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Analysis of discrete vibratory systems with parameter uncertainties

Lee, Chungda, Ph.D.
The Ohio State University, 1992
ANALYSIS OF DISCRETE VIBRATORY SYSTEMS
WITH PARAMETER UNCERTAINTIES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Chungda Lee, B.S.M.E., M.S.M.E.

* * * *

The Ohio State University
1992

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Department of Mechanical Engineering
To my parents and my wife
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Specialty Areas: Dynamics and Vibration, Acoustics, and Mechanical Design.
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<tr>
<td>$a,b$</td>
<td>random variables</td>
</tr>
<tr>
<td>$A,B$</td>
<td>random parameter matrices</td>
</tr>
<tr>
<td>$c_i$</td>
<td>modal damping</td>
</tr>
<tr>
<td>$C,C$</td>
<td>damping</td>
</tr>
<tr>
<td>Cov</td>
<td>Covariance</td>
</tr>
<tr>
<td>cs</td>
<td>column transformation</td>
</tr>
<tr>
<td>$dy$</td>
<td>incremental of $y$</td>
</tr>
<tr>
<td>$D$</td>
<td>direct product of two deterministic parameter matrices</td>
</tr>
<tr>
<td>$e$</td>
<td>exponential</td>
</tr>
<tr>
<td>$F,F$</td>
<td>force amplitude</td>
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<td>$g$</td>
<td>jointly density function</td>
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<td>$G,G$</td>
<td>dynamic stiffness</td>
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<td>$G_1,G_2$</td>
<td>perturbed terms of $G^{-1}$</td>
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<tr>
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<td>time</td>
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<td>a function</td>
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<td>variance</td>
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\( x \)  
\( X,X \)  
\( y,y \)  
\( y_1,y_2,y_3 \)  
\( \delta \)  
\( \sigma \)  
\( \omega \)  
\( \Omega \)  
\( \Phi \)  
\( \zeta \)  
\( r \)  
\( R,R \)  
\( \text{Re} \)  
\( s_j \)  
\( S \)  
\( \text{Si} \)  
\( t \)  
\( T \)  
\( u,u \)  
\( \xi(t) \)  
\( \text{Var} \)  
\( X,X \)  
\( y,y \)  
\( y_1,y_2,y_3 \)  
\( Y,Y \)  
\( g \)  
\( \mathcal{G} \)  
\( h \)  
\( \mathcal{H} \)  
\( \alpha,\beta \)  
\( \mu \)  
\( \rho \)  
\( \delta \)  
\( \lambda \)  
\( \gamma \)
\( \sigma \) standard deviation
\( \tau, \nu \) time
\( \omega \) undamped natural frequency (rad/s)
\( \Omega \) excitation frequency (rad/s)
\( \Phi \) eigenvector
\( \zeta \) damping ratio

Superscripts

* conjugate
-1 inverse
\( T \) transpose
\( \otimes 2 \) second power of direct product
\( \otimes 3 \) third power of direct product
. first time derivative
.. second time derivative
' derivative terms
+ upper bound
- lower bound
~ fluctuation or random component
- deterministic component

Subscripts

\( A, B \) random parameter matrices
\( c_i \) modal damping
\( cs \) column transformation
\( C \) damping
\( d \) damped
\( F \) force amplitude
\( G \) dynamic stiffness
\( i,j \) modal indices
\( I \) identity matrix
\( k_i \) modal stiffness
\( K \) stiffness
\( m_i \) modal mass
\( M \) mass
\( \phi_i \) \( i \)th normalized eigenvalue
\( P \) variance by the proposed method
\( u \) a function
\( X \) generalized coordinate
\( y \) random function
\( Y \) normal coordinate
\( \gamma \lambda \) eigenvalue
\( \Phi \) eigenvector
\( \omega \) undamped natural frequency (rad/s)

Operators

\( \text{max} \) maximum value
\( \text{abs} \) absolute value
\( D \) derivative
\( \Delta \) incremental
\( \langle \rangle \) expectation operator
\( \int \) integral
\( \otimes \) direct or Kronecker product
\( ! \) factorial
\( \| \| \) norm operator
\( \| \mid \) magnitude or absolute value
\( | \) operating point
CHAPTER I
INTRODUCTION

1.1. Motivation

Analysis of parameter uncertainties has fascinated researchers for several decades as evident from recent review articles by Ibrahim [1.1] and Benaroya and Rehak [1.2]. This problem is of interest in many disciplines including vibrations, physical chemistry and mathematics [1.1-1.8]. Random differential equations have also been studied extensively [1.9-1.12]. However, a number of unresolved issues remain. One such issue is the estimation of random eigensolutions. Another research issue is the estimation of impulse response or frequency response of a linear time-invariant vibratory system with uncertain system parameters. Based on the earlier assessments by prior investigators [1.1] and [1.2] and a review of existing literature [1.13,1.14] it is clear that new or improved analytical techniques are definitely needed to overcome some of the deficiencies of existing methods which are as follows. The first order perturbation methods [1.4-1.7, 1.15-1.17] are severely limited in their applications and sometimes they can not predict system response adequately. The Monte-Carlo numerical simulation technique [1.18-1.19] is often time consuming and may not yield any improved understanding of system behavior. A detailed review of these and other methods can be found in references [1.1, 1.2].
1.2 Problem Statement

The major objective of this study is to propose new analytical methods for the eigensolutions and forced response of a linear time-invariant, proportionally damped vibratory system of dimension \( N \) with uncertain parameter matrices \( M, C \) and \( K \) representing mass, damping and stiffness respectively; each matrix is assumed to be symmetric and positive definite. The excitation amplitude, \( MF \), is also considered to be randomly distributed but the time history \( \xi(t) \) is deterministic and arbitrary for forced response studies. The random differential equation can be given in the matrix form as follows; see LIST OF SYMBOLS for the identification of symbols. Since only the eigensolutions and forced response is of interest, initial displacement \( X(0) \) and velocity \( \dot{X}(0) \) are assumed to be null vectors.

\[
M\ddot{X}(t) + C\dot{X}(t) + KX(t) = MF\xi(t) \quad ; \quad X(0) = \dot{X}(0) = 0
\]

(1.1)

Only the first two moments of the ensuing eigensolutions and forced response are estimated analytically by making a few simplifying assumptions: (i) random matrices and vectors of equation (1.1) can be given by the sum of deterministic or mean (identified by bar) and random or fluctuating components (identified by tilde) e.g. \( M = \bar{M} + \tilde{M} \); (ii) expected means of system matrices e.g. \( \bar{M} = \langle M \rangle \) and excitation amplitude \( \langle F \rangle = \bar{F} \) are known; (iii) probability distributions of \( M, C, K \) and \( F \) are of the same type and are known; (iv) means of a random parameter matrix and \( \bar{F} \) are equal to null e.g. \( \langle \bar{M} \rangle = 0 \), \( \langle \bar{F} \rangle = 0 \); (v) parameter fluctuations are much smaller compared to the deterministic values i.e. \( \|\tilde{M}\| \ll \|\bar{M}\| \); (vi) co-variances of parameter fluctuations are known in the form of cross-correlation matrices such as \( R_{M,M} = \langle \tilde{M} \otimes \tilde{M} \rangle = \text{Var}(M) \) and \( R_{M,K} = \langle \tilde{M} \otimes \tilde{K} \rangle = \text{Cov}(M, K) \); and (vii) \( F \) is uncorrelated with system parameter matrices i.e.
\[ R_{f,m} = R_{f,c} = R_{f,k} = 0. \] Additional assumptions will be specified as a part of the analytical development. Both single and multi-degree-of-freedom system examples are considered. Various cases of parameter uncertainties are investigated analytically and numerically. The proposed methods are validated by comparing it with the Monte-Carlo numerical simulation [1.19] which is considered as the benchmark. Predictions yielded by the first order perturbation technique are also given when appropriate. The scope is limited to the following specific studies:

a. Eigensolutions: Both undamped and proportionally damped systems are considered.

b. Impulse Response: Consider \( \xi(t) \) to be an unit impulse; proportionally damped system problems are considered.

c. Frequency response: \( \xi(t) \) is assumed to be a sinusoidal function and the assumption of proportional damping is relaxed.

1.3. Scope And Organization

Each chapter is self-sufficient with its own objectives, methodology, examples, results and conclusions. However, nomenclature is common.

a. Eigensolutions (Chapter II): A new analytical method is proposed for the estimation of eigensolutions for undamped and proportionally damped discrete vibratory systems when the system parameters are uncertain or random variables. Given several simplifying assumptions, a direct product technique is used to estimate the first and second moments of eigensolutions in terms of the moments of system parameter matrices. Existing methods such as the first order perturbation method and the Monte-Carlo simulation are also used. Application of the new method is demonstrated through several single and
multi-degree-of-freedom systems. Cases with moderately large random fluctuations are also considered.

b. **Impulse Response (Chapter III):** This chapter is a continuation of Chapter II where a new analytical method was proposed to estimate the first and second moments of eigenvalues. Now, force amplitude is also considered as a random variable but its time history is assumed to be deterministic. Based on certain simplifying assumptions and given probabilistic eigenvalue solutions, the first two moments of response in both modal and physical domains are estimated. First, an impulse excitation is considered and single and two-degree-of-freedom system examples are used to illustrate the proposed method. The proposed method is compared with the benchmark Monte-Carlo simulation and the first order perturbation technique. Second, a convolution integral formulation is developed for harmonic and other excitations. One example case is considered to illustrate and validate the proposed approach.

c. **Frequency Response (Chapter IV):** The primary objective of this study is to develop a new analytical method for estimating the frequency response characteristics of a damped whether proportionally or non-proportionally system. Excitation amplitude is also considered to be random but the frequency is deterministic. Given several simplifying assumptions, a different direct product technique is proposed to estimate the mean and standard deviation of the steady state displacement response at the excitation frequency. Application of this theory is demonstrated by several single and multi-degree of freedom examples. In order to verify the proposed analytical technique, predictions are compared with the results obtained by the Monte-Carlo simulation and/or perturbation methods. Spectral coupling issues for a multi-degree-of-freedom system are investigated.
References


CHAPTER II
EIGENSOLUTIONS

2.1. Introduction

The subject of parameter uncertainties has fascinated researchers for several decades as evident from extensive review articles by Ibrahim [2.1] and Benaroya and Rehak [2.2]. This problem is of interest in many disciplines including vibrations and physical chemistry [2.1-2.7]. Random differential equations have also been studied extensively [2.8-2.11]. However, several research issues still remain unresolved [2.1,2.2]. One such issue is the estimation of eigensolutions for viscously damped systems which is addressed in this chapter. Yet another issue deals with impulse response characteristics which forms the basis of Chapter III.

A few analytical techniques have been used to solve real or undamped eigenvalue problems. For instance, the transfer matrix method has been applied by Kener [2.12] and Soong and Bogdanoff [2.13] to study disordered periodic systems. Another method which has been used widely is the perturbation method [2.4,2.9]. Pierre [2.6] has however modified this perturbation method and applied it to disordered periodic systems. Typically first order perturbations are included in the analysis since higher orders may make the analytical and numerical problems tedious. Finally, Monte-Carlo technique can be employed to find eigensolutions numerically [2.14,2.15]. However, such a simulation is computationally intensive since a large number of iterations is required to estimate the probability distributions. Based on this literature review and earlier assessments by prior
investigators [2.1, 2.2, 2.6 & 2.10], it is evident that new or improved analytical techniques are definitely required to overcome the deficiencies of existing methods, and still yield reasonably accurate results.

2.2. Problem Formulation

Random differential equations for a linear time-invariant, proportionally damped vibratory system of dimension $N$ can be given in the matrix form as follows:

$$M\ddot{X}(t) + C\dot{X}(t) + KX(t) = 0$$

where $M$, $C$ and $K$ are system parameter matrices representing mass, viscous damping and stiffness, respectively. (Also refer to LIST OF SYMBOLS for the identification of symbols.) The following simplifying assumptions are made to develop new solution methodology: (i) random matrices and vectors of equation (2.1) can be given by the sum of deterministic or mean (identified by bar) and random or fluctuating components (identified by tilde) i.e. $M = \bar{M} + \tilde{M}$, $C = \bar{C} + \tilde{C}$, $K = \bar{K} + \tilde{K}$ and $X(t) = \bar{X}(t) + \tilde{X}(t)$; (ii) expected means of system matrices $\bar{M} = <M>$, $\bar{C} = <C>$ and $\bar{K} = <K>$ are known where $<$ > is the expectation operator; (iii) probability distribution of $M$, $C$, and $K$ are of the same type and are known; (iv) mean of a random parameter matrix is equal to zero e.g. $< \tilde{M} > = 0$; (v) parameter fluctuations are much smaller compared to the deterministic values i.e. $\|\tilde{M}\| < \|\bar{M}\|$ where $\| \|$ is the norm operator; and (vi) co-variances of random fluctuations are known in the form of cross-correlation matrices such as $R_{M,M} = \langle \tilde{M} \otimes \tilde{M} \rangle$ and $R_{M,K} = \langle \tilde{M} \otimes \tilde{K} \rangle$ where $\otimes$ is the direct product of two matrices.

The scope of this chapter is limited to the proportional damping case. For the sake of convenience, consider the Rayleigh damping model given by $C = \alpha M + \beta K$ where
\(a = \bar{a} + \tilde{\alpha}\) and \(\beta = \bar{\beta} + \tilde{\beta}\) are random scalar variables. Depending upon \(\alpha\) and \(\beta\), one can obtain various cases where \(C\) is uncorrelated, partially correlated and fully correlated with \(M\) and/or \(K\). The theory, based on a direct product technique presented in this chapter, is developed first for the undamped case i.e. \(\alpha=\beta=0\). Then the proportionally damped system is examined. Finally, an undamped mass-spring system with randomly distributed masses [2.6,2.13] is investigated but periodic disordered systems [2.6,2.16] are considered beyond the scope of this study. Of primary interest here is the prediction of standard deviation of natural frequencies and damping ratios. In each case, theory is illustrated through numerical examples and validated by comparing standard deviation predictions with the results yielded by either existing analytical techniques (whenever applicable) and/or Monte-Carlo simulation technique. The assumption of very small random fluctuations is relaxed in numerical examples. For instance in several cases, normalized standard deviation values of upto 0.5 are selected for analysis. Solutions yielded by all methods are compared.

### 2.3. Undamped System

#### 2.3.1 Theory

Consider the undamped system given by equation (2.1) with \(C=0\). The eigensolution is given by \((\lambda_i I - G)\Phi_i = 0\) where \(I\) is an identity matrix, \(G = M^{-1}K\), \(\lambda_i = \omega_i^2\) is the \(i\)th eigenvalue, \(\omega_i\) is the natural frequency and \(\Phi_i\) is the corresponding eigenvector. Note that \(N\) eigenvalues \(\lambda_i\) of \(G\) are assumed to be distinct and \(M\) is considered to be non-singular. Deterministic eigenvalues \(\bar{\lambda}_i\) and eigenvectors \(\bar{\Phi}_i\) are determined by \((\bar{\lambda}_i I - \bar{G})\bar{\Phi}_i = 0\). Now express \(\lambda_i\) as \(\lambda_i = \bar{\lambda}_i + \tilde{\lambda}_i = \Phi_i^T G \Phi_i / \Phi_i^T \Phi_i\). If \(\tilde{\lambda}_i\) is estimated using deterministic \(\bar{\Phi}_i\) the covariance between two eigenvalues is found to be
as follows where $D$ is the direct product of two deterministic matrices e.g. $D_{\Phi,\Phi} = \Phi \otimes \Phi$:

$$
\text{Cov}(\lambda_i, \lambda_j) = \langle \lambda_i \otimes \lambda_j \rangle - \langle \lambda_i \rangle \otimes \langle \lambda_j \rangle
$$

$$
= \frac{D_{\Phi,\Phi}^T((G \otimes G) - \langle G \rangle \otimes \langle G \rangle) D_{\Phi,\Phi}}{D_{\Phi,\Phi}^T D_{\Phi,\Phi} D_{\Phi,\Phi}} = \frac{D_{\Phi,\Phi}^T R_{G,G} D_{\Phi,\Phi}}{D_{\Phi,\Phi}^T D_{\Phi,\Phi} D_{\Phi,\Phi}}; \quad i,j=1,2,\ldots, N.
$$

$$
\langle \lambda_i \rangle = \Phi_i^T \langle G \rangle \Phi_i / \Phi_i^T \Phi_i;
$$

$$
R_{G,G} = (G \otimes G) - \langle G \rangle \otimes \langle G \rangle;
$$

$$
\langle G \otimes G \rangle = \langle (M \otimes M)^{-1} (K \otimes K) \rangle;
$$

$$
\langle G \rangle = \langle M^{-1} K \rangle \quad (2.2a-e)
$$

It is possible that in many real mechanical systems $M$ and $K$ are uncorrelated i.e. $R_{M,K} = 0$ which should simplify equation (2.2). First, we identify the variance generated by the uncertainties of $M^{-1} = (\overline{M} + \tilde{M})^{-1}$. The Taylor matrix expansion yields the following (refer to Appendix A for a brief review of matrix direct products):

$$
M^{-1} = \overline{M}^{-1} + D_{\text{cs}(M)^T} M^{-1} \left[ \text{cs}(\tilde{M}) \otimes I_N \right] + \frac{1}{2} D_{\text{cs}(M)^T} M^{-1} \left[ \text{cs}(\tilde{M})^2 \otimes I_N \right] + \ldots
$$

$$
= \overline{M}^{-1} + \overline{M}^{-1} M \left[ N_2 \otimes \overline{M}^{-1} \right] \left[ \text{cs}(\tilde{M}) \otimes I_N \right] + \overline{M}^{-1} M^\prime \left[ N_4 \otimes \overline{M}^{-1} \right] \left[ \text{cs}(\tilde{M})^2 \otimes I_N \right] + \ldots \quad (2.3)
$$
where $M'$ and $M''$ etc. are the derivative terms. After some manipulation, equation (2.3) can be rewritten as

$$M^{-1} = \bar{M}^{-1} + \bar{M}^{-1}M'\left[\text{cs} \left( \hat{M} \right) \otimes I_N \right] \bar{M}^{-1} + \bar{M}^{-1}M''\left[\text{cs} \left( \hat{M} \right) \otimes I_N \right] \bar{M}^{-1} + \ldots$$

$$= \bar{M}^{-1} + \bar{M}^{-1}M_i \bar{M}^{-1} + \bar{M}^{-1}M_2 \bar{M}^{-1} + \ldots$$

$$= \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1}$$  \hspace{1cm} (2.4)

