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Modeling slender viscoelastic jets and fibers with torsion

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The Ohio State University, 1992
MODELING SLENDER VISCOELASTIC JETS AND FIBERS WITH TORSION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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Slender free surface fluid jets are associated with many industrial processes such as printing, rock cutting, optical fiber drawing, and especially textile fiber spinning. The fiber spinning process is an important process in the textile industry and has received considerable attention in the literature. (The process of continuous drawing of polymeric fluid jets to form fibers is shown schematically in Figure 20.) A filament is extruded from a small hole in the spinneret plate into air at a temperature below the solidification temperature of the polymer. The solidified fiber is then taken up at a draw velocity which is considerably higher than the average velocity at the spinneret. This results in a drawing of the extrudate into a thin filament. It is of interest to find the profile of the fiber, the velocity distribution within the fiber, the take up force, the maximal draw ratio which is defined as the ratio of the take-up velocity to the velocity at the spinneret, and the optimal operating conditions.

Since the first slender jet models by Kase & Matsuo [17] and Matovich & Pearson [18] 1960’s, more than twenty slender jet models have been developed by a number of researchers. For example, Pearson & Shah [21], Weinberger & Goddard [27], Denn, Petrie Avenas [12], Fisher & Denn [13], Baid & Metzner [1], Waters et al. [29], White & Ide [28], Tanner [24], Gupta et al. [15], Phan-Thien [19], Beris & Liu [9],
Patel & Bogue [20], and Ting & Keller [25] etc. Bechtel, Forest & Lin [8] has a complete list of all these models. Importantly, all these models are torsionless and just leading order asymptotical model. Torsional effects and asymptotical validity of the leading order solutions have never been considered. It is interesting, as shown in this dissertation, that it is necessary to consider higher-order corrections in order to examine the coupling of torsional effects in the axisymmetric jet.

The slender jet modeling has been both productive and successful. It also raises some important questions. These models were proposed by a number of different researchers under certain special circumstances (different dominant physical effects and/or different constitutive equations to model different fluids). What is the instinct unity of all these different models? Schultz [23] claimed that all viscoelastic slender jet models were invalid, while most published researches have never mentioned the validity of the asymptotic solutions of the model. They simply assumed all the asymptotic solutions were valid without any checking. In fact, no equations have even been derived for the higher order corrections. Ignoring the validity of the asymptotic solutions has caused some misunderstanding of some widely referenced jet models such as Denn, Petrie & Avenas [12] model. Can the corrections be “safely” neglected? In other words, are these leading order models really asymptotically valid? Although torsional effects are observed in the laboratory and are of physical importance in free surface jet flows of viscoelastic, all existing slender jet theories model only torsionless slender jets. Can a torsionless slender jet actually exist? Can torsional effects always be neglected? If not, how does one model torsional effects in a slender jet
flow? Bechtel, Forest & Lin [8] summarizes the slender jet modeling and proposes a comprehensive higher-order perturbation theory which provides a uniform derivation of all existing (torsionless) slender jet models.

My doctoral research further generalizes this perturbation theory to include torsional effect, and applies this higher-order perturbation theory to several physical examples motivated by engineering applications. The essence of slender-jet modeling is to make use of symmetries and the slender geometry to reduce the spatial dimensions from three to one. In Bechtel, Forest & Lin [8], the dimension reduction is achieved by axisymmetry combined with averaging of the 3-D governing equations and their moments over the jet cross-section. In my doctoral research, however, the dimension reduction is achieved by combining axisymmetry with pointwise radial expansions in the plane orthogonal to the jet axis of symmetry. For the first time, perturbation equations for higher order corrections are derived for several Newtonian as well as viscoelastic models. For many practical examples, higher order corrections are computed to determine whether the leading order prediction is really a valid approximation to the 3-D problem. The effects of torsion on the overall behavior of a jet are investigated.

The general methodology can be summarized as follows:

1. Determine a **rheological model** for the fluid. Johnson-Segalman constitutive model is chosen in this dissertation. It contains as special cases the Oldroyd fluid B, Maxwell fluid, inviscid fluid, second order fluid, and hypoelasticity.

2. Determine the **relative importance of physical effects** in the fluid. Four ex-
amples are considered: Newtonian in this dissertation: Newtonian with weakly elastic effect, viscosity & elasticity dominant, and viscosity, elasticity, gravity, inertia, surface tension dominant.

3. Choose appropriate physical scales (e.g. the jet length $z_0$, the initial jet radius $r_0$) to nondimensionalize the problem.

4. Exploit the slenderness — Quantify the relative importance of physical effects and the allowable ranges for all model parameters in the terms of the slenderness ratio ($\varepsilon \equiv r_0/z_0$).

5. Derive 1-D "master" equations from which and perturbation expansions of all field variables follows a specific model.

6. Check the compatibility of initial-boundary conditions of the physics with those of the model. It should be emphasized that one cannot study a steady state problem in isolation. Boundary conditions for a steady state problem must be determined in context of the corresponding time-dependent problem.

7. Solve the 1-D perturbation equations for the first three (or more if necessary) orders to get approximate solutions to the 3-D problem; check the validity of the approximate solutions.

8. Analyze the solution as a function of dimensionless as well as dimensional parameters for engineering applications, e.g. to "optimize" operating conditions.
In this dissertation, several examples are used to illustrate the above methodology, the necessity and applications of this higher-order perturbation theory.

In this dissertation, we present in Chapter VI several time-dependent jet models which include torsion for the leading order quantities and the first two orders of corrections. The time-dependence is necessary for the determination of the well-posed initial and boundary conditions, and for the study of spatial and temporal stability of the steady state solutions. We then compute the leading order solutions and two orders of corrections for several leading order models, due to either boundary fluctuations and/or weak physical effects suppressed from the leading order equations, to check the asymptotic validity of the leading order solutions. It will be shown that the formally higher order corrections when computed may be so large that they invalidate the asymptotics, though some other models (such as the DPA model) are robust to both boundary fluctuations and weak physical effects.

It is found that, depending on the model, either (1) the higher order corrections are small, thereby confirming that the behavior predicted by the jet model is a valid asymptotic representation of the full 3-D fiber behavior, or (2) the formally higher order terms when computed are in fact large, thereby invalidating in that problem the predictions of the jet model. Importantly, we find that solutions from jet models which include elasticity can be valid leading order descriptions of fiber behavior, disproving claims of Schultz (1987). In particular we show that the DPA model is robust to both boundary perturbations and the weak physical effects of inertia, gravity, and surface tension (i.e., computed higher order corrections due to these terms remain small), so
that it is a valid asymptotic model for fiber spinning behavior.

In this dissertation, we also investigate the effect of torsion on the validity of the asymptotics and the stability of solutions, and whether torsion can be used to stabilize a jet or speed up fiber spinning process. Though effects of torsion in free jets have not been addressed in the existing literature, we find examples which show that they can not be neglected in general even for Newtonian jets.

For one jet model which includes all viscoelastic effects except retardation time, it is found that: (1) If torsion starts from zero, then it stays zero. (2) Torsional effects for a slender jet only enter as corrections to the leading order equations. (3) Numerical experiments show that torsion can have significant impact on the asymptotic validity of the torsionless components. (4) Mathematically, torsion can advance the change-of-type of the linear or quasilinear PDE systems. In other words, the full system which includes torsion can change type before the subsystem which excludes the torsion. The change-of-type (from hyperbolic to elliptic) of a PDE system is important to the stability of its solutions.
CHAPTER II

The 3-D Formulation for Non-Newtonian Flows

In this chapter, we first present the governing equations for unbounded, 3-D, viscoelastic flows. Then we derive an invariant subspace of the full equations which, through separation of variables, reduces the 3-D equations to 1-D. We remark that our perturbation scheme is built on this separation of variables structure.

The 3-D dynamical field equations governing the viscoelastic fluid consist of:

\textit{incompressibility}

\[ \text{div } \mathbf{v} = 0; \quad (2.1) \]

\textit{balance of linear momentum}

\[ \rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \text{div } \mathbf{T} - \nabla p + \rho \mathbf{g}; \quad (2.2) \]

\textit{conservation of angular momentum}

\[ \mathbf{T} = \mathbf{T}' . \quad (2.3) \]

An equation for \( \mathbf{T} \) required to close this system. For Newtonian fluids the “extra stress” \( \mathbf{T} \) is posited as proportional to the symmetric part of velocity gradient; however, and the Johnson-Segalman viscoelastic constitutive model

\[ \mathbf{T} + \lambda_1 \frac{D_{\mathbf{T}}}{D_t} \mathbf{T} = 2\eta \left( \mathbf{D} + \lambda_2 \frac{D_{\mathbf{D}}}{D_t} \mathbf{D} \right), \quad (2.4) \]
where the rate operator is defined by

$$\frac{D}{Dt}(\cdot) = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right)(\cdot) + (\cdot)\mathbf{W} - \mathbf{W}(\cdot) - a [(\cdot)\mathbf{D} + \mathbf{D}(\cdot)], \quad (2.5)$$

with

$$\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T), \quad \mathbf{W} = \frac{1}{2} (\mathbf{L} - \mathbf{L}^T), \quad \mathbf{L} = \nabla \mathbf{v}. \quad (2.6)$$

In equations (2.1-2.6) \( \mathbf{v} \) is the fluid velocity, \( \rho \) is the fluid density, \( \mathbf{T} - p\mathbf{I} \) is the Cauchy stress (with \( \mathbf{T} \) the extra stress and \( p \) the constraint pressure), \( g \) is the acceleration of gravity, and \( \mathbf{D} \) and \( \mathbf{W} \) are the symmetric and skew symmetric parts of the velocity gradient. The coefficients \( \lambda_1, \eta, \lambda_2 \) and \( a \) are material constants in the Johnson-Segalman constitutive model, namely the relaxation time, zero strain rate viscosity, retardation time, and slip parameter, respectively.

We note that the Johnson-Segalman model is valid for finite deformation flows, and includes as special cases: Maxwell fluids \((\lambda_2 = 0)\), Newtonian flows \((\lambda_1 = \lambda_2 = 0)\), inviscid fluids \((\lambda_1 = \lambda_2 = \eta = 0)\), second order fluids \((\lambda_1 = 0)\), and hypoelastic solids (only the \( \lambda_1 \) and \( \lambda_2 \) terms nonzero, i.e., \( \frac{1}{\lambda_1} = \frac{\eta}{\lambda_2} = 0, \frac{\eta \lambda_2}{\lambda_1} \neq 0 \)).

In this chapter, we present an invariant subspace of flows which: (1) generalizes von Kármán’s [26] invariant subspace from Navier-Stokes flows to Johnson-Segalman flows; (2) shows torsion (or swirl) is coupled strongly to these flows, although it is a “homogeneous” effect in that one may assume torsion is zero in the 3-D field equations (2.1-2.6) without violating invariance of the subspace; and (3) establishes the mathematical structure that underlies the perturbation theory. These results are an axisymmetric reduction of a result in [7].
The reduction from functions of \((r, \theta, z, t)\) to functions of \((r, z, t)\) by assuming all field variables do not depend on \(\theta\), i.e., \(\frac{\partial}{\partial \theta} = 0\).

In cylindrical polar coordinates, the velocity field \(v\) and the extra stress tensor \(T\) are represented by

\[
v = v_r(r,z,t)e_r + v_\theta(r,z,t)e_\theta + v_z(r,z,t)e_z,
\]

and

\[
T = \begin{bmatrix}
T_{rr} & T_{r\theta} & T_{rz} \\
T_{r\theta} & T_{\theta\theta} & T_{\theta z} \\
T_{rz} & T_{\theta z} & T_{zz}
\end{bmatrix}
\]

respectively. If one assumes the velocity vector in these coordinates takes the special form

\[
v = r\zeta(z,t)e_r + r\psi(z,t)e_\theta + \nu(z,t)e_z,
\]

and pressure \(p\) takes the form

\[
p = p_1(z,t) + r^2 p_2(z,t),
\]

then the field equations of the axisymmetric 3-D Johnson-Segalman viscoelastic flow (2.1-2.6) reduce to ((·),\(z\) \(\equiv \frac{\partial}{\partial z}(·)\), and ((·),\(t\) \(\equiv \frac{\partial}{\partial t}(·)\)):

\[
2\zeta + \nu,_{z} = 0,
\]

\[
\rho r \left(\zeta,_{t} + \zeta^2 - \psi^2 + \nu,_{z}\right) = T_{rr},_{r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + T_{rz},_{z} - 2rp_2 + \rho g_r,
\]

\[
\rho r \left(\psi,_{t} + 2\zeta\psi + \nu,_{z}\right) = T_{r\theta},_{r} + \frac{2}{r}T_{r\theta} + T_{\theta z},_{z} + \rho g_\theta,
\]

\[
\rho \left(\nu,_{t} + \nu,_{z}\right) = T_{rz},_{r} + \frac{1}{r}T_{rz} + T_{zz},_{z} - p_1,_{z} - r^2 p_2,_{z} + \rho g_z,
\]
\[(\varepsilon \phi + \varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]

\[(\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda) \cdot \frac{d}{dL} = \left[\frac{\varepsilon \phi \eta \nu \gamma + \varepsilon \phi \eta \lambda + \varepsilon \phi \nu \gamma + \varepsilon \phi \lambda}{d + 1}\right] \cdot \frac{d}{dL}
\]
\[ \eta \left( 2\nu_z + \lambda_2 \left[ (1 - a)r^2(\zeta_z^2 + \psi_z^2) + 2\nu_{zt} + 2\nu\nu_{zz} - 4\alpha\nu_z^2 \right] \right). \]  

(2.20)

The components \( \phi, T_{\theta\theta}, T_{\theta z} \) can be identified with axisymmetric torsion. Note that equations (2.11-2.20) admit the solution

\[ \psi = T_{\theta\theta} = T_{\theta z} \equiv 0, \]  

(2.21)

in which case the equations are torsionless. Therefore torsion is a homogeneous effect in equations (2.11-2.19).

For Newtonian flows,

\[ \lambda_1 = \lambda_2 = 0, \quad \eta \neq 0, \]  

(2.22)

equation (2.4) reduces to the Navier-Stokes relation

\[ \mathbf{T} = 2\eta \mathbf{D}, \]  

(2.23)

and (2.15-2.20) simplify into algebraic relations,

\[ T_{rr} = T_{\theta\theta} = 2\eta \zeta, \quad T_{\theta\theta} = 0, \]  

(2.24)

\[ T_{rz} = r\eta \zeta_z, \quad T_{\theta z} = r\eta \psi_z, \quad T_{zz} = 2\eta \nu_z. \]  

(2.25)

Substituting these relations into equations (2.12-2.14) yields

\[ \rho(\zeta, \zeta^2 - \psi^2 + \nu \zeta_z) = \eta \zeta_{zz} - 2p_2, \]  

(2.26)

\[ \rho(\psi, \zeta + 2\psi + \nu \psi_z) = \eta \psi_{zz}, \]  

(2.27)

\[ \rho(\nu, \nu + \nu \nu_z) = \eta \nu_{zz} - p_{1,z} - r^2 p_{2,z} + \rho g. \]  

(2.28)

Since all functions in equation (2.28) depend \( z \) and \( t \), the last equation implies that \( p_{2,z} = 0 \); that is, \( p_2(z, t) \equiv p_2(t) \) depends on \( t \) only. \( p_2(t) \) is assumably prescribed.
Since
\[ \zeta = -\frac{1}{2} \nu_z \] (2.29)
from equation (2.11) follows a closed system of PDEs, which together with (2.24) and (2.25) govern this invariant subspace:
\begin{align*}
\rho \left( \zeta_t + \zeta^2 - \psi^2 + \nu \zeta_z \right) &= \eta \zeta_{zz} - 2p_2(t) + \rho g_r, \\
\rho \left( \psi_t + 2\zeta \psi + \nu \psi_z \right) &= \eta \psi_{zz} + \rho g_z, \\
\rho \left( \nu_t + \nu \nu_z \right) &= \eta \nu_{zz} - p_1, + \rho g_z.
\end{align*}
(2.30) (2.31) (2.32)
If one further ignores the gravity force \( \rho g \), the steady reduction of equations (2.29)-(2.32) yields equations (22)-(23) of von Kármán [26]. This steady reduction with appropriate boundary conditions at \( z = 0 \) and \( z = \infty \) yields von Kármán's 3-D problem.

In more general viscoelastic flows \( \lambda_1 \neq 0 \), and/or \( \lambda_2 \neq 0 \). Remarkably, one may further assume that \( T \) is a quadratic polynomial in \( r \), with scalar coefficients \( a_k(z,t) \), \( k = 1, \ldots, 8 \):
\begin{align*}
T_{rr} &= a_1 + a_2r^2, \\
T_{\theta\theta} &= a_5 + a_6r^2, \\
T_{zz} &= a_8,
\end{align*}
(2.33) (2.34)
T_{r\theta} = a_3r^2, \\
T_{rz} = a_4r,
(2.33) together with the same velocity and pressure assumptions (2.9) and (2.10), and still find an invariant closed system of PDEs for these scalar functions of \( z \) and \( t \). In particular, we emphasize that the torsional components \( T_{r\theta}, T_{\theta z}, \) and \( v_\theta \) are strongly coupled in this invariant subspace of flows. One therefore can generalize previous approximation theory for free surface jets of viscoelastic fluids, which is effectively
based on this invariant structure, to include torsional (or swirling) effects.

Remarks:

1. Linear terms in $r$ for $T_{rr}$, $T_{r\theta}$, $T_{\theta\theta}$ are zero by axisymmetry.

2. There is no other higher-order finite truncation in $r$ which yields closure.
From now on in this dissertation, it will be demanded that the stress, pressure, and velocity variables satisfy boundary conditions on a lateral free surface defining the jet, in addition to the field equations (2.1)–(2.6).

The free jet is assumed axisymmetric along the direction of gravity. A cylindrical polar coordinate system is used, with the $z$ axis coincident with the jet axis of symmetry and the jet flowing in the positive $z$ direction. Then, the gravity force becomes

$$\rho g = \rho g e_z,$$

where $g$ is the gravitational acceleration.

The free surface of the axisymmetric jet is given by

$$F(r, z, t) \equiv \phi(z, t) - r = 0,$$

where $\phi$ is a function of $(z, t)$ to be determined. The free surface obeys the kinematic boundary condition

$$\left[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) F \right]_{\theta} = 0,$$

where "$|_{\theta}$" denotes that all quantities are evaluated on the free surface (3.2).
The kinetic boundary condition is

$$\left[(T_a - T + p) n\right]_\theta = \sigma \kappa n,$$

where $n$ is the unit outward normal to the free surface (3.2), $T_a$ the stress in the ambient atmosphere, which is further assumed to be passive,

$$T_a = -p_a I,$$

where $p_a$ is a specified constant ambient pressure, and

$$\kappa = \frac{1}{\phi (1 + \phi_r^2)^{1/2}} - \frac{\phi_{zz}}{(1 + \phi_r^2)^{3/2}}$$

is the mean curvature of the free surface (3.2).

*With the addition of the boundary conditions (3.2)-(3.6), the closure results of Chapter 2 break down.* This compels one to allow for a more general radial dependence than $O(r^0, r^1, r^2)$ (physically, the flow must adjust to the presence of the free surface), which then leads to coupling between all orders — the classic nonlinear closure problem. A recovery of closure, however, may be achieved by exploiting a *slender geometry.* This issue has been analyzed in detail in [7] for the leading order integrated theory, and then to all orders in [8] again for an integrated theory. This dissertation presents an alternative, pointwise theory.

Now we may proceed directly to the formulation of the pointwise or local $(\varepsilon, r)$ perturbation formalism to all orders. This formalism builds approximate solutions to the full 3-D problem in principle to any desired accuracy. It should be emphasized that slender perturbation theory has been applied in many circumstances to get a closed
set of leading order equations. The perturbation formulation consists of the double perturbation expansions in $\varepsilon$ and $r$ of the field variables and the master equations (reflect the relative importance of all the physical effects) from which particular special models follow. The extension to higher order in the perturbation is essential in order to use perturbation methods to analyze torsional effects in a viscoelastic free jet, as the torsional coupling is a formally higher order effect, which will be shown later in this dissertation.

The following physical scales are adopted in this dissertation:

$$r_0 = \text{transverse length scale}, \quad z_0 = \text{axial length scale},$$

$$t_0 = \text{characteristic time}, \quad f_0 = \text{characteristic force},$$

to nondimensionalize the 3-D problem. The dimensionless small parameter, the slenderness ratio, is defined by

$$\varepsilon = \frac{r_0}{z_0} \ll 1. \quad (3.7)$$

Note that $\varepsilon$, which will serve as the small parameter in the perturbation theory, is a fixed, finite positive number.

The dimensionless coordinates $\tilde{r}, \tilde{z}, \tilde{t}$ are defined by

$$r = r_0 \tilde{r}, \quad z = z_0 \tilde{z}, \quad t = t_0 \tilde{t}. \quad (3.8)$$

The scaling process can be expressed as follows

$$r \mapsto r_0 \tilde{r}, \quad z \mapsto z_0 \tilde{z}, \quad t \mapsto t_0 \tilde{t}, \quad \kappa \mapsto \frac{1}{r_0} \tilde{\kappa}, \quad (3.9)$$

$$v \mapsto \frac{z_0}{t_0} \tilde{v}, \quad T \mapsto \frac{f_0}{r_0^2} \tilde{T}, \quad p \mapsto \frac{f_0}{r_0^2} \tilde{p}. \quad (3.10)$$
After scaling, the dimensionless equations given by equations (2.1)–(3.6) involve the following dimensionless parameters

\[
\begin{align*}
\varepsilon &= \frac{\rho_a}{\rho_0}, \\
B &= \frac{\rho_f \rho_0^2}{\rho_0^2 z_0^2}, \\
F &= \frac{\rho_0 g}{\sigma}, \\
W &= \frac{\rho g \alpha a^2}{\sigma^2}, \\
\Lambda_1 &= \frac{\Lambda_1}{t_0}, \\
\Lambda_2 &= \frac{\Lambda_2}{t_0}, \\
P_a &= \frac{\rho g a}{\sigma}, \\
Z &= \frac{\rho g a^2}{\sigma^2 t_0},
\end{align*}
\]

(3.11)

where \( F, W, \) and \( \Lambda_1 \) are called the Froude, Weber, and Weissenberg numbers, respectively; \( \frac{1}{B^2} \) is the Reynolds number; \( Z, \Lambda_2, \) and \( P_a \) are the nondimensionalized zero strain rate viscosity, relaxation time, retardation time, and ambient pressure, respectively.

The scaled equations from incompressibility (2.1) and conservation of linear momentum (2.2) are (now with all tildes dropped)

\[
\begin{align*}
u_{r,r} + \frac{1}{r} v_r + \varepsilon v_{z,z} &= 0; \\
v_{r,t} + \frac{v_r}{\varepsilon} v_{r,r} - \frac{1}{\varepsilon r} v_\theta^2 + v_z v_{r,z} &= B \left[ \frac{1}{\varepsilon} T_{rr,r} + \frac{1}{\varepsilon r} (T_{rr} - T_{\theta\theta}) + T_{rz,z} - \frac{1}{\varepsilon} p_r \right], \\
v_{\theta,t} + \frac{v_r v_{r,r}}{\varepsilon} + \frac{1}{\varepsilon r} v_r v_\theta + v_z v_{\theta,z} &= B \left( -\varepsilon T_{r\theta,r} + \frac{2}{\varepsilon r} T_{r\theta} + T_{\theta z,z} \right), \\
v_{z,t} + \frac{v_r v_{z,r}}{\varepsilon} + v_z v_{z,z} &= B \left[ \frac{1}{\varepsilon} T_{rz,r} + \frac{1}{\varepsilon r} T_{rz} + T_{zz,z} - p_z \right] - \frac{1}{F}.
\end{align*}
\]

(3.12) \( (3.13) \) \( (3.14) \) \( (3.15) \)

The six dimensionless scalar equations from the Johnson-Segalman constitutive relation (2.4) are

\[
\begin{align*}
T_{rr} + \Lambda_1 \left[ T_{rr,t} + \frac{v_r}{\varepsilon} T_{rr,r} + v_z T_{rr,z} - \frac{2a}{\varepsilon} v_{r,r} T_{rr} + \frac{(1-a)}{\varepsilon} \left( v_{\theta,r} - \frac{v_\theta}{r} \right) T_{r\theta} -
(1+a) v_{r,z} T_{rz} + \frac{(1-a)}{\varepsilon} v_{z,r} T_{rz} \right] &= 2 \frac{Z}{\varepsilon} v_{r,r} + 2Z \Lambda_2 \left[ \frac{v_{r,t}}{\varepsilon} + \frac{v_r v_{r,r}}{\varepsilon^2} + \frac{v_z}{\varepsilon} v_{r,r} - 2a \left( \frac{v_{r,r}}{\varepsilon} \right)^2 \\
+ \frac{1-a}{2\varepsilon^2} \left( v_{\theta,r} - \frac{v_\theta}{r} \right)^2 - \frac{1}{2} \left( v_{r,r} \frac{v_z}{\varepsilon} \right)^2 - \frac{a}{2} \left( v_{r,z} + \frac{v_z, r}{\varepsilon} \right)^2 \right],
\end{align*}
\]

(3.16)
\[ T_{\theta\varphi} + \Lambda_1 \left[ T_{\theta\varphi,t} + \frac{v_r}{\varepsilon} T_{\theta\varphi,r} + v_z T_{\theta\varphi,z} - \frac{1}{2\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) (1 + a) T_{\varphi\varphi} - (1 - a) T_{\theta\theta} \right] \\
\left. \right. - \frac{a}{\varepsilon r} (v_{r,r} + \frac{v_r}{r}) - \frac{1 + a}{2} \left( v_{\theta,z} T_{r\varphi} + v_{\varphi,z} T_{\theta \varphi} \right) + \frac{1 - a}{2\varepsilon} v_{x,r} T_{\theta \varphi} \right] \\
= \frac{Z}{\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) + Z \Lambda_2 \left[ \frac{1}{\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r})_t + \frac{v_r}{\varepsilon^2} (v_{\theta,rr} - \frac{v_{\theta}}{r})_r \right] \\
\left. \right. + \frac{v_z}{\varepsilon} (v_{\theta,z} - \frac{v_{\theta}}{r}) - \frac{1 + 2a}{\varepsilon^2} v_{r,r} (v_{\theta,r} - \frac{v_{\theta}}{r}) + \\
\frac{1 - 2a}{\varepsilon^2} v_r (v_{\theta,r} - \frac{v_{\theta}}{r}) - v_{r,z} v_{\theta,z} - a v_{\theta,z} (v_{r,z} + \frac{1}{\varepsilon} v_{z,r}) \right], \\
(3.17) \\

