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Stationary subsets of $[\mathbb{N}_\omega]^{<\omega_n}$

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The Ohio State University, 1992
STATIONARY SUBSETS OF $[\mathcal{N}_\omega]^{<\omega_1}$

DISSERTATION

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CHAPTER I

Introduction

Let $\kappa \leq \lambda$ be cardinals with $\kappa$ regular. Let $[\lambda]^{<\kappa} = \{x \subseteq \lambda : |x| < \kappa\}$. A set $C \subseteq [\lambda]^{<\kappa}$ is called closed unbounded if

1. for all $x \in [\lambda]^{<\kappa}$ there exists $y \in C$ with $x \subseteq y$ and
2. for all $X \subseteq C$, if $|X| < \kappa$ and $X$ is directed under inclusion then $\bigcup X \in C$.

We say that a set $S \subseteq [\lambda]^{<\kappa}$ is stationary if it intersects every closed unbounded subset of $[\lambda]^{<\kappa}$.

In his study of the closed unbounded sets in $[\lambda]^{<\kappa}$, Baumgartner [1] has studied the sets $S(\kappa, \lambda, X, f)$ defined as follows:

Let $\kappa \leq \lambda$ be cardinals with $\kappa$ regular. Let $X$ be a set of regular cardinals contained in the interval $(\kappa, \lambda] = \{\alpha : \kappa < \alpha \leq \lambda\}$. Suppose $|X| < \kappa$.

Let $f : X \to \kappa + 1$ be a function such that for all $\alpha \in X$, $f(\alpha)$ is a regular cardinal. We define

$$S(\kappa, \lambda, X, f) = \{A \in [\lambda]^{<\kappa} : \forall \alpha \in X, cf(A \cap \alpha) = f(\alpha)\}.$$  

In [1], Baumgartner has shown that if $X$ is finite and $\kappa \notin X$ then $S(\kappa, \lambda, X, f)$ is stationary for any $f$ as in above. But if $X$ is infinite, the problem is rather complicated.
and very little is known. In [1], Baumgartner has also shown that $S(\kappa, \lambda, X, f)$ is stationary if $\lambda = \sup\{\mu_n : n \in \omega\}$, $\kappa = \omega_1$, $< \mu_n : n \in \omega >$ is an increasing sequence of measurable cardinals, $X = \{\mu_n : n \in \omega\}$ and $f$ is any function from $X$ to $\{\omega, \omega_1\}$. In this paper, we will establish some results about the stationary sets in $[\aleph_\omega]^{\omega+n+1}$.

Throughout this paper, $j : V \to M_1$ will be a fixed elementary embedding and $\kappa_0 = \text{crit}(j)$, i.e., the critical point of $j$ which is the first ordinal moved by $j$. Let $\kappa_{n+1} = j(\kappa_n)$ and $\kappa_\omega = \sup\{\kappa_n : n \in \omega\}$. We will also assume that $V_{\kappa_\omega} \subseteq M_1$. In this paper, we will prove the following theorem:

**Theorem 1** Assume GCH and that there is an elementary embedding $j : V \to M_1$ such that $V_{\kappa_\omega} \subseteq M_1$. Let $1 < n < \omega$ and $X = \{R_i : n < i < \omega\}$. Then there is a forcing extension $V[G]$ of $V$ such that in $V[G]$, GCH holds and $S(\omega_n, R_\omega, X, f)$ is stationary in $[\aleph_\omega]^{\omega+n+1}$ for some $f : X \to \{R_k : k \leq n\}$ assuming the values $R_n$ and $R_{n-1}$ infinitely often.

In the theorem above, the function $f$ there assumes the two values $R_n$ and $R_{n-1}$ infinitely often. We can actually prove that (in the same generic extension $V[G]$ above) for any $k_1, k_2 < n$ there is some $g : X \to \{R_{k_1}, R_{k_2}\}$ such that $S(\omega_n, R_\omega, X, g)$ is stationary and $g$ assumes the values $R_{k_1}$ and $R_{k_2}$ infinitely often. But we will not prove this claim in this paper.

Let $f : X \to \{R_k : k \leq n\}$ with $X$ as in the theorem above. If $|\{\lambda \in \text{range}(f) : f^{-1}\{\lambda\} \text{ is infinite}\}| = m$ and $S(\omega_n, R_\omega, X, f)$ is stationary, we say that we have $m$ cofinalities. Theorem 1 says that we can get two cofinalities together with GCH. Recently, Shelah has proved that if $\sup(\text{pcf}([\aleph_m : m \in \omega])) > \aleph_{\omega+n}$ (In particular,
GCH fails.), then you get $n$ different cofinalities. He has also proved that under the existence of infinitely many measurable cardinals, there is a generic extension of $V$ in which we can get two different cofinalities with one of the two cofinalities being $\omega$.

Magidor also proved that if $f : \{\aleph_m : m \geq 1\} \to \{\omega, \omega_1\}$ is not eventually constant and $S(\omega_1, \aleph_\omega, X, f)$ is stationary in $[\aleph_\omega]^{\omega_2}$, then there is an inner model with infinitely many measurable cardinals.

I would like to thank my thesis advisor Professor Foreman for allowing me to use some of his ideas concerning the consistency of $\aleph_\omega$-Jonsson. He has given me enormous encouragement and guidance during the years of my graduate study at Ohio State.
CHAPTER II

Notations and Preliminaries

For a partial ordering $P$, if $p, q \in P$, $p \leq q$ will mean that $p$ is a stronger condition than $q$. For partial orderings $P, Q$, a map $i : P \rightarrow Q$ is a complete embedding if $i$ is order preserving and for any maximal antichain $A \subseteq P$, $i''A$ is a maximal antichain in $Q$.

For a partial ordering $P$, we let $B(P)$ be the Boolean completion of $P$. If $P$ is separative, there is a one-to-one complete embedding from $P$ to $B(P)$. So we will view $P$ as a subordering of $B(P)$. If $i : P \rightarrow Q$ is a complete embedding, then there is a complete embedding $i' : B(P) \rightarrow B(Q)$ extending $i$. By abuse of notation, we still use $i$ to denote $i'$. We define the natural projection map $\pi : B(Q) \rightarrow B(P)$ by

$$\pi(q) = \bigwedge \{p \in B(P) : i(p) \geq q\}$$

Note that $i \circ \pi(q) \geq q$ and $\pi \circ i$ is the identity map. In this paper We will use both partial order and Boolean algebra terminology interchangeably.

In this paper, all embeddings between partial orderings involved are one to one.

If $P$ is a partial ordering, we let $V^P$ denote the class of $P$-names. If $i : P \rightarrow Q$ is a complete embedding, then $i$ induces a map $i_*$ from $V^P$ to $V^Q$, defined by recursion
on \( \tau \in V^P \):

\[
i_*(\tau) = \{< i_*(\sigma), i(p) > : < \sigma, p > \in \tau \}
\]

If \( P \) is a partial ordering and \( G \) is \( P \)-generic over \( V \) then, for \( \tau \in V^P \), \((\tau)_G\) is the interpretation of the term \( \tau \) under \( G \).

**Lemma 2.0.1** Suppose \( P, Q, i \) and \( i_* \) are as above. Then we have:

(a) If \( G \) is \( Q \)-generic over \( V \), then \((\tau)_i - i_G = (i_*(\tau))_G\) for each \( \tau \in V^P \).

(b) if \( \phi(x_1, \ldots, x_n) \) is a \( \Delta_1^{ZF} \)-formula, then for \( p \in P, p \models_P \phi(\tau_1, \ldots, \tau_n) \) iff \( i(p) \models_Q \phi(i_*(\tau_1), \ldots, i_*(\tau_n)) \).

(c) if \( \phi(x_1, \ldots, x_n) \) is a \( \Delta_1^{ZF} \)-formula, then \( i(\|\phi(\tau_1, \ldots, \tau_n)\|_B(p)) = \|\phi(i_*(\tau_1), \ldots, i_*(\tau_n))\|_B(Q) \).

**Proof.** (a) is easy by induction on \( \text{rank}(\tau) \).

(b) is standard. Now we proceed to prove the equality in (c).

Let \( b_1 = \|\phi(\tau_1, \ldots, \tau_n)\|_B(p) \) and \( b_2 = \|\phi(i_*(\tau_1), \ldots, i_*(\tau_n))\|_B(Q) \). Then \( b_1 = \Sigma\{i(p) : p \in P, p \models \phi(\tau_1, \ldots, \tau_n)\} \) and \( b_2 = \Sigma\{q \in Q : q \models \phi(i_*(\tau_1), \ldots, i_*(\tau_n))\} \). So it's clear that \( b_1 \leq b_2 \) by (b). Similarly, \( -b_1 \leq -b_2 \). Hence \( b_1 = b_2 \). \( \Box \)

Now, let's turn to the iterated forcing. We will adapt the definition for two stage iterations in Kunen's book, Kunen[7].

**Definition 2.0.1** (1) If \( P \) is a partial ordering and \( P \models \text{"\( \tau \) is a partial ordering}" \),

then we define the two-stage iteration \( P \ast \tau \) as follows:

\[
P \ast \tau = \{< p, \sigma > : p \in P \land \sigma \in \text{dom}(\tau) \land p \models_P \neg \sigma \in \tau \}
\]
and $P \times \tau$ is ordered by $< p_1, \sigma_1 > < p_2, \sigma_2 >$ iff $p_1 \leq p_2$ and $p_1 \models \sigma_1 \leq \sigma_2$.

(2) We say $<< P_\beta : \beta \leq \alpha >, < \tau_\beta : \beta < \alpha >>$ is an iterated forcing construction of length $\alpha$ (or $P_\alpha$ is an $\alpha$-stage iteration) iff $P_\alpha$ is a set of $\alpha$-sequences satisfying the following conditions:

(a) If $\alpha = 0$ then $P_0 = \{< >\}$ is the trivial forcing.

(b) If $\alpha = \beta + 1$ for some $\beta$, then $P_\beta = \{p|\beta : p \in P_\alpha\}$ is a $\beta$-stage iteration, $P_\beta \models \tau_\beta$ is a partial ordering, and $p \in P_\alpha$ iff $p|\beta \in P_\beta$, $p(\beta) \in \text{dom}(\tau_\beta)$ and $p|\beta \models p(\beta) \in \tau_\beta$. Moreover, $p \leq q$ iff $p|\beta \leq q|\beta$ and $p|\beta \models p(\beta) \leq q(\beta)$. Thus, $P_\alpha$ is isomorphic to $P_\beta \times \tau_\beta$.

(c) If $\alpha$ is a limit ordinal then for all $\beta < \alpha$ $P_\beta = \{p|\beta : p \in P_\alpha\}$ is a $\beta$-stage iteration, and

(i) $\vec{1} \in P_\alpha$, where $\vec{1}(\beta) = 1_{\tau_\beta}$ for all $\beta < \alpha$. ($1_{\tau_\beta}$ is the maximal element of $\tau_\beta$.)

(ii) if $\beta < \alpha$, $p \in P_\alpha$, $q \in P_\beta$ and $q \leq p|\beta$, then $r \in P_\alpha$, where $r|\beta = q$ and $r(\gamma) = p(\gamma)$ for $\beta \leq \gamma < \alpha$.

(iii) for all $p, q \in P_\alpha$, $p \leq q$ iff for all $\beta < \alpha$ $p|\beta \leq q|\beta$.

(3) Let $P_\alpha$ be an $\alpha$-stage iteration with $\alpha$ a limit ordinal. We say $P_\alpha$ is the direct limit of $< P_\beta : \beta < \alpha >$ iff $p \in P_\alpha$ iff there $\beta < \alpha$ such that $p|\beta \in P_\beta$ and for all $\gamma$ with $\beta \leq \gamma < \alpha$, $p(\gamma) = 1_{\tau_\gamma}$. We say $P_\alpha$ is the inverse limit of $< P_\beta : \beta < \alpha >$ if $p \in P_\alpha$ iff for all $\beta < \alpha$ $p|\beta \in P_\beta$.

(4) For $p \in P_\alpha$, the support of $p$ is the set $\text{supp}(p) = \{\beta < \alpha : p(\beta) \neq 1_{\tau_\beta}\}$.