Specifically, $M_0 = \bar{M}$ and $M_1 = \hat{M}$. If the distribution of $M$ is assumed to be symmetric then odd moments of $\hat{M}$ are null matrices. The expected mean and variance of $M^{-1}$ are given as

$$\langle M^{-1} \rangle = \left\langle \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1} \right\rangle$$

$$\text{Var}(M^{-1}) = \langle M^{-1} \otimes M^{-1} \rangle - \langle M^{-1} \rangle \otimes \langle M^{-1} \rangle$$

$$= \left\langle \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1} \otimes \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1} \right\rangle - \left\langle \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1} \right\rangle \otimes \left\langle \sum_{i=0}^{\infty} \bar{M}^{-1}M_i \bar{M}^{-1} \right\rangle$$

$$= \left\langle \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} D^{-1}_{M,M_i} M_i \otimes M_j D^{-1}_{M,M_j} \right\rangle - \left\langle \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} D^{-1}_{M,M_i} M_{2i} \otimes M_{2j} D^{-1}_{M,M_j} \right\rangle$$  \hspace{1cm} (2.5a,b)
where \( M_i \otimes M_j = 0 \) when \( i+j=\text{odd} \). Two reasonable approximations of equation (2.5b) are presented as follows:

\[
\text{Var}(M^{-1}) = \left(D_{M,M} - R_{M,M} - \langle M_2 \otimes M_2 \rangle \right)^{-1} - \left(D_{M,M} - \langle M_2 \rangle \otimes \langle M_2 \rangle \right)^{-1} \tag{2.6}
\]

which can be simplified as follows

\[
\text{Var}(M^{-1}) = \left(D_{M,M} - M_1 \otimes M_1 \right)^{-1} - D_{M,M}^{-1} = \left(D_{M,M} - R_{M,M} \right)^{-1} - D_{M,M}^{-1} \tag{2.7}
\]

Consequently, \( R_{G,G} \) of equation (2.2) is now defined as follows by using equation (2.6)

\[
R_{G,G} = \langle G \otimes G \rangle - \langle G \rangle \otimes \langle G \rangle = \langle (M \otimes M)^{-1} (K \otimes K) \rangle - \langle M^{-1} K \rangle \otimes \langle M^{-1} K \rangle
\]

\[
= \left( D_{M,M} - R_{M,M} - \langle M_2 \otimes M_2 \rangle \right)^{-1} \left( D_{K,K} + R_{K,K} \right) - \left( D_{M,M} - \langle M_2 \rangle \otimes \langle M_2 \rangle \right)^{-1} D_{K,K} \tag{2.8}
\]

A simplified form of equation (2.8) is when equation (2.7) is used

\[
R_{G,G} \approx \left( D_{M,M} - R_{M,M} \right)^{-1} \left( D_{K,K} + R_{K,K} \right) - D_{M,M}^{-1} D_{K,K} \tag{2.9}
\]

Equation (2.9) can be simplified further when the parameter fluctuations are much smaller than the corresponding deterministic values: \( R_{G,G} \approx D_{M,M}^{-1} R_{M,M} \bar{\lambda}_i^2 + D_{M,M}^{-1} R_{K,K} \). Note that this is compatible with the first order perturbation method.

Now let us go back to equation (2.2) and define the standard deviations of \( \lambda_i \) and \( \omega_i \) as
\[ \sigma(\lambda_i) = \left( \langle \lambda_i \otimes \lambda_i \rangle - \langle \lambda_i \rangle^2 \right)^{1/2} = \left( \frac{D_{\phi_i,\phi_i}^T \left( \langle G \otimes G \rangle - \langle G \rangle \otimes \langle G \rangle \right) D_{\phi_i,\phi_i}}{D_{\phi_i,\phi_i}^T D_{\phi_i,\phi_i}} \right)^{1/2} = \left( \frac{D_{\phi_i,\phi_i}^T R_{G,G} D_{\phi_i,\phi_i}}{D_{\phi_i,\phi_i}^T} \right)^{1/2} \]

\[ \sigma(\omega_i) = 0.5 \sigma(\lambda_i) / \bar{\omega}_i \]  

(2.10a,b)

Similarly, the expected value of eigenvalue formulation \( \Phi_i = G\Phi_i / \lambda_i \), estimated by using deterministic \( \Phi_i \) and \( \bar{\lambda}_i \), yields \( \langle \Phi_i \rangle = \langle G \rangle \bar{\Phi}_i / \bar{\lambda}_i \) which is then used to find the covariance between \( \Phi_i \) and \( \Phi_j \) in the matrix form as

\[
\text{Cov}(\Phi_i,\Phi_j) = \langle \Phi_i \otimes \Phi_j \rangle - \langle \Phi_i \rangle \otimes \langle \Phi_j \rangle = \left( \langle G \otimes G \rangle - \langle G \rangle \otimes \langle G \rangle \right) D_{\phi_i,\phi_j} / \bar{\lambda}_i \bar{\lambda}_j = R_{G,G} D_{\phi_i,\phi_j} / \bar{\lambda}_i \bar{\lambda}_j
\]

(2.11)

The covariance between \( \lambda_i \) and \( \Phi_j \) is found to be as follows by using the same technique

\[
\text{Cov}(\lambda_i,\Phi_j) = \langle \lambda_i \otimes \Phi_j \rangle - \langle \lambda_i \rangle \otimes \langle \Phi_j \rangle = D_{\phi_i,\phi_j}^T \left( \langle G \otimes G \rangle - \langle G \rangle \otimes \langle G \rangle \right) D_{\phi_i,\phi_j} / \bar{\lambda}_i \bar{\lambda}_j = D_{\phi_i,\phi_j}^T R_{G,G} D_{\phi_i,\phi_j} / \bar{\lambda}_i \bar{\lambda}_j
\]

(2.12)

### 2.3.2 Example I: Single-Degree-of-Freedom System

Consider the undamped system shown in Figure 2.1a with \( \bar{\lambda}_i = \bar{\omega}_i^2 = 1 \). Various random perturbations in mass (upto \( \sigma_M = 0.4 \)) are investigated. Table 2.1 lists the results
Figure 2.1 Physical systems used to illustrate theory. (a) single degree-of-freedom system used for Examples I and III. (b) visco-elastic model considered for Example IV. (c) two degree-of-freedom system used for Example II and V. (d) periodic system with randomly distributed masses used for Example VI.
Table 2.1 Standard deviation of $\omega_1$ for Example I (Fig. 2.1a)

<table>
<thead>
<tr>
<th>$\sigma_M$</th>
<th>Monte-Carlo Simulation</th>
<th>Proposed Method</th>
<th>First Order Perturbation Method [2.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\sigma(\omega_1)$ and associated error $\epsilon^*$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Simplified Eqs.(2.9&amp;2.10)</td>
<td>Exact Eqs.(2.8&amp;2.10)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0504</td>
<td>0.0503</td>
<td>0.0504</td>
</tr>
<tr>
<td></td>
<td>($\epsilon=0.2%$)</td>
<td>($\epsilon=0%$)</td>
<td>($\epsilon=0.8%$)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.106</td>
<td>0.102</td>
<td>0.104</td>
</tr>
<tr>
<td></td>
<td>($\epsilon=3.8%$)</td>
<td>($\epsilon=1.9%$)</td>
<td>($\epsilon=5.7%$)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.176</td>
<td>0.157</td>
<td>0.163</td>
</tr>
<tr>
<td></td>
<td>($\epsilon=10.8%$)</td>
<td>($\epsilon=7.4%$)</td>
<td>($\epsilon=20.5%$)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.271</td>
<td>0.218</td>
<td>0.235</td>
</tr>
<tr>
<td></td>
<td>($\epsilon=19.6%$)</td>
<td>($\epsilon=13.3%$)</td>
<td>($\epsilon=26.2%$)</td>
</tr>
</tbody>
</table>

$\epsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte Carlo simulation}$
Table 2.2 Standard deviation of $\omega_1$ and $\omega_2$ for Example II (Fig. 2.1c)

<table>
<thead>
<tr>
<th>Mode $i$</th>
<th>$R_{M,M}$</th>
<th>$R_{K,K}$</th>
<th>$\sigma(\omega_i)$ and error $\varepsilon^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Monte-Carlo Simulation</td>
</tr>
<tr>
<td>$i=1$</td>
<td>$0.01D_{M,M}$</td>
<td>0.0</td>
<td>0.0234</td>
</tr>
<tr>
<td></td>
<td>$0.01D_{M,M}$</td>
<td>0.01$D_{K,K}$</td>
<td>0.0334</td>
</tr>
<tr>
<td></td>
<td>$0.04D_{M,M}$</td>
<td>0.0</td>
<td>0.0496</td>
</tr>
<tr>
<td>$i=2$</td>
<td>$0.01D_{M,M}$</td>
<td>0.0</td>
<td>0.0755</td>
</tr>
<tr>
<td></td>
<td>$0.01D_{M,M}$</td>
<td>0.01$D_{K,K}$</td>
<td>0.1074</td>
</tr>
<tr>
<td></td>
<td>$0.04D_{M,M}$</td>
<td>0.0</td>
<td>0.1599</td>
</tr>
</tbody>
</table>

*$\varepsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte Carlo simulation}$
predicted by simplified (2.9,2.10) and exact (2.8,2.10) version of the proposed method as well as those yielded by the first order perturbation method [2.4]. All predictions are compared with those obtained by Monte-Carlo simulation which is considered as the benchmark. It is seen from Table 2.1 that the proposed methodology is in better agreement with the Monte-Carlo simulation than the first order perturbation method. Observe the discrepancy between two analytical methods when \( \sigma_M \) is high. It is because the proposed methodology, unlike the perturbation method, maintains a few higher order terms of the mass randomness. It shows that the proposed method can indeed handle larger uncertainties of the system parameters reasonably well.

2.3.3 Example II: Two-Degree-of-Freedom System

The undamped system \((C=0)\) of Figure 2.1c with \( \bar{\omega}_1 = 0.468 \) and \( \bar{\omega}_2 = 1.510 \) is investigated next. The randomness of mass or stiffness matrix being fully correlated is considered and the corresponding results for \( \sigma(\omega_1) \) and \( \sigma(\omega_2) \) are presented in Table 2.2. Again, the proposed theory matches with the Monte-Carlo simulation better than the first order perturbation method [2.4]. Like Example I, the error for the case \( R_{MM} = 0.04 \, D_{MM} \) is associated with the higher order terms which are obviously ignored by the first order perturbation method.

2.4. Proportionally Damped System

2.4.1 Theory

Equation (2.1) of dimension \( N \) can be rewritten as follows in accordance with the assumptions stated earlier:

\[
\ddot{X}(t) + B\dot{X}(t) + AX(t) = 0;
\]
\[ A = M^{-1}K \; ; \; B = M^{-1}C \]  \hfill (2.13a-c)

Now let \( X(t) = \Phi Y(t) \) where \( \Phi \) is the modal matrix of the undamped system described earlier in Section 2.3.1 and \( Y(t) \) is the principal coordinate vector. Further, premultiply (2.13a) by \( \Phi^T \) to yield the following characteristic equation for the damped system

\[ \gamma_i^2 \Phi_i^T \Phi_i + \gamma_i \Phi_i^T B \Phi_i + \Phi_i^T A \Phi_i = 0; \; i = 1, 2, \ldots, N \]  \hfill (2.14)

Here \( \gamma_i = -\zeta_i \omega_i \pm j\omega_di = -\zeta_i \omega_i \pm j\omega_i \sqrt{1 - \zeta_i^2} \) is the \( i \)th complex valued eigenvalue corresponding to real eigenvector \( \Phi_i \); \( \zeta_i \) is the damping ratio, \( \omega_di \) is the damped natural frequency and \( j = \sqrt{-1} \). Rewrite equation (2.14) as

\[ \gamma_i = \frac{-\Phi_i^T B \Phi_i \pm j\sqrt{4(\Phi_i^T \Phi_i A \Phi_i) - (\Phi_i^T B \Phi_i)^2}}{2\Phi_i^T \Phi_i} = \text{Re}(\gamma_i) + j\text{Im}(\gamma_i); \]  \hfill (2.15)

where

\[ \text{Re}(\gamma_i) = \frac{-\Phi_i^T B \Phi_i}{2\Phi_i^T \Phi_i}; \]

\[ [\text{Im}(\gamma_i)]^2 = \omega_{di}^2 = \frac{4\Phi_i^T \Phi_i A \Phi_i - \Phi_i^T B \Phi_i \Phi_i^T B^T \Phi_i}{4(\Phi_i^T \Phi_i)^2} \]  \hfill (2.16a,b)

If \( \gamma_i \) is estimated using \( \Phi_i \), expected value of \( \gamma_i \) yields the following

\[ \langle \text{Re}(\gamma_i) \rangle = -0.5\Phi_i^T \langle B \rangle \Phi_i / d_i; \]
\[
\left\langle \left[ \text{Im}(\gamma_i) \right]^2 \right\rangle = \langle \omega_{\alpha_i}^2 \rangle = \left( 4d_i \bar{\Phi}_i \langle A \rangle \bar{\Phi}_i - \bar{\Phi}_i \langle B \bar{\Phi}_i B^\dagger \bar{\Phi}_i \rangle \right) / (4d_i^2);
\]

\[d_i = \bar{\Phi}_i \bar{\Phi}_i\]  

(2.17a-c)

Since \( \gamma_i \) is complex valued, the standard deviation of \( \gamma_i \) is complex valued as well: \( \sigma_{\gamma_i} = \sigma(\text{Re}(\gamma_i)) + j \sigma(\text{Im}(\gamma_i)) \). It is determined by using the covariance matrices of \( A \) and \( B \).

Further, covariances of \( \gamma_i \) and \( \omega_{\alpha_i} \) can be given as

\[
\text{Cov}(\text{Re}(\gamma_i), \text{Re}(\gamma_j)) = \langle \text{Re}(\gamma_i) \otimes \text{Re}(\gamma_j) \rangle - \langle \text{Re}(\gamma_i) \rangle \otimes \langle \text{Re}(\gamma_j) \rangle;
\]

\[
\text{Cov}(\omega_{\alpha_i}^2, \omega_{\alpha_j}^2) = \langle \omega_{\alpha_i}^2 \otimes \omega_{\alpha_j}^2 \rangle - \langle \omega_{\alpha_i}^2 \rangle \otimes \langle \omega_{\alpha_j}^2 \rangle
\]

(2.18a,b)

**2.4.2 Rayleigh Damping Model**

Two limiting cases of the Rayleigh damping model \( C = \alpha M + \beta K \) are investigated next: (a) \( C \) is uncorrelated with \( M \) and \( K \) i.e. \( \alpha \) and \( \beta \) are random variables, and (b) \( C \) is fully correlated with \( M \) and \( K \) i.e. \( \alpha = \bar{\alpha} \) and \( \beta = \bar{\beta} \) are deterministic.

**Case (a):** If \( C \) is uncorrelated with \( M \) and \( K \), define the standard deviation of \( \gamma_i \) by using equations (2.16-2.18)

\[
\sigma(\text{Re}(\gamma_i)) = \left( \langle \text{Re}(\gamma_i) \otimes \text{Re}(\gamma_i) \rangle - \langle \text{Re}(\gamma_i) \rangle \otimes \langle \text{Re}(\gamma_i) \rangle \right)^{1/2}
\]

\[= 0.5 \left( D_{\phi_i,\phi_i}^T R_{B,B} D_{\phi_i,\phi_i} + \left( D_{\phi_i,\phi_i}^T D_{\phi_i,\phi_i} \right) \right)^{1/2};\]

\[R_{B,B} = \left( D_{M,M} - R_{M,M} \right)^{-1} \left( D_{C,C} + R_{C,C} \right) - D_{M,M}^{-1} D_{C,C}
\]

(2.19a,b)
And the standard deviation of $\omega^2_{di}$ is

$$\sigma(\omega^2_{di}) = (\langle \omega^2_{di} \otimes \omega^2_{di} \rangle - \langle \omega^2_{di} \rangle \otimes \langle \omega^2_{di} \rangle)^{1/2}$$

$$= \left( D_{\Phi,\Phi}^T \left( 16d_i^2 R_{A,A} - 4d_i \left( R_{A,B\tilde{\Phi},\tilde{\Phi}^T} + R_{B\tilde{\Phi},\tilde{\Phi}^T,A} \right) + R_{B\tilde{\Phi},\tilde{\Phi}^T,B\tilde{\Phi},\tilde{\Phi}^T} \right) D_{\Phi,\Phi}, (16d_i^4) \right)^{1/2}$$

(2.20)

Note that the following terms appearing in equation (2.20) can be simplified as:

$$R_{A,A} \approx (D_{M,M} - R_{M,M})^{-1} (D_{K,K} + R_{K,K}) - D_{M,M}^2 D_{K,K};$$

$$R_{A,B\tilde{\Phi},\tilde{\Phi}^T} = R_{M^{-1},B\tilde{\Phi},\tilde{\Phi}^T} D_{K,i};$$

$$D_{\phi,\phi}^T R_{B\tilde{\Phi},\tilde{\Phi}^T,B\tilde{\Phi},\tilde{\Phi}^T} D_{\phi,\phi}, (16d_i^4) = 4 \left[ \text{Re}(\gamma_i) \right]^2 \text{Var(Re(\gamma_i))}$$

(2.21a-c)

