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Dynamical properties of Josephson junction arrays

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The Ohio State University, 1991
DYNAMICAL PROPERTIES OF
JOSEPHSON JUNCTION ARRAYS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of the Ohio State University

by

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The Ohio State University
1991

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To My Parents
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CHAPTER I

INTRODUCTION

The physical properties of granular superconductors have been of interest since the pioneering work of B. Abeles and his collaborators in the early 1970's[1]. Interest has grown in recent years for a variety of reasons. Most of all, it has been realized that many nominally homogeneous superconductors in fact show many attributes of granularity. In particular, there is considerable evidence that even nominally single crystal samples of the high-$T_c$ superconductors discovered by Bednorz and Müller[2] are often studded with grain and twin boundaries which behave very much like superconducting weak links[3]. These weak links may have something to do with the strange time-dependent magnetization of high-temperature superconductors, which has been variously attributed to glassy behavior[4], giant flux creep[5], and flux lattice melting[6]. Secondly, with recent advances in microfabrication technology, it has been possible to make artificial "granular materials," with controlled particle sizes, disorder, etc., which permit one to study systematically the physical properties of such disordered materials.

In this chapter, we will briefly review the fundamental background for the
dynamical response of a single junction, for granular superconductors viewed as collections of coupled weak links, and for the high-temperature cuprate superconductors.

The rest of the chapters are organized as follows. This thesis can be divided into two parts: The first part deals with a two-dimensional network of superconducting grains in Chapters II, III, and IV. The second part of the thesis, which consists of Chapters V and VI, deals with a three-dimensional array, which is somewhat closer to real granular superconductors.

In Chapter II, we present a calculational method for finding the dynamical properties of an array of resistively-shunted Josephson junctions. This method is used to solve various models presented in each subsequent chapter. The first part of Chapter III deals with pinning and depinning mechanisms of a single vortex in the periodic pinning potential of a square array. In the following part of the chapter, we present harmonic generation and microwave absorption by finite clusters of Josephson junctions in the presence of a combined d.c. and a.c. magnetic field. Chapter IV is devoted to quantized voltage plateaus, known as fractional giant Shapiro steps, for an array subject to a combined d.c. and a.c. current and a transverse magnetic field. Coherent motion of the vortices on the Shapiro steps is discussed. We also study the power emission on and off the steps as a function of an applied d.c. current.

We study the critical currents of composite superconductors in Chapter V. We offer a possible explanation for the saturation of the critical currents with increasing magnetic field found in recent experiments on high-temperature superconductors. We propose a new scaling interpretation of the phase transition near the percolation threshold of a three-dimensional array. In Chapter VI, we
discuss the influence of a magnetic field in terms of a vortex glass model and and flux flow resistivity. In order to model the behavior of high-temperature superconductors, we study a model involving an array of Josephson junctions. Scaling behaviors near the vortex glass transition and the transition in a three-dimensional classical XY system at zero magnetic field are also discussed to extract the experimentally relevant critical exponents.

1.1 Dynamics of a Single Junction

A Josephson junction consists of two superconducting grains (electrodes) separated by a thin insulator, a normal metal, or another superconductor having a lower transition temperature. Each superconducting grain can be characterized by the complex superconducting order parameter, \( \psi = |\psi| \exp(i\phi) \), where \( \phi \) is the phase of the order parameter. The problem of the single junction is a special example of quantum-mechanical tunneling through a barrier, in this case involving tunneling Cooper pairs of charge \( 2e \). Josephson analyzed this situation and discovered the interesting phenomena that bears his name[7]. The key results are the relations for the Josephson (pair) current and the voltage:

\[
I = I_c(T) \sin(\phi_1 - \phi_2), \tag{1.1}
\]

\[
V = \frac{\hbar}{2e} \frac{d(\phi_1 - \phi_2)}{dt}, \tag{1.2}
\]

where \( \phi_i \) (\( i = 1, 2 \)) is the phase of the order parameter of grain \( i \), and \( I_c(T) \) is called the critical current. Equation (1.1) describes the sinusoidal current flow from grain 1 to grain 2, while Eq. (1.2) relates the voltage drop across the junction to the time-derivative of the phase difference.

The critical current \( I_c(T) \) depends on the geometry and material of the junction, temperature, and other factors. If the material of the junction is an insu-
lator, the critical current for identical superconducting grains is given by [8]

\[ I_c(T) = \frac{\pi \Delta(T)}{2eR_N} \tanh \left( \frac{\Delta(T)}{2k_B T} \right), \] (1.3)

where \( \Delta(T) \) is the equilibrium value of the energy gap and \( R_N \) is the resistance of the junction in a normal state. For this superconducting-insulating-superconducting (SIS) type of tunnel junction, the critical current of the junction is far less than that of the superconducting grains.

On the other hand, if a layer of normal metal is interposed between two superconducting grains, the critical current is governed by the well-known proximity effect [9].

\[ I_c(T) = C(T_c - T)^2 e^{-r/\xi_N(T)}, \] (1.4)

where the constant \( C \) depends on the geometry and material of the junction, \( T_c \) is the transition temperature of the grains, and \( \xi_N(T) \) is the coherence length of the normal metal. The proximity effect in this superconducting-normal-superconducting (SNS) junction lies in the fact that some Cooper pairs will penetrate into the normal metal from the superconductor. This leads to a nonzero order parameter within the normal metal, which exponentially decreases over a distance of the order of \( \xi_N(T) \).

A typical tunnel junction has a large intrinsic capacitance. On the other hand, the effect of the capacitance in the proximity-effect junction is negligible. In a simplified equivalent circuit, the resistance effectively shunts the conductance in this SNS-type junction:

\[ I = I_c \sin(\phi_1 - \phi_2) + \frac{V}{R_N}, \] (1.5)

where \( R_N \) is the shunt resistance. This equation is known as the resistively-shunted junction (RSJ) model [10] In this model, the time dependence of the
phase satisfies a simple first-order differential equation. The voltage across the
junction connected to a d.c. current source I [obtained by solving Eqs. (1.5) and
(1.2) simultaneously] can be shown to be given analytically by

\[
V(t) = \frac{R_N(I^2 - I_c^2)}{I + I_c \sin(\omega_J t + \alpha)}; \\
\langle V(t) \rangle = R_N \sqrt{I^2 - I_c^2},
\]

(1.6) \hspace{1cm} (1.7) \hspace{1cm} (1.8)

where \(\omega_J\) is the Josephson frequency \(2e(V)/\hbar\), \(\alpha\) is a constant depending on the
initial conditions, and \(\langle \ldots \rangle\) denotes a time average. The time-dependent voltage
in Eq. (1.6) represents the Josephson oscillation and its Fourier analysis shows
many higher harmonics beyond the fundamental Josephson frequency:

\[
V(t) = \langle V \rangle + \sum_{m=1}^{\infty} V_m \cos(m\omega_J t);
\]

\[
V_m = 2R_N \sqrt{I^2 - I_c^2} \left( \frac{I}{I_c} - \sqrt{\left( \frac{I}{I_c} \right)^2 - 1} \right)^m,
\]

(1.9) \hspace{1cm} (1.10)

where \(V_m\) is the amplitude of \(m^{th}\) harmonic component.

1.2 Dynamics of Granular Superconductors

Studies of superfluid and superconducting films, and planar arrays of coupled
weak links[11], have been intensified by the experimental evidence for a phase
transition, known as a Kosterlitz-Thouless (KT) transition, which was first pre­
dicted for an abstract Hamiltonian known as the two-dimensional XY model[12].
The main features of the KT transition can be deduced from a vortex-unbinding
picture in a two-dimensional system which intrinsically has a relatively large
magnetic penetration length. In a superconducting film, the flux flow resistance[13] and the nonlinear I-V characteristics around the transition temperature $T_c$ are governed by the temperature-dependent coherence length, and more directly by the density of free vortices[14]. The density of free vortices is very sensitive to changes in the current. For $I \rightarrow 0$, Halperin and Nelson[14] found nonlinear behavior in the voltage:

$$V \propto I^{a(T)},$$

(1.11)

where, for $T < T_c$, $a(T) = 1 + 1/(2\eta(T))$ is a power-law exponent, and $\eta(T)$ ($0 \leq \eta(T) \leq 1/4$) is the phase-phase (spin-spin) correlation function. Above $T_c$, the presence of thermally-activated free vortices implies a linear I-V characteristics for small current. Since $\eta(T)$ has the value of $1/4$ at $T = T_c[15]$, the exponent $a(T)$ jumps from 1 to 3 as the temperature is lowered from $T_{c+}$ to $T_{c-}$. The direct measurement of the exponent $a(T)$ was a key experiment which helped to demonstrate the existence of the universal jump at $T_c$ in superconducting thin films and arrays[16,17].

In the presence of a magnetic field, the phases arrange themselves into a relatively lower energy state to form a flux lattice. In a film the vortices form a triangular lattice in the ground state whose melting causes the phase transition. For an array of Josephson junctions, the lattice introduces a pinning potential so that the vortices are constrained to lie at the pinning sites. The phases of the order parameter at a finite magnetic field are “frustrated” in a sense that they are different from all parallel phases of the ground-state phases at zero magnetic field. The frustration $f$ in a closed area surrounded by superconducting grains is defined as the ratio of the external magnetic flux $\Phi$ to the flux quantum $\Phi_0 =$
\[ f = \frac{\Phi}{\Phi_0}. \]  

A number of experimental[16-19] and theoretical works[20,21] have been devoted to static and dynamic properties of an array of Josephson junctions in the presence of an applied magnetic field. Among these studies is a systematic study of fractional giant Shapiro steps[21] in the time-averaged voltage.

Thus far, we have briefly reviewed two-dimensional systems. However, experimental and theoretical work has not been limited to 2D arrays. Recently, three-dimensional arrays[22] have drawn attention because of the existence of weak links in the high-temperature cuprate superconductors[3]. We are interested in type-II superconductors because the large magnetic penetration depth and the relatively short coherence length for the high-temperature superconductors suggest modeling these systems as arrangements of weak links.

Figure 1.1(a) shows the schematic H–T phase diagram for conventional type-II superconductors which exhibit relatively much smaller thermal fluctuations than do the high-T\textsubscript{c} superconductors. The vortex lattice (Abrikosov) phase occurs between the lower (H\textsubscript{c1}) and the upper (H\textsubscript{c2}) critical fields. In the absence of pinning centers, the vortex lattice phase has a finite flux-flow resistance[13], i.e., energy dissipation through normal cores of moving vortices. In the presence of disorder, the flux-flow resistance is reduced by the pinning of vortices. However, the vortex lattice loses its long-range order through random pinning produced by the disorder[23] and it possibly shows glassy behavior[4,24].

In the high-temperature superconductors, strong thermal fluctuations lead to somewhat different phase diagram[25] from the low-T\textsubscript{c} superconductors, as shown in Fig. 1.1(b). The thermal fluctuations are enhanced by several factors:
Figure 1.1:
(a). Schematic phase diagram of a type-II superconductor as a function of the magnetic field $H$ and the temperature $T$.  (b). Schematic phase diagram of a three-dimensional type-II superconductor with strong thermal fluctuations. Without random pinning a vortex lattice phase is present while with random pinning the vortex lattice is replaced with a vortex glass phase. After Ref. [25].
a high transition temperature, a large magnetic penetration length, a short coherence length, and a large anisotropy (quasi-two dimensionality). With random pinning, the vortex glass phase appear in three dimensions (not in 2D) with a long-range order phase coherence that destroys the resistance. This is in sharp contrast to the flux creep model[26] which predicts nonzero resistance except at $T = 0$. Thus, the normal to superconducting transition occurs at the phase boundary between the vortex glass phase and the vortex fluid phase. Below the glass transition temperature, the voltage $V$ is predicted to exhibit an extremely nonlinear current dependence ($V \propto \exp(-\mu J^\mu)$)[25], where $\mu$ is a glass exponent, and satisfies $0 < \mu \leq 1$. Evidence for the existence of a vortex glass was reported from measurement of nonlinear I-V characteristics[27] and complex impedance[28].

The phase boundary between the normal phase and the vortex liquid phase occurs near the mean field transition $H^{MF}$, but it is not a sharp transition. The vortex liquid phase is a fully disordered phase but there is local pairing, while the normal phase does not have any local pairing. Recent measurements of resistivity versus temperature at several constant fields[29] and resistivity (or Hall resistivity) versus magnetic field at constant temperatures[30] showed a broad resistive transition. One of the main issue was the existence of a crossover temperature $T_c$ in the vortex fluid phase, which separates two different regimes in the resistive transition. For $T > T_c$, the flux flow governs the unpinned vortices in the vortex fluid phase. Below $T_c$ the vortex fluid can move by a thermally activated flux flow since it is in the pinned state, and thus the resistivity drops exponentially: $\rho \sim \exp(-U_0/T)$, where $U_0$ is a magnetic field-dependent activation energy.
CHAPTER I REFERENCES


[18] For recent reviews and an extensive list of references, see, e. g., C. J. Lobb, in Physica (Amsterdam) 126B+C, 319 (1984); or the articles in Physica (Amsterdam) 152B+C (1988).


[21] See, e.g., references in Chapter IV.


CHAPTER II

CALCULATIONAL METHOD

2.1 Introduction

Prior to our own work, several calculations had already investigated the behavior of a few coupled Josephson junctions[1]. These calculations were usually motivated by an interest in chaotic response by groups of junctions, and have usually included capacitive terms in the junction equations[1,2]. Recently many groups have been investigating large arrays of resistively-shunted junctions at finite temperature and magnetic field[3-8]. In this chapter, we describe our own method for carrying out dynamical calculations for arrays of resistively-shunted Josephson junctions.

2.2 Dynamical Equations

We consider a (possibly disordered) square collection of $N \times N$ superconducting grains. Figure (2.1) shows a schematic diagram for $8 \times 8$ superconducting grains. The $i^{th}$ grain is assumed to be coupled to the $j^{th}$ by a Josephson junction in
Figure 2.1:
Schematic of an array of resistively shunted Josephson junctions, with boundary conditions as used in these calculations. Current $I$ flows into each of the grains in the bottom row and out of each grain on the top row. Top and bottom edges are free, while left and right sides are treated either free or periodic boundary conditions. (a) an ordered array of $8 \times 8$ superconducting grains; (b) a disordered array with a missing superconducting grain.
parallel with a shunt resistance. Thus the current from grain $i$ to grain $j$ is given by

$$I_{ij} = I_{cij} \sin(\phi_i - \phi_j) + \frac{V_{ij}}{R_{ij}}$$

(2.1)

where $\phi_i$ is the (time-dependent) phase of the superconducting order parameter on the $i^{th}$ grain, $I_{cij}$ is the critical current of the junction between grains $i$ and $j$, $V_{ij} = V_i - V_j$ is the voltage drop between grains $i$ and $j$, and $R_{ij}$ is the shunt resistance between these grains. We assume that capacitive terms in the equations of motion can be neglected; this is probably a reasonable approximation for superconducting-normal-superconducting (SNS) arrays, but will be less accurate for superconducting-insulating-superconducting (SIS) systems. We also assume weak coupling, so that the induced magnetic field generated by currents in the array can be neglected. This will be a reasonable approximation provided that the supercurrents in the array are sufficiently small. In practical terms, it is probably adequate that the Josephson penetration depth of the array be large in comparison to the array dimensions. The voltage drop across the junction is related to the phase difference by the Josephson relation,

$$\frac{d}{dt}(\phi_i - \phi_j) = \frac{2e(V_i - V_j)}{\hbar}.$$  

(2.2)

These equations must be supplemented by Kirchhoff's equation, which states that current at each node is conserved:

$$\sum_j I_{ij} = I_{i\text{ext}},$$

(2.3)

where $I_{i\text{ext}}$ is the external current fed into grain $i$. (In practice, $I_{i\text{ext}}$ will generally be zero except at the boundaries of the array.)

Given the external currents, Eqs. (2.1) - (2.3) may be conveniently combined
into a set of $N^2$ coupled non-linear first order differential equations in the phases:

$$
\sum_j [g_{ij} V_{ij} + I_{cij} \sin(\phi_i - \phi_j)] = I_{iext},
$$

(2.4)

where $g_{ij} = 1/R_{ij}$. With the introduction of the N-component vector $F(t)$ defined by

$$
F_i(t) = I_{iext} - \sum_j I_{cij} \sin(\phi_i - \phi_j),
$$

(2.5)

and the matrix $M$ with components defined by

$$
M_{ij} = \begin{cases} 
\sum_j 1/R_{ij} & \text{if } i = j \\
-1/R_{ij} & \text{if } i \neq j
\end{cases}
$$

(2.6)

we can rewrite eq. (2.4) as

$$
M \frac{d\vec{\phi}}{dt} = \frac{2e}{\hbar} F
$$

(2.7)

where $\vec{\phi}$ is a column vector with components $\phi_i$. Equation (2.7) constitutes a set of $N^2$ coupled first-order differential equations for the $N^2$ unknown phases. Disorder in critical currents can be explicitly included in $\{I_{cij}\}$ of Eq. (2.5). The matrix $M$ in Eq. (2.6) is independent of critical current, but depends on the values of the shunt resistances, meaning that the matrix elements are expressed differently according to the boundary conditions, dimensionality, and disorder in shunt resistances.

It would seem that Eq. (2.7) could be rewritten by inverting the matrix $M$ and expressing $d\vec{\phi}/dt$ in terms of $F$. However, $M$ is not invertible as it stands. The singularity of $M$ is due to the existence of a zero eigenvalue with eigenvector $V = V_0 (1, 1, ..., \equiv (\hbar/2e)d\vec{\phi}/dt$, where $V_0$ is a constant. This singularity originates from the $U(1)$ symmetry of the system: a uniform rotation of all the phases leaves the equations of motion invariant. To remedy this problem, one can fix
one of the phases, which amounts to deleting an arbitrary (say $k^{th}$) row and the $k^{th}$ column from $M$. Physically, this has the effect of fixing the value of the phase on the $k^{th}$ grain. Equation (2.7) can then be inverted to give

$$V' = G'F',$$

where $G' = M'^{-1}$, and $M'$ is the matrix constructed by deleting one column and one row from $M$. $V'$ and $F'$ are the column vectors corresponding to $V$ and $F$ but with the $k^{th}$ components removed.

We iterate this set of equations by simple first-order integration with time step $\Delta t$, using the Josephson relations Eq. (2.2) to give the phases $\phi_i(t)$ and the voltages $V_i(t)$. Use of more sophisticated second-order or fourth-order Runge-Kutta procedures has a negligible effect on the results. Typically, we use $\Delta t \approx 0.01 - 0.05t_0$ where $t_0 = \hbar/(2eR_0I_c)$, $R$ being the shunt resistance and $I_c$ the critical current of each individual Josephson junction. Since the matrix $G'$ is long ranged, with matrix elements varying as $\ln|z_i - z_j|$ in two dimensions and as $1/|z_i - z_j|$ in three dimensions, this iteration can still be relatively time-consuming.

As an output from the calculation, we take the difference between the average potential on the grains of the top row and that of the bottom row. The time response of the voltage is conveniently expressed in terms of the power spectrum $S_V(\omega)$ for the voltage:

$$S_V(\omega) = \lim_{T_0 \to \infty} \left| \int_0^{T_0} V(t)e^{i\omega t} dt \right|^2.$$  \hspace{1cm} (2.9)

Both the time-dependent voltage and its power spectrum are good tools to study the periodicity, intermittency (one of the typical routes of transition from a periodic state to chaos), and chaos in the nonlinear dynamical systems [9]. In
Figure 2.2:
Typical time-dependent voltages just above the critical current for very small values of the frustration. [See Eq. (1.12) for the definition.] Note the regular appearance of spikes of nearly equal amplitude and shape, interspersed with occasional bursts of seemingly chaotic voltage pulses. (a) $f = 1/64$, $9 \times 9$ lattice; (b) $f = 1/120$, $9 \times 9$ lattice.
Ref. [8], we have shown qualitatively that resistive-shunted Josephson junction arrays can show these behaviors even without any capacitive term. Figure 2.2 shows examples of characteristic voltage patterns resembling the intermittent regime at very small values of an applied magnetic field[10]. The method for including an applied magnetic field is discussed in the next section.

