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Borel diagonalization theorems and second-order arithmetic

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The Ohio State University, 1992
Borel Diagonalization Theorems and Second-Order Arithmetic

Dissertation

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By

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Abstract

In a 1981 paper, H. Friedman presented a series of statements, known as the Borel diagonalization theorems, and proved that they require unexpectedly large amounts of set theory to prove. The most basic one of these theorems cannot be proved in second order arithmetic, but can be proved in a subsystem of second-order arithmetic when restricted to Borel functions of any fixed finite rank. The aim of this dissertation is to obtain sharp upper and lower bounds for the amount of second order arithmetic required to prove these theorems when restricted in this manner. Our methods include a detailed analysis of the ramified analytical hierarchy of sets, and of forcing methods in the context of weak fragments of second order arithmetic. The upper and lower bounds do not precisely match yet, and we plan to do further research in this direction.
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CHAPTER I

Introduction

1.1 What is in this paper?

In chapter I, we define the usual subsystems of \( Z_2 \) and formally introduce well-known concepts such as "coded structure", "satisfaction predicate" and (code for ) " ramified analytical hierarchy". In chapter II, we develop the ramified analytical hierarchy theory in very weak subsystems of \( Z_2 \). Results here are then used in chapter IV to obtain the lower bounds. Many results in this chapter involve technical subtleties of independent interest. In chapter III, we obtain a tight upper bound for \( A^\omega_1(X) \) (see below for definition) as well as other related results.

Let \( N \) be the set of nonnegative integers and \( X \) be among Baire space \( N^N \), Cantor space \( 2^N \), the space of reals \( R \) or closed unit interval \( I \) (as a subspace of \( R \)), all with the usual topologies. We want to study Borel functions from \( X^N \) to \( X \) ( or from \( X^N \) to \( X^N \) ) which have finite Baire rank and satisfy certain invariant properties. A finite permutation \( \sigma \) of \( N \) is a bijection from a finite subset of \( N \) onto that finite subset. For \( \bar{x} \in X^N \), a finite permutation \( \sigma \), \( \bar{x}^\sigma \) is the sequence whose \( i^{th} \)-term is the \( \sigma^{-1}(i)^{th} \)-term of \( \bar{x} \) for those \( i \) such that \( \sigma^{-1}(i) \) is defined, while the \( i^{th} \)-term of \( \bar{x}^\sigma \) is the \( i^{th} \)-term of \( \bar{x} \) if \( \sigma(i) \) is undefined. A \( \pi \in N^N \) is called a permutation of \( N \) if it is
a bijection. For $\bar{x} \in X^N$ and $\pi$ a permutation, we define $\bar{x}^\pi$ similar to the case when $\pi$ is a finite permutation. For $\bar{x}, \bar{y} \in X^N$, we say they are finite permutations of each other if there exists a finite permutation $\sigma$ such that $\bar{y}$ is $\bar{x}^\sigma$, permutations of each other if there is a permutation $\pi$ such that $\bar{y}$ is $\bar{x}^\pi$. We say that $\bar{x}$ and $\bar{y}$ have the same range if every term of $\bar{x}$ is a term of $\bar{y}$ and vice versa.

A function $F$ from $X^N$ to $X$ is said to have the first (second, third) invariant property if whenever $\bar{x}$ and $\bar{y}$ are finite permutations of each other (permutations of each other, having the same range) then $F(\bar{x}) = F(\bar{y})$. And a function from $X^N$ to $X^N$ is said to have the quasi-first (second, third) invariant property if whenever $\bar{x}$ and $\bar{y}$ are finite permutations of each other (permutations of each other, having the same range) then $F(\bar{x})$ and $F(\bar{y})$ have the same range.

Let us use $A^n_i(X)$ ($i = 1, 2$ or $3$, $n$ a nonnegative integer) to denote the following statement:

**Statement A.** If $F : X^N \to X$ is a Borel function of Baire rank $n$ and has the $i^{th}$-invariant property, then there is an $\bar{x} \in X^N$ such that $F(\bar{x})$ is a term of $\bar{x}$.

Similarly, we will use $B^n_i(X)$ ($i = 1, 2$ or $3$) to denote the following statement:

**Statement B.** If $F : X^N \to X^N$ is a Borel function of Baire rank $n$ and has the quasi-$i^{th}$ invariant property, then there is an $\bar{x} \in X^N$ such that $F(\bar{x})$ is a subsequence of $\bar{x}$.

The above two statements can both be coded as $\Pi^1_2$-sentences in the language of second-order arithmetic, $L^*_2$ (with second-order variables ranging over functions) (see section 3.2). Let us use the same symbols to denote their coded version as well.
The major results of this dissertation are included in the following two theorems.

**Theorem 1.** For any $n$ and $i$ ($n \geq 1 \leq i \leq 3$) $\Delta^1_n$-CA is enough to prove $A_i^n(N^N)$, $A_i^{n+1}(2^N)$, $A_i^{n+1}(R)$ and $A_i^{n+1}(I)$, whereas $\Delta^1_{n-3}$-CA ($n \geq 9$) is not enough to prove $A_i^{n+1}(2^N)$ and $\Delta^1_{n-4}$-CA ($n \geq 10$) is not enough to prove any of the other three statements.

**Theorem 2.** For any $n \geq 1$ and $i=2$ or $3$ $\Pi^1_n$-CA is enough to prove $B_i^n(N^N)$, $B_i^{n+1}(2^N)$, $B_i^{n+1}(R)$ and $B_i^{n+1}(I)$, whereas $\Pi^1_{n-1}$-CA ($n > 1$) is not enough to prove any of them.

**Remark.** For any $n > 1$, ACA proves $\neg B_i^n(X)$, where $X$ is among Baire, Cantor, R and I. See lemma 3.6.2 for detail.

Let $T$ be some theory in $L_2$ (or $L_2^*$ depending on which space we are talking about). The lower bound stated in Theorem 1 is two to three levels below being perfect. One way to remedy the situation is to strengthen Statement A to

**Statement $A(X, T)$.** If $F : X^N \to X$ is a Borel function of Baire rank $n$ and has the $i^{th}$-invariant property, then there is an $\bar{x} \in X^N$ such that $F(\bar{x})$ is a term of $\bar{x}$ and $\bar{x}$ codes a model\(^1\) for $T$.

We define $A^n_n(X, T)$ similar to $A^n_n(X)$, then we have

**Theorem 1*. For any $n$ and $i$ ($n \geq 1 \leq i \leq 3$) $\Delta^1_n$-CA is enough to prove $A_i^n(N^N, \Pi^1_{n-1}$-CA) and $A_i^{n+1}(2^N, \Pi^1_{n-1}$-CA), $A_i^{n+1}(R, \Pi^1_{n-1}$-CA), $A_i^{n+1}(I, \Pi^1_{n-1}$-CA), whereas $\Delta^1_{n-1}$-CA ($n \geq 8$) is not enough to prove any of $A_i^n(N^N, ACA)$, $A_i^{n+1}(2^N, ACA)$, $A_i^{n+1}(R, ACA)$ and $A_i^{n+1}(2^N, ACA)$.

\(^1\)for a precise meaning of being a model see section 3.3
1.2 Languages of second-order arithmetic $L_2$ and $L_2^*$

There are two languages, $L_2$ and $L_2^*$, frequently in use when studying second-order arithmetic. Each has its own advantage in different context. In chapter III and IV $L_2^*$ is used, and in other parts $L_2$ is used. What follows in this section is a brief review on subsystems of the formal theory of second-order arithmetic.

1.2.1 The language $L_2$

Let us use “PA” to denote the theory of full Peano arithmetic. Its language ($L(PA)$) contains a function symbol for each primitive recursive function and a predicate symbol for each primitive recursive relation (i.e., whose characteristic function is primitive recursive). The defining formulas for these functions and predicates are part of PA. Let $PA^-$ be PA without the induction axiom scheme. The atomic formulas of $L(PA)$ are the equalities between terms and predicate symbols applied to terms.

We choose to use $2^{i_1+1} \cdot 3^{i_2+1} \cdots p_k^{i_k+1}$ to code the sequence $i_1, \ldots, i_k$. Such numbers are called sequence numbers. $ls(n)$ denotes the length of the sequence coded by $n$ if $n$ is a sequence number and 0 if $n$ is not. $(n)_i (i < ls(n))$ denotes the $i$-th term of the sequence coded by $n$.

$L_2 = L(PA) + \{\in\}$ is a 2-sorted language with its first order variables range over non-negative integers and its second-order variables range over sets of non-negative integers. We usually use $m, n, l \cdots$ as numerical variables and $x, y, z, X, Y, Z \cdots$ as set variables.

Terms of $L_2$ are the terms of $L(PA)$, called first-order terms, and the second-order
variables, called second-order terms.

Atomic formulas of $L_2$ are the atomic formulas of $L(PA)$ plus formulas of the form $s \in x$, where $s$ is a first-order term of $L(PA)$ and $x$ is a second-order variable.

Formulas are built up from atomic formulas by applying the usual propositional connectives, numerical and set quantifications.

We will write $(\forall m \leq t)\phi$ and $(\exists m \leq t)\phi$ for $(\forall m)((m \leq t) \to \phi)$ and $(\exists m)((m \leq t) \land \phi)$ respectively, where $t$ is a first-order term and $m$ does not appear in $t$. These are called bounded quantifiers.

When a formula does not involve any set quantifier, we call it an arithmetic formula. Note that an arithmetic formula may contain free set variables.

$\Sigma_0^0=\Pi_0^0$ is the set of formulas built up from atomic formulas using propositional connectives and bounded quantifications.

And for any $n$, 

\[
\Sigma_{n+1}^0 = \{(\exists m)\phi : \phi \in \Pi_n^0\}. \\
\Pi_{n+1}^0 = \{(\forall m)\phi : \phi \in \Sigma_n^0\}. \\
\Sigma_\infty^0 = \Pi_\infty^0 = \bigcup_{n=0}^\infty \Sigma_n^0. \\
\Sigma_0^1 = \Pi_0^1 = \Sigma_\infty^0.
\]

For any $n$, 

\[
\Sigma_{n+1}^1 = \{(\exists x)\phi : \phi \in \Pi_n^1\}. \\
\Pi_{n+1}^1 = \{(\forall x)\phi : \phi \in \Sigma_n^1\}. \\
\Sigma_n^1 \text{ (} \Pi_n^1, \cdots \text{)} \text{, we call } \phi \text{ a } \Sigma_n^1\text{-formula (} \Pi_n^1\text{-formula, } \cdots \text{).}
$$S\Sigma_0^1 = \Pi_2^0, \ S\Pi_0^1 = \Sigma_2^0.$$  

For any n,

$$S\Sigma_{n+1}^1 = \{(\exists x)\phi : \phi \in S\Pi_n^1\}.$$  
$$S\Pi_{n+1}^1 = \{(\forall x)\phi : \phi \in S\Sigma_n^1\}.$$  

Formulas in $S\Sigma_n^1 (S\Pi_n^1)$ are called strict $\Sigma_n^1 (\Pi_n^1)$ formulas.

The theory of $\Sigma_0^0-CA_0$ consists of the axioms of $PA^-$, the axiom scheme of $\Sigma_0^0$-comprehension: the universal closure of

$$(\exists x)(\forall m)(m \in x \leftrightarrow \phi(m)),$$

where $\phi$ is $\Sigma_0^0$ and $x$ does not occur free in $\phi$, and the induction axiom:

$$(\forall x)((0 \in x) \land (\forall i)((i \in x) \rightarrow (i + 1 \in x)) \rightarrow (\forall i)(i \in x)).$$

The theory of $\textbf{RCA}_0$ consists of the axioms of $PA^-$ plus

- Recursive comprehension axiom scheme: the universal closure of

$$(\forall i)(\phi(i) \leftrightarrow \psi(i)) \rightarrow (\exists x)(\forall i)(i \in x \leftrightarrow \phi(i)),$$

where $\phi \in \Sigma_1^0$, $\psi \in \Pi_1^0$ and $x$ does not occur free in either one of the two formulas;

- $\Sigma_1^0$-induction scheme: the universal closure of

$$\phi(0) \land \forall i(\phi(i) \rightarrow \phi(i + 1)) \rightarrow \forall i\phi(i)$$

where $\phi \in \Sigma_1^0$. 

The theory of $\textbf{ACA}_0$ consists of the axioms of $\textbf{RCA}_0$ plus Arithmetic comprehension axiom scheme: the universal closure of

$$(\exists x)(\forall i)(i \in x \leftrightarrow \phi(i)),$$

where $\phi \in \Sigma^0_\infty$ and $x$ does not occur free in $\phi$.

For each natural number $n$, the theory of (bold-face) $\Sigma^1_n$-$\textbf{CA}_0$ consists of the axioms of $\textbf{RCA}_0$ plus the $\Sigma^1_n$-Comprehension axiom scheme: the universal closure of

$$(\exists x)(\forall i)(i \in x \leftrightarrow \phi(i)),$$

where $\phi \in S\Sigma^1_n$ and $x$ does not occur free in $\phi$.

For each natural number $n$, the theory of (bold-face) $\Delta^1_n$-$\textbf{CA}_0$ consists of the axioms of $\textbf{RCA}_0$ plus the $\Delta^1_n$-Comprehension axiom scheme: the universal closure of

$$\forall i(\phi(i) \leftrightarrow \psi(i)) \rightarrow (\exists x)(\forall i)(i \in x \leftrightarrow \phi(i)),$$

where $\phi \in S\Sigma^1_n$, $\psi \in S\Pi^1_n$ and $x$ does not occur free in either one of the two formulas.

Similarly, we may define $\Pi^1_n$-$\textbf{CA}_0$.

The theories $\Sigma^0_\infty$-$\textbf{CA}$, $\textbf{RCA}$, $\textbf{ACA}$, $\Sigma^1_n$-$\textbf{CA}$, $\Delta^1_n$-$\textbf{CA}$ and $\Sigma^1_n$-$\textbf{CA}$ are

$\Pi^0_\infty$-$\textbf{CA}_0$, $\textbf{RCA}_0$, $\textbf{ACA}_0$, $\Sigma^1_n$-$\textbf{CA}_0$, $\Delta^1_n$-$\textbf{CA}_0$ and $\Sigma^1_n$-$\textbf{CA}_0$ plus the full induction axiom scheme: the universal closure of

$$\phi(0) \land \forall i(\phi(i) \rightarrow \phi(i + 1)) \rightarrow \forall i\phi(i)$$

where $\phi$ is any formula of $L_2$.

We define the so-called (light-face) $\Pi^1_n$-$\textbf{CA}$ ($\Delta^1_n$-$\textbf{CA}$ and $\Sigma^1_n$-$\textbf{CA}$) from the corresponding “bold-face” theories by simply requiring the formula $\phi$ in all the axioms to not involve any free set variable (but free numerical variables are allowed).
Working in $\Sigma_0^0$-CA, we may define the concepts of "pairing" and "components". Given $X$, $Y$ and $i$, the pair $(X, Y)$ is defined as

$$\langle X, Y \rangle = \{(0, i) : i \in X\} \cup \{(1, j) : j \in Y\},$$

and the $i$-th component $X_i$ of $X$ is defined as

$$X_i = \{j : (i, j) \in X\}.$$

Remark. The only symbols which are allowed in a formal statement are those in $L_2$ as defined above or $L_2^*$ defined in the next subsection. Notations such as $(X, Y)$ and $X_i$ are introduced purely for the ease of elaboration. They, along with any other defined notation, can be eliminated from every formal statement we will be discussing. Hence, when we say that a theory of $L_2$ proves a statement involves these notations, we actually mean the theory proves the statement after those defined symbols are eliminated.

The theory of $\Sigma^1_n$-$AC_0$ consists of the axioms of $ACA_0$ and the $\Sigma^1_n$ choice scheme: the universal closure of

$$(\forall i)(\exists x)\phi(i, x) \rightarrow (\exists x)(\forall i)\phi(i, x_i),$$

where $\phi$ is $S\Sigma^1_n$.

The theory of $\Sigma^1_n$-$DC_0$ consists of the axioms of $ACA_0$ and the $\Sigma^1_n$ dependent choice scheme: the universal closure of

$$(\forall x)(\exists y)\phi(x, y) \rightarrow (\forall x)(\exists Y)((Y_0 = x) \land (\forall i)\phi(Y_i, Y_{i+1})), $$

where $\phi$ is $S\Sigma^1_n$. 
The theory of Wk-$\Sigma^1_n$-AC$_0$ consists of the axioms of ACA$_0$ and the axiom scheme of Weak-$\Sigma^1_n$-choice: the universal closure of

$$(\forall i)(\exists x)\phi(i,x) \rightarrow (\exists X)(\forall i)(\exists j)\phi(i,X_j),$$

where $\phi$ is $S\Sigma^1_n$.

$\Sigma^1_n$-AC, $\Sigma^1_n$-DC and Wk-$\Sigma^1_n$-AC are $\Sigma^1_n$-AC$_0$, $\Sigma^1_n$-DC$_0$ and Wk-$\Sigma^1_n$-AC$_0$ plus the full induction axiom scheme respectively.

The theory of second-order arithmetic, $\mathbb{Z}_2$ is the union of all $\Sigma^1_n$-CA ($n=1,2,3,\cdots$).

The theory of parameterless second-order arithmetic, $p\mathbb{Z}_2$ is the union of all $\Sigma^1_n$-CA ($n=1,2,3,\cdots$).\(^2\)

1.2.2 The language $L^*_2$

The language $L^*_2$ is a 2-sorted language whose first-order variables ranging over non-negative integers and whose second-order variables ranging over functions from non-negative integers to non-negative integers. Its function and relation symbols are exactly those of PA.

Terms of $L^*_2$ consist of first-order terms and function variables (called second-order terms). And the first order terms are obtained as follows:

1. terms of PA are terms;

2. if “t” is a term and “f” is a function variable, then “f(t)” is a term;

3. if “$t_1, \cdots, t_n$” are terms and “g” is a n-ary function constant, then “$g(t_1, \cdots, t_n)$” is a term;

\(^2\)The term $p\mathbb{Z}_2$ was first introduced by Friedman [1], where “p” stands for “parameterless”.
4. only those built from 1,2 or 3 are terms.

Atomic formulas of $L_2^*$ are built up by using the predicate symbols of $L(PA)$ and the first-order terms of $L_2^*$ in the usual way.

Formulas are built up from atomic formulas by applying the usual propositional connectives, numerical and function quantifications.

We will write $(\forall m \leq t) \phi$ and $(\exists m \leq t) \phi$ for $(\forall m)((m \leq t) \rightarrow \phi)$ and $(\exists m)((m \leq t) \wedge \phi)$ respectively, where $t$ is a first-order term and $m$ does not appear in $t$. These are called bounded quantifiers.

We define the hierarchy of arithmetical formulas $\Sigma_0^0$, $\Pi_0^0$, $\Sigma_n^0$- and $\Pi_n^0$-formulas and the hierarchy of analytical formulas $\Sigma_0^1$, $\Pi_0^1$, $\Pi_n^1$- and $\Sigma_n^1$-formulas in the same way as with $L_2$.

$$S\Sigma_1^0 = \Pi_1^0, \quad S\Pi_0^1 = \Sigma_1^0.$$  

For any $n,$

$$S\Sigma_{n+1}^1 = \{ (\exists x) \phi : \phi \in S\Pi_n^1 \}. $$

$$S\Pi_{n+1}^1 = \{ (\forall x) \phi : \phi \in S\Sigma_n^1 \}. $$

Formulas in $S\Sigma_n^1$ ($S\Pi_n^1$) are called strict $\Sigma_n^1$ ($\Pi_n^1$) formulas.

Note the difference of the strict $\Sigma_n^1$ ($\Pi_n^1$) formulas in $L_2^*$ and the strict $\Sigma_n^1$ ($\Pi_n^1$) formulas in $L_2$.

We will use $\langle a_1, \ldots, a_n \rangle$ to denote the sequence number of the sequence $a_1, \ldots, a_n$ if $n \geq 2$, and when $n = 1$, we will let $\langle a_1 \rangle = a_1$, namely, the identity function.
The theory of $Prim^*-CA_0$; i.e., the Primitive Recursive Comprehension axioms system, consists of the axioms of $PA^-$ plus the following

1. The axiom for projection function:

$$(\exists f)(\forall i_1, \ldots, i_n)(f((i_1, \ldots, i_n)) = i_j),$$

where $n \geq j \geq 1$ (note that when $n = 1$, we get the identity function);

2. The axiom for the successor function:

$$(\exists f)(\forall i)(f(i) = Si),$$

where "S" is the successor function symbol of PA;

3. The composition axiom scheme:

$$(\forall G, g_1, \ldots, g_n)(\exists F)(\forall i)(F((i)) = G((g_1((i)), \ldots, g_n((i))))),$$

where $\bar{i}$ is a block of $m$-many numerical quantifiers and $m, n \geq 1$;

4. Primitive recursion axiom scheme:

$$\forall G, H \exists F \forall i(F((0, i)) = H((i)) \land \forall nF((n + 1, i)) = G((n + 1, \bar{i}, F((n, \bar{i})))),$$

where $\bar{i}$ is a block of $n$ many numerical variables and $n = 1, 2, 3, \ldots$.

5. The induction axiom:

$$(\forall f)(f(0) = 1 \land (\forall i)(f(i) = 1 \rightarrow f(i + 1) = 1) \rightarrow (\forall i)(f(i) = 1)).$$

The theory of $RCA_0^*$ consists of the axioms of $PA^-$ plus
• Recursive comprehension axiom scheme: the universal closure of

\( (\forall i)(\exists j)\phi(i,j) \rightarrow (\exists f)(\forall i)\phi(i,f(i)) \),

where \( \phi \in \Sigma_1^0 \) and \( f \) does not occur free in \( \phi \);

• \( \Sigma_1^0 \)-induction scheme: the universal closure of

\( \phi(0) \land \forall i (\phi(i) \rightarrow \phi(i+1)) \rightarrow \forall i \phi(i) \)

where \( \phi \in \Sigma_1^0 \).

The theory of \( \text{ACA}_0^* \) consists of the axioms of \( \text{RCA}_0^* \) plus Arithmetic comprehension axiom scheme: the universal closure of

\( (\exists f)(\forall i)(f(i) = 1 \leftrightarrow \phi(i)) \),

where \( \phi \in \Sigma_\infty^0 \) and \( f \) does not occur free in \( \phi \).

For each natural number \( n \geq 1 \), the theory of (bold-face) \( (\Sigma_n^1)^*-CA_0 \) consists of the axioms of \( \text{RCA}_0^* \) plus the \( \Sigma_n^1 \)-Comprehension axiom scheme: the universal closure of

\( (\exists f)(\forall i)(f(i) = 1 \leftrightarrow \phi(i)) \),

where \( \phi \in S\Sigma_n^1 \) and \( f \) does not occur free in \( \phi \).

For each natural number \( n \geq 1 \), the theory of (bold-face) \( (\Delta_n^1)^*-CA_0 \) consists of the axioms of \( \text{RCA}_0^* \) plus the \( \Delta_n^1 \)-Comprehension axiom scheme: the universal closure of

\( (\forall i)(\exists j)\phi(i,j) \rightarrow (\exists f)(\forall i)(\phi(i,f(i))) \),
where $\phi \in S\Sigma^1_n$ and $f$ does not occur free in $\phi$.

Similarly, we may define $(\Pi^1_n)^*-CA_0$.

The theories $Prim^*-CA$, $RCA^*$, $ACA^*$, $(\Sigma^1_n)^*-CA$, $(\Delta^1_n)^*-CA$ and $(\Sigma^1_n)^*-CA$ are $(Prim^*)^*-CA_0$, $RCA^*$, $ACA^*$, $(\Sigma^1_n)^*-CA_0$, $(\Delta^1_n)^*-CA_0$ and $(\Sigma^1_n)^*-CA_0$ plus the full induction axiom scheme: the universal closure of

$$\phi(0) \land i(\phi(i) \to \phi(i + 1)) \to \forall i\phi(i)$$

where $\phi$ is any formula of $L^*_2$.

We define the so-called (light-face) $(\Pi^1_n)^*-CA$ ($(\Pi^1_n)^*-CA$ and $(\Sigma^1_n)^*-CA$) from the corresponding “bold-face” theories by simply requiring the formula $\phi$ in all the axioms to not involve any free second-order variable (but free numerical variables are allowed).

Given $f$ and $i$, the $i$-th component of $f$, $f_i$, is defined by

$$\forall k)(f_i(k) = f((i, k)))$$

The theory of $(\Sigma^1_n)^*-DC_0$ consists of the axioms of $ACA^*_0$ and the $\Sigma^1_n$ dependent choice scheme: the universal closure of

$$(\forall f)(\exists g)\phi(f, g) \to (\forall f)(\exists G)(G_0 = f \land (\forall i)\phi(G_i, G_{i+1})),$$

where $\phi$ is $S\Sigma^1_n$ and $G$ is not free in $\phi$.

And $(\Sigma^1_n)^*-DC$ is the theory $(\Sigma^1_n)^*-DC_0$ plus the full induction axiom schemes.

The theory of second-order arithmetic in $L^*_2$, $\mathbb{Z}^*_2$, and the theory of parameterless arithmetic in $L^*_2$, $p\mathbb{Z}^*_2$ are defined in a similar way as in $L_2$. 

The following are two well-known facts about the subsystems of $\mathbb{Z}_2$ and $\mathbb{Z}_2^*$. 

**Fact 1. (Normal Form Theorem).** If $\phi$ is an arithmetic formula of $L_2$, then there is an $S\Sigma_1^1$-formula $\psi$ with the same free variables as $\phi$ such that

$$(\text{ACA}_0) \vdash (\phi \leftrightarrow \psi);$$

and the same can be said about $L_2^*$ and $\text{ACA}_0^*$. 

Hence it follows that any $\Sigma^1_n$-formula is provably (in $\text{ACA}_0$) equivalent to a $S\Sigma^1_n$-formula.

**Fact 2.** Let $n \geq 0$, $T$ be among $\text{RCA}$, $\text{RCA}_0$, $\text{ACA}$, $\text{ACA}_0$, $\Sigma^1_n$-$\text{CA}$, $\Sigma^1_n$-$\text{CA}_0$, $\Delta^1_n$-$\text{CA}$ and $\Delta^1_n$-$\text{CA}_0$, and let $T^*$ be its corresponding theory in $L_2^*$. Then $T$ and $T^*$ are mutually interpretable. Similarly, if $T$ and $T^*$ is the corresponding light-face theory, then $T + \text{RCA}_0^*$ and $T^* + \text{RCA}_0^*$ are mutually interpretable.

For the remaining part of this chapter, we will work in $L_2$ exclusively. Of course, all the results we obtain may be established in $L_1$ as well, via “Fact 2”.

### 1.3 Coded structures and satisfaction predicates

The primary goal of this section is to formalize the concept of “satisfaction predicate” for a “coded structure” in $L_2$, and verify the fact that given a coded structure, its satisfaction predicate provably exists in $\Delta^1_1$-$\text{CA}$.

Informally speaking, given any set $X$, the “natural structure” (for $L_2$) coded by $X$ is

$$(\omega, \{X_i : i \in \omega\}, 0, S, +, x, <, \epsilon)$$

Unfortunately, this natural structure does not suit our needs. For our purpose, we
would like to think of $X$ as a set of triples of the form $([\psi], 0, 0)$, though $X$ may contain members of different form, where $[\psi]$ is the Gödel number of $\psi$ and $\psi$ is a sentence of $L_2$ (suppose a Gödel numbering for $L_2$ is given). If $\phi$ is a formula whose only free variable is "$m$" and is numerical, then $X_{[\phi]}$ denote the following set

$$X_{[\phi]} = \{ i \in \omega : ([\phi(m/k_i)], 0, 0) \in X \},$$

where $k_i$ represents the normal term $S \cdots S 0$.

Hence $X$ also codes the following structure

$$(\omega, \{ X_{[\phi]} : (\phi \in A) \}, 0, S, +, \times, <, \in)$$

where $A$ denotes the set of formulas whose only free variable is "$m$". Hence we may use those $[\phi]$ ($\phi \in A$) to represent set elements of the coded structure. It follows that we may use those non-negative integers $k$ satisfying: $(\forall i < ls(k))((k)_i = [\phi]$ for some $\phi \in A)$, to code set assignments. In other words, the $i$-th set variable is assigned with the set $X_{(k)_i}$. Similarly and more obviously, we may also use non-negative integers to code numerical assignment. Now, the satisfaction predicate for the structure coded by $X$, denoted by $\text{Sat}(X)$, consists of triples $([\theta], n, k)$ such that:

- $n$ codes a numerical assignment,
- $k$ codes a set assignment,
- every numerical variable in $\theta$ is assigned by $n$,
- every set variable in $\theta$ is assigned by $k$ and the coded structure satisfies the formula $\theta$ under the given assignment.

Now, we formalize the above notions in $\Delta_1^1$-CA (actually, most of which may be formalized in $\Sigma_0^0$-CA).

Let $m_0, m_1, \cdots$ list the numerical variables of $L_2$, and $x_1, x_2, \cdots$ list the set variables
of $L_2$. When there seems to be no confusion, we also use $i, j, k, l, s, t \cdots$ to denote numerical variables and $U, V, W, X, Y, Z, u, v, w, x, y, z, \cdots$ to denote set variables.

Fix a natural arithmetization of $L_2$. We use $[u]$ to denote the Gödel number of $u$ for $u$ a formula or term.

We give a brief definition of some useful sets and functions.

**Useful primitive recursive sets:**

\[
\overline{\text{Term}} = \{[t] : t \text{ is a term}\};
\]
\[
\overline{\text{Forml}} = \{[\phi] : \phi \text{ is a formula}\};
\]
\[
\overline{\text{fr}^1} = \{(\phi, n) : n\text{-th numerical variable appears free in } \phi\};
\]
\[
\overline{\text{fr}^2} = \{(\phi, n) : n\text{-th set variable appears free in } \phi\};
\]
\[
\overline{\text{Forml}^0} = \{[\phi] : \phi \text{ has at most one free variable : } m_0 \text{ (numerical)}\};
\]
\[
\overline{\text{Stassgn}} = \{k : (\forall i < ls(k))(k_i \in \overline{\text{Forml}^0})\}; \text{ (This is for set assignment)}
\]
\[
\overline{\text{Forml}^2} = \{[\phi] : \phi \text{ has at most one free variable : } x_2 \text{ (set variable)}\};
\]
\[
\overline{\Sigma}_n^0 = \{[\phi] : \phi \text{ is a } \Sigma_n^0 \text{ formula}\};
\]
\[
\overline{\Pi}_n^0 = \{[\phi] : \phi \text{ is a } \Pi_n^0 \text{ formula}\};
\]

**Useful primitive recursive functions:**

\[
|\phi| = \text{number of symbols in } \phi;
\]
\[
\overline{\text{Num}}(n) = [S \cdots S0];
\]
\[
\overline{\text{Sub}}([\phi], [m], [s]) = [\phi(m/s)];
\]
\[
\overline{\text{Sub}^0}([\phi], [s]) = \overline{\text{Sub}}([\phi], [m_0], [s]);
\]
\[\overline{\text{val}}([t], n) = [s] \text{ iff } [s] \text{ is the Gödel number of the term obtained from } t \text{ by replacing the } i\text{-th } (i < ls(n)) \text{ numerical variable by the term coded by } (n)_i.\]

Let Term, Frml, \(fr^1, fr^2, Frml_1^0, Frml_2^0, \Sigma_n^0, \Pi_n^0, \text{Stassgn}, \cdots,||,\text{Num, Sub, Sub}^0,\text{Val,} \cdots \text{be the corresponding symbols in } L_2 \text{ (actually in PA).}

In the remaining part of this chapter, unless otherwise specified, we will always assume we have a base theory at least as strong as \(\Sigma_0^0\)-CA.

**Lemma 1.3.1.** \(\Sigma_0^0\)-CA proves that

\[(\forall X)(\forall e \in Frml_0)(\exists! y)(\forall i)(i \in y \leftrightarrow (\text{Sub}^0(e, \text{Num}(i)), 0, 0) \in X)^3.\]

**Definition 1.** We use \(\Lambda(X, e) = y\) to denote the following formula,

\[(\forall i)(i \in y \leftrightarrow (\text{Sub}^0(e, \text{Num}(i)), 0, 0) \in X).\]

We will use \(x \in \text{set}^*(X)\) to abbreviate the formula:

\[(\exists e \in Frml_0)(\Lambda(X, e) = x).\]

Intuitively, for any \(X\), we think it as coding a structure for \(L_2:\)

\[(\omega, \{x : x \in \text{set}^*(X)\}, 0, S, +, \times, <, \in)\]

We next introduce the *satisfaction* predicate for this structure.

\(^3\)"\(x=y\)" is treated as an abbreviation for the formula \((\forall i)(i \in x \leftrightarrow i \in y)\). The uniqueness referred to in this lemma and all the other lemmas is subject to this interpretation of "\(=\)" among sets. Similar remark applies when "\(=\)" is used among functions.
Let Ext be a function defined as follows:

\[(\forall n, i, j, m), Ext(n, i, j) = m \leftrightarrow \]

\[ (ls(m) = \max\{ls(n), i\}) \land (\forall k < ls(m))(m)_k = \begin{cases} 
(n)_k & \text{if } k < ls(n) \land k \neq i \\
(j) & \text{if } k = i \\
0 & \text{if } ls(n) \leq k < i 
\end{cases} \]

Obviously "Ext" is primitive recursive, hence we may assume that Ext ∈ L(\(PA\)).

We will write \(n^i_j\) for Ext\((n,i,j)\).

In the definition 2, we will use primitive recursive function symbols (in \(PA\)):

\(F_\leq, F_\geq, F_\epsilon, F_\sigma, F_\forall, F_\exists\) and \(F_\exists^2\). Informally, they are defined as follows:

\[g_1(i) = [m_i] \text{ and } g_2(i) = [x_i];\]
\[F_\leq([s], [t]) = [s \leq t];\]
\[F_\geq([s], [t]) = [s = t];\]
\[F_\epsilon([s], g_2(i)) = [s \in x_i];\]
\[F_\sigma([\phi]) = [\neg \phi];\]
\[F_\forall([\phi], [\psi]) = [(\phi \land \psi)];\]
\[F_\exists(g_1(i), [\phi]) = [(\exists m_i)\phi];\]
\[F_\exists^2(g_2(i), [\phi]) = [(\exists x_i)\phi].\]

**Definition 2.** Let \(A^*((e, n, l), X, Y)\) be the disjunction of the following clauses:

1. \((\exists a, b \in Term)(e = F_\leq(a, b) \land val(a, n) \leq val(b, n));\)
2. \((\exists a, b \in Term)(e = F_\geq(a, b) \land val(a, n) = val(b, n));\)
3. \((\exists a \in Term)(\exists i)(e = F_\epsilon(a, g_2(i)) \land val(a, n) \in \Lambda(X, (l)_i)).\)
4. \((\exists e' \in Frlm)(e = F_\text{e}(e') \land (e', n, l) \not\in Y)\);

5. \((\exists e', e'' \in Frlm)(e = F_\text{e}(e', e'') \land (e', n, l) \in Y \land (e'', n, l) \in Y)\);

6. \((\exists i)(\exists e' \in Frlm)(e = F_3(g_i(i), e') \land (\exists j)((e', n, l) \in Y))\);

7. \((\exists i)(\exists e' \in Frlm)(e = F_3(g_2(i), e') \land (\exists j \in Stassgn)((e', n, l) \in Y)).\)

We define \(A^*(k,(e, n, l), X, Y)\) as \(((|e| \leq k) \land A((e, n, l), X, Y)).\)

**Definition 3.** \(A(X, Y)\) is the conjunction of the following two formulas:

1. \((\forall e, n, l)((e, n, l) \in Y \equiv \forall j((e, j) \in fr^1 \rightarrow j < ls(n)) \land \forall j((e, j) \in fr^2 \rightarrow j < ls(l))) \land A^*((e, n, l), X, Y).\)

2. \((\forall i)(i \in Y \rightarrow (\exists e \in Frml)(\exists n)(\exists l \in Stassgn)(i = (e, n, l))).\)

We define \(A(k, X, Y)\) in exactly the same way except we use \(A^*(k, (e, n, l), X, Y)\) instead of \(A^*((e, n, l), X, Y).\)

**Remark:** Informally speaking, if \(e\) codes formula \(\phi\), \(n\) codes numerical assignment \(\alpha\), \(l\) codes set assignment \(\beta\) and if every variable in \(\phi\) are assigned, then

\[ A(X, Y) \rightarrow ((e, n, l) \in Y \leftrightarrow (set^*(X) \models \phi[\alpha, \beta])).\]

Therefore any \(Y\) such that \(A(X, Y)\) would be the satisfaction predicate for the structure

\[ \langle \omega, \{x : x \in set^*(X)\}, 0, S, +, \times, <, \in \rangle, \]

while any \(Y\) such that \(A(k, X, Y)\) would be the satisfaction predicate for the structure for formulas of length up to \(k\).
Lemma 1.3.2 \((\text{ACA}_0 + \Sigma^1_1\text{-IND})\). \(\forall X \forall k \exists! Y \ A(k, X, Y)\).

**Proof**: We only give a sketch of the proof here. Fix an arbitrary \(X\), we prove the existence of \(Y\) such that \(A(k, X, Y)\) by induction on \(k\). Note that \(A(0, X, \emptyset)\) and if \(A(0, X, Y)\) then \(Y = \emptyset\) are obvious. Suppose \(A(k, X, Y)\) for some \(Y\). We define \(Z\) by

\[
(m \in Z \iff (\exists e \in \text{Frml})(\exists n)(\exists l \in \text{Stassgn})(m = (e, n, l) \wedge A(k+1, (e, n, l), X, Y)))
\]

Then by using the definition, we can prove that if \(e \in \text{Frml}\) is such that \(|e| \leq k\) then

\[
(\forall n, l)((e, n, l) \in Z \iff (e, n, l) \in Y).
\]

From this it is easy to check that \(A(k+1, X, Z)\).

For the uniqueness, let us suppose that for some \(k\), \(Y\) and \(Z\), we have both \(A(k, X, Y)\) and \(A(k, X, Z)\). We conclude that \(Y\) must be \(Z\) by formalizing an induction argument on formulas (recall that \(e\) codes a formula) to verify the following statement:

\[
(\forall n, l)((e, n, l) \in Y \iff (e, n, l) \in Z).
\]

Lemma 1.3.3 \((\Delta^1_1\text{-CA}_0 + \Sigma^1_1\text{-IND})\). \(\forall X \exists! Z \ A(X, Z)\).

**Proof**: Let us define a set \(Z\) as follows:

\[
(\forall e, n, l)((e, n, l) \in Z \iff (\exists Y)(A(|e|, X, Y) \wedge (e, n, l) \in Y)).
\]

By the uniqueness in Lemma 1.3.2, we also have

\[
(\forall e, n, l)((\exists Y)(A(|e|, X, Y) \wedge (e, n, l) \in Y) \iff (\forall Y)(A(|e|, X, Y) \rightarrow ((e, n, l) \in Y))).
\]
Hence $Z$ exists by $\Delta^1_1$-CA. Similar to the proof of Lemma 1.3.2, one may check that $A(X,Z)$ and that $Z$ is unique.

**Remark.** We will write "$Y=\text{k-Sat}(X)$" for $A(k, X, Y)$ and "$Y=\text{Sat}(X)$" for $A(X, Y)$.

**Definition 4.** For each theory $T$ of $L_2$ such that the set of Gödel numbers for axioms of $T$ may be represented by a primitive recursive predicate symbol $A_T$ (of PA), we use "$\text{set}^*(X) \models T$" to denote the following formula:

$$(\exists Y)((Y = \text{Sat}(X)) \land (\forall e)(e \in A_T \rightarrow (e, 0, 0) \in Y))$$

Intuitively, $\text{set}^*(X) \models T$ if and only if the structure coded by $X$ satisfies the theory $T$.

Let $\Phi_n(e, i, j)$ be universal for all $S\Sigma^1_n(i, j)$-formulas with free variables as shown and let $\Phi_n(e, i, X)$ be universal for all $S\Sigma^1_n(i, X)$-formulas with free variables as shown.

**Definition 5.** For each $n \geq 1$, we use $x = \text{Def}_{\Phi_n(e, i, j)}$ to denote

$$(\forall i)(i \in x \leftrightarrow \Phi_n(e, i, j))$$

and call $x$ $\Sigma^1_n$-definable if $\exists e, j \ x = \text{Def}_{\Phi_n(e, i, j)}$;

we use $x = \text{Sngl}_{\Phi_n(e, i, X)}$ to denote

$$(\forall X)(X = x \leftrightarrow \Phi_n(e, i, X))$$

and we call $x$ a $\Sigma^1_n$-singleton if $\exists e, i \ (x = \text{Sngl}_{\Phi_n(e, i, X)})$.

**Definition 6.** We use $\text{Def}^2(n, x, X)$ to denote

$$\exists Y(Y = \text{Sat}(X) \land (\exists m \in (\text{Frlm}^0)_n)(x = \Lambda(Y, m)))$$
and $\text{sng}l^2(n, x, X)$ to denote
\[(\exists e \in \text{Frml}^0)(x = \Lambda(X, e) \land (\exists e' \in (\text{Frml}^2)_n)(\exists Y)(Y = \text{Sat}(X) \land ((e', 0, 0^2) \in Y)) \land (\forall f \in \text{Frml}^0)(((e', 0, 0^2) \in Y) \rightarrow \Lambda(X, e) = \Lambda(X, f))),(\]
where $(\text{Frml}^0)_n$ and $(\text{Frml}^2)_n$ represent the set of $S\Sigma^1_n$ formulas in $\text{Frml}^0$ and $\text{Frml}^2$ respectively.

**Remark.** For fixed $X$, we call those $x$ such that $\text{Def}^2(n, x, X)$ ($\text{sng}l^2(n, x, X)$) $\Sigma^1_n$-definable over $\text{set}^*(X)$ without set parameter ($\Sigma^1_n$-singleton over $\text{set}^*(X)$ without set parameters).

### 1.4 Hierarchy codes

For any $u$, we may informally think of $u$ as coding a relation $R_u$:

\[R_u = \{(i, j) : (i, j) \in u\}.\]

We say $u$ is a linear ordering (well-ordering) if and only if $R_u$ is a linear-ordering (well-ordering).

Definition 7 is a formalization of this notion. Let us use $i \in \text{dom}(u)$ to denote the formula $\langle i, i \rangle \in u$.

**Definition 7.**

1. "LO($u$)" denotes the formula
\[(\forall i, j, k \in \text{dom}(u))((\langle i, j \rangle \in u \lor \langle j, i \rangle \in u) \land ((\langle i, j \rangle \in u \land \langle j, k \rangle \in u) \rightarrow \langle i, k \rangle \in u) \land ((\langle i, j \rangle \in u \land \langle j, i \rangle \in u \rightarrow i = j));\]
2. "WO(u)" denotes

\[ LO(u) \land (\forall x)(x \subseteq \text{dom}(u) \land u \neq \emptyset \rightarrow \exists i \in x \forall j \in x((i, j) \in u)). \]

Those u's such that LO(u) (WO(u)) are called linear orderings (well-orderings).

If u is a linear ordering, we define its limit points (denoted by \( \text{lim}(u) \)) in the usual way. For \( i \in \text{dom}(u) \), we usually use \( i +_u 1 \) to denote the immediate successor of \( i \), \( i +_u 2 \) to denote the immediate successor of the immediate successor of \( i \) if they exist.

3. "\( x \in \text{set}(i, u, X) \)" denotes

\[
[i \notin \text{lim}(u) \land x \in \text{set}^*(X_i)] \lor [i \in \text{lim}(u) \land \exists j <_u i \ (x \in \text{set}^*(X_{j+u2}) \lor ((\forall j \in \text{dom}(u))((i \leq_u j) \land \forall j_1, j_2 <_u i(j_1 - j = j_2 + u2)) \land x = \emptyset)]; \text{ in another words, set(i, u, X) = set}^*(X_i) \text{ if i is not a limit of u and when i is a limit}
\]

\[
\text{set}(i, u, X) = \bigcup_{j <_u i} \text{set}^*(X_{j+u2}),
\]

if there is some \( j' <_u i \) such that \( j' +_u 2 \) exists, otherwise

\[
\text{set}(i, u, X) = \{ \emptyset \}.
\]

**Remark 1**: Some notations we will use for the rest of the paper: Given \( u, X \) such that \( u \in LO \), we may think of the pair \(< u, X >\) as coding a hierarchy \( \{ \text{set}(i, u, X) \}_{i \in \text{dom}(u)} \). Since it will be clear from the context whether \( i \) is a limit or a successor of \( u \), we will write \( x \in \text{set}(X_i) \) instead of \( x \in \text{set}(i, u, X) \) for simplicity. Finally, we will write \( x \in \overline{\text{set}}(X) \) to indicate

\[
(\exists i \in \text{dom}(u))(x \in \text{set}(X_i)).
\]
Remark 2: Since $\text{set}(X_i) = \text{set}^*(X_i)$ when $i$ is not a limit of $u$, the satisfaction and $k$-satisfaction predicates for the structure $\text{set}(X_i)$ are $\text{Sat}(X_i)$ and $k\text{-Sat}(X_i)$ respectively. But when $i$ is a limit, $\text{set}(X_i)$ is not the same as $\text{set}^*(X_i)$, actually they have nothing to do with each other. Hence, we must define the “satisfaction predicates” of the structure separately. We will give a brief informal discussion only since the formal treatment is similar to the case of $\text{set}^*(X)$.

Fix $i \in \text{lim}(u)$, if $\text{set}(X_i) = \{\emptyset\}$, we define $\text{Sat}(X_i) = \text{Sat}(\emptyset)$, where $\text{Sat}(\emptyset)$ is as defined in last section. Now suppose there is some $j <_u i$ such that $j = j' +_u 2$ for some $j' \in \text{dom}(u)$. Then evidently,

$$\text{set}(X_i) = \{\Lambda(X_j, e) : (e \in \text{Frlm}^0) \land (j <_u i) \land (\exists j')(j = j' +_u 2)\}.$$

Hence we may use “$l$” such that $\forall n < \text{ls}(l), (l)_n$ is a pair $(j, e)$ such that $(j <_u i) \land ((\exists j')(j = j' +_u 2))$ and $(e \in \text{Frlm}^0)$ to code set assignment to this structure. Then $\text{Sat}(X_i)$ consists of triples “$(m, n, l)$” such that $m$ codes a formula, $n$ codes a numerical assignment, $l$ codes a set assignment and every variable of the formula coded by $m$ is assigned, and the formula coded by $m$ is satisfied by the structure $\text{set}(X_i)$ under the assignment coded by $n$ and $l$. We may give a similar discussion about $k\text{-Sat}(X_i)$. The existence of these predicates can be shown in $\text{ACA}_0 + \Sigma^1_1$-$\text{IND}$ or $\Delta^1_1$-$\text{CA}_0 + \Sigma^1_1$-$\text{IND}$ (see Lemma 1.3.2 and 1.3.3). Again because the context will make it clear if $i$ is a limit or successor, we will continue to use $\text{Sat}(X_i)$ to denote the satisfaction predicate of $\text{set}(X_i)$. One important reason we did not introduce two notations for them is that the formula $Y = \text{Sat}(X_i)$ is arithmetical in both cases and that seems to be what we really care about. However one should always keep in mind
that the meaning of $\text{Sat}(X_i)$ depends on whether $i$ is a limit or a successor.

**Definition 8.** We use $Y_i = Y_v^i$ to abbreviate

$$(\forall k)(k \in Y_i \leftrightarrow ls(k) = 2 \land (k)_0 < v \land (k)_1 \in Y_{(k)_0}).$$

Intuitively, $Y_i = Y_v^i$ states that $Y_i$ is the union of those $\{k\} \times Y_k$ such that $k < v \cdot i$.

**Definition 9 (Hierarchy code).** We use $RA(v,Y)$ to denote the conjunction of the following clauses:

1. $v \in WO$;

2. $Y = \bigcup_{i \in \text{dom}(v)} \{i\} \times Y_i$ (i.e., $\forall k (k \in Y \rightarrow (k)_0 \in \text{dom}(v))$);

3. $\forall i \in \text{dom}(v)(\exists j (i = j + v \cdot 1 \land Y_i = \text{Sat}(Y_j)) \lor (i \in \text{lim}(v) \land Y_i = Y_v^j))$;

4. $(\forall i,j)(i < u \land j \rightarrow \text{set}(X_i) \subset_{\text{proper}} \text{set}(X_j))$.

Where $\text{set}(X_i) \subset_{\text{proper}} \text{set}(X_j)$ should be considered as an abbreviation of the obvious formula in $L_2$ which asserts that property.

When $RA(v,Y)$, we call $(v,Y)$ a hierarchy (code).

To see the motivation behind this definition, let us assume $v$ is $\omega$. Then it is easy to see that

$$\text{set}(Y_0) = \{\emptyset\},$$
and for any $n$,

$$\text{set}(Y_{n+1}) = \{x : x \in L_2\text{-definable (without set parameter) over set}(Y_n)\}.$$
Hence the hierarchy coded by \( Y \) is the same as the hierarchy obtained by iterating the "second-order-definable-without-set-parameter operation \( \Gamma \)" along \( \omega \) starting with \( \{ \emptyset \} \). However, we will prove that iterating \( \Gamma \) along a well-ordering actually results in the "Ramified Analytical Hierarchy" (see definition 14 in section 2.1) along that well-ordering.