Definition 2.0.2 Let $P$ and $\tau$ be as above and $\delta$ a regular cardinal.
(i) We say \( \tau \) is \textbf{full for decreasing \( < \delta \)-sequences} if whenever \( p \in P, \eta < \delta,\) \( \sigma_\alpha \in \text{dom}(\tau)(\alpha \in \eta) \) and \( p \Vdash ((\sigma_\alpha \in \tau) \land (\sigma_{\alpha+1} \leq \sigma_\alpha)) \) for each \( \alpha, \) then there is a \( \sigma \in \text{dom}(\tau) \) such that \( p \Vdash \sigma \in \tau \) and \( p \Vdash \sigma \leq \sigma_\alpha \) for each \( \alpha. \)

(ii) \( \tau \) is called \textbf{a nice name} if \( \tau = \text{dom}(\tau) \times \{1_P\} \) and whenever \( p \in P, \sigma \in V^P \) and \( p \Vdash \sigma \in \tau, \) there is a \( \sigma' \in \text{dom}(\tau) \) such that \( p \Vdash \sigma' = \sigma. \)

(iii) Let \( \langle \langle P_\beta : \beta \leq \alpha \rangle, \langle \tau_\beta : \beta < \alpha \rangle \rangle \) be an iterated forcing construction. We say that \( \langle \langle P_\beta : \beta \leq \alpha \rangle, \langle \tau_\beta : \beta < \alpha \rangle \rangle \) has \textbf{\( \delta \)-support} if for all limit ordinal \( \beta < \alpha, \)

(a) if \( \text{cf}(\beta) < \delta \) then \( P_\beta \) is the inverse limit of \( \langle P_\beta : \beta < \alpha \rangle; \)

(b) if \( \text{cf}(\beta) \geq \delta \) then \( P_\beta \) is the direct limit of \( \langle P_\beta : \beta < \alpha \rangle. \)

Remark.

(1) If \( \tau \) is a \( P \)-name for a partial ordering, then there is a nice name \( \tau' \) such that \( P \Vdash \tau' = \tau. \)

(2) If \( \tau \) is a nice name for a partial ordering and \( P \Vdash (\tau \text{ is } \delta \text{-closed}), \) then \( \tau \) is \textbf{full for decreasing \( < \delta \)-sequences.}

(3) Let \( \langle \langle P_\beta : \beta \leq \alpha \rangle, \langle \tau_\beta : \beta < \alpha \rangle \rangle \) be a \( \delta \)-support iterated forcing construction with \( \alpha \) limit. If \( p \) is such that for all \( \beta < \alpha p|\beta \in P_\beta \) and \( |\text{supp}(p)| < \delta, \) then \( p \in P_\alpha. \) Moreover, for each \( p \in P_\alpha, |\text{supp}(p)| < \delta. \)

Lemma 2.0.2 Let \( \langle \langle P_\beta : \beta \leq \alpha \rangle, \langle \tau_\beta : \beta < \alpha \rangle \rangle \) be a \( \delta \)-support iterated forcing construction, and suppose that for each \( \beta < \alpha, \tau_\beta \) is \( P_\beta \)-full for decreasing \( < \delta \)-sequences. Then \( P_\alpha \) is \( \delta \)-closed.
Proof. Let \( \langle \gamma < \eta \rangle \) with \( \eta < \delta \) be a decreasing sequence in \( P_\alpha \). We need to find \( p \in P_\alpha \) such that \( p \leq p_\gamma \) for all \( \gamma < \eta \). We will define \( p|\beta \) by induction on \( \beta \leq \alpha \) such that \( p|\beta \in P_\beta \), \( p|\beta \leq p_\gamma|\beta \) for all \( \gamma < \eta \) and \( \text{supp}(p|\beta) \subseteq \bigcup \{ \text{supp}(p_\gamma) : \gamma < \eta \} \).

Suppose \( \beta < \alpha \) and we have defined \( p|\beta \). Let's define \( p|\beta \). If \( \beta \not\in \bigcup \{ \text{supp}(p_\gamma) : \gamma < \eta \} \), let \( p(\beta) = 1_{\tau_\beta} \). Otherwise, since \( \langle \gamma < \eta \rangle \) is decreasing and \( p|\beta \leq p_\gamma|\beta \) for \( \gamma < \eta \), \( p|\beta \upharpoonright \tau_\beta \models \tau_\beta \land p_\gamma(\beta) \leq p_\gamma(\beta) \) for all \( \gamma < \gamma' < \eta \). Also, \( p_\gamma(\beta) \in \text{dom}(\tau_\beta) \) for all \( \gamma < \eta \) by definition of \( P_\alpha \). Hence there is \( \sigma \in \text{dom}(\tau_\beta) \) such that \( p|\beta \models \sigma \land \forall \gamma < \eta \, \sigma \leq p_\gamma(\beta) \) since \( \tau_\beta \) is \( P_\beta \)-full for decreasing \( < \delta \)-sequences. Let \( p(\beta) = \sigma \). Then \( p|\beta + 1 \in P_{\beta+1} \) and \( p|\beta + 1 \leq p_\gamma|\beta + 1 \) for all \( \gamma < \eta \). It's also clear that \( \text{supp}(p|\beta + 1) \subseteq \bigcup \{ \text{supp}(p_\gamma) : \gamma < \eta \} \).

Now, if \( \beta \) is limit and we have defined \( p|\beta' \) for all \( \beta' < \beta \), then \( p|\beta \) is already defined and clearly \( \text{supp}(p|\beta) \subseteq \bigcup \{ \text{supp}(p_\gamma) : \gamma < \eta \} \) by induction hypothesis. We only need to see that \( p|\beta \in P_\beta \). But this follows from the above remark and \( |\bigcup \{ \text{supp}(p_\gamma) : \gamma < \eta \}| < \delta \) since \( \delta \) is regular, \( \eta < \delta \) and \( |\text{supp}(p_\gamma)| < \delta \) for each \( \gamma \). \( \square \)

\( S(\kappa, \lambda, X, f) \) will always be defined as in §1. If there is no confusion in the context, we will write \( S \) for \( S(\kappa, \lambda, X, f) \). Suppose \( \mathcal{A} \) is a structure of some language. By abuse of notation, we also denote the universe of \( \mathcal{A} \) by \( \mathcal{A} \) itself. Suppose \( X \subseteq \mathcal{A} \) and \( \mathcal{A} \) is skolemized. Let \( \text{sk}^{\mathcal{A}}(X) \) denote the skolem hull of \( X \) in \( \mathcal{A} \).

If \( \kappa \leq \lambda \) are cardinals and \( f \) is a function on \( \lambda \), then \( f|\kappa \) will denote the restriction of \( f \) to \( \kappa \).

Let \( \kappa \leq \lambda \) be ordinals. Let \( \mathcal{A} = \langle \lambda, \kappa, \ldots \rangle \) be a structure. We define \( \mathcal{A}|\kappa \) in the usual way, i.e., the universe of \( \mathcal{A}|\kappa \) is \( \kappa \); for any \( n \)-ary function symbol \( f \),
$f^{A\kappa}(x) = f^A(x)$ if $f^A(x) \in \kappa$ and $f^{A\kappa}(x) = 0$ otherwise; for any $n$-ary relation symbol $R$, $R^{A\kappa} = R^A \cap \kappa^n$.

We now establish some preliminary results which will be used later to prove the main theorem stated in §1.

**Lemma 2.0.3** Let $S = S(\kappa, \lambda, X, f)$ be as in §1. Let $\theta > (\lambda^\kappa)^+$ be a regular cardinal. Let $M \prec < H_\theta, \in, S, \triangleleft, \kappa, \lambda, \ldots >$, $M \supseteq \lambda + 1$ and $|M| = \lambda$, where $\triangleleft$ is a well-ordering of $H_\theta$. Then the following are equivalent:

1. $S$ is stationary in $[\lambda]^{<\kappa^+}$.

2. For any structure $A = < \lambda, \kappa, \ldots >$ with a countable language, there is a $B \prec A$ such that $|B| = \kappa$, $\kappa \subseteq B$ and $B \in S$.

3. there is $N \prec M$ such that $|N \cap \lambda| = \kappa$, $\kappa \subseteq N$, $|N| = \kappa$ and for all $\alpha \in X$, $\text{cf}(N \cap \alpha) = f(\alpha)$.

4. Let $F : M \to \lambda$ be a bijection. Let $\{g_i : i \in \omega\}$ be a list of all skolem functions for $M' = < M, F >$ closed under compositions. Let $N' = < \lambda, \kappa, g_i | \lambda >_{i<\omega}$.

Then there is $B \prec N'$ such that $\kappa \subseteq B$ and $B \in S$.

**Proof.** 1 and 2 are equivalent is standard. 3 implies 4 is obvious.

To see 4 implies 3, we will first show that for all $B \prec N'$, $\text{sk}^{M'}(B) \cap \lambda = B$. Let $\beta \in \text{sk}^{M'}(B) \cap \lambda$, then there exist $i < \omega$ and $\vec{\delta} \in B$ such that $g_i(\vec{\delta}) = \beta$ since $g_i$'s are closed under compositions. But $B \prec N'$, so $\beta \in B$ since $\vec{\delta} \in B$.

Now, let $B \prec N'$ be such that $\kappa \subseteq B$ and $B \in S$ by 4. Let $N = \text{sk}^{M'}(B)$. Then $N \prec M$, $\kappa \subseteq N$, $|N| = \kappa$ and $|N \cap \lambda| = \kappa$ since $B \subseteq N$. But for all $\alpha \in X$, $\text{cf}(B \cap \alpha) =$
\( f(\alpha) \) and \( N \cap \lambda = B \), so for all \( \alpha \in X \), \( cf(N \cap \alpha) = cf(N \cap \lambda \cap \alpha) = cf(B \cap \alpha) = f(\alpha) \). so we have proven 3.

We now show that 3 implies 2. Suppose 2 is false. Then \( < H_\theta, \in, S, <, \kappa, \lambda, \ldots > \models \text{"} \exists A = < \lambda, \ldots > \) with skolem functions closed under compositions such that for all \( B < A \) with \( \kappa \subseteq B \), \( B \) is not in \( S \). Let \( A \) be the \( < \)-least such structure. Then \( A \) is definable in \( < H_\theta, \in, S, <, \kappa, \lambda, \ldots > \). Therefore \( A \in M \). By 3, there is \( N < M \) such that \( |N| = \kappa, \kappa \subseteq N, |N \cap \lambda| = \kappa \) and for all \( \alpha \in X \), \( cf(N \cap \alpha) = f(\alpha) \). Note that \( A \in N \). Let \( B = < A \cap N, \ldots > \). Then \( A \cap N = N \cap \lambda \). Now let \( h \) be any skolem function of \( A \). Then \( h \in N \) since \( A \in N \). So \( N \cap \lambda \) is closed under \( h \). Hence \( B = N \cap \lambda \prec A \). But then \( \kappa \subseteq B \) and \( B \in S \) — a contradiction.

2 implies 4 is easy. □

**Remark.** By the lemma, in order to show \( S \) is stationary, it suffices to consider just one structure on \( \lambda \), namely the structure \( N' \) in 4 above.

Before we proceed for the rest of the section, let’s state a theorem of Baumgartner in Baumgartner[1]:

**Theorem (Baumgartner)** Suppose \( \omega < \kappa \leq \lambda \) and \( \kappa \) is regular. Let \( X \subseteq \{ \mu : \kappa < \mu \leq \lambda \text{ and } \mu \text{ is a regular cardinal} \} \) be such that \( |X| < \kappa \). Suppose also that \( n < \omega \) and \( \langle \mu_i : i \leq n \rangle \) is an increasing sequence of regular cardinals with \( \mu_0 = \kappa \) and \( \mu_n = \lambda^+ \), and that \( \kappa_i < \kappa \) is regular for all \( i < n \). Let

\[
S = \{ x \in [\lambda]^{<\kappa} : \forall i \forall \mu \in X (\mu_i \leq \mu < \mu_{i+1} \rightarrow cf(\sup(x \cap \mu)) = \kappa_i) \}
\]

Then \( S \) is stationary in \( [\lambda]^{<\kappa} \).

For completeness, let’s duplicate the proof of the theorem in Baumgartner[1] here.
Proof. We go by induction on $\alpha$. If $\alpha = \kappa$ then we may assume $n = 1$ and the theorem is obvious. Suppose $\lambda > \kappa$. Let $\mu = \mu_{n-1}$ and suppose $f : [\lambda]^{<\omega} \to \lambda$ is given. We need to find $x \in S$ such that $x$ is closed under $f$ and $x \cap \kappa \in \kappa$. We first construct an increasing sequence $< A_\alpha : \alpha < \kappa_{n-1} >$ of subsets of $\lambda$ such that

1. $X \cap \mu \subseteq A_0$
2. $A_\alpha \cap \mu \in \mu$ and $|A_\alpha \cap \mu| = |A_\alpha|
3. for all $\delta \in X$ if $\mu \leq \delta \leq \lambda$ then $\sup(A_\alpha \cap \delta) < \sup(A_{\alpha+1} \cap \delta)$ for all $\alpha
4. A_\alpha$ is closed under $f$.