T_{\varphi z} + \Lambda_1 \left[ T_{\varphi z,t} + \frac{v_r}{\varepsilon} T_{\varphi z,r} + v_z T_{\varphi z,z} + \frac{1 - a}{2} \left( v_{\varphi,z} T_{r\varphi} + \frac{v_{z,r}}{\varepsilon} T_{\varphi z} \right) - \right. \\
\frac{1 + a}{2} \left( \frac{v_{z,r}}{\varepsilon} T_{rr} + v_{\varphi,z} T_{z\varphi} \right) - a (\frac{v_{r,r}}{\varepsilon} + v_{z,z}) T_{\varphi z} + \frac{1 - a}{2\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) T_{\theta z} \\
+ \frac{1 - a}{2} \varepsilon^2 v_{\theta,z} T_{\theta \varphi} \right] = Z (v_{r,z} + \frac{v_{z,r}}{\varepsilon}) + Z \Lambda_2 \left[ v_{r,z} + \frac{v_{z,r}}{\varepsilon} + v_r (v_{r,z} + \frac{v_{z,r}}{\varepsilon}) \right] \\
+ v_z (v_{r,z} + \frac{v_{z,r}}{\varepsilon}) + \left( \frac{v_{r,r}}{\varepsilon} - v_{z,z} \right) (v_{r,z} - \frac{v_{z,r}}{\varepsilon}) \right. \\
\left. \right. - 2a (\frac{v_{r,r}}{\varepsilon} + v_{z,z}) (v_{r,z} + \frac{v_{z,r}}{\varepsilon}) + \frac{1 - a}{\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) v_{\theta,z} \right], \\
(3.18) \\

T_{\theta z} + \Lambda_1 \left[ T_{\theta z,t} + \frac{v_r}{\varepsilon} T_{\theta z,r} + v_z T_{\theta z,z} - \frac{1}{\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) T_{\varphi \varphi} - 2a \frac{v_r}{\varepsilon^2} T_{\theta \varphi} \right] = \\
\left. \right. \left( \frac{v_{r,t}}{\varepsilon} + \frac{v_r}{\varepsilon^2} \left( \frac{v_r}{r} \right)_r + \frac{v_z}{\varepsilon} v_{r,z} \right) + \left( \frac{v_{\theta,r}}{\varepsilon} - \frac{v_{\theta}}{r} \right)^2 - \frac{1 + a}{2\varepsilon} v_{\theta,zz} - 2a (\frac{v_{\theta}}{\varepsilon^2} r^2) \right], \\
(3.19) \\

T_{\varphi z} + \Lambda_1 \left[ T_{\varphi z,t} + \frac{v_r}{\varepsilon} T_{\varphi z,r} + v_z T_{\varphi z,z} - \frac{1}{2\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) T_{rr} + \right. \\
\left. \right. \frac{1 + a}{2\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) T_{rr} + \frac{1 - a}{\varepsilon} (v_{\theta,z} + \frac{v_{z,r}}{\varepsilon}) T_{\theta z} \right].
\[
\begin{align*}
\frac{1}{2} (v_{r,z} - v_{z,r}) T_{r\theta} + \frac{1}{2} v_{\theta,z} T_{\theta\theta} - a(v_{z,z} + \frac{v_r}{r}) T_{\theta z} \\
- \frac{1}{2} v_{\theta,z} T_{zz} = Z v_{\theta,z} + Z \Lambda_2 \left[ v_{\theta,z t} + \frac{v_r}{\varepsilon} v_{\theta,z r} + v_z v_{\theta,z z} \right] \\
\frac{1}{\varepsilon} (v_{\theta,r} - \frac{v_{\theta}}{r}) (v_{r,z} - \frac{1}{\varepsilon} v_{z,r}) + v_{\theta,z} [(1 - 2a) \frac{v_r}{\varepsilon} - (1 + 2a) v_{z,z}] \right], \quad (3.20)
\end{align*}
\]

\[
\begin{align*}
T_{z z} + \Lambda_1 \left[ T_{z z, t} + \frac{v_r}{\varepsilon} T_{z z, r} + v_z T_{z z, z} + [(1 - a) v_{r,z} - (1 + a) v_{z,r}] T_{z z} + \\
(1 - a) v_{\theta,z} T_{\theta z} - 2 a v_{z,z} T_{z z} \right] = 2 Z v_{z,z} + Z \Lambda_2 \left[ 2(v_{z,z t} + \frac{v_r}{\varepsilon} v_{z,z r} + v_z v_{z,z z}) \right] \\
+ a^2 T_{r z} - (\frac{v_r}{\varepsilon})^2 + (1 - a) v_{\theta,z}^2 - a(v_{r,z} + \frac{v_z}{\varepsilon})^2 - 4 a v_{z,z}^2 \right]. \quad (3.21)
\end{align*}
\]

At the free surface, the scaled kinematic boundary condition (3.3) reads

\[
\phi, t + v_z \phi, z - \frac{1}{\varepsilon} v_r = 0, \quad (3.22)
\]

and the kinetic boundary condition (3.4) yields the following three dimensionless scalar equations:

\[
\begin{align*}
T_{rr} - \varepsilon \phi_z T_{rz} = p - \frac{1}{BW} (P_a + \kappa), \quad (3.23) \\
T_{r\theta} - \varepsilon \phi_z T_{\theta z} = 0, \quad (3.24) \\
T_{rz} - \phi_z T_{z z} = \varepsilon \phi_z \left[ \frac{1}{BW} (P_a + \kappa) - p \right] \quad (3.25)
\end{align*}
\]

where

\[
\kappa = \frac{1}{\phi (1 + \varepsilon^2 \phi_z^2)^{1/2}} - \frac{\varepsilon^2 \phi_{zz}}{(1 + \varepsilon^2 \phi_z^2)^{3/2}}. \quad (3.26)
\]

This initial-boundary value problem involves eleven variables, \( \phi, v_r, v_\theta, v_z, T_{rr}, T_{r\theta}, T_{rz}, T_{\theta\theta}, T_{z z}, \text{ and } p \). These variables can be divided into the torsionless components \( \phi, v_r, v_z, T_{rr}, T_{rz}, T_{\theta\theta}, T_{z z}, p \), and the torsional components \( v_\theta, T_{r\theta}, T_{\theta z} \).
We note that this complete system generalizes the dimensionless torsionless problem by adding four new equations, (3.14), (3.17), (3.20), and (3.24), three field equations and one boundary condition, and coupling the torsion components $v_\theta$, $T_{r\theta}$, and $T_{\theta z}$ into five of the seven "old" equations, namely (3.13), (3.16), (3.18), (3.19), and (3.21).

Torsionless theories a priori set torsional components to zero. This can be justified with equations (3.12)-(3.25) above, as $v_\theta$, $T_{r\theta}$, $T_{\theta z}$ identically zero satisfies all field equations and boundary conditions. Therefore, torsionless flows are invariant in the free surface jet. (It is already noted in Chapter 2 that the torsional components are a homogeneous effect in the 3-D field equations (3.12)-(3.25). It is now clear that the boundary conditions (3.22)-(3.25) preserve this property.)

Now we can pose the perturbation ansatz. As discussed above, the exact truncation of Chapter 2 becomes overconstrained by the free surface boundary conditions. To recover closure of infinite sets of variables one may posit double series expansions in $r$ and $\varepsilon$ for all the field variables, which are then inserted into the scalar equations (3.12)-(3.25). The perturbation expansions are based the underlying geometrical assumption that the cross-section of the jet changes slowly, with the kinematical consequence that the velocity, stress and pressure change slowly across any cross-section. In other words, the radial dependence enters the perturbation expansions only as higher-order corrections.

PERTURBATION ANSATZ:
All field variables are prescribed as follows:

\[ v_r(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+1}r^{2n+1}v_r^{n,m}(z,t), \quad (3.27) \]

\[ v_\theta(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+1}r^{2n+1}v_\theta^{n,m}(z,t), \quad (3.28) \]

\[ v_z(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m}r^{2n}v_z^{n,m}(z,t), \quad (3.29) \]

\[ T_{rr}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m}r^{2n}T_{rr}^{n,m}(z,t), \quad (3.30) \]

\[ T_{r\theta}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+2}r^{2n+2}T_{r\theta}^{n,m}(z,t), \quad (3.31) \]

\[ T_{r\phi}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+1}r^{2n+1}T_{r\phi}^{n,m}(z,t), \quad (3.32) \]

\[ T_{\theta\phi}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m}r^{2n}T_{\theta\phi}^{n,m}(z,t), \quad (3.33) \]

\[ T_{\phi\phi}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+1}r^{2n+1}T_{\phi\phi}^{n,m}(z,t), \quad (3.34) \]

\[ T_{z\theta}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m+1}r^{2n+1}T_{z\theta}^{n,m}(z,t), \quad (3.35) \]

\[ T_{zz}(r,z,t) = \sum_{n,m \geq 0} \varepsilon^{2n+m}r^{2n}T_{zz}^{n,m}(z,t), \quad (3.36) \]

\[ p(r,z,t) = p_a + \sum_{n,m \geq 0} \varepsilon^{2n+m}p^{n,m}(z,t), \quad (3.37) \]

\[ \phi(z,t) = \sum_{m \geq 0} \varepsilon^m \phi^{(m)}(z,t), \]

where the scalar coefficients in these expansions, \( v_r^{n,m}, v_\theta^{n,m}, \ldots, p^{n,m}, \phi^{(m)}, \forall n, m \geq 0, \) are functions only of \( z \) and \( t \). Note that \( p - p_a \) instead of \( p \) is expanded in the perturbation, because \( p_a \) may be of higher order than \( O(1) \) while \( p - p_a \) can always be assumed to be of order \( O(1) \), \( p_a = \frac{1}{B_W} P_a \).

**Comment 1:** This ansatz assumes analyticity in \( r \) or at least the existence of a formal power series expansion in \( r \). Physically, this assumption is quite reasonable since the models we seek are based on von Kármán ansatz [26], which is polynomial
in \( r \). It is an open problem, of current investigation, to determine whether there is a nonzero radius of convergence for special solutions. In the absence of such a analysis, this theory must be viewed as purely formal.

*Comment 2:* The expansion in \( \varepsilon \) is likewise a formal expansion. The higher-order expansions in \( \varepsilon \) are new to this subject. We presume they are formal asymptotic expansions in \( \varepsilon \); the stronger conditions of analyticity in \( \varepsilon \) is most likely an unattainable result for such complicated PDEs. We shall be content with an investigation of the asymptotic ordering, through three orders, of terms in the \( \varepsilon \)-expansion. Again, the analysis of corrections to all orders is an interesting problem of current study.

Expansions (3.27)-(3.37) are carefully chosen to obtain a consistent perturbation theory in the sense that it allows one to determine a closed set of equations not only for the leading order prediction but also for each of the higher order corrections. These expansions clearly reflect two basic features of the jet: slenderness and axisymmetry. One may classify these field variables in three groups according to their relative magnitudes. Group I consists of six \( O(1) \) variables: \( v_2, T_{rr}, T_{\theta\theta}, T_{zz}, \phi, \) and \( p - p_a \); group II consists of four \( O(\varepsilon) \) variables: \( v_r, v_\theta, T_{r\theta}, \) and \( T_{rz} \); group III has only one \( O(\varepsilon^2) \) variable \( T_{r\theta} \).

In order to balance the powers of \( \varepsilon \) in perturbation expansions (3.12)-(3.25), one must give relative orders of \( B, F, W, Z, \Lambda_1, \) and \( \Lambda_2 \) with respect to \( \varepsilon \). This is achieved by positing these parameters in powers of \( \varepsilon \):

\[
B = \bar{B} \varepsilon^b, \quad F = \bar{F} \varepsilon^f, \quad W = \bar{W} \varepsilon^w, \\
Z = \bar{Z} \varepsilon^z, \quad \Lambda_1 = \bar{\Lambda}_1 \varepsilon^{\lambda_1}, \quad \Lambda_2 = \bar{\Lambda}_2 \varepsilon^{\lambda_2},
\]

(3.38)

where \( \varepsilon < \bar{B}, ..., \bar{\Lambda}_2 < \varepsilon^{-1} \), and the exponents are fixed integers (positive, negative,
or zero, and odd or even) depending on the physical application.

For notational convenience, we will also drop the tilde overbar on the nondimensional parameters (3.38) when a specific choice is implemented. *A set of the particular integer exponent values of b, f, w, g, \( \lambda_1 \), and \( \lambda_2 \) in (3.38) defines a regime of slender viscoelastic free jet behavior.* For instance, the special choice of exponents

\[
f = w = z = 0, \quad \lambda_1 = \lambda_2 = -b = 3
\]  

(3.39)

describes jets that are *Newtonian* through leading three orders of approximation and dominated by gravity, viscosity and surface tension. This particular set of equations is analyzed in [5]. In Chapter VI, we will analyze a regime in which the leading order equations are *viscoelastic* so that the flows are *strongly elastic*.

With a particular specified choice of regime defined by equation (3.38), the scalar equations (3.12)–(3.25) are called *master equations*. From the master equations and the perturbation ansatz above follow the equations for the leading order predictions as well as higher order corrections. In fact, one simply inserts the field variable expansions above into the master equations and equates the terms at each order \( r^n, \varepsilon^m \), and collects the equations. For properly chosen regimes, one finds *closure*: finite sets of \( k \) equations in \( k \) unknowns decouple at each order \( (n, m) \); the number \( k \) may vary from order to order. The set of equations involving the \( (0,0) \) unknowns is called the \( (0,0) \) system, the set of equations involving the \( (0,1) \) unknowns is called the \( (0,1) \) system, and so forth. The \( (0,0) \) system is also referred to as the leading order system, since from the perturbation ansatz (3.27)–(3.37) it is seen that the \( (0,0) \) variables \( v_r^{0,0}, v_\theta^{0,0}, v_z^{0,0} \), etc., give the leading order behavior of the free jet flow. The
(0,1) system then determines the first order correction to this leading order behavior, and the (0,2) and (1,0) systems determine the second order corrections; the (1,0) corrections allow for the first radially dependent corrections.
CHAPTER IV

1-D Slender Jet Models in Selected Regimes

In a particular regime the orders of magnitude in \( \varepsilon \) of the dimensionless groups \( B \), \( F \), \( W \), \( Z \), \( \Lambda_1 \), \( \Lambda_2 \) are fixed (recall equations (3.8) and (3.9)). Infinitely many coupled PDEs for the coefficients in expansions (3.11), (3.12), (3.13) are obtained (by equating powers of \( \varepsilon \) and \( \varepsilon \)) from each of the scalar components of the field equations (2.1), (2.2), (2.3) and free surface boundary conditions (2.7), (2.8) of the 3-D problem. As indicated in Chapter III, these coupled PDEs vary as the regime varies.

For many regimes it is found that these infinitely many coupled PDEs in infinitely many unknowns decouple at each order in the asymptotics into closed sets of \( n \) equations for \( n \) unknowns. This result, in specific regimes, establishes consistency of the perturbation scheme.

With closure through higher order, we can proceed to try to verify the asymptotic validity of leading order slender jet or "fiber spinning" solutions such as those which have appeared in the literature since 1969. As noted earlier, we shall in this work be content with a study of the leading three orders. The analysis to higher order is currently under study.
4.1 Regime 1: Viscosity, Elasticity, Inertia, Gravity, and Surface Tension Dominated Jets

In this section, we are going to derive the asymptotic equations up to order 2 for Regime 1: $B, F, W, \Lambda_1, Z = O(1)$, and $\Lambda_2 = O(\varepsilon^2)$, that is, $B = \tilde{B}$, $F = \tilde{F}$, $\Lambda_1 = \tilde{\Lambda}_1$, $W = \tilde{W}$, $Z = \tilde{Z}$, and $\Lambda_2 = \tilde{\Lambda}_2\varepsilon^2$.

It can be shown that this system is closed at each order. The $(0,0)$ system can be reduced to a system of six first order quasilinear PDE's, and the $(0,1)$ and $(0,2)$ systems can be reduced to systems of six first order linear PDE's with the same coefficient matrix as the reduced $(0,0)$ system. It is not shown that the high order systems, $(0,3), (0,4), \ldots$ share the same coefficient matrix. This immediately implies that all of the higher order systems share the same characteristics as the $(0,0)$ system. The $(1,0)$ system is reduced four linear PDE's which can be integrated formally and depend on the $(0,0)$ solutions only.

(In this regime the physical effects of inertia, gravity, surface tension, viscosity, and elastic relaxation are all important in the determination of the leading order jet behavior, whereas retardation is of lesser relative importance. The PDEs for this regime decouple into the following finite sets of closed equations.)

The Leading Order Problem

Eleven equations for the eleven unknowns $v_z^{0,0}, v_r^{0,0}, v_\theta^{0,0}, T_r^{0,0}, T_\theta^{0,0}, T_z^{0,0}, T_r^{0,0}, T_\theta^{0,0}, T_z^{0,0}, \phi^{(0)}, T_{r\theta}^{0,0}$ and $T_{r z}^{0,0}$ decouple from all the other equations and unknowns. This problem is called the leading order problem, to indicate its solution allows for determination of the leading order term $\phi^{(0)}(z, t)$ in the expansion (3.12) of the free surface profile.
The (0,0) equations consist of a closed system of the 11 linear or quasilinear PDE's. The (0,0) system may be reduced to a coupled system of six quasilinear PDEs for $\phi^{(0)}$, $v_{z}^{0,0}$, $T_{rr}^{0,0}$, $T_{zz}^{0,0}$, $v_{\theta}^{0,0}$ and $T_{\theta z}^{0,0}$, which can be summarized as follows:

REGIME 1: Reduced Leading Order Problem

$$u_{,ij}^{0,0} + M(u^{0,0})u_{,z}^{0,0} = P(u^{0,0})u^{0,0} + f^{0,0},$$  \hspace{1cm} (4.1a)

with

$$u^{0,0} = [\phi^{(0)}, v_{z}^{0,0}, T_{rr}^{0,0}, T_{zz}^{0,0}, v_{\theta}^{0,0}, T_{\theta z}^{0,0}]^T,$$  \hspace{1cm} (4.1b)

$$M(u^{0,0}) = \begin{bmatrix}
v_{z}^{0,0} & \frac{1}{2}\phi^{(0)} & 0 & 0 & 0 & 0 \\
M_{21} & v_{z}^{0,0} & B & -B & 0 & 0 \\
0 & aT_{rr}^{0,0} + \frac{Z}{\Lambda_{1}} & v_{z}^{0,0} & 0 & 0 & 0 \\
0 & -2(aT_{zz}^{0,0} + \frac{Z}{\Lambda_{1}}) & v_{z}^{0,0} & 0 & 0 & 0 \\
-\frac{4B}{\phi^{(0)}}T_{\theta z}^{0,0} & -v_{\theta}^{0,0} & 0 & 0 & v_{z}^{0,0} & -B \\
0 & \frac{-1+\alpha T_{\theta z}^{0,0}}{2} & 0 & 0 & 0 & M_{65} \ v_{z}^{0,0}
\end{bmatrix}.$$  \hspace{1cm} (4.1c)

$$M_{21} = \frac{2B}{\phi^{(0)}}(T_{rr}^{0,0} - T_{zz}^{0,0}) - \frac{1}{W\phi^{(0)}},$$  \hspace{1cm} (4.1d)

$$M_{65} = \frac{1}{2} - \frac{a}{2} T_{rr}^{0,0} - \frac{1}{\Lambda_{1}} T_{zz}^{0,0} - \frac{Z}{\Lambda_{1}},$$  \hspace{1cm} (4.1e)

$$P = \text{diag} \begin{bmatrix} 0, 0, -\frac{1}{\Lambda_{1}}, -\frac{1}{\Lambda_{1}}, 0, -\frac{1}{\Lambda_{1}} \end{bmatrix},$$  \hspace{1cm} (4.1f)

$$f^{0,0} = [0, 1/F, 0, 0, 0, 0]^T.$$  \hspace{1cm} (4.1g)

Note that the $6 \times 6$ matrix $M$ depends on the physical parameters $B$, $F$, $W$, $Z$, $\Lambda_{1}$, in addition to the unknowns $\phi^{(0)}$, $v_{z}^{0,0}$, $T_{rr}^{0,0}$, $T_{zz}^{0,0}$, $v_{\theta}^{0,0}$, and $T_{\theta z}^{0,0}$. The diagonal matrix $P$ depends only on the Weissenberg number $\Lambda_{1}$, and the vector $f^{0,0}$ depends only
on the Froude number $F$. The retardation parameter $\Lambda_2$ is absent from the leading order problem of this regime.

$\mathbf{M}$ has the following six eigenvalues:

\begin{equation}
\begin{aligned}
s_1 &= s_2 = v_{z,0}^0, \\
n_3 &= v_{z,0}^0 + \sqrt{\text{disc}_1}, \\
n_4 &= v_{z,0}^0 - \sqrt{\text{disc}_1}, \\
n_5 &= v_{z,0}^0 + \sqrt{\text{disc}_2}, \\
n_6 &= v_{z,0}^0 - \sqrt{\text{disc}_2},
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
\text{disc}_1 &= B(a + 1)T_{rr}^{0,0} + B(2a - 1)T_{zz}^{0,0} + \frac{3BZ}{\Lambda_1} - \frac{1}{2W(0)}, \\
\text{disc}_2 &= B\left(\frac{a - 1}{2}T_{rr}^{0,0} + \frac{a + 1}{2}T_{zz}^{0,0} + \frac{Z}{\Lambda_1}\right).
\end{aligned}
\end{equation}

The characteristics

\begin{equation}
\frac{dz}{dt} = s_j, \quad j = 1, 2, \ldots, 6,
\end{equation}

determine the type of initial boundary conditions which are proper to impose on (4.1a). Note that $\text{disc}_1$ and $\text{disc}_2$ depend explicitly on

\begin{equation}
\phi^{(0)}, \quad T_{rr}^{0,0}, \quad \text{and} \quad T_{zz}^{0,0}
\end{equation}

only. Note that $\mathbf{M}$ is a lower block triangular matrix, and the first four components of $u^{0,0}$: $\phi^{(0)}$, $v_{z,0}^{0,0}$, $T_{rr}^{0,0}$, $T_{zz}^{0,0}$ do not depend on the last two components $v_{\theta,0}^{0,0}$, and $T_{\theta z}^{0,0}$.

In this regime which includes elasticity as a leading order physical effect, the stresses are primitive dynamical unknowns, with their own evolution equations (the
third, fourth and sixth scalar equations in the matrix form (4.1a)). Initial and boundary values must be specified for stresses, as well as for free surface location and axial velocity; precisely which boundary values may be specified is addressed in Chapter V and VI.

The remaining leading order unknowns are determined from the solution of the reduced leading order problem (4.1) through

\[ v_r^{0,0} = -\frac{1}{2} v_{zz}, \]
\[ T_{\theta\theta}^{0,0} = T_{rr}^{0,0}, \]
\[ p^{0,0} = T_{rr}^{0,0} + \frac{1}{BW\phi^{(0)}}, \]
\[ T_{r\theta}^{0,0} = \frac{\phi^{(0)}}{\phi^{(0)}} T_{\theta z}^{0,0}, \]
\[ T_{rz}^{0,0} = \frac{\phi^{(0)}}{\phi^{(0)}} (T_{zz}^{0,0} - T_{rr}^{0,0}). \]

**Remarks on the Structure and Validity of These Equations**

We emphasize that these models describe flows with long length scales in \( z \). Therefore, sharp gradients and shocks are not allowable with these approximations. If sharp gradients develop, we must abandon the model and seek to add new physical effects which control the gradients.

**The First Order Corrections**

Eleven equations for the eleven unknowns \( v_z^{0,1}, v_r^{0,1}, v_{\theta}^{0,1}, T_{rr}^{0,1}, T_{\theta\theta}^{0,1}, T_{rr}^{0,1}, T_{zz}^{0,1}, T_{rz}^{0,1}, p^{0,1}, \phi^{(1)}, T_{r\theta}^{0,1}, \) and \( T_{\theta z}^{0,1} \) likewise decouple. This set has variable coefficients prescribed by the previously determined leading order solution. It is called the first order problem, as it involves the first order free surface correction \( \phi^{(1)} \).
Just as in the leading order problem above, using algebraic constraints, the first order problem can be simplified to the following coupled system of six linear homogeneous PDEs for $\phi^{(1)}, v_z^{0,1}, T_{rr}^{0,1}, T_{zz}^{0,1}, v_{\theta}^{0,1}$, and $T_{\theta z}^{0,1}$:

**REGIME 1: Reduced Problem for First Order Corrections**

$$u_{zz}^{0,1} + M(u^{0,0}) u_{zz}^{0,1} = Q(u^{0,0}) u^{0,1}, \quad (4.3a)$$

with $M(u^{0,0})$ given in (4.1c), and

$$u^{0,1} = [\phi^{(1)}, v_z^{0,1}, T_{rr}^{0,1}, T_{zz}^{0,1}, v_{\theta}^{0,1}, T_{\theta z}^{0,1}]^T, \quad (4.3b)$$

$$Q(u^{0,0}) = \begin{bmatrix} -\frac{1}{2} v_{z,z}^{0,0} & -\phi^{(0)} & 0 & 0 & 0 & 0 \\ Q_{21} & -v_{z,z}^{0,0} & -\frac{2 B\phi^{(0)}}{\phi^{(0)}} & \frac{2 B\phi^{(0)}}{\phi^{(0)}} & 0 & 0 \\ 0 & -T_{rr,z}^{0,0} & -av_{z,z}^{0,0} - \frac{1}{\lambda_1} & 0 & 0 & 0 \\ 0 & -T_{zz,z}^{0,0} & 2a v_{z,z}^{0,0} - \frac{1}{\lambda_1} & 0 & 0 & 0 \\ Q_{51} & -v_{\theta,z}^{0,0} & 0 & 0 & v_{z,z}^{0,0} & -\frac{4 B\phi^{(0)}}{\phi^{(0)}} \\ 0 & -T_{\theta z,z}^{0,0} & \frac{a-1}{2} v_{\theta,z}^{0,0} & \frac{1+a}{2} v_{z,z}^{0,0} & 0 & -\frac{\lambda_1}{\lambda_1} \end{bmatrix}, \quad (4.3c)$$

with

$$Q_{21} = \frac{2 B\phi^{(0)}}{\phi^{(0)}} \left[T_{zz}^{0,0} - T_{rr}^{0,0} + \frac{1}{BW\phi^{(0)}} \right], \quad (4.3d)$$

and

$$Q_{51} = -\frac{4 B\phi^{(0)}}{\phi^{(0)}} T_{\theta z}^{0,0}. \quad (4.3e)$$

$M$ and $Q$ depend on $u^{0,0}$ only. Note that the $(0,1)$-system is linear and homogeneous. In other words, $u^{0,1} \equiv 0$ can be a solution — 1st order correction can be identically zero.