**Lemma 1.4.1 (\( \Pi^1_1 \text{-CA} \))** \( \forall u (u \in WO \rightarrow \exists X \text{RA}(u, X)) \lor (\exists X)(X \models Z_2) \).

If \( \langle u, X \rangle \) is a hierarchy and \( k \in \text{dom}(u) \), \( \langle u|_{<k}, X|_{<k} \rangle \), is the portion of \( \langle u, X \rangle \) **below** stage \( k \) and \( \langle u|_k, X|^k \rangle \) is the portion of \( \langle u, X \rangle \) **below and including** stage \( k \)

**Definition 10.** We define two formulas: \( \text{Equiv}(u, X, v, Y) \) and \( \text{Comp}(u, X, v, Y) \) as follows:

1. \( \text{Equiv}(u, X, v, Y) \) is the **conjunction** of the following two clauses:
   
   \begin{itemize}
   \item \( \forall i \in \text{dom}(u) \)(\( \exists j \in \text{dom}(v) \)(\text{set}(X_i) = \text{set}(Y_j)));
   \item \( \forall j \in \text{dom}(v) \)(\( \exists i \in \text{dom}(u) \)(\text{set}(X_i) = \text{set}(Y_j)));
   \end{itemize}

2. \( \text{Comp}(u, X, v, Y) \) is the **disjunction** of the following three clauses:

   \begin{itemize}
   \item \( \exists k \in \text{dom}(v) \)(\text{Equiv}(u, X, v|_k, Y|^k) \lor \text{Equiv}(u, X, v|_{<k}, Y|^k));
   \item \( \exists k \in \text{dom}(u) \)(\text{Equiv}(v, Y, u|_k, X|_k) \lor \text{Equiv}(v, Y, u|_{<k}, X|_{<k}));
   \item \text{Equiv}(u, X, v, Y).
   \end{itemize}

\(^4\text{this is actually provable in a system called } ATR_0, \text{ which is weaker than } \Pi^1_1 \text{-CA. See } [2] \text{ for definition.}\)
**Remark.** We will write \( (u, X) \equiv (v, Y) \) when "\( \text{Equiv}(u, X, v, Y) \)"', \( (u, X) \leq (v, Y) \) when \( (\exists k \in \text{dom}(v))(\langle u, X \rangle \equiv \langle v, Y \rangle \upharpoonright_k \forall (u, X) \equiv \langle v\upharpoonright_{<k}, Y\upharpoonright_{<k} \rangle) \), and \( (u, X) < (v, Y) \) if \( (u, X) \leq (v, Y) \land (u, X) \neq (v, Y) \).

**Lemma 1.4.2 (ACA, Comparison Lemma).** If RA(u, X), RA(v, Y) then Comp(u, X, v, Y), moreover, there is a function \( f \) (i.e, its graph exists) which is either an isomorphic embedding of \( u \) onto an initial segment of \( v \), or an isomorphic embedding of \( v \) onto an initial segment of \( u \), or an isomorphism between \( u \) and \( v \).

**Proof:** We give an informal proof of this Lemma.

Let

\[ F = \{(i, j) : i \in \text{dom}(u) \land j \in \text{dom}(v) \land \text{set}(X_i) = \text{set}(Y_j)\} \]

**Claim 1:** \((\forall i)(\exists (at most one) j)(<, i, j) \in F)\).  
Suppose \( \langle i, j_1 \rangle \in F \) and \( \langle i, j_2 \rangle \in F \) and \( j_1 <_v j_2 \). Then \( \text{set}(X_i) = \text{set}(Y_{j_1}) \subset \text{proper} \text{set}(Y_{j_2}) = \text{set}(X_i) \), which is a contradiction.

**Claim 2:** \((\forall j)(\exists (at most one) i)(<, i, j) \in F)\).  
Same as Claim 1.

**Claim 3:** \((\forall \langle i_1, j_1 \rangle, \langle i_2, j_2 \rangle \in F)(i_1 <_u i_2 \leftrightarrow j_1 <_v j_2)\).  
Suppose \( i_1 <_u i_2 \), then \( \text{set}(X_{i_1}) \subset \text{proper} \text{set}(X_{i_2}) \). It follows that \( \text{set}(Y_{j_1}) \subset \text{proper} \text{set}(Y_{j_2}) \), hence \( j_2 \not\in_v j_1 \), namely, \( j_1 <_v j_2 \). The converse is similar.

Let

\[ A = \{i : (\forall j)(\langle i, j \rangle \not\in F)\}. \]

**Case 1:** \( A \neq \emptyset \).
Claim 4. If $\forall i < u \exists j < v j_0 ((i, j) \in F) \land \forall j < v \exists i < u i_0 ((i, j) \in F)$, then $(i_0, j_0) \in F$.

We first assume $i_0 \in \text{lim}(u)$. Then we clearly have $j_0 \in \text{lim}(v)$. It follows directly from definition that $\text{set}(X_{i_0}) = \text{set}(Y_{j_0})$; i.e., $(i_0, j_0) \in F$. Now if $i_0 = i_1 + u 1$ for some $i_1$, then $j_0 = j_1 + v 1$ for some $j_1$. It follows easily from our assumption that $\text{set}(X_{i_1}) = \text{set}(Y_{j_1})$. Hence again $\text{set}(X_{i_0}) = \text{set}(Y_{j_0})$; i.e., $(i_0, j_0) \in F$.

Let $i_0$ be the $u$-minimum of $A$, and

$$B = \{ j : (\exists i < u i_0)((i, j) \in F) \}.$$  

Claim 5 : $B = \text{dom}(v)$.

Otherwise, let $j_0$ be the $v$-least not in $B$. Hence for all $j < v j_0$, there is an $i < u i_0$ such that $(i, j) \in F$. Suppose that for all $i < u i_0$, there is an $j < u j_0$ such that $(i, j) \in F$. Then by claim 4, $(i_0, j_0) \in F$, which is a contradiction. Therefore there is an $i < u i_0$ such that for some $j > v j_0$, we have $(i, j) \in F$. Let $i'_0$ be the $u$-least such $i$.

Subclaim : $(i'_0, j_0) \in F$.

By claim 4, it suffices to note that $\forall j < v j_0 \exists i < u i'_0$ such that $(i, j) \in F$. Since otherwise, there will be some $j_1 < v j_0$ and $i'_0 < u i_1 < u i_0$ such that $\text{set}(X_{i_1}) = \text{set}(Y_{j_1})$. By the choice of $i'_0$, there is an $i'_0 \leq u i < u i_1$ and $j \geq v j_0$ such that $(i, j) \in F$, which contradicts the monotonicity of Claim 3.

But because of the choice of $j_0$, the Subclaim is impossible. Hence we must have $B = \text{dom}(v)$. It follows that the inverse of $F$ is an isomorphic embedding from $v$ into $u$.

Case 2 : $A = \emptyset$. 


As in Case 1, we may prove that $F$ is an isomorphic embedding from $u$ into $v$.

Finally, Lemma 1.4.2 follows from Claim 1 to 5.

**Lemma 1.4.3 (ACA).** For any $u, v, X, Y$ and $f$ such that $RA(u, X), RA(v, Y)$ and $f$ an embedding of $u$ onto an initial segment of $v$ or onto $v$ itself, and for any $i$ in $\text{dom}(u)$, we have

1. if $i + u_2$ exists then $X_{i+u_2} = Y_{f(i)+v_2}$;

2. $\text{set}(X_i) = \text{set}(Y_{f(i)})$.

**Proof:** We prove 1 and 2 by a transfinite induction on $i \in \text{dom}(u)$.

Suppose both are true for all $i' <_u i$.

**Case 1.** $i = i_0 + u_1$ for some $i_0$.

Since $\text{set}(X_{i_0}) = \text{set}(Y_{f(i_0)})$, it is easy to see that

$$(\forall e)(\langle e, 0, 0 \rangle \in \text{Sat}(X_{i_0}) = X_i \leftrightarrow \langle e, 0, 0 \rangle \in \text{Sat}(Y_{f(i_0)}) = Y_{f(i)}).$$

Hence by definition $\text{set}(X_i) = \text{set}(Y_{f(i)})$. Also

$$X_{i+u_2} = \text{Sat}(X_{i_0+u_2}) = \text{Sat}(Y_{f(i_0)+v_2}) = Y_{f(i)+v_2}.$$

**Case 2.** $i \in \text{lim}(u)$.

By induction hypothesis, $\text{set}(X_i) = \text{set}(Y_{f(i)})$, hence

$$(\forall e)(\langle e, 0, 0 \rangle \in X_{i+u_1} = \text{Sat}(X_i) \leftrightarrow \langle e, 0, 0 \rangle \in Y_{f(i)+v_1} = \text{Sat}(Y_{f(i_0)})).$$

Then it is easy to see from the definition of the “satisfaction predicate” (when $i \not\in \text{lim}(u)$) that

$$X_{i+u_2} = \text{Sat}(X_{i+u_1}) = \text{Sat}(Y_{f(i)+v_1}) = Y_{f(i)+v_2}.$$
If \( RA(u, X) \), \( x \in \overline{set}(X) \) and \( x = L(X, e) \), we will call \( \langle i, e \rangle \) a \( u \)-representative of \( x \) in \( \langle u, X \rangle \). We will write \( \langle i, e \rangle_x \) to indicate that
\[
x = L(X, e) \land \forall \langle i', e' \rangle \ (x = L(X, e') \rightarrow i <_u i' \lor i = i' \land e < e').
\]
If \( \langle i, e \rangle_x \) we call \( \langle i, e \rangle \) the \( u \)-least representative of \( x \) in \( \langle u, X \rangle \). It is called the \( u \)-least because it is the least such pair in the "lexicographical" ordering on \( dom(u) \times \omega \). Note that if \( \langle i, e \rangle_x \), \( i \) must be a successor.

Now we define an ordering \( <_u \) on \( \overline{set}(X) \).

**Definition 13.** We write \( x <_u y \) for
\[
RA(u, X) \land (x, y \in \overline{set}(X)) \land (\exists i_1, i_2, e_1, e_2)(\langle i_1, e_1 \rangle_x <_u \langle i_2, e_2 \rangle_y).
\]

**Lemma 1.4.4 (ACA).** If \( RA(u, X) \), then \( \overline{set}(X) \) is well-ordered by \( <_u \); i.e,
\[
(\forall y)((\forall i)(y_i \in \overline{set}(X)) \rightarrow (\exists i)(\forall j)(y_i <_u y_j \lor y_i = y_j)).
\]

**Proof:** Fix \( y \) such that \( (\forall i)(y_i \in \overline{set}(X)) \). Let
\[
A = \{ i \in dom(u) : (\exists j)(y_j \in set(X_i)) \}.
\]
Let \( i_0 \) be the \( u \)-least of \( A \). Let
\[
B = \{ e \in Frm^0 : (\exists j)(y_j = L(X, e)) \}.
\]
And let \( e_0 \) be the least element of \( B \) and \( j_0 \) be such that \( y_{j_0} = L(X, e_0) \). It is straightforward to check that \( y_{j_0} \) is the \( u \)-least element of \( \{ y_i \}_{i \in \omega} \).
Corollary (ACA). If \( RA(u, X), RA(v, Y), u, v \in WO, x, y \in \text{set}(X) \cap \text{set}(Y) \) then

\[ x <_u y \text{ if and only if } x <_v y. \]

where \( x, y \in \text{set}(X) \cap \text{set}(Y) \) is an abbreviation for \( x \in \text{set}(X) \land x \in \text{set}(Y) \).

Proof: By comparability, we may assume that \( (u, X) \leq (v, Y) \). Let \( \langle i_0, e_0 \rangle_x \) and \( \langle j_0, f_0 \rangle_x \) be the \( u \)-least and the \( v \)-least representatives of \( x \) respectively, and similarly choose \( \langle i_1, e_1 \rangle_y \) and \( \langle j_1, f_1 \rangle_y \).

Suppose that we have

\[ \langle i_0, e_0 \rangle_x <_u \langle i_1, e_1 \rangle_y \text{ and } \langle j_1, f_1 \rangle_y <_v \langle j_0, f_0 \rangle_x. \]

Note that \( i_0, i_1, j_0, j_1 \) are all successors.

Case 1. \( i_0 = i_1 \). By comparability, there exists \( j^* \) such that \( \text{set}(X_{i_0}) = \text{set}(Y_{j^*}) \).

By the choice of \( j_0, j_1 \), we must have that \( j_0, j_1 \leq_v j^* \). Hence, there are \( i_0^*, i_1^* \leq_u i_0 \) such that \( \text{set}(X_{i_0^*}) = \text{set}(Y_{j_0}) \) and \( \text{set}(X_{i_1^*}) = \text{set}(Y_{j_1}) \). But by the choice of \( i_0 \), we have \( i_0^* \geq i_0 \), Therefore, \( i_0 = i_0^* \). Similarly, \( i_0 = i_1^* \). It follows that \( \text{set}(X_{i_0}) = \text{set}(Y_{j_0}) = \text{set}(Y_{j_1}) \). Note that \( i_0, j_0 \) and \( j_1 \) all are successors. By the definition of \( <_u \) and \( <_v \) (at successor stages), we must have \( e_0 < e_1 \leftrightarrow f_0 < f_1 \).

Case 2. \( i_0 \neq i_1 \). The argument is similar.

Lemma 1.4.5 (ACA, the first Glue-up lemma). If for all \( i \), \( RA((C_i)_0, (C_i)_1) \) and \( (C_i)_0 \in WO \), then there is a \( (u, X) \) such that \( RA(u, X), u \in WO, \) and for all \( i \), \( ((C_i)_0, (C_i)_1) \leq (u, X) \).

Proof: Let us write \( u^i \) and \( X^i \) for \( (C_i)_0 \) and \( (C_i)_1 \) respectively.
Define:

\[ A = \{(i, j) : j \in \text{dom}(u_i)\}. \]

\[ A^* = \{(i, j) \in A : \forall (i', j') \in A (\text{set}((X^i)_j) = \text{set}((X^{i'})_{j'}) \rightarrow i \leq i')\}. \]

\[ u = \{\{(i, j), (i', j')\} \in (A^*)^2 : \text{set}((X^i)_j) \subseteq \text{set}((X^{i'})_{j'})\}. \]

Then \( A, A^* \) and \( u \) exist by ACA.

**Claim.** \( u \) is a well-ordering whose order type is the union of \( \{\text{ot}(u_i)\}_{i \in \omega} \).

**Proof of the claim:** Let \( a \subseteq A^* \).

Fix \( (i, j) \in a \), let \( b = \{j \in \text{dom}(u_i) : \exists (i', j') \in a (\text{set}((X^i)_j) = \text{set}((X^{i'})_{j'}))\}. \)

Note that \( j \in b \neq \emptyset \). Let \( j_0 \) be the \( u_i \)-least element of \( b \). Let \( (i', j') \in a \) be such that \( \text{set}((X^{i'})_{j'}) = \text{set}((X^i)_{j_0}) \).

We may easily conclude that \( (i', j') \) is the \( u \)-least element of \( a \).

Given an \( i \), to see that \( u^i \leq u \), consider

\[ F = \{(j, (i', j')) : (i', j') \in A \land \text{set}((X^i)_j) = \text{set}((X^{i'})_{j'})\}. \]

\( F \) is an isomorphic embedding of \( u^i \) into \( u \).

On the other hand, it is easy to see that if \( (i, j) \in \text{dom}(u) \), then the initial segment of \( u \) below \( (i, j) \) can be embedded into \( u^i \).

Similarly, we may define \( X \) from \( \{X^i\}_{i \in \omega} \), such that \( RA(u, X) \).

Intuitively, \( n \in X \) if and only if it has the form \( (\langle i, j \rangle, k) \) for some \( k \) and \( (i, j) \in A^* \), and it satisfies the following conditions depending on which case it falls in:

**Case 1.** If for some \( j' \in \text{dom}(u^i), j = j' +_u 2 \), then \( k \in (X^i)_j \).
Case 2. If for some $j' \in \lim(u^i)$, $j = j' + u^i 1$, there must be some $e \in Frml,m$ and $l$ such that $k = \langle e, m, l \rangle$ and for any $f < ls(l)$ $(l)_f = \langle \langle i_f, j_f \rangle, m_f \rangle$ for some $m_f \in Frml^0$ and $\langle i_f, j_f \rangle \in dom(u)$ such that $(i_f, j_f) <_u (i, j)$ and $(\exists j')(j_f = j' + 1)$. Moreover if for any $f < ls(l)$, $j'_f$ is such that $\text{set}(X^{i^j}_{j'_f}) = \text{set}(X^{i^j}_{j'_f})$ and "l" is such that $ls(l') = ls(l)$ and for any $f < ls(l')$, $(l')_f = \langle j'_f, m_f \rangle$, then $(e, m, l') \in (X^i)_j$.

Case 3. If $j \in \text{limit}(u^i)$, then $k = \langle \langle i', j' \rangle, m \rangle$ for some $m$ and $\langle i', j' \rangle$ such that $\langle i', j' \rangle = \langle i'', j'' \rangle + u^2$ for some $\langle i'', j'' \rangle \in dom(u)$ and $m \in (X^i)_j$.

One may check straightforwardly that the above informal definition of $X$ can be translated into a formal definition of $X$ by an arithmetical formula and $RA(u, X)$.
CHAPTER II

The Ramified Analytical Hierarchy

2.1 R.A. Hierarchy in ZFC

The following is a formal definition of the Ramified Analytical Hierarchy in set theory.

Definition 14. For each ordinal $\alpha$, we have a set $A_\alpha$ defined by transfinite induction on $\alpha$:

$A_0 = \{0\}$;

$A_{\alpha+1} = \{x \subseteq \omega : x \text{ is definable over } A_\alpha\}$;

where $x$ is definable over $A_\alpha$ means $x$ is definable over the model

$A_\alpha = <\omega, A_\alpha, 0, 1, +, \cdot, \epsilon, a >_{a \in A_\alpha}$

in the language $L_2$.

$A_\lambda = \cup_{\alpha < \lambda} A_\alpha$ if $\lambda$ is a limit ordinal.

We sometimes also call this R.A. Hierarchy.

Theorem 2.1.1 (ZFC). For any ordinal $\alpha$, if $A_\alpha \neq A_{\alpha+1}$ then:

1. there is a well-ordering $<_\alpha$ of $A_\alpha$ which is definable over $A_\alpha$ itself;
2. there is an integer $n$ such that for any $x \in A_\alpha$, $x$ is a $\Sigma^1_n$-definable singleton over $A_\alpha$.

In the above, "$x$ is a $\Sigma^1_n$-definable singleton over $A_\alpha$" means that $A_\alpha \models (\exists e)(\forall y)(y = x \iff \Phi_n(e, y))$, where $\Phi_n(e, y)$ is universal for all $\Sigma^1_n(y)$-formulas whose only free variable is $y$.

To prove this by induction, we need a somewhat stronger induction hypothesis: if $A_\alpha \neq A_{\alpha+1}$ then the following are true:

1. there is a well-ordering $<_\alpha$ of $A_\alpha$ which is definable over $A_\alpha$ itself;
2. there is an integer $n$ such that $\forall x \in A_\alpha$, $x$ is a $\Sigma^1_n$-definable singleton over $A_\alpha$;
3. there is an element $(v, Y) \in A_{\alpha+1}$ such that $v \in WO$, $RA(v, Y)$ and $ot(v) = \alpha + 1$.

Let us name the above induction hypothesis by $(\#)$.

Lemma 2.1.1 (ZFC). If the induction hypothesis, $(\#)$, is true for ordinals less than $\alpha$ and $(u, X)$ is such that $RA(u, X)$ and $ot(u) = \gamma$ is no bigger than $\alpha$, then $set(X) = A_{\gamma-1}$, where $\gamma-1$ is $\gamma$ if $\gamma$ is a limit, is $\gamma - 1$ if otherwise.

Proof: Let $f$ be an isomorphism between $u$ and $\gamma$. Then it is easy to prove by induction on $i \in dom(u)$ that

$$set(X_i) = A_{f(i)},$$

by taking into account of the fact that for some $n$, every $x \in A_{f(i)}$ is $\Sigma^1_n$-definable over $A_{f(i)}$. 

Lemma 2.1.2 (ZFC). If $A_\alpha \neq A_{\alpha+1}$ and the induction hypothesis, (#), is true for ordinals less than $\alpha$, then for all $\langle v, Y \rangle \in A_\alpha$, if $(RA(v, Y))^A_\alpha$ then $RA(v, Y)$; i.e., being a hierarchy is absolute with respect to $A_\alpha$.

Proof: Case 1. $\alpha$ is a limit ordinal. Hence $\langle v, Y \rangle \in A_\beta$ for some $\beta < \alpha$. By induction hypothesis, there is a $\langle u, X \rangle \in A_{\beta+1} \subset A_\alpha$ such that $RA(u, X)$ and $\overline{set}(X) = A_\beta$. Since $A_\alpha$ satisfies ACA, by the Comparison Lemma 1.4.2, $\langle v, Y \rangle$ and $\langle u, X \rangle$ are comparable in $A_\alpha$. If $\langle v, Y \rangle$ is taller than $\langle u, X \rangle$, then there is an $i \in \text{dom}(v)$ such that $Y_i \in A_\beta$ and $set(Y_i) = A_\beta$. Since every element of $A_{\beta+1}$ is definable over $A_\beta$, it follows that every element of $A_{\beta+1}$ is arithmetical in $Y_i$. Applying ACA in $A_\beta$, it follows that $A_\beta = A_{\beta+1}$, which is a contradiction.

If $\langle v, Y \rangle$ is no taller than $\langle u, X \rangle$ then

$$(\exists f)(f \text{ embeds } v \text{ onto an initial segment of } u \text{ or } u \text{ itself}).$$

Since $u \in WO$, it follows that $v$ must also be a well-ordering.

Case 2. $\alpha = \gamma + 1$.

Fix $\langle u, X \rangle \in A_\alpha$ such that

$$A_\alpha \models RA(u, X).$$

By induction hypothesis, there is a $\langle v, Y \rangle \in A_\alpha$ such that $RA(v, Y)$ and $\overline{set}(Y) = A_\gamma$. Applying ACA in $A_\alpha$, we may compare $\langle u, X \rangle$ with $\langle v, Y \rangle$.

Subcase 1: $\langle u, X \rangle > \langle v, Y \rangle$.

Then there is some $i \in \text{dom}(u)$ such that $A_\gamma = set(X_i)$. First we observe that we may assume $i$ is not the top of $u$, since otherwise, being only one level higher than a
well-ordering $v$, $u$ must also be a well-ordering, hence we have nothing more to prove. Thus $i+u1$ exists and by induction hypothesis $A_\alpha = \text{set}(X_{i+u1})$. On the other hand, $X_{i+u1} \in A_\alpha$. It follows that every element of $A_{\alpha+1}$, being arithmetical in $X_{i+u1}$, must be in $A_\alpha$ since $A_\alpha$ satisfies ACA, which is impossible by assumption.

subcase 2 : $(u, X) \leq (v, Y)$.

In this case, we may easily conclude that $u$ is a well-ordering.

Lemma 2.1.3 (ZFC). If $A_\alpha \neq A_{\alpha+1}$ and the induction hypothesis, (#), is true for ordinals less than $\alpha$, then there is a well-ordering $<_\alpha$ of $A_\alpha$, which is definable over $A_\alpha$, and for some $n$, every element of $A_\alpha$ is a $\Sigma^1_n$-singleton over $A_\alpha$.

Proof: Case 1. $\alpha$ is a limit.

By the above two Lemmas, we know

$$A_\alpha \models \forall x \exists u, X(RA(u, X) \land x \in \text{set}(X)).$$

We may then define the well-ordering $<_\alpha$ (in $A_\alpha$) as follows

$$x <_\alpha y \text{ iff } \exists v, Y(RA(v, Y) \land (x, y \in \text{set}(Y)) \land x <_v y).$$

$<_\alpha$ is obviously $\Sigma^1_n$-definable over $A_\alpha$. The fact $<_\alpha$ is indeed a well-ordering follows from Lemma 1.4.4 and its Corollary.

Since $A_\alpha \neq A_{\alpha+1}$, there must be an $m$ and some $\Sigma^1_m$-formula $\psi(t, a)$ with parameter $a$ from $A_\alpha$ and $t$ being its only free variable such that

$$b = \{k : A_\alpha \models \psi(k, a)\}$$
is not in $A_\alpha$. In another word,

\[(*) \quad A_\alpha \models \forall y (y \neq \{ k : \psi(k, a) \}),\]

where $y \neq \{ k : \psi(k, a) \}$ is an abbreviation for the formula

\[(\exists k)((k \in y \land \neg \psi(k, a)) \lor (k \not\in y \land \psi(k, a))).\]

Let "$a$" be the $<_\alpha$-least element of $A_\alpha$ such that above is true; i.e,

\[A_\alpha \models \forall y (y \neq \{ k : \psi(k, a) \}) \land \forall x <_\alpha a \exists y (y = \{ k : \psi(k, a) \}).\]

Then $a$ is a $\Sigma^1_{m+3}$-definable singleton of $A_\alpha$. Hence there is a parameterless $\Sigma^1_{m+3}$-formula $\psi(t)$ such that

\[(**) A_\alpha \models \forall y (y \neq \{ k : \psi(k) \}).\]

Let $l \geq 2$ be an integer such that there is some $a$ and $\psi \in \Sigma^1_1$ such that (**) is true and let $n = l + 1$.

Next we do a Skolem hull argument. Let $\theta_1$ denote the formula

\[\forall x \exists (v, Y)(RA(v, Y) \land x \in \text{set}(Y))\]

and $\theta_2$ be a $\Sigma^1_1$-formula asserting $\text{ACA}_0$. ($\text{ACA}_0$ can be axiomatized by a $\Sigma^1_1$-formula. See Simpson [2] for more information. And also note that $RA(v, X)$ includes the condition $v \in WO$.)

Now, let us work outside of $A_\alpha$. Let $\{ \psi_k(x, y) \}_{k \in K}$ list all the proper subformulas of $\varphi(t)$, $\theta_1$ and $\theta_2$. For each $k \in K$, suppose $\psi_k$ has $m_k$ many free variable, we define
an $m_k$-ary function $f_k$ from $A_\alpha$ union $\omega$ to $A_\alpha$ by letting $f_k(\bar{a}) = \text{the } \prec_\alpha\text{-least } b \text{ in } A_\alpha$ such that $A_\alpha \models \psi_k(\bar{a}, b)$ if such $b$ exists, otherwise $f_k(\bar{a}) = \emptyset$. We define a sequence \( \{H_k\}_{k \in \omega} \) of subsets of $A_\alpha$ as follows:

\[
H_0 = \{\emptyset\};
\]

\[
H_{n+1} = \text{the image of } H_n \text{ under } \{f_k\}_{k \in K} \cup H_n.
\]

Finally, we let $H = \bigcup_{k \in \omega} H_k$.

**Claim.** Every element of $H$ is a $\Sigma^1_n$-singleton of $A_\alpha$.

We actually prove that all the $H_n$’s have this property by induction on $n$. Suppose $H_n$ has the property. For any $b \in H_{n+1}$, by our construction, there is some $k$ and $a \in H_n$ such that $f_k(a) = b$ (where we assume that $f_k$ is unary, but the argument is general). Then the following statement is true in $A_\alpha$:

\[
\forall y (y = b \leftrightarrow \psi_k(a, y) \land \forall z <_\alpha y \land \neg \psi_k(a, z)).
\]

This shows that $b$ is a $\Sigma^1_n$-singleton of $A_\alpha$.

According to our construction, $H$ is a $\beta$-submodel\footnote{If $M$ and $N$ are two structures of second-order arithmetic, we say $M$ is a $\beta$-submodel of $N$ if $M$ is a substructure of $N$ with the same first order part as $N$ and any $\Sigma^1_1$-sentence with parameters from $M$ true in $N$ is also true in $M.$} of $A_\alpha$ which satisfies

\[
\forall x (\exists v, Y)(RA(v, Y) \land x \in \text{set}(Y)).
\]

Let

\[
\gamma = \sup\{ot(v) : H \models (\exists Y)RA(v, Y)\}.
\]
We may easily show that $H = A_\gamma$. Also by our construction,

$$b = \{ k : A_\alpha \models \varphi(k) \} = \{ k : A_\gamma \models \varphi(k) \}$$

Therefore, $b$ must be in $set(A_{\gamma+1})$. It follows that $\alpha = \gamma$; i.e., $H = A_\alpha$.

**Case 2.** $\alpha = \gamma + 1$.

Note that, in the proof of Lemma 2.1.2, we have actually verified that if $(v, Y) \in A_\alpha$ is such that $set(Y_i) = A_\gamma$ for some $i \in dom(v)$ then $i$ must be the top of $v$ and $(v, Y)$ is no shorter than any other hierarchy in $A_\alpha$. As a matter of fact, any hierarchy in $A_\alpha$, which has a top and is no shorter than any other hierarchy, must code $A_\gamma$ (actually its top level alone already codes $A_\gamma$). This makes it possible to define $<_\alpha$ over $A_\alpha$. For $x, y \in A_\alpha$, we define $x <_\alpha y$ by cases:

1. if $(\exists u, X)(x \in \overline{\text{set}}(X) \land y \notin \overline{\text{set}}(X))$, then $x <_\alpha y$;

2. if there is a hierarchy $(u, X)$ such that both $x$ and $y$ are in $\overline{\text{set}}(X)$, and any hierarchy shorter than $(u, X)$ contains neither $x$ nor $y$, then $x <_\alpha y$ if and only if $x <_u y$;

3. neither $x$ nor $y$ is coded in any hierarchy. In this case, we fix a hierarchy $(u, X)$ no shorter than any other hierarchy. Note that $u$ must have a top in this case. Then $x$ and $y$ are definable over $\overline{\text{set}}(X_i)$ without set parameter by induction hypothesis (hence the definition doesn't depend on the choice of $(u, X)$), where $i$ is the top of $u$. We define $x <_\alpha y$ if and only if the following is true $(\exists e \in Frml^0, Z)(Z = |e| \cdot \text{Sat}(X_i) \land x = \Lambda(Z, e) \land (\forall e' \in Frml^0, Z')(Z' = |e'| \cdot \text{Sat}(X_i) \land y = \Lambda(Z', e') \rightarrow e < e')$. 
It is easy to see that $<_\alpha$ is $\Delta^1_3$-definable over $A_\alpha$. The fact that $<_\alpha$ is indeed a well-ordering follows from Lemma 1.4.4 and its Corollary.

Now, Let $\langle u_0, X_0 \rangle$ be the $<_\alpha$-least hierarchy in $A_\alpha$ no shorter than any other hierarchy in $A_\alpha$. Then $\langle u_0, X_0 \rangle$ is a $\Sigma^1_4$-singleton over $A_\alpha$. Since all the elements of $A_\alpha$ are arithmetical in $\langle u_0, X_0 \rangle$, they are all $\Sigma^1_4$-singletons over $A_\alpha$.

**Lemma 2.1.4.** If $A_\alpha \neq A_{\alpha+1}$ and the induction hypothesis, (#), is true for ordinals less than $\alpha$, then there is a $\langle u, X \rangle \in A_{\alpha+1}$ such that $RA(u, X)$, $\langle u, X \rangle$ has a top, and $\text{set}(X) = A_\alpha$.

**Proof:** Case 1. $\alpha$ is a limit.

Let $\Psi^1_n(e, t)$ be universal for $\Sigma^1_n$-formulas with a numerical variable $t$ being the only free variable.

Define a set $B$ by

$$B = \{ \langle e, i \rangle : A_\alpha \models \Psi^1_n(e, i) \land (\exists v, Y)(\langle v, Y \rangle = \{ j : \Psi^1_n(e, j) \}) \land RA(v, Y) \}.$$ 

Then $B$ is $\Sigma^1_{n+2}$-definable over $A_\alpha$. Hence $B$ is in $A_{\alpha+1}$.

We obviously have the following:

**Claim.** $(\forall v, Y \in A_\alpha)([A_\alpha \models RA(v, Y)] \rightarrow (\exists e)(\langle v, Y \rangle \equiv (\langle B_e \rangle_0, \langle B_e \rangle_1)))$ and $A_{\alpha+1} \models (\forall e)RA(\langle B_e \rangle_0, \langle B_e \rangle_1)$.

In other words, $B$ codes all the hierarchies in $A_\alpha$.

Glue $B$ into a hierarchy $\langle u^*, X^* \rangle$ inside $A_{\alpha+1}$ using the first Glue-up lemma 1.4.5. Then by our construction, $\langle u^*, X^* \rangle$ is such that $RA(u^*, X^*)$ and its order type is $\alpha$. We use $\text{ACA}$, inside $A_{\alpha+1}$, to obtain a $u$ by adding a top to $u^*$. We obtain
$X$ from $X^*$ similarly. Obviously $RA(u,X)$ is true and if $i$ is the top of $u$, then $set(X_i) = \overline{set}(X^*) = A_\alpha$.

**Case 2.** $\alpha = \gamma + 1$.

To find a $(u,X) \in A_{\alpha+1}$ such that $RA(u,X)$ and $ot(u) = \alpha + 1$, we note that we already have some $(v,Y) \in A_\alpha$ such that $RA(v,Y)$, where $v$ has a top $i$ and $set(Y_i) = A_\gamma$. Since $Sat(Y_i)$ is definable over $A_\alpha$, it is an element of $A_{\alpha+1}$. All we have to do is to use ACA in $A_{\alpha+1}$ to attach one more element "$i_0$" to the top of $v$ and define $Y_{i_0}$ to be $Sat(Y_i)$.

**Corollary (ZFC).** If $(u,X)$ is such that $RA(u,X)$ and $f$ is an isomorphism between $u$ and its order type then

$$(\forall i \in dom(u))(set(X_i) = A_{f(i)}).$$

**Theorem 2.1.2 (ZFC).** If $A_\alpha = A_{\alpha+1}$, then $A_\alpha$ has every element definable over itself without parameters, $\alpha$ is a limit ordinal, and the least such $\alpha$ is countable.

**Proof:** Without loss of generality, we assume that $\alpha$ is the least ordinal such that $A_\alpha = A_{\alpha+1}$. We first show that $\alpha$ is a limit ordinal. Suppose otherwise. Then $\alpha = \gamma + 1$ for some $\gamma$. Hence by the proof of theorem 2.1.1, there is a $(u,X) \in A_\alpha$ which codes $A_\gamma$. Since $A_\alpha \models Z_2$, $Sat(X)$ is in $A_\alpha$. But since $set(Sat(X)) = A_\alpha$, all the elements of $A_\alpha$ are recursive in $Sat(X)$. Choose a set $b$ such that $b$ is arithmetic in $Sat(X)$ but not recursive in it. Then $b$ is in $A_\alpha$, since it satisfies $Z_2$. But since $b$ is not recursive in $Sat(X)$, we have a contradiction.

Therefore, $\alpha$ is a limit ordinal. By a proof similar to that of theorem 2.1.1, we
know that \( A_\alpha \) has definable well-ordering \(<_\alpha \). Consider

\[
D = \{ x \in A_\alpha : x \text{ is definable over } A_\alpha \}. 
\]

It is easy to see that \( D \) is an elementary substructure of \( A_\alpha \) by using the definable well-ordering \(<_\alpha \). If \( D \neq A_\alpha \), then \( D = A_\lambda \) for some \( \lambda < \alpha \). But then

\[
A_\lambda = A_{\lambda+1},
\]

which contradicts the choice of \( \alpha \).

Let us denote this least \( \alpha \) by \( \beta_0 \). Note that at each stage earlier than \( \beta_0 \), at least one new element is added. Hence, the cardinality of \( A_{\beta_0} \) is at least \( |\beta_0| \). Now, by theorem 2.1.2, we know that all the elements of \( A_{\beta_0} \) are definable over \( A_{\beta_0} \) (with no set parameters). The cardinality of \( \beta_0 \) is at most countable. Hence \( \beta_0 \) is a countable ordinal.

### 2.2 R.A. Hierarchy in \( Z_2 \)

By lemma 2.1.4, each initial segment of \( A_{\beta_0} \) is coded by some hierarchy code. By discussing these codes in the language \( L_2 \), we may prove many results about the Ramified Analytical Hierarchy in weak subsystems of \( Z_2 \). In this section, we work in \( Z_2 \). In the next two sections, we work in weaker theories.

We first point out that the restatement of Theorem 2.1.1 in terms of hierarchy codes is actually provable in ACA.

**Theorem 2.2.1.** If \( RA(u, X) \), \( i \in \text{dom}(u) \) and \( \text{set}(X_i) \neq \text{set}(X_{i+1}) \) then
1. there is a well-ordering (subject to the interpretation stated in lemma 1.4.4) \( < \)
of set\( (X_i) \) which is definable over set\( (X_i) \) itself;

2. there is an integer \( n \) such that \( \forall x \in \text{set}(X_i), x \) is a \( \Sigma^1_n \)-definable singleton over set\( (X_i) \);

3. There is an element \( (v, Y) \in \text{set}(X_{i+u_1}) \) such that \( (RA(v, Y))^{\text{set}(X_{i+u_1})} \) and
\( (v, Y) \equiv (u|_{i}, X^i) \).

**Proof:** This theorem will be proved in section 2.4.

The above theorem has great significance in second-order arithmetic. Many important results can be established through some further work following this line. We state a few in theorem 2.2.2 for our own reference without proof. Thorough proofs of them can be found in Simpson [2] through a somewhat different approach.

**Definition 15.** We use \( x \in \overline{M} \) to denote the following formula:

\[
(\exists u, X)(RA(u, X) \land (x \in \overline{\text{set}}(X))).
\]

**Theorem 2.2.2.**

1. For \( n \geq 1 \), \( \Pi^1_n \)-CA proves \( (\phi)^\overline{M} \) for \( \phi \in \Pi^1_n \)-CA;

2. for \( n \geq 1 \), \( \Pi^1_n \)-CA proves \( \exists M(M \models \Sigma^1_n DC) \);

3. for \( n \geq 1 \), \( \Delta^1_{n+1} \)-CA proves \( (\phi)^\overline{M} \) for \( \phi \in \Delta^1_{n+1} \)-CA;

4. for \( n \geq 1 \), and any \( \Pi^1_2 \) sentence \( \phi \) in \( L_2 \), if \( \Sigma^1_n DC \) proves \( \phi \), then so does \( \Delta^1_n CA \). In other words, \( \Sigma^1_n DC \) is conservative over \( \Delta^1_n CA \) for \( \Pi^1_2 \) sentences.\(^2\)

\(^2\)When \( n=1 \), the traditional approach does not seem to work. The result for this case was proved
Some explanation is needed for formulas such as $M \models \Sigma^1_n DC$. It is different from
the model notation as defined in definition 4. Any set $M$ codes a natural structure for
$L_2$, whose second order part is $\{M_i\}_{i \in \omega}$ (see the beginning of section 1.3). This is the
kind of structure we are talking about at this particular moment. Hence $x \in M$ is
an abbreviation for $\exists i \forall k (k \in x \leftrightarrow \langle i, k \rangle \in M)$. Similar to before, we may define the
satisfaction predicate $Sat(M)$ for this structure. Let $\bar{A}$ the set of Gödel numbers for
sentences in $\Sigma^1_n DC$. Then $\bar{A}$ is primitive recursive (i.e., its characteristic function is
primitive recursive). Hence $\bar{A}$ is represented by some relation symbol $A$ in $L_2$. Then
$M \models \Sigma^1_n DC$ is an abbreviation for $\exists Y (Y = Sat(M) \land \forall m (m \in A \rightarrow \langle m, 0, 0 \rangle \in Y))$. Similar remarks apply to other theories.

2.3 R.A. Hierarchy in $\Delta^1_n$-CA plus $\Sigma^0_0$-CA $(n \geq 6)$

In this section, unless otherwise specified, we will work in $\Delta^1_n$-CA plus $\Sigma^0_0$-CA with
$n \geq 5$. Note that $\Delta^1_n$-CA is light-faced and $\Sigma^0_0$-CA is bold-faced. All the theorems
stated are assumed to be provable from this theory unless otherwise specified. Since
we have $\Sigma^0_0$-CA, most notations we developed in Chapter I can be used here. It
should be noted that, according to our definition, numerical parameters are allowed
in the axioms of $\Delta^1_n$-CA.

Let $\sigma(e, i, x)$ be universal for all $\Sigma^0_1(i, x)$-formulas with free variables shown. The
Turing jump of $x$, $TJ(x)$, is defined by

$$(\forall m) (m \in TJ(x) \leftrightarrow ls(m) = 2 \land \sigma((m)_0, (m)_1, x)),$$

by Harvey Friedman ([6]) using the idea of pseudohierarchies. When $n$ is greater than 1, the
conservativity may be improved to for $\Pi^1_4$ sentences.
More formally speaking, we are introducing an abbreviation "y = TJ(x)" for the formula:

$$(\forall m)(m \in y \leftrightarrow ls(m) = 2 \land \sigma((m)_0, (m)_1, x)).$$

We will also use "y = TJ(\omega)" to denote the formula

$$(\forall i)((y)_i+1 = TJ(y_i) \land y_0 = x).$$

Obviously, both "y = TJ(x)" and "y = TJ(\omega)(x)" are $\Pi^0_2$ formulas. Sometimes "y = TJ(\omega)(x)" is written as "y = x^{(\omega)}".

As usual, we define $TJ^0(x) = x$ and $TJ^{k+1}(x) = TJ(TJ^k(x))$. We will write $x^{(k)}$ for $TJ^{(k)}(x)$.

**Lemma 2.3.1.** If x is a $\Sigma^1_n$-singleton, then its Turing jump and its $\omega$-th Turing jump exist; i.e,

$$(\exists e, i)(x = sng\phi_{\nu(e,i,x)}) \rightarrow (\exists y)(y = TJ(x)) \land (\exists y)(y = TJ^{(\omega)}(x)),$$

where $x = sng\phi_{\nu(e,i,x)}$ is as defined in definition 5.

**Proof:** Using induction on $k$ we may prove that, for all $k$, the $k$-th jump of $x$, $x^{(k)}$ exists and is unique. Hence $x^{(k)}$ is a $\Sigma^1_n$-singleton for all $k$. Then apply $\Delta^1_n$-CA to show the $\omega$-th Turing jump exists.

**Lemma 2.3.2.** $\Sigma^0_0$-CA proves $(\exists e_0)(\forall x)(\forall k)(TJ(x_k) = ((TJ(x))_{i_0})_k)$.

**Proof:** Let $A(i, x)$ be a $\Sigma^0_1(i,x)$-formula such that

$$(\forall m, k)(m \in TJ(x_k) \leftrightarrow A(\langle k, m \rangle, x)).$$
Hence there is some $i_0$ such that

$$(\forall k)(\forall m)(A((k, m), x) \leftrightarrow \sigma(i_0, (k, m), x)).$$

But $\sigma(i_0, (k, m), x)$ is equivalent to $m \in ((T J(x))_k)_k$.

**Corollary.** $\Sigma_0^0$-CA proves that if $(\exists y)(y = T J^{(\omega)}),$ then $(\forall m, k)(\exists Y)(Y = TJ^m(x_k))$.

**Lemma 2.3.3.** $\Sigma_0^0$-CA proves that if $\exists y(y = T J^{(\omega)}(x))$ then $(\forall k)(\exists Z)(Z = T J^{(\omega)}(x_k))$.

**Proof:** We apply Lemma 2.3.2. Consider the primitive recursive function symbol in the language of PA with defining equation

$$f(0, k, i) = (k, i) \land f(n + 1, k, i) = \langle e, \langle e, \cdots, \langle e, f(n, k, i) \rangle \cdots \rangle \rangle.$$ 

Repeatly applying Lemma 2.3.2, it is easy to check that

$$(\forall n, k, i)(i \in TJ^n(x_k) \leftrightarrow f(n, k, i) \in TJ^n(x)).$$

Hence if $Y$ is such that $Y = T J^{(\omega)}(x)$ and $Z$ is defined by

$$(\forall i)(i \in Z \leftrightarrow ls(i) = 2 \land f(\langle i \rangle_0, k, \langle i \rangle_1) \in Y(\langle i \rangle_0),$$

then evidently, $Z = T J^{(\omega)}(x_k)$. Note that the existence of $Z$ follows from $\Sigma_0^0$-CA.

**Lemma 2.3.4.** If $\phi(i, x)$ is an arithmetical formula with no free set variables other than $x$, then $\Sigma_0^0$-CA proves that

$$\exists y(y = T J^{(\omega)}(x)) \rightarrow (\exists z)(\forall i)(i \in z \leftrightarrow \phi(i, x)).$$

Namely, we may prove in $\Sigma_0^0$-CA that if the $\omega$-th jump of $x$ exists, then any set arithmetical in $x$ also exists.
Proof: As a matter of fact, we will show by induction on \( k > 0 \) that, for each 
\( \phi(i, x) \in \Sigma^0_k(i, x) \) with all its free variables shown, \( \Sigma^0_k \cdot CA \) proves that

\[
\exists y (y = T J(x)) \rightarrow (\exists ! e)(\forall i)(\phi(i, x) \leftrightarrow (i \in (T J^k(x))_e)).
\]

When \( k = 1 \), it is the definition of Turing jump.

Now, suppose that \( \phi(i, x) \in \Sigma^0_{k+1}(i, x) \). Write \( \phi(i, x) \) as \( (\exists m)(\neg \psi(i, m, x)) \). By induction hypothesis, \( \Sigma^0_k \cdot CA + \exists y (y = T J(x)) \) proves that

\[
(\exists e)(\forall i, m)(\psi(i, m, x) \leftrightarrow ((i, m) \in (T J^k(x))_e)).
\]

Hence, reasoning in \( \Sigma^0 \cdot CA + \exists y (y = T J(x)) \), we have

\[
\phi(i, x) \leftrightarrow (\exists m)(\neg \psi(i, m, x)) \leftrightarrow (\exists m)((e', (i, m)) \notin T J^k(x)) \leftrightarrow (e, i) \in T J^{k+1}(x).
\]

Given \( x \) and \( y \), we will call \( y \) arithmetical in \( x \) if there is an arithmetical formula \( \phi(i, x) \) with no free set variables other than \( x \) such that

\[
(\forall i)(i \in y \leftrightarrow \phi(i, x)).
\]

We write \( y \leq_\phi x \) to indicate this fact.

Lemma 2.3.5. For any arithmetical formulas \( \psi(y_1, \ldots, y_k), \phi_1(i, x), \ldots, \phi_k(i, x) \) with all its free variables shown, there is a \( \Pi^0_2 \)-formula \( \theta(y, y_1, \ldots, y_k) \) with all its free variables shown such that \( \Sigma^0 \cdot CA \) proves that

\[
\exists z (z = T J(x)) \rightarrow (\forall y_1 \leq_{\phi_1} x, \ldots, y_k \leq_{\phi_k} x)(\psi \leftrightarrow (\exists y)\theta(y)).
\]

and

\[
\exists z (z = T J(x)) \rightarrow (\forall y_1, \ldots, y_k)((\exists y)\theta(y) \rightarrow \psi).
\]
Proof: Write $\psi$ in its prenex form with blocks of similar quantifiers contracted. We prove the Lemma by induction on the number of existential quantifiers in $\psi$.

If $\psi$ can be converted to a $\Pi^0_2$-formula through quantifier contraction, we are evidently done.

If otherwise, then $\psi$ may be written as

$$\forall m \exists n \psi'(m,n,y_1,\cdots,y_k),$$

by adding dummy universal variable if necessary, such that $\psi'$ has one less existential quantifier. Fix formulas $\phi_1,\cdots,\phi_k$.

Assume $\exists z(z = T J^\omega(x))$. Fix $y_1 \leq \phi_1 x, \cdots, y_k \leq \phi_k x$. Let $\phi_{k+1}(i,x)$ be the following formula,

$$\psi'((i)_0, (i)_1) \land (\forall n' < (i)_1) - \psi'((i)_0, n') \land ls(i) = 2.$$

Let $y_{k+1}$ be such that $y_{k+1} \leq \phi_{k+1} x$. Clearly we have

$$y_{k+1} = \{(m,n) : \psi' \land (\forall n' < n) - \psi'(m,n')\}.$$

**Case 1.** $\psi'$ can be written as $\forall m' \exists n' A(m,n,m',n',y_1,\cdots,y_k)$. In this case we may claim

$$\psi \leftrightarrow (\exists y)(\forall m, k, k'(m,k) \in y \land (m,k') \in y \rightarrow k = k') \land (\forall m,m')(\exists n,n')(m,n) \in y \land A(m,n,m',n',y_1,\cdots,y_k)).$$

Let $\psi^*$ be the formula on the right hand side. It is clear that if $\psi^*$ then $\psi$. For the other direction, we may let $y$ be $y_{k+1}$.
Now write $\psi^*$ as $\exists y \forall m \exists n B(m, n, y_1, \ldots, y_k, y)$. Applying induction hypothesis to $\forall m \exists n B$, we get a formula $\theta'$ such that

$$\forall m \exists n B(m, n, y_1, \ldots, y_k, y_{k+1}) \leftrightarrow \exists y' \theta'(y', y_1, \ldots, y_k, y_{k+1}).$$

Let $\theta(y, y_1, \ldots, y_k)$ be $\theta'(y_0, y_1, \ldots, y_k, y)$. It is straightforward to check that $\theta$ satisfies the condition of the Lemma.

