$, $E(C_1) = \{v_0p, pw_2, w_2w, w_1w, w_1a_{k-2}, a_{k-2}v_0\}$. Since $H'$ is 3-connected, besides $a_{k-2}, v_0, p \in A(H', B)$, one of $w, w_1, w_2$ must also belong to $A(H', B)$. Therefore either $H'$
- \(a_{k-1}a_{k-2}\) or \(H' - a_{k-1}p\) has a unique planar representation. Hence \(H'\) is edge-reconstructible. Therefore \(G\) is edge-reconstructible.

(b) We obtain a reconstruction \(H'\) of \(G\), which is isomorphic to \(H\), from \(G - v_0v_1\) by adding the edge \(va_{k-1}\). In this case, \(v_0\) is the 3-vertex only incident to 4-faces (see Figure 5.6).
Figure 5.6

Clearly, $a_{k-1}, v, v_4$ and $v_5$ belong to the same outer bridge $B$ of the circuit $C_1$, where $V(C_1) = \{a_{k-1}, v, v_4, v_5, b_1, p\}$, $E(C_1) = \{a_{k-1}v, vv_4, v_4v_5, v_5b_1, b_1p, p_{a_{k-1}}\}$. This implies that $H' - v_0v_5$ has a unique planar representation. Therefore $H'$ is edge-reconstructible. Hence $G$ is edge-reconstructible.

(c) The last possibility is to add the edge $va_1$ in $G - v_0v_1$ to get a reconstruction of $G$. We claim that we can not obtain a reconstruction $H'$ of $G$, which is isomorphic to $H$, from $G - v_0v_1$ by adding the edge $va_1$. Assume that we obtain $H'$ from $G - v_0v_1$ by adding the edge $va_1$. Then $v_1$ is a 3-vertex only incident to 4-faces in $H'$. Hence $d_G(v_1) = 4$, $d_G(a_1) = 3$. $a_1$ and $v_2$ are neighbors of $v_1$. Let $w$ be the other neighbor of $v_1$. Then $w$ belongs either to the outer bridge $B_3$ of the circuit $C$ or to the outer bridge $B_1$ of the circuit $C$. First we assume that $w \in B_3$. Clearly, $w \neq v_3$ and $w$ can not be adjacent to both $v_3$ and $v_2$. Let $w_1$ be a neighbor of $v_2$ and $w$ such that $v_1w_1v_2v_1$ is the boundary of one of the 4-faces to which $v_1$ is incident. Clearly edge $v_1a_1$ belongs to $B_1$. Let
v_1a_1w_2wv_1 be the boundary of another face to which v_1 is incident. Clearly, w_2 \notin A(G, B_3). Hence w_2 can not be adjacent to w. This is a contradiction. Now we assume w \in B_1. Since w_1 \notin B_1, w_1 can not be adjacent to w. This is also a contradiction. Therefore the proof of case 1 follows.

Case 2. t > 2. In this case, we first show that if H is any reconstruction of G, which is constructed from G - vv_0 by adding an edge joining v and some vertex belonging to V(C'), then f_4(H) = f_4(G). If this is not true, then f_4(H) \leq f_4(G) - 1. Since G has more than eight 3-vertices which are only incident to 4-faces, then H contains a 3-vertex u which is incident only to 4-faces. Let u_0, u_1, u_2 be the neighbors of u. Since f_4(H - uu_0) = [f_4(H) - 2] \leq [f_4(G) - 3] and since it is impossible to construct three more 4-faces by adding one edge, we can not obtain G from H - uu_0. This is a contradiction. Therefore f_4(H) = f_4(G). Hence the face list of G is edge-reconstructible. Now we show that S(G, A) is also edge-reconstructible. Assume that S(G, A) is not edge-reconstructible. Let H be any reconstruction of G, which is constructed from G - vv_0 by adding an edge joining v and some vertex belonging to V(C). Clearly S(H, A) = [S(G, A) - 1]. Since G contains at least [8 + n_5 + ... + (\Delta - 4)n_\Delta] 3-vertices which are only incident to 4-faces and at least [24 + n_5 + ... (2\Delta - 9)n_\Delta] edges which belong to E_{34} and are incident to these 3-vertices which are only incident to 4-faces, then H contains a 3-vertex w which is incident only to 4-faces and is adjacent to a 4-vertex w_0. Since G is a reconstruction of H, we construct G from H - ww_0 by adding an edge. Since S(H - ww_0, A) = [S(H, A) - 1] = [S(G, A) - 2], it is impossible to construct two more A type subgraphs by adding an edge in this case. Hence S(G, A) is edge-reconstructible.