In this strongly elastic regime (i.e., elastic effects present in leading order) the stress corrections $T_{rr}^{0,1}$, $T_{zz}^{0,1}$ are primitive dynamical unknowns, determined by evolu-
tion equations (the third, fourth and sixth scalar equations in (4.3a)) and demanding initial and boundary values.

The remainder of the first order unknowns are determined from the solution of the reduced first order problem (4.3) and the previously obtained leading order solution through

$$v_{r,1}^{0,1} = -\frac{1}{2} v_{z,z}^{0,1}, \quad (4.4a)$$

$$T_{\theta z}^{0,1} = T_{rr}^{0,1}, \quad (4.4b)$$

$$p^{0,1} = T_{rr}^{0,1} - \frac{1}{BW} \phi^{(1)} \quad (4.4c)$$

$$T_{r\theta}^{0,1} = \frac{\phi^{(0)} z}{\phi^{(0)}} T_{\theta z}^{0,1} + \frac{T_{\theta z}^{0,0}}{\phi^{(0)}} [\phi^{(0)} \phi^{(1)} z_{z}^{(0)} - \phi^{(0)} \phi^{(1)}], \quad (4.4d)$$

$$T_{rz}^{0,1} = \frac{\phi^{(1)}}{\phi^{(0)}} (T_{zz}^{0,0} - T_{rr}^{0,0}) + \frac{\phi^{(0)} z}{\phi^{(0)}} (T_{rr}^{0,0} - T_{zz}^{0,0}) + \frac{\phi^{(0)}}{\phi^{(0)}} (T_{zz}^{0,0} - T_{rr}^{0,0}). \quad (4.4e)$$

The Second Order Corrections

Ten equations decouple for the ten unknowns $v_{z}^{1,0}, v_{r}^{1,0}, v_{\theta}^{1,0}, T_{rr}^{1,0}, T_{\theta \theta}^{1,0}, T_{zz}^{1,0}, T_{rz}^{1,0}, p^{1,0}, T_{r\theta}^{1,0}$, and $T_{\theta z}^{1,0}$. These terms give the $O(\varepsilon^2)$ parabolic radial corrections to the axial velocity, normal stress, and constraint pressure distributions, and the $O(\varepsilon^3)$ cubic radial corrections to the radial velocity and shear stress distributions (refer to the expansions (4.1) and (4.3)). This system has coefficients prescribed by the leading order solution.

Eleven equations arise for the eleven unknowns $v_{z}^{0,2}, v_{r}^{0,2}, v_{\theta}^{0,2}, T_{rr}^{0,2}, T_{\theta \theta}^{0,2}, T_{zz}^{0,2}, T_{rz}^{0,2}, p^{0,2}, \phi^{(2)}$, $T_{r\theta}^{0,2}$, and $T_{\theta z}^{0,2}$. This system is forced by the "0,0", "0,1", and "1,0" solutions.
The proceeding two sets, together comprised of 21 equations for 21 unknowns are called the second order problem, since their solution allows for the determination of \( \phi^{(2)} \) in expansion (4.2).

The first decoupled part of the second order problem reduces through algebraic constraints to the following four uncoupled linear nonhomogeneous PDEs for \( T_{rr}^{1,0} \), \( T_{\theta\theta}^{1,0} \), \( T_{zz}^{1,0} \) and \( T_{\theta z}^{1,0} \).

REGIME 1: Reduced Second Order Problem — Radially Dependent Corrections

\[
\begin{align*}
\mathbf{u}_t^{1,0} + v_x^{0,0} \mathbf{u}_z^{1,0} &= \mathbf{R}(\mathbf{u}^{0,0}) \mathbf{u}^{1,0} + \mathbf{f}^{1,0}(\mathbf{u}^{0,0}), \\
\mathbf{u}^{1,0} &= [T_{rr}^{1,0}, T_{\theta\theta}^{1,0}, T_{zz}^{1,0}, T_{\theta z}^{1,0}]^T,
\end{align*}
\]

where

\[
\mathbf{R}(\mathbf{u}^{0,0}) = \begin{bmatrix}
(1 - a)v_{z,z}^{0,0} - \frac{1}{\Lambda_1} & 0 & 0 & 0 \\
0 & (1 - a)v_{z,z}^{0,0} - \frac{1}{\Lambda_1} & 0 & 0 \\
0 & 0 & (1 + 2a)v_{z,z}^{0,0} - \frac{1}{\Lambda_1} & 0 \\
0 & \frac{2-a-1}{2}v_{\theta,z}^{0,0} & \frac{a+1}{2}v_{\theta,z}^{0,0} & \frac{1+a}{2}v_{z,z}^{0,0} - \frac{1}{\Lambda_1}
\end{bmatrix},
\]

\[
\mathbf{f}^{1,0}(\mathbf{u}^{0,0}) = [f_1, f_2, f_3, f_4]^T,
\]

\[
\begin{align*}
f_1 &= [2(a - 1)T_{rr}^{0,0} - T_{rr}^{0,0}]v_{z,z}^{1,0} - \frac{3}{2}(aT_{rr}^{0,0} + \frac{Z}{\Lambda_1})v_{z,z}^{1,0} - \frac{1+a}{2}v_{z,z,z}T_{rr}^{0,0}, \\
f_2 &= -T_{rr,z}^{0,0}v_{z,z}^{1,0} - \frac{1}{2}[aT_{rr}^{0,0} + \frac{Z}{\Lambda_1}]v_{z,z}^{1,0}, \\
f_3 &= [2(1 + a)T_{rr}^{0,0} - T_{zz,z}^{0,0}]v_{z,z}^{1,0} + 2[aT_{zz}^{0,0} + \frac{Z}{\Lambda_1}]v_{z,z}^{1,0} - \frac{1-a}{2}v_{z,z,z}T_{rr}^{0,0}, \\
f_4 &= \frac{a - 1}{2}T_{rr}^{0,0} + \frac{a + 1}{2}T_{zz}^{0,0} + \frac{Z}{\Lambda_1}v_{\theta,z}^{1,0} - T_{0,0}^{0,0}v_{z,z}^{1,0} + \frac{1+3a}{4}T_{\theta z}^{0,0}v_{z,z}^{1,0} + \frac{4}{2}v_{z,z}^{0,0} + (1 + a)v_{z}^{1,0}T_{\theta z}^{0,0}.
\end{align*}
\]
The coefficient \( v_{1,0} \) of the radially parabolic axial velocity correction which appears in equations (4.5e-h) is determined completely from the previously obtained leading order solutions by

\[
v_{1,0} = v_{1,0}(u^{0,0}) = \left[ T_{rs,t}^{0,0} + v_{s,0}^{0,0}T_{ss,s}^{0,0} + \left( \frac{1}{\Lambda_1} - \frac{1 + a}{2} v_{s,s}^{0,0} \right) T_{t,t}^{0,0} + \frac{1}{4} [(a - 1) T_{rr}^{0,0} + (a + 1) T_{xx}^{0,0} + \frac{2Z}{\Lambda_1}] \right]^{-1}.
\]

The remaining "1,0" corrections are given by

\[
u_r^{1,0} = -\frac{1}{4} v_{s,s}^{1,0},
\]

\[
p_{r}^{1,0} = \frac{3}{2} T_{rr}^{1,0} - \frac{1}{2} T_{\theta \theta}^{1,0} + \frac{1}{8B} \left( 2v_{r,s}^{0,0} - v_{r,s}^{0,2} + 2v_{s,0}^{0,0}v_{r,s}^{0,0} + 4v_{s}^{0,0} \right)
\]

\[+ \frac{1}{2\phi(0)} \left[ (T_{zz}^{0,0} - T_{rr,z}^{0,0}) \phi(0) + (T_{zz}^{0,0} - T_{rr}^{0,0}) (\phi(0) - \frac{\phi(0)^2}{\phi(z)^2}) \right],
\]

\[T_{r}^{1,0} = \frac{1}{4} (p_{r}^{1,0} - T_{zz}^{1,0}) + \frac{1}{4B} (v_{s,t}^{1,0} + v_{s,0}^{0,0}v_{s,s}^{1,0}),
\]

\[v_{\theta}^{1,0} = \frac{1}{2} \left[ T_{r}^{0,0} + \frac{1}{\Lambda_1} (a - 1) v_{r,0}^{0,0} + (1 - a) v_{r,0}^{0,0} \right] - \frac{1}{6} T_{zz}^{1,0},
\]

\[v_{\phi}^{1,0} = \frac{1}{2} \left[ T_{r}^{0,0} + \left( \frac{1}{\Lambda_1} (a - 1) v_{r,0}^{0,0} + (1 - a) v_{r,0}^{0,0} \right) \cdot \left\{ aT_{rr}^{0,0} + \frac{Z}{\Lambda_1} \right\}^{-1}.
\]

The remainder of the second order problem can be reduced using algebraic constraints to the following coupled system of four linear nonhomogeneous PDEs for \( \phi(2) \), \( v_{0,2} \), \( T_{rr,0} \), and \( T_{zz,0} \):

**REGIME 1: Reduced Second Order Problem — Radially Independent Corrections**

\[u_{t}^{0,2} + M(u^{0,0})u_{z}^{0,2} = Q(u^{0,0})u^{0,2} + f^{0,2}(u^{0,0}, u^{0,1}, u^{1,0}),
\]

\[\text{(4.7a)}
\]
with \(M(u_{0,0}^0)\) given in (4.1c), \(Q(u_{0,0}^0)\) given in (4.3c), and

\[
\mathbf{u}_{0,2} = [\phi(1), v_{2,0}^2, T_{0,2}, T_{0,0}^2, v_{\theta,2}^2, T_{\theta,2}^0]^T, \tag{4.7b}
\]

\[
f_{0,2}(u_{0,0}^0, u_{0,1}^0, u_{1,0}^0) = [g_1, g_2, g_3, g_4, g_5, g_6]^T, \tag{4.7c}
\]

where

\[
g_1 = -v_{z,2}^0 \phi_{,2}^{(1)} - \frac{1}{2} v_{z,2}^1 \phi_{,2}^{(1)} - \phi(0) \left[ v_{z,2}^1 \phi_{,2}^{(0)} + \frac{1}{4} v_{z,2}^1 \phi_{,2}^{(0)} \right], \tag{4.7d}
\]

\[
g_2 = B \left[ (3 \phi(0)^2 + \phi_{,2}^{(0)} \phi_{,2}^{(0)}) T_{0,0}^0 + T_{0,0}^0 \phi_{,2}^{(0)} \phi_{,2}^{(0)} + 2(T_{0,1}^0 - T_{0,1}^0) \phi_{,2}^{(1)} \phi_{,2}^{(1)} - 2 T_{0,1}^1 \phi_{,2}^{(1)} \phi_{,2}^{(1)} + 2(T_{1,0}^0 - 2 T_{1,0}^0 + p_{,2}^{(0)}) \phi_{,2}^{(0)} \phi_{,2}^{(0)} + (p_{,2}^{(0)} - T_{0,2}^0) \phi_{,2}^{(0)} \phi_{,2}^{(0)} - 2 T_{0,2}^0 \phi_{,2}^{(0)} \phi_{,2}^{(0)} \right] + \frac{1}{W} \left[ \phi_{,zz}^{(0)} - \phi_{,zz}^{(0)} \right] - v_{z,2}^1 v_{z,2}, \tag{4.7e}
\]

\[
g_3 = -v_{z,2}^0 T_{0,1}^0 - av_{z,2}^0 v_{z,2}^0 - \frac{\alpha 2}{\Lambda_1} \left[ v_{z,2}^0 + v_{z,2}^0 v_{z,2}^0 + av_{z,2}^2 \right], \tag{4.7f}
\]

\[
g_4 = -v_{z,2}^0 T_{0,1}^0 + 2av_{z,2}^0 T_{0,1}^0 + 2 \frac{\alpha 2}{\Lambda_1} \left[ v_{z,2}^0 + v_{z,2}^0 v_{z,2}^0 - 2av_{z,2}^0 \right], \tag{4.7g}
\]

\[
g_5 = v_{z,2}^0 v_{z,2}^0 - v_{z,2}^0 v_{z,2}^0 - 4B \left[ \phi_{,2}^{(0)} T_{0,1}^0 + \frac{2 \phi_{,2}^{(1)}}{\phi_{,2}^{(0)}} T_{0,1}^0 + \phi_{,2}^{(0)} T_{0,1}^0 \right] + \frac{4B}{\phi_{,2}^{(0)}} \left[ T_{0,0}^0 \phi_{,2}^{(1)} \phi_{,2}^{(1)} + T_{0,0}^0 \phi_{,2}^{(1)} \phi_{,2}^{(1)} + T_{0,1}^1 \sum_{l=0}^{1} \phi_{,2}^{(1-l)} \right], \tag{4.7h}
\]

\[
g_6 = \frac{1}{2} \left( v_{z,2}^2 T_{0,1}^0 + v_{z,2}^2 T_{0,1}^0 \right) + \frac{a - 1}{2} v_{z,2}^0 T_{0,1}^0 - v_{z,2}^0 T_{0,1}^0 + 2 \frac{\alpha 2}{\Lambda_1} \left[ v_{z,2}^0 + v_{z,2}^0 v_{z,2}^0 + v_{z,2}^0 v_{z,2}^0 - (a + \frac{3}{2}) v_{z,2}^0 v_{z,2}^0 \right]. \tag{4.7i}
\]

The remainder of the second order unknowns are determined from the solution of the reduced second order problem (4.7) and the previously obtained "0,0", "0,1", "1,0" solutions through

\[
v_{r,2}^0 = -\frac{1}{2} v_{z,2}^0, \tag{4.7j}
\]
\[ T_{02}^{0} = T_{rr}^{0}, \]

\[ \rho_{02}^{0} = T_{rr}^{0} + \phi_{0}^{(0)}(T_{rr}^{00} - p^{1}) - \phi_{0}^{(0)} \phi_{zz}^{(0)} T_{rr}^{00}, \]

\[ - \frac{1}{BW} \left[ \phi_{0}^{(2)} - \frac{\phi_{0}^{(1)} \phi_{0}^{(0)}}{\phi_{0}^{(0)} + \phi_{0}^{(0)} + \phi_{0}^{(0)}} \right], \]  

\[ T_{02}^{0} = -\phi_{0}^{(0)} T_{rr}^{10} - \frac{2 \phi_{0}^{(1)}}{\phi_{0}^{(0)}} T_{rr}^{01} - \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} [T_{rr}^{02} + \phi_{0}^{(0)} T_{rr}^{10}] \]

\[ + \frac{1}{\phi_{0}^{(0)}} \left( \sum_{i=0}^{2} \phi_{0}^{(i)} T_{rr}^{00} - \phi_{0}^{(i)} T_{rr}^{00} \phi_{0}^{(2-i)} + T_{rr}^{01} \sum\phi_{0}^{(i)} \phi_{0}^{(1-i)} \right), \]

\[ T_{rr}^{02} = \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} (T_{rr}^{00} - T_{rr}^{00}) - \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} T_{zz}^{00} + \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} (T_{zz}^{02} - T_{rr}^{01}) - \phi_{0}^{(0)} T_{rr}^{10}, \]

\[ + \phi_{0}^{(0)} \phi_{0}^{(0)} (T_{zz}^{10} - T_{rr}^{10}) - \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} T_{rr}^{01} + \frac{\phi_{0}^{(0)}}{\phi_{0}^{(0)}} (T_{zz}^{01} - T_{rr}^{01}) + \phi_{0}^{(0)} T_{rr}^{00}. \]

Note that the retardation parameter \( \tilde{\Lambda}_{2} \), which is absent from the "0,0", "0,1", and "1,0" problems, first appears in the problem (4.7) for the "0,2" corrections as a coefficient in the nonhomogeneous forcing term \( f_{0}^{(0)}(u^{00}, u^{01}, u^{10}) \). Retardation in this regime is therefore called a "weak effect", since it does not enter into the determination of the leading order jet behavior, but contributes to the determination of higher order corrections to this leading order flow. The \( O(1) \) quantity \( \tilde{\Lambda}_{2} \) in equations (4.7f,g) is defined from the \( O(\varepsilon^{2}) \) parameter \( \Lambda_{2} \) by

\[ \Lambda_{2} = \tilde{\Lambda}_{2} \varepsilon^{2}, \quad \varepsilon < \tilde{\Lambda}_{2} < \varepsilon^{-1}. \]  

In Chapter V, the steady torsionless reduction of these equations through the leading three orders in the asymptotics (namely equations (4.1), (4.3), (4.5), and (4.7)) will be numerically integrated to produce nontrivial steady solutions for a slender viscoelastic fluid jet with dominant physical effects of viscosity, elastic relaxation, inertia, gravity, and surface tension. The torsional effects will addressed in Chapter
VI.

4.2 Regime 2: The Denn, Petrie & Avenas Model: Viscosity and Elasticity Dominated Jets

For the second example we consider flows in which the effects of viscosity and elasticity are dominant, whereas inertia, gravity, and surface tension effects are negligible to leading order. This condition fixes only the leading order equations, which form a closed system (4.11) as displayed below. All succeeding order systems are closed, independently of which order the presumed weak effects of inertia, gravity and surface tension first enter. Therefore, we have the freedom to study a variety of corrections to the dominant viscous-elastic behavior, in which each weak effect (inertia, gravity and surface tension) individually enters as a first, second or higher order correction.

To be specific, we consider the regime where surface tension first appears as a second order "weak effect", and inertia and gravity are neglected through second order in the asymptotics. This balance among all competing effects is accomplished by the following parameter ordering:

\[
B = \tilde{B} \varepsilon^{-3}, \quad F = \tilde{F} \varepsilon^0, \quad W = \tilde{W} \varepsilon^1, \\
Z = \tilde{Z} \varepsilon^0, \quad \Lambda_1 = \tilde{\Lambda}_1 \varepsilon^0, \quad \Lambda_2 = \tilde{\Lambda}_2 \varepsilon^3,
\]

where \(\tilde{B}, \tilde{F}, \tilde{W}, \tilde{Z}, \tilde{\Lambda}_1, \tilde{\Lambda}_2\) are \(O(1)\).

The leading (zeroth) order problem in this regime reduces through algebraic constraints to the following coupled system of four quasilinear PDEs and two ODE
constraints:

**REGIME 2: Reduced Leading Order Problem**

\[
\text{Nu}_{i,i} + \dot{M}(u^{0,0})u_{i,i}^{0,0} = P(u^{0,0})u^{0,0}, \tag{4.11a}
\]

with

\[
u^{0,0} = [\phi^{0}, v_{z}^{0,0}, T_{r r}^{0,0}, T_{z z}^{0,0}, v_{\theta,z}^{0,0}, T_{\theta z}^{0,0}]^T, \tag{4.11b}
\]

\[
N = \text{diag}[1, 0, 1, 1, 0, 1], \tag{4.11c}
\]

\[
\dot{M}(u^{0,0}) = \begin{bmatrix}
v_{z}^{0,0} & \frac{1}{2} \phi^{(0)} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & a T_{r r}^{0,0} + \frac{Z}{\Lambda_1} & v_{z}^{0,0} & 0 & 0 & 0 \\
-\frac{4}{\phi^{(0)}} T_{\theta z}^{0,0} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1 + a}{2} T_{\theta z}^{0,0} & 0 & 0 & M_{65} & v_{z}^{0,0}
\end{bmatrix} \tag{4.11d}
\]

\[
\dot{M}_{21} = \frac{2}{\phi^{(0)}} (T_{r r}^{0,0} - T_{z z}^{0,0}), \tag{4.11e}
\]

\[
M_{65} = \frac{1 - a}{2} T_{r r}^{0,0} - \frac{1 + a}{2} T_{z z}^{0,0} - \frac{Z}{\Lambda_1}. \tag{4.11f}
\]

\[
P = \text{diag}[0, 0, -\frac{1}{\Lambda_1}, -\frac{1}{\Lambda_1}, 0, -\frac{1}{\Lambda_1}]. \tag{4.11g}
\]

This system displays a critical difference when compared with the leading order equations (4.1) of Regime 1: since inertia is neglected, there is no evolution equation for the axial velocity \(v_{z}^{0,0}\) and \(v_{\theta}^{0,0}\), and the second and fifth equations of (4.11a) are not PDEs but rather ODE constraints. The importance of this remark to the question of boundary conditions will be made clear in the next chapter. Also, we note that since
the only leading order physical effects are viscosity and elasticity, the only parameters in the leading order problem are $Z$, $\Lambda_1$, and the slip parameter $a$.

The remaining leading order unknowns can be produced from the solution of (4.11) through

$$v_r^{0,0} = -\frac{1}{2} v_z^{0,0}, \quad (4.12a)$$

$$T_{\theta\theta}^{0,0} = p^{0,0} = T_{rr}^{0,0}, \quad (4.12b)$$

$$T_{rz}^{0,0} = \frac{\phi_r^{(0)}}{\phi^{(0)}} (T_{zz}^{0,0} - T_{rr}^{0,0}), \quad (4.12c)$$

$$T_{r\theta}^{0,0} = \frac{\phi_z^{(0)}}{\phi^{(0)}} T_{\theta z}^{0,0}. \quad (4.12d)$$

**REMARK: Contact with Denn, Petrie, & Avenas [12]**

When the characteristic scales $z_0, r_0, t_0, f_0$ in (3.1), (3.2), (3.4), (3.5) are chosen as $z_0 = L, r_0 = (Q/\pi v_0)^{1/2}, t_0 = L/v_0, f_0 = F/\pi$, (where $L$ is the distance to the take-up spool, $Q$ is the volumetric flow rate, $v_0$ is the axial velocity at $z = 0$, and $F$ is the applied take-up force), the rate parameter $a$ is set to one, and time-dependence is suppressed, the torsionless reduction of the leading order problem (4.11) of this regime implies the thin-filament model for an upper convected Maxwell fluid of Denn, Petrie, & Avenas [12] (their equations (12)-(15) with $\nu = 0$).

The *first order problem* in this regime reduces to

**REGIME 2: Reduced Problem for First Order Corrections**

$$Nu_z^{0,1} + \tilde{M}u_z^{0,1} = \tilde{Q}(u^{0,0})u^{0,1}, \quad (4.13a)$$
with $M(u^{0,0})$ and $N$ given in equations (4.11c,d),

$$u^{0,1} = [\phi^{(1)}, v_x^{0,1}, T_r^{0,1}, T_{zz}^{0,1}, v_\theta^{0,1}, T_{\theta z}^{0,1}]^T,$$

(4.13b)

and

$$\dot{Q} = \begin{bmatrix}
-1 v_{z,z} & -\phi^{(0)} & 0 & 0 & 0 & 0 \\
-\frac{2e^{(0)}}{\phi^{(0)}} & -2\psi^{(0)} & 0 & 0 & 0 & 0 \\
0 & -T_r^{0,0} - 2\psi v_{z,z} + \frac{1}{\Lambda_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{4\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

(4.13c)

$$\dot{Q}_{21} = \frac{2\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} (T_r^{0,0} - T_r^{0,0}).$$

(4.13d)

$$\dot{Q}_{51} = -\frac{4\phi^{(0)} v_{z,zz}}{\phi^{(0)} + 1} T_r^{0,0}.$$  

(4.13e)

As in the leading order problem (4.11), the first order problem (4.13) is a coupled system of four PDEs and two ODE constraints, all homogeneous, involving only the parameters $Z$, $\Lambda_1$, and $a$; here, however, the system is linear, albeit with variable coefficients.

From the solutions of (4.13) and (4.11), one can then determine

$$v_r^{0,1} = -\frac{1}{2} v_{z,z}^{0,1},$$

(4.14a)

$$T_{\theta \theta}^{0,1} = T_r^{0,1},$$

(4.14b)

$$p^{0,1} = T_r^{0,1},$$

(4.14c)

$$T_r^{0,1} = \frac{\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} T_r^{0,1} + \frac{T_{zz}^{0,0}}{\phi^{(0)} + 1} [\phi^{(0)} v_{z,z} - \phi^{(0)} \phi^{(1)}],$$

(4.14d)

$$T_{r z}^{0,1} = \frac{\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} (T_r^{0,0} - T_r^{0,0}) + \frac{\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} (T_{r z}^{0,0} - T_{r z}^{0,0}) + \frac{\phi^{(0)} v_{z,z}}{\phi^{(0)} + 1} (T_{zz}^{0,0} - T_{zz}^{0,0}).$$

(4.14e)
The second order problem in this regime reduces to four uncoupled linear nonhomogeneous PDEs for $T_{rr}^{1,0}$, $T_{\theta\theta}^{1,0}$, and $T_{zz}^{1,0}$, driven by the leading order solution, and a coupled nonhomogeneous system of four quasilinear PDEs and two ODE constraints for $\phi^{(2)}$, $v_{z}^{0,2}$, $T_{rr}^{0,2}$, and $T_{zz}^{0,2}$ driven by the leading and first order solutions, and the "1,0" second order solution above.

