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Repetitive control of closed loop material testing

Shaw, Feng-Rong, Ph.D.

The Ohio State University, 1991
REPETITIVE CONTROL OF CLOSED LOOP MATERIAL TESTING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Feng-Rong Shaw, M.S. M.E.

******

The Ohio State University

1991

Dissertation Committee
Dr. K. Srinivasan
Dr. C-H. Menq
Dr. K. Ishii

Approved by

Advisor

Department of Mechanical Engineering
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Finally, I thank my wife, Huei-Hua, and my parents in Taiwan for their constant support and encouragement.
VITA

April 18, 1961 ........ Born in Taipei, Taiwan

June 1983 ............ B.S., Mechanical Engineering, National Taiwan University

June 1988 ............ M.S., Mechanical Engineering, The Ohio State University

PUBLICATIONS


FIELDS OF STUDY

Major Field: Mechanical Engineering

System Dynamics, Measurement and Controls, Automatic Control of Hydraulic Systems
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NOMENCLATURE

A          Polynomial
AAF        Anti-aliasing filter
Ap         Actuator cross-sectional area
a, a_2     Characteristic root real part estimate
B          Controller matrix
B          Polynomial
B^+        Polynomial with cancellable zeros
B^-        Polynomial with uncancellable zeros
Bi         Bilinear transform of argument
Bs         Viscous damping coefficient
b          Filter in repetitive controller
b_c        Constant
C          System output vector
C_d        Discharge flow coefficient
C_ep       External leakage coefficient
C_ip       Internal leakage coefficient
C_1, C_2   Constants
c          System output
D          Disturbance input vector
d          Disturbance input
E          Error vector
e          Error
F          Load
F_c         Coulomb friction in actuator
F_ref       Reference load signal
G_{uc}     Uncompensated plant matrix
G_p        Compensated plant transfer matrix
G, G_0     Closed loop transfer function and nominal value
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<td>Fourier transform of $r_i(t)$</td>
</tr>
<tr>
<td>$r_i(t)$</td>
<td>Regenerated signal</td>
</tr>
<tr>
<td>$r_q$</td>
<td>Controller parameter</td>
</tr>
<tr>
<td>$S$</td>
<td>Sensitivity matrix</td>
</tr>
<tr>
<td>$S_0$</td>
<td>Sensitivity matrix, no repetitive control</td>
</tr>
<tr>
<td>$S$</td>
<td>Sensitivity function</td>
</tr>
<tr>
<td>$S_0$</td>
<td>Sensitivity function, no repetitive control</td>
</tr>
<tr>
<td>$s$</td>
<td>Laplace variable</td>
</tr>
<tr>
<td>$T$</td>
<td>Complementary sensitivity matrix</td>
</tr>
<tr>
<td>$T_0$</td>
<td>Complementary sensitivity matrix, no repetitive control</td>
</tr>
<tr>
<td>$T$</td>
<td>Complementary sensitivity function</td>
</tr>
<tr>
<td>$T_C$</td>
<td>Controller computation time</td>
</tr>
<tr>
<td>$T_D$</td>
<td>Time delay, period of input signal</td>
</tr>
<tr>
<td>$T_G$</td>
<td>Equivalent time delay</td>
</tr>
<tr>
<td>$T_s$</td>
<td>Sampling interval</td>
</tr>
<tr>
<td>$(t_0)_{ij}$</td>
<td>Element of $T_0$</td>
</tr>
<tr>
<td>$u$</td>
<td>Input voltage to amplifier</td>
</tr>
<tr>
<td>$V$</td>
<td>Reference input vector</td>
</tr>
<tr>
<td>$V_1, V_2$</td>
<td>Fluid volume in either chamber of linear actuator</td>
</tr>
<tr>
<td>$V_t$</td>
<td>Fluid volume in actuator</td>
</tr>
</tbody>
</table>
\( v \) Reference input signal

\( W \) Width of servovalve opening

\( X \) State vector

\( X_p \) Piston position

\( X_v \) Servovalve position

\( Y \) Output vector

\( Z \) z-transform of argument

\( z \) z-transform operator

\( z_i \) Characteristic root of discrete-time repetitive control system

\( \alpha \) real part of complex variable

\( \alpha_{k_i} \) Characteristic root real part

\( \alpha_{\text{max}} \) Real part of dominant pole

\( \beta \) Imaginary part of complex variable

\( \beta_e \) Fluid bulk modulus

\( \Delta \) Multiplicative uncertainty in plant

\( \Delta_G \) Multiplicative uncertainty in \( G \)

\( \delta_1, \delta_2, \delta_3, \delta_4 \) Underlap in servovalve

\( \varepsilon \) Strain in specimen

\( \gamma(\cdot) \) Loop Gain

\( \lambda_i() \) Eigenvalue of matrix argument

\( \theta \) Angular degree

\( \rho \) Fluid density

\( \rho_e \) Characteristic root magnitude estimate

\( \rho_i \) Characteristic root magnitude

\( \sigma \) Largest singular value

\( \sigma \) Smallest singular value

\( \tau_b \) Time advance in \( b \)

\( \tau_q \) Time advance in \( q \)

\( \tau_{sv} \) Servovalve time constant

\( \omega \) Frequency

\( \omega_b \) Controller parameter

\( \omega_h \) Hydraulic resonant frequency

\( \omega_k, \omega_i \) Imaginary part of characteristic root

\( \omega_q \) Natural frequency in denominator of \( q \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}$</td>
<td>Laplace transform</td>
</tr>
<tr>
<td>$\mathcal{L}^{-1}$</td>
<td>Inverse Laplace transform</td>
</tr>
<tr>
<td>$\zeta_b$</td>
<td>Damping ratio in denominator of $b$</td>
</tr>
<tr>
<td>$\zeta_h$</td>
<td>Hydraulic damping ratio</td>
</tr>
<tr>
<td>$\zeta_q$</td>
<td>Damping ratio in denominator of $q$</td>
</tr>
</tbody>
</table>
1.1 Introduction

Most electrohydraulic material testing machines currently rely on manual adjustment of their controller parameters. The adjustments may include proportional control gain tuning, rate feedback gain tuning or differential pressure feedback gain tuning. Rational design procedures for feedback control need quantitative measures of system properties such as accuracy, stability, robustness etc., along with a model of the system dynamics. The problems in controller design unique to closed loop material testing are that: (1) Specimen characteristics change from test to test if the specimens change. Also, dynamic models of complex structures may be difficult to obtain. Therefore, the appropriate controller varies from test to test. Hence, the machine manufacturer can not preset the controller parameters. (2) The operator of the material testing machine is usually not well versed in control theory which makes the selection of appropriate controller parameters, based on control theory, more difficult. (3) Some tests, such as low cycle fatigue testing, involve plastic deformation of the material and hence require nonlinear models of system behavior which are difficult to accommodate in controller design procedures. Material testing situations involve other nonlinear phenomena because of their heavy reliance on valve controlled hydraulic actuation. The pressure-flow relationships needed to describe valve flow are nonlinear because the flow is usually turbulent. Since available controller design procedures are better suited for linear system models, these nonlinear effects complicate the controller design procedure.
In view of these difficulties, manufacturers of material testing machines support simple types of controllers only. The machine user then has to select a small number of controller parameters. In most material testing machines currently in use, two or three controller parameters have to be adjusted to get good performance. Selection of the proper controller parameters is aided by guidelines suggested by the manufacturer. Trial and error procedures are necessary to ensure that the desired accuracy is achieved and that the control system is stable and robust.

Self-tuning control is an obvious approach to automation of the parameter setting procedure. A typical self-tuning controller performs two basic functions, system identification and controller parameter selection. The controller first identifies the system model parameters by using past input control signals and past output response signals, and then determines the appropriate controller parameters within a selected controller form based on the identified system parameters. Besides eliminating the manual controller tuning referred to above, the self-tuning controller enables the use of alternative, more powerful, controller forms. Furthermore, if the system parameters change due to a change in the test specimen or conditions of test, the self-tuning controller can identify the new system parameters and adjust controller parameters accordingly. Such a self-tuning scheme has been implemented successfully on a testing machine (Lee and Srinivasan, 1989).

The current limitation of self-tuning control, as implemented for this application, is that identification schemes for more accurate or more complex system models need input and output signals that are rich enough in information content or number of frequency components to get good estimates of the system parameters. However, the signals available for model identification are determined by the nature of the test being conducted and may not be rich enough, resulting in estimation error. Such estimation error would degrade system response. Thus, the current approach relies on low order or
simple system models. System nonlinearities contribute additionally to the modeling error, but this contribution has not been characterized quantitatively. The self-tuning control system accommodates the resulting model error by sacrificing accuracy and the closed loop speed of response. Additional research needs to be done in determining more appropriate controller forms that can achieve a better tradeoff between accuracy, robustness and stability.

A new type of controller called the repetitive controller has been devised by researchers and applied to robot control (Omata et al, 1987) and disk head positioning control (Chew et al, 1990). The repetitive controller can track periodic signals with high accuracy and reject periodic disturbances effectively. The idea behind repetitive control is that the controller can 'learn' from errors of previous periods to adjust the feedback control action for the next cycle. Consequently, repetitive controllers use very high feedback gains at the fundamental and harmonic frequencies to obtain extremely high accuracy. This controller is potentially useful in material testing situations involving periodic excitation. Also, this controller relies less on the accuracy of the plant model; therefore, it is a more appropriate controller form for self-tuning control of material testing.

Repetitive control is subject to the same tradeoff between accuracy, stability and robustness mentioned above earlier. However, since the accuracy improvement is sought only at discrete frequencies, a more favorable resolution of the tradeoff is possible. Quantitative measures of accuracy, stability and robustness are necessary for the evolution of rational design procedures. Since repetitive controllers rely on a time delay term to achieve their objectives, quantitative measures of the degree of stability have not been available in the past and this has hindered rational design of repetitive control systems.
1.2 Introduction to Repetitive Control

Any periodic signal of a known period can be generated by a time delayed feedback system as shown in Figure 1.1. A control system including such a signal generator is a repetitive control system. The repetitive controller was first proposed by Inoue et al (1981). They showed, based on the Internal Model Principle (Francis and Wonham, 1975), that asymptotic tracking of any periodic signal of a known period can be achieved by implementing such a periodic signal generator.

![Figure 1.1 A Periodical Signal Generator](image)

The system shown in Figure 1.1 has infinite number of poles along the imaginary axis and is therefore a marginally stable system. The wave form of any periodic signal generated from this system is the same as the initial function stored in the system itself. The signal of the next period can be updated if an appropriate input is injected in the present period. This characteristic forms the basis for repetitive control systems. Figure 1.2 shows a continuous repetitive control system proposed by Srinivasan and Shaw.
(1990). The error signal in such a repetitive control system will effectively update the signal generator until the output response is close to the command signal. When the error signal is zero, the signal generator will provide the exact driving signal to the plant so that the desired output response can be maintained. In other words, the system 'learns' from the error signals of previous periods to refine the control action for the next period. $q(s)$ and $b(s)$ are controller parameters chosen to guarantee accuracy, fast transient response, and stability, i.e., a better design of $q(s)$ and $b(s)$ means a better learning scheme.

The system gain of the signal generator shown in Figure 1.1 is infinite at the fundamental frequency and harmonic frequencies. From feedback control theory, high accuracy and excellent disturbance rejection can be achieved using high controller gain provided that the control system is stable. Therefore, very high accuracy and excellent disturbance rejection can be expected for the repetitive control system at these discrete frequencies. However, the controller gain will be greatly reduced at other frequencies. Consequently, the repetitive controller is effective for periodic signal tracking or periodic disturbance rejection only.

![Figure 1.2 A SISO Repetitive Control System Structure](image)
The inherent infinite dimensional characteristic of the repetitive controller complicates the design and analysis of such systems. Moreover, higher gain implies poorer stability. The design objective is to maintain high accuracy while the stability of the system is guaranteed. Quantitative measures of system properties such as stability robustness, transient response, and performance robustness are necessary for more rational design procedures for such systems. Unfortunately, available research results as discussed in the next section provide only limited information for design and analysis of repetitive control systems.

The objectives of this research are (i) to establish measures of stability robustness, performance robustness and transient response for repetitive control systems and (ii) to implement repetitive control systems for closed loop material testing applications.

1.3. Literature Review

Sufficient conditions for the stability of single-input single-output repetitive control systems were derived by Hara et al (1985). The stability results were also extended to multivariable systems using the small gain theorem (Desoer and Vidyasagar, 1975) by the same researchers (Hara et al, 1988). Two synthesis methods were also proposed, based on the stability criteria, for multivariable repetitive control systems. For minimum phase plants, a method using Kalman filters with perfect regulation (Kimura, 1981) can be utilized to guarantee a stable repetitive control system. For non-minimum phase plants, a factorization approach (Glover, 1984) can be used to find a stable rational controller which results in a stable repetitive control system.

Nonlinear repetitive controllers for trajectory control of manipulators have also been investigated (Omata, 1987). A sufficient stability condition for a class of nonlinear repetitive control systems, suitable for manipulators, has been derived using the passivity
theorem (Desoer and Vidyasagar, 1975). Nonlinear compensation of nonlinear
dynamics, such as gravity forces, centrifugal and coriolis forces, has been shown to
effectively improve the system performance. Accurate models of these nonlinear
dynamics are desired but not strictly required since repetitive control can correct the
control action by learning from previous cycles.

Analysis and design procedures for discrete-time repetitive controllers were
proposed by Tomizuka et al (1988), wherein a specific plug-in form of a digital repetitive
controller was added to a pre-existing control system. The method of discrete-time zero-
phase compensation (Tomizuka, 1987) with appropriate gain scaling was used to
guarantee the stability of the repetitive control system. This approach could automatically
relax the requirement of a modified repetitive controller (Hara et al, 1988) for strictly
proper plants. Asymptotic tracking property could be obtained without violating the
stability criterion.

Successful applications of repetitive control systems, for periodic reference signal
tracking or/and periodic disturbance signal rejection, have also been reported by
researchers. For example, high accuracy machining was obtained for noncircular
machining (Tsao and Tomizuka, 1988), and significant periodic disturbance rejection in
disk read/write head positioning systems was achieved by Tomizuka et al (1988). The
trade-off between accuracies at different frequencies for the disk head positioning
systems was further explored assuming a stochastic disturbance model (Chew and
Tomizuka, 1990). This issue becomes important when repetitive control systems are
subjected to input frequencies different from the frequencies at which they are designed to
be effective.

Even with the theoretical development and these successful applications, clear
measures for the tradeoff between accuracy, stability, transient response, and robustness
have not been established. The inherent infinite dimensional characteristic of repetitive
control systems imposes some difficulties in searching for a rational design procedure for repetitive control systems. One of the objectives of this dissertation is to find a rational procedure for designing repetitive control systems, by considering the tradeoff between stability, accuracy, transient response and robustness of such control systems.

1.4. Contributions of the Research

This thesis addresses the repetitive control of a closed loop material testing machine which is characterized by strong nonlinearities. The system properties of linear repetitive control systems, such as transient response, relative stability, stability robustness, are fully investigated. Also, repetitive control is shown to be suitable for effective closed loop control of material testing.

The contributions of the thesis are in five areas:

(1) Quantitative measures of the degree of stability, transient response and performance robustness for single-input single-output repetitive control systems are established.

(2) The development above is extended to multi-input multi-output repetitive control systems. The motivation for this extension is that there are multi-input multi-output control system applications, for example, automobile structure response to tire loads or multi-axis material testing.

(3) The study leads to new interpretations of a frequency domain function, the regeneration spectrum, for large time delay systems. Also, a quantitative measure for the largeness of the time-delay is established.

(4) Development of procedures to describe or model nonlinear effects in hydraulic systems used in material testing in terms amenable to controller design. This will enable the development of controller forms which can tolerate the attendant modeling error most effectively.

(5) Design and implementation of a repetitive controller on material testing hardware.
1.5 Organization of the Thesis

The remaining chapters of the thesis are briefly outlined here.

In Chapter 2, a small scale material testing machine is described. Some simplifications are made to obtain an appropriate model for computer simulation. Linearized models are then obtained analytically and numerically to characterize the system dynamics of the material testing machine. Also, open loop experimental test results are used to verify the linearized models.

In Chapter 3, the definition of the regeneration spectrum for a class of SISO time delay systems is given. A new interpretation of the regeneration spectrum is also discussed, which helps its usefulness in stability analysis. The regeneration spectrum concept is then applied to SISO repetitive control systems. System properties such as sensitivity and complementary sensitivity functions for such control systems are discussed. Based on these measures, a rational design procedure for SISO repetitive control systems is developed and shown by a design example.

In Chapter 4, the regeneration spectrum concept is extended to continuous-time MIMO systems. This relative stability measure along with other available measures of MIMO control system performance are used to develop guidelines for MIMO repetitive control system design. An example system is illustrated.

In Chapter 5, discrete time versions of the regeneration spectrum and repetitive control system design are discussed. Extensive simulations and experimental tests on the material testing machine are conducted. Important features and test results from this repetitive control application are then summarized.

Conclusions and recommendation for further work are given in Chapter 6.

In order to help the reader to concentrate on the basics of repetitive control systems and the application of the repetitive control to closed loop material testing, most of the related theorems, proofs and details of the work are given in the appendices.
2.1 Introduction

Electrohydraulic servomechanisms are very commonly used in material testing machines. Such hydraulically powered machines can provide the large load amplitudes and high bandwidths needed for material testing applications. However, the dynamics of the control loop are dependent on test specimen characteristics and testing conditions. Also, the design of the controller is complicated by the nonlinearity inherent in the electrohydraulic systems. A small scale material testing machine using an electrohydraulic servomechanism will be investigated here in order to study the basic characteristics of hydraulically controlled material testing machines. A nonlinear model will be used for digital simulation of the testing machine. Linearized models are also derived to be used as a basis for control system design and analysis. Experimental tests are conducted to get more accurate characterization of this material testing machine.

2.2 Dynamics of a Material Testing Machine

A small scale material testing machine is depicted in Figure 2.1. It has been used as a test rig for various self-tuning control algorithms (Lee, 1988). Figure 2.2 shows the control schematic of the material testing machine. Three control parameters, proportional gain, derivative feedback gain and pressure difference feedback gain, are to be manually tuned according to procedures suggested by the manufacturer (MTS System Co.) Some
Figure 2.1 Hydraulic System Schematic for Small Scale Material Testing Machine
Figure 2.2 Controller Schematic for Small Scale Material Testing Machine

- Set Point
- ΔP gain
- Servo amplifier
- ΔP transducer
- Servovalve
- Actuator and specimen
- Extensometer
- Load cell
- Position transducer
- 406.11 controller

Set Point

ΔP gain

proportional gain

rate gain

low-pass filter & differentiator

servo amplifier

ΔP transducer

servovalve

actuator and specimen

extensometer

load cell

position transducer

406.11 controller
Table 2.1. Numerical Values of System Parameters (Lee, 1988)

<table>
<thead>
<tr>
<th>Parameter Type</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Servovalve Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$C_d$</td>
<td>0.61</td>
</tr>
<tr>
<td>$K_{sv}$</td>
<td>0.0008 in/ma</td>
</tr>
<tr>
<td>$W$</td>
<td>0.881 in</td>
</tr>
<tr>
<td>$\tau_{sv}$</td>
<td>0.00187 sec</td>
</tr>
<tr>
<td>$\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$ in (critically centered)</td>
<td></td>
</tr>
<tr>
<td><strong>Actuator Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$A_p$</td>
<td>1.17 in$^2$</td>
</tr>
<tr>
<td>$B_s$</td>
<td>0 lbf-sec/in</td>
</tr>
<tr>
<td>$C_{ep}$</td>
<td>0</td>
</tr>
<tr>
<td>$C_{ip}$</td>
<td>$2.5 \times 10^{-5}$ (in$^3$/sec)/psi</td>
</tr>
<tr>
<td>$M$</td>
<td>0.256 lbf-sec$^2$/in</td>
</tr>
<tr>
<td>$V_t$</td>
<td>8.424 in$^3$</td>
</tr>
<tr>
<td>$F_c$</td>
<td>30 lbf</td>
</tr>
<tr>
<td><strong>Hydraulic Fluid Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$P_s$</td>
<td>1000 psi</td>
</tr>
<tr>
<td>$P_o$</td>
<td>0 psi</td>
</tr>
<tr>
<td>$\beta_c$</td>
<td>120,000 psi</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$7.8 \times 10^{-5}$ lbf-sec$^2$/in$^4$</td>
</tr>
<tr>
<td><strong>Specimen Parameters</strong></td>
<td></td>
</tr>
<tr>
<td>(Stainless steel specimen of diameter 0.339&quot; and gage length 1.75&quot;)</td>
<td></td>
</tr>
<tr>
<td>$K_{sp}$</td>
<td>$1.5 \times 10^6$ lbf/in (theoretical value of the specimen)</td>
</tr>
<tr>
<td>$K_{st}$</td>
<td>$3.0 \times 10^5$ lbf/in (estimated structure compliance)</td>
</tr>
<tr>
<td>$K_s = (K_{sp} \cdot K_{st}) / (K_{sp} + K_{st})$</td>
<td>$2.5 \times 10^5$ lbf/in (equivalent system stiffness)</td>
</tr>
<tr>
<td><strong>Transducer Gains</strong></td>
<td></td>
</tr>
<tr>
<td>$y/X_p$ = stroke transducer gain</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2.5$ v/in, 5.0 v/in or 10.0 v/in depending on controller setting</td>
</tr>
<tr>
<td>$y/F$ = load transducer gain</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.0005$ v/lbf, $0.001$ v/lbf or $0.002$ v/lbf depending on controller setting</td>
</tr>
<tr>
<td>$y/\varepsilon$ = extensometer gain</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$200$ v/in - 2000 v/in depending on gain setting</td>
</tr>
</tbody>
</table>
of the important system parameters are listed in table 2.1. More details of the experimental setup are described in the reference (Lee, 1988). A dynamic model of a typical servovalve controlled linear actuator is available from standard textbooks on hydraulic control (Merrit, 1967). The dynamic model of the small scale material testing machine will be investigated through simulation and experimental tests to provide a basis for control system design and analysis.

2.2.1 A Nonlinear Model for Computer Simulation

The dynamics of the material testing machine can be decomposed into three parts: servovalve dynamics, linear actuator dynamics and specimen characteristics. The dynamics of the servovalve itself are quite complicated since the valve is driven by an electric motor and also by a hydraulic pressure feedback force. It is modeled here as a first order system for simplicity as is usually the case. This is justified as the servovalve is usually chosen to be much faster than the closed loop system bandwidth. Its time constant and the system gain are estimated from the experimental frequency response data provided by the manufacturer. The dynamics of test specimens are also complicated in some cases such as low cycle large amplitude fatigue tests and are dependent on material properties. The specimen is simply modeled here as a spring with constant spring rate and appropriate damping, since the deformation is restricted to be in the elastic range. Different specimens have different spring rates.

Figure 2.3 shows the diagram of a valve-controlled linear actuator. The dynamic behavior of the servovalve is modeled as

\[
\frac{X_v(s)}{i_{rv}} = \frac{K_{rv}}{\tau_{rv} s + 1}
\]  

(2.1)
Figure 2.3 Diagram of Valve-Controlled Linear Actuator
The pressure/flow characteristic of the valve is described by

\[ Q_1 = C_d W \cdot U(\delta_1 + X_v) \cdot \sqrt{\frac{2(P_s - P_1)}{\rho}} \]  
\[ Q_2 = C_d W \cdot U(\delta_2 + X_v) \cdot \sqrt{\frac{2(P_2 - P_0)}{\rho}} \]  
\[ Q_3 = C_d W \cdot U(\delta_3 - X_v) \cdot \sqrt{\frac{2(P_3 - P_2)}{\rho}} \]  
\[ Q_4 = C_d W \cdot U(\delta_4 - X_v) \cdot \sqrt{\frac{2(P_4 - P_0)}{\rho}} \]

and

\[ Q_1 - Q_4 = A_p \dot{X}_p + C_p (P_1 - P_2) + C_{\phi} P_1 + \frac{V_1}{\beta_1} \dot{P}_1 \]  
\[ Q_2 - Q_3 = A_p \dot{X}_p + C_p (P_1 - P_2) - C_{\phi} P_2 - \frac{V_2}{\beta_2} \dot{P}_2 \]

where

\[ U(X) = \begin{cases} X, & X > 0 \\ 0, & X \leq 0 \end{cases} \]

\(P_s\) is the supply pressure and \(P_0\) the return pressure to the reservoir. \(W\) is the width of valve opening. \(\delta_1, \delta_2, \delta_3\) and \(\delta_4\) represent underlap (positive) of the servovalve.

