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A methodology to create a digital cadastral overlay through upgrading digitized cadastral data

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The Ohio State University, 1992
A METHODOLOGY
TO CREATE A DIGITAL CADAstral OVERLay
THROUGH UPGRADING DIGITIZED CADAstral DATA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Najeh Sadiq Tamim, B.S., M.S.

The Ohio State University
1992

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To My Parents
ACKNOWLEDGMENTS

Many people have contributed to the creation and production of this major milestone in my life. A few of them, however, deserve to be mentioned.

I would like to express my sincere appreciation and indebtedness to my advisor, Professor Burkhard Schaffrin, for his invaluable encouragement, fruitful discussions and input, as well as moral support. My deep thanks and gratitude go also to my co-advisor, Professor Grenville Barnes, for his advice, support, and the constructive comments and suggestions which strengthened this research. Professor John Bossler is also acknowledged for his valuable comments as a member of my reading committee.

I would like to acknowledge the Center for Mapping (Center for the Commercial Development of Space Division) at The Ohio State University for supporting this research through a seed grant. Special thanks to the Department of Geodetic Science and Surveying at The Ohio State University for offering me the financial support during my Ph.D. study.

I would like to express my personal thanks to my parents for their support and encouragement to search for excellency. Last, but not least, I offer special thanks to my wife for her patience, understanding and invaluable support, especially when I did not have enough time to spend with her. Her assistance in typing and producing this dissertation is greatly appreciated.
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Major Field: Geodetic Science and Surveying
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<th>Description</th>
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<tr>
<td>AM/FM</td>
<td>Automated Mapping/Facilities Management</td>
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<tr>
<td>COGO</td>
<td>Coordinate Geometry</td>
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<tr>
<td>DLG</td>
<td>Digital Line Graph</td>
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<tr>
<td>GIS</td>
<td>Geographic Information System</td>
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<tr>
<td>GMM</td>
<td>Gauss-Markov Model</td>
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<tr>
<td>GPS</td>
<td>Global Positioning Systems</td>
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<tr>
<td>GRF</td>
<td>Geodetic Reference Framework</td>
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<tr>
<td>IAAAO</td>
<td>International Association of Assessing Officers</td>
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<tr>
<td>LIS</td>
<td>Land Information System</td>
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<tr>
<td>MMS</td>
<td>Measurement Management System</td>
</tr>
<tr>
<td>MPC</td>
<td>Multipurpose Cadastre</td>
</tr>
<tr>
<td>MPLIS</td>
<td>Multipurpose Land Information System</td>
</tr>
<tr>
<td>NRC</td>
<td>National Research Council (USA)</td>
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<tr>
<td>PCID</td>
<td>Parcel Corner Identifier</td>
</tr>
<tr>
<td>PID</td>
<td>Parcel Identifier</td>
</tr>
<tr>
<td>PLS</td>
<td>Public Land Survey</td>
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<tr>
<td>UME</td>
<td>Useful Matrix Equalities</td>
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<tr>
<td>USGS</td>
<td>United States Geological Survey</td>
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<td>WLRC</td>
<td>Wisconsin Land Records Committee</td>
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CHAPTER I
INTRODUCTION

1.1 Problem Statement

In the last decade, there has been an increasing trend towards the modernization and computerization of land records (such as maps, property deeds and plats) for the purpose of building a multipurpose cadastral (NRC, 1980; NRC, 1983) and creating integrated and coordinated land information systems (LIS). In the United States, it has been estimated that in the 14 year period from 1986-2000 as much as $90 billion will be spent by local authorities and public utilities on the collection and management of spatially-related information (Dale and McLaughlin, 1988, p.2). Part of this is due to the rapid advances in mapping and information technologies, particularly geographic information systems (GIS) which deal with spatially-related data and information. GIS can provide significant benefits on both the institutional and technical fronts for the public as well as the private sectors (WLRC, 1987).

One major question in the computerization of land records and the creation of an automated LIS deals with the "best" or "most appropriate" approach for the creation of a digital property ownership map (to be called cadastral layer or cadastral overlay) which is spatially accurate, legally supportive, operationally efficient, economically feasible, and implementable over a short time period. From a surveying standpoint, we have tended to regard the "most accurate" as the "best", even though this accurate approach may be the "worst" in terms of cost, implementation time, dispute resolution, and other
factors (Barnes and Tamim, 1991). This issue of digital cadastral overlay creation has also been expressed by Hintz and Onsrud (1990, p.2) as:

One difficult issue facing geographic information systems (GIS) developers in the United States today is the current inability to create spatially accurate, legally supportive and operationally efficient land ownership databases.

Several approaches exist for the creation of a digital cadastral overlay. These range from the simple digitization of tax maps (which are currently the only widely available, parcel-specific, county-wide parcel maps in the US), to the extensive field resurvey of all cadastral parcels. While the digitizing approach is relatively inexpensive, and can be implemented in a short time period, it does not provide sufficient accuracy to satisfy all land information users. On the other hand, even though the resurvey approach achieves high accuracy which will satisfy most of the land information community, it requires high up-front finances (which are usually not available) and takes a long time to be completed (which users of all kinds are not willing to wait for). Local governments need for high accuracy, low cost, and short implementation time of the cadastral overlay has been expressed by Totten (1989, p.81) as follows:

Governments want their parcel maps accurate and they want them now and they would prefer not to spend their entire GIS budget in the process.

As a result, there is a current need for a new approach for building the cadastral overlay which gives greater weight to the efficiency and cost factors, at least in the initial phase. The approach described in this research makes use of coordinates digitized from tax maps and incrementally upgrades their accuracy using geometric constraints in the initial phase of implementation, followed by new accurate parcel data as they become available.
1.2 Research Objective

This research aims to lay out a convenient means for building a digital cadastral overlay from spatial data derived from tax maps. This data is then upgraded using geometric constraints and spatial data derived from real property surveys. It responds to the challenge laid down by Hintz and Onsrud (1990, pp: 9-10):

The current challenge for the surveying and GIS academic community is to develop a means of upgrading the spatial information derived from tax maps to spatial information derived from real property surveys.

In a mathematical integral form, this approach can be portrayed as:

\[
\int_{\text{now}}^{t} \left( \text{Upgrading digitized tax map coordinates with new and higher quality survey data} \right) \cdot dt = \left( \text{Accurate, economically feasible and legally supportive cadastral overlay at some future time } t \right) \ldots (1.1)
\]

1.3 Significance of this Research

This research will provide the following benefits to the land information community:

1) Provide a new technique to construct the cadastral overlay with relatively low front-end cost. This offers major cost savings for the creation and maintenance of the multipurpose cadastre.

2) Allow the immediate implementation of the cadastral overlay, simply by digitizing the widely available tax maps. No extensive remonumentation and resurvey of land parcels is required. The accuracy of the digitized coordinates will be upgraded initially through the application of geometric constraints, and over time through the incorporation of new survey data as they become available.
3) Facilitate the continuous updating of the cadastral layer through the continuous entry of new measurements (such as new land conveyances, subdivisions and resurvey data) into the system.

4) Make parcel mapping a continuous dynamic process which reflects the current status of a changing situation, rather than a one-time static activity.

5) Due to the continuing accuracy improvements which can be achieved over time, the numeric dimensions of a parcel will compare closely to the legally described dimensions. This will help create a cadastral overlay which reflects the legal status of cadastral parcels.

6) In the long run, it facilitates the sharing of cadastral data between the various organizations who might have differing accuracy needs. As the cadastral layer accuracy is increased over time, more user group requirements will be met. This will enhance the communication between the different land information users and increase their willingness to share in an integrated database.

1.4 Organization

This study is organized into seven major chapters. Chapter II discusses the cadastral overlay as a component of the multipurpose cadastre (MPC or LIS) and establishes the need for developing a new approach for building a digital cadastral overlay. Related literature and methods for implementing the cadastral overlay are reviewed and weighed in terms of accuracy, cost and time of implementation. In the final part of the chapter, the status of tax maps in the USA and their suitability as a basis for building a digital cadastral overlay are discussed.
Chapter III outlines a convenient approach for building a digital cadastral overlay and establishes the mathematical background behind it. It addresses the two major phases of implementation which are: the initial upgrading of the digitized tax map coordinates using geometric constraints and available survey data, and the continuous upgrading of these coordinates using higher quality survey data whenever they become available.

Chapter IV discusses the database design and development in terms of the digitization process, the transformation of the digitized coordinates into real world coordinates, and the required topology files structure. It also suggests a numbering system for cadastral blocks and parcel corners.

The test results of this methodology are addressed in Chapter V. In particular, this chapter discusses the choice of test samples, accuracy improvement indicators, number and location of coordinated points required in the affine transformation and adjustment process, choice of weights for the geometric constraints, as well as ways to detect and strengthen the weak parts of the cadastral maps.

Due to the computational and storage limitations involved in the simultaneous adjustment of the parcels contained in a large area like a county or a state, this approach deals with the adjustment of one map at a time. Chapter VI discusses alternatives for joining the individually adjusted maps so that the accuracy achieved by the adjustment process will not be destroyed by the horizontal matching of these maps.

The final chapter summarizes the conclusions that can be drawn from this study, and lists some recommendations for the successful implementation of this approach.
CHAPTER II
THE CADASTRAL OVERLAY

2.1 Background

The call for implementing the multipurpose cadastre by the National Research Council Reports (NRC, 1980; NRC, 1983) has and is encountering support and response from different federal, state and local governments. Many of these governments are in the process of improving and creating a modernized land information system as suggested by the NRC '83 report (Figure 2.1) and modified by Wilcox (1984) and Chrisman and Niemann (1985) into a layer-based LIS (Figure 2.2). Examples of these local governments in the United States include: Forsyth County, North Carolina (Ayers, 1984); Dane County, Wisconsin (WLRC, 1987); Prince William County, Virginia (Marshall, 1988; Phillips, 1987); Wyandotte County, Kansas (Rhodes and Crane, 1984). As the Wisconsin Land Records Committee (WLRC, 1987) has reported, improvements to land records through computerization can lead to significant quantifiable and non-quantifiable benefits on both the institutional and technical sides for both public as well as private sectors. These include: better land conveyancing; more equitable taxation; faster and more accurate responses to land information requests or inquiries; lower maintenance and storage costs; higher staff productivity; shorter processing times; a decrease in redundant activities; more effective communication and sharing of data between different agencies as well as many other benefits.
FIGURE 2.1: Components of a multipurpose cadastre (in heavy outline) as the foundation for Land Information Systems (NRC, 1983, p.16).

FIGURE 2.2: Multipurpose land information systems (MPLIS) concept (based on Chrisman and Niemann, 1985)
A land information system (LIS), or multipurpose cadastre (MPC) as it is interchangeably called, is defined as "a combination of people, technology and a long-term strategy to organize the relationships between owners and what they own, including the nature and extent of their holdings" (Keating, 1990). It may also be viewed as a framework that supports continuous, readily available and comprehensive land-related information. Figure 2.1 shows the major components of the MPC which are: a geodetic reference framework (GRF), base map, cadastral map, parcel identifiers (PID) and parcel attributes (NRC, 1980; NRC, 1983). Among these, the cadastral layer is an essential component which delineates the current status of property ownership and is a key link for the independent land agencies to cooperate in order to create a universal land data system (Anderson, 1985; NRC 1983, p.55). This is due to the fact that the majority of available land information in these agencies is directly referenced to the cadastral parcel.

2.2 General Requirements for a Digital Cadastral Overlay

The International Association of Assessing Officers (IAAO, 1988, p.12) defined a cadastral map as:

A map showing the boundaries of subdivisions of land, usually with the bearing and lengths thereof and the areas of the individual tracts, for the purposes of describing and recording ownership. A cadastral map may also show culture, drainage, and other features relating to the value and use of the land.

This definition applies basically to manual graphic cadastral maps. A digital cadastral overlay, on the other hand, is a computerized, seamless cadastral map. Its fundamental objective is to store in digital coordinate form, and to provide or display on request, the current shape, position, identity and relationship of the individual land parcels within its area of coverage. It differs from its manual graphical counterpart in that it is not segmented on separate map sheets, and that it is virtually scale independent (Cyprus
Department of Lands and Surveys, 1991).

Several requirements need to be fulfilled when developing a digital cadastral overlay which forms the basic building unit for an automated LIS, and which can be used and shared by all land agencies. These are (the first three requirements are adopted from Hintz and Onsrud, 1990):

1) The cadastral overlay should be **spatially accurate**. Since the cadastral overlay is used as evidence for the delineation of property rights, it should be precise and accurate down to the level that surveying measurements allow.

2) The cadastral overlay should be **operationally efficient**. A secretary should be able to easily input survey data into the system, and a surveying technician should be able to make use of it and detect input and measurement blunders.

3) The cadastral overlay should be **legally supportive**. This means that it should be based on the documents that carry legal weight in the event conflicts between adjoining parcels arise.

4) The cadastral overlay should be **economically feasible**. The elected government officials, with public demand to minimize taxes and expenditures, find it very hard to make a large investment when the benefits may not be perceptible to the public even after several years, as compared to other needs such as roads and health expenditures (Bauer, 1982). As a result, it should be economically justifiable and cost effective.

5) The cadastral overlay should be **implementable in a relatively short period of time**. The cadastral overlay users will not wait 50 years for its perfect completion (Crane, 1990). Moreover, there is a tendency for systems to lose credibility when a long lead time exists before operational status is achieved, resulting in users becoming impatient and seeking interim solutions (Ducker and Kjerne, 1988).
Before exploring the different methods which have been followed for the creation of digital cadastral overlays, an overview of the criteria that are crucial to this process will be given.

2.3 Criteria for Cadastral Overlay Creation

An ideal digital cadastral overlay will possess the five general requirements mentioned in the previous section. However, when it comes to implementation, the proper choice of the cadastral overlay creation method is directly affected by the following three main criteria (Figure 2.3):
1) Spatial geometric accuracy of the cadastral information
2) Cost of implementation, and
3) Time needed for the cadastral overlay to be functionally operational.

FIGURE 2.3: Factors affecting choice of the cadastral overlay creation method.
2.3.1 Accuracy

Accuracy in the context of this research refers to the horizontal component associated with the spatial accuracy of cadastral information. It can be defined as the nearness of quantities (observations, computations, estimates, etc.) to their true values, the error equaling the difference between these quantities and their true values (Bolstad et al, 1990). Positional accuracy refers to the closeness of the determined coordinate values to their true values. If it is stated in relation to the origin of the coordinate system used, it is called absolute positional accuracy, while it is termed relative accuracy if it is stated in relation to other points.

The accuracy issue is important for several reasons which include:

- Accuracy of the cadastral database is one of the principal factor which determine its usefulness and the extent of its possible applications. In particular, accuracy is essential for the proper identification of parcel ownership, tax liabilities, and the relationship of the cadastral overlay to other LIS layers.

- The accuracy level of a database has a significant impact on the ultimate cost of its creation. Figure 2.4 shows how dramatically the cost rises in relation to an increase in the accuracy level of property boundaries.

The question which, therefore, needs to be addressed is: how much spatial accuracy is needed in order to maximize the use of the cadastral overlay? One organization might require high quality ground control and accuracy, another might be satisfied with relatively low accuracy (Mclaurin, 1987). In the latter case, the digitization of available maps of uncertain quality like the tax maps may be adequate. Where is the truth? Who is right? How much accuracy is needed? What is the accuracy of available maps? What is the users perception of accuracy?
To answer some of these questions, Croswell (1987) asked several map users through survey forms to describe the maps currently being used, and to indicate their accuracy. The users included utility groups, public works departments, city and county planning departments, property assessment offices, and others. Answers were categorized into: "fair", "poor", "90%", "as accurate as possible", "to scale", "varies", "low", "unknown", and "close". Rarely did the respondents use numerical values. Croswell reports that the responses indicate:

1) The concept of map accuracy is misunderstood.
2) The accuracy of maps is not known, and
3) Inconsistent methods of map compilation and update have occurred resulting in accuracy that is inconsistent and unmeasurable.
In other research done by McLaurin (1987), he found the following land base accuracy requirements to support specific AM/FM projects of different user groups:

1) For the telephone industry, a graphic land base of 20' accuracy is satisfactory.
2) For other utilities, an accuracy of 5'-10' is satisfactory in most cases, and 1'-5' accuracy is occasionally needed.
3) For public works and land records, accuracy of 1'-5' is needed. Sometimes accuracy of less than 1' may be required.

In a study done by Dueker et al (1986), the most frequent land data users of Multnomah County, Oregon, were interviewed and found to have the accuracy requirements summarized in Table 2.1. The results of both studies show that the required accuracy depends largely on the intended use of the land database. Most of the users are satisfied with an accuracy of around 5 feet, and few need higher accuracy than that.

TABLE 2.1
Accuracy requirements of land data users in Multnomah County, Oregon (Dueker et al, 1986)

<table>
<thead>
<tr>
<th>User Type</th>
<th>Accuracy Requirement (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Title Insurance Companies</td>
<td>Not stated*</td>
</tr>
<tr>
<td>2. Facilities Management and Construction</td>
<td></td>
</tr>
<tr>
<td>Private Utility</td>
<td>1 - 10</td>
</tr>
<tr>
<td>Public Utility and Transportation</td>
<td>1</td>
</tr>
<tr>
<td>Engineering, Surveying and Photogrammetry</td>
<td>1</td>
</tr>
<tr>
<td>3. Planning and General Administration</td>
<td>1 - 100</td>
</tr>
<tr>
<td>4. Public Safety</td>
<td>100</td>
</tr>
</tbody>
</table>

* Title insurance companies did not specify a certain value for the accuracy, because it is not a major issue for them.
2.3.2 Cost

The second factor which is closely related to accuracy, is the cost of cadastral overlay creation. The need for highly accurate land information is recognized, but this information is not available because it is perceived that we cannot afford to pay for its collection. The land information officials find it difficult to compete for resources with roads, police, health, and other needs within the county government. Often these other units have problems that are more acute and visible than the need to computerize and modernize land records (Dueker, 1987). The big question becomes: How much money should be spent for how much accuracy?

In an attempt to study the relationship between accuracy and associated costs, McLaurin (1987) estimated the following costs for different accuracy ranges for creating a digital land base which includes utility lines (telephone, gas, etc.), cadastral data, as well as other land-based information (Table 2.2).

<table>
<thead>
<tr>
<th>Land Base Category</th>
<th>Horizontal Accuracy</th>
<th>Approx. Cost ($/mile(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Telephone Facilities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Digitize USGS Quads</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Urban</td>
<td>10'-40'</td>
<td>400-500</td>
</tr>
<tr>
<td>Rural</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>B. Use USGS DLG Data</td>
<td></td>
<td>20-30</td>
</tr>
<tr>
<td>2. Utility Facilities</td>
<td>5'-10'</td>
<td>4000-5000</td>
</tr>
<tr>
<td>Urban</td>
<td></td>
<td>2000</td>
</tr>
<tr>
<td>Rural</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Public Works and Land Records</td>
<td>1'-5'</td>
<td>40,000-50,000</td>
</tr>
<tr>
<td>Urban</td>
<td></td>
<td>20,000</td>
</tr>
<tr>
<td>Rural</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In another example, which is consistent with the above mentioned study, Baikie (1987) reports the following costs (see Table 2.3) for mapping and creating a digital land base for a 242 square mile area in Milwaukee, Wisconsin. Baikie does not go into detail on how each of the estimates was made, but he does indicate that the 40 feet option was selected for the project.

**TABLE 2.3**
Land base cost versus accuracy for Milwaukee

<table>
<thead>
<tr>
<th>Accuracy (ft)</th>
<th>40</th>
<th>10</th>
<th>5</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($)</td>
<td>157,000</td>
<td>181,000</td>
<td>393,000</td>
<td>756,000</td>
<td>1,966,000</td>
</tr>
</tbody>
</table>

Tables 2.2 and 2.3 show in an unambiguous way how drastically the cost of creating a digital land base increases as more accuracy is demanded. Moreover, even though high accuracy is needed for the reasons stated earlier (see section 2.3.1), the lack of finances might lead to the choice of lower accuracy methods of data collection, such as in the Milwaukee case.

2.3.3 Time of Implementation

The third dimension affecting the cadastral overlay creation, and which cannot be dealt with separately from cost and accuracy, is the implementation time needed for the digital cadastral database to become operational. Some cadastral overlay creation methods require less effort and lower accuracy, but experience has shown that these methods cannot handle unanticipated user requirements, resulting in subsequent cadastral layer redesign and discarding of much of the earlier work (Dueker and Kjerne, 1988). Other methods may strive for higher accuracy and more complete land database, but the long time needed before the implementation is complete jeopardizes the viability of the
land information system. As mentioned earlier, users are not willing to wait many years, and the system tends to lose credibility when a long lead time exists before operational status is achieved.

The previous discussion shows a very close interaction between the three factors affecting cadastral overlay implementation. Governments want their cadastral overlays and maps accurate, and they want them now and would prefer not to spend most of their budget in the process (Totten, 1989). Some methods are more accurate than others, but does the extra accuracy justify the time, money and effort spent for that purpose? In an investigation done by Dueker and Kjeme (1988), it was found that some organizations were performing much better with a relatively less sophisticated and less accurate database than others that had spent enormous amounts of time and money. Others argue that inaccurate information in the LIS will make it self-defeating (Thapa and Burtch, 1990), and will fail to satisfy the needs of land information users. It has also been observed that limited funding leads to the design of small, single-purpose LIS systems which will serve a limited number of users (Palmer, 1989). In the long run, this will be more expensive than building a more ambitious multipurpose system which can be shared by most users.