The uncoupled PDEs for $T_{rr}^{1,0}$, $T_{\theta\theta}^{1,0}$, and $T_{zz}^{1,0}$, and the algebraic constraints for $v_{z}^{1,0}$ are the same as in Regime 1 and are given by equation (4.5). The remaining "1,0" corrections are determined from

\begin{align}
\frac{\partial v_{r}^{1,0}}{\partial r} &= -\frac{1}{4}v_{z,z}^{1,0}, \\
\frac{\partial p^{1,0}}{\partial r} &= \frac{3}{2}T_{rr}^{1,0} - \frac{1}{2}T_{\theta\theta}^{1,0} + \frac{1}{2\phi^{(0)}}\left[(T_{zz}^{0,0} - T_{rr}^{0,0})\phi^{(0)} + (T_{zz}^{0,0} - T_{rr}^{0,0})(\phi^{(0)} - \frac{\phi^{(0)}_{z}^{2}}{\phi^{(0)}_{r}})\right], \\
T_{rz}^{1,0} &= \frac{1}{4}(p_{r}^{1,0} - T_{zz}^{1,0}), \\
T_{r\theta}^{1,0} &= -\frac{1}{6}T_{\theta z,z}, \\
v_{\theta}^{1,0} &= \frac{1}{2}\left[T_{r\theta}^{0,0} + \left((a - 1)v_{z}^{0,0} + \frac{1}{A_{1}}\right)v_{r}^{1,0} + (1 - a)v_{z}^{1,0}T_{\theta z}^{0,0} - \frac{1 + a}{2}(v_{\theta}^{0,0}T_{rz}^{0,0} + v_{r}^{0,0}T_{\theta z}^{0,0}) + v_{z}^{0,0}T_{r\theta}^{0,0}\right]^{-1} \left\{aT_{rr}^{0,0} + \frac{Z}{A_{1}}\right\},
\end{align}

The remainder of the reduced second order problem is

**REGIME 2: Reduced Second Order Problem — Radially Independent Corrections**

\begin{align}
\text{Nu}_{t}^{0,2} + \hat{\mathcal{M}}(u^{0,0})u_{r}^{0,2} &= \hat{Q}(u^{0,0})u_{0,2} + \hat{f}^{0,2}(u^{0,0}, u^{0,1}, u^{1,0}), \quad (4.16a)
\end{align}

with $\hat{\mathcal{M}}(u^{0,0})$ given in (4.11c), $\hat{Q}(u^{0,0})$ given in (4.13c), and

\begin{align}
\frac{\partial u^{0,2}}{\partial r} &= \left[\phi^{(2)}, v_{z}^{0,2}, T_{rr}^{0,2}, T_{zz}^{0,2}\right]^{T}, \\
\hat{f}^{0,2}(u^{0,0}, u^{0,1}, u^{1,0}) &= \left[g_{1}, g_{2}, g_{3}, g_{4}, g_{5}, g_{6}\right]^{T},
\end{align}
where

\[ g_1 = -v_{zz}^{0,1} \phi_{zz}^{(1)} - \frac{1}{2} v_{zz}^{0,1} \phi_{zz}^{(1)} - \phi^{0)^2}_{zz} \left[ v_{zz}^{1,0} \phi^{(0)}_{zz} + \frac{1}{4} v_{zz}^{1,0} \phi^{(0)} \right], \]

\[ \hat{g}_2 = (3 \phi_{zz}^{0,2} + \phi^{0)^2}_{zz} + T_{zz}^{0,0} \phi_{zz}^{(0)} \phi_{zz}^{(0)} + 2(T_{zz}^{0,1} - T_{zz}^{0,1}) \phi_{zz}^{(1)} \phi_{zz}^{(0)} - 2T_{zz}^{0,1} \phi^{0,1}_{zz} \phi^{(1)} + 2(T_{zz}^{0,1} - 2T_{zz}^{1,0} + pT^{1,0}) \phi_{zz}^{(0)} \phi_{zz}^{(0)} + (p^{1,0} - T_{zz}^{1,0}) \phi^{(0)^2}_{zz} - 2T_{zz}^{1,0} \phi^{(0)^2} \]

\[ + \frac{\phi_{zz}^{(0)}}{B W \phi^{(0)^2}} \]

\[ \hat{g}_3 = -v_{zz}^{1,0} T_{zz}^{0,1} - a v_{zz}^{0,1} T_{zz}^{0,1} - \frac{A_2}{A_1} \left[ v_{zz}^{0,0} + v_{zz}^{0,0} v_{zz}^{0,0} + a v_{zz}^{0,2} \right], \]

\[ \hat{g}_4 = -v_{zz}^{0,1} T_{zz}^{0,1} + 2a v_{zz}^{0,1} T_{zz}^{0,1} \]

\[ \hat{g}_5 = -4 \left( \frac{\phi^{(0)^2}_{zz} T_{zz}^{1,0} + 2 T_{zz}^{0,1} + \phi^{(0)}_{zz} \phi^{(0)} T_{zz}^{1,1}}{\phi^{(0)}_{zz}} \right) \]

\[ + \frac{4}{\phi^{(0)}_{zz}} \left[ T_{zz}^{0,0} \phi_{zz}^{(1)} \phi^{(1)} - T_{zz}^{0,0} \phi^{(1)^2} + T_{zz}^{0,1} \sum_{i=0}^{1} \phi_{zz}^{(i)} \phi^{(1-i)} \right], \]

\[ \hat{g}_6 = \frac{1 + a}{2} (v_{zz}^{0,1} T_{zz}^{0,1} + v_{zz}^{0,1} T_{zz}^{0,1}) + \frac{a - 1}{2} v_{zz}^{0,1} T_{zz}^{0,1} - v_{zz}^{0,1} T_{zz}^{0,1}. \]

The other "0,2" unknowns are then given by

\[ v_{zz}^{0,2} = -\frac{1}{2} v_{zz}^{0,2} \]

\[ T_{zz}^{0,2} = T_{zz}^{0,2} \]

\[ p^{0,2} = T_{zz}^{0,2} + \phi^{(0)^2}_{zz} (T_{zz}^{0,1} - T_{zz}^{1,0}) - T_{zz}^{0,1} - \phi^{(0)}_{zz} T_{zz}^{0,0} \]

\[ T_{zz}^{0,2} = -\phi^{(0)^2}_{zz} T_{zz}^{1,0} - \frac{2 T_{zz}^{0,1}}{\phi^{(0)}_{zz}} - \frac{\phi^{(0)}_{zz}}{\phi^{(0)}_{zz}} [T_{zz}^{0,2} + \phi^{(0)^2}_{zz} T_{zz}^{1,0}] \]

\[ + \frac{1}{\phi^{(0)^2}_{zz}} \left( \sum_{i=0}^{2} [\phi_{zz}^{(i)} T_{zz}^{0,0} - \phi^{(0)}_{zz} T_{zz}^{0,0}] \phi^{(2-i)} + T_{zz}^{0,1} \sum_{i=0}^{1} \phi_{zz}^{(i)} \phi^{(1-i)} \right), \]

\[ T_{zz}^{0,2} = \frac{\phi^{(2)}_{zz}}{\phi^{(0)^2}_{zz}} (T_{zz}^{0,0} - T_{zz}^{0,0}) - \frac{\phi^{(2)}_{zz}}{\phi^{(0)^2}_{zz}} T_{zz}^{0,0} + \frac{\phi^{(2)}_{zz}}{\phi^{(0)^2}_{zz}} (T_{zz}^{0,0} - T_{zz}^{0,0}) - \phi^{(0)^2}_{zz} T_{zz}^{1,0} \]

\[ + \frac{\phi^{(0)}_{zz}}{\phi^{(0)^2}_{zz}} T_{zz}^{1,0} - \frac{\phi^{(1)}_{zz}}{\phi^{(0)^2}_{zz}} T_{zz}^{0,1} + \frac{\phi^{(1)}_{zz}}{\phi^{(0)^2}_{zz}} (T_{zz}^{1,0} - T_{zz}^{1,0}) + \frac{\phi^{(0)^2}_{zz}}{\phi^{(0)^2}_{zz}} T_{zz}^{0,0}. \]
Note that the surface tension parameter $\hat{B}\hat{W}$ appears, for the first time, in the second order problem. Surface tension in this regime is a "weak effect", which first appears as a coefficient in the nonhomogeneous forcing term $\hat{f}^{0,2}(u^{0,0}, u^{0,1}, u^{1,0})$ in the problem (4.16) for the "0,2" corrections.

It has been shown that the asymptotic expansions (3.11), (3.12), (3.13), together with the particular scaling (4.10) of physical effects, produce the time-dependent, arbitrary tensor rate generalization of the viscoelastic thin-filament model of Denn, Petrie, & Avenas [12] as its leading order problem. Furthermore, with this balance of physical effects the theory yields closed systems of equations for corrections to the leading order slender jet flow to any order. Therefore, the time-dependent generalization of the Denn, Petrie, & Avenas model may be regarded as the leading order approximation in a formally valid asymptotic theory.

4.3 Regime 3: A Weakly Elastic, Viscosity Dominated Jet

We now consider a jet in which the length, time, and force scales of the flow and material properties of the fluid are such that

\begin{align}
B &= \tilde{B}\epsilon^{-3}, & F &= \tilde{F}\epsilon^0, & W &= \tilde{W}\epsilon^0, \\
Z &= \tilde{Z}\epsilon^0, & \Lambda_1 &= \tilde{\Lambda}_1\epsilon^1, & \Lambda_2 &= \tilde{\Lambda}_2\epsilon^3,
\end{align}

where $\tilde{B}$, $\tilde{F}$, $\tilde{W}$, $\tilde{Z}$, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$ are $O(1)$. In this physical application the effects of inertia, gravity, surface tension, retardation time, and ambient pressure are negligible.
compared to the effects of viscosity and elasticity; furthermore, the effect of elasticity is small compared to that of viscosity.

The leading order problem in this regime reduces through algebraic constraints to the quasilinear coupled PDE and ODE constraints for $\phi^{(0)}$, $v_z^{0,0}$ and $v_\theta^{0,0}$:

**REGIME 3: Reduced Leading Order Problem**

\[
(\phi^{(0)})^2\varepsilon + (\phi^{(0)} v_z^{0,0}),_z = 0, \quad (4.19a)
\]

\[
(\phi^{(0)} v_z^{0,0}),_z = 0. \quad (4.19b)
\]

\[
(\phi^{(0)} v_\theta^{0,0}),_z = 0. \quad (4.19c)
\]

The remaining leading order unknowns are algebraically determined from the solution of equations (4.19) through

\[
v_r^{0,0} = -\frac{1}{2} v_z^{0,0}, \quad (4.20a)
\]

\[
T_{\theta\theta}^{0,0} = T_{rr}^{0,0} = p^{0,0} = -Z v_z^{0,0}, \quad (4.20b)
\]

\[
T_{zz}^{0,0} = 2Z v_z^{0,0}, \quad (4.20c)
\]

\[
T_{xz}^{0,0} = -\frac{3}{2} Z v_{z,z}^{0,0}, \quad (4.20d)
\]

\[
T_{r\theta}^{0,0} = -\frac{1}{4} Z v_{\theta,z}^{0,0}, \quad (4.20e)
\]

\[
T_{\theta z}^{0,0} = Z v_{\theta,z}^{0,0}. \quad (4.20f)
\]

The leading order problem in this regime involves only viscosity and is hence Newtonian.
Note that in this regime, unlike the previous two regimes, stresses are not primitive unknowns, but are instead determined algebraically through equations (4.20). This is because elasticity is not a leading order effect. Specification of initial and boundary values on stresses is not necessary.

The first order problem in this regime reduces through algebraic constraints to the quasilinear coupled PDE and ODE constraints for $\phi^{(1)} v_z^{0,1}$ and $\phi_{\theta}^{0,1}$:

**REGIME 3: Reduced Problem for First Order Corrections**

\[
\phi_{,z}^{(1)} + \phi_{,z} v_z^{0,0} + \frac{1}{2} v_{z,z}^{0,0} \phi^{(1)} + \frac{1}{2} v_{z,z} \phi_{,z}^{(1)} + v_{z}^{0,1} \phi^{(0)} = 0, \quad (4.21a)
\]

\[
2v_{z,z}^{0,0} \phi_{,z}^{(1)} + 2v_{z,z}^{0,1} \phi_{,z}^{(0)} + v_{z,z}^{0,0} \phi_{,z}^{(1)} + \phi^{(0)} v_{z,z}^{0,1} = h^{0,1}(\phi^{(0)}, v_z^{0,0}), \quad (4.21b)
\]

\[
4v_{\theta,z}^{0,0} \phi_{,z}^{(1)} + 4v_{\theta,z}^{0,1} \phi_{,z}^{(0)} + v_{\theta,z}^{0,0} \phi_{,z}^{(1)} + \phi^{(0)} v_{\theta,z}^{0,1} = q^{0,1}(\phi^{(0)}, v_{\theta}^{0,0}), \quad (4.21c)
\]

with

\[
h^{0,1}(\phi^{(0)}, v_z^{0,0}) = \tilde{\Lambda}_1 \phi^{(0)} [v_{z}^{0,0} + v_{z}^{0,0} v_{z}^{0,0} + 2a v_{z}^{0,0} v_{z}^{0,0} - \alpha(v_{z}^{0,0})^2]. \quad (4.21d)
\]

\[
q^{0,1}(\phi^{(0)}, v_z^{0,0}, v_{\theta}^{0,0}) = 4\tilde{\Lambda}_1 \phi^{(0)} [v_{\theta}^{0,0} + (2 + a) \phi_{,z}^{(0)} - (2 + a) \phi_{,z}^{0,0} - \phi_{,z}^{0,0} \phi_{,z}^{0,0} + v_{z}^{0,0} v_{z}^{0,0} - (2 + a) v_{z}^{0,0} v_{z}^{0,0} + v_{z}^{0,0} v_{z}^{0,0}] \quad (4.21e)
\]

Note that the first order corrections in this regime include elastic effects through the nonhomogeneous forcing terms $h^{0,1}(\phi^{(0)}, v_z^{0,0})$ and $q^{0,1}(\phi^{(0)}, v_{\theta}^{0,0})$. From the solutions of equations (4.19) and (4.21) one can algebraically determine

\[
v_r^{0,1} = -\frac{1}{2} v_{z,z}^{0,1}, \quad (4.22a)
\]

\[
T_{rr}^{0,1} = T_{\theta\theta}^{0,1} = p^{0,1} = -Z v_{z,z}^{0,1} + Z \tilde{\Lambda}_1 [v_{z}^{0,0} + v_{z}^{0,0} v_{z}^{0,0} + \alpha(v_{z}^{0,0})^2], \quad (4.22b)
\]
The second order problem decouples into a “1,0” part and a “0,2” part. In this regime, the “1,0” problem for the radially dependent second order corrections is Newtonian and strictly algebraic:

\[ T_{rz}^{0,1} = -\frac{3}{2} Z v_{x,zz}^{0,0} + \frac{3}{2} Z \tilde{\Lambda}_1 \left[ v_{x,zzt}^{0,0} + v_{x,zz}^{0,0} + (1 - 2a) v_{x,z}^{0,0} v_{x,zzz}^{0,0} \right], \quad (4.22d) \]

\[ T_{rr}^{0,1} = -\frac{1}{4} Z v_{\theta,zz}^{0,0}, \quad (4.22e) \]

\[ T_{\theta z}^{0,1} = Z v_{\theta,z}^{0,0}, \quad (4.22f) \]

The remainder of the second order problem reduces to a coupled PDE and ODE constraints for \( \phi^{(2)} \) and \( v_z^{0,2} \):

\[ v_z^{1,0} = -\frac{1}{2} Z v_{x,zz}^{0,0}, \]
\[ v_r^{1,0} = \frac{1}{8} Z v_{x,zz}^{0,0}, \]
\[ v_\theta^{1,0} = -\frac{1}{8} v_{\theta,zz}^{0,0}, \]
\[ T_{rr}^{1,0} = \frac{3}{4} Z v_{x,zz}^{0,0}, \]
\[ T_{\theta\theta}^{1,0} = p^{1,0} = \frac{1}{4} Z v_{x,zz}^{0,0}, \quad (4.23) \]
\[ T_{zz}^{1,0} = -Z v_{x,zz}^{0,0}, \]
\[ T_{rz}^{1,0} = \frac{5}{16} Z v_{x,zz}^{0,0}, \]
\[ T_{r\theta}^{1,0} = -\frac{1}{6} Z v_{\theta,zz}^{1,0}, \]
\[ T_{\theta z}^{1,0} = Z v_{\theta,z}^{1,0}. \]

REGIME 3: Reduced Second Order Problem — Radially Independent Corrections
\[ \phi^{(2)} + \frac{1}{2} v_{x,z}^{2} \phi^{(2)} + v_{x,z}^{0} \phi^{(0)} + v_{x,z} \phi^{(2)} + \frac{1}{2} v_{x,z}^{0} \phi^{(0)} = -\frac{1}{2} v_{x,z}^{0} \phi^{(1)} + v_{x,z} \phi^{(1)} + \phi^{(2)} \left( \frac{1}{4} v_{x,z}^{0} \phi^{(0)} + v_{x,z} \phi^{(0)} \right), \]  
\tag{4.24a}

\[ 2v_{x,z}^{0} \phi^{(2)} + v_{x,z}^{0} \phi^{(2)} + v_{x,z}^{0} \phi^{(0)} + 2v_{x,z}^{0} \phi^{(0)} = -2v_{x,z}^{0} \phi^{(1)} - v_{x,z}^{0} \phi^{(1)} \]
\[ +\frac{1}{2} v_{x,z}^{0} \left[ \phi^{(0)} + 3 \phi^{(0)} \phi^{(0)} \right] 2v_{x,z}^{0} \phi^{(0)} + \phi^{(0)} + \frac{3}{8} v_{x,z}^{0} \phi^{(0)} \phi^{(0)} \]
\[ + h^{0,2} \left( \phi^{(0)}, v_{x,z}^{0}, \phi^{(1)}, v_{x,z}^{0} \right), \]  
\tag{4.24b}

\[ 4 \phi^{(0)} \phi^{(2)} + 2 \phi^{(2)} \left[ 2 \phi^{(0)} v_{x,z}^{0} + \phi^{(0)} v_{x,z}^{0,0} \right] + 4 \phi^{(0)} \phi^{(0)} v_{x,z}^{0} + \phi^{(0)} v_{x,z}^{0,0} = q^{0,2} \]  
\tag{4.24c}

with

\[ h^{0,2} \left( \phi^{(0)}, v_{x,z}^{0}, \phi^{(1)}, v_{x,z}^{0} \right) \]
\[ = \tilde{\Lambda}_{1} \left\{ \frac{1}{3} \phi^{(0)} \left[ v_{x,z}^{0,0} v_{x,z}^{0,1} + (1 - 10 \alpha) (v_{x,z}^{0,0} v_{x,z}^{0,1} + v_{x,z}^{0,0} v_{x,z}^{0,1}) + v_{x,z}^{0,0} v_{x,z}^{0,1} \right] \right. \]
\[ + \phi^{(0)} \left[ v_{x,z}^{0,0} + (1 - 2 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} + v_{x,z}^{0,0} v_{x,z}^{0,0} \right] + 2 \phi^{(1)} \left[ v_{x,z}^{0,0} + v_{x,z}^{0,0} v_{x,z}^{0,0} \right] \]
\[ - \frac{1}{3} \left\{ \phi^{(0)} \left[ v_{x,z}^{0,0} + 2 v_{x,z}^{0,0} v_{x,z}^{0,0} + (2 - 15 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} \right] \right. \]
\[ + (1 - 15 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} + v_{x,z}^{0,0} v_{x,z}^{0,0} + (1 - 15 \alpha + 21 \alpha^{2}) (v_{x,z}^{0,0}) v_{x,z}^{0,0} \]
\[ + (1 - 15 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} + v_{x,z}^{0,0} v_{x,z}^{0,0} + (1 - 15 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} \]
\[ - \frac{2}{3} \left\{ \phi^{(0)} \left[ v_{x,z}^{0,0} + 2 v_{x,z}^{0,0} v_{x,z}^{0,0} - 15 \alpha v_{x,z}^{0,0} v_{x,z}^{0,0} + v_{x,z}^{0,0} v_{x,z}^{0,0} \right] \right. \]
\[ + (v_{x,z}^{0,0})^{2} v_{x,z}^{0,0} + (1 - 15 \alpha) v_{x,z}^{0,0} v_{x,z}^{0,0} + 7 \alpha^{2} (v_{x,z}^{0,0})^{3} \right\}. \]  
\tag{4.24d}

\[ q^{0,2} \left( \phi^{(0)}, v_{x,z}^{0,0}, \phi^{(1)}, v_{x,z}^{0,0} \right) = -f_{3} - \tilde{\Lambda}_{1} f_{3} \]  
\tag{4.24e}
The radially independent "0,2" second order corrections in this regime involve elasticity, explicitly through the nonhomogeneous forcing terms \( h^{0,2}(\phi^{0}, v_{z}^{0,0}, \phi^{1}, v_{z}^{0,1}) \), \( q^{0,2}(\phi^{0}, v_{\theta}^{0,0}, \phi^{1}, v_{\theta}^{0,1}) \) and implicitly through the elastic first order corrections \( \phi^{(1)} \), \( v_{z}^{0,1} \), and \( v_{\theta}^{0,1} \). From the solutions of equations (4.19), (4.21), and (4.24) one can
algebraically determine

\[ v_r^{0,2} = -\frac{1}{2} v_{z,z}^{0,2}, \]

\[ T_{rr}^{0,2} = T_{\theta\theta}^{0,2} = -Z v_{z,z}^{0,2} - \tilde{\Lambda}_1 \left[ T_{rrz}^{0,1} + v_{z}^{0,0} T_{rr}^{0,1} + v_{z}^{0,1} T_{rr}^{0,0} + a (v_{z}^{0,0} T_{rr}^{0,1} + v_{z}^{0,1} T_{rr}^{0,0}) \right], \]

\[ T_{zz}^{0,2} = 2Z v_{z,z}^{0,2} - \tilde{\Lambda}_1 \left[ T_{zzz}^{0,1} + v_{z}^{0,0} T_{zz}^{0,1} + v_{z}^{0,1} T_{zz}^{0,0} - 2a (T_{zz}^{0,1} + v_{z,z}^{0,1} T_{zz}^{0,0}) \right], \]

\[ p^{0,2} = T_{rr}^{0,2} + \frac{Z}{2} \left( v_{z,zzz}^{0,0} \phi^{(0)}(0)^2 + 3v_{z,z}^{0,0} \phi^{(0)}(0) \phi^{(0)}_r \right), \] \hspace{1cm} (4.25)

\[ T_{r\theta}^{0,2} = \frac{1}{2} (p_{r}^{0,2} - T_{zz}^{0,2}), \]

\[ T_{r\theta}^{0,2} = -\frac{1}{4} T_{z,z}^{0,2}, \]

\[ T_{z,z}^{0,2} = Z v_{z,z}^{0,2} - \Lambda_1 \left( T_{z,z}^{0,1} + \sum_{l=0}^{1} \left[ v_{z}^{0,l} T_{z,z}^{0,1-l} + \frac{1-a}{2} v_{z,z}^{0,l} T_{rr}^{0,1-l} \right. \right. \]

\[ \left. \left. - \frac{1}{2} \left[ v_{z,z}^{0,l} T_{z}^{0,1-l} + v_{z,z}^{0,l} T_{zz}^{0,1-l} \right] \right] \right). \]

Although these first and second order corrections are elastic (all corrections depend on the elastic parameter $\tilde{\Lambda}_1$), the stress corrections in this weakly elastic regime are not primitive unknowns, but rather are determined algebraically through equations (4.22), (4.23), and (4.25) from the leading order velocity and the velocity corrections. The stress corrections do not demand initial or boundary values.

The torsionless steady reduction of the slender jet equations in this regime will be integrated exactly through three orders in the asymptotics (equations (4.20)-(4.25)) in Chapter V.
4.4 Regime 4: The Newtonian Slender Jet

We now consider a regime in which only viscous effects are important through three orders in the asymptotics, and the effects of elasticity, inertia, surface tension, and gravity are assumed to be negligible. Therefore this regime dictates

\[ B = \bar{B} \varepsilon^{-3}, \quad F = \bar{F} \varepsilon^0, \quad W = \bar{W} \varepsilon^0, \]

\[ Z = \bar{Z} \varepsilon^0, \quad \Lambda_1 = \bar{\Lambda}_1 \varepsilon^3 \quad \Lambda_2 = \bar{\Lambda}_2 \varepsilon^3, \] (4.26)

where \( \bar{B}, \bar{F}, \bar{W}, \bar{Z}, \bar{\Lambda}_1, \bar{\Lambda}_2 \) are \( O(1) \).

The "0,0" and "1,0" problems in this regime are identical to those of the regime 3, and are given by equations (4.19) and (4.23), respectively. The "0,1" and "0,2" problems are given by equations (4.21) with \( h^{0,1} = q^{0,1} = 0 \), equation (4.24) with \( h^{0,2} = q^{0,2} = 0 \), and equations (4.22) and (4.25) with \( \bar{\Lambda}_1 = 0 \). The problems for the leading order behavior and first and second order corrections are Newtonian, involving only the single parameter \( Z \).

The complete solution will be given in Chapter V through three orders in the asymptotics of the steady form of these Newtonian slender jet equations.
CHAPTER V

Torsionless Steady Solutions Through Three Orders in the Perturbation Expansion

In this chapter solutions to the torsionless steady equations for the four regimes of Chapter 4 are presented. The torsional effects will be discussed in the next chapter. In this chapter, for simplicity, we will consider the $4 \times 4$ torsionless subsystem of the $6 \times 6$ complete system which includes torsion. Numerical (in Regimes 1 and 2) and closed form (in Regimes 3 and 4) solutions are given through three orders in the perturbation expansions of the free surface, velocity, stresses, and pressure. The essential focus in each physical regime is the predicted slender jet behavior, beginning with the leading order solution, and then proceeding to the role of the corrections in adjusting (and perhaps invalidating) the prediction of the dominant balance equations.

The 1-D slender jet models for Regimes 1-4 have been selected both for their potential relevance to engineering processes, and to illustrate some critical issues concerning physically relevant solutions of these 1-D steady models. In particular, the issues of asymptotic validity of leading order solutions to these models and appropriate boundary conditions for steady state solutions are raised. Elements of this comprehensive, higher order perturbation theory will be used to address these practical issues.

Before proceeding to solutions, the following remarks seem appropriate.
**Remark 1: Asymptotic validity of the leading-order solutions**

The formal perturbation theory outlined in the previous chapters allows one to conclude that the equations are *formally* consistent for slender jet approximations in Regimes 1-4. Formal consistency, however, does not guarantee that the corresponding solutions remain asymptotically valid. This condition has to be checked on a solution-by-solution basis. A check on asymptotic validity has not been done before for slender jets; as it is illustrated here, however, both the higher order equations and their solutions are now accessible so that one can reasonably check this condition.

Leading order solutions will be computed first, and then for each solution first and second order corrections are computed to verify whether the assumed asymptotic expansion is valid uniformly along the length of the jet. As will be illustrated, the corrections may be influenced by a variety of physical effects. In each case we proceed as follows:

(i) We first calculate the simplest corrections, which assume that the corrections are not subject to boundary fluctuations, and that only the leading order physical effects are present in the corrections (so that no new weak effects enter into the corrected solutions).

(ii) If these corrections remain uniformly small relative to the leading-order solution, I proceed to compute corrections due to boundary fluctuations, still without weak effects.

(iii) If these corrections remain ordered, we then proceed to compute corrections due to new weak effects, absent in the leading order model but entering to first or second
order. These corrections are more complicated, but physically quite important.

We shall present solutions to illustrate that these studies are not purely academic. Some leading order steady solutions will be shown to be asymptotically valid even in the presence of small boundary perturbations and in the presence of weak physical effects. For applications to engineering processes such as fiber spinning, if solutions remain asymptotically valid for all types of corrections described here, one confidently can predict these steady states will be robust (i.e., able to survive physical perturbations and therefore physically realizable). On the other hand, we also give an example where the asymptotic validity is destroyed by a new but weak physical effect.

**Remark 2:** Why is it computed to third order?

This somewhat arbitrary choice is made for two reasons. First, it is reasonable and yet not trivial to check not just one, but two orders of corrections. Second, this is the minimum order necessary to capture radial variations in axial velocity, normal stresses, and pressure. It turns out that the radial variation is essential for the perturbation expansions to capture the some important feature of original 3-D model. (One example is that the inviscid jet is stable for large wave numbers. See [6] for details.)