2.3 Finite Magnetic Field

Thus far, we have discussed the case of no vector potential and zero temperature. In the presence of a vector potential (resulting, e.g., from an applied magnetic field), the Josephson current between grains $i$ and $j$ is expressed in the gauge-invariant form $I_{cij} \sin(\phi_i - \phi_j - A_{ij})$ where

$$A_{ij} = \frac{2\pi}{\Phi_0} \int_{\bar{z}_i}^{\bar{z}_j} \vec{A} \cdot d\bar{l}.$$  \hfill (2.10)

Here $\bar{z}_i$ denotes the center of grain $i$, and $\Phi_0 = \hbar c/2e$ is a flux quantum. The phase factor $A_{ij}$ satisfies the constraint that the sum around any unit cell of a square array is given by

$$A_{ij} + A_{jk} + A_{kl} + A_{li} = 2\pi f = 2\pi B a^2/\Phi_0,$$ \hfill (2.11)

where $\vec{B} = \vec{\nabla} \times \vec{A} = B \hat{z}$ is the external magnetic field and $a$ is an array lattice constant.

2.4 Finite Temperature

A finite temperature may be included by introducing into the $ij$th shunt resistance a fluctuating noise current $L_{ij}(t)$ whose ensemble average satisfies

$$\langle L_{ij}(t) \rangle_s = 0$$
\[ \langle L_{ij}(t+\tau)L_{kl}(t) \rangle_e = \frac{2k_BT}{R_{ij}} \delta(\tau)\delta_{ijkl} \]  

(2.12)

where \( \langle \cdots \rangle_e \) denotes an ensemble average. With this assumption, the noise currents in different shunt resistances are uncorrelated, and the noise current within a given bond has a zero correlation time, corresponding to white noise (i.e. Johnson noise). Note that Eq. (2.12) is a generalization of the noise current assumed by Ambegaokar and Halperin\cite{11} in their discussion of a single Josephson junction with thermal noise. Combining this noise current with the expressions for the other currents, we obtain an equation for \( F_i(t) \) in the presence of both a vector potential and thermal noise currents:

\[ F_i(t) = I_{i_{ext}} - \sum_j [I_{c_{i,j}} \sin(\phi_i - \phi_j - \Delta_{ij}) + L_{ij}(t)]. \]  

(2.13)

To describe the random noise currents \( L_{ij}(t) \) within each time step, \( \Delta t \), we use either a Gaussian distribution or a uniform distribution. In the former case, we select values of the \( L_{ij}(t) \)'s within each time step from a Gaussian distribution with a standard deviation \( \sigma_L/\sqrt{\Delta t} \), where \( \sigma_L \) is chosen so as to satisfy Eq. (2.12). In latter case, the noise currents \( L_{ij}(t) \) are randomly selected from a uniform distribution in the range between \(-\sigma_L\sqrt{3/\Delta t}\) and \(\sigma_L\sqrt{3/\Delta t}\), where the prefactor \( \sqrt{3} \) is chosen so that the random currents satisfy Eq. (2.12). Both the Gaussian and the uniform distributions have the same means and the same two-point correlation functions, i.e.,

\[ \langle L \rangle^G_e = \langle L \rangle^U_e, \]

\[ \langle L(t)L(t') \rangle^G_e = \langle L(t)L(t') \rangle^U_e. \]  

(2.14)

The Gaussian distribution is believed to be more physical than the uniform distribution. However, we have found no significant difference between the two in
any of our numerical studies throughout this work. In order to reduce computing time (usually in a vector machine), we have usually chosen the uniform distribution for the random noise currents[12].

2.5 Initial Conditions

We have considered a variety of initial conditions for Eq. (2.8): the ground state phase configuration, parallel phases, a single vortex phase configuration, random phases, etc.... At zero temperature and zero magnetic field, parallel phases are usually selected. To examine a special type of initial condition, the first part of Chapter III is devoted to the motion of a single vortex in an array. The initial conditions are chosen to be appropriate to a single vortex. This is discussed further below.

At zero temperature and finite magnetic field, the ground state phase configuration[13,14] is generally used for the initial phases. To obtain the ground state phase configuration at a given flux per plaquette, $f = \Phi/\Phi_0$, we minimize the Josephson energy in the absence of the external current,

$$H = -E_J \sum_{<ij>} \cos(\phi_i - \phi_j - A_{ij}),$$

(2.15)

where $E_J$ is the Josephson coupling energy, $E_J = \hbar I_c/2e$. Figures (2.3)(a)-(e) show examples of ground state phase configurations for several values of f's. In some cases there are other degenerate ground states which are not shown. The arrow at each node indicates the phase angle while the arrow at each junction represents the direction of Josephson current. The corresponding ground state energy is plotted in Fig. (2.3)(f) as a function of 1/f. Note that the energy is symmetric around $f=1/2$ because all the physical quantities are invariant under the transformation $f \rightarrow f+1$ and $f \rightarrow -f$. 
Figure 2.3:
Examples of ground state phase configurations calculated by minimizing Eq. (2.15). (a). $f=1/2$, (b). $f=1/3$, (c). $f=1/4$, (d). $f=1/5$, (e). $f=2/5$. The arrow at each node indicates the phase angle. The arrow on each junction represents the Josephson current. A square with a plus sign contains a vortex; (f). The ground state energy as a function of $1/f$. 

frustration, $f=p/q$

- ENERGY

2.0

1.8

1.6

1.4

1.2

0.0  0.5  1.0
At finite temperature which corresponds to most of our calculations, we have used as an initial condition the last phase configuration of the previous value of current (or temperature). This is a convenient choice if we wish to step the current (or temperature) up or down. Such a procedure is useful for studying hysteresis in the glasslike state which is thought to occur irrational values of $f$ [15,16].

2.6 Discussion

The methods described in this chapter have a very broad range of applicability. As indicated above, they can be used to treat ordered and disordered arrays with and without an applied magnetic field. While the disordered arrays are more conveniently treated under the assumption that the shunt resistance is identical for each junction, this simplification is made only for calculational convenience and in no way inhibits the application of the method to more complicated disordered media. The same approach can also be used to study the finite frequency response of arrays (and, by implication, granular media)[17]. These calculations would have numerous important applications, including the studies of microwave absorption and magnetoresistance in arrays and granular systems as a function of temperature. Many phenomena have been observed in such arrays, particularly in high-temperature superconductors[18].

A potentially important result of the present calculations is the existence of both periodic and "chaotic" voltage power spectra, depending on the magnetic field and the applied current. These regimes would correspond to "coherent" and "incoherent" responses of the network. Evidently as will be seen later, both can exist at 0 K. A coherent response would suggest the possibility of coherent
radiation by the network at the response frequency, and perhaps, the possibility of coherent detection of radiation. The present method of studying the dynamics of Josephson systems can be used to systematically examine such coherent and incoherent response. In the coming chapters, it proves convenient to interpret this behavior in terms of mode locking and other complex responses of coupled nonlinear systems.
CHAPTER II REFERENCES


[10] The behavior shown in Fig. 2.2 differs from intermittent chaos in that the spikes between the chaotic bursts are apparently not periodically spaced.

For the present application, which deals with low-frequency properties, this property would explain the similarity between results for the Gaussian and uniform distributions. For further discussion, see, e.g., N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, New York, 1981), pp. 237ff.


W. Y. Shih and D. Stroud, Phys. Rev. B28, 6575 (1983);


See, e.g., Chapter IV and references therein.

3.1 Introduction

The high-temperature ceramic superconductors have exceptionally broad resistive transitions which are highly sensitive to even very weak magnetic fields[1,2]. Their magnetization shows strong hysteresis, and decays very slowly with time, often changing over a period of hours or even days[3]. At higher (microwave) frequency, they show a strong nonresonant absorption which is magnetic-field dependent, nonlinear in the incident intensity, and hysteretic[4], which might be explained principally in terms of viscous damping of fluxons. At lower frequencies, they show strong generation of higher harmonics when they are disturbed by radiofrequency pulses[5-8].

Much of this behavior has been attributed to the presence of weak links. Such weak links have been shown to form in grain and twin boundaries[9,10], where, because of the extremely short coherence lengths of the ceramic superconductors (of the order of 10 Å)[1], the order parameter can easily be suppressed. It is apparently very difficult to prepare ceramic superconductors without the existence
of weak links. (Thus, the dynamical properties of these materials may well be described in terms of random networks of weak links.)

In the first part of this chapter, we describe some recent work on the dynamical properties of two-dimensional networks of weak links. Previous calculations using similar methods have been presented by several authors[11-16]. These calculations consider specifically the critical energy for depinning of vortices below the critical current. The results of these calculations can in principle be compared to experiments on artificially prepared two-dimensional networks. Even more of interest, however, is the possible connection to random granular superconductors, and to high-temperature superconductors, where many of the phenomena we discuss may occur naturally.

In the second part of the chapter, we present the a.c. response of a cluster of weak links. The grain and twin boundaries in ceramic superconductors act as weak links. The model can be viewed as a primitive representation of a granular high-Tc superconductor exposed to a combination of d.c. and a.c. magnetic fields.

3.2 Motion of a Single Vortex

3.2.1 Periodic Pinning Potential

A single resistively-shunted junction (RSJ), that is driven by a d.c. current larger than a characteristic critical current $I_c$, shows a periodic time-dependent voltage $V(t)$. The period diminishes as the current increases. When $I < I_c$, $V(t)$ falls off exponentially with time and the time average $\langle V(t) \rangle$ vanishes.

The behavior of a single RSJ is sometimes interpreted in terms of a "particle" moving through a "washboard potential". The equation of motion Eqs. (2.1)
and (2.2) are combined to give

\[ \phi = -\frac{1}{\tau_0} \frac{\partial W}{\partial \phi} \]  

\[ W(\phi) = -\cos \phi - i \phi \]  

where \( i = \frac{I}{I_c} \) is a normalized current, \( \phi \) is the phase difference across the junction, and \( \tau_0 = \frac{\hbar}{2eR_0I_c} \) is a characteristic time. \( W(\phi) \) is the periodic washboard potential. For \( i < 1 \), the “particle” settles in one of the minima of the potential, i.e., the phase approaches a constant value and the voltage dies away exponentially. When \( i > 1 \), the particle slides down the washboard, approaching a constant average speed (i.e., constant voltage). The voltage has periodic maxima at those times when the particle moves through the regions of the potential with the greatest slope. Because there is no capacitance term, the particle is massless.

In an ordered array of \( N \times N \) superconducting grains, when the phases are initially all parallel, the array can behave just like a single junction. The parallel initial state is the ground state configuration of the potential energy function for the \( N^3 \) phases

\[ W(\phi_1, ..., \phi_{N^2}) = -\frac{\hbar}{2e} \sum_{i\neq j} J_{cij} \cos(\phi_i - \phi_j), \]  

in the absence of an external current.

If we start from a non-ground-state configuration, we find very different behavior. A simple non-ground state configuration is a single “vortex” in an array. We introduce a vortex at some point (usually at the center of a plaquette near the middle of the array) within the array, and define the vortex configuration to be an arrangement of phases in which each phase (as denoted by an arrow) points radially outward from the vortex. The potential energy of such a phase
configuration can be plotted as a function of the vortex location \((x,y)\). [For convenience, we choose the center of a grain at lower left corner as the origin of the coordinates in two-dimensional geometry, and assume that the current flows toward the positive \(y\)-axis.] The resulting potential, denoted \(W(x,y)\), looks very much like an "egg-crate" and is plotted in Fig. 3.1(a). The plotted array of \(8 \times 8\) superconducting grains contains a single missing grain at site \((3,4)\). (The critical currents \(I_{tij}\) between nearest neighbors, except bonds connecting the missing grain, are assumed to be identical.) Note that each energy cusp corresponds to the location of the superconducting grain, meaning that the center of a plaquette is relatively preferred position for a single vortex. This is the two-dimensional analog of the washboard potential discussed above. Because of the free boundary conditions in two directions, the potential is not quite periodic but bends outward towards the edges of the array.

If there is an external current, the energy of Eq. (3.2) has another term,

\[
W(\phi_1, ..., \phi_{N^2}) = -\frac{\hbar}{2e} \sum_{<ij>} I_{tij} \cos(\phi_i - \phi_j) - \sum_i I_{\text{ext}} \phi_i, \tag{3.3}
\]

where \(I_{\text{ext}}\) is the external current fed into grain \(i\): +I for grains on the bottom row, -I for grains on the top row of the array, and zero otherwise. When a defect is introduced into the array in the form of a missing superconducting grain, the potential has a deep minimum near the defect [Figs. 3.1(a) and (3.1)(b)], superimposed on the two-dimensional pattern of the periodic array. Figure 3.1(b) shows an example of the tilted potential, Eq. (3.3), due to an applied current in the presence of a single defect.

The phase configuration of a defect-free array of \(16 \times 16\) superconducting grains containing a single vortex is shown in Fig. 3.2. Initially, the vortex is at the center of the array, as shown in Fig. 3.2(a). Depending on the applied
Figure 3.1:
"Egg-crate" potential energy function $W(x,y)$ for a single vortex in (a) an array of $8 \times 8$ superconducting grains with a single missing grain at site (3,4); and (b) the same array but with a current in the direction denoted by an arrow.
Figure 3.2:
Phase configuration of an ordered lattice (16 x 16) containing a single vortex at the point indicated by the black dot. (a) Initial position of vortex; (b) position of vortex at time $t = 69.25(\hbar/2eR_0I_c)$ after application of current $i = 0.15$. 
current, the vortex may be driven towards the left-hand edge of the array by the “Magnus” (Lorentz) force, $\vec{F}$, between the current and magnetic field of the vortex:

$$\vec{F} = \vec{J} \times \frac{\Phi_0}{c}, \quad (3.4)$$

where $\Phi_0$ is the flux quantum in the direction of the magnetic field, $\vec{J}$ is the current density, and $c$ is the speed of light. [The tilt angle in Fig. 3.1(b) is determined by the magnitude of the current density $\vec{J}$.] A typical later configuration of the vortex is shown in Fig. 3.2(b) for one such applied current. The position of the vortex in both instances is marked by a dot.

The voltage pattern for an ordered 16x16 array is shown in Fig. 3.3 for several values of the applied current, starting from the initial vortex configuration of Fig. 3.2(a). For small currents, the vortex remains pinned in the center of the array and the voltage dies away exponentially. When the current exceeds a depinning valued $i_{c1}$ (in this particular array, $i_{c1} \approx 0.0967$), the vortex escapes from the center and is driven off the left edge of the array, as shown in Fig. 3.3(b) for several values of the applied current. The ripples in the $V(t)$ curve correspond to the vortex rolling over the hills and valleys of the egg-crate potential. There is a slight acceleration, corresponding to the outward warp of this potential surface, and the vortex produces its maximum voltage pulse as it escapes from the edge of the array. For $i > i_{c2}$, there is a continuous generation of vortices from left to right, initially with a periodic motion but eventually with no periodicity.

3.2.2 Pinning and Depinning of a Vortex Near Defects

The vortex depinning current is strongly affected by the presence of defects. Consider a defect in the form of a missing grain, by assuming that each nearest
Figure 3.3:
Voltage traces for an ordered 16x16 lattice at several values of the applied current, starting from the single-vortex phase configuration shown in Fig. 3.2(a). (a) \( i < i_{c1} \). Vortex is pinned in the center of the array. (b) \( i_{c1} < i < i_{c2} \). Vortex escapes from the center and leaves array from left-hand edge. For \( i > i_{c2} \), a continuous stream of vortices moves from right to left (not shown). In this particular array, \( i_{c1} \approx 0.0967 \) and \( i_{c2} \approx 0.956 \).
junction is connected only by a shunt resistance neglecting Josephson current at the junction. Figure 3.4 shows the current-voltage characteristics of an array with a single defect located at a coordinate (3,8). Once again, we assume an initial condition with one vortex in the center (8.5, 8.5) of the array of 16 × 16 grains. Note that the location of the defect is a half lattice constant below the line of the vortex motion. In this case, there are five regimes, separated by four "critical currents" $i_{c1}$, $i_{c2}$, $i_{c3}$, and $i_{c4}$. [In this example, $i_{c1} \approx 0.090$, $i_{c2} \approx 0.216$, $i_{c3} \approx 0.644$, and $i_{c4} \approx 0.866$.] For $i < i_{c1}$, the vortex remains trapped in the center. When $i_{c1} < i < i_{c2}$, the vortex escapes from the center, only to be retrapped near the defect. For $i_{c2} < i < i_{c3}$, the vortex escapes from the center, pauses near the defect, and moves off the left-hand edge of the array. Between $i_{c3}$ and $i_{c4}$, the vortex escapes from the center, moves off the left edge, where an antivortex is created which moves off the right-hand edge of the array. Finally, above $i_{c4}$, we have a nonzero $\langle V(t) \rangle$ produced by a continuous train of vortices and antivortices moving perpendicular to the current pattern. This sequence is shown in Fig. 3.4(d).

The current $i_{c1}$ for an ordered array is of special interest as the "depinning" current required to drive a trapped vortex out of the array. This current depends on the size of the array, as is shown in Fig. 3.5 for an ordered $N \times N$ array, but asymptotically approaches a value of about 0.10 per incoming junction, for a large array. This is near the value predicted by Lobb et al from a static calculation, and is about $2e/\hbar$ times the energy barrier for vortex motion estimated by Lobb, Abraham, and Tinkham[17] from static considerations.

Note that the detrapping current $i_{c1}$ for a single defect at (3,8) is smaller than that for the periodic array, while $i_{c2}$ is larger than that value. Both features can
Figure 3.4:
Voltage traces for a 16×16 lattice with a single missing superconducting grain at (3,8), starting from a single-vortex initial configuration. (a). $i_{c1} < i < i_{c2}$. Vortex escapes from the center but is pinned near the defect. (b). $i_{c2} < i < i_{c3}$. Vortex escapes from the center and leaves array from left-hand edge. The vortex produces its maximum voltage pulse as it escapes from the edge of the array. (c). $i_{c3} < i < i_{c4}$. Vortex escapes from center, leaves array at left; antivortex is generated near left edge of array and leaves array at right. (d). $i > i_{c4}$. Critical current of array exceeded; there is a continuous stream of vortices.
Figure 3.5:

$i_{c1}$, the critical current for depinning a vortex from the center of an $N \times N$ ordered array, plotted against $1/N$. The asymptotic limit as $N \to \infty$ is $i_{c1} \approx 0.10$.

Figure 3.6:

$V(t)$ for a $16 \times 16$ array with two defects symmetrically located at $(3,8)$ and $(14,8)$. Pattern is shown at current $i = 0.65$. Initial configuration is a single vortex as in Fig. 3.2(a). Each hump with three apparent peaks corresponds to back-and-forth motion of the vortex and antivortex.
be understood from the vortex potential energy function $W(x,y)$ shown in Fig. 3.1. The defect provides a deep potential well in which a vortex can be more readily trapped than in the periodic case. At the same time, the vortex is more easily *detrapped* from the center of the array when there is a defect nearby. Both effects tend to show the critical current can be increased by introducing defects into an otherwise periodic array, a feature which may be of practical importance in producing high-critical-current granular materials.

One of the most striking features of the single-defect array is the apparent transformation of a vortex into an antivortex in the regime $i_{c3} < i < i_{c4}$. This transformation seems to occur near the edge of the array, and is seemingly catalyzed by the defect: if no defect is present, no antivortex is generated. A logical next step is to consider two defects. If a single defect can cause a vortex to turn into an antivortex, and reverse direction then it seems plausible that a vortex set in motion between two defects can, under certain applied currents, "ping-pong" back and forth between the two defects, being transformed from a vortex to an antivortex with each reversal in direction. We have confirmed this speculation for $16 \times 16$ array in which two defects consisting of missing sites are symmetrically arranged at (3,8) and (14,8). The critical current for the onset of the back-and-forth motion, in this particular geometry, is about 0.6385. At currents slightly above this value, we have a periodic back-and-forth motion of the vortex and antivortex phase disturbance, as shown in Fig. 3.6.