One could conclude that the ideal cadastral overlay implementation method is one which achieves the highest possible accuracy that will still satisfy all cadastral layer users, and which has a relatively low implementation costs and time (Line 1 in Figure 2.3). Since this ideal situation is hard to achieve, an alternative which has higher accuracy than the digitizing approach, and a lower cost and time requirement than the resurvey approach, needs to be developed. Since the temptation to use tax maps is high due to their wide availability, and digitizing is very efficient and inexpensive compared
to most other data acquisition methods, it is reasonable to start with digitized tax maps. While cost and time are given the highest priority initially, accuracy can be upgraded over time using geometric constraints and by including additional higher quality data (survey measurements, control points, etc.). Before exploring this alternative (line 2, Figure 2.3), it is useful to provide some background about other techniques which have been employed for building a digital cadastral overlay.

2.4 Digital Cadastral Overlay Creation Methods

A digital cadastral overlay stores, maintains and provides access to information about cadastral parcels. This information is comprised of two basic components; the first is a geometrical representation of administrative and legal boundaries, and the second is some attribute coding for the parcels in the system (Pullar, 1989). The primary sources of publicly available geometric information relating to parcel boundaries in the USA are as follows: legal descriptions in the registered property deeds (specifically metes and bounds descriptions), tax maps, recorded surveys, and subdivision plats.

In a manual environment, the cadastral map may serve as a graphical record which may have a recognized legal status for property parcel description, or it may merely serve as an index to other legal records (Morgensten et al, 1989; Dale and McLaughlin, 1988, p.13). In a computerized environment, the ideal cadastral overlay, as mentioned earlier, is expected to have a legal status for description and to be spatially accurate so that it can correctly register to other layers in the system like the base map, especially when performing overlay operations. This will be achieved through the continuing incremental accuracy improvements made to the parcel dimensions which will make them compare closely to the legally described ones (for more details about the legality of the created cadastral overlay, see Chapter VII).
The National Research Council study on the *Procedures and Standards for a Multipurpose Cadastre* (NRC, 1983) identified two general approaches for the construction of the cadastral layer (overlay):

1) Comprehensive: This involves remonumentation and resurvey of all cadastral parcels in a jurisdiction, followed by the input of the surveying measurements into the computer, which are then processed by adjustment and coordinate geometry (COGO) techniques to produce the required cadastral layer. Although this approach gives the highest accuracy, resurvey of land parcels is impractical because:
   a. It is expensive and time consuming.
   b. It requires the pre-establishment of a geodetic reference framework (if not already established) and the restoration of lost monuments. Craig (1983) states that:

   The price to relocate Minnesota's 155,000 lost PLS corners (one-half of all corners) is $10 million without determining coordinates. Even if the cost of completing [the 155,000 lost PLS corners] were low, the time before it could be completed is so long that we cannot wait for its completion.

2) Iterative: This approach is based initially on the existing information base, like the simple digitization of the widely available tax maps, but improved over a period of time as higher quality information becomes available. This method is relatively quick and cost effective, but of low spatial accuracy due to the uncertainty in the available tax maps.

Figure 2.5 shows these cadastral layer implementation methods in a continuum. Other methods will lie between these two extremes. Dueker and Kjeme (1986) compared some of these methods in terms of accuracy, rate of compilation and overall usefulness of
the cadastral layer. The result of their study is summarized in Table 2.4, which represents a matrix with columns identifying different categories of base layer content (whether the cadastral layer is spatially tied to geodetic control, planimetric data or free-floating), and rows identifying two different methods of entering cadastral location data. They found that the problem in most of these methods is that once the cadastral overlay is compiled,

<table>
<thead>
<tr>
<th>Iterative approach</th>
<th>Comprehensive approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Digitization of Tax Maps</td>
<td>Field Resurvey of All Cadastral Boundaries</td>
</tr>
</tbody>
</table>

ALTERNATIVES:
- COGO
- MMS
- etc.

*low accuracy
*low cost
*short implementation time

*high accuracy
*high cost
*long implementation time

FIGURE 2.5: Options for cadastral overlay creation methods.

TABLE 2.4
Base layer content versus cadastral layer compilation method

<table>
<thead>
<tr>
<th>Base Layer Content</th>
<th>Base Layer Content</th>
<th>Base Layer Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geodetic control</td>
<td>Planimetric data</td>
<td>No base layer</td>
</tr>
<tr>
<td>high accuracy</td>
<td>highest accuracy</td>
<td>low accuracy</td>
</tr>
<tr>
<td>slow rate of compilation</td>
<td>medium rate of compilation</td>
<td>slowest rate of compilation</td>
</tr>
<tr>
<td>good framework</td>
<td>best framework</td>
<td>fair framework</td>
</tr>
<tr>
<td>Digitization of maps</td>
<td>rapid rate of compilation</td>
<td>rapid rate of compilation</td>
</tr>
<tr>
<td>good framework</td>
<td>good framework</td>
<td>fair framework</td>
</tr>
<tr>
<td>good framework</td>
<td>good framework</td>
<td>fair framework</td>
</tr>
</tbody>
</table>
it is considered to be a final product of a series of computations and judgments. Hence, no updating or upgrading mechanisms were involved.

One of these methods, in particular, which has been used in some counties, like Franklin county in Ohio, is the manual keyboard entry of available legal descriptions from deeds and subdivision plats using coordinate geometry (COGO). However, it was estimated that it generally takes three to four times longer to enter data using COGO as it does to digitize the data from existing maps (Dangermond 1989; Werle 1984). Moreover, COGO methods are expensive and time consuming, and force the cadastral mapper to address the problem of gaps and overlaps in parcel descriptions which are difficult to resolve.

To address the issue of updating digital cadastral data, others have suggested the concept of measurement management systems in which measurements are stored (Hintz et al, 1988; Hintz and Onsrud, 1990; Elfick, 1989). The primary aim of these systems is to manage effectively the huge amount of cadastral measurements, and periodically provide an updated set of parcel corner coordinates using all related measurements. Buyong (1992), on the other hand, introduced the concept of measurement-based systems. In these systems, measurements are stored and considered to be the first class objects, and when coordinates of parcels are required, a local adjustment is performed using all the measurements in the area of interest. The idea behind these “measurement-based or measurement management systems” is that the cadastral database is dynamic and should be upgraded on an ongoing basis as more measurements are added. These systems seem feasible from a theoretical point of view, however, when it comes to implementation, they have the following weaknesses:
1) An initial large scale implementation is difficult and time consuming because they require either new measurements to be done, which is costly (especially if boundary identification is involved), or researching through old plat books to extract existing measurements of unknown reliability and precision which is needed in the adjustment computation process.

2) Available measurements might not form a continuous coverage of the area for which the cadastral layer is to be implemented, which will impose difficulties in the creation of a continuous digital cadastral overlay.

3) These systems provide coordinates by adjusting all related measurements every time coordinates are needed. This means that the calculation of the normal equations of the old measurements is repeated every time the adjustment is performed. This is a waste of computer storage and processing.

Others (Haag, 1988; Morgenstern et al., 1989) have discussed the application of some geometric constraints for upgrading coordinates digitized from non-homogeneous cadastral maps. They considered constraints like orthogonality, collinearity, parallelism, and circular continuity, but no results have been published about the practicality of this application. Moreover, due to the technical complications involved in the simultaneous adjustment of a large number of maps, like those of a county or a state, a mechanism is needed to deal with the processing and joining of individual maps. This issue has not been addressed by any of these authors.

The approach described in this research integrates the ideas of Haag and Morgenstern (mainly the application of geometric constraints) to improve the accuracy of coordinates digitized from tax maps during the early stages of implementation, and the ideas of Buyong, Hintz, Onsrud and others to continuously upgrade the coordinate
accuracy by the incorporation of new survey data as they become available. This approach uses all types of possible geometric conditions which may exist in a cadastral map (area, collinearity, rectangularity, distance, tangency, etc.), and employs the sequential adjustment technique for the upgrading process (see Chapter III). It deals with the adjustment of one map at a time, and in order to create a seamless database, alternatives for joining the bordering maps are discussed (Chapter VI). Other aspects of major concern for the successful implementation of this methodology will be discussed in the following chapters as well.

Before exploring the details of this approach, some background on the quality and function of tax maps will be provided.

2.5 Tax Maps

2.5.1 Background

Tax maps, or assessor's maps, represent the effort by assessors to find a means to fairly and equitably carry out their taxation functions. Therefore they represent an attempt to discover and correctly represent the true status of rights in land. Legality is generally reserved to the deeds and other recorded instruments which together define the title status of individual land parcels. Some of the current uses of these tax maps are tax assessment, zoning, planning, permit granting and flood plain insurance assessment (Petersohn et al, 1982).

In order for taxation to be equitable, knowledge of the areas and dimensions of land parcels is essential. To achieve this goal in the past, boundaries of these land holdings were surveyed and mapped and parcel areas were then calculated from these
rudimentary tax maps (Palmer, 1988). In other situations, tax maps were produced originally by enlarging available planimetric maps and then drafting on approximate parcel boundaries obtained from registry of deed descriptions (Keating, 1990). These maps usually contain several kinds of information, among which is the map number, street names, subdivision names, block numbers, parcel numbers, parcel areas, certain parcel dimensions, as well as other hand written notes. These maps help determine the location of property, indicate the size and shape of each parcel, and reveal geographic relationships that affect property value (IAAO, 1988).

Tax maps, therefore, represent a graphic portrayal of parcel locations and dimensions rather than a highly accurate cartographic representation of the legal tenure situation. Several factors have caused the relatively low level of reliability of these maps. These include: the temporal origin of parcel data, the lack of referencing to geodetic reference frameworks, the different surveying methods employed, different tax map scales, poor draftsmanship and lack of systematic updating (Keating, 1990; Morgenstem et al, 1989). While most tax maps are suitable for assessment purposes, using these maps for representing the cadastral situation within a GIS/LIS environment is questionable, unless they are modified to enhance their accuracy.

In an effort to investigate the feasibility of merging various source map data by fitting digitized map coordinates to surveyed ground control points, Petersohn and Vonderohe (1982) studied the accuracy of coordinates digitized from existing tax maps. For this investigation, they used several tax maps at a scale of 1:4800 for Dane County (Wisconsin). These maps, which were originally produced in the 1930's and updated from time to time, were digitized and then transformed to the State Plane Coordinate System using conformal Helmert, affine and projective transformation models. A set of
36 evenly distributed test points which are located in the tax maps area were selected and surveyed accurately to determine their coordinates. These coordinates were then compared with their digitized counterparts resulting from the three transformation models. Petersohn and Vonderohe found that the affine transformation gave the smallest residuals and that the sample average and standard positional errors were of the order of 2.37 and 2.90 m (7.9 and 9.7 ft) respectively when using five control points for the affine transformation. This study sheds some light on the accuracy of tax maps, and shows that some user groups are satisfied with this accuracy as stated in Table 2.2.

2.5.2 Are Tax Maps Suitable for Cadastral Overlay Creation?

Several factors make tax maps the logical start to implement and create a digital cadastral overlay in the USA. These are:

1) The tax maps are, at the present time, the only available parcel-specific, county-wide parcel base maps which makes them an attractive and convenient choice as the cadastral base for most local land information systems now being assembled in the USA (Hintz and Onsrud, 1990; Dueker et al, 1986).

2) Although the accuracy of tax maps is not high enough to satisfy some user groups, like the engineering and surveying community, many other users, such as private utilities, title insurance companies, public safety and planning, and general administration, can still be satisfied with the tax map accuracy of 7-10 ft as determined by Petersohn and Vonderohe (1982). If a methodology can be developed to upgrade digital tax map coordinates, and improve their accuracy over time by integrating them with accurate survey data and information, other users will be satisfied as well.

3) They can be transferred into digital form in a relatively short time period and in a cost-effective way by using manual digitization, or scanning and vectorization.
Actually, digitizing existing tax maps is a quick way to "jump-start" the implementation and automation of the property ownership maps.

4) They are already used by the land information community and are familiar to most users. In fact, Davis county (Utah) and San Bernardino county (California) used tax maps as the basis for the creation of their digital cadastral overlays (Werle, 1984).

Several authors have realized some of these advantages and agree that tax maps form the basis for a good start to computerize land records and to create the cadastral overlay. Palmer (1988, p.2) states that:

Without the active support of assessors, it is questionable if multi-purpose land information systems can be routinely implemented at the local level.

Hintz and Onsrud (1990) acknowledge that parcel information digitized from tax maps is valuable and sufficient for many intended purposes. However, they see that this information is not enough by itself to be a basis for a computerized land information system, and raised a challenge for the surveying and GIS community to come up with a means for upgrading this information with spatial information derived from real property surveys (see section 1.2).

The preceding discussion shows that tax maps form a convenient starting point for the creation of a digital cadastral overlay which is relatively inexpensive and implementable over a short period of time. However, a methodology needs to be designed to improve their accuracy in the initial stage of implementation, and over time as higher quality information becomes available. The following chapters will discuss and lay out such a methodology.
CHAPTER III
DEVELOPING A DIGITAL CADAstral OVERLAY

As established in Chapter II, coordinates digitized from tax maps form a feasible and efficient starting point for building a digital cadastral overlay. This chapter will lay out and establish the mathematical background behind an approach which utilizes the least squares adjustment principle on these coordinates.

3.1 Methodology Layout

This research envisions the accuracy improvement of coordinates digitized from tax maps in two complementary phases. These are:

Phase I: Initial improvement of the accuracy of the digitized coordinates, so that the resulting cadastral overlay will be readily available for use. This is carried out by a least squares adjustment of the coordinates using the method of observation equations on the following two types of data:

1) Geometric constraints (as stochastic constraints or pseudo-observations). These include:
   - Parcel areas
   - Street widths (wherever mentioned, this refers to the street right of way)
   - Line-curve and curve-curve tangency, and
   - Angles which are multiples of 45°, including rectangularity (90° angles) and collinearity (180° angles).
Figure 3.1 shows examples of these geometric constraints. Angle and tangency constraints are derived automatically by using a user-specified angular tolerance (e.g. 2°). Area and street width constraints are usually created manually because of their subjective nature.

2) Any other data which might be written on tax maps like angles and lengths of parcel boundary lines.

FIGURE 3.1: Examples of geometric constraints
Phase 2: The continuous upgrading of the spatial cadastral data accuracy through the addition of survey data as portrayed on subdivision plats, deeds and other publicly recorded documents, as well as field survey data as they become available. This means that the coordinates obtained from phase 1 will be sequentially adjusted to fit the new measurements which are assumed to be more precise than the coordinates. This will be performed as follows:

1) Retain the coordinates as well as the normal equations matrix \((\mathbf{N} = \mathbf{A}^\top \mathbf{P} \mathbf{A})\) from the prior adjustment. This matrix contains the geometry of the cadastral map as represented by the geometric constraints and any survey data utilized in the previous adjustments. The reason for storing \(\mathbf{N}\) rather than the variance-covariance matrix of the coordinates, is that \(\mathbf{N}\) is a symmetric and extremely sparse matrix, and only half the non-zero elements needs to be stored. This can save as much as 90\% of the storage needed for the variance-covariance matrix.

2) Perform a sequential adjustment on the initially adjusted coordinates using the stored \(\mathbf{N}\) matrix as well as the new survey data.

The rest of this chapter deals with the mathematical background needed for the implementation of these two phases. The first phase, which lays the groundwork for the second phase is dealt with in section 3.2, while the second phase (which is the most important for the long term upgrading of the coordinate accuracy) is dealt with in section 3.3. Figure 3.2 shows a graphic representation of these two interrelated phases.
FIGURE 3.2: Layout of the upgrading methodology of coordinates digitized from tax maps
3.2 Phase 1: Initial Accuracy Improvement Using Geometric Constraints

3.2.1 The Gauss-Markov Model (GMM)

The GMM is a linear mathematical model consisting of functional and stochastic relations (Caspary, 1987, p.4). It relates the stochastic observations $L^b$ to the fixed parameters $X^*$.

In matrix notation, it takes the following form:

$$L^* = E[L^b] = F(X^*) = AX^* \quad \text{or} \quad L^b - \varepsilon = AX^* \quad \text{with} \quad \sum_n \varepsilon = \sigma_e^2 Q = \sigma_e^2 P^{-1}$$

To apply this theoretical model, the estimation of parameters from real observed data is performed such that the following relation holds:

$$L^b + V = A\hat{X}^* \quad \text{for} \quad V^T P V = \min (L^b - AX^*)^T P (L^b - AX^*) \quad \text{---------------- (3.2)}$$

where

- $L^b \quad = n$ - dimensional vector of observed values of observables
- $L^* \quad = n$ - dimensional vector of theoretical values of observables
- $E[.] \quad = \text{expectation operator}$
- $X^* \quad = u$ - dimensional vector of unknown parameters (theoretical values)
- $\hat{X}^* \quad = u$ - dimensional vector of estimates of unknown parameters
- $A \quad = n \times u$ - matrix of known coefficients, usually $\text{rank}(A) = u$
- $\varepsilon \quad = n$ - dimensional vector of true (real) errors
- $V \quad = n$ - dimensional vector of residuals (for which $V^T P V$ becomes a minimum)
- $\sigma_e^2 \quad = \text{unknown variance of unit weight (or variance component)}$
- $Q \quad = n \times n$ - cofactor matrix of observations
- $P \quad = n \times n$ - weight matrix of observations
- $\sum_n b \quad = \text{dispersion matrix of the observations}$
In most situations, like that of geodetic networks, the original observation equations are usually non-linear. For example, the distance equation is:

\[ E[D] = \sqrt{\Delta x^2 + \Delta y^2} \]  

(3.3)

To obtain the linear form as required in the GMM, a linearization through restriction to the first two terms in a Taylor series is carried out. Equation (3.1) becomes:

\[ L^b - \varepsilon = F(X^*) = F(X^0) + \frac{\partial F}{\partial X^i} \Bigg|_{X^* = X^0} (X^* - X^0) + \text{higher order terms} \]

(3.4)

where \( X^0 \) = \( u \)-dimensional vector of approximate values of parameters
\( X = X^* - X^0 \) = \( u \)-dimensional vector of parameter increments, and
\( A = \frac{\partial F}{\partial X^i} \Bigg|_{X^* = X^0} \) = the Jacobian of \( F \)

Neglecting the high order terms, the following two relations can be obtained:

\[ -\varepsilon = AX + F(X^0) - L^b, \quad X = X^* - X^0 \]

(3.5)

or

\[ V = A\hat{X} + F(X^0) - L^b, \quad \hat{X} = \hat{X}^* - X^0 \]

Denoting \( L^0 = F(X^0) \) and \( L = L^0 - L^b \), we get the following system of observation equations:

\[ -\varepsilon = AX + L \]  

(3.6)

The effect of neglecting the higher order terms in equation (3.4) is substituted through an iterative solution of the parameters, using the current estimates in the new Taylor series expansion.
It must be expected that all observations contain errors of some kind in which case it is impossible to get the true values of the parameters \((X^*)\). The number of observations in most cases is larger than the number of unknown parameters in the GMM. Equations (3.1) and (3.6) then represent an over-determined system of linear equations. If we assume that errors are random with expectation zero, the minimum variance solution among the unbiased estimates for \(X\) in (3.6) becomes:

\[
\hat{X} = -(A^T P A)^{-1} A^T P L
\]

(3.7)

This is referred to as BLUE (best linear unbiased estimator). Under the previous assumption, the estimated variance of unit weight \((\hat{\sigma}_o^2)\) and the variance-covariance matrix of adjusted parameters \((\Sigma_{\hat{X}})\) can be calculated from the following formulae:

\[
\hat{\sigma}_o^2 = \frac{V^T P V}{n-u}
\]

(3.8)

\[
\hat{\Sigma}_{\hat{X}} = \hat{\sigma}_o^2 (A^T P A)^{-1}
\]

(3.9)

provided that the inverse of \((A^T P A)\) exists. Further details about the derivation of the previous equations can be found in Uotila (1986).

The next section gives the derivation of the formulae needed in the case when constraints are imposed on the unknown parameters.