Since the face list and S(G, A) are edge-reconstructible, there are only two ways to add an edge in G - vv_0 to construct a reconstruction of G, one is to add the edge vv_1 and the other is to add the edge vv_{t-1}. But in these cases, v_0 = v_5 and v_t = v_1 must be
3-vertices. This contradicts our assumption that \( d(v_1), d(v_2) \geq 4 \). Therefore the proof of our theorem 5.1.1 follows.
6.1. Introduction

In this chapter, we consider the edge-reconstruction of 3-connected planar graphs with minimum valency 4 under conditions different from Chapter II. For a k-vertex \( v \in G \), we denote by \( N_k(G, v) \) the induced subgraph consisting of the vertex \( v \) and the vertices adjacent to \( v \). For graphs \( F \) and \( G \), the number of subgraphs of \( G \) isomorphic to \( F \) is denoted by \( \text{S}(F, G) \). By Kelly's Lemma (the edge version), \( \text{S}(F, G) \) is edge-reconstructible. In this chapter, we will prove the following theorem.

**THEOREM 6.1.1.** Let \( G \) be a 3-connected planar graph with minimum valency 4. If for any 4-vertex \( v \in G \), \( N_4(G, v) \) is a \( k_{1,3} \)-free graph, then \( G \) is edge-reconstructible.

Throughout the rest of this chapter, \( G \) always stands for a 3-connected planar graph with minimum valency 4 and satisfies the condition that for any 4-vertex \( v \in G \), \( N_4(G, v) \) is a \( k_{1,3} \)-free graph. Since \( G \) is trivially edge reconstructible from \( G_{uv} \) for any two adjacent 4-vertices \( u \) and \( v \) in \( G \), we will assume that there do not exist any two adjacent 4-vertices in \( G \). By [3], \( G \) is edge-reconstructible if \( |V(G)| < 10 \). Hence we will
assume that $|V(G)| \geq 10$. Thus, the planarity of $G$, by [8], is recognizable from its edge-deck. This means that all reconstructions of $G$ are planar graphs. We will often use this fact without a further explanation. By chapter II, if $G$ contains a graph in Figure 6.1 as a subgraph, $G$ is edge-reconstructible. Hence we will assume that $G$ does not contain any graph in Figure 6.1 as a subgraph.

![Figure 6.1](image-url)

6.2. The proof of theorem 6.1.1

Before we prove our theorem, we need several lemmas.

LEMMA 6.2.1. If $H$ is a 3-connected planar graph with minimum valency 4 and contains only one 4-vertex, then $H$ is edge-reconstructible.

Proof. By chapter II, if the 4-vertex $v$ is not adjacent to any 5-vertex, then $H$ is edge-reconstructible. Therefore, we assume that the 4-vertex $v$ is adjacent to a 5-vertex $u$ in $H$. Since in $H - vu$, we can recognize $v$, $u$ and $(d(v), d(u))$ is edge-reconstructible,
there is only one way to add an edge $e$ in $H - vu$ to get a reconstruction of $H$. Therefore, $H$ is edge-reconstructible. 

**Lemma 6.2.2.** If $H$ is a reconstruction of $G$, then for any 4-vertex $v \in H$, $N_4(H, v)$ is a $k_{1,3}$-free graph.

**Proof.** It is clear that if $G$ is edge-reconstructible, our conclusion is true. Hence we assume that $G$ is not edge-reconstructible. First, we show that there exists at most one 4-vertex $x \in H$ such that $N_4(H, x)$ is not a $k_{1,3}$-free graph. Let $v \in G$ be a 4-vertex. Since $N_4(G, v)$ is a $k_{1,3}$-free graph, it is clear that $v$ is incident to a 3-face in $G$. Let $v_0, v_1$ be the other two vertices incident to this 3-face. By [15], if $v_0, v_1$ are both 5-vertices, then $G$ is edge-reconstructible. Hence, we can assume $d(v_0) \geq 6$. Consider $G - vv_0$. Since $G$ does not contain two adjacent 4-vertices, it is clear that for any 4-vertex $u \in G - vv_0$, $N_4(G - vv_0, u)$ is a $k_{1,3}$-free graph. Since we can identify $v$ in $G - vv_0$ and $(d(v), d(v_0))$ is edge-reconstructible, we know that only $v \in H$ could satisfy that $N_4(H, v)$ is not a $k_{1,3}$-free graph by adding an edge $e$ in $G - vv_0$ to obtain the reconstruction $H$ of $G$.