3.3 Electrodynamics: Microwave Properties in a Granular Superconductors

In the previous section, we have discussed a very simplified case: the motion of a single vortex in an array. From this study, we have shown a fact that the pinning
centers from the imperfection in an array play an important role in impeding the motion of fluxons. In this section, we turn to the nonlinear electrodynamics of an ordered array in a combined d.c. and a.c. magnetic field. As already mentioned in Section 3.1, the high temperature superconductors turn out to have many grain and twin boundaries that can be assumed to act as weak links[9,10]. The main goal of this section is to describe the observed nonlinear electrodynamics in the high-temperature superconductors in terms of the dynamic equations for the resistively-shunted junctions and the finite response time to the external frequency.

3.3.1 Formalism

We consider a collection of $N \times N$ superconducting grains, coupled together by resistively-shunted Josephson junctions, and subjected to an external electric and magnetic fields $\vec{E}_{ext}(\vec{z}, t)$ and $\vec{B}_{ext}(\vec{z}, t)$. For simplicity, we assume that these fields are such that they can be described by a vector potential $\vec{A}(\vec{z}, t)$ alone, with the corresponding scalar potential $V(\vec{z}, t) = 0$, so that

$$\vec{B}_{ext} = \nabla \times \vec{A}_{ext}$$

$$\vec{E}_{ext} = -\frac{1}{c} \frac{\partial \vec{A}_{ext}}{\partial t}.$$  \hspace{1cm} (3.5)

Such a choice of gauge is generally possible provided that there are no sources present[18].

The response of the network to the external vector potential is described by a set of coupled nonlinear equations which was already presented in Chapter II. However, the voltage difference $V_{ij}$ appearing in Eq. (2.1) is not expressed as in Eq. (2.2). Instead, because of the potential induced by the oscillating field, it is
expressed as

\[ V_{ij} = \frac{\hbar}{2e} \frac{d}{dt}(\phi_i - \phi_j - A_{ij}). \]  \hspace{1cm} (3.7)

Equation (3.7) represents the Josephson equation, which expresses the voltage drop in terms of the gauge invariant phase difference across the junction.

In our treatment, we neglect the capacitive terms in the equations of motion — an approximation which is reasonable for SNS arrays but less accurate for superconducting-insulating-superconducting (SIS) systems. We also assume weak coupling, so that the vector potential is assumed entirely generated by the external electric and magnetic fields, and is unaffected by the induced currents in the array. This will be a reasonable approximation provided that the supercurrents in the array are sufficiently small. In practical terms, it is probably adequate that the Josephson penetration depth of the array be large in comparison to the array dimensions. Finally we are also assuming “point” grains in the sense that each grain can be described by a single, definite phase which does not vary within a grain.

To describe a transverse magnetic field \( \vec{B} = (B_0 + B_1 \cos(\omega t))\hat{z} \) applied to a square array of \( N \times N \) superconducting grains with free boundary conditions, we use a symmetric gauge with an origin at the center of the array:

\[ \vec{A} = \frac{1}{2} \vec{B} \times \vec{r} \]
\[ = \frac{1}{2} (-y\hat{x} + x\hat{y})(B_0 + B_1 \cos(\omega^* t^*)), \]  \hspace{1cm} (3.8)

where the dimensionless quantities \( \omega^* \) and \( t^* \) are defined by

\[ t^* = t/\tau_0, \]
\[ \omega^* = \omega/\omega_0, \]
\[ \tau_0 = 1/\omega_0 = \hbar/(2eR_0 I_c). \]  \hspace{1cm} (3.9)
We also introduce a dimensionless voltage $V_{ij}^*$ by

$$V_{ij}^* = \frac{V_{ij}}{R_0 I_c}. \tag{3.10}$$

Circulating currents in the array are induced by the oscillating magnetic field described by parameters $f_0 = B_0 a^2/\Phi_0$, $f_1 = B_1 a^2/\Phi_0$, and frequency $\omega$, where $\Phi_0 = hc/2e$ and $a$ is the lattice constant of the array.

By the definition of the magnetic phase $A_{ij}$ in Eq. (2.10), we have in the symmetric gauge,

$$A_{ij} = \pi (f_0 + f_1 \cos \omega t)(-n_y^i s \delta_{\tilde{\rho}_{ij}},eas + n_y^i s \delta_{\tilde{\rho}_{ij}},eas), \tag{3.11}$$

($n_x^i, n_y^i$) are the Cartesian coordinates of grain $i$, in units of the lattice constant $a$, and $s$ is a simple sign defined according to the direction from grain $i$ to grain $j$,

$$s = \begin{cases} +1 & \text{if } \tilde{\rho}_{ij} \parallel \hat{x} \text{ or } \tilde{\rho}_{ij} \parallel \hat{y}, \\ -1 & \text{if } \tilde{\rho}_{ij} \parallel -\hat{x} \text{ or } \tilde{\rho}_{ij} \parallel -\hat{y}. \end{cases} \tag{3.12}$$

Putting Eq. (3.10) into the voltage difference Eq. (3.7), we obtain the dimensionless voltage difference across a junction:

$$V_{ij}^* = \left( \frac{d\phi_i}{dt^*} - \frac{d\phi_j}{dt^*} \right) + \pi f_1 \omega^* \sin(\omega^* t^*) \left(-n_y^i s \delta_{\tilde{\rho}_{ij}},eas + n_y^i s \delta_{\tilde{\rho}_{ij}},eas\right). \tag{3.13}$$

Given the external currents, the dynamical equations can be combined into $N^2$ coupled nonlinear first-order differential equations for the phases. These equations have been described in Chapter II for the case of a time-independent vector potential. For the present case, they take the form

$$\sum_j M_{ij} \frac{d\phi_j}{dt^*} = I_{ext} - \sum_j \{ \sin(\phi_i - \phi_j - A_{ij})$$

$$+ \pi f_1 \omega^* \sin(\omega^* t^*)(-n_y^i s \delta_{\tilde{\rho}_{ij}},eas + n_y^i s \delta_{\tilde{\rho}_{ij}},eas) \}, \tag{3.14}$$

$$+ \pi f_1 \omega^* \sin(\omega^* t^*)(-n_y^i s \delta_{\tilde{\rho}_{ij}},eas + n_y^i s \delta_{\tilde{\rho}_{ij}},eas) \}.$$
where $M$ is the same matrix defined in Eq. (2.6). In this section, we solve the above equations by setting the temperature $T = 0$ and external current $I_{\text{ext}} = 0$, because we are interested in only the response of an array to the oscillating field.

Given the solution to equations of motion, we can calculate several quantities of experimental interest. The first of these is the magnetic moment $\bar{m}(t) \equiv m(t)\dot{\varepsilon}$, which we compute from

$$m(t) = \frac{1}{2c} \sum_{<ij>} \vec{r}_{ij} \times \vec{I}_{ij}$$

(3.15)

where $\vec{r}_{ij}$ is a vector drawn from the origin to the center of a bond between grain $i$ and grain $j$. We also compute the electromagnetic absorption of the array, $P(t)$. $P(t)$ is produced by power dissipation in the shunt resistances and is given by

$$P(t) = \sum_{<ij>} \frac{V_{ij}^2}{R_{ij}}$$

(3.16)

Since both $m(t)$ and $P(t)$ are periodic in time, they can be expressed as Fourier series of the form

$$m(t) = m_0 + \sum_{n=1}^{\infty} [m_{n1} \cos(n\omega t) + m_{n2} \sin(n\omega t)]$$

(3.17)

$$P(t) = P_0 + \sum_{n=1}^{\infty} [P_{n1} \cos(n\omega t) + P_{n2} \sin(n\omega t)]$$

(3.18)

where $m_0$ and $P_0$ are the time-averaged moments and power absorption.

3.3.2 Higher Harmonics: AC-Amplitude Dependence

Figure 3.7 represents the experimental power spectrum as reported for powdered YBa$_2$Cu$_3$O$_7$ by Jeffries et al in Ref. [5]. At zero magnetic field, the power
spectrum shows extensive generation of all odd harmonics up to at least \( n = 23 \). When a d.c. field \( H_0 = 1.0 \text{G} \) is applied, even harmonics also appear. Our calculation qualitatively produces the same features of the harmonic intensities.

Figure 3.8 shows the power spectrum \( S_m(\omega) \) of the magnetic moment \( m(t) \) at frequency \( \omega = 0.01\omega_0 \) in an array of 9 \( \times \) 9 superconducting grains, as induced by two magnetic fields: (a) \( f_0 = 0, f_1 = 1/5 \) (\( f_0 = B_0a^2/\Phi_0, f_1 = B_1a^2/\Phi_0 \), where \( a^2 \) is the area of an elementary plaquette), and (b) \( f_0 = 1/3, f_1 = 1/15 \). Like the experimental results in Fig. 3.7, only odd harmonics are present for \( f_0 = 0 \), while for \( f_0 \neq 0 \) even harmonics also appear in the spectrum because of the symmetry breaking by the non-zero static field. The existence of the higher harmonics in our array system suggests that the ceramic high-temperature superconductors may contain loops of weakly coupled superconducting grains. Such loops were first suggested to exist in the high-temperature superconductors by Müller et al [3]. In our calculations, however, even harmonics are sometimes seen even for zero \( f_0 \) and larger \( f_1 \) (say, \( f_1 \approx 1/2, 1/3 \)) at zero temperature and sufficiently large frequency. This may be due to the small size of the square array on which we carry out our calculations, to our neglect of the magnetic field generated by the induced current, or to several other assumptions that we have already mentioned in the previous section. These unexpected even harmonics become absent when we consider finite temperature or smaller frequency. Figure 3.9(a) shows the unexpected even harmonics on 9 \( \times \) 9 array at \( f_0 = 0 \) and \( f_1 = 1/2 \). These even harmonics disappear in the presence of temperature as shown in Fig. 3.9(b).

Figure 3.10(a) shows the \( f_1 \)-dependence of the power absorption [Eq. (3.16)]
Figure 3.7:

Power spectra for powdered YBa$_2$Cu$_3$O$_7$ at $T=77K$. (a) d.c. field $H_0 = 0.0$ G; (b) d.c. field $H_0 = 1.0$ G. After Ref. [5].
Figure 3.8:

Power spectrum $S_m(\omega)$ of the magnetic moment $m(t)$ of a $9 \times 9$ array in an external field (a) $f_0 = 0, f_1 = \frac{1}{5}$, and (b) $f_0 = \frac{2}{3}, f_1 = \frac{1}{18}$, and frequency $\omega = 0.01\omega_0$. 
Figure 3.9:

Power spectrum $S_m(\omega)$ of the magnetic moment of a $9 \times 9$ array at an applied frequency $\omega = 0.1\omega_0$ in an external field $f_0 = 0, f_1 = 1/2$ (a) at zero temperature, and (b) at temperature $0.02 E_J/k_B T$, where $E_J$ is the Josephson coupling energy, $E_J = \hbar I_c/2e$. Unexpected even harmonics in (a) disappear at finite temperature.
at several values of $f_0$ ($f_0 = 0, 1/2, 1/3,$ and $1/4$) and at $\omega = 0.1 \omega_0$. The data points at lower frequencies, not shown in Fig. 3.10(a), are shifted toward lower absorption with almost the same slope in the logarithmic plot. (The frequency dependence of the absorption will be discussed in the next section.) As expected from Eq. (3.16), the power absorption does not depend on the d.c. magnetic field $f_0$, but it depends on $f_1$ roughly quadratically, $P \propto f_1^2$.

### 3.3.3 Higher Harmonics: Frequency Dependence

The magnetic moment exhibits many harmonics at low frequencies, fewer at high frequencies [Fig. 11(a) (b)]. This may be understood as follows: As the field slowly changes, the phase configuration rearranges itself adiabatically in response. At higher frequencies, the vortex configuration cannot follow the rapidly varying a.c. field and there is far less harmonic intensity. If the a.c. field is oscillating much faster than the characteristic frequency of the array, the array will vary little from its d.c. ground state and there will be correspondingly few higher harmonics. On the other hand, if the frequency $\omega$ is much less than the characteristic frequency, we expect that the array will follow the ground state of the array adiabatically, and we wind up with a great deal of harmonic content. The crossover frequency, for the network shown, is of order $0.1 \omega_0$. [See Fig. 3.10(b)]

The corresponding power absorption is caused at low frequencies by flux slips induced by the a.c. magnetic field. At higher frequencies, absorption is due primarily to the a.c. electric field across the shunt resistances. The limiting slope at high frequency of Fig. 3.10(b) is about two, following the law $P \propto \omega^2$ as expected from Eq. (3.16). The absorption depends strongly on d.c. magnetic
Figure 3.10:

Time-averaged power absorbed, in units of $R_0I_0^2$: (a) as a function of $f_1$, for an applied frequency $\omega = 0.1 \omega_0$, and for $f_0 = 0$ (+), $f_0 = 1/2$ (*), $f_0 = 1/3$ (o), and $f_0 = 1/4$ (x); (b) as a function of frequency, for $f_1 = \frac{1}{15}$, and for $f_0 = 0$ (triangles) and $f_0 = \frac{1}{3}$ (circles).
Figure 3.11:

Power spectra for the magnetic moment at two different frequencies: (a) $\omega = 0.2 \omega_0$, and (b) $\omega = 0.02 \omega_0$. Spectrum at low frequency of (b) exhibits many harmonics because of the adiabatic change of ground state in response to the external a.c. field.
field at low frequencies, as is observed experimentally [5].

3.4 Discussion

In the first part of the chapter, we have shown the motion of a single vortex in a resistively-shunted superconducting array. Although this feature is shown for a very simplified case, it still has a number of implications. First, even an ordered array, and still more a disordered one, can sit for a long time in a metastable energy minimum and, indeed, can be dislodged only with the application of a sufficient current. In a realistic array, or a granular superconductor, we can imagine creating such a state by, e.g., cooling the array in a magnetic field, then turning the field off. The vortices thus generated will be dislodged only by the application of a sufficient current. Another way to remove these vortices would be to raise the temperature sufficiently to excite them over the energy barriers that hold them in position.

In the second part of the chapter, we have presented a model of weakly coupled superconducting grains to describe the nonlinear electrodynamics of high-temperature superconductors. The results suggest that the Josephson system gives a reasonable picture of some novel electrodynamic phenomena observed in the high-\(T_c\) superconductors. However, more detailed calculations in larger arrays in the presence of the finite temperature and disorder would be desirable to test for detailed agreement with experiment.
CHAPTER III REFERENCES


CHAPTER IV

QUANTIZED VOLTAGE PLATEAUS IN JOSEPHSON JUNCTION ARRAYS

In this chapter we present a numerical study of quantized voltage plateaus in a square array of $N \times N$ plaquettes of resistively-shunted Josephson junctions subjected to a combined d.c. and a.c. applied current $I_{dc} + I_{ac}\sin(2\pi ft)$, and a transverse magnetic field equal to $p/q = f$ flux quanta per plaquette ($p$ and $q$ integers with no common divisor). We use periodic boundary conditions in the direction perpendicular to the applied current. With these conditions, we find plateaus at all voltages satisfying $\langle V \rangle = nNh\nu/(2eq)$, where $n$ is an integer, and $\langle \ldots \rangle$ denotes a time average. With free transverse boundary conditions, additional steps at $\langle V \rangle = Nh\nu/(4eq)$ sometimes appear. For $f = 1/5$ and $2/5$, we study the motion of the vortex lattice on the steps. At both fields, on every step, the lattice moves an integer number of array lattice constants per cycle of the a.c. field. For both zero and finite applied transverse magnetic field, the width of the steps varies sinusoidally with $I_{ac}$, in a manner reminiscent of that seen in single Josephson junctions. As the temperature increases, steps are found to narrow,
become smoother, and finally to vanish above a characteristic temperature at which the underlying vortex lattice melts. The time-dependent voltage across the array is found to be periodic on the steps, aperiodic off the steps. Its power spectrum reveals strong harmonics at multiples of the fundamental on the steps, and an apparently broad band with possible subharmonic structure off the steps.

4.1 Introduction

Superconducting arrays have been intensively studied in recent years, and exhibit much complex behavior[1]. Such arrays can be prepared artificially by photolithographic techniques[1]. When one applies a transverse magnetic field $B$ which is a rational fraction of a flux quantum per plaquette, the vortices form a lattice at low temperature, which is commensurate with the array[2]. This lattice manifests itself experimentally in a variety of ways. For example, the superconducting transition temperature of the array, $T_c(B)$, is strongly field dependent, as is the resistivity above $T_c(B)$. Weakly coupled Josephson systems are also useful models for high-$T_c$ superconductors[3]. These materials, especially in their polycrystalline form, often contain many weak links between superconducting grains, and hence may sometimes be modeled by coupled Josephson systems in three dimensions.

When a combined d.c. and a.c. external current $I_{dc} + I_{ac}\sin(2\pi ft)$ is applied, the current-voltage (I-V) characteristics of ordered two-dimensional arrays exhibit quantized voltage plateaus. In a square array of $N \times N$ plaquettes, with an applied transverse magnetic field of magnitude $f = p/q$ flux quanta per plaquette, where $p$ and $q$ are integers with no common divisor, these plateaus occur at voltages $nNh\nu/(2eq)$, where $n$ is an integer[4]. These plateaus are generalizations
of similar phenomena, called Shapiro steps, long familiar in single resistively-shunted Josephson junctions[5]. In the arrays at finite fields, the plateaus in a field are called fractional giant Shapiro steps[4]; at zero field, they are known as integer giant steps[6].

Several theoretical studies of fractional giant Shapiro steps have appeared in the literature. Benz et al, in their original experimental paper[4], propose that the vortex lattice moves rigidly in response to the combined d.c. and a.c. applied current. In this picture, the steps occur when the vortex lattice is able to lock onto the underlying “egg-carton” vortex potential formed by the Josephson junction lattice. Lee et al[7] showed numerically that the fractional steps emerged naturally from a model of coupled resistively-shunted Josephson junctions (RSJ’s). They also found (in a calculation with free transverse boundary conditions) additional steps beyond those envisioned by Benz et al. Free et al[8], used a model of coupled RSJ’s and, for the first time, periodic boundary conditions. They also found fractional giant steps and, by studying voltage drops across individual junctions, provided evidence in favor of rigid vortex lattice motion. Halsey[9] considered special values of transverse magnetic field, at which the so-called staircase phase configuration[10] might be the ground state, and proposed that, for certain directions, amplitudes, and frequencies of the applied currents, there might be subharmonic steps beyond those proposed by Benz and collaborators. Recently Sohn et al discussed the existence of fractional and subharmonic Shapiro steps in diagonal arrays[11]. According to their experiment and their phenomenological explanation, the fractional steps should disappear in diagonal arrays because the vortices move in the direction of Lorentz force and toward “next” nearest neighbor cells.
This chapter reports detailed numerical studies of quantized voltage plateaus in RSJ arrays. We study how the presence of steps is affected by boundary conditions, and show that several “anomalous” steps found with free transverse boundary conditions disappear when periodic boundary conditions are used. We find that the step widths exhibit an oscillatory Bessel-function-like dependence on a.c. current amplitude and frequency, similar to that of single junctions. We show that the steps disappear at a critical temperature which depends on magnetic field. The current motions on the steps are studied in detail for several values of the applied field. At flux $f = 1/5$, the vortex pattern moves in phase with the applied current, as proposed in Ref. [4]. This is confirmed by a series of “snapshots” of the vortex configuration at various times within a cycle of the a.c. current. For $f = 2/5$, the motion of the vortex lattice is more complex. Nevertheless, on each step, the vortex pattern is always translated rigidly by a certain number of plaquettes per cycle. Finally, we examine the power spectrum of the voltage on and off the steps. Not surprisingly, this voltage on the step is not only periodic, but has many higher harmonics. These could lead, in principle, to coherent radiation from the junction array at appropriate frequencies, or perhaps also to coherent detection[12].

4.2 Formalism

Our calculation proceeds by directly solving the equations for a square network of resistively coupled Josephson junctions in the limit of zero capacitance and negligible array self-inductance[13-19]. We also assume that a magnetic field is applied perpendicular to the plane of the array, of magnitude $f = p/q$ flux quanta per plaquette.
The main dynamical equations are the same as in Chapter II. As an input current, we impose current \( I = I_{dc} + I_{ac} \sin(2\pi \nu t) \) to calculate the response of an array of \( N \times N \) plaquettes of resistively-shunted Josephson junctions. In a transverse d.c. magnetic field of \( p/q \equiv f \) flux quanta per plaquette of area, we find fractional giant Shapiro steps in the time-averaged voltage \( \langle V \rangle \) at values \( \langle V \rangle = nNh\nu/(2eq), n = 1, 2, 3, \ldots \), in agreement with the measurements of Benz et al. We include finite temperature by adding to each junction a parallel Langevin noise current source defined in Chapter II.