3.2.2 Observations with Constraints

Let us assume that two sets of observations, \(L^b\) and \(L^c\), which are functions of the same parameters \(X^*\) have been collected. The mathematical models will be:

\[
L^b = F_1(X^*) = AX^*
\]

(3.10)

\[
L^c = F_2(X^*) = CX^*
\]
where \( A = \frac{\partial F_1}{\partial X^*} \mid_{X^* = X^o} \); \( C = \frac{\partial F_2}{\partial X^*} \mid_{X^* = X^o} \)

Let
\[
P = \sigma^2 \sum_{n=1} P_n; \quad L^o = F_i(X^o); \quad L = L^o - L^b
\]
\[
P_e = \sigma^2 \sum_{n} P_n; \quad L_e = F_i(X^e); \quad L_e = L_e - L^b
\]
\[
X^* = X^o + X
\]

Then, using the Taylor series expansion, we get the following sets of observation equations:

\[
-\epsilon = AX + L
\]
\[
-\epsilon_e = CX + L_e
\]

In order to get the minimum variance unbiased solution, we may equivalently minimize the following expression:

\[
\Phi = \epsilon^T P \epsilon + \epsilon_e^T P_e \epsilon_e
\]

By substituting equations (3.11) into (3.12),

\[
\Rightarrow \Phi = (AX + L)^T P (AX + L) + (CX + L_e)^T P_e (CX + L_e)
\]

\[
= X^T A^T PA X + X^T A^T PL + L^T PAX + L^T PL + X^T C^T P_e CX
\]

\[
+ X^T C^T P_e L_e + L_e^T P_e CX + L_e^T P_e L_e
\]

Let \( N = A^T PA \), \( U = A^T PL \)

\[
\Rightarrow \frac{\partial \Phi}{\partial X} = 2NX + 2U + 2C^T P_e CX + 2C^T P_e L_e
\]

set
\[
2N\dot{X} + 2U + 2C^T P_e \dot{X} + 2C^T P_e L_e = 0
\]

\[
\Rightarrow \text{Normal equations:}
\]

\[
(N + C^T P_e C) \dot{X} = -(U + C^T P_e L_e)
\]

\[
\Rightarrow \dot{X} = -(N + C^T P_e C)^{-1} (U + C^T P_e L_e)
\]
Equation (3.13) can be expressed in another form as:

$$\hat{X} = -\frac{(N + C^T P_e C)^{-1}}{\delta} U - \frac{(N + C^T P_e C)^{-1} C^T P_e L_e}{\delta} \quad \ldots \quad (3.14)$$

From Useful Matrix Equalities (UME), (Uttila, 1986, p.159, equations 24a and 29a)

$$(a) \Rightarrow (N + C^T P_e C)^{-1} = N^{-1} - N^{-1} C^T (P_e^{-1} + C N^{-1} C^T)^{-1} C N^{-1}$$

$$(b) \Rightarrow (N + C^T P_e C)^{-1} C^T P_e = N^{-1} C^T (P_e^{-1} + C N^{-1} C^T)^{-1}$$

Substitute (a) and (b) into equation (3.14):

$$\Rightarrow \hat{X} = \frac{-N^{-1} U}{\delta} + \frac{N^{-1} C^T (P_e^{-1} + C N^{-1} C^T)^{-1} (C N^{-1} U - L_e)}{\delta} \quad \ldots \quad (3.15)$$

Part (c) is the solution if only the first set of observations $L^b$ was made. Part (d) is the disturbance caused by making the observations $L_e$.

Now, if $P_e^{-1} \rightarrow 0$ in equation (3.15), this results in the case of observations with fixed constraints on the unknown parameters and part (d) becomes the disturbance due to fixed constraints. Fixed constraints can be envisioned as observations which are completely free of error and carry an infinite weight. If these constraints are relaxed to carry some error by giving them a high weight instead of infinite weight, the situation of stochastic constraints (or "pseudo-observations") results. Stochastic constraints are, in many situations, preferable when dealing with real world problems which carry the possibility of error like rectangular and collinear parcel boundaries where angles might not be exactly 90 or 180 degrees. It also makes it easier to invert the normal equations matrix by avoiding ill-conditioning and numerically unstable situations.

Equation (3.15) can be applied successfully for the case of fixed constraints or weighted constraints with very large weights. However, equations (3.13) and (3.14) are more suitable for the case of stochastic constraints and are used in this research through
the application of four types of geometric constraints to coordinates digitized from tax maps. These constraints are parcel area, distance (street widths), angle (multiples of 45°) and tangency constraints.

In equations (3.10) through (3.15), the first set of observations is considered to be the digitized coordinates themselves. The second set of observations is the geometric constraints. The next sections will outline the mathematical models and give the derivation of the observation equations for both the digitized coordinates as well as the geometric constraints.

3.2.3 Observed (Digitized) Coordinates

Mathematical Model:

For point i:

\[ F_1(x_i) = x_i^b - \varepsilon_{x_i} = x_i \]
\[ F_2(y_i) = y_i^b - \varepsilon_{y_i} = y_i \] .................................(3.16)

where \( x_i \) and \( y_i \) are the theoretical unknown coordinates of point i, and 
\( x_i^b \) and \( y_i^b \) are the observed coordinates of point i.

The partial derivatives are:

\[ \frac{\partial F_1}{\partial x_i} = 1 \] .................................(3.17)
\[ \frac{\partial F_2}{\partial y_i} = 1 \]

The observation equations for the coordinates of point i are:

\[ -\varepsilon_{x_i} = x_i^b - x_i^b + \Delta x_i \] .................................(3.18)
\[ -\varepsilon_{y_i} = y_i^b - y_i^b + \Delta y_i \]
where \( x_i^p, y_i^p \) are the approximate coordinates of point \( i \).

This results in \( A = I_{2n \times 2n} \) and \( N = P_{2n \times 2n} \) in equations (3.10) through (3.15), where \( I \) is the identity matrix of size \( 2n \times 2n \), \( P \) is the weight matrix of the observed coordinates, and \( n \) is the number of observed points. Since no prior information about the accuracy of the digitized coordinates is available, the weight matrix \( P \) of these coordinates was considered to be \( I_{2n \times 2n} \).

### 3.2.4 Parcel Area Constraints

For simplicity, let us assume the two configurations of parcels shown in Figure 3.3.

![Figure 3.3: Examples of two different configurations of parcel boundaries.](image)

FIGURE 3.3: Examples of two different configurations of parcel boundaries.
Let $A_i$ represent the area surrounded by traverse $123451$, and $A_s$ represent the area of the shaded circular segment.

The total area $A$ of the land parcel will be:

$$A = A_i + A_s$$

(3.19)

where $A_i$ is positive in Figure 3.3a, and the corresponding curve is said to be positive, and $A_s$ is negative in Figure 3.3b, and the corresponding curve is said to be negative.

1) The area $A_i$ of the closed traverse $123451$ can be represented as:

$$A_i = \frac{1}{2} \left[ y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_4 - x_2) + y_4(x_5 - x_3) + y_5(x_1 - x_4) \right]$$

(3.20)

For a closed traverse with $n$ vertices, equation (3.20) can be generalized as follows:

$$A_i = \frac{1}{2} \left[ y_1(x_2 - x_n) + y_2(x_3 - x_1) + \ldots + y_{n-1}(x_n - x_2) + y_n(x_1 - x_{n-1}) \right]$$

(3.21)

2) The segment area $A_s$ is:

$$A_s = \frac{R^2}{2} (\Delta - \sin \Delta)$$

(3.22)

where $\Delta$ is the opening angle of the circular curve in radians.

$R$ is the radius of the circular curve.
Let \( K \) be the chord length of the circular curve, then using the cosine law:

\[
K^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2 = R^2 + R^2 - 2R^2 \cos \Delta
\]

\[\Rightarrow \ \Delta = \arccos\left(1 - \frac{K^2}{2R^2}\right) \tag{3.23}\]

**Mathematical Model:**

When combining equations (3.21) and (3.22), the following mathematical model for parcel areas results:

\[
A_k = \frac{1}{2} \left[ y_1(x_2 - x_n) + y_2(x_3 - x_1) + \ldots + y_{n-1}(x_n - x_{n-2}) + y_n(x_1 - x_{n-1}) \right] + \sum_{i=1}^{m} \pm \frac{R_i^2}{2} (\Delta_i - \sin \Delta_i) \tag{3.24}
\]

where \( A_k \) = area of parcel \( k \) with \( n \) corners and \( m \) circular arcs.

The observation equations are:

\[
-\epsilon = CX + L \tag{3.25}
\]

where \( X = X^* - X^o \)

\[
C_k = \frac{\partial A_k}{\partial X^o} \quad \text{as typical row in } C
\]

\( X^o = X^o \)

The element \( L_k \) of the vector \( L \) is calculated as:

\[
L_k = A_k(x^o) - A_k^b \tag{3.26}
\]

where \( A_k(x^o) \) = calculated area of parcel \( k \) using \( X^o \)

\( A_k^b \) = given area constraint value of parcel \( k \).
The elements of matrix C are calculated as follows:

1) Assume points i and i+1 are the beginning and end of a positive circular curve (Figure 3.3a) respectively, then:

\[
\begin{align*}
\frac{\partial A_k}{\partial x_i} &= \frac{1}{2}(y_{i+1} - y_{i+2}) - \frac{(1 - \cos \Delta_{i,i+1})(x_{i+1} - x_i)}{Z} \\
\frac{\partial A_k}{\partial y_i} &= \frac{1}{2}(x_{i+1} - x_{i+2}) - \frac{(1 - \cos \Delta_{i,i+1})(y_{i+1} - y_i)}{Z} \\
\frac{\partial A_k}{\partial x_{i+1}} &= \frac{1}{2}(y_{i+1} - y_{i+2}) + \frac{(1 - \cos \Delta_{i,i+1})(x_{i+1} - x_i)}{Z} \\
\frac{\partial A_k}{\partial y_{i+1}} &= \frac{1}{2}(x_{i+2} - x_i) + \frac{(1 - \cos \Delta_{i,i+1})(y_{i+1} - y_i)}{Z}
\end{align*}
\]  

(3.27)

where \( \Delta_{i,i+1} \) is the opening angle of the circular arc between points i and i+1.

\[
Z = 2\sqrt{1 - \left(1 - \frac{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}{2 R_{i,i+1}^2}\right)^2}
\]

2) Points i and i+1 are beginning and end of a negative circular curve (Figure 3.3b): the same as equations (3.27) with the sign of the second term in each equation reversed.

3) Points i and i+1 are normal edges (corners) which are not located on circular arcs, but are edges of straight lines: the same as equations (3.27) with the second term in each equation becoming zero.

**Note:** All the previous partial derivatives are estimated at \( X^0 \).
3.2.5 Distance Constraints

Mathematical Model

If the distance (e.g.: street widths) between two points i and j whose coordinates are observed is $S$, then the following mathematical model exists:

$$S = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \quad \quad \quad \quad \quad \quad \quad \quad (3.28)$$

The partial derivatives are:

$$\frac{\partial S}{\partial x_i} = \frac{-(x_j - x_i)}{S} \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.29)$$
$$\frac{\partial S}{\partial y_i} = \frac{-(y_j - y_i)}{S} \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.29)$$
$$\frac{\partial S}{\partial x_j} = \frac{(x_j - x_i)}{S} \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.29)$$
$$\frac{\partial S}{\partial y_j} = \frac{(y_j - y_i)}{S} \quad \quad \quad \quad \quad \quad \quad \quad \quad (3.29)$$

The observation equation of distance $S_i$ is:

$$-e_s = S_i - S_i^0 - \frac{(x_j^0 - x_i^0)}{S_i^0} \Delta x_i - \frac{(y_j^0 - y_i^0)}{S_i^0} \Delta y_i$$
$$+ \frac{(x_j^0 - x_i^0)}{S_i^0} \Delta x_j + \frac{(y_j^0 - y_i^0)}{S_i^0} \Delta y_j \quad \quad \quad \quad \quad \quad \quad \quad (3.30)$$

where $S_i^0 = \text{given distance constraint value of distance } S_i$

$$S_i^0 = \sqrt{(x_j^0 - x_i^0)^2 + (y_j^0 - y_i^0)^2}$$
3.2.6 Angle Constraints

If the angle between three points i, j and k (Figure 3.4) whose coordinates are observed is $\alpha_{jk}$, then the following mathematical model exists:

$$
\alpha_{jk} = \arctan \left( \frac{x_k - x_j}{y_k - y_j} \right) - \arctan \left( \frac{x_i - x_j}{y_i - y_j} \right)
$$

(3.31)

The partial derivatives are:

$$
\frac{\partial \alpha_{jk}}{\partial x_i} = -\frac{(y_i - y_j)}{S_{ij}^2}
$$

$$
\frac{\partial \alpha_{jk}}{\partial y_i} = \frac{(x_i - x_j)}{S_{ij}^2}
$$

$$
\frac{\partial \alpha_{jk}}{\partial x_j} = \frac{(y_k - y_j)}{S_{jk}^2} + \frac{(y_i - y_j)}{S_{ij}^2}
$$

$$
\frac{\partial \alpha_{jk}}{\partial y_j} = \frac{(x_k - x_j)}{S_{jk}^2} - \frac{(x_i - x_j)}{S_{ij}^2}
$$

$$
\frac{\partial \alpha_{jk}}{\partial x_k} = \frac{(y_k - y_j)}{S_{jk}^2}
$$

$$
\frac{\partial \alpha_{jk}}{\partial y_k} = -\frac{(x_k - x_j)}{S_{jk}^2}
$$

(3.32)
where $S_j = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$

$$S_k = \sqrt{(x_k - x_i)^2 + (y_k - y_j)^2}$$

The observation equation of angle $\alpha_{ik}$ is:

$$-e_{\alpha_{ik}} = \alpha_{ik}^o - \alpha_{ik}^b + \frac{\partial \alpha_{ik}}{\partial x_i} \Delta x_i + \frac{\partial \alpha_{ik}}{\partial y_i} \Delta y_i + \frac{\partial \alpha_{ik}}{\partial x_j} \Delta x_j + \frac{\partial \alpha_{ik}}{\partial y_j} \Delta y_j$$

$$+ \frac{\partial \alpha_{ik}}{\partial x_k} \Delta x_k + \frac{\partial \alpha_{ik}}{\partial y_k} \Delta y_k$$

$$.............................(3.33)$$

where $\alpha_{ik}^b = \text{given angle constraint value of angle } \alpha_{ik} \text{ in radians, and}$

$$\alpha_{ik}^o = \arctan\left(\frac{x_k^o - x_i^o}{y_k^o - y_j^o}\right) - \arctan\left(\frac{x_i^o - x_j^o}{y_i^o - y_j^o}\right)$$

and the partial derivatives are those given in equations (3.32) and estimated at $X^o$.

**Special Cases:**

1) When the angle equals $90^\circ$. Figure 3.5 portrays two examples where this situation occurs.

![Diagram](FIGURE 3.5: Types of right angles in cadastral maps.)
The mathematical model will be derived for case (b) of Figure 3.5 and then specialized to case (a).

Lines \( ij \) and \( kl \) are perpendicular to each other,

\[ \Rightarrow (\text{Slope of } ij) \cdot (\text{Slope of } kl) = -1 \]

\[ \Rightarrow \frac{(y_j - y_i)}{(x_j - x_i)} \cdot \frac{(y_k - y_l)}{(x_k - x_l)} = -1 \]

\[ \Rightarrow \text{Mathematical model is:} \]

\[ F_{ikl} = (y_j - y_i) (y_k - y_l) + (x_j - x_i) (x_k - x_l) = 0 \quad (3.34) \]

The partial derivatives are:

\[
\begin{align*}
\frac{\partial F_{ikl}}{\partial x_i} &= x_k - x_t \\
\frac{\partial F_{ikl}}{\partial y_i} &= y_k - y_t \\
\frac{\partial F_{ikl}}{\partial x_j} &= x_t - x_k \\
\frac{\partial F_{ikl}}{\partial y_j} &= y_t - y_k \\
\frac{\partial F_{ikl}}{\partial x_k} &= x_1 - x_j \\
\frac{\partial F_{ikl}}{\partial y_k} &= y_1 - y_j \\
\frac{\partial F_{ikl}}{\partial x_l} &= x_j - x_1 \\
\frac{\partial F_{ikl}}{\partial y_l} &= y_j - y_1 
\end{align*}
\]

\[ (3.35) \]
The observation equation is:

\[-\epsilon_{iJ} = F^o_{iJ} + (x^o_i - x^o_J)\Delta x_i + (y^o_i - y^o_J)\Delta y_i + (x^o_J - x^o_i)\Delta x_J + (y^o_J - y^o_i)\Delta y_J\]

\[+ (x^o_j - x^o_k)\Delta x_k + (y^o_j - y^o_k)\Delta y_k + (x^o_k - x^o_j)\Delta x_k + (y^o_k - y^o_j)\Delta y_k\]

..........................(3.36)

where \( F^o_{iJ} = (y^o_i - y^o_J)(y^o_J - y^o_i) + (x^o_i - x^o_J)(x^o_J - x^o_i) \)

When points \( j \) and \( k \) are the same point, as in case (a) of Figure 3.5, the mathematical model becomes:

\[ F^o_{iJ} = (y_j - y_i)(y_i - y_j) + (x_j - x_i)(x_i - x_j) = 0 \]

..................(3.37)

and the observation equation becomes:

\[-\epsilon_{iJ} = F^o_{iJ} + (x^o_i - x^o_J)\Delta x_i + (y^o_i - y^o_J)\Delta y_i + (x^o_J - x^o_i - 2x^o_i)\Delta x_J + 2y^o_i \Delta y_j \]

.........(3.38)

where \( F^o_{iJ} = (y^o_i - y^o_J)(y^o_J - y^o_i) + (x^o_i - x^o_J)(x^o_J - x^o_i) \)

2) When the angle equals 0° or 180°, as in the case of collinear and parallel lines.

Figure 3.6 portrays these two cases:

\( I \quad j \quad I \)

\( \quad i \quad \quad k \quad l \)

(a) (b)

FIGURE 3.6: Geometry of collinear (a) and parallel (b) lines.

The mathematical model will be derived for case (b) of Figure 3.6 and then specialized to case (a).
Lines $ij$ and $k\ell$ are parallel,

$$\Rightarrow \text{ (Slope of } ij) = \text{ (Slope of } k\ell) \Rightarrow \frac{(y_j - y_i)}{(x_j - x_i)} = \frac{(y_k - y_\ell)}{(x_k - x_\ell)}$$

$$\Rightarrow \text{ The mathematical model is:}$$

$$F_{ijkl} = (x_j - x_i)(y_\ell - y_k) - (x_\ell - x_k)(y_j - y_i) = 0 \hspace{1cm} (3.39)$$

The partial derivatives are:

$$\begin{align*}
\frac{\partial F_{ijkl}}{\partial x_i} &= y_k - y_\ell \\
\frac{\partial F_{ijkl}}{\partial y_i} &= x_\ell - x_k \\
\frac{\partial F_{ijkl}}{\partial x_j} &= y_\ell - y_k \\
\frac{\partial F_{ijkl}}{\partial y_j} &= x_k - x_\ell \\
\frac{\partial F_{ijkl}}{\partial x_k} &= y_j - y_i \\
\frac{\partial F_{ijkl}}{\partial y_k} &= x_i - x_j \\
\frac{\partial F_{ijkl}}{\partial x_\ell} &= y_i - y_j \\
\frac{\partial F_{ijkl}}{\partial y_\ell} &= x_j - x_i
\end{align*}$$

\hspace{1cm} \text{ (3.40)}$$

The observation equation is:

$$-\varepsilon_{ijkl} = F'_{ijkl} + (y^*_\ell - y^*_i)\Delta x_i + (x^*_i - x^*_\ell)\Delta y_i + (y^*_i - y^*_j)\Delta x_j + (x^*_j - x^*_i)\Delta y_j + (y^*_j - y^*_k)\Delta x_k + (x^*_k - x^*_j)\Delta y_k + (y^*_k - y^*_\ell)\Delta x_\ell + (x^*_\ell - x^*_k)\Delta y_\ell$$

$$\hspace{1cm} \text{ (3.41)}$$

where

$$F'_{ijkl} = (x_j - x_i)(y_\ell - y_k) - (x_\ell - x_k)(y_j - y_i)$$
When points j and k are the same point as in the collinearity case (Part (a) of Figure 3.6), the mathematical model becomes:

\[ F_{ij} = (y_t - y_j)(x_t - x_j) - (x_t - x_j)(y_j - y_t) \] ...................................................... (3.42)

and the observation equation becomes:

\[-\varepsilon_{i} = F_{ij}^0 + (y_{j}^0 - y_j^0)\Delta x_i + (x_{j}^0 - x_j^0)\Delta y_i + (y_{j}^0 - y_j^0)\Delta x_j + (x_{j}^0 - x_j^0)\Delta y_j + (x_{i}^0 - x_i^0)(y_{j}^0 - y_j^0)\] ...................................................... (3.43)

where \[ F_{ij}^0 = (y_{i}^0 - y_j^0)(x_{i}^0 - x_j^0) - (x_{i}^0 - x_j^0)(y_{j}^0 - y_j^0) \]

The above special cases are preferred to be used due to the instability of trigonometric functions around angles 0°, 90° and 180°.

3.2.7 Tangency Constraints

Two types of tangency constraints occur in parcel boundaries. These are:

1) Curve meeting a curve (curve-curve tangency)
2) Line meeting a curve (line-curve tangency).

Before the mathematical models for these types of tangency conditions are derived, a derivation for the center of a circular arc, as well as, the slope of the tangent to the circular arc at point i \((x_i, y_i)\) will be given.
3.2.7.1 Center Coordinates of a Circular Arc

\[ C(x_c, y_c) \]

**FIGURE 3.7:** Geometry of circular curves.

Let

- \( R \) = radius of the circular curve
- \( K \) = chord length between points 1 and 2
- \( o \) = middle point of chord \( K \)
- \( m \) = perpendicular distance to the chord from the center \( C(x_c, y_c) \)
- \( \vec{V}_1 \) = vector from \( o \) to \( C \)
- \( \vec{V}_2 \) = vector from 1 to 2

1, 2 be the points of beginning and end of the circular curve respectively when the curve is viewed in a counterclockwise direction. The coordinates of points 1 and 2 are assumed to be known.