Now, we prove that for any 4-vertex $v \in H$, $N_4(H, v)$ is a $k_{1,3}$-free graph. Assume that it is not true. Then there exists a 4-vertex $v \in H$ such that $N_4(H, v)$ is not a $k_{1,3}$-free graph. By Lemma 6.2.1 and the above discussions, there exists another 4-vertex $u \in H$ such that $N_4(H, u)$ is a $k_{1,3}$-free graph. It is clear that $u$ is incident to a 3-face in $H$. Therefore, $u$ is adjacent to a vertex $u'$ with $d(u') \geq 6$. Consider $H - uu'$. Since we can recognize $u$ and $(d(u), d(u'))$ is edge-reconstructible, we can construct a reconstruction $G'$ of $H$ from $H - uu'$, which is isomorphic to $G$, by adding an edge incident to $u$ and a vertex $u'' \neq v$. Clearly, $N_4(G', v)$ is not a $k_{1,3}$-free graph in $G'$. This is a contradiction since $G \cong G'$. 

#
LEMMA 6.2.3. Let $G$ contain the graph in Figure 6.2 as a subgraph. If $G - v_0$ is not 3-connected, then \{v_0, v_j, v_j\} for some $j \in \{i + 1, \ldots, t - 1\}$ is a cut set of $G$, $G - v_j$ is 3-connected and there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_j, v_j, \ldots, v_{t-1}\}$, where $V(C) = \{v_k; k = 0, 1, \ldots, t - 1\}$, $E(C) = \{v_kv_{k+1}; k = 0, 1, \ldots, t - 1, v_t = v_0\}$. (Note: We do not exclude the case $i = 2$).

![Figure 6.2](image)

**Proof.** Since $G - v_0$ is not 3-connected, there is a proper two-vertex separation of $G - v_0$ such that

$$G - v_0 = H_1 + H_2$$

in which $E(H_1) \cap E(H_2) = \emptyset$ and $V(H_1) \cap V(H_2) = \{x, y\}$. Clearly, $v \not\in \{x, y\}$. This is because if $\{v, w\}$ were a cut set of $G - v_0$, then it would also be a cut set of $G$, which is impossible, since $G$ is 3-connected. It is also clear that $\{x, y\} \subset V(C)$. Since if one of $x$ and $y$ does not belong to $V(C)$, then $V(C)$ and $v$ belong to the same $H_i$, say $i = 1$. This implies that for any vertex $z$ in $H_2$, where $z \not\in \{x, y\}$, a chain from $z$ to $w$, where $w \in$
V(C) - \{x, y\}, must pass through x or y. Therefore in G, any chain from z to w must pass through x or y. This contradicts the fact that G is 3-connected.

Since \{x, y\} \subset V(C) and \{x, y\} is contained neither \{v_1, v_2, ..., v_i\} nor \{v_{i+1}, v_{i+2}, ..., v_0\}, \{x, y\} must be equal to \{v_1, v_j\} for some \(j \in \{v_k; k = i + 1, ..., t - 1\}\).

Therefore \{v_0, v_1, v_j\} is a cut set of G and there is an outer bridge B of the circuit C in G with \(A(G, B) = \{v_0, v_1, v_j, ..., v_{t-1}\}\).

Now, we prove that G - vv is 3-connected. Assume that this is not true. Then, by using the same argument as we used above, we can prove that \{v_0, v_1, v_k\} for some \(k \in \{2, ..., i\}\) is a cut set of G and there is an outer bridge D of the circuit C in G with \(A(G, D) = \{v_0, v_1, ..., v_k\}\). Therefore \(A(G, B) \cap A(G, D) = \{v_0, v_1\}\), which contradicts the fact that G is a planar graph.

\[\text{Figure 6.3}\]

Where in Figure 6.3, \(v_t = v_0\), \(v_0\), \(v_2\) are adjacent by an edge and \(t - 2 \geq i \geq 4\).
LEMMA 6.2.4. If $G$ contains the graph in Figure 6.3 as a subgraph, then $G$ is edge-reconstructible.