We iterate the set of dynamical equations for ordered arrays (including the Langevin noise current[20]) by simple first-order time integration with time step \( \Delta t \), using the Josephson relation Eq. (2.2) to give the phases \( \{ \phi(t) \} \). We usually consider intervals of \( I_{dc}/I_c \) ranging from 0.01 to 0.05. To obtain average voltages, we time-average over an interval from 400 - 800 \( \tau_0 \). We have tested a variety of initial phase configurations (phases parallel, random initial phases, and ground state phase configuration). Generally these do not make a large difference in the resulting current-voltage characteristic. We have also considered two different methods of ramping up or down the applied d.c. current: re-randomizing after each increase of d.c. current, and using the final phase configuration of the previous current as the initial configuration of the new current. This choice also seems to have little effect on the resulting I-V characteristics.

4.3 Effects of Boundary Conditions

Figure 4.1(a) shows representative voltage traces for \( 12 \times 12 \) and \( 10 \times 10 \) arrays having free boundaries at several values of the flux per plaquette \( f \), measured in units of \( \Phi_0 \), using \( I_{ac} = I_c \), and \( \nu/\nu_0 = 0.1 \), where \( \nu_0 = 2eR_0Ic/\hbar \). This value
of $I_{ac}$ was chosen to correspond to the experimental conditions of Ref. [4]; but we found numerically that the same Shapiro steps were also present for other, greater or lesser, values of $I_{ac}$, as well as in disordered samples with random $I_c$'s. The time-averaged voltage $\langle V \rangle$ shown is the difference between the mean voltages along the top and the bottom rows, typically averaged over the time interval of $800\tau_0$. The I-V characteristic is calculated at intervals of $0.015I_c$, the current being ramped up after each I-V point is evaluated.

For all fields shown, there are characteristic plateaus in $\langle V \rangle$, generally with spacings of $Nh\nu/(2eq)$. The $q = 1$ plateaus are usually much wider than those at higher $q$. Also, the principal ($n=1$) giant step is far broader than the higher-$n$ plateaus. For $f = 1 - g$, where $g = (\sqrt{5} - 1)/2$ is the golden mean, the differential resistance $dV/dI$ is nonmonotonic, and seems to show a precursor of an integer giant step at $Nh\nu/2e$. When $f = 1/5$, $2/5$, and $1/3$, we find "anomalous" half-integer steps at $(N/2)h\nu/(2e)$. These steps are not artifacts of the calculation, as is shown in the inset of Fig. 4.1 where we show an enlarged portion of the I-V characteristic for $f = 1/5$.

Figure 4.1(b) shows the calculated current-voltage characteristics of a $10 \times 10$ array at $T = 0$, field $f = 1/5$, and with two types of transverse boundary conditions: periodic and free. In both cases, $I_{ac}/I_c = 1.0$ and $\nu/\nu_0 = 0.1$, where $I_c$ is the critical current of each individual Josephson junction, and $\nu_0 = 2eR_0I_c/h$ is the natural unit of frequency. Changing the boundary conditions has a striking effect. With periodic boundary conditions, there are clear steps at $Nh\nu/(2eq)$, for every integer $n \geq 1$, and no other steps. [These steps correspond to $\langle V \rangle/(NR_0I_c) = (n/q)0.1(2\pi) = (n/q)0.628$ for our choice of frequency.] By contrast, when free transverse boundary conditions are used, one sees conspicu-
Figure 4.1:

Time-averaged voltages versus d.c. current in an $N \times N$ array. In all cases, $I_{ac} = I_c$ and $\nu = 0.1(2eR_0I_c/h)$. (a), with free boundary condition. $N = 12$ for all curves shown except $f = 1/5$ and $2/5$, for which $N = 10$. The notation "gm" refers to $f = 1-(\sqrt{5}-1)/2$. All except the $f = 0$ curve are horizontally displaced. Inset: expansion of half-integer step at $f = 1/5$; (b), with both periodic (pbc) and free (fbc) boundary conditions in the transverse direction. $N = 10$ for both curves at $f = 1/5$. The plot for fbc is displaced horizontally to the right by one unit (indicated by a tick mark).
ous "integer" and "half-integer" giant steps at \( N\nu/(2e) \) and \( (N/2)\nu/(2e) \), but no other clear steps. We have seen such anomalous half-integer giant steps for most values of \( f = p/q \) with \( q \) odd ("odd-denominator frustration"), whenever we use free boundary conditions, but they disappear with periodic boundary conditions. Our results are thus consistent with the findings of Ref. [8] that the anomalous steps are absent at \( f = 1/3 \) with periodic boundary conditions. The anomalous half-integer steps are always characterized by a periodic time-dependent voltage just as in Ref. [7], and they exist under a variety of initial conditions (phases parallel, random initial phases, and ground state phase configuration).

The reasons for these anomalous steps are not understood. Most likely, they arise from an irregular motion of the vortex lattice near the free boundaries. This complicated vortex motion near the boundaries seems to be related to the relatively lower egg-carton potential in which the vortices move near the boundaries. In an array with periodic boundary conditions, this egg-carton potential is more nearly periodic, and the extra steps are absent.

Experiments are generally done in large arrays with free transverse boundary conditions. Because of the size of the experimental arrays (typically 300 x 300 or larger), it seems reasonable to model experiment with periodic rather than free boundary conditions. This would imply that the extra steps should be absent in experimental arrays, as they appear to be in the data so far published[4]. Further experimental studies may shed more light on this point.
4.4 Coherent Vortex Motion on the Shapiro Steps.

To visualize the coherent vortex motion that produces the Shapiro steps, we have carried out detailed calculations for two fields: $f = 1/5$ and $f = 2/5$. These calculations (and all subsequent ones, unless otherwise stated) are carried out on a $10 \times 10$ array with periodic boundary conditions in the lateral directions and uniform current injection in the vertical direction. The case $f = 2/5$ represents a particularly interesting test case: it is the simplest non-trivial fraction which cannot be reduced to the form $1/q$. (Because of the symmetry of the square lattice, the fractions $f$ and $1-f$ are equivalent, so that one need not consider values of $f > 1/2$.) It is therefore of interest whether or not all predicted voltage steps of the form $\langle V \rangle = nN\hbar\nu/(2eg)$ actually appear in this case.

The time-dependent variation of the phases of the Josephson junctions can be concisely represented in terms of "vortex motion," as is shown in Fig. 4.2(a) for $f = 1/5$. A square marked with a plus sign holds a vortex (i.e., a plaquette of positive, or counterclockwise, "vorticity"). A blank square contains an antivortex. The vorticity, in turn, is defined at the center of each plaquette as the sum of the supercurrents through the four junctions bounding the plaquette, the sum being taken in the counterclockwise direction.

$$\Omega = I_{ij} + I_{jk} + I_{kl} + I_{li},$$

where the $I_{ij}, I_{jk}, \ldots$ denote the supercurrents through the four junctions bounding the plaquette. Each rectangle of plaquettes represents a "snapshot" of part of the array at the time shown. Time increases in the downward direction. In order to make the motion clearer, we have "tagged" one of the vortices with a circle. From examining many vorticity snapshots, we infer that the tagged vor-
tex moves as shown. In general, on the $n^{th}$ step at $f = 1/q$, the vortex pattern moves $n$ times faster than for $n= 1$. This is shown schematically in the diagrams of Fig. 4.2(a) for $f = 1/5$, but we have obtained similar results for $f = 1/3$.

Our interpretation of the $f = 2/5$ steps is shown in Fig. 4.2(b). As in Fig. 4.2(a), time advances downwards. Once again, our simulations suggest that the motion of the vortex lattice on the $n^{th}$ step is roughly $n$ times faster than on the $n = 1$ step. On each of the steps, the vorticity pattern is identical at the beginning and end of an a.c. cycle, except for a uniform displacement to the right. This displacement is (i) by three array lattice constants to the right on the $n = 1$ step; (ii) by one lattice constant on the $n = 2$ step; (iii) by four lattice constants at $n = 3$; and (iv) by two lattice constants at $n = 4$. On the $n = 5$ step, the vorticity pattern is identical at the beginning and end of each a.c. cycle. The tagged vortices are meant to suggest how the various parts of the pattern move during a cycle.

4.5 Step Widths.

We turn next to a discussion of the step widths in these Josephson arrays. In a single Josephson junction, it is well known that some of the Shapiro steps disappear at certain amplitudes of the a.c. driving current[21]. A similar effect occurs in arrays. Figures 4.3(a) and 4.3(b) show the step widths and critical current as functions of $I_{ac}$ for two different fields and a frequency $\nu = 0.1(2eR_0I_c/h)$, obtained numerically. At $f = 0$, the oscillating behavior of the step widths and the critical current is identical to that of a single junction, as calculated by Russer[21]. (note that the variable $\xi$ defined by Russer is related to our variable $\nu$ by $\xi = 2\pi\nu/\nu_0$.) The minima of the critical current occur roughly at the
A schematic diagram showing the motion of the vortex pattern at (a) $f = 1/5$ and (b) $f = 2/5$. A square containing a plus sign contains a 'vortex' (i.e., a region of positive vorticity); an empty square holds an 'antivortex'. Each diagram of connected plaquettes represents a snapshot of the vorticity configuration at a particular time. The arrows represent the extent of motion of the vortex lattice during one cycle of the a.c. field, when the d.c. voltage corresponds to step $n$. In both pictures, we have 'tagged' several vortices by enclosing them with circles or triangles. From examining many vorticity snapshots on all the steps, we infer that the tagged vortices move as shown.
maxima in the widths of the first step. All step widths are zero at $I_{ac} = 0$. A similar oscillatory behavior is also found at other frequencies (not shown; but see Ref. [22]). At $f = 1/2$, the oscillatory patterns of step widths are compressed relative to $f = 0$. The critical current is about $0.34 I_c$ at $I_{ac} = 0$. At small values of $I_{ac}$, the step widths seem to increase quadratically with $I_c$ for $n = 2$ rather than varying linearly with $I_{ac}$ as they do for $n = 1$. Note that the plots shown are interpolations of our calculated points, which are spaced about $0.25 I_c$ apart. The general features of the oscillatory behavior of a square array are similar to an analytical result by Rzchowski et al [23] and a numerical result by Octavio et al [24].

4.6 Effects of Finite Temperature

Both the integer and the fractional giant Shapiro steps are caused by a coherent variation of all the phases in the array in step with the applied a.c. current. Such a variation can occur only if the array is in a phase coherent state in the absence of an applied current. This phase coherent state, in turn, can exist only below a certain field-dependent temperature, above which the lattice "melts". When the temperature exceeds this melting temperature, the steps are expected to disappear. This is confirmed by the present calculations.

Figures 4.4(a) and 4.4(b) show the current-voltage characteristics of a $10 \times 10$ array with periodic boundary conditions as a function of temperature at $f = 0$ and $f = 1/2$. On both sets of curves, the Shapiro steps "melt" with increasing temperature, beginning with the higher-order steps and proceeding to the lowest steps. At $f = 0$, the last step has disappeared by a temperature of $0.4hI_c/(k_Be)$ ($= 0.8E_J/k_B$), where $E_J = \hbar I_c/(2e)$ is the Josephson coupling en-
Figure 4.3:
Critical currents $I_{ac}^{crit}$ and step widths $\Delta I_n$ computed as a function of $I_{ac}$ at $\nu = 0.1(2eR_0I_c/\hbar)$ and (a) $f = 0$; (b) $f = 1/2$. 
Figure 4.4:
Variation of step width with temperature at two different magnetic fields: (a). $f = 0$ (no magnetic field); (b). $f = 1/2$ (one-half flux quantum per plaquette). Both graphs are obtained using periodic boundary conditions in the lateral direction, uniform current injection at the top, uniform current removal at the bottom, and the ground state phase configuration as the initial state. In all cases, $I_{dc}/I_c = 1$ and $\nu = 0.1(2eR_0I_c/\hbar)$. The various plots of $\langle V \rangle$ are all displaced horizontally; the tick marks on the horizontal axis denote the positions of zero current for the corresponding curve. All calculations are carried out for a $10 \times 10$ array. Temperature in units of $\hbar I_c/(k_B e)$. 
ergy. At $f=1/2$, the corresponding temperature is $0.2hI_c/(k_Be) (= 0.4E_J/k_B)$. These temperatures are near or slightly lower than the melting temperatures at zero applied current (as discussed, e.g., by Teitel and Jayaprakash[2]), which are $\approx 0.95E_J/k_B$ and $\approx 0.4E_J/k_B$. Since we are, in effect, calculating the melting temperatures in the presence of finite a.c. and d.c. currents, these finite-current melting-temperatures cannot be higher than the zero-current values. Extrapolating from these numerical results, we anticipate that the fractional giant Shapiro steps at other values of the applied field will disappear at temperatures no higher than the corresponding vortex lattice melting temperature (i.e., the superconducting/normal transition temperature) in zero applied current.

4.7 Power Spectrum

As noted earlier, the voltage drop $V(t)$ is periodic on the Shapiro steps, and aperiodic at other values of the current. This periodicity is reflected in the power spectrum $S_V(\omega)$ of the voltage, which is defined in Eq. (2.9).

4.7.1 On the Step

Figure 4.5 shows the power spectrum $S_V(\omega)$ for $I_{dc}/I_c = 0, 1.0, 1.5, 2.15$, and $2.732$ (corresponding to $n = 0, 1, 2, 3, 4$, respectively) at $f = 0$. (Similar spectra for $n = 2 - 5$ at $f = 2/5$ are shown in Ref. [22].) The points denote $S_V(m\omega)$, and the lines are merely to guide the eye and have no other significance. We have studied a variety of initial conditions on the phases (phases parallel, phases in the ground state configurations, and random initial phases) and for the most part obtain very similar power spectra, after the initial transients have died out. Since the voltage is periodic on the steps, the power spectrum consists of
Figure 4.5:

Voltage power $S_V(\omega)$ in a $10 \times 10$ array at $f = 0$ and several different values of $I_{dc}$. In all cases, $I_{dc}/I_c = 1.0$, $\nu/\nu_0 = 0.1$, and temperature $T = 0$. Shown are $I_{dc} = 0$; $I_{dc}/I_c = 1.0$ ($n = 1$ step); $I_{dc}/I_c = 1.5$ ($n = 2$ step); $I_{dc}/I_c = 2.15$ ($n = 3$ step); and $I_{dc}/I_c = 2.732$ ($n = 4$ step). The vertical scale is logarithmic, but otherwise in arbitrary units. The power spectra for successive values of $n$ are displaced vertically by one decade relate to the previous step. Solid lines are merely to guide the eye. Only integer multiples of the fundamental appear in each case (for $I_{dc} = 0$, only odd integers appear). $\omega/2\pi$ is plotted in units of $2eR_0I_c/\hbar$. 
a series of sharp harmonics which fall off roughly algebraically with increasing order m of the harmonic. At \( I_{dc} = 0 \), only odd harmonics appear in the power spectrum. Figure 4.5 shows that, whereas the harmonics fall off monotonically with m for the \( n = 1 \) step, they show an oscillatory dependence on m on the higher steps. For example, on the \( n = 2 \) and \( n = 3 \) steps at \( f = 0 \), \( S_\nu(m\omega) \) has a secondary maximum at the second and third harmonic, respectively, with further oscillations at higher harmonics. This oscillatory behavior seems to be in qualitative agreement with the preliminary experimental results of Hebboul and Garland[25] for a 300 × 300 array. In the limit \( I_{ac}/I_c \gg 1 \), an analytic expression can be derived for the oscillatory dependence of the stepwidth of a single junction on the parameter \( I_{ac}/\nu \). This expression, given in the Appendix, gives an excellent fit to our calculations for arrays at \( f = 0 \).

4.7.2 Off the Step

At d.c. currents off the steps, \( V(t) \) is aperiodic and \( S_\nu(\omega) \) is a quasicontinuous function which does not have period \( 1/\nu \). However, the time-dependent voltage \( V(t) \) sometimes is periodic at certain values of currents. A representative power spectrum is shown in Figs. 4.6(a) and 4.6(b). The period-doubling and period-tripling shown resemble behavior often seen in the power spectra of nonlinear dynamical variables. At other values of the voltage, we have found other multiples of the period (e.g. period-octupling) as well as spectra which suggest the coexistence of sharp lines with a continuum.

The existence of the multiples of the period (or subharmonic structures in frequency domain) can be explained by a “beating” effect in a single junction. The effect comes from the competition between driving frequency and its Josephson
Figure 4.6:
Power spectra at $f = 0$ (logarithmic scale), as in Fig. 4.5, but for (a) $I_{dc}/I_c = 1.37$ (between the first and the second Shapiro step) showing period-doubling; (b) $I_{dc}/I_c = 1.40$ (between $n = 1$ and $n = 2$) showing period-tripling. Since these power spectra show subharmonic structure, we show a continuous plot of $S_V(\omega)$. 
response. Consider the general case in which the driving a.c. frequency has a
relation \( n_a T_a = n_j T_j \), where \( T_a \) is the period of a.c. current, \( T_j \) is the corre-
sponding Josephson period \( T_j = \frac{2\pi n_j \sqrt{(I_{dc}/I_c)^2 - 1}}{y_j(I_{dc}/I_c)^2 - 1} \), and \((n_a, n_j)\) are integers
having no common divisor. Then the combined voltage signal \( V(t) \) of both
\( V_a(t) = V_a(t + T_a) \) and \( V_j(t) = V_j(t + T_j) \) can be expressed in the form

\[
V(t + n_a T_a) = V_a(t + n_a T_a) + V_j(t + n_a T_a)
= V_a(t) + V_j(t + n_j T_j)
= V_a(t) + V_j(t)
= V(t).
\]

Thus, the combined period is \( T = n_a T_a \) (\( = n_j T_j \)), so that subharmonic struc-
tures are present at frequency \( \nu = 1/T = \nu_a/n_a \) (\( = \nu_j/n_j \)). This is consistent
with our numerical observations.

4.8 Phase Locking in Disordered Array

We have shown in Ref. [7] that the edges of the Shapiro steps are rounded by
disorder in the critical currents. Disorder in the shunt resistances also rounds
off the edges of the steps. In spite of the presence of defects, one would expect a
phase locking when a sufficient a.c. current is supplied to the array. To check this
tendency to go to a coherent state from an incoherent state, we assign uniform
random critical currents between 0.5 \( I_c \) and 1.5 \( I_c \) to all the junctions, and ramp
up the magnitude of a.c. current from zero to higher values at a fixed d.c.
current. Figure 4.7(a) shows the I-V characteristics for a particular realization
of disorder in the critical currents on an array of 6 \( \times \) 6 plaquettes. As the a.c.
current increases, Shapiro steps appear. A step at 5/6 is due to the special
distribution of the critical currents in this particular realization of the disorder. [In this example, at certain values of the applied d.c. current, the critical currents for all the junctions except one in each column exceed the applied current, so that only five out of six junctions in the direction of macroscopic current participate in the phase locking and contribute to the finite voltage drop.]

Figure 4.7(b) shows the power spectrum at a fixed d.c. current of 1.1 $I_c$ and increasing magnitude of a.c. currents. For a small a.c. current, the power spectrum has a broad background and does not show any conspicuous peaks. As the a.c. current further increases, several peaks start to develop. For a sufficient a.c. current, the peaks in the power spectrum are finally sitting at integer multiples of the applied frequency. Above this current, the vortices may move coherently out of the random pinning potential constructed by defects.

4.9 DC-Current Dependence of Power Spectrum

In an experiment, it is relatively easy to measure the amplitudes of many harmonics of the voltage signals produced by a combined d.c. and a.c. current. The harmonic components contain full information about the power spectrum. To compare our calculated spectral response to the experimental results, we turn next to the dc-current dependence of the power spectrum.