The modal damping ratio $\zeta_i$ of the system is defined as $\zeta_i = 0.5c_i / \sqrt{m_i k_i}$ where $c_i = \Phi_i^T C \Phi_i$, $m_i = \Phi_i^T M \Phi_i$, and $k_i = \Phi_i^T K \Phi_i$ are modal parameters. The corresponding standard deviations are

$$\sigma(\zeta_i) = \frac{1}{2} \left( \frac{R_{c,c}}{m_i k_i} + \frac{\overline{c_i}^2 R_{k,k}}{4m_i k_i^3} + \frac{\overline{c_i}^2 R_{m,m}}{4m_i^3 k_i} \right)^{0.5};$$

$$R_{c,c} = D_{\phi,\phi}^T R_{C,C} D_{\phi,\phi};$$
If the system is lightly damped, the variance of $\zeta_i$ has to be much smaller than unity. Accordingly, $\text{Im}(\gamma_i)$ is estimated by assuming $\sqrt{1 - \zeta_i^2}$ to be deterministic. Thus

$$\left[\text{Im}(\gamma_i)\right]^2 = \left< \omega^2_i \right> = \left(1 - \zeta_i^2\right) \Phi_0^T (A) \Phi_0 / \left(\Phi_0^T \Phi_0\right)$$

and the standard deviation of $\omega_i$ is calculated by using

$$\sigma(\omega_i) = 0.5 \sigma(\omega^2_i) / \bar{\omega}_i$$

$$\sigma(\omega^2_i) = \left< \omega^2_i \otimes \omega^2_i \right> - \left< \omega^2_i \otimes \left< \omega^2_i \right> \right>^{1/2}$$

$$= \left(1 - \zeta_i^2\right) \left(D_{\Phi_0,\Phi_0}^T R_{A,A} D_{\Phi_0,\Phi_0} / \left(D_{\Phi_0,\Phi_0}^T D_{\Phi_0,\Phi_0}\right)\right)^{1/2}$$

**Case (b):** If $C$ is fully correlated with $M$ and $K$, standard deviation of $\gamma_i$ can be given in terms of $\sigma(\omega_i)$ since $C = \alpha M + \beta K$. The eigenvalue has the following form

$$\text{Re}(\gamma_i) = -0.5 \left(\alpha + \beta \omega_i^2\right);$$

$$\left[\text{Im}(\gamma_i)\right]^2 = \omega_i^2 - \left[\text{Re}(\gamma_i)\right]^2$$
The standard deviation of $\gamma_i$ is defined as

$$\sigma(\text{Re}(\gamma_i)) = 0.5\bar{\beta}\sigma(\omega_i^2);$$

$$\sigma\left[\text{Im}(\gamma_i)\right]^2 = \left(1 - \bar{\zeta}_i^2\right)\sigma(\omega_i^2) \quad (2.26a,b)$$

The modal damping ratio $\zeta_i$ is defined as follows along with its standard deviation

$$\zeta_i = 0.5\left(\frac{\alpha}{\omega_i} + \bar{\beta}\omega_i\right);$$

$$\sigma(\zeta_i) = 0.5\text{abs}\left(\bar{\beta} \cdot -\frac{\alpha}{\omega_i^2}\right)\sigma(\omega_i) \quad (2.27a,b)$$

A few specific cases of interest are considered next:
(i) if $\alpha=0$ or $C = \bar{\beta} K$, $\sigma(\zeta_i) = 0.5\bar{\beta}\sigma(\omega_i)$
(ii) if $\beta=0$ or $C = \alpha M$, $\sigma(\zeta_i) = 0.5\alpha\frac{\sigma(\omega_i)}{\omega_i^2}$, and
(iii) if $\alpha = \bar{\beta}\omega_i^2$ then $\sigma(\zeta_i) = 0$.

One interesting case arises when $C = \bar{\beta} K$ but only $M$ is uncertain. The expression obtained for this fully correlated case should be the same that was found previously for the uncorrelated case i.e.

$$\sigma(\zeta_i) = \frac{1}{4}\left(\frac{\bar{\zeta}_i^2 R_{m_i.m_i}}{m_i^3 k_i}\right)^{0.5} \quad (2.28)$$
Similarly, each eigenvector is represented as follows

\[ \Phi_i = A \Phi_i / \omega_i^2 \]  
(2.29)

Using the proposed direct product technique, the covariance matrix between eigenvectors is estimated by using deterministic \( \omega_i^2 \) and \( \Phi_i \) as

\[ \text{Cov}(\Phi_i, \Phi_j) = < \Phi_i \otimes \Phi_j > - < \Phi_i > \otimes < \Phi_j > = R_{AA} D_{\Phi_i, \Phi_j} / (\omega_i^2 \omega_j^2) \]  
(2.30)

It must be noted that all covariances between the real and imaginary parts of \( \gamma_i \) can also be calculated by using the proposed methodology.

2.4.3 Example III: Single-Degree-of-Freedom System

For the damped system of Figure 2.1a with \( \gamma = -0.05 \pm 0.99875j \), the standard deviation of \( \gamma \) is calculated by using equations (2.19) and (2.24). Results are given in Table 2.3 for \( R_{C,C} = 0 \) and 0.0004; in the second case \( C \) is assumed to be uncorrelated with \( M \) and \( K \). It can be seen from Table 2.3 that the proposed methodology is in excellent agreement with the Monte-Carlo simulation (sample size=2,000).

2.4.4 Example IV: Visco-Elastic System

A first order system \( C\dot{X}(t) + KX(t) = 0 \), as shown in Figure 2.1c with deterministic time constant \( \tau = C / K = 0.1 \), is investigated. When randomness is introduced in the spring, \( \tau \) is a random variable. Given \( R_{C,C} = 0.01D_{C,C} \), the proposed method finds \( \sigma(\tau) \) to be 0.01. This compares very well with \( \sigma(\tau) = 0.0099 \) yielded by the
Table 2.3 Standard deviation of $\text{Re}(\gamma_i)$ for Example III (Fig. 2.1a)

<table>
<thead>
<tr>
<th>Mode $i$</th>
<th>$R_{M,M}$</th>
<th>$R_{C,C}$</th>
<th>$R_{K,K}$</th>
<th>$\sigma(\text{Re}(\gamma_i))$ and error $\epsilon^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Monte Carlo Simulation</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.01</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0004</td>
<td>0.01</td>
<td>0.0101</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.01</td>
<td>0.0500</td>
</tr>
<tr>
<td>2</td>
<td>0.0</td>
<td>0.0004</td>
<td>0.01</td>
<td>0.0497</td>
</tr>
</tbody>
</table>

*\(\epsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte Carlo simulation}\)*
Table 2.4 Standard deviation of $\gamma_1$ and $\gamma_2$ for Example V (Fig. 2.1c)

<table>
<thead>
<tr>
<th>$R_{M,M}$</th>
<th>$R_{C,C}$</th>
<th>$R_{K,K}$</th>
<th>$\gamma_i$</th>
<th>$\sigma(\gamma_i)$ and error $\varepsilon^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Re}(\gamma_1)$</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Im}(\gamma_1)$</td>
<td>0.0235</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Re}(\gamma_2)$</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Im}(\gamma_2)$</td>
<td>0.0756</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.01D_{K,K}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Re}(\gamma_1)$</td>
<td>0.00606</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Im}(\gamma_1)$</td>
<td>0.0235</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Re}(\gamma_2)$</td>
<td>0.0163</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Im}(\gamma_2)$</td>
<td>0.0753</td>
</tr>
</tbody>
</table>

*$\varepsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte Carlo simulation}$
Table 2.5 Standard deviation of $\zeta_i$ for Example V (Fig. 2.1c)

<table>
<thead>
<tr>
<th>Mode $i$</th>
<th>$\bar{\alpha}$</th>
<th>$\bar{\beta}$</th>
<th>$\bar{\zeta}_i$</th>
<th>$\sigma(\zeta_i)$ and error $\varepsilon^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Monte Carlo Simulation Proposed Method Eq. (2.28)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1000 samples 2000 samples</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0107</td>
<td>0.00784 0.00774 0.00762 ($\varepsilon=1.6%$)</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.1</td>
<td>0.0203</td>
<td>0.00176 0.00170 0.00167 ($\varepsilon=1.8%$)</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.1</td>
<td>0.1302</td>
<td>0.00615 0.00605 0.00595 ($\varepsilon=1.7%$)</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0331</td>
<td>0.00243 0.00240 0.00235 ($\varepsilon=2.1%$)</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.1</td>
<td>0.0755</td>
<td>0.00550 0.00550 0.00534 ($\varepsilon=2.9%$)</td>
</tr>
<tr>
<td></td>
<td>0.1</td>
<td>0.1</td>
<td>0.1086</td>
<td>0.00307 0.00310 0.00299 ($\varepsilon=3.5%$)</td>
</tr>
</tbody>
</table>

*Monte-Carlo simulation with 2000 samples is taken as the benchmark for error,
$\varepsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte Carlo simulation}$
Monte-Carlo simulation based on 1,000 samples. Results of this case (with \(\epsilon = 1\%\)) verify the proposed theory.

### 2.4.5 Example V: Two-Degree-of-Freedom Damped System

Consider the damped system of Figure 2.1c with \(\bar{\gamma}_1 = -0.06096 \pm 0.464j\) and \(\bar{\gamma}_2 = -0.1640 \pm 1.5013j\). Table 2.4 compares results obtained by using equations (2.19) and (2.24) with those yielded by the Monte-Carlo simulation based on 2,000 samples. Again, the proposed methodology is found to be in excellent agreement with simulation.

Now we re-examine this physical system by using the Rayleigh damping model with the assumption that \(C\) is fully correlated with \(M\) and/or \(K\); recall case (b) from Section 2.4.2.

Table 2.5 presents results for \(\sigma(\zeta_1)\) and \(\sigma(\zeta_2)\) given the following data set: \(R_{M,M} = 0.01D_{M,M}, R_{K,K} = 0.01D_{K,K}, \bar{\omega}_1 = 0.468\) and \(\bar{\omega}_2 = 1.510\). Standard deviations of \(\omega_1\) and \(\omega_2\) were found earlier in Table 2.2 by using the proposed methodology: \(\sigma(\omega_1) = 0.0333\) and \(\sigma(\omega_2) = 0.1070\). For this system, the Monte-Carlo simulation is run twice, once with 1,000 samples and the second time with 2,000 samples. As expected, the proposed method matches better with the numerical simulation based on 2,000 samples as evident from \(\epsilon \leq 3.5\%\), and a smaller sample size Monte-Carlo simulation results deviate slightly more from the proposed method.

### 2.5. System with Only Randomly Distributed Masses

#### 2.5.1 Theory

Next we examine the parameter case when only one matrix is randomly distributed. For the sake of illustration, consider an undamped system with randomly distributed masses. If the expected mean mass matrix is described as \(\bar{M} = \bar{M}I\) [2.13], the covariance mass matrix can be given as
\[
R_{M,M} = \sigma_M^2 S = \sigma_M^2 \begin{bmatrix}
\sigma_{11}^2 & r_{12} S_1 S_2 & 0 \\
r_{12} S_1 S_2 & \ddots & \ddots \\
0 & \ddots & r_{N1} S_1 S_N \\
0 & \ddots & \ddots \\
0 & \ddots & r_{NN} S_N^2
\end{bmatrix}
\]

where \( r_{jk} = \langle \tilde{M}_j \tilde{M}_k \rangle / \sigma_{M_j} \sigma_{M_k} \) is the correlation coefficient between \( \tilde{M}_j \) and \( \tilde{M}_k \) terms; \( s_j = \sigma_{M_j} / \sigma_M \) and \( \sigma_M = \max(\sigma_{M_i}) \); note that \(-1 \leq r_{jk} \leq 1\) and \(0 \leq s_j \leq 1\). The standard deviation of random eigensolutions can be simplified by using equations (2.7-2.12).

Substituting (2.31) into (2.10) and (2.11) we get

\[
\sigma(\lambda_i) = \left( \frac{D_{\Phi_i,\Phi_i}^T \left[ (D_{l,l} - \sigma_M^2 S)^{-1} - D_{l,l} \right] D_{\Phi_i,\Phi_i}}{D_{\Phi_i,\Phi_i}^T D_{\Phi_i,\Phi_i}} \right)^{1/2} \bar{\lambda}_i;
\]

\[
\sigma(\Phi_{ij}) = \left( \frac{1}{1 - \sigma_M^2 s_j^2} - 1 \right)^{1/2} \bar{\Phi}_{ij}
\]

(2.32a,b)

where \( \sigma_M = \sigma_M / \bar{M} \). The standard deviation of the \( i \)th normalized eigenvalue \( \lambda_{oi} = \lambda_i / \bar{\lambda}_o \) can be expressed as follows given that \( \bar{\lambda}_o = \bar{K} / \bar{M} \), \( \bar{\lambda}_{oi} = \bar{\lambda}_i / \bar{\lambda}_o \) and \( \lambda_{oi} = \bar{\lambda}_{oi} + \tilde{\lambda}_{oi} \)

\[
\sigma(\lambda_{oi}) = \left( \frac{D_{\Phi_i,\Phi_i}^T \left[ (D_{l,l} - \sigma_M^2 S)^{-1} - D_{l,l} \right] D_{\Phi_i,\Phi_i}}{D_{\Phi_i,\Phi_i}^T D_{\Phi_i,\Phi_i}} \right)^{1/2} \bar{\lambda}_{oi}
\]

(2.33)
The effect of randomness of stiffness matrix alone can be estimated in a similar manner. Also, all covariances of eigenvalues and eigenvectors can also be calculated by using the methodology described earlier in Section 2.3.

2.5.2 Limiting Cases of Random Mass Matrix

Case (a): If the randomness of each diagonal mass element is identical and fully correlated with each other i.e. \( S=I \), equations (2.33) and (2.32b) reduce to

\[
\sigma(\lambda_{oi}) = \left( \frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\lambda}_{oi} ;
\]

\[
\sigma(\Phi_{ij}) = \left( \frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\Phi}_{ij} \quad (2.34a,b)
\]

Case (b): If the randomness of each diagonal mass element is identical but uncorrelated with each other i.e. \( r_{jk} = \delta_{jk} \) where \( \delta_{jk} \) is Kronecker delta and \( s_j = s_k = 1 \), then equations (2.33) and (2.32b) reduce to

\[
\sigma(\lambda_{oi}) = \left( \frac{\sum_{j=1}^{N} \Phi_{ij}^4}{D_{\Phi,\Phi,\Phi,\Phi}} \right)^{1/2} \left( \frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\lambda}_{oi} ;
\]

\[
\sigma(\Phi_{ij}) = \left( \frac{1}{1 - \bar{\sigma}_M^2} - 1 \right)^{1/2} \bar{\Phi}_{ij} \quad (2.35a,b)
\]
When the random fluctuations are very small i.e. $\bar{\sigma}_M << 1$, these solutions can be approximated by using the first order perturbation technique. For instance, for case (b), equation (2.35) is simplified as

$$\sigma(\lambda_{oi}) = \left( \frac{\sum_{j=1}^{N} \phi_{ij}^4}{D_{\phi_i,\phi_i}} \right)^{1/2} \bar{\sigma}_M \lambda_{oi};$$

$$\sigma(\Phi_{ij}) = \bar{\sigma}_M \Phi_{ij} \quad \text{(2.36a,b)}$$

2.5.3 Example VI: Multi-Degree-of-Freedom System

An undamped system of Figure 2.1d with random masses is studied to illustrate the proposed theory. The randomness of the masses being uncorrelated, case (b) as described in Section 2.5.2, is investigated first for $N=5$. Equation (2.35) is used to obtain results for $\sigma(\lambda_1)$ given various $\sigma_M$ values. Results are compared in Table 2.6 with predictions yielded by the Monte-Carlo simulation (sample size=1,000) and the first order perturbation method [2.6]. When $\sigma_M << 1$, all four predictions are virtually identical. However as $\sigma_M$ increases, first order perturbation method begins to deviate from the Monte-Carlo simulation. The proposed method as described by equation (2.35) gives excellent results up to $\sigma_M=0.3$, using Monte-Carlo simulation as a benchmark. When $\sigma_M$ is as high as 0.5 only a discrepancy of $\varepsilon=6.6\%$ is seen, as opposed to $\varepsilon=19.2\%$ associated with the first order perturbation method. This example demonstrates clearly that the proposed method is superior to the existing analytical techniques based on the first order perturbations.
Table 2.6 Standard deviation of $\lambda_i$ for Example VI (Fig. 2.1d) with $N=5$

<table>
<thead>
<tr>
<th>$\sigma_M$</th>
<th>Monte-Carlo Simulation</th>
<th>Proposed Method Eq.(2.17)</th>
<th>First Order Perturbation Method [2.5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.00217</td>
<td>0.00212</td>
<td>0.00211</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\varepsilon=2.3%$)</td>
<td>($\varepsilon=2.8%$)</td>
</tr>
<tr>
<td>0.15</td>
<td>0.00645</td>
<td>0.00642</td>
<td>0.00634</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\varepsilon=0.5%$)</td>
<td>($\varepsilon=1.7%$)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.01358</td>
<td>0.01331</td>
<td>0.01269</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\varepsilon=2%$)</td>
<td>($\varepsilon=6.6%$)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.02616</td>
<td>0.02443</td>
<td>0.02115</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($\varepsilon=6.6%$)</td>
<td>($\varepsilon=19%$)</td>
</tr>
</tbody>
</table>

$\varepsilon=(\text{Monte Carlo simulation} - \text{analytical method})/\text{Monte-Carlo simulation}$
Table 2.7 Standard deviation of $\lambda_{oi}$ for Example VI(Fig. 2.1d) with $N=10$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\bar{\lambda}_{oi}$</th>
<th>$\sigma(\lambda_{oi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed Method: Eq. (2.11)</td>
<td>First Perturbation Method: [2.6]</td>
</tr>
<tr>
<td>1</td>
<td>0.0223</td>
<td>0.00849</td>
</tr>
<tr>
<td>2</td>
<td>0.1981</td>
<td>0.07523</td>
</tr>
<tr>
<td>3</td>
<td>0.5339</td>
<td>0.20203</td>
</tr>
<tr>
<td>4</td>
<td>1.000</td>
<td>0.3799</td>
</tr>
<tr>
<td>5</td>
<td>1.555</td>
<td>0.5906</td>
</tr>
<tr>
<td>6</td>
<td>2.1495</td>
<td>0.8164</td>
</tr>
<tr>
<td>7</td>
<td>2.7307</td>
<td>1.0373</td>
</tr>
<tr>
<td>8</td>
<td>3.247</td>
<td>1.233</td>
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<tr>
<td>9</td>
<td>3.652</td>
<td>1.387</td>
</tr>
<tr>
<td>10</td>
<td>3.911</td>
<td>1.486</td>
</tr>
</tbody>
</table>
Standard deviation of normalized eigenvalues associated with random masses of Figure 2.1d with \( N=10 \) is shown in Table 2.7. Results obtained by using equation (2.30) are slightly higher than those obtained earlier by Pierre [2.6] and Soong and Bogdanoff [2.13]. This is expected since the proposed method includes the higher order terms.