Then,

\[ K = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

\[ m = \sqrt{R^2 - \frac{K^2}{4}} = \frac{1}{2} \sqrt{4R^2 - K^2} \]

\[ x_o = \frac{(x_1 + x_2)}{2} \]

\[ y_o = \frac{(y_1 + y_2)}{2} \]
\[ \vec{V}_1 = (x_o - x_0)\hat{i} + (y_o - y_0)\hat{j} \]
\[ \vec{V}_2 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} \]
\[ \hat{i}, \hat{j} = \text{base vectors} \]

\[ \vec{V}_1 \text{ and } \vec{V}_2 \text{ are perpendicular}, \]
\[ \Rightarrow \vec{V}_1 \cdot \vec{V}_2 = 0 \]
\[ \Rightarrow (x_o - x_0)(x_2 - x_1) + (y_o - y_0)(y_2 - y_1) = 0 \]
\[ \Rightarrow \left( \frac{y_o - y_0}{x_o - x_o} \right) = -\left( \frac{x_2 - x_1}{y_2 - y_1} \right) \] .............. (3.44)

Also,
\[ (x_o - x_o)^2 + (y_o - y_o)^2 = m^2 \] .............. (3.45)

Substitute the value of \( y_o - y_o \) from equation (3.44) into (3.45)
\[ \Rightarrow (x_o - x_o)^2 + (y_o - y_o)^2 \left( \frac{(x_2 - x_1)^2}{(y_2 - y_1)^2} \right) = m^2 \]
\[ \Rightarrow (x_o - x_o)^2 \left[ (y_2 - y_1)^2 + (x_2 - x_1)^2 \right] = m^2 (y_2 - y_1)^2 \]

which leads to:
\[ x_o = x_o \pm m \frac{(y_2 - y_1)}{\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right]^{1/2}} \] .............. (3.46)

Substitute (3.46) into (3.44),
\[ \Rightarrow y_o = y_o \mp m \frac{(x_2 - x_1)}{\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right]^{1/2}} \] .............. (3.47)

Substitute the values of \( x_o, y_o \) and \( m \) into (3.46) and (3.47),
\[ x_0 = \frac{(x_1 + x_2)}{2} + C_x \frac{(y_2 - y_1)}{2} \left( \frac{4 R^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} - 1 \right)^{1/2} \]
\[ y_0 = \frac{(y_1 + y_2)}{2} + C_y \frac{(x_2 - x_1)}{2} \left( \frac{4 R^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2} - 1 \right)^{1/2} \]...

where \( C_x \) and \( C_y \) take the values of -1 and +1 or vice versa. Two different cases here can be identified:

1) When the circular curve is positive (i.e., the circular segment adds to the parcel area when viewed in a counterclockwise direction, like that shown in Figure 3.3a), then:

\( C_x = -1, \quad C_y = +1 \)

2) When the circular curve is negative (i.e., the circular segment subtracts from the parcel area when viewed in a counterclockwise direction, like that shown in Figure 3.3b), then:

\( C_x = +1, \quad C_y = -1 \)

3.2.7.2 Slope of the Tangent to a Circular Arc

In Figure 3.7, assume that the slope of the tangent to the circular curve extending between points 1 and 2 is to be calculated at point \( i (x_i, y_i) \). The equation of the circular arc is:

\[ (x_i - x_e)^2 + (y_i - y_e)^2 = R^2 \] .......................................................... (3.49)

where \( (x_i, y_i) \) are the coordinates of any point \( i \) located on the arc. Differentiating (3.49) with respect to \( x_i \),

\[ \Rightarrow 2(x_i - x_e) + 2(y_i - y_e) \frac{dy_i}{dx_i} = 0 \]
Equation (3.50) represents the slope of the tangent to the circular arc at point \( i (x_i, y_i) \).

After establishing the required mathematical background, the next two sections will provide the derivation of the mathematical model and observation equations for the curve-curve and line-curve tangency conditions.

### 3.2.7.3 Curve-Curve Tangency

Figure 3.8 illustrates the geometry of the curve-curve tangency condition. If the two circular curves 1 and 2 meet tangentially at point 2, then the slope of the tangent for both curves at point 2 must be the same. This means:

\[
\frac{x_2 - x_{c_1}}{y_2 - y_{c_1}} = \frac{x_2 - x_{c_2}}{y_2 - y_{c_2}},
\]

which can be re-arranged as:

\[
(x_2 - x_{c_1})(y_2 - y_{c_2}) - (y_2 - y_{c_1})(x_2 - x_{c_2}) = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.51)
\]
Substitute the values of \(x_v, y_v, x_c, \& y_c\) from equation (3.48) into (3.51):

\[ F = \left[ x_2 - x_1 - C_1 (y_2 - y_1)D_1 \right] \left[ y_2 - y_3 - C_2 (x_2 - x_3)D_2 \right] \]
\[ - \left[ y_2 - y_1 - C_1 (x_2 - x_1)D_1 \right] \left[ x_2 - x_3 - C_2 (y_2 - y_3)D_2 \right] = 0 \quad \ldots \ldots (3.52) \]

where
\[ D_1 = \left( \frac{4R_i^2}{(x_2 - x_1)^2 + (y_2 - y_1)^2 - 1} \right)^{1/2} \]
\[ D_2 = \left( \frac{4R_i^2}{(x_2 - x_3)^2 + (y_2 - y_3)^2 - 1} \right)^{1/2} \]

\(C_1\) and \(C_2\) refer to the curve 1
- If curve 1 is positive \(\Rightarrow C_1 = -1, C_2 = +1\)
- If curve 1 is negative \(\Rightarrow C_1 = +1, C_2 = -1\)

\(C_3\) and \(C_4\) refer to the curve 2
- If curve 2 is positive \(\Rightarrow C_3 = +1, C_4 = -1\)
- If curve 2 is negative \(\Rightarrow C_3 = -1, C_4 = +1\)

For curve definition being positive or negative, refer to section (3.2.7.1) and Figure 3.3.

For simplicity, the mathematical model can be written as:
\[ F = E_1 \cdot E_2 - E_3 \cdot E_4 = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3.53) \]

where \(E_1, E_2, E_3, \& E_4\) are as indicated in equation (3.52).
With \( L_1 = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right]^{1/2} \)
\( L_2 = \left[ (x_2 - x_3)^2 + (y_2 - y_3)^2 \right]^{1/2} \),

the partial derivatives are:

\[
\begin{align*}
\frac{\partial F}{\partial x_1} &= E_2 \left[ -1 - C_{x_1} \frac{4R_1^2(x_2 - x_1)(y_2 - y_1)}{D_1L_1^4} \right] - E_4 \left[ C_{y_1}D_1 - C_{y_1} \frac{4R_1^2(x_2 - x_1)^2}{D_1L_1^4} \right] \\
\frac{\partial F}{\partial y_1} &= E_2 \left[ C_{x_1}D_1 - C_{x_1} \frac{4R_1^2(y_2 - y_1)^2}{D_1L_1^4} \right] - E_4 \left[ -1 - C_{y_1} \frac{4R_1^2(x_2 - x_1)(y_2 - y_1)}{D_1L_1^4} \right] \\
\frac{\partial F}{\partial x_2} &= E_2 \left[ 1 + C_{x_1} \frac{4R_1^2(x_2 - x_1)(y_2 - y_1)}{D_1L_1^4} \right] + E_1 \left[ -C_{y_2}D_2 + C_{y_2} \frac{4R_2^2(x_2 - x_3)^2}{D_2L_2^4} \right] \\
\frac{\partial F}{\partial y_2} &= E_2 \left[ -C_{x_1}D_1 + C_{x_1} \frac{4R_1^2(y_2 - y_1)^2}{D_1L_1^4} \right] + E_1 \left[ 1 + C_{y_2} \frac{4R_2^2(x_2 - x_3)(y_2 - y_3)}{D_2L_2^4} \right] \\
\frac{\partial F}{\partial x_3} &= E_1 \left[ C_{y_2}D_2 - C_{y_2} \frac{4R_2^2(x_2 - x_3)^2}{D_2L_2^4} \right] - E_3 \left[ 1 - C_{x_2} \frac{4R_3^2(x_2 - x_3)(y_2 - y_3)}{D_2L_2^4} \right] \\
\frac{\partial F}{\partial y_3} &= E_1 \left[ -1 - C_{y_2} \frac{4R_2^2(x_2 - x_3)(y_2 - y_3)}{D_2L_2^4} \right] - E_3 \left[ C_{x_2}D_2 - C_{x_2} \frac{4R_3^2(y_2 - y_3)^2}{D_2L_2^4} \right]
\end{align*}
\]
The observation equation of the curve-curve tangency at point 2 is:

\[-c = F^0 + \frac{\partial F}{\partial x_1} \Delta x_1 + \frac{\partial F}{\partial y_1} \Delta y_1 + \frac{\partial F}{\partial x_2} \Delta x_2 + \frac{\partial F}{\partial y_2} \Delta y_2 + \frac{\partial F}{\partial x_3} \Delta x_3 + \frac{\partial F}{\partial y_3} \Delta y_3\]

\[...........................(3.55)\]

where \( F^0 \) is calculated from equation (3.52) using \( X^0 \).

Also, all the partial derivatives in equations (3.54) are calculated using \( X^0 \).

### 3.2.7.4 Line-Curve Tangency

![Diagram of line-curve tangency](image.png)

**FIGURE 3.9:** Geometry of the line-curve tangency condition.

Figure 3.9 illustrates the geometry of the line-curve tangency condition. If the line 1-2 meets tangentially with the circular curve 2-3, then, the slope of the line 1-2 and the slope of the tangent to the curve at point 2 must be the same. The same situation also applies to curve 2-3 and line 3-4 at point 3. For simplicity, only the derivation of the tangency condition at point 2 will be given here:

Slope of line 1-2 = Slope of the tangent to the curve at 2

\[\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{-x_2 - x_3}{y_2 - y_3},\]

which can be re-arranged as:

\[(y_2 - y_1)(y_2 - y_3) + (x_2 - x_1)(x_2 - x_3) = 0 \quad ...........................(3.56)\]
Substitute the values of \(x_e\) and \(y_e\) from equation (3.48) into (3.56):

The mathematical model for the line-curve tangency is:

\[
F = \frac{(y_2 - y_1)(y_2 - y_3 - C_y(x_2 - x_3)D)}{\hat{u}_s} + \frac{(x_2 - x_1)(x_2 - x_3 - C_x(y_2 - y_3)D)}{\hat{u}_s}
\]

.................................(3.57)

Where \(D = \left(\frac{4R^2}{K^2} - 1\right)^{1/2}\)

\(K = \left[(x_2 - x_3)^2 + (y_2 - y_3)^2\right]^{1/2}\)

\(E_2\) and \(E_4\) are as indicated in equation (3.57)

If the curve is positive \(\Rightarrow C_x = -1, \ C_y = +1\)

If the curve is negative \(\Rightarrow C_x = +1, \ C_y = -1\) (See section 3.2.7.1)

The partial derivatives are:

\[
\frac{\partial F}{\partial x_1} = - E_4
\]

\[
\frac{\partial F}{\partial y_1} = - E_2
\]

\[
\frac{\partial F}{\partial x_2} = E_4 + (y_2 - y_1) \left[ -C_yD + C_y \frac{4R^2(x_2 - x_3)^2}{DL^4} \right] + (x_2 - x_1) \left[ 1 + C_x \frac{4R^2(x_2 - x_3)(y_2 - y_3)}{DL^4} \right]
\]

\[
\frac{\partial F}{\partial y_2} = E_2 + (y_2 - y_1) \left[ 1 + C_y \frac{4R^2(x_2 - x_3)(y_2 - y_3)}{DL^4} \right] + (x_2 - x_1) \left[ -C_xD + C_x \frac{4R^2(y_2 - y_3)^2}{DL^4} \right]
\]
\[ \frac{\partial F}{\partial x_3} = E_4 - \frac{\partial F}{\partial x_2} \]
\[ \frac{\partial F}{\partial y_3} = E_2 - \frac{\partial F}{\partial y_2} \]

........................(3.58)

The observation equation of the line-curve tangency at point 2 is:

\[ -e_p = F^0 + \frac{\partial F}{\partial x_1} \Delta x_1 + \frac{\partial F}{\partial y_1} \Delta y_1 + \frac{\partial F}{\partial x_2} \Delta x_2 + \frac{\partial F}{\partial y_2} \Delta y_2 + \frac{\partial F}{\partial x_3} \Delta x_3 + \frac{\partial F}{\partial y_3} \Delta y_3 \]

........................(3.59)

where \( F^0 \) is calculated for equation (3.57) using \( X^0 \).

Also, all the partial derivatives in equation (3.58) are calculated using \( X^0 \).

3.3 Phase 2: Cadastral Overlay Maintenance (Upgrade and Update)

From the time of spatial data capture, it is essential to keep the data up-to-date. It would also be ideal for the database to be kept free of all known errors, mainly blunders and systematic errors, and contain the best available cadastral information. This involves performing two processes on the database which are upgrading and updating.

Upgrading deals with the correction of errors and the improvement of coordinate accuracy through the use of high quality cadastral data. Updating, on the other hand, involves the modification of the database which is associated with all approved changes to the cadastre. Generally, these relate to the removal (deletion) of one or more parcels and replacement with one or more new parcels, and are typically associated with land subdivision, consolidation and resurvey. This research mainly addresses the upgrade problem.
3.3.1 Upgrading the Accuracy of Digital Cadastral Data

As stated in section 3.1, the upgrading of the spatial cadastral data accuracy will be done through the continuous addition of survey data as portrayed on subdivision plats, deeds and other publicly recorded documents, as well as field survey data as they become available. These data include distances, control point measurements, GPS vectors between parcel corners, angles, as well as azimuths. The coordinates initially adjusted using the geometric constraints will be subsequently re-adjusted to fit the new survey data which are assumed to have higher quality than the coordinates, in general.

At least, two approaches can be followed for upgrading the accuracy of digital cadastral data. These are:

1) Performing a simultaneous adjustment of the digitized coordinates using both the geometric constraints and new survey data. This requires the accumulative storage of the geometric constraints and survey data which will result in a huge database which is difficult to manage. Moreover, it involves an unjustifiable repetition of the calculation and building of the normal equations matrices of equation (3.15), resulting in an unnecessary waste of resources.

2) Performing a sequential adjustment of the coordinates resulting from previous adjustments using only the new survey data at the time of upgrading and the normal equations matrix saved from the previous adjustment. This approach is computationally less intensive and is preferred over the first approach, and thus is used in this research. The next sections will give the mathematical background behind this approach.
3.3.2 Sequential Adjustment

The derivation of the appropriate formulae from a computer programming perspective will start here with equation (3.15) which can be rewritten as:

\[ \hat{X} = \hat{X}^* - N_i^1A_2^T(P_{2^1} + A_2N_i^1A_2^T)^{-1}(A_2\hat{X}^* + L_2) \] .................(3.60)

where the subscripts 1 and 2 refer to the first and second sets of observations, respectively.

\[ \hat{X}^* = -N_i^1U_1, \] the solution if only the first set of observations was made.

If \( N_i^1 \) and \( \hat{X}^* \) of the previous adjustment were stored, equation (3.60) works well enough to find the final solution \( \hat{X} \) after adding the second set of observations. However, for the case of cadastral maps, storing \( N_i^1 \) for all the points in one map (700 points on the average) requires a storage space of approximately 4 MB using single precision. This is a forbidding task, especially when all the maps of a county are involved. Therefore, alternatives to storing \( N_i^1 \) were sought. Experimentation has been done using two different approaches:

1) Fitting a mathematical function through the elements of \( N_i^1 \) so that these elements can be derived from this formula for subsequent adjustments. High degree polynomials were tried and failed to produce satisfactory results.

2) Performing spectral analysis on \( N_i^1 \). This was found to be extremely time consuming and subject to human judgment concerning the number of harmonics to be used. For more details about these two approaches, the reader is referred to Davis (1986).

As a result, a third more appropriate alternative was reached by modifying equation (3.60).
Using equations (39) and (40) of UME (Uotila, 1986, p.160),
\[ (P^2_1 + A_2 N_1^1 A^T_2 P_2) = P_2 (I + A_2 N_1^1 A^T_2 P_2)^{-1} \]
Substitute this equality into (3.60)
\[ \Rightarrow \hat{X} = \hat{X}^* - N_1^1 A^T_2 P_2 (I + A_2 N_1^1 A^T_2 P_2)^{-1} (A_2 \bar{X}^* + L_2) \] ........................ (3.61)

Again, using equations (42) and (43) of UME (Uotila, 1986, p.160),
\[ N_1^1 A^T_2 P_2 (I + A_2 N_1^1 A^T_2 P_2)^{-1} = (I + N_1^1 A^T_2 P_2 A_2) N_1^1 A^T_2 P_2 \]
Substitute this equality into (3.61) and put \( N_2 = A^T_2 P_2 A_2 \),
\[ \Rightarrow \hat{X} = \hat{X}^* - (I + N_1^1 N_2) N_1^1 (N_2 \bar{X}^* + A^T_2 P_2 L_2) \] ........................ (3.62)

Also, from equation (29a) of UME (Uotila, 1986, p.159),
\[ (I + N_1^1 N_2)^{-1} N_1^1 = (N_1 + N_2)^{-1} \]
Substitute the above equality into (3.62), and introduce \( U_2 = A^T_2 P_2 L_2 \),
\[ \Rightarrow \hat{X} = \hat{X}^* - (N_1 + N_2) N_1^1 (N_2 \bar{X}^* + U_2) \] ........................ (3.63)

Equation (3.63) provides the mechanism for the upgrading process followed in this research. Compared to equation (3.60), it can be seen that \( N_1 \) from the previous adjustment needs to be stored rather than \( N_1^1 \). \( N_1 \) is a symmetric and extremely sparse matrix in the case of cadastral maps, and only half of the non-zero elements needs to be stored. This can save as much as 90% of the storage needed for the \( N_1^1 \) matrix.

For simplicity, the equations for \( \sigma_o^2 \), \( V^TPV \), and \( \hat{X} \) will be given without derivation:
\[
V^TPV = V^TPV^* + U_1^T (\hat{X} - \hat{X}^*) + U_2^T \hat{X} + L_2^T  \] ........................ (3.64)
\[
\sigma_o^2 = \frac{V^TPV}{r} \] ........................ (3.65)
\[ \hat{\mathbf{x}}^* = \sigma_x^2 (N_1 + N_2)^{-1} \]  \hspace{1cm} (3.66)

where \( \mathbf{V}^* \) = Residuals vector resulting from the previous adjustment.

\( r = \text{redundancy} = \text{summation of the total number of observations in the current and previous adjustments excluding the digitized coordinate observations.} \)

Note: In equation (3.63), if the coordinates resulting from the prior adjustment are used to calculate \( N_2 \) and \( A_2 \) rather than \( X^0 \), then \( \hat{X}^* \) can be considered to be zero.

To illustrate how the geometric constraints used in the initial adjustment work together with the new survey data employed to upgrade the parcel corner coordinates through a sequential adjustment using equation (3.63), let us consider the simple block in Figure 3.10. Assume that this block has been initially adjusted using the line-curve tangency constraints at points 1, 5, 6, 8, 9, 13, 14 and 16, as well as all the 90° and 180° angle constraints which can be derived from this block, and that the resulting \( N_1 \) matrix has been stored. Assume also that after this adjustment, distances 4-19, 3-18 and 2-17 were each equal to 58 ft and that distances 10-19, 11-18 and 12-17 were each equal to 62 ft. If later, distances 4-19 and 10-19 were both measured and found to be 60 ft each, then

\[ \begin{array}{cccccc}
13 & 12 & 11 & 10 & 9 & 8 \\
5 & 17 & 18 & 19 & 7 & \\
6 & 1 & 2 & 3 & 4 & 5 \\
\end{array} \]

**FIGURE 3.10:** Geometry of a simple block of parcels.
utilizing these values in a sequential adjustment using equation (3.63) and appropriate weights will result in all distances 4-19, 10-19, 3-18, 11-18, 2-17 and 12-17 being approximately equal to 60 ft. The word approximately is used here due to the random errors which are difficult to eliminate completely. Moreover, this will also result in almost equal distances 6-7, 7-8, 14-15 and 15-16.

3.3.3 Correlation between Survey Data

When performing the sequential adjustment, the correlation between the incoming new survey data can be ignored for the following reasons:

1) Most of the incoming measurements are taken from deeds or subdivision plats which usually do not carry the correlation or even the variance data.

2) For survey data which are coming from actual field surveys, land parcels are usually surveyed at different times (except subdivisions) and often by different surveyors. This means that angles at a single node, 17 for example in Figure 3.10, will not be correlated. This applies also to other kinds of measurements like distances.

3) The complexity of structuring the measurements database, as well as the large computer space required in the random access memory (RAM) of the computer to store the weight matrix $P$, which makes programming more difficult.