The continuity equation applied to the linear actuator chambers yields
where

\[ V_1 = \frac{V_t}{2} + X_p A_p \]  
\[ V_2 = \frac{V_t}{2} - X_p A_p \]

\( A_p \) is the cross-sectional area of the piston and \( V_t \) is the total volume of the actuator. \( C_{ip} \) and \( C_{ep} \) are internal and external leakage coefficients of the actuator. \( \beta_e \) is the bulk modulus of the fluid.

The dynamics of the piston are given by

\[ M\ddot{X}_p + B_p \dot{X}_p + K_p X_p = (P_1 - P_2)A_p - \text{sgn}(\dot{X}_p)F_c \]

Numerical values of the system parameters are listed in Table 2.1 and details of the simulation program are listed in Appendix B.3.

### 2.2.2 Linearized Models of Material Testing Machine Dynamics

The control system design to be developed in this thesis is based on the framework of linear system theory. Therefore, a linearized model of the hydraulic servomechanism is needed as a basis for control system design. Two methods are used to obtain linearized models of the material testing machine. The first method is to obtain an analytical model using small perturbation analysis. This approach is useful especially in choosing proper hydraulic system parameters and understanding the effect of these parameters on the system dynamics. The second approach is to obtain linear forms of the system equation using numerical methods. The latter can be used in conjunction with
digital simulation and can be used easily to obtain linearized models under a variety of operating conditions.

2.2.2.1 Small perturbation analysis

A linearized model of the material testing machine is obtained from small perturbation analysis at a specified equilibrium condition. A number of simplifications have been made in arriving at the linearized model. The servovalve is assumed to be symmetric and critically centered. The resulting transfer functions are listed here.

For load control

\[
F(s) = \frac{2K_p A_r K_v K_i}{(1 + \tau_v s) \left\{ \frac{MV}{2\beta_e} + \frac{B_v V}{2\beta_e} + (2A_p^2 + 2B_v K_{\text{co}} + \frac{K_v V}{2\beta_e})s + 2K_v K_{\text{co}} \right\}}
\]

(2.12)

For stroke control

\[
\frac{X_p}{u}(s) = \frac{2K_p A_r K_v K_i}{(1 + \tau_v s) \left\{ \frac{MV}{2\beta_e} + \frac{B_v V}{2\beta_e} + (2A_p^2 + 2B_v K_{\text{co}})s + 2K_v K_{\text{co}} \right\}}
\]

(2.13)

where

\[
K_{\text{co}} = K_c + C_{e_p} + C_{i_p} / 2
\]

(2.14)

\[
K_c = -\frac{\partial Q_I}{\partial P_L} = \frac{C_i X_v}{\sqrt{P_e}}
\]

(2.15)
Load flow = \( \frac{Q_l + Q_3}{2} \)  

\[ Q_l \triangleq \text{Load flow} = \frac{Q_l + Q_3}{2} \quad (2.17) \]

and

Load pressure = \( P_L = (P_1 - P_2) \)  

\[ P_L \triangleq \text{Load pressure} = (P_1 - P_2) \quad (2.18) \]

\( C_{sp} \) and \( C_{ip} \) are generally negligible since the leakage in most hydraulic actuators is negligible compared to the valve flows. In load control applications, the valve opening \( X_v \) is small. From equations (2.14) and (2.15), the coefficient \( K_{ce} \) is very small. Hence equation (2.12) can be reduced to

\[
F_u(s) \equiv \frac{2K_q A_p K_v K_i}{(1 + \tau_v s)^2 + \left( 2MK_{ce} + \frac{B_q V_1}{2\beta_e} \right) s + \left( 2A_p^2 + 2B_q K_{ce} + \frac{K_j V_1}{2\beta_e} \right)}
\]

\[ \quad (2.19) \]

Therefore, the actuator dynamics for load control applications can be approximately modeled as an integrator cascaded with a second order system. This dynamic model under load control (2.19) is therefore close to the one under stroke control (2.13). Note that the denominator of equation (2.19) has one more term, \( (K_q V_1 / 2\beta_e) \), than that of equation (2.13). Hence, the resonant frequency for load control is higher than that for stroke control. Also, the system gains are different for the two control systems.
From the above equations, the parameters are subject to large variation as the applied load and valve opening change within a testing cycle. Valve overlap in the actual system leads to further variation in the linearized valve coefficients, especially when the valve is operated near the null position as in load or strain control tests. The complexity of the neglected effects obviously prevents quantitative description of the modeling error bounds based on analytical considerations alone.

2.2.2.2 Determination of linearized models using numerical methods

A numerical procedure is proposed here to obtain a linear dynamic model of the material testing machine. The nonlinear model of the material testing machine is simulated using the digital computer simulation package ACSL. The ACSL package has a linear analysis capability which allows users to obtain linearized models for a nonlinear system numerically. The linear analysis method is as follows.

Consider a nonlinear system,

\[ \dot{X} = f(X, U) \]
\[ Y = g(X, U) \]  \hspace{1cm} (2.20)

where \( Y \) and \( U \) are output and input vectors respectively. \( X \) is the state vector of the system. The nonlinear system can be linearized about a specified operating condition to yield the state space equations,

\[ \dot{X} = AX + BU \]
\[ Y = CX + DU \]  \hspace{1cm} (2.21)
where

\[
\begin{align*}
A &= \frac{\partial f}{\partial X} \equiv \frac{\Delta f}{\Delta X} & B &= \frac{\partial f}{\partial U} \equiv \frac{\Delta f}{\Delta U} \\
C &= \frac{\partial g}{\partial X} \equiv \frac{\Delta g}{\Delta X} & D &= \frac{\partial g}{\partial U} \equiv \frac{\Delta g}{\Delta U}
\end{align*}
\] (2.22)

A, B, C and D are Jacobian matrices of the nonlinear system. These matrices, i.e. the partial derivatives of the nonlinear functions f and g, are functions of states and inputs and can be calculated using small numerical perturbation at any specified operating condition. Therefore, various linearized models (2.21) of the nonlinear system can be obtained at different operating conditions. The ACSL run-time commands allow users to specify the input and output variables and then the values of these matrices are obtained numerically. Details of the ACSL commands for the analysis of nonlinear systems are given in the ACSL reference manual (1987). Frequency responses of those models (2.21) can then be calculated to analyze the nonlinear system.

Open loop load response of the hydraulic machine is simulated using ACSL with the parameter values listed in Table 2.1. A 10 Hz sinusoidal signal is input to the open loop system. Figure 2.4 shows the valve opening of the material testing machine under the open loop conditions. Figure 2.5 shows the frequency responses of the linearized models, each curve corresponding to the operating point depicted in Figure 2.4. When the valve opening is small, such as points 4 and 8, the system response is close to that of an integrator cascaded with a lightly damped second order system. This result agrees with equation (2.19). When the valve opening is large, as at some other points 2 and 6, the integrator part of the transfer function is replaced by a first order lag term. Note that the resonant frequency does not change much. The system gain also varies significantly within a cycle as indicated by the dc gain variation in Figure 2.5.
Figure 2.4 Valve Opening of the Open Loop Load Control System
Figure 2.5 Frequency Responses of Linearized Models – Open Loop Load Control System
Figure 2.6 Frequency Responses of Linearized Models – Open Loop Stroke Control System
Figure 2.7 Valve Opening for the Closed Loop Load Control System
Figure 2.8  Frequency Responses of Linearized Models – Closed Loop Load Control System
Figure 2.9 Valve Opening for the Closed Loop Stroke Control System
Figure 2.10 Frequency Responses of Linearized Models – Closed Loop Stroke Control System
Figure 2.6 depicts the frequency responses of the linearized models of the system under open loop stroke control, each curve corresponding to the operating point depicted in Figure 2.4. The specimen is removed from the machine in the simulation of the stroke control system. A 10 Hz sinusoidal signal is input to the servomechanism, and linearized models are calculated at different operating points within the cycle. The frequency responses of the open loop system match well with the dynamic equation (2.13). The resonant frequency does not change much and, as indicated before, its value is lower than that of the open loop load control system. The system gain varies significantly within a cycle as before.

The closed loop system response can also be obtained in a similar manner. The servosystem is simulated under proportional control. A 10Hz triangular signal reference signal is applied and the linearized models of the closed loop load control system are calculated numerically. Figure 2.7 shows the valve opening under the closed loop load control. Figure 2.8 shows the frequency responses of the linearized closed loop models calculated at different operating points as depicted in Figure 2.7. The variation of the load control system at low frequencies, up to 25 rad/sec, is reduced by the proportional feedback control, but the variation at higher frequencies is still significant though less than in Figure 2.5. Note that the steady state gain in Figure 2.8 is not unit, this indicates that equation (2.19) can not replace (2.12) in the low frequency range. Figure 2.9 and Figure 2.10 show the valve opening and the frequency responses of the linearized models of the closed loop stroke control system. The variation of system response is reduced at low frequencies, up to 10 rad/sec, but is still significant at high frequencies.

Figure 2.5 and Figure 2.6 show the amount of variation in linear models of the system dynamic behavior as a result of the valve related nonlinearity. The corresponding frequency responses can be used to come up with parametric models of such variation,
such as uncertainty bounds as functions of frequency, which can be effectively used for controller design. Figure 2.8 and Figure 2.10 show that the linearizing effect of feedback is stronger primarily at low frequencies. They also show that simple control algorithms such as proportional control result in the significant variability of closed loop system performance at frequencies as low as 20 rad/sec. More elaborate control algorithms will improve the closed loop system performance at these frequencies and will reduce the variability. However, one objective of the current research is to achieve more effective control for periodic reference input signals by taking explicit account of the periodicity of the signal.

2.3 Experimental Tests

The small scale material testing machine shown in Figure 2.1 and Figure 2.2 has strong nonlinear dynamic effects such as valve pressure/flow relationships and valve overlap characteristics. The overlap characteristics in the actual machine are not included in the above analysis and simulation. There are other sources of uncertainty in the actual material testing machine, such as fluid density, fluid bulk modulus, servovalve dynamics, etc. Therefore, an experimental testing procedure is designed to investigate the actual testing machine and also to obtain a more accurate nominal model for control system design. This procedure is as follows. The parameters of the small scale testing machine are selected as close as possible to the nominal values listed in Table 2.1. The load control loop is operated only under analog proportional control, as shown in Figure 2.11, for sinusoidal reference input signals at different frequencies and magnitudes. The amplitude ratio and phase shift relating the components of F and u at the input signal frequency are obtained based upon computed Fourier transforms. Higher harmonic components are neglected. Each set of amplitude ratio and phase shift values at the
fundamental signal frequency represents one point on the frequency response curve. The reference input signal magnitude is varied and the frequency response curve computed again as described above. A typical test result is shown in Figure 2.12. The valve opening is small in closed loop load control applications, as shown in Figure 2.7, so the dynamic response in the frequency range from 2 rad/sec to 100 rad/sec is close to those of an integrator. This indicates the appropriateness of the form of equation (2.19) simplified from equation (2.12). The significance of the modeling error is seen from the frequency response variation noted in the figure. Wider valve opening results in higher system gain. Also, a sharp cut off in the magnitude response at high frequencies is seen in the figure. This indicates that a first order model as in equation (2.1) probably does not accurately represent the dynamics of the servovalve. Some additional nonlinear effects are probably significant at these higher frequencies.
Figure 2.12 Frequency Responses of $10^{-3} \text{F/u}$, Experimental Tests
2.4 Summary

The dynamic response of an electrohydraulic material testing machine is investigated in this chapter. The test specimen is modeled as a pure spring with constant spring rate and the servovalve dynamics are assumed to be of first order. Even with these simplifications, the system still has significant nonlinearities. The linearized models calculated numerically from digital simulation provide some insight into variation in the linear system dynamics. An experimental testing procedure is also used to investigate the variations of linear models of the system behavior. The strong nonlinearity shown by the experimental tests suggests that improved closed loop control is going to place demanding requirements on the controller.
3.1 Introduction

The analysis and synthesis of repetitive control systems are complicated by the fact that there is an inherent time delay in such systems, introduced by the controller. Hence, a class of time delay systems, which includes most repetitive control systems, is investigated first. The relative stability of such time delayed systems is examined using a function of frequency termed the regeneration spectrum. The regeneration spectrum is related to important features of the characteristic root distribution of such systems, for large values of the time delay. An additional meaning of regeneration spectrum in terms of the transient response of the system is also offered here.

Absolute and relative stability of single variable repetitive control systems is then analyzed using the regeneration spectrum. The regeneration spectrum is combined with other frequency domain measures of control system performance such as the sensitivity and complementary sensitivity functions, to provide deeper insight into the trade-offs in repetitive control system design. The result is a more rational approach to repetitive control system design and is illustrated by an example.

3.2 The Regeneration Spectrum: Definition and Use

A relative stability measure for a class of linear, time delayed systems, which has been successfully used in the past to analyze the dynamic stability of a variety of
machining situations (Srinivasan and Nachtigal, 1978; Srinivasan, 1981; Srinivasan 1982), will be used to analyze the stability of repetitive control systems. This relative stability measure is based on an easily evaluated function of frequency termed the regeneration spectrum. The class of linear time delayed systems for which the regeneration spectrum is useful as a relative stability measure is characterized by 'large' values of the time delay. A more precise characterization of a 'large' time delay is deferred until later in the section. For such a system, the gain margin and phase margin and important features of the characteristic root distribution have been shown to be accurately inferred from the regeneration spectrum, with great savings in computation as compared to their direct evaluation. The ease of evaluation of these relative stability measures based on the regeneration spectrum has enabled effective use of the regeneration spectrum in the design of active machine tool chatter control systems (Srinivasan and Nachtigal, 1978) and in the analysis of chatter in precision grinding operations (Srinivasan, 1981; Srinivasan, 1982).

3.2.1 Regeneration Spectrum and Characteristic Roots

Consider the characteristic equation of a continuous-time, time-invariant, time delayed system with a single time delay $T_D$, as shown in Figure 3.1, is given by

$$P(s) - Q(s)e^{-sT_D} = 0$$  \hspace{1cm} (3.1)

where $P(s)$ and $Q(s)$ are polynomials in the Laplace variable $s$. $P(s)$ is assumed to have no roots in the right half of the s-plane.
The regeneration spectrum for such a system is defined as a plot of $R(\omega)$ versus the frequency $\omega$.

$$R(\omega) = \left| \frac{Q(j\omega)}{P(j\omega)} \right|$$

(3.2)

The relationship of the regeneration spectrum to the absolute stability of the system is established by the amplitude-phase method of stability analysis (El'sgol'ts, 1966), which is essentially an application of the Nyquist criterion to time delayed systems. Briefly stated, if the polynomial $P(s)$ has no zeros in the right half of the $s$ plane and if

$$R(\omega) < 1$$

(3.3)

the closed loop system in Figure 3.1 is stable for all values of the time delay. The condition (3.3) is thus a sufficient condition for system stability. The regeneration spectrum is independent of the time delay $T_D$ and is easily evaluated.
The usefulness of the regeneration spectrum for stability analysis and controller design for time delayed system rests not so much on the absolute stability result stated above, but on the relationship of the degree of stability of such systems, for 'large' values of the time delay. The time delay $T_D$ is considered to be 'large' if it satisfies

$$T_D >> \frac{1}{|\alpha_{\max}|} \quad (3.4)$$

where $\alpha_{\max}$ is the real part of the zero of the polynomial $P(s)$ closest to the imaginary axis. The requirement (3.4) for a 'large' time delay was simplified, somewhat arbitrarily, by Srinivasan and Nachtigal (1978) to

$$T_D \geq \frac{5}{|\alpha_{\max}|} \quad (3.5)$$

For systems with a large time delay in this sense, Srinivasan and Nachtigal (1978) showed that the regeneration spectrum bore a very simple relationship to the dominant characteristic roots of the equation (3.1).

$$\alpha_{i,-i} = \frac{\ln(R(\omega_i))}{T_D} \quad (3.6)$$

where $\alpha_{i,-i}$ are the real parts of complex conjugate roots with their imaginary parts equal to $\omega_i$ in magnitude. Since $\omega_i$ are not known beforehand, the practical implication of equation (3.6) is that the continuum of complex numbers $\ln(R(\omega)) / T_D \pm j\omega$ for $\omega$ varying from 0 to $\infty$ includes the characteristic root locations of the system (3.1) closest to the imaginary axis. In view of the fact that the
transcendental characteristic equation (3.1) has an infinite number of roots and would require considerable computational effort for its numerical solution, the regeneration spectrum offers a convenient alternative to estimate significant features of the system dynamic response. In controller design applications, shaping of the regeneration spectrum by appropriate compensator selection is an effective way to alter the characteristic root distribution of the system and hence its transient response.

The equation (3.6) is merely an approximate version of the characteristic equation (3.1) expressed as

\[ \alpha_{i,-1} = \frac{\ln \frac{Q(\alpha_i + j\omega_i)}{P(\alpha_i + j\omega_i)}}{T_D} \]  

(3.7)

The requirement (3.4) for a large time delay results in one category of roots of the equation (3.1) being close to the imaginary axis. For stable systems, these roots would be in the left half of the complex plane. Being close to the imaginary axis, they are important for system stability and have been termed as the regenerative effect poles by Bartalucci and Lisini(1969) because of their critical dependence on the time delay $T_D$. For these roots, setting $\alpha_i$ to zero on the right hand side of equation (3.7) yields the equation (3.6). A generalization of the regeneration spectrum to improve the goodness of approximation of equation (3.7) by equation (3.6) has also been described by Srinivasan(1981) and involves an iterative solution of equation (3.7). It requires knowledge of the time delay $T_D$ and in effect, sacrifices ease of computation in order to improve the accuracy of approximation.

The above paragraphs summarize how the regeneration spectrum has been developed so far. In the following two sections, a quantitative definition of a large time
delay will be derived first, followed by another interpretation of the relationships of the regeneration spectrum to the transient response of the system.

3.2.2 A Quantitative Criterion for Largeness of Time delay

Equations (3.4) and (3.5) are quantitative conditions under which the closed loop poles are seen to map closely onto the regeneration spectrum. Explicit justification for these conditions has not been determined previously. Such an explicit justification is sought here. Consider the characteristic equation (3.1) of the time delayed system shown in Figure 3.1 and let \( \alpha + j\beta \) be a characteristic root of the system. We have

\[
e^{\alpha T_d} = |R(\alpha + j\beta)|
\] (3.8)

Let \( a(\omega) \) be a function of frequency which satisfies

\[
e^{a(\omega)T_d} = |R(\omega)|
\] (3.9)

If the time delay is large enough, the function \( a(\omega) \), when evaluated at \( \beta \), is close to \( \alpha \) in value. From equations (3.8) and (3.9), we have

\[
e^{(\alpha - a(\beta))T_d} \equiv \frac{|R(\alpha + j\beta)|}{|R(\beta)|} \equiv \frac{|R(a(\beta) + j\beta)|}{|R(\beta)|}
\] (3.10)

By taking natural logarithms,

\[
\alpha - a(\beta) = \frac{1}{T_d} \ln \left( \frac{|R(\alpha + j\beta)|}{|R(\beta)|} \right) = \frac{1}{T_d} \ln \left( \frac{|R(a(\beta) + j\beta)|}{|R(\beta)|} \right)
\] (3.11)
and

\[
\frac{\alpha - a(\beta)}{\alpha} = \frac{1}{T_d \alpha} \ln \left( \frac{|R(\alpha + j\beta)|}{|R(\beta)|} \right) = \frac{1}{T_d a(\beta)} \ln \left( \frac{|R(a(\beta) + j\beta)|}{|R(\beta)|} \right)
\]

(3.12)

The left hand side of equation (3.12) is the normalized discrepancy between the actual real part of the characteristic root and its estimate from the regeneration spectrum. The quotient within the parentheses on the right hand side of equation (3.12) can be easily evaluated, its denominator being the regeneration spectrum and its numerator the magnitude of \( R(s) \) evaluated at the estimated characteristic root. Note that these are estimates of the error resulting from replacing \( \alpha \) on the right hand side of equation (3.10) by \( a(\beta) \). Whether or not these estimates are conservative depends on the nature of the function \( R(s) \) in the vicinity of the roots of the characteristic equation. Section 3.2.4 considers this question in greater detail.

An application of the regeneration spectrum to machine tool chatter control systems (Srinivasan and Nachtigal, 1978) is used here to demonstrate the use of equation (3.11) and (3.12). Two examples of characteristic equations for the machine tool chatter control systems are

\[
(s^2 + 26.3122s + 197470) - 3872e^{-0.8s} = 0
\]

(3.13)

and

\[
(s^2 + 26.3122s + 205216) - 11616e^{-0.8s} = 0
\]

(3.14)

The regeneration spectra of the two equations (3.13) and (3.14) are shown in Figure 3.2 and Figure 3.3 respectively, along with some of the actual characteristic roots. The characteristic root locations appear to closely match with the regeneration spectrum in Figure 3.3, but less so in Figure 3.2. Figure 3.4 and Figure 3.5 show the calculated
Figure 3.2 Regeneration Spectrum Comparison With Characteristic Root Locations System (3.13)

Figure 3.3 Regeneration Spectrum Comparison With Characteristic Root Locations System (3.14)
Figure 3.4 Discrepancy of Closed Loop Pole Locations, Equation (3.11)

Figure 3.5 Normalized Discrepancy of Closed Loop Pole Locations, Equation (3.12)
Figure 3.6 Discrepancy of Closed Loop Pole Locations for Different Values of the Time Delay, Equation (3.11)

Figure 3.7 Normalized Discrepancy of Closed Loop Pole Locations for Different Values of the Time Delay, Equation (3.12)
discrepancy (3.11) and normalized discrepancy (3.12) between actual characteristic root locations and estimates based on the regeneration spectrum. Figure 3.4 indicates that the estimated discrepancy using equation (3.11) is a conservative one, that is, it exceeds the actual discrepancy. The reason for the conservativeness of the estimation error is related to the nature of the function $R(s)$ in the vicinity of the imaginary axis, as described in section 3.2.4. Note that the normalized discrepancy estimates for both cases are quite similar as shown in Figure 3.5 and are also more accurate.

Figure 3.6 and Figure 3.7 show the discrepancy and normalized discrepancy of the estimated pole locations for the first system (3.13) for different values of the time delay. The acceptability of estimated closed loop pole locations based on the regeneration spectrum is a matter of judgement. For this example, a time delay of 1.2 second may be said to be large enough, since the normalized discrepancy in Figure 3.7 is less than 6%. If a lower error level is desired, the time delay must be larger. Therefore, the usefulness of the error estimates (3.11) and (3.12) is that they help predict trends even if they are conservative. Also, they can be used to help determine when the time delay is large enough for the regeneration spectrum to be used effectively for determining pole locations. The resulting criterion for largeness of the time delay is certainly an improvement on equation (3.5) since it gives an associated error bound on which to base acceptability of the estimate. However, it leaves open the question of the conditions that must be satisfied for the error bounds to be conservative, or upper, bounds. Section 3.2.4 addresses this issue in greater details.