Case 2. $\psi'$ is $\forall m' A(m, n, m', y_1, \ldots, y_k)$, where $A$ is quantifier free. We then let $\theta(y, y_1, \ldots, y_k)$ be the $\Pi^0_2$ prenex form of the following formula,

$$\forall m, k_1, k_2 ((\langle m, k_1 \rangle \in y \land \langle m, k_2 \rangle \in y) \rightarrow k_1 = k_2) \land \forall m, m' \exists n (\langle m, n \rangle \in y \land A(m, n, m', y_1, \ldots, y_k)).$$

Corollary. If $\phi(x)$ is an arithmetical formula, where $x$ is its only free set variable, then there is a $\Sigma^1_1$-formula $\psi(x)$ such that $\Sigma^0_0$-CA proves that

$$\exists z(z = TJ^{(\omega)}(x)) \rightarrow (\phi(x) \leftrightarrow \psi(x)).$$

Definition 17. We use $RA^*(u, X)$ to denote the conjunction of the following clauses:

1. $u \in WO$ and $u$ has a top which is a successor;

2. $Y = \bigcup_{i \in \text{dom}(v)} \{i\} \times Y_i$ (i.e, $\forall k (k \in Y \rightarrow (k)_0 \in \text{dom}(v))$);

3. $\exists y (y = \langle u, X \rangle^{(\omega)})$;

4. $\forall i \in \text{dom}(u) (\exists j (i = j + v \land Y_i = \text{Sat}(Y_j)) \lor (i \in \text{lim}(u) \land Y_i = Y_i^j))$;

5. $\forall i \in \text{dom}(u)$ the conjunction of the following:

   - $(\forall j >_u i) (\text{set}(X_i) \subset \text{proper set}(X_j))$;
• $\forall (v, Y) \in \text{set}(X_i)((RA(v, Y))^{\text{set}(X_i)} \rightarrow (3j < u i)(v, Y) \equiv \langle u|_j, X|_i^j\rangle$;

• $(3n)(\forall x \in \text{set}(X_i))(x \text{ is a } \Sigma_n\text{-singleton over } \text{set}(X_i))$; i.e., $(3n)(\forall x \in \text{set}(X_i))\text{sgl}^2(n, x, X_i)$(see definition 6);

• if $i$ is not the top of $u$ then $(3(v, Y) \in \text{set}(X_{i+u_1}))(\text{RA}(v, Y))^{\text{set}(X_{i+u_1})} \land \langle v, Y \rangle \equiv \langle u|_i, X|i^i\rangle$, where $(\text{RA}(v, Y))^{\text{set}(X_{i+u_1})}$ is the relativization of $\text{RA}(v, Y)$ to $\text{set}(X_{i+u_1})$; this can be written as an arithmetic formula with free variables $i, u$ and $X$.

Remark. It is obvious that if $\text{RA}^*(u, X)$ then $\text{RA}(u, X)$.

Lemma 2.3.6. There is an $S\Sigma^*_1$-formula $\Theta(u, X)$ with free variable as shown such that $\Sigma^0_0$-CA proves that

$$(\forall u, X)(\Theta(u, X) \land u \in WO \leftrightarrow \text{RA}^*(u, X)).$$

Proof: Apply Lemma 2.3.5.

Lemma 2.3.7. For any $\phi \in \text{ACA}$, $\Sigma^0_0$-CA proves that

$$(\forall u, X)(\forall i)(\text{RA}^*(u, X) \land i \in \text{dom}(u) \rightarrow (\phi)^{\text{set}(X_i)}).$$

In another words, “$\text{set}(X_i) \models \text{ACA}$” for any $i \in \text{dom}(u)$ and $i$ not the top of $u$.

Proof: It suffices to show that $\text{set}(X_i) \models \text{ACA}$ when $i$ is a successor. We give an informal argument. Suppose $i = j +_u 1$. By definition, $\text{set}(X_i)$ is the set of all those $x$‘s such that $x$ is $L_2$-definable over $\text{set}(X_j)$ without any set parameter. Let $a_1, \cdots, a_k \in \text{set}(X_i)$ and $\phi(l, a_1, \cdots, a_k)$ be an arithmetical sentence with set
parameters shown (let us assume that 1 is the first numerical variable). We want to show that

\[ b = \{ i : \text{set}(X_i) \models \phi(i, a_1, \ldots, a_k) \} \]

exists in \text{set}(X_i). By replacing \( a_1 \) through \( a_k \) by their defining formulas over \text{set}(X_i), we get a formula \( \psi(l) \) without set parameters such that

\[ \forall l (\text{set}(X_i) \models \phi(l, a_1, \ldots, a_k) \leftrightarrow \text{set}(x_j) \models \psi(l)). \]

Let \( e \in Frlm^0 \) be the Gödel number of \( \psi(l) \). Then evidently \( b = \Lambda(X_i, e) \in \text{set}(X_i) \), which exists by \( \Sigma^0_0 \)-CA.

**Lemma 2.3.8.** \( \Sigma^0_0 \)-CA proves the following:

\[ \forall u, X, i, j, v, Y \ (i \prec u \wedge (v, Y) \in \text{set}(X_i) \wedge RA^*(u, X) \wedge \text{set}(X_i) \models RA(v, Y) \rightarrow \text{set}(X_j) \models RA(v, Y)) \]

**Proof:** Straightforward.

We write \( i = TL(u) \) for the fact "\( i \) is the \( u \)-largest limit point of \( u \) and \( i +_u 1 \) exists".

**Definition 18.** We define two formulas \( WRA^*(u, X) \) and \( SRA^*(u, X) \) as follows:

- \( WRA^*(u, X) \) is \( RA^*(u, X) \wedge \forall v, Y, i \ (RA^*(v, Y) \wedge i = TL(v) \wedge \forall j < v \ i(j \notin \lim(v) \rightarrow \exists k \notin \lim(u)(\text{set}(Y_j) = \text{set}(X_k)) \rightarrow \exists k \notin \lim(u)(\text{set}(Y_{i+1}) \equiv \text{set}(X_k)))) \),

- \( SRA^*(u, X) \) is

\[ WRA^*(u, X) \wedge (\forall v, Y)(WRA^*(v, Y) \rightarrow \text{Comp}(u, X, v, Y)). \]
Remark. We will call \((u, X)\) a “well-founded hierarchy” if \(WRA^*(u, X)\) is true and a (“strong hierarchy”) if \((SWA^*(u, X))\) is true.

Let us use \(\text{set}(X_i) \equiv \text{set}(Y_j)\) to denote the following formula:

\[
(\forall e \in Frlm^0)(\forall k)((\text{Sub}^0(e, \text{num}(k)), 0, 0) \in X_i \leftrightarrow (\text{Sub}^0(e, \text{num}(k)), 0, 0) \in Y_j).
\]

This is clearly a \(\Pi^0_1\) formula.

Let \(Equiv^*(u, X, v, Y)\) be the conjunction of the following two clauses:

1. \(\forall i \in \text{dom}(u)(i \notin \text{lim}(u) \rightarrow \exists j \in \text{dom}(v)(j \notin \text{lim}(v) \land \text{set}(X_i) \equiv \text{set}(Y_j)))\);

2. \(\forall i \in \text{dom}(v)(i \notin \text{lim}(v) \rightarrow \exists j \in \text{dom}(u)(j \notin \text{lim}(u) \land \text{set}(Y_i) \equiv \text{set}(X_j)))\).

Lemma 2.3.9. \(\Sigma^0_0\)-CA proves that

\[
\forall u, X, v, Y ((RA^*(u, X) \land RA^*(v, Y)) \rightarrow (Equiv^*(u, X, v, Y) \leftrightarrow Equiv(u, X, v, Y))).
\]

Proof: Fix \(u, X, v, Y\) such that \(RA^*(u, X)\) and \(RA^*(v, Y)\). We first assume that \(Equiv(u, X, v, Y)\). Let \(i \in \text{dom}(u)\) be a successor. By assumption, there is \(j \in \text{dom}(v)\) such that \(\text{set}(X_i) = \text{set}(Y_j)\). It is easy to see that \(j\) must also be a successor. Let \(i = i_0 +_u 1\) and \(j = j_0 +_v 1\). By the definition of \(RA^*\), there is some \((w, Z) \in \text{set}(X_i)\) such that

\[
\text{set}(X_i) \models (RA(w, Z) \land (\forall x (x \text{ is definable over } \text{set}(Z)))).
\]

Since \(\text{set}(X_i) = \text{set}(Y_j)\), the above formula is still true after we replace \(X_i\) by \(Y_j\). It follows from the definition of \(RA^*\) that

\[
(\text{set}(Z) = \text{set}(X_{i_0})) \text{ and } (\text{set}(Z) = \text{set}(Y_{j_0})).
\]
Therefore we have $\text{set}(X_{i_0}) = \text{set}(Y_{j_0})$. Again from the definition of $RA^*$, it follows that $\text{set}(X_i) = \text{set}(Y_j)$.

Now let us assume $\text{Equiv}^*(u, X, v, Y)$. Suppose that $\text{Equiv}(u, X, v, Y)$ does not hold. Then there is an $i \in \lim(u)$ such that for any $j \in \text{dom}(v)$ $\text{set}(X_i) \neq \text{set}(Y_j)$.

By the definition of $RA^*(u, X)$, $i$ must not be the top of $u$. Hence $i' = i + u.1$ exists. By assumption, $\text{set}(X_{i'}) = \text{set}(Y_{j+1})$ for some $j$. Then it is easy to see that $\text{set}(X_i) = \text{set}(Y_j)$, which is a contradiction.

Since $\text{Equiv}^*(u, X, v, Y)$ is $\Pi_3^0$, by lemma 2.3.9 we may assume that $\text{Equiv}(u, X, v, Y)$ is a $\Pi_3^0$-formula. Therefore $\text{Comp}(u, X, v, Y)$ is $\Sigma_4^0$.

**Lemma 2.3.10.** There is a $S\Pi_2^1$-formula $\Psi^W(u, X)$ and a $S\Pi_3^1$-formula $\Phi^S(u, X)$ such that $\Sigma_0^0$-CA proves that

$$(\forall u, X)(\text{WRA}^*(u, X) \leftrightarrow \Psi^W(u, X)),$$

and

$$(\forall u, X)(\text{SRA}^*(u, X) \leftrightarrow \Phi^S(u, X)).$$

**Proof:** Use Lemma 2.3.9 and note that $i = TL(v)$ is equivalent to a $\Pi_2^0$ formula. It is then straightforward to get a $S\Pi_2^1$-formula $\Psi^W(u, X)$ with the required property.

Then we may easily construct a $S\Pi_3^1$-formula $\Phi^S(u, X)$ from $\Psi^W(u, X)$ by using the previous lemma.

**Lemma 2.3.11.** $\Sigma_0^0$-CA plus $\Delta^1_n$-CA proves that

$$\forall u \in WO, e, j, i \in \text{dom}(u), v \in LO, F (u = \text{sneg}(e, j, X) \land F \in \text{isom}(v, u|_i) \rightarrow v \in WO),$$
where $F \in isom(v, u|_i)$ abbreviates the formula which asserts that $F$ is an isomorphism between $v$ and $u|_i$.

**Proof:** Fix $u$ as above. We prove the following by transfinite induction on $u$ (This can be done since $u$ is a $\Sigma^1_n$-singleton.),

$$\forall i \in dom(u), v \in LO, F (F \in isom(v, u|_i) \rightarrow v \in WO).$$

Suppose it is true for all $i < u i_0$. Fix $v \in LO, F \in isom(v, u|_{i_0})$ and a nonempty $a \subseteq dom(v)$. Let $j_1 \in a$ be arbitrary. If $j_1$ be the $v$-least of $a$, then we are done. We suppose that it is not. Let $i_1 < u i_0$ be such that $(j_1, i_1) \in F$, and define $v|_{j_1}, F|_{j_1}, a|_{j_1}$ as follows,

$$v|_{j_1} = \{ (j', j'') \in v : j'' \leq_v j_1 \},$$

$$F|_{j_1} = \{ (j, i) \in F : j \leq_v j_1 \},$$

and

$$a|_{j_1} = \{ j \in a : j \leq_v j_1 \}.$$

These exist by $\Sigma^0_0$-CA. It is easy to check that $F|_{j_1} \in isom(v|_{j_1}, u|_{i_1+1})$. Since $i_1 + u 1 < u i_0$ (otherwise we may make $j_1$ $v$-smaller) and $a|_{j_1} \subseteq dom(v|_{j_1})$, we can use induction hypothesis to show that there is a $j^* \in a|_{j_1}$ which is the $v$-least of $a|_{j_1}$. Obviously, this $j^*$ is also the $v$-least of $a$.

**Lemma 2.3.12.** If $SRA^*(u, X)$, $RA^*(v, Y)$ and $(v, Y) \leq (u, X)$, then $SRA^*(v, Y)$.

**Proof:** Fix $m_0 \in dom(u)$ such that $(v, Y) \equiv (u|_{m_0}, X|^{m_0})$

Fix $(w, Z)$ such that $RA^*(w, Z)$ and $i = TL(w)$. Suppose

$$\forall j < u i \exists k \in dom(v) (set(Z_{j+u1}) = set(Y_{k+u1})).$$
Since \((v, Y) \leq (u, X)\), the above is also true with \((v, Y)\) replace by \((u, X)\). Hence by the well-foundedness of \((u, X)\), there is an \(m \in \text{dom}(u)\) such that \(\text{set}(Z_{j+\omega_1}) = \text{set}(X_m)\). If \(m \leq_u m_0\) then we are done. Otherwise \(m_0 <_u m\). We want to draw a contradiction. By definition of \(RA^*\), there must be some \((v', Y') \in \text{set}(X_m) = \text{set}(Z_{i+\omega_1})\) such that \((v', Y') \equiv (v, Y) \equiv (u|_{m_0}, X|^{m_0})\). Again by definition, \((v', Y') \equiv (w|_l, Z|_l)\) for some \(l <_w i +_w 1\). Since \(m_0\) is a successor, \(l\) must also be a successor. Hence \(l <_w i\). It follows from our assumption that

\[\exists k \in \text{dom}(v), m <_u m_0 k \notin \text{lim}(v) \land \text{set}(X_{m_0}) = \text{set}(Z_i) = \text{set}(Y_k) = \text{set}(X_{m'})\]

which is a contradiction. It follows that \((v, Y)\) is well-founded. It is trivial to check that \((v, Y)\) is also strong.

**Corollary 1.** \(\Delta^1_n\)-\(CA\) plus \(\Sigma^0_\delta\)-\(CA\) proves that if \(SRA^*(u, X)\), \((u, X)\) is a \(\Sigma^1_n\) singleton and \((v, Y) \in \text{set}(X)\) has a top which is not a limit, then

\[\text{set}(X) \models RA(v, Y) \rightarrow SRA^*(v, Y)\]

**Proof:** Fix \(u, X, v, Y\) satisfying the above assumption. All required properties of \((v, Y)\) can be derived in a way similar to the proof of lemma 2.3.12, except for \(v\) being a well-ordering which follows from lemma 2.3.11.

**Corollary 2.** \(\Sigma^0_\delta\)-\(CA\) plus \(\Delta^1_n\)-\(CA\) proves that, for any \((u, X)\) and \(i \in \text{dom}(u) \land i \notin \text{lim}(u)\),

- \(RA^*(u, X) \rightarrow RA^*(u|_i, X|_i)\);
- \(WRA^*(u, X) \rightarrow WRA^*(u|_i, X|_i)\);
\[ SRA^*(u, X) \rightarrow SRA^*(u_i, X_i). \]

**Proof:** Fix \( \langle u, X \rangle \) and \( i \) a successor of \( u \) as described. Let us check that \( RA^*(u, X) \rightarrow RA^*(u_i, X_i) \). The only nontrivial part is \( \exists Y (Y = TJ^{(\omega)}(\langle u_i, X_i \rangle)) \). We know that \( \langle u_i, X_i \rangle \) is arithmetical in \( \langle u, X \rangle \). By the proof of lemma 2.3.4, there is an \( e \) and \( k \) such that \( \langle u_i, X_i \rangle = (TJ^{(k)}(\langle u, X \rangle))_e \). Obviously \( TJ^{(\omega)}(TJ^{(k)}(\langle u, X \rangle)) \) exists. Hence, by lemma 2.3.3, \( TJ^{(\omega)}((TJ^{(k)}(\langle u, X \rangle))_e) \) exists.

**Lemma 2.3.13.** \( \Delta^{1}_n \)-CA plus \( \Sigma^{0}_0 \)-CA proves that if \( RA^*(u, X) \) and \( a \in \overline{set}(X) \), then the \( u \)-least representative of \( a \) in \( \langle u, X \rangle \), \( \langle i, m \rangle_a \) exists.

**Proof:** If \( \overline{set}(X) \models (\exists v, Y)(RA(v, Y) \land a \in \overline{set}(Y)) \), then working in \( \overline{set}(X) \) (ACA available), we may get the \( v \)-least representative \( \langle j, e \rangle_a \) of \( a \). By the definition of \( RA^* \), there is an \( i \in dom(u) \) such that \( set(Y_j) = set(X_i) \). We may readily check that \( \langle i, e \rangle_a \) is the \( u \)-least representative of \( a \).

Hence, we may assume that \( \overline{set}(X) \not\models (\exists v, Y)(RA(v, Y) \land a \in \overline{set}(Y)) \). Let \( e \in Frml^0 \) be the least such that \( a = \Lambda(X_{i_0}, e) \), where \( i_0 \) is the top of \( u \). Then \( \langle i_0, e \rangle_a \) will be the \( u \)-least representative of \( a \).

**Lemma 2.3.14.** \( \Delta^{1}_n \)-CA plus \( \Sigma^{0}_0 \)-CA proves that if \( SRA^*(u, X) \), \( RA^*(v, Y) \) and \( \langle u, X \rangle < \langle v, Y \rangle \) then for all \( x, y \in \overline{set}(X) \),

\[ (x \leq u y) \leftrightarrow (x \leq_v y). \]

**Proof:** The proof is similar to the corollary of lemma 1.4.4.

We use \( x \leq_L y \) to denote the following formula

\[ \exists \langle u, X \rangle (SRA^*(u, X) \land x, y \in \overline{set}(X) \land x \leq_L y). \]
Corollary. $\Delta^1_n$-CA plus $\Sigma^0_0$-CA proves that if $RA^*(u, X)$ and $x, y \in \overline{\text{set}}(X)$ then 

$$\exists(v, Y) \in \overline{\text{set}}(X) \ (RA(v, Y)\overline{\text{set}}(X) \land x, y \in \overline{\text{set}}(Y)) \rightarrow ((x \preceq_L y)\overline{\text{set}}(X) \leftrightarrow x \preceq u y).$$

In another word, the internally defined ordering of $\overline{\text{set}}(X)$ coincides with the externally defined ordering.

Lemma 2.3.15. $\Delta^1_n$-CA plus $\Sigma^0_0$-CA proves that if $SRA^*(u, X)$, and $(u, X)$ is a $\Sigma^1_n$-singleton, then there is another $\Sigma^1_n$-singleton $(v, Y)$ such that $(SRA^*(v, Y)\wedge(u, X) < (v, Y))$ or $(\overline{\text{set}}(X) \models Z_2)$.

Proof: Let $i_0 \in \text{dom}(u)$ be the top of $u$. First let us assume that $\text{Sat}(X_{i_0})$ exists and is a $\Sigma^1_n$-singleton.

Without loss of generality, we may assume that $0 \notin \text{dom}(u)$. Putting 0 on top of $u$, we get a $v \in WO$. $v$ is one level taller than $u$ and is clearly a $\Sigma^1_n$-singleton.

Then, we define $Y$ such that $Y_i = X_i$ for $i \in \text{dom}(u)$ and $Y_0 = \text{Sat}(X_{i_0})$. Evidently $(v, Y)$ is a $\Sigma^1_n$-singleton. $(v, Y)^{(\omega)}$ exists by $\Delta^1_n$-CA. Exactly the same as the proof of theorem 2.1.1 for the successor case, we may check that $(v, Y)$ also satisfies the other conditions of $RA^*$ hierarchy.

We may prove that $(v, Y)$ is well-founded easily. It is straightforward to check that $(v, Y)$ is also strong.

Hence to prove the lemma, it suffices to show that $\text{Sat}(X_{i_0})$ exists.

We may do so by employing the same idea as used in the proof of Lemma 1.3.3, except that $\Delta^1_n$-CA will be used everywhere $\Delta^1_n$-CA is used.

Definition 19. We define a formula $M^*(x)$ and a formula $M^{**}_n(x)$ for each $n$ as
follows:

1. \( M^*(x) \) is \((\exists u, X)(SRA^*(u, X) \land (x \in \overline{\text{set}}(X)))\);

2. \( M^*_n(x) \) is \((x \in M^* \land (x \text{ is a } \Sigma^1_n\text{-singleton}))\).

We will write \( x \in M^* \) and \( x \in M^*_n \) for \( M^*(x) \) and \( M^*_n(x) \) respectively.

**Lemma 2.3.16.** \( \Delta^1_n\text{-CA plus } \Sigma^0_n\text{-CA proves that if } \forall x \,(x \not\in Z_2) \text{ then } (\forall x \in M^*_n)(\exists (v, Y) \in M^*_n)(SRA^*(v, Y) \land x \in \overline{\text{set}}(Y)) \), in particular, \([(\forall x)(\exists v, Y)(RA(v, Y) \land x \in \overline{\text{set}}(Y)))]^{M^*_n}.

**Proof:** Fix \( a \in M^*_n \).

**Case 1:** There is a \((u, X)\) such that \( SRA^*(u, X) \) and \( \overline{\text{set}}(X) \models (\exists v, Y)(RA(v, Y) \land a \in \overline{\text{set}}(Y)) \). Let \( i_0 \) be the top of \( u \), and \((j_0, e_0)_a\) be the \( u \)-least representative of \( a \). Then \( j_0 + u \leq u i_0 \).

Let \( e \in Frml^0 \) be the least such that \( \Lambda(X_{j_0+u1}, e) = (v, Y) \) for some \((v, Y)\) and \( \overline{\text{set}}(X) \models (RA(v, Y) \land a \in \overline{\text{set}}(Y)) \) (\( e \) exists by induction axioms). The \((v, Y)\) represented by \((j_0+u1, e)\) is a \( \Sigma^1_n\)-singleton since \((\forall w, Z)((w, Z) = (v, Y) \iff (\exists u, X)(SRA^*(u, X) \land (w, Z) \in \overline{\text{set}}(X) \land \overline{\text{set}}(X) \models (RA(w, Z) \land a \in \overline{\text{set}}(Z) \land \forall (w', Z') <_u (w, Z)(RA(w', Z') \rightarrow a \notin \overline{\text{set}}(Z')))\)).

Hence \((v, Y) \in M^*_n \).

**Case 2:** Case 1 does not hold.

Note that in this case, if \( SRA^*(u, X) \), \( a \in \overline{\text{set}}(X) \) and \((i_0, e_0)_a\) is the \( u \)-least representative of \( a \), then \( i_0 \) must be the top of \( u \). Let \( e \) be the least so that, for some \((v, Y)\), \( \Lambda(X_{i_0}, e) = (v, Y) \) and \( \overline{\text{set}}(X) \models (RA(v, Y) \land \forall k(k \in a \iff \overline{\text{set}}(Y) \models) \).
Sub^0(e, Num(k))). The \langle v, Y \rangle represented by this \langle i_0, e \rangle exists. Furthermore \langle v, Y \rangle is a \Sigma^1_n-singleton for a similar reason as before. Also SRA^*(v, Y) by Lemma 2.3.12. Hence we may extend \langle v, Y \rangle by two levels to \langle v + 2, Y^* \rangle such that SRA^*(v + 2, Y^*).

(Also see the proof of lemma 2.3.23.) Then

\[ \overline{\text{set}}(Y^*) \models (\exists v, Y)(RA(v, Y) \land a \in \overline{\text{set}}(Y)), \]

which contradicts our hypothesis.

**Lemma 2.3.17.** For every \( \phi \in \text{ACA} \), \( \Sigma^0_0 \cdot \text{CA} \) plus \( \Delta^1_n \cdot \text{CA} \) proves \( (\phi)^{M_n**} \). In other words, \( M_n** \) satisfies \( \text{ACA} \).

**Proof:** Let \( a_1, a_2 \in M_n** \), \( \psi(t, a_1, a_2) \) be arithmetical with all its parameters shown.

By lemma 2.3.16, there is a \( \langle v, Y \rangle \in M_n** \) such that \( SRA^*(v, Y) \) and \( a_1, a_2 \in \overline{\text{set}}(Y) \).

By applying \( \text{ACA} \) in \( \overline{\text{set}}(Y) \), it follows that there is a \( b \in \overline{\text{set}}(Y) \subseteq M_n** \) such that

\( b = \{ k : \overline{\text{set}}(Y) \models \psi(k, a_1, a_2) \} \).

But \( \forall k(\overline{\text{set}}(Y) \models \psi(k, a_1, a_2) \leftrightarrow M_n** \models \psi(k, a_1, a_2)) \).

Hence the lemma follows.

**Lemma 2.3.18.** \( \Sigma^0_0 \cdot \text{CA} \) plus \( \Delta^1_n \cdot \text{CA} \) proves that if \( (RA(v, Y))^{M_n**} \) and \( v \) has a top which is not a limit then \( SRA^*(v, Y) \).

**Proof:** By Lemma 2.3.16, there is a \( \langle u, X \rangle \in M_n** \) such that \( SRA^*(u, X) \) and \( \langle v, Y \rangle \in \overline{\text{set}}(X) \). It is easy to check directly that we must have \( \overline{\text{set}}(X) \models RA(v, X) \).

Then \( SRA^*(v, Y) \) follows from Lemma 2.3.12.

**Lemma 2.3.19.** \( \Sigma^0_0 \cdot \text{CA} \) proves that

\[ (\forall u)(((u \in \text{WO}) \land (u \text{ has no top})) \rightarrow (\forall m)(\exists \alpha) \text{ embd}(\alpha, m, u)). \]
In the above, \( \text{embd}(\alpha, m, u) \) abbreviates the formula which asserts that \( \alpha \) embeds \( m + 1 \) to an initial segment of \( u \), more precisely the formula:

\[
(ls(\alpha) = (m + 1)) \land (\forall i \leq m)((m)_i \in \text{dom}(u)) \land (\forall i, j \leq m)(i < j \leftrightarrow (m)_i <_{u (m)_j}) \land (\forall j \leq m)(\forall k <_{u (m)_j}(\exists i \leq j)((m)_i = k)).
\]

**Proof:** Fix an infinite \( u \in WO \). Then argue by induction on \( m \).

It is true when \( m = 0 \) since \( u \) has a least element. Suppose it is true for \( m \). Let \( \alpha_m \) be such that \( \text{embd}(\alpha_m, m, u) \). Suppose \( \alpha_m(m) = i_m \). Let \( i_{m+1} \) be the immediate successor of \( i_m \). Then define \( \alpha \) such that \( \alpha(k) = \alpha_m(k) \) if \( k \leq m \), and \( \alpha(k) = i_m + 1 \) if \( k = m + 1 \). Then obviously \( \text{embd}(\alpha, m + 1, u) \).

**Lemma 2.3.20.** \( \Sigma^0_0 \)-CA plus \( \Delta^1_n \)-CA proves the following:

1. \((\forall m > 1)(\exists! X)RA^*(m + 1, X)\), Where \( RA^*(m + 1, X) \) means \( \exists u \forall i, j((i, j) \in u \leftrightarrow (i \leq j \leq m)) \land RA^*(u, X) \);

2. \((\forall m, X) (RA^*(m + 1, X) \rightarrow SRA^*(m + 1, X))\).

**Proof:** We argue by induction on \( m \). It is obviously true when \( m = 0 \). Assume it is true for \( m \). Let us consider the case of \( m + 1 \).

Fix \( X \) such that \( RA^*(m + 1, X) \). By the uniqueness, we know that \( X \) is a \( \Sigma^1_n \)-singleton. It then follows from lemma 2.3.15.

**Corollary.** \( \Sigma^0_0 \)-CA plus \( \Delta^1_n \)-CA proves \((\exists x)(x \in M_n^{**})\).

**Lemma 2.3.21 (Second-Glue-up Lemma).** If \( n \geq 5 \), then \( \Sigma^0_0 \)-CA plus \( \Delta^1_n \)-CA proves the following: If \( C \) is a \( \Sigma^1_n \)-singleton and \( \forall i SRA^*(((C)_i)_0, ((C)_i)_1) \), then there
is a \( (u, X) \) such that \( SRA^*(u, X) \) and for all \( i \) \( ((C_i)_0, (C_i)_1) \leq (u, X) \) or there is an \( X \) such that \( X \models Z_2 \).

**Proof:** Let us write \( (u_i, X_i) \) for \( ((C_i)_0, (C_i)_1) \). We may divide the proof into following two cases.

**Case 1:** \( \{ (u_i, X_i) \}_{i \in \omega} \) has a tallest one, say, \( (u_0, X_0) \). Then we just choose \( (u_0, X_0) \) to be our \( (u, X) \).

**Case 2:** \( \{ (u_i, X_i) \}_{i \in \omega} \) has no tallest element.

They may not be in a strictly increasing order. We first introduce a few abbreviations for use in this proof. \( Hd_1(j, C) \) is the formula \( \forall l < j \ (u_l, X_l) < (u_j, X_j) \), and \( Hd_2(K, C) \) is the formula \( \exists j \ Hd_1(j, C) \land K = (u_j, X_j) \). Let \( D^* \) be such that

\[
D^* = \{ (i, j) : (\exists H)(\forall k \leq i)(Hd_2(H_k, C) \land \forall k_1 < k_2 \leq i \ (H_{k_1} < H_{k_2}) \land \forall j (Hd_1(j, C) \rightarrow (\exists k \leq i \ (u_j, X_j) = H_k \lor \forall k \leq i \ (H_k < (u_j, X_j)))) \land j \in H_i) \}.
\]

Intuitively \( D^*_i \) is the \( i \)th hierarchy in \( C \) with the property of being taller than all earlier hierarchies. \( D^* \) exists by \( \Delta_{<\omega} - CA \). And it is obviously strictly increasing and cofinal in \( C \).

Let us write \( ((D^*_i)_0, (D^*_i)_1) \) as \( (u^*_i, X^*_i) \) for simplicity. Now, for any \( i \), we have

\[
(u^*_i, X^*_i) < (u^*_{i+1}, X^*_{i+1}).
\]

Let

\[
A^* = \{ (i + 1, j) : (u^*_i, X^*_i) < (u^*_{i+1})_{|j}, (X^*_{i+1})_{|j}) \} \cup \{ (0, j) : j \in \text{dom}(u^*_0) \}.
\]

Then \( A^* \) exists by \( \Delta_{<\omega}^1 - CA \). Let

\[
u^* = \{ ((i_1, j_1), (i_2, j_2)) \in (A^*)^2 : \text{set}((X^*_{i_1})_{j_1}) \subseteq \text{set}((X^*_{i_2})_{j_2}) \}.
\]
Then obviously $u^*$ is a $\Sigma^1_n$-singleton.

**Claim.** $u^* \in WO$.

To verify this, let us fix an arbitrary $a \subseteq A^*$ such that $a \neq \emptyset$. Let $i_0$ be the least (in the natural order of integers) such that there is some $j$ so that $(i_0, j) \in a$. Such $i_0$ must exist, since otherwise we may prove

$$\forall i, j \ (i, j) \notin a$$

by induction on $i$, which would be a contradiction. Let

$$a_{i_0} = \{ j \in \text{dom}(u^*_{i_0}) : (i_0, j) \in a \}.$$

Then $a_{i_0} \subseteq \text{dom}(u^*_{i_0})$. Since $u^*_{i_0} \in WO$, there is a $j_0 \in a_{i_0}$ which is the $u^*_{i_0}$-least of $a_{i_0}$. It is straightforward to check that $(i_0, j_0)$ is the $u^*$-least of $a$.

Now $u^*$ clearly has no top. To add a top to it, we need to enlarge its domain. Let

$$A = \{ 1, 2, (0, i) : i \in A^* \}$$

and

$$u = \{ (i, j) : (i, j) \in u^* \lor i \neq 2 \land (j = 1 \lor j = 2) \lor i = j = 2 \}.$$

It is easy to see that $u$ is also a $\Sigma^1_n$-singleton and $u$ has a top which is not a limit.

When we define $X$, $X_i$ can be obtained in almost the same way as in the proof of Lemma 1.4.5 if $i$ is not 1 or 2 (the top two elements of $u$). $X_1$ is as usual the union of those $\{i\} \times X_i$ such that $i <_u 1$ and such that $i$ is not a limit or a successor of a limit. $X_2$ is $\text{Sat}(X_1)$, which exists by $\Delta^1_n$-CA.
When we verify that $RA^*(u, X)$, the only non-straightforward part is the condition
\[ \forall i \in \text{dom}(u) \exists n \forall x \in \text{set}(X_i) \text{ sngl}^2(n, x, X_i), \]
which requires a Skolem-hull argument similar to that of the proof of lemma 2.1.3.
We may do this because all parameters involved are $\Sigma^1_n$-singletons. Moreover, we will be in a similar situation in section 2.4, in which a detailed treatment is given. (see the proof of theorem 2.4.1.)

Now, let us show that $(u, X)$ is well-founded.

Fix $(v, Y)$ and $i$ such that $RA^*(v, Y)$, $i = TL(v)$ and
\[ \forall j < v \ i(j \notin \text{lim}(v) \rightarrow \exists k \in \text{dom}(u) \ (Y_j \equiv X_k)). \]

We want to conclude that there is a $k \in \text{dom}(u)$ such that $Y_{i+1} \equiv X_k$.

We consider two cases.

Case 1. For some $m$, we have
\[ \forall j < v \ i(j \notin \text{lim}(v) \rightarrow \exists k \in \text{dom}(u_m) \ (Y_j \equiv (X_m)_k)). \]

This case is of course trivial since $(u^*_m, X^*_m)$ is well-founded.

Case 2. If case 1 does not hold, then we must actually have
\[ \forall j < v \ i(j \notin \text{lim}(v) \rightarrow \exists k < u \ 1 \ (Y_j \equiv X_k)) \wedge \forall k < u \ 1(k \notin \text{lim}(u) \exists j < v \ i \ (Y_j \equiv X_k)), \]
where "1" is the top limit of $u$ by our definition. Since otherwise there must be a $j_0 < v \ i$ such that $j_0 \notin \text{lim}(v)$ and $Y_{j_0} \equiv X_2$, where "2" is the top element of $u$. Then $Y_{i+1} \neq X_k$ for any $k \in \text{dom}(u)$, which is a contradiction. Hence we must have $\text{set}(Y_i) = \text{set}(X_1)$, which implies that $Y_{i+1} \equiv X_2$. 
Finally, let us verify that \( (u, X) \) is a strong hierarchy. Fix \( (v, Y) \) such that \( WRA^*(v, Y) \). We want to show that \( (u, X) \) and \( (v, Y) \) are comparable with each other.

If \( (v, Y) \leq (u^*_i, X^*_i) \) for some \( i \), then we are done. Otherwise, Since all the \( (u^*_i, X^*_i) \)'s are strong hierarchies, we must have the following situation:

\[
\forall i < u \left( i \notin \text{lim}(u) \rightarrow \exists k \in \text{dom}(v) (X_i \equiv Y_k) \right).
\]

Hence by the well foundedness of \( (v, Y) \), there is some \( k \in \text{dom}(v) \) such that \( X_2 \equiv Y_k \).

Firstly \( k \) can not be a limit point of \( v \), since otherwise we would have

\[
\text{set}(Y_k) \models \forall x \exists (w, Z) \left( x \in \overline{\text{set}}(Z) \land RA(w, Z) \right)
\]

and

\[
\text{set}(X_2) \models \exists x \forall (w, Z) \left( RA(w, Z) \rightarrow x \notin \overline{\text{set}}(Z) \right),
\]

which is a contradiction.

Let \( (w, Z) \in \text{set}(X_2) \) be such that \( \text{set}(X_2) \models RA(w, Z) \) and \( (w, Z) \equiv (u|_1, X|_1) \).

It is easy to check that \( (w, Z) \equiv (v|_{k-1}, Y|^{k-1}) \), which immediately gives \( (u|_1, X|_1) \equiv (v|_{k-1}, Y|^{k-1}) \), where \( k-1 \) denotes the immediate predecessor of \( k \) in \( v \). Hence \( (u, X) \) and \( (v, Y) \) are indeed comparable with each other.

**Lemma 2.3.22.** For any \( \Sigma_1 \)-formula \( \exists x \theta(x, y) \), \( \Sigma_0 \)-CA plus \( \Delta^1_n\)-CA proves

\[
\forall y \in M^{**} ((\exists x \theta(x, y))^{M^{**}} \leftrightarrow \exists (u, X) \left( SRA^*(u, X) \land y \in \overline{\text{set}}(X) \land \exists x \in \overline{\text{set}}(X) \theta(x, y) \right)).
\]

In other words,

\[
\forall y \in M^{**} ((\exists x \theta(x, y))^{M^{**}} \leftrightarrow \exists x \in M^* \theta(x, y)).
\]
**Proof:** Fix $y \in M_{n}^{**}$ and $u, X$ such that

$$(SRA^{*}(u, X) \land y \in \overline{\text{set}}(X) \land \exists x \in \overline{\text{set}}(X) \theta(x, y)).$$

**Case 1.**

$$\overline{\text{set}}(X) \models \exists (v, Y) (RA(v, Y) \land y \in \overline{\text{set}}(Y) \land \exists x \in \overline{\text{set}}(Y) \theta(z, y)).$$

Working inside $\overline{\text{set}}(X)$, one can easily see that

$$\overline{\text{set}}(X) \models \exists x (\theta(x, y) \land \forall z <_{L} x \neg \theta(z, y)).$$

By the corollary of lemma 2.3.14, we know that

$$\exists x (\theta(x, y) \land \forall z <_{L} x \neg \theta(z, y)).$$

Now since $y$ is a $\Sigma_{n}^{1}$-singleton, so is $x$. Hence $x \in M_{n}^{**}$.

**Case 2.** If case 1 does not hold, then intuitively we know that $(u, X)$ is among the "shortest" hierarchy which contains an $x$ such that $\theta(x, y)$. Let $i_{0}, e_{0}$ be such that:

1. $i_{0}$ is the top of $u$, and
2. $\theta(\Lambda(X_{i_{0}}, e_{0}), y)$ and for any $e < e_{0}$ (in the usual order of integers) $\neg \theta(\Lambda(X_{i_{0}}, e), y)$.

Then obviously

$$\theta(\Lambda(X_{i_{0}}, e_{0}), y) \land \forall z <_{L} \Lambda(X_{i_{0}}, e_{0}) \neg \theta(\Lambda(X_{i_{0}}, e), y).$$

Hence $\Lambda(X_{i_{0}}, e_{0}) \in M_{n}^{**}$ since $\Lambda(X_{i_{0}}, e_{0})$ is also a $\Sigma_{n}^{1}$-singleton as we have demonstrated above.

**Lemma 2.3.23.** $\Sigma_{0}^{0}$-CA plus $\Delta_{n}^{1}$-CA proves $\phi^{M_{n}^{**}}$ or $\exists x (x \models Z_{2})$, where $\phi \in \Sigma_{1}^{1}$-CA.

**Proof:** Fix an arbitrary $\Sigma_{1}^{1}$-formula $\exists x \psi(i, x, y)$ with all its free set variables as shown. Fix an arbitrary $a \in M_{n}^{**}$. Then

$$b = \{i : \exists x \in M^{*} \psi(i, x, a)\}$$
exists and is a $\Sigma^1_n$-singleton. Let $\psi^*(i, x, y)$ be the formula

$$\psi(i, x, y) \land \forall z <_L x \neg \psi(i, z, y).$$

As we demonstrated in the proof of the previous lemma, the following is true:

$$\forall i \ (\exists x \in M^* \psi(i, x, a) \leftrightarrow \exists x \psi^*(i, x, a)).$$

We should keep in mind that any $x$ such that $\psi^*(i, x, a)$ must also be an element of $M^*_{\Sigma^1_n}$ since $x$ is unique.

Now it is obvious that

$$\forall i \exists! x((i \in b \land \psi^*(i, x, a)) \lor (i \notin b \land x = 0)).$$

Let

$$C = \{(i,j) : i \in b \land \exists y(\psi^*(i, y, a) \land j \in y)\}.$$  

$C$ exists by $\Delta^1_n$-CA. It is also a $\Sigma^1_n$-singleton since $y = C$ is equivalent to

$$\forall i \ ((i \in b \land \psi^*(i, C, a)) \lor (i \notin b \land C = 0)).$$

Intuitively, for all $i \in b$, $C_i$ is the $<_L$-least element such that $\psi^*(i, C_i, a)$.

Next we let $D = \{(i, j) : (i \in b) \land (j \text{ is in the } _{<_L} \text{-least } (u, X) \text{ such that } C_i \in \overline{\text{set}}(X) \land SRA^*(u, X))\}$. Then $D$ exists by $\Delta^1_n$-CA. Also $D$ is a $\Sigma^1_n$-singleton since $y = D$ is equivalent to

$$\forall i([i \in b \rightarrow (\exists (D_i) \in M^* \land SRA^*(D_i) \land C_i \in \overline{\text{set}}((D_i)_2)) \land \forall y < D_i(SRA^*(y) \rightarrow C_i \notin \overline{\text{set}}(y_2))] \land [i \notin b \rightarrow (D_i) = 0)).$$
Then by applying the Second Glue-up lemma, we get a \( (u, X) \) such that: (1) \( (u, X) \) is a \( \Sigma^1_n \)-singleton, (2) \( SRA^*(u, X) \) and (3) there is some \( (v, Y) \in \overline{\text{set}}(X) \subseteq M_n^{**} \) such that \( (v, Y) \) is "taller" than all the \( D_i \)'s.\(^3\)

Finally,

\[ b = \{ i : (\exists x \in \overline{\text{set}}(Y) \psi(i, x, a))^{M_n^{**}} \}. \]

Hence \( b \in M_n^{**} \) by ACA.

For any \( \Sigma^1_1 \)-formula \( \exists x \forall m \exists n \theta(x, m, n) \), where \( \theta \) is \( \Pi^0_6 \), it follows from the normal form theorem that there is a \( \Pi^0_6 \) formula \( \bar{\theta}(\sigma, m, n) \) such that

\[ \forall x \left( \forall m \exists n \theta(x, m, n) \leftrightarrow \forall m \exists n \bar{\theta}(\bar{x}[s(m, n)], m, n) \right), \]

where \( s \) is some binary function symbol of PA and \( \bar{x}[k] \) denotes the finite sequence \( \sigma \) such that

\[ \text{ls}(\sigma) = k \land \forall i < k ((i < x \leftrightarrow (\sigma)_i = 1) \land ((\sigma)_i = 0 \lor (\sigma)_i = 1)) \].

The formula \( \forall m \exists n \bar{\theta}(\bar{x}[s(m, n)], m, n) \) is clearly equivalent to \( \forall m \exists n, k(\bar{\theta}(\bar{x}[k], m, n) \land k = s(m, n)) \). The latter formula may be written as \( \forall m \exists n \theta^*(\bar{x}[(n)_0], m, (n)_1) \) for some bounded formula \( \theta^* \).

Hence

\[ \exists x \forall m \exists n \theta(x, m, n) \leftrightarrow \exists x \forall m \exists n \theta^*(\bar{x}[(n)_0], m, (n)_1). \]

Let

\[ T_\theta = \{ (\sigma, \tau) \in 2^{<\omega} \times \omega^{<\omega} : \text{ls}(\sigma) = \max_{j < \text{ls}(\tau)} \{ (\tau_j)_0, \text{ls}(\tau) \} \land \forall i < \text{ls}(\tau) \theta^*(\sigma[(\tau_i)_0], i, (\tau_i)_1) \}. \]

\(^3\)See the proof of the second glue-up lemma, where we derived this fact for the case when there is no "tallest" one among the \( D_i \)'s. However if there is a "tallest" one, we may then apply lemma 2.3.15 to get a "taller" hierarchy to satisfy our requirement.
We write \( \langle \sigma_1, \tau_1 \rangle \subseteq \langle \sigma_2, \tau_2 \rangle \leftrightarrow \sigma_1 \subseteq \sigma_2 \land \tau_1 \subseteq \tau_2 \). We say \( T_\emptyset \) is not well-founded if there is a \( y \subseteq T_\emptyset \) such that

\[
\forall (\sigma_1, \tau_1), (\sigma_2, \tau_2) \in y ((\sigma_1, \tau_1) \subseteq (\sigma_2, \tau_2) \lor (\sigma_2, \tau_2) \subseteq (\sigma_1, \tau_1)) \land \exists i \exists (\sigma, \tau) \in y (ls(\sigma) \geq i).
\]

Otherwise it is called well-founded. For \( (i_1, n_1), (i_2, n_2) \in 2 \times \omega \), we say \( (i_1, n_1) \prec_0 (i_2, n_2) \) if \( n_1 < n_2 \lor (n_1 = n_2 \land i_1 < i_2) \). \( \prec_0 \) is a well-ordering of \( 2 \times \omega \). Let \( KB(T_\emptyset) \) be the usual Kleene-Brouwer ordering of \( T_\emptyset \) constructed from \( \prec_0 \). Both \( T_\emptyset \) and \( KB(T_\emptyset) \) exist by \( \Pi^0_0 \)-CA.

It is easy to see that

\[
(T_\emptyset \text{ is well-founded} \iff KB(T_\emptyset) \in WO)^{M^{**}}
\]

and

\[
(\exists x \forall m \exists n \theta^*(\bar{x}[\langle n \rangle_0], m, \langle n \rangle_1) \leftrightarrow T_\emptyset \text{ is not well-founded})^{M^{**}}.
\]

**Corollary.** For any \( S \Sigma^1_1 \)-formula \( \phi(y) \), \( \Sigma^0_3 \)-CA plus \( \Delta^1_n \)-CA proves that if \( \forall x \phi(x) \not\in Z_2 \) then

\[
\forall y \in M^{**} ((\phi(y))^{M^{**}} \leftrightarrow \phi(y)).^4
\]

**Proof:** Write \( \phi \) as

\[
\exists x \forall m \exists n \theta(y, x, m, n).
\]

Fix \( a \in M^{**} \). Obviously, if \( (\theta(a))^{M^{**}} \), then \( \theta(a) \). Thus it suffices to show that

\[
-(\theta(a))^{M^{**}} \implies -\theta(a).
\]

Suppose \( -(\theta(a))^{M^{**}} \). Inside \( M^{**} \), define \( T \) and \( KB(T) \). As indicated earlier,

\[
(KB(T) \in WO)^{M^{**}}.
\]

---

^4For the definition of \( S \Sigma^1_n \) and \( S \Pi^1_n \) formulas, see section 1.2.1.
By lemma 2.3.23 and lemma 1.4.1,

\[(\exists X RA(KB(T_0), X))^{M_n^{**}}.\]

Fix such an \(X \in M_n^{**}\). By lemma 2.3.16, there is a \(\Sigma_n^1\)-singleton \((v, Y)\) such that (1) \(SRA^*(v, Y)\) holds, (2) \(T_0, KB(T_0), X \in set(Y)\), and (3) \(set(Y) \models RA(KB(T_0), X)\). Hence for some \(j_0 \in dom(v)\),

\[\langle KB(T_0), X \rangle \equiv \langle v|_{j_0}, Y|_{j_0} \rangle.\]

Let

\[F = \{(i, j) : i \in T_0 \land j \leq v j_0 \land set(X_i) = set(Y_j)\}.\]

\(F\) exists by \(\Sigma_0^0\)-CA plus \(\Delta_0^1\)-CA. It is straightforward to check that \(F\) is an isomorphic embedding of \(KB(T_0)\) to \(v|_{j_0}\). By lemma 2.3.11, \(KB(T_0) \in WO\). But this contradicts the following claim.

Claim. \(\Sigma_0^0\)-CA plus \(\Delta_0^1\)-CA proves that if \(\exists x \forall m \exists n \theta^*\), then \(KB(T_0) \notin WO\).

Proof of the claim: Fix \(x_0\) such that \(\forall m \exists n \theta\). Let \(A\) be the set of all \((\sigma, \tau)\) such that the following hold:

- \(\forall i \leq l_\theta(\sigma) (\sigma_i = 1 \leftrightarrow i \in x_0)\);
- \(l_\theta(\sigma) = \max_{j < l_\theta(\tau)} \{(\tau_j)_0, l_\theta(\tau)\}\);
- \(\forall i < l_\theta(\tau)(\theta^*([\sigma_{(\tau)}], i, \tau_i) \land \forall j < \tau_i\theta([\sigma_{(\tau)}], i, j))\).

\(A\) exists by \(\Sigma_0^0\)-CA. It is easy to check that this \(A\) witnesses the non-well-foundedness of \(KB(T_0)\).
Theorem 2.3.1. There is a translation "*" of formulas of $L_2$ such that the following holds:

- if $\phi$ is $S\Sigma_1$, then $\phi^* = \phi$ and $\Sigma^0_0$-CA plus $\Delta^1_n$-CA proves that
  \[
  \forall \bar{y} \in M^{**} (\phi^{M**} \leftrightarrow \phi^*) \lor \exists x (x \models Z_2),
  \]
  where $\bar{y}$ is the block of quantifiers free in $\phi$;

- if $\phi = \exists x \theta(x) \in S\Sigma_k$ for some $2 \leq k \leq n-2$ then $\phi^* = \exists \psi$ for some $\psi \in S\Pi_{k+1}$, $\phi^*$ has the same free variables as $\phi$ and moreover $\Sigma^0_0$-CA plus $\Delta^1_n$-CA proves either $\exists x (x \models Z_2)$ or the following
  \[
  1. \forall x, z (\psi(x, \bar{y}) \land \psi(z, \bar{y}) \rightarrow x = z);
  
  2. \forall \bar{y} \in M^{**} (\phi^{M**} \leftrightarrow \phi^*),\text{ where } \bar{y}\text{ is the block of quantifiers free in } \phi;
  
  3. \forall \bar{y} \in M^{**}, x (\psi(x, \bar{y}) \rightarrow \theta^{M**}(x_1, \bar{y})).
  \]

Proof: We define "*" on $\Sigma^1_k$ formulas by induction on $k$.