However, if correlation was available and needs to be taken into consideration, then pre-processing of the data will simplify the problem. This is done using the homogenization technique which can be described as follows (Koch, 1988, pp. 183-184):

Given the observation vector $L^b$, where:

$$E[L^b] = AX^b, \quad \sum_{d_b} = \sigma^2 P^{-1}.$$
Use the Cholesky decomposition to break the weight matrix $P$ into lower and upper triangular matrices $G$ and $G^T$,

$$P = G G^T$$  \hspace{1cm} \text{(3.67)}

Construct a new observations vector $L_0$ to replace $L_b$, such that:

$$L_0 = G^T L_b$$  \hspace{1cm} \text{(3.68)}

Equations (3.69) and (3.70) are easier to handle in an adjustment computation program, because the large weight matrix $P$ has been reduced to a diagonal matrix with the same constant value $\sigma^2$ on its diagonal.

This chapter has dealt with the design and study of the mathematical background of a more appropriate approach for building a digital cadastral overlay. The next chapter will deal with the cadastral data acquisition and organization necessary for the implementation of this approach.
CHAPTER IV
ACQUISITION AND ORGANIZATION OF CADAstral DATA

4.1 Introduction

Tax maps, and cadastral maps in general, can be viewed as a network of short intersecting line segments, which could be straight or curvilinear. These line segments, in turn, define the "skeleton" of parcel boundaries. A digital representation of a cadastral map can then be created by assigning coordinates to the points of intersection (parcel corners) and establishing the topological relationships between the line segments which join these points. Assigning coordinates is a result of a digitization process, while topological relationships are established either manually or through the digitizing process if the software has this capability.

4.2 A Numbering system for Parcel Boundary Corners

A parcel corner can be defined as "a point of change of direction of the boundary of real property. It may be marked by a monument, fence, or other physical object, or it may not be marked at all" (Brown, 1969, p.132). A land parcel is defined as "the smallest registered unit of land, continuous in both area and ownership, and capable of being separately conveyed" (Sedunary, 1983). The term "block" will be used here to indicate any aggregate of adjoining parcels which are surrounded by streets on all sides. Figure 4.1 shows two examples of such blocks, 1 and 2.
FIGURE 4.1: Relationship between parcel corners, parcels and blocks.

A significant element in the application of the methodology outlined in Chapter III, is the design and assignment of parcel corner identifiers (PCID) for accessing, processing and linking the data. This parcel corner identification system should exhibit the following characteristics:

1) **Uniqueness.** Uniqueness refers to a one-to-one relationship between parcel corners and their identifiers.

2) **Permanence.** An identifier should not be changed unless there is a change in the boundaries of the parcel to which it is assigned.

3) **Simplicity.** Corner identifiers should be easy to understand, use and process in computers.

One parcel corner identification system which achieves these characteristics and which proved to work very well for this methodology is the sequential numbering
system. Under this system, each corner is assigned an abstract numeric hierarchical number which is composed of two parts:

1) A left integer string of one or more digits which represents the block number where that corner is located.

2) A right integer string of up to three digits that represents a random serial number assigned to the corner. Several tax maps have been inspected and found to have much fewer than 1000 corners per block. This makes three digits sufficient for introducing a unique parcel corner identification number (PCID).

Examples of PCIDs in Figure 4.1 are shown in Table 4.1:

<table>
<thead>
<tr>
<th>Block</th>
<th>Corner</th>
<th>PCID</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1002</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>1014</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>2008</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>2011</td>
</tr>
</tbody>
</table>

One important issue in this numbering system is that blocks crossing the boundary between two or more adjoining maps should be given the same number. This will facilitate the process of joining and edge matching individually processed maps as will be explained in Chapter VI.

The identification system outlined above can be expanded to include three parts instead of two, where the third part represents the map number. By limiting the block numbers in each map to two digits (since it is unlikely that more than 99 blocks per map
can exist), keeping identical numbers for those blocks which cross the map boundary, the structure of the PCID becomes:

map# - block# - serial corner#

Example: The PCID for corner 103 in block 8 of map 98 will be 9808103.

4.3 Digitization of Cadastral Maps

Digitization is the process of converting existing graphical documents into digital form. This may be done in different ways with the aid of various instruments, ranging from a simple manual digitizer to an automatic scanner. Regardless of the method used for digitization, the end result is assumed to be a file listing the machine coordinates x and y of discrete parcel corners contained in each map.

The basic difficulty in the digitization of cadastral maps using the digitizing software currently available in the market, is the inability of this software to build topology which is explicitly known to the user. For example, during the digitization process, the user should be able to assign the PCIDs which have been introduced in the numbering process, rather than the automatic numbers assigned by the digitizing software, and which exist in internal tables that are difficult to be accessed and understood by the user. The reason for this, is that the adjustment program requires the PCIDs of those corners which form the boundary of a particular parcel, so that their coordinates can be easily accessed. Several brands of digitizing software and GIS packages including ARC/INFO, GeoVision and Roots have been examined and found to lack this capability.
In order to overcome this problem, Doytsher and Shmutter (1984) suggested three procedures for the digitization process:

a) "Pick-up" of discrete points accompanied by manually prepared topology files which establish the relationships between points and lines.

b) "Pick-up" of sequences of points forming closed polygons, followed by data identifying each polygon.

c) "Pick-up" of individual segments of lines between junctions with subsequent definition of the lines.

All three methods are based on the interaction between man and machine which leaves room for blunders, mainly in the manual preparation of topology files. Therefore, until software which addresses this problem is developed, careful consideration should be given to the construction of these topology files.

Doytsher and Shmutter (1984) preferred alternative (b) over the two other choices because it requires less effort in the manual preparation of topology files. However, this alternative causes an increase in the volume of digitization, for a point lying on a line shared by several parcels is being digitized more than once. This, in turn, requires preprocessing of the digitized data to reduce the several pairs of coordinates associated with a point to one single pair.

Alternative (a) seems to be the most promising, in my opinion, and has been used throughout this research. Although it involves the manual preparation of polygon topology files which makes it subject to human errors, some checks can still be performed to limit this problem and uncover any mistakes at an early stage. A computer program has been developed to run a check on the topology file as well as the digitized
coordinates. This program reads the polygon topology and digitized coordinates file and generates a drawing file in a SCRIPT format which can be read by AUTOCAD to produce a graphic drawing. This drawing can be compared visually against the original paper map to detect any possible blunders. The next section will explain the structure of the polygon topology file.

4.4 Polygon Topology

Topology is concerned with establishing the location of objects, identified by points, lines, polygons and surfaces, with respect to each other in a non-metric relational structure (Dale and McLaughlin, 1988). The identification of topological relationships is needed in this methodology for several purposes, which include: the automatic generation of angle and tangency constraints, the adjustment of areas and the construction of the drawing files explained in the previous section. The following is a description of the structure of the parcel polygon topology file used here:

1) Each cadastral map has its own parcel polygon topology file.
2) Each file consists of records, where each record represent one parcel.
3) Each record consists of the following fields:
   - Block number
   - Parcel number
   - An integer (n) representing the total number of corners and curves radii forming the parcel
   - Listing of the parcel PCIDs and radii in a counterclockwise direction.
   (A PCID which is a beginning of a circular curve is given a negative sign, so is the radius of a negative curve.)

Table 4.2 shows the polygon topology file structure for block 3 in Figure 4.2.
FIGURE 4.2:  Geometry of a typical cadastral block

TABLE 4.2  
An example of a polygon topology file structure

<table>
<thead>
<tr>
<th>Block #</th>
<th>Parcel #</th>
<th>Number of entries (n)</th>
<th>Listing of the PCIDs and curve radii in a counterclockwise direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>100</td>
<td>6</td>
<td>1  2  17  15 -16  20.0</td>
</tr>
<tr>
<td>3</td>
<td>101</td>
<td>6</td>
<td>2 -3 25.0 4 5 17</td>
</tr>
<tr>
<td>3</td>
<td>102</td>
<td>5</td>
<td>-5 -50.0 6 18 17</td>
</tr>
<tr>
<td>3</td>
<td>103</td>
<td>5</td>
<td>-6 -50.0 7 19 18</td>
</tr>
<tr>
<td>3</td>
<td>104</td>
<td>4</td>
<td>7  8  9 19</td>
</tr>
<tr>
<td>3</td>
<td>105</td>
<td>4</td>
<td>9 10 11 19</td>
</tr>
<tr>
<td>3</td>
<td>106</td>
<td>4</td>
<td>11 12 18 19</td>
</tr>
<tr>
<td>3</td>
<td>107</td>
<td>4</td>
<td>12 13 14 18</td>
</tr>
<tr>
<td>3</td>
<td>108</td>
<td>4</td>
<td>14 15 17 18</td>
</tr>
</tbody>
</table>

As can be seen from Table 4.2, the PCID 3 of parcel 101 was given a negative sign because it is the beginning point of a curve, so is 16 for parcel 100, 5 for parcel 102 and 6 for parcel 103. Even though 6 was given a negative sign because it is a beginning point for parcel 103, it was given a positive sign for parcel 102 because it is considered an end point of curve. Radii 20' and 25' were given positive signs because they are the radii of positive curves which add to the areas of parcels 100 and 101. The radius 30'
was given a negative sign because it is the radius of a negative curve which subtracts from the areas of parcels 102 and 103.

This polygon topology file is later read by a computer program which derives a line topology file. For the polygon topology file shown in Table 4.2, the following line topology file (Table 4.3) is produced:

**TABLE 4.3**  
An example of a line topology file structure

<table>
<thead>
<tr>
<th>From point</th>
<th>To point</th>
<th>Radius (if the line segment is a curve)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3001</td>
<td>3002</td>
<td></td>
</tr>
<tr>
<td>3002</td>
<td>3017</td>
<td></td>
</tr>
<tr>
<td>3017</td>
<td>3015</td>
<td></td>
</tr>
<tr>
<td>3015</td>
<td>-3016</td>
<td></td>
</tr>
<tr>
<td>-3016</td>
<td>3001</td>
<td>20.0</td>
</tr>
<tr>
<td>END</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3002</td>
<td>-3003</td>
<td></td>
</tr>
<tr>
<td>-3003</td>
<td>3004</td>
<td>25.0</td>
</tr>
<tr>
<td>3004</td>
<td>3005</td>
<td></td>
</tr>
<tr>
<td>3005</td>
<td>3017</td>
<td></td>
</tr>
<tr>
<td>END</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3004</td>
<td>3006</td>
<td>-50.0</td>
</tr>
<tr>
<td>3006</td>
<td>3018</td>
<td></td>
</tr>
<tr>
<td>3018</td>
<td>3017</td>
<td></td>
</tr>
<tr>
<td>END</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3006</td>
<td>3007</td>
<td>-50.0</td>
</tr>
<tr>
<td>3007</td>
<td>3019</td>
<td></td>
</tr>
</tbody>
</table>
4.5 Transformation of Digitized Coordinates into Real World Coordinates

The coordinates resulting from the digitization process are related to the reference frame of the digitizer. In order to be usable for later processing, these coordinates should be transformed to the state plane coordinate system (in the USA), or to any general reference coordinate system which covers the area under consideration.

Let \( P_1, P_2, \ldots, P_n \) be a group of points whose real world coordinates \( X_i \) and \( Y_i \) are known with respect to the chosen coordinate system. If the coordinates of these points relative to the reference frame of the digitizer are \( x_i \) and \( y_i \), then the relation between the coordinates in both systems can be represented by a linear affine transformation as follows:

\[
X_i = A_0 + x_i A_1 + y_i A_2 \\
Y_i = B_0 + x_i B_1 + y_i B_2
\] ................................. (4.1)

Where \( A_0, A_1, A_2, B_0, B_1 \) and \( B_2 \) are coefficients which take into account the effect of scaling, rotation, translation, and shear between the two systems. Substituting the coordinates of points \( P_1, P_2, \ldots, P_n \) into equations (4.1) yields a system of observation equations which can be solved by the adjustment computation technique to determine the coefficients \( A_i \) and \( B_i \). Three points \( P_1 \) will lead to an exact solution for the six coefficients, however, more than three points is usually recommended to provide for an independent check. Knowing the coefficients \( A_i \) and \( B_i \), the machine coordinates of all the digitized map points can be transformed into world coordinates (ft or m) by substituting them into equations (4.1).

Figure 4.3 portrays in a general way the flow of the processes explained in chapters III and IV.
Edit tax maps:
- Assign block numbers
- Assign parcel corner identifiers
- Get missing curve radii
- Locate control or coordinated points

Digitize tax maps in point mode

Perform affine transformation

Transformed (World) coordinates

Derive and build geometric constraint files which include: distance (street widths), area, angle, and tangency constraints

Available survey data

New survey data

Adjusted coordinates

Build the parcel polygon topology file

Derive line topology

Generate drawing files

Graphic representation of the cadastral overlay

FIGURE 4.3: Sequence of operations for developing a digital cadastral overlay
CHAPTER V

METHODOLOGY TESTING

Research has been done to study the performance and feasibility of the approach outlined in chapters III and IV on both simulated as well as real tax maps. This chapter provides the findings of this research.

5.1 Choice of Sample Maps for Testing

Two main criteria needed to be fulfilled when choosing sample maps for testing. These are:

1) The maps should possess a geometric configuration which resembles the geometry of a representative tax map.

2) These areas covered by these maps should have accurate and consistent survey data (like subdivision plats and deed descriptions) which can be used to derive accurate coordinates for parcel corners to facilitate the comparison of the digitized and adjusted coordinates, and hence measure the amount of numerical accuracy improvement which has been achieved.

In order to fulfill these criteria, two samples have been chosen. These are:

a) A simulated tax map (Figure 5.1): This is a hypothetical parcel map whose dimensions (coordinates, lengths and bearings) are correctly known (free of error). This map has been manually drawn to a scale of 1:3000 and then digitized...
FIGURE 5.1: Geometry of a simulated tax map

using the ARC/INFO digitizing software. The resulting coordinates are expected to have drafting and digitization errors.

b) Real tax maps: Two adjoining representative tax maps for Cross Creek village in Franklin County, Ohio were used. The most accurate survey data for the area covered by these two maps were obtained from available subdivision plats. These tax maps were then digitized using ARC/INFO to derive digitized coordinates which are expected to be in error due to the digitization process, drafting, instability of the paper maps, etc.
5.2 Accuracy Improvement Indicators

In order to measure the numerical accuracy improvement of the adjusted coordinates as compared to the digitized coordinates, three statistics have been used. These are:

1) **Percentage of improved points (P_i)**. This is defined as the ratio of the total number of points which experienced accuracy improvement (N_i) to the total number of adjusted points in the map (N). This can be represented mathematically as:

\[ P_i = \frac{N_i}{N} \times 100\% \]  \hspace{1cm} \text{(5.1)}

A point is said to have gained accuracy if the error in the adjusted coordinates as compared to the correct coordinates is less than the error in the digitized coordinates, or mathematically:

\[ \left[ (x_{c} - x_{d})^2 + (y_{c} - y_{d})^2 \right]^{1/2} < \left[ (x_{a} - x_{c})^2 + (y_{a} - y_{c})^2 \right]^{1/2} \]  \hspace{1cm} \text{(5.2)}

where \( x_{c}, y_{c} \) = the correct point coordinates
\( x_{a}, y_{a} \) = the adjusted point coordinates
\( x_{d}, y_{d} \) = the digitized point coordinates

2) **Average point error (E_{avg})**. This is defined as:

\[ E_{avg} = \frac{\sum_{i=1}^{n} \left[ (x_{i} - x_{a_{i}})^2 + (y_{i} - y_{a_{i}})^2 \right]^{1/2}}{n} \]  \hspace{1cm} \text{(5.3)}

where \( x_{a_{i}}, y_{a_{i}} \) = correct coordinates of point \( i \)
\( x_{i}, y_{i} \) = could be the digitized or adjusted coordinates of point \( i \).

3) **Point standard error (\sigma)**. This is defined as:

\[ \sigma = \sqrt{\frac{\sum_{i=1}^{n} \left[ (x_{i} - x_{a_{i}})^2 + (y_{i} - y_{a_{i}})^2 \right]}{n}} \]  \hspace{1cm} \text{(5.4)}
The average and standard errors are calculated for both the digitized and adjusted coordinates, and then compared.

5.3 Choice of Control Points

Any accurate mapping project usually requires the establishment of a system of survey control which consists of a network of points whose horizontal and vertical positions and interrelationships have been accurately established by field surveys. In digital cadastral mapping, control points are needed to serve two main functions:
1) Transformation of the digitized machine coordinates into real world coordinates.
2) Providing a framework for the adjustment, positioning and checking of cadastral map details.

5.3.1 Optimum Number of Control Points

When choosing the number of control points required in a single cadastral map, the following factors need to be considered:
1) The number of control points per map should be kept low to reduce the total cost and effort of coordinating these points.
2) Three control points at least are needed in each map to solve for the affine transformation parameters, however, more than three points is recommended to provide for an independent check.
3) The number of control points should be sufficient to ensure an improvement in accuracy when adjusting the transformed coordinates, and to facilitate joining the bordering maps.
4) Control points should be well distributed throughout the area concerned to facilitate the collection of survey data when needed. The National Research Council Panel on Procedure and Standards for a Multipurpose Cadastre (NRC,
1983, p. 24) recommended the following spacing between control points for cadastral mapping purposes:

- **Urban areas:** 0.2 to 0.5 miles (0.3 to 0.8 km)
- **Rural areas:** 1.0 to 2.0 miles (1.6 to 3.2 km)

When studying the preceding four factors, it was found that four to six control points per map is most suitable. Four points are good enough for maps covering an area of around 1500 x 2000 ft², while six points are recommended for maps covering larger areas. This conclusion was also supported by the results shown in Table 5.1 in the next section which addresses the location and distribution of these points.

### 5.3.2 Location and Distribution of Control Points

Several factors control the location and distribution of control points over the cadastral map area. These include:

1. **The points should be sufficiently close to the boundary of the map.** This will serve two purposes:
   a. Control points will be shared by the adjoining maps which will minimize the total number of required control points which, in turn, will reduce the total cost.
   b. When performing the adjustment on the transformed coordinates, this will minimize the amount of error in the adjusted coordinates of the boundary points and, hence, reduce the variation between the coordinates of the same points appearing on two adjoining maps. This, in turn, will facilitate the edge matching of these maps.

2. **Control points should be located at the edges of blocks, mainly, near street intersections.** This location is preferred for measurement and maintenance
purposes and results in better accuracy improvements than when using other locations. Figure 5.1 shows typical locations for four control points 1, 2, 3 and 4.

In the case of six points, research has been done to find the configurations which optimize the accuracy improvement resulting from the adjustment. Five configurations (Figure 5.2) have been studied on both simulated and real tax maps. Table 5.1 shows the amount of accuracy improvement which resulted from the application of geometric constraints only on the digitized coordinates using the respective configurations of Figure 5.2.

![Figure 5.2: Different configurations for the distribution of control points.](image)
### TABLE 5.1
Effect of the adjustment improvement using the control point configurations of Figure 5.2.

<table>
<thead>
<tr>
<th>Control points configuration</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accuracy improvement indicators</td>
<td>%*</td>
<td>E\textsubscript{i}**</td>
<td>σ\textsubscript{j}***</td>
<td>%</td>
<td>E\textsubscript{i}</td>
</tr>
<tr>
<td>Simulated data (rectangular area)</td>
<td>98.6</td>
<td>1.78</td>
<td>1.88</td>
<td>98.6</td>
<td>1.90</td>
</tr>
<tr>
<td>Real data (rectangular area)</td>
<td>76.9</td>
<td>0.85</td>
<td>1.03</td>
<td>83.6</td>
<td>1.15</td>
</tr>
<tr>
<td>Real data (square area)</td>
<td>76.9</td>
<td>0.86</td>
<td>1.04</td>
<td>84.5</td>
<td>1.32</td>
</tr>
</tbody>
</table>

* % = Percentage of improved points.
** E\textsubscript{i} = Average error before minus after adjustment (in ft)
*** σ\textsubscript{j} = Standard error before minus after adjustment (in ft)
As can be seen from the results in Table 5.1, the two configurations (b) and (c) gave the best results in the three data samples (one square area and two rectangular areas). A justification for this result can be sought by analogy between the behavior of geodetic networks and structural elements. Jäger (1989) noticed that the deflection pattern of residuals in geodetic networks is analogous to the stiffening of structural elements, like beams and trusses, in the mechanical sense. To simplify the problem, let us assume a structure composed of four beams which are connected at the fixed supports 1, 2, 3 and 4 (Figure 5.3). If beam 1-2 is subjected to uniform loading, the maximum deflection will be experienced by point p (Figure 5.3b). The effect of strengthening the structure by the dashed elements and the fixed supports 5 and 6 will be the reduction of the deflection at point p and the result of a more uniform deflection pattern which looks

![Diagram](image)

**FIGURE 5.3:** Analogy between structural elements and geodetic networks.
like that of Figure 5.3c. Supports 5 and 6 will act as spring supports for the beam 1-2 especially if other minor structural elements were to connect beam 1-2 with beams 2-5, 5-6 and 6-1 resulting in a truss-like structure, which resembles a cadastral map structure. This also applies to beams 2-3, 3-4 and 4-1.

If a geodetic network of similar shape to that of Figure 5.3a exists, and assuming that line 1-2 was subjected to a strain (example: distance error), the maximum error is expected to be experienced by points located in the vicinity of point p. Introducing two control points at locations 5 and 6 will ensure more control on the cadastral map and result in a better accuracy and more uniform error distribution all over the map.