4.9.1 Zero Magnetic Field

Figure 4.8 illustrates the dependence of the power spectrum on the d.c current $I_{dc}$. Figure 4.8(a) shows the I-V characteristic of a 10 x 10 array at zero magnetic field, using the parameters $I_{ac} = 15.8 I_c$, $\nu = 0.222 \nu_0$, where $\nu_0 = 2eR_0I_c/h$. The values of the above parameters are the same as those used in Fig. 2 of Ref. [26],
Figure 4.7:

(a). I-V characteristics of an array of 6 x 6 plaquettes for a particular realization of disorder in critical currents. Each junction has a uniform random value of critical current in the range between 0.5 $I_c$ and 1.5 $I_c$. As the a.c. current increases, Shapiro steps develop and become stronger. (b). Power spectra at several values of a.c. current for the same array of (a). For a sufficient a.c. current, the power spectrum shows many harmonics indicating coherent motion of vortices, in spite of the presence of disorder in the critical currents.
except that the experimental results were obtained at finite temperature in the presence of an inevitable amount of disorder. Figure 4.8(b) and 4.8(c) are the spectral intensities at $2\nu$ and $3\nu$. The large values of these intensities across the width of the step can be explained by the fact that both odd and even harmonics are present on the steps. The intensities are found to change substantially with only a slight variation of the a.c. amplitude or frequency. Thus, sometimes the emission peaks are located at the edges of each step as in Fig. 4.8(b) and 4.8(c), but sometimes they reside near the middle of steps as in Ref. [26]. In any case, the emission peaks start at the left edge of a step and end at the right edge of the step. The apparent sharp increase and decrease of the peaks at the edge are rounded by disorder and finite temperature. The finite value at the third harmonic peak at zero d.c. current is consistent with the presence of only odd harmonics at $I_{dc}=0$ that has discussed in previous section.

For subharmonic intensities, $S_{\nu}^{(m=1/3)}(\omega)$ and $S_{\nu}^{(m=1/3)}(\omega)$, shown in Figs. 4.9(d) and 4.9(e), we do not see any emission peak on the steps because only harmonics exist on the step. However, there are sharp and narrow subharmonic peaks off the Shapiro steps around d.c. currents corresponding to the time-averaged voltage $\langle V \rangle = \langle nm \rangle N\hbar \nu/2eq$, where $N$ is the number of junctions in the direction of macroscopic current flow, $q$ is the denominator of the frustration, $n$ is an integer, and $m$ indicates the subharmonic number. These subharmonics have already been explained in a previous section by invoking a beating effect. Note that these subharmonic peaks do not imply the existence of substeps of the current in I-V characteristics.

In the presence of disorder in the critical currents (not shown), we have found that the intensities of harmonics are slightly reduced while those of subharmonics
Figure 4.8:

(a). I-V characteristic of a 10 x 10 array at zero magnetic field (f=0);  (b) and (c): The second ($S^{(m=3)}(\omega)$) and the third ($S^{(m=3)}(\omega)$) harmonic intensities as a function of a d.c. current;  (d) and (e): Subharmonic spectral intensities, $S^{(m=1/3)}(\omega)$ and $S^{(m=1/3)}(\omega)$, as a function of d.c. current. In all cases, $I_{ac} = 15.8 I_c$, and $\nu = 0.222 \nu_0$. 
are severely reduced. For disorder in both critical currents and shunt resistances, intensities of both harmonics and subharmonics are almost equally affected by disorder and severely reduced (not shown).

4.9.2 Finite Magnetic Field: $f=1/3$

In the previous section, we have discussed the dc-current dependence of the power spectrum at zero magnetic field. In this section, we discuss the spectrum at finite magnetic field, specifically, at $f=1/3$. Other values of magnetic field lead to similar results and can be explained in a similar way.

Figures 4.9(a)-(c) show the I-V characteristics, and the second and the third harmonic intensities as a function of d.c. current. We have used $I_{ac} = 1.5 I_c$ and $\nu = 0.1\nu_0$ for a $6 \times 6$ array with a periodic boundary condition. Both integer and fractional integer Shapiro steps are clearly seen in the I-V characteristics. The harmonic signals $S^{(m=2)}_V$ and $S^{(m=3)}_V$, shown in Figs. 4.9(b) and 4.9(c), show slowly varying background as a function of the applied d.c. current, with broad superimposed resonances when the currents correspond to Shapiro steps. The resonance peaks on the steps are analogous to those found at zero magnetic field. [For other value of magnetic field, we also found that a broad background emission exists. The maximum intensity of the background for $f=1/2$ is smaller than that for $f=1/3$ at the same parameters.]

The presence of narrow peaks in subharmonic spectral intensities, $S^{(m=1/3)}_V$ and $S^{(m=2/3)}_V$, at currents off the Shapiro steps, is explained by the beating effect discussed previously [Fig. 4.9(d) and (e)]. Note that Fig. 4.9 was obtained from the voltage difference between average voltages on bottom and top rows of grains. However, if we Fourier analyze the voltage drop across a single junction near the
Figure 4.9:

(a). I-V characteristic of a 6 x 6 array at a magnetic field of magnitude f = 1/3 flux quanta per quanta per plaquette; (b) — (e): $S_V^{(m=2)}$, $S_V^{(m=3)}$, $S_V^{(m=1/3)}$, and $S_V^{(m=2/3)}$, as a function of d.c. current. In all cases, $I_{ac} = 1.5 I_c$, and $\nu = 0.1 \nu_0$. 
middle of the array, the subharmonic emission spectra are quite different. In this latter case, there are broad intensity peaks on fractional steps, as shown in Fig. 4.10. The narrow off-step peaks are not apparent for the particular parameters shown. [When \( f = 1/2 \), the narrow off-step peaks in \( S_{y}^{(m=1/2)} \) and \( S_{y}^{(m=3/2)} \) can be clearly seen, superimposed on broad peaks which occur on the fractional steps.]

The reason why the broad intensity peaks seen in a single junction are absent from the whole array is related to the coherent motion of vortices. As an example, consider the first fractional step \( (n = 1/q) \) in a magnetic field \( f = 1/q \). This field has a vortex lattice with a \( q \times q \) unit cell. As already mentioned in connection with Fig. 4.2, this vortex lattice needs \( q \) cycles of an a.c. field in order to return to an identical vortex pattern. Thus, at a junction \( m \), we have a periodic voltage signal \( V_{m}(t) \) with a period \( qT_{a} \), where \( T_{a} \) is the period of the applied a.c. field:

\[
V_{m}(t) = V_{m}(t + qT_{a}), \quad (4.3)
\]

\[
V_{m}(t + T_{a}) = V_{m+1}(t). \quad (4.4)
\]

Equation (4.3) reflects the coherent motion of vortices, while Eq. (4.4) implies the period of the vortex lattice \( T = qT_{a} \) (or \( \nu = \nu_{a}/q \)). The implication of the combined Eqs. (4.3) and (4.4) is that subharmonic intensities can exist at frequency \( \nu_{a}/q \).

If we consider an average voltage over \( q \) adjacent junctions (or over the whole commensurate array), \( V(t) = \sum_{i=1}^{q} V_{m+i}/q \), the period of this voltage is just the same as that of the applied a.c. field, \( T = T_{a} \):

\[
V(t + T_{a}) = (V_{m}(t + T_{a}) + V_{m+1}(t + T_{a}) + \cdots + V_{m+q-1}(t + T_{a}))/q,
\]

\[
= (V_{m+1}(t) + V_{m+2}(t) + \cdots + V_{m+q}(t))/q.
\]
Figure 4.10:

(a). I-V characteristic of a $6 \times 6$ array at a magnetic field of magnitude $f = 1/3$ flux quanta per quanta per plaquette; (b) and (c): The same subharmonics intensities as in Figs. 4.9(d) and 4.9(e), as a function of d.c. current, but for a voltage across a single junction near the middle of the array; (d). subharmonic intensity $S_V^{(m=4/3)}$ for the voltage across the same junction.
\[ V(t) = V(t). \] (4.5)

Therefore, the average over the whole commensurate array should not show any subharmonic structure on the fractional steps.

4.10 Discussion.

We have presented a numerical study of Shapiro steps in an array of resistively shunted Josephson junctions. We have verified that the time-dependent voltage is periodic on the steps, with many higher harmonics. The presence or absence of steps is sensitively affected by the transverse boundary conditions. The step widths are found to be oscillating functions of the amplitude and frequency of the a.c. driving currents.

At finite magnetic field of strength \( p/q \) flux quanta per plaquette, we find generally that all steps of the form \( N\hbar \nu/(2e\omega) \) appear in the I-V characteristics. At all fields investigated, the vortex lattice is translated an integer number of plaquette lattice constants per cycle of the a.c. field. This seems consistent with the model of Benz et al, as well as with a recent model of Shapiro steps due to Kvale and Hebboul[27], which describes these steps in terms of only two degrees of freedom: a "vortex" and a "phase." At each magnetic field, the steps melt and disappear at temperatures no higher than the corresponding zero-current superconducting transition temperatures of the array.

Several groups[28] have suggested the possibility of "anomalous" half-integer steps of the form \( \langle V \rangle = N\hbar \nu/(4e) \) at extremely low magnetic fields. We have no explanation for these anomalies, except to note that we see similar half-integer steps at \( f = 1/5 \) with free boundary conditions. We may speculate that such anomalies may occur whenever the lattice is such that the vortex
pinning potential is significantly aperiodic. This will occur with free boundary conditions, or with disorder in plaquette areas or coupling strengths. Another possibility is that even a periodic vortex pinning potential will not be purely sinusoidal but will have higher harmonics. Such harmonics [analogous to terms of the form $\cos (2\phi)$ in the coupling energy of a single Josephson junction] produce subharmonic steps in single junctions. These speculative possibilities could be further investigated, if the anomalous half-integer steps are confirmed by further experiment.

The existence of the narrow off-step peaks can be explained by a beating effect. Since such peaks are found to be sensitive to disorder and finite temperature, they may not be seen in an experiment if there is no substep at the corresponding d.c. current[26]. Broad spectral peaks for the harmonics exist at d.c. currents on the Shapiro steps at zero and finite magnetic field. This feature qualitatively is in good agreement with the results observed by Hebboul and Garland[26]. At a finite magnetic field of $f = 1/q$ flux quantum per plaquette, Fourier analysis of the voltage drop across a single junction in an array also shows such broad peaks for the subharmonics at currents on the fractional giant Shapiro steps. The absence of the peaks from the whole array confirms the coherent motion of vortices on the Shapiro steps.
CHAPTER IV REFERENCES


5.1 Introduction.

In previous chapters, we have studied the dynamical properties in two-dimensional Josephson junction arrays. In this chapter, we turn to three-dimensional networks of resistively-shunted junctions. First, we extract possible static and dynamical properties from this percolative phase transition, and then we focus on the critical currents of real granular superconductors to connect our results to plateaus of critical currents as observed, for example, by Sato et al [1].

A high critical current is one of the most desirable practical properties of superconductors[2]. The high-temperature superconductors may possess a large intrinsic, or depairing critical current, if they are prepared in single-crystal form, but this large nominal critical current may be of little practical use in possible high-current applications, because single crystals are brittle and may have poor thermal properties. For this reason, there has been much recent research in the use of composite superconductors for current-carrying applications[3]. Such
composites should ideally be prepared of high-temperature superconductor and a normal metal which is a good conductor. Both components should form an infinite connected cluster extending throughout the sample, so that the superconducting component can carry a finite critical current below the superconducting transition temperature $T_c$, while the normal metal can carry a reasonable current in the normal state and contribute mechanical strength.

In this chapter, we describe calculations of current-voltage characteristics in normal-superconducting composites, modeled as a random coupled collection of Josephson junctions. We also present a qualitative analysis of the behavior to be expected of such composites, which is similar to the behavior we find numerically. Finally, we discuss our results in the light of recent measurements on Ag/YBa$_2$Cu$_3$O$_{7-x}$ composites[4] and on Ag-sheathed Bi-based high-temperature superconducting wires in a magnetic field[1]. These latter experiments show a $J_c$ as high as nearly $10^5$ amps/cm$^2$ at zero field, and, even more strikingly, a plateau in $J_c$ as a function of magnetic field, persisting to a field of 10 tesla or even higher, at a value of $J_c$ 10 to 50 % of the zero-field value. We suggest a possible explanation for this plateau, based on a concept of saturation of frustration in a composite superconductor. Evidence for this picture is provided by our model calculations carried out on disordered arrangements of Josephson junctions.

5.2 Models

To model critical currents, we consider a simple cubic array of $N \times N \times N$ superconducting grains, spaced a distance $a$. The grains are assumed connected by resistively-shunted Josephson junctions, each junction having the same critical
current $I_{0}$ and same shunt resistance $R_{0}$. A current $I$ is injected into each grain on one face of the cube, and extracted from each grain on the other face. We add additional layer of superconducting grains on each face of electrode. Periodic boundary conditions are imposed in the two transverse directions.

The procedure for the calculation of this three-dimensional array is the same as in two-dimensional calculation described in Chapter II.

\[ \sum_{i} \frac{\hbar}{2eR_{ij}} \frac{d\phi_{i}}{dt} = I_{i,ext} - \sum_{j} I_{ij} \sin(\phi_{i} - \phi_{j} - A_{ij}), \]  

(5.1)

where $G_{ij} = -\frac{1}{R_{ij}}$ for $i \neq j$; and $G_{ii} = -\sum_{j \neq i} G_{ij}$. We solve these equations using a straightforward approach described previously[5-10] and in our own earlier chapters.

To introduce randomness, we have considered two different models: a site diluted array (Model I) and a positional disordered array (Model II).

5.2.1 A Site Diluted Array: Model I

This model involves site dilution: grains are present or absent with probabilities $p$ or $1 - p$, with the occupation probabilities of neighboring grains assumed uncorrelated. In the absence of a grain at a given site, only normal current is assumed to flow between that site and its nearest neighbors. For convenience of calculation in this model, we assume that shunt resistances between all nearest neighbor sites, whether or not occupied by grains, are identical. If $p$ is sufficiently small, one expects no superconducting path connecting one face to the other face of the cubic array. As the concentration $p$ is increasing, the first infinite connected cluster of superconductors can be formed at the percolation threshold $p_{c}$. The value of $p_{c}$ depends on the dimensionality and the type of lattice being
considered. Table 5.1 shows some values of the percolation threshold in several lattices\[11\].

Near the percolation threshold, some quantities such as the order parameter $P$ and correlation length $\xi_p$ follow a power law [11]:

$$P \sim (p - p_c)^\beta,$$

$$\xi_p \sim |p - p_c|^{-\nu},$$  \hspace{1cm} (5.2)

where $\beta$ and $\nu$ are the exponents for the percolation order parameter (corresponding to magnetization in thermal phase transition) and the correlation length, respectively. The percolation order parameter $P$ is defined as the percolation probability that an arbitrary selected grain belongs to the infinite cluster of the system. Both $\beta$ and $\nu$ are related to the geometrical aspect of percolation.

In a mixture of normal conducting materials and superconducting materials, like our model, the conductivity for $p > p_c$ becomes infinite due to the presence of superconducting paths. For $p < p_c$, the conductivity diverges as percolation threshold of the superconductors is approached [12]:

$$\sigma \sim (p - p_c)^{-s},$$ \hspace{1cm} (5.3)

where $s$ is the conductivity exponent characterizing the diverging behavior. [In a metal-insulator mixture, the conductivity vanishes below the critical volume fraction of metals (i.e. $p < p_c$). For $p > p_c$, the conductivity follows a power law: $\sigma \sim (p - p_c)^t$, where $t$ is the exponent for the conductivity of the network.]

The critical current density in this mixture can be defined in the superconducting state and is expected to follow a power law near the percolation threshold [12]:

$$J_c \sim (p - p_c)^\gamma,$$  \hspace{1cm} (5.4)
where $\nu$ is the exponent for the critical current density. The exponent $\nu$ is thought to be related to the correlation-length exponent $\nu$, for example, by $\nu = (d - 1)\nu$ in a d-dimensional lattice percolation[11]. (Continuum system such as the “Swiss-cheese” model has a different relation[12]. This model is defined as a uniform d-dimensional superconducting medium which contains d-dimensional spherical non-superconducting regions of radius $a$.) The values of the exponents for two dimensions, three dimensions, and Bethe lattices are listed in Table 5.2 [11,12,13].

5.2.2 Randomly Displaced Array: Model II

We generate the second model (Model II) starting from an ordered simple cubic lattice of resistively-shunted Josephson junctions such that all critical currents and shunt resistances are equal. We then move the $i$th grain a random distance $\delta x_i = (\delta x_i, \delta y_i, \delta z_i)$, keeping all the $I_e$'s and $R_0$'s unchanged, such that $\delta x_i, \delta y_i$, and $\delta z_i$ are chosen from independent Gaussian distributions of width $\sigma$ - that is, the probability that $\delta x_i$ lies between $x$ and $x + dx$ is $P(x)dx$, where

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}.$$  \hspace{1cm} (5.5)

This model is a generalization of a two-dimensional model studied theoretically and experimentally by several authors [14-16].

In order to analyze the behavior of this model, we have to consider the effect of random plaquette area on the magnetic phase factors $A_{ij}$. If we consider the general direction of the external magnetic field relative to the array, the magnetic field has three components, ($B_x, B_y, B_z$).

$$\bar{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z} = \vec{\nabla} \times \bar{A},$$

$$\bar{A} = B_y x \hat{x} + B_z x \hat{y} + B_y y \hat{z},$$  \hspace{1cm} (5.6)
Table 5.1:
Values of $p_c$ for bonds and site percolation on various lattices. After Ref. [11]

<table>
<thead>
<tr>
<th>Lattice</th>
<th>Bond Percolation</th>
<th>Site Percolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>0.5000</td>
<td>0.5928</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.3473</td>
<td>0.5000</td>
</tr>
<tr>
<td>Diamond</td>
<td>0.388</td>
<td>0.428</td>
</tr>
<tr>
<td>Honeycomb</td>
<td>0.6527</td>
<td>0.6962</td>
</tr>
<tr>
<td>Simple cubic</td>
<td>0.2492</td>
<td>0.3117</td>
</tr>
<tr>
<td>Body-centered cubic</td>
<td>0.1785</td>
<td>0.245</td>
</tr>
<tr>
<td>Face-centered cubic</td>
<td>0.119</td>
<td>0.198</td>
</tr>
</tbody>
</table>

Table 5.2:
Values of the exponents $\beta, \nu, t, \text{and } a$ for $d = 2, d = 3$, and Bethe lattices. After Refs. [11-13].

<table>
<thead>
<tr>
<th>Exponent</th>
<th>$d=2$</th>
<th>$d=3$</th>
<th>Bethe</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$5/36$</td>
<td>0.4</td>
<td>1</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$4/3$</td>
<td>0.9</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$t$</td>
<td>1.3</td>
<td>1.9</td>
<td>3</td>
</tr>
<tr>
<td>$s$</td>
<td>1.3</td>
<td>0.8</td>
<td>0</td>
</tr>
</tbody>
</table>
where $\vec{A}$ is the vector potential in Landau gauge. Suppose $B_x = 0$ so that the direction of the magnetic field is perpendicular to the $x$-axis. Using Eq. (2.10), we get

$$A_{ij} = \frac{2\pi \Phi_0}{B_x} \left[ B_x \frac{(x_j + x_i)}{2} (y_j - y_i) + B_y \frac{(x_j + x_i)}{2} (x_j - x_i) \right],$$

where $\Phi_0$ is the flux quantum $hc/2e$, and $(x_i, y_i, z_i)$ is the location of a grain $i$ displaced from its original site of $(x_i^0, y_i^0, z_i^0)$. If we assign the direction of the magnetic field as an angle between $y$- and $z$-axis, we use $B_y = B \cos \theta, B_z = B \sin \theta$ to get

$$A_{ij} = \frac{2\pi B}{\Phi_0} \left[ \frac{(x_j + x_i)}{2} (y_j - y_i) \sin \theta + \frac{(x_j + x_i)}{2} (x_j - x_i) \cos \theta \right].$$

The present chapter deals specially with a magnetic field along the $z$-axis ($\theta = 90^\circ$), so that only the first term of the above equation is important throughout this chapter. [Angular dependence of the current-voltage characteristics will be discussed by using the above equation on the next chapter.]