Let $A = \cup \{A_\alpha : \alpha < \kappa_{n-1}\}$. Then $\text{cf}(\sup(A \cap \delta)) = \kappa_{n-1}$ for all $\delta \in X$ with $\mu \leq \delta \leq \lambda$. For each such $\delta$ let $a_\delta \subseteq A \cap \delta$ be a cofinal set of size $\kappa_{n-1}$. Let $a$ be the union of all such $a_\delta$.

By (2) we have $A \cap \mu \in \mu$. Let $\lambda' = |A|$. If $\lambda' < \kappa$ then $A \in S$ and $\mu = \kappa$ and we are done. So suppose $\kappa \leq \lambda'$.

Let $\pi : \lambda' \to A$ be a bijection, and let $C$ be the set of all $x \in [\lambda']^{<\kappa}$ such that if $y$ is the smallest subset of $\lambda$ such that $x \cup a \subseteq y$, $y \cap \kappa \in \kappa$, and $y$ is closed under $\pi$, $\pi^{-1}$ and $f$, then $y \cap \lambda' = x$. It is clear that $C$ is closed unbounded in $[\lambda']^{<\kappa}$, so by inductive hypothesis there is $x \in C$ such that for all $\delta \in X$ if $\mu_i \leq \delta < \mu_{i+1}$ then $\text{cf}(\sup(x \cap \delta)) = \kappa_i$. But now if $y$ is the smallest subset of $\lambda$ satisfying the conditions above, the $y \cap \lambda' = x$. So $\text{cf}(\sup(y \cap \delta)) = \text{cf}(\sup(x \cap \delta)) = \kappa_i$ for $i < n - 1$, while $a \subseteq y \subseteq A$ so $\text{cf}(\sup(y \cap \delta)) = \kappa_{n-1}$ for $\delta$ with $\mu_{n-1} \leq \delta \leq \lambda$. This completes the proof. □

The following lemma uses an idea of Shelah.
Lemma 2.0.4 Let $\kappa < \lambda$ be cardinals and $2^\kappa = \kappa^+$. Let $\mathcal{A} = \langle \lambda, \kappa, \in, (F_n)_{n<\omega}, \ldots \rangle$ be a structure of some countable language, where $F_n : \kappa^+ \times \kappa^n \rightarrow \kappa$ is such that $< F_n(\alpha, *) : \alpha < \kappa^+ >$ is a list of all the functions from $\kappa^n$ to $\kappa$. Suppose $\mathcal{A}$ is skolemized and the skolem functions of $\mathcal{A}$ are closed under compositions. Let $\mathcal{B} < \mathcal{A}|\kappa^+$ and $\mathcal{C} < \mathcal{A}$ be such that $\mathcal{C} \cap \kappa^+ \subseteq \mathcal{B}$. Then $\text{sk}^\mathcal{A}(\mathcal{C} \cup (\mathcal{B} \cap \kappa)) \cap \kappa = \mathcal{B} \cap \kappa$.

Proof. Let $f$ be any skolem function of $\mathcal{A}$. Let $\bar{c} \in [\mathcal{C}]^{<\omega}$. Suppose $f(\bar{c}, \ast)$ is an $n$-ary function. Let $\varphi(\alpha, \bar{c})$ be the following formula in the language of $\mathcal{A}$:

$$\forall \alpha \in \kappa^n((f(\bar{c}, x) \prec \kappa \rightarrow F_n(\alpha, x) = f(\bar{c}, x)) \wedge (f(\bar{c}, x) \geq \kappa \rightarrow F_n(\alpha, x) = 0))$$

Then $\mathcal{A} \models \exists \alpha \varphi(\alpha, \bar{c})$. Since $\mathcal{C} < \mathcal{A}$, there's $\alpha \in \mathcal{C}$ be such that $\mathcal{C} \models \varphi(\alpha, \bar{c})$. But then $\alpha < \kappa^+$ and $\alpha \in \mathcal{C} \cap \kappa^+ \subseteq \mathcal{B} \subseteq \mathcal{A}|\kappa^+$. So $\mathcal{A}|\kappa^+ \models \varphi(\alpha, \bar{c})$ since $\mathcal{A} \models \varphi(\alpha, \bar{c})$.

But $\mathcal{B} < \mathcal{A}|\kappa^+$, so $\mathcal{B} \models \varphi(\alpha, \bar{c})$. Now, since $\mathcal{A}|\kappa^+ \models \forall \alpha \in \kappa^n \exists \gamma F_n(\alpha, x) = \gamma$, so $\mathcal{B} \models \forall \alpha \in \kappa^n \exists \gamma F_n(\alpha, x) = \gamma$. Hence if $f(\bar{c}, x) < \kappa$ and $x \in (\mathcal{B} \cap \kappa)^n$, then $F_n(\alpha, x) = f(\bar{c}, x) = \gamma \in \mathcal{B} \cap \kappa$ for some $\gamma$. Therefore, $\text{sk}^\mathcal{A}(\mathcal{C} \cup (\mathcal{B} \cap \kappa)) \cap \kappa = \mathcal{B} \cap \kappa$.

The above lemma leads to the following lemma which will be used in the proof of our main theorem.

Lemma 2.0.5 Let $< \mu_n : n \in \omega >$ be an increasing sequence of regular cardinals such that $2^{\mu_n} = \mu_n^+$ for each $n$ and $\mu_{n+1} = \mu_n^{+k_n}$ for some $0 < k_n < \omega$. Let $\mu = \sup \{ \mu_n : n < \omega \}$. Let $Z$ be the set of all regular cardinals in the interval $[\mu_0, \mu)$. Let $\eta < \mu_0$ be a regular cardinal. Let $\mathcal{A} = \langle \mu, \in, (\lambda)_{\lambda \in Z}, \eta, (F_{nm})_{n,m<\omega}, \ldots \rangle$ be a fully skolemized structure with skolem functions closed under composition, where $F_{nm} : \mu_n^+ \times \mu_n^m \rightarrow \mu_n$ is such that $< F_{nm}(\alpha, *) : \alpha < \mu_n^+ >$ lists all the functions from $\mu_n^m$ to $\mu_n$. 

Let $< B_n : n \in \omega >$ be such that

(a) $B_n \prec A|\mu_n^+; |B_n| = \eta^+$ and $|B_n \cap \mu_n| = \eta$;

(b) $B_{n+1} \cap \mu_n^+ \subseteq B_n$;

(c) For all regular $\lambda$ with $\mu_{n-1} \leq \lambda \leq \mu_n$, $cf(B_n \cap \lambda) \leq \eta$.

Then there is a structure $B \prec A$ satisfying that $|B| = \eta$, $\eta \subseteq B$ and for all regular cardinals $\mu_{n-1} < \lambda \leq \mu_n$, $cf(B \cap \lambda) = cf(B_n \cap \lambda)$.

**Proof.**

First we define $< B'_n : n < \omega >$ as follows:

(i) $B'_0 = B_0$

(ii) for all $n$, $B'_{n+1} = sk^A|\mu^+_{n+1}(B_{n+1} \cup (B'_n \cap \mu_n))$.

Notice that $B'_{n+1} \cap \mu_n = B'_n \cap \mu_n$, since $B_{n+1} \cap \mu_n^+ \subseteq B_n \subseteq B'_n$ and Lemma 2.0.4.

**Claim 1:** For all $n$, $cf(B'_n \cap \lambda) = cf(B_n \cap \lambda)$ for regular $\lambda$ with $\mu_{n-1} < \lambda \leq \mu_n$.

**Proof of claim 1.** For $n = 0$, since $B'_0 = B_0$, Claim 1 is clearly true. For $n > 0$, we show $sup(B'_n \cap \lambda) = sup(B_n \cap \lambda)$ for regular cardinals $\mu_{n-1} < \lambda \leq \mu_n$. It is clear that $sup(B_n \cap \lambda) \leq sup(B'_n \cap \lambda)$ since $B_n \subseteq B'_n$.

On the other hand, let $\beta \in B'_n \cap \lambda$. Then there exists a skolem function $g$, and $\bar{a} \subseteq B_n$, $\bar{x}_0 \subseteq B'_{n-1} \cap \mu_{n-1}$ such that $\beta = g(\bar{a}, \bar{x}_0)$. If $\bar{x}_0 = \emptyset$ then $\beta = g(\bar{a}) \in B_n$, so $\beta < sup(B_n \cap \lambda)$. If $\bar{x}_0 \neq \emptyset$, let $m$ be the length of $\bar{x}_0$. We define the function $h : \mu^m_{n-1} \rightarrow \lambda$ by

$$h(\bar{x}) = \begin{cases} g(\bar{a}, \bar{x}) & \text{if } g(\bar{a}, \bar{x}) < \lambda \\ 0 & \text{otherwise} \end{cases}$$
Then $h$ is definable in $A|\mu_n^+$ from $\bar{a}$. So $A|\mu_n^+ \models (\exists \alpha (\alpha < \lambda \land \alpha = sup(h''\mu_{n-1}^n)))$.

But $B_n \prec A|\mu_n^+$, so there exists $\alpha \in B_n$ such that $B_n \models \alpha = sup(h''\mu_{n-1}^n)$. But then $A|\mu_n^+ \models \alpha = sup(h''\mu_{n-1}^n)$. Since $A|\mu_n^+ \models \beta = g(\bar{a}, \bar{x}_0) = h(\bar{x}_0)$, $\beta < \alpha \in B_n \cap \lambda$. Hence $sup(B_n' \cap \lambda) \leq sup(B_n \cap \lambda)$. □

For each $n$, let $X_n = B_n' \cap \mu_n$. Let $B' = sk^A(\cup_{n<\omega} X_n)$. Note that $B' = \cup_{n<\omega} sk^A(X_n)$ since $X_n = B_n' \cap \mu_n = B'_{n+1} \cap \mu_n \subseteq X_{n+1}$.

Claim 2: $B' = \cup_{n<\omega} X_n$.

Proof of claim 2. We first show that $sk^A(X_n) \cap \mu_n = X_n$. It is clear that $X_n \subseteq sk^A(X_n) \cap \mu_n$. On the other hand, if $\beta \in sk^A(X_n) \cap \mu_n$, then there is a skolem function $f$ and $\bar{\alpha} \subseteq X_n$ such that $f(\bar{\alpha}) = \beta$. But then $\bar{\alpha} \subseteq X_n = B_n' \cap \mu_n$.

Since $B_n' \prec A|\mu_n^+$, $\beta \in \mu_n$, and $A|\mu_n^+ \models f(\bar{\alpha}) = \beta$, so $A|\mu_n^+ \models \exists \delta f(\bar{\alpha}) = \delta$. So $B_n' \models \exists \delta f(\bar{\alpha}) = \delta$. Let $\delta \in B_n'$ be such that $B_n' \models f(\bar{\alpha}) = \delta$, then $\beta = f(\bar{\alpha}) = \delta \in B_n'$. So $sk^A(X_n) \cap \mu_n \subseteq B_n' \cap \mu_n = X_n$. Hence $sk^A(X_n) \cap \mu_n = X_n$.

Now, for each $\alpha \in B'$, $\alpha \in sk^A(X_n)$ for some $n$ since $B' = \cup_{n<\omega} sk^A(X_n)$. But if $m$ is such that $\alpha < \mu_m$ and $n \leq m$, then $\alpha \in sk^A(X_n) \subseteq sk^A(X_m)$. So $\alpha \in sk^A(X_m) \cap \mu_m = X_m$.

Hence, $B' = \cup_{n<\omega} X_n$. We are done. □.

Claim 3:

(a) For all $n$, $B' \cap \mu_n = X_n$

(b) For all $n > 0$, $cf(B' \cap \lambda) = cf(B_n \cap \lambda)$ for regular $\mu_{n-1} < \lambda \leq \mu_n$.

Proof of claim 3. For (a), we first notice that for all $n$, $X_{n+1} \cap \mu_n = (B_{n+1} \cap \mu_n$
\( \mu_{n+1} \cap \mu_n = B_{n+1}' \cap \mu_n = B_n' \cap \mu_n = X_n. \)

Now, fix \( n \). We show \( X_n \cap \mu_n = X_n \) by induction on \( m > n \).