5.4 Determination of Geometric constraint weights

As stated in section 3.2.2, geometric constraints were first treated as fixed conditions which are free of error and carry an infinite weight. This resulted in an unstable and ill-conditioned system of equations which could not easily be solved. The linear dependency between the constraints was identified using Cholesky decomposition, which resulted in zero elements in the diagonal Cholesky matrix for those constraints which are linearly dependent on other constraints. These identified constraints were excluded from further calculations, but there was no easy way to get rid of the ill-conditioning even with double precision calculations. The alternative was to relax the constraints to carry the possibility of some error by giving them a proper weight instead of infinite weight. This resulted in what is called stochastic constraints (Schaffrin, 1990) or "pseudo-observations". The question which needed to be answered then was: What is the best combination of weights for the constraints to optimize the adjusted coordinate accuracy? The following sections will address this question and report the results which have been achieved.
5.4.1 Approach for Determining Geometric Constraint Weights

Seven kinds of geometric constraints were identified in this study. These are:

1) Control points
2) Distances (e.g., street widths)
3) Angles which are multiples of 90°
4) Angles which are not multiples of 90°, such as 45° and 135°
5) Areas
6) Line-curve tangency, and
7) Curve-curve tangency.

The sensitivity of the adjusted coordinates to these types of constraints depends on the certainty (level of confidence) of these constraints. This can be explained as follows:

1) Control points: these points are usually measured or calculated with highly accurate procedures and are expected to provide the supporting framework for the adjustment process. Therefore, they are expected to carry the highest weight.

2) Distances: since the distance constraints (e.g., street widths) are user specified and not derived from the digitized coordinates like the angle constraints, they are expected to carry the next highest weight due to their high certainty.

3) Angles: angle constraints are, in most cases, automatically detected or derived by calculating all angles that are within a user-specified tolerance from any of the 45° multiples. Because of the angular tolerance range, some angles might be constrained to a value, say 90°, while in reality they are different from that (e.g. 89.5°). As a result, angle constraints are expected to have less weight than distance constraints and control points.
4) Areas: two aspects about areas are:
   a. They are computationally dependent on angles and distances (Brown, 1969, p.167). Therefore, they are expected to have less priority or importance than both angles and distances.
   b. Areas appearing on land deeds sometimes carry a "more or less" statement which indicates a low level of certainty. Moreover, these areas are usually given in acres which is of very low resolution when dealing with ft². These two factors suggest that area constraints should have lower weights than the preceding constraints.

5) Line-curve tangency: to assure smooth continuity, lines and curves are usually constructed so that they will be tangent to each other. Therefore, this kind of constraint is expected to have a fairly high weight.

6) Curve-curve tangency: sometimes curves might meet without being completely tangent to each other, although they meet smoothly and seem to be tangent to each other. This, in turn, suggests that curve-curve tangency constraints will have lower weights than the line-curve tangency constraints.

These arguments about the weights of geometric constraints were indeed supported by actual adjustments performed on the digitized coordinates of both simulated and real data. To explain the process of weight determination, refer to equation (5.5) which expresses the accuracy of adjusted coordinates (A) as a function of the constraint weights:

\[ A = F(W_{CP}, W_D, W_{a90}, W_{at}, W_{AR}, W_{LC}, W_{CC}) \] \hspace{1cm} (5.5)

where
\[ W_{CP} \] = weight of control points
\[ W_D \] = weight of distance constraints
\[ W_{a90} \] = weight of angle constraints which are multiples of 90 degrees
$W_{ot}$ = weight of other angle constraints (e.g., 45° and 135°)
$W_{ar}$ = weight of area constraints
$W_{lc}$ = weight of line-curve tangency constraints
$W_{cc}$ = weight of curve-curve tangency constraints

Initial approximate weights $W^o$ were chosen for these constraints. The adjustment was then performed by fixing $W^o$ for all the constraints except for the control points whose weight was varied over a range of 100 to 90000 ft$^3$. The accuracy improvement was then plotted versus the weight of control points (Figure 5.4), and the weight which gave the best accuracy was chosen (3600 ft$^3$ in this case). In order to determine the weight for distance constraints, the process was repeated by fixing the already determined weight of the control points and $W^o$ of the other constraints, and changing the weight of the distance constraints over a range of 36 to 900 ft$^3$. The accuracy improvement was again plotted versus the weight of distance constraints (Figure 5.5), and the weight which gave the best accuracy improvement was selected. The weight of angles which are multiples of 90° was determined by fixing the already determined weights of the control points and distance and $W^o$ of the rest of the constraints, and changing the weight of the angle constraints (Figure 5.6). The process continued in this sequence until the weights of all the constraints were determined. Table 5.2 summarizes the finally chosen values of these geometric constraint weights, while figures 5.4 through 5.9 give the details of the selection process. Angle constraints which are not multiples of 90° were excluded here because very few of these constraints existed which made it difficult to measure their effect. However, they were given a weight of $4.25 \times 10^4$ radians$^2$ which is equivalent to a standard error of 10".
### TABLE 5.2
Weights of the geometric constraints

<table>
<thead>
<tr>
<th>Constraint type</th>
<th>Weight</th>
<th>Equivalent σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Control points</td>
<td>$3600.0 \text{ ft}^2$</td>
<td>±0.017 ft = ±0.5 cm</td>
</tr>
<tr>
<td>2. Distances (like street widths)</td>
<td>$100.0 \text{ ft}^2$</td>
<td>±0.100 ft = ±3.0 cm</td>
</tr>
<tr>
<td>3. Angle (multiples of 90°)</td>
<td>$10.0 \text{ ft}^4$</td>
<td>±0°00'15&quot;</td>
</tr>
<tr>
<td>4. Areas</td>
<td>$0.01 \text{ ft}^4$</td>
<td>±10.0 ft$^2$</td>
</tr>
<tr>
<td>5. Line-curve tangency</td>
<td>$10.0 \text{ ft}^4$</td>
<td>±0°00'15&quot;</td>
</tr>
<tr>
<td>6. Curve-curve tangency</td>
<td>$0.10 \text{ ft}^4$</td>
<td>±0°03'30&quot;</td>
</tr>
</tbody>
</table>
- Weight of control points:

![Graph showing accuracy improvement versus weight of measured control points weight]

**FIGURE 5.4:** Accuracy improvement versus weight of measured control points

Distance weight = 100.0 ft²
Area weight = 1.0 ft⁴
Angles (90° multiples) weight = 10.0 ft⁴
Angles (different from 90° multiples) weight = 4.25x10⁴ radians²
Line-curve tangency weight = 10.0 ft⁴
Curve-curve tangency weight = 0.10 ft⁴
- Weight of distance constraints:

![Graph A] Percentage of points which gained accuracy versus distance weight

![Graph B] Amount of accuracy and standard error (RMSE) improvement versus distance weight

FIGURE 5.5: Accuracy improvement versus weight of distance constraints.

Measured control points weight = 3600.0 ft²
Area weight = 1.0 ft⁻¹
Angles (90° multiples) weight = 10.0 ft⁻¹
Angles (different from 90° multiples) weight = 4.25x10⁴ radians⁻³
Line-curve tangency weight = 10.0 ft⁻¹
Curve-curve tangency weight = 0.10 ft⁻¹
- Weight of angle constraints:

![Graph A](image)

A) Percentage of points which gained accuracy versus angles (90 multiples) weight

![Graph B](image)

B) Amount of accuracy and standard error (RMSE) improvement versus angles (90 multiples) weight

**FIGURE 5.6:** Accuracy improvement versus weight of angle constraints (multiples of 90°)

- Measured control points weight = 3600.0 ft²
- Distance weight = 100.0 ft³
- Area weight = 1.0 ft⁴
- Angles (different from 90° multiples) weight = 4.25x10⁸ radians²
- Line-curve tangency weight = 10.0 ft⁴
- Curve-curve tangency weight = 0.10 ft⁴
- Weight of area constraints:

![Graph A: Percentage of points which gained accuracy versus area weight](image)

![Graph B: Amount of accuracy and standard error (RMSE) improvement versus area weight](image)

FIGURE 5.7: Accuracy improvement versus weight of area constraints

Measured control points weight = 3600.0 ft²
Distance weight = 100.0 ft²
Angles (90° multiples) weight = 10.0 ft⁴
Angles (different from 90° multiples) weight = 4.25x10⁴ radians²
Line-curve tangency weight = 10.0 ft⁴
Curve-curve tangency weight = 0.10 ft⁴
- Weight of line-curve tangency constraints:

![Graph A](image1.png)

A) Percentage of points which gained accuracy versus line-curve tangency weight

![Graph B](image2.png)

B) Amount of accuracy and standard error (RMSE) improvement versus line-curve tangency weight

FIGURE 5.8: Accuracy improvement versus weight of line-curve tangency constraints

Measured control points weight = 3600.0 ft$^2$
Distance weight = 100.0 ft$^2$
Area weight = 0.01 ft$^4$
Angles (90° multiples) weight = 10.0 ft$^4$
Angles (different from 90° multiples) weight = 4.25x10$^8$ radians$^2$
Curve-curve tangency weight = 0.10 ft$^4$
- Weight of curve-curve tangency constraints:

FIGURE 5.9: Accuracy improvement versus weight of curve-curve tangency constraints

Measured control points weight = 3600.0 ft²
Distance weight = 100.0 ft²
Area weight = 0.01 ft⁴
Angles (90° multiples) weight = 10.0 ft⁴
Angles (different from 90° multiples) weight = 4.25x10⁸ radians²
Line-curve tangency weight = 10.0 ft⁴
5.4.2 Equivalent Standard Error Associated with the Weight of Constraints

If the weight of geometric constraints \((W_i)\) is defined to be the inverse of the square of the standard deviation \(\sigma_i\), then \(\sigma_i\) will be:

\[
\sigma_i = \pm \frac{1}{\sqrt{W_i}} \tag{5.6}
\]

Table 5.2 shows the equivalent \(\sigma\) values associated with the previously determined weights. These values conform very well with existing accuracy standards (e.g., Federal Geodetic Control Committee Standards), and are within the accuracy range which can be achieved using currently available surveying equipment (GPS, total stations, etc.).

The equivalent standard error for control points, distances and areas is easily calculated using equation (5.6). However, for angle and tangency constraints, equation (5.6) results in standard error values in \(\text{ft}^2\) or \(\text{m}^2\). In order to transform these values into seconds of arc, a rigorous mathematical manipulation is needed. Appendix A gives the derivation for the required transformation formulae.

5.5 Influential Analysis

Using the geometric constraint weights as determined earlier, the approach developed in this research (see Chapter III) has been tested on both simulated as well as real tax maps. Two types of accuracy improvement have been observed:

1) Qualitative: This is reflected by the rectification of the geometry of the cadastral overlay. The resulting overlay is aesthetically pleasing and reflects the intended geometric relationships between parcel corners and lines which have been lost due to the various errors in the digitized coordinates, are now restored. Streets
appear with the intended uniform width and straight edges, rectangularity between parcel lines is maintained, lines connect tangentially with curves especially at street intersections, and curves connect smoothly with other curves.

2) Quantitative. This is reflected by the improvement in the positional accuracy of the parcel corner coordinates and is measured by the accuracy improvement indicators discussed in section 5.2, mainly percentage of improved points, the average error and the standard error.

The least-squares adjustment was done on the two samples using all the constraints. In addition, the adjustment was done by excluding one type of constraints at a time to study the influence of the individual constraints on the accuracy improvement. The adjusted coordinates were then compared against the digitized and correct coordinates by calculating the percentage of improved points (%), the average error before and after adjustment ($E_b$ and $E_a$) and the standard error before and after adjustment ($\sigma_b$ and $\sigma_a$). Table 5.3 shows the results of this process.

As can be seen from this table, there is a significant improvement in the accuracy of the digitized coordinates when all the constraints are involved. This is indicated by the noticeable reduction in the average and standard errors, and the high percentage of points which gained accuracy. This accuracy improvement was enhanced even further when other high quality data were incorporated into the system through a sequential adjustment as explained in section 3.3.2.
TABLE 5.3
Accuracy before and after adjustment for different constraint combinations

<table>
<thead>
<tr>
<th>Used constraints</th>
<th>Simulated data</th>
<th></th>
<th>Real data</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_n = 2.16$ ft</td>
<td>$\sigma_n = 2.47$ ft</td>
<td>$E_n = 1.40$ ft</td>
<td>$\sigma_n = 1.60$ ft</td>
</tr>
<tr>
<td></td>
<td>%</td>
<td>$E_A$ (ft)</td>
<td>$\sigma_A$ (ft)</td>
<td>%</td>
</tr>
<tr>
<td>1. All</td>
<td>98.64</td>
<td>0.40</td>
<td>0.59</td>
<td>86.30</td>
</tr>
<tr>
<td>3. All except distances</td>
<td>80.54</td>
<td>1.16</td>
<td>1.49</td>
<td>78.37</td>
</tr>
<tr>
<td>2. All except areas</td>
<td>80.09</td>
<td>1.54</td>
<td>1.94</td>
<td>71.69</td>
</tr>
<tr>
<td>4. All except angles</td>
<td>70.59</td>
<td>1.82</td>
<td>2.07</td>
<td>74.43</td>
</tr>
<tr>
<td>5. All except tangency</td>
<td>68.78</td>
<td>1.62</td>
<td>1.91</td>
<td>75.32</td>
</tr>
</tbody>
</table>

It is difficult to conclude from these results which constraints have a larger relative influence on the adjustment of the digitized coordinates. The reason for this is that the adjustment is sensitive to the number of constraints of a particular kind (e.g., distances) as compared to the total number of all kinds of constraints involved. For example, if only two distance constraints exist among 200 other area, angle and tangency constraints, then the effect of excluding these two distance constraints on the adjustment will be barely noticeable.

5.6 Reliability of Geometric Constraints

Two measures for the quality of spatial observations can be distinguished. These are the precision and the reliability of an observation (Caspary, 1987, p. 85). The precision is expressed by the standard error associated with the observation. On the other hand, the reliability, or controllability as interchangeably called, is determined by examining the residuals of the observations resulting from the least-squares adjustment.
This examination is one of the most important and effective means for quality control and helps in identifying and isolating blunders or outliers in the observations. Since no prior precision information (standard errors) are available about the geometric constraints, especially that most of these constraints are automatically detected with a user-specified tolerance, the reliability measure only is used to study the quality of these constraints. The following discussion is based on the Gauss-Markov model explained in section (3.2.1).

Let

\[ V_{xy} = \text{Residual vector of the observed coordinates} \]
\[ V_c = \text{Residual vector of the stochastic constraints} \]
\[ L^{b}_{xy} = \text{Vector of observed coordinates} \]
\[ L^{b}_c = \text{Vector of stochastic constraint values} \]
\[ A = \text{Coefficient matrix of the observed coordinates} = I_{2n \times 2n} \]
\[ C = \text{Coefficient matrix of the constraints} \]
\[ P_{xy} = \text{Weight matrix of the observed coordinates} \]
\[ P_c = \text{Weight matrix of the constraints} \]
\[ \hat{X} = \text{Vector of estimated coordinate corrections (increments)} \]

Then, from equation (3.11) of section (3.2.2), we get:

\[ V_{xy} = AX + L_{xy} \]
\[ V_c = CX + L_c \]

or

\[ V = B\hat{X} + L \]

where, \( V = \begin{bmatrix} V_{xy} \\ V_c \end{bmatrix} \), \( B = \begin{bmatrix} A \\ C \end{bmatrix} \)

\[ L = \begin{bmatrix} L^{b}_{xy} \\ L_c \end{bmatrix} = \begin{bmatrix} L^{o}_{xy} - L^{b}_{xy} \\ L^{o}_c - L^{b}_c \end{bmatrix} \]
The solution for $\hat{X}$ is:

$$\hat{X} = -N^tU$$ .......................................................... (5.9)

where, $N = B^T PB = (P_x + C^T P_s C)$

$$U = B^T PL = (P_x L_x + C^T P_s L_s)$$

$$P = \begin{bmatrix} P_x & 0 \\ 0 & P_s \end{bmatrix}$$

Substitute equation (5.9) into (5.8),

$$V = B\hat{X} + L = -B(N^{-1}B^TPL) + L$$

$$= (I - BN^{-1}B^T P)L .............................................. (5.10)$$

The variance-covariance matrix ($\Sigma_v$) of the residuals is:

$$\Sigma_v = (I - BN^{-1}B^T P) \cdot \Sigma_L \cdot (I - BN^{-1}B^T P)^T$$

$$= (I - BN^{-1}B^T P) \cdot \sigma_o^2 P^{-1} \cdot (I - BN^{-1}B^T P)^T$$

$$\Rightarrow \Sigma_v = \sigma_o^2 (P^{-1} - BN^{-1}B^T) ................................................. (5.11)$$

Define the standardized residual vector to be $V_s = \frac{1}{\sigma_o} P^{1/2} V$ (this is possible because $P$ is a positive diagonal matrix),

$$\Rightarrow \Sigma_{v_s} = \frac{1}{\sigma_o^2} P^{1/2} \Sigma_v (P^{1/2})^T$$

$$= \frac{1}{\sigma_o^2} P^{1/2} \sigma_o^2 (P^{-1} - BN^{-1}B^T)P^{1/2}$$

$$\Rightarrow \Sigma_{v_s} = (I - P^{1/2}BN^{-1}B^TP^{1/2}) ........................................ (5.12)$$
Substitute the matrices $P$, $B$ and $N$ into (5.12),

$$
\sum_{v} = \left\{ I_1 - \begin{bmatrix} P^{1/2}_y & 0 \\ 0 & P^{1/2}_c \end{bmatrix} \begin{bmatrix} I_2 \\ C \end{bmatrix} (P_y + C^T P_c C)^{-1} \begin{bmatrix} I_2 \\ C^T \end{bmatrix} \begin{bmatrix} P^{1/2}_y & 0 \\ 0 & P^{1/2}_c \end{bmatrix} \right\}
$$

where, $I_2$ is of size $(2n \times 2n)$

$I_1$ is of size $(2n + m) \times (2n + m)$

$n = \text{number of points involved in the adjustment}$

$m = \text{total number of geometric constraints}$

The diagonal elements of the matrix of equation (5.13) carry information about the reliability (controllability) of both the observed coordinates and the geometric constraints. They are called redundancy numbers (RN) and range in value from 0.0 to 1.0. The redundancy number of an observation indicates the contribution of that observation to the overall degree of freedom in the system. If the RN of a certain constraint is high (say > 0.7), then that constraint is well-controlled. If that RN is low (say < 0.3), then the corresponding constraint is poorly-controlled and any gross error (blunder) in that constraint will not easily be detected. This, in turn, means that some control measurements might need to be added near the locations of these constraints. For more information about the interpretations of the redundancy number, the reader is referred to Kavouras (1982), Caspary (1987) or Dawod (1991).
The RN was calculated for both simulated and real data using equation (5.13). Table 5.4 shows the RN ranges which have been observed. As can be seen from this table, geometric constraints including control points showed a very low controllability with the exception of a few distance constraints. As a rule, it is not very dangerous to have an observation which has low controllability if the weight of that observation is low. However, this situation becomes critical if the observation or constraint is of a high weight such as control points. Table B.1 in Appendix B shows the RNs obtained when the adjustment was performed on the simulated data.

**TABLE 5.4**
Redundancy number for observed coordinates and geometric constraints

<table>
<thead>
<tr>
<th>Observation or constraint type</th>
<th>RN Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Digitized coordinates</td>
<td>0.0 - 1.00, the majority of points have RN range from 0.75 - 1.00</td>
</tr>
<tr>
<td>• Control points</td>
<td>0.0 - 0.20</td>
</tr>
<tr>
<td>• Distance constraints</td>
<td>0.5 - 0.90</td>
</tr>
<tr>
<td>• Area constraints</td>
<td>0.0 - 0.50</td>
</tr>
<tr>
<td>• Angle constraints</td>
<td>0.0 - 0.90, the majority &lt; 0.50</td>
</tr>
<tr>
<td>• Tangency constraints</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Attempts were made to find ways to increase the controllability of control points. This was done by inspecting the part of equation (5.13) relating to the controllability of coordinates which is \( \left\{ I - P_y^{1/2} \left( P_y + C^T P_z C \right)^{-1} P_y^{1/2} \right\} \). If we think of this part as a scalar, and substitute \( P_y = 3600 \) ft\(^2\) for the control points as determined earlier, the RN for these points becomes:

\[
RN = 1 - \left( 1 + \frac{C^T P_z C}{3600} \right)^{-1}
\] ................................. (5.14)

The only way to increase the RN is by making \( C^T P_z C \) very large. This can be done by making repeated measurements at the control points involving angles, distances and maybe GPS observations. This process has been tried on the simulated data by adding two distances and four angles between the control points. The RN for these points increased only from around 0.10 to 0.20 (see Table B.1, Appendix B). In order to increase this value to somewhere close to 1.0, a large number of observations involving the control points might be needed.