5.3 Results

5.3.1 Model I, Zero Magnetic Field

Figure 5.1 shows the calculated I-V characteristics of a $7 \times 7 \times 7$ array at temperature $T = 0$, with boundary conditions as described above, at zero magnetic field and for several concentrations of superconducting grains. Each point shown represents a single realization of the disorder. Since the array is small, different realizations of the disorder lead to different percolation threshold. However, we expect that the apparent critical exponents near the percolation threshold would change little from one realization to another. The I-V characteristics are obtained by averaging over a time interval of approximately 1000 $\tau_0$, where
Figure 5.1:
I-V characteristics of a $7 \times 7 \times 7$ cubic array of grains connected by resistively-shunted junctions, with periodic boundary conditions, and randomly site-diluted so that a fraction $p$ of the sites are occupied by superconducting grains. A current $I$ is injected into each grain on one face of the sample, and removed from the opposite face. $(V)$ is the time-averaged voltage drop across the sample, $I_{c0}$ is the critical current and $R_0$ is the shunt resistance of a single junction. The percolation threshold for this particular realization of site percolation is $p_c = 0.370$. 

\[
\begin{align*}
\frac{\langle V \rangle}{NR_0} &> 0.6 \\
\frac{I}{I_{c0}} &> 0.6 \
\end{align*}
\]
\( \tau_0 = \hbar/(2eRI_c) \) is the natural microscopic time unit for this problem, after the first 200 \( \tau_0 \) of voltages are discarded. The curves show the expected behavior. A percolation threshold occurs at approximately \( p_c \approx 0.370 \), near the known site-percolation threshold \( p_c = 0.312 \) of an infinite simple cubic lattice\[11\]. Above this concentration, the composite is superconducting at \( T = 0 \), with a finite critical current.

Figure 5.2(a) shows that the critical current density \( J_c(p) \) roughly follows a power-law dependence of the form \( J_c(p) \propto (p - p_c)^v \) with \( v \approx 1.7 \). This is in agreement with the prediction of a purely geometrical model of critical current in a percolating superconductor\[12\], but is obtained here on the basis of a dynamical model with time-dependent current elements. Below \( p_c \), the composite has a finite resistivity at \( T = 0 \), but the I-V characteristics are nonlinear, a result of finite superconducting clusters present at these concentrations. In the limit of small currents, the resistivity in this regime \( \rho(p) = L^{d-2}dV/dI \) is expected to vary as \( (p_c - p)^s \) with \( s \approx 0.8 \) (\( d = 3 \) is the dimensionality, \( L = \) sample dimension)\[13\]. Within our rather limited range of concentrations and sample sizes studied, this expectation is confirmed [Fig. 5.2(b)].

These I-V characteristics seem to be reasonably well described by a scaling form analogous to one proposed by Fisher et al for the phase transition from a normal state to the vortex glass state at finite temperatures\[17\]:

\[
E(\xi_p, J) = J\xi_p^{d-2-s}E_{\pm}(J\xi_p^{d-1}/I_c).
\tag{5.9}
\]

Here \( d = 3 \) is the spatial dimensionality of the system, \( J \) is the d.c. current density, \( \xi_p \) is a correlation length appropriate for the problem, \( I_c \), the critical current of a single link, is the characteristic microscopic current of the problem, and \( s \) is an appropriate dynamical critical exponent, corresponding to a relaxation time.
Figure 5.2:
(a). Critical current density $J_c(p)$ of the array of Fig. 5.1, for several values of $p$. The slope of the straight line shown is 1.73, comparable to the predicted value of $v \approx 2\nu$, where $\nu = 0.9$ is the correlation length exponent. Inset: I-V characteristics for the same samples. (The curves, from left to right, are $p=0.431$, 0.446, 0.472, 0.504, 0.534, 0.577, and 0.636.)
(b). Resistivity $\rho \equiv (V)/(NI)$ for several values of $p$, evaluated at $I = 0.03I_c$, for the sample of Fig. 5.1, shown on a log-log plot. The slope of the dashed line is $s = 0.84$. (The predicted value[16] is 0.8 for $d = 3$). Inset: I-V characteristics of the same samples. (The curves, from bottom to top, denote $p=0.315$, 0.300, 0.274, 0.213, and 0.140.)
\( \tau_s \propto \xi_p^s \) which would diverge at \( p_c \). For the present problem, \( \xi_p \) is the percolation correlation length, which, in an infinite system, is believed to diverge on either side of the critical concentration \( p_c \) as \( \xi_p \propto |p - p_c|^{-\nu} \) where \( \nu \approx 0.9 \) in \( d = 3 \) [18]. \( E_\pm \) denotes two different scaling functions which apply above and below \( p_c \). [In the vortex glass transition, the scaling equation has the same form as Eq. (5.9), but \( \xi_p \) is replaced by the vortex glass correlation length, and \( I_c \) by a different characteristic current appropriate to that problem. See the next chapter.]

We can estimate some features of the scaling functions from the generally accepted behavior[11-13] of a percolating superconductor in several limiting regimes. For \( p < p_c \), we expect Ohmic behavior, with a resistivity \( \rho \propto |p - p_c|^s \propto \xi_p^{-s/\nu} \). This implies that \( E_-(x) \rightarrow \text{constant as } x \rightarrow 0 \), and, further, that

\[
d - 2 - z = -s/\nu. \tag{5.10}
\]

Secondly, for \( p > p_c \), we expect that, in the limit of an infinite sample, the composite should exhibit a critical current varying as \( |p - p_c|^{\nu} \propto \xi_p^{-s/\nu} \). A finite critical current is achieved by requiring that \( E_+(x) \) vanish for \( x < x_c \). The value of \( \nu \) implied by the scaling relation is then \( \nu = (d-1)\nu \). This value is believed to be a reasonable approximation in both \( d = 2 \) and \( d = 3 \) [12].

Given these relations, one test of the scaling relation is to consider the I-V characteristics precisely at \( p = p_c \), where \( \xi_p \) becomes infinite. At this concentration, therefore, the scaling form must become independent of \( \xi_p \). This requires that both \( E_+(x) \) and \( E_-(x) \) must vary as \( x^a \) for large \( x \), with \( a = (z-d+2)/(d-1) \), or equivalently, that at \( p = p_c \)

\[
E \propto J^{1+a} \propto J^{(z+1)/(d-1)} \propto J^{1+a/[(d-1)\nu]} \tag{5.11}
\]

Using the accepted values \( s \approx 0.8, \nu \approx 0.9 \) in \( d = 3 \), we find \( E \propto J^{1.44} \) in \( d = 3 \).
We have tested this relation by finite size scaling[19] as applied at $p = p_c$, where the correlation length $\xi_p$ is infinite. Under such conditions, relation Eq. (5.9) continues to apply, provided $\xi_p$ is replaced by $L \equiv Na$, the linear dimension of the sample. In Fig. 5.3(a), we plot current-voltage characteristics for several sample sizes at $p = p_c$, using $s/\nu = 0.89$. The various curves all fall on the same curve, as predicted, and exhibit, in particular, the power-law behavior of Eq. (5.9) [20].

Figure 5.3(b) is a plot of the scaling functions of Eq. (5.9). The data points in insets of Fig. 5.2(a) and 5.2(b) were used to deduce the scaling functions by varying the exponents $\nu$ and $s$. The exponents $\nu \approx 1.7$, $s \approx 1.0$ are fairly consistent with the previous values obtained from the slopes of Fig. 5.2(a) and 5.2(b). Considering that our calculation is carried out for a single realization of a sample of only moderate size, our results are in reasonable agreement with the predicted scaling behavior.

5.3.2 Models I and II, Finite Magnetic Field.

In the presence of a magnetic field, the expected qualitative behavior of the critical current in these models can be understood from the schematic of Fig. 5.4. The key idea is frustration. To define frustration in the present model, we consider some particular closed loop (say the $a_{ih}$ loop) of junctions. The decomposition of the lattice into closed loops is, of course, not unique in three dimensions, but this is not relevant to the qualitative argument. Now introduce $f_\alpha$, the flux through this loop, in units of a superconducting flux quantum $\Phi_0 = \hbar e/2e$. $f_\alpha$ can always be written as the sum of an integer and a fractional part $\delta f_\alpha$, which is defined as the frustration. By definition, $0 \leq \delta f_\alpha < 1$. At zero
Figure 5.3:
(a). Finite-size scaling plot of \((V/N)^{d+z}N^z\) versus \(N^z\) for an \(N \times N \times N\) simple cubic lattice of Josephson junctions for several values of \(N\), precisely at the percolation threshold for the particular realization of the lattice shown. \(V\) is the voltage drop across the sample, \(I\) is the current injected into each node, \(d = 3\) is the space dimension, and \(z\) is the dynamical critical exponent. We use \(z = d-2+s/\nu\) with \(s/\nu = 0.89\), as predicted theoretically for \(d = 3\). The various points collapse satisfactorily onto a single curve.

(b). Plot of the scaling function near the percolation threshold on a \(7 \times 7 \times 7\) lattice, as obtained from the data points in insets of Fig. 5.2(a) and 5.2(b). The plots shown are the best fits obtained by varying \(\nu\) and \(s\).
Schematic diagram of a percolating superconductor in a magnetic field, indicating the meaning of various terms discussed in the text. The percolating structure is depicted in terms of the "nodes-links" model, according to which superconducting links (or chains of Josephson junctions) of length $\xi_p$, connect one node to another. The flux through the $\alpha^{th}$ plaquette (of area $\xi_p^2$) is $f_\alpha \Phi_0$, where $\Phi_0 = \hbar c/2e$ is a flux quantum. $f_\alpha$ can be written as an integer plus a fractional part $\delta f_\alpha$ which is defined as the frustration of the $\alpha^{th}$ plaquette.
magnetic field, the frustration is zero in all loops (disregarding any screening flux which may be induced by the applied currents). If the loops have random areas, and are randomly oriented with respect to the applied field, then at sufficiently strong fields the frustration will be approximately uniformly distributed between 0 and 1, and this distribution will change little with further increases in field. This composite will then closely approximate the superconducting glass state discussed by Shih et al [21], or the model vortex glass described by Huse and Seung[22]. The cross-over field will be of the order of $H_X = \Phi_0 / \Delta S$, where $\Delta S$ is the rms fluctuation in loop area. For applied fields $H \gg H_X$, we expect the critical current to saturate at a constant value. Thus, this model is expected to produce the critical current plateau exhibited by composite superconductors in a magnetic field.

Given the plateau, the fractional depression of the critical current by the field depends not only on the frustration, but also on the size of the frustrated loop. The size effect is analogous to that which governs the amplitude of the Little-Parks $T_c$ oscillations in superconducting loops. Just as the amplitude of those oscillations is smallest for large loops[23], so the depression of critical current by a magnetic field is expected to be smallest near $p_c$, where the size of typical loops in the infinite superconducting cluster is largest.

To estimate the depression of $J_c$ crudely for our model, consider a single loop of $N$ grains, injecting current $I$ into one grain and removing it from a grain at the opposite side of the loop, such that the loop is divided into two parts each containing $N/2$ junctions ($N$ even). The critical current of this loop will be symmetric about $\delta f_\alpha = 1/2$, where $\delta f_\alpha$ is the frustration of the loop. For $0 \leq \delta f_\alpha < 1/2$, it is easily shown that the critical current of the loop in this
Table 5.3:

Suggested dependence of several properties on the percolation correlation length $\xi_p$ in $d = 3$. Shown are the critical current density $J_c$, the cross-over field $H_X$ at which $J_c$ starts to develop a plateau, and the fractional decrease $1-J_c(\infty)/J_c(0)$ in critical current density with magnetic field.

<table>
<thead>
<tr>
<th>Property</th>
<th>Dependence on Coherence Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_c$</td>
<td>$1/\xi^2_p$</td>
</tr>
<tr>
<td>$H_X$</td>
<td>$1/\xi^2_p$</td>
</tr>
<tr>
<td>$1-J_c(\infty)/J_c(0)$</td>
<td>$1/\xi^2_p$</td>
</tr>
</tbody>
</table>

configuration at $T = 0$ is $2I_c\cos(2\pi\delta f_a/N)$. In the limit of a large loop, the maximum fractional depression of critical current occurs in the "fully frustrated loop" ($\delta f_a = 1/2$) and is $\pi^2/(2N^2) \propto 1/\xi^2_p$ for a loop of typical linear dimension $\xi_p$. Thus, although the critical current at zero field is smallest near the percolation threshold, the fractional reduction in critical current density by the field is also smallest near $p_c$. (A summary of these qualitative predictions is given in Table 5.3.)

We have confirmed these arguments by calculations for both Model I and Model II in finite magnetic fields. In model II, the couplings between grains are unaffected by the disorder, but the various plaquettes have random areas. Thus, there is effectively no disorder at zero field, but the effective disorder becomes stronger and stronger as the field is increased, because the fluctuations
in frustration become larger and larger. At sufficiently large fields, the frustration through the various closed loops is approximately uniformly distributed between 0 and 1, and further increases in the field do not affect the critical current. This behavior is clearly exhibited in Fig. 5.5, which shows the critical current for Model II, with $\sigma = 0.12$ and several integer values of the average flux per plaquette $f_{av} \equiv H a^2 / \Phi_0$. The cross-over field $H_x$ is approximately $\approx 5 \approx 1/\sigma$. The fractional depression of critical current on the plateau is about 80%. Note that in this model, superimposed on the monotonic decrease of $J_c$ with increasing integer values of $f_{av}$, there are also expected to be oscillations in $J_c(f)$ with period unity, which are not shown in the figure. These oscillations arise because, when $f_{av}$ takes on fractional values (e.g., $f_{av} = 1/2$), the average frustration per plaquette is larger than at $f_{av} = 1$, because fluctuations in area are quite small at $\sigma = 0.12$. For larger $\sigma$, the fluctuations in frustration are so great that such oscillations in $J_c(f_{av})$ are not expected to occur. We have not, however, tested this conjecture numerically.

Figures 5.6(a) and 5.6(b) show the current-voltage characteristics of a $4 \times 4 \times 4$ percolating sample (Model I) at full occupancy ($p = 1.00$) and just above the percolation threshold ($p = 0.328$) at several fields. Although the sample size is extremely small, the results are consistent with the qualitative arguments given above. In particular, as predicted, the I-V characteristics are significantly field-dependent in the ordered case, but nearly field-independent near the percolation threshold, where the loop size is larger. (In this model, since the grains lie on the sites of a lattice, all properties are periodic in field with period $f = 1$.)
Figure 5.5:
Current-voltage characteristics of a $7 \times 7 \times 7$ array, with Model II disorder and disorder parameter $\sigma = 0.12$, plotted as a function of magnetic field, for several values of the average flux $f \Phi_0$ per plaquette. The field is oriented perpendicular to the applied current (similar results are obtained with field and current parallel). The solid line is the "Huse-Seung limit" (infinite-field limit), in which the phase factors $A_{ij}$ are chosen independently and randomly from a uniform distribution on the interval $-\pi \leq A_{ij} < \pi$. Inset: critical current versus field for the curves of Fig. 5.5, at several integer values of $f$. 
Figure 5.6:
Current-voltage characteristics of a $4 \times 4 \times 4$ array with Model I disorder, plotted as a function of field ($f=1/2, f=1/3, f=1/4$) at a concentrations (a) $p = 1.00$ (ordered lattice); (b) $p = 0.328$. The I-V characteristics, strongly field-dependent in the fully occupied lattice, are little affected by the magnetic field in case (b), which is just above the percolation threshold for this realization.
5.4 Discussion.

Two main conclusions can be drawn from the present work. First, in a percolating network of Josephson junctions, the low-temperature critical current will exhibit a roughly power-law dependence on volume fraction of superconductor, \( J_c(p) \propto (p - p_c)^v \) with \( v \approx 1.7 \). This result is in agreement with previous static work, and with unpublished experiments of Calabrese et al [4] on random three-dimensional composite of Ag and YBa\(_2\)Cu\(_3\)O\(_{7-\delta}\). In these experiments, the critical current is of order \( 10^2 \) amps/cm\(^2\), suggesting control by weak links, and the temperature dependence of \( J_c \) for a particular volume fraction is \( J_c(T) \propto (T_c - T)^3 \), suggesting that the relevant weak links are S-N-S proximity junctions[24]. Since our calculations involve only a single realization of the disorder, a very precise power-law cannot be extracted for the concentration dependence. Our predictions for the magnetic-field dependence of the critical current remain to be tested.

We have also suggested a new scaling relation [Eq. (5.9)] which we have tested numerically by finite-size scaling arguments. This relation would be more difficult to test experimentally at different concentrations, since each concentration would require the preparation of a new composite. However, one aspect of the scaling theory — namely, the prediction of an anomalous power law precisely at \( p = p_c \) — could be readily tested experimentally.

Our second prediction has to do with the plateau in the I-V characteristic as a function of a magnetic field. We suggest one possible explanation in terms of the saturation of frustration, and, using the language of a random array of coupled weak links, estimate both the field at which the saturation begins to set in, and the fractional reduction in critical current arising from this saturation. While the
model is couched in terms of weak links, the essential element is randomness and frustration, rather than the weak links themselves. That is, even in a random but multiply connected network of strongly linked superconductor (i.e. a network containing many holes), random frustration would produce a plateau in the I-V characteristics at strong magnetic fields[25]. This plateau would persist until single grain effects (e.g., driving some of the grains normal) would again cause a fall-off in the critical current. Thus, our model might offer a possible explanation for the results of Sato et al [1], who find transport critical currents exceeding $10^6$ amps/cm² in tapes of Ag-clad BSCCO superconductors, prepared by cycles of sintering and mechanical deformation, with very little drop-off in critical current up to magnetic fields of 30-40 Tesla.

Our model describes point junctions. In a real composite superconductor, the junctions have finite areas, which may be randomly oriented relative to the applied field. The present model could still be applied to networks of such junctions, provided the single-junction current, $I_{c0}$, were replaced by $I_{c0}(H)$, the usual single-slit-pattern oscillating critical current for a single junction. Since each such junction would have a different projected area normal to the field, the superposition of such single-slit patterns could also lead to a sharp but monotonic fall-off in the critical current with field, followed by a plateau. Some proposed explanations, which involve the physics of such single connections between the BSCCO crystallites, but not the network effects discussed here, could offer an alternative means of understanding a plateau in the critical current[26].

Finally, although this work describes I-V characteristics at $T = 0$, we speculate that a similar picture might describe a highly inhomogeneous transition at finite $T$. For example, if the Josephson couplings, instead of having a bimodal
distribution as assumed here, were described by a much broader distribution, then one might imagine an inhomogeneous, or percolation, transition at finite $T$, in which each junction, of critical current $I_c$, "turned on" at its own characteristic temperature $T_c = \frac{\hbar I_c}{2k_B e}$. Such a picture has been found by Deutscher and collaborators[27], to give a reasonable description of the specific heat in granular Al. In the present model, a similar picture would lead to a temperature dependent fraction of Josephson junctions. The predictions of the present chapter, and in particular the scaling hypothesis, would then continue to hold valid, provided that the volume fraction $p$ is replaced by temperature $T$. Such a phase transition might occur in a highly disordered system, and might lead to percolation critical exponents for the I-V characteristics near $T_c$ at zero magnetic field, rather than those predicted by scaling for a homogeneous transition.
CHAPTER V REFERENCES


[20] Duxbury and collaborators [see, e. g., P. M. Duxbury, P. D. Beale, and P. L. Leath, Phys. Rev. Lett 57, 1052 (1986)], using arguments based on the statistics of extremes, have emphasized that breakdown fields and currents at any concentration of defects should vanish in the limit of an infinite sample. Their arguments, which are particularly applicable at small concentrations of defects, are analogous to the Lifshitz arguments used to discuss exponential band tails in semiconductor alloys. Our scaling analysis, which emphasizes behavior near a percolation threshold at fixed
volume, is presumably consistent with that of Duxbury et al, although we have not attempted a direct comparison of the two.


[25] Previous discussions of composite superconductors in a magnetic field have been given by S. Alexander, Phys. Rev. B27, 1541 (1983), who did not analyze the magnetic field dependence of the critical current.