**Remark 3:** The selection of steady boundary conditions

It is important from the viewpoint of engineering relevance to impose well-posed boundary conditions for the 1-D steady models. Choices of boundary conditions made from viewing the steady equations in isolation run the risk of unknowingly predicting highly unstable and thus physically unattainable steady states. As presented in
Chapter IV, the steady 1-D models are embedded in time-dependent 1-D models. One can use the time-dependent PDEs to infer the well-posed steady boundary conditions and with them calculate only steady solutions which have a chance to be temporally stable. This observation was first applied to 1-D slender viscoelastic jet equations by Boris & Liu [9].

The number of conditions which must be imposed at the left and/or right boundaries, as well as the nature of these conditions, may be deduced for Regimes 1-4 of Chapter IV. (See Forest & Wang [14] for details.) At each order and for each regime in Chapter IV, the time-dependent problems consist of a system of quasilinear PDEs. The method of characteristics allows us to classify the equations based on the values and signs of the characteristics. Recall that the four characteristic velocities for the 4 × 4 leading order PDE system (5.1) of the strongly elastic Regime 1 are explicitly

\[ \frac{dz}{dt} = s_j, \quad j = 1, 2, 3, 4, \]

\[ s_1 = s_2 = v_z^{0,0}, \]

\[ s_{3,4} = v_z^{0,0} \pm \sqrt{\text{disc}_1}, \]

\[ \text{disc}_1 = B(a + 1)T_{rr}^{0,0} + B(2a - 1)T_{zz}^{0,0} + \frac{3BZ}{\Lambda_1} - \frac{1}{2W\phi^{(0)}}. \]

When all characteristics are real (and the system is not degenerate), the equations are genuinely hyperbolic and of evolutionary type. Positive characteristics carry information propagating in the positive axial direction (to the right by the convention). It follows that for each positive characteristic, one condition has to be specified at the left boundary. Likewise, each negative characteristic requires boundary data at
the right in order to define the solution in the interior for future times. A well-posed steady problem must have boundary conditions consistent with the PDE problem. Precisely which data can be posed at each boundary is also deduced from the characteristic analysis.

For applications to a "take-up" fiber spinning process, a slender jet model is required to have both positive and negative characteristics so that boundary conditions are allowed at the nozzle and at the take-up spool. The strongly elastic Regimes 1 and 2 will be studied below with boundary conditions relevant for a take-up spinning process.

To focus on fiber spinning applications, examples have been selected in Regime 1 for which the leading order 4 × 4 subsystem of PDE system (5.1) is hyperbolic of type (3,1). This means that there are three positive and one negative characteristic speeds. We have carefully explored for parameter and boundary values which predict steady solutions that are of hyperbolic type (3,1) everywhere along the length of the jet. In other words, the "good" solutions reported here are the result of a careful study.

It is noted that for different choices of parameter and boundary values the characteristics (5.1) may all be positive, so that the slender jet equations are of hyperbolic type (4,0). The corresponding steady solutions would be unstable to imposed take-up conditions at the right boundary, and thus inappropriate for fiber spinning processes. Such solutions would be appropriate instead for extrusion processes.

There is another caution about the risk of ignoring the time-dependent equations
when imposing boundary conditions for steady solutions. Two of the characteristics (5.1) may be complex, in which case the system is said to be of mixed hyperbolic-elliptic type. If so, steady states will be catastrophically unstable. We give an example in Figure 6 of a steady solution which appears to be quite physically reasonable, with the corrections remaining small, but for which the PDE characteristics are complex valued. A quasilinear PDE system with complex characteristics is ill-posed as an evolutionary system; the corresponding steady solutions are catastrophically unstable and hence physically unrealizable.

Remark 4: Exact integration of the steady mass flux conservation equations

The steady systems, at each order in the asymptotics and in any regime, can be reduced by one equation from the time-dependent systems. This is because the first equation in each of the "0,n" problems, n = 0, 1, ..., is the same for all regimes and is a mass flux conservation law, derived from the incompressibility constraint (2.1) and the kinematic free surface boundary condition (2.7). Therefore, the spatial derivatives can be given as exact derivatives and the steady forms can be integrated. For instance, the first equation of the "0,0" problem in each of the four regimes in Chapter IV is

$$\phi_{t}^{(0)} + v_{z}^{0,0} \phi_{z}^{(0)} + \frac{1}{2} \phi^{(0)} v_{z,z}^{0,0} = 0,$$

the first equation of the "0,1" problem in each regime is

$$\phi_{t}^{(1)} + \phi_{z}^{(1)} v_{z}^{0,0} + \frac{1}{2} v_{z,z}^{0,1} \phi^{(0)} + \frac{1}{2} v_{z,z}^{0,0} \phi^{(1)} + v_{z,z}^{0,1} \phi^{(0)} = 0.$$
and the first equation in each regime of the "0,2" problem is

\[
\phi^{(2)}_{zz} + \frac{1}{2} v_z^{0,0} \phi^{(2)} + v_z^{0,0} \phi^{(2)}_{zz} + \frac{1}{2} v_z^{0,2} \phi^{(0)} = -\frac{1}{2} v_z^{0,1} \phi^{(1)} + v_z^{0,1} \phi^{(1)} + \left( \frac{1}{4} v_z^{1,0} \phi^{(0)} + v_z^{1,0} \phi^{(0)} \right) \phi^{(0)}^2.
\]  

(5.4)

In the steady state case these equations (5.1), (5.2), (5.3) can be integrated to yield

\[
v_z^{0,0} \phi^{(0)} = C_0, \quad (5.5)
\]

\[
v_z^{0,1} \phi^{(0)} + 2 v_z^{0,0} \phi^{(0)} \phi^{(1)} = C_1, \quad (5.6)
\]

\[
v_z^{0,2} \phi^{(0)}^2 + 2 v_z^{0,0} \phi^{(0)} \phi^{(2)} + v_z^{0,0} \phi^{(1)} = C_2, \quad (5.7)
\]

where \(C_0, C_1,\) and \(C_2\) are constants. These kinematical algebraic constraints allow us in steady problems to eliminate \(\phi^{(0)}, \phi^{(1)},\) and \(\phi^{(2)},\) in favor of \(v_z^{0,0}, v_z^{0,1},\) and \(v_z^{0,2},\) or vice versa.

In the following solutions the constants of integration (5.5), (5.6), (5.7) are necessary to obtain the analytical solutions (§5.3 and §5.4), but are employed only as checks to the numerical integrations (§5.1 and §5.2).

### 5.1 Solutions in Regime 1

The first examples consist of solutions in Regime 1. Consistent with the characteristic analysis above, we exhibit solutions which satisfy "take-up" fiber spinning boundary conditions. The domain of the fiber in all solutions will be taken from \(z = 0\) to \(z = 1.\) This choice in effect corresponds to selecting the axial length scale \(z_0\) in (3.1) to be the length of the fiber. From the characteristic analysis it is found that the
following particular conditions are well-posed for solutions of hyperbolic type (3.1).
Steady boundary conditions for the leading order problem (5.1) of Regime 1

left: \( v_z^{0,0}(0), \phi(0), T_{zz}^{0,0}(0), \)

right: \( v_z^{0,0}(1). \)  \hspace{1cm} (5.8)

In all integrations,

\[ \phi(0) = 1, \quad v_z^{0,0}(0) = 1, \quad Z = 1 \]  \hspace{1cm} (5.9)

are specified. These conditions in effect select the transverse length scale \( r_0 \) in (3.1),
the characteristic velocity \( v_0 \) in (3.6), and the characteristic force \( f_0 \) in (3.5) to be
the radius of the fiber at its left boundary, the cross-sectional average axial velocity
at the left boundary, and the viscous force \( \eta r_0^2 / t_0 \), respectively.

The remaining two free boundary conditions in (5.8) consist of an initial axial
stress \( T_{zz}^{0,0}(0) \) and the take-up velocity \( v_z^{0,0}(1) \). The modeling of the fiber spinning
process now reduces to studying the variation in steady solutions due to variations
in \( T_{zz}^{0,0}(0) \) and \( v_z^{0,0}(1) \) and remaining material parameters, subject to the constraint
that the characteristics (5.1) consist of three positive and one negative directions for
all \( 0 < z < 1 \).

Figure 1 corresponds to leading order solutions for the choice

\[ T_{zz}^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 1.10, \]  \hspace{1cm} (5.10a)

\[ B = F = \Lambda_1 = W = a = 1, \quad \bar{\Lambda}_2 = 0. \]  \hspace{1cm} (5.10b)
We plot the leading order solutions of \( \phi^{(0)}(z) \), \( v_z^{0,0}(z) \), and \( F_z^{(0)}(z) \), which are the jet radius, axial velocity, and axial force, respectively. In general, the axial force \( F_z \) is defined by

\[
F_z = \int_{\text{cross section}} (T_{zz} - p) da. \tag{5.11}
\]

To leading order in dimensionless form this becomes

\[
F_z^{(0)} = (T_{zz}^{0,0} - p^{0,0}) \phi^{(0)}^2. \tag{5.12}
\]

Next we study the higher order corrections for the solution with boundary and parameter values (5.10), depicted in Figure 1. Well-posed boundary conditions for these corrections follow from the leading order solutions because equations (5.3) and (5.7) have the same characteristics as the leading order problem (5.1). In particular, we choose the following boundary conditions:

**Boundary conditions for the first order "0,1" problem**

left: \( v_z^{0,1}(0) \), \( \phi^{(1)}(0) \), \( T_{zz}^{0,1}(0) \),

right: \( v_z^{0,1}(1) \). \tag{5.13}

**Boundary conditions for the second order "1,0" and "0,2" problems**

Problem (5.5) for the radially dependent "1,0" corrections

\[
T_{rr}^{1,0}(0), \quad T_{\theta\theta}^{1,0}(0), \quad T_{zz}^{1,0}(0). \tag{5.14}
\]
(This 3 x 3 system has the single positive characteristic \( u_z^{0,0} \), so these are the only allowable conditions.)

Problem (5.7) for the radially independent "0,2" corrections

left: \( u_z^{0,2}(0), \phi^{(2)}(0), T_{zz}^{0,2}(0) \),

right: \( u_z^{0,2}(1) \).

In all computations,

\[ \varepsilon = 0.1 \]  

is specified. Selecting the slenderness ratio \( \varepsilon = r_0/z_0 \) to be one tenth corresponds to a fiber whose length at take-up is ten times its initial radius. It is noted that in industrial fiber spinning processes the ratio of take-up length to initial radius is much larger than ten. To model these processes one must either take \( \varepsilon \) much smaller than 0.1 or integrate beyond one unit in the dimensionless coordinate \( z \).

The higher order quantities displayed in Figures 2, 3, and 4 are:

a) The corrected free surface profile, computed to \( O(\varepsilon^2) \),

\[ \phi(z) \sim \phi^{(0)}(z) + \varepsilon \phi^{(1)}(z) + \varepsilon^2 \phi^{(2)}(z). \]  

b) The higher order correction \( \phi_{\text{correction}}(z) \) to the leading order prediction \( \phi^{(0)}(z) \) of Figure 1,

\[ \phi_{\text{correction}}(z) = \phi(z) - \phi^{(0)}(z). \]  

c) The area-averaged axial velocity over the jet cross section at location \( z \).

From the expansion (4.1a) of \( v_z(r,z) \), to \( O(\varepsilon^2) \) this is easily shown to be

\[ v_z^{av}(z) \sim v_z^{0,0}(z) + \varepsilon v_z^{0,1}(z) + \varepsilon^2 v_z^{0,2}(z) + \frac{1}{2} v_z^{1,0}(z)[\phi^{(0)}(z)]^2. \]
(For comparison we also plot the leading order axial velocity \( v_z^{0,0} \).)

d) The difference \( v_{\text{correction}}^{\text{avg}}(z) \) between the corrected \( v_z^{\text{avg}}(z) \) and the leading order axial velocity \( v_z^{0,0}(z) \) from Figure 1,

\[
v_{\text{correction}}^{\text{avg}}(z) = v_z^{\text{avg}}(z) - v_z^{0,0}(z).
\]

(5.20)

e) The axial force \( F_z \) through \( O(\varepsilon^2) \), which from definition (5.5) is given by

\[
F_z \sim F_z^{(0)} + \varepsilon F_z^{(1)} + \varepsilon^2 F_z^{(2)}
\]

\[
= (T_{zz}^{0,0} - p^{0,0})\phi^{(0)} + \varepsilon \left[ (T_{zz}^{0,0} - p^{0,0})2\phi^{(0)} + (T_{zz}^{0,1} - p^{0,1})\phi^{(1)} \right] \\
+ \varepsilon^2 \left[ (T_{zz}^{0,0} - p^{0,0})(2\phi^{(0)}\phi^{(2)} + \phi^{(1)}) + (T_{zz}^{0,1} - p^{0,1})2\phi^{(0)}\phi^{(1)} \right] \\
+ (T_{zz}^{0,2} - p^{0,2})\phi^{(0)} + \frac{1}{2}(T_{zz}^{1,0} - p^{1,0})\phi^{(1)}).
\]

(5.21)

(For comparison the leading order axial force \( F_z^{(0)} \) is also plotted.)

In Figure 2, the corrections are computed with \( \tilde{A}_2 = 0 \) in the "0,2" problem (5.7) and with zero boundary perturbations, defined for the purposes by

\[
\phi_{\text{correction}}(0) = 0,
\]

\[
v_{\text{correction}}^{\text{avg}}(0) = v_{\text{correction}}^{\text{avg}}(1) = 0,
\]

\[
T_{zz}^{0,1}(0) = T_{zz}^{0,2}(0) = T_{rr}^{1,0}(0) = T_{\theta\theta}^{1,0}(0) = T_{zz}(0) = 0.
\]

(5.22)

(Alternatively, one could define kinetic boundary perturbations in terms of forces, e.g., \( F_{\text{correction}}(0) \), instead of the above stress perturbations.)

In Figure 3, the corrections are computed, again with \( \tilde{A}_2 = 0 \), but with nonzero boundary perturbations specified by

\[
\phi_{\text{correction}}(0) = 0,
\]
In Figure 4, the corrections are computed with zero boundary perturbations, but \( A_2 = 1 \); i.e., we study the corrections due to a weak retardation effect.

**Interpretation of Regime 1 Results**

With the calculations exhibited in Figures 2, 3, and 4 (and many additional calculations not shown), we have demonstrated that the leading order steady jet solution of Figure 1 is asymptotically valid and robust to boundary perturbations and weak retardation effects. *Therefore one can infer that the steady jet behavior shown in Figure 1 is physical.* Recall that viscosity, elasticity, inertia, gravity, and surface tension are all dominant influences on this slender jet.

We have presented an examination of one leading order solution, although the studies have produced and examined many more. As stated earlier, the leading order modeling of the fiber spinning process corresponds to studying the changes in the solution of Figure 1 as the axial stress at the nozzle and take-up velocity (boundary conditions (5.10)) and process parameters \( \Lambda_1, B, W, F \) are varied. To illustrate the possible spectrum of leading order behavior due to variations in the process variables, Figure 5 presents the analog of Figure 1 corresponding to

\[
T_{zz}^{0,0}(0) = 0.50, \quad \nu_z^{0,0}(1) = 1.25, \quad B = \Lambda_1 = a = 1, \quad F = W = 5, \quad \Lambda_2 = 0. \quad (5.24)
\]
We simply state that it is found that this solution is asymptotically valid with and without boundary perturbations or weak retardation effects, and the detailed graphs are omitted.

We note the qualitative difference in the jet radius $\phi^{(0)}$, axial velocity $v_z^{0,0}$, and axial force $F_z^{(0)}$ between the two leading order solutions, Figures 1 and 5, each predicted by the theory to be a physically realizable steady state.

As a final example in this regime, we present a calculation that predicts a steady solution which appears acceptable, when viewed in the context of the 1-D ODE steady model in isolation, but is in fact unphysical, in the sense that the steady solution cannot be realized in an experiment. Figure 6 exhibits a leading order solution which is asymptotically valid, since the corrections are small. However, for the values of $\phi^{(0)}$, $v_z^{0,0}$, $T_{rr}^{0,0}$, and $T_{zz}^{0,0}$ in this calculation, two of the four characteristics (5.1) are complex on the left side of the domain (see Figure 6i). The “0,0”, “0,1”, and “0,2” systems are then of mixed hyperbolic-elliptic type, and all steady solutions are catastrophically unstable (short wavelength perturbations have unbounded growth rates).

5.2 Solutions in Regime 2

The next computations are in Regime 2. The leading order equations (5.11) in this regime describe viscosity and elasticity dominated slender jets; their torsionless steady form reproduces the well-known Denn, Petrie, & Avenas (DPA) fiber spinning equations. The corrections in Regime 2 have been selected to incorporate surface tension at second order, so we can test asymptotic validity of DPA solutions in the presence of this weak effect.
Before computing solutions, it is necessary to determine the allowable boundary conditions. As discussed in Remark 3 above, one must appeal to the PDE system to define well-posed boundary conditions. However, the time-dependent generalization (5.11) of the DPA model is singular, since one equation lacks a time derivative. As a result, it is possible to infer only three of the four required boundary conditions from this isolated model. The following further analysis is necessary in order to determine well-posed boundary conditions. A detailed discussion of well-posed boundary conditions for this regime can be found in [4]. The boundary conditions chosen for Regime 2 here are similar to those of Regime 1 except at the right take-up force instead of take-up velocity is imposed.

The first example from Regime 2 is a solution to the leading order steady equations (5.11) equipped with boundary conditions (5.8). Again we fix \( \psi^{(0)}(0) = 1, \) \( v_z^{0,0}(0) = 1, \) and \( Z = 1, \) as in (5.9). The process is modeled at leading order by specification of axial stress, take-up velocity, and the remaining process parameters. Figure 7 corresponds to the choice

\[
T_{zz}^{0,0}(0) = 3.50, \quad F_z^{(0)}(1) = 2.22, \quad \Lambda_1 = a = 1.
\]

(5.25)

As in Figures 1 and 5, we plot the leading order predictions for free surface position, axial velocity, and axial force. We note that in the steady form of Regime 2, the leading order axial force \( F_z^{(0)} \) is a constant of integration.

We now compute the corrections to this leading order solution, so as to address the question:
Are solutions to the Denn, Petrie, & Avenas model asymptotically valid approximations to full 3-D solutions through three orders in a slenderness expansion?

We address asymptotic validity of the solution of Figure 7 in the same manner as in Regime 1: First, we compute corrections through $O(\varepsilon^2)$ without boundary perturbations and without any weak effects (Figure 8). Second, we compute corrections due to nonzero boundary perturbations, still in the absence of weak effects (Figures 9 and 10). Finally, we compute the corrections due to a weak effect, here surface tension (Figure 11). The same quantities are plotted as in Figures 2, 3, and 4. We again take $\varepsilon = 0.1$.

Interpretation of Regime 2 Results

With the calculations of Figures 8, 9, and 10 (and many additional perturbation calculations not shown), it has been demonstrated that for the leading order solution of the DPA model in Figure 7 the answer to the above question is affirmative: The solution is valid through three orders both without (Figure 8) and with (Figures 9 and 10) boundary perturbations. It also has been found that the solution is likewise robust to the weak effects of surface tension (shown in Figure 11) and inertia, gravity, and retardation (through calculations similar to Figure 11, not exhibited).

To illustrate more clearly how weak effects are modeled in the theory, we present the following discussion. Suppose the material properties and flow conditions are such that

$$\Lambda_1 = \frac{\lambda_1}{t_0} = 1.00, \quad (BW)^{-1} = \frac{\sigma t_0^2}{r_0 z_0^2} = 0.05.$$  \hspace{1cm} (5.26)

The nondimensional parameters in (5.26) must be scaled in the slenderness parameter
\( \varepsilon; \) when \( \varepsilon = 0.1 \) there are two possibilities:

\[
\Lambda_1 = \tilde{\Lambda}_1 \varepsilon^0, \quad (BW)^{-1} = (\tilde{B}\tilde{W})^{-1} \varepsilon^2,
\]

(5.27)

with

\[
\tilde{\Lambda}_1 = 1, \quad (\tilde{B}\tilde{W})^{-1} = 5
\]

or

\[
\Lambda_1 = \tilde{\Lambda}_1 \varepsilon^0, \quad (BW)^{-1} = (\tilde{B}\tilde{W})^{-1} \varepsilon^1
\]

(5.28)

with

\[
\tilde{\Lambda}_1 = 1, \quad (\tilde{B}\tilde{W})^{-1} = 0.5
\]

In alternative (5.27), the weak surface tension first appears in the "0,2" problem. (This is Regime 2 of Chapter IV). In alternative (5.28), surface tension first appears in the "0,1" problem. Not surprisingly, the predictions of the velocity corrections by both models are similar; both are shown in Figure 11. With no boundary perturbations to the leading order profile, it is observed that the corrections due to weak surface tension are very small. In addition, without weak surface tension the higher order corrections increase the velocity (\( v_{\text{correction}}^{\text{avg}} > 0 \) in Figure 8d), but with weak surface tension the corrections decrease the leading order velocity prediction (\( v_{\text{correction}}^{\text{avg}} < 0 \) in Figure 11b).

We have demonstrated the asymptotic validity and robustness of one DPA solution; other solutions can be examined on a solution-by-solution basis. We comment that all of the many different DPA solutions we have studied have proved asymptotically valid and robust to boundary perturbations and weak effects.
5.3 Newtonian Slender Jet Behavior

The steady equations for the slender Newtonian jet (Regime 4 of Chapter IV) admit closed form integration to any order in the perturbation. We present the complete torsionless solution through three orders in this section.

**Leading order solution**

Recall that the time dependent problem for leading order torsionless Newtonian jet behavior consists of one quasilinear PDE and one ODE for $v^0_\varphi$ and $\phi^{(0)}$, equations (5.19a,b). In the steady case, equation (5.19a) can be integrated to give the algebraic constraint (5.5). Using this constraint to eliminate $\phi^{(0)}$, the remaining equation of the Newtonian leading order problem, equation (5.19b), becomes

$$v^0_{zz}v^0_{zz} - (v^0_{zz})^2 = 0. \quad (5.29)$$

This can be integrated to give the solution

$$v^0_z(z) = A_0 e^{\alpha_0 z}, \quad (5.30a)$$

where $A_0$ and $\alpha_0$ are constants. The remainder of the leading order solution then follows from (5.20) and (5.5):

$$\phi^{(0)} = \left(\frac{C_0}{A_0}\right)^{1/2} e^{-\alpha_0 z/2}, \quad v^0_r = -\frac{1}{2} \alpha_0 A_0 e^{\alpha_0 z},$$

$$T^0_{rr} = T^0_{\theta\theta} = p^0 = -Z \alpha_0 A_0 e^{\alpha_0 z}, \quad (5.30b-h)$$

$$T^0_{zz} = 2Z \alpha_0 A_0 e^{\alpha_0 z}, \quad T^0_{r\theta} = -\frac{3}{2} Z \alpha_0^2 A_0 e^{\alpha_0 z}.$$

Note that the complete steady Newtonian leading order solution (5.30) involves the three constants of integration $A_0$, $\alpha_0$, and $C_0$, and single material parameter $Z$. In
the following we will retain this parameter, but we note that without loss of generality the force scale \( f_0 \) in (3.5) can be chosen as the viscous force \( \eta r_0^2/t_0 \). With this choice of \( f_0 \) we have \( Z \equiv 1 \).

First order corrections

Recall that the time dependent problem for the Newtonian \( O(\varepsilon) \) corrections \( v_z^{0,1} \) and \( \phi^{(1)} \) to the leading order Newtonian flow consists of the two coupled PDEs, equations (4.21a,b) with \( h^{0,1} \) set to zero. The functions \( v_z^{0,0} \) and \( \phi^{(0)} \) are now given by the leading order solution (5.30), so that the coefficients in these equations are known functions of \( z \) and the constants \( A_0, \alpha_0, \) and \( C_0 \).

In the steady case the algebraic constraint (5.6) reduces the first order problem to a single ODE for \( v_z^{0,1} \), which integrates to

\[
v_z^{0,1}(z) = (A_1 + B_1 z)e^{\alpha_0 z}, \quad (5.31a)
\]

where \( A_1 \) and \( B_1 \) are constants. The remainder of the first order solution then follows as

\[
\phi^{(1)} = \left[ \frac{C_1}{2(C_0 A_0)^{1/2}} - \frac{C_0^{1/2}}{A_0^{3/2}}(A_1 + B_1 z) \right] e^{-\alpha_0 z/2}, \quad (5.31b)
\]

\[
v_r^{0,1} = -\frac{1}{2} \left[ \alpha_0 A_1 + (1 + \alpha_0 z)B_1 z \right] e^{\alpha_0 z}, \quad (5.31c)
\]

\[
T_{rr}^{0,1} = T_{\theta\theta}^{0,1} = p^{0,1} = Z \left[ \alpha_0 A_1 + (1 + \alpha_0 z)B_1 z \right] e^{\alpha_0 z}, \quad (5.31d-f)
\]

\[
T_{zz}^{0,1} = 2Z \left[ \alpha_0 A_1 + (1 + \alpha_0 z)B_1 z \right] e^{\alpha_0 z}, \quad (5.31g)
\]

\[
T_{rz}^{0,1} = -\frac{3}{2} Z \left[ \alpha_0 A_1 + (1 + \alpha_0 z)B_1 z \right] \alpha_0 e^{\alpha_0 z}. \quad (5.31h)
\]

Note that the complete steady Newtonian first order solution (5.31) involves the three constants \( A_0, \alpha_0, \) and \( C_0 \) from the leading order solution and the three additional
constants $A_1$, $B_1$, $C_1$.