Proof. Since $v_0$, $v_2$ are adjacent, $N_4(G, v)$ contains $k_4$ as a subgraph with vertices $v, v_0, v_1$ and $v_2$. By our assumption, $v_i$ is neither adjacent to $v_0$ nor $v_2$. Since $v_0$, $v_2$ are adjacent by an edge and $G$ is a 3-connected planar graph with minimum valency 4, it is clear that there is an outer bridge $B$ of the circuit $C$ in $G$ with $A(G, B) = \{v_0, v_1, v_2\}$, where $V(C) = \{v_k; k = 0, 1, ..., t - 1\}$ and $E(C) = \{v_kv_{k+1}; k = 0, 1, ..., t - 1, v_t = v_0\}$. By Lemma 6.2.3, it is clear that $G - vv_1$ is not 3-connected. Hence $G - vv_0$ is 3-connected. Since $G - vv_0$ is 3-connected and $v$ is the only 3-vertex in $G - vv_0$, any reconstruction of $G$ can be obtained by adding an edge which joins $v$ and a vertex $v_j \in V(C)$. Since $S(k_4, G)$ is edge-reconstructible and $S(k_4, G - vv_0) = S(k_4, G) - 1$, we must construct one $k_4$ by adding the new edge $vv_j$ in $G - vv_0$. It is clear that $v, v_j$ are two vertices of $k_4$. Let $x$ and $y$ be the other two vertices of $k_4$. Since $x$ and $y$ must be adjacent to $v$, $\{x, y\} \subset \{v_1, v_2, v_3\}$. Since $G$ is a 3-connected planar graph and $v_0$ is adjacent to $v_2$, we know that $v_1$ and $v_2$ can not be adjacent to $v_i$ in $G$. Hence $\{x, y\} = \{v_1, v_2\}$. Since $v_j$ is adjacent to both $v_1, v_2$ in $K_4$ and $v_1$ can not be adjacent to $v_k \in V(C)$ except $k = 0$ and 2, we must have $j = 0$. Therefore $G$ is edge-reconstructible.

By our assumptions and above discussions, if $G$ is not edge-reconstructible, $G$ must contain a graph in Figure 6.4 as a subgraph.
Figure 6.4

Where in Figure 6.4, \( v_t = v_0 \), in (i) \( t \geq 6 \) and in (ii) \( i \geq 3, t \geq i + 3 \).

Like in [7], let \( v \) be a \( k \)-vertex of \( G \). Let the faces incident to \( v \) be \( F_0, ..., F_{k-1} \) such that \( F_i \) is adjacent to \( F_{i-1} \) (modulo \( k \)), and let \( F_i \) be an \( (a_i + 2) \)-face with \( V(F_i) = \{ v_{a_0} + ... + v_{a_{i-1}}, v_{a_0} + ... + v_{a_i - 1} + 1, ..., v_{a_0} + ... + a_i, v \} \), where \( v_{a_0} + ... + a_{k-1} = v_0 \), \( a_{-1} = 0 \). Then \( W(G, v) = (a_0, a_1, ..., a_{k-1}) \) is called the wheel sequence of \( v \) in \( G \). Each \( a_i \) is called a term of the wheel sequence. Clearly, the wheel sequence of \( v \) is unique up to the choice of an initial term and an orientation. By this definition and above discussion, there are two types of wheel sequences for each 4-vertex \( v \) of \( G \), one is \((1, 1, 1, a_v)\), called a type I wheel sequence, where \( a_v \) is a integer \( \geq 3 \), the other is \((1, a_v, 1, b_v)\), called a type II wheel sequence, where \( a_v, b_v \) are integers \( \geq 2 \). We claim that if \( G \) is not edge-reconstructible, there is only one reconstruction \( H \) of \( G \) which is not isomorphic to \( G \). First, we consider the case that \( G \) contains a 4-vertex \( v \) with a wheel sequence \((1, 1, 1, a)\). Since one of \( G - vv_1 \) and \( G - vv_2 \) (say \( G - vv_1 \)) is 3-connected,
it has a unique planar embedding. Since we can recognize $v$ in $G - vv_1$ and since by Lemma 6.2.2, for any 4-vertex $u$ of a reconstruction $H$ of $G$, which is not isomorphic to $G$, $N_4(H, u)$ is a $k_13$-free graph, the only way to construct the reconstruction $H$ of $G$ is to add the edge $vv_{a+2}$ ($v_0 = v_{a+3}$). Clearly, $H$ contains the 4-vertex $v$ with the wheel sequence $(1, 2, 1, a - 1)$. In this case, we changed the type of the wheel sequence of the 4-vertex $v$ from $G$ to $H$. Similarly we can consider the case that $G$ contains a 4-vertex $v$ with a wheel sequence $(1, a, 1, b)$ and prove our claim in this case.

**Lemma 6.2.5.** Let $G$ contain a 4-vertex $v$ with a type I wheel sequence $(1, 1, 1, a)$. If one of $G - vv_0$ and $G - vv_3$ is 3-connected, then $G$ is edge-reconstructible.