Case 1: \( m = n + 1 \). \( X_{n+1} \cap \mu_n = X_n \) by the observation above.

Case 2: \( m \geq n + 1 \). Assume we have shown \( X_m \cap \mu_n = X_n \). Then \( X_{m+1} \cap \mu_n = (X_{m+1} \cap \mu_m) \cap \mu_n = X_m \cap \mu_n = X_n \). So we proved (a).

Now we prove (b). For all \( \mu_{n-1} < \lambda \leq \mu_n \), \( cf(B' \cap \lambda) = cf(B' \cap \mu_n \cap \lambda) = cf(X_n \cap \lambda) = cf(B_n' \cap \lambda) = cf(B_n \cap \lambda) \) by Claim 1. \( \square \)

**Claim 4:** \( |B'| = \eta. \)

**Proof of claim 4.** Suppose \( |B'| > \eta \), then \( |B'| = \eta^+. \) Let \( \lambda \) be the least regular cardinal such that \( |B' \cap \lambda| = \eta^+ \). Then there exists \( n \) such that \( \mu_{n-1} < \lambda \leq \mu_n \). But \( B' \cap \mu_n = X_n = B_n' \cap \mu_n \). So \( |B_n' \cap \lambda| = \eta^+. \) Since \( cf(B_n' \cap \lambda) \leq \eta \), there is \( \alpha \in B_n' \cap \lambda \) such that \( |B_n' \cap \alpha| = \eta^+. \)

Suppose \( |\alpha| = \lambda' \). Then \( A|\mu_n^+| = "\exists F : \alpha \rightarrow \lambda' \text{ is a bijection}". (Notice that \( F \) can not be a member of \( A|\mu_n^+| \). But we can use the \( F_{nm} \)'s to "code" \( F \). We will just say \( A|\mu_n^+| = "\exists F \ldots", \) by abuse of language.) Let \( F \) be such that \( B_n' = "F : \alpha \rightarrow \lambda' \text{ is a bijection}". Then \( F''(B_n' \cap \alpha) \subseteq B_n' \cap \lambda'. \) So \( |B' \cap \lambda'| = |B' \cap \mu_n \cap \lambda'| = |X_n \cap \lambda'| = |B_n' \cap \mu_n \cap \lambda'| = |B_n' \cap \lambda'| = \eta^+ \), which is a contradiction since \( \lambda' < \lambda \). \( \square \)

We are now almost done with the proof of the lemma except that we don’t know if \( \eta \subseteq B' \). If \( \eta \subseteq B' \), then just let \( B = B' \) which will be as desired. If not, let \( B = sk^A(B' \cup \eta) \). Let \( \lambda \) be regular such that \( \mu_{n-1} < \lambda \leq \mu_n \). We must show that \( cf(B \cap \lambda) = cf(B' \cap \lambda) \). we show this by showing that \( sup(B \cap \lambda) = sup(B' \cap \lambda) \).

It suffices to show that \( sup(B \cap \lambda) \leq sup(B' \cap \lambda) \). So let \( \delta \in B \cap \lambda \). Then there
are $\bar{b} \subseteq B'$, $\bar{a} \subseteq \eta$ and a skolem function $f$ such that $\delta = f(\bar{b}, \bar{a})$. Let $m = |a|$. Define $g : \eta'' \to \lambda$ by letting $g(\bar{x}) = f(\bar{b}, \bar{x})$ if $f(\bar{b}, \bar{x}) < \lambda$ and letting $g(\bar{x}) = 0$ otherwise. Then $g$ is definable from $\bar{b}$ in $B'$. So there is $\alpha \in B' \cap \lambda$ such that $B' \models \"\alpha = \sup(g''(\eta))\"$. But then $\delta = g(\bar{a}) < \alpha \in B' \cap \lambda$ since $B' \prec B$, we are done. □

We now turn to the iterations of the elementary embedding $j$ mentioned above.

**Definition 2.0.3** Let $j : V \to M_1$ be such that $V_{\kappa_\omega} \subseteq M_1$. Let $j_{01} = j$. We can define $j_{12}$ to be $j(j)$, which is the relativization of the definition of $j$ to $M_1$, and $M_2$ the relativization of the definition of $M_1$ to $M_1$. In general, we define $j_{n,n+1} = j_{0n}(j)$ to be the relativization of the definition of $j$ to $M_n$, and $M_{n+1}$ to be the relativization of the definition of $M_1$ to $M_n$. We also define $j_{mn} = id$, and $j_{m} = j_{n-1} \circ j_{n-2,n-1} \circ \ldots \circ j_{m+1,m+2} \circ j_{m,m+1}$ for all $m < n$. We also define $M_\omega$ to be the direct limit of the directed system $\{M_n, j_{mn} \mid m, n \in \omega \text{ and } m \leq n\}$.

**Remark.** We can also use iterated ultrapower construction to obtain a sequence of embeddings $j_{mn}$ with properties stated in the next lemma.

The following lemma is standard and its proof is omitted. For those who are interested in the details, see Martin[9].

**Lemma 2.0.6**

1. For all $x \in V$, $j_{0n}(x) = j^n(x)$. In particular, $j_{kn}(\kappa_m) = \kappa_{m+n-k}$ for $k, m, n \in \omega$, $k \leq n$ and $k \leq m$.

2. For all $n \in \omega$, $\kappa_n = \text{crit}(j_{n,n+1})$.

3. $M_\omega$ is well founded.

4. For all $n < \omega$, $V_{\kappa_n}^{M_n} = V_{\kappa_m}$ for all $m \in \omega$. 
CHAPTER III

The Forcing Notion

In this section, we are going to define the forcing notion which we are going to use to prove our main theorem stated in Chapter I.

Definition 3.0.4 We define the Silver collapse $Sc(\mu_1, \mu_2)$ for cardinals $\mu_1 < \mu_2$ as follows: for any $p, p \in Sc(\mu_1, \mu_2)$ iff

(i) $p$ is a partial function from $\mu_2 \times \mu_1$ to $\mu_2$;

(ii) For all $(\gamma, \beta) \in \text{dom}(p)$, $p(\gamma, \beta) \in \gamma$

(iii) $|p| \leq \mu_1$ and $\bigcup\{\beta < \mu_1 : \exists \gamma < \mu_2, (\gamma, \beta) \in \text{dom}(p)\} < \mu_1$.

$Sc(\mu_1, \mu_2)$ is ordered by $p_1 \leq p_2$ iff $p_2 \subseteq p_1$.

Remark. $Sc(\mu_1, \mu_2)$ is $\mu_1$-directed closed if $\mu_1$ is regular. If $\mu_2$ is inaccessible, then $Sc(\mu_1, \mu_2)$ is $\mu_2$-c.c.

If $Q$ is another forcing notion, we use $Sc^Q(\mu_1, \mu_2)$ to denote the Silver collapse $Sc(\mu_1, \mu_2)$ defined in $V^Q$. If there is no confusion, we will write $Q \ast Sc^Q(\mu_1, \mu_2)$ as $Q \ast \tilde{Sc}(\mu_1, \mu_2)$.

Definition 3.0.5 Let $\kappa$ be a Mahlo cardinal. Let $\delta < \kappa$ be a regular cardinal. We call a forcing notion $P$ a $(\delta, \kappa)$-universal collapse if $P$ is a forcing notion satisfying:
(1) \(|P|=\kappa, P \subseteq V_\kappa\) and \(P \models \kappa = \delta^+;\)

(2) \(P\) has \(\kappa\)-c.c. and is \(<\delta\)-closed;

(3) For all \(\delta < \lambda < \kappa\) with \(\lambda\) inaccessible, for all \(Q \subseteq V_\lambda\), if there is a complete embedding \(i: Q \to P\), then there are complete extensions of \(i\), \(\overline{i}: Q \ast Sc^Q(\lambda, \kappa) \to P\) and \(\overline{i}: Q \ast Sc^Q(\lambda^{++}, \kappa) \to P\), where \(Sc^Q(\lambda, \kappa)\) and \(Sc^Q(\lambda^{++}, \kappa)\) are nice \(Q\)-names.

If \(\delta\) is clear from context, we will simply call a \((\delta, \kappa)\)-universal collapse a \(\kappa\)-universal collapse.

Before we prove the existence of universal collapses, we prove the following:

**Lemma 3.0.7** Let \(P, Q\) be partial ordering and \(i: Q \to P\) be a complete embedding. Let \(i_*: V^Q \to V^P\) be the map induced by \(i\). Let \(\tau\) be a nice \(Q\)-name for a partial ordering. We define \(\overline{i}: Q \ast \tau \to P \ast i_*(\tau)\) by \(\overline{i}(<q, \sigma>) = <i(q), i_*(\sigma)>\). Then \(\overline{i}\) is a complete embedding extending \(i\). Furthermore, if \(\tau\) is \(Q\)-full for decreasing \(<\delta\)-sequences, then \(i_*(\tau)\) is \(P\)-full for decreasing \(<\delta\)-sequences.

**Proof.** It's easy to see that \(\overline{i}\) is order-preserving and incompatibility-preserving. In order to see that \(\overline{i}\) is complete, it suffices to show that for all \(<p, i_*(\sigma)>\in P \ast i_*(\tau), \exists <q, \sigma_0>\) such that for all \(r \leq <q, \sigma_0>, \overline{i}(r)\) and \(<p, i_*(\sigma)>\) are compatible. Let \(<p, i_*(\sigma)>\in P \ast i_*(\tau)\). Then \(\exists q \in Q, \forall q' \leq q, i(q')\) and \(p\) are compatible since \(i\) is a complete embedding. Then for all \(<q', \sigma'> \leq <q, \sigma>\), we have \(q' \leq q\) and \(q' \models \neg\sigma' \leq \sigma\). By Lemma 2.0.1, we have \(i(q') \models \neg p \ast i_*(\sigma') \leq i_*(\sigma)\).

But \(i(q')\) and \(p\) are compatible, so if \(p' \leq i(q') \land p\) then \(p' \models \neg i_*(\sigma') \leq i_*(\sigma)\). So \(<p', i_*(\sigma') > \leq <i(q'), i_*(\sigma') > \land <p, i_*(\sigma)>\). Hence \(\overline{i}(<q', \sigma>)\) and \(<p, i_*(\sigma)>\) are compatible. So \(\overline{i}\) is complete.
Now, suppose $\tau$ is $Q$-full for decreasing $< \delta$-sequences, we want to show that $i_\ast(\tau)$ is $P$-full for decreasing $< \delta$-sequences. Notice that the domain of $i_\ast(\tau)$ is $i_\ast dom(\tau)$ by the definition of $i_\ast(\tau)$. Let $p \in P$, $\eta < \delta$, $\sigma_\alpha \in dom(\tau)$ for each $\alpha$ and $p \models i_\ast(\sigma_\alpha) \in i_\ast(\tau) \land i_\ast(\sigma_{\alpha'}) \leq i_\ast(\sigma_\alpha)$ for each $\alpha < \alpha'$. Let $\pi : B(P) \to B(Q)$ be the natural projection map. Then $p \leq \|i_\ast(\sigma_{\alpha'}) \leq i_\ast(\sigma_\alpha)\|_{B(P)}$. So $\pi(p) \leq \pi(\|i_\ast(\sigma_{\alpha'}) \leq i_\ast(\sigma_\alpha)\|_{B(P)}) = \pi \circ i(\|\sigma_{\alpha'} \leq \sigma_\alpha\|_{B(Q)}) = \|\sigma_{\alpha'} \leq \sigma_\alpha\|_{B(Q)}$ by Lemma 2.0.1. So $\pi(p) \models Q \sigma_{\alpha'} \leq \sigma_\alpha$. Since $\tau$ is a nice $Q$-name for a partial ordering and is $Q$-full for decreasing $< \delta$-sequences, there exists $\sigma \in dom(\tau)$ such that $\pi(p) \models \sigma \leq \sigma_\alpha$ for each $\alpha$. So $i(\pi(p)) \models p \models i_\ast(\sigma) \leq i_\ast(\sigma_\alpha)$ for each $\alpha$ by Lemma 2.0.1. But $p \leq i(\pi(p))$, so $p \models \sigma_\alpha \leq i_\ast(\sigma_\alpha)$ for each $\alpha$. Hence $i_\ast(\tau)$ is $P$-full for decreasing $< \delta$-sequences. □

The following construction of universal collapses is similar to the construction of Kunen[6]. The particular construction here and the chain condition argument is very close to that of Laver[8].