Second order corrections

Recall from Chapter IV that the time dependent problem for the Newtonian $O(\varepsilon^2)$ corrections decouples into the algebraic problem (5.23) for the $O(\varepsilon r^2)$ (or "1,0") corrections, and the two coupled PDEs, equations (5.24a,b) with $\psi^{0,2}$, for the $O(\varepsilon r^0)$ (or "0,2") corrections of $v_x^{0,2}$ and $\phi^{(2)}$. From equations (5.23) and the Newtonian leading order solution (5.30) it follows

$$v_x^{1,0} = -\frac{1}{2} \alpha_0^2 A_0 e^{\alpha_0 z}, \quad v_r^{1,0} = \frac{1}{8} \alpha_0^3 A_0 e^{\alpha_0 z},$$

$$T_{rr}^{1,0} = \frac{3}{4} Z \alpha_0^3 A_0 e^{\alpha_0 z}, \quad T_{\theta\theta}^{1,0} = p^{1,0} = \frac{1}{4} Z \alpha_0^3 A_0 e^{\alpha_0 z}, \quad (5.32)$$

$$T_{zz}^{1,0} = -Z \alpha_0^3 A_0 e^{\alpha_0 z}, \quad T_{rr}^{1,0} = \frac{5}{16} Z \alpha_0^4 A_0 e^{\alpha_0 z}.$$  

In the steady case the algebraic constraint (5.7) reduces the "0,2" problem to a single ODE for $v_x^{0,2}$, which integrates to

$$v_x^{0,2} = \frac{1}{8} \alpha_0^2 C_0 + \left[ A_2 + B_2 z + \frac{B_1^2}{2A_0} z^2 \right] e^{\alpha_0 z}. \quad (5.33a)$$

The remainder of the "0,2" solutions follow from this as

$$\phi^{(2)} = \frac{\alpha_0^2 C_0^{3/2}}{16 A_0^{3/2}} e^{-3\alpha_0 z/2} + \frac{1}{8 A_0^{5/2} C_0^{1/2}} \left[ C_0 B_1^2 z^2 - 2(2A_0 B_2 C_0 + A_0 C_1 B_1 - 3A_1 C_0 B_1) z - 4A_0(A_2 C_0 + A_0 C_2) \right] e^{-\alpha_0 z/2}, \quad (5.33b)$$

$$v_r^{0,2} = -\frac{1}{4 A_0} \left[ 2A_0 A_2 \alpha_0 + 2A_0 B_2 + 2A_0 \alpha_0 B_2 z + 2B_1^2 z + \alpha_0 B_1^2 z^2 \right] e^{\alpha_0 z}, \quad (5.33c)$$

$$T_{rr}^{0,2} = T_{\theta\theta}^{0,2} =$$

$$-\frac{Z}{2A_0} \left[ 2A_0 A_2 \alpha_0 + 2A_0 B_2 + 2A_0 \alpha_0 B_2 z + 2B_1^2 z + \alpha_0 B_1^2 z^2 \right] e^{\alpha_0 z}. \quad (5.33d, e)$$
Note that the complete steady Newtonian second order solution (5.32), (5.33) involves the nine constants $A_0$, $\alpha_0$, $C_0$, $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$.

**General solution for the Newtonian jet through three orders in the asymptotics**

The complete solution for the Newtonian jet through $O(\varepsilon^2)$ can now be assembled from the solutions (5.30)-(5.33), and the perturbation expansions (4.1), (4.2), and (4.3). In particular, the axial velocity and free surface radius are (in dimensionless form):

\[
v(r, z) = v_r^{0,0}(z) + \varepsilon v_r^{0,1}(z) + \varepsilon^2 \left[ r^2 v_r^{1,0}(z) + v_r^{0,2}(z) \right] + O(\varepsilon^3)
\]

\[
= \varepsilon^{\alpha z} \left\{ A_0 + \varepsilon (A_1 + B_1z) + \varepsilon^2 \left[ -\frac{1}{2} \alpha_0^2 A_0 r^2 
\right. 
\right. 
\left. 
+ (A_2 + B_2z + \frac{B_1^2}{2A_0} z^2 + \frac{1}{8} \alpha_0^2 C_0 e^{-\alpha z}) \right] + O(\varepsilon^3) \right\}, 
\]

**5.34a**

\[
\phi(r, z) = \phi^{(0)}(z) + \varepsilon \phi^{(1)}(z) + \varepsilon^2 \phi^{(2)}(z) + O(\varepsilon^3)
\]

\[
= e^{-\alpha z/2} \left\{ \left( \frac{C_0}{A_0} \right)^{1/2} + \varepsilon \left[ \frac{C_1}{2(C_0 A_0)^{1/2}} - \frac{C_0^{1/2}}{A_0^{3/2}} (A_1 + B_1z) \right] 
+ \varepsilon^2 \left( \frac{\alpha_0^2 C_0^{3/2}}{16 A_0^{3/2}} e^{-\alpha z} + \frac{1}{8 A_0^{5/2} C_0^{1/2}} \right] C_0 B_1^2 z^2 - 2(2A_0 B_2 C_0 
\]

Note that the complete steady Newtonian second order solution (5.32), (5.33) involves the nine constants $A_0$, $\alpha_0$, $C_0$, $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$. 

**General solution for the Newtonian jet through three orders in the asymptotics**

The complete solution for the Newtonian jet through $O(\varepsilon^2)$ can now be assembled from the solutions (5.30)-(5.33), and the perturbation expansions (4.1), (4.2), and (4.3). In particular, the axial velocity and free surface radius are (in dimensionless form):

\[
v(r, z) = v_r^{0,0}(z) + \varepsilon v_r^{0,1}(z) + \varepsilon^2 \left[ r^2 v_r^{1,0}(z) + v_r^{0,2}(z) \right] + O(\varepsilon^3)
\]

\[
= \varepsilon^{\alpha z} \left\{ A_0 + \varepsilon (A_1 + B_1z) + \varepsilon^2 \left[ -\frac{1}{2} \alpha_0^2 A_0 r^2 
\right. 
\left. 
+ (A_2 + B_2z + \frac{B_1^2}{2A_0} z^2 + \frac{1}{8} \alpha_0^2 C_0 e^{-\alpha z}) \right] + O(\varepsilon^3) \right\}, 
\]

**5.34a**

\[
\phi(r, z) = \phi^{(0)}(z) + \varepsilon \phi^{(1)}(z) + \varepsilon^2 \phi^{(2)}(z) + O(\varepsilon^3)
\]

\[
= e^{-\alpha z/2} \left\{ \left( \frac{C_0}{A_0} \right)^{1/2} + \varepsilon \left[ \frac{C_1}{2(C_0 A_0)^{1/2}} - \frac{C_0^{1/2}}{A_0^{3/2}} (A_1 + B_1z) \right] 
+ \varepsilon^2 \left( \frac{\alpha_0^2 C_0^{3/2}}{16 A_0^{3/2}} e^{-\alpha z} + \frac{1}{8 A_0^{5/2} C_0^{1/2}} \right] C_0 B_1^2 z^2 - 2(2A_0 B_2 C_0 
\]
\[ +A_0 C_1 B_1 - 3A_1 C_0 B_1 \right) z - 4A_0 (A_2 C_0 + A_0 C_2) \right] + O(z^3) \right) \quad (5.34b). \]

**Solutions of Newtonian jet boundary value problems**

In a particular application the constants \( A_0, \alpha_0, C_0, A_1, B_1, C_1, A_2, B_2, C_2 \) in the general solution (5.34) of the steady Newtonian slender jet equations are determined from boundary conditions on axial velocity and free surface radius. The form (5.34) of the general solution dictates the class of admissible boundary conditions: the solution of the steady *three-dimensional* Newtonian jet problem can be approximated by the asymptotic form (5.34) *only if* the boundary conditions in the three dimensional problem are consistent with the form (5.34).

Schultz & Davis [22] also model slender Newtonian jets. Their analysis differs from the analysis here in that they solve a particular boundary value problem, as opposed to obtaining the general solution. The boundary conditions they impose are

\[ \text{at } z = 0 : \quad \phi = 1, \quad v_z^{av} = \int_{\text{cross section}} v_z \, da = 1, \quad (5.35a,b) \]

\[ \text{at } z = 1 : \quad v_z^{av} = \int_{\text{cross section}} v_z \, da = e^\alpha. \quad (5.35c) \]

The boundary conditions (5.35a,b) amount to the selection of length scale \( r_0 \) and velocity scale \( v_0 \) in (3.1) and (3.6). The boundary condition (5.35c) models the take-up problem in the industrial process of fiber spinning. By specifying the take-up location to be at dimensionless axial position \( z = 1 \) they have selected the length scale \( z_0 \) in (3.2). Schultz & Davis [22] also set the material parameter \( Z = 1 \), which is equivalent to selecting the force scale \( f_0 \) of (3.5) to be the viscous force \( \eta r_0^2 / t_0 \).
The boundary conditions (5.35) imply, through the perturbation expansions (4.1) and (4.2),
\[
\begin{align*}
\phi^{(0)}(0) &= 1, \quad v_x^{0,0}(0) = 1, \quad v_x^{0,0}(1) = e^\alpha, \quad (5.36a-c) \\
\phi^{(1)}(0) &= 0, \quad v_x^{0,1}(0) = 0, \quad v_x^{0,1}(1) = 0, \quad (5.36d-f) \\
\phi^{(2)}(0) &= 0, \quad v_x^{0,2}(0) + \frac{1}{2} v_x^{1,0}(0) [\phi^{(0)}(0)]^2 = 0, \quad (5.36g-i) \\
\end{align*}
\]

From the leading order solution (5.30) one can see that the boundary conditions (5.36a-c) demand
\[
A_0 = 1, \quad \alpha_0 = \alpha, \quad C_0 = 1; \quad (5.37a-c)
\]
from the first order solution (5.31) and the boundary conditions (5.36d-f),
\[
A_1 = 0, \quad B_1 = 0, \quad C_1 = 0; \quad (5.37d-f)
\]
and from the second order solution (5.10), (5.33) and the boundary conditions (5.36g-i) it follows
\[
A_2 = \frac{\alpha^2}{8}, \quad B_2 = \frac{\alpha^2}{8}(e^{-\alpha} - 1), \quad C_2 = 0. \quad (5.37g-i)
\]

When the values (5.37) are inserted into the general solutions (5.34) I obtain exactly the same solutions as in Schultz & Davis [22]; in particular, the assembled axial velocity (5.34a) and the free surface position (5.34b) reduce identically to Schultz & Davis’s equations (28a), (28b):
\[
\begin{align*}
v_x(r, z) &= e^\alpha \left[1 + \epsilon^2 \alpha^2 \left\{-\frac{1}{2} r^2 + \frac{1}{8} \left[1 + (e^{-\alpha} - 1)z + e^{-\alpha z} \right] \right\} + O(\epsilon^3) \right], \\
\phi(z) &= e^{-\alpha z/2} \left\{1 + \frac{1}{16} \epsilon^2 \alpha^2 \left[-1 + (1 - e^{-\alpha})z + e^{-\alpha z} \right] + O(\epsilon^3) \right\}. \quad (5.38)
\end{align*}
\]
In Figure 12a, we plot the cross-sectional averaged axial velocity, \( v_{z}^{\text{avg}} \), including corrections up to third order, defined in (5.19), for the range \( 0 \leq z \leq 1 \) from (5.38a) with \( \alpha = \ln 20 \) and \( \epsilon = 0.1 \). On the same plot we also exhibit for comparison the leading order solution,

\[
v_{z}^{0,0}(z) = e^{\alpha z}. \tag{5.39}
\]

In Figure 12b we plot just the correction \( v_{z}^{\text{corr}} = v_{z}^{\text{avg}} - v_{z}^{0,0} \) to the leading order order flow (5.39).

Note from Figure 12 that the leading order solution (5.39) of this particular boundary value problem, i.e., with boundary conditions (5.36), is an asymptotically valid solution to the 3-D boundary value problem, since \( |v_{z}^{\text{corr}}(z)| \) remains small over the domain of interest \( 0 \leq z \leq 1 \). Also note that the velocity correction is positive over the length of the jet, so that the leading order solution slightly underestimates the jet velocity.

Many physically motivated perturbations of the Schultz & Davis solution (5.38) can be described by the general solution (5.34) when values of the constants \( A_0, \alpha_0, C_0, A_1, B_1, C_1, A_2, B_2, C_2 \) different from those in (5.37) are considered.

For instance, suppose the velocity at the nozzle \( z = 0 \) is perturbed to \( O(\epsilon) \), but the nozzle dimensions and take-up conditions are held fixed:

\[
\begin{align*}
\text{at } z = 0 : & \quad \phi = 1, \quad v_{z}^{\text{avg}} = \int_{\text{cross section}} v_z \, da = 1 + \epsilon \gamma, \\
\text{at } z = 1 : & \quad v_{z}^{\text{avg}} = \int_{\text{cross section}} v_z \, da = e^{\alpha}, \tag{5.40}
\end{align*}
\]
where \( \gamma \) is a constant. These boundary conditions imply

\[
\phi^{(0)}(0) = 1, \quad v_x^{0,0}(0) = 1, \quad v_x^{0,0}(1) = e^\alpha,
\]

\[
\phi^{(1)}(0) = 0, \quad v_x^{0,1}(0) = \gamma, \quad v_x^{0,1}(1) = 0,
\]

\[
\phi^{(2)}(0) = 0, \quad v_x^{0,2}(0) + \frac{1}{2} v_x^{1,0}(0) \phi^{(0)}(0)^2 = 0,
\]

\[
v_x^{0,2}(1) + \frac{1}{2} v_x^{1,0}(1) \phi^{(0)}(1)^2 = 0,
\]

from which follow

\[
A_0 = 1, \quad \alpha_0 = \alpha, \quad C_0 = 1;
\]

\[
A_1 = \gamma, \quad B_1 = -\gamma, \quad C_1 = \gamma;
\]

\[
A_2 = \frac{\alpha^2}{8}, \quad B_2 = \frac{\alpha^2}{8} (e^{-\alpha} - 1) - \frac{\gamma^2}{2}, \quad C_2 = 0.
\]

In Figures 13a and 14a we plot the cross-sectional averaged velocity \( v_x^{\text{avg}}(z) \) for this perturbed solution with \( \alpha = \ln 20, \varepsilon = 0.1 \), and two values of the nozzle velocity perturbation \( \gamma \), together with the leading order solution (5.39). In Figures 13b and 14b we plot the velocity correction \( v_x^{\text{correction}}(z) \) for the two values of \( \gamma \). Note that for both values of \( \gamma \), the corrections in the interior \( 0 < z < 1 \) are bounded on the order of the boundary perturbations, so that the leading order solution (5.39) is robust with respect to small perturbations at the boundary. These examples show that \( v_x^{\text{correction}} \) can be either positive or negative; if both inlet and outlet conditions are perturbed, it is possible that \( v_x^{\text{correction}} \) can change sign along the length of the jet.
5.4 Weakly Elastic Slender Jet Behavior

The steady equations for the weakly elastic jet (Regime 3 of Chapter IV) also admit closed form integration to any order in the asymptotics.

**Leading order solution**

The reduced leading order problem for the weakly elastic jet is identical to the reduced leading order problem for the Newtonian jet. The general solution of this Newtonian reduced leading order problem is given in (5.31), in terms of the three constants of integration $A_0$, $\alpha_0$, and $C_0$.

**First order corrections**

Recall from Chapter IV that in the weakly elastic Regime 3 the problem (4.21) for the $O(\varepsilon)$ corrections $v^{0,1}_2$ and $\phi^{(1)}$ to the leading order Newtonian flow is nonhomogeneous, since the elastic forcing term $h^{0,1}$ in equation (4.21b) is nonzero. With the functions $v^{0,0}_2$ and $\phi^{(0)}$ in (4.21) now given by the leading order solution (5.31), the general solution of the first order problem is

\[
v^{0,1}_2(z) = (A_1 + B_1 z)e^{\alpha z} + \tilde{\Lambda}_1(1 - a)\frac{A_0^2}{C_0}e^{2\alpha_0 z},
\]

\[
\phi^{(1)} = \left[\frac{C_1}{2(C_0 A_0)^{1/2}} - \frac{C_0^{1/2}}{A_0^{3/2}}(A_1 + B_1 z)\right]e^{-\alpha_0 z/2} + \tilde{\Lambda}_1(a - 1)\frac{A_0^{1/2}}{2C_0^{1/2}}e^{\alpha_0 z/2},
\]

\[
v^{0,1}_r = -\frac{1}{2}[\alpha_0 A_1 + (1 + \alpha_0 z)B_1]e^{\alpha_0 z} + \tilde{\Lambda}_1(a - 1)\frac{A_0^2}{C_0}\alpha_0 e^{2\alpha_0 z},
\]

\[
T^{0,1}_{rr} = T^{0,1}_{\theta\theta} = p^{0,1} = -Z[\alpha_0 A_1 + (1 + \alpha_0 z)B_1]e^{\alpha_0 z} + \tilde{\Lambda}_1 Z\frac{A_0^2}{C_0}\alpha_0 [2(a - 1) + C_0\alpha_0(a + 1)]e^{2\alpha_0 z},
\]

\[ (5.43) \]
Note that the first order corrections are viscoelastic, since they involve not only the viscous material parameter $Z$, but also the elastic material parameter $\tilde{A}_1$. The first order solution also depends on six constants of integration: $A_0$, $\alpha_0$, $C_0$ from the leading order solution, and the three additional constants $A_1$, $B_1$, $C_1$.

**Second order corrections**

The "1,0" corrections (4.23) in the weakly elastic Regime 3 depend only on its leading order Newtonian solution, and hence are the same as the "1,0" corrections for the Newtonian jet, given by equation (5.32). Therefore the leading order radially dependent corrections in the weakly elastic regime are determined only by viscous effects. The general steady solution of the problem (4.24) for the radially independent "0,2" corrections is readily obtainable, but is not exhibited here due its extreme length. It involves three new constants, $A_2$, $B_2$, and $C_2$.

**Steady solution through three orders**

The steady solution for the slender jet through three orders can be constructed from the leading, first, and second order solutions and the asymptotic expansions (3.27)-(3.37). Through three orders, in dimensionless form,
\begin{align*}
v_z(r, z) &= v_z^{0,0}(z) + \varepsilon v_z^{0,1}(z) + \varepsilon^2 \left[ r^2 v_z^{1,0}(z) + v_z^{0,2}(z) \right], \\
v_r(r, z) &= \varepsilon v_r^{0,0}(z) + \varepsilon^2 r v_r^{0,1}(z) + \varepsilon^3 \left[ r^3 v_r^{1,0}(z) + r v_r^{0,2}(z) \right], \\
\phi_z(r, z) &= \phi^{(0)}(z) + \varepsilon \phi^{(1)}(z) + \varepsilon^2 \phi^{(2)}(z), \\
T_{rr}(r, z) &= T_{rr}^{0,0}(z) + \varepsilon T_{rr}^{0,1}(z) + \varepsilon^2 \left[ r^2 T_{rr}^{1,0}(z) + T_{rr}^{0,2}(z) \right], \\
T_{zz}(r, z) &= T_{zz}^{0,0}(z) + \varepsilon T_{zz}^{0,1}(z) + \varepsilon^2 \left[ r^3 T_{zz}^{1,0}(z) + T_{zz}^{0,2}(z) \right], \\
T_{\theta\theta}(r, z) &= T_{\theta\theta}^{0,0}(z) + \varepsilon T_{\theta\theta}^{0,1}(z) + \varepsilon^2 \left[ r^3 T_{\theta\theta}^{1,0}(z) + T_{\theta\theta}^{0,2}(z) \right], \\
T_{rz}(r, z) &= \varepsilon T_{rz}^{0,0}(z) + \varepsilon^2 r T_{rz}^{0,1}(z) + \varepsilon^3 \left[ r^3 T_{rz}^{1,0}(z) + r T_{rz}^{0,2}(z) \right], \\
p(r, z) &= p_a + p^{0,0}(z) + \varepsilon p^{0,1}(z) + \varepsilon^2 \left[ r^2 p^{1,0}(z) + p^{0,2}(z) \right].
\end{align*}

In the weakly elastic Regime 3 we can obtain the general solution for (5.44). It involves nine constants of integration and two \(O(1)\) material parameters, \(Z\) and \(\Lambda_1\).

In what immediately follows we set the viscous parameter \(Z = 1\) without loss of generality, by selecting the force scale \(f_0 = \eta r_0^2/t_0\).

**Solutions of weakly elastic jet boundary value problems**

As with the previous example, in a particular physical application the nine constants of integration in the general solution through three orders for the weakly elastic problem are determined from the boundary conditions.

When we adopt the boundary conditions (5.35) (or equivalently, (5.36)) pertinent to the fiber spinning problem with no boundary perturbations, the general solution
(5.44) reduces through evaluation of constants to

\[ v_{z}(r, z) = e^{\alpha z} + \varepsilon \tilde{\Lambda}_1 \alpha (a - 1) \{ e^{\alpha z} [1 + (e^{\alpha} - 1) z] - e^{2\alpha z} \} \]
\[ + \varepsilon^2 \left[ r^2 \left[ -\frac{\alpha^2}{2} e^{\alpha z} + \frac{\alpha^2}{8} \right] + \left[ 1 + (e^{\alpha} - 1) z \right] e^{\alpha z} + 1 \right] \]
\[ + \tilde{\Lambda}_1^2 \alpha e^{\alpha z} \left\{ \frac{\alpha}{2} \left[ (a^2 - 6a + 3)e^{2\alpha z} + 2(a^2 e^{2\alpha} - 2a + 1) z + 3a^2 - 2a + 1 \right] \right. \]
\[ + (a - 1)^2 \left. \left\{ (1 - e^{\alpha})(2\alpha z + 1) - 2\alpha \right\} e^{\alpha z} + \frac{\alpha}{2} \left( e^{\alpha} - 1 \right)^2 z^2 \]
\[ + \left[ \alpha^2 (e^{\alpha} - 2) + (e^{\alpha} - 1)^2 \right] z + e^{\alpha} - 1 \right\} \}, \tag{5.45a} \]

\[ \phi(z) = e^{-\alpha z/2} + \varepsilon \tilde{\Lambda}_1 \frac{\alpha}{2} \left\{ e^{-\alpha z/2} [1 + (e^{\alpha} - 1) z] - e^{2\alpha z/2} \right\} \]
\[ + \varepsilon^2 \left[ \frac{\alpha^2}{16} \left[ -1 + (1 - e^{-\alpha} z + e^{-\alpha z}) \right] e^{-\alpha z/2} \right. \]
\[ + \tilde{\Lambda}_1^2 \frac{\alpha}{8} e^{-\alpha z/2} \left\{ \frac{\alpha}{2} \left[ (a^2 + 6a - 3)e^{2\alpha z} - 4(a^2 e^{2\alpha} - 2a + 1) z - 3a^2 - 2a + 1 \right] \right. \]
\[ + (a - 1)^2 \left. \left\{ 2[(e^{\alpha} - 1)(2\alpha z + 2) + \alpha] e^{\alpha z} + \alpha(e^{\alpha} - 1)^2 z^2 \right\} \right. \]
\[ + 2 \left[ \alpha^2 (e^{\alpha} + 1) - 2(e^{\alpha} - 1)^2 \right] z + 4(1 - e^{\alpha}) \right\} \}, \tag{5.45b} \]

\[ v_{r}(r, z) = \varepsilon \left[ -\frac{\alpha}{2} e^{\alpha z} \right] + \varepsilon^2 r \tilde{\Lambda}_1 \frac{\alpha^2}{2} \left( 1 - a \right) \left\{ e^{\alpha z} [1 + (e^{\alpha} - 1) z] - 2 e^{2\alpha z} \right\} \]
\[ + \varepsilon^3 \left[ r^2 \left[ \frac{\alpha^3}{8} e^{\alpha z} \right] - \frac{1}{2} v_{z,zz}^2 (z) \right], \tag{5.45c} \]

\[ T_{rr}(r, z) = -\alpha e^{\alpha z} + \varepsilon \tilde{\Lambda}_1 \alpha \left\{ \alpha (3a - 1)e^{\alpha z} + (1 - a)(\alpha + D(\alpha z + 1)) \right\} e^{\alpha z} \]
\[ + \varepsilon^2 \left[ r^2 \left[ \frac{3}{4} \alpha^2 e^{\alpha z} \right] - v_{z,zz}^2 (z) + \tilde{\Lambda}_1^2 \alpha^2 \left[ 2(a^2 - 1)(D + \alpha + D\alpha z) \right. \right. \]
\[ - (5a^2 + 4a - 3)\alpha e^{\alpha z} \right\] e^{2\alpha z} \right], \tag{5.45d} \]

\[ T_{\theta\theta}(r, z) = T_{rr}(r, z) - \varepsilon^2 r^2 \frac{1}{2} \alpha^3 e^{\alpha z} \tag{5.45e} \]
\[ T_{zz}(r,z) = 2\alpha e^{\alpha z} + \varepsilon^2 \tilde{\Lambda}_1 \alpha \{ \alpha^2 e^{\alpha z} + (a - 1)[\alpha + D(\alpha z + 1)] \} e^{\alpha z} \]
\[ + \varepsilon^2 \left[ r^2 \left( -\alpha^3 e^{\alpha z} \right) + 2v^0_{z,z}(z) + 2(a - 1)\tilde{\Lambda}_1 \alpha^2 [(2a - 1)(D + \alpha + D\alpha z) \right. \]
\[ \left. + (3 - 4a)\alpha e^{\alpha z} \right] e^{2\alpha z} \], \quad (5.45f) \]

\[ T_{rr}(r,z) = \varepsilon r \left[ -\alpha e^{\alpha z} \right] + \varepsilon^2 r^2 \tilde{\Lambda}_1 \alpha^2 \{ \alpha(a - 1)e^{\alpha z} + (1 - a)[D + \frac{\alpha}{2}(\alpha z + 1)] \} e^{\alpha z} \]
\[ + \varepsilon^3 r \left[ r^2 \left( \frac{5}{16} \alpha^4 e^{\alpha z} \right) - \frac{3}{2} v^0_{z,z}(z) - \frac{3}{2} \tilde{\Lambda}_1^2 \alpha^3 [2(a - 1)^2(3D + 2\alpha + 2D\alpha z) \right. \]
\[ \left. - (a^2 - 6a + 3)\alpha e^{\alpha z} \right] e^{2\alpha z} \], \quad (5.45g) \]

\[ p(r,z) = p_0 + T_{\theta\theta}(r,z) - \frac{1}{4} \varepsilon^2 \alpha^2, \quad (5.45h) \]

where \( D = e^\alpha - 1 \), and

\[ v^0_{z,z}(z) = \frac{\alpha^2}{8} \left( [1 + (e^{-\alpha} - 1)z]e^{\alpha z} + 1 \right) \]
\[ + \tilde{\Lambda}_1^2 \alpha e^{\alpha z} \left\{ \frac{\alpha}{2} [(a^2 - 6a + 3)e^{2\alpha z} + 2(a^2 e^{2\alpha} - 2a + 1)z + 3a^2 - 2a + 1] \right. \]
\[ \left. + (a - 1)^2 \left\{ [(1 - e^\alpha)(2\alpha z + 1) - 2\alpha] e^{\alpha z} + \frac{\alpha}{2} (e^\alpha - 1)^2 z^2 \right. \]
\[ \left. + \left[ \alpha(e^\alpha - 2) + (e^\alpha - 1)^2 \right] z + e^{\alpha - 1} \right\} \}. \quad (5.45i) \]

Schultz [23] solved the same asymptotic weakly elastic fiber spinning boundary value problem by a different approach. Schultz's solution agrees with (5.45), except for a number of typographical errors in the Schultz equations (e.g., his elastic parameter \( De \) and a factor \(-2\) missing from his equation (2b), a factor \(-1\) missing from his (2c), the term \(-zE^2\) instead of \(-z^2E\) in his (3b), several errors in his (3c), and a sign error in his (4c)).
The solution (5.45) allows for variations in \( \alpha \) (the fiber speed at the take-up spool), \( \varepsilon \) (the initial radius of the fiber divided by the distance to the take-up spool), \( a \) (the rate parameter in the constitutive model (3)) and \( \Lambda_1 \) (the elasticity of the fiber relative to its viscosity). In the weakly elastic regime \( \Lambda_1 \) must be \( O(\varepsilon) \), i.e.,

\[
\Lambda_1 = \tilde{\Lambda}_1 \varepsilon, \tag{5.46}
\]

where \( \tilde{\Lambda}_1 \) is \( O(1) \).