**Proof.** Assume that $G$ is not edge-reconstructible. First we consider the case in which $G - vv_3$ is 3-connected. Since $G - vv_3$ is 3-connected, it has a unique planar representation. Since we can identify $v$ in $G - vv_3$, the only reconstruction $H$ of $G$, which is not isomorphic to $G$, can be uniquely constructed from $G - vv_3$ by adding the edge $vv_{a+2}$ ($v_0 = v_{a+3}$). Clearly, by our construction, the face list of $H$ is the same as one of $G$, which implies that the face list of $G$ is edge-reconstructible. Since one of $H - vv_0$ and $H - vv_1$ (say $H - vv_1$) is 3-connected and the face list is edge-reconstructible, there is only one way to add an edge in $H - vv_1$ to obtain a reconstruction of $H$. $H$ is edge-reconstructible. Hence $G$ is edge-reconstructible. Similarly, we can consider the case in which $G - vv_0$ is 3-connected and prove that $G$ is also edge-reconstructible. #

By Lemma 6.2.5, we can assume that if there is a 4-vertex $v$ with a type I wheel sequence in $G$, then $G - vv_0$ and $G - vv_3$ are not 3-connected.
LEMMA 6.2.6. If $G$ contains at least two 4-vertices $v, u$ with type I wheel sequences, then $G$ is edge-reconstructible.

Proof. Assume that $G$ is not edge-reconstructible. Let the two wheel sequences of $v$ and $u$ be $(1, 1, 1, a_v)$ and $(1, 1, 1, a_u)$ respectively. By Lemma 6.2.5, $G - vv_1$ and $G - vv_2$ are 3-connected. Since $v_1$ and $v_2$ can not be both 5-vertices, we can assume $d(v_1) \geq 6$. Consider $G - vv_1$. Since we can recognize $v$ in $G - vv_1$ and since by Lemma 6.2.2 for any 4-vertex $x$ of a reconstruction $H$ of $G$, $N_4(H, x)$ is a $k_{1,3}$-free graph, the only way to construct the reconstruction $H$ of $G$, which is not isomorphic to $G$, from $G - vv_1$, is to add the edge $vv_{a_v} + 2 (v_0 = v_{a_v} + 3)$. Clearly, $u$ is still a 4-vertex with a type I wheel sequence in $H$. Since $H - uu_1$ is 3-connected, we can construct a reconstruction $G'$ of $H$ from $H - uu_1$, which is isomorphic to $G$, by adding the edge $uu_{a_u} + 2 (u_0 = u_{a_u} + 3)$. Since $f_3(G) = f_3(H) + 1$ and $f_3(H) = f_3(G') + 1$, we have $f_3(G) = f_3(G') + 2$, which implies that $G$ has two different planar embeddings. This contradicts the fact that a 3-connected planar graph has a unique planar embedding. Therefore $G$ is edge-reconstructible. 

LEMMA 6.2.7. If $G$ contains a 4-vertex with a type I wheel sequence, then $G$ is edge-reconstructible.