Lemma 3.0.8 There exists a $(\delta, \kappa)$-universal collapse for $\kappa$ a Mahlo cardinal and $\delta < \kappa$ regular.

Proof. We will build a $< \delta$-support iteration of forcings, $< P_\xi : \xi \leq \kappa >$ such that

(i) for all $\xi$, $|P_\xi| \leq \kappa$, $P_\xi$ is $\kappa$-c.c. and $P_\xi \subseteq V_\kappa$;

(ii) If $\xi$ is limit and $cf(\xi) \geq \delta$, $P_\xi$ is the direct limit of $< P_\eta : \eta < \xi >$;

(iii) If $\xi$ is limit and $cf(\xi) < \delta$, $P_\xi$ is the inverse limit of $< P_\eta : \eta < \xi >$.

Let $\pi : \kappa \times \kappa \to \kappa$ be a bijection such that for all $\alpha, \beta < \kappa$ $\pi(\alpha, \beta) \geq \alpha$. We will define $P_\xi$ and $< (i_{\xi\beta}, Q_{\xi\beta}, \lambda_{\xi\beta}) : \beta < \kappa >$ by induction on $\xi < \kappa$ such that
(1) for each $\beta$, $Q_{\xi \beta} \subseteq V_{\lambda_{\xi \beta}}$, $\lambda_{\xi \beta} < \kappa$ is inaccessible and $i_{\xi \beta} : Q_{\xi \beta} \rightarrow P_{\xi}$ is a complete embedding.

(2) $<(i_{\xi \beta}, Q_{\xi \beta}, \lambda_{\xi \beta}) : \beta < \kappa>$ is a list of all possible triples satisfying (1) above. (Note there are only $\kappa$ many such triples since $\kappa$ is inaccessible.)

Let $P_0 = \{1\}$ be the trivial forcing. Let $P_1 = Sc(\delta, \kappa)$. Let $<(i_{1 \beta}, Q_{1 \beta}, \lambda_{1 \beta}) : \beta < \kappa>$ be an enumeration of all the triples satisfying (1) above with 1 in place of $\xi$ there.

Suppose we have constructed $P_{\xi}$ and $<(i_{\xi' \beta}, Q_{\xi' \beta}, \lambda_{\xi' \beta}) : \beta < \kappa>$ for $\xi' < \xi$.

Let $<\alpha, \beta >$ be such that $\pi(\alpha, \beta) = \xi$. We consider the complete embedding $i_{\alpha \beta} : Q_{\alpha \beta} \rightarrow P_{\alpha} \subseteq P_{\xi}$. Then $i_{\alpha \beta}$ is a complete embedding from $Q_{\alpha \beta}$ to $P_{\xi}$ also. Let $Q = Q_{\alpha \beta}$, $i = i_{\alpha \beta}$, $\lambda = \lambda_{\alpha \beta}$. Let $i_* : V^Q \rightarrow V^{P_{\xi}}$ be the map induced by $i$. Let $\tau \in V^Q$ be such that $Q \models \tau = Sc^Q(\lambda, \kappa)$ and $\tau$ is a nice $Q$-name. Notice that the $\tau$ here can be chosen to be such that $\tau \subseteq V_{\kappa}$. Let $P_{\xi + 1}' = P_{\xi} * i_*(\tau)$. Then there is a natural complete embedding $\tilde{i} : Q * \tau \rightarrow P_{\xi + 1}'$ such that $\tilde{i}(<q, \sigma>) = <i(q), i_*(\sigma)>$ for $<q, \sigma> \in Q * \tau$. Then $\tilde{i}$ is an extension of $i$ by Lemma 3.0.7. Since $\tau$ is a nice $Q$-name and $Q \models \tau$ is $<\delta$-closed, $\tau$ is $Q$-full for decreasing $< \delta$-sequences. Therefore $i_*(\tau)$ is $P_{\xi}$-full for decreasing $< \delta$-sequences by Lemma 3.0.7.

Now, let $\tau' \in V^Q$ be such that $Q \models \tau' = Sc^Q(\lambda^{++}, \kappa)$ and $\tau'$ is $Q$-nice. Let $P_{\xi + 1} = P_{\xi + 1} * i_*(\tau')$. Then we similarly have that there’s a complete embedding $\tilde{i}' : Q * \tau' \rightarrow P_{\xi + 1}$ extending $i$ and $i_*(\tau')$ is full for decreasing $< \delta$-sequences.

Let $<(i_{\xi + 1 \beta}, Q_{\xi + 1 \beta}, \lambda_{\xi + 1 \beta}) : \beta < \kappa>$ be a list of all triples satisfying (1) above with $\xi + 1$ in place of $\xi$ there. This completes the construction of $<P_{\xi} : \xi \leq \kappa>$. 
By Lemma 2.0.2, for all $\xi \leq \kappa$, $P_\xi$ is $\delta$-closed. So $\delta$ is not collapsed in $V^{P_\kappa}$. It's also clear that $P_\xi \subseteq V_\kappa$ for each $\xi < \kappa$. Since $P_\kappa$ is the direct limit of $< P_\xi : \xi < \kappa >$, we can identify each $p \in P$ with $p|\alpha$ if $\text{supp}(p) \subseteq \alpha$ and $\alpha < \kappa$. Therefore, we may assume $P_\kappa \subseteq V_\kappa$. It remains to show that $P_\kappa$ has $\kappa$-c.c. By Theorem 2.2 in Baumgartner[2], it suffices to show that for all $\alpha < \kappa$, $P_\alpha$ has $\kappa$-c.c.

Claim: For all $\alpha < \kappa$, $P_\alpha$ has $\kappa$-c.c.

Proof of Claim. Notice that we can view $P_\alpha$ as an $\alpha$-stage iteration with $\delta$-support such that for each $\xi < \alpha$, $P_{\xi+1} = P_\xi \ast (i_\xi)_*(\tau_\xi)$ where either $Q_\xi \models \tau_\xi = Sc^{Q_\xi}(\lambda_\xi, \kappa)$ or $Q_\xi \models \tau_\xi = Sc^{Q_\xi}(\lambda_\xi^{++}, \kappa)$, and $i_\xi : Q_\xi \rightarrow P_\xi$ is a complete embedding. Now let $\{p_\gamma : \gamma < \kappa\} \subseteq P_\alpha$. Since $\kappa$ is inaccessible, $|P(\alpha)| < \kappa$. So we can assume for all $\gamma_1, \gamma_2$, $\text{supp}(p_{\gamma_1}) = \text{supp}(p_{\gamma_2})$. Let $X \subseteq \alpha$ be this common support.

For each $\xi \in X$, $P_{\xi+1} = P_\xi \ast i_\xi(\tau_\xi)$, where $i = i_\xi$ and $\tau_\xi$ is a $P_\xi$-nice name such that $Q_\xi \models \tau_\xi = Sc^{Q_\xi}(\lambda_\xi', \kappa)$, where either $\lambda_\xi' = \lambda_\xi^{++}$ or $\lambda_\xi' = \lambda_\xi$. Let $p_\gamma(\xi) = (i_\xi)_*(\sigma_\gamma)$. Since $Q_\xi$ has $\lambda_\xi^{++}$-c.c. by the fact that $Q_\xi \subseteq V_{\lambda_\xi}$ and $\lambda_\xi$ is inaccessible, for each $\gamma < \kappa$, we can find a set $D_{\gamma_\xi} \subseteq \kappa$ such that

\begin{enumerate}
  \item $|D_{\gamma_\xi}| \leq \lambda_\xi'$;
  \item $Q_\xi \models \text{dom}(\sigma_\gamma) \subseteq D_{\gamma_\xi} \times \lambda_\xi'$.
\end{enumerate}

Let $\eta = \sup\{\lambda_\xi' : \xi \in X\}$. Then $\eta < \kappa$. We define $f : \kappa \setminus (\eta \cup \alpha) \rightarrow \kappa$ by $f(\gamma) = \sup((\bigcup_{\xi \in X} D_{\gamma_\xi}) \cap \gamma)$. If $\gamma$ is inaccessible, then $f(\gamma) < \gamma$ since $|\bigcup_{\xi \in X} D_{\gamma_\xi}| < \gamma$. Since $\kappa$ is Mahlo, there's a stationary $S \subseteq \kappa$ and $\beta < \kappa$ such that for all $\gamma \in S$, $f(\gamma) \leq \beta$. Choose $S' \subseteq S$ of size $\kappa$ such that for all $\gamma_1, \gamma_2 \in S'$, $(\bigcup_{\xi \in X} D_{\gamma_{1\xi}}) \cap (\bigcup_{\xi \in X} D_{\gamma_{2\xi}}) \subseteq \beta$. (W.l.o.g., we may assume $\beta > \eta$ and $\beta$ is inaccessible.) For each $\gamma \in S'$, we build $q_\gamma$ by letting,
for \( \xi \in X \), \( q_\gamma(\xi) = (i_\xi)_*(\sigma'_{\gamma \xi}) \), where \( \sigma'_{\gamma \xi} \in \text{dom}(\tau_\xi) \) and \( Q_\xi \models \sigma'_{\gamma \xi} = \sigma_{\gamma \xi}\big| D_{\gamma \xi} \cap \beta \times \lambda_\xi \).

Note that such a \( \sigma'_{\gamma \xi} \) exists by the niceness of \( \tau_\xi \). So \( q_\gamma \) is well-defined. Notice that \( Q_\xi \models \sigma'_{\gamma \xi} \in S_c Q^z(\lambda_\xi, \beta) \) since \( Q_\xi \models \sigma'_{\gamma \xi} \subseteq \beta \times \lambda_\xi \times \beta \). Since \( |Q_\xi| = \lambda_\xi \), there are at most \( 2^{\beta \cdot \lambda_\xi} \) distinct \( \sigma'_{\gamma \xi} \) up to equivalence in \( V^{Q_\xi} \). (Note that two \( Q_\xi \)-terms \( \sigma \) and \( \sigma' \) are equivalent if \( Q_\xi \models \sigma = \sigma' \).) So

\[
|\{< q_\gamma(\xi) : \xi \in X > : \gamma \in S'\}| \leq |\{< \sigma'_{\gamma \xi} : \xi \in X > : \gamma \in S'\}| \leq (2^{\beta})^{\lambda_\xi} < \kappa
\]

Therefore there must be \( \gamma_1 \) and \( \gamma_2 \) in \( S' \) such that \( < q_{\gamma_1}(\xi) : \xi \in X > = < q_{\gamma_2}(\xi) : \xi \in X > \) (up to equivalence).

Now we show that \( p_{\gamma_1} \) and \( p_{\gamma_2} \) are compatible. It will suffice to show that for all \( \xi \in X \), \( Q_\xi \models \sigma_{\gamma_1 \xi} \) and \( \sigma_{\gamma_2 \xi} \) are compatible. Let \( D \) be such that \( Q_\xi \models D = \text{dom}(\sigma_{\gamma_1 \xi}) \cap \text{dom}(\sigma_{\gamma_2 \xi}) \). Then \( Q_\xi \models D \subseteq (D_{\gamma_1 \xi} \cap D_{\gamma_2 \xi}) \times \lambda_\xi \subseteq (D_{\gamma_1 \xi} \cap D_{\gamma_2 \xi} \cap \beta) \times \lambda_\xi \). So \( Q_\xi \models \sigma_{\gamma_1 \xi} \mid D = \sigma'_{\gamma_1 \xi} \mid D \) by the definition of \( \sigma'_{\gamma_1 \xi} \), where \( i = 1, 2 \). But \( Q_\xi \models \sigma'_{\gamma_1 \xi} = \sigma'_{\gamma_2 \xi} \). Thus \( Q_\xi \models \sigma_{\gamma_1 \xi} \) and \( \sigma_{\gamma_2 \xi} \) are compatible since \( \sigma_{\gamma_1 \xi} \) and \( \sigma_{\gamma_2 \xi} \) are forced to be equal everywhere in the intersection of their domains. But for \( \xi \in X \), \( p_{\gamma_1}(\xi) = (i_\xi)_*(\sigma_{\gamma_1 \xi}) \) and \( p_{\gamma_2}(\xi) = (i_\xi)_*(\sigma_{\gamma_2 \xi}) \). So \( P_\xi \models p_{\gamma_1}(\xi) \) and \( p_{\gamma_2}(\xi) \) are compatible by Lemma 2.0.1. Therefore, \( p_{\gamma_1} \) and \( p_{\gamma_2} \) are compatible. This completes the proof of Claim.