The results of this investigation show that it is extremely expensive to control the control points as well as other constraints. This suggests that other techniques for blunder detection need to be incorporated into the system to detect any bad data and isolate them. One of these techniques is the use of the robust adjustment method where the residuals of the constraints and other observations are inspected after each iteration in the adjustment and excluded from subsequent iterations if they are found to be too large (Pope, 1976; Aduol and Schaffrin, 1986; Gao et al, 1992).
5.7 Detecting and Strengthening Weak Areas in the Cadastral Overlay

One of the important issues in the implementation and building of a digital cadastral overlay using this approach, is the detection and strengthening of weak parts in the overlay so that supplemental measurements and survey data can be added first to these areas in the upgrading process. Two ways may be followed to identify the weaknesses in the cadastral overlay: visual and analytical.

A) Visual:

This is usually done by inspecting the geometry of the overlay with an "expert eye". However, general guidelines which were realized as a result of comparing the adjusted and digitized coordinates against the theoretically correct coordinates can be given. These include:

1. Points where two curves meet, or where a line does not meet a curve tangentially, were found to be weak. In most cases, these points showed no or very little improvement in accuracy when applying the geometric constraints. Examples of such points are 17, 18, 19, 20 and 21 in Figure 5.10. The accuracy of these points can be enhanced by adding survey data, such as angles and distances associated with these points.

2. Points which are not involved in any geometric constraint were also found to be weak. Examples of these points are 35, 39 and 41 in Figure 5.10. These points do not have any angle, distance or tangency constraints which support them. Involving these points in later survey data will positively affect their accuracy.

3. Blocks which have cul-de-sac curves are weak along the lines which have these curves. Figure 5.10 illustrates such a situation. With the absence of area constraints for the block of Figure 5.10, the cul-de-sac curve can move freely in the adjustment process along the line extending between points 10 and 28.
FIGURE 5.10: Weak areas in a typical cadastral block.

(x-direction), and still all other angle, tangency and distance constraints are completely satisfied. Moreover, the cul-de-sac itself is weak in the y-direction along the line 14-16. Two distance measurements for the lines 10-13 and 14-16 will significantly improve the accuracy of this block. In general, if a block line has n cul-de-sac curves branching from it, then adding a similar number of distances along that line between the cul-de-sac curves will take care of the weaknesses in that block. An example of such a situation exists for block 3 in Figure 5.1.

4. In general, this research showed that new survey data added along the boundaries of parcel blocks have the greatest effect on the overall accuracy of the cadastral overlay.
B) Analytical:

The detection of weak areas in the cadastral overlay is achieved by examining the diagonal elements of the variance-covariance matrix of the adjusted coordinates. The weak points were found to have large variances. In order to find the direction and value of the largest weakness of a given point, the standard error ellipse parameters were calculated. These include $\sigma_u$, $\sigma_v$ and $\theta$ (direction of the semi-major axis of the standard error ellipse) as illustrated below in Figure 5.11.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure511.png}
\caption{The standard error ellipse of constant probability}
\end{figure}

\begin{align*}
\text{Given } Q_{uv} & = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}, \\
\text{Looking for } Q_{uv} & = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}, \text{ and } \theta \text{ as rotation angle to the principal axes } u \text{ and } v
\end{align*}

These quantities can be calculated from the following formulae:

\begin{align*}
\theta & = \frac{1}{2} \arctan \left( \frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} \right) \quad \ldots \quad (5.15) \\
\sigma_u^2 & = \frac{1}{2} \left[ \sigma_x^2 + \sigma_y^2 + \sqrt{4\sigma_{xy}^2 + (\sigma_x^2 - \sigma_y^2)^2} \right] \quad \ldots \quad (5.16)
\end{align*}
\[ \sigma_v^2 = \frac{1}{2} \left[ \sigma_x^2 + \sigma_y^2 - \sqrt{4\sigma_x^2 + (\sigma_x^2 - \sigma_y^2)} \right] \]  

(5.17)

For the derivation of these formulae and for more information about the standard error ellipse, the reader is referred to Wolf (1980, pp. 207-222) or Mikhail and Gracie (1982, pp. 224-230).

One way of identifying the weak points automatically in a computer environment is to scan the \( \sigma_u \) values of the adjusted points and locate the two points with the smallest and largest \( \sigma_u \)'s (\( \sigma_{u_{\text{min}}} \) and \( \sigma_{u_{\text{max}}} \)). If a limit between these two extremes is set so that all the \( \sigma_u \)'s above this limit will be considered large, then the weak points in an adjusted map can be automatically displayed. An example of such a limit which was in fact used here is:

\[ \text{Limit} = \sigma_{u_{\text{min}}} + c \left( \sigma_{u_{\text{max}}} - \sigma_{u_{\text{min}}} \right) \]  

(5.18)

where \( c \) is a coefficient ranging from 0.0 to 1.0, preferably 0.5 to 0.75.

By adding a few measurements which involve the detected weak points, the overall accuracy will be greatly enhanced, and the variance of the weak points will be significantly reduced. This approach has been tried on the simulated data with a value of \( c=0.7 \). Several angles and distances were added at the weak locations and a sequential adjustment was then performed. The result was that the percentage of improved points increased from 98.6 to 100%, and the average and standard errors were reduced from 0.40 and 0.59 to 0.24 and 0.33 respectively.
CHAPTER VI
EDGE MATCHING AND JOINING BORDERING CADAstral MAPS

6.1 Introduction

Due to the computation and storage limitations involved in the simultaneous
adjustment of the parcels contained in a large area like a county or a state, the approach
outlined in the previous chapters deals with adjusting one cadastral map at a time. If the
neighboring maps were strictly compatible and free of errors, the border lines would
match each other exactly, and continuity of lines passing from one map to another will be
assured. However, due to the various sources of errors in the original digitized
coordinates, maps fall short of this goal even after adjustment because it is difficult to
eliminate all types of error completely.

Several approaches have been followed for joining individually digitized map
sheets to form a "sheetless" database. Among these are the rubber-sheeting and localized
edge matching techniques (Beard and Chrisman, 1988), which are employed in most
existing GIS systems (e.g. ARC/INFO). These techniques are used mainly for joining
digitized topographic maps where the continuity of lines across the map borders is
traditionally given higher priority over the positional accuracy and relative relationships
between boundary line and interior map points. Another approach which has been used
by Shmutter and Doytsher (1991) to join digitized cadastral maps, is the block
adjustment technique, often used in photogrammetry. This approach, again, "emphasizes
the continuity of cadastral objects across the boundary lines as opposed to the positional
accuracy, and fails to maintain geometric relationships between parcel boundary corners, which makes it unsuitable for this research, where the positional accuracy is of a major concern. More information about joining digitized cadastral maps can also be found in a German document by Weins (1986). This chapter provides an alternative for zipping the individually adjusted bordering maps while keeping the geometric figure and accuracy improvements achieved by the individual adjustment of these maps unchanged.

6.2 Detection of Common Points on the Borderline of Two Adjoining Maps

One of the basic issues in the zipping of adjoining cadastral maps is the identification of common points located at the borderline separating two neighboring maps. At least, two approaches can be utilized for the detection of these points:

1) By calculating the distance between each point on the first map and all the points on the second map. The smallest distance, then, will most likely represent the error between the two pairs of coordinates of the same point which is supposed to be located at the boundary between the two maps, given that this distance is less than a user specified tolerance (5' for example). This procedure is computationally intensive, and might lead to errors, since the smallest distance does not necessarily imply that the two points are really the same.

2) By comparing the parcel corner identifications numbers (PCIDs) of those points on the first map with the PCIDs of the points on the second map. Since points on the borderline are given exactly the same PCID (as explained in Chapter IV), then the match will be perfect and the possibility of errors is non-existent. This procedure does not involve calculations and is much faster in a computer environment and, hence, is recommended over the first procedure.
6.3 The Zipping Process

This section will describe an exact technique for joining the individually adjusted neighboring maps without destroying the accuracy improvement achieved in the adjustment process. For simplicity, the solution will be derived for only two maps, but can easily be extended to five maps (the central map and the four bordering maps).

![Diagram of two adjoining cadastral maps](image)

**FIGURE 6.1:** Configuration of two adjoining cadastral maps

In Figure 6.1, let:

- $X_{1b}^b$ = coordinate vector of the borderline points located on map 1
- $X_1^b$ = coordinate vector of the interior points of map 1
- $X_{2b}^b$ = coordinate vector of the borderline points located on map 2
- $X_2^b$ = coordinate vector of the interior points of map 2

and let $N = \begin{bmatrix} N_{bb} & N_{bi} \\ N_{bi}^T & N_{ii} \end{bmatrix}$ = normal equations matrix resulting from the adjustment of the first map $(A_i^T P_i A_i)$ divided by the estimated variance component of the first map $(\sigma_{o1}^2)$
M = \begin{bmatrix} M_{bb} & M_{b2} \\ M_{b2}^T & M_{22} \end{bmatrix} = \text{normal equations matrix resulting from the adjustment of the second map } (A_2^T P_2 A_2) \text{ divided by the estimated variance component of the second map } (\hat{\sigma}^2_{e2})

The following mathematical model can be written,

\[ L^* = x_1 \quad X \quad x_2 \]

\[ X_b, X_1, X_2 \text{ are the theoretical unknown coordinates. Then the observation equations become:} \]

\[ X_{1b}^b - \varepsilon_{1b} = X_b \]
\[ X_1^b - \varepsilon_1 = X_1 \]
\[ X_{2b}^b - \varepsilon_{2b} = X_b \]
\[ X_2^b - \varepsilon_2 = X_2 \]

or

\[ L^b - \varepsilon = AX \]

where

\[ L^b = \begin{bmatrix} X_{1b}^b \\ X_1^b \\ X_{2b}^b \\ X_2^b \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{1b} \\ \varepsilon_1 \\ \varepsilon_{2b} \\ \varepsilon_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_b \\ X_1 \\ X_2 \end{bmatrix} \]
The solution for $X$ is:

$$X = (A^T P A)^{-1} (A^T P L^T)$$ ............................................... (6.4)

where $P = \sigma^2 \begin{bmatrix} N^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix} = \sigma^2 \begin{bmatrix} N_{bb} & N_{b1} & 0 & 0 \\ N_{b1}^T & N_{11} & 0 & 0 \\ 0 & 0 & M_{bb} & M_{b2} \\ 0 & 0 & M_{b2}^T & M_{22} \end{bmatrix}$

$$\Rightarrow (A^T P A) = \begin{bmatrix} 1 & 0 & I & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_{bb} & N_{b1} & 0 & 0 \\ N_{b1}^T & N_{11} & 0 & 0 \\ 0 & 0 & M_{bb} & M_{b2} \\ 0 & 0 & M_{b2}^T & M_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (N_{bb} + M_{bb}) & N_{b1} & M_{b2} \\ N_{b1}^T & N_{11} & 0 \\ M_{b2}^T & 0 & M_{22} \end{bmatrix} .................................. (6.5)$$

Assume $(A^T P A)^{-1} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix}$

then,

$$\begin{bmatrix} (N_{bb} + M_{bb}) & N_{b1} & M_{b2} \\ N_{b1}^T & N_{11} & 0 \\ M_{b2}^T & 0 & M_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ............... (6.6)$$

By performing the matrix multiplication in equation (6.6), a system of 9 equations results, which when solved leads to the following inverse:
\[(A^T P A)^{-1} = \begin{bmatrix}
Q_{11} & -Q_{11}N_{b1}N_{i1}^T & -Q_{11}M_{b2}M_{22}^T \\
-N_{b1}^TQ_{11}N_{i1}^T & (I + N_{b1}^TQ_{11}N_{b1}N_{i1}^T) & N_{i1}^TQ_{11}M_{b2}M_{22}^T \\
-M_{b2}^TM_{b1}^TQ_{11} & M_{b2}^TM_{b1}^TQ_{11}N_{b1}N_{i1}^T & M_{b2}^T(I + M_{b2}^TQ_{11}M_{b2}M_{22}^T)
\end{bmatrix}\]

With
\[Q_{11} = [(N_{bb} + M_{bb}) - N_{b1}N_{i1}^TN_{i1}^T - M_{b2}M_{22}^TM_{b2}^T]^{-1}\]

The full derivation of the previous inverse is given in Appendix C.

\[A^T P L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
N_{bb} & N_{b1} & 0 & 0 \\
N_{b1}^T & N_{i1} & 0 & 0 \\
0 & 0 & M_{bb} & M_{b2} \\
0 & 0 & M_{b2} & M_{22}
\end{bmatrix}\begin{bmatrix}
x_{b1}^b \\
x_{b1}^b \\
x_{b2}^b \\
x_{b2}^b
\end{bmatrix}\]

\[= \begin{bmatrix}
N_{b1}X_{b1}^b + N_{b1}X_{b1}^b + M_{bb}X_{b2}^b + M_{b2}X_{b2}^b \\
N_{b1}X_{b1}^b + N_{i1}X_{b1}^b \\
M_{b2}X_{b2}^b + M_{22}X_{b2}^b
\end{bmatrix} = \begin{bmatrix}
U_1 \\
U_2 \\
U_3
\end{bmatrix}\]

Finally,
\[
\begin{bmatrix}
\hat{x}_b \\
\hat{x}_1 \\
\hat{x}_2
\end{bmatrix} = \begin{bmatrix}
Q_{11}(U_1 - N_{b1}N_{i1}^TU_2 - M_{b2}M_{22}^TU_3) \\
N_{i1}^T(-N_{b1}^TQ_{11}U_1 + (I + N_{b1}^TQ_{11}N_{b1}N_{i1}^TU_2 + N_{b1}^TQ_{11}M_{b2}M_{22}^TU_3) \\
M_{22}^T(-M_{b2}^TQ_{11}U_1 + M_{b2}^TQ_{11}N_{b1}N_{i1}^TU_2 + (I + M_{b2}^TQ_{11}M_{b2}M_{22}^TU_3)
\end{bmatrix}\]

where \(\hat{x}_b, \hat{x}_1\) and \(\hat{x}_2\) are the coordinates of the two joined maps.

This technique has been tried on two neighboring tax maps and found to have the following merits:
1) Only one iteration is needed to perform the adjustment. This is due to the fact that the mathematical model is already linear and no linearization process is needed.

2) It provides an updated version of the normal equations matrix \((A^T PA)\) to be used in later adjustments when more accurate survey data become available.

3) It not only preserves the geometric figure and accuracy improvement achieved in the previous adjustments, but also enhances the accuracy even further. This is caused by the fact that information is propagated from one map to another.

Table 6.1 shows the results of the application of this technique on the zipping of two bordering tax maps. Also tables D.1 and D.2 in Appendix D show examples of the following coordinates for the two neighboring tax maps: the theoretically correct coordinates, the digitized coordinates, the adjusted coordinates before zipping and the coordinates after zipping the two maps.

| TABLE 6.1 |
| Accuracy before and after the zipping of two cadastral maps |

<table>
<thead>
<tr>
<th></th>
<th>Digitized Coordinates</th>
<th>Accuracy after Initial Adjustment</th>
<th>Accuracy after Map Zipping</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E* (ft)</td>
<td>σ** (ft)</td>
<td>%***</td>
</tr>
<tr>
<td>MAP 1</td>
<td>1.94</td>
<td>2.46</td>
<td>77.3</td>
</tr>
<tr>
<td>MAP 2</td>
<td>1.40</td>
<td>1.60</td>
<td>86.3</td>
</tr>
</tbody>
</table>

* E = Average error
** σ = Standard error
*** % = Percentage of improved points
6.4 Simplified Techniques for Joining Cadastral Maps

The previously described technique for joining individually adjusted tax or cadastral maps provides an exact solution and works extremely well when five or less maps are involved in the process. However, when dealing with tens or hundreds of maps, like the situation in a county or a state, an iterative solution is needed. Figure 6.2 illustrates the problem. Map 2 is first joined simultaneously with maps 1, 3 and 7. Then map 6 is joined with 1, 7 and 11. This changes the coordinates of maps 1 and 7 and makes them fail to join exactly with map 2 even though the difference between the coordinates of points on the boundary lines 1-2 and 2-7 will now be smaller. Therefore another iteration to join 2 with 1, 3 and 7 might be needed if the coordinates of the borderline points differ by more than a specified tolerance. Again, this changes the coordinates of maps 1 and 11 and makes them fail to join exactly with map 6. This situation worsens even more when a larger number of maps is involved.

In order to avoid this problem, research has been done on three other simple alternatives for joining bordering cadastral maps. These alternatives have something in common which is that a weighted average for the coordinates of the borderline points

```
1  2  3  4  5
6  7  8  9 10
11 12 13 14 15
16 17 18 19 20
```

FIGURE 6.2: Expanding the zipping process beyond two cadastral maps
between each two neighboring maps was calculated using only the variances and covariances of the two sets of coordinates on the two maps belonging to these points. This means that the correlation between the borderline points and interior points was neglected in the averaging process. The residuals between these averaged coordinates and the coordinates resulting from the original adjustment (using geometric constraints and other available survey data) were then calculated for the points of each borderline on each map. After that, the following three methods were used in order to distribute these residuals to the interior map points:

a) The weighted coordinates of the borderline points were considered as new observed point data and hence, a sequential adjustment was performed as explained in section 3.3.2. Equation (3.63) was used for this purpose which takes into account the geometry represented by the already stored normal equations matrix resulting from prior adjustments. This approach succeeded to some extent in keeping the relative relationships and geometric conditions between parcel boundary corners, but failed to preserve the numerical accuracy (represented by the average and standard errors) attained from previous adjustments.

b) An affine transformation like that described in section 4.5 was fitted through the weighted borderline coordinates and the affine transformation parameters were determined (Shmutter and Doytsher, 1991). After that, the new modified coordinates of the interior points were calculated using equation (4.1). When these coordinates were, in turn, compared with the known correct coordinates, it was found that a significant amount of the accuracy achieved in previous adjustments had been lost.

c) The x and y residuals for the coordinates of the interior map points were interpolated from the residuals of the borderline points using Bjerhammars's
model (Bjerhammar, 1973). This model can be described mathematically as follows:

\[
V_{x_i} = \frac{\sum_{j=1}^{c} \frac{V_{x_j}}{r_{ij}^k}}{\sum_{j=1}^{c} \frac{1}{r_{ij}^k}} \quad \cdots \quad (6.10)
\]

\[
V_{y_i} = \frac{\sum_{j=1}^{c} \frac{V_{y_j}}{r_{ij}^k}}{\sum_{j=1}^{c} \frac{1}{r_{ij}^k}} \quad \cdots \quad (6.11)
\]

where \( V_{x_i} \) and \( V_{y_i} \) are the interpolated \( x \) and \( y \) residuals of the interior points \( i \).

\( V_{x_j} \) and \( V_{y_j} \) are the \( x \) and \( y \) residuals of the borderline point \( j \).

\( k \) = power of prediction

\( r_{ij}^2 = d_{ij}^2 + C^2 \)

\( C \) = smoothing quantity

\( d_{ij} \) = distance between the borderline points \( j \) and the wanted interior points \( i \).

When applying this approach, the following logical assumption was made: If the borderline point and the required interior point belong to the same block of parcels, then the \( x \) and \( y \) residuals of this borderline point will carry a higher weight than the residuals of other borderline points located in different blocks. This is due to the fact that points located in the same block will most likely be correlated through a measurement or a geometric constraint. This higher weight was given by assigning a smaller smoothing quantity \( (C) \) for the corresponding borderline points.
This Bjerhammar model was applied on the two tax maps using a range between 0 and 10 for the power of prediction \((k)\) and a range of 0 to 1000 ft for the smoothing quantity \((C)\). Again, like the two previous approaches, it failed to give satisfactory results.

An investigation has been done to find the reason behind the failure of these three approaches. When the averaged coordinates of the borderline points were compared with the known correct coordinates of these points, it was found that more than 50% of these points lost accuracy as compared to the coordinates resulting from prior adjustments before averaging. So when using these bad coordinates as a basis for interpolating the residuals of the interior points, it is logical to expect that interior points will also lose accuracy, regardless of the approach used for interpolation process. This demonstrates that the correlation between the borderline points and interior map points is very important, and cannot simply be ignored in the calculation of weighted coordinates for the borderline points. This, in turn, leaves the door open for researchers to think of different techniques for the zipping of cadastral maps.
CHAPTER VII
CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary of the Approach

This research aimed to design and test a feasible methodology for building a digital cadastral overlay. The methodology starts with coordinates digitized from tax maps and incrementally upgrades their accuracy through adjustment computation techniques using "stochastic" geometric constraints (areas, angles, distances, circular continuity, etc.) as well as accurate survey data as they become available. Due to their wide availability and familiarity to the land information community in the USA, tax maps were considered as the most viable basis to "jump-start" the implementation of a digital cadastral overlay. In other countries, existing paper cadastral maps will serve this purpose equally well.

Several issues which are necessary for the implementation of this approach have been studied. These include:

- The development of the mathematical background (mathematical models and observation equations) for the geometric constraints; mainly area, angle and tangency constraints. This includes the derivation of simplified mathematical models for the rectangularity and collinearity conditions which do not include trigonometric functions, and thus minimize the numerical instability in the system.
• Geometric constraints were first dealt with as fixed conditions which are free of error and carry an infinite weight. This resulted in a numerically unstable and ill-conditioned system of equations which could not be solved. This difficulty was overcome by relaxing the constraints and giving them a high weight instead of infinite weight, thus minimizing the effect of linear dependency between them. This agrees with real world situations which are stochastic in nature, as rectangular or collinear parcel boundary lines will not generally be exactly 90° or 180° in reality. Extensive research has been done to find the most suitable weights for geometric constraints which will optimize the adjusted coordinate accuracy.