Proof. Assume that $G$ is not edge-reconstructible. Let $v$ be the 4-vertex with a type I wheel sequence $(1, 1, 1, a)$. Let $H$ be a reconstruction of $G$ which is not isomorphic to $G$. Then $H$ can be obtained by $(G - vv_1) + vv_{a} + 2$, where $d(v_1) \geq 6$ and $v_{a} + 3 = v_0$. Clearly, any 4-vertex of $G$ is also a 4-vertex of $H$ and $H$ contains no 4-vertex with a type I wheel sequence. It is also clear that $f_3(G) = f_3(H) + 1$. We claim that all 4-vertices of $H$ will have the wheel sequence $(1, 2, 1, 2)$. First we consider the vertex
v in H. Since G - vv₀ and G - vv₃ are not 3-connected, by Lemma 6.2.3 there are outer bridges B, B' of the circuit C(v) in G with A(G, B) = {v₀, v₁, vₐ₊₂, ..., vᵢ} and A(G, B') = {v₂, v₃, v₄, ..., vⱼ} where V(C(v)) = {vₖ; k = 0, 1, ..., a + 2}, E(C(v)) = {vₖvₖ₊₁; k = 0, 1, ..., a + 2} and 4 ≤ j ≤ i ≤ a + 2. By the construction of H, H - vv₃ is not 3-connected. Then H - vv₂ is 3-connected. Since H - vv₂ is 3-connected, the reconstruction G' of H which is isomorphic to G can be constructed by G' = (H - vv₂) + vv₄. Since G' ∼ G and f₃(G) = f₃(H) + 1, we must construct two 3-faces by adding the edge vv₄ in H - vv₂ to obtain G'. Therefore a - 1 = 2. Now, we consider the other 4-vertices of H. Let u be a 4-vertex of H distinct from v with a wheel sequence (1, au, 1, bu). Without loss of generality, we assume that H - uu₀ is 3-connected. Since H - uu₀ is 3-connected, the reconstruction G' of H which is isomorphic to G can be obtained by adding the edge uu₂ in H - uu₀. Since G' ∼ G and f₃(G) = f₃(H) + 1, we must construct two 3-faces by adding the edge uu₂ in H - uu₀ to obtain G'. Therefore au = 2. Since G' - uu₄, G' - uu₁ can not be 3-connected by Lemma 6.2.5, there are outer bridges B, B' of the circuit C(u) in G' with A(G', B) = {v₃, v₄, ..., vₖ}, A(G', B') = {v₂, v₁, v₀, ..., vⱼ} where V(C(u)) = {vᵢ; i = 0, 1, ..., bᵤ + 3}, E(C(u)) = {vᵢvᵢ₊₁; i = 0, 1, ..., bᵤ + 3, vᵢvₒ + 4 = v₀} and 5 ≤ k ≤ j ≤ bᵤ + 4. Clearly, H - uu₄ can not be 3-connected. Therefore H - uu₃ is 3-connected. By using the same argument, we can show bu = 2. We claim that H can not contain any graph in Figure 6.5 as a subgraph. This is because (1) one of H - vv₁ and H - vv₀ (say H - vv₁) is 3-connected; (2) we can recognize v in H - vv₁ by its wheel sequence of v; (3) up to isomorphism, there are only two reconstructions of H, one is G and the other is H; (4) f₃(G) = f₃(H) + 1. It is clear that if H contains a graph in Figure 6.5 as a subgraph, the face list of a reconstruction of H which is isomorphic to G and constructed from H - vv₁ is different from one of G. This is a contradiction, since a 3-connected planar graph has a unique planar embedding. Therefore by [22], H contains at least twelve 4-vertices with the wheel sequence (1, 2, 1,
2). Let \( v \) be one of them. Assume that \( H - vv_0 \) and \( H - vv_3 \) are 3-connected. We claim that \( v_0, v_3 \) cannot both be 5-vertices. This is because that if \( v_0, v_3 \) are 5-vertices, \( v_2, v_5 \) would be 4-vertices. By [4], \( H \) is edge-reconstructible. Hence \( G \) is edge-reconstructible. This contradicts our assumption. Therefore we assume that \( d(v_0) \geq 6 \).

We construct a reconstruction \( G' \) of \( H \) which is isomorphic to \( G \) by \( G' = (H - vv_0) + vv_2 \). It is clear that there exists at least one 4-vertex (call it \( u \)) distinct from \( v \) in \( G' \) with the wheel sequence \((1, 2, 1, 2)\). Since one of \( G' - uu_0 \) and \( G' - uu_1 \) (say \( G' - uu_0 \)) is 3-connected, we can construct a reconstruction \( H' \) of \( G' \) which is isomorphic to \( H \) by \( H' = (G' - uu_0) + uu_2 \). Then \( H' \) contains the 4-vertex \( u \) with a type I wheel sequence. This is a contradiction, since \( H \) does not contain any 4-vertex with a type I wheel sequence. 

\[ \]

\( \)

Figure 6.5

**Proof of Theorem 6.1.1.** By Lemma 6.2.6, we can assume that all 4-vertices in \( G \) have type II wheel sequences. First we consider the case that there exists a 4-vertex \( u \) with a wheel sequence \((1, 2, 1, b)\). Clearly, if one of \( G - uu_0 \) and \( G - uu_4 \) is 3-
connected, the reconstruction H of G would contain a 4-vertex with a type I wheel sequence. By Lemma 6.2.7, H is edge-reconstructible. Hence G is edge-reconstructible. Therefore, we assume that G - uu0 and G - uu4 are not 3-connected. This means that G - uu1 and G - uu3 are 3-connected. Since both G - uu1 and G - uu3 are 3-connected, for the same reason, we can assume \( b \geq 4 \). Let H be constructed from G - uu1 by adding the edge uu\( b + 3 \) (\( u_0 = u_{b + 4} \)). Clearly, we have \( f_3(G) = f_3(H) \), and \( f_4(G) = f_4(H) + 1 \).