3.2.3 Another Interpretation of the Regeneration Spectrum

The transient response of the linear time delayed system shown in Figure 3.1 can be estimated by examining the real part of the characteristic roots, which can be calculated
from the regeneration spectrum. Consequently, the regeneration spectrum can be used to predict the transient response of the system. A new interpretation of the regeneration spectrum is introduced here to provide a deeper understanding of the transient response of the time delayed system for large values of the time delay. This interpretation also suggests the motivation for using the term "regeneration" spectrum to describe $R(\omega)$ given by equation (3.2).

The transient response of a system is characterized by its impulse input response. The Laplace transform of the impulse input response of a system is equal to the system transfer function. The Laplace transform of the impulse input response of the time delayed system at the input location shown in Figure 3.1 is

$$h(s) = \frac{1}{1 - e^{-T_d s} R(s)}$$

(3.15)

and its power series expansion is

$$h(s) = 1 + e^{-T_d s} R(s) + e^{-2T_d s} R^2(s) + e^{-3T_d s} R^3(s) + ...$$

(3.16)

If the region of convergence of the power series includes the imaginary axis in the s-plane, the inverse Laplace transform of the power series exists. It is shown in Appendix A.1 that the region of convergence includes the imaginary axis in the s-plane if the regeneration spectrum is less than one, and $R(s)$ is analytical in the right half plane. Under these two conditions, which are reasonable, the impulse input response is

$$h(t) = \sum_{i=0}^{\infty} \xi(t - iT_d)$$

(3.17)
where

\[ r_i(t) = \mathcal{L}^{-1}\{R_i(s)\} \]  \hspace{2cm} (3.18)

The impulse response is thus a summation of an infinite number of time delayed functions. Each successive function is equal to the preceding function filtered by \( R(s) \) and delayed by \( T_D \). Note that \( r_i(t) \) is not a function of the time delay. The implication of equations (3.17) and (3.18) is that the transient response of the time delayed system is only determined by the regeneration spectrum if the time delay is large in the sense that successive functions \( r_i(t) \) do not overlap each other. Thus, the time delay \( T_D \) is considered large if the functions \( r(t) \) decay to zero within the time delay period. If this were to be true, the functions \( r_i(t) \) may be viewed as being re-generated from one time delay period to the next, \( R(s) \) governing the regeneration. Therefore, from the viewpoint of the frequency domain, the regeneration spectrum can be regarded as the attenuation rate of the corresponding frequency component from one time delay to the next. Thus, another criterion for a time delay \( T_D \) to be considered large is that it satisfies

\[ r_i(t) \equiv 0, \ t \geq T_D \]  \hspace{2cm} (3.19)

where \( r_i(t) \) is defined by equation (3.18). If the regeneration spectrum has the significance assigned to it here, namely, that it describes the regeneration of the corresponding frequency component from one time delay period to the next, then

\[ R(\omega) = \frac{|R_i(\omega)|}{|R_{i-1}(\omega)|} \]  \hspace{2cm} (3.20)
where

\[
R_i(\omega) = \int_0^\infty r_i(t) e^{-j\omega t} dt = \int_0^{T_D} r_i(t) e^{-j\omega t} dt
\]  

(3.21)

The approximation in equation (3.21) is very accurate if equation (3.19) is valid. If the time delay \( T_D \) is much larger than the dominant time constant of the transfer function, it is clear that the equation (3.19) would be valid. Such a relationship is in fact proposed in equation (3.5). It is clear that the criterion (3.5) leads to the result that successive \( r_i(t) \) functions do not overlap in time. However, the criterion (3.5) is not directly related to the validity of the approximation (3.6). There is obviously a link between the criterion (3.5) and the error in the estimated root locations (3.11). However, that link needs to be clarified further.

Figure 3.8 depicts the impulse response of the system described by equation (3.13). In this figure, the unit impulse at zero second is not plotted, so that the response functions (3.18) can be seen clearly. The frequency content of the first five functions over each time period is calculated using the fast Fourier transform. The attenuation rate of any frequency component of these functions is found to be very close to the values of the regeneration spectrum at that frequency. Regeneration spectrum values less than unity indicate a decay of the signal component, lower values indicating a more rapid decay.
Figure 3.8 Impulse Response of the Time Delayed system
3.2.4 Elaboration on a Quantitative Criterion for Largeness of the Time Delay

The appropriateness of the approximation (3.10) can be clarified by evaluating the function $|R(s)|$ and the function $|e^{T_D s}|$, for the time delayed system shown in Figure 3.1, over the left half of the s-plane. Figure 3.9 shows a three dimensional plot of $|R(s)|$ evaluated over the upper left quarter of the s-plane and Figure 3.10 shows a two dimensional plot of $|R(\alpha+j\omega)|$ as a function of $\omega$, $R(s)$ being computed for the system (3.13). The function $|e^{T_D (\alpha+j\omega)}|$ is independent of $\omega$ and increases exponentially with respect to $\alpha$. The characteristic roots satisfy equation (3.8), i.e. the values of the functions $|e^{T_D s}|$ and $|R(s)|$ are equal. If $T_D$ is large enough, the values of $\alpha$ at which the two functions intersect are very close to the imaginary axis, i.e. $|R(\alpha+j\omega)|$ is very close to the regeneration spectrum $|R(j\omega)|$. Since $|e^{T_D (\alpha+j\omega)}|$ is independent of $\omega$, it is convenient to begin the investigation by examining the two functions $e^{T_D \alpha}$ and $|R(\alpha+j\omega)|$, for a constant value of $\omega$, in the upper left quarter of the s-plane. Figure 3.11 shows the two functions evaluated at a constant value of $\omega$, while varying the real value $\alpha$. The real part of a characteristic root $\alpha_1$ of equation (3.8) must be at a value $\alpha$ for which the two curves intersect. Note that all such intersections for different $\omega$ values, are not necessarily valid roots of the characteristic equation (3.8) since the phase angles for both sides of the equation must also be equal for any such intersection to be a root of the characteristic equation. The intersections obviously include all the roots of the characteristic equation. The exact and approximate discrepancies in the real parts of the estimated characteristic roots, the latter given by the right hand side of equation (3.11), are also shown in Figure 3.11. If points $x_1$ and $x_2$ are on opposite sides of the curve $e^{\alpha_1 T_D}$, i.e. $\alpha_1$ is between $a$ and $a_2$, the estimated discrepancy (3.11) is an upper bound on the exact discrepancy as indicated in the figure. One special condition for the points $x_1$ and $x_2$ to be on opposite sides of the curve $e^{\alpha T_D}$ is that the function $|R(\alpha+j\omega)|$ is
Figure 3.9 $|R(\alpha+j\omega)|$ Versus $\alpha$ and $\omega$, System (3.13)

Figure 3.10 $|R(\alpha+j\omega)|$ Versus $\omega$, System (3.13)
Figure 3.11 Monotonically Decreasing $|R(\alpha + j\omega)|$ Versus $\alpha$ and the Corresponding Discrepancy (3.11) and Normalized Discrepancy (3.12)
Figure 3.12 $|R(\alpha + j\omega)|$ Versus $\alpha$ and $\omega$, $R(s)$ With Zero at -35 and Poles at $-25 \pm j25.2$

Figure 3.13 $|R(\alpha + j\omega)|$ Versus $\omega$, $R(s)$ With Zero At -35 and Poles At $-25 \pm j25.2$
Figure 3.14  \(|R(\alpha+j\omega)|\) Versus \(\alpha\) and \(\omega\), \(R(s)\) With Zero at -15 and Poles at \(-25 \pm j25.2\)

Figure 3.15  \(|R(\alpha+j\omega)|\) Versus \(\omega\), \(R(s)\) With Zero At -15 and Poles At \(-25 \pm j25.2\)
Figure 3.16 General Cases of $|R(\alpha +j\omega)|$ Versus $\alpha$ and the Corresponding Discrepancy (3.11) and Normalized Discrepancy (3.12)
monotonically decreasing with respect to $\alpha$. If $R(s)$ has no zero, the function $|R(\alpha+j\omega)|$ does decrease monotonically everywhere to the right of the dominant poles of $R(s)$. For example, $|R(\alpha+j\omega)|$ for the example system (3.13), as shown in Figure 3.9, does decrease monotonically in the region where $\alpha$ is larger than -13.1 rad/sec, i.e. $\alpha_{\text{max}}$, the real part of the dominant poles. This explains the results noted in the latter part of section 3.2.2, i.e. the estimated discrepancy (3.11) for the system (3.13) is a conservative, or upper, bound on the actual discrepancy.

If $R(s)$ has zeros, $|R(\alpha+j\omega)|$ may not be monotonically decreasing with respect to $\alpha$ for all $\alpha$ larger than $\alpha_{\text{max}}$, the real part of the dominant poles of $R(s)$. Figure 3.12 shows the three dimensional plot of $|R(\alpha+j\omega)|$, for $R(s)$ having one zero at -35 and two poles at $-25 \pm j25.2$. Figure 3.13 shows the plot of $|R(\alpha+j\omega)|$ as a function of $\omega$ for different values of $\alpha$ from 0 to -9. The zero's location is the left of the poles' location, and $|R(\alpha+j\omega)|$ is seen to decrease monotonically with respect to $\alpha$ for all $\alpha$ larger than -9. Figure 3.14 shows the three dimensional plot of $|R(\alpha+j\omega)|$, for $R(s)$ having one zero at -15 and two poles at $-25 \pm j25.2$. Figure 3.15 shows the plot of $|R(\alpha+j\omega)|$ as a function of $\omega$ for different values of $\alpha$ from 0 to -9. $|R(\alpha+j\omega)|$ is seen to increase monotonically with respect to $\alpha$ for low values of $\omega$, but decreases monotonically with respect to $\alpha$ for higher values of $\omega$. It is clear from Figures 3.12 - 3.15 that zeros of $R(s)$, especially the ones to the right of the dominant poles, may affect the values of $|R(\alpha+j\omega)|$ to the extent of preventing it from decreasing monotonically with respect to $\alpha$. In Figure 3.15, for example, the effect of the zero at -15 is dominant for low $\omega$, the region where $|R(\alpha+j\omega)|$ increases monotonically with respect to $\alpha$. It should also be noted however, that even if $|R(\alpha+j\omega)|$ does not decrease monotonically with respect to $\alpha$, the discrepancy in the estimated pole locations given by equation (3.11) may still be an upper bound.
Consider two more general cases of the function $|R(\alpha+j\omega)|$ as shown in Figure 3.16. In these two cases, $|R(\alpha+j\omega)|$ is not monotonically decreasing with respect to $\alpha$. These curves depict situations where $R(s)$ may have zeros to the right of dominant poles. Both the points $x_1$ and $x_2$ are on the opposite sides of the curve $e^{\alpha T_D}$ in the two figures, i.e. $\alpha_1$ is between $a$ and $a_2$. Hence, the discrepancy estimate (3.11) is still an upper bound on the exact discrepancy. For $x_1$ and $x_2$ to be on the opposite sides of the curve $e^{\alpha T_D}$, one of the following conditions must hold:

$$e^{\alpha T_D} - |R(a + j\omega)| > 0 \quad \text{and} \quad e^{\alpha_2 T_D} - |R(a_2 + j\omega)| < 0$$

(3.22)

as in Figure 3.16(a) or

$$e^{\alpha T_D} - |R(a + j\omega)| < 0 \quad \text{and} \quad e^{\alpha_2 T_D} - |R(a_2 + j\omega)| > 0$$

(3.23)

as in Figure 3.16(b). Combining the above two sets of equations yields

$$(e^{\alpha T_D} - |R(a + j\omega)|)(e^{\alpha_2 T_D} - |R(a_2 + j\omega)|) < 0$$

(3.24)

where $a$ is defined by equation (3.9) and

$$a_2 = \frac{1}{T_D} \ln(|R(a + j\omega)|)$$

(3.25)

Equation (3.24) can therefore be used to verify if equation (3.11) is a conservative bound on the discrepancy or error in the estimated real parts of the characteristic roots.
Equation (3.24) is a general condition but it needs to be evaluated numerically. Such a result, while it does not give a clear statement of the characteristics of $R(s)$ that result in equation (3.24) being valid, is nevertheless of great practical value. The numerical evaluation of (3.24) is also a straightforward one. Furthermore, most time delayed systems such as the repetitive control systems and machine tool chatter control systems investigated here are seen to have zeros far to the left of their dominant poles. Moreover, taking Figure 3.15 as an example, $|R(\alpha+j\omega)|$ tends to decrease monotonically with respect to $\alpha$ in the critical range of $\omega$ where the values of the regeneration spectrum are high. In view of observations above, we can expect that equations (3.11) and (3.12) will be quite useful in practice to help judge when the time delay is large enough for estimates based on the regeneration spectrum to be accurate enough.

3.3 Regeneration Spectrum Application to Repetitive Control Systems

Figure 3.17 shows the block diagram of a SISO repetitive control system. $G_p(s)$ is the uncompensated or compensated plant transfer function, $q(s)$ is a low-pass filter needed to guarantee system stability (Hara et al., 1988), and $b(s)$ is a compensator transfer function. $T_D$ is the period of the reference/disturbance input signal. The block diagram configuration in Figure 3.17 is equivalent to the SISO repetitive control system configuration considered by Hara et al. (1985), but has the advantage that the effect of the design parameter $b(s)$ on system stability and transient response is very clearly illustrated as discussed later.

3.3.1 Relative Stability and the Regeneration Spectrum

The repetitive control system shown in Figure 3.17 has a time delay within the system, the time delay being equal to integer multiple of the time period of the reference
signal and/or the disturbance signal. Hence, the relative stability of the repetitive control system can be studied using the regeneration spectrum. The characteristic equation of the closed loop system is

\[ 1 + G_p(s) + q(s)\left[b(s)G_p(s) - G_p(s) - 1\right]e^{-\tau_0} = 0 \]  

(3.26)

and the regeneration spectrum is

\[ R(\omega) = \left| q(j\omega)\left(1 - b(j\omega)\left\frac{G_p(j\omega)}{1 + G_p(j\omega)}\right\right) \right| \]  

(3.27)
The expression for the regeneration spectrum illustrates very clearly the effect of different aspects of the controller design on the stability of the repetitive control system. If the equation

\[ 1 + G_p(s) = 0 \]  

(3.28)

has no roots in the right half of the complex s-place, and if the regeneration spectrum (3.27) is less than one in magnitude at all frequencies, the repetitive control system is stable. The first condition suggests therefore that the closed loop system should be stable in the absence of the repetitive control action. If compensation is needed to achieve this, the compensator transfer functions are included appropriately in \( G_p(s) \). The condition on the regeneration spectrum indicates very clearly the need for a low-pass filter \( q(s) \). The expression within parentheses in equation (3.27) tends to unity as \( \omega \to \infty \) since \( G_p(j\omega) \to 0 \) at high frequencies for physical systems. Thus \( q(j\omega) \) must be lower than one at high frequencies to ensure system stability. It should be noted that the stability condition is a sufficient one and is valid for all values of the time delay \( T_D \). The effect of the compensator \( b(s) \) on system stability is also clear from equation (3.27). Choice of \( b(j\omega) \) to compensate for the amplitude and phase of the frequency response \( G_p(j\omega) / (1 + G_p(j\omega)) \) would keep the magnitude of the term within parentheses in equation (3.27) close to zero for a wider range of frequencies and hence improve the stability of the closed loop system.

For large value of time delay \( T_D \), the sufficient stability condition tends to become a necessary and sufficient one. Furthermore, the regeneration spectrum (3.27) can be used to determine the locus of the dominant characteristic roots of equation (3.26) to a very good degree of accuracy using equation (3.6).
The assumption of a large value of the time delay \( T_D \) should be checked using equation (3.11) for each specific application of interest. However, it is reasonable to expect that the assumption of a large time delay will be valid for many repetitive control applications. The right hand side of the expression (3.4) is the dominant or slowest time constant of the system in the absence of the repetitive control action. Hence, the condition (3.4) merely states that the time delay \( T_D \) should be much larger than the dominant time constant of the system in the absence of repetitive control. At any rate, the compensated plant transfer function \( G_p(s) \) can be selected so as to satisfy the condition (3.4) during the first stage of the repetitive controller design procedure described later in the chapter. Even if the condition (3.4) is violated, the absolute stability results based on the regeneration spectrum would continue to be valid.

More conventional relative stability measures such as gain and phase margins can be determined by applying the Nyquist criterion to the characteristic equation (3.26) restated as

\[
1 + G_p(s) \frac{1 + q(s)(b(s) - 1)e^{-sT_o}}{1 - q(s)e^{-sT_o}} = 0
\]

(3.29)

The gain and phase margins thus determined would indicate the amount of gain increase or additional phase lag that can be accommodated by the plant \( G_p(s) \) without resulting in system instability. However, the usefulness of these measures in controller design situations is limited since the effect of changing the design variables \( b(s) \) or \( q(s) \) on these measures is not easy to infer. In contrast, equation (3.27) indicates very clearly the effect of \( q(j\omega) \) on system stability. Lower magnitudes of \( q(j\omega) \) at high frequencies improve the degree of the system stability, by resulting in lower \( R(\omega) \). \( b(j\omega) \) is chosen to compensate the dynamics \( G_p(j\omega) / (1 + G_p(j\omega)) \).
As an example of the use of the regeneration spectrum, it is applied to a SISO repetitive control system design given by Hara et al (1988). For the example, the compensated plant is

\[
G_p(s) = \frac{112.65s^2 + 242.61s + 185.45}{s^6 + 12.263s^5 + 75.227s^4 + 273.14s^3 + 408.10s^2 + 345.13s + 146.22} \tag{3.30}
\]

Also, \( b(s) = 1 \)

and \( q(s) = \frac{1}{1+s} \) \tag{3.31}

The regeneration spectrum (3.27) is graphed in Figure 3.18 as the continuous curve. The roots of the characteristic equation (3.26) for a time delay \( T_D \) of 12 seconds are computed by explicit numerical solution and plotted on Figure 3.18 as the points \( (\omega_i, e^{\alpha_i T_b}) \), where the computed characteristic roots are \( \alpha_i \pm j\omega_i \). Since the characteristic roots of (3.28) are all in the left-half plane for this example, being \(-4.636, -2.322 \pm 4.0223j, -1.4913, -0.7457 \pm 1.2915j \text{ rad/sec}\), and since the time delay satisfies the inequality

\[
T_D = 12 \text{ sec} \geq \frac{5}{|\alpha_{\max}|} = 6.705 \tag{3.32}
\]

equation (3.6) should be valid to a good degree of approximation and the points \( (\omega_i, e^{\alpha_i T_b}) \) should lie very close to the regeneration spectrum. This expectation is confirmed by the results in Figure 3.18. In fact, the normalized discrepancy in the computed real parts of the roots, giving by equation (3.12), is within 5% for time delay values larger than 4.5 sec.
Figure 3.18 Regeneration Spectrum Comparison With Characteristic Roots
3.3.2 Transient Response of Repetitive Control Systems

The transient response, or the error convergence rate, of repetitive control systems is governed by the real part of the dominant poles, which can be approximately estimated by the regeneration spectrum. As indicated in section 3.2.3, the transient response of time delay systems can be done easily by directly examining the regeneration spectrum, if the time delay is large. If an impulse input is applied to the repetitive control system shown in Figure 3.17, the Laplace transform of the output response of the system is

\[ h(s) = \frac{1}{1 + [1 + \frac{q(s)e^{-T_D}}{1 - q(s)e^{-T_D}} b(s)]G_p(s)} \]  

(3.33)

It is shown in the Appendix A.2 that \( h(s) \) has the power series expansion,

\[ h(s) = S(s) - S(s)[b(s)G(s)q(s)e^{-T_D}][1 + e^{-T_D}R(s) + e^{-2T_D}R^2(s) + e^{-3T_D}R^3(s) + ...] \]  

(3.34)

where

\[ R(s) = q(s)[1 - b(s)G(s)] \]  

(3.35)

\[ G(s) = \frac{G_p(s)}{1 + G_p(s)} \]  

(3.36)

and

\[ S(s) = \frac{1}{1 + G_p(s)} \]  

(3.37)
Note that $S_0(s)$ is both the Laplace transform of the impulse response and the sensitivity function of the system, both quantities being evaluated without repetitive control action. As shown in Section 3.2.3, if the regeneration spectrum is less one and $R(s)$ is analytical in the right half plane, the inverse Laplace transform of equation (3.27) exists.

$$h(t) = h_0(t) - \sum_{i=1}^{\infty} \tau_i(t - iT_D)$$

(3.38)

where

$$h_0(t) = \mathcal{L}^{-1}\{S_0(s)\}$$

(3.39)

and

$$\tau_i(t) = \mathcal{L}^{-1}\{S_0(s)b(s)G(s)q(s)R^{i-1}(s)\} \equiv \mathcal{L}^{-1}\{S_0(s)R^{i-1}(s)\}$$

(3.40)

The approximation in equation (3.40) is valid in the low frequency range, because $b(s)$ is chosen to compensate $G(s)$ and $q(s)$ is typically a lowpass filter. It is clear that the regeneration spectrum value at each frequency determines the attenuation rate of the corresponding frequency component in the signal transient $h_0(t)$. Therefore, if the regeneration spectrum of the repetitive control system is small at all frequencies, fast transient response and a rapid error decay can be expected of the system.

### 3.4 Frequency Domain Analysis of Repetitive Control Systems

Frequency domain measures of the repetitive control system performance will be used here as they are compatible with the regeneration spectrum, which is also based in the frequency domain.
3.4.1 Sensitivity Function

The sensitivity function is used as a measure of disturbance rejection and sensitivity to plant modeling errors and parameter variations. Low values of the sensitivity function magnitude are desired, especially in the low frequency range where these aspects of performance are significant. Moreover, for the unity feedback system in Figure 3.17 being considered here, low values of the sensitivity function magnitude result in low values of the error signal in response to the reference input $v(t)$.

The sensitivity function for the system in Figure 3.17 can be shown to be

$$S(s) = \frac{1}{1+G_p} \cdot \frac{1}{1 + \frac{q e^{-ST_0}}{1 - q e^{-ST_0}} \cdot \frac{bG_p}{1 + G_p}}$$  \hspace{1cm} (3.41)

$$\Delta \left( \frac{1}{1 + G_p} \right) \cdot M_s$$  \hspace{1cm} (3.42)

where the expression within parentheses on the right hand side is the sensitivity function for the system without repetitive control. The remainder of the expression represents the multiplying factor modifying the sensitivity function as a result of the repetitive control action, and is labeled here as $M_s$.

The implications of the repetitive control action for the sensitivity function are clearer if the multiplying $M_s$ factor is examined at low frequencies.

$$M_s(j\omega) = \frac{1 - q(j\omega)e^{-j\omega T_0}}{1 - R(j\omega)e^{-j\omega T_0}} \equiv 1 - q(j\omega)e^{-j\omega T_0}$$  \hspace{1cm} (3.43)
at the low frequencies for which

\[
\frac{b(j\omega)G_p(j\omega)}{1 + G_p(j\omega)} \equiv 1
\]  

(3.44)

The latter assumption is justified by reasoning that choice of \( b(s) \) to satisfy equation (3.44) for a wider range of frequencies leads also to a reduction in the value of the regeneration spectrum, according to equation (3.27).

Equation (3.43) indicates that the sensitivity function can be reduced to very low values at specific frequencies. However, because of the cyclical nature of the multiplying factor \( M_\star \), this improvement in the sensitivity function is achieved at the expense of degradation at intermediate frequencies. For example, if \( q(j\omega) \) is chosen to be close to unity at low frequencies, the sensitivity function and hence the error signal is reduced to nearly zero at frequencies which are integral multiples of the input signal fundamental frequency \( 2\pi / T_D \) by adding repetitive control action, but is nearly doubled at some intermediate frequencies. This tradeoff in the sensitivity function values at different frequencies is a specific instance of analytic constraints on compensator design for linear systems (Freudenberg and Looze, 1988).