Now let \( P = P_\kappa \). In order to see \( P \) is \((\delta, \kappa)\)-universal, we need to check condition (3) of Definition 3.0.5 is satisfied. If \( Q \subseteq V_\lambda \) for some inaccessible \( \lambda < \kappa \) and \( i : Q \to P \) is a complete embedding, then there is \( \xi < \kappa \) such that \( \text{range}(i) \subseteq P_\xi \). So \( i = i_{\xi \beta} \) for some \( \xi' \leq \xi \) and \( \beta < \kappa \). So the desired extensions \( \overline{i} \) and \( \overline{\xi} \) of \( i \) are obtained at some stage \( \xi' \geq \xi \). Hence \( P \) is a \((\delta, \kappa)\)-universal collapse. \( \square \)
Now, we are ready to define the forcing notion $R$. Let $1 \leq n_0 < \omega$ be fixed. Let $P \subseteq V_{\kappa_0}$ be a $(\omega_{n_0}, \kappa_0)$-universal collapse. Let $R_0 = P$ and $R_1 = R_0 \ast \dot{Q}_1$, where $\dot{Q}_1$ is a nice $R_0$-name for $S_{c^{R_0}}(\kappa_0^{++}, \kappa_1)$. Having defined $R_{n-1}$ for $n > 1$, let $\dot{Q}_n$ be a nice $R_{n-1}$-name such that

$$R_{n-1} \models \dot{Q}_n = \begin{cases} S_{c^{R_{n-1}}}(\kappa_{n-1}, \kappa_n) & \text{if } n \text{ is odd} \\ S_{c^{R_{n-1}}}(\kappa_{n-1}^{++}, \kappa_n) & \text{otherwise} \end{cases}$$

Define $R_n = R_{n-1} \ast \dot{Q}_n$. Finally, let $R$ be the inverse limit of $\langle R_n : n < \omega \rangle$. 
CHAPTER IV

The Main Theorem

In this section, we will prove the main theorem of this paper, which is stated in §1. First, let us explore some of the properties of the forcing \( R \).

Following Baumgartner[2], we define, for \( n \in \omega \), \( R_{n\omega} = \{ p : \exists q \in R, q|(\omega \setminus (n + 1)) = p \} \). For any \( G_n \subseteq R_n \) generic over \( V \), for any \( p, q \in R_{n\omega} \), we define \( p \leq q \) iff \( (\exists r \in G_n) r^\upharpoonright p \leq r^\upharpoonright q \). Then \( R \) is isomorphic to a dense subset of \( R_n \times R_{n\omega} \), where \( R_{n\omega} \) is a full name for \( (R_n, \leq) \) defined in \( V^{R_n} \).

Lemma 4.0.9 Assume GCH. Then we have the following:

1. for all \( n \) \( R_n \) has the \( \kappa_n \)-c.c. and \( |R_n| = \kappa_n \) and \( R_n \subseteq V_{\kappa_n} \);
2. for all \( n \) \( R_n \models \) \( \dot{R}_{n\omega} \) is \( < \kappa_n \)-directed closed if \( n \) is even, and \( \dot{R}_{n\omega} \) is \( < \kappa_n^{++} \)-directed closed if \( n \) is odd";
3. for all \( n \) \( R_n \models \) \( \dot{R}_{n\omega} \) is the inverse limit of \( < R_{nm} : n \leq m < \omega > \);
4. \( R \) is \( < \omega_{n_0} \)-closed;
5. \( R \models \) GCH holds;
6. \( R \models \kappa_0 = \aleph_{n_0+1} \) and \( R \models \forall n \geq 1 \kappa_{2n} = \aleph_{4n+n_0+3}, \kappa_{2n+1} = \aleph_{4n+n_0+4} \) and \( \kappa_\omega = \aleph_\omega \).
Proof. (1), (2) and (3) are standard, e.g., see Baumgartner[2]. For (4), notice that $R_0 = P$ is $<\omega_{\omega_0}$-closed.

For (5), if $n$ is even, $\mathcal{P}^V R(\kappa_n) \subseteq V R_{n+1}$ by (3). So $(2^{\kappa_n})^V R \leq (2^{\kappa_n})^V R_{n+1} \leq (|R_{n+1}|^{\kappa_{n+1}})^\kappa_n = \kappa_n^{\kappa_n} = |R_n|^{\kappa_n} = (\kappa_n^+)^{V R}$. If $n$ is odd, then $\mathcal{P}^V R(\kappa_n) \subseteq V R_n$ by (3). So $(2^{\kappa_n})^V R \leq (2^{\kappa_n})^V R_n \leq (|R_n|^{\kappa_n})^{\kappa_n} = \kappa_n^{\kappa_n} = \kappa_n^+$. (6) is clear by easy computation. □

**Lemma 4.0.10** There is a sequence of complete embeddings $< h_n : n < \omega >$ such that

1. $h_0 = id$ and for all $n < \omega$, $h_n : R_n \to j_0n P$ is a complete embedding;
2. for all $m < n < \omega$, the following diagram commutes:

\[
\begin{array}{ccc}
R_m & \xrightarrow{h_m} & j_0m P \\
| & id & |j_{mn}|
\end{array}
\]

\[
\begin{array}{ccc}
R_n & \xrightarrow{h_n} & j_0n P
\end{array}
\]

Proof. First, note that $j_{mn}|(j_0n P) = id$ for $m < n$. Let $h_0 : R_0 \to j_00 R_0 = R_0$ be the identity map. Since $P$ is $\kappa_0$-universal and $P \subseteq V_{\kappa_0}$, so in $M_1$, $j_01 P$ is $\kappa_1$-universal and $j_01 P \subseteq V_{\kappa_1}$. But $V_{\kappa_2} \subseteq M_1$, so $j_01 P$ is $\kappa_1$-universal in $V$. But $j_01|P : P \to j_01 P$ is a $\kappa_0$-complete embedding and $P$ is $\kappa_0$-c.c., so $j_01|P = id$ is a complete embedding. So there exists $h_1 : R_1 \to j_01 P$ such that $h_1$ extends $j_01|P = id = h_0$. 
Having obtained \( h_n : R_n \to j_{0n}P \), we can extend \( h_n \) to a complete embedding \( h_{n+1} : R_{n+1} = R_n* \hat{Q}_{n+1} \to j_{0,n+1}P \) since \( j_{0,n+1}P \) is \( \kappa_{n+1} \)-universal in \( M_{n+1} \) and therefore in \( V \) since \( V_{\kappa_{n+2}} \subseteq M_{n+1} \). This completes the construction of \( \langle h_n : n < \omega \rangle \).

Since each \( h_n \) is an extension of \( h_{n-1} \) and \( j_{n,n+1}j_{0n}P = id \) is a complete embedding, the diagram in (2) commutes. □

**Lemma 4.0.11** There is a sequence \( \langle p_n : n < \omega \rangle \) such that

1. For all \( n > 0 \), \( p_n \in j_{0n}R_n \) is a master condition for \( R_n, h_n \) and \( j_{0n} \), i.e., for all \( q \in R_n, p_n \models \neg h_n(q) \in G_{j_{0n}P} \to j_{0n}q \in G_{j_{0n}R_n} \);
2. for all \( n < m, j_{nm}p_n \leq p_m|n + 1 \).

**Proof.** Notice that we can view \( h_i \) as a complete embedding from \( R_i \) to \( j_{0n}P \) for all \( n \geq i \) by the previous lemma.

Now, we define \( p_n|i + 1 \) (\( i \leq n \)) by induction on \( i \) such that

(a) \( p_n|i + 1 \in j_{0n}R_i \) and

(b) for all \( q \in R_i, p_n|i + 1 \models \neg h_i(q) \in G_{j_{0n}P} \to j_{0n}q \in G_{j_{0n}R_i} \).

For \( i = 0 \), let \( p_n(0) = 1 \in j_{0n}P \). Since \( h_0 = id \), (a) and (b) are clearly true.

Now, suppose we have defined \( p_n|i \). We want to define \( p_n(i) \) so that \( p_n|i + 1 \) satisfies (a) and (b). Let \( G'_i = M_{i+1} \)-generic over \( j_{0n}R_{i-1} \) and \( p_n|i \in G'_i \). Since \( j_{0n}R_{i-1} \in V_{\kappa_{i+1}}^M = V_{\kappa_{i+1}}, \) so \( G'_i \) is also \( V \)-generic. Let \( G'_0 = \{ p|1 : p \in G'_1 \} \), then \( G'_0 \) is \( V \)-generic over \( j_{0n}P \). Let \( G_i = h_i^{-1}(G'_0) \). Then \( G_i \) is \( V \)-generic over \( R_i \). Now, let \( G_{i-1}, H_i \) be such that \( G_i = G_{i-1} * H_i \).
Now, we work in $V[G'_{i-1}]$. By inductive hypothesis, we can extend $j_{0n}$ to $j'_{0n} : V[G'_{i-1}] \to M_n[G'_{i-1}]$. Since $H_i \in V[G'_{i-1}]$, so $j'_{0n}H_i \in V[G'_{i-1}]$. But $j''_{0n}H_i \in V[G'_{i-1}] = V[G'_{i-1}] = V[G'_{i-1}] = V[G'_{i-1}] \subseteq M_n[G'_{i-1}]$ and $j''_{0n}H_i = \kappa_i$, so we have $j''_{0n}H_i \in M_n[G'_{i-1}]$ and $M_n[G'_{i-1}] \models j''_{0n}H_i = \kappa_i$. We must show that $M_n[G'_{i-1}] \models j''_{0n}H_i$ is a condition in $j_{0n}Q_i$.

Notice that $\kappa_{n+i+1} \geq \kappa_i$ since $i \leq n$. By the definition of $Q_i$, $j_{0n}Q_i = Sc(\kappa_{n+i+1}, \kappa_{n+i})$, where $l = 0$ if $i > 1$ is odd and $l = 2$ if $i$ is even or $i = 1$. So $j_{0n}Q_n$ is $\kappa_i$-directed closed. Hence $\cup j''_{0n}H_i \in j_{0n}Q_n$.

Now, let $p_n(i) \in M_{n+1}R_i$ be a canonical name for $\cup j''_{0n}H_i$. Then we can easily check that $p_n|i + 1$ satisfies (a) and (b):

Let $G'_i$ be $M_n$-generic over $j_{0n}R_i$ and $p_n|i + 1 \subset G'_i$. Let $q \in R_i$. Then if $h_i(q) \in G'_0$ then $q \subset G_i = h_i^{-1}G'_0$. So $q|i \subset G_{i-1} = h_{i-1}^{-1}G'_0$ and $< i, h(i) > \in G_i$. Therefore $j_{0n}(q|i) \subset G'_{i-1}$ by induction hypothesis and $(q(i))_{G'_{i-1}} \in H_i$. Since $p_n|i + 1 \subset G'_i$, so $(j_{0n}q(i))_{G'_{i-1}} \in H'_i$, where $H'_i$ is such that $G'_i = G'_{i-1} \star H'_i$.

Now we proceed to show (2). We will show $j_{nm}p_n|i + 1 \leq p_m|i + 1$ by induction on $i \leq n$. For $i = 0$, this is true since $j_{nm}p_n(0) = p_m(0) = 1$.

Suppose we have shown $j_{nm}p_n|i \leq p_m|i$, we want to show $j_{nm}p_n|i + 1 \leq p_m|i + 1$, i.e., we want to show, $j_{nm}p_n|i \models \overline{M_n} j_{0n}p_n(i) \leq p_m(i)$. By the definition of $p_m(i)$, it suffices to show that for each $q \in R_i$,

$$j_{nm}p_n|i \models \overline{M_n} h_i(q) \in \hat{G}\dot{j}_{0m}P \rightarrow j_{nm}p_n(i) \leq j_{0n}q(i)$$

Now, suppose $j_{nm}p_n|i \models \overline{M_n} h_i(q) \in \hat{G}\dot{j}_{0m}P$. Since $i \leq n$, $h_i(q) \in j_{0n}P$. So $j_{nm}h_i(q) = h_i(q)$. Therefore $j_{nm}p_n|i \models \overline{M_n} j_{nm}h_i(q) \in \hat{G}\dot{j}_{0m}P$. In $M_n$, $G\dot{j}_{0m}P$ is the canonical name
for the generic ultrafilter over \( j_{on}P \). So, by elementarity, \( M_m \models j_{nm}(\dot{G}_{j_{on}P}) \) is the canonical name for the generic ultrafilter over \( j_{nm}(j_{on}P) = j_{on}P \). So \( j_{nm}(\dot{G}_{j_{on}P}) = \dot{G}_{j_{on}P} \) since in \( M_m \), \( \dot{G}_{j_{on}P} \) is the canonical name for the generic ultrafilter over \( j_{on}P \). But then \( j_{nm}p_n|i \models j_{nm}h_i(q) \in j_{nm}(\dot{G}_{j_{on}P}) \). By elementarity, we have \( p_n|i \models h_i(q) \in \dot{G}_{j_{on}P} \). By the definition of \( p_n(i) \). we have \( p_n|i \models p_n(i) \leq j_{on}q(i) \).