By the construction of H, H - uu3 is still 3-connected. Since a reconstruction G' of H which is isomorphic to G can be constructed by G' = (H - uu3) + uu5, this implies that we must construct one 4-face by adding uu5 in H - uu3. Therefore \( b = 4 \). Clearly, we have \( f_3(G) = f_3(H) \), \( f_4(G) = f_4(H) + 1 \), \( f_5(G) = f_5(H) - 2 \), \( f_6(G) = f_6(H) + 1 \) and \( f_i(G) = f_i(H) \) \( i \geq 7 \). Let v be a 4-vertex of H distinct from u. We claim that the wheel sequence of v must be (1, 3, 1, 3). Assume that our claim is not true. Let the wheel sequence of v be (1, \( a_v \), 1, \( b_v \)). Since one of H - vv0 and H - vv1 (say H - vv0) is 3-connected, we can construct a reconstruction G' of H from H - vv0, which is isomorphic to G, by adding the edge vv2. Since G is a 3-connected planar graph and G' \( \cong G \), then \( f_i(G) = f_i(G') \) \( i \geq 3 \). Therefore, we must construct one 4-face by adding the edge vv2 in H - vv0. This means \( a_v = 3 \). Since G' - vv5 and G' - vv1 are not 3-connected and G' is a 3-connected planar graph, there are outer bridges B, B' of the circuit C in G' with A(G', B) = \{v4, v5, ..., v_k\}, A(G', B') = \{v2, v1, v0, ..., v_j\} where V(C) = \{v_i; i = 0, 1, ..., b_v + 4\}, E(C) = \{v_{iv_i + 1}; i = 0, 1, ..., b_v + 4, v_{b_v} + 5 = v_0\} and \( 6 \leq k \leq j \leq b_v + 5 \). Hence H - vv5 is not 3-connected. Therefore H - vv4 is 3-connected. From the above discussion, we know that \( b_v \) is equal to 3 too. We claim that H cannot contain any graph in Figure 6.5 as a subgraph. If it is not true, then H contains a graph in Figure 6.5 as a subgraph.

By using the facts: (1) one of H - vv0 and H - vv1 (say H - vv1) is 3-connected; (2) we can recognize v in H - vv1 by its wheel sequence; (3) up to isomorphism, there are only two reconstructions of H, one is G and the other is H itself; (4) \( f_3(G) = f_3(H) \), \( f_4(G) = \)
It is clear that if \( H \) contains a graph in Figure 6.5 as a subgraph, the face list of a reconstruction of \( H \) which is isomorphic to \( G \) and constructed from \( H - vv_1 \) is different from one of \( G \). This is a contradiction, since a 3-connected planar graph has a unique planar embedding. Therefore, we assume that \( H \) contains no graph in Figure 6.5 as a subgraph. By [22], \( H \) contains at least thirty 4-vertices with the wheel sequence \((1, 3, 1, 3)\). From \( H \), we construct \( G \) by \( G = (H - uu_7) + uu_1 \). Clearly, there exists at least one 4-vertex (call it \( v \)) distinct from \( u \) in \( G \) with the wheel sequence \((1, 3, 1, 3)\) such that \( v \notin V(C(u)) \) where \( C(u) \) is a circuit with \( V(C(u)) = \{v_i; i = 0, 1, ..., 7\} \) and \( E(C(u)) = \{v_iv_{i+1}; i = 0, 1, 7; v_8 = v_0\} \). Since one of \( G - vv_0 \) and \( G - vv_1 \) is 3-connected, we can obtain a reconstruction \( H' \) of \( G \) which is isomorphic to \( H \) such that \( H' \) contains two 4-vertices with the wheel sequence \((1, 2, 1, 4)\). This is a contradiction.