3.4.2 Complementary Sensitivity Function

The complementary sensitivity function is used as a measure of noise rejection and robustness in the presence of unstructured multiplicative uncertainty. Low values of the complementary sensitivity function magnitude are desired for improved noise response and robustness, especially in the high frequency range where these aspects of performance are significant (Freudenberg and Looze, 1988).
The complementary sensitivity function for the system in Figure 3.17 can be shown to be

\[ T(s) = \frac{G_p}{1 + G_p} \cdot \frac{1}{1 + G_p} - \frac{q e^{-s T_D} b}{1 + q e^{-s T_D} (b - 1)} \cdot \frac{1}{1 + G_p} \]  

(3.45)

\[ \Delta \left( \frac{G_p}{1 + G_p} \right) \cdot M_T \]  

(3.46)

where the expression within parentheses on the right hand side is the complementary sensitivity function for the system without repetitive control. The multiplying factor \( M_T \) represents the modification of the complementary sensitivity function as a result of the repetitive control action. At high frequencies, the expression for \( M_T \) simplifies to

\[ M_T \cong 1 + \frac{q(j\omega)e^{-j\omega T_D}}{1 - q(j\omega)e^{-j\omega T_D}} b(j\omega) \]  

(3.47)

if \( G_p(j\omega) \ll 1 \). As for the case of the sensitivity function, the complementary sensitivity function magnitude is lower at some frequencies as compared to other frequencies. The difference here is that it is not particularly clear that the complementary sensitivity function magnitude can be lowered by the addition of the repetitive control action. In fact, the case may be just the opposite. For instance, the complementary sensitivity function is degraded at frequencies which are integral multiples of the input signal fundamental frequency \( 2\pi / T_D \), if \( q(j\omega) \) is close to unity. Thus, the need for \( q(j\omega) \) to have low pass characteristics to minimize degradation of robustness to
multiplicative uncertainties is clear. Also, equation (3.48) indicates a need to limit the magnitude of \( b(j\omega) \) at high frequencies.

3.4.3 **Stability Robustness Criterion**

A stability robustness condition for multiplicative uncertainty can be derived using the regeneration spectrum. Consider the uncertainty in the pre-existing control system,

\[
G(s) = G_0(s)(1 + \Delta_\alpha(s))
\]  

(3.48)

\( G_0(s) \) is the nominal value of the pre-existing closed loop control system transfer function defined in equation (3.36). \( \Delta_\alpha(s) \) is the multiplicative unstructured uncertainty which is formulated in the frequency domain to include all possible variations of the system characteristics, such variation being determined experimentally if necessary. The repetitive control system is stable if the regeneration spectrum

\[
R(\omega) = |q(j\omega)[1 - b(j\omega)G_0(j\omega)(1 + \Delta_\alpha(j\omega))]|
\]  

(3.49)

is less than unity for all frequencies. We omit the arguments for notational convenience below. Note that

\[
|q[1 - b(G_0 + G_0\Delta_\alpha)]| \leq 1
\]  

(3.50)

if

\[
|q(1 - bG_0)| + |qbG_0\Delta_\alpha| < 1
\]  

(3.51)

or

\[
R_\alpha + |qbG_0\Delta_\alpha| < 1
\]  

(3.52)
where \( R_0 \) is the nominal value of the regeneration spectrum. Alternatively, including the arguments omitted above,

\[
|\Delta_0(j\omega)| \leq \frac{1 - R_s(\omega)}{|q(j\omega)b(j\omega)G_s(j\omega)|}
\]

(3.53)

The result is a conservative one firstly because the requirement that \( R(\omega) \) be less than unity for stability is a conservative one, and secondly because of the simplification used in equation (3.51) which yields an upper bound for equation (3.50). Since the multiplicative uncertainty \( |\Delta_0(j\omega)| \) is usually high at high frequencies, the right hand side of the inequality in (3.53) has to be large for the system to be stable. This suggests that the magnitudes of \( q(j\omega) \) and \( b(j\omega) \) have to be small at high frequency for stability robustness.

3.5 A Design Procedure and an Example

A rational design procedure should allow the designer to consider different aspects of a control system performance so that tradeoffs among system properties such as performance robustness, stability robustness, transient response, noise attenuation, etc., can be handled rationally during the design. With help of the regeneration spectrum and the other frequency domain measures discussed above, a rational design procedure is developed and proposed in this section.
3.5.1 Overview of the Design Procedure

As a prelude to the design of repetitive control systems, the tradeoffs inherent in repetitive control system design are summarized here. The sensitivity and complementary sensitivity functions for the system in the absence of repetitive control play an important role in the repetitive control system performance, as indicated by the equations (3.42) and (3.46). Consequently, the compensated plant characteristic $G_p(j\omega)$ should be determined using the available degrees of design freedom effectively using classical procedures such as those described by Horowitz (1963). It is also important that the closed loop frequency response for the system without repetitive control action be flat and close to unity in magnitude for a wide range of frequencies. This requirement is motivated by equation (3.27) which indicates that the regeneration spectrum will then be lower in magnitude indicating an improved degree of stability. It should be noted, however, that the accuracy of the repetitive control system is not tied too closely to the closed loop bandwidth of the system in the absence of repetitive control.

Following the design of the compensated plant $G_p(s)$, the repetitive control system is to be designed and requires the selection of the design parameters $q(s)$ and $b(s)$. $q(s)$ is chosen primarily based on the accuracy requirements, using equation (3.43) as a guideline. $b(s)$ is then chosen, based on the nominal value of $G_p(j\omega)/(1+G_p(j\omega))$, to minimize the peak of the regeneration spectrum (3.27). The resulting robustness to unstructured multiplicative uncertainties and the noise attenuation in the high frequency range can be examined using the complementary sensitivity function, equation (3.46) and the simplified robustness result (3.53). Some iterations in the choice of $q(s)$ and $b(s)$ will probably be necessary to modify system accuracy, stability and robustness as desired.
3.5.2 Design of the Precompensator

Consider the precompensator design shown in Figure 3.19.

![Precompensation of the System Without Repetitive Control Action](image)

Figure 3.19 Precompensation of the System Without Repetitive Control Action

The uncompensated plant \( G_{uc}(s) \) usually has uncertainty,

\[
G_{uc}(s) = G_{w0}(s)(1 + \Delta(s)) \tag{3.54}
\]

where \( \Delta(s) \) is the multiplicative uncertainty of the uncompensated plant. It is shown in Appendix A.3 that the multiplicative uncertainty \( \Delta_G \) of the closed loop control system is much smaller than the multiplicative uncertainty \( \Delta \) of the open loop control system within the bandwidth of the closed loop system.

\[
|\Delta_0| = |\Delta||1 - G_o| << |\Delta| \tag{3.55}
\]
It should be noted that the bandwidth of $G_0(s)$ is limited by the stability robustness criterion

$$|G_0(j\omega)| < \frac{1}{|\Delta(j\omega)|} \quad (3.56)$$

It is clear that the uncertainty of the uncompensated plant can be reduced by feedback control. Thus one objective of precompensator design is to reduce uncertainty in the system, for example, by making $\Delta_Q(s)$ small by keeping $G_0(s)$ very close to unity without violating the stability robustness condition (3.56). It is also desirable that $G_0(s)$ have a flat magnitude response and a linear phase shift versus frequency since such a $G_0$ can be compensated effectively by $b$, reducing the regeneration spectrum $R(\omega)$ as indicated in section 3.3.1.

3.5.3 Design of Repetitive Controller Parameters $b$ and $q$

Considering multiplicative uncertainty (3.48) in the system, the regeneration spectrum (3.27) is modified to

$$R(\omega) = |q(j\omega)(1-b(j\omega)G_0(j\omega)(1+\Delta_0(j\omega)))| \quad (3.57)$$

$b(s)$ is chosen to compensate the nominal $G_0(j\omega)$ in equation (3.48), so that the regeneration spectrum can be as small as possible. If perfect compensation of the nominal plant is chosen, $bG_0$ is equal to unity and the regeneration spectrum becomes

$$R(\omega) = |q(j\omega)(-\Delta_0(j\omega))| \quad (3.58)$$
Thus, the regeneration spectrum would be determined by the multiplicative uncertainty in $G_0$. The development in the preceding section indicates that $\Delta_0$ is low in magnitude within the bandwidth of $\rG_0$. It would however be higher at high frequencies and could result in the regeneration spectrum exceeding unity at these frequencies, indicating potential instability. Note that $\rq(j\omega)$ must be unity for improved accuracy, according to the equation (3.43). It can be made low at high frequencies to improve the robustness of the system, at the expense of accuracy. Of course, the inverse of $\rG_0$ may not be exactly realizable either, in which case the regeneration spectrum is given by the more general expression (3.57). The preceding development indicates that making $b$ equal to the inverse of $\rG_0$ places additional requirements on $\rq(j\omega)$. It seems more preferable, however, to place the burden of robustness improvement as a constraint on the selection of $b(j\omega)$ as well as of $\rq(j\omega)$. $b(j\omega)$ will therefore be a low pass filter which compensates for $\rG_0(j\omega)$ in the lower frequency range.

The choice of $\rq(j\omega)$ is constrained by the need to guarantee the stability of the repetitive control system while keeping $q$ as close to unity as possible for improved accuracy. The accuracy is determined by the sensitivity function (3.43) at the discrete frequencies, i.e.,

$$S(s_n) = \left( \frac{1}{1 + \rG_p(s_n)} \right) \frac{1 - \rq(s_n)}{1 - \rR(s_n)}, \quad s_n = \frac{2\pi n j}{T_D}$$

(3.59)

Note that the accuracy also depends on $\rR(s_n)$, so a better design of $b$, making the regeneration spectrum small, also improves the system accuracy. Thus $\rq(j\omega)$ will also be a lowpass filter with as nearly flat an amplitude ratio characteristic as possible.
Finally, it should be remembered that the lower the regeneration spectrum, the faster the error decays to zero and the better the transient response. The burden of achieving a low $R(\omega)$ at the lower frequencies falls primarily on $b(j\omega)$.

3.5.4 A Design Example

As an example of repetitive control system design, an application to closed loop servohydraulic material testing is considered. Periodic reference inputs are encountered often in such applications, for example, fatigue testing. Since these reference inputs are electrically synthesized, the period of the input signal is precisely known, which is essential for successful application of repetitive control.

Servohydraulic material testing systems have some characteristic features which make accurate closed loop control at high input signal frequencies difficult. Hydraulic systems are lightly damped because of the absence of inherent mechanisms for energy dissipation other than friction. Furthermore, the hydraulic resonant frequency resulting from load inertia and the fluid compliance is usually not well known because of the dependence of the fluid compliance on factors such as air entrapment, and hose/tubing compliance which depend on operational practices and are not easily characterized in practice. The hydraulic resonant frequency also changes with the piston position, in the case of linear actuators (Merritt, 1967).

The dynamic model of the servohydraulic system to be considered here is a linearized model of a system used in prior analytical and experimental work on servohydraulic material testing (Lee and Srinivasan, 1989a; Lee and Srinivasan 1989b). Closed loop control of piston position is considered here, the uncompensated plant transfer function being given by
Nominal parameter values and uncertainties in parameter estimates are given in Table 3.1. \(X_p\) is the piston position and \(u\) the servoamplifier input. Nonlinear aspects of servovalve operation are important in general, but are omitted from explicit consideration here since the emphasis is on the use of analytical tools such as the regeneration spectrum for the analysis and design of repetitive control systems in general.

Table 3.1  Plant Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau_{nv})</td>
<td>0.00187 ± 0.00019 sec</td>
</tr>
<tr>
<td>(\omega_h)</td>
<td>618 ± 62 rad/sec</td>
</tr>
<tr>
<td>(\zeta_h)</td>
<td>0.4 ± 0.1</td>
</tr>
</tbody>
</table>

A 62.83 rad/sec (10 Hz) reference input signal is chosen for consideration as the input signal frequency. It is a significant fraction of the hydraulic resonant frequency of 618 rad/sec (nearly 100 Hz). In fact, a triangular input signal has significant signal content even at five times the fundamental frequency, which would be nearly one-half the hydraulic resonant frequency in the above case. A peak error of less than five percent is sought here as it represents a level of accuracy difficult to achieve without repetitive control. Along with the accuracy specification given here, it is desirable to maintain system stability over the range of plant parameter variations indicated in Table 3.1.
order to get an idea of the desired stability robustness, the multiplicative uncertainty in the uncompensated plant model is computed as follows.

\[ \Delta(s) = \frac{G_p'(s) - G_p(s)}{G_p(s)} \]  

where \( G_p'(s) \) and \( G_p(s) \) being the exact and nominal uncompensated plant transfer functions respectively. \( \Delta(s) \) thus computed is stable and satisfies the inequality

\[ |\Delta(j\omega)| \leq M(\omega) \quad \forall \omega \]  

(3.62)

\( M(\omega) \) is the upper bound determined by explicit computation of \( \Delta(s) \) for a number of combinations of parameter values and is shown graphically in Figure 3.20. If the complementary sensitivity function satisfies

\[ |T(j\omega)| \leq \frac{1}{M(\omega)} \quad \forall \omega \]  

(3.63)

the closed loop system is stable for the range of parameter variations assumed in the calculation of \( M(\omega) \) (Doyle and Stein, 1981).

The first step in the control system design is the design of the compensated plant \( G_p(s) \). For the lightly damped electrohydraulic system, it is known that cascade compensation in the form of phase lag or phase lead compensation is not effective and neither is velocity feedback. Dynamic pressure feedback involving feedback of the differential pressure across the actuator is known to be effective (Doebelin, 1985), but is not pursued here. Rather, simple proportional control is employed and the gain \( K \) in
Figure 3.20 Upper Bound on Multiplicative Uncertainty in Plant Model
equation (3.44) chosen such that the effective damping ratio of the closed loop system in Figure 3.19 is 0.707. This criterion also results in a closed loop phase shift characteristic that is nearly linear with frequency, a feature that is shown to be definitely advantageous in subsequent stages of the repetitive control system design. The resulting value of $K$ is $150 \text{ sec}^{-1}$ and the closed loop bandwidth for the system in Figure 3.19 is 400 rad/sec. The linear phase shift characteristic of the closed loop system in Figure 3.19 results in an equivalent time delay of 0.007 second.

The regeneration spectrum is applicable to this example since the system in Figure 3.19 is stable. Furthermore, the condition (3.5) of a large time delay is satisfied since the time delay is 0.1 second and $\alpha_{\text{max}}$ for the closed loop system in Figure 3.19 is 235 rad/sec. Note that the dominant poles of $G(s)$ are also the dominant poles of $R(s)$, since the poles of $q(s)$ is generally to the left of the $G(s)$. Since $G_p/(1+G_p)$ has a flat amplitude characteristic and a nearly linear phase characteristic at low frequencies, the regeneration spectrum (3.27) is very effectively reduced if

$$b(s) = e^{\tau_b s} = e^{0.007 s} \quad (3.64)$$

the magnitude of the time advance $\tau_b$ being chosen to correspond to the equivalent time delay for $G_0$. Such a time advance to compensate for plant delays has been previously employed by Arimoto et al (1984) in learning control systems and by Tsao and Tomizuka (1988) in repetitive control systems. However, the relationship of the time advance to the compensated plant transfer function for effective stability improvement has been clarified here for the first time, based on the use of the regeneration spectrum. Note also that the above choice of $b(s)$ attempt to compensate only for the phase of $G$ and not for the amplitude. Consequently, $|b(j\omega)G(j\omega)|$ goes to low values at high frequencies.
simply due to the characteristics of $G(j\omega)$. This choice of $b(s)$ therefore does not improve robustness but does not degrade it either.

The regeneration spectrum (3.27) and the robustness equation (3.53) indicate the need for a low pass characteristic of $q(j\omega)$ for improved stability and robustness to multiplicative uncertainties, whereas equation (3.59) indicates that a broader band $q(j\omega)$ would result in improved accuracy and sensitivity. The following form of $q(s)$ is used here.

$$q(s) = \frac{e^{\tau_q s}}{\left(\frac{s^2}{\omega_q^2} + \frac{2\zeta_q s}{\omega_q} + 1\right)}$$  \hspace{1cm} (3.65)

A damping ratio $\zeta_q$ of 0.707 is chosen to ensure a flat amplitude characteristic and a linear phase shift characteristic of the denominator term alone. The time advance $\tau_q$ is chosen to equal the effective time delay resulting from the denominator term, namely, $2\zeta_q/\omega_q$. Such a form of $q(s)$ ensures that the minima of the sensitivity function (3.43) occur almost exactly at integral multiples of the input signal fundamental frequency $2\pi/T_D$. The natural frequency $\omega_q$ is chosen here as 1000 rad/sec. Larger value of $\omega_q$ result in improved accuracy but degrade stability robustness.

The repetitive control system is shown in Figure 3.21. Note that the implementation of the time advance in $b(s)$ and $q(s)$ is convenient since the time delay $T_D$ in the repetitive controller exceeds these time advances in magnitude. The regeneration spectrum for the system is shown in Figure 3.22. The repetitive control system is stable since the peak of the regeneration spectrum is well below unity. The sensitivity function for the control system is given in Figure 3.23, and indicates that the sensitivity function magnitude is very small at integral multiples of the input frequency. In fact, it is smaller than 0.04 at the sixth harmonic. The sensitivity function magnitude in the absence of
Figure 3.21 Example Design of Repetitive Control System

Figure 3.22 Regeneration Spectrum for Repetitive Control System
Figure 3.23 Sensitivity Function Magnitude With and Without Repetitive Control Action

Figure 3.24 Complementary Sensitivity Function Magnitude With and Without Repetitive Control Action
Figure 3.25 Repetitive Control System Error Signal - Sinusoidal Input

Figure 3.26 Repetitive Control System Error Signal - Triangular Input
repetitive control action is indicated by the dashed line and illustrates clearly the
degradation in sensitivity and accuracy of the repetitive control system at intermediate
frequencies, such degradation being needed to achieve the improved performance at
integral multiples of the fundamental frequency of the input signal. The complementary
sensitivity function for the control system is given in Figure 3.24, and shows a peak
value of nearly 2. Comparison of Figures 3.23 and 3.20 indicates that equation (3.47) is
satisfied, ensuring stability of the system in the presence of the parameter variations of
Table 3.1. The dashed line curve in Figure 3.24 indicates the complementary sensitivity
function magnitude in the absence of repetitive control. In general, the complementary
sensitivity function magnitude is higher for the repetitive control system at the higher
frequencies, illustrating another aspect of the price to be paid for the improved accuracy
of repetitive control systems at specific frequencies.

Figure 3.25 shows the error $e(t)$ for the repetitive control system in Figure 3.21
for a sinusoidal signal of unit amplitude and 62.83 rad/sec frequency, obtained by
simulation. The error transients decay rapidly, within two cycles of the input signal, as a
result of the good relative stability indicated by the regeneration spectrum. The steady
error level of one percent is consistent with the sensitivity function in Figure 3.23.
Figure 3.26 shows the error signal $e(t)$ for the same system for a triangular input signal
of unit amplitude and the same frequency as before, again obtained by simulation. The
error level is higher, about 3%, because of the significance of the input signal harmonics,
but is good considering that these harmonics occur at frequencies comparable to the
hydraulic resonant frequency. Table 3.2 shows the sensitivity function values (3.59) and
the corresponding error components at odd harmonic frequencies. The sum of the error
signal components listed in the table is 1.34%. The repetitive control system is unable to
track higher frequency components, above 816 rad/sec, which account for additional
1.84% error. It is clear that equation (3.59) can predict steady-state error of the repetitive control system. The error transients decay rapidly as a consequence of the good relative stability of the repetitive control system. The values of the regeneration spectrum are 0.024 and 0.042 at the signal frequency 62.83 rad/sec and first harmonic frequency 125.6 rad/sec respective. The error transient is therefore attenuated rapidly. Finally, it should be noted that the non-zero steady-state error in these figures results from the fact that the frequency response $q(j\omega)$ is not exactly one, as indicated also by equation (3.59). Since higher frequency harmonics are present in the triangular signal, the low pass nature of $q(j\omega)$ resulted in a higher steady state error level.

Table 3.2

<table>
<thead>
<tr>
<th>Frequency rad/sec</th>
<th>Reference Signal Components</th>
<th>Sensitivity Function</th>
<th>Error signal Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>62.8</td>
<td>0.8106</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>188</td>
<td>0.0901</td>
<td>0.0039</td>
<td>0.0004</td>
</tr>
<tr>
<td>314</td>
<td>0.0324</td>
<td>0.0296</td>
<td>0.0010</td>
</tr>
<tr>
<td>440</td>
<td>0.0165</td>
<td>0.1099</td>
<td>0.0018</td>
</tr>
<tr>
<td>565</td>
<td>0.0100</td>
<td>0.2909</td>
<td>0.0029</td>
</tr>
<tr>
<td>690</td>
<td>0.0067</td>
<td>0.5438</td>
<td>0.0036</td>
</tr>
<tr>
<td>816</td>
<td>0.0048</td>
<td>0.7612</td>
<td>0.0037</td>
</tr>
</tbody>
</table>

* Sum of error components = 1.34%
In order to show the usefulness of the regeneration spectrum, the repetitive controller parameters b(s) and q(s) are modified to provide a better understanding the relationship of the regeneration spectrum to the features of the repetitive control system response. First, b(s) is changed as indicated below.

\[ b(s) = e^{0.005s} \]  \hspace{1cm} (3.66)

while using the same q(s) as in equation (3.65). The regeneration spectrum for the modified repetitive control system is shown in Figure 3.27. The new b(s) does not have enough phase advance to compensate for the phase lag of G(s), so the relative stability is poorer than the original case. The maximum value of the regeneration spectrum is increased from 0.80 to 0.86. Figure 3.28 and Figure 3.29 show the sensitivity and the complementary sensitivity functions of the repetitive control system. The sensitivity function has higher peaks than the one shown in Figure 3.23. The values of the sensitivity function at signal frequencies are about the same as those in Figure 3.23. The complementary sensitivity function also has higher peaks than those in Figure 3.24. Some of the peaks violate the stability criterion (3.63), indicating possible instability of the system with the unstructured uncertainty listed in Table 3.1. However, extensive simulation of the repetitive control system, with quite a few possible combinations of parameter values listed in Table 3.1, does not show the existence of instability, though the error transient is more oscillatory in some cases. This indicates that equation (3.63) is very conservative stability condition. When the stability of the repetitive control system is checked using equation (3.27), the regeneration spectrum value is found to be less than unity for all the plant parameter variations listed in Table 3.1. Figure 3.30 shows the
Figure 3.27 Regeneration Spectra for the Repetitive Control Systems
Figure 3.28 Sensitivity Function Magnitude of the Repetitive Control System, 
\( \tau_b = 0.005 \text{ sec}, \omega_q = 1000 \text{ rad/sec} \)

Figure 3.29 Complementary Sensitivity Function Magnitude of the Repetitive Control System, 
\( \tau_b = 0.005 \text{ sec}, \omega_q = 1000 \text{ rad/sec} \)
Figure 3.30 Repetitive Control System Error Signal,
\( \tau_b = 0.005 \text{ sec}, \omega_q = 1000 \text{ rad/sec} \)

Figure 3.31 Repetitive Control System Error Signal,
\( \tau_b = 0.007 \text{ sec}, \omega_q = 500 \text{ rad/sec} \)
Figure 3.28 Sensitivity Function Magnitude of the Repetitive Control System, $\tau_b=0.007$ sec, $\omega_q = 500$ rad/sec

Figure 3.29 Complementary Sensitivity Function Magnitude of the Repetitive Control System, $\tau_b=0.007$ sec, $\omega_q = 500$ rad/sec
error transient of the repetitive control system with the same triangular reference signal used in Figure 3.26. The transient response is poorer than that in the Figure 3.26, which agree with the higher regeneration spectrum values at the signal frequencies of 62.83 rad/sec and the first few harmonic frequencies. The steady state accuracy is about the same.