Note that the solution (5.38) to the Newtonian problem subject to the same boundary conditions (5.36) is obtained from the weakly elastic solution (5.45) by setting \( \tilde{\Lambda}_1 = 0 \), so that the Newtonian solution is a regular limit of the weakly elastic solution.

**The effect of weak elasticity on the slender jet**

One can now use the asymptotic solution (5.45) through three orders to determine the effect of weak elasticity in the fluid (in the sense that elasticity is small compared to viscosity) on the leading order Newtonian behavior of the jet.

We specify take-up speed \( e^\alpha = 20 \), slenderness ratio \( \varepsilon = 0.1 \), and the rate parameter \( a = 1 \) (upper convected rate). (Note from (5.45) that for the upper convected rate the elastic components vanish from the first order corrections to the axial velocity \( v_z \) and free surface \( \phi \), but not from the first order corrections to the stresses and pressure. Elastic contributions survive for \( a = 1 \) in all second order corrections, however.)

Figure 12a gives the cross-sectional average of axial velocity from the purely Newtonian solution ((5.38a), or (5.45a) with \( \tilde{\Lambda}_1 = 0 \), together with the Newtonian leading
order solution. Recall that in this Newtonian case the leading order solution is a valid asymptotic solution to the corresponding 3-D boundary value problem, since the solution including two orders of asymptotic corrections is virtually indistinguishable from the leading order solution. The leading order solution only slightly underestimates the jet velocity.

In Figure 15, we plot the cross-sectional axial velocity $v_2^{av}(z)$ including first and second order corrections for several values of the elastic parameter $\tilde{\Lambda}_1$. For comparison we overlay the leading order Newtonian solution $v_2^{0,0}(z) = e^{(ln20)z}$.

The plot of $v_2^{av}(z)$ for $\tilde{\Lambda}_1 = 0$ recalls the information of Figure 12a.

When $\tilde{\Lambda}_1 = 0.1$, so that $\Lambda_1 = \tilde{\Lambda}_1 \varepsilon = 0.01$, the Newtonian leading order solution is still reasonably accurate.

When $\tilde{\Lambda}_1 = 0.2 (\Lambda_1 = 0.02)$, the asymptotic validity of the Newtonian leading order solution is questionable, since the assembled axial velocity $v_2^{av}$ including first and second order corrections differs significantly from the leading order solution (i.e., the viscoelastic corrections do not remain small).

When $\tilde{\Lambda}_1 = 0.3 (\Lambda_1 = 0.03)$ the Newtonian leading order solution is certainly not a valid asymptotic solution to the 3-D boundary value problem.

Figure 16 summarizes the dependence of the solution (5.45a) on the elastic parameter $\tilde{\Lambda}_1$. The curves $v_2^{av}(z)$ and $v_2^{0,0}(z)$ in Figure 15 are the intersections of the surfaces in Figure 16 with the planes $\tilde{\Lambda}_1 = 0, 0.1, 0.2, 0.3$, respectively. Note that by the value $\tilde{\Lambda}_1 = 0.3$ the two surface $v_2^{av}$ and $v_2^{0,0}$ depart significantly at some $z$ locations.
According to the perturbation theory, the parameter $\tilde{\Lambda}_1$ in Regime 3 must be $O(1)$, i.e., one must have

$$\varepsilon = 0.1 < \tilde{\Lambda}_1 < \varepsilon^{-1} = 10.$$  \hspace{1cm} (5.47)

We have shown that for almost all of this allowable range, the presumed asymptotic ordering of the solution expansion is violated. Therefore, the weakly elastic model in Regime 3, although formally valid, is not relevant for physical applications. We have reached the most dramatic result of this study: A formally valid leading order solution has been shown to be invalidated by a formally weak effect, through the violation of the assumed ordering in the asymptotic expansion on which the perturbation theory is based. Obviously, this result cannot be deduced from the leading order equations alone.

5.5 Conclusion

In this chapter we have demonstrated, through the representative models and explicit solutions, the practical usefulness of a high order perturbation theory for slender viscoelastic jets and fibers.

In this chapter we apply the Denn, Petrie, & Avenas fiber spinning equations, and many generalizations of DPA, which allow physical effects besides viscosity and elasticity in the dominant equations. We then analyze the asymptotic corrections to solutions of these leading order models.

The generalizations of DPA are useful in the analysis of industrial processes. The asymptotic scaling described in §4 (and demonstrated explicitly in Appendices 1 and
2) dictates which physical effects are dominant in a particular process. These effects, and only these effects, must be included in the leading order model. This ordering selects the appropriate leading order model from the 128 candidates (one of which is DPA).

The generalizations of DPA can also be useful in the design of industrial processes. Material parameters and process conditions can be conceptually varied to alter the asymptotic ordering of physical effects. For instance, surface tension, inertia, retardation, and/or gravity can be added or removed as dominant effects, producing alternative governing models for the process. By analyzing the solutions to these alternative models, one can arrive at a near optimal industrial process (in terms of throughput or stability, for instance) before constructing an actual prototype.

In addition to presenting formally consistent strongly elastic equations to all orders in a slenderness perturbation expansion, we have verified that representative steady solutions to these equations remain ordered as they should be within the bounds of the asymptotic approximation. We have validated the ordering in these solutions through three orders; the solutions have been found to be asymptotically valid both with and without small perturbations to the boundary values. These representative approximate solutions are therefore likely to be physically realizable processes.

Moreover, we have explored the asymptotic validity of steady leading order solutions in the presence of weak effects, i.e., those physical effects which are absent from the leading order equations but present at some higher order. We found that weak effects of surface tension, retardation, inertia, viscosity, and gravity can produce only
small corrections to the leading order strongly elastic behavior. Therefore, if these physical effects are small then in practice they can be neglected from the leading order problem.

However, a weak elasticity effect is found to be potentially dangerous in the slender jet expansion. The formal asymptotic ordering of an explicit Newtonian leading order solution is destroyed by weak elasticity that appears only in the higher order corrections. We emphasize that in my experience weak relaxation is the only higher order effect that clearly violates the assumption of a slender perturbation expansion for steady solutions. We infer from this observation that elastic relaxation is the only singular effect in these slender jet models. If we change the order of $\Lambda_1$ so that elasticity enters in the dominant equations, then the results from Regimes 1 and 2 indicate valid, robust solutions.

One could be even more confident that the steady solutions of the strongly elastic Regimes 1 and 2 are physically stable if their time-dependent stability were analyzed. These analyses will be reported elsewhere, but we note that all necessary machinery is made available here. It is shown, however, that some preliminary results are accessible before performing time-dependent stability analyses. For example, in Regimes 1 and 2 calculation of characteristic speeds allows us \textit{a priori} to avoid steady states that are guaranteed to be highly temporally unstable.
CHAPTER VI

Solutions with Torsion

The leading three orders of equations of Regime 1 are derived in Chapter IV. This is an example of an axisymmetric slender jet dominated by viscosity, inertia, gravity, and surface tension, whose torsional effects are analyzed in this chapter. The results in this chapter generalize the torsionless studies of Chapter V and allow one to assess the influence of torsion on previous results relating to: (1) the mathematical structure of 1-D thin-filament models; (2) catastrophic instabilities in these models; and (3) classes of steady solutions, both numerical and exact.

This chapter focuses on the torsional components ($v_{\theta}^{0,0}$, $T_{\theta\theta}^{0,0}$, $T_{\theta z}^{0,0}$, $v_{\theta}^{0,1}$, $T_{\theta\theta}^{0,1}$, $T_{\theta z}^{0,1}$, $v_{\theta}^{1,0}$, $T_{\theta\theta}^{1,0}$, $T_{\theta z}^{1,0}$, $v_{\theta}^{0,2}$, $T_{\theta\theta}^{0,2}$, $T_{\theta z}^{0,2}$) of solutions and their effects on the torsionless components. Recall that the leading order torsional problem for $v_{\theta}^{0,0}$, $T_{\theta\theta}^{0,0}$, $T_{\theta z}^{0,0}$ decouples from the torsionless problem for $v_{r}^{0,0}$, $v_{z}^{0,0}$, $T_{rr}^{0,0}$, $T_{\theta\theta}^{0,0}$, $T_{rz}^{0,0}$, $T_{zz}^{0,0}$, $p^{0,0}$, $\phi^{(0)}$, and that the analogous situation occurs also in the next order problem for the (0,1) corrections to the leading order flow. In fact, the first torsional coupling is the appearance of $v_{\theta}^{0,0}$, $T_{\theta z}^{0,0}$ as forcing functions in the problems for $T_{\theta\theta}^{1,0}$, $p^{1,0}$ and $T_{rz}^{1,0}$, which in turn drive the entire (0,2) problem. It is this coupling of the leading order torsional problem to the $O(\varepsilon^2)$ corrections in the torsionless flow that will be examined here through specific solutions. It is found that torsion may or may not have significant impact on
the non-torsional part for these flows.

In this chapter, the following examples are presented: (1) weak torsional coupling, (2) strong torsional coupling which destroys or re-establishes the asymptotic validity of torsionless leading order predictions, (3) torsion advances change-of-type, and (4) two examples where an initial-boundary condition which is well-posed for the torsionless problem but ill-posed for the complete problem.

Since only real characteristics can lead to stable steady state solutions, well-posed boundary conditions demand all eigenvalues $s_k$ of the coefficient matrix $M$ given by (4.1) be real. For simplicity, it is further assumed that all eigenvalues are non-zero. Then $s_1$, $s_2$, $s_3$, and $s_5$ are always positive; and four boundary conditions should be prescribed on the left boundary ($z = 0$). The remaining two boundary conditions should be given as follows

1. both at the left boundary if $s_4 > 0, s_6 > 0$;
2. one at the left boundary and one at the right boundary if $s_4 s_6 < 0$;
3. both at the right boundary ($z = 1$) if $s_4 < 0, s_6 < 0$.

The examples presented in this chapter all have $s_4 < 0$ and $s_6 < 0$. Therefore, there must be two boundary data imposed at the right, which are chosen as $v_{z}^{0,0}(1)$ and $T_{x}^{0,0}(1)$ for the leading order problem and likewise for the higher-order corrections.

The boundary conditions imposed in this chapter can be summarized as follows:

**Steady boundary conditions for the leading order problem**

left: $v_{z}^{0,0}(0), \phi^{(0)}(0), T_{x}^{0,0}(0), v_{\theta}^{0,0}(0)$,
right: \( v_z^{0,0}(1), \ T_{\theta z}^{0,0}(1); \)  

boundary conditions for the first order "0,1" problem

left: \( v_z^{0,1}(0), \ \phi^{(1)}(0), \ T_{zz}^{0,1}(0), \ v_{\theta}^{0,1}(0), \)

right: \( v_z^{0,1}(1), \ T_{\theta z}^{0,1}(1); \)  

boundary conditions for the second order "1,0" problems

\[
T_{rr}^{1,0}(0), \ T_{\theta \theta}^{1,0}(0), \ T_{zz}^{1,0}(0), \ T_{\theta z}^{1,0}(0);
\]  

(6.3)

boundary conditions for the second order "0,2" problems

left: \( v_z^{0,2}(0), \ \phi^{(2)}(0), \ T_{zz}^{0,2}(0), \ v_{\theta}^{0,2}(0), \)

right: \( v_z^{0,2}(1), \ T_{\theta z}^{0,2}(1). \)  

(6.4)

6.1 An Example of Weakly Torsional Coupling

In Figure 17, what is presented is an asymptotically valid leading order prediction of the axial force \( F_z^{(0)} \) and the prediction of the cross-sectional averaged axial force including the first and second corrections, defined in (5.21), \textit{without torsion} (all torsional components set to zero) and \textit{with torsion}. Note that the axial force is a torsionless quantity. This example is a solution with physical parameters

\[
B = F = W = Z = a = 1.0, \quad \Lambda_1 = 1.1, \quad \varepsilon = 0.1.
\]  

(6.5)

The boundary conditions imposed to produce the torsionless problem are:

\[
\phi^{(0)}(0) = v_z^{0,0}(0) = 1.0, \quad T_{zz}^{0,0}(0) = 3.2, \quad v_z^{0,0}(1) = 1.3,
\]  

(6.6)
\[
\phi^{(1)}(0) = v_z^{0,1}(0) = T_{zz}^{0,1}(0) = v_z^{0,1}(1) = 0.0, \\
T_{\theta\theta}^{1,0}(0) = T_{rr}^{1,0}(0) = T_{zz}^{1,0}(0) = T_{\theta z}^{1,0}(0) = 0.0,
\]
\[
\phi^{(2)}(0) = v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = T_{zz}^{0,2}(0) = 0.0, \quad v_z^{0,2}(1) = 1.0,
\]
\[
v_{\theta}^{0,0}(0) = T_{\theta z}^{0,0}(1) = v_{\theta}^{0,1}(0) = T_{\theta z}^{0,1}(1) = T_{\theta z}^{1,0}(0) = v_{\theta}^{0,2}(0) = T_{\theta z}^{0,2}(1) = 0.0.
\]

The homogeneous boundary conditions (6.10) on the torsional components imply that the solution to this boundary value problem is torsionless to this order.

The boundary conditions imposed for the torsional problem are as given in the same the torsionless problem, except for nonzero boundary values for the torsional components:

\[
v_{\theta}^{0,0}(0) = 1.0, \quad T_{\theta z}^{0,0}(1) = 3.0, \quad v_{\theta}^{0,1}(0) = T_{\theta z}^{0,1}(1) = 0.0,
\]
\[
T_{\theta z}^{1,0}(0) = 0.0, \quad v_{\theta}^{0,2}(0) = 8.0, \quad T_{\theta z}^{0,2}(1) = 0.7.
\]

It is verified that \(\Delta_1(z) > 0, \Delta_2(z) > 0, s_4(z) < 0\), and \(s_\theta(z) < 0\).

From Figure 17 it can be seen that the difference between the two curves is small. That is, in this case, torsion does not qualitatively affect the non-torsional part.

### 6.2 An Example of Strong Torsional Coupling which Destroys the Asymptotic Validity in the Torsionless Subspace

Figure 18 is an example where the torsional effects in the second order correction disorders the perturbation expansions.

The physical parameters in this computation are the same as those of Figure 17.
The boundary conditions imposed to produce the torsionless problem are also the same as those of Figure 17.

The boundary conditions imposed for the torsional problem are as given in the same the torsionless problem, except for nonzero boundary values for the torsional components:

\[ v_\theta^{0,0}(0) = 8.0, \quad T_{\theta z}^{0,0}(1) = 9.9, \quad v_\theta^{0,1}(0) = T_{\theta z}^{0,1}(1) = 0.0, \]  \hspace{1cm} (6.13)

\[ T_{\theta z}^{1,0}(0) = 0.0, \quad v_\theta^{0,2}(0) = 9.0, \quad T_{\theta z}^{0,2}(1) = 1.4. \]  \hspace{1cm} (6.14)

It is verified that \( \Delta_1(z) > 0, \Delta_2(z) > 0, s_4(z) < 0, \) and \( s_6(z) < 0. \)

Note that, comparing Figure 18 with 17, the change of the torsional boundary conditions dramatically changes the torsional effect while all the physical parameters and torsionless boundary conditions remain the same. In Figure 17, torsion does not affect the asymptotic validity of the leading order prediction for axial force at all. In Figure 18, torsion invalidates the leading order prediction for axial force.

6.3 An Example Where Torsional Effects Reduces the Magnitude of the Higher Corrections of a Torsionless Variable

An interesting question arises as to whether torsional effects can “stabilize” jet flows. Figure 19 presents an example where without torsion the second order correction is so large that the asymptotic validity of the leading order prediction is questionable, and with torsion the second order correction becomes qualitatively small so that the leading order prediction is asymptotically valid.
The physical parameters are

\[ B = F = W = Z = a = 1.0, \quad \Lambda_1 = 1.2, \quad \varepsilon = 0.1. \quad (6.15) \]

The boundary conditions imposed for the torsionless problem are:

\[ \varphi^{(0)}(0) = \psi_z^{0,0}(0) = 1.0, \quad T_{zz}^{0,0}(0) = 3.6, \quad v_z^{0,0}(1) = 1.5, \quad (6.16) \]
\[ \varphi^{(1)}(0) = 1.0, \quad v_z^{0,1}(0) = 1.2, \quad T_{zz}^{0,1}(0) = -3.0, \quad v_z^{0,1}(1) = 1.8, \quad (6.17) \]
\[ T_{\theta \theta}^{1,0}(0) = 2.0, \quad T_{rr}^{1,0}(0) = -2.0, \quad T_{zz}^{1,0}(0) = 2.0, \quad (6.18) \]
\[ \varphi^{(2)}(0) = v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0) [\varphi^{(0)}(0)]^2 = T_{zz}^{0,2}(0) = 0.0, \quad v_z^{0,2}(1) = 0.2, \quad (6.19) \]
\[ v_\theta^{0,0}(0) = T_{\theta z}^{0,0}(1) = v_\theta^{0,1}(0) = T_{\theta z}^{0,1}(1) = T_{\theta z}^{1,0}(0) = v_\theta^{0,2}(0) = T_{\theta z}^{0,2}(1) = 0. \quad (6.20) \]

The boundary conditions imposed for the torsional problem are (6.16–6.19) and the nonhomogeneous torsional boundary values:

\[ v_\theta^{0,0}(0) = 6.0, \quad T_{\theta z}^{0,0}(1) = 9.5, \quad v_\theta^{0,1}(0) = 9.0, \quad T_{\theta z}^{0,1}(1) = 7.7, \quad (6.21) \]
\[ T_{\theta z}^{1,0}(0) = 3.0, \quad v_\theta^{0,2}(0) = 9.0 \quad T_{\theta z}^{0,2}(1) = 2.4. \quad (6.22) \]

It is verified that \( \Delta_1(z) > 0, \Delta_2(z) > 0, s_4(z) < 0, \) and \( s_6(z) < 0. \)

### 6.4 An Initial Condition on \( v_z^{0,0}, T_{rr}^{0,0} \) and \( T_{zz}^{0,0} \) Is Well-posed for the Torsionless Subsystem, but Ill-posed for the Torsional System.

If gravity is suppressed, that is, \( 1/F = 0, \) then the reduced (0,0)-system (4.1) has the following uniformly damped traveling-wave solutions, similar to those found in [14]:

\[ \varphi^{(0)} \equiv c_0, \quad v_z^{0,0} \equiv c, \quad (6.23) \]
where \( f(\cdot) \) is an arbitrary smooth function, and \( c_0, c, c_1, c_2, c_3, \) and \( c_4 \) are arbitrary constants to be determined by proper initial and boundary conditions.

For this particular solution, it follows

\[
\Delta_2 = B e^{-t/\Lambda_1} [af(z - ct) - \frac{a + 1}{2}c_1] + \frac{BZ}{\Lambda_1}, \tag{6.26}
\]

\[
\Delta_1 = B e^{-t/\Lambda_1} [3af(z - ct) - (2a - 1)c] + \frac{3BZ}{\Lambda_1} - \frac{1}{2Wc_0}, \tag{6.27}
\]

For simplicity, one may choose

\[
c_0 = a = B = 1, \quad c_1 > 0. \tag{6.28}
\]

Then, for \( f \equiv 0 \), it follows

\[
\Delta_1(z, 0) = \frac{3Z}{\Lambda_1} - \frac{1}{2W} - c_1, \quad \Delta_2(z, 0) = \frac{Z}{\Lambda_1} - c_1. \tag{6.29}
\]

If

\[
\frac{1}{2W} < \frac{2Z}{\Lambda_1},
\]

then there exists \( c_1 \) such that

\[
\frac{Z}{\Lambda_1} < c_1 < \frac{3Z}{\Lambda_1} - \frac{1}{2W}; \tag{6.30}
\]

that is,

\[
\Delta_1(z, 0) > 0, \quad \Delta_2(z, 0) < 0. \tag{6.31}
\]

In other words, the same initial condition on \( v^0_z, T^0_{zz}, \) and \( T^0_{zz} \) is well-posed for the torsionless subsystem, but ill-posed for the entire system which includes torsion. As
in [14], explicit choices of the function \( f \) can be used to display various change-of-type regions in \( z, t \) space.

### 6.5 An Example of Torsion Advancing Change-of-Type

Consider

\[
f(z - ct) = \gamma - e^{2(t-z/c)/\Lambda_1},
\]

where \( \gamma \) is a constant, \( \gamma \geq c_1 \). Then

\[
\Delta_1 = (3\gamma - c_1)e^{-t/\Lambda_1} - 3e^{-2z/(c\Lambda_1)}e^{t/\Lambda_1} + \frac{3Z}{\Lambda_1} - \frac{1}{2W};
\]

\[
\Delta_2 = (\gamma - c_1)e^{-t/\Lambda_1} - e^{-2z/(c\Lambda_1)}e^{t/\Lambda_1} + \frac{Z}{\Lambda_1}.
\]

For each fixed \( z \), \( \Delta_1(z, t) = 0 \) at one unique moment

\[
t = t_1(z) = \Lambda_1 \ln \left| \frac{Z}{2\Lambda_1} - \frac{1}{12W} + \sqrt{\left(\frac{Z}{2\Lambda_1} - \frac{1}{12W}\right)^2 + \left(\gamma - \frac{c_1}{3}\right)e^{-2z/(c\Lambda_1)}} \right|;
\]

whereas \( \Delta_2(z, t) = 0 \) at a different unique moment

\[
t = t_2(z) = \Lambda_1 \ln \left| \frac{Z}{2\Lambda_1} + \sqrt{\left(\frac{Z}{2\Lambda_1}\right)^2 + \left(\gamma - c_1\right)e^{-2z/(c\Lambda_1)}} \right|.
\]

Consider solutions given by equations (6.23-6.25), with

\[
f(z - ct) = \gamma - e^{2(t-z/c)/\Lambda_1},
\]

where \( \gamma \) is a constant, \( \gamma \geq c_1 \). Then

\[
\Delta_2 = (\gamma - c_1)e^{-t/\Lambda_1} - e^{-2z/(c\Lambda_1)}e^{t/\Lambda_1} + \frac{Z}{\Lambda_1};
\]

\[
\Delta_2(z, \mp \infty) = \pm \infty;
\]

(6.38)
\[
\frac{\partial \Delta_2(z, t)}{\partial t} = -\frac{1}{\Lambda_1} e^{-2z/(c \Lambda_1)} e^{t/\Lambda_1} - \frac{\gamma - c_1}{\Lambda_1} e^{-t/\Lambda_1} < 0. \tag{6.40}
\]

For each fixed \(z\), \(\Delta_2(z, t) = 0\) at a unique moment \(t\), where
\[
t = t_2(z) = \Lambda_1 \ln \left| \frac{Z}{2\Lambda_1} + \sqrt{\left(\frac{Z}{2\Lambda_1}\right)^2 + \left(\gamma - c_1\right) e^{-2z/(c \Lambda_1)}} \right|, \tag{6.41}
\]
since \(\gamma - c_1 > 0\), and \(\Delta_2(z, t)\) is monotonically decreasing in \(t\).

From (6.41) it is obvious that if
\[
Z \geq 2\Lambda_1, \tag{6.42}
\]
then
\[
t_2(z) > \Lambda_1 \ln 2 > 0, \quad \forall t \geq 0. \tag{6.43}
\]

On the other hand,
\[
\Delta_1 = (3\gamma - c_1) e^{-t/\Lambda_1} - 3e^{-2z/(c \Lambda_1)} e^{t/\Lambda_1} + \frac{3Z}{\Lambda_1} - \frac{1}{2W}; \tag{6.44}
\]
\[
\Delta_1(z, \mp \infty) = \pm \infty; \tag{6.45}
\]
\[
\frac{\partial \Delta_1}{\partial t}(z, t) = -\frac{3}{\Lambda_1} e^{-2z/(c \Lambda_1)} e^{t/\Lambda_1} - \frac{3\gamma - c_1}{\Lambda_1} e^{-t/\Lambda_1} < 0. \tag{6.46}
\]

For each fixed \(z\), \(\Delta_1(z, t) = 0\) at a unique moment \(t\), where
\[
t = t_1(z) = \Lambda_1 \ln \left| \frac{Z}{2\Lambda_1} - \frac{1}{12W} + \sqrt{\left(\frac{Z}{2\Lambda_1} - \frac{1}{12W}\right)^2 + \left(\gamma - \frac{c_1}{3}\right) e^{-2z/(c \Lambda_1)}} \right|, \tag{6.47}
\]
since \(\gamma - c_1/3 > 0\), and \(\Delta_1(z, t)\) is monotonically decreasing in \(t\).

Since \(\gamma - c_1/3 \geq 2c_1/3\), it is straightforward to verify that if
\[
c_1 > \left[ \frac{Z}{\Lambda_1 W} + \frac{1}{W^2} \right] e^{2z/(c \Lambda_1)}, \tag{6.48}
\]
then
\[ t_2(z) < t_1(z), \quad 0 \leq z \leq 1. \quad (6.49) \]

Therefore, for the physical parameters which satisfy relations (6.28) and (6.42), there exist solutions to system (4.1) such that \( \Delta_2(z,t) \) changes sign at \( t = t_2(z) \) which is an earlier time than \( t = t_1(z) \) at which \( \Delta_1(z,t) \) changes sign.

In summary, we have shown that if
\[
Z \geq 2\Lambda_1 \quad \text{and} \quad c_1 > \frac{Z}{\Lambda_1 W} + \frac{1}{W^2} e^{2/\chi_1}, \quad (6.50)
\]
then
\[ t_2(z) < t_1(z), \quad 0 \leq z \leq 1. \quad (6.51) \]

Therefore, for the physical parameters which satisfy relations (6.50), there exist solutions to system (4.1) such that \( \Delta_2(z,t) \) changes sign at \( t = t_2(z) \) which is an earlier time than \( t = t_1(z) \) at which \( \Delta_1(z,t) \) changes sign. Granted, these solutions are rather artificial and contrived, with the free surface constrained somehow to be constant in the presence of nonzero viscoelastic stresses. However, this example suggests that torsional effects may enhance the onset of small scales which imply departure from the slowly varying slender flows that this model describes.