2.6. Concluding Remarks

A new analytical technique for random eigensolutions of undamped and proportionally damped discrete systems has been presented and validated by using two existing techniques, namely the first order perturbation method and the Monte-Carlo simulation. It is potentially a powerful technique because of its reasonable accuracy coupled with computational ease. Yet another key advantage of the proposed method is its application to the determination of impulse and frequency response characteristics; it is the subject of Chapter III. The proposed method has been found to be more accurate than the commonly used first order perturbation methods especially when random fluctuations are no longer very small. Also, it is computationally faster than the Monte-Carlo simulation. For instance, the computational speed ratio between the proposed method and Monte-Carlo simulation for Example V is of the order of \( 10^2 \) to \( 10^3 \). Such a ratio is expected to be even higher for a complex vibratory system of higher \( N \); this is due to the fact that a large number of iterations is required to estimate the probability distribution in a stochastic simulation. However, it should be noted that there are uncertainties for various numerical examples associated with Monte-Carlo predictions. Therefore, comparisons given by \( \varepsilon \) values should be considered valid within the confidence interval associated with the sample size.

Major weaknesses of the proposed analytical method come from various simplifying assumptions made during the development. Some of these are as follows: (i)
This method loses accuracy when random fluctuations are very high, say that, normalized standard deviation is higher than 0.5; (ii) it can not address localization effects in periodic disordered systems; and (iii) non-proportionally damped systems can not be analyzed. Research is underway to overcome these as well as extensions of proposed method to continuous systems.
References


CHAPTER III
IMPULSE RESPONSE

3.1. Introduction

Impulse response of a single or multi-degree-of-freedom system with uncertain mass, stiffness and damping parameters has been the subject of several investigations [3.1-3.5]. Primarily the first order perturbation method has been used but with limited success [3.1-3.3,3.5]. It has been seen that such perturbation methods are valid only for a short time duration since the standard deviation of transient response becomes unbounded with time especially for a virtually undamped system. Stochastic finite element methods have the same limitations since they are also based on the theory of first order perturbations [3.6,3.7]. Such problems including mathematical singularities arise due to the presence of secular terms in analytical solutions. Accordingly the first main objective of this chapter is propose a new analytical method which does not include any secular terms in the impulse response expression. It should be noted that direct Monte-Carlo simulation can always be employed for these problems [3.8]. However, this simulation is computationally intensive since a large number of iterations is required to estimate the probabilistic distributions. Hence, there is interest in developing new or improved analytical or semi-analytical methods which overcome the disadvantages of existing methods and still yield reasonably accurate solutions. Review articles by Ibrahim [3.9] and Benaroya and Rehak [3.10] discuss these and other related research issues.
In a parallel development we have proposed a new direct product technique which estimates the statistical frequency response of a damped vibratory system [3.11]. This method has been found to be efficient and reasonably accurate at and near resonance(s), unlike the first order perturbation method [3.13]. However, the system matrix is inverted numerically which poses computational problems especially when the damping ratio is very low. Also, this method does not use or predict random eigensolutions. Therefore in this chapter an alternate technique, based on the eigensolutions formulated in Chapter II [3.12] and the impulse response characteristics given in this chapter, will be proposed to estimate the frequency response. Even though this paper is a continuation of Chapter II, it is written such that it is self-sufficient. However, for the sake of brevity, reader is asked to refer to Chapter II for a few items.

3.2. Problem Formulation

The major objective of this chapter is to develop a new analytical method for the dynamic response of a linear time-invariant, proportionally damped vibratory system of dimension $N$ with uncertain parameter matrices $M$, $C$ and $K$; each matrix is assumed to be symmetric and positive definite. The amplitude of the excitation, $MF$, is also randomly distributed but the time history $\xi(t)$ is deterministic and arbitrary; impulse and sinusoidal functions are chosen here to illustrate the method. The random differential equation can be given in the matrix form as follows; see LIST OF SYMBOLS for the identification of symbols. Since only the forced response is of interest, initial displacement $X(0)$ and velocity $\dot{X}(0)$ are assumed to be null vectors.

$$M\ddot{X}(t) + C\dot{X}(t) + KX(t) = MF\xi(t); \quad X(0) = \dot{X}(0) = 0$$  (3.1)
Specifically, the first two moments of the ensuing response are estimated by making a few simplifying assumptions and from the knowledge of statistical eigenvalues of Chapter II. Both single and two degree-of-freedom system examples are considered. The proposed method is validated by comparing it with the Monte-Carlo numerical simulation [3.10] which is considered as the benchmark. Predictions yielded by the first order perturbation technique are also given when appropriate. Like Chapter II, the following assumptions are made to develop the new solution methodology: (i) random matrices and vectors of equation (3.1) can be given by the sum of deterministic or mean (identified by bar) and random or fluctuating components (identified by tilde) i.e. \( M = \bar{M} + \tilde{M} \), \( C = \bar{C} + \tilde{C} \), \( K = \bar{K} + \tilde{K} \), \( F = \bar{F} + \tilde{F} \) and \( X = \bar{X} + \tilde{X} \); (ii) expected means of system matrices \( \bar{M} = \langle M \rangle \), \( \bar{C} = \langle C \rangle \), \( \bar{K} = \langle K \rangle \) and excitation amplitude \( \langle F \rangle = \bar{F} \) are known; (iii) probability distributions of \( M, C, K, F \) are of the same type and are known; (iv) means of a random parameter matrix and \( F \) are equal to null e.g. \( \langle \tilde{M} \rangle = 0 \), \( \langle \tilde{F} \rangle = 0 \); (v) parameter fluctuations are much smaller compared to the deterministic values i.e. \( \| \tilde{M} \| << \| \bar{M} \| \); (vi) co-variances of parameter fluctuations are known in the form of cross-correlation matrices such as \( R_{M,M} = \langle \tilde{M} \otimes \tilde{M} \rangle = \text{Var}(M) \) and \( R_{M,K} = \langle \tilde{M} \otimes \tilde{K} \rangle = \text{Cov}(M,K) \); and (vii) \( F \) is uncorrelated with system parameter matrices i.e. \( R_{F,M} = R_{F,C} = R_{F,K} = 0 \). Additional assumptions will be specified as a part of the analytical development.

3.3. Proposed Analytical Method

3.3.1 Eigenvalues

Even though theory presented in Chapter II is summarized here, it is written in a manner which facilitates further analytical development. Assuming \( F = 0 \), equation (3.1) is expressed as follows in accordance with the assumptions stated earlier
\[ \ddot{X}(t) + B\dot{X}(t) + AX(t) = 0; \]

\[ A = M^{-1}K; \quad B = M^{-1}C \quad (3.2a-c) \]

Now let \( X(t) = \Phi Y(t) \) where \( \Phi \) is the modal matrix of the undamped system and \( Y(t) \) is the normal coordinate vector. Further, premultiply (3.2a) by \( \Phi^T \) to yield the following characteristic equation for the proportionally damped system

\[ \gamma_i^2 \Phi_i^T \Phi_i + \gamma_i \Phi_i^T B \Phi_i + \Phi_i^T A \Phi_i = 0 \quad ; \quad i = 1,2,\ldots,N \quad (3.3) \]

Here \( \gamma_i = -\zeta_i \omega_i \pm j\omega_i = -\zeta_i \omega_i \pm j\omega_i \sqrt{1-\zeta_i^2} \) is the \( i \)th complex valued eigenvalue corresponding to real eigenvector \( \Phi_i \), \( \zeta_i \) is the damping ratio, \( \omega_i \) is the undamped natural frequency, \( \omega_{di} \) is the damped natural frequency and \( j = \sqrt{-1} \). The eigenvalues are assumed to be distinct. Rewrite equation (3.3) as

\[ \gamma_i = \frac{-\Phi_i^T B \Phi_i \pm j\sqrt{4(\Phi_i^T \Phi_i \Phi_i^T A \Phi_i) - (\Phi_i^T B \Phi_i)^2}}{2\Phi_i^T \Phi_i} = \text{Re}(\gamma_i) + j\text{Im}(\gamma_i) \quad (3.4) \]

where

\[ \text{Re}(\gamma_i) = \frac{-\Phi_i^T B \Phi_i}{2\Phi_i^T \Phi_i}; \]

\[ [\text{Im}(\gamma_i)]^2 = \omega_{di}^2 = \frac{4\Phi_i^T \Phi_i \Phi_i^T A \Phi_i - \Phi_i^T B \Phi_i \Phi_i^T B \Phi_i}{4(\Phi_i^T \Phi_i)^2} \quad (3.5a,b) \]
If the system is lightly damped, the variance of $\zeta_i$ has to be much smaller than unity. Accordingly, $\text{Im}(\gamma_i)$ is estimated by assuming $\sqrt{1-\zeta^2_i}$ to be deterministic. Thus

$$\langle \left[\text{Im}(\gamma_i)\right]^2 \rangle = \langle \omega_{ai}^2 \rangle = (1-\zeta_i^2)\langle \omega_i^2 \rangle = (1-\zeta_i^2)\Phi_i^T\langle A \rangle \Phi_i / (\Phi_i^T\Phi_i)$$

(3.6)

If $\gamma_i$ is estimated using the deterministic eigenvector $\Phi_i$, expected value of $\gamma_i$ yields the following

$$\langle \text{Re}(\gamma_i) \rangle = -0.5\Phi_i^T(B)\Phi_i / (\Phi_i^T\Phi_i) ;$$

$$\langle \left[\text{Im}(\gamma_i)\right]^2 \rangle = \langle \omega_{ai}^2 \rangle = (1-\zeta_i^2)\langle \omega_i^2 \rangle = (1-\zeta_i^2)\Phi_i^T\langle A \rangle \Phi_i / (\Phi_i^T\Phi_i)$$

(3.7a,b)

Since $\gamma_i$ is complex valued, its standard deviation is complex valued as well: $\sigma_{\gamma_i} = \sigma(\text{Re}(\gamma_i)) + j\sigma(\text{Im}(\gamma_i))$. It is determined by using the covariance matrices of $A$ and $B$. Further, covariances of $\text{Re}(\gamma_i)$ and $\omega_{ai}$ can be given as

$$\text{Cov}\left(\text{Re}(\gamma_i),\text{Re}(\gamma_j)\right) = \langle \text{Re}(\gamma_i) \otimes \text{Re}(\gamma_j) \rangle - \langle \text{Re}(\gamma_i) \rangle \otimes \langle \text{Re}(\gamma_j) \rangle ;$$

$$\text{Cov}\left(\omega_{ai},\omega_{aj}\right) = \langle \omega_a \otimes \omega_j \rangle - \langle \omega_a \rangle \otimes \langle \omega_j \rangle$$

(3.8a,b)

### 3.3.2 Moments of Modal Response

Equation (3.1) in the normal coordinates can be given as follows where $\Phi Y(t) = X(t)$.
\[ \Phi^T \Phi \dot{Y}(t) + \Phi^T B \Phi \dot{Y}(t) + \Phi^T A \Phi Y(t) = \Phi^T F \xi(t) \]  

Since the system is assumed to be proportionally damped, equation (3.9) becomes uncoupled and takes the following form

\[ \ddot{Y}(t) + \begin{bmatrix} 2 \zeta_i \omega_i & \ldots \end{bmatrix} \begin{bmatrix} \omega_i^2 \\ \omega_i^2 \\ \ldots \end{bmatrix} Y(t) = \frac{\Phi^T F}{\Phi^T \Phi} \xi(t) \]  

Like the deterministic system, equation (3.10) yields \( N \) uncoupled equations, each describing a single-degree-of-freedom in the modal domain. The response \( Y_i(t) \) for the \( i \)th mode is determined by

\[ Y_i(t) = \int \frac{\Phi_i^T F}{\Phi_i^T \Phi_i} e^{-a_i t} \sin b_i \tau \frac{\xi(t - \tau)}{b_i} d\tau , \quad t \geq 0 \]  

where \( a_i = \text{Re}(\lambda_i) \) and \( b_i = \text{Im}(\lambda_i) \) are random variables. The corresponding response of the deterministic system is

\[ \overline{Y}_i(t) = \int \frac{\overline{\Phi}_i^T F}{\overline{\Phi}_i^T \overline{\Phi}_i} e^{-\overline{a_i} t} \sin \overline{b}_i \tau \frac{\xi(t - \tau)}{\overline{b}_i} d\tau , \quad t \geq 0 \]  

The statistical behavior of the eigenvalues is found by using the direct product technique based on the covariance matrices of system parameters in Section 3.3.1; also refer to Chapter II. Without losing any accuracy, the first two moments are estimated based solely on the knowledge of deterministic modal vector \( \Phi_i \). Therefore, the expected mean and mean-square values of \( Y_i(t) \) are found to be
\[
\langle Y_i(t) \rangle = \int \rho_i \int e^{-a_{i,\tau}} \frac{\sin b_i \tau}{b_i} p(a_i, b_i) da_i db_i \xi(t-\tau) d\tau;
\]

\[
\langle Y_i^2(t) \rangle = \int \mu_i \int e^{-a_{i,(\tau+\nu)}} \frac{\sin b_i \tau \sin b_i \nu}{b_i^2} p(a_i, b_i) da_i db_i \xi(t-\tau) \xi(t-\nu) d\tau d\nu
\]  \hspace{1cm} (3.13a,b)

where \( p(a_i, b_i) \) is the joint density function, and \( \rho_i \) and \( \mu_i \) are the expected modal participation factors defined as

\[
\rho_i = \frac{\langle \Phi_i^T F \rangle}{\langle \Phi_i^T \Phi_i \rangle} = \Phi_i^T \Phi_i
\]

\[
\mu_i = \frac{D_{\Phi_i,\Phi_i}^T \langle F \otimes F \rangle}{\langle \Phi_i^T \Phi_i \rangle^2} = \Phi_i^T \Phi_i \left( D_{F,F} + R_{F,F} \right)
\]  \hspace{1cm} (3.14a,b)

### 3.3.3 Uncorrelated Damping

The first limiting case of the Rayleigh damping model \( C = \alpha M + \beta K \) is investigated next when \( C, M \) and \( K \) are statistically independent i.e. \( \alpha \) and \( \beta \) are random variables; recall Chapter II [12]. Define the standard deviation of \( \text{Re}(\gamma_i) \) by using equation (3.8a)

\[
\sigma(\text{Re}(\gamma_i)) = \left( < \text{Re}(\gamma_i) \otimes \text{Re}(\gamma_i) > - < \text{Re}(\gamma_i) > \otimes < \text{Re}(\gamma_i) > \right)^{1/2}
\]

\[
= 0.5 \left( D_{\Phi_i,\Phi_i}^T R_{B,B} D_{\Phi_i,\Phi_i} / \left( D_{\Phi_i,\Phi_i}^T D_{\Phi_i,\Phi_i} \right) \right)^{1/2};
\]

\[
R_{B,B} = \left( D_{M,M} - R_{M,M} \right)^{-1} \left( D_{C,C} + R_{C,C} \right) - D_{M,M}^{-1} D_{C,C}
\]  \hspace{1cm} (3.15)
where $D$ is the direct product of two deterministic matrices e.g. $D_{M,M} = \overline{M} \otimes \overline{M}$. The standard deviation of $\omega_{di}$ is calculated by using

$$\sigma(\omega_{di}) = 0.5\sigma(\omega_i^2) / \overline{\omega}_{di}; \quad \sigma(\omega_i) = 0.5\sigma(\omega_i^2) / \overline{\omega}_i;$$

$$\sigma(\omega_{di}^2) = (\left< \omega_{di}^2 \right> - \left< \omega_{di} \right>^2)^{1/2}$$

$$\approx (1 - \overline{\zeta_i^2})\sigma(\omega_i^2) = (1 - \overline{\zeta_i^2})\left(D_{\phi_i,\phi_i}^T R_{A,A} D_{\phi_i,\phi_i} / \left(D_{\phi_i,\phi_i}^T R_{A,A} D_{\phi_i,\phi_i}\right)\right)^{1/2}$$

$$R_{A,A} = \left(D_{M,M} - R_{M,M}\right)^{-1} \left(D_{K,K} + R_{K,K}\right) - D_{M,M}^{-1} D_{K,K}$$

(3.17a-d)

If $M$ is deterministic but $K$ is still random, $a_i$ and $b_i$ must be uncorrelated. Therefore mean and mean-square values of $Y_i(t)$ are

$$\langle Y_i(t) \rangle = \int p_a \int e^{-a_i \tau} p(a_i) da_i \int \frac{\sin b_i \tau}{b_i} p(b_i) db_i \xi(t - \tau) d\tau$$

$$\langle Y_i^2(t) \rangle = \int \mu \int \int e^{-a_i(\tau + \nu)} \frac{\sin b_i \tau \sin b_i \nu}{b_i^2} p(a_i) da_i p(b_i) db_i \xi(t - \tau) \xi(t - \nu) d\tau d\nu$$

(3.18a,b)

where $p(a_i)$ and $p(b_i)$ are marginal density functions.