So \( j_{nm}p_n|i \models j_{nm}j_{on}q(i) = j_{on}q(i) \) by elementarity. We are done. □

From now on, let \( < p_n : n < \omega > \) be a sequence of master conditions as in Lemma 4.0.11

Now, for each \( n < \omega \), let \( l_n = 2n + 1 \). Let \( \dot{A} \) be an \( R \)-name such that

\[
\mathcal{R} \models \ " \dot{A} = \langle \kappa_\omega, (\kappa_n)_{n < \omega}, (f_i)_{i < \omega}, \ldots > \text{ is a structure of some countable language with skolem functions closed under compositions}" .
\]

**Lemma 4.0.12** There is a \( p \in \mathcal{R} \) and a sequence \( \langle \dot{A}_n : n \in \omega \rangle \) such that

for all \( n \), \( p|l_n + 1 \models R_{l_n} \dot{A}_n \) is a structure on \( \kappa_{l_n}^{+} \) and \( p|(l_n, \omega) \models (\dot{A}_n)^\mathcal{V} = \dot{A} |_{\kappa_{l_n}^{+}} \)

**Remark.** Technically, \( \dot{A} \) is not an \( R_{l_n \omega} \)-name, but we will still use it to denote the same object in \( V[G_{R_{l_n}}][G_{R_{l_n \omega}}] \) as the object \( (\dot{A})_{G_{R_{l_n}} * G_{R_{l_n \omega}}} \) in \( V[G_{R_{l_n}} * G_{R_{l_n \omega}}] \), where \( G_{R_{l_n \omega}} \) is \( R_{l_n \omega} \)-generic over \( V[G_{R_{l_n}}] \).

**Proof.** We build a sequence \( < q_n : n \in \omega > \) by induction on \( n \). For \( n = 0 \), since \( R_{l_0} \models (R_{l_0})^{\mathcal{V}} \models \dot{A} |_{\kappa_{l_0}^{+}} \in V[G_{R_{l_0}}] \), there is \( r_0 \) and \( \dot{A}_0 \) in \( V[R_{l_0}] \) such that \( \models R_{l_0} \dot{A}_0 \) is a structure on \( \kappa_{l_0}^{+} \) and \( r_0 \models (\dot{A}_0)^\mathcal{V} = \dot{A} |_{\kappa_{l_0}^{+}} \). But \( R \) is isomorphic to a dense subset of \( R_{l_0} * R_{l_0 \omega} \) in a canonical way, so there are \( q_0 \in R \) such that \( q_0 \leq (1, r_0) \). But then \( q_0|(l_0 + 1) \models R_{l_0} \dot{A}_0 \) is a structure on \( \kappa_{l_0}^{+} \) and \( q_0|(\omega \setminus (l_0 + 1)) \models (\dot{A}_0)^\mathcal{V} = \dot{A} |_{\kappa_{l_0}^{+}} \).
Suppose we have obtained \( q_n, \hat{A}_n \) such that \( q_n|((l_n + 1)) \models R_{l_n} \hat{A}_n \) is a structure on \( \kappa_n^+ \) and \( q_n|((\omega \setminus (l_n + 1))) \models (\hat{A}_n)^\gamma = \hat{A}|\kappa_n^+ \). We now construct \( q_{n+1} \).

Since \( q_n|((l_{n+1} + 1)) \models R_{l_{n+1}} (q_n|((\omega \setminus (l_{n+1} + 1))) \models (\hat{A}_n)^\gamma \in V[G^B_{R_{l_{n+1}}}] \), there is \( r, \hat{A}_{n+1} \in V^{R_{l_{n+1}}+\omega} \) such that \( q_n|((l_{n+1} + 1)) \models R_{l_{n+1}} \hat{A}_{n+1} \) is a structure on \( \kappa_{n+1}^+ \), \( r \leq q_n|((l_{n+1} + 1)) \) and 
\[
 r \models (\hat{A}_{n+1})^\gamma = \hat{A}|\kappa_{n+1}^+. \]
Again, since \( R \) is isomorphic to a dense subset of \( R_{l_{n+1}} * R_{l_{n+1}}^{\omega} \) in a canonical way, there are \( q_{n+1} \leq q_n|((l_{n+1} + 1)), r > \) such that

1. \( q_{n+1}|((l_{n+1} + 1)) \models R_{l_{n+1}} \hat{A}_{n+1} \) is a structure on \( \kappa_{n+1}^+ \) and
2. \( q_{n+1}|((l_{n+1} + 1)) \models R_{l_{n+1}} q_{n+1}|((\omega \setminus (l_{n+1} + 1))) \models (\hat{A}_{n+1})^\gamma = \hat{A}|\kappa_{n+1}^+ \),

Note that \( q_{n+1} \leq q_n \). (As a remark, \( \hat{A}_{n+1} \) is member of \( V^{R_{l_{n+1}}} \). So \( (\hat{A}_{n+1})^\gamma \) stands for the canonical name of \( \hat{A}_{n+1} \) in the forcing language of \( R_{l_{n+1}}^{\omega} \) inside \( V^{R_{l_{n+1}}} \).)

Now, since \( R \) is countably closed, there is \( p \in R \) such that \( p \leq q_n \) for each \( n \).

Then \( p \) and \( \langle \hat{A}_{n} \colon n \in \omega \rangle \) are as desired. □

Let \( p \) and \( \langle \hat{A}_{n} \colon n \in \omega \rangle \) be fixed as in Lemma 4.0.12.

**Definition 4.0.6** Let \( < T(\hat{A}), \leq_T > \) be a tree defined as follows. For any \( t, t \in T(\hat{A}) \)
iff \( t = \langle (q_i, C_i) : i < n > \) for some finite sequence \( \langle (q_i, C_i) : i < n > \) such that for all \( i < n \),

1. \( C_i \subseteq \kappa_i^+ \), \( |C_i| = \kappa_0^+ \) and \( |C_i \cap \kappa_i| = \kappa_0 \);
2. \( C_i \cap \kappa_{i-1}^+ \subseteq C_{i-1} \);
3. \( cf(C_i \cap \kappa_i) = \kappa_0 \) and \( \omega_{\kappa_0} \leq cf(C_i \cap \lambda) < \kappa_0 \) for \( \lambda \in \{ \kappa_{i-1}^+, \kappa_{i-1}^{++}, \kappa_{i-1+1} \} \);
(4) \( q_i \in R_i, q_i \leq p|(l_i + 1) \) and \( q_i|l_{i+1} + 1 \leq q_{i+1}; \)

(5) \( q_i \models "\mathcal{C} \text{ is the universe of some elementary substructure of } \mathcal{A}_i". \)

We order \( T(\mathcal{A}) \) by end extension, i.e., \( t \preceq s \) iff \( s|m = t, \) where \( m \) is the length of \( t.\)

Lemma 4.0.13 \( T(\mathcal{A}) \) is not well founded.

Proof. We are going to show that \( j_{0\omega}\mathcal{T}(\mathcal{A}) \) is not well founded in \( V. \) Since \( M_\omega \) is well founded, so \( j_{0\omega}\mathcal{T}(\mathcal{A}) \) is not well founded in \( M_\omega \) either. By elementarity, we conclude that \( T(\mathcal{A}) \) is not well founded.

Let \( < p_n : n < \omega > \) be as in Lemma 4.0.11, and \( p, < A_n : n < \omega > \) be as in Lemma 4.0.12. Let \( C_n = j_{0\omega}^n \kappa_{i_n}^+. \) Notice that \( C_n \subseteq j_{0\omega}^n \kappa_{i_n}^+ \) and \( C_n \in M_{i_n}. \)

Let \( < r_n : n < \omega > \) be defined as follows:

\[
(1) \ r_0 = < h_0(p|l_0 + 1), p_{i_0}(1) >. \text{ Note that } r_0 \in j_{0\omega} R_{i_0};
\]

\[
(2) \text{ for all } n > 0, r_n = < h_n(p|l_n + 1), j_{i_{n-1}} l_n r_{n-1}(1), \ldots, j_{i_{n-1}} l_n r_{n-1}(l_{n-1}),
\]
\[
p_{i_n}(l_{n-1} + 1), p_{i_n}(l_n) >
\]

We now let \( B = \{ < (j_{l_{k-1}} r_{k-1}, j_{l_{k-1}} C_{k-1}), \ldots, (j_{l_{k-1}} r_{k-1}, j_{l_{k-1}} C_{k-1}) > : k < \omega \}. \)

We are going to show that \( B \) is an infinite branch of \( j_{0\omega}\mathcal{T}(\mathcal{A}). \) (Note that this branch is only in \( V. \)) By the definition of \( T(\mathcal{A}) \) and the elementarity of \( j_{l_{k-1}} \)'s, it suffices to show the following claim:

Claim: For all \( n, \)

\[
(1) \ C_n \subseteq j_{0\omega} \kappa_{i_n}^+, \ |C_n| = j_{0\omega} \kappa_0^+ \text{ and } |C_n \cap j_{0\omega} \kappa_{i_n}| = j_{0\omega} \kappa_0;
\]
(2) $C_n \cap j_0 \kappa_{l_{n-1}}^+ \subseteq j_{l_{n-1}} C_n$;

(3) $\text{cf}(C_n \cap j_0 \kappa_n) = j_0 \kappa_0$ and $\omega_n \leq \text{cf}(C_n \cap j_0 A) < j_0 \kappa_0$

for $\lambda \in \{\kappa_n^+, \kappa_{l_{n-1}}, \kappa_{l_{n-1}+1}\}$;

(4) $r_n \in j_0 R_n$, $r_n \leq j_0 p \cdot (l_n + 1)$ and $r_n |l_{n-1} + 1 \leq j_{l_{n-1}} r_{n-1}$;

(5) $r_n \models ^X C_n$ is the universe of some elementary substructure of $j_0 A_n$.

**Proof of Claim.** (1) and (2) easily follow from the definition of $C_n$ and the properties of $j_0 A_n$ and $j_{l_{n-1}}$.

To see (3), just notice that $R$ is $\omega_n$-closed and

$$
cf(C_n \cap j_0 A) = \text{cf}(j_0 A) = \lambda \begin{cases} = j_0 \kappa_0 & \text{if } \lambda = \kappa_n \\ < j_0 \kappa_0 & \text{if } \lambda \in \{\kappa_n^+, \kappa_{l_{n-1}}, \kappa_{l_{n-1}+1}\} \end{cases}
$$

We now verify (4). First note that for all $n < m$, $h_m(p|m + 1) \leq h_n(p|n + 1)$ by Lemma 4.0.10. So $r_n(0) \leq r_{n-1}(0) = j_{l_{n-1}} r_{n-1}(0)$. So $r_n |l_{n-1} + 1 \leq j_{l_{n-1}} r_{n-1}$ by the definition of $r_n$. Now, we show that $r_n \leq p_{l_n}$ by induction. By the definition of $r_0$, it is clear that $r_0 \leq p_0$. Suppose $r_{n-1} \leq p_{l_{n-1}}$. We show $r_n \leq p_{l_n}$. By the above remark, $r_n(0) \leq j_{l_{n-1}} r_{n-1}(0)$. By induction hypothesis, $j_{l_{n-1}} r_{n-1} \leq j_{l_{n-1}} p_{l_{n-1}}$. But $j_{l_{n-1}} p_{l_{n-1}} \leq p_{l_n} |l_{n-1} + 1$ by Lemma 4.0.11. So $r_n |l_{n-1} + 1 \leq p_{l_n} |l_{n-1} + 1$ by the definition of $r_n$.