6.6 A Final Note on Torsional Solutions in this Regime

In previous slender jet theories only torsionless jets are considered, that is, \( v_\theta, T_{r\theta} \), and \( T_{\theta z} \) are all assumed to be zero. Recall that these quantities can all be set to zero without any contradictions; the equations are homogeneous in these variables.
Here we note that only certain subsets of swirling variables can be nonzero: From the equations derived in the Appendix, one can show that

$$T_{\theta x}^{n,m} = 0 \implies T_{\theta \theta}^{n,m} = 0, \quad v_{x}^{n,m} = 0,$$

(6.52)

that is, if $T_{\theta x}^{n,m}$ is zero, then all torsional components must be zero, but it is possible that $T_{\theta x}^{n,m}$ is nonzero when $T_{\theta \theta}^{n,m}$ and $v_{x}^{n,m}$ vanish, i.e.

$$T_{\theta x}^{n,m} = 0, \quad v_{x}^{n,m} = 0, \quad \iff \quad T_{\theta x}^{n,m} = 0.$$

(6.53)

For example,

$$v_{r}^{0,0} = v_{\theta}^{0,0} = v_{z}^{0,0} = g(t),$$

(6.54)

$$T_{r \theta}^{0,0} = T_{\theta z}^{0,0} = 0,$$

(6.55)

$$p^{0,0} = T_{r r}^{0,0} = T_{\theta \theta}^{0,0} = C_{1}e^{-t/A_{1}},$$

(6.56)

$$T_{\theta x}^{0,0} = C_{2}e^{-t/A_{1}}, \quad T_{zz}^{0,0} = C_{3}e^{-t/A_{1}},$$

(6.57)

$$\phi^{(0)} = C_{4},$$

(6.58)

is a solution of the $(0,0)$-system (4.1) with $\frac{1}{p} = 0$, where $C_{i}$'s are arbitrary constants and $g(t)$ is an arbitrary smooth function of $t$. Therefore it is possible for the jet to support a nonzero torque

$$T = \int_{\text{cross section}} rT_{\theta z} da = \frac{1}{2} \varepsilon^{2} \phi^{(0)} 4 T_{\theta z}^{0,0} + O(\varepsilon^{3})$$

(6.59)

without a swirling velocity field, but in the absence of a torque there will be no swirling flow.

We conclude this dissertation with some open problems that I have been or will be working on in the near future:
1. Asymptotic and convergent properties of this formal perturbation theory;

2. coupling of thermal effects;

3. recovery of stability of the 3-D problem through higher-order equations;

4. relaxing axisymmetry;

5. using the same methodology to model thin-film blowing process.
Fig.1: A strongly viscoelastic jet model; leading order solution

Regime 1 with $B = F = A_1 = W = Z = a = 1$, $A_2 = 0$,

$$v_x^{0,0}(0) = \phi^{(0)}(0) = T_x^{0,0}(0) = 1.00, \quad v_x^{0,0}(1) = 1.10 :$$

(a) leading order free surface radius $\phi^{(0)}(z)$,

(b) leading order axial velocity $v_x^{0,0}(z)$,

(c) leading order axial force $F_x^{(0)}(z)$. 

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Figure 1
Figure 1 (continued)

![Graph showing axial force $F_z$ vs axial coordinate $z$. The graph depicts a decreasing curve from left to right, with $F_z$ values ranging from 1.5 to 0.5 and $z$ values ranging from 0.0 to 1.0.](image)
Fig. 2: Asymptotic corrections to Figure 1 without boundary perturbations

Regime 1 with $B = F = \Lambda_1 = W = Z = a = 1, \Lambda_2 = 0, \varepsilon = 0.1,$

\begin{align*}
v_z^{0,0}(0) &= \phi^{(0)}(0) = T_{zz}^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 1.10, \\
v_z^{0,1}(0) &= \phi^{(1)}(0) = T_{zz}^{0,1}(0) = v_z^{0,1}(1) = 0, \\
T_{rr}^{1,0}(0) &= T_{yy}^{1,0}(0) = T_{zz}^{1,0}(0) = 0, \\
v_z^{0,2}(0) + \frac{1}{2}v_z^{1,0}(0)[\phi^{(0)}(0)]^2 &= \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0, \\
v_z^{0,2}(1) + \frac{1}{2}v_z^{1,0}(1)[\phi^{(0)}(1)]^2 &= 0.
\end{align*}

(a) leading order free surface radius $\phi^{(0)}(z)$ (-----) vs. $\phi(z)$ including first and second order corrections (--------),

(b) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(c) leading order axial velocity $v_z^{0,0}(z)$ (-----) vs. $v_z^{\text{avg}}(z)$ including first and second order corrections (--------),

(d) $v_{\text{correction}}^{\text{avg}} = v_z^{\text{avg}} - v_z^{0,0}$,

(e) leading order axial force $F_z^{(0)}(z)$ (-----) vs. $F_z(z)$ including first and second order corrections (--------).

Note that the corrections $\phi_{\text{correction}}, v_{\text{correction}}^{\text{avg}},$ and $F_{\text{correction}}$ remain small along the length of the jet. Hence the leading order solution is asymptotically valid.

99
Figure 2
Figure 2 (continued)

**2c**

**2d**
Figure 2 (continued)
Fig. 3: Asymptotic corrections to Figure 1 due to boundary perturbations

Regime 1 with $B = F = \Lambda_1 = W = Z = a = 1$, $\Lambda_2 = 0$, $\varepsilon = 0.1$,

$$v_z^{0,0}(0) = \phi^{(0)}(0) = T_{zz}^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 1.10,$$

$$v_z^{0,1}(0) = \phi^{(1)}(0) = T_{zz}^{0,1}(0) = v_z^{0,1}(1) = 0,$$

$$T_{rr}^{0,0}(0) = T_{\theta\theta}^{0,0}(0) = T_{zz}^{1,0}(0) = 0,$$

$$v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = -3.00, \quad \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0,$$

$$v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0 :$$

(a) leading order free surface radius $\phi^{(0)}(z)$ (—–) vs. $\phi(z)$ including first and second order corrections (———),

(b) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(c) leading order axial velocity $v_z^{0,0}(z)$ (—–) vs. $v_z^{\text{avg}}(z)$ including first and second order corrections (———),

(d) $v_{\text{correction}}^{\text{avg}} = v_z^{\text{avg}} - v_z^{0,0}$,

(e) leading order axial force $F_z^{(0)}(z)$ (—–) vs. $F_z(z)$ including first and second order corrections (———).

This is one particular choice of boundary perturbation values. We have investigated many other boundary perturbations, with similar results that the corrections remain small. We conclude that the strongly viscoelastic leading order solution of Figure 1 is robust to boundary perturbations.
Figure 3
Figure 3 (continued)

**3c**

**3d**
Figure 3 (continued)
Fig. 4: Asymptotic corrections to Figure 1 due to weak retardation

Regime 1 with $B = F = \Lambda_1 = W = a = 1$, $\Lambda_2 = 0.02$, $\varepsilon = 0.1$,

- $v_z^{0,0}(0) = \phi^{(0)}(0) = T_{zz}^{0,0}(0) = 1.00$, $v_z^{0,0}(1) = 1.10$,
- $v_z^{0,1}(0) = \phi^{(1)}(0) = T_{zz}^{0,1}(0) = v_z^{0,1}(1) = 0$,
- $T_{rr}^{1,0}(0) = T_{\theta\theta}^{1,0}(0) = T_{zz}^{1,0}(0) = 0$,
- $v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0$,
- $v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0$.

(a) leading order free surface radius $\phi^{(0)}(z)$ (-----) vs. $\phi(z)$ including first and second order corrections (------),

(b) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(c) leading order axial velocity $v_z^{0,0}(z)$ (-----) vs. $v_z^{\text{avg}}(z)$ including first and second order corrections (------),

(d) $v_z^{\text{correction}} = v_z^{\text{avg}} - v_z^{0,0}$,

(e) leading order axial force $F_z^{(0)}(z)$ (-----) vs. $F_z(z)$ including first and second order corrections (------).

The leading order solution is robust to a weak retardation effect.
Figure 4
Figure 4 (continued)

4c

4d
Figure 4 (continued)
Fig. 5: A strongly viscoelastic jet model; leading order solution

Regime 1 with $B = A_1 = Z = a = 1$, $F = W = 5$, $\Lambda_2 = 0$,

$v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00$, $T_{zz}^{0,0}(0) = 0.50$, $v_z^{0,0}(1) = 1.25$:

(a) leading order free surface radius $\phi^{(0)}(z)$,

(b) leading order axial velocity $v_z^{0,0}(z)$,

(c) leading order axial force $F_z^{(0)}(z)$.

This solution is qualitatively different from the leading order solution of Figure 1, with the axial velocity increasing monotonically, and the axial force increasing instead of decreasing. This difference is achieved by varying the boundary conditions $T_{zz}^{0,0}(0)$ and $v_z^{0,0}(1)$, and the process parameters $F$ and $W$.  

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Figure 5
Figure 5 (continued)

![Graph showing the axial force $F_z$ against the axial coordinate $z$.](image)
Fig. 6: An asymptotically valid but catastrophically unstable steady solution

Regime 1 with $B = \Lambda_1 = Z = a = 1$, $F = 5$, $1/W = 0$, $\Lambda_2 = 0$,

\begin{align*}
\nu_z^{0,0}(0) &= \phi^{(0)}(0) = 1.00, \quad T_{zz}^{0,0}(0) = -1.50, \quad \nu_z^{0,0}(1) = 1.50, \\
\nu_z^{0,1}(0) &= \phi^{(1)}(0) = T_{zz}^{0,1}(0) = \nu_z^{0,1}(1) = 0, \\
T_{rr}^{1,0}(0) &= T_{\theta\theta}^{1,0}(0) = T_{zz}^{1,0}(0) = 0, \nu_z^{0,2}(0) + \frac{1}{2} \nu_z^{1,0}(0)[\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0, \\
\nu_z^{0,2}(1) + \frac{1}{2} \nu_z^{1,0}(1)[\phi^{(0)}(1)]^2 &= 0.
\end{align*}

(a) leading order free surface radius $\phi^{(0)}(z)$,

(b) leading order axial velocity $\nu_z^{0,0}(z)$,

(c) leading order axial force $F_z^{(0)}(z)$,

(d) leading order free surface radius $\phi^{(0)}(z)$ (——) vs. $\phi(z)$ including first and second order corrections (-----),

(e) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(f) leading order axial velocity $\nu_z^{0,0}(z)$ (——) vs. $\nu_z^{\text{avg}}(z)$ including first and second order corrections (-----),

(g) $\nu_z^{\text{avg}_{\text{correction}}} = \nu_z^{\text{avg}} - \nu_z^{0,0}$,

(h) leading order axial force $F_z^{(0)}(z)$ (——) vs. $F_z(z)$ including first and second order corrections (-----),

(i) characteristics defined in equation (6.1): $\text{Res}_3$ (——), $\text{Res}_4$ (----), $\text{Im}_3$ (-----), and $\text{Im}_4$ (………..).

Note that the imaginary parts of $s_3$ (-----) and $s_4$ (………..) are nonzero for $0 \leq z < 0.25$. The equations are therefore of mixed hyperbolic-elliptic type, and steady solutions are catastrophically unstable.
Figure 6
Figure 6 (continued)

6c
Figure 6 (continued)

6d

6e
Figure 6 (continued)

6f

6g
Figure 6 (continued)

[Graph showing axial force $F_z$ vs. axial coordinate $z$]

[Graph showing characteristics vs. axial coordinate $z$]

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Fig. 7: A solution to the Denn, Petrie & Avenas steady leading order model

Regime 2 with $\Lambda_1 = Z = a = 1$, $1/W = 0$,

$v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00$, $T_{xx}^{0,0}(0) = 3.50$, $F_z^{(0)}(1) = 2.22$

(a) leading order free surface radius $\phi^{(0)}(z)$,

(b) leading order axial velocity $v_z^{0,0}(z)$,

(c) leading order axial force $F_z^{(0)}(z)$. 

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Figure 7
Figure 7 (continued)

axial coordinate $z$

$7c$
Regime 2 with $\Lambda_1 = Z = a = 1, \ 1/W = 0, \ \varepsilon = 0.1,$
\[ v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00, \quad T_{zz}^{0,0}(0) = 3.50, \quad F_z^{(0)}(1) = 2.22, \]
\[ v_z^{0,1}(0) = \phi^{(1)}(0) = T_{zz}^{0,1}(0) = F_z^{(1)}(1) = 0, \]
\[ T_{rr}^{1,0}(0) = T_{\theta\theta}^{1,0}(0) = T_{zz}^{1,0}(0) = 0, \]
\[ v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0, \]
\[ v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0 : \]

(a) leading order free surface radius $\phi^{(0)}(z)$ (-----) vs. $\phi(z)$ including first and second order corrections (-----),

(b) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(c) leading order axial velocity $v_z^{0,0}(z)$ (-----) vs. $v_z^{\text{avg}}(z)$ including first and second order corrections (-----),

(d) $v_z^{\text{avg}} = v_z^{\text{avg}} - v_z^{0,0}$,

(e) leading order axial force $F_z^{(0)}(z)$ (-----) vs. $F_z(z)$ including first and second order corrections (-----).

Note that the corrections $\phi_{\text{correction}}, v_z^{\text{avg}}_{\text{correction}},$ and $F_{\text{correction}}$ remain small along the length of the jet. Hence the Denn, Petrie & Avenas solution of Figure 7 is asymptotically valid.
Figure 8
Figure 8 (continued)

8c

8d
Figure 8 (continued)

![Diagram showing axial force $F_z$ as a function of axial coordinate $z$. The graph is flat, indicating a constant value of axial force across the axial coordinate.]

8e
Fig. 9: Asymptotic corrections to the Denn, Petrie & Avenas solution of Figure 7 due to boundary perturbations

Regime 2 with $\Lambda_1 = Z = a = 1, 1/W = 0, \varepsilon = 0.1,$

$$v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00, \quad T_{zz}^{0,0}(0) = 3.50, \quad F_z^{(0)}(1) = 2.22,$$

$$v_z^{0,1}(0) = 0.50, \quad \phi^{(1)}(0) = T_{zz}^{0,1}(0) = F_z^{(1)}(1) = 0,$$

$$T_{rr}^{1,0}(0) = T_{g\phi}^{1,0}(0) = T_{zz}^{1,0}(0) = 0,$$

$$v_z^{0,2}(0) + \frac{1}{2}v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0,$$

$$v_z^{0,2}(1) + \frac{1}{2}v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0 :$$

(a) leading order free surface radius $\phi^{(0)}(z)$ (-----) vs. $\phi(z)$ including first and second order corrections (-----),

(b) $\phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z)$,

(c) leading order axial velocity $v_z^{0,0}(z)$ (-----) vs. $v_z^{\text{avg}}(z)$ including first and second order corrections (-----),

(d) $v_{\text{correction}}^{\text{avg}} = v_z^{\text{avg}} - v_z^{0,0}$,

(e) leading order axial force $F_z^{(0)}(z)$ (-----) vs. $F_z(z)$ including first and second order corrections (-----).
Figure 9
Figure 9 (continued)

9c

9d
Figure 9 (continued)
Fig. 10: Asymptotic corrections to the Denn, Petrie & Avenas solution of Figure 7 due to boundary perturbations

Regime 2 with \( \Lambda_1 = Z = \alpha = 1, \ 1/W = 0, \ \varepsilon = 0.1, \)
\[
v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00, \quad T_{zz}^{0,0}(0) = 3.50, \quad F_z^{(0)}(1) = 2.22,
\]
\[
v_z^{0,1}(0) = -0.50, \quad \phi^{(1)}(0) = T_{zz}^{0,1}(0) = F_z^{(1)}(1) = 0,
\]
\[
T_{rr}^{1,0}(0) = T_{\theta\theta}^{1,0}(0) = T_{zz}^{1,0}(0) = 0,
\]
\[
v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0) [\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0,
\]
\[
v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1) [\phi^{(0)}(1)]^2 = 0:
\]

(a) leading order free surface radius \( \phi^{(0)}(z) \) (——) vs. \( \phi(z) \) including first and second order corrections (----),

(b) \( \phi_{\text{correction}} = \phi(z) - \phi^{(0)}(z) \),

(c) leading order axial velocity \( v_z^{0,0}(z) \) (——) vs. \( v_z^{\text{avg}}(z) \) including first and second order corrections (----),

(d) \( v_{\text{correction}}^{\text{avg}} = v_z^{\text{avg}} - v_z^{0,0} \),

(e) leading order axial force \( F_z^{(0)}(z) \) (——) vs. \( F_z(z) \) including first and second order corrections (----).

Figures 9 and 10 consider two different choices of boundary perturbation values. We have investigated many others, with similar results that corrections remain small. We conclude that the Denn, Petrie & Avenas solution of Figure 7 is robust to boundary perturbations.

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Figure 10
Figure 10 (continued)

10c

10d
Figure 10 (continued)
Fig. 11: Asymptotic corrections to the Denn, Petrie & Avenas solution of Figure 7 due to the weak presence of surface tension

Regime 2 with $\Lambda_1 = Z = a = 1$, $1/W = 0.05 = 0.5\varepsilon = 5.0\varepsilon^2$, $\varepsilon = 0.10$,

\[ v_z^{0,0}(0) = \phi^{(0)}(0) = 1.00, \quad T_{zz}^{0,0}(0) = 3.50, \quad F_z^{(0)}(1) = 2.22, \]

\[ v_z^{0,1}(0) = \phi^{(1)}(0) = T_{zz}^{0,1}(0) = F_z^{(1)}(1) = 0, \]

\[ T_{rr}^{1,0}(0) = T_{\theta\theta}^{1,0}(0) = T_{zz}^{1,0}(0) = 0, \]

\[ v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = \phi^{(2)}(0) = T_{zz}^{0,2}(0) = 0, \]

\[ v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0 : \]

(a) leading order axial velocity $v_z^{0,0}(z)$ (-----) vs. $v_z^{avg}(z)$ including first and second order corrections for $1/W = 0.0$ (-----), $1/W = 0.5\varepsilon$ (-----),

and $1/W = 5.0\varepsilon^2$ (-----),

(b) $v_z^{avg} = v_z^{avg} - v_z^{0,0}$ for $1/W = 0.0$ (-----), $1/W = 0.5\varepsilon$ (-----),

and $1/W = 5.0\varepsilon^2$ (-----).

The Denn, Petrie & Avenas model is robust to a weak surface tension effect.
Figure 11
Fig.12: The Newtonian jet: asymptotic corrections to the leading order solution without boundary perturbations

Regime 4 with $\varepsilon = 0.1$,

$$\phi^{(0)}(0) = v_z^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 20.0,$$

$$\phi^{(1)}(0) = v_z^{0,1}(0) = v_z^{0,1}(1) = 0,$$

$$\phi^{(2)}(0) = v_z^{0,2}(0) + \frac{1}{2}v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = v_z^{0,2}(1) + \frac{1}{2}v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0 :$$

(a) leading order solution $v_z^{0,0}(z) = e^{(\ln 20)z}$ (-----) vs. $v_z^{avg}(z)$ including first and second order corrections (-----),

(b) $v_z^{avg} = v_z^{avg} - v_z^{0,0}$.

Note that the velocity correction $v_z^{avg}$ remains small along the length of the jet. Hence the leading order solution is asymptotically valid.
Figure 12

\hspace{1cm}

Figure 12
Regime 4 with $\varepsilon = 0.1,$

\[
\begin{align*}
\phi^{(0)}(0) &= v_z^{0,0}(0) = 1.00, & v_z^{0,0}(1) &= 20.0, \\
\phi^{(1)}(0) &= 0, & v_z^{0,1}(0) &= 3.00, & v_z^{0,1}(1) &= 0, \\
\phi^{(2)}(0) &= v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0) \left[ \phi^{(0)}(0) \right]^2 = v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1) \left[ \phi^{(0)}(1) \right]^2 = 0:
\end{align*}
\]

(a) leading order solution $v_z^{0,0}(z) = e^{(\ln 20)z} \text{ (-----)}$ vs. $v_z^{avg}(z)$ including first and second order corrections (-----),

(b) $v_z^{\text{correction}} = v_z^{avg} - v_z^{0,0}$.
Figure 13
Fig.14: The Newtonian jet: asymptotic corrections to the leading order solution of Figure 12 due to boundary perturbations

Regime 4 with \( \varepsilon = 0.1\),

\[\phi^{(0)}(0) = v_z^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 20.0,\]

\[\phi^{(1)}(0) = 0, \quad v_z^{0,1}(0) = -3.00, \quad v_z^{0,1}(1) = 0,\]

\[\phi^{(2)}(0) = v_z^{0,2}(0) + \frac{1}{2}v_z^{1,0}(0)[\phi^{(0)}(0)]^2 = v_z^{0,2}(1) + \frac{1}{2}v_z^{1,0}(1)[\phi^{(0)}(1)]^2 = 0:\]

(a) leading order solution \( v_z^{0,0}(z) = e^{(\ln 20)z} \) (-----) vs. \( v_z^{avg}(z) \) including first and second order corrections (-----),

(b) \( v_z^{correction} = v_z^{avg} - v_z^{0,0}. \)

Figures 13 and 14 exhibit the corrections to the solution of Figure 12 due to two choices of boundary perturbation values. We have investigated many other boundary perturbations, with the similar result that the corrections remain small. We conclude that the leading order solution of Figure 12 is robust to boundary perturbations.
Figure 14
Fig. 15: Weakly elastic corrections to the Newtonian leading order solution

Regime 3 with $Z = a = 1$, $\Lambda_1 = \bar{\Lambda}_1 \varepsilon$, $\varepsilon = 0.1$,

$$\phi^{(0)}(0) = v_z^{0,0}(0) = 1.00, \quad v_z^{0,0}(1) = 20.0,$$

$$\phi^{(1)}(0) = v_z^{0,1}(0) = v_z^{0,1}(1) = 0,$$

$$\phi^{(2)}(0) = v_z^{0,2}(0) + \frac{1}{2} v_z^{1,0}(0) [\phi^{(0)}(0)]^2 = v_z^{0,2}(1) + \frac{1}{2} v_z^{1,0}(1) [\phi^{(0)}(1)]^2 = 0 :$$

(a) leading order solution $v_z^{0,0}(z) = e^{(\ln 20)z}$ (---) vs. $v_z^{avg}(z)$ including elastic perturbation for several values of $\bar{\Lambda}_1$: $\bar{\Lambda}_1 = 0.0$ (-----), $\bar{\Lambda}_1 = 0.1$ (.........), $\bar{\Lambda}_1 = 0.2$ (------), $\bar{\Lambda}_1 = 0.3$ (-----).

(b) $v_z^{correction} = v_z^{avg} - v_z^{0,0}$, for several values of $\bar{\Lambda}_1$: $\bar{\Lambda}_1 = 0.0$ (-----), $\bar{\Lambda}_1 = 0.1$ (.........), $\bar{\Lambda}_1 = 0.2$ (------), $\bar{\Lambda}_1 = 0.3$ (-----).

The formal asymptotic theory assumes that $0.1 < \bar{\Lambda}_1 < 10$. Here we show that for most of this range the presumed asymptotic ordering of the solution expansion is violated (the corrections for $\bar{\Lambda}_1 > 0.2$ are of the order of leading order solution). Hence the weakly elastic model of Regime 3, although formally valid, is usually not physically relevant.
Figure 15
Fig. 16: The weakly elastic jet

Regime 3 with $Z = a = 1$, $\Lambda_1 = \tilde{\Lambda}_1 \varepsilon$, $0.0 \leq \tilde{\Lambda}_1 \leq 0.5$, $\varepsilon = 0.1$,

$\phi^{(0)}(0) = u_x^{0,0}(0) = 1.00, \quad v_x^{0,0}(1) = 20.0,$

$\phi^{(1)}(0) = u_x^{0,1}(0) = v_x^{0,1}(1) = 0,$

$\phi^{(2)}(0) = u_x^{0,2}(0) + \frac{1}{2} v_x^{1,0}(0) [\phi^{(0)}(0)]^2 = v_x^{0,2}(1) + \frac{1}{2} v_x^{1,0}(1)[\phi^{(0)}(1)]^2 = 0$:

Upper surface: cross-sectional averaged axial velocity $u_x^{\text{avg}}$ including weakly elastic first and second order corrections,

Lower surface: leading order Newtonian solution $v_x^{0,0}(z) = e^{(\ln 20)z}$.

Note that for $\tilde{\Lambda}_1 > 0.2$ the two surface are widely separated at some $z$ locations.

\section*{\textit{Fig. 16: The weakly elastic jet}}

Regime 3 with $Z = a = 1$, $\Lambda_1 = \tilde{\Lambda}_1 \varepsilon$, $0.0 \leq \tilde{\Lambda}_1 \leq 0.5$, $\varepsilon = 0.1$,

$\phi^{(0)}(0) = u_x^{0,0}(0) = 1.00, \quad v_x^{0,0}(1) = 20.0,$

$\phi^{(1)}(0) = u_x^{0,1}(0) = v_x^{0,1}(1) = 0,$

$\phi^{(2)}(0) = u_x^{0,2}(0) + \frac{1}{2} v_x^{1,0}(0) [\phi^{(0)}(0)]^2 = v_x^{0,2}(1) + \frac{1}{2} v_x^{1,0}(1)[\phi^{(0)}(1)]^2 = 0$:

Upper surface: cross-sectional averaged axial velocity $u_x^{\text{avg}}$ including weakly elastic first and second order corrections,

Lower surface: leading order Newtonian solution $v_x^{0,0}(z) = e^{(\ln 20)z}$.

Note that for $\tilde{\Lambda}_1 > 0.2$ the two surface are widely separated at some $z$ locations.
Figure 17. Weakly Torsional Effect upon The Axial Force $F_z$. Leading order prediction $F_z^{(0)}$ (-----) vs. the predictions including the first and second corrections without torsion (-----) and with torsion (······). Here the torsional effect is small along the entire length of jet. The asymptotic validity of the leading order solution is not affected by torsion.
Figure 18. Strongly Torsional Effect upon The Axial Force $F_z$. Leading order prediction $F_z^{(0)}$ (——) vs. the predictions including the first and second corrections without torsion (-----) and with torsion (······). Note that, in this example, the torsional effect is much larger than the example of Figure 17. The correction due to torsional effects is so large that the leading order solution is not a valid asymptotic prediction, so that torsion invalidates the asymptotics in this example.
Figure 19. Another Example of Strongly Torsional Effect upon The Axial Force $F_z$. Leading order prediction $F_z^{(0)}$ (-----) vs. the predictions including the first and second corrections without torsion (------) and with torsion (...........). Note that, in this example, the torsional correction is much smaller that the torsionless correction.
Figure 20. A Schematic Diagram of A Fiber Spinning Process
BIBLIOGRAPHY