### 3.3.4 Fully Correlated Damping

The second limiting case is examined by assuming that $C$ is fully correlated with $M$ and $K$. From Chapter II [12] that the standard deviation of $\gamma_i$ can be given in terms of $\sigma(\omega_i)$ since $C = \overline{\alpha} M + \overline{\beta} K$. The eigenvalue has the following form
\[ \text{Re}(\gamma_i) = -0.5(\overline{\alpha} + \overline{\beta}\omega_i^2) \]  
\[ [\text{Im}(\gamma_i)]^2 = \omega_i^2 - [\text{Re}(\gamma_i)]^2 = \omega_i^2 (1 - \overline{\zeta_i}^2) \omega_i^2 \]  

The standard deviation of \( \gamma_i \) is defined as

\[ \sigma(\text{Re}(\gamma_i)) = 0.5\overline{\beta}\sigma(\omega_i^2) \]

However, \( \sigma(\omega_i^2) \) is still given by equation (3.17). Since \( a_i \) and \( b_i \) are fully correlated, they can be given in terms of a single random variable, say \( \omega_i \). Expected first and second moments of \( Y_i(t) \) are consequently given as follows where \( p(\omega_i) \) is the density function

\[ \langle Y_i(t) \rangle = \int_{-\infty}^{\infty} e^{-0.5(\alpha+\beta\omega_i^2)t} \frac{\sin(1 - \overline{\zeta_i}^2\omega_i)t}{\sqrt{1 - \overline{\zeta_i}^2\omega_i}} \int_{-\infty}^{\infty} p(\omega_i) \omega_i \xi(t - \tau)d\tau \]

\[ \langle Y_i^2(t) \rangle = \int \int_{-\infty}^{\infty} e^{-0.5(\alpha+\beta\omega_i^2)(t+\nu)} \frac{\sin(1 - \overline{\zeta_i}^2\omega_i)t\sin(1 - \overline{\zeta_i}^2\omega_i)\nu}{(1 - \overline{\zeta_i}^2\omega_i)^2} \int_{-\infty}^{\infty} p(\omega_i) \omega_i \xi(t - \tau)\xi(t - \nu)d\tau d\nu \]

3.4. Impulse Response

3.4.1 Moments of Green's Function: Analytical Development

Equations (3.22) and (3.23) illustrate that \( \langle Y_i(t) \rangle \) and \( \langle Y_i^2(t) \rangle \) must be evaluated numerically because of the double integrals and probabilistic density functions involved.
In order to illustrate the proposed method, a simplified case is presented analytically. Consider $\xi(t)$ to be an unit impulse function. Assume that the random variables $a_i$ and $b_i$ are uniformly distributed and uncorrelated with each other. The expected mean of impulse response or Green's function is obtained from equations (3.18a) as

$$\langle Y_i(t) \rangle = \rho_i \rho_i \left( e^{-a_i t} \frac{\sin b_i t}{b_i} \right) = \rho_i \frac{p(a_i) p(b_i)}{-t} e^{-a_i t} \left[ \psi(\alpha_i) \right] _{b_i}^{b_i} ;$$

$$p(a_i) = \frac{1}{(2 \sqrt{3} \sigma_{a_i})} ; \quad p(b_i) = \frac{1}{(2 \sqrt{3} \sigma_{b_i})} \quad (3.24a-c)$$

where $\psi(b_i) = \int \frac{\sin b_i t}{b_i} \frac{db_i}{b_i}$ is the sine integral, $a_i^+ = \alpha_i + \sqrt{3} \sigma_{a_i}$, $a_i^- = \alpha_i - \sqrt{3} \sigma_{a_i}$, $b_i^+ = \beta_i + \sqrt{3} \sigma_{b_i}$, and $b_i^- = \beta_i - \sqrt{3} \sigma_{b_i}$. The expected mean-square value of impulse response or Green's function is expressed as

$$\langle \dot{Y}_i^2(t) \rangle = \mu_i \dot{Y}_i(t) = \mu_i \left( e^{-2a_i t} \frac{\sin^2 b_i t}{b_i^2} \right) = \mu_i p(a_i) p(b_i) \frac{e^{-2a_i t}}{-2t} \left[ \psi(\alpha_i) \right] _{b_i}^{b_i} \int \frac{\sin^2 b_i t}{b_i^2} \frac{db_i}{b_i}$$

$$= \mu_i p(a_i) p(b_i) \frac{e^{-2a_i t}}{-2t} \left[ \psi(\alpha_i) \right] _{b_i}^{b_i} \left( \frac{-1 + \cos 2b_i t}{b_i} + 2t \psi(2b_i t) \right)$$

$$\psi(2b_i t) = \int \frac{\sin 2b_i t}{b_i} \frac{db_i}{b_i} \quad (3.25a,b)$$

And the standard deviation of the impulse response $Y_i(t)$ as
\[ \sigma_r(t) = \left( \langle Y_i^2(t) \rangle - \langle Y_i(t) \rangle^2 \right)^{1/2} \] (3.26)

which can be determined from equations (3.24) and (3.25). The moments of impulse response in terms of the generalized coordinate can now be estimated by the deterministic modal matrix \( \Phi \) i.e. \( X(t) = \Phi Y(t) \).

### 3.4.2 Approximate Solution

Observe that the sine integrals of equations (3.24) and (3.25) can be evaluated analytically or numerically. However to demonstrate the procedure again, we find an approximate analytical solution by assuming first order perturbations of only \( b \), not \( a \).

Therefore the expected first and second moments are as follows, from equations (3.24) and (3.25); rest of the procedure remain the same.

\[
\langle Y_i(t) \rangle = \rho_i h(t) = \rho_i \left( \frac{2b_i - b_i}{b_i^2} e^{-a_i t} \sin b_i t \right) \\
= \rho_i p(a_i) p(b_i) \left. \frac{e^{-a_i t}}{-t} \left( \frac{2b_i - b_i}{b_i^2} \cos b_i t + \frac{\sin b_i t}{t^2} \right) \right|_{b_i}^{b_i'} (3.27)
\]

\[
\langle Y_i^2(t) \rangle = \mu_i g(t) = \mu_i \left( \frac{2b_i^2 - b_i^2}{b_i^4} e^{-2a_i t} \sin^2 b_i t \right) = \mu_i p(a_i) p(b_i) \left. \frac{e^{-2a_i t}}{-2t} \right|_{a_i}^{a_i'} \]

\[
* \frac{1}{b_i^4} \left( \frac{b_i^2}{2} - \frac{\sin 2b_i t}{4t} - \frac{b_i^3}{6} - \frac{b_i^2 \sin 2b_i t}{4t} + \frac{b_i \cos 2b_i t}{8t^2} + \frac{\sin 2b_i t}{8t^3} \right) \right|_{b_i}^{b_i'} (3.28)
\]
since

\[ \frac{1}{\bar{b}_i} = \frac{1}{\bar{b}_j + \bar{b}_j} = \frac{\bar{b}_j - \bar{b}_j}{\bar{b}_j^2} = \frac{2\bar{b}_j - \bar{b}_j}{\bar{b}_j^2} \]

\[ \frac{1}{\bar{b}_j^2} = \frac{1}{(\bar{b}_j + \bar{b}_j)^2} = \frac{\bar{b}_j^2 - 2\bar{b}_j\bar{b}_j + \bar{b}_j^2}{\bar{b}_j^4} = \frac{2\bar{b}_j^2 - \bar{b}_j^2}{\bar{b}_j^4} \]  

(3.29a, b)

**3.4.3 Example I: Undamped Single-Degree-of-Freedom System**

The parameters of the system shown in Figure 3.1a are chosen to be: \( \bar{M} = 1 \), \( \bar{C} = 0 \), \( \bar{K} = 1 \) and hence \( \bar{\gamma}_j = -0 \pm 1 j \). Excitation is unit impulse with \( F = 1 \) and \( R_{x,F} = 0 \). Random variation is considered only in the stiffness and it is assumed to be uniformly distributed i.e. \( R_{x,K} = 0.01 \), \( R_{M,M} = R_{C,C} = 0 \) and \( \sigma(\omega_{di}) = 0.05 \). The corresponding expected mean \( \langle \gamma_j(t) \rangle \), standard deviation \( \sigma_{\gamma_j}(t) \) and upper-bound \( \sigma_{\gamma_j}(t) + \langle \gamma_j(t) \rangle \) time histories, as yielded by the proposed method both in the exact form by using equations (3.24, 3.25) and in the simplified form (3.27, 3.28) are given in Figures 3.2-3.4. Results are also compared with predictions yielded by Monte-Carlo simulation (sample size=400) and the first order perturbation method [3.1]. From Figures 3.2a and 3.2b, it can be seen that the results of \( \langle \gamma_j(t) \rangle \) as predicted by the proposed method, unlike the first order perturbation [3.1], match well with the Monte-Carlo simulation. Observe that both exact and simplified forms yield virtually the same answers except in the vicinity of \( t=0 \). Our method, unlike the first order perturbation technique [3.1], shows a decay in amplitudes in an otherwise undamped system. It must be noted that the Monte-Carlo simulation has its uncertainties as well since the sample size is finite. We observe from Figures 3.3a and 3.3b that standard deviations predicted by the proposed method (equations (3.24, 3.25) or (3.27, 3.28)) are in reasonably good agreement with the Monte-
Carlo simulation except that the oscillations in the numerical simulation results die out slowly. It is seen that the standard deviation amplitude, as yielded by the first order perturbation method [3.1], grows boundlessly as time increases because of the secular terms which are present inherently. Note that the simplified form of our method as given by equations (3.27) and (3.28) does not show the presence of any secular terms even though first order perturbations of one random variable were assumed in Section 3.4.2. The results of the upper-bound response, $\sigma_y(t) + \langle y(t) \rangle$, are shown in Figure 3.4; again the proposed method matches well with the numerical simulation.

3.4.3. Example II: Damped Single-Degree-of-Freedom System

Reconsider the physical system of Example I with $C = 0.2$ and $\gamma_i = -0.1 \pm 0.995j$. Two cases of randomness are considered.

1. $R_{k,k} = 0.01$ and $\sigma_{\omega_{a1}} = 0.04975$: We observe from Figures 3.5-3.7 that predictions obtained by the proposed method (equations (3.24, 3.25) or (3.27, 3.28)) are in virtual agreement with the results of Monte-Carlo simulation. It is seen again that the first order perturbation method [3.1] deviates from the Monte-Carlo simulation.

2. $R_{k,k} = 0.01$, $R_{C,C} = 4 \times 10^{-4}$, $\sigma_{\gamma_i} = 0.02$ and $\sigma_{\omega_{a1}} = 0.04975$: Figures 3.8 and 3.9 show the results predicted by equations (3.24, 3.25) and those obtained by the existing methods. The proposed method is more accurate than the first order perturbation method [3.1] when compared with the Monte-Carlo simulation. A comparison of both damping cases demonstrates that a small fluctuation in damping ratio does not affect the impulse response significantly.
Figure 3.1 Physical systems used to illustrate and validate the proposed method: (a) single degree-of-freedom system (Example I,II and IV); (b) two degree-of-freedom system (Example III). Deterministic system parameters are given here. Also the locations of force are shown.
Figure 3.2. Expected mean of impulse response for Example I shown in Figure 3.1a: (a) $\langle Y_1(t) \rangle$ by using equation (3.24); (b) $\langle Y_2(t) \rangle$ by using equation (3.27). Key: —— proposed method; - - - - first order perturbation technique [3.1]; and +++ Monte-Carlo simulation.
Figure 3.3. Standard deviation of impulse response $\sigma(Y_i(t))$ for Example I shown in Figure 3.1a: (a) exact form yielded by equations (3.24) and (3.25); (b) approximate solution yielded by equations (3.27) and (3.28). Key as Figure 3.2.
Figure 3.4. Upperbound of impulse response \( (Y_i(t)) + \sigma(Y_i(t)) \) for Example 1: (a) exact form yielded by equations (3.24) and (3.25); (b) approximate solution yielded by equations (3.27) and (3.28). Key as Figure 3.2.
Figure 3.5. Expected mean of impulse response for a lightly damped system with $C = 0.2$ shown in Figure 3.1a (Example II): (a) $\langle Y(t) \rangle$ by using equation (3.24); (b) $\langle Y(t) \rangle$ by using equation (3.27). Key as Figure 3.2.
Figure 3.6. Standard deviation of impulse response $\sigma(y_i(t))$ for Example II: (a) exact form yielded by equations (3.24) and (3.25); (b) approximate solution yielded by equations (3.27) and (3.28). Key as Figure 3.2.
Figure 3.7. Upperbound of impulse response $\langle Y_i(t) \rangle + \sigma(Y_i(t))$ for Example II: (a) exact form yielded by equations (3.24) and (3.25); (b) approximate solution yielded by equations (3.27) and (3.28). Key as Figure 3.2.
Figure 3.8. Effect of damping fluctuation for a lightly damping system with \( \bar{C} = 0.2 \) and \( R_{c,c} = 4 \times 10^{-4} \) shown in Figure 3.1a (Example II): (a) \( \langle Y_1(t) \rangle \) by using equation (3.24); (b) \( \sigma(Y_1(t)) \) by using equations (3.24) and (3.25). Key as Figure 3.2.
Figure 3.9. Effect of damping fluctuation on upperbound impulse response \( \langle Y(t) \rangle + \sigma(Y(t)) \) for a lightly damped system shown in Figure 3.1a (Example II) with \( \bar{C} = 0.2 \) and \( R_{c,c} = 4 \times 10^{-4} \). Key as Figure 3.2.
3.4.4 Example III: Two Degree-of-Freedom System

Consider the physical system shown in Figure 3.1b with $\tilde{\gamma}_1 = -0.061 \pm 0.464 j$ and $\tilde{\gamma}_2 = -0.164 \pm 1.501 j$. We consider randomness in the stiffness matrix only and choose it as $R_{k,k} = 0.01 D_{k,k}$ i.e. $\sigma_{\omega_1} = 0.0232$ and $\sigma_{\omega_2} = 0.07506$. Statistical moments of impulse response in terms of the generalized coordinate vector $X(t)$ (Figures 3.13-3.15) are estimated from the knowledge of the statistical response in the normal mode vector $Y(t)$ form (Figures 3.10-3.12) and deterministic modal matrix $\Phi$. It is seen from Figures 3.10 and 3.13 that the expected mean values as predicted by the proposed method (equations (3.24, 3.25)) are in very good agreement with the Monte-Carlo simulation. The predicted standard deviations and upper-bound responses are also reasonably accurate, for both $Y(t)$ and $X(t)$ evident from Figures 3.11, 3.12 and 3.14, 3.15 respectively.

3.5. Frequency Response

3.5.1 Arbitrary Excitation

The first two moments of the dynamic response $Y_i(t)$ due to an arbitrary but deterministic excitation $\xi(t)$ can be obtained from equation (3.18) as

$$\langle Y_i(t) \rangle = \int \rho_i h_i(\tau) \xi(t-\tau)d\tau;$$

$$\langle Y_i^2(t) \rangle = \iint \mu_i g_i(\tau, \nu) \xi(t-\tau)\xi(t-\nu)d\tau d\nu$$

(3.30a,b)

where $h_i(\tau)$ and $g_i(\tau, \nu)$ are the first and second moments of Green's function defined as follows by using equations (3.24) and (3.25)
Figure 3.10. Expected mean of impulse response in the normal mode form for Example III shown in Figure 3.1b. (a) $\langle Y_1(t) \rangle$; (b) $\langle Y_2(t) \rangle$. Key: —— proposed method and + + + Monte-Carlo simulation.
Figure 3.11. Standard deviation of impulse response in the normal mode form for Example III shown in Figure 3.1b: (a) $\sigma(Y_1(t))$; (b) $\sigma(Y_2(t))$. Key as Figure 3.10.
Figure 3.12. Standard deviation of impulse response in the normal mode for Example III shown in Figure 3.1b: (a) $\langle Y_1(t) \rangle + \sigma(Y_1(t))$; (b) $\langle Y_2(t) \rangle + \sigma(Y_2(t))$. Key as Figure 3.10.
Figure 3.13. Expected mean of impulse response in the generalized vector form for Example III shown in Figure 3.1b: (a) $\langle X_1(t) \rangle$; (b) $\langle X_2(t) \rangle$. Key as Figure 3.10.
Figure 3.14. Standard deviation of impulse response in the generalized vector form for Example III shown in Figure 3.1b: (a) $\sigma(X_1(t))$; (b) $\sigma(X_2(t))$. Key as Figure 3.10.
Figure 3.15. Standard deviation of impulse response in the generalized vector form for Example III shown in Figure 3.1b. : (a) \(X_1(t) + \sigma(X_1(t))\); (b) \(X_2(t) + \sigma(X_2(t))\). Key as Figure 3.10.
\( f_i(\tau) = \left( \frac{2b_i - b_i^2}{b_i^4} e^{-a_i \sin b_i \tau} \right) = p(a_i) p(b_i) \left( e^{-a_i e^{\sin b_i \tau}} - \frac{1}{b_i^2} \left( \frac{(2b_i - b_i) \cos b_i \tau + \sin b_i \tau}{\tau^2} \right) \right) \); \\
\( g_i(\tau, u) = \left( \frac{2b_i^2 - b_i^2}{b_i^4} e^{-a_i (\tau + u)} \sin b_i \tau \sin b_i u \right) = p(a_i) p(b_i) \left( e^{-a_i (\tau + u)} \frac{1}{\tau + u} \right) \);  \\
\left( \frac{2b_i^2 (\sin b_i (\tau - u))}{(\tau - u)} - \frac{2b_i (\tau - u) \cos b_i (\tau - u) + ((\tau - u)^2 b_i^2 - 2) \sin b_i (\tau - u)}{(\tau - u)^3} \right) \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau + u))}{(\tau + u)} + \frac{2b_i (\tau + u) \cos b_i (\tau + u) + ((\tau + u)^2 b_i^2 - 2) \sin b_i (\tau + u)}{(\tau + u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau - u))}{\tau - u} - \frac{2b_i (\tau - u) \cos b_i (\tau - u) + ((\tau - u)^2 b_i^2 - 2) \sin b_i (\tau - u)}{(\tau - u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau + u))}{\tau + u} + \frac{2b_i (\tau + u) \cos b_i (\tau + u) + ((\tau + u)^2 b_i^2 - 2) \sin b_i (\tau + u)}{(\tau + u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau - u))}{\tau - u} - \frac{2b_i (\tau - u) \cos b_i (\tau - u) + ((\tau - u)^2 b_i^2 - 2) \sin b_i (\tau - u)}{(\tau - u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau + u))}{\tau + u} + \frac{2b_i (\tau + u) \cos b_i (\tau + u) + ((\tau + u)^2 b_i^2 - 2) \sin b_i (\tau + u)}{(\tau + u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau - u))}{\tau - u} - \frac{2b_i (\tau - u) \cos b_i (\tau - u) + ((\tau - u)^2 b_i^2 - 2) \sin b_i (\tau - u)}{(\tau - u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau + u))}{\tau + u} + \frac{2b_i (\tau + u) \cos b_i (\tau + u) + ((\tau + u)^2 b_i^2 - 2) \sin b_i (\tau + u)}{(\tau + u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau - u))}{\tau - u} - \frac{2b_i (\tau - u) \cos b_i (\tau - u) + ((\tau - u)^2 b_i^2 - 2) \sin b_i (\tau - u)}{(\tau - u)^3} \right) \) \left( \frac{2b_i^2 (\sin b_i (\tau + u))}{\tau + u} + \frac{2b_i (\tau + u) \cos b_i (\tau + u) + ((\tau + u)^2 b_i^2 - 2) \sin b_i (\tau + u)}{(\tau + u)^3} \right) \)