CHAPTER VI

MODEL VORTEX GLASS AND FLUX FLOW RESISTIVITY

6.1 Introduction

In the previous chapter, we have considered a three-dimensional percolative and positionally disordered array of weakly coupled Josephson junctions to study critical currents of composite superconductors. In the first part of this chapter, we turn to a vortex glass transition in a strong magnetic field in a three-dimensional disordered array. Following this, the angular dependence of the flux flow resistivity in a cubic ordered array is discussed in the next part of the chapter. Both parts deal with dynamical scaling functions near the vortex glass transition and the corresponding three-dimensional classical XY phase transition.

6.1.1 Background for the Vortex Glass Model

Fisher and collaborators[1,2] have recently developed a theory for a vortex-fluid to vortex-glass transition in a disordered superconductor in a strong magnetic field. Unlike the "giant flux creep" model[3] of Yeshurun and Malozemoff, which was proposed to describe the "irreversibility line" seen in high temperature su-
perconductors[4], this theory involves a true phase transition with static and
dynamic scaling behavior. Also, in contrast to the vortex-fluid to vortex-lattice
theory of Nelson and coworkers[5], disorder plays an essential role in the fluid-
to-glass transition. Several recent experiments have described high-field current-
voltage characteristics[6,7] and complex impedance measurements[8] in thin films
of high-temperature superconductors which are consistent with the Fisher et al
scaling theory[1,2] over a wide range of parameters.

On the numerical side, Huse and Seung[9] have carried out careful Monte
Carlo simulations, based on a simplified model of a vortex glass. This model,
which is a refinement of one originally proposed for granular superconductors[10,11],
may be in the same universality class as that proposed by Fisher et al. By cal-
culating size-dependent fluctuations in a spin-glass order parameter, Huse and
Seung suggested a glass transition temperature $T_g$ of about $0.6-0.7 E_J/k_B$, where
$E_J$ is a characteristic interaction energy. The same model was investigated in
greater detail by Reger et al [12], and by Gingras[12], who considered the Monte
Carlo size-dependence of a suitably defined domain wall energy. They found
the three-dimensional Huse-Seung vortex glass to lie at or just above the lower
critical dimension (defined to be the dimension below which no phase transition
takes place), with $T_g \approx 0.45E_J/k_B$.

In the first part of this chapter, we calculate the I-V characteristics of a dynamical
model vortex glass. Our model can be thought of as describing a gran-
ular superconductor in the limit of a strong magnetic field, in which the grains
are coupled by resistively-shunted Josephson junctions. Alternatively, it might
be viewed more abstractly, as a representation of a disordered (not necessarily
granular) superconductor in a strong magnetic field, in which the correlation
length characterizing the geometry is called the "spacing between grains" and
the dissipative dynamics is simulated by resistively shunted junctions. In the
spirit of critical phenomena, it is possible that the response of this highly simpli-
fied dynamical model is in the same universality class as a more realistic model
describing experiments. Because of numerical limitations, it is not possible to
say definitively that a phase transition does take place at the assumed vortex
glass transition temperature, or that, if there is a transition, we have numerically
reached the critical region. Nevertheless, the resulting I-V characteristics both
above and below the assumed $T_c$ appear consistent with the scaling theory of
the vortex glass, with a dynamical critical exponents $z \approx 4$, $\nu \approx 0.5$.

6.1.2 Background for Flux Flow Resistivity

In conventional superconductors, the transition temperature for the magnetoresis-
tivity is shifted to lower temperature with the increasing magnetic field, with
relatively small broadening. However, the high-temperature superconductors
have been reported to have a broad resistive transition[13-15] in the vortex fluid
phase. Kwok and collaborators[15] have recently studied the resistivity $\rho_{ab}$ of
single-crystal YBa$_2$Cu$_3$O$_{7-\delta}$ with both current density $\vec{J}$ and magnetic field $\vec{B}$
applied parallel to the $ab$ plane. In untwinned single crystals, for temperatures
$T < T_c$, where $T_c$ is the temperature at which superconducting fluctuations
first become substantial, they found a roughly $\sin^2 \theta$ angular dependence on the
angle $\theta$ between $\vec{B}$ and $\vec{J}$:

$$
\rho_{ab}(B, T, \theta) \approx \rho_{ab}(B, T, 0) + \Delta \rho(B, T) \sin^2 \theta. \quad (6.1)
$$

with $\Delta \rho > 0$. It is not unexpected that the resistivity would be greatest for field
and current perpendicular, since this configuration maximizes the Lorentz force
acting on the flux lines. However, even for fields parallel to the currents, several groups found a substantial magnetoresistance \( \Delta \rho_{ab}(B,T,0) = \rho_{ab}(B,T,0) - \rho_{ab}(0,T,0) \), the origin of which is unexplained[13-15].

In the second part of this chapter, we present the results of a simulation of flux flow resistivity in a highly simplified model of a "high-temperature superconductor." We are able to qualitatively reproduce some of the features of the flux flow resistivity observed by Kwok et al, including the substantial magnetoresistance with \( B \parallel J \). In zero magnetic field, we find that the current-voltage characteristics satisfy a scaling relation proposed by Fisher, Fisher and Huse[2], with dynamical critical exponent \( z \approx 1.5 \); correlation length exponent \( \nu \approx 0.67 \) at \( T_c \approx 2.21E_J/k_B \) of a three dimensional classical XY model.

6.2 Model "Vortex Glass"

6.2.1 Formalism

Our model vortex glass is a simple cubic \( N \times N \times N \) array of superconducting grains, spaced a distance \( a \). The grains are assumed connected by resistively-shunted Josephson junctions, each junction having the same critical current \( I_c \) and same shunt resistance \( R_0 \). A current \( I \) is injected into each grain on one face of the cube (taken as the plane at \( Y = 0 \)), and extracted from each grain on the other face (at \( Y = Na \)). Periodic boundary conditions are imposed in the two transverse directions.

We introduce randomness into our model vortex glass by using a dynamic generalization of the Huse-Seung Hamiltonian. The phase factor \( A_{ij} \) is chosen to be uniformly distributed in the interval \([-\pi, \pi]\), with \( A_{ij} \)'s corresponding to different bonds assumed uncorrelated. Since the flux through a given plaquette,
$\Phi \equiv \Phi_0 f$, is given by

$$\Phi = \frac{\Phi_0}{2\pi} \sum A_{ij},$$

(6.2)

where the sum runs around the plaquette, this model introduces a random flux through each plaquette. The model has therefore no preferred direction, whereas in the Fisher vortex glass there is a preferred direction of the flux imposed by the applied field. Also, disorder is introduced by random fields in an ordered geometry, whereas a more realistic model might involve a spatially uniform field and plaquettes of random areas.

The dynamical equations for the model glass are[16] the same as in Chapter II. We take the Langevin noise currents at a finite temperature[17]. The Langevin noise currents associated with different junctions are assumed uncorrelated. At each time step, they are selected from a uniform distribution in a range chosen to satisfy Eq. (2.12). [In initial smaller-scale simulations, we also used a Gaussian distribution with parameters satisfying Eq. (2.12). Since this distribution appeared to give numerically the same results as the uniform distribution, while requiring about triple the computer time on the CRAY Y-MP 8/864, we have used the uniform distribution in the results presented here[18].] We have generally used $\Delta t = 0.05\tau_0$, where $\tau_0 = h/(2eR_0 I_c)$. We obtain time averages typically over 500-600 $\tau_0$ after discarding the voltages over the first 200 $\tau_0$. In general, at a given temperature, we begin with a random phase configuration at a large current, reducing the current in intervals and using the final phase configuration of the previous current as the initial one for the new current[19,20].
6.2.2 Scaling Behavior

The scaling at and near a second-order superconducting-to-normal phase transition induces nonlinear current-voltage characteristics. At a given temperature, for an infinite lattice, the current-voltage characteristic is expected to obey the scaling relation [1,2],

\[ E \propto \xi^{1-z} F_{\pm}(J/J_0), \]

\[ J_0 = \frac{c k_B T}{\xi^{d-1} g_0}, \]

where \( E \) is the electric field, \( J \) is the current density, \( c \) is the speed of light, \( d \) is the dimensionality, \( T \) is the temperature, and \( \xi \) is the correlation length characterizing the supposed glass transition (\( \xi \) is expected to vary as \((|T - T_g|/T_g)^{-\nu}\) near \( T_g \)). \( J_0 \) is the characteristic current density. For \( T < T_g \), \( J_0 \) is the critical current density of superconducting phase. To have a finite voltage in Eq. (6.3) at \( T = T_g \), we should have \( F_{\pm}(x) \sim x^{(1+z)/(d-1)} \) for \( x \to \infty \), which gives a power-law I-V characteristic,

\[ E \approx J^{(z+1)/(d-1)} \text{ at } T = T_g. \] (6.4)

Figure 6.1(a) show I-V curves of epitaxial thin films of YBa\(_2\)Cu\(_3\)O\(_y\) measured at \( H = 4 \) T by Koch et al [6]. For \( T < T_g \), the calculated I-V curves have negative curvatures which are inconsistent with the flux-creep model, \( V \sim \sinh(I/J_0) \). Using Eq. (6.4), they obtained the dynamical exponent, \( z = 4.8 \pm 0.2 \), which was extracted directly from the I-V data.

Figure 6.1(b) shows the I-V characteristics of a 7 x 7 x 7 model vortex glass at several temperatures. The curves shown are linear averages over 5 realizations of the \( A_{ij} \)'s, with random initial conditions in each case. The error bars shown
Figure 6.1:
(a). I-V curves measured on epitaxial thin films of YBa$_2$Cu$_3$O$_y$ at H = 4T. After Ref. [6]; (b). The calculated I-V characteristics for a 7 × 7 × 7 model vortex glass at several temperatures. Each point shown is an average over five realizations of the random phase factors $A_y$; the error bars represent the root-mean-square standard deviation. $\langle V \rangle$ denotes the difference between the time-averaged voltages averaged over the planes at Y = 0 and Y = Na. The straight line is the ohmic limit, $\langle V \rangle = NR_0I_c$. 
are rms deviations among the five realizations. According to Fisher et al [1,2], the glass temperature $T_g$ should be marked by a power-law dependence of voltage on current, separating regimes of concave-up and concave-down I-V curves. Since it is difficult to determine $T_g$ accurately from Fig. 6.1(b), using this criterion, we have adopted a different procedure: we assume $T_g = 0.45(hI_c/2k_B e) \equiv 0.45E_J/k_B$, as estimated by Reger et al [12]. Visual inspection of Fig. 6.1(b) suggests that a $T_g$ outside the range $0.3 < k_B T_g/E_J < 0.6$ would be difficult to justify.

Given this $T_g$, we estimate the critical exponent $z$ of the model by finite-size scaling. In a finite-size system of edge $L$, $E$ should depend on an additional scaling variable, $L/\xi$. Exactly at $T_c$, $\xi$ becomes infinite, and $\xi$ in Eq. (6.3) must be replaced by $L = Na$. At $T = T_g$, we have

$$L^{1+z}E(J) \approx G(JL^{d-1}), \quad (6.5)$$

where $G$ is the scaling function.

In order to verify scaling of Eq. (6.5), and determine $z$, we therefore graph the function $(V)N^{z+1}/N$ versus $JN^{d-1}$, where $(V)$ is the voltage drop across the sample, averaged over the X and Y directions. If scaling is valid, this function should fall on a universal curve for all $N$. In Fig. 6.2, we show such a plot for values of $N$ from 4 to 7. For $z \approx 4.0$, the fit is excellent over a restricted range of the variable $IN^2$. The fit is not quite as good for $z = 3$ and 5, and much worse for other values of $z$ (not shown). We therefore estimate $z = 4 \pm 1$. This is (perhaps surprisingly) in the range of other estimates of $z$ obtained from experimental measurements of IV characteristics and of the phase angle describing the complex impedance at $T_c$ of thin film high-temperature superconductors in a strong magnetic field[6-8]. It is also close to the mean-field value predicted for
Figure 6.2:
I-V characteristics for an $N \times N \times N$ model vortex glass at a temperature $T = 0.45 \, E_J/k_B$. We have plotted the quantity $\langle V \rangle/(N R_0 I_c) \times N^{1+z}$ versus $IN^2/I_c$, for several values of $z$ and several values of $N$. If scaling is valid, these curves should all fall atop one another for the correct value of $z$. The best fit is obtained with $z \approx 4$. 
conventional spin glasses[21]. The fit is best over currents in the restricted range $IN^2 = 2 - 10 I_c$. For larger values of $IN^2$, the function is no longer universal. For small values of $I$, there is considerable size dependence, possibly because a "current length" $\xi_j \approx [ck_BT/(J\Phi_0)]^{1/2}$ introduced by Fisher et al approaches $L$ for some values of $L$. In our model,

$$\xi_j \approx a [k_BT/(2\pi iE_J)]^{1/2},$$

(6.6)

$$E_J = \hbar I_c/(2e),$$

where $E_J$ is the Josephson coupling energy, and $i = I/I_c$ is a dimensionless current. For the simulations shown in Fig. 6.1(b), when $i$ is in the range of 0.01 - 0.1, $\xi_j$ is of order $1 - 4 a$. This length should be considered only as an order of magnitude estimate of the scale over which the magnus forces produced by the currents can move the vortices. Evidently, for $i \leq 0.02$, $\xi_j$ will be larger than the dimensions of at least some of the Monte Carlo cells shown in Fig. 6.1(b), so that scaling cannot be verified in this regime.

It is also of interest to consider the times probed by these simulations. The characteristic time, which would diverge at the glass transition, is $\tau = \tau_0(\xi/a)^z$. Taking $z = 4$ and $\xi/a = 7$, we find that the longest time probed is about 2000 $\tau_0$. That is, one would expect a block of grains of linear dimension $7a$ to undergo a phase slip as a whole in a time of order $2000\tau_0$. This is greater than the longest time considered in our time averages. These considerations show that, in order to investigate currents smaller than $\approx 0.01I_c$ per junction, we might need to consider not only larger samples, but also substantially longer time averages.

A more thorough test of scaling is actually to compute the scaling functions, but collapsing all the IV curves shown in Fig. 6.1(b) onto two universal curves, for $T > T_g$ and $T < T_g$. Such a collapse would lead to an estimate of the
correlation length exponent $\nu (\xi \sim t^{-\nu}$, where $t = |T - T_g|/T_g)$. From Eq. (6.3), we have

$$E(J)t^{-\nu(1+z)} \approx F_{\pm} \left( J t^{-\nu(d-1)} / c k_B T \right).$$

(6.7)

Figure 6.3 shows such a plot for $N = 7$ and several choices of $T_g$, $z$, and $\nu$. Of the parameters we have investigated, it appears that the best collapse occurs for $T_g \approx 0.4 - 0.5$, $z \approx 4$, and $\nu \approx 0.5$. In general, the lower $T_g$, the larger $z$, since if there is a power-law I-V characteristic at $T_g$, that power would have to be larger at lower temperatures. Although the predicted scaling behavior thus appears to be quite well obeyed, it is quite difficult to obtain a value of $\nu$ as large as unity, for our range of currents, irrespective of what is assumed for $z$ and $T_g$. We believe that the apparent low value of $\nu$ indicates that the applied current lies in a range outside the critical region of the assumed glass transition; it corresponds, as noted above, to a "current length" of only a few times the lattice constant. It is amusing to note that the values we obtain ($z \approx 4, \nu \approx 0.5$) are those predicted by a mean field theory of phase transitions in an Ising spin glass[19]. This is further evidence that we are in the mean-field regime of the vortex-glass transition. Experiments[6-8] seem to suggest a somewhat larger value of $\nu$ than obtained here, perhaps because the experiments probe a region where the effective current length is much larger than that considered here. The finite-size static scaling analysis of Reger et al [12] also seems to yield a larger value of $\nu$. We speculate that this difference arises once again because our calculations probe a range of current lengths outside the asymptotic critical regime.
Figure 6.3:

I-V characteristics for a $N \times N \times N$ model vortex glass with $N = 7$. We have plotted $(V)t^{-(1+z)}/(NR_0I_c)$ versus $I_{dc}t^{-2\nu}$, where $t = |(T - T_g)/T_g|$, assuming several values of $z$, $\nu$, and $k_B T_g/E_J$, respectively, as follows: (a) 4, 0.5, 0.45; (b) 4, 1.0, 0.45; (c) 4, 0.5, 0.6; (d) 5, 0.5, 0.3. $R$ is the shunt resistance; $I_{dc}$ is the d.c. current per bond. Note that with parameters (d) one of the isotherms actually scales onto the wrong branch.
6.3 Flux Flow Resistivity in Model High Temperature Superconductors

In the previous section, we have discussed the model vortex glass by using uniformly distributed values of the phase factors $A_{ij}$ in our dynamical equations for a three-dimensional array. As shown in Chapter V, the current-voltage characteristics of this model are very similar to those of a model granular superconductor in a strong magnetic field. In this section, we turn to a resistive transition, magnetoresistivity as a function of temperature. In order to study this flux flow resistivity in a vortex fluid phase, we select an ordered array as a highly simplified model of high-temperature superconductors.

6.3.1 Formalism

Our model consists of a simple cubic network of $N = (L/a)^3$ superconducting grains coupled together by resistively-shunted Josephson junctions. Current $I$ is uniformly injected into each node on one face ($Y = 0$) of the network, parallel to one of the planes of the network and extracted from the opposite face (at $Y = L$). Temperature is simulated by a Langevin noise current\[17\] of the appropriate strength, added in parallel to each junction. A magnetic field $\vec{B}$ is applied in the $YZ$ plane, making an angle $\theta$ with respect to the direction of macroscopic current flow along $Y$-axis.

The dynamical equations can be solved by straightforward iteration in time, as described in previous chapters. Periodic boundary conditions are imposed in the two transverse directions ($X$ and $Z$ axes). The generalized magnetic phase factor $A_{ij}$ in this case has already been expressed in Chapter V. Without any displacement of position of the grains (i.e., $x_i = x_0^i$, $y_i = y_0^i$, and $z_i = z_0^i$), we
have from Eq. (5.8)

\[ A_{ij} = \frac{2\pi B}{\Phi_0} \left( \frac{(x_j^0 + x_i^0)}{2} (y_j^0 - y_i^0) \sin \theta + \frac{(x_j^0 + x_i^0)}{2} (z_j^0 - z_i^0) \cos \theta \right). \]  

(6.8)

With the definitions \((x_i^0 = n_w a, y_i^0 = n_y a, z_i^0 = n_z a)\), the phase factor along each axis takes on the following simple forms:

\[ A_{x_{ij}} = 2\pi f n_{x_i^0} \cos \theta \]
\[ A_{y_{ij}} = 2\pi f n_{y_i^0} \sin \theta \]
\[ A_{z_{ij}} = 0, \]  

(6.9)

where \(A_{x_{ij}}\) is the magnetic phase factor for a junction connecting grain \(i\) and grain \(j\) along the \(X\) axis, and \(f\) is the strength of the magnetic flux per plaquette in units of the flux quantum \(\Phi_0 = \hbar c/2e\). We assume that all critical currents and shunt resistances are the same and equal to \(I_c\) and \(R_0\). Resistivities are obtained by computing the voltage differences across the cubic sample, averaged over the sample faces and averaged over a time interval of order \(500 - 1000\ \tau_0\), where \(\tau_0 = h/(2eR_0 I_c)\) is the natural unit of time. The equations of motion are iterated in time steps of typically \(0.05\ \tau_0\), starting from random initial phase configurations for each temperature, current, and magnetic field.

6.3.2 Scaling at Zero Magnetic Field

Figure 6.4(a) shows the resistivity of \(8 \times 8 \times 8\) array, calculated as a function of temperature at zero magnetic field with several different d.c. current levels from \(0.1 I_c\) to \(0.4 I_c\). The resistivity \(\rho\) drops sharply (to nearly zero) near \(T \approx 2 E_J/k_B\), where \(E_J = \hbar I_c/(2e)\) is the Josephson coupling energy for a single junction. This compares with the temperature \(T_c(B = 0) = 2.21 E_J/k_B\) where
a phase transition to a superconducting state is expected in this model in the
limit of zero current[22]. (In zero current, the present model is equivalent to
the three-dimensional (d=3) classical XY model, which for this geometry has
a transition into a phase-ordered state at $T_c = 2.21 \frac{E_J}{k_B}$. The effect of the
finite current is thus to reduce the transition temperature (as measured by the
sharp drop in resistivity) slightly below its zero current value. The nonlinear
current-voltage characteristics in this transition should also scale as

$$E(J) \approx J \xi_f^{d-2-z} E_{\pm} \left( J \xi_f^{d-1} \Phi_0 / c k_B T \right),$$

$$\rho \approx \xi_f^{d-2-z} E_{\pm} \left( J \xi_f^{d-1} \Phi_0 / c k_B T \right),$$

(6.10)

where $\xi_f$ is the coherence length which diverges at $T_c$, and $z$ is the dynamical
critical exponent for this transition.