In the second case, the parameter \( b(s) \) is set to be the same as in equation (3.64), but the bandwidth in \( q(s) \) is reduced from 1000 rad/sec to 500 rad/sec.

\[
q(s) = \frac{e^{0.0014s}}{s^2 + \frac{1.4s}{500} + 1}
\]

(3.67)

The resulting regeneration spectrum of the repetitive control system is shown in Figure 3.27. The regeneration spectrum values are not changed much at frequencies below 300 rad/sec but are largely reduced at high frequencies, the maximum value being reduced from 0.8 to 0.43. Hence, this system has better stability robustness. Figure 3.31 shows the error transient of the repetitive control system for the same reference signal used in Figure 3.26. The transient response is similar to that in Figure 3.26, since the regeneration spectrum values are similar at frequencies below 300 rad/sec, approximately five times the signal frequency. The steady state accuracy is poorer because \( q(s) \) has a smaller bandwidth. A accuracy is thus sacrificed for better stability robustness. Figure 3.32 and Figure 3.33 show the sensitivity and the complementary sensitivity functions of the repetitive control system respectively. The sensitivity function here is quite similar to that in Figure 3.23, except that the values at the signal frequencies are higher than those in Figure 3.23. This verifies the poor steady state accuracy shown in Figure 3.31. The complementary sensitivity function is also similar to that in Figure 3.24 at low
frequencies. However, the values of the complementary sensitivity function are lower than those in Figure 3.24 at high frequencies, indicating better stability robustness.

3.6 Summary

The regeneration spectrum for a class of time delayed systems has been introduced here along with the derivation of quantitative criteria for the time delay to be considered large. Alternative interpretations of the regeneration spectrum have also been offered to provide deeper insight into the use of the regeneration spectrum.

The usefulness and the validity of the regeneration spectrum as a relative stability measure for continuous-time SISO repetitive control systems has been established here. It has been shown that the condition of a large time delay, needed for effective use of the regeneration spectrum, is usually satisfied for repetitive control systems. Important features of the characteristic root distribution of the system are shown to be accurately inferred from the regeneration spectrum for such systems, with a great saving in computational effort. When combined with other frequency domain measures for control system performance, such as the sensitivity function and the complementary sensitivity function, improved insight is obtained into the trade-offs inherent in repetitive control system design. The result of such improved insight into design trade-offs is a more rational approach to repetitive control design, as shown by an example above.
CHAPTER IV
MULTI-INPUT-MULTI-OUTPUT REPETITIVE CONTROL SYSTEMS

4.1 Introduction

The regeneration spectrum concept is extended here to continuous-time MIMO systems. The control issues to be addressed here are the development of a relative stability measure for MIMO repetitive control systems, and the use of this performance measure along with other available measures of MIMO control system performance to develop guidelines for MIMO repetitive control system design.

A MIMO repetitive control system structure is introduced first, followed by stability analysis of MIMO repetitive control systems. Regeneration spectra of MIMO repetitive control systems are defined and used as a measure of relative stability of such systems for 'large' values of the time delay. Guidelines for MIMO repetitive controller design are formulated based on the regeneration spectra and other performance measures, such as sensitivity and complementary sensitivity functions, and applied to an example system.

4.2 A MIMO Repetitive Control System Structure

A MIMO repetitive control system structure is shown in Figure 4.1. This structure is an extended version of the SISO repetitive control system structure discussed in Chapter 3. $G_p(s)$ is the compensated plant transfer matrix.
where \( G_{uc}(s) \) is the uncompensated plant transfer matrix and \( K(s) \) is the precompensator. \( G_p(s) \) is assumed to be square. If the plant \( G_{uc}(s) \) is not square, \( K(s) \) can be chosen with appropriate dimensions to make \( G_p(s) \) a square matrix. The system matrix

\[
T_0(s) = (I + G_p(s))^{-1} G_p(s)
\]

is called the system matrix of the pre-existing control system. The pre-existing control system is the control system with the repetitive control block within the dashed lines removed. \( T_0 \) is restricted to be stable, i.e. the control system without repetitive control action is stable. \( Q(s) \) and \( B(s) \) are square rational transfer matrices and are parameters of
the repetitive controller. They are assumed here to be stable, that is, analytic in the closed right half complex s-plane, and proper, that is, finite as $s \to \infty$. The assumptions stated here are reasonable since a stable pre-existing control system $T_0$ and stable controller parameters $B$ and $Q$ are desirable. The MIMO repetitive control system structure described here is more general than that considered by Hara et al (1988), in that $Q(s)$ and $B(s)$ are not restricted to be diagonal matrices. Furthermore, the parametrization of the repetitive controller in the form of $Q(s)$ and $B(s)$ is also unique here and simplifies analysis and design procedures.

4.3 Stability Analysis of MIMO Repetitive Control Systems

Repetitive control systems are infinite-dimensional systems, i.e. they have an infinite number of state variables because of the presence of the time delay in the system. Feedback control theories involving finite state feedback are not suitable for systems with infinite number of states. Different approaches to control system design have to be taken in order to deal with such systems. Stability analysis techniques are needed for developing repetitive controller design procedures. In this section, two stability analysis methods, the Small Gain Theorem (Desoer and Vidyasagar, 1975) and the Generalized Nyquist Stability Criterion (Smith, 1981), are applied to repetitive control systems to obtain relevant stability conditions. In order to use the stability analysis methods, an equivalent system for the one in Figure 4.1 is derived first (see Appendix A.4) and depicted in Figure 4.2.
4.3.1 Stability Analysis Using the Small Gain Theorem

Stability analysis of MIMO repetitive control systems using the small gain theorem was proposed by Hara et al (1988). The small gain theorem is a very general theorem which can be applied to linear or nonlinear systems. The criterion states that a bounded input produces a bounded output if the open loop gain of the system is less than unity (Desoer and Vidyasagar, 1975). However, this stability criterion is only a sufficient one and it may be conservative.

Consider the loop gain of the system in Figure 4.2,

$$
\gamma(- (I - T_0(s)B(s))Q(s)) \cdot \gamma(Ie^{-T_0s}) = \sigma((I - T_0(j\omega)B(j\omega))Q(j\omega))
$$

(4.3)

where \( \gamma() \) denotes the gain of the system within parentheses. Since \( T_0, Q(s), \) and \( B(s) \) are stable, the largest singular value \( \sigma \) on the right hand side of equation (4.3) is finite. The repetitive control system is stable, according to the small gain theorem, if
\[ \sigma(R(j\omega)) < 1, \quad \forall \omega \] 

\[ (4.4) \]

where

\[ R(j\omega) = (I - T_0(j\omega)B(j\omega))Q(j\omega) \] 

\[ (4.5) \]

This result is equivalent to the result proposed by Hara et al (1988).

4.3.2 Stability Analysis Using the Generalized Nyquist Stability Criterion

A stability analysis method called the Generalized Nyquist Stability Criterion has been developed for multivariable control systems recently (Smith, 1981), and is a generalization of the classical SISO Nyquist stability analysis technique. The criterion is applied to multivariable repetitive control systems and the result leads to a less conservative stability condition than that given by the small gain theorem. Moreover, it also leads to the definition of regeneration spectra for multivariable repetitive control systems.

The characteristic equation of the closed loop system in Fig. 4.1 is obtained from the equivalent system in Figure 4.2 as

\[ \det[I - (I - T_0(s)B(s))Q(s)e^{-\tau_d s}] = 0 \] 

\[ (4.6) \]

The return ratio for the system is expressed as

\[ -R(s)e^{-\tau_d s} = -(I - T_0(s)B(s))Q(s)e^{-\tau_d s} \] 

\[ (4.7) \]
where $T_0$, $B$ and $Q$ are exponentially stable, that is, proper and having no Smith-McMillan poles in the closed right half $s$-plane (Maciejowski, 1989). This restriction ensures the absence of any hidden unstable modes. Also, the return ratio (4.7) will then have no poles in the closed right half $s$-plane. For such a system, if there are no hidden unstable modes, the Generalized Nyquist Stability Criterion indicates that the closed loop system is stable if and only if

(i) the characteristic loci do not pass through the $(-1, j0)$ point on the complex plane and

(ii) the number of encirclements of the $(-1, j0)$ point by the characteristic loci is zero.

The characteristic loci are graphs of the eigenvalues of $-R(s)e^{-T_0s}$ in equation (4.7) as $s$ goes around the Nyquist contour encircling the closed right half $s$-plane. They are therefore obtained by rotating the characteristic loci of $-R(s)$ by $e^{-T_0s}$.

The result above suggests the following sufficient condition for stability of the repetitive control systems of interest. If $T_0$, $B$ and $Q$ are as indicated above, the repetitive control system in Figure 4.1 is stable if

\[
|\lambda_i(R(j\omega))| < 1 \quad \forall \omega \quad , \quad i = 1, n_s
\]  

(4.8)

where $n_s$ is the order of $R$. $\lambda_i(R(j\omega))$ denote the eigenvalues of $R(j\omega)$, and their magnitudes are the characteristic gains of $R(j\omega)$. The inequality (4.8) ensures that the characteristic loci of $R(j\omega)e^{-j\omega T_0}$ do not pass through or encircle the $(-1, j0)$ point, since the multiplying factor $e^{-j\omega T_0}$ merely rotates the eigenvalues of $R(j\omega)$ clockwise by $\omega T_0$ radians.
The condition (4.8) is less conservative than the stability condition (4.4) reported by Hara et al (1988), if we recall that the maximum singular value of a square matrix is always greater than or equal to the maximum eigenvalue magnitude (Maciejowski, 1989).

4.4 Regeneration Spectra and Relative Stability of Repetitive Control Systems

The second stability condition discussed above is a generalization of the result for SISO repetitive control systems. The simplification in stability analysis for 'large' time delay values noted in Chapter 3 holds here as well. For large time delay values, the sufficient condition (4.8) becomes close to a necessary condition as well. This simplification would be indicated here by the fact that the multiplying factor $e^{-j\omega T_d}$ results in rotation of the characteristic loci of $R(j\omega)$ by $360^\circ$ for an increment in frequency $\Delta \omega$ given by

$$\Delta \omega = \frac{2\pi}{T_d}$$

(4.9)

The requirement of a 'large' time delay ensures simply that over this frequency increment the characteristic gains of $R(j\omega)$ change by small amounts. In such a case, violation of the condition (4.8) would also result in enclosure of the $(-1, j0)$ point by the characteristic loci $\lambda_i(R(j\omega)e^{-j\omega T_d})$, indicating closed loop system instability.

The characteristic gains $|\lambda_i(R(j\omega))|$ are therefore referred to as the regeneration spectra for multivariable repetitive control systems. The simplification of the relative stability analysis results from the fact that the closed loop pole locations and transient response features can be quite accurately predicted from the characteristic gains
as indicated below. Consequently, synthesis and design procedures for MIMO repetitive control systems can be derived more rationally.

4.4.1 Regeneration Spectra and Characteristic Roots

The regeneration spectra for the MIMO repetitive control system shown in Figure 4.1 are defined by the characteristic gains,

\[ |\lambda_i(R(j\omega))| = |\lambda_i((I - T_0(j\omega)B(j\omega))Q(j\omega))| \] (4.10)

The regeneration spectra are related simply to the characteristic root locations of the multivariable repetitive control system, for large values of time delay. Such a correlation has been shown for SISO repetitive control systems in Chapter 3 and is of great practical value here also as it enables pole placement, roughly speaking, by shaping the regeneration spectra.

The large time delay condition in effect results in closed loop pole locations which are much closer to the imaginary axis than are the poles of \( R(s) \). Equation (4.6) for the characteristic roots simplifies therefore to

\[ \det[I - e^{-T_0(\alpha_k+j\omega_k)}R(j\omega_k)] = 0 \] (4.11)

if \( R(\alpha_k+j\omega_k) \) is replaced by \( R(j\omega_k) \), \( \alpha_k+j\omega_k \) indicating the closed loop pole locations. Comparing equation (4.11) to that for the characteristic loci \( |\lambda_i(R(j\omega))| \),

\[ \det[I\lambda_i - R(j\omega)] = 0 \] (4.12)
we get

\[ |\lambda_i(R(j\omega_k))| \equiv |e^{-T_D(\alpha_k+j\beta_k)}| = |e^{T_D(\alpha_k)}| \quad \text{(4.13)} \]

or

\[ \alpha_k = \frac{1}{T_D} \ln |\lambda_i(R(j\omega_k))| \quad \text{(4.14)} \]

Of course, since \( \omega_k \) is not known, the practical usefulness of the result (4.14) is that all the characteristic roots map closely on the regeneration spectra \( |\lambda_i(R(j\omega))| \), the spacing of these roots being approximately \( 2\pi / T_D \) on the \( \omega \)-axis. Note in this case that there will be \( n_s \) regeneration spectra, \( n_s \) being the number of inputs (or outputs) of the square plant \( G(s) \). Equation (4.14) indicates that the regeneration spectra for a MIMO repetitive control system should be less than unity in magnitude at all frequencies, lower magnitudes indicating characteristic root locations farther to the left of the imaginary axis in the complex plane and a higher degree of system stability.

Quantitative estimates of discrepancy and the normalized discrepancy between the actual pole location and the estimated one from the regeneration spectra can be obtained, in a manner similar to that described in conjunction with equation (3.11) and equation (3.12).

\[
\alpha - a(\beta) = \frac{1}{T_D} \ln \left( \frac{|\lambda_i(R(\alpha+j\beta))|}{|\lambda_i(R(\beta))|} \right) \equiv \frac{1}{T_D} \ln \left( \frac{|\lambda_i(R(a+j\beta))|}{|\lambda_i(R(\beta))|} \right) \quad \text{(4.15)}
\]

\[
\frac{\alpha - a(\beta)}{\alpha} = \frac{1}{T_D a(\beta)} \ln \left( \frac{|\lambda_i(R(\alpha+j\beta))|}{|\lambda_i(R(\beta))|} \right) \equiv \frac{1}{T_D a(\beta)} \ln \left( \frac{|\lambda_i(R(a+j\beta))|}{|\lambda_i(R(\beta))|} \right) \quad \text{(4.16)}
\]
where \(a(\beta)\) is the estimation of \(\alpha\),

\[
a(\beta) = \frac{1}{T_0} \ln(\lambda_1(\mathbf{R}(\beta)))
\]

(4.17)

In the MIMO case, each eigenlocus is used separately to obtain the discrepancy and the normalized discrepancy for each chain of closed loop poles. Note that there are \(n_s\) chains of closed loop poles for MIMO systems, as will be shown in a latter section, \(n_s\) being the number of inputs or outputs. Note also that the approximate discrepancy and normalized discrepancy in equations (4.15) and (4.16) are generally conservative estimates if the zeros of \(\mathbf{R}(s)\) are far to the left of poles of \(\mathbf{R}(s)\) is the s-plane as indicate in section 3.2.4.

4.4.2 Transient Response of the System

The transient response of the MIMO repetitive control system is determined directly by the real part of the dominant poles of the closed loop system and hence can be inferred from the regeneration spectra. A frequency domain analysis of the change in the impulse response of the repetitive control system, similar to the one in section 3.3.2, can be performed for the MIMO case. A unit impulse disturbance input is applied to the repetitive control system in Figure 4.1 and the output response of the system is examined. The Laplace transform of the output response is equivalent to the sensitivity function of the system (see Appendix A.4).

\[
h(s) = S(s) = \left[ I - (I - T_0(s)B(s))Q(s)e^{-Ts} \right]^{-1}S_0(s)(I - Q(s)e^{-Ts})
\]

(4.18)
Note that \( h(s) \) is a matrix, each column vector being the output response vector of the system due to a unit impulse disturbance at the corresponding input channel or axis. The response matrix \( h(s) \) has the power series expansion,

\[
h(s) = [I + R(s)e^{-T_0s} + R^2(s)e^{-2T_0s} + R^3(s)e^{-3T_0s} + \ldots] S_0(s) (I - Q(s)e^{-T_0s})
\]

\[
= S_0(s) - [I + R(s)e^{-T_0s} + R^2(s)e^{-2T_0s} + R^3(s)e^{-3T_0s} + \ldots] S_0(s) (B(s)T_0(s)Q(s)) \tag{4.19}
\]

where

\[
S_0(s) = (I + G_p(s))^{-1} \tag{4.20}
\]

Since \( T_0(s) \), \( Q(s) \) and \( B(s) \) are stable, \( R(s) \) is analytical in the right half \( s \)-plane. If the regeneration spectra, or the characteristic gains of \( R(s) \), are less than one, the region of convergence of the power series in (4.19) includes the imaginary axis in the \( s \)-plane and hence the inverse Laplace transform of equation (4.19) exists. Therefore, the impulse response function (4.19) can be expressed in time domain, using inverse Laplace transforms, as

\[
h(t) = h_0(t) - \sum_{i=1}^{\infty} r_i(t - iT_0) \tag{4.21}
\]

where

\[
h_0(t) = \mathcal{L}^{-1}[S_0(s)] \tag{4.22}
\]

and

\[
r_i(t) = \mathcal{L}^{-1}[R^{i-1}(s)S_0(s)B(s)T_0(s)Q(s)] \equiv \mathcal{L}^{-1}[R^{i-1}(s)S_0(s)] \tag{4.23}
\]
Note that equations (4.21)-(4.23) are extensions of equations (3.38)-(3.40). The approximation in equation (4.23) is appropriate in the low frequency range, because $B(s)$ is chosen to compensate $T_0(s)$ and $Q(s)$ is typically a lowpass filter as explained in Section 4.6. It is clear that $R(s)$ determines the attenuation rate of each successive signal function $r_1(t)$. Therefore, if the regeneration spectra of the repetitive control system are small, fast transient response can be expected for the system.

4.5 Frequency Domain Analysis of MIMO Repetitive Control Systems

Frequency domain measures of repetitive control system performance will be used here as they are compatible with the frequency domain based regeneration spectra. The sensitivity function evaluated at the plant output is obtained after some algebraic manipulations as

$$S(s) = [I - (I - T_0(s)B(s))Q(s)e^{-T_0s}]^{-1}S_0(s)(I - Q(s)e^{-T_0s})$$

(4.24)

where $S_0(s)$ and $T_0(s)$ are the sensitivity function and the closed loop transfer matrix of the system in the absence of repetitive control and are expressed by equations (4.20) and (4.2) respectively. The sensitivity function is used here as a measure of the rejection of disturbances at the plant output, and of tracking accuracy, for the unity feedback system in Figure 4.1. Low maximum singular values of the sensitivity function are desired, especially at the low frequencies where these aspects of system performance are important (Freudenberg and Looze, 1988). The maximum singular values of the sensitivity functions with and without repetitive control are related by

$$\overline{\sigma}(j\omega) \leq \overline{\sigma}_1(j\omega)\overline{\sigma}_2(j\omega) / \sigma_1(j\omega)$$

(4.25)
where \( \sigma_1 \) and \( \sigma_2 \) are the minimum and maximum singular values of the matrices \( (I-(I-T_0(B)Q_0)e^{-j\omega T_d}) \) and \( (I-Q_0e^{-j\omega T_d}) \) respectively in the equation (4.25). \( \sigma_{\text{max}} \) is the maximum singular value of \( S_0(j\omega) \). The effectiveness of repetitive control in improving the sensitivity function is indicated by the fact that, if \( Q(j\omega) \) is nearly equal to the identity matrix at low frequencies, the singular values \( \sigma_2 \) are much smaller than unity at integral multiples of the input signal frequency. It will be pointed out subsequently that \( \sigma_1(j\omega) \) is nearly unity at low frequencies, as a result of the design criterion for the parameter \( B \). Thus, at integral multiples of the input signal frequency, the singular values \( \sigma_1(j\omega) \) will be much smaller than the singular values \( \sigma_{\text{max}} \), resulting in excellent disturbance rejection and tracking accuracy at these signal frequencies. It follows also that, given the near periodic nature of \( Q(j\omega)e^{-j\omega T_d} \), the singular values \( \sigma_2(j\omega) \) are also nearly periodic and reach maximum values of nearly two at odd multiples of \( \pi/T_D \). Thus, as in the SISO repetitive control case, the sensitivity function is improved at the discrete signal frequencies at the expense of the function values at other frequencies. This aspect of the tradeoff in sensitivity function values for time delayed systems in general has also been examined in greater detail by Freudenberg and Looze(1987).

The complementary sensitivity function evaluated at the plant output is obtained (see Appendix A.5) as

\[
T(s) = [I - (I - T_0(s)B(s))Q(s)e^{-\tau_s}]^{-1}T_0(s)[I - (I - B(s))Q(s)e^{-\tau_s}]
\]  
(4.26)

where \( T_0(s) \) is also the complementary sensitivity function at the plant output in the absence of repetitive control. Since the sensitivity and complementary sensitivity functions are evaluated at the same location, they satisfy
\[ T(s) + S(s) = I \]  
(4.27)

Consequently, the sensitivity function improvement is limited to low frequencies, whereas the complementary sensitivity function is improved or made "small" at high frequencies, the maximum singular values of \( T(j\omega) \) being used as a measure of size. Smaller maximum singular values of \( T(j\omega) \) ensure system stability in the presence of larger uncertainty in the plant transfer characteristics, such uncertainty being expressed as multiplicative unstructured uncertainty referred to the plant output (Doyle and Stein, 1981). Now

\[ \overline{\sigma}_T(j\omega) \leq \overline{\sigma}_{\tau}(j\omega) \overline{\sigma}_s(j\omega) / \overline{\sigma}_s(j\omega) \]  
(4.28)

where \( \overline{\sigma}_{\tau} \) and \( \overline{\sigma}_s \) are the maximum singular values of \( T_0 \) and \( (I - (I - B)Qe^{-j\omega T_0}) \) respectively. At the higher frequencies for which plant uncertainty is significant, the elements of the closed loop transfer matrix \( T_0(j\omega) \) of the system in the absence of repetitive control tend to zero in magnitude and

\[ (I - (I - T_0 B)Qe^{-j\omega T_0}) \rightarrow (I - Qe^{-j\omega T_0}) \]  
(4.29)

Consequently, \( Q(j\omega) \) becomes small at frequencies which are integral multiples of the signal fundamental frequency \( 2\pi / T_D \), and \( \sigma_T(j\omega) \) becomes large at these frequencies, if \( Q(j\omega) \) is close to the identity matrix. Thus, to maintain stability robustness to unstructured multiplicative uncertainties, \( Q(j\omega) \) must have low pass characteristics if it is a diagonal matrix. A nondiagonal matrix at these frequencies appears also to potentially
help improve robustness. It is also clear from the inequality (4.28) that low pass characteristics of B would improve robustness by lowering $\bar{\omega}_3$ at integral multiples of $2\pi/T_p$.

Examination of expressions for the sensitivity and complementary sensitivity functions at the plant output indicates therefore that low pass characteristics of the elements of $B(s)$ and $Q(s)$ will improve the stability robustness of the repetitive control system to unstructured multiplicative uncertainties in the plant transfer characteristics, such uncertainties being referred to the plant output. On the other hand, a diagonal Q matrix at low frequencies, nearly equal to the identity matrix, helps improve system accuracy. Evaluation of sensitivity and complementary sensitivity functions at the plant input and/or other characterizations of plant modeling error may lead to other guidelines for selecting the B and Q matrices.

4.6 A Design Procedure for MIMO Repetitive Control Systems

A good design procedure should allow control system designers to consider, as thoroughly as possible, the following control system properties: (1) stability robustness (2) performance robustness (3) transient response. In previous sections, it is shown that the regeneration spectra and the frequency domain measures can be used to determine these properties. A rational design procedure for MIMO repetitive control systems is proposed here to deal with the tradeoffs between these properties.

4.6.1 Overview of the Design Procedure

The tradeoffs in MIMO repetitive control system design are similar to the SISO system case. The sensitivity and complementary sensitivity functions for the system in the absence of repetitive control play an important role in repetitive control system
performance, as indicated by the equations (4.24) and (4.26). Consequently, the compensated plant characteristic $G_p(j\omega)$ can be determined using available multivariable design procedures. The closed loop frequency response for the system without repetitive control action should be flat and close to unity in magnitude for a wide range of frequencies. It should be noted, however, that the accuracy of the repetitive control system is not governed by the closed loop bandwidth of the pre-existing system.