(3.31a,b)

### 3.5.2 Harmonic Excitation

Consider harmonic excitation as \( \xi(t) = \sin \Omega t \). The frequency response of a discrete system is, in general, given by \( Y(\Omega) = A(\Omega) + jB(\Omega) \) where \( A \) and \( B \) are coincident and quadrature components. The first and second moments of \( Y(\Omega) \) can be estimated by using the Fourier series expansion of \( f_i(\tau) \) and \( g_i(\tau, u) \) as follows, irrespective of the analytical method used

\[
\langle Y(\Omega_1) \rangle = \rho, A(\Omega_1)
\]

\[
\langle Y(\Omega_1)Y(\Omega_2) \rangle = \mu, G(\Omega_1, \Omega_2)
\]

(3.32a,b)
where \( H_i(\Omega_i) \) and \( G_i(\Omega_1, \Omega_2) \) are the Fourier transforms of \( h_i(\tau) \) and \( g_i(\tau, \nu) \). Similarly, the auto-power spectrum \( \langle Y_i(\Omega_1)Y_i^*(\Omega_2) \rangle \) can be obtained. The one-dimensional and two-dimensional Fourier transforms of the sampled time data are computed as

\[
H_i(m\Delta\Omega_1) = \Delta t_1 \sum_{k=0}^{M-1} h_i(k\Delta t_1) \left[\cos(w_1) - j\sin(w_1)\right]
\]

\[
G_i(m\Delta\Omega_1, n\Delta\Omega_2) = \Delta t_1 \Delta t_2 \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g_i(l\Delta t_1, k\Delta t_2) \left[\cos(w_1) - j\sin(w_1)\right] \left[\cos(w_2) - j\sin(w_2)\right]
\]

\(3.33a,b\)

where \( m, n = 1, 2, 3, \ldots \); \( \Delta t_1 = T_1 / M \); \( \Delta t_2 = T_2 / N \); \( \Delta \Omega_1 = 2\pi / T_1 \); \( \Delta \Omega_2 = 2\pi / T_2 \); \( w_1 = m\Delta\Omega_1 k\Delta t_1 \); \( w_2 = n\Delta\Omega_2 l\Delta t_2 \); \( M \) and \( N \) are total number of points sampled in time domain; and \( T_1 \) and \( T_2 \) are time windows. And the variance and covariance of the real and imaginary parts of \( Y_i(\Omega) \) can be obtained from its second moment and auto-power spectrum definitions. For instance [3.14],

\[
2\langle A_i^2(\Omega) \rangle = \text{Re}\langle Y_i(\Omega)Y_i^*(\Omega) \rangle + \text{Re}\langle Y_i(\Omega)Y_i^*(\Omega) \rangle;
\]

\[
\langle A_i(\Omega) \rangle = \text{Re}\langle Y_i(\Omega) \rangle
\]

\(3.34a,b\)

Therefore, the standard deviation of amplitude at \( \Omega \) as effected by the parameter uncertainties, e.g. variance and covariance of mass and stiffness, can be found as

\[
\sigma(A_i(\Omega)) = \left(\langle A_i^2(\Omega) \rangle - \langle A_i(\Omega) \rangle^2\right)^{0.5}
\]

\(3.35\)
3.5.3 Example IV: Single-Degree-of-Freedom System

Consider the single degree-of-freedom system of Figure 3.1a again with $\bar{M} = 1$, $\bar{C} = 0.4$, $\bar{K} = 1$, $\bar{F} = 1$ and $\xi(t) = \sin \Omega t$; hence $\bar{y}_t = -0.2 \pm 0.98 j$. We consider the uncertainty in stiffness only and assume it to be $R_{k,k} = 0.01$ i.e. $\sigma(\omega_d) = 0.049$. The one and two-dimensional FFT algorithms of CTRL-C software [3.15] are used to calculate the mean and mean-square spectra through equations (3.30) and (3.31). For this study the following sampling parameters were chosen: $M = N = 64$ and $T_1 = T_2 = 20\pi$. It is seen from Figure 3.16 that the expected mean and standard deviation of $Y_i(\Omega)$ spectra as predicted by the Green's function approach presented here and the direct product method discussed elsewhere in parallel development [3.11] are in very good agreement. It must be noted that the selections of sampling parameters influence the accuracy of FFT algorithm. Therefore prediction should improve for higher values of $M$ and $N$. But given computationally intensive nature of the proposed approach, the alternate method of Reference [3.11] is preferred which can also handle non-proportionally damped systems.

3.6. Concluding Remarks

A new analytical method for the solutions of impulse and frequency response of a linear time-invariant, proportionally damped, discrete vibratory system with uncertain system parameters and force amplitude has been developed and validated through single and two-degree-of-freedom system examples. Even though only uncorrelated damping and uniform distribution cases are presented analytically for the sake of illustration, the proposed method is applicable for other cases as well. The proposed method overcomes some of the deficiencies of the existing methods. It found to be clearly superior to the commonly used first order perturbation technique [3.1-3.3] since it removes the effect of secular terms generated by the perturbation method. Also, the proposed method is
Figure 3.16. Frequency response spectra of Example IV shown in Figure 3.1a with $C = 0.4$: (a) $\langle |Y_i(j\Omega)| \rangle$; (b) $\sigma(\langle |Y_i(j\Omega)| \rangle) / |\tilde{V}(j\Omega)|$. Here $|\tilde{V}(j\Omega)|$ is the amplitude of the corresponding deterministic system. Key: —— direct product method [3.11]; ◇ ◇ ◇ proposed method of Section 3.5.2.
computationally faster than the Monte-Carlo simulation since a large number of iterations is not required. Limitations of the proposed approach are already documented in Chapter II. Future efforts will be focused on overcoming some of the shortcomings of the method formulated here.


CHAPTER IV
FREQUENCY RESPONSE

4.1. Introduction

Analysis of parameter uncertainties remains a viable area of research as evident from recent review articles by Ibrahim [4.1] and Benaroya and Rehak [4.2]. Particularly, forced response characteristics of linear dynamic systems must be understood in order to solve design problems associated with random fluctuations in parameters due to manufacturing variations, measurement uncertainties or modeling inaccuracies. One of the unresolved issues is the estimation of response moments under harmonic excitation [4.1-4.6]. In this chapter we address this specific issue by developing a new analytical technique for discrete vibratory systems with uncertain inertial, elastic and damping properties. Excitation amplitude is also considered to be random but the frequency is deterministic.

The first order perturbation methods have been used commonly to determine the frequency response characteristics with limited success [4.1-4.6]. Previous investigators [4.4-4.6] have pointed out that in the neighborhood of resonant frequencies the standard deviation of displacement response becomes quite large, indicating greater uncertainty in such regions. This is obviously due to the existence of secular and near secular terms. Stochastic finite element methods have also been developed but these are based on the theory of first order perturbation [4.7-4.9]. Additionally, the Monte-Carlo simulation
method [4.10] can always be employed. However such a simulation is computationally intensive since a large number of iterations is needed to estimate the probability distributions. Based on this literature review and earlier assessments by prior investigators [4.1-4.3], it is clear that new analytical techniques are definitely needed to overcome the deficiencies of existing methods. The method proposed here is a step in this direction.

4.2. Problem Formulation

The scope of this chapter is limited to a linear time-invariant, viscously damped vibratory system of dimension $N$ with uncertain parameter matrices $M$, $C$ and $K$; each matrix is assumed to be symmetric and positive definitive. For harmonic excitation at a given deterministic frequency $\Omega$ but with randomly varying complex amplitude vector $F(\Omega)$ of dimension $N$, the random differential equations can be given in the matrix form as follows; see LIST OF SYMBOLS for the identification of symbols.

$$M\dddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) = F(\Omega)e^{i\Omega t}$$

(4.1)

Since only the steady state harmonic response $x(t) = X(\Omega)e^{i\Omega t}$ is of interest, initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$. Here $X(\Omega)$ is the complex valued response given by the dynamic stiffness matrix $G(\Omega)$

$$X(\Omega) = G(\Omega)^{-1}F(\Omega) ; \quad G(\Omega) = (-\Omega^2M + j\Omega C + K)$$

(4.2a,b)

The primary objective of this chapter is to determine the frequency response characteristics of single and multi-degree-of-freedom systems. Specifically, mean $\langle |X_i(\Omega)| \rangle$ and standard deviation $\sigma(|X_i(\Omega)|)$ of displacement magnitude $|X_i(\Omega)|$ are
examined and compared with the deterministic response \( \langle X \rangle \) over the applicable frequency range of interest. A new analytical technique is proposed and several numerical examples are taken to illustrate theory. Predictions are compared with the results yielded by the commonly used first order perturbation technique [4.4, 4.5] and the Monte-Carlo simulation [4.10]. For the Monte-Carlo method, equation (4.2) is used to simulate the probabilistic behavior given random fluctuation in \( M, C, K, \) and \( F \). Given the sample size a random ensemble of parameters is generated by a digital computer. Its distribution must be transformed appropriately; for instance numerical simulation of a uniform distribution assumes a range from 0 to 1. Simulation is then executed to yield \( \langle X(\Omega) \rangle \) and \( \sigma(X(\Omega)) \) at a given value of \( \Omega \). Even though the Monte-Carlo simulation is considered here as the bench-mark method, caution must be exercised in interpreting its results since accuracy depends on the finite ensemble size and prediction may vary from trial to trial [4.10].

The following simplifying assumptions are made to develop the new solution methodology: (i) random matrices and vectors of equation (4.1) or (4.2) can be given by the sum of deterministic or mean (identified by bar) and random or fluctuating components (identified by tilde) i.e. \( M = M + \tilde{M} \), \( C = C + \tilde{C} \), \( K = K + \tilde{K} \), \( F = F + \tilde{F} \) and \( X = X + \tilde{X} \); (ii) expected means of system matrices \( M=<M> \), \( C=<C> \), \( K=<K> \) and excitation amplitude \( \langle F \rangle = \bar{F} \) are known; (iii) probability distributions of \( M, C, K \) and \( F \) are of the same type and are known; (iv) means of a random parameter matrix and \( \tilde{F} \) are equal to null e.g. \( <\tilde{M}> = 0 \), \( \langle \tilde{F} \rangle = 0 \); (v) parameter fluctuations are much smaller compared to the deterministic values i.e. \( \|\tilde{M}\| < \|M\| \); (vi) co-variances of parameter fluctuations are known in the form of cross-correlation matrices such as \( R_{M,M} = \langle \tilde{M} \otimes \tilde{M} \rangle = \text{Var}(M) \) and \( R_{M,K} = \langle \tilde{M} \otimes \tilde{K} \rangle = \text{Cov}(M,K) \); and (vii) matrix \( G(\Omega) \) can be inverted numerically.
4.3. Single-Degree-of-Freedom System

4.3.1 Theory

The frequency response of one dimensional system with parameter uncertainties as given by equation (4.2) is

$$X(\Omega) = \bar{X}(\Omega) + \tilde{X}(\Omega) = G^{-1}(\Omega)F(\Omega) = (\bar{G}(\Omega) + \tilde{G}(\Omega))^{-1}(\bar{F}(\Omega) + \tilde{F}(\Omega))$$

$$= (-\Omega^2(\bar{M} + \tilde{M}) + j\Omega(\bar{C} + \tilde{C}) + \bar{K} + \tilde{K})^{-1}(\bar{F}(\Omega) + \tilde{F}(\Omega))$$

(4.3)

and the response of the corresponding deterministic system of natural frequency \(\bar{\omega}_1 = (\bar{K} / \bar{M})^{0.5}\) and damping ratio \(\bar{\zeta}_1 = 0.5\bar{C} / (\bar{K}\bar{M})^{0.5}\) is given by the following well known expression

$$\bar{X}(\Omega) = \bar{G}^{-1}(\Omega)\bar{F}(\Omega) ; \bar{G}^{-1} = (-\Omega^2\bar{M} + j\Omega\bar{C} + \bar{K})^{-1}$$

(4.4a,b)

From this point on we drop \(\Omega\) for sake of brevity. The magnitude of deterministic response is

$$|\bar{X}| = |\bar{G}|^{-1}\bar{F} = \left[(\bar{K} - \Omega^2\bar{M})^2 + \Omega^2\bar{C}^2\right]^{-0.5} \bar{F}$$

(4.5)

Before proceeding further we need to examine the first and second moments of an arbitrary function \(h(y) = h(\bar{y} + \tilde{y})\) of random variable \(y\). Expected value of \(h(y)\) is approximated by using the Taylor series expansion as
\[ \langle h(y) \rangle = \left( \tilde{h} + \frac{d\tilde{h}}{dy} \tilde{y} + \frac{1}{2!} \frac{d^2\tilde{h}}{dy^2} \tilde{y}^2 + \frac{1}{3!} \frac{d^3\tilde{h}}{dy^3} \tilde{y}^3 \ldots \right) \]

\[ = \tilde{h} + \frac{1}{2!} \frac{d^2\tilde{h}}{dy^2} \langle \tilde{y}^2 \rangle + \frac{1}{3!} \frac{d^3\tilde{h}}{dy^3} \langle \tilde{y}^3 \rangle \]

And variance is given as follows by considering the first four central moment terms

\[ \text{Var}(h(y)) = \langle (h - \langle h \rangle)^2 \rangle \]

\[ = \left( \left( \frac{d\tilde{h}}{dy} \tilde{y} + \frac{1}{2!} \frac{d^2\tilde{h}}{dy^2} \tilde{y}^2 + \frac{1}{3!} \frac{d^3\tilde{h}}{dy^3} \tilde{y}^3 \right) - \left( \frac{1}{2!} \frac{d^2\tilde{h}}{dy^2} \langle \tilde{y}^2 \rangle + \frac{1}{3!} \frac{d^3\tilde{h}}{dy^3} \langle \tilde{y}^3 \rangle \right) \right)^2 \]

\[ = \left( \frac{d\tilde{h}}{dy} \right)^2 R_{y,y} + \frac{d\tilde{h}}{dy} \frac{d^2\tilde{h}}{dy^2} \tilde{y}^3 + \frac{1}{4} \left( \frac{d^2\tilde{h}}{dy^2} \right)^2 \left( \tilde{y}^4 - \langle \tilde{y}^2 \rangle^2 \right) + \frac{2}{3!} \frac{d\tilde{h}}{dy} \frac{d^3\tilde{h}}{dy^3} \tilde{y}^4 \ldots \] \hspace{1cm} (4.6a,b)

Suppose that a deterministic variable \( u = \bar{u} \) is given such that \( h = y^{-1}u = (\bar{y} + \bar{y})^{-1} \bar{u} \).

Equation (4.6b) can be rewritten as

\[ \text{Var}(h(y,u)) = \left[ \left( \bar{y}^2 - \left( R_{y,y} + 2 \bar{y}^{-1} \langle \bar{y}^3 \rangle + \bar{y}^{-2} \langle 3\bar{y}^4 - 2\langle \bar{y}^2 \rangle^2 \rangle \right) \right)^{-1} - \bar{y}^{-2} \right] \bar{u}^2 \] \hspace{1cm} (4.7)

If we retain only the first and second central moment terms, equations (4.6a) and (4.7) are then reduced to

\[ \langle h(y,u) \rangle = \left( y - R_{y,y} \right)^{-1} \bar{u} \]

\[ \text{Var}(h(y,u)) = \left[ \left( \bar{y}^2 - R_{y,y} \right)^{-1} - \bar{y}^{-2} \right] \bar{u}^2 \] \hspace{1cm} (4.8a,b)
Taking the moments of equation (4.3) and by using equation (4.8b), the variance terms of response amplitude $X$ associated with random fluctuations individually in $M$, $C$, $K$ or $F$ are found to be as follows; here it is assumed that various co-variance terms such as $R_{M,K}$ and $R_{M,F}$ are zero.

\[
P_{XM} = \left| \left( X^2 \right) - \left( X \right)^2 \right| = \left| \left( G^2 - R_{G,G} \right)^{-1} - \bar{G}^{-2} \right| F^2 ; \quad R_{G,G} = \Omega^2 R_{M,M}.
\]

\[
P_{XC} = \left| \left( X^2 \right) - \left( X \right)^2 \right| = \left| \left( \bar{G}^2 - (j\Omega)^2 R_{C,C} \right)^{-1} - \bar{G}^{-2} \right| F^2
\]

\[
P_{XK} = \left| \left( X^2 \right) - \left( X \right)^2 \right| = \left| \left( \bar{G}^2 - R_{X,K} \right)^{-1} - \bar{G}^{-2} \right| F^2
\]

\[
P_{XF} = \left| \left( X^2 \right) - \left( X \right)^2 \right| = \left| \bar{G}^{-2} R_{F,F} \right|
\]

(4.9a-e)

The total standard deviation of response amplitude is

\[
\sigma(|X|) = \left[ \text{Var}(|X|) \right]^{0.5} = \left[ P_{XM} + P_{XC} + P_{XK} + P_{XF} \right]^{0.5}
\]

(4.10)

The corresponding expected mean from (4.8a) is given as follows; note that $\langle |X| \rangle \neq |\bar{X}|$

\[
\langle |X| \rangle = \left| \langle |\bar{X} + \bar{X}| \rangle \right| = \left| \left( \bar{G} - \bar{G}^{-1} \text{Var}(G) \right)^{-1} \right| F
\]

(4.11)

All expressions can be normalized by taking the corresponding deterministic system response as a reference. Accordingly, Equations (4.10) and (4.11) are normalized as follows

\[
\frac{\sigma(|X|)}{|\bar{X}|} = \left[ P_{XM} + P_{XC} + P_{XK} + P_{XF} \right]^{0.5} / |\bar{X}|
\]
4.3.2 Limitations and Modifications