As in the previous section of vortex glass, we plot the above function of Eq.
(6.10) to get two universal curves for $T > T_c$ and $T < T_c$ by adjusting two
parameters $z$ and $\nu$, by fixing the transition temperature at $T_c = 2.21 E_J / k_B$:

$$\rho t^{\nu(d-2-z)} \approx E_{\pm} \left( J t^{-\nu(d-1)} \Phi_0 / c k_B T \right),$$

(6.11)

where $t$ is a reduced temperature, $t = |T - T_c| / T_c$. Figure 6.4(b) shows the best
collapse for all curves in Fig. 6.4(a) onto two universal curves corresponding to
$T > T_c$ and $T < T_c$. The dynamical critical exponent and the coherence length
exponent at the collapse are $z \approx 1.5$ and $\nu \approx 0.67$, respectively. These estimates
are in good agreement with the accepted values[22-24] of the classical three-
dimensional XY model. [In planar magnet and superfluid helium, the dynamical
exponent $z$ is known to be $z = (d + \tilde{\alpha} / \nu) / 2$, where $\tilde{\alpha} = \max(\alpha, 0)$ and $\alpha$ is the
specific heat exponent that is negative for $d = 3$ [23].]
Figure 6.4:

(a). Resistivity $\rho(T)$ of a 8 x 8 x 8 sample at zero magnetic field with several current levels, plotted as a function of temperature. $\rho$ is defined as the voltage drop per junction, divided by the applied current per grain. Temperature is given in units of $E_j/\hbar_B$, where $E_j$ is the coupling energy of the junction, $E_j = \hbar I_c/(2e)$; (b). Scaled resistivity of curves in (a). We have plotted $\rho_1 t^{-(z-1)}$ versus $l_{dc}t^{-2\nu}/l_c$, where $\rho_1$ is the resistivity at a current level $I_1$, and $t = |T - T_c|/T_c$. The best estimates are $z = 1.5, \nu = 0.67$, and $T_c = 2.21E_j/\hbar_B$. 
6.3.3 Angular Dependence of Flux Flow Resistivity

Figure 6.5(a) shows the resistive transition curves, measured by Kwok et al.\textsuperscript{[15]}, of single-crystal YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7-δ} with both current density $\vec{J}$ and magnetic field $\vec{B}$ applied parallel to the $ab$ plane. The application of a magnetic field leads to broad resistive transition. It is not unexpected that the resistivity would be greatest for field and current perpendicular, since this configuration maximizes the Lorentz force acting on the flux lines. However, even for fields parallel to the currents, they found a substantial magnetoresistance, the origin of which is unexplained. For sufficiently high temperature, they found a roughly $\sin^2\theta$ angular dependence[Fig. 6.5(b)] on the angle $\theta$ between $\vec{B}$ and $\vec{J}$. The sharp drop in resistivity at low temperature, as shown in Fig 6.5(b), implies that the twin planes act as strong pinning centers in the flux flow regime when they are aligned parallel with the magnetic field.

Figure 6.6(a) shows the resistivity of $8 \times 8 \times 8$ array, calculated as a function of temperature for a magnetic field $f = 1/8$ (1/8 of a flux quantum per plaquette) at a current level of 0.1 $I_c$. Note that the right-most curve is identical to the curve at the same current level in Fig. 6.4(a). The more interesting results of Fig. 6.6(a) are the magnetoresistivities with fields parallel ($\theta = 0$) and perpendicular ($\theta = 90^\circ$) to the current.

These magnetoresistivities are shown as a function of angle for several temperatures in Fig. 6.6(b). It may be seen that the magnetoresistance does roughly follow the behavior of Eq. (6.1). Such behavior occurs because the application of a field considerably lowers the transition temperature of the cubic network. In our model, this lowering is due to the "frustration" of the Josephson coupling in the direction perpendicular to the magnetic field. The coupling energy
Figure 6.5:
Resistivity measured by Kwok et al [15] of single-crystal YBa$_2$Cu$_3$O$_{7-\delta}$ which has twin planes. (a). Resistive transition in zero field and 1.5 T for B $\parallel$ J and B $\perp$ J along the ab plane; (b). Angular dependence of the resistivity at three temperatures. The sharp drop in resistivity at low temperature implies that the twin planes act as strong pinning centers in the flux flow regime when they are aligned parallel with the magnetic field.
Figure 6.6:

(a). Resistivity $\rho(B,I,T,\theta)$ of a $8 \times 8 \times 8$ sample at a current level of $0.1 I_c$, at zero magnetic field, and at a field of magnitude $f = 1/8$ (1/8 of a flux quantum per square plaquette of the network), plotted as a function of temperature. The right-most curve is at zero magnetic field. The middle curve is at a parallel ($\theta = 0$) field $f = 1/8$ to the current, while the left one at a perpendicular ($\theta = 90^\circ$) field; (b). $\rho(B,I,T,\theta)$, plotted at several temperatures (from top to bottom, $T = 1.4, 1.0,$ and $0.6$ in units of $E_J/k_B$) as a function of angle $\theta$ for $I = 0.1 I_c$, and $f = 1/8$. The lines are merely to guide the eye.
between nearest neighbors is $-E_J \cos(\phi_i - \phi_j - A_{ij})$. In the absence of a magnetic field, $A_{ij} = 0$ so the coupling energy of all the junctions can be minimized simultaneously, by a parallel arrangement of phases. Correspondingly, the superconducting transition temperature of the network is maximized for zero field. When a magnetic field is present, it is no longer possible to simultaneously maximize all the Josephson coupling energies for all the junctions. This "frustration" causes the ground state energy to be less negative in a field than at zero field. The transition temperature is correspondingly reduced.

Since $T_c$ is reduced in the presence of a field, the resistivity is nonzero no matter what the direction of the current relative to the field. For a given $B$ and $T$, $\rho$ is larger when $B$ and $I$ are perpendicular because of the extra resistivity produced by flux flow arising from the Lorentz force. But even when $B$ and $I$ are parallel, there is still a kind of Lorentz-force-driven flux flow resistivity. This arises because above the freezing temperature of the flux lattice, the vortex lines are mobile and floppy. Hence, although oriented parallel to the applied field on average, they also have components of their length at an angle to the applied field. They therefore experience a Lorentz force even when current and field are nominally parallel.

At relatively high temperatures ($k_BT \geq 1.0E_J$ in Fig. 6.6(b)) the resistivity has a roughly $\sin^2\theta$ angular dependence, as seen experimentally. At lower temperatures ($k_BT = 0.6E_J$), $\rho$ is very small except for $\theta \approx 90$ deg. We interpret this as indicating that $k_BT = 0.6E_J$ is below $T_c$ but that the flux lines are only weakly pinned by the periodic lattice at this temperature. Hence, the lattice can be set in motion by the relatively small current of 0.1 $I_c$ provided the current is perpendicular to the magnetic field lines.
All our remaining results are consistent with the picture described above. Doubling the field to $f = 1/4$, shown in Fig. 6.7, lowers the transition temperature somewhat and hence further broadens the resistivity curve. Similarly, doubling the current level to $0.2I_c$ lowers the apparent zero-field transition and broadens the low-temperature tails at finite fields. Especially at the lowest temperatures, the dissipation can be seen to be quite non-Ohmic.

6.4 Discussion

In the first part of the chapter, we have presented numerical calculations of the I-V characteristics in a model vortex glass at finite temperatures. Our results are consistent with a glass transition at a temperature $T_g \approx 0.45E_J/k_B$. The I-V characteristics above and below $T_g$ collapse onto two scaling curves, as suggested by Fisher et al with dynamical critical exponents $z \approx 4, \nu \approx 0.5$, in agreement with the mean-field values predicted for a glass transition in an Ising spin glass[21]. Our value of $z$ is in the range of those found experimentally and suggested theoretically for the vortex glass transition, but the value of $\nu$ is somewhat lower. Our results suggest that there may be a vortex glass transition in our model, with the expected scaling characteristics, but that we have not yet reached the asymptotic current region where the critical, rather than mean field, exponents will become manifest.

In the second part of the chapter, we have presented the calculations of flux flow resistivity in a highly simplified model of a “high-temperature superconductor”. Many features of our results are qualitatively consistent with the experimental results of Kwok et al [13-15]. We note particularly the differences between transverse and longitudinal magnetoresistivity, the nonzero longitu-
Figure 6.7:
(a). Resistivity \( \rho(B, I, T, \theta) \) of the same sample as in Fig. 6.6, but at a field of magnitude \( f = 1/4 \), plotted as a function of temperature;  
(b). \( \rho(B, I, T, \theta) \), plotted at several temperatures (from top to bottom, \( T = 1.4, 1.0, \) and \( 0.6 \) in units of \( E_J/k_B \)) as a function of angle \( \theta \) for \( I = 0.1 I_c \), and \( f = 1/4 \). The lines are merely to guide the eye.
nal magnetoresistivity, the lowering of the transition temperature with increasing current and field, and the approximately $\sin^2 \theta$ angular dependence observed at certain temperatures. Our broadening occurs over a far greater relative temperature range than does the experimental broadening of Kwok et al, and our $\sin^2 \theta$ angular dependence is much less clearly observed in our calculations than in the measurements. On the other hand, our model bears only a rough resemblance to the experimental sample, which has an anisotropy of perhaps $10^3$ between the coupling in the $ab$ plane and the coupling between the planes, and which also has a temperature-dependent order parameter (not directly considered in our model, which considers only the phase degrees of freedom) [25].

Further work on the dissipation mechanism in high-temperature superconductors might be possible by using more refined model of a superconducting network of Josephson junctions. As Chien et al [26] recently reported from measurements of magnetoresistivity $\rho_{xx}$ and Hall resistivity $\rho_{xy}$ in single crystals of $\text{YBa}_2\text{Cu}_3\text{O}_7-\delta$, one may test the model of coupled weak links to investigate the Lorentz-force dominant regime above a characteristic temperature $T_k$ and the pinning-force dominant regime below $T_k$. Both regimes are in the vortex fluid phase and the two forces govern the flux flow and thermally activated flux flow resistance, respectively.
CHAPTER VI REFERENCES


[18] In a single-variable Langevin equation, any distribution satisfying Eq. (2.12) is guaranteed to have the same two-point correlation functions $< V(t)V(t') >$; presumably the same is true for coupled Langevin equations. For the present application, which deals with low-frequency properties, this property would explain the similarity between results for the Gaussian and uniform distributions. For further discussion, see, e.g., N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, New York, 1981), pp. 237ff.

[19] We have recently used the same procedure to treat an ordered three-dimensional array with no magnetic field [See Section 6.3 in this chapter.]. At applied currents of order 0.05$I_c$ per bond, we find a sharp drop to near zero voltage at a temperature $\approx 2.1E_f/k_B$, very near the temperature at which the static version of this model is known to have a phase transition.
This is further evidence that our method of including the effects of temperature via Langevin noise currents correctly leads to a phase transition in models where one is known to occur.

[20] We have confirmed that the I-V characteristics of our model are indeed very similar to those of a model granular superconductor in a strong magnetic field [See Section 5.3.2 in Chapter V.]. We modeled the latter as a three-dimensional simple cubic lattice, in which the grains undergo random Gaussian displacements from their nominal lattice positions, and a magnetic field is applied transverse to the current. (The high field limit is unchanged with magnetic field parallel to the current.) In the high-field limit, in which the random fluctuations in flux through difference plaquettes is much larger than $\Phi_0$, we found that the I-V characteristics are nearly indistinguishable from those of the vortex glass model.

[21] See, for example, K. Binder and A. P. Young, Rev. Mod. Phys. 58, 801 (1986) and references cited therein. A. T. Dorsey, M. Huang, and M. P. A. Fisher (preprint) have analytically obtained mean-field exponents $\nu = 2$, $\nu = 0.5$ for a model vortex glass.


[25] An anisotropic calculation would lead to the same results, but over a much narrow temperature range. The argument is as follows. Well above $T_c$ (the transition temperature of the network) there is coupling only in the plane. The coherence length in the plane grows as the two-dimensional transition temperature is approached. Regions of area $\xi^2$ within the plane are essentially phase locked. They couple to adjacent layers as single units. Thus the interlayer coupling is greatly enhanced, and sufficiently near $T_c$, becomes comparable to the intralayer coupling. In this temperature range, our isotropic model is reasonable.

APPENDIX

The equation of motion for a single resistively-shunted Josephson junction subjected to a current $I = I_{dc} + I_{ac}\sin(\omega t)$ can be expressed in the form

$$\frac{d\phi}{d\tau} = I_1 + I_2 \sin(z\tau) - \sin \phi,$$  \hspace{1cm} (A.1)

where $\tau = 2eR_0I_{dc}/\hbar$ is a reduced time variable, and $z = \omega/(2eR_0I_{c}/\hbar)$ is a reduced frequency. $I_1$ and $I_2$ are the amplitudes of the applied d.c. and a.c. currents normalized to $I_c$, the critical current of the junction.

We obtain an approximate solution to Eq. (A.1) in the limit of large $I_2$, $(I_2 \gg 1)$, following the method of Kvale and collaborators[1]. To leading order in $I_2$, we neglect the last term in Eq. (A.1) and integrate this equation to obtain

$$\phi = I_1 \tau - \frac{I_2}{z} \cos(z\tau) + \Delta,$$ \hspace{1cm} (A.2)

where $\Delta$ is a constant of integration. To obtain the next order approximation, we substitute Eq. (A.2) back into the original Eq. (A.1) to get

$$\frac{d\phi}{d\tau} = I_1 + I_2 \sin(z\tau) - \sin \left( I_1 \tau - \frac{I_2}{z} \cos(z\tau) + \Delta \right).$$ \hspace{1cm} (A.3)

The last term can be expanded using the Bessel function identities

$$\cos(a \cos(x)) = J_0(a) + 2 \sum_{m=1}^{\infty} (-1)^m J_{2m}(a) \cos(2mx)$$ \hspace{1cm} (A.4)

$$\sin(a \cos(x)) = -2 \sum_{m=1}^{\infty} (-1)^m J_{2m-1}(a) \cos((2m-1)x)$$ \hspace{1cm} (A.5)
to yield

\[
\frac{d\phi}{d\tau} = I_1 + I_2 \sin(\pi T) - J_0 \left( \frac{I_2}{z} \right) \sin(I_1 \tau + \Delta) \tag{A.6}
\]

\[
- \sum_{m=1}^{\infty} (-1)^m J_{2m} \left( \frac{I_2}{z} \right) \left( \sin(I_1 \tau + \Delta + 2m \pi T) + \sin(I_1 \tau + \Delta - 2m \pi T) \right)
\]

\[
+ \sum_{m=1}^{\infty} (-1)^m J_{2m-1} \left( \frac{I_2}{z} \right) \left( \cos(I_1 \tau + \Delta + (2m - 1) \pi T) + \cos(I_1 \tau + \Delta - (2m - 1) \pi T) \right).
\]

By time-averaging this equation in Ref. [1], Kvale and collaborators were able to explain the widths of the Shapiro steps in a single resistively-shunted Josephson junction, in the limit of large a.c. currents.

We first show that, in the absence of a d.c. current, the power spectrum deduced from Eq. (A.6) exhibits odd harmonics only. First, we note that, in order for the time-averaged voltage \(\langle d\phi/d\tau \rangle\) in Eq. (A.6) to equal zero, we must have \(\Delta = k \pi\), where \(k\) is an integer. Using this relation, and \(I_1 = 0\), we can rewrite Eq. (A.6) as

\[
\left( \frac{d\phi}{d\tau} \right)_{I_1=0} = I_2 \sin(\pi T) \tag{A.7}
\]

\[
+ (-1)^k \sum_{m=1}^{\infty} 2(-1)^m J_{2m-1} \left( \frac{I_2}{z} \right) \cos[(2m - 1) \pi T]
\]

The first term in the right hand side represents the applied a.c. current while the second term is generated from the nonlinear behavior of the junction current. This equation shows that only odd harmonics appear for \(I_1=0\).

The prefactors \(2J_{2m-1}(I_2/z)\) in Eq. (A.7) are directly related to the power spectrum of the voltage. Since \(d\phi/d\tau\) is periodic in time with period \(1/f_0 = 2\pi/z\), it can be expressed in a Fourier series of the form

\[
\frac{d\phi}{d\tau} = \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos(m \pi T) + b_m \sin(m \pi T)] \tag{A.8}
\]
where the power in the $m^{th}$ harmonic is

$$S_V(m f_0) = \frac{1}{4} \left( a_m^2 + \delta_m^2 \right). \quad (A.9)$$

From this definition, when $I_1 = 0$, the power in the odd harmonics is

$$S_V(m f_0) = \frac{1}{4} I_2^2 \delta_{m,1} + J_m^2 \left( \frac{I_2}{z} \right). \quad (A.10)$$

with $m = 1, 3, 5, ...$ The circles in Fig. A.1(a) are the values of the $m^{th}$ harmonics as calculated from this equation for $I_{ac} = 200 I_c$. As can be seen, the power spectrum for the array at zero magnetic field is well fitted by this prediction for a single junction.

In the limit $I_2 \gg 1$, Shapiro steps in a single junction occur at values of the d.c. current $I_1 = nz$, with $n = 1, 2, 3, ...$. We can readily use Eqs. (A.6), (A.8), and (A.9) to obtain formulas for the power of the $m^{th}$ harmonic on the $n^{th}$ voltage step. For $n = 1$, in order to have a time-averaged voltage $\langle d\phi/d\tau \rangle = z$, we choose $A = \pi(2k - 1)/2$ where $k$ is an integer. Substituting this choice and $I_1 = z$ into Eq. (A.6), we obtain for the power spectrum on the first step

$$S_V^{(1)}(m f_0) = \frac{1}{4} \left( I_2^2 + (J_2 - J_0)^2 \right) \quad \text{for } m = 1 \quad (A.11)$$

$$= \frac{1}{4} (J_{m+1} + J_{m-1})^2 \quad \text{for } m \text{ even and } \geq 2 \quad (A.12)$$

$$= \frac{1}{4} (J_{m+1} - J_{m-1})^2 \quad \text{for } m \text{ odd and } \geq 3. \quad (A.13)$$

In the same way, for the second Shapiro step ($n = 2$), we find, using $\Delta = k\pi$ with $k$ an integer, that

$$S_V^{(2)}(m f_0) = \frac{1}{4} \left( I_2^2 + (J_1 - J_3)^2 \right) \quad \text{for } m = 1 \quad (A.14)$$

$$= \frac{1}{4} (J_{m-2} - J_{m+2})^2 \quad \text{for } m \text{ even and } \geq 2 \quad (A.15)$$

$$= \frac{1}{4} (J_{m-2} + J_{m+2})^2 \quad \text{for } m \text{ odd and } \geq 3. \quad (A.16)$$
Figure A.1:

Voltage power $S_\nu(\omega)$ in an array of $10 \times 10$ plaquettes at $f = 0$ and several different values of $I_{dc}$. In all cases, $I_{ac}/I_c = 200$, $\nu/\nu_0 = 0.1$, and temperature $T = 0$. (a) $I_{dc}/I_c = 0$; (b) $I_{dc}/I_c = 0.628$ ($n = 1$ step); (c) $I_{dc}/I_c = 1.257$ ($n = 2$ step). The circles represent the predictions of the asymptotic Bessel function formulas for single junctions, Eqs. (A.10) – (A.16).
In Figs. A.1(b) and A.1(c) these predictions for a single junction are compared to the calculated power spectrum on the first and second step of a $10 \times 10$ array in the limit of large a.c. current. Agreement with the predictions is very good. We do not understand the fact that our asymptotic expressions seem to depend on the choice of initial phases. Nevertheless, the resulting agreement between our asymptotic expressions and the numerical calculations for arrays is quite convincing.

APPENDIX REFERENCES

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