If $\phi$ is $S\Sigma_1$, let $(\phi)^* = \phi$. In this case, the theorem is the same as the corollary to lemma 2.3.23.

We now consider the case when $\phi = \exists x \theta(x, \bar{y})$ and the least $k$, such that $\phi$ is $S\Sigma_k$, is greater than 1. If this $k$ is greater than $(n-2)$, we then define $(\phi)^*$ arbitrarily.

There is nothing to prove. We hence assume that $k \leq (n-2)$.

For simplicity, we assume that $\bar{y}$ is actually a single variable $y$.

If $k=1$, then $\phi^*$ is already defined in (a).
Let $A(x,u,X)$ denote the following formula:

$$
\exists \langle v,Y \rangle (RA^*(v,Y) \land \exists k (k = top(v) \land \langle u,X \rangle \in set(Y_k) \land \langle v|_{k=1},Y|_{k-1} \rangle) \land \\
\forall \langle w,Z \rangle \in set(Y_k) ((set(Y_k) \models RA(w,Z) \land \langle w,X \rangle \equiv \langle u,X \rangle) \rightarrow \langle w,Z \rangle \geq_{lex(v)} \langle u,X \rangle) \land \\
x \in \overline{set}(X)).
$$

By the corollary to lemma 3.2.5 and lemma 3.2.6, $A(x,u,X)$ is equivalent to a $S\Sigma^1_2$-formula. Without loss of generality, we assume that $A(x,u,X)$ is a $S\Sigma^1_2$-formula.

What $A(x,u,X)$ says is this: (1) $\langle u,X \rangle$ is equivalent to the portion of $\langle v,Y \rangle$ below (not include) the top level $Y_k$; (2) $\langle u,X \rangle$ is an element of $set(Y_k)$; (3) $x \in \overline{set}(X)$; and (4) $\langle u,X \rangle$ is the $v$-least element with these properties. The implications are:

(1) $\langle v,Y \rangle$ would be a strong hierarchy rather than just a hierarchy as asserted by $A(x,u,X)$ provide that $\langle u,X \rangle$ is a strong hierarchy and (2) $\langle u,X \rangle$ would be the $\leq_L$-least element with said properties, hence a $\Sigma^1_n$-singleton.

Now when $k = 2$, we simply let $\phi^*$ be

$$
\exists x, \langle u,X \rangle (SRA^*(u,X) \land A(x,u,X) \land \theta^*(x) \land \forall z \in \overline{set}(X) (z <_u x \rightarrow \neg \theta^*(z,y))).
$$

Since by lemma 2.3.10, $SRA^*(u,X)$ is $S\Pi^1_3$, and since by definition for the case of $k = 1$, $\theta^*(x)$ is $S\Pi^1_1$, it is easy to see that $\phi^*$ is $S\Sigma^1_4$.

The fact that $\phi^*$ satisfies the condition of the theorem will be verified together with the next case.

So we assume $k \geq 2$.

We first analyze the same formula which defines $\phi^*$ when $k = 2$, except now $\theta^*(z,y)$ is, by induction hypothesis, $S\Pi^1_{k+1}$. Also by induction hypothesis, we may write $\neg \theta^*(z,y)$ as $\exists ! \tilde{z} \sigma(\tilde{z},z,y)$, where $\sigma$ is a $S\Pi^1_k$ formula.
We write the following formula,
\[ \exists x, u, X, Z \ (SRA^*(u, X) \land A(x, u, X) \land \theta^*(x, y) \land \forall i, e (x \leq u \ \Lambda(X_i, e) \land Z(i, e) = \emptyset \lor \Lambda(X_i, e) < u \ x \land \sigma(Z(i, e), \Lambda(X_i, e), y))) \],

in the form \( \exists x, u, X, Z \ B(x, (u, X), Z, y) \).

Obviously, we may do this in a way such that \( B \) is a \( S\Sigma^1_{k+2} \)-formula. Finally let \( \phi^* \) be
\[ \exists x (B(x_0, x_1, x_2, y) \land \forall i \in x (ls(i) = 2 \land (i)_0 < 3)). \]

Then \( \phi^* \) is in \( S\Sigma^1_{k+2} \).

Now let us verify that \( \phi^* \) satisfies all three of the required conditions.

(1) (Uniqueness). Let \( \psi(x, y) \) be \( B(x_0, x_1, x_2, y) \land \forall i \in x \ ls(i) = 3 \). Suppose that \( \psi(x) \) and \( \psi(x^*) \). We want to conclude that \( x = x^* \). Evidently, it suffices to show that \( x_0 = x_0^*, x_1 = x_1^* \) and \( x_2 = x_2^* \).

Write \( x_1 \) as \( (u, X) \) and \( x_1^* \) as \( (u^*, X^*) \). Then from the definition of \( \phi^* \), one can see that
\[ SRA^*(u, X) \land x_0 \in \bar{set}(X) \land \theta^*(x_0, y) \land \forall z < u \ x_0 - \theta^*(z, y) \]
and
\[ SRA^*(u^*, X^*) \land x_0^* \in \bar{set}(X^*) \land \theta^*(x_0^*, y) \land \forall z < u^* \ x_0^* - \theta^*(z, y) \]

From the comparability of \( (u, X) \) and \( (u^*, X^*) \), we may easily conclude that \( x_0 = x_0^* \). Now since \( x_1 \) is the \( <_L \)-least strong hierarchy that contains \( x_0 \) and \( x_1^* \) is the \( <_L \)-least strong hierarchy that contains \( x_0^* \), we must also have \( x_1 = x_1^* \). Similar we may check that \( x_2 = x_2^* \).\\

If we were discussing the case when \( k=2 \), then we only have \( x_0 \) and \( x_1 \) to deal with, which is a strictly simpler situation. The same remark applies to other parts of the verification.
(2) (⇒). Fix an \( a \in M_n^{**} \). Suppose that \( (\exists x \theta(x, a))^{M_n^{**}} \). Fix a \( b \in M_n^{**} \) such that \( \theta(b, a)^{M_n^{**}} \). By induction hypothesis, we have \( \theta^*(a, b) \). By an earlier lemma, there is some \( \langle u, X \rangle \in M_n^{**} \) such that \( SRA^*(u, X) \) and \( b \in \overline{set}(X) \). Let
\[
C = \{ i \in \text{dom}(u) : \exists x (x \in \text{set}(X_i) \land \theta^*(x, a)) \}.
\]
Since \( \theta^*(x, a) \) may be written as \( \neg \exists ! \bar{z} \sigma(\bar{z}, x, a) \) with \( \sigma \in \Pi_1^1 \) and since \( \langle u, X \rangle \) and \( a \) are \( \Sigma_n^1 \)-singleton and \( \exists x \) is essentially a numerical quantifier, we know \( C \) exists by \( \Delta_n^1 \)-CA. Let \( i_0 \) be the \( u \)-least of \( C \) and \( e_0 \) be the least such that \( \theta^*(\Lambda(X_{i_0}, e_0), a) \).

Then \( \bar{x} = \Lambda(X_{i_0}, e_0) \) is the \( <_L \)-least element such that \( \theta^*(\bar{x}, a) \). Let \( \langle u, X \rangle \) be the \( <_L \)-least strong hierarchy such that \( \bar{x} \in \overline{set}(X) \). Then it is straightforward to check
\[
(SRA^*(u, X) \land A(\bar{x}, u, X) \land \theta^*(\bar{x}) \land \forall i, e \exists ! z(\bar{x} \leq_u \Lambda(X_i, e) \land z = \emptyset \lor \sigma(z, \Lambda(X_i, e), y))).
\]
Since all the parameters involved in \( \forall i, e \exists ! z(\bar{x} \leq_u \Lambda(X_i, e) \land z = \emptyset \lor \sigma(z, \Lambda(X_i, e), y)) \) are \( \Sigma_n^1 \)-singletons, we may apply \( \Delta_n^1 \)-CA to get a \( Z \) such that
\[
(SRA^*(u, X) \land A(\bar{x}, u, X) \land \theta^*(\bar{x}) \land \forall i, e (\bar{x} \leq_u \Lambda(X_i, e) \land Z(i, e) = \emptyset \lor \sigma(Z(i, e), \Lambda(X_i, e), y))).
\]
Finally let \( x \) be such that \( x_0 \) is \( \bar{x} \), \( x_1 \) is \( \langle u, X \rangle \) and \( x_2 \) is \( Z \). Obviously this \( x \) is a witness to the truth of \( \phi^* \).

(⇐). Conversely we assume \( \phi^* \) is true. Let \( x \) be a witness to \( \phi^* \). Then by uniqueness, \( x \) is a \( \Sigma_n^1 \)-singleton. In particular \( x_0 \) is a \( \Sigma_n^1 \)-singleton. So by the definition of \( \phi^* \), \( x_0 \in M_n^{**} \) and \( \theta^*(x_0, a) \). By induction hypothesis, \( (\theta(x_0, a))^{M_n^{**}} \). Thus \( \phi^{M_n^{**}} \).

(3). This is by the definition of \( \phi^* \).
Lemma 2.3.24. For all \((k \leq (n - 2))\) and \(\phi \in Wk-\Sigma_k^1-AC\), \(\Delta_n^1-CA\) plus \(\Sigma_0^0-CA\) proves that \((\phi)^{M_n^{**}} \vee (\exists x)(x \models Z_2)\).

Proof: Let \(\theta \in \Pi^1_{k-1}\) and \(a \in M_n^{**}\) be such that

\[M_n^{**} \models \forall i \exists x \theta(i, x, a).\]

Then by Theorem 2.3.1, we have

\[\forall i(\exists x \theta(i, x, a))^*.\]

Write this formula as

\[\forall i \exists! x \psi(i, x, a),\]

where \(\psi(i, x, a)\) is \(S\Pi^1_{k+1}\). Recall that this is done in the proof of Theorem 2.3.1.

Let

\[y = \{(i, j) : \exists x(\psi(i, x, a) \land j \in x_0)\}.\]

Then \(y\) exists by \(\Delta_n^1-CA\) and \(y\) is also a \(\Sigma_n^1\)-singleton. By theorem 2.3.1,

\[\forall i \ (y_i \in M^{**} \land \psi(i, y_i, a)).\]

Let \(D\) be such that for each \(i\), \(D_i = \langle u_i, X_i \rangle\) is the \(<_L\)-least strong hierarchy such that \(y_i \in \overline{set}(X_i)\). Then \(D\) exists by \(\Delta_n^1-CA\) and \(D\) is also a \(\Sigma_n^1\)-singleton. If \(\overline{set}(X) \models Z_2\) for some \(X\), then we are done. Otherwise by using the second-glue-up lemma, we may get a strong hierarchy \(\langle w, Z \rangle \in M_n^{**}\) such that \(\langle w, Z \rangle\) is a \(\Sigma_n^1\)-singleton which codes the union of \(\{(u_i, X_i)\}\). It is obvious that

\[\forall i \exists k, e \phi^*(i, \Lambda(Z_k, e), a),\]
where $\phi$ is $\exists x \theta$. And since $(\forall k, e) \Lambda(Z_k, e) \in M_n^{**}$, it follows from Theorem 2.3.1 that

$$M_n^{**} \models \forall i \exists k, e \phi(i, \Lambda(Z_k, e), a).$$

Let $Y$ be such that

$$Y = \{(e, k, j) : \text{Sub}^0(e, \text{Num}(j)) \in Z_k\}.$$

Then $Y \in M_n^{**}$ by ACA. Obviously,

$$M_n^{**} \models \forall i \exists j \phi(i, Y, a).$$

**Lemma 2.3.25.** $Wk-\Sigma_k^1$-AC plus ACA proves $\Delta_k^1$-CA.

**Proof:** We argue by induction on $k$.

Assume the lemma is true for $k - 1$.

Let $\phi(i) \in \Sigma_k^1$ and $\psi(i) \in \Pi_k^1$ be such that

$$\forall i (\phi(i) \leftrightarrow \psi(i)).$$

Write $\phi(i)$ as $\exists x \theta_1(i, x)$ and $\psi$ as $\forall x \theta_2(i, x)$. Hence

$$\forall i \exists x (\theta_1(i, x) \lor \theta_2(i, x)).$$

Apply quasi-$\Sigma_k^1$-CA once to get an $x$ such that

$$\forall i \exists k (\theta_1(i, (x)_k) \lor \theta_2(i, (x)_k)).$$

Then obviously

$$\forall i, k (\theta_1(i, (x)_k) \leftrightarrow \theta_2(i, (x)_k)).$$
So by $\Delta^1_{k-1}$-CA, the set

$$a = \{(i, k) : \theta_1(i, (x)_k)\}$$

exists. It follows that

$$\{i : \phi(i)\} = \{i : (\exists k)((i, k) \in a)\}$$

also exists.

Note: $\text{ACA}_0$ is used when we handle the case when $k = 1$.

**Theorem 2.3.2.** For $\phi \in \Delta^1_{n-2}$-CA,

$\Delta^1_n$-CA plus $\Sigma^b_0$-CA proves $(\phi)^{M^{**}} \lor (\exists x(x \models Z_2))$.

**Proof:** Apply the above two lemmas.

### 2.4 R.A. hierarchy in $\Delta^1_n$-CA plus ACA($n \geq 6$)

In the last section, we have built many strong properties into the definition of hierarchy codes for us to eventually establish Theorem 2.3.2. The reason behind this is the lack of ACA. In this section we will see that, for the purpose of constructing the ramified analytical hierarchy, $\Delta^1_n$-CA plus ACA behaves very much like the corresponding bold-face theory. So let us go back to the original definition of hierarchy. Again note that $\Delta^1_n$-CA is always light-faced in this section.

**Definition.** $RA(v, Y)$ is the conjunction of the following clauses:

1. $v$ codes a well-ordewring;

2. $Y = \bigcup_{i \in \text{dom}(v)} \{i\} \times Y_i$;
3. for any $i \in \text{dom}(v)$

- if $i = j +_v 1$ then $Y_i = \text{Sat}(Y_j)$;

- if $i$ is a limit then $Y_i = \cup_{j < i} \{j +_v 2\} \times Y_{j +_v 2}$.

4. $\forall i, j (i <_v j \rightarrow \text{set}(Y_i) \subseteq \text{proper set}(Y_j))$.

**Theorem 2.4.1.** ACA proves that if $\text{RA}(u, X), i \in \text{dom}(u)$ and $i$ is not the top of $u$ then

1. there is a well-ordering (subject to the interpretation of lemma 1.4.4) $<_i$ of $\text{set}(X_i)$ which is definable over $\text{set}(X_i)$ itself;

2. there is an integer $n$ such that $\forall x \in \text{set}(X_i), x$ is a $\Sigma^1_n$-definable singleton over $\text{set}(X_i)$;

3. there is an element $(v, Y) \in \text{set}(X_{i+u1})$ such that $\text{set}(X_{i+u1}) \models \text{RA}(v, Y)$ and $(v, Y) \equiv (u|_i, X|_i)$.

Clearly, the statement we want to prove in ACA can be written as

$$\text{RA}(u, X) \rightarrow \forall i \in \text{dom}(u) \; A(i, u, X)$$

for some arithmetical formula $A(i, u, X)$ with all its free variables shown.

**Proof of theorem 1:** Fix a $(u, X)$ such that $\text{RA}(u, X)$. We will prove, in ACA, that

$$\forall i \in \text{dom}(u) \; A(i, u, X)$$

by transfinite induction on $i$. 
First let us point out that the third clause of the theorem really implies that RA(v, Y) is true since (1) v must be a well-ordering for it is isomorphic to u; and (2) the rest of the properties asserted by RA(v, Y) all follow from that set(X_{i+1}) \models RA(v, Y) because those properties are all arithmetical.

All the arguments used in the proof of theorem 2.1.1 can also be used here with one single exception: the Skolem-hull argument needs to be modified so that it can be carried out in ACA.

We proceed as follows.

The corresponding situation in our current case is that we have \( i \in \text{lim}(u) \) and the following:

- \( \text{set}(X_i) \models \forall x \exists v, Y (RA(v, Y) \land x \in \text{set}(Y)) \);

- there is a \( \Sigma^1_k \)-formula \( \phi(t) \) with no free set variables such that

\[
\begin{align*}
b &= \{ k : \text{set}(X_i) \models \phi(k) \} \not\in \text{set}(X_i); \end{align*}
\]

- the induction hypothesis is true for all \( j < u; i \);

- \( \text{set}(X_i) \not\models Z_2 \).

We want to conclude that, for some \( m \) and for all \( x \in \text{set}(X_i) \), \( x \) is a \( \Sigma^1_m \)-singleton over \( \text{set}(X_i) \). Let \( l \) be the least such that \( \text{set}(X_i) \not\models \Sigma^1_l \)-CA. \( l \) exists because \( \text{set}(X_i) \not\models Z_2 \).

Let \( \Phi(e, t, x) \equiv \exists y \theta(e, t, x, y) \) be universal for \( \Sigma^1_l \)-formulas with the distinguished free numerical variable \( t \) and with \( x \) as its only free set variable.
Let \( \{\exists y_1 \sigma_1, \exists y_2 \sigma_2, \ldots\} \) be the (finite) set of formulas which are either subformulas of \( \exists y \theta(e, t, x, y) \) or their negation are subformulas of \( \exists y \theta(e, t, x, y) \). We want to produce a skolem hull in \( \text{set}(X_i) \) for this finite many formulas. For the sake of simplicity, we assume that \( \{\exists y_1 \sigma_1, \exists y_2 \sigma_2, \ldots\} \) only contains a single element, say, \( \exists y \theta(e, t, x, y) \).

Let \( \psi((j, k), (u, X), i) \) be the following formula:

\[
(j \leq u \leq i)(\forall e, t \forall x \in \text{set}(X_j)(\text{set}(X_i) \models \exists y \theta(e, t, x, y) \rightarrow \exists y \in \text{set}(X_k)\text{set}(X_i) \models \theta(e, t, x, y))).
\]

Then \( \psi((j, k), (u, X), i) \) is an arithmetical formula. Obviously, we have \( \forall j \exists k \psi((j, k), (u, X), i) \).

Hence \( \forall j \exists k \psi((j, k), (u, X), i) \land (\forall k^* < u \psi((j, k^*), (u, X), i)) \). Let \( \sigma(j, k, i, u, X) \) denote \( \psi((j, k), (u, X), i) \land (\forall k^* < u \psi((j, k^*), (u, X), i)) \). Let \( i_0 \) be the least element of \( u \).

**Claim 1.** \( \forall k \exists t((t : (k + 1) \rightarrow \text{dom}(u)) \land t(0) = i_0 \land (\forall m < k)(\sigma(t(m), t(m + 1), i, u, X)) \)

The proof of the claim is straightforward by induction on \( k \).

Now, write the formula in Claim 1 as \( \forall n \exists t \sigma^*(i, t, n, u, X) \). Let

\[
f = \{(j, k) : (\exists t)(\sigma^*(i, t, j + 1, u, X) \land t(j) = k)\}.
\]

Then obviously

\[
f(0) = i_0 \land \forall j \sigma(f(j), f(j + 1), i, u, X).
\]

**Claim 2.** For any \( m \) there is a \( (v_m, Y_m) \in \text{set}(X_i) \) such that \( RA(v_m, Y_m) \), \( v_m \) has a top element, \( (v_m, Y_m) \equiv (\text{set}(X_i) \models \sigma \land (\forall m \exists t \exists t^* (t^* < t \rightarrow \sigma(t^*) \land t^* = \emptyset)) \). Let \( (v_m, Y_m) \) be a \( \Sigma^1_{i+3} \)-singleton over \( \text{set}(X_i) \) or \( \forall x \in \text{set}(X_i) (x \) is a \( \Sigma^1_{i+3} \)-singleton over \( \text{set}(X_i) \))

**Proof:** We show this claim by induction on \( m \).
When \( m = 0 \), \( \text{set}(X_{f(0)}) = \text{set}(X_0) = \{\emptyset\} \). \( \emptyset \) is a \( \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \).

Assume the claim is true for \( m \). If we already have \( \forall x \in \text{set}(X_i) \ (x \text{ is a } \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \)\), we are done. Otherwise, fix \( \langle v_m, Y_m \rangle \in \text{set}(X_i) \) such that \( \langle v_m, Y_m \rangle \equiv \langle u_{f(m)}, X^{f(m)} \rangle \).

Case 1. \( f(m + 1) = i \). In this case, we may conclude that \( \forall x \in \text{set}(X_i) \ (x \text{ is a } \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \)\).

Fix \( x \in \text{set}(X_i) \). Then for some \( j <_{ \text{u} } i \), \( x \in \text{set}(X_j) \). Fix \( a \in \text{set}(Y_m) \) so that \( \text{set}(X_i) \models \exists y \sigma(e, t, a, y) \) and the \( \prec_i \)-least \( b \) for which \( \text{set}(X_i) \models \sigma(e, t, a, b) \) is such that the \( \text{u} \)-least \( k \) for which \( b \in \text{set}(X_k) \) is \( \text{u} \)-larger than \( j \). Let \( \langle v, Y \rangle \) be the \( \prec_i \)-least hierarchy in \( \text{set}(X_i) \) such that \( \langle v, Y \rangle \equiv \langle u_{k}, X^{k} \rangle \). Then the following shows that \( \langle v, Y \rangle \) is a \( \Sigma_{n+3}^1 \)-singleton over \( \text{set}(X_i) \):

\[
\forall \langle w, Z \rangle (\langle w, Z \rangle = \langle v, Y \rangle \iff \text{RA}(w, Z) \land \langle w, Z \rangle \geq \langle v_m, Y_m \rangle \land \exists b \in \text{set}(Z)(\text{set}(X_i) \models \theta(e, t, a, b)) \land \forall \langle \bar{w}, \bar{Z} \rangle < \langle w, Z \rangle \forall b \in \text{set}(\bar{Z})(\text{set}(X_i) \not\models \theta(e, t, a, b))).
\]

Now, \( x \in \text{set}(X_j) \subseteq \text{set}(Y) \). Let \( j_0 \in \text{dom}(v) \) and \( e_0 \in Frml^0 \) be such that

\[
x = \Lambda(Y_{j_0}, e_0).
\]

Since \( \langle v, Y \rangle \) is a \( \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \), it follows that \( x \) is also a \( \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \).

Case 2. \( f(m + 1) <_{ \text{u} } i \).

As in Case 1, the \( \prec_i \)-least hierarchy \( \langle v, Y \rangle \) in \( \text{set}(X_i) \) such that \( \langle v, Y \rangle \equiv \langle u_{f(m)+1}, X^{f(m)} \rangle \) is a \( \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \). Let \( \langle v_{m+1}, Y_{m+1} \rangle = \langle v, Y \rangle \). This obviously satisfies the conditions of the claim.
In any case, we know that

\[ \forall m \forall x \in \text{set}(X_{f(m)})(x \text{ is a } \Sigma_{i+3}^1 \text{ singleton over set}(X_i)). \]

Thus if \( f(m) = i \) for some \( m \), then, as in case 1, we are done. So we may assume that \( \forall m(f(m) < u i) \). If \( f \) has a maximum value, say \( f(m_0) \), then

\[ b = \{ t : \text{set}(X_i) \models \phi(t) \} = \{ t : \text{set}(X_{f(m_0)}) \models \phi(t) \}. \]

Hence \( b \in \text{set}(X_i) \), which is a contradiction.

So we may assume that \( f \) is strictly increasing. Let \( i^* \leq u i \) be the supremum of \( f \). If \( i^* < u i \) then again

\[ b = \{ t : \text{set}(X_i) \models \phi(t) \} = \{ t : \text{set}(X_{i^*}) \models \phi(t) \} \in \text{set}(X_i), \]

which is a contradiction. Hence we must have \( i^* = i \), which implies that every element of \( \text{set}(X_i) \) is a \( \Sigma_{i+3}^1 \)-singleton over \( \text{set}(X_i) \).

Q.E.D.

With the above theorem, it is easy to see that if \( RA(u, X) \), then ACA proves that \( \langle u, X \rangle \) (provided that \( u \) has a top which is not a limit) has all the properties of \( SRA^*(u, X) \) except \( \exists y y = T.J^{(\omega)}(\langle u, X \rangle) \). This property is built into the definition of \( RA^* \) purely for the purpose of being able to use lemma 2.3.5, which is no longer needed when ACA is given. It is never used in proving any of the results regarding \( SRA^* \) hierarchies. It follows that all those properties proved in section 2.3 would still be valid with \( SRA^* \) replaced by RA (provided that the ordering has a top which is not a limit) and \( \Pi_0^0 \)-CA replaced by ACA.
Definition:

\[ M^\ast = \{ x \in M : \exists (u, X)(RA(u, X) \land x \in \text{set}(X)) \}; \]

\[ M^\ast_n = \{ x \in M^\ast : x \text{ is a } \Sigma^1_n\text{-singleton} \}. \]

Note that, formally speaking, what we are doing here is to define two formulas \( M^\ast(x) \) and \( M^\ast_n \) like in section 2.3. The sets in \( M^\ast \) are called the RA sets. Hence \( M^\ast_n \) consists of all the \( \Sigma^1_n\)-singleton RA sets.

We now define some operations on hierarchies. They will only be used in the proof of theorem 2.4.2.

If \( \langle u, X \rangle \) is a hierarchy, then \( \{k\} \times \langle u, X \rangle \) denotes the hierarchy \( \langle u^\ast, X^\ast \rangle \) such that

- \( \text{dom}(u^\ast) = \{k\} \times \text{dom}(u); \)

- \( \langle k, i \rangle \preceq^\ast \langle k, j \rangle \iff i \preceq_j j; \)

- \( X^\ast \) is such that (1) \( X^\ast_{\langle k, i \rangle} \) is \( X_i \) when \( i \) is \( j +_u 2 \) for some \( j \); (2) if \( i \) is a limit,
  then \( X^\ast_{\langle k, i \rangle} \) is the union of all \( \{\langle k, j \rangle\} \times X_j \) such that \( j <_u i \) and \( j = j' +_u 2 \) for some \( j' \); (3) if \( i = j +_u 1 \) for some limit \( j \), then

\[ X^\ast_{\langle k, i \rangle} = \{\langle e, n, l^\ast \rangle : (e, n, l) \in X_i \}, \]

where \( l^\ast \) is obtained from \( l \) by replacing, for each \( m < \text{ls}(l) \), \( l_m = \langle j, e_0 \rangle \) with \( \langle\langle k, j \rangle, e_0 \rangle \).

If \( \langle u, X \rangle \) and \( \langle u^\ast, X^\ast \rangle \) are two hierarchies such that \( \langle u, X \rangle \leq \langle u^\ast, X^\ast \rangle \) and \( \text{dom}(u) \cap \text{dom}(v) \neq \emptyset \) then \( \langle u, X \rangle + \langle u^\ast, X^\ast \rangle \) denotes the hierarchy \( \langle \hat{u}, \hat{X} \rangle \) defined as follows:
\begin{itemize}
  \item \(\text{dom}(\hat{u}) = \text{dom}(u) \cup \{ j \in \text{dom}(u^*) : \forall i \in \text{dom}(u) \text{set}(X_i) \subseteq \text{proper set}(X_j^*)\}\);
  \item \((i \leq_u j) \leftrightarrow ((i \leq_u j) \lor (i \in \text{dom}(u) \land j \in \text{dom}(u^*)) \lor (i \leq_u j))\);
  \item for each \(i \in \text{dom}(u^*)\),
    \begin{enumerate}
      \item \(\hat{X}_i\) is \(X_i\) if \(i \in \text{dom}(u)\) and \(i = j +_u 2\) for some \(j\), or \(X_i^*\) if \(i \not\in \text{dom}(u)\) and \(i = j +_u 2\) for some \(j\);
      \item if \(i\) is a limit and \(i \in \text{dom}(u)\), then \(\hat{X}_i = X_i\), or if \(i\) is a limit and \(i \not\in \text{dom}(u)\), then \(\hat{X}_i\) is the union of all \(\{j\} \times X_j\) such that \(j <_u i, j = j' +_u 2\) for some \(j'\) together with all \(\{j\} \times X_j^*\) such that \(j <_u i, j = j' +_u 2\) for some \(j'\) and \(j \in \text{dom}(\hat{u})\); \(\)
      \item if \(i = j +_u 1\) for some limit \(j\) and \(i \in \text{dom}(u)\) then \(\hat{X}_i = X_i\); otherwise \(\hat{X}_i = \{(e, n, l^*) : (e, n, l) \in X_i\}\), where \(l^*\) is obtained from \(l\) by replacing, for each \(m < l_3(l)\), \(l_m = (j, e_0)\) with \((j', e_0)\) if \(j \not\in \text{dom}(\hat{u})\) and \(\text{set}(X_{j'}) = \text{set}(X_j^*)\). \(\)
    \end{enumerate}
\end{itemize}

**Theorem 2.4.2.** There is a translation "*" of formulas of \(L_2\) such that if \(\phi\) is \(\Sigma_1\), then \(\phi^* = \phi\), and if \(\phi = \exists x \theta(x, y) \in \Sigma^1_k\) for some \(2 \leq k \leq n\), then \(\phi^* = \exists x \psi \in \Sigma^1_k\), where \(\psi \in \Pi^1_{k-1}\), and \(\phi^*\) has the same free variables as \(\phi\), such that \(\Delta^1_n\text{-CA plus ACA}\) proves the following

\begin{enumerate}
  \item \(\forall x, z (\psi(x, y) \land \psi(z, y) \rightarrow x = z)\);
  \item \(\forall \bar{y} \in M^{**} (\phi^{M^{**}} \leftrightarrow \phi^*)\), where \(\bar{y}\) is the block of quantifiers free in \(\phi\); \(\)
\end{enumerate}
3. \( \forall \bar{y} \in M^{**}, x (\psi(x, \bar{y}) \rightarrow \theta^{M^{**}}((x)_{1}, \bar{y})). \)

**Proof:** As in the definition given in the proof of theorem 2.3.1, we define \( \phi^* \) to be \( \phi \) itself when it is \( \Sigma_1^{1} \). When \( k \geq 2 \), we need to consider the two cases, \( k = 2 \) and \( k > 2 \).

In the rest of the proof, we assume \( \bar{y} \) is actually a single variable \( y \).

**Case 1.** When \( k=2 \), by definition \( (\theta)^* = \theta \). Let us write \( \neg \theta \) as

\[
\exists z A(x, y, z),
\]

where \( A \) is an arithmetical formula.

Consider the arithmetical formula \( B(x, y, z, u, X) \):

there are \( i, j \in \text{dom}(u) \) such that the conjunction of the following clauses:

1. \( \langle u|_{i}, X|^{i} \rangle = \{0\} \times \langle v_{1}, Y_{1} \rangle \), where \( \langle v_{1}, Y_{1} \rangle \in \text{set}(X_{i+u}) \) is the \( u \)-least hierarchy such that \( x \in \overline{\text{set}}(Y_{1}) \);  

2. \( \langle u|_{j}, X|^{i} \rangle = \{0\} \times \langle v_{1}, Y_{1} \rangle + \{1\} \times \langle v_{2}, Y_{2} \rangle \), where \( \langle v_{2}, Y_{2} \rangle \in \text{set}(X_{j+u}) \) is the \( u \)-least hierarchy such that \( x, z \in \overline{\text{set}}(Y_{2}) \);  

3. the top of \( u \) is "2" and \( j + u 1 = 2 \);  

4. \( \forall e \in Frlm^0 (\Lambda(X, e) \geq u x \land z \in \emptyset \lor \Lambda(X, e) < u x \land A(\Lambda(X, e), y, z)) \), and \( z \) is the \( u \)-least element with this property.

Let \( \psi(x, \langle u, X \rangle, z, y) \) be the following formula

\[
\exists x, \langle u, X \rangle, z (RA(u, X) \land B(x, y, z, u, X) \land \theta(x, y)).
\]
Recall that $RA(u, X)$ is a $\Pi^1_1$ formula in conjunction with an arithmetic formula. We define $\phi^*$ to be

$$\exists x(\psi(x_0, x_1, x_2, y) \land \forall m \in x \ i(m) = 3).$$

We now show that this definition satisfies all the required conditions.

(1) (Uniqueness). Fix $x$ and $x^*$ such that $\psi(x_0, x_1, x_2, y)$ and $\psi(x_0^*, x_1^*, x_2^*, y)$. Write $x_1$ as $(u, X)$ and $x_1^*$ as $(u^*, X^*)$. Firstly, $B(x_0, y, x_2, u, X)$ and $B(x_0^*, y, x_2^*, u^*, X^*)$ assert that $x_0$ and $x_0^*$ are both the $<_L$-least $t$ such that $\theta(t, y)$. Hence $x_0 = x_0^*$. Let $(v_1, Y_1)$ be the $<_L$-least hierarchy such that $x_0 \in \text{set}(Y_1)$. There are $i \in \text{dom}(u)$ and $i^* \in \text{dom}(u^*)$ such that $(u|_i, X|_i) = \{0\} \times \langle v_1, Y_1 \rangle = (u^*|_{i^*}, X^*|_{i^*})$. In particular, $i = i^*$. Now by the definition of $Z$, $x_2 = Z$ is the $<_L$-least element satisfying

$$\forall e \in \text{Frml}^0 (\langle \Lambda(X_i, e) <_u x \land A(\Lambda(X_i, e), y, Z_e) \rangle \lor (\Lambda(X_i, e) \geq u x \land Z_e = \emptyset)), $$

which is exactly the same as

$$\forall e \in \text{Frml}^0 (\langle \Lambda(X_i^*, e) <_{u^*} x \land A(\Lambda(X_i^*, e), y, Z_e) \rangle \lor (\Lambda(X_i^*, e) \geq_{u^*} x \land Z_e = \emptyset)), $$

which in turn is the definition of $Z^*$. Hence $Z = Z^*$; i.e, $x_2 = x_2^*$.

Finally let $(v_2, Y_2)$ be the $<_L$-least hierarchy such that $x_0, x_2 \in \text{set}(Y_2)$. Then

$$\langle u|_j, X|_j \rangle = \{0\} \times \langle v_1, Y_1 \rangle + \{1\} \times \langle v_2, Y_2 \rangle = (u^*|_j, X^*|_j),$$

where $j$ and $j^*$ are the second "highest" points of $u$ and $u^*$ respectively. We also required both $u$ and $u^*$ have the same top; i.e, "2". Hence we must have $(u, X) = (u^*, X^*)$; i.e, $x_1 = x_1^*$. Since both $x$ and $x^*$, by definition, consists of triples we must be able to deduce $x = x^*$ from $x_0 = x_0^*, x_1 = x_1^*$ and $x_2 = x_2^*$. 

(2) \( \Rightarrow \). Fix \( a \in M_n^{**} \) such that \((\exists \theta(x,a))^{M_n^{**}}\). Working inside \( M_n^{**} \) and applying \( \Pi^1_1 \)-CA (lemma 2.3.23), we may get the \( <L \)-least element \( b_0 \) such that \( \theta(b,a)^{M_n^{**}} \), which, by lemma 2.3.16, lemma 2.3.18 and the corollary of lemma 2.3.14, is actually the \( <L \)-least element such that \( \theta(b,a) \). While still in \( M_n^{**} \), we get a \( <L \)-least hierarchy \( \langle v_1,Y_1 \rangle \) such that \( b_0 \in \overline{\text{set}}(Y_1) \). Let \( \langle u,X \rangle \) be a hierarchy such that \( \langle u,|X| \rangle = \{0\} \times \langle v_1,Y_1 \rangle \). Then clearly,

\[
\forall e \ (\Lambda(X_i,e) < u b_0 \rightarrow \exists z \Lambda(\Lambda(X_i,e),a,z)).
\]

Hence

\[
\forall e \exists z \ ((\Lambda(X_i,e) < u b_0 \wedge \Lambda(\Lambda(X_i,e),a,z)) \vee (\Lambda(X_i,e) \geq u \wedge z = \emptyset)).
\]

Following the proof of lemma 2.3.23, we may get a \( z \in M_n^{**} \) such that\(^6\)

\[
\forall e \ ((\Lambda(X_i,e) < u b_0 \wedge \Lambda(\Lambda(X_i,e),a,z)) \vee (\Lambda(X_i,e) \geq u \wedge z = \emptyset)).
\]

Let \( b_1 \) be the \( <L \)-least element satisfying this property (it exists in \( M_n^{**} \)). Let \( \langle v_2,Y_2 \rangle \) be the \( <L \)-least hierarchy such that \( b_0, b_1 \in \overline{\text{set}}(Y_2) \). Let \( \langle u,X \rangle \) be a hierarchy whose top is "2" and

\[
\langle u|_{2^{-u_1}},|X|^{2^{-u_1}} \rangle = \{0\} \times \langle v_1,Y_1 \rangle + \{1\} \times \langle v_2,Y_2 \rangle = \langle u^*|_{j},|X^*|^{j} \rangle.
\]

Finally, let \( x \) be defined as follows

\[
x = \{ \langle i,j,k \rangle : i \in b_0 \wedge j \in \langle u,X \rangle \wedge k \in b_1 \}.
\]

\(^6\)Now, this is the reason why we have to separate the case when \( k=2 \) with the case when \( k > 2 \). When \( k > 2 \), we would generally not be able to get such a \( z \) inside \( M_n^{**} \), but we may apply the induction hypothesis, which says among other things that the \( z' \) such that \( A(x,y,z') \) must be unique if it exists, to get a unique \( z \) outside \( M_n^{**} \) (using \( \Delta^1_n \)-CA).
It is straightforward to check that this $x$ witnesses the truth of $\phi^*$.

($\Leftarrow$) Conversely let us assume $\phi$ is true. Let $x$ be a witness to $\phi^*$. Then by uniqueness, $x$ is a $\Sigma^1_n$-singleton. In particular $x_0$ is a $\Sigma^1_n$-singleton. So by the definition of $\phi^*$, $x_0 \in M'^{**}$ and $\theta^*(x_0, a)$. By induction hypothesis, $(\theta(x_0, a))^{M'^{**}}$. Thus $\phi^{M'^{**}}$.

(3) The proof is trivial.

Case 2. In this case, by induction hypothesis $(-\theta)^*$ may be written as

$$\exists ! z \sigma(x, y, z)$$

for some $\sigma \in \Pi^1_{k-1}$.

The definition is similar and easier.

Consider the arithmetical formula $B(x, y, z, u, X)$:

there is an $i \in \text{dom}(u)$ such that the conjunction of the following clauses hold:

1. $\langle u \rangle_i, X \rangle_i \rangle = \{0\} \times \langle v_1, Y_1 \rangle$, where $\langle v_1, Y_1 \rangle \in \text{set}(X_{i+1})$ is the $u$-least hierarchy such that $x \in \overline{\text{set}(Y_1)}$;

2. the top of $u$ is "1" and $j + u 1 = 1$;

3. $\forall e \in Frlm^0 ((\Lambda(X_i, e) \geq_u x \wedge z_e = \emptyset) \vee (\Lambda(X_i, e) <_u x \wedge \sigma(\Lambda(X_i, e), y, z_e)))$.

Let $\psi(x, (u, X), z, y)$ be the following formula

$$\exists x, (u, X), z (RA(u, X) \wedge B(x, y, z, u, X) \wedge \theta^*(x, y))$$

Let $\phi^*$ be

$$\exists x \psi(x_0, x_1, x_2, y).$$
Then $\phi^*$ is in $\Sigma^1_k$.

We may follow the same idea as in the proof of theorem 2.3.1 to check that this definition satisfies all the required conditions. The only difference as we noted before is that we need to use the condition that $(\neg \theta)^*$ may be written as

$$\exists ! \sigma(x, y, z).$$

Use this uniqueness and $\Delta^1_n$-CA to collect all those witnesses together to form a single element. Then this new element would also be unique.

Q.E.D.

**Corollary.** ACA plus $\Delta^1_n$-CA proves $\phi^{M^{**}}$, where $\phi \in \Delta^1_n$-CA.

**Proof:** Exactly the same as lemma 2.3.24, we may prove that $M^{**}$ is a model of WK-$\Sigma^1_n$-AC. Then apply lemma 2.3.25.
CHAPTER III

Upper bounds

3.1 Introduction

We are going to formalize the Borel diagonalization theorems and give a proof regarding the upper bounds of those theorems in the formal language $L_2$. Though theorems, lemmas as well as corollaries are stated in a strictly formal sense, proofs and remarks are less so as it is practically impossible to treat everything totally formally.

A partial list of abbreviations often used in this chapter is given below. The others will be defined when they are first used.

"$a \subseteq_1 b$" means

$$(ls(a) \leq ls(b)) \land (\forall i < ls(a))(a)_i = (b)_i;$$

"$a \subseteq_2 b$" means

$$(ls(a) \leq ls(b)) \land (\forall i < ls(a))(a)_i \subseteq_1 (b)_i;$$

"$g(i)(j) = k$" means

$$g(i, j) = k \text{ or } g(2^{i+1}3^{j+1}) = k.$$ 

"$g_k$" designates the function satisfying

$$\forall i, j \ (g_k(i) = j \leftrightarrow g(\langle k, i \rangle) = j).$$
“$g|_k$” designates the finite sequence of functions such that

$$\forall m((g|_k)_m = g_m).$$

“$a = g|_k$” means

$$(ls(a) = k) \land (\forall i < k)(ls((a)_i) = k) \land (\forall i, j \leq k)((a)_i)_j = g((i, j))).$$

### 3.2 Complete separable metric space and Borel functions

#### 3.2.1 Complete separable metric spaces

To formally talk about the Borel diagonalization theorems we discussed in the introduction, we need to formalize concepts such as complete separable metric space, Borel function, permutation, etc. Though similar treatment can be readily found in the literature, we decide to give a brief and somewhat informal description of the formalization for the benefit of self-containment and, more importantly, to lay an unambiguous foundation for our further discussion. We try to stay as close to the notation in Simpson [2] as is convenient.

\(\text{ACA}^*\) is the weakest theory we have any interest in for this work. Hence we implicitly assume \(\text{ACA}^*\) when results are proved and formalizations are laid out in the sequel, though many of which can actually be done in still weaker theories.

To start with, we have the integers:

$$Z = \{ (m, n) : m, n \in N \}.$$

The number represented by \((m, n)\) is \(m - n\). Then we have the rationals

$$Q = \{ (\langle m_1, n_1 \rangle, \langle m_2, n_2 \rangle) : m_1, m_2, n_1, n_2 \in N \land m_2 \neq n_2 \}.$$
The number represented by \((\langle m_1, n_1 \rangle, \langle m_2, n_2 \rangle)\) is \(\frac{m_1 - n_1}{m_2 - n_2}\). Once we know the intended meaning of these representations, we may define operations

\[+z, +q, -z, -q, \cdot z, \cdot q, /q\]

and relations

\[=z, =q, \leq z, \leq q\]

accordingly. What we need to keep in mind about is that they are all primitive recursive, hence represented by symbols in PA. In the sequel, we will use \(Q^+\) to denote the set of nonnegative rationals.

**Definition 19.** We use \((K_A, d_A) \in CSM\) (CSM stands for complete separable metric space) to denote the arithmetical formula (in \(L^2\)) asserting the following property: \(K_A\) is the characteristic function of a set \(A\) such that

1. \(\forall a \in A \ d_A(\langle a, a \rangle) = Q 0\);

2. \(\forall a, b \in A \ d_A(\langle a, b \rangle) = Q d_A(\langle b, a \rangle) \geq Q 0\);

3. \(\forall a, b, c \in A \ d_A(\langle a, c \rangle) \leq Q d_A(\langle a, b \rangle) + Q d_A(\langle b, c \rangle)\).

In the above definition, we used \(+Q, \leq Q, =Q\) to distinguish those operations and relations from their more usual counterparts on the natural numbers. Since the meanings of these operations and relations will always be clear from the context, we will not keep on this practice unless absolutely necessary.

We will write \((A, d_A)\) for \((K_A, d_A)\) and \(d_A(a, b)\) for \(d_A(\langle a, b \rangle)\).
A pair \((A, d_A) \in CSM\) is called a code for a complete separable metric space. We write \(x \in \hat{A}\) to mean

\[ x \in A^\omega \land \forall k, i \ d_A(x_k, x_{k+i}) \leq 2^{-k}, \]

where \(x \in A^\omega\) means \(\forall k(x(k) \in A)\).

To formalize the real numbers, we let \(d_Q(a, b) = |a - b|\). We may extend \(d_Q\) to a function \(d_\hat{Q} : \hat{Q} \times \hat{Q} \rightarrow \hat{Q}\) by letting

\[ d_\hat{Q}(x, y)(k) = |x(k) - y(k)|. \]

Clearly, we may identify \(\mathbb{R}\) with \((\hat{Q}, d_\hat{Q})\). We may also introduce the concept of "convergence" in \((\hat{Q}, d_\hat{Q})\).

In general, for \((A, d_A) \in CSM\), We may extend \(d_A\) to \(d_\hat{A}\) by letting

\[ d_\hat{A}(x, y) = \lim_k d_A(x(k), y(k)). \]

We write \(x =_A y\) for \(d_A(x, y) = 0\). Note that \(x =_A y\) is equivalent to \(\forall k d_A(x_k, y_k) \leq 2^{-k+1}\), which is a \(\Pi^0_1\) formula in \(L^*_2\). Sometimes we write \(\hat{A} \in CSM\) for \((A, d_A) \in CSM\) when \(d_A\) is obvious.

We say that \(\langle x_i : i \in \omega \rangle \in \hat{A}^\omega\) converges to \(x \in \hat{A}\) if

\[ \forall n \exists k \forall i \geq k \ d_A(x_i, x) \leq \frac{1}{n}. \]

It turns out that the spaces we are interested in: \(\omega^\omega, 2^\omega, \mathbb{R}, I\), all have perfectly natural codes. We may identify \(\mathbb{R}\) with \((\hat{Q}, d_\hat{Q})\) as is pointed out earlier. Obviously "\(I\)" may be identified with \((\hat{Q}_1, d_\hat{Q})\), where \(Q_1 = \{q \in Q : 0 \leq q \leq 1\}\). Let \(A = \omega^{<\omega}\). For \(s \in A\), let \(s^* \in \omega^\omega\) be the extension of \(s\) with 0's. Now if \(s, t \in A\), we define
\(d_A(s,t)\) to be 0 if \(s^* = t^*\), and \(2^{-i}\), where \(i\) is the least \(k\) such that \(s^*(k)\) and \(t^*(k)\) differ, if \(s^* \neq t^*\). We may then identify the Baire space \(N^N\) with \((\hat{A}, d_{\hat{A}})\) and the Cantor space with \((2^{\omega}, d_{\hat{A}})\).

The usual order relation on \(\mathbb{R}\) is defined as follows:

\[x \leq_R y \iff (\forall k)(x(k) \leq_Q y(k) + 2^{-k}), \text{ and } x <_R y \iff \neg(y \leq_R x).\]

Hence \(x \leq_R y\) is a \(\Pi^0_1\) relation and \(x <_R y\) is a \(\Sigma^0_1\) relation in \(L_2^\ast\).

### 3.2.2 Borel functions

If \(\hat{A} \in CSM\), \(a \in A, r \in Q\), we use \(x \in B_A(a,r)\) to mean \(d_A(x,a) < r\). \(B_A(a,r)\) hence represents the open ball of radius \(r\) with center \(a\). \(B_A(a,r) \subseteq B_A(b,s)\) abbreviates the fact for all \(x \in B_A(a,r), x \in B_A(b,s)\).

**Definition 20.** A \(\Pi^0_i\) \((i=0,1)\) formula \(\phi\) is called an \(\hat{A}\)-formula and is written as \(\phi \in \hat{A}\)-formula if it satisfies the condition:

\[\forall x, y \in \hat{A}^\omega(\phi(x) \land x =_A y \rightarrow \phi(y)).\]

And a formula \(\phi \equiv \forall m_0 \forall m_2 \cdots \forall m_n \theta \in \Pi^0_{n+1}\) is called an \(\hat{A}\)-formula if \(\forall m_n \theta\) is a \(\hat{A}\)-formula. A \(\Sigma^0_n\)-formula \(\phi\) is called an \(\hat{A}\)-formula if \(\neg \phi\) is an \(\hat{A}\)-formula.

For each arithmetical formula \(\theta(x,b,s)\) (possibly with other free variables), let \(C_\theta(A,B)\) be the following formula: \(\forall b \in B, s \in Q, x, y \in \hat{A} (\theta \in \hat{A}\text{-formula}) \land \forall x \in \hat{A}, m \exists b \theta(x,b,2^{-m}) \land \forall b_1, b_2 \in B, r_1, r_2 \in Q ((d_B(b_1, b_2) < r_2 - r_1 \land \theta(x,b_1,r_1)) \rightarrow \theta(x,b_2,r_2)) \land ((\theta(x,b_1,r_1) \land \theta(x,b_2,r_2)) \rightarrow B(b_1,r_1) \cap B(b_2,r_2) \neq \emptyset)).\)

This formula says that \(\theta(x,b,s)\) defines a function from \(\hat{A}\) to \(\hat{B}\).
Let $\theta_2^0(e, x, b, s, f)$ be universal for all $\Sigma_2^0(x, b, s, f)$ formulas with free variables among shown.