• The traditional sequential adjustment formula (see equation 3.60) which is found in most adjustment computation literature requires the storage of the variance-covariance matrix \( N_i^1 \) resulting from the previous adjustment. When used in a computer environment, this formula has two drawbacks:

1) It requires a large storage space in the random access memory (RAM) of the computer which is currently not affordable in most existing computers, and also requires longer processing time, as more mathematical operations are involved.

2) Storing the variance-covariance matrix \( N_i^1 \) for all the points in one cadastral map (700 points on the average) requires a storage of approximately \( (1400 \times 1400)/2 \) values. This is a forbidding task, especially when all the maps of a county or a state are involved.

To overcome these two difficulties, this formula has been modified and simplified to take a form which is a function of \( N_i \) instead of \( N_i^1 \) (see equation...
This new form of the equation, which is used for the continuous upgrading of the cadastral database accuracy using new survey data, is much easier to use from a computer programming perspective, as less processing time and computer storage are needed. Storing $N_i$ can save as much as 90% of the storage needed for the $N_i^{-1}$ matrix.

- The study of the optimum number, location and distribution of geodetic control points over the cadastral map area.

- The development of an easy to implement and maintain parcel corner identification system.

- The identification of the weak parts in the cadastral overlay so that supplemental measurements and survey data can be added first to these areas in the upgrading process.

- The reliability of geometric constraints has also been examined. This is considered as one of the most important and effective means for quality control and helps identify and isolate outliers or blunders among the geometric constraints.

- Performing a simultaneous adjustment of all the cadastral maps of a county or a state is technically difficult due to the storage and computational limitations of existing computing facilities. Therefore, the approach outlined in this research deals with adjusting one map at a time. A technique has been presented for
joining the individually adjusted maps so that the achieved accuracy improvement within each map will be preserved or even further improved.

7.2 Major Findings of the Research

Certain general conclusions and findings can be identified as a result of the actual testing of this methodology on both simulated and real tax maps. These findings are:

- The impact of this approach on the geometric figure and accuracy improvement of the digitized coordinates is very encouraging and demonstrates that this is a viable alternative for creating a cadastral overlay. Incorporating this as a module in existing GIS systems should significantly enhance the capabilities of these systems, particularly in the area of land records modernization.

- Geodetic control or coordinated points are needed to serve two purposes:
  1) To transform digitized machine coordinates into real world coordinates, and
  2) To provide the spatial support framework for the adjustment process.

However, the absence of coordinated or control points should not hinder the implementation of the cadastral overlay using this approach. Although it is not as accurate as using geodetic control, digitizer tablet coordinates can still be transformed into real world units (ft or m) relative to a local coordinate system using the paper map scale or any average scale derived from measuring a few distances or even getting them from available deeds, subdivision plats or the tax maps themselves. A free network adjustment is then performed on these transformed coordinates. Control point measurements can be incorporated later
through a sequential adjustment in the same way as other high quality data are incorporated into the system.

- Geometric constraints proved to be of a great value for the improvement of digitized coordinate accuracy, especially in urban areas where many of these constraints exist. The adjustment, using only geometric constraints, managed to eliminate at least 50% of the errors in the digitized coordinates of both simulated and real data.

- This approach provides a mechanism for constructing the cadastral overlay with relatively low front-end cost. It also allows the immediate implementation of this overlay, simply by digitizing the widely available tax or cadastral maps. No extensive remonumentation and resurvey of land parcels is required. The accuracy of parcel corner coordinates will be upgraded over time with higher quality survey data as they become available. This makes parcel mapping a continuous dynamic process which reflects the current status of a changing situation, rather than a one-time static process (Anderson, 1985).

- Due to the continuing accuracy upgrading which will be achieved over time, the numeric dimensions of parcels will compare closely with the legally described ones. This helps create a cadastral overlay which is legally supportive. However, this is difficult to achieve without institutional and legal reforms of some sort as will be explained in the next section.
7.3  Recommendations

The work done in this research represents a major step in developing a convenient approach for building a digital cadastral overlay. However, other requirements should be considered for the successful implementation of this approach. These can be categorized into six groups:

1. Legal Issues:

The approach developed here deals with the technical upgrading of parcel corner coordinate accuracy using geometric constraints and accurate survey data as they become available. However, the vast majority of existing survey data have been acquired with traditional techniques that are far less accurate than present day technology, and recorded in conventional ways that tend to hide or obscure their inherent inaccuracies and inconsistencies (Werle, 1984). These data are kept in legal records which cannot be altered on merely technical grounds, without going through some sort of judicial procedure. In case of conflict over the location of parcel corners, these recorded data may have precedence as legal evidence over recent measurements performed with higher precision equipment. If these old data with errors imbedded in them are used to update the digitized coordinates, the resulting cadastral overlay might carry a legal status, but fails to represent reality as it is. On the other hand, if more recent accurate survey data are used to upgrade the coordinate accuracy, the resulting overlay will be spatially accurate, but may lack the legal status. Therefore, until these legal issues are resolved, it may be difficult to produce a cadastral overlay which is both accurate and legally supportive.
2. Polygon Topology:

A basic difficulty in the digitization of cadastral maps using the digitizing software currently available in the market, is the inability of this software to build line and polygon topology which is explicitly known to the cadastral mapper. For example, during the digitization process, the user should be able to assign the PCIDs which have been introduced in the numbering process rather than the numbers assigned by the digitizing software and exist in internal tables which are difficult to be accessed and understood by the user. The reason for this is that the adjustment program and the programs which derive angle and tangency constraints need to know which corners form the boundary of a particular parcel, so that their coordinates can be easily accessed. Several digitizing software including ARC/INFO, GeoVision and Roots have been examined and found to lack this capability. This difficulty was overcome temporarily in this research by manually building the polygon topology files for the tested maps and writing a computer program which runs a consistency check on these files as explained in Chapter IV.

This issue of polygon topology raises a question for GIS developers to come up with digitizing software which can allow one of two solutions:

a) Building line and polygon topology which is explicitly known to the user while allowing the use of a user-controlled numbering system for the parcel corners, or

b) Extend the capabilities of existing digitizing software to generate unique PCIDs for parcel corners which can be easily accessed by the user.
3. Reliability of geometric constraints:

Research has revealed that geometric constraints and other survey data including control points have low controllability, and any blunder contained in these data and constraints will not likely be detected. This suggests that other techniques for outlier detection need to be incorporated into the adjustment process to identify and isolate any bad data. One of these techniques is the use of the robust adjustment method where the residuals of the constraints and other observations are inspected after each iteration in the adjustment and excluded from subsequent iterations if they are found to be too large (Pope, 1976; Aduol and Schaffrin, 1986; Gao et al, 1992).

4. Joining bordering cadastral maps:

To avoid the technical difficulties involved in the simultaneous adjustment of a large number of tax maps, this approach deals with adjusting one map at a time. A technique has been described in Chapter VI for joining the individually adjusted maps which preserves the geometric figure and accuracy improvement achieved in the adjustment. However, this technique requires an iterative process which is time consuming when a large number of maps is involved. Experimentation has been done on three other approaches for zipping these maps together, but failed to give satisfactory results. This leaves the door open for researchers to try to come up with a simpler and less time consuming solution for matching the individually adjusted maps while keeping the results of the adjustment undestroyed.

5. Updating the digital cadastral overlay:

This research has dealt only with the upgrading problem, where the digitized coordinate accuracy is improved through the application of geometric constraints, and the incorporation of new survey data which involve the digitized points. In order to develop
a comprehensive system for building a digital cadastral overlay, the updating of the cadastral database with new land subdivisions and consolidations should be taken into consideration. This usually involves the deletion or addition of new parameters (parcel corners) to the database, and as a result, the appropriate mathematical formulae need to be used to accommodate these changes.

6. Availability of Survey Data:

As outlined in Chapter 3, this approach upgrades the accuracy of the digitized map coordinates incrementally through the incorporation of new survey data as they become available. Therefore, if this approach is to be useful and implementable, a strategy needs to be developed to make this data available and collect it. Since a large scale resurvey of all parcels is not feasible, upgrading should start with reliable information obtained from publicly registered property deeds, recorded surveys and subdivision plats. However, to support the continuing improvement of the digital cadastral overlay accuracy, it is required to pass state legislation that would make the recording of survey plans for conveyance or subdivision mandatory, and require that all new deeds be based on a reliable survey (NRC, 1983, p.57). Moreover, a cooperation between county governments and private surveyors and other agencies which collect survey data will facilitate the process.

7.4 Closing Statement

A more feasible approach for developing a digital cadastral overlay has been designed and tested. This approach gives greater weight to the efficiency and cost factors, at least in the initial phase of implementation, while the accuracy of the cadastral overlay is upgraded incrementally with geometric constraints and new survey data as they become available. The approach is very suitable for urban areas, in particular, where land
values are high, and an accurate representation of parcel boundaries is stringent, as compared to rural areas. By considering the recommendations mentioned in the previous section, it will be possible to build an integrated system for digital cadastral mapping which can be successfully incorporated in an automated LIS/GIS environment.
LIST OF REFERENCES

Aduol, F.W.O. and B. Schaffrin (1986), "On Outlier Identification in Geodetic Networks Using Principal Component Analysis", Presented at the conference on Influential Data Analysis, University of Sheffield, Germany, April 8-11, 28 p.


Bjerhammar, Arne (1973), General Model for Linear Filtering, The Royal Institute of Technology, Division of Geodesy, Stockholm, 12 p.


National Research Council (1980), The Need for a Multipurpose Cadastre, National Academy Press, Washington, D.C.

National Research Council (1983), Procedures and Standards for a Multipurpose Cadastre, National Academy Press, Washington, D.C.


Uotila, Urho A. (1986), Notes on Adjustment Computation : Part I. Department of Geodetic Science and Surveying, The Ohio State University, Columbus, Ohio 168 p.


APPENDIX A

EQUIVALENT STANDARD ERROR ASSOCIATED WITH THE WEIGHT OF ANGLE AND TANGENCY CONSTRAINTS
EQUIVALENT STANDARD ERROR ASSOCIATED WITH THE WEIGHT OF
ANGLE AND TANGENCY CONSTRAINTS

For angle and tangency constraints, equation (5.6) results in standard error values
in \( \text{ft}^2 \) or \( \text{m}^2 \). In order to transform these values into seconds of arc, a rigorous
mathematical manipulation is needed. The following is a derivation for the required
transformation formulae.

1) Right angles (90°):

\[
\text{FIGURE A.1: Geometry of a right angle}
\]

In Figure A.1, assume that the angle \( \text{ijk} \) differs from \( \pi/2 \) by an \( \epsilon \),

\[
AZ_{ik} = AZ_{ij} + \pi/2 + \epsilon, \quad \text{where } AZ \text{ is an abbreviation for azimuth}
\]

\[
\tan AZ_{ij} \cdot \tan AZ_{ik} = \tan AZ_{ij} \cdot \tan(AZ_{ij} + \pi/2 + \epsilon)
\]

\[
= \tan AZ_{ij} \cdot \frac{\tan (AZ_{ij} + \pi/2) + \tan \epsilon}{1 - \tan(AZ_{ij} + \pi/2) \tan \epsilon}
\]

which can be reduced to:
\[
\tan AZ_{\mu} \cdot \tan AZ_{\nu} = -1 + \frac{\left(\tan^2 AZ_{\mu} + 1\right) \tan \epsilon}{\tan AZ_{\mu} + \tan \epsilon} \quad \cdots \text{(A.1)}
\]

As given in equation (3.37), the mathematical model for the right angle is:
\[
F_{ik} = (y_i - y_j)(y_k - y_j) + (x_i - x_j)(x_k - x_j) = 0 \quad \cdots \text{(A.2)}
\]

If \( F_{ik} \) is allowed to have an error of \( \sigma = \frac{1}{\sqrt{W}} \), then equation (A.2) becomes:
\[
(y_i - y_j)(y_k - y_j) + (x_i - x_j)(x_k - x_j) = \sigma \quad \cdots \text{(A.3)}
\]

Divide equation (A.3) by \((y_i - y_j)(y_k - y_j)\), and re-arrange terms:
\[
\Rightarrow \tan AZ_{\mu} \cdot \tan AZ_{\nu} = -1 + \frac{\sigma}{(y_i - y_j)(y_k - y_j)} \quad \cdots \text{(A.4)}
\]

Equate (A.1) and (A.4), and solve for \( \epsilon \),
\[
\Rightarrow \epsilon = \arctan \left[ \frac{x_i - x_j}{\sqrt{W} \left( y_k - y_j \right) \left[ (x_i - x_j)^2 + (y_i - y_j)^2 \right] - (y_i - y_j)} \right] \quad \cdots \text{(A.6)}
\]

Substituting \( \tan AZ_{\mu} = \frac{x_i - x_j}{y_i - y_j} \) and \( \sigma = \frac{1}{\sqrt{W}} \) in (A.5),
2) 180° angles:

\[ \text{FIGURE A.2: Geometry of a 180° angle} \]

In Figure A.2, if angle \( ijk \) differs from \( \pi \) by an \( \varepsilon \),
\[
\Rightarrow AZ_{ik} = AZ_{ij} + \pi + \varepsilon
\]

\[
\tan AZ_{ik} = \tan(AZ_{ij} + \pi + \varepsilon) = \frac{\tan AZ_{ij} + \tan \varepsilon}{1 - \tan AZ_{ij} \cdot \tan \varepsilon}
\]

\[
\Rightarrow \tan AZ_{ik} = \tan AZ_{ij} + \frac{\tan \varepsilon (1 + \tan^2 AZ_{ij})}{1 - \tan AZ_{ij} \cdot \tan \varepsilon} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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3) Curve-curve tangency:

At perfect tangency, angle $c_1p_2$ should be $180^\circ$. If the angle $c_1p_2$ is allowed to have an error $\varepsilon$ so that the two curves are not perfectly tangent to each other, then this case becomes similar to the $180^\circ$ situation discussed above, and the transformation equation will be:

$$
\varepsilon = \arctan \left[ \frac{(y_{c_1} - y_p)}{\sqrt{W(y_{c_1} - y_p)\{(x_{c_1} - x_p)^2 + (y_{c_1} - y_p)^2\} + (x_{c_1} - x_p)}} \right]
$$

...... (A.12)
4) Line-curve tangency:

At perfect tangency, angle \( \angle ijc \) should be 90°. If this angle is allowed to have an error \( \varepsilon \) so that the line \( ij \) is not perfectly tangent to the curve, then this case becomes similar to that of the 90° discussed before, and the transformation equation will be:

\[
\varepsilon = \arctan \left[ \frac{(x_i - x_j)}{\sqrt{W (y_i - y_j)} \left\{ (x_i - x_j)^2 + (y_i - y_j)^2 \right\} - (y_i - y_j)} \right]
\]  
\[ A.13 \]

Equations (A.6), (A.11), (A.12) and (A.13) give the relationship between the angular error \( \varepsilon \) (in radians) and the respective weight \( W \) (in \( \text{ft}^4 \) or \( \text{m}^4 \)).
APPENDIX B

REDUNDANCY NUMBERS OF GEOMETRIC CONSTRAINTS
REDUNDANCY NUMBERS OF GEOMETRIC CONSTRAINTS

The following table (B.1) shows the redundancy numbers (RNs) of the geometric constraints which can be derived from the simulated map in Figure B.1. The redundancy numbers for area, distance, angle and tangency constraints did not change significantly when two distances and four angles were added to increase the controllability of the control points. Only the RN for control points increased by a maximum value of 0.1. The last part of Table B.1 shows these values.

FIGURE B.1: Geometry of a simulated tax map
### TABLE B.1

Redundancy numbers (RNs) of geometric constraints derived from Figure B.1

<table>
<thead>
<tr>
<th>Constraint #</th>
<th>Parcel #</th>
<th>Area (ft²)</th>
<th>Redundancy Number (RN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>14214.2</td>
<td>0.18</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>13114.2</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>13200.0</td>
<td>0.20</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>13200.0</td>
<td>0.20</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>13114.2</td>
<td>0.20</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>14214.2</td>
<td>0.18</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>14300.0</td>
<td>0.18</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
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<td>0.18</td>
</tr>
<tr>
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<td>9</td>
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<td>0.02</td>
</tr>
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<td>90</td>
<td>15600.0</td>
<td>0.46</td>
</tr>
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<td>91</td>
<td>15600.0</td>
<td>0.46</td>
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<td>15600.0</td>
<td>0.46</td>
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<td>15600.0</td>
<td>0.46</td>
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<tr>
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<td>94</td>
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<tr>
<td>97</td>
<td>97</td>
<td>20714.2</td>
<td>0.43</td>
</tr>
</tbody>
</table>

2) Distance constraints:

<table>
<thead>
<tr>
<th>Constraint #</th>
<th>Point From #</th>
<th>Point To #</th>
<th>Distance (ft)</th>
<th>Redundancy Number (RN)</th>
</tr>
</thead>
<tbody>
<tr>
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Table B.1 (Continued):

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6) Digitized points:

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<th>Redundancy Number (RN_a)***</th>
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* Redundancy number before adding additional control measurements involving the control points
** Redundancy number after adding additional control measurements involving the control points
*** Redundancy numbers in the directions of x, y, and minor axis of the controllability ellipses, respectively
APPENDIX C

INVERSE OF A PARTITIONED SYMMETRIC MATRIX
INVERSE OF A PARTITIONED SYMMETRIC MATRIX

Given a symmetric matrix of the form \[
\begin{bmatrix}
A & B & C \\
B^T & D & 0 \\
C^T & 0 & E
\end{bmatrix}
\]
like that of equation (6.5), the inverse of this matrix can be found as follows:

Let the inverse be the symmetric matrix of form \[
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12}^T & Q_{22} & Q_{23} \\
Q_{13}^T & Q_{23}^T & Q_{33}
\end{bmatrix}
\], then,

\[
\begin{bmatrix}
A & B & C \\
B^T & D & 0 \\
C^T & 0 & E
\end{bmatrix} \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12}^T & Q_{22} & Q_{23} \\
Q_{13}^T & Q_{23}^T & Q_{33}
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] ............................................................ (C.1)

By expanding this matrix using regular matrix multiplication, we get:

\[
AQ_{11} + BQ_{12}^T + CQ_{13}^T = I \] .................................................. (C.2)
\[
AQ_{12} + BQ_{22} + CQ_{23}^T = 0 \] .................................................. (C.3)
\[
AQ_{13} + BQ_{23} + CQ_{33} = 0 \] .................................................. (C.4)
\[
B^TQ_{11} + DQ_{12}^T = 0 \] .................................................. (C.5)
\[
B^TQ_{12} + DQ_{22} = I \] .................................................. (C.6)
\[
B^TQ_{13} + DQ_{23} = 0 \] .................................................. (C.7)
\[
C^TQ_{11} + EQ_{13}^T = 0 \] .................................................. (C.8)
\[
C^TQ_{12} + EQ_{23}^T = 0 \] .................................................. (C.9)
\[
C^TQ_{13} + EQ_{33} = I \] .................................................. (C.10)
From (C.5):

\[ \Rightarrow Q_{12}^T = -D^{-1}B^TQ_{11} \] ................................. (C.11)

\[ \Rightarrow Q_{12} = -Q_{11}BD^{-1} \] ................................. (C.12)

From (C.8):

\[ \Rightarrow Q_{13}^T = -E^{-1}C^TQ_{11} \] ................................. (C.13)

\[ \Rightarrow Q_{13} = -Q_{11}CE^{-1} \] ................................. (C.14)

Substitute (C.11) and (C.13) into (C.2):

\[ \Rightarrow Q_{11} = (A - BD^{-1}B^T - CE^{-1}C^T)^{-1} \] .................................(C.15)

From (C.6):

\[ \Rightarrow Q_{12} = D^{-1} + D^{-1}B^TQ_{11}BD^{-1} \] .................................(C.16)

From (C.10):

\[ \Rightarrow Q_{33} = E^{-1} + E^{-1}C^TQ_{11}CE^{-1} \] .................................(C.17)

From (C.7):

\[ \Rightarrow Q_{23} = D^{-1}B^TQ_{11}CE^{-1} \] .................................(C.18)

Putting these results together,

\[
\begin{bmatrix}
A & B & C \\
B^T & D & 0 \\
C^T & 0 & E
\end{bmatrix}^{-1} =
\begin{bmatrix}
Q_{11} & -Q_{11}BD^{-1} & -Q_{11}CE^{-1} \\
-D^{-1}B^TQ_{11} & D^{-1}(I + B^TQ_{11}BD^{-1}) & D^{-1}B^TQ_{11}CE^{-1} \\
-E^{-1}C^TQ_{11} & E^{-1}C^TQ_{11}BD^{-1} & E^{-1}(I + C^TQ_{11}CE^{-1})
\end{bmatrix}
\]

................................. (C.19)
APPENDIX D

COORDINATES OF TWO TAX MAPS BEFORE AND AFTER ZIPPING
Figure D.1: Example of two bordering tax maps.
### TABLE D.1
Examples of Map 1 coordinates (Figure D.1) before and after zipping the two maps 1 and 2

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**TABLE D.2**

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