By above discussion, we can assume that \( G \) does not contain any 4-vertex with the wheel sequence \((1, 2, 1, b)\). Therefore, we assume that \( G \) contains a 4-vertex with a wheel sequence \((1, 3, 1, b)\). Let \( v \) be a 4-vertex of \( G \) with a wheel sequence \((1, 3, 1, b)\). Clearly, if one of \( G - vv_0 \) and \( G - vv_5 \) (say \( G - vv_0 \)) is 3-connected, the reconstruction \( H \) of \( G \) which is not isomorphic to \( G \), can be obtained by \( H = (G - vv_0) + vv_2 \). Therefore, \( H \) contains the 4-vertex \( v \) with the wheel sequence \((1, 2, 1, b + 1)\). By above discussion, \( H \) is edge-reconstructible. Hence \( G \) is edge-reconstructible. Therefore, we assume that \( G - vv_0 \) and \( G - vv_5 \) are not 3-connected. Hence \( G - vv_1 \) and \( G - vv_4 \) are 3-connected. Since both \( G - vv_1 \) and \( G - vv_4 \) are 3-connected, for the same reason, we can assume \( b \geq 5 \). Let \( H \) be constructed from \( G - vv_1 \) by adding the edge \( vv_{b+4} \) (\( v_0 = v_{b+5} \)). By the construction of \( H \), clearly, \( f_2(G) = f_2(H) \), \( f_4(G) = f_4(H) \), \( f_5(G) = f_5(H) + 1 \). Since a reconstruction \( G' \) of \( H \) which is isomorphic to \( G \) can be constructed by \( G' = (H - vv_4) + vv_6 \), this implies that we must construct one 5-face in \( H - vv_4 \) by adding the edge \( vv_6 \). Therefore \( b = 5 \). Let \( u \) be a 4-vertex of \( H \) distinct from \( v \). We claim that the wheel sequence of \( u \) must be \((1, 4, 1, 4)\). Assume that our claim is not true. Let the
wheel sequence of \( u \) be \((1, a_u, 1, b_u)\). Since one of \( H - uu_0 \) and \( H - uu_1 \) (say \( H - uu_0 \)) is 3-connected, we can construct a reconstruction \( G' \) of \( H \) from \( H - uu_0 \), which is isomorphic to \( G \), by adding the edge \( uu_2 \). Since \( G \) is a 3-connected planar graph and \( G' \equiv G \), \( f_i(G) = f_i(G') \) \( i \geq 3 \). Therefore, we must construct one 5-face by adding the edge \( uu_2 \) in \( H - uu_0 \). This means \( a_u = 4 \). Since \( G' - uu_6 \) is not 3-connected, \( H - uu_6 \) is not 3-connected. Therefore \( H - uu_5 \) is 3-connected. From the above discussion, we know that \( b_v \) is equal to 4 too. By [22], \( H \) must contain a graph in Figure 6.5 as a subgraph. Since \( f_3(G) = f_3(H) \), \( f_4(G) = f_4(H) \) and \( f_5(G) = f_5(H) + 1 \) and one of \( H - vv_1 \) and \( H - vv_2 \) (say \( H - vv_1 \)) is 3-connected, it is clear that the face list of a reconstruction of \( H \) which is isomorphic to \( G \) and constructed from \( H - vv_1 \) is different from one of \( G \). This is a contradiction. Therefore \( G \) is edge-reconstructible in this case.

Now, we assume that \( G \) does not contain any 4-vertex with a wheel sequence either \((1, 2, 1, b)\) or \((1, 3, 1, b)\). Therefore, by [22], \( G \) contains a graph in Figure 6.5 as a subgraph. Since \( G \) contains 4-vertices and for a 4-vertex \( v \) of \( G \), we assume that the wheel sequence of \( v \) is \((1, a_v, 1, b_v)\), as we did above, we can prove that \( f_3(G) \) is edge-reconstructible. Assume that \( G \) contains the graph (i) in Figure 6.5 as a subgraph. Since one of \( G - vv_0 \) and \( G - vv_1 \) (say \( G - vv_1 \)) is 3-connected and we can recognize \( v \) in \( G - vv_1 \), clearly, \( G \) is edge-reconstructible. By using that \( f_3(G) \) is edge-reconstructible and one of \( G - vv_1 \) and \( G - vv_2 \) is 3-connected, we can also prove that if \( G \) contains the graph (iii) in Figure 6.5 as a subgraph, then \( G \) is edge-reconstructible. Therefore, we assume that \( G \) contains the graph (ii) as a subgraph. We prove that \( G \) is also edge-reconstructible in this case. If \( G - vv_1 \) is not 3-connected, by Lemma 6.2.3, \( G - vv_0 \) and \( G - vv_2 \) are 3-connected. Since we can recognize \( v \) in \( G - vv_i \) \( (i = 0, 2) \), we can construct reconstructions \( H, H' \) of \( G \) as follows: \( H = (G - vv_0) + vv_5 \) and \( H' = (G - vv_2) + vv_5 \), where \( H \equiv H' \), \( H \) is not isomorphic to \( G \). Clearly, \( H - vv_2 \) is still 3-connected. This allows us to construct \( H' \) from \( H - vv_2 \) by \( H' = (H - vv_2) + vv_0 \). Since
G is not edge-reconstructible, H is not edge-reconstructible. The reconstruction $H'$ of $H$ can not be isomorphic to $H$. This is a contradiction. If $G - vv_1$ is 3-connected, then one of $G - vv_2$ and $G - vv_3$ is 3-connected. Therefore we can use the same argument as we did above to prove that $G$ is edge-reconstructible. 

#
REFERENCES


[14] J Lauri, Edge reconstruction of planar graphs with minimum valency 5, J. Graph Theory 3 (1979), 269-286.


[18] B. D. Mckay, Computer reconstruction of small graphs. J. Graph Theory 1, 281-283.