In the following definition, we write $(a, r)u(b, s)$ to mean

$$\exists n \left((n, a, r, b, s) = 0\right).$$

**Definition 21.** We define a sequence of pairs of formulas,

$$(BC_n(u, A, d_A, B, d_B), \phi_n(u, x, b, s, A, d_A, B, d_B)) \text{ for } n=0, 1, 2, \ldots$$

by induction on $n$. For simplicity, we suppress the mention of $A, d_A, B, d_B$ in the following definition.

$BC_0(u)$ is the conjunction of the following clauses:

1. $u(0) = 0$;

2. $\forall a, b \in A, c \in B, r_1, r_2, r_3 \in Q^+(((a, r_1)u(c, r_3) \land d(a, b) \leq r_1 - r_2) \rightarrow (b, r_2)u(c, r_3))$;

3. $\forall a \in A, b, c \in B, r_1, r_2, r_3 \in Q^+(((a, r_1)u(b, r_2) \land (a, r_1)u(c, r_3)) \rightarrow B(b, r_2) \cap B(c, r_3) \neq \emptyset)$;

4. $\forall a \in A, b, c \in B, r_1, r_2, r_3 \in Q^+(((a, r_1)u(c, r_3) \land d(b, c) \leq r_2 - r_3) \rightarrow (a, r_1)u(b, r_2))$;

5. $\forall x \in \hat{A}, m \exists a \in A, b \in B, r \in Q^+ \exists a \in A, b \in B, r \in Q^+ (x \in B(a, r) \land (a, r)u(b, s))$;

6. $\forall x \in \hat{A}, a \in A, b \in B, r_1, r_2 \in Q^+ ((x \in B(a, r_1) \land (a, r_1)u(b, r_2)) \rightarrow \exists a', r', s \in Q^+ (x \in B(a', r') \land (a', r')u(b, s) \land s < r_2))$.

$\phi_0(u, x, b, s)$ is

$$u(0) = 0 \land (\exists a \in A, r \in Q^+ (x \in B(a, r) \land (a, r)u(b, s))).$$
$BC_1(u)$ is

$$u(0) = 1 \land C_{\theta_2}(e/u(1), f/(u)_0).$$

$\phi_1(u, x, b, s)$ is

$$(u(0) = 1 \land \theta_2^0(u(1), x, b, s, (u)_0)) \lor \phi_0(u, x, b, s).$$

For $n \geq 1$, $BC_{n+1}(u)$ is the conjunction of the following clauses:

1. $u(0) = n + 1$;
2. $\forall i (\forall k \leq n BC_k(u_i))$;
3. $(\forall x \in \hat{A})(\forall m)(\exists b \in B, s)(\forall i \geq s)(\phi_n(u_i, x, b, \frac{1}{m}))$;

Let $D(u, x, b, s)$ be

$$\exists r < Q, s, m \forall i \geq m, a \in B, t \in Q^+((d_B(a, b) \geq t + r) \rightarrow -\phi_n(u_i, x, a, t)).$$

$\phi_{n+1}(u, x, b, s)$ is

$$(u(0) = n + 1 \land D(u, x, b, s)) \lor \phi_n(u, x, b, s).$$

The intended meaning of $BC_n(u, A, d_A, B, d_B)$ is that $u$ codes a (total) Borel function of rank $n$ from $\hat{A}$ to $\hat{B}$ (rank 0 corresponds to the continuous function), which will be abbreviated as $u \in BC_n(\hat{A}, \hat{B})$. And the intended meaning of $\phi_n(u, x, b, s)$ is that the value of $u$ at $x$ is in $B(b, s)$, which we will write as $u(x) \in B(b, s)$.

**Lemma 3.2.1.** If $n \geq m \geq 0$, then
1. \textbf{ACA}^* proves that
\[ \forall \hat{A}, \hat{B} \in \text{CSM}, u \in BC_m(\hat{A}, \hat{B}), x \in \hat{A}, b \in B, s \in Q(\phi_m(u, x, b, s) \leftrightarrow \phi_n(u, x, b, s)); \]

2. there is a \( \hat{\phi}_n(u, x, b, s) \in \Sigma_{n+1}^0 \) such that \textbf{ACA}^* proves that
\[ \forall \hat{A}, \hat{B} \in \text{CSM}, u \in BC_m(\hat{A}, \hat{B}), x \in \hat{A}, b \in B, s \in Q(\hat{\phi}_n(u, x, b, s) \leftrightarrow \phi_n(u, x, b, s)); \]

3. there is a \( P_n(u, A, d_A, B, d_B) \in \Pi_1^1 \) with free variables shown such that \textbf{ACA}^* proves that
\[ (\forall u, \hat{A}, \hat{B})(u \in BC_n(\hat{A}, \hat{B}) \leftrightarrow P_n(u)). \]

\textbf{Remark}: In light of the above lemma, from now on, we will assume that \( BC_n \) is a \( \Pi_1^1 \) formula and \( \phi_n \) is a \( \Sigma_{n+1}^0 \)-formula. We will usually write \( u(x) \in_n B(b, s) \) instead of \( \phi_n(u, \bar{x}, i, j) \), and \( u \in BC_n \) instead of \( BC_n(u) \).

Given a formula \( \phi(i) \) with a distinguished free numerical variable \( i \) and \( u \) a function variable, we write \( u = \text{Def}_\phi \) for the formula
\[ \forall j, k \ (u(j) = k \iff \phi((j, k))). \]

In other words, \( u \) is defined by \( \phi \).

We next present two simple facts with only informal proofs. They will be repeatedly used in the sequel either explicitly or implicitly.

\textbf{Fact 1}. For any \( \phi \in \Sigma_1^0 \), there is a \( \psi \in \Sigma_2^0 \), such that \textbf{ACA}^* proves that
\[ \phi \in \hat{\text{A}}\text{-formula} \rightarrow (\psi \in \hat{\text{A}}\text{-formula} \land (\phi \iff \psi)). \]
Proof: This is just the formal version of a well-known fact about separable metric spaces which says that any open set is a union of countably many closed sets.

Remark: An immediate implication of fact 1 is that any Boolean combination of $\hat{A}$-formulas is an $\hat{A}$-formula.

Fact 2. ACA* proves the following

$$(X = \omega^\omega \vee X = 2^\omega \vee Y = R) \rightarrow \forall u \in BC_1(X,Y) \exists v \forall i ((v)_i \in BC_0(X,Y) \land u = \lim_{i \in \omega} v_i),$$

where $u = \lim_{i \in \omega} v_i$ is an abbreviation for the formula

$$\forall x \in X, b \in B, s \in Q^+ \ (u(x) \in B(b,s) \leftrightarrow \exists s' \exists k \forall j \geq k \ v_j(x) \in B(b,s')).$$

Proof: This is just a restatement of a well-known fact from descriptive set theory in ACA*. We will only present an informal proof\(^1\) for the case when $Y = R$ and leave to the reader to verify that the argument presented here can indeed be carried out in ACA*. When $X = \omega^\omega$, see [4], and the case when $X = 2^\omega$ may be treated similarly.

It is known that Tietze's extension lemma can be proved in ACA* (see Simpson [2] for details).

Let $\{(q_n,r_n)\}_{n \in \omega}$ list all the rational intervals of $R$. Fix a $\Sigma^0_2$-measurable function $f$. Then for each $n$, $f^{-1}(q_n,r_n)$ may be written as $\bigcup_{m=1}^{\infty} C_m^{(q_n,r_n)}$, where each $C_m^{(q_n,r_n)}$ is a closed set, and is a subset of $C_{m+1}^{(q_n,r_n)}$.

For each $i \geq 1$, we define a function $f_i$ such that

$$(*) \quad \forall n \leq i, x \in C_i^{(q_n,r_n)} (f_i(x) \in (q_n,r_n)).$$

\(^1\)The author learned this proof from Prof. Randy Dougherty.
This would insure the condition that requires that \( f \) is the limit of \( f_i \). For any \( x \) and \( \epsilon > 0 \), choose \((q_n, r_n)\) such that it contains \( f(x) \) and \( r_n - q_n < \epsilon \). If \( i > n \) is large enough, then \( x \in C_i^{(q_n, r_n)} \). Hence for all \( i > n \) large enough, \( f_i(x) \in (q_n, r_n) \). It follows that \( |f_i(x) - f(x)| < \epsilon \).

Now fix an \( i \geq 1 \). For any \( S \subseteq \{0, \cdots, i\} \), let \( C_S^{n,i} \) be the closed set \( \cap_{j \in S} C_i^{(q_n, r_n)} \).

Clearly if \( T \subseteq S \) then \( C_S^{n,i} \subseteq C_T^{n,i} \) and when \( S = \emptyset \), \( C_S^{n,i} = X \). Also note that if \( C_S^{n,i} \neq \emptyset \) then \( \cap_{n \in S}(q_n, r_n) \neq \emptyset \). Hence it must be a non-empty interval. We will define \( f_i^S \) on \( C_S^{n,i} \) by a reverse induction on \( S \) so that \( f_i^S \) is continuous with domain \( C_S^{n,i} \) and the condition (*) together with the condition: if \( s \subseteq s' \) then \( f_i^s \supseteq f_i^{s'} \), are maintained for \( f_i^S \). Suppose that \( f_i^T \) has been defined for all the \( S \subset_{\text{proper}} T \).

Let \( \hat{f}_i^S \) be the union of all those \( f_i^T \)'s. Note that \( \hat{f}_i^S \) is a continuous function with domain \( \hat{C}_S^{n,i} = \cup_{S \subset_{\text{proper}} T \subseteq \{0, \cdots, i\}} C_T^{n,i} \) and range contained in \( \cap_{n \in S}(q_n, r_n) \). Since \( \hat{C}_S^{n,i} \) is a closed subset of \( C_S^{n,i} \), by applying Tietze's theorem, we may extend \( \hat{f}_i^S \) to a continuous function from \( C_S^{n,i} \) to \( \cap_{n \in S}(q_n, r_n) \). Let \( f_S^{n,i} \) be this extension. Clearly, the inductive conditions are maintained.

Finally, let \( f_i \) be \( f_\emptyset^{n,i} \).

**Lemma 3.2.2.** For any arithmetical formula \( \phi(i) \) in which \( u, w \) are not free, and arithmetical formula \( \psi(i) \) in which \( v, w \) are not free (both may contain other free variables), there is an arithmetical formula \( \theta(i) \) whose free variables are exactly those of \( \phi(i) \) plus those of \( \psi(i) \) such that \( \text{ACA}^* \) proves the following:

\[ \forall u \in BC_m(\hat{X}, \hat{Y}), v \in BC_0(\hat{Y}, \hat{Z}) \exists w \in BC_m(\hat{X}, \hat{Z})(u = \text{Def}_\phi \land v = \text{Def}_\psi \rightarrow w = \text{Def}_\theta \land w = v \circ u), \]
where \( w = v \circ u \) abbreviates the formula \( \forall x \in X, c \in Z, t \in Q(w(x) \in_m B(c, t) \iff \forall y \in Y(\forall i(u(x) \in_m B(y(i), \frac{1}{2^{i-1}}) \land d_Y(y(i), y(i+1)) \leq 2^{-i-1}) \rightarrow v(y) \in_0 B(c, t))) \); i.e., \( w \) codes the composition of the function coded by \( u \) and the function coded by \( v \).

**Proof:** We show this by induction on \( m \).

First let us consider the case \( m = 0 \). We fix \( \phi(i) \) and \( \psi(i) \). Let \( \theta(i) \) be

\[
ls(i) = 2 \land ((i_0 = i_1 = 0) \lor (ls(i_0) = 5 \land ((A(i) \land i_1 = 0) \lor (\neg A(i) \land i_1 = 1)) \lor (ls(i_0) \neq 5 \land i_1 = 1)),
\]

where \( A(i) \) is the following formula

\[
(i_0)_1 \in X \land (i_0)_2 \in Q \land (i_0)_3 \in Z \land (i_0)_4 \in Q \land \exists b \in Y, s \in Q, m (\phi(\langle (i_0)_0, (i_0)_1, (i_0)_2, b, s, i_1 \rangle)) \land \\
(\psi(\langle m, b, s, (i_0)_3, (i_0)_4, i_1 \rangle))).
\]

Then working in \( \text{ACA}^* \), we fix \( u \in BC_0(X, Y) \) and \( v \in BC_0(Y, Z) \) such that \( u = Def_{\phi} \) and \( v = Def_{\psi} \). Then \( w = Def_{\theta} \) satisfies the following conditions:

1. \( w(0) = 0 \);

2. for all \( k, a \in X, c \in Z, r, t \in Q \)

\[
w(\langle k, a, r, c, t \rangle) = 0 \iff \exists b \in Y, s \in Q, m (u(\langle k, a, r, b, s \rangle) = 0 \land v(\langle m, b, s, c, t \rangle) = 0).
\]

This \( w \) satisfies our requirement.

Another base case is when \( m = 1 \). To avoid unnecessary technical detail, we will keep the rest of the argument somewhat informal.

Fix \( \phi, \psi, u \in BC_1(X,Y) \) and \( v \in BC_0(Y,Z) \), such that \( u \) and \( v \) are defined by \( \phi \) and \( \psi \) respectively.
By definition

\[ v \circ u(x) \in B(c, t) \iff \exists b \in B, s \in Q^+ (u(x) \in_1 B(b, s) \land (b, s)v(c, t)). \]

Let us write the right hand side as \( \theta^*(x, c, t, u, v) \). Then again by definition, \( \theta^* \) may be written as

\[ \exists b \in B, s \in Q^+ (\theta^0_2(u(1), x, b, s) \land (b, s)v(c, t)). \]

Clearly the above formula is \( \Sigma_2^0 \) and \( C_\theta \) holds (for the definition of \( C_\theta \), see the paragraph before Definition 21). It is easy to see that there is some \( w^* \) definable from \( u \) and \( v \) via an arithmetical formula and some \( e \) such that

\[ \theta^*(x, c, t, u, v) \iff \theta^0_2(e, x, c, t, w^*). \]

Then define \( w \) such that \( w(0) = 1, w(1) = e, (w)_0 = w^* \) and at all the other places \( w \) takes on the value 0. Then \( w \) satisfies the requirement.

Now suppose \( m > 1 \) and the lemma has been proved for all \( k < m \).

Fix \( u \in BC_m(\hat{X}, \hat{Y}) \), \( v \in BC_0(\hat{Y}, \hat{Z}) \), \( u = Def_\phi \) and \( v = Def_\psi \). Let \( \phi^*(i) \) be the formula \( \phi[i/(j, i)] \), where \( j \) is a numerical variable which does not appear in \( \psi \) or \( \phi \).

By the definition of Borel code, we know that

\[ \forall j \exists n < m \exists u^* \in BC_n(\hat{X}, \hat{Y}) (u^* = Def_{\phi^*(i)}). \]

Hence by induction hypothesis, there is a \( \theta^*(i) \) whose free variables are those of \( \phi^* \) plus those of \( \psi \) (in particular it includes the free variable \( j \)) such that

\[ \exists w^* (w^* = Def_{\theta^*(i)} \land w^* = v \circ u^*). \]
Let $\theta(i)$ be the following formula

$$ls(i) = 2 \land (i_0 = 0 \land i_1 = m \lor i \neq 0 \land \theta^*[i/((i_0)_0,i_1), j/(i_0)_1]).$$

Then $\theta$ is arithmetical and if $w = Def\theta$, then $w \in BC_m(\hat{X}, \hat{Z})$ and $w = v \circ u$.

**Lemma 3.2.3.** If $\psi(x, i, j) \in \Sigma_{n+1}^0$ (possibly with other free variables), then there is an arithmetical formula $\phi$ whose free variables are those of $\psi$ excluding $x$ such that $\text{ACA}^*$ proves that

$$(\forall x \in \hat{X}, i \exists! j \psi(x, i, j) \land \psi \in X\text{-formula}) \rightarrow \exists u \in BC_n(\hat{X}, \omega^\omega)(u = Def_\phi \land \forall x \in \hat{X}, b \in \omega^\omega(\phi_n(u, x, b, 2^{-ls(b)+1}) \leftrightarrow \forall i < ls(b)\psi(x, i, b))).$$

**Proof:** We prove this by induction on $n$.

When $n = 0$, we fix such a $\psi(x, i, j)$. Working in $\text{ACA}^*$, by an argument similar to Lemma 5.7 of Simpson’s [2], we know there is an arithmetical formula $A(i)$ such that

$$\forall i, j \forall x \in \hat{X} (\psi(x, i, j) \leftrightarrow (\exists a \in X, r \in Q, k (A((k,a,r,i,j),0) \land x \in B(a,r))).$$

Let $u$ be such that $u(0) = 0$ and $\forall (n,a,r,b,s) (u((n,a,r,b,s)) = 0 \leftrightarrow (\exists a', b', r' \forall i < ls(b')(A((n,a,r,i,b),0) \land B(b', 2^{-ls(b')+1}) \subseteq B(b,s) \land B(a,r) \subseteq B(a', r')))).$

It is easy to check that $u \in BC_0(\hat{A}, \hat{B})$ and $u$ can be defined by an arithmetical formula. Let $\phi$ be the formula which defines $u$. Then $u$ and $\phi$ satisfy the requirement of the lemma.

The case $n = 1$ follows easily from the definition of $BC_1(\hat{X}, \omega^\omega)$.

Now suppose that $\psi \in \Sigma_{n+1}^0$ for some $n > 1$ and $\forall x \in \hat{X}, i \exists! j \psi(x, i, j)$. Write $\psi$
as

\[ \exists m \forall k \theta(x, i, j, m, k), \]

where \( \theta \in \Sigma^0_{n-1} \). Let \( A(s, x, i, j) \) be the formula \( \exists t (\forall k \leq s \theta(x, i, (t)_0, (t)_1, k) \land \forall t' < t \exists k \leq s \neg \theta(x, i, (t)_0, (t)_1, k) \land j = (t)_0). \)

It is easy to check that \( \forall x \in \hat{\hat{X}}, s, i, \exists j A(s, x, i, j) \) holds and that \( A \in \Sigma^0_n \). Moreover, by fact 1, \( A \) is also an \( \hat{\hat{X}} \)-formula. Hence by induction hypothesis, there is an arithmetical formula \( \phi(i, s) \) (possibly with other free variables) and a \( u \in BC_{n-1}(\hat{\hat{X}}, \omega^\omega) \) such that \( u = Def_{\phi(i)} \) and

\[ \forall x \in \hat{\hat{X}}, b (\phi_n(u, x, b, 2^{-ls(b)+1}) \leftrightarrow \forall i < ls(b) A(s, x, i, b_i)). \]

Let \( w \) be such that \( w(0) = n + 1 \) and \( \forall s w_s = Def_{\phi(s, k)} \). Obviously, \( w \) may be defined by an arithmetical formula \( \phi^*_w \) and satisfies other conditions.

Lemma 3.2.1 roughly says that each \( u \in BC_n(\hat{A}, \hat{B}) \) codes a function definable by a \( \Sigma^0_{n+1} \) formula. It turns out that the reverse is also true; i.e, if a function from \( \hat{A} \) to \( \hat{B} \) is definable by a \( \Sigma^0_{n+1} \) formula, then it is coded by some \( u \in BC_n(\hat{A}, \hat{B}) \). We state this precisely in the following lemma.

**Lemma 3.2.4.** \( \theta(x, b, s) \) is \( \Sigma^0_{n+1} \) (possibly with other parameters) then there is an arithmetical formula \( \phi(i) \) with a distinguished free variable \( i \) and other free variables among those of \( \theta \) excluding \( x \) such that \( ACA^* \) proves that

\[ C_\theta(A, B) \rightarrow \exists u \in BC_n(\hat{A}, \hat{B}) \forall b \in B, s \in Q^+, x \in \hat{A} ((u(x) \in B(b, s) \leftrightarrow \theta(x, b, s)) \land u = Def_{\phi(i)}). \]

**Proof:** Let us argue by induction on \( n \).
When $n = 0$, the proof is similar to that of lemma 3.2.3.

When $n = 1$, it is just the definition. Let $h_Y$ enumerate $Y$ in an increasing order (in the natural order of integers). Write $\theta(x, b, s)$ as $\exists m A(x, b, s, m)$. Consider the formula $\psi(x, i, j)$ defined as follows:

$$\exists t(A(x, h_Y((t)_0), 2^{-i-1}, (t)_1) \land \forall s < t \neg A(x, h_Y((s)_0), 2^{-i-1}, (s)_1) \land j = (t)_0).$$

Then $\psi \in \Sigma^0_{n+1}$ and $\forall x \in \hat{X}, \exists! j \psi(x, i, j)$. By lemma 3.2.3, there is a $\phi^*(i)$ and $v \in BC_n(\hat{X}, \omega^\omega)$ such that $v = Def_{\phi^*(i)}$ and

$$\forall x \in \hat{X}, b (\phi_n(v, x, b, 2^{-ls(b)+1}) \leftrightarrow \forall i < ls(b) \psi(x, i, b_i)).$$

Let $B(x, b, s)$ be the formula asserting the following property:

There exists $i, (a_0, \cdots, a_i) \in Q^{i+1}$ such that

- $a_0 = h_Y(x(0));$
- for all $j < i$, if $(d_Y(a_j, h_Y(x(j + 1)))) \leq 2^{-j-1})$ then $a_{j+1} = h_Y(x(j) + 1)$; otherwise $a_{j+1} = a_j;$
- $d_Y(b, a_i) < s - 2^{-i}.$

It is straightforward to check that $B(x, b, s)$ is $\Sigma^0_1$ and satisfies the condition of this lemma. Hence by the base case of the induction, there is a $\psi^*(i)$ and $\pi \in BC_0(\omega^\omega, \hat{Y})$ such that $\pi = Def_{\psi^*(i)}$ and

$$\forall b \in Y, s \in Q, x \ (\pi(x) \in_0 B(b, s) \leftrightarrow B(x, b, s)).$$
Finally, applying lemma 3.2.2, we may find a $\phi(i)$ and $u \in BC_n(\hat{X}, \hat{Y})$ such that $u = Def\phi(i)$ and $u = \pi \circ v$. It is straightforward (but tedious) to check that

$$\forall x \in \hat{X}, b \in Y, s \in Q \left( u(x) \in_n B(b, s) \leftrightarrow \theta(x, b, s) \right).$$

The following is a generalization of lemma 3.2.2.

**Corollary.** ACA* proves that $\forall u \in BC_m(\hat{X}, \hat{Y}), v \in BC_n(\hat{Y}, \hat{Z}) \exists w \in BC_{m+n}(\hat{X}, \hat{Z})(w = v \circ u)$.

Here $w = v \circ u$ abbreviates the formula $\forall x \in \hat{X}, c \in Z, t \in Q(w(x)) \in_{m+n} B(c, t) \leftrightarrow \forall y \in \hat{Y} \left( \forall i (u(x) \in_m B(y(i), \frac{1}{2^{i-1}}) \land dy(y(i), y(i+1)) \leq 2^{-i-1}) \rightarrow v(y) \in_n B(c, t) \right)$; i.e., $w$ codes the composition of the function coded by $u$ and the function coded by $v$.

**Proof:** Fix a listing $\{(b_i, s_i) : i \in \omega\}$ of $Y \times Q^+$. Write $v(y) \in_n B(c, t)$ as

$$\exists k_1, \ldots, Q_n k_n A(k_1, \ldots, k_n, y, c, t, v),$$

where $A$ is either a $\Pi^0_1$ or a $\Sigma^0_1$ formula and $Q_n$ is either "$\forall$" or "$\exists$". Without loss of generality, assume that $A$ is $\Sigma^0_1$ (otherwise, consider $\neg A$). For each fixed $v$, we may find a $g$ (see the proof of lemma 5.7 of [2]) such that

$$A(k_1, \ldots, k_n, y, c, t, v) \leftrightarrow \exists i \left( y \in B(b_g((k_1, \ldots, k_n, c, t, i)), s_g((k_1, \ldots, k_n, c, t, i)) \right).$$

Write $u(x) \in_m B(b_g((k_1, \ldots, k_n, c, t, i)), s_g((k_1, \ldots, k_n, c, t, i)))$ as

$$\exists l_1, \ldots, Q_{m+1} l_{m+1} C(l_1, \ldots, l_{m+1}, k_1, \ldots, k_n, i, g, c, t),$$

where $C$ is bounded and $Q_{m+1}$ is either "$\forall$" or "$\exists$". Hence the formula $\forall k_1, \ldots, \forall k_n, \exists i (u(y) \in_m B(b_f((k_1, \ldots, k_n, c, t)), s_f((k_1, \ldots, k_n, c, t))))$ can be written as

$$\exists k_1, \ldots, \forall k_n, \exists i, \exists l_1, \ldots, Q_{m+1} l_{m+1} C(l_1, \ldots, l_{m+1}, k_1, \ldots, k_n, i, g, c, t).$$
Let us denote this last formula by $\phi$. It is clear that $\phi$ is a $\Sigma_{m+n+1}^0$-formula and $C_\phi$ holds. Then apply lemma 3.2.4.

We now turn to the discussion of Cantor space.

Let $y = 1\text{-limit}(x)$ be the following formula

$$\forall j \exists k \forall i \geq k \ (x_i(j) = x_{i+1}(j) \land y(j) = x_k(j));$$

and $y = 2\text{-limit}(x)$ be the following formula

$$\forall j_1, j_2 \exists k \forall i \geq k \ (x_i(j_1)(j_2) = x_{i+1}(j_1)(j_2) \land y(j_1)(j_2) = x_k(j_1)(j_2)).$$

Clearly we have the following:

**Lemma 3.2.5.** Let $(A, d_A)$ be the code for Cantor space as defined at the beginning of this section. Then $\text{ACA}^*$ proves

$$\forall x \in \hat{A} (\hat{A}^\omega) \exists! y(y = 1\text{-limit}(x)(y = 2\text{-limit}(x)))$$

and

$$\forall y \exists x \in \hat{A} (\hat{A}^\omega)(y = 1\text{-limit}(x)(y = 2\text{-limit}(x))).$$

To better utilize the simplicity of Cantor space, we want to slightly modify the definition of Borel codes for this space to reflect its close relation with the intended structure of $L_2^*$ (which is the Baire space). For instance, the basic open sets in Cantor space, are more familiarly coded by finite sequences instead of the way we adopted above.

Here is the modified version.
Definition 22. We define a sequence of pairs of formulas \( \{(\overline{BC}_n(\overline{u}, \phi^*_n(\overline{u}, \overline{x}, s)) : n \in N \} \) by induction as follows:

\( \overline{BC}_0(\overline{u}) \) is

1. \( u(0) = 0; \)

2. \( \forall s_1, s_2 \in (2^{<\omega})^{<\omega}, t \in 2^{<\omega}((s_1 \subseteq s_2 \land \exists n u((n, s_1, t)) = 0) \rightarrow \exists n u((n, s_2, t)) = 0); \)

3. \( \forall s \in (2^{<\omega})^{<\omega}, t_1, t_2 \in 2^{<\omega}((t_2 \subseteq t_1 \land \exists n u((n, s, t_1)) = 0) \rightarrow \exists n u((n, s, t_2)) = 0); \)

4. \( \forall s \in (2^{<\omega})^{<\omega}, t_1, t_2 \in 2^{<\omega}((\exists n u((n, s, t_1)) = 0 \land \exists n u((n, s, t_2)) = 0) \rightarrow (t_1 \subseteq t_2 \lor t_2 \subseteq t_1)); \)

5. \( \forall g \in (2^\omega)^{<\omega} \forall k \exists t \in 2^k \exists m \exists n u((n, g|m|, t)) = 0). \)

\( \phi^*_0(\overline{u}, \overline{x}, s) \) is

\[ \exists m, n (u((n, x|_m^m, s)) = 0). \]

\( \overline{BC}_{n+1}(\overline{u}) \) is

1. \( u(0) = n + 1; \)

2. \( \forall i \cup_k \leq n \overline{BC}_k(u_i); \)

3. \( \forall s \in 2^{<\omega}, x \exists m \forall i \geq m (\phi^*_n(u_i, x, s) \leftrightarrow \phi^*_n(u_{i+1}, x, s)). \)

\( \phi_{n+1}^*(\overline{u}, \overline{x}, s) \) is

\[ (u(0) = n + 1 \land \exists m \forall i \geq m, t \in 2^{<\omega}(\phi^*_n(u_i, x, s) \rightarrow s \subseteq t \lor t \subseteq s)) \lor \phi^*_n(\overline{u}, \overline{x}, s). \]
We will write \( u \in \overline{BC}_n \) for \( \overline{BC}_n(u) \). For \( u \in \overline{BC}_{n+1} \), what \( \phi^*_{n+1}(u, \bar{x}, b) \) says is that \( u(\bar{x}) \) extends the finite sequence \( b \) or \( u(\bar{x}) \) belongs to the basic open set decided by \( b \). We may clearly prove a lemma similar to lemma 3.2.1. Moreover, the following lemma asserts that the two version of Borel codes we have defined for Cantor space are essentially equivalent.

**Lemma 3.2.6.** \( \text{ACA}^* \) proves the following:

For any \( u \in BC_n((2^\omega)^\omega, 2^\omega) \), there is a \( v \in \overline{BC}_n \) such that for any \( x \in (2^\omega)^\omega \) and any \( s \in 2^{<\omega} \), we have

\[
\phi_n(u, x, s, \frac{1}{2|b|}) \leftrightarrow \tilde{\phi}^*_n(v, 2\text{-limit}(x), s).
\]

**Proof:** We may argue by an easy induction on \( n \). All steps follow easily from the definitions of \( BC_n \) and \( \overline{BC}_n \) except when \( n = 1 \), we also need to use Fact 2 with \( X = 2^\omega \). (see the part following lemma 3.2.1.)

Hence for any \( u \in BC_n((2^\omega)^\omega, 2^\omega) \), there is a \( v \in \overline{BC}_n \) such that \( u \) and \( v \) code the same function. From now on, when we write \( u \in BC_n((2^\omega)^\omega, 2^\omega) \) we mean \( u \in \overline{BC}_n \).

**Remark.** Clearly, we may similarly modify the definition of Borel codes for Borel functions from \((2^N)^N \) to \((2^N)^N \). Let us assume that we have done so.

We now quite informally define a few Borel functions for later use. A formal definition is possible but would be very long.

**Lemma 3.2.7.** \( \text{ACA}^* \) proves the following Borel codes \( F^B_C, \tilde{F}^B_C \in BC_0(\omega^\omega, 2^\omega) \), \( F^C_B \in BC_0(2^\omega, \omega^\omega) \), \( \tilde{F}^C_B \in BC_1(2^\omega, \omega^\omega) \), \( F^B_I \in BC_0(\omega^\omega, I) \), \( F^I_B \in BC_1(I, \omega^\omega) \), \( F^R_I \in BC_0(R, I) \), \( F^I_R \in BC_0(I, R) \) exist and satisfy (in informal terms) the following:

\[
\forall f, i, j \ (F^B_C(f)(i) = j \leftrightarrow ((j = 0 \land f(i) = 0) \lor (j = 1 \land f(i) \neq 0)));
\]
\[ \forall f, i, j \, (\tilde{F}^B_G(f)(i) = 1 \leftrightarrow (ls(i) = 2 \land f((i)_0) = (i)_1)); \]
\[ \forall f \in 2^{<\omega} \, (F^G_B(f) = f); \]
\[ \forall f \in 2^{<\omega}, i, j \, (\tilde{F}^B_G(f)(i) = j \leftrightarrow ((f((i, j)) = 1 \land \forall j' < j f((i, j')) \neq 1) \lor \forall s f((i, s)) \neq 1 \land j = 0)); \]
\[ \forall f, F^B(f)^2 \text{ is the dyadic irrational:} \]
\[ \begin{array}{cccc}
(0) \text{ many} & (1) \text{ many} & (2) \text{ many} \\
11 \cdots 1 & 00 \cdots 0 & 11 \cdots 1 \ 0 \cdots; \\
\end{array} \]

for all \( x \) in I, (1) if \( x \) is a dyadic irrational then \( F^I_B(x) = (F^B_I)^{-1}(x) \), (2) if \( x = 1 \) then \( F^I_B(x) \) is the function \( f \) such that \( \forall i \, f(i) = 0 \), and (3) if \( x \) is a dyadic rational other than "1", then for any \( k \), \( F^I_B(x)(k) = (F^B_I)^{-1}(x)(k) \) if \( (F^B_I)^{-1}(x)(k) \) is defined, otherwise \( F^I_B(x)(k) = 0 \), where \( g \) is the binary representation of \( x \) that only has finitely many 1’s;
for all \( x \) in R, \( F^R_I(x) = x \) if \( 0 \leq_R x \leq_R 1 \), 1 if \( x >_R 1 \), 0 if \( x <_R 0 \);
for all \( x \) in I, \( F^I_H(x) = x \).

**Remark.** We do not intend to formally introduce these symbols (or any new symbols at all) into \( L^*_2 \). Formula such as \( F^B_B(x) = f \) is nothing but an abbreviation for the formula of \( L^*_2 \) which asserts that property. The subscripts \( B,C,R,I \) stand for Baire space, Cantor space, the Real line and the unit interval I respectively. Hence \( F^X_Y \) designates a function from \( X \) to \( Y \).

Since \( F^I_B \) and \( F^B_I \) will be used frequently, we specially name them as FR (Function Representation) and BIR (Binary Irrational Representation) resp.. Furthermore we

\[ ^2 \text{This function is actually a homeomorphism between Baire space and the Binary Irrationals (as a subspace of I). We used this fact in the definition of } F^I_B. \text{ A real number is a binary irrational iff its binary expansion has both infinitely many 0's and infinitely many 1's. For more detail, see Peter Hinman [3] p19.} \]
Lemma 3.2.8. ACA* proves that

1. \( \forall f \in 2^\omega \, F_B^C \circ F_C^B(f) = f \);

2. \( \forall f \, \tilde{F}_B^C \circ \tilde{F}_C^B(f) = f \);

3. \( \forall x \in I \, F_i^R \circ F_i^L(x) = x \);

4. \( \forall f \, FR \circ BIR(f) = f \).

3.3 Relations among different spaces

This section aims at comparing the relative strength of Borel diagonalization theorems when stated on different spaces: Baire space, Cantor space, Real line and the Unit interval. At the end of this section we also make effort to generalize these results to arbitrary (perfect) complete separable metric spaces. With our approach, it turns out that "\( \sigma \)-compactness" is the deciding factor contributes to that strength. Baire space, being the only one among the four without this property, requires the highest upper bound.

We may also easily formalize the notion of permutation of \( N \) and finite permutation of \( N \). We usually use \( \pi, \gamma,...,etc \) to range over permutations and use \( \sigma, \tau \,... \, etc \) to range over finite permutations. If \( \pi \) is a permutation and \( f \in X^\omega \), we use \( f^\pi \) to denote the sequence whose i-th term is the \( \pi^{-1}(i) \)-th term of \( f \). We similarly define \( f^\sigma \) when \( \sigma \) is a finite permutation, taking \( \sigma(i) = i \) when undefined. More formally speaking, when we write \( f^\pi(i)(j) = k \), we actually mean the formula \( f(\pi^{-1}(i))(j) = k \).
If $T = \{\phi_1, \cdots, \phi_k\}$ is a finite set of sentences in the language $L_2$, we write $f \models^*_C T$ to mean

$$f \in 2^\omega \land (\phi_1)^f \land \cdots \land (\phi_k)^f,$$

where $(\phi_i)^f$ is the formula obtained from $\phi$ by relativizing its second-order quantifiers to $\{f_i : i \in \omega\}$ and replacing atomic formulas of the form $t \in x$ by $x(t) = 1$.

Similarly, if $T = \{\phi_1, \cdots, \phi_k\}$ is a finite set of sentences in the language $L_2^*$, we write $f \models^*_B T$ to mean

$$(\phi_1)^f \land \cdots \land (\phi_k)^f,$$

$f \models^*_I T$ to mean

$$FR(f) \models^*_B T,$$

and $f \models^*_R T$ to mean

$$FR \circ F^R_I(f) \models^*_B T.$$

The following lemma is obvious.

**Lemma 3.3.1.** If $T$ is the theory $\Pi^1_n CA_0$ ($n \geq 1$), then there are formulas $\phi_C \in \Pi^0_{n+2}$ in $L_2$ and $\phi_X \in \Pi^0_{n+1}$ ($X=B,I,R$) in $L_2^*$ such that $ACA$ proves

$$\forall f \in Y (f \models^*_Y T \leftrightarrow \phi_Y(f)),$$

where $Y$ is among $C,B,I$ and $R$.

**A Remark On Induction:** Since the formal system we work in (at least as strong as $ACA^*$) always has full induction, for any $f$ and any formula $\phi(i)$, the following is provable by induction

$$(\phi(0) \land \forall i (\phi(i) \rightarrow \phi(i + 1)) \rightarrow \forall i \phi)^f.$$
In other words, any $f$ satisfies the "full induction axiom scheme". Because of this, formulas such as $f \models^c \text{ACA}$ make sense because it is equivalent to $f \models^c \text{ACA}_0$.

Now we are ready to present the statements $A^n_i(X,T)$, $B^n_i(X,T)$ and etc, strictly in the language $L^*_2$. For instance, $A^n_i(R^N,T)$ may be formally stated as follows:

**Statement $A^n_i(R^N,T)$**: $\forall u \in BC_n(R^N,R) \ (\forall \bar{x} \in R^N, a \in Q,k,\sigma \ (\phi_n(u,\bar{x},a,\frac{1}{k}) \leftrightarrow \phi_n(u,\bar{x}\sigma,a,\frac{1}{k})) \rightarrow \exists \bar{x} \in R^N, \forall a \in Q,k \ ((\phi_n(u,\bar{x},a,\frac{1}{k}) \leftrightarrow \bar{x}_i \in B(a,\frac{1}{k})) \land \bar{x} \models^c R T)).$

Obviously, this formula is provably (in weak systems such as $\text{ACA}$) equivalent to a $\Pi^1_2$ formula. The same may be said about the formal version of other statements. Thus we will assume all these statements are themselves $\Pi^1_2$ formulas in the language $L^*_2$.

Finally if $u$ codes a function from $X$ to $Y$, then it is clear that we may find a Borel code $\bar{u}$ of same rank as $u$, which codes a function from $X^\omega$ to $Y^\omega$ such that $(\bar{u}((\bar{x}))_i$ is $u((\bar{x})_i)$. We will use this fact in the following lemma.

**Lemma 3.3.2.** For $i=1,2$ and 3, $\text{ACA}^*$ proves $\forall u \in BC_m(\hat{X},\hat{Y}), v \in BC_n(\hat{Y},\hat{X})((\forall y \in \hat{Y} \ (u \circ v(y) = y) \land \forall \bar{x} \in \hat{X}^\omega \ (\bar{x} \models^c_{\hat{X}} T \rightarrow \bar{u}(\bar{x}) \models^c_{\hat{Y}} T)) \rightarrow (((\hat{A}^m_{i+n+k}(\hat{X},T) \rightarrow \hat{A}^k_{i}(\hat{Y},T)) \land (B^i_{n+m+k}(\hat{X}) \rightarrow B^k_{i}(\hat{Y}))))), \text{ where } \bar{u}(\bar{x}) \text{ is the sequence whose } i\text{-th term is } u((\bar{x})_i).

**Proof:** We only prove the case when $i = 1$. The other two cases are similar. Fix $u \in BC_m(\hat{X},\hat{Y}), v \in BC_n(\hat{Y},\hat{X})$ and some $F \in BC_k(\hat{Y}^\omega,\hat{Y})$ such that the condition of the lemma is met and $F$ has the first invariant property. By the corollary to lemma 3.2.4, there is some $G \in BC_{m+n+k}(\hat{X}^\omega,\hat{X})$ such that $G = v \circ F \circ \bar{u}$. 
It is easy to check that $G$ also has the first invariant property. By $\tilde{A}_1^{m+n+k}(\tilde{X}, T)$, there is some $\tilde{x} \in \tilde{X}^\omega$ such that for some $i$, $G(\tilde{x}) = \tilde{x}_i$. Hence $u(G(\tilde{x})) \in \tilde{u}(\tilde{x})$. But $u(G(\tilde{x}))$ is $F(\tilde{u}(\tilde{x}))$. It follows that $\tilde{A}_1^k(\tilde{Y}, T)$.

**Corollary 1.** Let $i=1, 2, 3$ and $T$ be among the "empty theory", $\text{ACA}$, $\Pi^1_n \text{-CA}, \Delta^1_n \text{-CA}$ or $\Sigma^1_n \text{-DC}$ for some $n \geq 1$, then $\text{ACA}^*$ proves the following:

1. $A^*_i(\omega^\omega, T)$ implies $A^*_i(2^\omega, T)$;

2. $A^{n+1}_i(2^\omega, T)$ implies $A^n_i(\omega^\omega, T)$;

3. $A^n_i(R, T)$ implies $A^n_i(I, T)$;

4. $A^{n+1}_i(I, T)$ implies $A^n_i(\omega^\omega, T)$.

**Proof:** The only condition in applying lemma 3.3.2 which still needs to be checked is that the functions $\tilde{F}^B_C$, $\tilde{F}^C_B$, $\tilde{F}^R_I$ and $\tilde{F}^I_B$ all preserve weak models of $T$, where we say a function $\tilde{F}^X_Y$ preserves weak models of $T$ if for any $\tilde{x}$ such that $\tilde{x} \models^X T$, $\tilde{F}^X_Y(\tilde{x}) \models^Y T$. But this follows from the definition of the concept of weak model at the beginning of the section.

**Corollary 2.** For $i=1, 2$ and $3$, $\text{ACA}^*$ proves the following:

1. $B^*_i(\omega^\omega)$ implies $B^*_i(2^\omega)$;

2. $B^{n+1}_i(2^\omega)$ implies $B^n_i(\omega^\omega)$;

3. $B^n_i(R)$ implies $B^n_i(I)$;

4. $B^{n+1}_i(I)$ implies $B^n_i(\omega^\omega)$. 
Proof: The proof is similar to the proof of the previous corollary, and simpler. By formalizing results from classical descriptive set theory, we may generalize corollary 1 and 2 to all interesting complete separable metric spaces. We call \((A, d_A) \in CSM\) "perfect" if it has no "isolated point", or put formally, if it satisfies the following:

\[ \forall a \in A, r \in Q^+ \exists b \in A (a \neq b \land b \in B(a, r)). \]

We write \(\hat{A} \in PCSM\) to indicate that \(\hat{A} \in CSM\) and \(\hat{A}\) has the additional property of being perfect.

**Corollary 3.** ACA\(^*\) proves that

\[ \forall \hat{X} \in PCSM ((\pi_{n+1}(\hat{X}) \rightarrow \pi_n(\omega)) \land (B_{i+1}(\hat{X}) \rightarrow B_i(\omega))). \]

Proof: Let \(F_X^C \in BC_0(2^n, \hat{X})\) code the function

\[ F: 2^n \rightarrow \hat{X}, \]

defined as follows: \(F(f)(i) = k\) if and only if there exists a sequence

\[ \langle \langle(a_0, r_0), (b_0, s_0)\rangle, \cdots, \langle(a_i, r_i), (b_i, s_i)\rangle \rangle \in ((X \times Q) \times (X \times Q))^{i+1}, \]

such that \(\forall j \leq i\) the conjunction of the following clauses holds:

1. \(r_j, s_j \leq 2^{-j};\)

2. \(\langle(a_0, r_0), (b_0, s_0)\rangle\) is the least pair (in their natural order) so that \(d_X(a_0, b_0) > r_0 + s_0;\)

3. for any \(0 < j < i\), the following hold:
• if \( f(j) = 0 \) then \( \langle \langle a_j, r_j \rangle, \langle b_j, s_j \rangle \rangle \) is the least pair (in their natural order) so that \( d_X(a_j, a_{j-1}) < r_{j-1} - r_j \), \( d_X(b_j, a_{j-1}) < r_{j-1} - s_j \) and \( d_X(a_j, b_j) > r_j + s_j \);

• if \( f(j) = 1 \) then \( \langle \langle a_j, r_j \rangle, \langle b_j, s_j \rangle \rangle \) is the least pair (in their natural order) so that \( d_X(a_j, b_{j-1}) < s_{j-1} - r_j \), \( d_X(b_j, b_{j-1}) < s_{j-1} - s_j \) and \( d_X(a_j, b_j) > r_j + s_j \);

4. if \( f(i) = 0 \) then \( k = a_i \), and if \( f(i) = 1 \) then \( k = b_i \).

This function \( F \) is really the standard function which embeds the Cantor space into a perfect polish space. Let \( F_X^B \in BC_0(\omega^\omega, X) \) be \( F_X^C \circ F_X^B \) (see lemma 3.2.7 for definition of \( F_X^B \)).

Let \( \phi' \) be the formula which asserts the following:

\( s \in 2^{i+1} \) and there is some

\[
\langle \langle \langle a_0, r_0 \rangle, \langle b_0, t_0 \rangle \rangle, \ldots, \langle \langle a_i, r_i \rangle, \langle b_i, t_i \rangle \rangle \rangle \in ((X \times Q) \times (X \times Q))^{i+1},
\]
such that for all \( j \leq i \), the “conjunction” of the following clauses holds:

1. \( r_j, t_j \leq 2^{-j} \);

2. \( \langle \langle a_0, r_0 \rangle, \langle b_0, t_0 \rangle \rangle \) is the least pair (in their natural order) so that \( d_X(a_0, b_0) > r_0 + t_0 \);

3. for \( 0 < j \leq i \), the following hold:

   • if \( s(j) = 0 \) then \( \langle \langle a_j, r_j \rangle, \langle b_j, t_j \rangle \rangle \) is the least pair (in their natural order)
so that $d_X(a_j, a_{j-1}) < r_{j-1} - r_j$, $d_X(b_j, a_{j-1}) < r_{j-1} - t_j$ and $d_X(a_j, b_j) > r_j + t_j$;

- if $s(j-1) = 1$ then $\langle (a_j, r_j), (b_j, t_j) \rangle$ is the least pair (in their natural order) so that $d_X(a_j, b_{j-1}) < t_{j-1} - r_j$, $d_X(b_j, t_{j-1}) < t_{j-1} - t_j$ and $d_X(a_j, b_j) > r_j + t_j$;

4. if $s(i) = 0$ then $x \in B(a_i, r_i)$, and if $s(i) = 1$ then $x \in B(b_i, t_i)$.

Clearly, $\phi'$ is $\Sigma^0_1$. So we may write it as $\exists \sigma \phi$ with $\phi$ bounded.

Let $F_B^X \in BC_1(\hat{X}, \omega^\omega)$ code the function

$$F : \hat{X} \rightarrow \omega^\omega$$

defined by: $F(x)(i) = j \leftrightarrow (\exists s (\exists \sigma \phi \land s(i, j)) = 1 \land \forall j' < j s((i, j')) \neq 1) \lor (\forall s, \sigma, j' (\phi \rightarrow s((i, j')) \neq 1) \land j = 0))$.

It is straightforward to check that $F_B^X \circ F_B^X$ is the identity on $B$. By lemma 3.3.2 with $T$ being the empty theory, we get corollary 3.

**Corollary 4.** For $i=1,2, and 3, ACA^*$ proves

$$\forall \hat{X} \in CSM ((A_i^{n+1}(\omega^\omega) \rightarrow A_i^n(\hat{X})) \land (B_i^{n+1}(\omega^\omega) \rightarrow B_i^n(\hat{X}))).$$

**Proof:** Let $F_X^B \in BC_0(\omega^\omega, \hat{X})$ be as defined in the proof of lemma 3.2.4 and let $F_B^X \in BC_1(\hat{X}, \omega^\omega)$ be such that

$$F_B^X(x)(i) = j \leftrightarrow (j \in A \land d_A(x, j) \leq 2^{-i-2} \land \forall j' < j (j' \in A \rightarrow \neg d_A(x, j') \leq 2^{-i-2})).$$

Then it is easy to check that $F_B^B \circ F_B^X$ is the identity on $X$. Hence again by lemma 3.2.2 with $T$ being the empty theory, we get corollary 4.
Corollary 5. \( \text{ACA}^* \) proves

\[ \forall \hat{X} \in P_{CSM}, \hat{Y} \in C_{SM} (A^n_{i+2}(\hat{X}) \rightarrow A^n_{i}(\hat{Y})). \]

**Proof:** Apply corollary 1, 3 and 4.

It seems difficult to improve corollary 5 in general. But between \( R \) and \( I \), we have the following perfect match.

**Lemma 3.3.3.** \( \text{ACA}^* \) proves

\[ A^n_{3}(I) \leftrightarrow A^n_{1}(R). \]

**Proof:** It suffices to show that \( A^n_{3}(I) \) implies \( A^n_{1}(R) \). In light of theorem 3.6.1, for fixed \( X \), \( A^n_{i}(X) \) (i=1,2,3) are all equivalent under \( \text{ACA}^* \). Hence we only need to show that \( A^n_{3}(I) \) implies \( A^n_{3}(R) \).

Let \( G^R_I \in B_{C_{0}}(R,I) \) code the function \( F \) with the following property: for any \( x \) if there is an integer \( n \) such that \( 2n \leq x \leq 2n + 1 \), then \( f(x) = x - 2n \) or \( 2n + 1 \leq x \leq 2n + 2 \), then \( F(x) = 2n + 2 - x \).

Let \( G^I_R \in B_{C_{0}}(I^\omega,R^\omega) \) code the function \( F \) defined as follows: Write \( \bar{x} \in I^\omega \) as \( (x_0, x_1, x_2, \ldots) \). Let \( F(\bar{x}) \) be the following sequence:

\[ (x_0, 2 - x_0, x_1, -x_0, 2 - x_1, -x_1, x_1 + 2, x_2, \ldots). \]

In another words, \( G^I_R(\bar{x}) \) is the canonical enumeration of \( (G^R_I)^{-1}(x_0) \cup (G^R_I)^{-1}(x_1) \cup (G^R_I)^{-1}(x_2) \cup \ldots \) (any primitive recursive enumeration would suffice).