Four cases when the theory of Section 4.3.1 must be modified are discussed next. First, equation (4.7) must be used when the third and fourth central moment terms are not relatively smaller than the second central moment term. Such a case arises for a virtually undamped system especially near the resonance. Second, when \( \Omega^2 \tilde{C}^2 \) is close to \( \Omega^4 R_{M,M} \) or \( R_{K,K} \), then equation (4.10) will not yield accurate answers at the resonance. This will be illustrated later through a numerical example. Third, equation (4.11) may not yield an accurate answer for a lightly damped system, say when \( \zeta = o(0.01) \), within the resonance regime because of numerical inversion problems. Such a regime can be given by \( \omega_{\text{di}} - \varepsilon \omega_{\text{di}} \leq \Omega \leq \omega_{\text{di}} + \varepsilon \omega_{\text{di}} \) where \( \varepsilon \) is a random variable because of the statistical nature of the system. For the sake of convenience, define \( \varepsilon = \sigma_{\omega_{\text{di}}} / \omega_{\text{di}} \) where the standard deviation \( \sigma_{\omega_{\text{di}}} \) can be found from the random eigensolution [4.11]. The standard deviation of damped natural frequency \( \omega_{\text{di}} \) associated with random fluctuations \( M \) and \( K \) is approximated as follows provided \( \text{var}(\zeta) \ll 1 \) [4.11]

\[
\sigma_{\omega_{\text{di}}} = 0.5 \omega_{\text{di}} \left( R_{M,M} / \bar{M}^2 - 2 R_{M,K} / (\bar{MK}) + R_{K,K} / \bar{K}^2 \right)^{0.5} \tag{4.13}
\]

In the resonance regime given by \( \omega_{\text{di}} - \varepsilon \omega_{\text{di}} \leq \Omega \leq \omega_{\text{di}} + \varepsilon \omega_{\text{di}} \), use the following expressions for \( \tilde{G}(\Omega) \) and \( \text{Var}(G) \) when computing \( \langle |X| \rangle \) by using equation (4.11)

\[
\tilde{G}(\Omega) = \tilde{G}(\omega_{\text{di}} - \sigma_{\omega_{\text{di}}}) = \left( \bar{K} - (\omega_{\text{di}}^2 - 2 \omega_{\text{di}} \sigma_{\omega_{\text{di}}} + \sigma_{\omega_{\text{di}}}^2) \bar{M} \right) + j(\omega_{\text{di}} - \sigma_{\omega_{\text{di}}}) \bar{C}
\]

\[
\langle |X| \rangle / \langle X \rangle = \left| (\tilde{G} - \tilde{G}^{-1} \text{Var}(G))^{-1} \right| / |\tilde{G}|
\tag{4.12a,b}
\]
\[ \text{Var}(G) = \text{Var} \left( G \left( \bar{\omega}_d - \sigma_{\omega_d} \right) \right) \]  

(4.14a,b)

Fourth, if system parameters and/or force amplitude are correlated with each other, then the covariance terms such as \( R_{M,K} \) and \( R_{M,F} \) must be considered. For instance, consider the case when \( M \) and \( K \) are fully correlated. Equation (4.9b) now is modified as follows

\[ R_{G,G} = \Omega^4 R_{M,M} - 2\Omega^2 \left( R_{M,M} R_{K,K} \right)^{0.5} + R_{K,K} \]  

(4.15)

4.3.3 First Order Perturbation Analysis

If \( |\bar{G}| \ll |G| \), equation (4.9) may be approximated by using the first order perturbation analysis to yield the individual response variances identified below by the \( Q \) terms; in each case only one randomness like equation (4.9) is considered.

\[ Q_{XM} = \Omega^4 \left( \bar{G}^{-2} R_{M,M} \bar{X}^2 \right) = \Omega^4 \left( \bar{G}^{-2} \delta_m^2 \bar{M}^2 \bar{X}^2 \right) ; \ R_{M,M} = \left( \bar{M}^2 \right) = \delta_m^2 \bar{M}^2 \]

\[ Q_{XC} = \Omega^4 \left( \bar{G}^{-2} \delta_c^2 \bar{C}^2 \bar{X}^2 \right) ; \ \delta_c^2 = \frac{R_{C,C}}{\bar{C}^2} \]

\[ Q_{XX} = \Omega^4 \left( \bar{G}^{-2} \delta_k^2 \bar{K}^2 \bar{X}^2 \right) ; \ \delta_k^2 = \frac{R_{K,K}}{\bar{K}^2} \]

\[ Q_{XF} = \left( \bar{G} \right)^{-2} \delta_F^2 \bar{F}^2 ; \ \delta_F^2 = \frac{R_{F,F}}{\bar{F}^2} \]

(4.16a-h)

Thus, the total standard deviation of response amplitude is

\[ \sigma(|X'|) = \left[ \text{Var}(|X'|) \right]^{0.5} = \left[ Q_{XM} + Q_{XC} + Q_{XX} + Q_{XF} \right]^{0.5} \]  

(4.17)
The expected mean of $X$ as defined by (4.11) is approximated as

$$\langle X \rangle = |\bar{X} + \langle \hat{X} \rangle| \approx \left| (\overline{G}^{-1} + \overline{G}^{-2}[\text{Var}(G)])F \right|$$

Equation (4.17) and (4.18) are normalized as

$$\sigma(|X|) / |\bar{X}| = [Q_{xm} + Q_{xc} + Q_{xx} + Q_{xf}]^{0.5} / |\bar{X}|$$

Equations (4.19a,b) yield results identical to those given by the Taylor series expansion or perturbation method [4.4-4.5]. Errors associated with the perturbation analysis can be determined by comparing $P$ terms of equation (4.9) with corresponding $Q$ terms of equation (4.16). Difference between any $P$ and $Q$ term is expected to be large near and at the resonance provided that $\overline{\zeta}$ is very small.

### 4.3.4 Example I: Viscoelastic System

The deterministic parameters of the system shown in Figure 4.1a are chosen to be $\overline{C} = 1$, $\overline{K} = 1$ and $\overline{F} = 1$. Random variation is considered only in the damper and uniformly distributed i.e. $R_{c,c} = \langle \overline{C}^2 \rangle = 0.01$ and $R_{k,k} = R_{f,f} = R_{f,k} = 0$. Excitation is considered from $\Omega = 0$ to 2 rad/s and the corresponding results for $\langle |X| \rangle$ and $\sigma(|X|) / |\bar{X}|$ spectra are shown in Figure 4.2. It is seen that the predictions yielded by the proposed method (equations (4.11, 4.12a)) are in virtual agreement with the results of existing method namely the perturbation method (equations (4.18, 4.19a)) and the Monte-Carlo
Figure 4.1. Physical systems used to illustrate and validate the proposed method: (a) viscoelastic system (Example I); (b) single degree-of-freedom system (Example II-IV); (c) two degree-of-freedom system (Example V-VIII); and (d) five degree-of-freedom periodic system (Example IX). Deterministic system parameters are given here. Also the locations of harmonic force are shown.
Figure 4.2. Comparison of frequency response spectra for Example I shown in Figure 4.1a: (a) mean $\langle |X(\Omega)| \rangle$; (b) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$. Key: --- proposed technique; ---- first order perturbation method; and + + + Monte-Carlo simulation.
simulation (sample size=400). This is expected since no resonance-like behavior leading to large displacements is seen in this case.

4.3.5 Example II: Very Lightly Damped System

Consider the physical system shown in Figure 4.1b with \( \bar{M} = 1, \bar{K} = 1, \bar{C} = 0.01 \) (or \( \bar{\zeta} = 0.005 \)), \( \bar{F} = 1 \) and \( \Omega = 0 \) to 2 rad/s. For illustration purposes, only the randomness in mass with uniform distribution is included i.e. \( R_{M,M} = \left( \bar{M}^2 \right) = 0.01 \) and \( R_{C,C} = R_{K,K} = R_{F,F} = R_{M,C} = R_{M,K} = R_{M,F} = R_{C,F} = R_{C,P} = 0 \). Equations (4.11), (4.12a) and (4.14) are used to predict \( \langle |X| \rangle, \sigma(|X|)/|X| \) and \( \langle |X| \rangle + \sigma(|X|) \) spectra as shown in Figures 4.3 and 4.4. Third and fourth central moment terms are included in equation (4.12a). In each case, results obtained by the perturbation method (equations (4.18, 4.19a)) and the Monte-Carlo simulation (sample size=400) are also plotted on the same graphs. It is seen that the perturbation method deviates from the Monte-Carlo simulation within the resonance regime because of secular or near secular terms. Our method, however, overcomes this deficiency and is in reasonable agreement with the numerical solutions except for the \( \langle |X| \rangle \) value at \( \Omega = \bar{\omega}_i \) in Figure 4.3a. This is due to the numerical inversion problem associated with \( G(\Omega) \). But this discrepancy seems to disappear when \( \langle |X| \rangle \) is predicted by equations (4.11 and 4.14) within the resonance regime and by equation (4.11) outside this regime as shown in Figure 4.3b. It should also be pointed out that there is some uncertainty associated with the Monte-Carlo simulation as well since the sample size is finite.

4.3.6 Example III: Effect of Damping

Now we increase \( \bar{C} \) from 0.01 to 0.1 (\( \bar{\zeta} = 0.05 \)) and 0.5 (\( \bar{\zeta} = 0.25 \)) and repeat Example II. Figure 4.5a shows \( \langle |X| \rangle \) results for \( \bar{C} = 0.1 \) of three methods and Figure 4.5b
Figure 4.3. Comparison of $\langle |X(\Omega)| \rangle$ results for a very lightly damped system shown in Figure 4.1b (Example II) with $\bar{\xi}_i = 0.005$ and $R_{M,M} = 0.01$. Key as Figure 4.2.
Figure 4.4. Frequency response spectra for Example II: (a) $\sigma(|X(\Omega)|)/|\mathcal{X}(\Omega)|$; (b) $|\mathcal{X}(\Omega)|$ vs. probabilistic $\langle|X(\Omega)|\rangle + \sigma(|X(\Omega)|)$. Key as Figure 4.2 except deterministic response $|\mathcal{X}(\Omega)|$. 
Figure 4.5. Comparison of $\langle |X(\Omega)| \rangle$ results for a lightly damped system (Example III) with $\bar{\zeta}_i = 0.05$. Key as Figure 4.2.
Figure 4.6. Frequency response spectra of $\sigma(|X(\Omega)|)/|X(\omega)|$ for Example III. Key as Figure 4.2.
Figure 4.7. Comparison of frequency response spectra for a damped system (Example III) with $\bar{\zeta}_1 = 0.25$: (a) mean $\langle |X(\Omega)| \rangle$; (b) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$. Key as Figure 4.2.
shows the modified expected mean $\langle |X| \rangle$. The perturbation method indicates a singularity at $\Omega = \bar{\omega}_i$ but this is again due to the numerical inversion of $G(\Omega)$. Figure 4.6 shows the results of $\sigma(|X|) / |X|$ where all methods are in close agreement except for the Monte-Carlo curve at the resonance; the precise reason for this dip is not obvious. Such discrepancies between three methods seem to vanish for a highly damped system with $\zeta_i = 0.25$ as illustrated by Figure 4.7. Because of high damping, random fluctuations in the displacement response are indeed reduced by about one order of magnitude as we compare Figures 4.3-4.7 for both $\langle |X| \rangle$ and $\sigma(|X|) / |X|$ spectra.

4.3.7 Example IV: Effect of Large Random Fluctuations

We go beyond the limits of our assumption of small random fluctuations and repeat Example III with $\bar{C} = 0.1$ but increase $R_{M,M}$ from 0.01 to 0.25. Such a fluctuation is indeed large as $\delta_M = 0.5$. Figure 4.8a shows results for $\sigma(|X|) / |X|$ and we observe that the Monte-Carlo curve is quite different from the perturbation method curve but our technique is closer to the Monte-Carlo simulation. This demonstrates that our method is capable of handling second and higher order perturbations even though earlier we had assumed that $\bar{M} \ll \bar{M}$. It is seen from Figure 4.8b as well where $\langle |X| \rangle + \sigma(|X|)$ spectra are compared; observe that our method predicts results in reasonable proximity of numerical solutions even though minor peaks and valleys do not match.

4.3.8 Example V: Correlated Mass and Stiffness

Next, we reconsider example II for the case when $M$ and $K$ are fully correlated i.e. $R_{M,M} = (0.2)^2$, $R_{K,K} = (0.1)^2$ and $R_{M,K} = (R_{M,M}R_{K,K})^{0.5}$. Using the analytical expression given by equation (4.15) response spectra are predicted and compared with two existing methods in Figure 4.9. Observe that the $\sigma(\langle X(\Omega) \rangle) / \langle |X(\Omega)| \rangle$ curve is different from
Figure 4.8. Effect of large random fluctuations examined by Example IV with $R_{M,M} = 0.25$ and $\xi_i = 0.05$: (a) standard deviation $\sigma(|X(\Omega)|) / |X(\Omega)|$; (b) probabilistic $\langle |X(\Omega)| \rangle + \sigma(|X(\Omega)|)$ vs. deterministic $|X(\Omega)|$ response characteristics. Key as Figure 4.2 except --- deterministic system.
Figure 4.9. Comparison of results for a fully correlated case (Example V) with $R_{m,k} = (R_{m,m} R_{k,k})^{0.5}$: (a) standard deviation $\sigma(|X(\Omega)|)/|\bar{X}(\Omega)|$; (b) probabilistic $\langle |X(\Omega)| \rangle + \sigma(|X(\Omega)|)$ vs. deterministic response $|\bar{X}(\Omega)|$. Key as Figure 4.2 except---
previously discussed results but predictions again match with the Monte-Carlo simulation.

4.4. Multi-Degree-of-Freedom-Systems

4.4.1 Theory

Mathematical expressions presented in Section 4.3.1 will now be generalized for a discrete system of dimension $N$. The frequency response of equation (4.2) is given as follows; again $(\Omega)$ is dropped.

\[ X = \bar{X} + \dot{X} = G^{-1} F = \left( -\Omega^2 (\bar{M} + \dot{M}) + j\Omega (\bar{C} + \dot{C}) + \bar{K} + \dot{K} \right)^{-1} \left( \bar{F} + \dot{F} \right) \] (4.20)

And, the response of the corresponding deterministic system is

\[ \bar{X} = \bar{G}^{-1} \bar{F} = \left( -\Omega^2 \bar{M} + j\Omega \bar{C} + \bar{K} \right)^{-1} \bar{F} \] (4.21)

Recall equation (4.6) which was used to develop moments of a random variable. Equation (4.6) is now generalized by taking an arbitrary random vector $h(y, \bar{u}) = y^{-1} \bar{u}$ where $y = \bar{y} + \tilde{y}$ is a symmetric random matrix of dimension $N$ and $u = \bar{u}$ is a deterministic vector of dimension $N$. The Taylor matrix expansion yields the following; also see Appendix A for a few direct product calculations.

\[ \langle h \rangle = \left\langle \bar{h} + D_{cs(y)} h \left[ cs(\tilde{y}) \right] + \frac{1}{2!} D_{cs(y)\tilde{y}^2} h \left[ cs(\tilde{y})^{\otimes 2} \right] + \frac{1}{3!} D_{cs(y)\tilde{y}^3} h \left[ cs(\tilde{y})^{\otimes 3} \right] \ldots \right\rangle \] (4.22)

After some manipulation equation (4.22) is approximated up to third order terms as
\[ \langle h(y) \rangle = \langle \bar{h} + \bar{y}^{-1}y_1\bar{y}^{-1}\bar{u} + \bar{y}^{-1}y_2\bar{y}^{-1}\bar{u} + \bar{y}^{-1}y_3\bar{y}^{-1}\bar{u} \rangle = \bar{h} + \bar{y}^{-1}\langle y_2 \rangle \bar{h} + \bar{y}^{-1}\langle y_3 \rangle \bar{h} \]  
\quad (4.23a)

And variance expression given below includes up to fourth central moment terms

\[
\text{Var}(h) = \langle h \otimes h \rangle - \langle h \rangle \otimes \langle h \rangle = D_{y,y}^{-1} \left[ R_{y,y} - \langle y_2 \rangle \otimes \langle y_2 \rangle + \langle y_1 \otimes y_2 \rangle + \langle y_2 \otimes y_1 \rangle \right] D_{y,y}^{-1} D_{u,u}
\]

\quad (4.23b)

where \( D \) is the direct product of two deterministic matrices e.g. \( D_{y,y} = \bar{y} \otimes \bar{y} \). See Appendix A for \( y_1, y_2 \) and \( y_3 \) definitions. Equation (4.23) is now rewritten by retaining only first and second central moment terms

\[ \langle h \rangle = (\bar{y} - y_2)^{-1} \bar{u} \]

\[ \text{Var}(h) = \left[ \left( D_{y,y} - R_{y,y} \right)^{-1} - D_{y,y}^{-1} \right] D_{u,u} \]  
\quad (4.24a-b)

Using equations (4.24a) and (4.20, 4.21), the expected mean of the response vector is obtained as

\[ \langle |X| \rangle = |\bar{X} + \langle \bar{X} \rangle| \approx \left| (\bar{G}_2)^{-1} \bar{F} \right| \]  
\quad (4.25)

For the sake of simplicity like Section 4.3.1, all cross-covariance matrices such as \( R_{M,K} \) and \( R_{M,F} \) are assumed to be null. Taking first and second moments of equation (4.20), the individual variance of the response amplitude associated individually with \( M, K, C \) or \( F \) can be determined by
Define the total variance and standard deviation of the displacement amplitude as

\[
\text{Var}(|X|) = P_{XM} + P_{XC} + P_{XX} + P_{XF}
\]

\[
\sigma(|X_i|) = \left( \langle X_i \otimes X_i \rangle - \langle X_i \rangle \otimes \langle X_i \rangle \right)^{0.5} = \left[ \text{Var}(|X_i|) \right]^{0.5}, \quad i=1,2, \ldots, N
\]