After the design of the compensated plant $G_p(j\omega)$, the sensitivity function of the pre-existing control system is reduced. So the uncertainty in the characteristics of the uncompensated plant is reduced and the resulting variation of $T_o(s)$ is small. The repetitive control system design then requires selection of the design parameters $Q(s)$ and $B(s)$. $B(s)$ is chosen, based on nominal value of $T_o(s)$, to minimize the peak of the regeneration spectra (4.10). $Q(s)$ can be chosen primarily based on the accuracy requirements, using equation (4.25) as a guideline and the relative stability criterion (4.10). Performance robustness against plant uncertainty is usually good for repetitive control systems since high gain is used at the signal frequencies. The resulting robustness to multiplicative uncertainties and the noise attenuation in the high frequency range can be examined using equation (4.26). The stability robustness of the system can also be examined by checking the regeneration spectra. Some iteration in the choice of $Q(s)$ and $B(s)$ will probably be necessary to modify system accuracy, stability and robustness as desired.

There are a few multivariable feedback control system design methods which can be used to choose the compensator $K(s)$ for the pre-existing control system. For example, the weighted sensitivity function method along with the $H_\infty$ design procedures such as the factorization approach(Francis, 1989) or state space approach(Dailey 1990), can be used to find appropriate compensators minimizing the weighted sensitivity
functions. However, the resulting compensated system tends to have a more complex form, which complicates the subsequent design of the repetitive controller parameters, $B(s)$ and $Q(s)$.

The controller design procedure to be described here has as one of its objectives the achievement of low interaction between the inputs $R$ and the outputs $C$, or equivalently, a closed loop transfer matrix $T(s)$ in equation (4.26) with a strong degree of diagonal dominance over the bandwidth of interest. Referring to equation (4.26), one way of achieving this objective is to restrict $B$ and $Q$ to be diagonal matrices, and to choose the precompensator $K$ so that the transfer matrix $T_0$ is also strongly diagonally dominant. If, in addition to these steps, the matrix $[I - (I - T_0B)Qe^{-i\omega T}]$ is made strongly diagonally dominant, the matrix $T$ in equation (4.26) is the product of matrices which are either diagonal or strongly diagonally dominant, and hence is itself strongly diagonally dominant. Restriction of $B$ and $Q$ to be diagonal results in the factor $[I - (I - B)Qe^{-i\omega T}]$ in equation (4.26) being diagonal. Furthermore, improvement of the degree of stability of the repetitive control system requires reducing the magnitudes of the regeneration spectra, or the characteristic gains of $(I - T_0(j\omega)B(j\omega))Q(j\omega)$ to values less than unity. Therefore, the degree of diagonal dominance of $[I - (I - T_0B)Qe^{-i\omega T}]$ depends on the extent to which the selection of $B$ and $Q$ is able to make the regeneration spectrum magnitudes small.

The guideline for selection of diagonal $B$ and $Q$ matrices to reduce regeneration spectrum magnitudes can be obtained using Gershgorin's theorem (Rosenbrock, 1974). If the $i^{th}$ diagonal elements of $B$ and $Q$ are $b_i$ and $q_i$ respectively, and $(t_0)_{ij}$ is the $ij^{th}$ element of $T_0$, we can easily apply Gershgorin's theorem to locate the eigenvalues of $R$, and hence the regeneration spectra.
Gershgorin's theorem indicates that the eigenvalues $\lambda_j$ of $R(j\omega)$ satisfy

$$|\lambda_j(R(j\omega)) - (1 - (t_0)_{ij} b_j)q_j| \leq \sum_{i=1}^{n} |(t_0)_{ij} b_j q_j|$$

(4.31)

This inequality may be simplified to

$$|\lambda_j(R(j\omega))| \leq \left\| (1 - (t_0)_{ij} b_j) + \sum_{i=1}^{n} |(t_0)_{ij} b_j| \right\| |q_j|$$

(4.32)

This inequality suggests that $b_j$ should be close to $1/(t_0)_{ij}$, and that $|q_j|$ should be small, to improve the degree of system stability.

The requirements of accuracy and robustness to unstructured multiplicative uncertainties on the $B$ and $Q$ matrices can be extended from the results for SISO systems in Chapter 3. $Q$ must be chosen to be close to the identity matrix at integral multiples of $2\pi/T$ to achieve high accuracy at the signal frequencies. Also, robustness to unstructured multiplicative uncertainty at high frequencies is improved by choosing $B$ and/or $Q$ to have significant attenuation at these frequencies.

A design procedure has thus been formulated for MIMO repetitive control systems. Available multivariable controller design procedures are used to select $K$ to
achieve a strongly diagonally column dominant $T_0$. $B$ and $Q$ are restricted to be diagonal matrices, the diagonal elements of $B$ being chosen to compensate the diagonal elements of $T_0$ in amplitude and phase over a wider bandwidth to improve the degree of system stability. The diagonal elements of $Q$ are chosen to be nearly unity over the bandwidth containing the signal frequencies of interest, and to have sharp attenuation at higher frequencies for improved stability robustness.

4.6.2 Design of Precompensator

The design procedure chosen here is to design the precompensator by aligning the input and output frames at low and high frequencies so that the resulting compensated plant matrix is close to a normal matrix (Pang and MacFarlane, 1988). Normality of the system matrix implies less sensitivity of the closed loop system response to the plant matrix variation. Moreover, this approach tends to result in a diagonally dominant system matrix $T_0$, which allows simpler forms of $Q(s)$ and $B(s)$, as explained above.

The MIMO plant chosen for illustration of the design procedure is from Pang and MacFarlane (1987).

$$G(s) = \begin{bmatrix} 32.6 + 16s + 2.15s^2 & 9.4 + 4s + 1.1s^2 \\ 6.2 + 4s + 1.05s^2 & 3 + 4s + s^2 \end{bmatrix} \frac{1}{(s + 1)(s + 2)(s + 3)} \quad (4.33)$$

The precompensator designed by Pang and MacFarlane is

$$K(s) = K_\infty \left( \frac{K_s}{s} + 2I \right) \quad (4.34)$$
where

\[ K_w = \begin{bmatrix} 2.7992 & -3.0737 \\ -2.9338 & 6.0126 \end{bmatrix} \quad (4.35) \]

\[ K_o = \begin{bmatrix} -0.2559 & 10.9761 \\ -2.1494 & 15.9995 \end{bmatrix} \quad (4.36) \]

\( K_w \) and \( K_o \) are chosen to align the input and output gain frames at high and low frequencies respectively and hence make the system normal at these frequencies. The principal gains are also balanced. A more detailed description of the design can be found in the literature (Pang and MacFarlane, 1987). The resulting closed loop system response \( T_0 \) is shown in Figures 4.3 (a)-(d), and indicates that the closed loop system is diagonally column and row dominant though the interaction is not negligible. The system bandwidth is about 1.7 Hz.

### 4.6.3 Design of Repetitive Controller Parameters \( B \) and \( Q \)

Ideally, \( B \) can be chosen to be the inverse of \( T_0 \) if it is invertible. But the number of state variables in the controller parameter \( B \) would then be as large as that in \( T_0 \). Also, if \( T_0 \) is a nonminimum phase matrix, the inverse of \( T_0 \) will have unstable poles. Moreover, as indicated in section 3.5.3, inverse dynamic compensation results in the stability of the closed loop being limited by the unstructured multiplicative uncertainty in \( T_0 \), which is an undesirable result since this uncertainty is high at high frequencies. Here a simpler procedure is used to overcome these problems.

Since \( T_0 \) is designed to be diagonally dominant, the design of \( B \) and \( Q \) can be based on the diagonal entries of \( T_0 \). The \( B \) matrix is chosen to be diagonal, its diagonal
Figure 4.3 Closed Loop Response of the Control System
Without Repetitive Control Action
Figure 4.3 Closed Loop Response of the Control System Without Repetitive Control Action
Figure 4.4 Regeneration Spectra and Characteristic Roots of the Repetitive Control System

Figure 4.5 Normalized Estimation errors of Closed Loop Pole Locations for Different Time-Delay Values
elements being chosen to compensate the diagonal elements of $T_0$ respectively. The diagonal elements of $T_0$ can be compensated effectively by time advance terms as in the SISO case if the phase lag of these diagonal elements varies quite linearly with frequency. The resulting $B$ matrix has identical diagonal elements since the diagonal elements of $T_0$ for the design example are similar.

$$B(s) = e^{0.05s} \cdot I \quad (4.37)$$

$Q(s)$ is also chosen to be diagonal, each diagonal element of $Q(s)$ approximately determining the tracking accuracy of each axis in $C$. The following form ensures reasonably broadband characteristics, the time advance terms in the numerators effectively compensating for the phase shifts introduced by the denominator terms. The damping ratio of 0.7 also results in a flat amplitude ratio characteristic. The undamped natural frequency of 120 rad/sec may be viewed as a tuning parameter, higher values resulting in better accuracy but poorer robustness and degree of stability.

$$Q(s) = \left( \frac{e^{0.012s}}{s^2 + 1.4s + 1} \right) \cdot I \quad (4.38)$$

The repetitive controller is designed for a sinusoidal input of frequency 2 Hz.

### 4.6.4 Evaluation of the Controller Design

The regeneration spectra $|\lambda_1(R(j\omega))|$ for the repetitive control system are shown in Figure 4.4. The maximum value of the regeneration spectra is 0.97 indicating that the repetitive control system is stable, though the degree of stability is not very great. Since
Gershgorin's Theorem (4.31) indicates that the eigenvalues of a complex matrix lie within the Gershgorin bands for that matrix, bounds corresponding to these bands for $R(j\omega)$ are also shown in Figure 4.4, and do bound the regeneration spectra as expected. These bounds are easier to calculate and can be used instead of the actual regeneration spectra, if they are spaced closely enough. For the example of interest however, the Gershgorin bands of $R(j\omega)$ are not narrow enough and could not have been used to determine the regeneration spectra accurately enough.

The characteristic roots given by equation (4.6) are computed numerically and are also shown in Figure 4.4 as $(\omega_k, e^{\alpha_k T_D})$. The characteristic roots are computed by providing initial guesses for the root locations to routines for nonlinear equation solution. The root locations obtained thus are shown in the figure. They are seen to lie closely on the regeneration spectra and hence to agree with equation (4.13), illustrating the simplification in stability analysis for large values of the time delay. Note that there are two chains of characteristic roots, each of them mapping closely on each regeneration spectrum. The largeness of the time delay for validation of equation (4.13) can be determined quantitatively using (4.15). The normalized discrepancies (4.15) for different values of time delay are plotted in Figure 4.5. For a time delay of 0.5 sec, the normalized estimation errors (4.15) for the characteristic roots, at 13 rad/sec, are 12% and 14% respectively. For a time delay of 1.2 sec, the normalized estimation errors become 5% and 6% respectively at the same frequency. The actual normalized discrepancies between the closed loop pole locations and the corresponding regeneration spectrum values for the case of $T_D=0.5$ sec are also plotted in the same figure. It is clear that discrepancy estimate (4.15) is very close to the actual one.

The singular values of the sensitivity function $S$ for the repetitive control system are shown in Figure 4.6 and indicate very low errors (less than five percent) for signal
Figure 4.6 Singular Values of the Sensitivity Function of the System With/Without Repetitive Control Action

Figure 4.7 Singular Values of the Complementary Sensitivity Function of the System With/Without Repetitive Control Action
Figure 4.8 Error Responses of the Repetitive Control System
frequencies of up to 6 Hz, at integral multiples of 2 Hz. This is an obvious improvement over the 3-db bandwidth of $T_c$ of about 1.7 Hz. Figure 4.6 also shows the singular values of the sensitivity function $S_o$ for the system in the absence of repetitive control, and indicates higher errors. The singular values of the complementary sensitivity function $T$ are shown in Figure 4.7 and indicate that the maximum singular value is almost always less than one, and is less than 0.75 for frequencies of a little over 10 Hz. The latter limit indicates good stability robustness to unstructured multiplicative uncertainties referred to the plant output. The repetitive control action results in greater frequency selectivity of the robustness property, the robustness being specially poorer at integral multiples of the signal frequency. Also, the singular values of $T$ are generally increased at the high frequencies by the addition of repetitive control action as indicated by Figure 4.7.

Figure 4.8 shows the error responses $E$ of the repetitive control system to 2 Hz sinusoidal input signals of unit amplitude. The steady state error of about one percent is very small, as predicted by the sensitivity function in Figure 4.6. The rate of decay of the error transients is related to the values of the regeneration spectra at the signal frequency. In this case, the regeneration spectra have values of about 0.75 and 0.71 at the signal frequency of 2 Hz. The error transients in the Figure 4.8 have approximately exponential decays especially in the latter stages, the exponents of the envelopes being -0.49 sec$^{-1}$ for first axis and -0.627 sec$^{-1}$ for the second axis. Equation (4.14) predicts $\alpha$'s of -0.57 sec$^{-1}$ and -0.69 sec$^{-1}$ respectively.

In order to further illustrate the usefulness of this design approach, modifications of $B(s)$ and $Q(s)$ are made to highlight some significant features of the control system design.
Figure 4.9 Regeneration Spectra and Characteristic Roots of the Repetitive Control System

Figure 4.10 Normalized Estimation Errors of Closed Loop Pole Locations for Different Time-Delay Values
Figure 4.11 Error Responses of the Repetitive Control System
Figure 4.12 Regeneration Spectra and Characteristic Roots of the Repetitive Control System

Figure 4.13 Normalized Estimation Errors of Closed Loop Pole Locations for Different Time-Delay Values
Figure 4.14 Error Responses of the Repetitive Control System
First, the design parameter $B(s)$ in (4.37) is changed to

$$B(s) = \begin{bmatrix} e^{0.05s} & 0 \\ 0 & e^{0.03s} \end{bmatrix}$$

(4.39)

i.e. the second diagonal element has less phase advance compensation than the one in (4.37). This change reduces the magnitude of the $(2,2)$ element of $R(s)$ at high frequencies, but increases it at low frequencies. The effect of the change can be seen from the regeneration spectra of the repetitive control system as shown in Figure 4.9. Slower transient response of the repetitive control system at the lower signal frequencies but better stability robustness against uncertainty in the $(t_o)_{22}$ at higher frequencies can be expected. The latter would be important when the uncertainty in $T_o$ is known to be large on the $(2,2)$ element at high frequencies. In this figure, the two chains of characteristic roots map closely on the regeneration spectra, though they are clearly separated in this case. The normalized estimation errors of the characteristic roots for different time delay values are plotted in Figure 4.10. The estimation errors for a time delay value of 0.5sec are 11% and 9% respectively at 13 rad/sec. The actual errors for this case is also plotted in the same figure, indicating that equation (4.15) is accurate. By comparing Figure 4.9 with Figure 4.4, only one of the regeneration spectra is seen to have noticeably changed as a result of the change of $b_2(s)$. This fact is due to the diagonal dominance of $T_o$ and diagonal nature of $B$ and $Q$. Figure 4.11 shows the error responses $e_1$ and $e_2$ of the repetitive control system to 2 Hz sinusoidal input signals of unit magnitude. The regeneration spectra predict $\alpha$'s of -0.61 sec$^{-1}$ and -0.304 sec$^{-1}$ respectively. The error transient in the second axis has an exponential decay as expected, the exponent of the envelope being -0.27 sec$^{-1}$. However, the transient in the first axis does not have a simple exponential decay, due to the interaction between the two axes of the control.
system. After the interaction, it does have an exponential decay, the exponent of the envelope being \(-0.288\text{s}^{-1}\). This decay time constant is not related to the first regeneration spectrum curve but to the second regeneration spectrum curve. Note that the steady-state error is not changed much.

The effect of \(Q(s)\) can also be examined by changing \(Q(s)\). The parameter \(B(s)\) is retained as in (4.37) but \(Q(s)\) is changed to

\[
Q(s) = \begin{bmatrix}
\frac{e^{0.012s}}{s^2 + \frac{1.4s}{120} + 1} & 0 \\
0 & \frac{e^{0.006s}}{s^2 + \frac{1.4s}{40} + 1}
\end{bmatrix}
\]

(4.40)

The purpose of this change is to reduce the magnitude of the (2,2) element of \(R(s)\) at higher frequencies so that the stability robustness against uncertainty in \((\Theta)_{22}\) at high frequencies is improved. The resulting regeneration spectra of the system are shown in Figure 4.12. As expected, one of the regeneration spectrum curves is pulled down in the high frequency range by the new low pass filter \(q_2(s)\). Note that the stability robustness is improved by degrading accuracy of the second axis, instead of degrading the transient response. The normalized estimation errors (4.15) of the characteristic roots are plotted in Figure 4.13. Comparing Figure 4.13 with Figure 4.5, we find that only one family of the estimation error curves is noticeably changed in the high frequency range. The estimation errors of the dominant poles are similar to the original design, 12% and 14% respectively for time delay of 0.5\text{s}. The actual errors, also shown in the figure, are very close to the error estimates using equation (4.15). The error responses of the system are shown in Figure 4.14. The transient responses with decay time constants, -0.48
sec\(^{-1}\) and -0.64 sec\(^{-1}\), are not much different from the original design as predicted by the regeneration spectra, which is only slightly changed in the dominant low frequency range. The steady-state error in the second axis is increased due to the smaller bandwidth of the filter \(q_2(s)\), but the one in the first axis remains unchanged.

The above three design examples demonstrate the tradeoffs between system properties, when the system parameters \(B(s)\) and \(Q(s)\) are constrained to have specific forms. More flexible forms of \(B(s)\) and \(Q(s)\) can certainly improve the overall system properties.

4.7 Summary

The regeneration spectrum concept has been extended to MIMO repetitive control systems. Application of the Generalized Nyquist Criterion leads naturally to the definition of the regeneration spectra for a MIMO system. The resulting sufficient stability condition is less conservative than that derived using the small gain theorem by Hara et al (1988). Simplifications in stability analysis that result for large values of the time delay for SISO repetitive control systems are shown to occur here as well.

The regeneration spectra are used in conjunction with other measures of system performance such as the sensitivity and complementary sensitivity functions to develop design guidelines for MIMO repetitive controller design. The resulting design procedure allows the designer more insight into the tradeoffs inherent in the design process and afford a more direct manipulation of this tradeoff than other design procedures (Hara et al, 1988). The design procedure is applied successfully to an example system.
CHAPTER V

REPETITIVE CONTROL APPLICATION TO AN ELECTROHYDRAULIC
MATERIAL TESTING MACHINE

5.1 Introduction

The definition of the regeneration spectrum and its usefulness for the stability
analysis of discrete-time repetitive control systems is described first. In particular, the
relationship of the regeneration spectrum to the closed loop characteristic roots is
established, and measures of stability robustness and performance robustness based on
the regeneration spectrum determined. A repetitive control system design procedure is
described next. The application of the controller design procedure to a material testing
situation is extensively investigated, including the experimental characterization of the
system modeling error and its use in controller design. Experimental results illustrating
the nature of the improvements achievable with repetitive control are presented.

5.2 Discrete-Time Repetitive Control Systems

Repetitive control systems are effective in feedback control applications involving
periodic reference inputs or disturbance inputs. The advantages of digital implementation
of repetitive control are obvious because of the need for inclusion of a time delay in the
repetitive controller. Moreover, controller structures can be versatile and easily changed
if they are implemented by digital computers. For example, the time delay in the
repetitive controller needs to be changed according to the period of reference signal.
Figure 5.1  Repetitive Control System Block Diagram - All Digital Module

Figure 5.2  Repetitive Control System Block Diagram - Plug-in Module
Hence, some important properties of discrete-time repetitive control systems are investigated here.

5.2.1 Regeneration Spectrum for Discrete-Time Repetitive Control Systems

Block diagrams of discrete-time repetitive control systems are shown in Figure 5.1 and Figure 5.2. Figure 5.1 shows an all-digital controller, whereas Figure 5.2 shows the repetitive controller in a plug-in form where it adds to the action of an existing servomechanism, as in the application reported by Chew and Tomizuka (1990). $G_p(s)$ is the transfer function of the plant, including any analog compensation. The anti-aliasing filter (AAF) and hold have transfer function $G_a(s)$ and $G_h(s)$ respectively. $r(s)$ is a periodic input signal of period $NT_s$, where $T_s$ is the sampling interval. The parametrization of the controller in the form of the transfer functions $q(z)$ and $b(z)$ is similar to that in Chapter 3.

The closed loop characteristic equations of the systems in Figure 5.1 and Figure 5.2 are noted below.

$$z^n = q(z) \left[ 1 - b(z)Z\left( G_a G_p G_h \right) \right]$$

(5.1)

$$z^n = q(z) \left[ 1 - b(z)Z\left( \frac{G_p}{1 + G_p G_h} \right) \right]$$

(5.2)

The regeneration spectrum $R(\omega)$ is defined in terms of the right hand sides of the equations above.
An application of the Nyquist stability criterion, described in Appendix A.6, indicates that if the regeneration spectrum \( R(\omega) \) is less than unity at all frequencies, the repetitive control system is stable if the closed loop system in the absence of the repetitive control action is stable. This stability condition is thus a sufficient one. This stability condition becomes close to a necessary one when the period, \( NT_s \), is sufficiently large.

The effect of the design parameters \( q(z) \) and \( b(z) \) on the regeneration spectrum is clear from equations (5.3) and (5.4). \( b(z) \) should be chosen to compensate for the effects of the multiplying transfer functions, \( Z[G_h G_p G_a] / (1 + Z[G_h G_p G_a]) \) for Figure 5.1 and \( Z[(G_p / (1 + G_p)) G_h G_a] \) for Figure 5.2. These transfer functions essentially represent the closed loop responses in the absence of the repetitive control action. \( q(z) \) must have a low-pass characteristic to ensure that \( R(\omega) \) is low even at higher frequencies. Choice of \( b(z) \) and \( q(z) \) as described here would ensure lower values of \( R(\omega) \) and hence improve the system stability.

Another aspect of the usefulness of the regeneration spectrum lies in the simplicity of its relationship to the characteristic root locations for the repetitive control system, for large values of the integer \( N \). The characteristic root locations for a repetitive control system satisfy

\[
z_i^N = R(z_i)
\]
where \( z_i \) is a characteristic root of the system. With the transformation,

\[
z_i = e^{T_D (\alpha_i + j\omega_i)}
\]

(5.6)

we can obtain the corresponding characteristic root \((\alpha_i + j\omega_i)\) in the analog domain setting. Then

\[
e^{N\alpha_i} = |R(e^{T_i (\alpha_i + j\omega_i)})|
\]

(5.7)

As the value of \( N \) increases to a sufficiently high value, the real parts of the most critical characteristic roots decrease in magnitude to very small values, allowing the following simplification to be made. It should be noted that the approximation below is better for the more critical characteristic root locations where \(|\alpha_i|\) is smaller.

\[
e^{\alpha R N} \equiv |R(e^{j\omega_1})| = R(\omega_1)
\]

(5.8)

Thus,

\[
\alpha_i = \frac{\ln(R(\omega_1))}{T_i N}
\]

(5.9)

or, alternatively, the continuum of complex numbers \(\ln(R(\omega))/T_i N + i\omega\) contains all the characteristic root locations, if \(\omega\) is allowed to assume positive and negative values. Equation (5.9) also suggests that the sufficient condition that \(R(\omega)\) be less than unity at all frequencies for the stability of the repetitive control system becomes more nearly
necessary and sufficient as \( N \) becomes large. For example, if the peak value of \( R(\omega) \) is unity, (5.9) predicts a maximum \( \alpha \) of zero, a marginally stable system.