Now fix \( G \in B_{C_{n}}(R^\omega,R) \) such that \( G \) has the third invariant property. Let \( G^* \in B_{C_{n}}(I^\omega,I) \) be such that \( G^* = G^R_I \circ G \circ G^I_R \). It is easy to see that \( G^* \) also has the
third invariant property. By $A_3^n(I)$, there is some $\bar{x} \in I^\omega$ and i, such that $G^*(\bar{x}) = x_i$.
Hence
\[ G \circ \bar{G}_R^I(\bar{x}) \in (G^R_I)^{-1}(x_i) \subseteq G_R^I(\bar{x}). \]

Let $\bar{y} = G_R^I(\bar{x})$. We hence have $G(\bar{y}) \in \bar{y}$.

### 3.4 Forcing Notion

From Corollary 1 of lemma 3.3.2 and theorem 3.6.1 (see section 3.6), to verify the upper bound part of theorem 1, we only need to show that $(\Delta^1_n)^*-CA$ proves $A^+_{n+1}(2^\omega, \Pi^1_{n-1}-CA)$. Our strategy is to define a forcing notion with conditions from $X^{<\omega}$ so that for each Borel code $u$ (for a Borel function from $X^\omega$ to $X$) of rank $n + 1$, some of the so-called $(u, n, X)$-generic sequences will witness the truth of $A^+_1(X, \Pi^1_{n-1}-CA)$ with respect to that $u$. The major challenge we are facing is to define the notion of forcing and genericity in a way so that the existence of such generic sequences is provable in $(\Delta^1_n)^*-CA$.

### 3.4.1 Basic definitions

Let us fix three distinct function variables "$f"", "u" and "p" from $L_2$ throughout this chapter.

Term symbols such as "$s", "t", "r", \ldots, unless otherwise specified, always stand for terms of PA.

\[ \Pi^0_\omega(u,f) \] is the set of formulas built up inductively as follows:

1. $p_0(t_1, \ldots, t_k), u(s) = t$ and $\hat{f}(s)(t) = r$ are in $\Pi^0_\omega(u,f)$, where $t_1, \ldots, t_k, s, t$ and $r$ are terms, and $p_0$ is a $k$-ary predicate symbol of PA;
2. if \( \phi, \psi \in \Pi_\infty^0(u, \dot{f}) \), then \((\neg \phi) \in \Pi_\infty^0(u, \dot{f})\) and \((\phi \land \psi) \in \Pi_\infty^0(u, \dot{f})\);

3. if \( \phi \in \Pi_\infty^0(u, \dot{f}) \) and "\( m \)" is a numerical variable, then \((\forall m) \phi \in \Pi_\infty^0(u, \dot{f})\);

4. all the formulas in \( \Pi_\infty^0(u, \dot{f}) \) are built up from 1, 2, or 3.

For any \( n \), we define the set of \( \Pi^0_n(u, \dot{f}) \)-formulas and \( \Sigma^0_n(u, \dot{f}) \)-formulas the same way as we define \( \Pi^0_\infty \)-formulas in \( L_\infty^\ast \) except the formulas are all built up as above.

\textbf{Remark.} From lemma 3.2.1, if \( u \in BC_n(X, Y) \), and if the codes for \( X \) and \( Y \) are primitive recursive symbols then \( u(\dot{f}) \in_n BY(a, r) \) is a \( \Sigma^0_n(u, \dot{f}) \)-formula. Moreover if \( u \in BC_n \), then \( u(\dot{f})(i) =_n j \) is a \( \Pi^0_{n+1}(u, \dot{f}) \)-formula as well. And the same is true when \( u \in BC_n((2^\omega)^\omega, (2^\omega)^\omega) \) (this fact will be repeatedly used), where \( u(\dot{f})(i) =_n j \) is the formula \( \exists b \in 2^{< \omega} \ (ls(b) > i \land \phi_n^*(u, \dot{f}, b) \land b(i) = j) \).

Let \( X \) stand for either \( 2^\omega \) or \( \mathbb{R} \), the reals. We write \( q \in X^{< \omega} \) for the property:

\[ \forall i < q(0)(q_i \in X) \land \forall i, j ((i \geq q(0) \rightarrow q((i, j)) = 0) \land (ls(i) \neq 2 \land i \neq 0 \rightarrow q(i) = 0)) \]

Given any \( q \in X^{< \omega} \), we think of it as coding a finite sequence of elements of \( X \), whose "length" is "\( q(0) \)" and whose i-th element is \( q_i \) for \( i < q(0) \). Obviously, \( q \in X^{< \omega} \) is a \( \Pi^0_1 \) property. We will mostly use \( p, q, r, \cdots \) etc to range over these conditions. Hence when we write \( \forall p \) or \( \exists p \), we actually mean that \( \forall p \in X^{< \omega} \) or \( \exists p \in X^{< \omega} \). When the complexity makes a difference, we will point it out.

We will write \("|p|\"\) for \( p(0) \) and \("p \leq q\"\) for the following:

\[ p(0) \leq q(0) \land \forall i < p(0) (p)_i = (q)_i \]

Hence \( p \leq q \) is also a \( \Pi^0_1 \) property.
Definition 23. (definition of forcing translation) For the fixed function variable "p", we define a translation of formulas from $\tilde{\Pi}_\infty^0(u, \hat{f})$ to $\Pi^1_\infty(u, p)$

$$\phi \mapsto (p \vdash \phi),$$

by induction on formulas, where $\Pi^1_\infty(u, p)$ is the set of formulas of $L^*_2$ whose free variables are among $u$ and $p$.

- $p \vdash \phi$ is $\phi$ if $\phi$ is either an atomic formula of PA or formula of the form $u(s) = t$;
- $p \vdash \hat{f}(i)(j) = k$ is $|p| \geq (i + 1) \land p(i)(j) = k$;
- $(p \vdash \neg \psi)$ is $((\forall q \geq p)(q \nvdash \psi))$, where $q$ is the first function variable which does not appear in $(p \vdash \psi)$;
- $(p \vdash (\psi_1 \land \psi_2))$ is $((p \vdash \psi_1) \land (p \vdash \psi_2))$;
- $(p \vdash (\forall n)\psi)$ is $((\forall n)(p \vdash \psi))$.

Remark. It is easy to see that $p \vdash \phi$ is equivalent (in $\text{ACA}^*$) to $\phi$ itself if $\phi$ does not involve $\hat{f}$.

Lemma 3.4.1. For $\phi \in \tilde{\Pi}_\infty^0(u, \hat{f})$, the relativization to $X^{<\omega}$ of following is provable in $\text{ACA}^*$:

$$\forall p, q((p \vdash \phi \land p \leq q) \rightarrow q \vdash \phi).$$

Proof: It can be done by an easy induction on $\phi$.

Lemma 3.4.2 (weak forcing lemma). For $\phi \in \tilde{\Pi}_\infty^0(u, \hat{f})$, the relativization to $X^{<\omega}$ of the following is provable in $\text{ACA}^*$:

$$\forall p((p \vdash \phi) \leftrightarrow \forall q \geq p \exists r \geq q \ r \vdash \phi).$$
Proof: We argue by induction on $\phi$.

The only interesting cases are when $\phi$ is $\dot{f}(s)(t) = l$, $\neg\psi$, $\phi_1 \land \phi_2$ or $\forall k\psi(k)$. We will discuss these three cases.

Case 1: When $\phi$ is $\dot{f}(s)(t) = l$, let us suppose $p \not\models \dot{f}(s)(t) = l$. There are two possibilities:

- if $ls(p) < s + 1$ we may then define $q \geq p$ such that $q(s)(t) \neq l$. Clearly, $(\forall r \geq q)(q \not\models \dot{f}(s)(t) = l)$.

- if $ls(p) \geq s + 1$ and $p(s)(t) \neq 1$ then $\forall r \geq p(r \not\models \dot{f}(s)(t) = l)$.

Case 2: If $\phi$ is $\neg\psi$, then by definition, $(p \models \neg\psi) \iff (\forall q \geq p)(q \not\models \psi)$. Hence $(\forall r \geq q)(r \not\models \psi)$.

Conversely, if $p \not\models \neg\psi$, then $(\exists q \geq p)(q \models \psi)$. By the previous lemma, $(\forall r \geq q)(r \not\models \psi)$. In particular, $r \not\models \neg\psi$.

Case 3: When $\phi$ is $\phi_1 \land \phi_2$, by applying definition and induction hypothesis, we know that

\[ p \models \phi_1 \land \phi_2 \iff (\forall q' \geq p)(\exists r' \geq q')(r' \models \phi_1) \land (\forall q'' \geq p)(\exists r'' \geq q'')(r'' \models \phi_2). \]

Now, let us fix an arbitrary $q \geq p$. First applying induction hypothesis on $\phi_1$, we have an $r' \geq q$ such that $r' \models \phi_1$. Then since $r' \geq p$, applying induction hypothesis on $\phi_2$, we have $r \geq r'$ such that $r \models \phi_2$. To see that $r$ also forces $\phi_1$, use lemma 3.4.1. It follows that $(\forall q \geq p)(\exists r \geq q)(r \models \phi)$. The other direction is easy.

Case 4: When $\phi$ is $\forall k\psi(k)$, let us assume that $p \models \forall k\psi(k)$. Fix a $q \geq p$. By lemma 3.4.1, $(q \models (\forall k)(\phi(k)))$. Hence $\exists r \geq q(r \models (\forall k)(\phi(k)))$. Conversely, if
$(\forall q \geq p)(\exists r \geq q)(r \vdash \forall k \phi)$, then for each fixed $k$, $(\forall q \geq p)(\exists r \geq q)(r \vdash \phi(k))$. By induction hypothesis, $(\forall k)(p \vdash \phi(k))$; i.e, $p \vdash \forall k \phi(k)$.

**Corollary.** For $\phi \in \Pi^0_1(u, \dot{f})$, ACA* proves the relativization to $X^{<\omega}$ of 

$$\forall p(p \vdash \neg \phi) \iff p \vdash \phi).$$

**Lemma 3.4.3.** For $\phi \in \Pi^0_1(u, \dot{f})$, ACA* proves the relativization to $X^{<\omega}$ of 

$$\forall p(p \vdash (\forall n < t) \phi \iff (\forall n < t) p \vdash \phi).$$

**Proof:** According to our notation, $(\forall n < t) \phi$ is thought of as an abbreviation for $(\forall n)(\neg (n < t \land \neg \phi))$. Hence by definition $p \vdash (\forall n < t) \phi$ is equivalent to

$$(\forall n)(\forall q \geq p)(q \not\vdash n < t \lor q \not\vdash \neg \phi),$$

which in turn is equivalent to

$$(\forall n)(n \not\in t \lor (\forall q \geq p)(\exists r \geq q)(r \not\vdash \phi)).$$

By the Weak-forcing lemma, the last formula is equivalent to

$$(\forall n)(n \not\in t \lor p \not\vdash \phi),$$

which is abbreviated as $(\forall n < t)(p \not\vdash \phi)$.

Let us consider a theory $PA(u, \dot{f})$ whose language is the language of PA plus two new function symbols $u$ and $\dot{f}$, and whose axioms are the same as that of PA except, in the induction axiom scheme, the formulas may involve special function symbols $u$ and $\dot{f}$. We have the following lemma.
Lemma 3.4.4. For any formulas $\phi, \psi \in \Pi^0_\infty(\dot{f}, u)$, $\text{ACA}^*$ proves the relativization to $X^{<\omega}$ of

$$\forall p ((p \models \phi \rightarrow \psi) \rightarrow (p \models \phi \rightarrow p \models \psi)).$$

Proof: Suppose $p \models \phi$ and $p \models \phi \rightarrow \psi$. Hence $(\forall q \geq p)(q \models \phi \vee q \not\models \neg \psi)$. Since $p \models \phi$, we know $(\forall q \geq p)(q \models \phi)$, which, when combined with the last statement, gives $(\forall q \geq p)(q \not\models \neg \psi)$. By weak forcing lemma, we have $p \models \psi$.

Lemma 3.4.5. If $\phi \in \Pi^0_\infty(\dot{f}, u)$ and $\text{PA}(u, \dot{f}) \vdash \phi$ then $\text{ACA}^* \vdash (\emptyset \models \phi)$.

Proof: Let us prove this by induction on the theorem of $\text{PA}(u, \dot{f})$.

Case 1. $\phi$ is an axiom of $\text{PA}(u, \dot{f})$.

If $\phi$ is not an instance of the induction axiom scheme or a logical axiom involving $u$ or $\dot{f}$, then it must be an axiom of PA, which is a part of $\text{ACA}^*$, hence provable from $\text{ACA}^*$. On the other hand, $p \models \phi$ is equivalent to $\phi$ since it does not involve the symbol $\dot{f}$.

If $\phi$ is a logical axiom involving $u$ or $\dot{f}$, the proof directly follows from the definition of forcing and logic axioms that do not involve $u$ or $\dot{f}$.

Let $\phi$ be an instance of induction scheme. For instance, $\phi$ is

$$\psi(0) \land \forall n(\psi \rightarrow \psi(n + 1)) \rightarrow \forall n \psi$$

for some $\psi \in \Pi^0_\infty(\dot{f}, u)$. To obtain a contradiction, we suppose $\emptyset \not\models \phi$. By definition, there is some $q \geq p$ such that

$$q \models \psi(0) \land \forall n(\psi \rightarrow \psi(n + 1)) \land \neg \forall n \psi,$$

\footnote{Note that in the forcing language, the formula $A \rightarrow B$ is an abbreviation for $\neg(A \land \neg B)$}

\footnote{Here we use the proof system presented in Chapter 2 of [5].}
which is equivalent to

\[ q \vdash \psi(0) \land q \vdash \forall n(\psi \rightarrow \psi(n + 1)) \land q \vdash \neg \forall n \psi. \]

By the previous lemma, the above implies

\[ (q \vdash \psi)[n/0] \land \forall n((q \vdash \psi) \rightarrow (q \vdash \psi)[n/n + 1]) \land q \vdash \neg \forall n \psi. \]

Applying induction axiom in ACA*, we have

\[ (\forall n)(q \vdash \psi) \land q \vdash \neg \forall n \psi; \]
i.e.,

\[ (q \vdash \forall n \psi) \land (q \vdash \neg \forall n \psi), \]

which is a contradiction.

**Case 2.** \( \phi \) follows from structure rules or the cut rule. The verification is straightforward.

**Case 3.** \( \phi \) is \( A \rightarrow \forall nB \) and it follows from \( A \rightarrow B \), where \( n \) is not free in \( A \).

Suppose that \( \emptyset \vdash A \rightarrow B \), but \( \emptyset \nvdash A \rightarrow \forall nB \). Then

\[ (\exists q \geq p)(q \vdash A \land q \vdash \neg \forall n B). \]

It follows that

\[ (\exists q \geq p)(q \vdash A \land q \nvdash \forall n B). \]

Hence

\[ (\exists n)(\exists q \geq p)(q \vdash A \land q \nvdash B). \]
Fix an \( n_0 \) and \( q_0 \geq p \) such that

\[ q_0 \models A \land q_0 \notmodels B[n/n_0]. \]

Now since \( 0 \models A \rightarrow B \), surely, \( 0 \models (A \rightarrow B)[n/n_0] \). Since \( n \) is not free in \( A \), 
\( (A \rightarrow B)[n/n_0] \) is \( A \rightarrow B[n/n_0] \). Hence we have both \( q_0 \models A \) and \( q_0 \notmodels A \rightarrow B \). By the previous lemma, we have \( q_0 \models B[n/n_0] \). But we already have \( q_0 \notmodels B[n/n_0] \). This is a contradiction.

Next, we want to show that "\( p \models \phi \)" is provably (in \( \text{ACA}^* \)) equivalent to a quantifier free formula if \( \phi \) is quantifier free.

For \( \psi \in \Pi^0_0(u, f) \), we use \( \{i^\psi_1, \ldots, i^\psi_{p_\psi} \} \) to denote the set of free numerical variables in \( \psi \) arranged in their natural order (variables are given in certain order in \( L^*_2 \)).

**Definition 24.** We define three terms, \( t^c_\psi \) \((c=1,2,3)\), of \( \text{PA} \) for each \( \psi \in \Pi^0_0(u, f) \) with free variables among \( i^\psi_1, \ldots, i^\psi_{p_\psi} \) by induction on \( \psi \):

1. if \( \psi \) is \( s < t \), \( s = t \) or \( u(s) = t \) then \( t^c_\psi \) is 0;

2. if \( \psi \) is \( f(s_1)(s_2) = s_3 \), then \( t^c_\psi \) is \( s_c \);

3. if \( \psi \) is \( \neg \phi \), then \( t^c_\psi \) is \( t^c_\phi \);

4. if \( \psi \) is \( \phi_1 \land \phi_2 \) then \( t^c_\psi \) is \( \max(m_1/t^c_{\phi_1}, m_2/t^c_{\phi_2}) \), where "max" is the binary primitive function symbol of \( \text{PA} \) which picks the larger one of its two arguments;

5. if \( \psi \) is \( (\forall m \leq t)\phi \) then \( t^c_\psi \) is \( h_{t^c_\phi}(t) \), where \( h_{t^c_\phi} \) is the primitive function symbol whose defining equation is provably (in \( \text{PA} \)) equivalent to

\[
h_{t^c_\phi}(0) = t^c_\phi(m/0) \land (\forall i)(h_{t^c_\phi}(i + 1) = \max\{t^c_\phi(m/i + 1), h_{t^c_\phi}(i)\}).
\]
To deal with the real numbers we need one more term. Let us introduce two more notations. They will only be used in the proofs of the lemma 3.4.6 and the claim prior to the lemma.

We use $R(\sigma)$ to denote the formula:

$$
\forall i < ls(\sigma), j < ls(\sigma_i) - 1 \ ((\sigma_i)_j \in Q \land d_Q((\sigma_i)_j, (\sigma_i)_{j+1}) \leq 2^{-j-1}).
$$

$R(\sigma)$ says that $\sigma$ is a finite sequence of initial segments of Cauchy sequences of rational numbers.

Fix a formula $\phi' \in \Pi^0_0(u, \dot{f})$. Let $m \geq t_1^\psi, n \geq t_2^\psi, k, l \geq t_3^\psi$. Let $\tau \in (\{0, \cdots, k\}^n)^m, \sigma \in \{0, \cdots, l\}^t_{\phi'+1}$. We will use $\tau|_{\psi} \subseteq 2 \sigma$ to denote the following formula:

$$(R(\tau) \rightarrow R(\sigma)) \land \forall i \leq t_1^\psi, j \leq t_2^\psi \ ((\tau(i))_j \leq t_3^\psi \rightarrow \tau(i)(j) = \sigma(i)(j)) \land (\tau(i)(j) > t_3^\psi \rightarrow \sigma(i)(j) > t_3^\psi)).$$

Let $t_\psi(i_1^\psi, \cdots, i_{\phi'+1}^\psi)$ be a term such that PA proves that

$$
\forall m \geq t_1^\psi, n \geq t_2^\psi, k \geq t_3^\psi \forall \tau \in (\{0, \cdots, k\}^n)^m (R(\tau) \rightarrow \exists \sigma \in (\{0, \cdots, t_\psi\})^t_{\phi'+1}(R(\sigma) \land \\
\tau|_{\psi} \subseteq 2 \sigma)).
$$

Claim. For any $\psi \in \Pi^0_0(u, \dot{f}), \text{ACA}^*$ proves that

$$
\forall m \geq t_1^\psi, n \geq t_2^\psi, k \geq t_3^\psi, \forall \tau \in (\{0, \cdots, k\}^n)^m (R(\tau) \rightarrow (\psi[\dot{f}/p^\tau] \leftrightarrow \exists \sigma \in (\{0, \cdots, t_\psi\})^t_{\phi'+1}(R(\sigma) \land \\
(\tau|_{\psi} \subseteq 2 \sigma \land \psi[\dot{f}/p^\tau])))),
$$

where $\psi[\dot{f}/p^\tau]$ is obtained from $\psi$ by replacing every appearance of $\dot{f}(s)(t) = r$ by $p(s)(t) = r$ if $|p| \geq s + 1$, otherwise by $\tau(s - |p|)(t) = r$.

Proof of the claim: We argue by induction on $\psi$. Whenever we use symbols such as $\sigma, \tau, \delta, \cdots$ etc, we implicitly assume that $R(\tau), R(\sigma), R(\delta), \cdots$ etc hold.

Case 1. $\psi$ is an atomic formula of the form $\dot{f}(s)(t) = r$. The proof follows easily from the definition of the term $t_\psi$. 


Case 2. \( \psi \) is \( \psi_1 \land \psi_2 \). Fix \( m, n, k \) and \( \tau \) as stated in the claim. Assume that \( \psi[\dot{f}/p^*\tau] \) holds. Then \( \psi_1[\dot{f}/p^*\tau] \) and \( \psi_2[\dot{f}/p^*\tau] \) both hold. Fix an arbitrary \( \sigma \in (\{0, \ldots, t_\psi\}^{t_\psi+1})_{t_\psi+1} \) such that \( \tau|_\psi \subseteq \sigma \). The existence of such \( \sigma \) follows from the choice of \( t_\psi \). By applying induction hypothesis on \( \psi_1 \), we may find a \( \delta \in (\{0, \ldots, t_\psi\}^{t_\psi+1})_{t_\psi+1} \) such that \( \tau|_{\psi_1} \subseteq \delta \) and \( \psi_1[\dot{f}/p^*\delta] \) holds. On the other hand, it is easy to check that \( \sigma|_{\psi_1} \subseteq \delta \). Hence by applying induction hypothesis on \( \psi_1 \) again, we may conclude that \( \psi_1[\dot{f}/p^*\sigma] \) also holds. Similarly, we may deduce that \( \psi_2[\dot{f}/p^*\sigma] \) holds. The converse is similar.

Case 3. \( \psi \) is \( \neg \phi \). Assume that \( \psi[\dot{f}/p^*\tau] \) holds. Fix a \( \sigma \in (\{0, \ldots, t_\psi\}^{t_\psi+1})_{t_\psi+1} \) such that \( \tau|_\psi \subseteq \sigma \). Suppose that \( \psi[\dot{f}/p^*\sigma] \) does not hold. In other words, \( \phi[\dot{f}/p^*\sigma] \) holds. Then by induction hypothesis, \( \phi[\dot{f}/p^*\tau] \) must also hold. This contradicts our assumption. Conversely, if there is some \( \sigma \in (\{0, \ldots, t_\psi\}^{t_\psi+1})_{t_\psi+1} \) such that \( \tau|_\psi \subseteq \sigma \) and \( \psi[\dot{f}/p^*\sigma] \) holds; i.e, \( \phi[\dot{f}/p^*\sigma] \) is not true. We can actually conclude from this that for any \( \delta \in (\{0, \ldots, t_\phi\}^{t_\phi+1})_{t_\phi+1} \), \( \phi[\dot{f}/p^*\delta] \) is not true. This is because that we have \( \delta|_\psi \subseteq \sigma \) for any such \( \delta \). In particular, for any \( \delta \in (\{0, \ldots, t_\phi\}^{t_\phi+1})_{t_\phi+1} \), if \( \tau_\psi \subseteq \delta \) then \( \phi[\dot{f}/p^*\delta] \) is not true. Hence by induction hypothesis \( \phi[\dot{f}/p^*\tau] \) holds.

Case 4. \( \psi \) is \( \forall k \leq t \ \phi \). The proof is similar to that of case 2.

Lemma 3.4.6. For any \( \psi \in \tilde{\Pi}_0^0(u, \dot{f}) \), ACA* proves that

1. \( \forall p \in R^{<\omega}((p \models \psi) R^{<\omega} \iff \forall \tau \in (\{0, \ldots, t_\psi\}^{t_\psi+1})_{t_\psi+1}(R(\tau) \to \psi[\dot{f}/p^*\tau])); \)

2. \( \forall p \in (2^{<\omega})^{<\omega}((p \models \psi)(2^{<\omega})^{<\omega} \iff \forall \tau \in (\{0, 1\}^{t_\phi+1})_{t_\phi+1}\psi[\dot{f}/p^*\tau])); \)

Proof: We give a proof to (1) by induction on \( \psi \). (2) is simpler.

Case 1. \( \psi \) is \( \dot{f}(s)(t) = r \).
If \( p \models \psi \), then by definition \( p(0) > s \) and \( p((s,t)) = r \). Hence \( \psi[\hat{f}/p] \), which implies that \( \forall \tau \psi[\hat{f}/p^* \tau] \). Conversely, if \( p \not\models \hat{f}(s)(t) = r \), then by definition either \( |p| \leq s \) or \( |p| \geq s + 1 \wedge p((s,t)) \neq r \). In either case, we may choose \( \tau \in \{0, \ldots, t_\psi\}^{t_\psi+1}\) such that \( R(\tau) \wedge \neg \psi[\hat{f}/p^* \tau] \).

**Case 2.** \( \psi \) is \( \psi_1 \wedge \psi_2 \).

If \( p \models \psi_1 \wedge \psi_2 \), then \( p \models \psi_1 \) and \( p \models \psi_2 \). For any \( \tau \in \{0, \ldots, t_\psi\}^{t_\psi+1}\), there is some \( \sigma \in \{0, \ldots, t_\psi\}^{t_\psi+1} \) such that \( \tau|_{\psi_1} \subseteq \sigma \). Since \( p \models \psi_1 \), by induction hypothesis we have \( \psi_1[\hat{f}/p^* \sigma] \). Then by the above claim, we also have \( \psi_1[\hat{f}/p^* \tau] \). Similarly, we may obtain \( \psi_2[\hat{f}/p^* \tau] \). Hence \( \psi[\hat{f}/p^* \tau] \) holds.

Conversely, suppose \( p \not\models \psi_1 \wedge \psi_2 \). Without loss of generality, assume \( p \not\models \psi_1 \). Then for some \( \sigma \in \{0, \ldots, t_\psi\}^{t_\psi+1} \), \( \neg \psi_1[\hat{f}/p^* \sigma] \) is true. Extend \( \sigma \) to a \( \tau \in \{0, \ldots, t_\psi\}^{t_\psi+1} \). By the above claim, we have \( \neg \psi_1[\hat{f}/p^* \tau] \) since \( \tau|_{\neg \psi_1} \subseteq \sigma \). Hence \( \neg \psi[\hat{f}/p^* \delta] \) is true.

**Case 3.** \( \psi \) is \( \neg \phi \).

First assume that \( p \not\models \psi \). Hence, by definition, there is some \( q \geq p \) such that \( q \models \phi \). By induction hypothesis, for any \( \sigma \in \{0, \ldots, t_\phi\}^{t_\phi+1} \), \( \phi[\hat{f}/q^* \sigma] \) holds. Fix an arbitrary such \( \sigma \) and a large integer \( m \). It is clear that we may choose \( \tau \in (m^{t_\phi+1})^{t_\phi+1} \) such that \( p^* \tau \) and \( q^* \sigma \) agree with each other at the places they are both defined and moreover \( \phi[\hat{f}/p^* \tau] \) holds. By the above claim, we may find some \( \delta \in \{0, \ldots, t_\phi\}^{t_\phi+1} \) such that \( \tau|_{\phi} \subseteq \delta \) and \( \phi[\hat{f}/p^* \delta] \) holds. In other words, \( \psi[\hat{f}/p^* \delta] \) does not hold.

Conversely, assume that for some \( \sigma \in \{0, \ldots, t_\phi\}^{t_\phi+1} \), \( \phi[\hat{f}/p^* \sigma] \) holds. Ex-
tend $p$ to a condition $q$ such that $|q| > t^b_\psi$ and $q$ agrees with $p^* \sigma$ everywhere except places where $p^* \sigma$ is undefined. Then it is easy to see that $\phi[\hat{f}/q]$ is also true. And furthermore for all $\sigma \in \{0, \cdots, t^2_\psi\}^{t^2_\psi}$, $\phi[\hat{f}/q^* \sigma]$ is true. It follows from induction hypothesis that $q \models R^* \phi$. Hence $p \models R^* \psi$.

Case 4. $\psi$ is $\forall n \leq t \phi$. This case is similar to case 2.

Lemma 3.4.7. For any $\phi \in \bar{\Pi}^0_\infty(u, \hat{f})$, ACA* proves that if $\phi$ is an $R$-formula then

$$\forall p, q \in R^{<\omega}((p \models \phi \land p =_R q) \rightarrow q \models \phi)^{R^{<\omega}}.$$  

Proof: We discuss two cases.

Case 1. $\phi \in \bar{\Pi}^0_\infty(u, \hat{f})$.

Fix $p, q \in R^{<\omega}$ such that $p =_R q$ and $p \models \phi$. By lemma 3.4.6, we know that

$$\forall \tau \in \{0, \cdots, t^2_\phi\}^{t^2_\phi+1} \phi[\hat{f}/p^* \tau].$$

Since $\phi$ is an $R$-formula, and for any $\tau \in \{0, \cdots, t^2_\phi\}^{t^2_\phi+1}$, $p^* \tau =_R p^* \sigma$, we also have

$$\forall \tau \in \{0, \cdots, t^2_\phi\}^{t^2_\phi+1} \phi[\hat{f}/p^* \tau],$$

which implies $q \models R^* \phi$.

Case 2. $\phi \equiv \forall n \psi$ is in $\Pi^0_1(u, \hat{f})$.

Then, $p \models R^* \phi \leftrightarrow \forall n p \models R^* \psi$. By lemma 3.4.6, $\forall n p \models R^* \psi$ is equivalent to

$$\forall n \forall \sigma \in \{0, \cdots, t^2_\psi\}^{t^2_\psi+1} (R(\sigma) \rightarrow \psi[\hat{f}/p^* \sigma]),$$

which in turn is equivalent to

$$\forall \sigma \forall n (\sigma \in \{0, \cdots, t^2_\psi\}^{t^2_\psi+1} \land R(\sigma) \rightarrow \psi[\hat{f}/p^* \sigma]).$$
One can easily see that this last formula is an R-formula from the fact that \( \forall n \psi \) is an R-formula.

The other cases follow straightforwardly by induction.

**Lemma 3.4.8.** For any \( \phi \in \Pi^0_1(u, j) \), there are formulas \( \psi_1, \psi_2 \in \Pi^0_2(u, p) \) such that \( \text{ACA}^* \) proves that

1. \( \forall p \in (2^\omega)^{<\omega}((p \models \neg \phi)(2^\omega)^{<\omega} \leftrightarrow \psi_1) \);

2. \( \phi \) is an R-formula \( \rightarrow \forall p \in R^{<\omega}((p \models \neg \phi)^{R^{<\omega}} \leftrightarrow \psi_2) \).

**Proof:** Let us prove (1) first.

Fix such a \( \phi \). Write \( \phi \) as \( \forall m \theta(m) \), where \( \theta \) is \( \Pi^0_0 \). Working in \( \text{ACA}^* \), we fix a \( p \in (2^\omega)^{<\omega} \). By definition, \( (p \models \neg \phi)(2^\omega)^{<\omega} \) is equivalent to

\[
\forall q \in (2^\omega)^{<\omega}(p \cdot q \not\models \forall m \theta(m)),
\]

which in turn is equivalent to

\[
\forall n, q \in (2^\omega)^n \exists m \ p \cdot q \not\models \theta(m).
\]

By lemma 3.4.6, there is some \( A(m, p, q, u) \in \Pi^0_0(m, p, q, u) \) such that \( (p \models \phi)(2^\omega)^{<\omega} \) is equivalent to

\[
\forall n, q \in (2^\omega)^n \exists m A(m, p, q, u).
\]

By the Weak Konig's lemma (see Schoenfield [5], p187), the above statement is equivalent to

\[
\forall n \exists m B(m, p, u)
\]
for some $B(m, p, u) \in \Pi_0^0(m, p, u)$.

Now, let us prove (2).

Fix $\phi$ as above. We know that $(p \models \neg \phi)^{\mathbb{R}^{<\omega}}$ is equivalent to

$$\forall n, k, q \in [-k, k]^n p \cdot q \not\models \forall m \theta(m).$$

Since $\forall m \theta$ is an $\mathbb{R}$-formula, by lemma 3.4.7 $p \cdot q \not\models \forall m \theta(m)$ is also an $\mathbb{R}$-formula. Write it as $\exists m A(m, p, q, u)$, where $A(m, p, q, u) \in \Pi_0^0(m, p, q, u)$. Thus $(p \models \neg \phi)^{\mathbb{R}^{<\omega}}$ is equivalent to

$$\forall n, k, q \in [-k, k]^n \exists m A(m, p, q, u).$$

Therefore it suffices to show that $\forall q \in [-k, k]^n \exists m A(m, p, q, u)$ is equivalent to a $\Sigma^0_1$ formula. For simplicity, we assume $n = 1$. By the normal form theorem, $\exists m A(m, p, q, u)$ is equivalent to a formula of the form

$$\exists m \theta(m, p, (q(0), \ldots, q(m)), u),$$

where $\theta \in \Pi_0^0(m, p, \sigma, u)$.

Now, since $\exists m A(m, p, q, u)$ is an $\mathbb{R}$-formula, so must be $\exists m \theta(m, p, (q(0), \ldots, q(m)), u)$.

Let $\theta^*(p, q, u)$ denote the following formula:

$$\exists m, \{a_0, \ldots, a_m\} \in Q^{m+1}(\theta(m, p, \{a_0, \ldots, a_m\}, u) \land (\forall i < m \ d_q(a_i, a_{i+1}) < q 2^{-i-1}) \land$$

$$(a_m - 2^{-m-1} <_R q <_R a_m + 2^{-m-1})).$$

It turns out the following claim is true.

**Claim.** $\text{ACA}^*$ proves that

$$\forall q \in R(\exists m A(m, p, q, u) \leftrightarrow \theta^*(p, q, u)).$$
For a proof, see Simpson[2] (lemma 5.7).

Hence \( \forall q \in [-k, k] \exists m A(m, p, q, u) \) is equivalent to \( \forall q \in [-k, k] \exists m, (a_0, \ldots, a_m) \in Q^{m+1}(\theta(m, p, (a_0, \ldots, a_m), u) \land \forall i < m \ d_Q(a_i, a_{i+1}) < q 2^{-i-1} \land a_m - 2^{-m-1} < q < a_m + 2^{-m-1}) \).

By the compactness of \([-k, k]\) (see [2] for a proof), the above is equivalent to

\[ \exists l, m_0, \ldots, m_i, a_0 \in Q^m_0, \ldots, a_i \in Q^m \forall i \leq l(\theta(k, m_i, a_i, p) \land \forall j \leq m_i | (a_i)_j - (a_i)_{j+1} | < q \ 2^{-j-1}) \land \forall q \in [-k, k] \exists i \leq l (a_i)_{m_i} - 2^{-m_i-1} < q \ < R (a_i)_{m_i} + 2^{-m_i-1}, \]

which in turn is equivalent to

\[ \exists l, m_0, \ldots, m_i, a_0 \in Q^m_0, \ldots, a_i \in Q^m \forall i \leq l(\theta(k, m_i, a_i, p) \land \forall j \leq m_i | (a_i)_j - (a_i)_{j+1} | < q \ 2^{-j-1}) \land \forall q \in [-k, k] \exists i \leq l (a_i)_{m_i} - 2^{-m_i-1} < q (a_0)_{m_0} - 2^{-m_0-1} \land \forall q (a_1)_{m_1} - 2^{-m_1-1} < q (a_0)_{m_0} + 2^{-m_0-1} \land \forall q \ldots (a_i)_{m_i} - 2^{-m_i-1} < q (a_{i-1})_{m_{i-1}} - 2^{-m_{i-1}-1} < q (a_i)_{m_i} + 2^{-m_i-1} > k). \]

This last formula is a \( \Sigma^0_1 \) formula.

**Corollary.** For \( X = R \) or \( 2^\omega \), \( \theta \in \Pi^0_3(u, f) \), there are formulas \( \theta^*_X \in \Pi^1_1(p, u) \) such that \( \text{ACA}^* \) proves the following:

1. \( \forall p \in X^{<\omega} ((p \models \theta)^{X^{<\omega}} \leftrightarrow \theta^*_X) \), where \( X \) is \( 2^\omega \);

2. \( \forall p \in X^{<\omega} (\theta \in R \text{-formula} \rightarrow ((p \models \theta)^{X^{<\omega}} \leftrightarrow \theta^*_X)) \), where \( X \) is \( R \).

**Proof:** The proof is by an easy induction on \( n \).

**Definition 25.** For each \( \psi \in \hat{\Pi}^0_\infty(u, f) \), we use "\( p \in D^X_\psi \)" to abbreviate

\[(p \models \psi)^{X^{<\omega}} \lor (p \models \neg \psi)^{X^{<\omega}} \] if \( \psi \) is quantifier free, or the formula

\[(p \models \psi)^{X^{<\omega}} \lor \exists k (p \models \varphi(k))^{X^{<\omega}} \] if \( \psi \) is \( \forall k \neg \varphi \) for some \( \varphi \), or the formula \((p \models \psi)^{X^{<\omega}} \lor \forall k (p \models \neg \varphi)^{X^{<\omega}} \) if \( \psi \) is \( \neg \forall k \neg \phi \).
Note that we only talk about formulas in prenex form here.

**Lemma 3.4.9.** For any $\psi \in \tilde{\Pi}_\omega^0(u, f)$, $\text{ACA}^*$ proves that

$$(\forall p \in X^{<\omega})(\exists q \in X^{<\omega})((q \geq p) \land q \in D_\psi)^{X^{<\omega}}.$$ 

**Proof:** The proof is obvious.

**Lemma 3.4.10.** If $\psi \in \tilde{\Pi}_{n+2}^0(u, f)$ then there is a formula $\phi_X \in \Pi_n^1(u, p)$ such that $\Sigma_n^1$-DC proves that

$$\forall p \in X^{<\omega}(p \in D_\psi \leftrightarrow \phi_X)^{X^{<\omega}}.$$ 

**Proof:** It follows easily from the definition.

Let $\Psi_1(e_1, \ldots, e_{n+1}, k, f, u)$ be universal for $\Pi_1^0(e_2, \ldots, e_{n+1}, k, f, u)$-formulas with the free variables shown, in the sense that for any formula $\phi$ in $\Pi_1^0(e_2, \ldots, e_{n+1}, k, f, u)$, there is an $e_1$ such that

$$PA(u, f) \vdash (\forall e_2, \ldots, e_{n+1}, k)(\Psi_1(e_1, \ldots, e_{n+1}, k, f, u) \leftrightarrow \phi).$$

For $1 < i \leq n + 1$, let $\Psi_i$ be $(\forall e_i)\neg \Psi_{i-1}$. Then it is easy to see that $\Psi_i$ is universal for $\Pi_i^0(e_{i+1}, \ldots, e_{n+1}, k, f, u)$-formulas (in the above sense). In particular, $\Psi_{n+1}$ is universal for $\Pi_n^0(k, f, u)$-formulas.

**Definition 26.** For each $n \geq 1$, we use $g \in G^u_n(2^\omega)$ to abbreviate the formula

$$g \in (2^\omega)^\omega \land (\forall e, k)(\exists p \subseteq g)(p \in D_{\Psi_{n+1}^u(e, k, f, u)}),$$

and $g \in G^u_n(R)$ to abbreviate the formula

$$g \in R^\omega \land (\forall e, k)(\Psi_{n+1} \text{ is } R\text{-formula} \rightarrow \exists p \subseteq g(p \in D_{\Psi_{n+1}^R(e, k, f, u)})).$$

**Remark:** We call any such $g \in G^u_n(X)$ a $(u, n, X)$-generic sequence.
Lemma 3.4.11. ACA* proves that

$$\forall f, g \in R^\omega \ (f \in G_n^u(R) \land f =^R g \rightarrow g \in G_n^u(R)).$$

Proof: This is an easy consequence of lemma 3.4.7.

Lemma 3.4.12. If $\phi \equiv \forall m \rightarrow \theta \in \tilde{\Pi}_{n+1}^0(k, u, f)$, where $\theta \in \Pi_n^0(m, k, u, f)$, then ACA* proves the following:

1. $\forall f \in (2^\omega)^\omega (f \in G_n^u(2^\omega) \rightarrow \exists p \subset f (p \models \phi \lor \exists m \ p \models \theta))^{(2^\omega)^{<\omega}}$;

2. $\forall f \in R^\omega (f \in G_n^u(R) \land \phi \in R\text{-formula} \rightarrow \exists p \subset f (p \models \phi \lor \exists m \ p \models \theta))^{R^{<\omega}}$.

Proof: We only consider the case concerning the reals. Fix an arbitrary R-formula $\phi$ as stated in the lemma. Fix $e_1, f$ such that $f \in G_n^u(R)$ and

$$PA(u, \dot{f}) \vdash (\forall e_2, k)(\theta(m/e_2, k, u, \dot{f}) \leftrightarrow \Psi_n(e_1, e_2, k, u, \dot{f})).$$

Then $\Psi_n$ is provably, in ACA* (actually in $PA(u, \dot{f})$), equivalent to an R-formula. Since $f \in G_n^u(R)$, for some $p \subset f$,

$$(p \models \Psi_{n+1}(e_1, k, u, \dot{f}) \lor \exists e_2 (p \models \Psi_n(e_2, k, u, \dot{f})))^{R^{<\omega}}.$$

By lemma 3.4.5,

$$(p \models \phi(k, u, \dot{f}) \lor \exists m (p \models \theta(m, k, u, \dot{f})))^{R^{<\omega}}.$$

3.4.2 Properties of generic sequences

Recall the definition of $f^\sigma$ from section 3.3, where $f$ is a function and $\sigma$ a finite permutation of $\omega$. For $p$ a condition, $\phi \in \tilde{\Pi}_\infty^0(u, \dot{f})$, we similarly define $p^\sigma$, $\phi^\sigma$ as
follows: $p^\sigma(s)(t) = p(\sigma^{-1}(s))(t)$ and $p^\sigma$ is the same as $p$ at all the other places, and
$
\phi^\sigma$ is obtained from $\phi$ by replacing all the appearances of $\dot{f}(s)(t) = l$ by $\dot{f}(\sigma(s))(t) = l$
.
Note that $p^\sigma$ may no longer be a condition even though $p$ is. Being a condition is
preserved if the domain of $\sigma$ is some number $k < ls(p)$, in which case the lengths of
the condition and the permuted one are the same. Because of this we will restrict our
consideration of finite permutations to those whose domain is some $k$. It is easy to
see that the restriction is by no means essential. We will also adopt the convention
of assuming $\sigma(i) = i$ if $i$ is not in the domain of $\sigma$.

**Lemma 3.4.13.** For $\psi \in \tilde{\Pi}_n^0(u, \dot{f})$, $X$ being $R$ or $2^\omega$, $\ ACA^*$ proves that

$$(\forall p \in X^{<\omega}, \sigma)(\text{dom}(\sigma) < ls(p) \rightarrow (p \models \psi \iff p^\sigma \models \psi^\sigma))^X^{<\omega}.$$  

**Proof:** We show this by induction on $\psi$. We will stop the mentioning of $X^\omega$,
since it makes no difference if we are talking about $R$ or $2^\omega$. Fix $p$ and $\sigma$ such that
$\text{dom}(\sigma) < ls(p)$.

**Case 1.** If $\psi$ is $\dot{f}(s)(t) = l$, then $\phi^\sigma$ is $\dot{f}(\sigma(s))(t) = l$.

$p \models \dot{f}(s)(t) = l \iff |p| \geq s \land p(s)(t) = l \iff |p| \geq s \land p(\sigma^{-1}\sigma(s))(t) = l
\iff |p^\sigma| \geq \sigma(s) \land p^\sigma(\sigma(s))(t) = l \iff p^\sigma \models \dot{f}(\sigma(s))(t) = l.$  

**Case 2.** $\phi$ is $\phi_1 \land \phi_2$, $\neg \psi$, or $(\forall t)\psi$. The verification is straightforward.

**Lemma 3.4.14.** Let $n$ be given and $X$ be either $R$ or $2^\omega$. Then $\ ACA^*$ proves that

$$(\forall f \in X^\omega, \sigma)(f \in G^w_n(X) \rightarrow f^\sigma \in G^w_n(X)).$$

**Proof:** Fix an arbitrary $f \in X^\omega$ and $\sigma$ such that $f \in G^w_n(X)$ and $\sigma$ is a finite
permutation. Let us consider the case when $X = 2^\omega$. For simplicity we will drop the
superscript $2^\omega$.

It is easy to see that $\Psi_n^{-1} \in \Pi^0_n(u,t)$. Hence by lemma 3.4.12, there is some $p \subset f$ such that

$$ p \models \Psi_n^{-1} \lor (\exists e_{n+1})(p \models \Psi_n^{-1}). $$

Now $\sigma \subset f^\sigma$, and by the above lemma we also have

$$ p^\sigma \models (\Psi_n^{-1})^\sigma \lor (\exists e_{n+1})(p^\sigma \models (\Psi_n^{-1})^\sigma). $$

Hence we have just shown in ACA* that

$$ (\forall e, k)(\exists q \subset f^\sigma)(q \models \Psi_n^{-1} \lor \exists e_{n+1} q \models \Psi_n), $$

namely, $f^\sigma \in G_n^u(X)$.

**Lemma 3.4.15.** For each $n \geq 1$, there is a formula $\phi_X \in \Sigma^1_n(g,u)$ such that $(\Sigma^1_n)^* \cdot DC$ proves that $(\forall g \in X^\omega)(g \in G_n^u(X) \leftrightarrow \phi_X)$.

**Proof:** By Lemma 3.4.10, there is a formula $\varphi_{n+1}^X(e,i,u,p) \in \Pi^1_{n-1}(u,p)$ such that

$$ (\forall p \in X^{\omega}, e, i)((p \in D_{\varphi_{n+1}^X(e,k,j,u)})X^{\omega} \leftrightarrow \varphi_{n+1}^X(e,i,u,p)). $$

Hence when $X = 2^\omega$, we have

$$ g \in G_n^u(X) \leftrightarrow (\forall e, k)(\exists p \subset g)\varphi_{n+1}^X(e,i,u,p); $$

and when $X$ is R, we have

$$ g \in G_n^u(X) \leftrightarrow (\forall e, k)(\Psi_{n+1}(e,k,t,u) is R-formula \rightarrow \exists p \subset g \varphi_{n+1}^X(e,i,u,p)). $$

But $(\forall e, k)(\exists p \subset g)\varphi(e,i,u,p)$ and $(\forall e, k)(\Psi_{n+1}(e,k,t,u) is R-formula \rightarrow \exists p \subset g \varphi_{n+1}^X(e,i,u,p))$ are evidently equivalent to some $\phi^{2^\omega}, \phi^R \in \Sigma^1_n(g,u)$ respectively.
because \((\Sigma^1_n)^*\)-AC is a consequence of \((\Sigma^1_n)^*-\text{DC}\). \(\phi^{2_\omega}, \phi^R\) will satisfy the condition of the Lemma.

**Lemma 3.4.16.** \((\Sigma^1_n)^*-\text{DC}\) proves \((\forall p \in X^{<\omega})(\exists f \in G^u_n(X))(p \subseteq f)\).

**Proof:** When \(X\) is \(2^\omega\), by the density of each \(D_{\psi(e,k)}\), we have

\[
(\forall p \exists q (p \subseteq q \land q \in D_{\psi(e,k)}))^{(2^\omega)^{<\omega}}.
\]

Then from \((\Sigma^1_n)^*\)-DC, it follows that

\[
\forall p \in (2^\omega)^{<\omega}\exists g \in ((2^\omega)^{<\omega})^\omega \forall e \ ((g)_0 = p \land |g_e| \geq e \land (g)_e \subseteq (g)_{e+1} \land (g)_e \in D_{\psi(e_0,e_1)}).
\]

Fix a \(p\) and fix a \(g\) as given in the above formula. Let \(f\) be the union of the sequence \(\{g_i\}_{i \in \omega}\). Then \(f\) satisfies the lemma.

When \(X\) is \(R\), again by density, we have

\[
(\forall p \exists q (\Psi_{n+1}(e,k,\hat{f},u) \text{ is } R\text{-formula} \rightarrow p \subseteq q \land q \in D_{\psi(e,k)}))^{R^{<\omega}}.
\]

From \((\Sigma^1_n)^*\)-DC, it follows that \(\forall p \in R^{<\omega}\exists g \in (R^{<\omega})^\omega \forall e \ ((g)_0 = p \land |g_e| \geq e \land g_e \subseteq g_{e+1} \land (\Psi_{n+1}(e,k,\hat{f},u) \text{ is } R\text{-formula} \rightarrow g_{e+1} \in D_{\psi(e_0,e_1)}))\). Fix \(p,g\) as above. Then let \(f\) be the union of \(\{g_e : e \in \omega\}\).

**Theorem 3.4.1 (Truth lemma).** For \(\psi \in \tilde{\Pi}^0_{n+1}(u,\hat{f})\), ACA* proves that

1. \(\forall g \in G^u_n(2^\omega)(\psi[\hat{f}/g] \leftrightarrow \exists p \subseteq g(p \mid \vdash \psi(\hat{f}))^{(2^\omega)^{<\omega}}\), where \(g\) is not \(u,\hat{f}\) or \(p\);

2. \(\forall g \in G^u_n(R)(\psi \in R\text{-formula} \rightarrow (\psi[\hat{f}/g] \leftrightarrow \exists p \subseteq g(p \mid \vdash \psi(\hat{f}))^{R^{<\omega}}))\), where \(g\) is not \(u,\hat{f}\) or \(p\);
Proof: We prove this by induction on $\psi$.