To finish the proof of (4), we need to show $r_n \leq j_0 p \cdot (l_n + 1)$. Notice that if $p_n' = h_{l_n}(p \cdot (l_n + 1)) = A p_{l_n}^1 (1), \ldots, p_{l_n}^1(l_n)$, then $p_n' \leq j_0 p \cdot (l_n + 1)$ by the definition of the master conditions $p_{l_n}$'s. But $r_n(0) = p_n'(0)$ and $r_n \leq p_n$, so $r_n \leq p_n' \leq j_0(p \cdot (l_n + 1))$.

We now prove (5). Let $G_{l_n}^n$ be $M_{l_n}$-generic over $j_0 R_n$ and $p_{l_n} \in G_{l_n}^n$. (Note that
$G_{l_n}$ is also $V$-generic.) Let $G'_0 = \{ p \in j_{0n} P : \exists q, p, q \in G'_{l_n} \}$. Let $G_{l_n} = h^{-1}_{l_n} G'_0$.

Then $G_{l_n}$ is $V$-generic over $R_{l_n}$.

Work in $V[G'_{l_n}]$ for the moment. Since $p_{n} \in G'_{l_n}$, we can extend $j_{0n}$ to $\bar{j}_{0l_n} : V[G_{l_n}] \to M_{l_n}[G'_{l_n}]$. Let $A_n = (\bar{A}_n)_{G_{l_n}}$, then $\bar{j}_{0l_n} A_n \in V_{\kappa_{2l_n+1}}[G'_{l_n}] = V_{\kappa_{2l_n+1}}[G''_{l_n}]$ $\subseteq M_{l_n}[G'_{l_n}]$ since $V_{\kappa_{2l_n}} = V_{\kappa_{2l_n}}^{M_{l_n}}$. So $\bar{j}_{0l_n} A_n = \bar{j}_{0l_n} A_{l_n+1} \ldots > \in M_{l_n}[G'_{l_n}]$. Since $\bar{j}_{0l_n}$ is an elementary embedding, $\bar{j}_{0l_n} A_n = \bar{j}_{0l_n} A_{l_n}$. But then $M_{l_n}[G'_{l_n}] = j_{0l_n} A_n 
\subseteq \bar{j}_{0l_n} A_n$.

We now go back to $M_{l_n}$. Let $B_n$ be a name for $\bar{j}_{0l_n} A_n$. Then we have $p_{l_n} \models \bar{\bar{A}}_n < j_{0l_n} A_n$ and the universe of $B_n$ is $C_{l_n}$. Therefore, $r_{l_n} \models \bar{\bar{C}}_n$ is the universe of some elementary substructure of $j_{0l_n} A_n$ since $r_{l_n} \leq p_{l_n}$ by the proof of (1). So we have completed the proof of the Claim. □

Now, by the above claim, $j_{0l_n} T(\hat{A})$ is not well founded in $V$. Hence $T(\hat{A})$ is not well founded by the remark at the beginning of the proof of this lemma. □

Now, we are ready to prove the following theorem:

**Theorem 2** Suppose there exists $j : V \to M_1$ such that $V_{\kappa_0} \subseteq M_1$. Assume GCH.

Let $R$ be as above. Then there is a $G$ $V$-generic over $R$, such that, in $V[G]$, if $X = \{ R_n : n_0 + 1 < n < \omega \}$, there is some $f : X \to \{ \omega_i : i \leq n_0 + 1 \}$ such that

$$f(\alpha) = \begin{cases} 
\omega_{n_0+1} & \text{if } \alpha = \kappa_{n} \text{ for some } n \\
\omega_{n_0} & \text{otherwise}
\end{cases}$$

and $S(\omega_{n_0+1}, \omega_{\omega}, X, f)$ is stationary in $[\omega_{\omega}]^{<\omega_{n_0+2}}$.

Proof. Let $\theta$ be a regular cardinal such that $R \models \bar{\bar{\forall}} \bar{\forall} \bar{\forall} \bar{\forall} \bar{\forall} \bar{\forall} \bar{\forall} (\kappa_{n_0}^+ \; \text{and} \; \bar{\forall} \theta \; \text{is regular})$.

Let $\hat{A}$ be an $R$-name such that $1$ forces the following:

1. $\hat{A} = \langle \kappa_\omega, (\kappa_n)_{n<\omega}, X, (F_{m,n})_{m,n<\omega}, \ldots \rangle$ is a structure of a countable language, where for all $m, n < \omega, F_{m,n} : \kappa_m^+ \times (\kappa_n) \to \kappa_m$ is a function
such that for all $f : \kappa_m^\omega \to \kappa_m$, \( \exists \alpha < \kappa^+_m \) (\( \forall x \in \kappa_m^\omega, f(x) = F_{m,n}(\alpha, x) \));

(2) there is $M < H_\theta, \in, \Delta, S, X, f, (\kappa_n)_{n<\omega}, \kappa_\omega >$ with $\kappa_\omega + 1 \subseteq M$ and $|M| = \kappa_\omega$ such that if $g : M \to \lambda$ is a bijection and \( \{ g_i : i < \omega \} \) is a list of all the skolem functions of the structure $M' = < M, g >$, then $\dot{A}$ is an expansion of the structure $< \lambda, (\kappa_n)_{n<\omega}, g_i|\kappa_\omega >_{i<\omega}$.

Let $p, < A_i : i \in \omega >$ be as in Lemma 4.0.12. Consider the tree $T = T(A)$ defined as in Definition 4.0.6. Then $T$ is not well founded by Lemma 4.0.13. Let $B = \{ < (p_i, B_i) : i < n \mid |n| < \omega \}$ be an infinite branch of $T$. Since $p_i|l_{i-1} + 1 \leq p_{i-1}$, if we let $p'_i = p^*_i < \bar{1} > \in R$ then $p'_i \leq p'_{i-1}$. But $R$ is countably closed, so there is $p' \in R$ such that for all $i < \omega$, $p'_i \leq p'_i$. By the definition of $T$, we have that for all $i$, $p_i \leq p|(l_i + 1)$. So for all $i$, $p'_i|(l_i + 1) \leq p|(l_i + 1)$. But then for all $i$, $p'|(l_i + 1) \leq p|(l_i + 1)$. So $p' \leq p$.

We will show that $p' \models S(\omega_{n_0+1}, R_\omega, X, f)$ is stationary in $[\kappa_\omega]^{<\omega_0}$. By Lemma 2.0.3 and the choice of $\dot{A}$, it suffices to show that $p' \models (\exists B : \dot{A} \kappa_0 \subseteq B \land B \subseteq S)$. So let $p' \in G$ be $V$-generic over $R$. We work in $V[G]$. Let $S = S(\omega_{n_0+1}, R_\omega, X, f)$. Let $A_n$ be the interpretation of $\dot{A_n}$. By abuse of notation, we will also use $B_n$ to denote the elementary substructure of $A_n$ mentioned in (5) of the definition of $T$. Then $A_n = A|\kappa_{n_0}^+$ for all $n < \omega$ by the choice of $p$ and $< A_n : n < \omega >$ and the fact that $p' \leq p$. So $< B_n : n < \omega >$ satisfies the following:

(a) $B_n < A|\kappa_{n_0}^+$, $|B_n| = \kappa_0^+$ and $|B_n \cap \kappa_{l_n}| = \kappa_0$;

(b) $B_{n+1} \cap \kappa_{l_n}^+ \subseteq B_n$;

(c) $cf(B_n \cap \lambda) = \omega_{n_0}$ for $\kappa_{l_{n-1}} < \lambda < \kappa_{l_n}$ and $cf(B_n \cap \kappa_{l_n}) = \kappa_0$. 
(Note: (a) and (b) are obvious from the definition of the tree $T$. (c) follows from (3) of the definition of $T$ and the fact that $R$ is $<\omega_{\alpha_0}$-closed.)

By Lemma 2.0.5, there is $B < A$ such that $|B| = \kappa_0$, $\kappa_0 \subseteq B$ and $B \in S$. This completes the proof of the theorem. □

**Remark** Theorem 2 immediately implies Theorem 1.

**Remark** In the model $V[G]$ of Theorem 2, we have $(\kappa_{2n+1}, \kappa_{2n}) \rightarrow (\aleph_{\alpha_0+1}, \aleph_{\alpha_0})$

**Proof.** Let $G_k'$ be generic over $j_0kP$, where $k = 2n + 1$. Let $G_k = h_k^{-1}G_k'$. Then we can extend $j_0k$ to $\bar{j}_0k : V[G_k] \rightarrow V[G_k']$. In $V[G_k]$, let $A = (\kappa_k, \kappa_{k-1}, \ldots)$ be a structure of type $(\kappa_k, \kappa_{k-1})$. Then $V[G_k']$ thinks that $\bar{j}_0kA$ has an elementary substructure of type $(\aleph_{\alpha_0+1}, \aleph_{\alpha_0})$, namely, the structure $j_0^\prime A$. By elementarity, $V[G_k]$ thinks that $A$ has an elementary substructure of type $(\aleph_{\alpha_0+1}, \aleph_{\alpha_0})$. So, $(\kappa_{2n+1}, \kappa_{2n}) \rightarrow (\aleph_{\alpha_0+1}, \aleph_{\alpha_0})$ holds in $V[G_k]$. But $V[G_k] \models "R_{\omega}$ is $< \kappa_k$-closed", hence $(\kappa_{2n+1}, \kappa_{2n}) \rightarrow (\aleph_{\alpha_0+1}, \aleph_{\alpha_0})$ is preserved. We are done. □

**Theorem 3** In the model $V[G]$ of Theorem 2, let $g : X \rightarrow \{\omega, \omega_{\alpha_0}, \omega_{\alpha_0+1}\}$ be defined by $g(\aleph_{\alpha_0+2}) = \omega$ and $g(\alpha) = f(\alpha)$ for $\aleph_{\alpha_0+2} < \alpha \in X$. Then $S' = S(\omega_{\alpha_0+1}, \aleph_\omega, X, g)$ is stationary in $[\aleph_\omega]^{\omega_{\alpha_0+2}}$.

**Proof.** We work in $V[G]$ of Theorem 2. Given $A =< \kappa_\omega, \ldots >$, we want to find $B < A$ such that $\aleph_{\alpha_0+1} \subseteq B$ and $B \in S'$.

We will construct $< B_n : n < \omega >$ by induction on $n$ such that for each $n$,

(a) $B_n \subseteq B_{n+1}$ and $\sup(B_n \cap \lambda) = \sup(B_{n+1} \cap \lambda)$ for $\aleph_{\alpha_0+2} < \lambda \in X$;

(b) $\sup(B_n \cap \aleph_{\alpha_0+2}) \in B_{n+1}$ and $|B_n| = \aleph_{\alpha_0+1}$;

(c) $\aleph_{\alpha_0+1} \subseteq B_0$. 

[End of document]
Suppose we have constructed \( < B_n : n < \omega > \) as above. Let \( B = \bigcup_{n<\omega} B_n \). Then it's easy to check that \( B \prec A, \aleph_{n_0+1} \subseteq B \) and \( B \in S' \).

Now we proceed to construct \( < B_n : n < \omega > \). First, let \( B_0 \prec A \) be such that \( B_0 \in S \) and \( \aleph_{n_0+1} \subseteq B_0 \). Having constructed \( B_n \), we let \( B_{n+1} = sk^A(B_n \cup \text{sup}(B_n \cap \aleph_{n_0+2})) \). This completes the construction of \( < B_n : n < \omega > \). Now it remains to show the sequence \( < B_n : n < \omega > \) satisfies (a) and (b) above. The proof is similar to the proof of Claim 1 in the proof of Theorem 2. □

**Remark.** By the proof of Theorem 3, we can easily prove the following:

In the model \( V[G] \) of Theorem 2, let \( n \geq 2 \) be any natural number and \( X_n = \{ \aleph_k : n \leq k < \omega \} \). Let \( g : X_n \rightarrow \{ \omega_i : i \leq n_0 + 1 \} \) be defined by \( g(\aleph_n) = \omega \) and \( g(\alpha) = f(\alpha) \) for \( \aleph_n < \alpha \in X \). Then \( S' = S(\omega_{n_0+1}, \aleph_\omega, X_n, g) \) is stationary in \( [\aleph_\omega]^<\omega_{n_0+2} \). In fact, for any finite variation \( g \) of \( f \), \( S' = S(\omega_{n_0+1}, \aleph_\omega, X_n, g) \) is stationary in \( [\aleph_\omega]^<\omega_{n_0+2} \).

As a concluding remark, we notice that there are still many problems open. For example, can we get the consistency of more than two cofinalities together with GCH?
BIBLIOGRAPHY