A quantitative measure for the approximation error in equation (5.8) can be inferred as in section 3.3.2.

\[
\frac{\alpha - a}{\alpha} = \frac{1}{NT_\alpha} \ln \left| \frac{R(e^{T_\alpha(\alpha + j\beta)_--})}{R(e^{P_\beta})} \right| \equiv \frac{1}{NT_\alpha} \ln \left| \frac{R(e^{T_\alpha(\alpha + j\beta)_+})}{R(e^{P_\beta})} \right|
\]

(5.10)

where \( a(\beta) \) is the estimation of \( \alpha \), i.e.,

\[
a(\beta) = \frac{1}{NT_\beta} \ln |R(\beta)|
\]

(5.11)

A larger \( N \) basically results if the input signal period is large relative to the time constant of the fastest significant mode of the plant response. In such a case, the sampling period \( T_\beta \) will be chosen to be sufficiently smaller than the shortest plant time constant of significance to achieve effective control over the corresponding mode of response. A upper bound on the achievable values of \( N \) is the computational power required for the control law implementation. Given the vast improvements in the computing speeds achievable by microelectronic devices, this limitation is not expected to be a restrictive one.

Note that the transform (5.6) allows the regeneration spectrum to be defined in the continuous time domain setting. For a completely digital system with the characteristic equation (5.5), the regeneration spectrum can be defined in a discrete time domain setting as \( |R(e^{j\theta})| \), which is a function of \( \theta \) in radians. The regeneration spectrum is related to the characteristic root, \( \rho e^{j\beta_1} \), by
The approximation is made based on the fact that \( \rho_i \) is close to one if \( N \) is large. Hence,

\[
\rho_i \equiv \left| R(e^{j\phi}) \right|^\frac{1}{N}
\]

(5.13)

The characteristic gain \( \rho_i \) determines the relative stability of the digital system. The stability condition that the regeneration spectrum is less than one implies that the characteristic gain is less than one. If \( \rho_i \) is closer to the origin, the system has better stability. The characteristic gain of the digital system (5.5) can be estimated from the regeneration spectrum (5.13) if \( N \) is large. Similarly, a quantitative measure of the estimation error can be derived as follows. The estimated characteristic gain is

\[
\rho_e \equiv \left| R(e^{j\phi}) \right|^\frac{1}{N}
\]

(5.14)

and the normalized estimation error is defined as

\[
\frac{\rho_i - \rho_e}{\rho_i} = 1 - \left( \frac{\left| R(\rho_i e^{j\phi}) \right|}{\left| R(e^{j\phi}) \right|} \right)^\frac{1}{N} \equiv 1 - \left( \frac{\left| R(\rho_i e^{j\phi}) \right|}{\left| R(e^{j\phi}) \right|} \right)^\frac{1}{N}
\]

(5.15)

Note that the value inside the parentheses in the above equation is close to one, so we can make a further approximation,
Equation (5.16) clearly shows that the normalized error is small when \( N \) is large.

Equation (5.9) or (5.13) represents a considerable simplification of the computations needed to determine characteristic root locations, especially for large values of \( N \). Consequently, its usefulness for design is greater. Shaping of the regeneration spectrum by appropriate controller design is therefore nearly equivalent to pole placement. Since there are only a discrete number of signal frequencies of interest for repetitive control systems, corresponding to the fundamental component of the input signal and significant harmonics, the values of the regeneration spectrum can also be used to estimate the decay rates of error components at these frequencies. Thus, the regeneration spectrum gives useful information on the transient response of repetitive control systems, if \( N \) is large.

### 5.2.2 Stability Robustness and Performance Robustness

Another use of the regeneration spectrum is that it allows the derivation of a stability robustness condition. Because of the simplicity of the relationship of the regeneration spectrum to system stability, it is reasonable to expect that the resulting condition will be simpler than other robustness results derived by other available methods. An example of the latter type of method would be the derivation of a robustness result based on the complementary sensitivity function and the well known stability robustness result of Doyle and Stein (1981) relating the complementary sensitivity function and unstructured multiplicative uncertainty in the loop transfer function. The simpler stability robustness result to be presented here would potentially be more useful for design.
A stability robustness result against additive uncertainty, based on the regeneration spectrum, is derived in Appendix A.6. The modeling uncertainty in the plant characteristic is represented as

$$G(z) = G_0(z) + \Delta G(z)$$

(5.17)

where $G_0$ is the nominal transfer function and $\Delta G$ is the unstructured modeling error. $G(z)$ is defined differently for the two block diagram in Figure 5.1 and Figure 5.2.

For Figure 5.1

$$G(z) \Delta \frac{Z(G_h G_s G_p)}{1 + Z(G_h G_s G_p)} (5.18)$$

for Figure 5.1 and

$$G(z) \Delta Z \left( \frac{G_p}{1 + G_p G_h G_s} \right) (5.19)$$

for Figure 5.2. For the repetitive control systems to be stable in the presence of modeling error, the sufficient condition derived in Appendix A.7 is that

$$\left| \frac{\Delta G(z)}{G_0(z)} \right|_{z=\omega_o} \leq \frac{1 - R_0(\omega)}{|q(z)b(z)G_0(z)|_{\omega=\omega_o}} (5.20)$$

and

$$R_0(\omega) < 1 (5.21)$$
where $R_0(\omega)$ is the regeneration spectrum for the system corresponding to the nominal transfer function $G_0(z)$.

The stability robustness result, equation (5.20), provides considerable insight into the effect of the design parameters $b(z)$, $q(z)$ and the transfer function $G_0(z)$ on the robustness of the system stability in the presence of the unstructured modeling error $\Delta G$. A low value of the nominal regeneration spectrum $R_0(\omega)$ does not, by itself, guarantee improved stability robustness. The design procedure should constrain the magnitude of the frequency response of $q(z)b(z)G_0(z)$ to low values also. Conversely, a high value of $R_0(\omega)$, close to unity, does not automatically result in poor stability robustness. If the frequency response of $q(z)b(z)G_0(z)$ has low magnitude in the frequency ranges where $R_0(\omega)$ is close to unity or where the multiplicative uncertainty $\Delta G / G_0$ is high, the inequality (5.21) may yet be satisfied, ensuring system stability.

There is also a practical advantage in stating the modeling error in terms of error in the transfer function $G(z)$. Equations (5.18) and (5.19) indicate that $G(z)$ is essentially a closed loop transfer function and includes the instantaneous feedback in Figure 5.1 and Figure 5.2. Because of the linearizing effect of this feedback it is feasible to represent the modeling error by frequency domain bounds even in many instances, such as in the hydraulic control system described later, where the underlying phenomena responsible for the model error or uncertainty are nonlinear. The error $\Delta G$ is also easier to determine experimentally as it refers to an error in a closed loop transfer function.

The sensitivity function of a control system is generally used as a measure of disturbance rejection. The exact sensitivity function of control systems with both digital and analog components, such as the ones shown in Figures 5.1 and 5.2 is cumbersome to obtain. However, an approximate sensitivity function can be obtained (see Appendix
A.8) assuming that the disturbance input is sampled and hold before it is injected to the system.

\[
S(z) = M_t(z) z [S_0(s) G_{hd}(s)]
\]  \hspace{1cm} (5.22)

where \( G_{hd}(s) \) is the fictitious sample and hold function for the disturbance input. \( S_0 \) is the sensitivity function of the pre-existing system in the absence of the repetitive control action, and the multiplying factor is

\[
M_t(z) = \frac{1 - q(z) z^{-N}}{1 - R(z) z^{-N}}
\]  \hspace{1cm} (5.23)

where \( R(z) \) refers to right hand side of equation (5.1) or (5.2). Note that the actual sensitivity function should be defined in a continuous time terms because of the continuous nature of the disturbance input signal and the system output signal. Here, the approximate sensitivity function is defined in discrete time terms and can only be used to characterize the response of the system at each sampled point. However, if the sampling rate is sufficiently fast, equations (5.22) and (5.23) give good estimates of the actual system sensitivity function. With this in mind, the tracking performance robustness against the uncertainty (5.17) can be examined by evaluating the multiplying factor at the signal frequencies,

\[
M_t(z_n) = \left[ \frac{1 - q(z_n)}{1 - R(z_n)} \right], \quad z_n = e^{j2\pi n T_s / T_0}
\]  \hspace{1cm} (5.24)
The magnitude of the multiplying factor is very small, if \( q(z) \) is close to one and the regeneration spectrum is small. Hence, the tracking performance of the system is robust if \( q(z_n) \) is very close to one, provided that all \( R(z_n) \) is not close to one over the range of uncertainty represented by equation (5.17). Clearly, small values of the regeneration spectrum at the signal frequencies improve the accuracy of the system.

The stability robustness and performance robustness of discrete-time repetitive control system against multiplicative uncertainty can thus be examined using equations (5.20) and (5.24), both of which are strongly dependent on the regeneration spectrum of the system.

\[
G_r(z) = \frac{z^B (z-1)}{A(z^{-1})}
\]

\[
G_p(z)
\]

\[r(z)\]

[Diagram]

Figure 5.3 Example System

5.2.3 Example of Regeneration Spectrum Use for Analysis

A discrete-time repetitive control design procedure was proposed and applied to hydraulic servosystems used for noncircular machining by Tsao and Tomizuka (1988). The application essentially required the servo to accurately follow a periodic reference input signal with high frequency components. Figure 5.3 shows a block diagram of the control system, the controller \( G_c(z) \) being given by the authors as
The polynomial $B^*(z^{-1})$ contains the cancellable zeros of the plant denominator polynomial $B(z^{-1})$, the uncancellable zeros being included in $B^-(z^{-1})$.

$$B(z^{-1}) = B^*(z^{-1})B^-(z^{-1})$$  \hspace{2cm} (5.26)

Also, $B^-(z)$ is $B^-(z^{-1})$ with $z^{-1}$ replaced by $z$. Finally

$$b_e = \max_{0 \leq \omega \leq \pi/T} |B^*(e^{-i\omega T})B^-(e^{i\omega T})|$$  \hspace{2cm} (5.27)

The controller described by equations (5.25)-(5.27) is a special case of the controller in Figure 5.1, and does include a reference signal generator as indicated by the second factor on the right hand side of equation (5.25). The controller results in a zero phase error characteristic of the transfer function $G_c(z)G(z)$. The regeneration spectrum for the system is easily shown to be

$$R(\omega) = \left| q(z) \left( 1 - \frac{k_r B^-(z)B^-(z^{-1})}{b_e} \right) \right|_{z=e^{i\omega T}}$$  \hspace{2cm} (5.28)

If $q(z)$ is unity, the equation (5.28) indicates that the parameter $k_r$ must satisfy $0 < k_r < 2$ for the regeneration spectrum to be less than unity and for the closed loop system to be stable. However, if $q(z)$ is not unity and has a low pass characteristic, equation (5.28) indicates that the condition $0 < k_r < 2$ is not necessary for stability. A less conservative
Figure 5.4 Regeneration Spectrum for Example System - Case a and b

Figure 5.5 Regeneration Spectrum and Characteristic Roots, Case b, N=200,50,20
Figure 5.6 Normalized Estimation Errors of Pole Locations
stability condition is obtained by merely requiring the regeneration spectrum in equation (5.28) to be less than unity. Thus, the use of the regeneration spectrum leads to less conservative stability results than those developed by Tomizuka et al (1989) for the same controller.

The use of the regeneration spectrum also indicates clearly the consequences of the zero phase error tracking constraint on the controller performance. Equation (5.28) indicates that if $k_r$ is unity, the regeneration spectrum $R(\omega)$ is close to zero in the frequency ranges where the product $|B^{-}(e^{-j\omega T_e})B^{-}(e^{j\omega T_e})|$ achieves its peak value. According to equation (5.9), error transients at these frequencies would then decay rapidly. Consequently, only if the input signal frequencies are close to the frequencies for which $|B^{-}(e^{-j\omega T_e})B^{-}(e^{j\omega T_e})|$ achieves its maximum value would the zero phase error tracking constraint and the choice of $k_r$ to be unity result in rapid decay of the error transients. For instance, if $B^{-}(z^{-1})$ has unstable zeros with negative real parts, the maximum of (5.27) is at zero frequency (See Appendix A.9). The controller using (5.25)-(5.27) results in low regeneration spectrum values at low frequencies and hence fast transient response. On the other hand, if $B^{-}(z^{-1})$ has unstable zeros with positive real part, the maximum of (5.26) is at $\pi / T_e$ and hence the scaling procedure tends to result in high values of regeneration spectrum at low frequencies. The transient response of such repetitive control systems would be poor.

Figure 5.4 gives the regeneration spectra for the experimental system studied by Tsao and Tomizuka. The identified plant model is

$$G(z) = \frac{z^{-3}(0.06 + 0.034z^{-1} + 0.071z^{-2})}{(1 - 0.606z^{-1} - 0.747z^{-2} + 0.519z^{-3})}$$ (5.29)
The sampling interval $T_s$ is 0.4 msec and $N$ is 250. $k_F$ is 1.0 and $q(z)$ is given by

$$q(z) = 1 \quad \text{Case a}$$

$$q(z) = 0.25z + 0.5 + 0.25z^{-1} \quad \text{Case b} \quad (5.30)$$

The regeneration spectra in the figure have peak values of 0.9957 and 0.5202 for case 'a' and case 'b' respectively. The use of a low pass filter $q(z)$ as in case 'b' did therefore improve the stability of the system. Experimentally, case 'b' was seen to be stable by Tsao and Tomizuka, whereas case 'a' was not. Good relative stability can thus avoid instability despite errors resulting from inaccurate modeling.

The validity of the equation (5.9) relating the regeneration spectrum to the characteristic root locations is illustrated in Figure 5.5 for a repetitive control system with the plant (5.29), and controller given by equations (5.25)-(5.27), $q(z)$ being given by case 'b' of equation (5.30). The regeneration spectrum is independent of $N$ and is plotted as a continuous curve. The characteristic root locations are computed numerically for different values of $N$ and plotted as points $(\omega_i, e^{i\omega_i T_s})$. The computed roots lie close to the curve of the regeneration spectrum even for values of $N$ as low as 20, confirming that the closeness of the computed characteristic roots to the regeneration spectrum is good for large values of $N$. Figure 5.6 shows the normalized estimation errors, using equations (5.10) and (5.16) respectively, of the characteristic roots for different values of $N$. The actual errors for two cases, $N$ being 50 and 100, are also plotted in the same figures. It is clear that equations (5.10) and (5.16) can predict the errors quite accurately.

Zero phase error compensation is a modified version of inverse dynamic compensation for non-minimum phase systems. Lack of explicit magnitude
compensation may lead to bad results such as poor robustness or bad transient response. As indicated earlier in Chapter 3, inverse dynamic compensation of $G(z)$, if $G(z)$ is strictly proper, may result in the stability being degraded by the uncertainty in the high frequency region. For systems which have unstable zeros with negative real parts, the zero phase compensation scheme results in magnitude attenuation at high frequency and is certainly a reasonable design. But for systems which have unstable zeros with positive real parts, either poor robustness or poor transient response can result from zero phase compensation. In the example chosen above, the unstable zeros have negative real parts, so low regeneration spectrum values at low frequencies are seen in Figure 5.4.

5.2.4 A Repetitive Controller Design Procedure

A discrete-time repetitive controller design procedure for the system in Figure 5.2 is described below, based on the use of the regeneration spectrum, and will be used for control of the small scale material testing machine introduced in Chapter 2. The design procedure is based in the frequency domain and relates the controller design parameters very explicitly to the stability, transient response, stability robustness and accuracy of the repetitive control system.

The first step in the design procedure is to design the analog compensation included in $G_p(s)$. Stability of the closed loop system in the absence of the repetitive control action is required for application of the regeneration spectrum. Furthermore, the robustness result, equation (5.20), indicates that $G_0(z)$ must be well damped to avoid degrading stability robustness. Equation (5.19) for $G_0(z)$ implies as a consequence that the closed loop system $G_p/(1+G_p)$ should also be well damped. In fact, if the closed loop system is dominantly second order, an effective damping ratio of 0.707 would be preferred since it would result in a nearly linear phase shift characteristic as a function of
frequency. Note that this is also a desired design objective of the pre-existing system without repetitive control action. This in turn simplifies the design of $b(z)$ as indicated below. Thus, the requirements on the design of the analog compensation correspond simply to conventional ideas of good design practice. The requirements are not particularly demanding either, since we are not requiring that the bandwidth of the resulting closed loop transfer function $G_p/(1+G_p)$ be high. In fact, one consequence of the inclusion of the repetitive control action is that the accuracy of the repetitive control system at the discrete signal frequencies of interest are not limited much by the bandwidth of $G_p/(1+G_p)$.

The choice of the controller parameter $b(z)$ is dictated by the need to achieve low values of the regeneration spectrum (5.4), as well as good stability robustness as indicated by equation (5.20). Both objectives are met by choosing $b(z)$ to be

$$b(z) = \text{Bi} \left\{ \frac{1}{s^2 + \frac{2\zeta_\omega s}{\omega_\omega} + 1} \right\} z^p$$

where $p$ is an integer and $\text{Bi}(\cdot)$ represent the bilinear transform of the argument, obtained by replacing $s$ by $2(1-z^{-1})/T_s(1+z^{-1})$. The term $z^p$ results in phase advance and is used to compensate for the phase lag introduced by $G(z)$ in equation (5.19) and by the second order low pass filter in equation (5.31). Such compensation is most effective when the damping ratio in equation (5.31) is 0.707 and closed loop transfer function $G_hG_aG_p/(1+G_p)$ is dominantly second order, also with a damping ratio of 0.707. The phase lag to be compensated is a nearly linear function of frequency in such a case, and
the amplitude ratio curve is flatter. p is chosen to satisfy equation (5.32) as closely as possible.

\[ pT_c \equiv T_0 + \frac{2\zeta_k}{\omega_b} + T_c \]  

(5.32)

where \( T_c \) is the controller computation time and \( T_G \) is the equivalent time delay of the closed loop system \( G_hG_aG_p/(1+G_p) \). \( \omega_b \) is chosen to be higher than the bandwidth of \( G(z) \) in equation (5.19), but lower than the frequencies for which the normalized error \( \Delta G / G_\infty \) is high. The compensated product \( b(z)G_\infty(z) \) in equation (5.4) is then more nearly equal to unity over a wide range of frequencies, resulting in lower \( R(\omega) \) values and hence better system stability and transient response. The product \( |b(z)G_\infty(z)| \) would also be low at the high frequencies where modeling error increases and hence, according to equation (5.20), improves the stability robustness of the system. Note that Butterworth filters of higher order may also be used in (5.31) if more computational power is available and the robustness is shown to be improved in such a case.

The choice of \( q(z) \) is dictated primarily by the need for accuracy, subject to the constraint on the stability and transient response as indicated by equation (5.4), and on stability robustness as indicated by equation (5.20). The approximate sensitivity function (5.22) of the repetitive control system in Figure 5.2 gives a proper measure of the accuracy of the system and indicates that \( q(z) \) should be as close to unity as possible for as wide a range of frequencies as possible. The regeneration spectrum constraint, equation (5.4), however, indicates the need for a low pass filter characteristic of \( q(z) \) and significant attenuation, especially at frequencies beyond the effective bandwidth of the product \( b(z)G_\infty(z) \). Also, such attenuation by \( q(z) \) at frequencies where the normalized modeling error is high improves stability robustness, according to equation (5.20).
Figure 5.7 Frequency Responses of $q(z)$ for Different $r_q$ Values
The form of \( q(z) \) proposed here is

\[
q(z) = \frac{(1+r_q)^2}{4} \frac{(1+z^{-1})^2}{(1+r_qz^{-1})^2} z^k, \quad 0 < r_q < 1
\]  

(5.33)

where \( r_q \) and \( k \) are design parameters. If \( r_q \) is close to unity, the bandwidth of \( q(z) \) is high and the phase lag low, as indicated in Figure 5.7. The term \( z^k \) results in phase advance and is used to compensate for the phase lag introduced by rest of the filter and thus reduce the overall phase shift characteristics. The filter \( q(z) \) thus has only one independent parameter, which makes experimental tuning easier. Its amplitude characteristic is also flatter with a sharper cutoff, as compared to alternative candidates, such as Butterworth filters (5.31). Thus the tradeoff between accuracy and other aspects of the control system performance may be resolved more effectively. Note that the sensitivity of the repetitive control system (5.22)-(5-24) also depends on \( R(z) \), so a flatter magnitude characteristic of \( q(z) \) can reduce the loss in accuracy caused by large variations of \( R(z) \).

Both the filters \( b(z) \) and \( q(z) \) proposed here are noncausal. However, since the phase advance in these filters is much lower than the time delay in the repetitive control loop, the plug-in repetitive controller in Figure 5.2 may be implemented equivalently in a causal form as shown in Figure 5.8.

![Figure 5.8 Digital Repetitive Controller in a Causal Form](image-url)
The design procedure described above is a simple one. The controller forms proposed have been derived from explicit consideration of system accuracy, stability, transient response and robustness. In contrast, the design procedure used by Hara et al (1988) uses an optimal control framework and obscures the effect of design parameters on these essential aspects of system performance. Also, the design procedure using zero-phase error tracking controller (Chew and Tomizuka, 1990) considers only one design tradeoff, namely, disturbance rejection at the discrete design frequencies versus that at other frequencies. The design procedure described here can be improved if more computational power is available. More complex forms for $b(z)$ and $q(z)$ can be designed based on the regeneration spectrum and stability performance robustness measures derived from it.

5.3 Discrete Time Repetitive Control of Closed Loop Material Testing

Repetitive control is certainly applicable for closed loop servohydraulic material testing applications involving periodic command signals. The improved accuracy is of obvious value, but so is the potential for improved ease of controller tuning. The latter attribute is of practical value given the dependence of the plant characteristics in such applications, on the test specimen and the nature of the test. The resulting variation of the plant characteristics from test to test poses problems for proper controller tuning, especially since material testing machine users may lack controls expertise. Some specific features of this class of applications are worthy of note as they distinguish it from other repetitive control applications, especially in terms of the performance requirements. The use of electrohydraulic actuation results in strongly nonlinear behavior related either to the turbulent fluid flow through valves or to manufacturing imperfections such as valve underlap or overlap, the terms referring to the land of the valve spool being respectively
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smaller or larger than the valve port width. The linear models used for controller design would consequently exhibit significant modeling error, as a result of these nonlinearities. It is important therefore to ensure adequate stability robustness and accuracy of the proposed control system despite the modeling error. The application of the repetitive control system here is intended to evaluate the practical benefits achievable in this instance, and also serves as a test bed to evaluate the proposed controller analysis and synthesis procedures.

5.3.1 Control System Setup of the Small Scale Testing Machine

The small scale material testing machine introduced in Chapter 2 will be used as a test bed for the repetitive control schemes. The control system application considered here is that of closed loop servohydraulic material testing, the controlled variable being stroke, load or strain. A hybrid control system for the hydraulically powered uniaxial test stand is depicted in Figure 5.9 and used for the experimental work here. For the load control tests considered here, the force exerted by the actuator on the test specimen is measured by a load cell and used to close the force feedback loop. Details of the material testing machine are given by Lee (1989).

An analog computer is used here to implement the analog precompensator \( G_c(s) \) for the pre-existing control system. The analog computer allows easy design changes for testing various compensators. An Intel 86/310 microcomputer with an Intel 8087 coprocessor is used for the implementation of the discrete-time repetitive controllers such as the one shown in Figure 5.8. The manufacturer of the material testing machine provides an analog controller, which also has signal conditioners for processing the measured signals. The reference signal is generated by the digital computer to synchronize with the action of the digital repetitive controller. A second order anti-
aliasing filter with a bandwidth of 247 Hz is used to avoid aliasing due to sampling, and also to attenuate the high frequency noise from the signal conditioners. The sampling time \( T_s \) of the discrete-time repetitive controller, which determines the smoothness of the reference signal and the bandwidth of the repetitive control system, is constrained by the computing ability and the input/output signal conversion rates. A sampling interval of 3 msec is used for the digital controller.