The case when $\psi \in \Pi^0_0(u, \dot{f})$ must be handled differently for (1) and (2). While (1) can be shown via a straightforward induction, (2) follows from the following sublemma.

**Sublemma.** For $\psi \in \Pi^0_0(u, \dot{f})$, $\text{ACA}^*$ proves that

$$\forall g \in R^e(\psi[\dot{f}/g]) \leftrightarrow \exists p \subset g(p \models \psi(\dot{f}))^{R^e},$$

where $g$ is not $u, \dot{f}$ or $p$.

The proof of the above sublemma is similar to the proof of lemma 3.4.6.

Now suppose that $\psi$ has unbounded quantifiers.

**Case 1:** $\psi$ is $\forall k \neg \phi$ for some $\phi \in \Pi^0_{n-1}(u, \dot{f})$.

Suppose $\neg \forall k \neg \phi[\dot{f}/g]$. Hence $\forall k \neg \phi[\dot{f}/g]$ is NOT true. By induction hypothesis, it is equivalent to $\neg (\exists p \subset g)(p \models \forall k \neg \phi)$. By the genericity of $g$, we have $(\exists p \subset g)(\exists k)(p \models \phi(k))$, which implies $p \models \forall k \neg \phi$.

Conversely, suppose $\exists p \subset g \models \forall k \neg \phi$. For a contradiction, let us assume that $\forall k \neg \phi[\dot{f}/g]$. By induction hypothesis, $\exists q \subset g \models \forall k \neg \phi$. Without loss of generality, we assume $p \leq q$. We then have $q \models \psi \land \neg \psi$, which is a contradiction.

**Case 2:** $\psi$ is $\forall k \neg \phi$ for some $\phi \in \Pi^0_{n}(u, \dot{f})$.

Suppose $\forall k \neg \phi[\dot{f}/g]$. By genericity, there exists $p \subset g$ such that either $(p \models \forall k \neg \phi$ or $(\exists k)(p \models \phi)$. If the former is true, then we are done. Otherwise, let us fix a $k_0$ such that $p \models \phi[k/k_0]$. By induction hypothesis we know $\phi[k/k_0, \dot{f}/g]$, which is a contradiction. Hence $p \models \forall k \neg \phi$. 
Conversely, suppose \( \exists p \subset g \forall k p \models \neg \phi \). If \( \forall k \neg \phi[\hat{f}/g] \) is not true, then \( \exists k \phi[\hat{f}/g] \) is true. By induction hypothesis, \( \exists k, q \subset g q \models \phi(\hat{f}) \), which is a contradiction.

**Corollary 1.** For \( \psi \in \Pi^0_{n+1}(u, \hat{f}) \), \((\Sigma^1_n)^*\)-DC proves

1. \( \forall p \in (2^\omega)^{<\omega} ((p \models \psi)^{2^\omega} \leftrightarrow \forall g \in G_n^u(2^\omega)(p \subset g \rightarrow \psi[\hat{f}/g])) \), where \( g \) is not \( p, u \) or \( \hat{f} \);

2. \( \forall p \in R^{<\omega} (\psi \in R\text{-formula} \rightarrow (p \models \psi)^{R^{<\omega}} \leftrightarrow \forall g \in G_n^u(R)(p \subset g \rightarrow \psi[\hat{f}/g])) \), where \( g \) is not \( p, u \) or \( \hat{f} \).

**Proof:** We will prove (2) only.

Fix such a \( \psi \). We work in \((\Sigma^1_n)^*\)-DC. Fix \( p \) and assume \( \psi \in R\text{-formula} \). When \( \psi \) is in \( \Pi^0_n(u, \hat{f}) \), this is essentially lemma 3.4.6. So we assume \( \psi \) may be written as \( \forall k \neg \theta \).

First suppose \( p \models \psi \). For any \( g \in G_n^u(R) \), if \( p \subset g \), then by Theorem 3.4.1, \( \psi[\hat{f}/g] \) is true.

Conversely, suppose \( p \not\models \psi \). Then \( \exists k p \not\models \neg \theta \); i.e,

\[
\exists k, q \geq p q \models \theta.
\]

Thus by induction hypothesis,

\[
\exists k, g \in G_n^u(R) (p \leq q \subset g \land \theta[\hat{f}/g]).
\]

It follows that

\[
\exists g \in G_n^u(R) (p \leq q \subset g \land \exists k \theta[\hat{f}/g]);
\]
i.e.,

\[ \exists g \in G^n_n(R) (p \subset g \land \forall k - \theta[k/g]). \]

**Corollary 2.** For \( \psi \in \tilde{H}^0_{n+2}(u, \dot{j}) \), \( (\Sigma^1_n)^*\)-DC proves the following:

1. \( \forall p \in (2^\omega)_{<\omega} ((p \models \psi)(2^\omega)_{<\omega} \leftrightarrow \forall g \in G_n^n(2^\omega)(p \subset g \rightarrow \psi[\dot{j}/g])) \);

2. \( \forall p \in R_{<\omega} (\psi \in R\text{-formula} \rightarrow ((p \models \psi)_{R_{<\omega}} \leftrightarrow \forall g \in G_n^n(R)(p \subset g \rightarrow \psi[\dot{j}/g]))) \);

3. \( \forall g \in G_n^n((2^\omega)_{<\omega}) (\neg \psi[\dot{j}/g] \rightarrow \exists p \subset g (p \models \neg \psi(2^\omega)_{<\omega})) \);

4. \( \forall g \in G_n^n(R_{<\omega}) ((\neg \psi[\dot{j}/g] \land \psi \in R\text{-formula}) \rightarrow \exists p \subset g (p \models \neg \psi)_{R_{<\omega}}) \).

In all the above cases \( g \) is not \( p, u \) or \( \dot{j} \).

**Proof:** We prove (2) and (4) only.

Fix \( \psi \in \tilde{H}^0_{n+2}(u, \dot{j}) \). Write \( \psi \) as \( \forall k - \theta \). Let us prove (2) first. Assume that \( \psi \) is an R-formula. Suppose \( p \models \psi \). Fix \( g \in G_n^n(R) \) and \( p \subset g \). Since \( \forall k p \models \neg \theta \), by the above lemma we know \( \forall k - \theta[k/g] \).

Conversely, suppose \( p \not\models \psi \). Then for some \( k_0 \), \( p \not\models \neg \theta[k/k_0] \). By the above lemma, we know there is some \( g \in G_n^n(R) \) such that \( \theta[k/k_0, \dot{j}/g] \). Hence \( \neg \forall k - \theta[k/k_0, \dot{j}/g] \).

For (4), we fix \( \psi \) as above. Fix \( g \in G_n^n(R) \) such that \( \neg \psi[\dot{j}/g] \); i.e., such that \( \exists k \theta[\dot{j}/g] \). Fix a \( k_0 \) such that \( \theta[k/k_0, \dot{j}/g] \). Again from the above lemma, it follows that \( \exists p \subset g p \models \theta[k/k_0] \). It is straightforward to check that, by definition, \( p \models \neg \psi \).

Q.E.D.
Let $\Phi_{n-1}^C(e, l, k, g, h)$ and $\Phi_{n-1}^R(e, l, k, g, h)$ be respectively provably universal in the theory $PA(x_1, \cdots, x_{n-1}, g, h) + \Sigma^1_\infty$-IND for all the $\Pi^1_{n-1}(l, k, g, h)$-formulas in $L_2$ and $L_2^*$ with the free variables shown, where $x_1, \cdots, x_{n-1}$ are the quantifiers in $\Phi_{n-1}^C$ and $\Phi_{n-1}^R$. Bear in mind the fact that $\Phi_{n-1}^C$ has $n-1$ alternating second-order quantifiers follows by two numerical quantifiers while $\Phi_{n-1}^R$ has $n-1$ alternating second-order quantifiers follows by only one numerical quantifiers.

Recall the concept of weak model (see section 3.3). Obviously, $f \models^* \Pi^1_{n-1}CA$ is equivalent to

$$(\forall i, j, e, l)(\exists m)(\forall k)(f_m(k) = 1 \leftrightarrow (\Phi_{n-1}^C(e, l, k, g/f_i, h/f_j))^f).$$

Also $f \models^* \Pi^1_{n-1}CA$ is equivalent to

$$(\forall i, j, e, l)(\exists m)(\forall k)(FR(f_m)(k) = 1 \leftrightarrow (\Phi_{n-1}^R(e, l, k, g/FR(f_i), h/FR(f_j)))^{\{FR(f_i) : i \in \omega\}}).$$

In the above, $(\Phi_{n-1}^C(e, l, k, g/f_i, h/f_j))^f$ and $(\Phi_{n-1}^R(e, l, k, g/FR(f_i), h/FR(f_j)))^{\{FR(f_i) : i \in \omega\}}$ stand for the relativizations of $\Phi_{n-1}^C(e, l, k, g/f_i, h/f_j)$ to $\{f_i : i \in \omega\}$

and $(\Phi_{n-1}^R(e, l, k, g/FR(f_i), h/FR(f_j)))$ to $\{FR(f_i) : i \in \omega\}$ respectively, where $FR \in BC_1(R, \omega)$ is (the code for) the function defined in lemma 3.2.7.

Clearly, there are $\Pi^0_{n+1}$-formula $\theta^X_{n-1}(e, k, l, i, j, f)$, $(X = R, C)$, such that $\text{ACA}^*$ proves that

$$\forall f \in X^\omega \ (f \models^*_X \Pi^1_{n-1}CA \leftrightarrow \forall i, j, e, l \exists m \forall k \ (H_m(k) = 1 \leftrightarrow \theta^X_{n-1}(e, k, l, i, j, f))),$$

where $H_m$ is $FR(f_m)$ if $X$ is $R$, and $f_m$ if $X$ is $C$. 
Theorem 3.4.2. \((\Sigma^1_n)^*-DC\) proves that

\[ \forall p \in X^{\leq \omega} \exists f \in G^u_n(X) \ (p \subset f \land f \models \varphi \Pi^1_{n-1} \neg \mathcal{A}) . \]

**Proof:** We will give a proof only for the case \(X = \mathbb{R}\). The proof for \(X = 2^\omega\) is similar and more straightforward.

The theorem will follow from the following two claims.

**Claim 1.** \((\Sigma^1_n)^*-DC\) proves that \((\forall i,j,e,l,k)(\forall p \in R^{\leq \omega})(|p| \geq i+1, j+1 \land \theta^R_{n-1}(e, k, l, i, j, f) \in R\text{-formula}) \rightarrow p \text{ decides } \theta^R_{n-1}(e, k, l, i, j, f)).\)

Proof of Claim 1: Assume the contrary. Then there are \(i, j, e, l, k\) and \(p\) such that \(|p| \geq i + 1, j + 1, \theta^R_{n-1} \in R\text{-formula} \) and \(p\) does not decide \(\theta^R_{n-1}(e, k, l, i, j, f)\).

By Corollary 1 to theorem 3.4.1, there are \(f, g \in G^u_n(R)\) such that \(p \subset f, g\) and \(\theta^R_{n-1}(e, k, l, i, j, f)\) and \(\neg \theta^R_{n-1}(e, k, l, i, j, g)\) are both true. Then by Corollary 2 to theorem 3.4.1, there are \(q \subset f\) and \(r \subset g\) such that \(q \models \theta^R_{n-1}(e, k, l, i, j, f)\) and \(r \models \neg \theta^R_{n-1}(e, k, l, i, j, f)\). We may assume that \(p \leq q, r\). Hence for some finite sequence (of functions) \(\tau_1\) and \(\tau_2\), \(q = p \tau_1\) and \(r = p \tau_2\). Apply Lemma 3.4.16 to get an \((u, n, R)\)-generic \(f\) such that \(p \tau_1 \tau_2 \subset f\). Then since \(q \models \theta^R_{n-1} \subset f, \theta^R_{n-1}(e, k, l, i, j, f)\) is true. Let \(\sigma\) be a finite permutation which switches the positions of \(\tau_1\) and \(\tau_2\). Then \(r = p \tau_2 = f^\sigma\), which implies \(\neg \theta^R_{n-1}(e, k, l, i, j, f^\sigma)\) is true. Since \(f\) and \(f^\sigma\) code the same model, it follows from the definition of the formula \(\theta^R_{n-1}\) and the condition \(|p| \geq i, j\) that \(\theta^R_{n-1}(e, k, l, i, j, f)\) and \(\theta^R_{n-1}(e, k, l, i, j, f^\sigma)\) are equivalent to each other. But this is a contradiction.
Now, let $A(e, i, j, l, p, h)$ denote the following formula:

$$(\forall k)(h(k) = 1 \leftrightarrow (p \models \theta_{n-1}^R(e, k, l, i, j, f))^{R^{<_\omega}}).$$

Then using $(\Sigma^1_n)^*-DC$, it is easy to see that $A(e, i, j, l, p, h)$ is equivalent to a $\Sigma^1_n$ formula with the free variables shown. Let us simply assume that $A(e, i, j, l, p, h)$ is a $\Sigma^1_n$-formula.

Note that, by Fact 1 of section 3.2.2, $\theta_{n-1}^R(e, k, l, i, j, f)$ is an $R$-formula. By the density of $D_{\theta_{n-1}^R(e, k, l, i, j, f)}$, we have $\forall p \exists q(p \leq q \land |q| \geq i + 1, j \land \exists x(A(p, x) \land BIR(x) \in q) \land (\Psi_{n+1} \in R\text{-formula} \to q \in D_{\Psi_{n+1}(e, k, l, i, j, f)})^{R^{<_\omega}}$. Here $x \in q$ stands for "$\exists i < |q|(x = q_i)$" and the function "$BIR \in BC_0(\omega^{\omega_0}, I)$" sending sequences of nonnegative integers to binary irrationals, is defined in lemma 3.2.7.

Then from $(\Sigma^1_n)^*-DC$, it follows that $\forall p \in R^{<_\omega}\exists g \in (R^{<_\omega})^{\omega}\forall e (g_0 = p \land g_e \subseteq g_{e+1} \land \exists x(A(e_0, e_1, e_2, e_3, g_e, x) \land BIR(x) \in g_{e+1}) \land |g_e| \geq e_1 + 1, e_2 + 1 \land (\Psi_{n+1} \in R\text{-formula} \to g_e \in D_{\Psi_{n+1}(e_0, e_3, e_4, e_2, f)})$.

We now continue the proof of Theorem 3.4.2. In the above statement, we take $p$ to be the condition fixed at the beginning of the proof. Let $g$ be given as in the above statement. Let $f$ be the union of the sequence $\{g_i\}_{i \in \omega}$. Then $f$ is evidently $(u, n, R)$-generic and Theorem 3.4.2 follows from the following Claim.

**Claim 2.** $f \models^*_R \Pi^1_{n-1}\text{-CA}$.

**Proof:** For any $i, j, e$ and $l$, let $s = (e, i, j, l, 0)$. Then

$$(\exists h)(A(e, i, j, l, g_s, h) \land BIR(h) \in g_{s+1}).$$

Let $f_m \in g_{s+1}$ be such that $A(e, i, j, l, g_s, FR(f_m))$. 

Since $ls(g_s) \geq i + 1 \geq j + 1$, by Claim 1 and the definition of formula $A$,

$$(\forall k)(FR(f_m(k)) = 1 \leftrightarrow g_s \models \theta^R_{n-1}(e, k, l, i, j, \hat{j})).$$

### 3.5 Upper bounds

Since the Borel diagonalization theorems in their coded form are all $\Pi^1_2$-sentences (see the statement following lemma 3.3.1) and $(\Sigma^1_n)^*-DC$ is conservative over $(\Delta^1_n)^*-CA$ for $\Pi^1_2$-sentences (one may establish this by applying 2.2.2 and fact 2 in section 1.1 or for details see [2]) it suffices to carry out the proof in $(\Sigma^1_n)^*-DC$.

#### 3.5.1 Upper bounds for $A^n_1(X)$

We will first establish the upper bound for $A^n_1(X)$ in this subsection. The upper bound for Statement $B^n_3(X)$ will be given in the next subsection.

**upper bound for Cantor space and Baire space**

At this point we want to use results from last section on the forcing notion and generic sequences with respect to $2^\omega$. Let us state what we are finally able to achieve:

**Theorem 3.5.1.** $(\Sigma^1_n)^*-DC$ proves $A^{n+1}_1(2^\omega)$.

The rest of this minisection is a verification of this theorem.

We fix a $u \in BC_{n+1}((2^\omega)^\omega, 2^\omega)$ with the first invariant property. Let $u(\hat{f})(i) =_{n+1} j$ denote the formula

$$\forall b \in \{0, 1\}^{i+1} (\phi^*_n(u, \hat{f}, b) \rightarrow b_i = j).$$
Then \( u(\hat{f})(i) = n+1 \) \( j \in \Pi^0_{n+2} \). Hence by the corollary to lemma 3.4.8, we may assume that \( p \models u(\hat{f})(i) = n+1 \) \( j \) is a \( \Pi^1_n \)-formula. Note that \( u(\hat{f})(i) = n+1 \) \( j \) is also equivalent to

\[
\exists b \in \{0,1\}^{i+1} (\phi^*_{n+1}(u, \hat{f}, b) \land b_i = j).
\]

**Lemma 3.5.1.** \( (\Sigma^1_n)^* - DC \) proves \( (\forall i)(\exists j)(p \models u(\hat{f})(i) = n+1 \) \( j \)).

**Proof:** Fix an arbitrary \( i_0 \), and choose a \( g_0 \in G_n \). Suppose \( u(g_0)(i_0) = n+1 \) \( j_0 \).

Claim: \( p \models u(\hat{f})(i_0) = n+1 \) \( j_0 \).

Proof of the claim: If otherwise, then, by corollary 2 of theorem 3.4.1 there is a \( g_1 \in G_n \) such that \( u(g_1)(i_0) = n+1 \) \( j_1 \neq j_0 \). Write \( u(\hat{f})(i) = n+1 \) \( j \) as

\[
\exists k \psi(\hat{f}, u, i, j, k),
\]

where \( \psi \in \Pi^0_{n+1} \). Then we know that both \( \exists k \psi(g_0, u, i_0, j_0, k) \) and \( \exists k \psi(g_1, u, i_0, j_1, k) \) are true. Let \( k_0 \) be a witness for the first sentence, and \( k_1 \) for the second one. Then by induction hypothesis, there are \( p_0 \subseteq g_0 \) and \( p_1 \subseteq g_1 \) such that

\[
p_0 \models \psi(\hat{f}, u, i_0, j_0, k_0)
\]

and

\[
p_1 \models \psi(\hat{f}, u, i_0, j_1, k_1).
\]

Choose \( g \in G_n \) such that \( p_0 \upharpoonright p_1 \) is an initial segment of \( g \). Then \( p_0 \subseteq g \) and \( p_1 \subseteq g^\sigma \) for some finite permutation \( \sigma \) of \( \omega \) (note that \( g^\sigma \in G^u_n(2^\omega) \) too). It follows that \( \psi(g, u, i_0, j_0, k_0) \) and \( \psi(g^\sigma, u, i_0, j_1, k_1) \) are both true. In particular, \( \exists k \psi(g, u, i_0, j_0, k) \) and \( \exists k \psi(g^\sigma, u, i_0, j_1, k) \) are true. Consequently, \( u(g)(i_0) = n+1 \) \( j_0 \) and \( u(g^\sigma)(i_0) = n+1 \)
This is impossible because $g^\sigma$ is a finite permutation of $g$, which in turn implies that $u(g)(i_0) = u(g^\sigma)(i_0)$.

Note that by lemma 3.5.1, $\models u(\dot{f})(i) =_{n+1} j$ is equivalent to $\forall k (\models u(\dot{f})(i) =_{n+1} k \rightarrow k = j)$. Hence the function $\alpha$ defined by

$$\alpha = \{ (i, j) : \models u(\dot{f})(i) =_{n+1} j \}$$

exists by $(\Delta^1_n)^*-CA$ (which is a consequence of $(\Sigma^1_n)^*-DC$). Choose $f \in G^\omega_n(2^\omega)$ such that $\alpha \in f$. Then $u(f) = \alpha \in f$, which is a solution to theorem 3.5.1.

Q.E.D.

By a result from section 3.2, we may make $f \in G^\omega_n(2^\omega)$ code a model of $\Pi^1_{n-1}-CA$.

Hence

**Corollary 1.** For $n \geq 1$, $(\Delta^1_n)^*-CA$ proves $A^{n+1}_1(2^\omega, \Pi^1_{n-1}-CA)$.

By corollary 1 to lemma 3.3.1, we also have

**Corollary 2.** For $n \geq 1$, $(\Delta^1_n)^*-CA$ proves $A^n_1(\omega^\omega, \Pi^1_{n-1}-CA)$.

**Theorem 3.5.2.** $(\Sigma^1_n)^*-DC$ proves $A^{n+1}_1(R)$.

We will use results on forcing notion and generic sequences with respect to $R$ in this subsection.

We work in $(\Sigma^1_n)^*-DC$.

Fix a $u \in BC_{n+1}(R^N, R)$ with the first invariant property. Then the formulas $u(f) < c$ and $u(f) > c$ are $\Sigma^0_{n+2}$, and $u(f) \leq c$ and $u(f) \geq c$ are $\Pi^0_{n+2}$, where $c$ is a

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Note: The image contains a page of text from a document, which appears to be a mathematical proof or explanation. The text is in English, and it seems to be discussing a theorem and its corollaries in the context of set theory and logic, specifically involving permutation and equivalence relations. The content involves formal logic and mathematical proofs, which are typically found in advanced mathematics or theoretical computer science contexts.
rational number. And, by Fact 1 of section 3.2.2, they are all $R$-formulas. Hence, by lemma 3.4.8, we may assume that

$$p \models u(f) \leq c \ (\geq c)$$

are $\Pi^1_n$.

**Lemma 3.5.2.** $(\forall a \in Q)((\emptyset \models u(f) \leq a) \lor (\emptyset \not\models u(f) \geq a)).$

**Proof:** Suppose for some $a, \emptyset \not\models u(f) \leq a$ and $\emptyset \not\models u(f) \geq a$. Then by Corollary 2 of theorem 3.4.1, there are $f, g \in G^u_n$ such that $u(f) > a$ and $u(g) < a$. Write $u(f) > a$ and $u(g) < a$ as

$$\exists k \theta_1(k, f, u, a) \text{ and } \exists k \theta_2(k, g, u, a)$$

respectively. Fix $k_1$ and $k_2$ such that $\theta_1(k_1, f, u, a)$ and $\theta_1(k_2, g, u, a)$. By the second part of that same Corollary, there are $p \subset f$ and $q \subset g$ such that

$$p \models \theta_1(k_1, f, u, a) \text{ and } q \models \theta_2(k_2, f, u, a).$$

Now as in the proof of lemma 3.5.1, we choose $(u, n, R)$-generic $f$ such that $p \ast q \subset f$. Then $\theta_1(k_1, f, u, a)$ and $\theta_2(k_2, f, u, a)$ are both true. In particular, $\exists k \theta_1(k, f, u, a)$ and $\exists k \theta_2(k, f, u, a)$ are both true. It follows that $u(f) > a$ and $u(f) < a$, which is a contradiction.

Q.E.D

Fix $f_0 \in G^u_n(R)$. Let $K_A$ be the characteristic function of

$$A = \{a \in Q : u(f_0) \geq a\}.$$
Then, by $\mathbf{ACA}^*$, $K_A$ exists. Clearly $A$ is bounded above since $u$ codes a total function.

Let $x$ be the least upper bound for $A$ (for the existence of such an upper bound, see [2]).

Claim. $\forall f \in G^n(R) \ u(f) =_R x$.

Proof: If $u(f) < x$, there must be an $a \in Q$ such that $u(f) < a < x$. Hence $\emptyset \models u(f) \leq a$. By the choice of $x$, $x \leq a$, which is a contradiction. We may get a similar contradiction by assuming $u(f) > x$.

Finally, applying theorem 3.4.2, we get a $(u, n, R)$-generic $f \in R^\omega$ such that $x \in f$ and $f \models \Pi_{n-1}^1$-CA.

Hence, we get

Corollary 1. $(\Delta_n^1)^\ast$-CA proves $A_i^{n+1}(R, \Pi_{n-1}^1$-CA) and $A_i^{n+1}(I, \Pi_{n-1}^1$-CA).

Proof: This follows from the above lemma and corollary 1 of lemma 3.3.1.

3.5.2 Upper bounds for $B^n_3(X)$

At this point, we need to discuss the forcing notion defined in section 3.4 in some weak model of $\Sigma^1_2$-DC. Though we really need to discuss both Cantor space and the real line, we will nonetheless only work with the Cantor space since by now it is clear how to translate results regarding one space to the other. Because of this restriction, we may drop unnecessary subscripts. For instance, $p \forces \psi$ really means $(p \forces \psi)^{(2^\omega)\times}$. The forcing translation we need here is an extended version of the forcing translation defined in section 3.4. More precisely, we extend the forcing translation to the set of all arithmetical formulas (in $L_\omega^2$) which do not involve $p$, instead of just $\Pi^0_\infty(u, f)$. 
Here $\hat{f}$ and $p$ are two special variables ranging over the generic sequences and forcing conditions respectively. Syntactically, this new translation is exactly the same as the earlier one except the first clause of definition 23 now reads:

1. $p \models p_0(t_1, \cdots, t_k), g(t_1) = t_2$ is $p(t_1, \cdots, t_k)$, $g(s) = t$ respectively, where $g$ is an arbitrary function variable other than $\hat{f}$ or $p$, and $p_0$ is a $k$-ary predicate symbol of PA and $t_1, \cdots, t_k$ are terms of PA.

We would like to point out that the weak forcing lemma and its corollary, the results regarding the forcing translation on the bounded formulas can be established without any significant change. So can the following lemma, which is restated here for reference:

**Lemma 3.5.3.** For each $\psi \in \Pi^0_{n+2}$ not involving $p$, there is an $\phi \in \Pi^1_n$ not involving $\hat{f}$ such that $\text{ACA}^*$ proves

$$\forall p \ ((p \models \psi) \leftrightarrow \phi).$$

Let $\Phi^0_{n+1}(e, k, p, M)$, (where $M$ is some function variable other than $p$ and $\hat{f}$), be universal for all $\Pi^0_{n+1}(k,p,M)$-formulas with the free variables shown.

**Definition 27.** We write $f \in G_n(M)$ for the formula $\forall i (f_i | e \in M \wedge f_i \in M \cap 2^\omega) \wedge \forall e, k \forall p \in M \exists q \in M ((p \leq q \wedge \Phi^0_{n+1}(e, k, q, M)) \rightarrow \exists i \Phi^0_{n+1}(e, k, f_i | M))$.

If $f \in G_n(M)$, we say $f$ is $n$-generic over $M$.

**Lemma 3.5.4.** $\text{ACA}^*$ proves

$$\forall M \in (2^\omega)^\omega \exists f (f \in G_n(M)).$$

The following simple property of $n$-generic objects over $M$ will be needed later on.
Lemma 3.5.5. ACA* proves that if $(M \models^* RCA)^5$ then

1. $\forall f \in G_n(M) (\text{range}(f) = \text{range}(M))$;

2. $\forall f \in G_n(M), i, k \exists j \geq k (M_i = f_j)$.

Proof: (1) Working in ACA*, we fix $M$ and $f \in G_n(M)$. By definition, we have that $\text{range}(f) \subseteq \text{range}(M)$. For the other direction, we fix an arbitrary $i_0$. Choose $e$ such that

$$\forall k \ (\Phi^0_{n+1}(e, k, p, M) \iff M_{i_0} \in P).$$

Applying RCA in $M$, we obviously have

$$\forall p \in M \exists q \in M \ (p \leq q \land M_{i_0} \in q).$$

Hence by genericity, for some $j_0$, we have $\Phi^0_{n+1}(e, k, f|_{j_0}, M)$; i.e., $M_{i_0} \in f|_{j_0} \subset f$.

(2) Fix $i_0, k_0$. We consider the formula

$$|p| > k_0 \land p_{l_3(p) - 1} = M_{i_0}.$$ 

Evidently

$$\forall p \in M \exists q \in M \ (p \leq q \land |q| > k_0 \land q_{l_3(q) - 1} = M_{i_0})).$$

Hence by genericity, there is some $j_0$ such that

$$|f|_{j_0} > k_0 \land (f|_{j_0})_{j_0-1} = f_{j_0-1} = M_{i_0}.$$

Corollary. ACA* proves that if $M \models^* RCA$, then

$$\forall f, g \in G_n(M) \exists \pi \ (f^\pi = g).$$

5See the definition at the beginning of section 3.3. This condition actually implies that $M \in (2^\omega)^\omega$. A similar remark applies to other lemmas for the remainder of this section.
Proof: Let $\pi$ be such that for any $i, j$, $\pi(i) = j$ iff for some $k$ and $a \in M$, $f_i = a$ is the $k$-th occurrence of $a$ in $\{f_0, \ldots, f_i\}$ and $g_j = a$ is the $k$-th occurrence of $a$ in $\{g_0, \ldots, g_j\}$.

It is straightforward to check that $\pi$ exists by $\text{ACA}^*$, $\pi$ is a bijection and $f^\pi = g$.

Lemma 3.5.6 (truth lemma). For any $\psi \in \Pi^0_{n+2}$ not involving $p$, $\text{ACA}^*$ proves that if $M \models C \Sigma^1_n \text{DC}$, then

$$\forall f \in G_{n+1}(M) (\psi[\hat{f}/f] \leftrightarrow \exists p \subset f (p \models M \psi^M)).$$

Proof: We argue by induction on $\psi$. The proof is essentially the same as that of theorem 3.4.1. There is a small difference when $\psi$ is $\forall k \neg \theta$ for some $\theta \in \Pi^0_{n+1}$.

Consider the formula $\varphi(p, M) :

\((p \models \psi)^M \vee (\exists k p \models \theta)^M).$$

By looking from inside $M$ and applying $\Sigma^1_n \text{DC}$, we know that $p \models \psi$ is $\Pi^1_n$ and $\exists k p \models \theta$ is $\Pi^1_{n-1}$. Hence we know $\varphi(p, M)$ is equivalent to a $\Pi^0_{n+2}$ formula. It is easy to check that

$$\forall p \in M \exists q \in M \varphi(p, M).$$

By the genericity of $f$, there is an $i$ such that $\varphi(f_i, M)$.

Now suppose $\psi[\hat{f}/f]$. Then we must have $f_i \models \psi$. Since if this is not true, we must have $\exists k f_i \models \theta$. It follows from induction hypothesis that $\exists \theta[\hat{f}/f]$; i.e., $\neg \psi[\hat{f}/f]$, which is a contradiction.

Conversely, if $\neg \psi[\hat{f}/f]$, then $\exists k f_i \models \theta$. Since if otherwise, we would have $\forall k f_i \models \neg \theta$. Then by induction hypothesis, it follows that $\forall k \neg \theta[\hat{f}/f]$. We have a
contradiction again.

**Corollary.** For any \( \psi \in \Pi_{n+2}^0 \) not involving \( p \), \( \text{ACA}^* \) proves that if \( M \models \Sigma_{n+2}^1 \text{DC} \) then

\[
\forall p \ (p \models \psi \iff \forall g \in G_n(M) \ (p \subseteq g \rightarrow \psi[\hat{f}/g])).
\]

**Proof:** This is the same as the proof of corollary 2 of theorem 3.4.1.

upper bounds for statement \( B \) on Cantor and Baire

**Theorem 3.5.3.** \( \text{ACA}^* \) proves that \( (\forall u)(\exists M) (u \in M \land M \models \Sigma_{n+2}^1 \text{DC}) \rightarrow B_3^n(\omega^\omega) \land B_3^{n+1}(2^\omega) \), where \( u \in M \) should be understood as the characteristic function of the graph of \( u \) is in \( M \).

**Proof:** We fix an \( u \in BC_{n+1}(2^\omega) \) with the quasi-third invariant property. By assumption there is an \( M \in (2^\omega)^\omega \) such that \( M \) is a weak model of \( \Sigma_{n+2}^1 \text{DC} \) and \( u \in M \).

As in the proof of Theorem 3.5.1, we may assume that the relation \( u(\dot{x})(s)(t) = k \)\(^6\) is \( \Pi_{n+2}^0 \). Hence

\[
p \models u(\dot{x})(s)(t) = k
\]

is equivalent to a \( \Pi_{n+1}^1 \)-formula in \( M \).

**Lemma 3.5.7.** \( \text{ACA}^* \) proves that

\[
(\forall \tilde{x})(\tilde{x} \in G_n(M) \rightarrow u(\tilde{x}) \subseteq \tilde{x}).
\]

Remark: Evidently, Theorem 3.5.3 follows from this lemma.

---

\(^6\)Here again we dropped the subscript \( n+1 \) to "=".
**Proof:** Let us argue by contradiction. Suppose there is an \( \bar{x} \in G_n(M) \) such that \( u(\bar{x}) \not\subseteq \bar{x} \). In other words, there is an \( \alpha \in u(\bar{x}) \) such that \( \alpha \not\in \bar{x} \).

**Case 1.** \( (\exists p, s)(\forall i)(p \models u(\bar{x})(s)(i) = \alpha(i))^M \).

Let \( \phi \) denote the formula \( (p \models u(\bar{x})(s)(i) = k) \). Then \( \phi \) is a \( \Pi^1_n \) formula. But in this case \( \phi \) is also equivalent (in \( M \)) to

\[
\forall j((p \models u(\bar{x})(s)(i) = j) \rightarrow k = j),
\]

which in turn is equivalent to a \( \Sigma^1_n \) formula in \( M \), since \( M \) is a model of \( \Sigma^1_n \cdot DC \). It follows that

\[
\alpha = \{(i, k) : p \models u(\bar{x})(s)(i) = k\}
\]

is in \( M \) (which implies that \( \alpha \in \bar{x} \) by lemma 3.5.5), because \( M \models \nabla^* \Delta^1_n - CA \). This contradicts our choice of \( \alpha \).

**Case 2.** We assume that Case 1 does not hold.

Let \( \Psi^0_{n+1}(e, k, p, M) \) be universal for \( \Pi^0_{n+1}(k, p, M) \) formulas (see definition 27).

**Sublemma.** \( \text{ACA}^* \) proves

\[
\forall i, s \exists j, l(M_j \geq M_i \land (M_j \models u(\bar{x})(s)(l) \neq \alpha(l)) \land (\forall p \exists q \geq p \Psi^0_{n+1}(q) \rightarrow \Psi^0_{n+1}(M_j))).
\]

Since if otherwise

\[
\exists s, i \forall l, j((M_j \geq M_i \land (\forall p \exists q \geq p \Psi^0_{n+1}(q) \rightarrow \Psi^0_{n+1}(M_j))) \rightarrow M_j \models u(\bar{x})(s)(l) = \alpha(l)).
\]

Hence \( (\forall l)(M_i \models u(\bar{x})(l) = \alpha(l)) \), which leads us back to case 1. Now suppose we have the above sublemma.

By \( \text{ACA}^* \), we have

\[
(\exists g)(\forall i)(\exists l)(g_{i+1} \geq g_i \in M \land (\forall p \exists q \geq p \Psi^0_{n+1}(q) \rightarrow \Psi^0_{n+1}(g_{i+1})) \land g_{i+1} \models u(\bar{x})(i)(l) \neq
\]
Finally, let \( \tilde{y} \) be the union of \( \{g_i : i \in \omega\} \). It is easy to see that \( \tilde{y} \in G_n(M) \) and \( \alpha \notin u(\tilde{y}) \). But by 1 of lemma 3.5.5, \( \text{range}(\tilde{x}) = \text{range}(\tilde{y}) \). Hence by \( B_3(2^\omega) \), \( \text{range}(u(\tilde{x})) = \text{range}(u(\tilde{y})) \), which implies that \( \alpha \notin u(\tilde{x}) \). But this contradicts our assumption.

We have now established our results regarding cantor space. The result on Baire space follows from corollary 2 of lemma 3.3.2.

upper bound for statement B on R and I

**Theorem 3.5.4.**\( \text{ACA}^* \) proves that

\[
(\forall u)(\exists M)(u \in M \land M \models R \Sigma^1_n \cdot DC) \rightarrow (B_3^{n+1}(R) \land B_3^{n+1}(I)).
\]

**Proof:** We fix a \( u \in BC_{n+1}(R^\omega, R^\omega) \) with the quasi-third invariant property. By assumption there is a weak model, \( M \), of \( (\Sigma^1_n)^* \cdot DC \) such that \( u \in M \). Modify the forcing relation "\models\" defined in last section inside \( M \) to being with respect to the reals. We write \( \tilde{x} \in G^R_n(M) \) for the fact that \( \tilde{x} \) is n-generic with respect to \( R \) over \( M \). All the interesting results may be proved similarly by requiring the formulas in discussion to be R-formulas at appropriate places. In particular, we have that

\[
p \models u(\tilde{x})(s) \leq c(\geq c)
\]

is equivalent to a \( \Pi^1_n \)-formula.

**Lemma 3.5.8.** \( \text{ACA}^* \) proves that

\[
(\forall \tilde{x})(\tilde{x} \in G^R_n(M) \rightarrow u(\tilde{x}) \subseteq \tilde{x}).
\]
Proof: Let us give a similar argument as in the proof of lemma 3.5.7. Suppose that there is an $\bar{x} \in G_n^R(M)$ such that $u(\bar{x}) \not\subseteq \bar{x}$. Namely, there is an $\alpha \in u(\bar{x})$ such that $\alpha \not\in \bar{x}$.

Case 1. $(\exists p, s)(\forall i)(p \models |u(\bar{x})(s)(i) - \alpha(i)| \leq 2^{-i+1})^M$. In other words, $(\forall \bar{y} \in G_n^R(M))(p \subseteq \bar{y} \rightarrow u(\bar{y})(s) =R \alpha)$.

Since $Q \subseteq M$, we have $\alpha \not\in Q$. In particular, if $g \supset p$ is $n$-generic over $M$, then $u(g)(s) = \alpha \neq c$ for any rational number $c$.

Hence we have:

Claim :

$$\forall p \in M, c \in Q(p \models x(s) \leq c \leftrightarrow \neg p \models x(s) \geq c)^M.$$ 

Let 

$$A = \{c \in Q: (p \models u(\bar{x})(s) \leq c)^M\}.$$ 

Then the characteristic function of $A, K_A$, is in $M$ by applying $(\Delta^1_n)^*-CA$ in $M$. Let $\beta$ be the least upper bound of $A$. Then $\beta \in M$, and it is easy to check that $\alpha = \beta$.

By applying lemma 3.5.5, we have $\alpha = \beta \in \bar{x}$, which is a contradiction.

Case 2. We assume that Case 1 does not hold. In this case, the proof is analogous to the Cantor space situation. Hence we have established the results regarding to $R$.

Then applying the corollary of Lemma 3.3.2, we also have the results for $I$.

Corollary. For $n \geq 1$, $(\Pi^1_n)^*-CA$ proves $B^n_1(\omega), B^{n+1}_1(2\omega), B^{n+1}_1(R)$ and $B^{n+1}_1(I)$.

Proof: Apply Theorem 3.5.4 and Theorem 2.2.4.
3.6 Relations among different invariant properties

So far we have only discussed the first-invariant property for statement A and the third invariant property for the statement B. The following important result tells us that we have not lost any generality.

**Theorem 3.6.1.** For $X$ among $\omega$, $2^\omega$, $R$ and $I$, $n \geq 1$, and $T$ among $\emptyset, ACA, \Sigma^1_n$-CA ($n \geq 1$), ACA* proves the following:

1. $A^n_1(X,T) \iff A^n_2(X,T) \iff A^n_3(X,T)$;

2. $B^n_2(X) \iff B^n_3(X)$.

The rest of this section is devoted to the proof of this theorem. Whenever we work in a formal system, we assume that ACA* is available.

It is obvious that $A^n_1(X,T), A^n_2(X,T)$ and $A^n_3(X,T)$ ($B^n_2(X), B^n_3(X)$) are successively weaker statement. We will verify that the reverse direction also goes through.

**Lemma 3.6.1.** ACA* proves that $A^n_3(X,T)$ implies $A^n_2(X,T)$ and $B^n_3(X)$ implies $B^n_2(X)$.

**Proof:** Consider a function $G \in BC_0(X^\omega, X^\omega)$ such that

$$G(\vec{x}) = (\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3 \cdots).$$

Evidently, there is a primitive recursive function $h$ such that

$$(\forall i)(G(\vec{x})_i = \vec{x}_{h(i)}).$$
Note that $G(\bar{x})$ repeats every term of $\bar{x}$ infinitely many times and $G(\bar{x})$ have the same range as $\bar{x}$. Hence they are permutations of each other (see the corollary to lemma 3.5.5).

Now, fix an arbitrary function $u \in BC_n(X^\omega, X)$ ($u \in BC_n(X^\omega, X^\omega)$) such that $u$ has the second invariant property (quasi second-invariant property). Let $v \in BC_n(X^\omega, X)$ ($v \in BC_n(X^\omega, X^\omega)$) be such that

$$v(\bar{x}) = u(G(\bar{x})).$$

From the remark following the definition of $G$, $v$ has the third invariant property (quasi third-invariant property). Hence by $A^\beta_3(X, T)$ ($B^\beta_3(X)$), there is an $\bar{x}$ such that $v(\bar{x}) \in \bar{x}$ and $\bar{x} \models T (v(\bar{x}) \subseteq \bar{x})$. Since $G(\bar{x})$ and $\bar{x}$ have the same range, we also have $u(G(\bar{x})) \in G(\bar{x})$ and $G(\bar{x}) \models T (u(G(\bar{x})) \subseteq G(\bar{x}))$.

Q.E.D.

**Lemma 3.6.2.** For $n > 0$, $\textsf{ACA}^*$ proves $\neg B^n_1(X)$, where $X$ is among Baire, Cantor, $R$ and $I$.

**Proof:** Let us take the case when $X = \omega^\omega$. The other cases are a little more complicated but can be done similarly. Since clearly $B^{n+1}_1(X)$ implies $B^n_1(X)$ for any $n$, it suffices to show $\neg B^0_1(X)$. To do this we need to find some $u \in BC_0(X^\omega, X^\omega)$ such that

$$\forall \bar{x}, \sigma (\text{range}(u(\bar{x})) = \text{range}(u(\bar{x}^\sigma)) \land u(\bar{x}) \not\subseteq \bar{x}).$$

Let us use the following notation: $x^i$ is $x^\sigma$ if $i$ is some finite permutation $\sigma$, and $x^i = x$ otherwise.
Let us fix \( v \in BC_0(X^\omega, X) \) such that \( v \) codes the function

\[
F(\bar{x})(i) = \bar{x}_i(i) + 1.
\]

Clearly, for any \( \bar{x} \), \( F(\bar{x}) \not\subseteq \bar{x} \). Then let \( u \in BC_0(\omega^\omega, X^\omega) \) code the following function

\[
F^*(\bar{x}) = \langle F(\bar{x}^0), F(\bar{x}^1), F(\bar{x}^2), F(\bar{x}^3), \cdots \rangle.
\]

Then \( u \) is the counterexample we are looking for.

To complete our argument, we once again need to define a forcing translation. In section 3.4, we essentially formalized the forcing notion of finite sequence of functions. Now we only need to formalize the forcing notion of finite sequence of nonnegative integers, which is much easier and can be done in \( \mathbf{ACA}^* \).

The forcing language \( \dot{L} \) consists of the language of the Peano arithmetic plus three special symbols: \( \dot{f}, \dot{x} \) and \( u \) standing for the generic object, a parameter from \( \omega^\omega \) and a Borel code for a Borel function from \( \omega^\omega \times \omega^\omega \) to \( X \), respectively.

We only consider atomic formulas of the following form:

\[
p(t_1, \cdots, t_k), u(s) = t, \dot{f}(s) = t, \dot{x}(s) = (t),
\]

where \( t_1, \cdots, t_k, s \) and \( t \) are terms of Peano arithmetic and \( p \) is a predicate symbol of \( \mathbf{PA} \).

The formulas in \( \dot{\Sigma}^0_\infty(u, \dot{f}, \dot{x}) \) are built up from atomic formulas by using \( \wedge, \neg \) and \( \exists n \). The set of \( \dot{\Sigma}^0_k(u, \dot{f}, \dot{x}) \) formulas are defined in the usual way. Note that here we use existential quantification instead of universal quantification as in the definition of \( \dot{\Pi}^0_\infty(u, \dot{f}) \) formulas.
\[ p \models p_0(t_1, \cdots, t_k), u(s) = t, \bar{x}(s) = t \text{ are } p_0(t_1, \cdots, t_k), u(s) = t, \bar{x}(s) = t \text{ respectively, and} \]
\[ p \models f(s) = t \text{ is } ls(p) > s \land p_s = t; \]
\[ p \models \varphi \lor \psi \text{ is } (p \models \varphi) \lor (p \models \psi); \]
\[ p \models \neg \varphi \text{ is } (\forall q \geq p)(q \not\models \varphi); \]
\[ p \models \exists n \varphi \text{ is } (\exists n)(p \models \varphi). \]

In the above definition, \( p \leq q \) stands for
\[ ls(p) \leq ls(q) \land \forall i < ls(p) p_i = q_i. \]

**Sublemma 1.** For \( k \geq 1 \) and \( \phi \in \bar{\Sigma}_k^0(u, \bar{f}, \bar{x}) \), there is a \( \psi \in \Sigma_k^0(u, p, \bar{x}) \) such that \( \text{ACA}^* \) proves that \( (p \models \phi) \leftrightarrow \psi \).

**Proof:** We may argue by induction on \( k \). The crucial case is when \( k = 0 \), which is handled in a way similar to lemma 3.4.6.

The reader may find this forcing notion is a generalization of the so-called “arithmetical forcing” first introduced by Solomon Feferman (see Peter Hinman [3], pp 124-131). With this forcing notion, it is known that the counterparts of lemma 3.4.2 and the corollary 1 of Theorem 3.4.1 are false. If we define genericity similar to definition 26, it does not seem possible for us to prove a counterpart of Theorem 3.4.1 either. To solve the latter problem, we need to redefine \((u, \bar{x}, k)\)-genericity as deciding all the \( \Sigma_k^0(u, \bar{f}, \bar{x}) \)-formulas instead of just a universal \( \Sigma_k^0(u, \bar{f}, \bar{x}) \)-formula.
For \( k \geq 1 \), let \( Tr^k(i, u, \bar{x}) \) be a formula with free variables shown such that \( \text{ACA}^* \) proves

\[
\phi(u, \bar{x}) \leftrightarrow Tr^k([\phi], u, \bar{x}),
\]

where \( \phi \in \Sigma^0_k(u, \bar{x}) \).

Let \( Fr^k(p, [\psi], u, \bar{x}) \) be the formula

\[
[\psi] \in [\Sigma^0_k(u, \hat{f}, \bar{x})] \land Tr^k([p \models \psi], u, \bar{x}),
\]

where \( [\Sigma^0_k(u, \hat{f}, \bar{x})] \) is the primitive predicate which represents the set of Gödel numbers of formulas in \( \Sigma^0_k(u, \hat{f}, \bar{x}) \). By formalizing the proof of sublemma 1 in \( \text{ACA}^* \), \( Fr^k(p, [\psi], u, \bar{x}) \) is clearly a \( \Sigma^0_k(u, \hat{f}, \bar{x}) \)-formula.

We call an \( f \in \omega^\omega (u, \bar{x}, k) \)-generic and write \( f \in G_k(u, \bar{x}) \) if

\[
\forall [\psi] \in [\Sigma^0_k(u, \hat{f}, \bar{x})] \exists p \subset f (Fr^{k+1}(p, [\psi], u, \bar{x}) \lor Fr^{k+1}(p, [\neg \psi], u, \bar{x})).
\]

The following sublemma is obvious.

**Sublemma 2.** For any \( k \geq 1 \), \( \text{ACA}^* \) proves

\[
\forall p \exists f (p \subset f \land f \in G_k(u, \bar{x})).
\]

**Sublemma 3.** (truth lemma): For any \( k \geq 1 \), \( \psi \in \Sigma^0_k \), \( \text{ACA}^* \) proves that

\[
(\forall f \in G_k(u, \bar{x}) (\psi(f, u, \bar{x}) \Leftrightarrow (\exists p \subset f) (Fr^k(p, [\psi(f, u, \bar{x})], u, \bar{x}))).
\]

**Proof:** This is similar to the proof of theorem 3.4.1.
Lemma 3.6.3. For $X$ among $\omega, 2^\omega, R$ and $I$, $n \geq 1$, ACA* proves that $A_2^n(X)$ implies $A_1^n(X)$.