The effect of cross-correlation matrices such as \( R_{M,K} \) and \( R_{M,C} \) can be included in the analysis in a similar manner; see Sections 4.3.2 and 4.3.8. When \( M \) and \( K \) are assumed to be fully correlated, use the following expression instead of equation (4.26b)

\[
R_{G,G} = \Omega^4 R_{M,M} - \Omega^2 \left( R_{M,K} + R_{K,M} \right) + R_{K,K}
\]

### 4.4.2 First Order Perturbation Analysis

Analytical expressions given by (4.25) and (4.26) are approximated by using the first order expansion as follows provided \( \| \tilde{G} \| \ll \| G \| \).
Like the single degree of freedom system theory, here $Q$ is the variance of the response amplitude given only one uncertainty e.g. $Q_{xM}$ considers only the effect of randomness in inertial properties. The total variance and standard deviation are given as

$$\text{Var}(|X|) = Q_{xM} + Q_{xc} + Q_{xx} + Q_{xF}$$

$$\sigma(|X|) = \left[\text{Var}(|X|)\right]^{0.5}, i=1,2,\ldots,N$$

Equations (4.29) and (4.30) yield the same results as those given by the Taylor matrix expansion or perturbation method [4.4, 4.5]; it will be illustrated later via a numerical example. The limitations of equations (4.29) and (4.30) are exactly the same as those associated with the existing perturbation methods [4.4, 4.5]. Like the single degree of freedom system, errors committed by the first order perturbation can be quantified by examining appropriate $P$ and $Q$ terms.
4.4.3 Example VI: Two Degree-of-Freedom System

Consider the physical system shown in Figure 4.1c with $\overline{K}_{12} = 2$ and a proportionally damped case ($C = 0.01\overline{K}$). Deterministic eigenvalue problem yields the following modal database: $\bar{\omega}_1 = 1.0$, $\bar{\omega}_2 = 2.236$, $\bar{\zeta}_1 = 0.005$, $\bar{\zeta}_2 = 0.011$, $\overline{\Phi}_1 = [1 1]$ and $\overline{\Phi}_2 = [1 -1]$. Like previous examples we consider randomness in masses only and set it as $R_{m,m} = 0.01D_{m,m}$. First, examine selected standard deviation results yielded by a conventional Monte-Carlo simulation as listed in Table 4.1. All results are for the case when the excitation frequency is fixed as 2.4 rad/s which places it in the vicinity of second resonance. We observe uncertainties which, as expected, depend on the sample size and the trial index. Accordingly numerical answers should be considered valid within a certain confidence interval, say about ±5% to 10% about any predicted value. For the sake of computational convenience we choose a sample size of 400 and run only one trial, like previous example, and execute simulation from $\Omega = 0$ to 3.45 rad/s. Analogous results obtained by using the proposed direct product technique (equations (4.23, 4.25 and 4.27)) and the first order perturbation method (equations (4.29 and 4.30)) are compared with numerical answers in Figures 4.10 and 4.11. All three methods match in the off-resonance regimes about $\bar{\omega}_1$ and $\bar{\omega}_2$. The proposed method seems to track all of the trends followed by the numerical simulation, even within the resonance regimes which are obviously well separated.

4.4.4 Example VII: Spectral Coupling Issues

For a deterministic system it is well known that spectral coupling is dictated by the separation between natural frequencies and the damping ratios. Now this concept is analysed for a two degree of freedom system with random parameters by bringing two natural frequencies closer. Table 4.2 lists 3 cases used for this study. We note a strong
### Table 4.1. Selected Monte-Carlo results for Example VI

<table>
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<tr>
<th>Sample size</th>
<th>Trial number</th>
<th>$\sigma(X_i)/\bar{X}_j$</th>
<th>$\sigma(x_{i4})/\bar{X}_4$</th>
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<td>1.69</td>
</tr>
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<td>800</td>
<td>1</td>
<td>1.077</td>
<td>1.51</td>
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<td>1600</td>
<td>1</td>
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</tr>
<tr>
<td>1600</td>
<td>2</td>
<td>1.13</td>
<td>1.58</td>
</tr>
<tr>
<td>1600</td>
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<td>1.24</td>
<td>1.74</td>
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</table>
Figure 4.10. Comparison of $|X_i(\Omega)|$ spectra for Example VI shown in Figure 4.1c with $\bar{\omega}_2 / \bar{\omega}_1 = 2.23$: (a) standard deviation $\sigma(|X_i(\Omega)|)/|X_i(\Omega)|$; (b) probabilistic $\langle |X_i(\Omega)| \rangle + \sigma(|X_i(\Omega)|)$ vs. deterministic $|X_i(\Omega)|$. Key: —— proposed technique; —— first order perturbation method, ++ + Monte-Carlo simulation and —— deterministic system.
Figure 4.11. Comparison of $|X_2(\Omega)|$ spectra for Example VI shown in Figure 4.1c. Key and format as Figure 4.10.
Table 4.2. Dataset used for spectral coupling study

<table>
<thead>
<tr>
<th>Case</th>
<th>Example</th>
<th>$K_{12}$</th>
<th>$\bar{\omega}_2 / \bar{\omega}_1$</th>
<th>Results in Figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>VI</td>
<td>2</td>
<td>2.23</td>
<td>10,11</td>
</tr>
<tr>
<td>b</td>
<td>VII</td>
<td>0.6</td>
<td>1.5</td>
<td>12,13</td>
</tr>
<tr>
<td>c</td>
<td>VIII</td>
<td>0.05</td>
<td>1.05</td>
<td>14,15</td>
</tr>
</tbody>
</table>
Figure 4.12. Comparison of $|X_1(\Omega)|$ spectra for Example VI shown in Figure 4.1c with $\bar{\omega}_2 / \bar{\omega}_1 = 1.5$. Key and format as Figure 4.10.
Figure 4.13. Comparison of $|X_2(\Omega)|$ spectra for Example VI shown in Figure 4.1c with $\bar{\omega}_2 / \bar{\omega}_1 = 1.5$. Key and format as Figure 4.10.
Figure 4.14. Comparison of $|X_1(\Omega)|$ spectra for Example VII shown in Figure 4.1c with $\bar{\omega}_2 / \bar{\omega}_1 = 1.05$. Key and format as Figure 4.10.
Figure 4.15. Comparison of $|X_2(\Omega)|$ spectra for Example VII shown in Figure 4.1c with $\bar{\omega}_2 / \bar{\omega}_1 = 1.05$. Key and format as Figure 10.
coupling between two resonances as $\bar{\omega}_2 \rightarrow \bar{\omega}_1$. Also two spectral shapes change. But unlike the proposed method, first order perturbation does not predict the broader and combined resonance very well. This example demonstrates that random fluctuations in $M$ and $K$ can also couple modes strongly provided they are moderately coupled for the corresponding deterministic problem.

4.4.5 Example VIII: Correlated Mass and Stiffness

Now we reconsider the Example VII (case c from Table 4.2) but with nonproportionally damping matrix $\bar{C} = [0.0205 \quad -0.0005; \quad -0.005 \quad 0.0005]$. Here mass and stiffness matrices are assumed to be fully correlated and chosen as $R_{M,M} = 0.05^2 D_{M,M}$, $R_{K,K} = 0.1^2 D_{M,M}$, and $R_{M,K} = 0.005 D_{M,M}$. Using equation (4.28), response spectra are predicted and compared with two existing methods in Figures 4.16 and 4.17. Predictions again match well with the Monte-Carlo simulation.

4.4.6 Example IX: Periodic System

The final case examines the periodic system of Figure 4.1d with $N=5$. Deterministic modal database is as follows: $\bar{\omega}_1 = 0.517$, $\bar{\omega}_2 = 1.0$, $\bar{\omega}_3 = 1.414$, $\bar{\omega}_4 = 1.732$, $\bar{\omega}_5 = 1.932$; $\bar{\zeta}_1 = 0.0026$, $\bar{\zeta}_2 = 0.005$, $\bar{\zeta}_3 = 0.007$, $\bar{\zeta}_4 = 0.009$ and $\bar{\zeta}_5 = 0.01$. Now we introduce randomness via $M$ and choose $R_{M,M} = 0.01 D_{M,M}$. Spectral characteristics of $X_1$ and $X_2$ are shown in Figures 4.18 and 4.19. Again, we note that our proposed technique matches well with the numerical answers (sample size=400) which are obviously valid within a certain confidence interval. Similar comparisons are found for $X_3$, $X_4$ and $X_5$, though not repeated here.
Figure 4.16. Comparison of $|X_i(\Omega)|$ spectra for Example VIII shown in Figure 4.1c with correlated mass and stiffness. Key and format as Figure 4.10.
Figure 4.17. Comparison of $|X_2(\Omega)|$ spectra for Example VIII shown in Figure 4.1c with correlated mass and stiffness. Key and format as Figure 4.10.
Figure 4.18. Comparison of $|X_1(\Omega)|$ spectra for Example IX shown in Figure 4.1d.

Key and format as Figure 4.10.
Figure 4.19. Comparison of $|X_2(\Omega)|$ spectra for Example IX shown in Figure 4.1d. Key and format as Figure 4.10.
4.5. **Concluding Remarks**

A new analytical methodology has been developed to compute the statistical frequency response characteristics of a linear time-invariant, discrete vibratory system with parameter uncertainties and/or random force amplitude. A direct product technique is proposed to estimate the mean and standard deviation of the displacement amplitude response at the deterministic excitation frequency. This theory is validated by comparing the results of several single and multi-degree of freedom examples with the Monte-Carlo simulation. Predictions yielded by the perturbation methods are also given. It is seen clearly that the perturbation method deviates from the Monte-Carlo simulation within the resonance regime(s) because of the secular or near secular terms. Our method, however, overcomes this deficiency reasonable well since predictions match reasonably well with the numerical solutions. Also, our method is computationally faster when compared with the numerical simulation especially for multi-degree of freedom systems. Although the proposed method is potentially powerful and overcomes a few deficiencies of the existing methods, more efforts are definitely required to overcome some of the limitations identified in Section 4.3.2. And, the proposed method, unlike the Monte-Carlo technique, can not be applied to a nonlinear problem.
References


CHAPTER V
CONCLUSION

5.1 Contribution

Chief contribution has been the development of a new analytical method of finding solution to a random differential equation. Its application includes problems in many disciplines but this study has focused on a linear time-invariant, discrete vibratory system. Only the mean and standard deviation of response variables have been predicted. The proposed method has been found to be clearly superior to the commonly used first order perturbation methods [5.1-5.6]. It is a very promising technique because of its reasonable accuracy coupled with computational ease. Perhaps of more fundamental importance has been the development of a mathematical framework which can be extended to study practical vibration and noise problems which arise due to parameter uncertainties. Specific contributions related to each of the studies conducted are as follows.

a. Eigensolutions: A new analytical method has been proposed to estimate the first two moments and standard deviation of eigensolutions in terms of the moments of system parameter matrices. Both undamped and proportionally damped problems are considered. A concise and reasonably accurate formulation for a multi-degree-of-freedom system with only randomly varying mass has also been presented. The proposed method has been found to be more accurate than the commonly used first order
perturbation methods [5.1-5.3] especially when random fluctuations are no longer very small. Also, it is computationally inexpensive when compared with the Monte-Carlo simulation [5.7, 5.8].

b. Impulse Response: A new analytical method has been developed to determine the statistical impulse response characteristics given uncertain system parameters and impulse amplitude. It is based on the convolution integral adapted from the deterministic system theory and it is formulated from the knowledge of statistical eigensolutions. This proposed method, unlike the first order perturbation methods commonly found in the literature, removes the secular terms in the impulse response expression and works well for long time durations. Previous investigators [5.4, 5.5] had not successful in achieving this goal. Predictions of the proposed method match fairly well with those yielded by the Monte-Carlo simulation. Also, the proposed method is computationally faster than the Monte-Carlo simulation since a large number of iterations is not required. It is therefore concluded that the proposed method overcomes some of the deficiencies of the existing methods while yielding reasonably accurate solutions.

c. Frequency Response: An alternate methodology for computing the statistical frequency response characteristics of a nonproportionally damped vibratory system with parameter uncertainties and random force amplitude has been developed. A direct product technique is employed to estimate the first two moments of displacement amplitude response at the deterministic excitation frequency. It has been found that the proposed method is again superior to the commonly used first order perturbation method [5.5, 5.6], especially within the vicinity of a resonance. And, predictions of this method match reasonably well with the results of Monte-Carlo numerical simulation. Yet
another contribution has been an improved understanding of frequency response characteristics. For the first time, general shapes of amplitude spectra in terms of mean, standard deviation or upper-bound response have been presented for single and multi-degree-of-freedom systems. These have been very helpful in explaining resonance characteristics and spectral coupling issues which were not discussed specifically in the literature.

5.2 Limitations and Future Research

The scope of the proposed method is restricted to a certain class of differential equations even though it can be applied to examine many vibration problems only two statistical moments are estimated. Specific shortcomings are as follows: (i) this method looses accuracy when random fluctuations are very high, say that, normalized standard deviation is higher than 0.5; (ii) it can not address localization effects in periodic disordered systems; and (iii) non-proportionally damped systems can not be analyzed. Other limitations are strictly related to the simplifying assumptions made during the theoretical development.

Several areas of further research are proposed here:

1. Verify theory through experimental measurements.

2. Extend the proposed method to non-proportionally damped discrete vibratory systems.

3. Extend the proposed method to continuous systems.

4. Incorporate the proposed method in stochastic finite element codes.

5. Extend the direct product method to a nonlinear problem.
References


REFERENCES


APPENDIX
Matrix Direct Products

A.1 Review of Key Concepts

Consider the direct product of two matrices $A(pxq)$ and $B(sxt)$, given by $A \otimes B$. It is a $ps \times qt$ dimensioned matrix defined as [2.18]

\[
A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \ldots & a_{1q}B \\
a_{21}B & \ldots & \ldots & \ldots \\
a_{p1}B & a_{p2}B & \ldots & a_{pq}B
\end{bmatrix}
\]  

(A1)

A few well known identities are as follows; also see [2.14] for more details.

\[1 \otimes A = A = A \otimes 1\]

\[(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \quad \text{(if } A^{-1} \text{ and } B^{-1} \text{ exist)}\]

\[(A \otimes B)^T = A^T \otimes B^T\]

\[AC \otimes BD = (A \otimes B)(C \otimes D)\]
\[(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D\]  \hspace{1cm} (A2a-e)

Selected matrix operations are described below:

(i) \(\text{cs}(A)\): column transformation of a matrix \(A(p \times q)\)

\[
\text{cs}(A) = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_q
\end{bmatrix}
\]  \hspace{1cm} (A3)

where \(A_j\) is the \(j\)th column of \(A\).

(ii) Matrix derivative

\[
\mathcal{D}_{\text{cs}(A)} B = \begin{bmatrix}
\frac{\partial B}{\partial a_{11}} & \frac{\partial B}{\partial a_{12}} & \cdots & \frac{\partial B}{\partial a_{pq}} \\
\frac{\partial^2 B}{\partial a_{11}^2} & \frac{\partial^2 B}{\partial a_{12} \partial a_{21}} & \cdots & \frac{\partial^2 B}{\partial a_{pq}}
\end{bmatrix}
\]  \hspace{1cm} (A4)

\[
\mathcal{D}^2_{\text{cs}(A)} B = \begin{bmatrix}
\frac{\partial^2 B}{\partial a_{11}^2} & \frac{\partial^2 B}{\partial a_{11} \partial a_{21}} & \cdots & \frac{\partial^2 B}{\partial a_{pq}} \\
\frac{\partial^2 B}{\partial a_{11} \partial a_{21}} & \frac{\partial^2 B}{\partial a_{11} \partial a_{21}} & \cdots & \frac{\partial^2 B}{\partial a_{pq}}
\end{bmatrix}
\]  \hspace{1cm} (A5)

(iii) Matrix Taylor expansion

\[
B(A) = B(\bar{A}) + \sum_{m=1}^{L} \frac{1}{m!} \mathcal{D}^m_{\text{cs}(A)} B \bigg|_{\bar{A}} \left[ \text{cs}(\bar{A}) \otimes_{m} I_q \right] + O(L+1)
\]  \hspace{1cm} (A6)

where \(O(L+1)\) is a remainder.
A.2 Example

Consider a two dimensional problem given by random matrix $y(2 \times 2)$ and deterministic vector $\bar{u} (2 \times 1)$ and define a random function $h(2 \times 1)$ such as $h = y^{-1}\bar{u}$. See Section 2.4.1 for its application. Using Taylor series we get

$$h(y, \bar{u}) = h(\bar{y}, \bar{u}) + \mathcal{D}_{cs(y)} h \left[ cs(\bar{y}) \right] + \frac{1}{2!} \mathcal{D}^2_{cs(y) \otimes \bar{y}} h \left[ cs(\bar{y})^{\otimes 2} \right] + \frac{1}{3!} \mathcal{D}^3_{cs(y) \otimes \bar{y}} h \left[ cs(\bar{y})^{\otimes 3} \right] + \cdots$$

(A7)

where

$$\mathcal{D}_{cs(y)} h \left[ cs(\bar{y}) \right] = \mathcal{D}_{cs(y)} y^{-1} \left[ cs(\bar{y}) \otimes I_2 \right] \bar{u}$$

(A8)

$$\frac{1}{2!} \mathcal{D}^2_{cs(y) \otimes \bar{y}} h \left[ cs(\bar{y})^{\otimes 2} \right] = \frac{1}{2!} \mathcal{D}^2_{cs(y) \otimes \bar{y}} y^{-1} \left[ cs(\bar{y})^{\otimes 2} \otimes I_2 \right] \bar{u}$$

$$= \bar{y}^{-1} \frac{1}{2!} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] y^{-1} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] y^{-1} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] y^{-1} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right]$$

$$= \bar{y}^{-1} y_2 \bar{y}^{-1} \bar{u} \quad (A9)$$

and $y_3$ can be determined similarly. If $h$ is approximated up to third order term

$$h(y, \bar{u}) = h + \bar{y}^{-1} y_1 \bar{y}^{-1} \bar{u} + \bar{y}^{-1} y_2 \bar{y}^{-1} \bar{u} + \bar{y}^{-1} y_3 \bar{y}^{-1} \bar{u}$$

(A10)