Proof: Fix $u \in BC_n(X^\omega, X)$ such that $u$ codes a Borel function $F : X^\omega \to X$ of Baire rank $n$ with the first invariant property. Let $u^* \in BC_n(X^\omega \times \omega^\omega, X)$ be a Borel code such that it codes the Borel function

$$F^* : X^\omega \times \omega^\omega \to X$$

defined by

$$F^*(\bar{x}, f) = F(\langle x_{f(0)}, x_{f(1)}, x_{f(2)}, \ldots \rangle).$$

Now, we need to be a little more specific. Let us take the case when $X = R$. Then the formulas

$$u^*(\bar{x}, f) < c, \quad u^*(\bar{x}, f) > c$$

are $\Sigma^0_{n+1}(u^*, \hat{f}, \hat{x})$-formulas and they obviously satisfy the conditions of lemma 3.2.4. Using this fact, sublemma 2, sublemma 3 (note that they are true for any $k \geq 1$) and the fact that $u$ has the first invariant property, it is easy to check that $\exists p \ (p \models u^*(\bar{x}, \hat{f}) < c)$ satisfies the condition of lemma 3.2.4. The crucial point here is that if there are $p$, $q$ and $c$ such that $p \models u^*(\bar{x}, \hat{f}) < c$ and $q \models u^*(\bar{x}, \hat{f}) \geq c$, then by choosing a $(u, \bar{x}, n)$-generic $f$ with $p \dot{\cup} q \subset f$, we would get a contradiction; i.e., both $u^*(\bar{x}, f) < c$ and $u^*(\bar{x}, f) \geq c$. Hence there is a Borel code $\bar{u} \in BC_n(R^\omega, R)$ such that

$$\forall \bar{x} \in R^\omega, c \in Q \ (\bar{u}(\bar{x}) < c \leftrightarrow \exists p \ (p \models u^*(\bar{x}, \hat{f}) < c)).$$

Next we want to prove that $\bar{u}$ has the second invariant property.
Sublemma 4. For any $k \geq 1$, $\mathbf{ACA}^*$ proves that

$$(\forall \pi \in \text{Bij}, p)(\exists f \supset p)(f \in G_k(u^*, \bar{x}) \land f^* \in G_k(u^*, \bar{x}^*)),$$

where $\pi \in \text{Bij}$ means $\pi$ is a bijection.

Proof: We only give a sketch. Fix $\pi \in \text{Bij}$. For a finite sequence $p$, we use $p^\pi$ to denote the finite sequence whose length is the least $k$ such that $\{\pi(0), \cdots, \pi(l_s(p) - 1)\} \subseteq k$ and for all $i < k$, $p^\pi(i)$ is $p(\pi^{-1}(i))$ if defined; 0 otherwise. This notation will only be used in this proof. Let $\{\phi_i\}_{i \in \omega}$ list all the $\Sigma^0_k(u, \bar{f}, \bar{x})$-formulas. We then use $\mathbf{ACA}^*$ to construct a sequence $\{(p_i, q_i) : i \in \omega\}$ such that: $p_0$ extends $p$; $q_i$ extends $p_i^\pi$; $p_{i+1}$ extends $q_i^\pi^{-1}$; $p_i$ decides $\phi_i$, and $q_i$ decides $\phi_i(\bar{\bar{x}}/\bar{x}^\pi)$. Finally, let $f$ be the union of the $p_i$’s and $g$ be the union of $q_i$’s. Then by the construction $f \in G_k(u, \bar{x})$, $f^* \in G_k(u, \bar{x}^\pi)$, and $f^* = g$.

Sublemma 5. $\mathbf{ACA}^*$ proves that

$$(\forall \pi \in \text{Bij}, \bar{x} \in \mathbb{R}^\omega, c \in \mathbb{Q})(\bar{u}(\bar{x}) < c \leftrightarrow \bar{u}(\bar{x}^\pi) < c).$$

Proof: Fix a $\pi \in \text{Bij}$ and $\bar{x} \in \mathbb{R}^\omega$.

Suppose that we have both $\bar{u}(\bar{x}) < c$ and $\neg \bar{u}(\bar{x}^\pi) < c$. By definition, there is some $p_0$ such that $p_0 \models u^*(\bar{f}, \bar{x}) < c$ and $\forall p (p \not\models u^*(\bar{f}, \bar{x}^\pi) < c)$. The latter is, by definition, equivalent to $\emptyset \models u^*(\bar{f}, \bar{x}^\pi) \geq c$. By sublemma 4, there is a $(u, \bar{x}, n + 2)$-generic $f$ such that $p_0 \subset f$ and $f^*$ is $(u, \bar{x}^\pi, n + 2)$-generic. By the truth lemma, we have

$$u^*(f, \bar{x}) < c, \text{ and } u^*(f^*, \bar{x}^\pi) \geq c.$$
However,

\[ u^*(f, \bar{x}) =_R F(< \bar{x}_{f^{-1}(0)}, \bar{x}_{f^{-1}(1)}, \bar{x}_{f^{-1}(2)}, \cdots>) \]

\[ =_R F(< \bar{x}_{\pi^{-1}(f^{-1}(0))}, \bar{x}_{\pi^{-1}(f^{-1}(1))}, \bar{x}_{\pi^{-1}(f^{-1}(2))}, \cdots>) \]

\[ =_R F(< \bar{x}_{\pi(f^{-1}(0))}, \bar{x}_{\pi(f^{-1}(1))}, \bar{x}_{\pi(f^{-1}(2))}, \cdots>) \]

\[ =_R u(f^\pi, \bar{x}^\pi) \]

\[ =_R \bar{F}(\bar{x}^\pi). \]

The last equality comes from the truth lemma. Hence

\[ u^*(f, \bar{x}) =_R u^*(f^\pi, \bar{x}^\pi), \]

which is a contradiction.

We have finally verified that \( \bar{u} \) has the second invariant property. By \( A^n_2(X, T) \), there is an \( \bar{x}^* \) such that \( \bar{u}(\bar{x}^*) \in \bar{x}^* \) and \( \bar{x} \models_x T \). Let \( \bar{y} \) be the sequence \( (x^*_{f^{-1}(0)}, x^*_{f^{-1}(1)}, \cdots) \).

Note that \( f \) is \((u, \bar{f}, \bar{x})\)-generic. Hence in particular \( f \) is onto. It follows that \( \bar{y} \models_x T \), and that \( F(\bar{y}) =_R \bar{u}(f, \bar{x}^*) \) is also a coordinate of \( \bar{y} \).
CHAPTER IV

Lower bounds

4.1 Lower bounds for statement \( A_1^n(X) \)

The theory of Subprim-CA consists of \( PA \) and the following axiom schemes together with the full induction axiom scheme.

1. If \( s, t \) are terms of \( PA \) with variables \( i_0, \ldots, i_k \), then we have the axiom

\[
\exists x \forall i_0, \ldots, i_k ((i_0, \ldots, i_k) \in x \iff s \leq t)
\]

and the axiom

\[
\exists x \forall i_0, \ldots, i_k ((i_0, \ldots, i_k) \in x \iff s = t).
\]

2. For each \( l, k, n \) \((k + l \geq n \geq k, l)\) and term \( t \) of \( PA \) with \( k \) many variables, we have the axiom

\[
\forall x \exists y \forall i_1, \ldots, i_n ((i_1, \ldots, i_n) \in y \iff \langle t(i_{p_1}, \ldots, i_{p_k}), i_{q_1}, \ldots, i_{q_l} \rangle \in x),
\]

where \( \{i_{p_1}, \ldots, i_{p_k}\} \cup \{i_{q_1}, \ldots, i_{q_l}\} = \{i_1, \ldots, i_n\} \).

3. \[
\forall x \exists y \forall i (i \in y \iff i \not\in x).
\]
4. For each $m, l, n \ (m + l \geq n \geq m, l)$, we have the axiom

$$\forall x_1, x_2 \exists y \forall i_1, \ldots, i_n \ ((i_1, \ldots, i_n) \in y \leftrightarrow (i_{p_1}, \ldots, i_{p_m}) \in x_1 \land (i_{q_1}, \ldots, i_{q_l}) \in x_2),$$

where $\{i_{p_1}, \ldots, i_{p_m}\} \cup \{i_{q_1}, \ldots, i_{q_l}\} = \{i_1, \ldots, i_n\}$;

5. For each $n$, we have the axiom $\forall x \exists y \forall i_0, \ldots, i_n \ ((0, i_0, \ldots, i_n) \in y \leftrightarrow (0, i_0, \ldots, i_n) \in x) \land \forall j \ ((j + 1, i_0, \ldots, i_n) \in y \leftrightarrow (j, i_0, \ldots, i_n) \in y \land (j + 1, i_0, \ldots, i_n) \in x)).$

**Lemma 4.1.1.** Subprim-CA proves $\Pi_0^0$-CA.

**Proof:** We prove by induction on $\phi(i_0, \ldots, i_n, x_1, \ldots, x_l) \in \Pi_0^0$ that Subprim-CA proves

$$\exists x \forall i_0, \ldots, i_n \ ((i_0, \ldots, i_n) \in x \leftrightarrow \phi).$$

**Case 1.** $\phi$ is atomic: $p(t_1, \ldots, t_k), s \in x$, where $s, t_1, \ldots, t_k$ are terms of PA and $p$ is a $k$-ary predicate of $PA$. This is by axiom scheme 1 and 2. Recall that $p(t_1, \ldots, t_k)$, by definition, is provably equivalent in PA to a formula of the form $t = 1$ for some term $t$ of PA.

**Case 2.** $\phi$ is $\neg \psi$. Apply induction hypothesis and axiom scheme 2 and 3.

**Case 3.** $\phi$ is $\phi_1 \land \phi_2$ with variables listed in order $i_1, \ldots, i_n$ such that $i_{p_1}, \ldots, i_{p_k}$ are the variables of $\phi_1$ and $i_{q_1}, \ldots, i_{q_l}$ are the variables of $\phi_2$. Then by induction hypothesis, there are $x_1, x_2$ such that

$$\forall i_{p_1}, \ldots, i_{p_k} \ ((i_{p_1}, \ldots, i_{p_k}) \in x_1 \leftrightarrow \phi_1)$$

and

$$\forall i_{q_1}, \ldots, i_{q_l} \ ((i_{q_1}, \ldots, i_{q_l}) \in x_2 \leftrightarrow \phi_2).$$
Then apply axiom scheme 4.

Case 4. $\phi$ is $\forall n \leq t \psi$.

List the variables of $\phi$ in order $i_1, \ldots, i_m$ such that $n, i_{p_1}, \ldots, i_{p_k}$ are variables of $\psi$, $i_{q_1}, \ldots, i_{q_l}$ are variables of $t$ and $\{i_{p_1}, \ldots, i_{p_k}\} \cup \{i_{q_1}, \ldots, i_{q_l}\} = \{i_1, \ldots, i_m\}$. By induction hypothesis there is some $x_\psi$ such that

$$\forall n, i_{p_1}, \ldots, i_{p_k} \ (\langle n, i_{p_1}, \ldots, i_{p_k}\rangle \in x_\psi \leftrightarrow \psi).$$

By axiom scheme 5. $\exists x \forall i_{p_1}, \ldots, i_{p_k} \ ((\langle 0, i_{p_1}, \ldots, i_{p_k}\rangle \in x \leftrightarrow \langle 0, i_{p_1}, \ldots, i_{p_k}\rangle \in x_\psi) \land \forall j \ ((j + 1, i_{p_1}, \ldots, i_{p_k}) \in x \leftrightarrow ((j + 1, i_{p_1}, \ldots, i_{p_k}) \in x_\psi \land \langle j, i_{p_1}, \ldots, i_{p_k}\rangle \in x))).$ Fix $x^*$ with the above property. Then, by axiom scheme 2,

$$\exists x \forall i_1, \ldots, i_m \ ((i_1, \ldots, i_m) \in x \leftrightarrow (t(i_{q_1}, \ldots, i_{q_l}), i_1, \ldots, i_k) \in x^*).$$

Fix an $x_{\forall n \leq t \psi}$ with the above property. Then it is easy to check that

$$\forall i_1, \ldots, i_m (\langle i_1, \ldots, i_m\rangle \in x_{\forall n \leq t \psi} \leftrightarrow \forall n \leq t \psi).$$

**Corollary.** Subprim-CA proves

$$\forall x, \sigma \exists y (y = x^\sigma),$$

where $\sigma$ range over finite permutations, and $y = x^\sigma$ abbreviates $\forall i, j \ ((i, j) \in y \leftrightarrow \langle \sigma_i, j\rangle \in x)$ (we let $\sigma_i = i$ when $i \geq ls(\sigma)$).

**Proof:** Consider the formula $\phi(i, \sigma, x)$:

$$((ls(i) \neq 2 \lor (ls(i) = 2 \land i_0 \geq ls(\sigma))) \land i \in x) \lor (ls(i) = 2 \land i_0 < ls(\sigma) \land (\sigma_{i_0}, i_1) \in x).$$
Evidently $\phi \in \Pi^0_0$. By $\Pi^0_0$-CA,

$$\exists y \forall i, \sigma \ ((\sigma, i) \in y \leftrightarrow \phi(i, \sigma, x)),$$

Fix $y$ satisfying the above. We may prove by induction on $\sigma$ that

$$\forall \sigma, \exists Y \forall i (i \in Z \leftrightarrow (\sigma, i) \in Y).$$

**Theorem 4.1.1.** $A_1^n(2^\omega)$ plus $\mathbf{ACA}^*$ proves that $(\exists f)(f \models^* \Sigma^1_{n-2}$-CA plus Subprim-CA).

We will try to deduce this theorem as a simple corollary of lemma 4.1.2 through lemma 4.1.5. There is a simple fact from classical recursion theory, which, informally, says that there is a recursive enumeration of all the primitive recursive functionals. Hence there is an $f$ which is recursive in some $g$ such that $f$ codes a model of primitive recursive comprehension scheme; i.e, $Prim^*-\mathcal{CA}$ (see section 1.2). For further reference, see Peter Hinman [3], pp. 29-37. Our $Subprim-\mathcal{CA}$ also shares this property. One may note that $Subprim-\mathcal{CA}$ is almost a paraphrase of $\Sigma^0_0$-CA. The reason we have taken this approach is that by observing the similarity between the definition of $Prim^*-\mathcal{CA}$ and $Subprim-\mathcal{CA}$, we see that there is a recursive model of the latter theory. We want to use the fact that this is provable within $\mathbf{ACA}^*$. We restate this formally as follows:

**Lemma 4.1.2.** There is a $\Pi^0_1$-formula $\phi(e, i, x)$ and a $\Sigma^0_1$-formula $\psi(e, i, x)$ with all free variables shown such that

1. $\mathbf{ACA}^*$ proves

$$(\forall e, i, x)(\phi(e, i, x) \leftrightarrow \psi(e, i, x)).$$
2. If we let \( y \in Pri(x) \) denote the formula

\[
(\exists e, m)(\forall i_1, \ldots, i_m)((i_1, \ldots, i_m) \in y \leftrightarrow \psi(e, (i_1, \ldots, i_m), x)),
\]

then

\[
\text{ACA}^* \vdash (\forall x)(\theta)^{Pri(x)},
\]

where \( \theta \in \text{Subprim-CA} \) and \( (\theta)^{Pri(x)} \) is the relativization of \( \theta \) to \( \{y : y \in Pri(X)\} \).

3. \text{ACA}^* \text{ proves that }

\[
(\forall \sigma, \vec{X}, e,)(\exists e')(\forall i)(\phi(e, i, \vec{X}) \leftrightarrow \phi(e', i, \vec{X}')),\]

in particular, \text{ACA}^* \text{ proves }

\[
(\forall \sigma, \vec{X}, y)(y \in Pri(\vec{X}) \leftrightarrow y \in Pri(\vec{X}')).
\]

**Remark.** We will call those \( f \)'s such that \( f \in Pri(x) \), *Subprimitive recursive* in \( x \). Hence the above lemma says that for any \( \vec{X} \), the class of all \( y \)'s that are Subprimitive recursive in \( \vec{X} \) forms a model of \( \text{Subprim-CA} \), and it is invariant under finite permutation. Intuitively, this is obvious.

Let \( \Theta_{n-2}(d, i) \) be provably universal in \( PA(x_1, \ldots, x_{n-2}) \) plus \( \Pi^1_{\infty}\)-IND, for \( \Sigma^1_{n-2}(i) \) formulas with all free variables shown, where \( x_1, \ldots, x_{n-2} \) are the second-order quantifiers in \( \Theta_{n-2} \). Note that \( \Theta_{n-2}(d, i) \) has no free set variable and has \( n-2 \) alternating set quantifiers followed by two alternating numerical quantifiers, which in turn is followed by a bounded formula.

Let \( \Theta^*_n(d, i, X) \) be the formula \( (\Theta_{n-2}(d, i))^{Pri(X)} \).
Lemma 4.1.3. There is a \( \Sigma^0_n \)-formula \( A(d,i,X) \) with all free variables shown such that \( \text{ACA}^* \) proves that

\[
(\forall d,i,X)(\Theta^*_{n-2}(d,i,X) \iff A(d,i,X)).
\]

**Proof:** Let us consider the case when the rightmost numerical quantifier in \( \Theta^*_{n-2}(d,i) \) is "existential". Without loss of generality, we assume that the matrix of \( \Theta^*_{n-2}(d,i) \) is built up from atomic formulas or the negation of atomic formulas by using \( \land, \lor \) and bounded numerical quantification. Then \( \Theta^*_{n-2}(d,i,X) \) is equivalent to the formula \( B(d,i,j,X) \) obtained from \( \Theta^*_{n-2}(d,i) \) by the following operations:

- replacing each set quantifier \( \exists y \) (or \( \forall y \)) with \( \exists e_y \) (or \( \forall e_y \));
- if \( \neg(s \in y) \) appears as a sub-formula in the matrix of \( \Theta^*_{n-2}(d,i) \), then replace that appearance by \( \neg \phi(e_y,s,X) \);
- if \( s \in y \) has an appearance in the matrix of \( \Theta^*_{n-2}(d,i) \) and that appearance is not a part of \( \neg(s \in y) \), then replace that appearance by \( \psi(e_y,s,X) \).

In the above, we use the subscript "y" in \( e_y \) to indicate its dependence on \( y \).

Using the fact that bounded quantifiers can change position with unbounded numerical quantifiers, we may easily see that \( B(d,i,X) \) is equivalent to some \( \Sigma^0_n \)-formula \( A(d,i,X) \).

Lemma 4.1.4. \( \text{ACA}^* \) proves that

\[
(\forall d,i,X,\sigma)(A(d,i,X) \iff A(d,i,X^\sigma)).
\]

**Proof:** This follows from Lemma 4.1.2.
Lemma 4.1.5. ACA* proves that there is \( u \in BC_n \) such that

\[
(\forall X, i, j)(u(X)(i) =_n j \iff ((j = 1 \land A((i)_0, (i)_1, X)) \lor (j = 0 \land \neg A((i)_0, (i)_1, X))).
\]

Proof: Consider the formula \( C(i, j, X) \):

\[
(j = 1 \land A((i)_0, (i)_1, X)) \lor (j = 0 \land \neg A((i)_0, (i)_1, X)).
\]

This is obviously a \( \Sigma^0_{n+1} \) formula. Furthermore, we have

\[
\forall X, \exists ! j \ C(i, j, X).
\]

By lemma 3.2.3,

\[
\exists u \in BC_n((2^\omega)^\omega, 2^\omega) \forall X, i, j \ (u(X)(i) =_n j \iff C((i)_0, (i)_1, j, X)).
\]

Fix such an \( u \). By lemma 4.1.4, \( u \) satisfies

\[
(\forall \sigma, \check{X}, i, j)(u(\check{X})(i) =_n j \iff u(\check{X}^\sigma)(i) =_n j).
\]

Applying \( A^*_1(2^\omega) \), we get an \( \check{X} \) such that \( u(\check{X}) \in \check{X} \). Let us fix this \( \check{X} \).

Claim. For any \( \phi \in \Sigma^1_{n-2}CA + Subprim-CA \), ACA* proves \( (\phi)^{Pri(\check{X})} \). In other words, ACA* proves that \( Pri(\check{X}) \) is a weak model of \( \Sigma^1_{n-2}CA \) plus Subprim-CA.

Proof: Fix \( \check{X}_0 \) such that \( \forall j \ (j \in \check{X}_0 \iff u(\check{X})(j) =_n 1) \). We already know that \( Pri(\check{X}) \) is a weak model of Subprim-CA. \( Pri(\check{X}) \) is a weak model of PA plus \( \Pi^1_\infty \)-IND. Hence inside \( Pri(\check{X}) \), \( \Theta_{n-2}(d, i) \) is also universal for all the \( \Sigma^1_{n-2} \)-formulas having no free set variables.
Now fix an arbitrary formula $\psi(i) \in \Sigma^1_{n-2}$ with no free set variable. Fix $d$ such that

$$\forall i \ (\Theta_{n-2}(d, i) \leftrightarrow \psi(i))^{Pri(\bar{X})}.$$  

Hence

$$\forall i \ ((\psi(i))^{Pri(\bar{X})} \leftrightarrow (d, i) \in \bar{X}_i).$$

Hence $Pri(\bar{X})$ is a weak model of $\Sigma^1_{n-2}$-CA by the following claim.

Claim. $Pri(\bar{X})$ satisfies

$$\forall d \forall x \exists y \forall i \ (i \in y \leftrightarrow (d, i) \in x).$$

Proof of the claim: The argument is straightforward by an induction on $d$.

**Theorem 4.1.2.** $(n \geq 8)$, $\Delta^1_{n-3}$-CA does not prove $A^{n+1}_1(2^\omega)$ and $\Delta^1_{n-1}$-CA does not prove $A^{n+1}_1(2^\omega, ACA)$.

Proof: We argue by contradiction. Suppose $\Delta^1_{n-3}$-CA proves $A^{n+1}_1(2^\omega)$. Working in $\Delta^1_{n-3}$-CA, by Theorem 4.1.1, we can get an $\bar{X}$ such that $Pri(\bar{X})$ is a weak model of $\Sigma^1_{n-1}$-CA plus $Subprim$-CA. It is easy to define an $f$ such that

$$\{f_i : i \in \omega\} = \{g : g \in Pri(\bar{X})\}.$$  

Obviously, $f \models \Sigma^1_{n-1}$-CA plus $Subprim$-CA.

Then by lemma 4.1.5, $f$ is also a weak model of $\Sigma^1_{n-1}$-CA plus $\Pi^0_2$-CA. We may repeat the process we carried out in Chapter II inside $\{f_i : i \in \omega\}$. This can be done because all the arguments presented in that chapter are "finitistic"; see section 4.3.3 for more elaboration. Hence by theorem 2.3.2, there is a $g$ such that $\{g_i : i \in \omega\} \subseteq \ldots$
{f_i : i \in \omega} and g is a weak model of $\Delta^1_{n-3}$-CA. But this contradicts the Gödel’s second incompleteness theorem. Similarly, if $\Delta^1_{n-1}$-CA proves $A_{i+1}^n((2^\omega, \text{ACA})$, then again by applying Theorem 4.1.1, we can get an f such that f is a model of $\Sigma^1_{n-1}$-CA plus ACA. Then apply the corollary of theorem 2.4.2. We get a g such that g is a model of $\Delta^1_{n-1}$-CA. This again contradicts Gödel’s second incompleteness theorem.

**Corollary 1.** For $n \geq 8$, none of $A_i^n(\omega^\omega), A_i^{n+1}(R), A_i^{n+1}(I)$ is provable in $(\Delta^1_{n-4})^*-CA$.

**Proof:** This follows from above theorem and the corollaries of lemma 3.3.2.

**Corollary 2.** For $n \geq 8$, none of $A_i^n(\omega^\omega, \text{ACA}), A_i^{n+1}(R, \text{ACA}), A_i^{n+1}(I, \text{ACA})$ is provable in $(\Delta^1_{n-1})^*-CA$.

**Proof:** The argument is entirely the same as that of theorem 4.1.1. We give a sketch here. For a contradiction, we suppose that $(\Delta^1_{n-1})^*-CA$ proves $A_i^n(\omega^\omega, \text{ACA})$. We fix $\Phi_{n-1}(d, i)$ with all free variables shown so that it is universal for all $\Sigma^1_{n-1}$-formulas whose only free variable is “i”. Let $A(d, i, f)$ be the relativization of $\Phi_{n-1}(d, i)$ to “f”. Then clearly we may assume $A(d, i, f)$ is $\Sigma^0_n$. Let $u \in BC_n((\omega^\omega,\omega^\omega)$ be such that $\forall f, i, j (u(f)(j) = n \iff j = 1 \land A(i_0, i_1, f) \lor j = 0 \land \neg A(i_0, i_1, f))$.

Then u has the first-invariant property. By $A_i^n(\omega^\omega, \text{ACA})$, there is an f such that f is a witness of $A_i^n(\omega^\omega, \text{ACA})$. It is easy to check that $f \models^*_B \text{ACA} + \Sigma^1_{n-1}$-CA. Evidently, if we translate the argument in section 2.4 to the language $L^*_2$, we will get a g such that $g \models_B \Delta^1_{n-1}$-CA. This of course violates Gödel’s second incompleteness
4.2 Lower bounds for $B^n_3(X)$

**Theorem 4.2.1.** For $n \geq 2$, none of $B^n_3(\omega^\omega), B^n_3(2^\omega), B^n_3(R)$ and $B^n_3(I)$ is provable in $\Pi^1_n$-CA.

**Proof:** We really only need to prove the result for Baire space, since the result for Cantor, R and I would follow from the corollaries of lemma 3.3.2.

Let $\Theta_{n-1}(d, i, h_1, h_2)$ be universal for $\Sigma^1_{n-1}(i, h_1, h_2)$-formulas in $L^*_2$ with all free variables shown.

Let $A(e, i, j, f)$ be the following formula:

$$(j = 1 \land (\Theta_{n-1}((e)_0, i, h_1/f(e)_1, h_2/f(e)_2))^j) \lor (j = 0 \land \neg (\Theta_{n-1}((e)_0, i, h_1/f(e)_1, h_2/f(e)_2))^j).$$

Let $B(i, j, h)$ be the following formula:

$$(h((i, j)) = 1 \land \forall j' < j \ h((i, j')) \neq 1) \lor (\forall j' \ h((i, j')) \neq 1 \land j = 0).$$

Obviously, this is the defining formula for the function $\tilde{F}_B^C$ (cf. lemma 3.2.7).

Let $C(k, i, j, f)$ be the following formula:

$$\exists e \ (k = 2e \land A(e, i, j, f)) \lor \exists e \ (k = 2e + 1 \land B(i, j, h/f_e)).$$

It is easy to see that $C(k, i, j, f) \in \Sigma^0_{n+1}$ and

$$\forall k, i, f \exists j C(k, i, j, f).$$

Hence by Lemma 3.2.3, we have

$$\exists u \in BC_n((\omega^\omega)^\omega, (\omega^\omega)^\omega) \forall k, i, j, f \ (u(f)(k)(i) = j \leftrightarrow C(k, i, j, f)).$$
Fix such a $u$. It is straightforward to check that

$$\forall f, g \ (\text{range}(f) = \text{range}(g) \rightarrow \forall k_1 \exists k_2 \forall i \ (u(f)(k_1)(i) = u(g)(k_2)(i))).$$

Applying $B_1^n(\omega^\omega)$, there is an $F$ such that

$$u(F) \subseteq F.$$  

Let us fix an $F$ with this property.

**Claim.** $F$ is a weak model of $(\Sigma^1_{n-1})^- \text{CA}$. 

Proof of the claim: It is easy to see that it suffices to prove that $F$ is a weak model of $\text{RCA}^-$. Fix an arbitrary $\Sigma^0_1$-formula $\phi(i, j, F_l, F_m)$ such that $\forall i \exists j \phi$ holds. We want to show that there is some $k$ such that $\forall i \phi(i, F_k(i), F_l, F_m)$, which would imply $\text{RCA}^-$. Note that here we assume that $\phi$ has only two function parameters without any loss of generality. Choose $e$ such that

$$\forall i, j \phi(i, j, F_l, F_m) \leftrightarrow \Theta_{n-1}(e, \langle i, j \rangle, h_1/F_l, h_2/F_m).$$

Choose number $s$ such that

$$\forall i, j \ (F_s(\langle i, j \rangle) = u(F)((e, l, m))(\langle i, j \rangle)).$$

Now, let $k$ be such that $\forall i \ F_k = u(F)(2s + 1)(i)$. Then it is easy to see from the definition that $\forall i \ \phi(i, F_k(i), F_l, F_m)$. 

### 4.3 Concluding remarks and open questions

In this final section, we plan to achieve four goals: (1) to apply the theorems we have proved so far to give a long overdue proof of Theorems 1, 2 and Theorem 1'; (2) to
present some independence results regarding subsystems of second-order arithmetic; (3) to give some brief comments on the classification of the arguments we used in proving theorems presented in this paper; and (4) to list a few relevant problems still open at this time.

4.3.1 Proof of Theorem 1,2,1*

Proof of Theorem 1 : Apply theorems 3.5.1, 3.5.2, 4.1.1, 4.1.2, Corollary 1 of theorem 4.1.2, theorem 3.6.1 and Corollary 1 of lemma 3.3.2.

Proof of Theorem 2 : Apply Theorem 3.5.3, 3.5.4, 3.6.1 and Corollary 2 of lemma 3.3.2 and theorem 4.2.1.

Proof of Theorem 1' : Apply Corollaries 1, 2 of Theorem 3.5.1, Corollary 1 of Theorem 3.5.2, Corollary 1 of lemma 3.3.2, corollary 2 of theorem 4.1.2 and theorem 3.6.1.

If we combine Theorem 1 with Corollaries 4 and 5 of lemma 3.3.2, we can get the following more general results:

**Theorem 3.** For $n \geq 1 \leq i \leq 3$, $\Delta^{1}_{n+1} \text{-CA}$ proves that

$$\forall X \in CSM \ A^n_i(X),$$

and for $n \geq 8$, $1 \leq i \leq 3$, $\Delta^{1}_{n-4}$ does not prove that

$$\exists X \ (X \in PCSM \land A^n_i(X)),$$

where $CSM$ and $PCSM$ stands for complete separable metric space and complete separable metric space without isolated points (see section 3.1.2) respectively.

Similarly we may generalize Theorem 2 to the following:
**Theorem 4.** For $n \geq 1, 2 \leq i \leq 3$, $\Pi^1_{n+1}$-CA proves

$$\forall X \in CSM \ B^n_i(X);$$

and for $n \geq 8$, $\Pi^1_{n-2}$-CA does not prove

$$\exists X \ (X \in PCSM \land A^n_i(X)).$$

**4.3.2 Simple independence results**

Recall the definition of Primitive Recursive Axioms System $Prim^*$-CA defined in section 1.2.2. Results in this section are related to this theory.

There is a well-known result from classical recursion theory, which roughly says that "there is a recursive function which lists all the primitive recursive functions". Similar facts can also be shown with "function" being replaced by "functional" (see Peter Hinman [3]). As a matter of fact, we already used a modified version of this fact in the proof of Theorem 4.1.1.

We here state a formal version of this result for our reference, which of course is very similar to lemma 4.1.2.

**Lemma 4.3.1.** There is a $\Pi^0_1$-formula $\phi(e,i,j,x)$ and a $\Sigma^0_1$-formula $\psi(e,i,j,x)$ with all free variables shown such that ACA$^*$ proves

1.

$$\forall x, e, i \exists j \psi(e, i, j, x);$$

2.

$$(\forall e, i, j, x)(\phi(e, i, j, x) \leftrightarrow \psi(e, i, j, x));$$
3. if we let $f \in Pri(x)$ denote the formula

$$(\exists e)(\forall i, j)(f(i) = j \leftrightarrow \psi(e, i, j, x)),$$

then

$$(\forall \bar{x})(\theta)^{Pri(\bar{x})},$$

where $\theta \in Prim^*-CA$ and $(\theta)^{Pri(\bar{x})}$ is the relativization of $\theta$ to $\{f : f \in Pri(\bar{x})\};$

4.

$$(\forall \sigma, \bar{x}, e, e')(\exists e')(\forall i, j)(\phi(e, i, j, \bar{x}) \iff \phi(e', i, j, \bar{x}'));$$

in particular

$$(\forall \sigma, \bar{x}, f)(f \in Pri(\bar{x}) \iff f \in Pri(\bar{x}')).$$

Remark. We will call those $f$'s such that $f \in Pri(\bar{x})$, primitive recursive in $\bar{x}$. Hence the above lemma says that for any $\bar{X}$, the class of all $f$'s primitive recursive in $\bar{X}$ forms a model of $Prim^*-CA$, and it is invariant under finite permutation. Intuitively, this is obvious.

By theorem 2.3.2, the corollary to theorem 2.4.1 and Gödel's second incompleteness theorem, we clearly have the following theorem:

**Theorem 4.3.1.** For $n \geq 8$, if $Con(Z_2)$ then

1. $\Delta^1_{n-2}-CA \vdash (\exists M)(M \models \Delta^1_n-CA + \Sigma^0_0-CA);$  
2. $\Delta^1_n-CA \vdash (\exists M)(M \models \Delta^1_n-CA$ plus $\text{ACA})$.

Here the notation "$\models$" is as defined in Definition 4.
**remark:** The reason we put the condition $Con(Z_2)$ in Theorem 4.3.1 is that we may prove the theorem "finitistically". For more information, see section 4.3.3.

In a way, this theorem says that the logical strength contributed by quantifiers may not come from parameters. We now present a set of independence results which shows that the logical strength contributed by parameters may not come from quantifiers either. Hence we may say that quantifiers and parameters may contribute to the logic strength of a formal system in somewhat "independent" ways.

**Theorem 4.3.2.** For $n \geq 1$, the following hold:

1. $(pZ_2)^* + Prim^*-CA \not\vdash RCA^*$;

2. $(pZ_2)^* + RCA^* \not\vdash ACA^*$;

3. $(pZ_2)^* + ACA^* \not\vdash \forall x \exists y (y = x^{(\omega)})$;

4. $(n \geq 1) (pZ_2)^* + \Delta^1_n-CA \not\vdash \Pi^1_n-CA$;

5. $(n \geq 1) (pZ_2)^* + \Pi^1_n-CA \not\vdash \Delta^1_{n+1}-CA$.

**Proof:** To show (1),(2) and (3), we prove a lemma first.

**Lemma 4.3.2.** Let $T_1$, $T_2$ be two theories of $L_2^*$ and $A(f,g)$ a $\Sigma^1_1$-formula. If they satisfy

1. $\forall f \exists ! g A(f,g) \land \forall g (A(f,g) \rightarrow f \subset g \land g \models T_1 \land g \not\models T_2)$, where $f \subset g$ means $\forall i \exists j f_i = g_j$;

2. $\forall x, y, y', \sigma (A(x,y) \land A(x^\sigma, y') \rightarrow \text{range}(y) = \text{range}(y'))$, 
then $\exists y \left((y \models pZ_2 + T_1) \land (y \not\models T_2)\right)$.

**Proof** of the lemma: Fix $T_1, T_2$ and $A(f, g) \in \Sigma_1$ such that the condition is true. Let $\theta(f, g, i, j)$ denote the following formula

$$A(f, g) \land ((j = 0 \land g \models pZ_2^*) \lor (g \not\models pZ_2^* \land ((j = 1 \land g \models \psi(i)) \lor (j = 0 \land g \not\models \psi(i))))),$$

where $\psi$ is the least formula whose only free variable is "$i$" and $g \not\models \exists h \forall i (h(i) = 1 \leftrightarrow \phi(i))$. It is easy to see that $\theta$ is a $\Sigma_1$-formula if we replace $g \models pZ_2^*$ by $\exists h (h = Sat^*(g) \land \forall m \in A_{pZ_2^*} h(m) = 1)$ and $g \not\models pZ_2^*$ by $\exists h (h = Sat^*(g) \land \exists m \in A_{pZ_2^*} h(m) \neq 1)$. Similarly $g \models \psi(i)$ and $g \not\models \psi(i)$ can also be replaced by $\Sigma_1$-formulas (See section 1.2 for definition of satisfaction predicates.).

Consider the function $F : (\omega^\omega)^{\omega^\omega}$ defined by

$$F(\tilde{f})(i) = j \text{ iff } \exists g \theta(\tilde{f}, g, i, j).$$

$F$ is Borel and invariant under finite permutations. By proposition C (of [1]), there is $\tilde{f}$ such that $F(\tilde{f}) \in \tilde{f}$. Fix $g$ such that $A(f, g)$. Then $g \models T_1$ and $g \not\models T_2$ by assumption. We claim that $g \models pZ_2^*$. Otherwise, let $\psi(i)$ be the least formula such that $g \not\models \exists h \forall i (h(i) = 1 \leftrightarrow \phi(i))$. Then it is easy to see by definition of $F$

$$\forall i, j \left(F(\tilde{f})(i) = j \leftrightarrow j = 1 \land g \models \psi(i) \lor j = 0 \land g \not\models \psi(i)\right).$$

Hence if $\tilde{f}_k = F(\tilde{f})$ then

$$\forall i \left(F(\tilde{f})(i) = 1 \leftrightarrow g \models \psi(i)\right),$$

which contradicts the choice of $\psi$. 

Proof of (1) of Theorem 4.3.2: Let $A(\bar{f}, g)$ be

$$\forall e, i, j \ g((e, i)) = j \leftrightarrow \phi(e, i, j, \bar{f}),$$

where $\phi$ is as in lemma 4.3.1. If $A(\bar{f}, g)$ holds then $\bar{f} \subseteq g, g \models \text{Prim}^*-\text{CA}, g \nvdash \text{RCA}^*$, and $g$ is unique and invariant under finite permutations.

Proof of (2) of Theorem 4.3.2: Let $A(\bar{f}, g)$ be the formula asserting that $g$ is the canonical enumeration of all the functions recursive in $f$. Let $T_1$ be $\text{RCA}^*$ and $T_2$ be $\text{ACA}^*$.

Proof of (3) of theorem 4.3.2: Let $A(\bar{f}, g)$ be the formula asserting that $g$ is the canonical enumeration of all the functions primitive recursive in $\bar{f}^{(n)}$ for some $n$. Let $T_1$ be $\text{ACA}^*$ and $T_2$ be $\forall x \exists y \ (y = x^{(\omega)})$.

Proof of (4) and (5) of theorem 4.3.2: We do a forcing argument.

Fix a countable transitive model $M$ of a large portion of ZFC. Let $P$ be $(\omega^\omega \cap M)^{<\omega}$, partially ordered by extension. We consider the $P$-forcing notion over $M$. We note that if $\bar{f}$ is $P$-generic over $M$, then $\{\bar{f}_i : i \in \omega\}$ is an enumeration of $P$. Let $\hat{f}$ be a canonical name for generic objects. And let $\hat{\delta}_n^1(\hat{f})$ and $\hat{x}_n^1(\hat{f})$ be canonical names such that

$$\emptyset \models (\hat{\delta}_n^1(\hat{f}) = \mu \beta \ (A_\beta(\hat{f}) \models \Delta_n^1 - \text{CA}) \land \hat{x}_n^1(\hat{f}) = \mu \beta \ (A_\beta(\hat{f}) \models \Pi_n^1 - \text{CA})),$$

where $A_\beta(x)$ is the $\beta$-th stage of the Ramified Analytical Hierarchy relativized to $x$, and $\mu \beta$—— is the operator which picks up the least $\beta$ such that ——. Clearly, if $f$ and $g$ are finite permutations of each other then $A_\beta(f)$ and $A_\beta(g)$ are the same for all $\beta > 0$. For any $F$, if $\alpha_n(F)$ is the least $\beta$ such that $A_\beta(F) \models \Delta_n^1 - \text{CA}$ and
\( \beta_n(F) \) is the least \( \beta \) such that \( A_\beta(F) \models \Pi^1_n \text{CA} \). It is known that \( A_{\alpha_n} \not\models \Pi^1_n \text{CA} \) and \( A_{\beta_n} \not\models \Delta^1_{n+1} \text{CA} \) (see Chapter 7 of [2] for further reference).

**Claim:** \( \emptyset \models (A_{\hat{h}_n}(\hat{f}) \models pZ_2^* \land A_{\bar{h}_n}(\hat{f}) \models pZ_2^*) \).

**Proof:** Let \( \psi \) be an arbitrary sentence in \( L_2^* \) without any function parameter. We want to show that \( \emptyset \) decides \( \psi_{A_{\hat{h}_n}}(\hat{f}) \), the relativization of \( \psi \) to \( A_{\hat{h}_n}(\hat{f}) \). Since otherwise, there would be two conditions \( p \) and \( q \) in \( P \) such that \( p \models \psi_{A_{\hat{h}_n}}(\hat{f}) \) and \( q \models \neg \psi_{A_{\hat{h}_n}}(\hat{f}) \). Let \( \bar{f} \) be \( P \)-generic over \( M \). Then there are finite permutations \( \sigma_1 \) and \( \sigma_2 \) such that \( p \subset f^{\sigma_1} \) and \( q \subset f^{\sigma_2} \). Note that \( f^{\sigma_1} \) and \( f^{\sigma_2} \) are both \( P \)-generic. Hence \( \psi_{A_{\hat{h}_n}}(f^{\sigma_1}) \) and \( \neg \psi_{A_{\hat{h}_n}}(f^{\sigma_2}) \). But \( A_{\hat{h}_n}(f^{\sigma_1}) \) and \( A_{\bar{h}_n}(f^{\sigma_2}) \) are the same set as remarked earlier, which is a contradiction.

Now for any \( \psi(t) \) in \( L_2^* \) whose only free variable is "\( t \)"., let

\[ h_\psi = \{ (i, j) : ((\emptyset \models \psi_{A_{\hat{h}_n}}(\hat{f})(i)) \land j = 1) \lor ((\emptyset \models \neg \psi_{A_{\bar{h}_n}}(\hat{f})(i)) \land j = 0) \}. \]

Then \( h_\psi \) is in \( M \). Hence it must also be in \( \bar{f} \) for any \( P \)-generic \( \bar{f} \). In particular, it must be in \( A_{\hat{h}_n} (\bar{f}) \) for any \( P \)-generic \( \bar{f} \). It is now easy to see that \( A_{\bar{h}_n} (\bar{f}) \models pZ_2^* \). We may similarly prove that the same is true for \( A_{\bar{h}_n} (\bar{f}) \).

### 4.3.3 A word about the metatheory

We would like to make some final remarks about the kind of arguments we employed in proving theorems presented in this paper. This issue is related to the theory PRA; i.e., the Primitive Recursive Arithmetic, which is essentially our \( PA^- \) (see section 1.2), together with an induction axiom scheme with respect to quantifier free formulas. This theory is used in characterizing the so-called "finitistic" arguments. More
precisely, it is generally agreed that any argument formalizable in PRA is qualified for being called "finitistic". We want to point out that all the theorems presented in this dissertation, with the exception of theorem 4.3.2, are proved in a rather "finitistic" fashion. A typical argument in this paper goes like this: We fix a formal system $T \ (\Delta_n^1\text{-CA}, \Pi_n^1\text{-CA}, \cdots)$, and fix a formula $\phi$ in the language of $T$, we then set out to show that $\phi$ can be deduced from the axioms of $T$ through formal deduction, which involves nothing more than manipulation of finite strings of concrete symbols following a finite set of concrete rules. This kind of argument was shown to be formalizable in a system still weaker than PRA; i.e, the so-called Robinson Arithmetic. Another kind of argument we use involves conservation results. For instance, since we know that $\Sigma_n^1$-DC is conservative over $\Delta_n^1$-CA for $\Pi_2^1$ sentences, to show that $\Delta_n^1$-CA proves a $\Pi_2^1$ sentence, it suffices to show that $\Sigma_n^1$-DC proves that sentence. The key point here is that the fact "$\Sigma_n^1$-DC is conservative over $\Delta_n^1$-CA for $\Pi_2^1$ sentences" may be established in a rather finitistic way. We define a formula $M(x)$ with only $x$ free so that it defines the Ramified Analytical Hierarchy (see section 2.2 for precise meaning). Then for any $\Pi_2^1$ sentence $\theta$ and $\phi \in \Sigma_n^1$-DC, we may construct formal proof trees of $\theta^{M(x)} \iff \theta$ and of $\phi^{M(x)}$ from $\Delta_n^1$-CA. Now if $\theta$ is a $\Pi_2^1$ sentence and we have a formal proof tree of $\theta$ from $\Sigma_n^1$-DC, by relativizing every formula in that tree to $M(x)$, we clearly get a proof tree of $\theta^{M(x)}$ from $\{\phi^{M(x)} : \phi \in \Sigma_n^1$-$DC\}$. Now replacing every appearance of $\phi^{M(x)}$ in that tree by its corresponding proof tree from $\Delta_n^1$-CA, we then obtain a proof tree of $\theta^{M(x)}$ from $\Delta_n^1$-CA. Combining this with the proof tree for $\theta^{M(x)} \iff \theta$, we get a proof tree of $\theta$ from $\Delta_n^1$-CA. This process can clearly be
formalized in PRA. Then there is the mutual interpretability between some theories of $L_2$ and $L_2^*$ that we need to use. It is clear this too can be formalized in PRA.

As for theorem 4.3.2, if we analyze the argument, it is not difficult to come up with the following formalized version.

Let $T$ be a theory (in $L_2^*$ in our case) represented (in PA) by some primitive recursive predicate $A_T$. There is a binary primitive recursive predicate $P_T$ such that for any sentence $\phi$ in $L_2^*$,

$$PA \vdash \phi \iff \exists k P_T(k, [\phi]).$$

Let $Thm_T$ be the formula $\exists k P_T$. Since PA can clearly be interpreted in ZFC, we may assume similar predicate and formulas are available in ZFC as well.

We now restate theorem 4.3.2 as follows:

**Theorem 4.3.2. (formal version) The following statements hold:**

1. $\Pi_1^1$-CA plus Proposition C proves the following:

   - $\exists m \in A_{RCA^*} (\neg Thm_T(m)), \text{ where } T = (pZ_2)^* + Prim^* \cdot CA$;
   - $\exists m \in A_{ACA^*} (\neg Thm_T(m)), \text{ where } T = (pZ_2)^* + RCA^*$;
   - $\exists m \in A_{\forall x \exists y (y=x^*)} (\neg Thm_T(m)), \text{ where } T = (pZ_2)^* + ACA^*$;

2. ZFC proves the following:

   - $\exists m \in A_{\Pi_1^1 \cdot CA} (\neg Thm_T(m)), \text{ where } T = \Delta_1^1 \cdot CA + pZ_2$;
   - $\exists m \in A_{\Delta_1^{1,1} \cdot CA} (\neg Thm_T(m)), \text{ where } T = \Pi_1^1 \cdot CA + pZ_2$. 
4.3.4 Open questions

There is an interesting problem which is related to Theorem 2. As usual, let $X$ be among Baire, Cantor, R and I. Consider the following statement:

**Statement D**($X$). *For any Borel function $F : X^N \to X^N$ such that for any $\bar{x} \in X^N$ and any finite permutation $\sigma$, $F(\bar{x})$ and $F(\bar{x}^\sigma)$ are finite permutation of each other, then there is an $\bar{x} \in X^N$ such that range($F(\bar{x})$) $\subseteq$ range($\bar{x}$).*

**Question 1.** Is $D(X)$ true? Is it provable in ZFC or any interesting extension of ZFC?

Of course we may again restrict $D(X)$ to Borel functions of some finite rank $n$ (call it $D^n(X)$).

**Question 2.** Is the statement “$\forall n D^n(X)$” true? Is it provable in ZFC or any interesting extension of ZFC? If not for what $n$ it is provable?

From Corollary 4 and 5 of lemma 2.3.2, we know that the statements $A^n_i(X)$'s, where $X$ ranges over perfect polish spaces, are logically related. Namely, $A^{n+2}_i(X_1)$ implies $A^n_i(X_2)$ if $X_1 \in PCSM$ (for the definition of $A^n_i(X)$ see section 1.1). The ultimate question we want to ask is

**Question 3.** Is “$\forall x_1 \in PCSM \forall x_2 \in CSM (A^n_i(x_1) \to A^n_i(x_2))$” true? Is it provable in weak systems (such as ACA)?

We can obviously formalize the concept of compactness in weak systems such as ACA. Let us use “CPCSM” to denote the “Compact Perfect Complete Separable Metric spaces”; i.e, compact perfect polish spaces. Since we have strong evidence
supporting a negative answer to this question, we also want to ask the following weaker question

Question 4. Is \( \forall X_1, X_2 \in CPCS M (A^n_i(X_1) \rightarrow A^n_i(X_2)) \) true? Is it provable in some weak system such as ACA?

Another interesting weakening of Question 3 is:

Question 5. Is \( \forall X_1 \in PCSMX_2 \in CPCS M (A^n_i(X_1) \rightarrow A^n_i(X_2)) \) true? Is it provable in weak systems such as ACA?

Of course we may ask these questions about statement B as well (see section 1.1 for definition). More precisely, we want to ask Question 2 through 5 with \( A^n_i \) being replaced by \( B^n_i \) everywhere and \( i \) restricted to 2 or 3. (Recall that \( B^n_i(X) \), when \( n \geq 1 \), is provably false in ACA.)

Note that the upper bound for \( B^n_i(N^N) \) is \( \Pi^1_n \)-CA while the upper bound for \( A^n_i(N^N) \) is \( \Delta^1_n \)-CA. Thus it makes sense to ask the following:

Question 6. Does \( \Delta^1_n \)-CA proves \( B^n_i(N^N) \) as well?

While we may clearly ask similar general questions about the lower bounds of these statements, we have questions more specific and fundamental yet to be answered.

Question 7. Does \( \Delta^1_{n-1} \)-CA proves any of \( A^n_i(N^N), A^{n+1}_i(2^N), A^{n+1}_i(R) \) and \( A^{n+1}_i(I) \)?

If we look at the proof of the lower bound part of Theorem 1, it is easy to see a “yes” answer to above question would follow from a “yes” answer to the following model-theoretical question
Question 8. Is it true that every model of $\Delta_n^1$-CA plus $\Sigma_0^0$-CA contains a definable submodel of $\Delta_n^1$-CA (let us assume $n \geq 7$)?

However a "yes" answer to Question 7 is much more likely than to Question 8 since $A^n_i(X)$ can potentially yield a lot more than a model of $\Delta_n^1$-CA plus $\Sigma_0^0$-CA.
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