5.3.2 Design and Analysis of the Pre-Existing Control System

In load control applications, the actuator differential pressure feedback is seen experimentally to be not as effective as in stroke control applications. This is due to the fact that the differential pressure signal is almost proportional to the load signal, the controlled variable, when the inertia force is negligible compared to the load. Therefore, only the load signal used for feedback control here. A lead-lag compensator is considered here. Experimental measurements of the frequency response \( F/u \) shown in Figure 2.12, and the model (2.14), are used for the design of the lead-lag compensator. After some iterations, the resulting controller is chosen to be

\[
G_c(s) = \frac{8(s+100)(s+200)}{(s+2)(s+600)}
\]

(3.34)

The controller has a steady state gain of 133.3 and a high frequency gain of 8. Figure 5.10 shows the frequency responses of the open loop system. The open loop gain is increased, especially at low frequencies, by the phase lag compensation with the attendant phase lag increase. Phase lead compensation reduces the phase lag of the system at frequencies above 200 rad/sec and hence allows higher gains at high frequencies. Figure 5.11 shows the Nyquist plots of the open loop system before and after compensation.
Figure 5.10 Frequency Responses of the Open Loop Plant $G_p(\omega)$
Figure 5.11 Nyquist Plots of $G_p(j\omega)$

- '-' With Lead-Lag Compensation
- '---' Without Lead-Lag Compensation
Figure 5.12 Closed Loop System Responses of the Load Control System

Figure 5.13 Sensitivity Functions of the Load Control System
Figure 5.14  
Output Load Signal and Error Signal of the Lead-Lag Control System
3Hz Triangular Reference Signal, Simulation Result
Figure 5.15  Output Load Signal and Error Signal of the Lead-Lag Control System
10Hz Triangular Reference Signal, Simulation Result
Figure 5.16  Output Load Signal and Error Signal of the Lead-Lag Control System
3Hz Triangular Reference Signal, Experimental Result

Figure 5.17  Output Load Signal and Error Signal of the Lead-Lag Control System
10Hz Triangular Reference Signal, Experimental Result
Figure 5.18  Frequency Responses of G - Experimental
Figure 5.19  Stability Robustness Evaluation (5.20)
The gain margin is reduced from 5.6 to 2.5 and the phase margin reduced from 85° to 45° respectively. The stability of the system is degraded by the compensator but the system still has the good gain margin and phase margin. Figure 5.12 and Figure 5.13 show the closed loop responses and the sensitivity functions of the system respectively. Sensitivity of the system is improved by the compensation, up to 240 rad/sec, but especially in the low frequency range where the disturbance signal is often dominant. For example, the effect of mechanical offset of the servovalve null can be attenuated. The closed loop response of the system also shows that the tracking performance of the system is improved. The above figures are calculated using experimental data points from Figure 2.10 and equation (5.34). The lead-lag controller together with the nonlinear model discussed in Chapter 2 are then simulated using ACSL, to evaluate the control system. Figure 5.14 and Figure 5.15 show the output load signal and the error signal of the system for 3 Hz and 10 Hz triangular reference signals respectively. Even though the bandwidth of the closed loop system is about 70 Hz from Figure 5.12, significant errors are seen in the peaks of the signals, 10% peak error for the 3 Hz and 34% for the 10 Hz signal. This is due to the significant nonlinearities in the hydraulic system and the poor sensitivity of the system at high frequencies, being greater than 0.1 above 5 Hz as indicated in Figure 5.13. Figures 5.16 and Figure 5.17 show the experimental results for 3 Hz and 10 Hz triangular signals. The experimental responses agree reasonably with the simulation result, but with greater peak errors, 22% for the 3 Hz signal and 41% for the 10 Hz signal. This is possibly due to additional nonlinearities, such as valve overlap which are shown to be significant experimentally but were not included in the simulation, and inaccurate modeling of the servovalve dynamics.
5.3.3 Design and Analysis of Discrete-time Repetitive controllers

The analog controller design may be viewed as the first step in the design of the repetitive control system, and the controller $G_c(s)$ in equation (5.34) included in the compensated plant $G_p(s)$ in Figure 5.2. The closed loop frequency response $F/F_{\text{ref}}$ is measured experimentally for different sinusoidal input signal amplitudes, combined with the frequency response of the anti-aliasing filter $G_a(s)$ and the zero order hold $G_h(s)$, and shown in Figure 5.18. In this case

$$G_a(s) = \frac{1}{(s/1700)^2 + 1.4s/1700 + 1}$$

and

$$G_h(s) = \frac{1-e^{-\tau_s}}{s}$$ (5.35)

The sampling interval $T_s$, being 3 milliseconds, is limited by the computing speed of the Intel microcomputer. The frequency response changes with the signal amplitude, indicating nonlinear system behavior. Sinusoidal frequency response data could not be obtained at higher frequencies because of excessive test stand vibration. The nominal closed loop frequency response $G_0$ over a wider frequency range is obtained using a random reference input signal and shown in the figure. Figure 5.18 indicates that the closed loop system in the absence of repetitive control is stable and well damped, as required for application of the regeneration spectrum and for good stability robustness as indicated by equation (5.20). The equivalent time delay $T_G$ for use in equation (5.32) is computed to be 6.5 milliseconds from the nominal phase response in Figure 5.18. The variability in the closed loop frequency response with signal amplitude in Figure 5.18 is considerably lower than that in the open loop frequency response in Figure 2.10, because
of the linearizing effect of the analog feedback. This fact is of practical significance here because it is much more feasible to represent the variation in Figure 5.18 using frequency domain based bounds on an additive model error $\Delta G$ as indicated in equation (5.17). Figure 5.19 is a graph of $|\Delta G/G_0|$, $G_0$ being the nominal closed loop frequency response derived from the random input signal testing, and indicates that the normalized model error increases at higher frequencies.

The design of the plug-in discrete-time repetitive controller in Figure 5.2 requires the design of $b(z)$ and $q(z)$ given by equations (5.31)-(5.33). $\omega_b$ is chosen to be 500 rad/sec and $\zeta_b$ to be 0.7. The repetitive control system would have better steady state accuracy if higher $\omega_b$ can be used. However, when higher $\omega_b$ is used, the system is seen experimentally to be unstable. Note that the uncompensated hydraulic system has a sharp loop gain drop near 600 rad/sec and large loop gain variations above 400 rad/sec, as shown in Figure 2.12. $T_c$, the controller processing time, is approximately 2.5 milliseconds. $pT_s$ is chosen to be 12 milliseconds according to equation (5.32), yielding an integer value of $p$ of 4. $r_q$ is chosen to be 0.98 in equation (5.33) and yields a 'k' of zero since the resulting phase lag is very small. Figure 5.20 and Figure 5.21 show the load output signal and the error signal of the repetitive control system, using ACSL simulation, for 3 Hz and 10 Hz triangular reference signals respectively. The digital repetitive control controller is turned on at the beginning of second period and starts to work at the third period. The errors for both cases are greatly reduced within two cycles, from 10% and 34% to 2.5% and 4% respectively. Figure 5.22 shows the details of the reference signal and the steady state output signal for the 10 Hz case. The reference signal is from the digital computer and is hence a stepwise signal. The digital repetitive controller tries to force the system to follow the reference signal at sampled values only. Fortunately, the resulting output signal is quite smooth. Figure 5.23 shows details of the
Figure 5.20  Output Load Signal and Error Signal of the Repetitive Control System
3Hz Triangular Reference Signal, Simulation Result
Figure 5.21  Output Load Signal and Error Signal of the Repetitive Control System
10Hz Triangular Reference Signal, Simulation Result
Figure 5.22  Details of Steady State Input/Output Signals - Simulation

Figure 5.23  Details of Continuous and Sampled Steady State Error Signals - Simulation
Figure 5.24  Output Load Signal and Error Signal of the Repetitive Control System
3Hz Triangular Reference Signal, Experimental Result

Figure 5.25  Output Load Signal and Error Signal of the Repetitive Control System
10Hz Triangular Reference Signal, Experimental Result
Table 5.1
Relative Magnitudes of Harmonic Components of Input and Output Signals

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>10 Hz</td>
<td>0.2951</td>
<td>0.3311</td>
<td>0.2951</td>
<td>0.2951</td>
</tr>
<tr>
<td>30 Hz</td>
<td>0.0305</td>
<td>0.0394</td>
<td>0.0309</td>
<td>0.0313</td>
</tr>
<tr>
<td>50 Hz</td>
<td>0.0122</td>
<td>0.0124</td>
<td>0.0126</td>
<td>0.0126</td>
</tr>
<tr>
<td>70 Hz</td>
<td>0.0050</td>
<td>0.0057</td>
<td>0.0053</td>
<td>0.0060</td>
</tr>
</tbody>
</table>

Table 5.2
Relative Phases of Harmonic Components of Input and Output Signals

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10 Hz</td>
<td>0</td>
<td>-9.5°</td>
<td>-5.1°</td>
<td>-5.1°</td>
</tr>
<tr>
<td>30 Hz</td>
<td>0</td>
<td>-59.2°</td>
<td>-15.7°</td>
<td>-15.8°</td>
</tr>
<tr>
<td>50 Hz</td>
<td>0</td>
<td>-93.0°</td>
<td>-28.3°</td>
<td>-26.0°</td>
</tr>
<tr>
<td>70 Hz</td>
<td>0</td>
<td>-173.0°</td>
<td>-45.6°</td>
<td>-41.0°</td>
</tr>
</tbody>
</table>
actual error signal and the sampled error signal. In order to assess system accuracy it is reasonable to use the sampled error signal instead of the 'saw-tooth' error signal, since we want a smooth triangular signal anyway. Therefore, sampled error signals are used in the Figure 5.20 and Figure 5.21 and will be used as a measure of tracking accuracy throughout this chapter.

Figure 5.24 and Figure 5.25 show the experimental output and error signals for 3 Hz and 10 Hz triangular reference input signals. The latter part of Figure 5.24 and Figure 5.25 shows the response of the control system when the plug-in controller is activated in addition to the analog lead-lag controller given by equation (5.34). The reduction of the error following the introduction of the repetitive control is quite rapid. The error levels are much lower and the output signal clearly shows reduced distortion. The peak error for both cases is reduced from 22% and 41% to 3% and 5% respectively and the experimental responses agree quite well with the simulations in Figure 5.20 and Figure 5.21.

The error signal of the system for the lead-lag control only is significant, but is not the most appropriate measure of control system performance for this application. In order to evaluate the performance of the control system more appropriately for the application, the harmonic components of the controlled signal are computed experimentally and compared with those of the reference input signal for a 10 Hz triangular reference input signal of amplitude 500 lbf. Columns 2 and 3 of Tables 5.1 and 5.2 show the results. The results indicate that the 10 Hz and 30 Hz components have amplitude gains different from unity and that their phase shifts are not proportional to the frequency, indicating distortion of the output. Figure 5.24 and Figure 5.25 indicate also that the valve overlap is primarily responsible for the distortion, since the error magnitude rises sharply when the valve crosses null and the load reverses its direction of change.
Figure 5.26  Nominal Regeneration Spectrum
Figure 5.27  Output Load Signal and Error Signal of the Repetitive Control System
3Hz Triangular Reference Signal, No Lead-Lag Compensation

Figure 5.28  Output Load Signal and Error Signal of the Repetitive Control System
10Hz Triangular Reference Signal, No Lead-Lag Compensation
The inclusion of the repetitive control makes the amplitude ratios equal to unity for the 10 Hz and 30 Hz components. Also, the phase shift is nearly linear with frequency when the repetitive controller is used. The equivalent time delay computed from the linear phase shift is about 1.48 msec, which is nearly half of the sampling interval.

The usefulness of the regeneration spectrum for quantitative analysis of the controlled system behavior is further illustrated by using it to estimate the decay rates of the error transients and the robustness of the system stability. The nominal regeneration spectrum of the system, based on the experimentally measured frequency response \( G_0 \), is shown in Figure 5.26 and indicates a value of \( R_0(\omega) \) of about 0.35 at the 10 Hz and 30 Hz frequencies dominant in the 10 Hz triangular input signal used in the experiments. Equation (5.9) indicates a characteristic root real part \( \alpha \) of -10.5 sec\(^{-1} \) which corresponds to the error signal decaying to a tenth of its initial value in about 0.22 second, a little over two periods of the input signal in this case. This explains the rapid decay of the error in Figure 5.25. \( N \) in this case is 33, which is a large enough value.

The large peak value of the regeneration spectrum in Figure 5.26 may at first suggest poor stability robustness. However, equation (5.20) is a more precise indicator of the stability robustness of the system. The right hand side of the equation is computed for the design and graphed in Figure 5.19, which also shows a graph of the normalized modeling error \( |\Delta G / G_0| \). The right hand side of the equation (5.20) remains moderately high, always above 0.4. This indicates that normalized modeling errors of at least 40 percent can be accommodated without violating the conservative stability robustness condition. The measured normalized modeling error \( |\Delta G / G_0| \) exceeds the allowed limit by a small amount in the frequency range 250-350 rad/sec, violating the conservative stability robustness condition. The fact that the experimental system is stable is probably due to the conservative nature of the stability robustness condition. When the
regeneration spectra are calculated individually for each of the curves in Figure 5.18, the resulting regeneration spectra of all these cases are seen to be less than one.

Finally, since ease of controller tuning has been stated to be an important requirement for material testing applications, it is appropriate to question how critical the analog compensator controller $G_c(s)$ in equation (5.34) is, and how sensitive the performance of the repetitive controller is to the analog controller design. Figure 5.27 and Figure 5.28 show the results of the repetitive control system with analog proportional control and the same plug-in repetitive controller as before with one change. The proportional controller results in a larger effective delay $T_G$, and $pT_s$ in equation (5.32) is chosen to be 15 milliseconds instead of 12 milliseconds. The performance of the analog proportional control, when used alone instead of the lead-lag controller (5.34), is much poorer, as indicated by comparing the earlier parts of Figure 5.25 and Figure 5.27. The reference signal input is the same in both cases. When the plug-in repetitive controller is activated, the error is reduced sharply along with the amount of distortion in the output signal. The results of harmonic analysis of the input and output signals are shown in columns 5 of Tables 5.1 and 5.2, and are very close to the results for the earlier repetitive controller.

These results should not be taken to mean that the plug-in repetitive controller performance is insensitive to the design of the analog controller. In fact, if the signal components at the higher harmonics were more significant, we would expect the difference between the performance levels of the two repetitive controllers to be greater. However, the results do show a good degree of performance robustness of the repetitive control system, most probably as a consequence of the fact that the plug-in repetitive controller gain is very high, essentially close to infinity, at the discrete set of signal frequencies.
5.3.4 Further Evaluation of Repetitive Control of the Material Testing Machine

Digital repetitive control has been shown to significantly improve the performance of the material testing machine. In order to explore its practical application further, different aspects of the control system are investigated through simulation and experiments. The results are given below.

5.3.4.1 Overlap and Underlap of Servovalve

The experimental results in Section 5.3 show that the error signal due to nonlinearities such as servovalve overlap characteristics can be greatly reduced. Since the exact overlap characteristics of the servovalve in the setup are not accessible, digital simulation is used to study how repetitive control can overcome the effect of such servovalve imperfections.

Figure 5.29 shows the simulation result for the control system corresponding to the experimental result shown in Figure 5.28. The servovalve in this simulation is critically centered, i.e. no overlap or underlap. The servovalve motion is also plotted and has a maximum displacement of 0.00018in. Figure 5.30 shows the simulation result for the control system with an overlap of 0.0002in on one side of the valve. The repetitive controller adjusts the servovalve motion to compensate the valve overlap characteristics in order to have high accuracy. The steady-state error in Figure 5.30 is almost the same as that in Figure 5.29. However, the transient response lasts slightly longer. Note that the servovalve displacement is automatically adjusted to be greater on the side of the overlap. Figure 5.31 shows the simulation result for the control system with an overlap of 0.0002in on both sides of the valve. The repetitive control system is seen to have almost the same steady state accuracy in this case. Again, the servovalve displacements are increased by 0.0002in on both sides of the null in this case, to achieve the same valve opening as in Figure 5.29. Further simulations indicate that the repetitive control system
performs well even with a valve overlap of 0.0005in on one side, as shown in Figure 5.32. The transient response is poorer and lasts longer in the presence of the overlap.

The electrohydraulic system seems less sensitive to underlap characteristics of the servovalve. Figure 5.33 shows the simulated responses of the control system with an underlap of 0.0002in on one side of the servovalve. Figure 5.34 shows the simulated responses of the system with a valve underlap of 0.0002in on both sides of the null. The transient response is not affected much in both cases, and neither the steady state errors. Repetitive control can therefore adjust valve motion to effectively overcome valve underlap characteristics. It performs well even for the case of a large underlap of 0.002in on one side of the servovalve, as shown in Figure 5.35. The transient responses is affected in this case and is degraded somewhat. The steady state accuracy is the same.

5.3.4.2 Sensor Noise

Most control systems are subject to sensor noise. Figure 5.36 shows the simulated responses of the repetitive control system with sensor noise in the load cell. The load signal is contaminated with a band limited noise signal with Gaussian distribution, having a bandwidth of 1000rad/sec, a standard deviation of 3 lbf, and a mean of zero. Due to the sensor noise, the error signal is slightly worse than that of the noise-free system shown in Figure 5.29. However, no significant difference is observed between the actual output signal shown in Figure 5.36 and that in Figure 5.29. Figure 5.37 shows the simulated response of the same control system but with a larger noise level, the standard deviation being 10 lbf. The error signal is larger, and the actual output signal is also less accurate. Note that the repetitive control system is not particularly effective in rejecting noise. In fact, the noise rejection of the control system is governed by the complementary sensitivity function of the system, low values of the latter resulting
in better noise rejection. In the actual setup, the load cell noise level is less than 3 lbf and also restricted to higher frequencies; hence, no significant error results from the load cell noise.

5.3.4.3 Bulk Modulus and Bandwidth of the Servovalve Dynamics

The bulk modulus of the fluid in the material testing machine is never known exactly. It varies with temperature and pressure and changes significantly if air is entrapped inside the fluid. The dynamics of the servovalve is complicated but simply modeled as a first order system in the simulation. It is important to assure that the repetitive control system can handle these two categories of uncertainties or variations in machine characteristics.

Figure 5.38 and Figure 5.39 show the simulated responses of the same repetitive control system except that the bulk modulus of the fluid is set to be 40,000psi and 360,000psi respectively instead of the nominal value of 120,000psi. Only proportional control is applied to the system in the early part of the responses in both figures. Significant error, due to the reduced system stiffness, is seen in the early part of Figure 5.38. The control system has better performance when bulk modulus is large as shown in Figure 5.39. The same repetitive controller, when applied to both systems, is seen to improve the system performance and achieve similar steady state accuracy without causing instability.

Figure 5.40 and Figure 5.41 show the simulated responses of the repetitive control system with the time constant of the servovalve set to be 0.0006 sec and 0.0054 sec respectively instead of the nominal value of 0.0018 sec. The repetitive controller can reduce the tracking error significantly in both cases. The servovalve with the smaller time constant has a larger system bandwidth; hence, its steady state error is slightly smaller
than that of the system with the larger time constant. Also, the transient response decays more quickly in Figure 5.40.

5.3.4.4 Stiffness of the Test Specimen

The stiffness of the specimen may vary from test to test. Repetitive control systems can achieve high accuracy as long as the regeneration spectrum of the system is less than unity for all frequencies. Hence, the repetitive control system designed here, with fixed controller parameters, is expected to have high accuracy despite the stiffness of the specimen changing from test to test.

Figure 5.42 through Figure 5.44 show results of experimental tests on three other specimens using the same repetitive controller. The stiffness of the specimen is approximately 1/3, 1/4, and 1/12 that of specimen #1 tested in Figure 5.28. The results show that the repetitive controller can accommodate a wide range of specimen stiffnesses. The testing machine itself is found to have some compliance, which is in series with the specimen. Therefore, the equivalent spring constant of the system should include the compliance of the machine structure itself. Experimental values of the equivalent system stiffness and theoretical specimen stiffness are listed in Table 5.3 for the specimen tested. The compliance in the machine structure, being relatively small, results in lower variation of the equivalent system stiffness. This compliance certainly helps the repetitive control system because the system dynamics is less affected by the specimen stiffness. The results in Figures 5.42 - 5.44 indicate however that the repetitive controller can accommodate a stiffness variation by a factor 3 to 1 while maintaining effective control.

Figure 5.45 and Figure 5.46 show the simulated responses of the same repetitive control system with the equivalent stiffness set at 30,000 lbf/in and 3,000,000 lbf/in respectively. The steady-state errors in both cases are quite similar, but the transient
Table 5.3

Theoretical Specimen Stiffness and Equivalent Stiffness of the System

<table>
<thead>
<tr>
<th>No.</th>
<th>Material</th>
<th>Diameter</th>
<th>Theoretical (lbf/in)</th>
<th>Experimental (lbf/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Steel</td>
<td>0.339&quot;</td>
<td>1,546,500</td>
<td>250,000</td>
</tr>
<tr>
<td>2</td>
<td>Aluminum</td>
<td>0.339&quot;</td>
<td>515,502</td>
<td>190,000</td>
</tr>
<tr>
<td>3</td>
<td>Steel</td>
<td>0.175&quot;</td>
<td>412,125</td>
<td>160,000</td>
</tr>
<tr>
<td>4</td>
<td>Aluminum</td>
<td>0.175&quot;</td>
<td>137,375</td>
<td>94,000</td>
</tr>
</tbody>
</table>

* The compliance of the test rig is estimated to be 300,000 lbf/in

response lasts longer for the system with less stiffness. The repetitive control system is
designed for specimen #1, with an equivalent stiffness of 250,000 lbf/in, but is still able
to accommodate a wider range of system stiffness than the stiffness range listed in Table
5.3. In fact, the stiffness varies by two orders of magnitude between Figure 5.45 and
Figure 5.46. Therefore, the repetitive control system can tolerate a wide range of
specimen stiffnesses. This fact makes the tuning of the repetitive controller easy or even
unnecessary in most cases. This is important in material fatigue testing, in which the
stiffness of the specimen degrades as cracks slowly grow in the specimen in the later
stages of the test.

5.3.4.5 Time Delay of the Repetitive Controller and the Period of the Reference Signal

In this material testing application, the period and the shape of reference signals
are assumed to be known in advance. The reference signals are generated by a digital
computer because it is easier to generate a variety of reference signals using digital
computers than by using analog signal generators. Consequently, the time delay in the
repetitive controller can be made exactly equal to the signal period. Nevertheless,
mismatch between the time delay in the digital repetitive controller and the period of the
reference signal could be a problem in the case where the reference signal is generated
from other unsynchronized sources.

A triangular signal, with a signal period of 0.099 sec, is generated from an analog
signal generator rather than from the digital computer. Figure 5.47 and Figure 5.48 show
the output and error signals of the repetitive control system with its time delay set to be
0.096 sec and 0.102 sec respectively. The mismatch in both cases significantly degrades
the effectiveness of the repetitive controller. This is expected since the repetitive
controller designed here results in high gain only at the reference signal frequencies. The
controller gain drops sharply at frequencies slightly away from the reference signal
frequencies. In applications where the reference or disturbance signal period is not
exactly known, controller parameters need to be modified to make the system
performance less sensitive to the period of the signals. This problem has been studied in
the literature (Chew and Tomizuka, 1990).

5.3.4.6 Time Differences Between the Input and the Output Signals

Due to digital/analog signal conversions and processing of the digital controller,
the sampled input and the sampled output do not occur at the same instant at which the
reference signal to the system is updated. In all the previous simulations, the time
differences have been included, i.e., the error signal is sampled 0.5 msec after the
computer updates the reference signal and the repetitive control signal is output to the
analog controller 2.5 msec after the reference signal update. This practical aspect differs
from the assumption made in this chapter for the analysis of the discrete-time repetitive
control system, i.e. sampled inputs and outputs occur at the same instant in time. Figure 5.49 shows details of the steady state response of the system simulated in Figure 5.29. Figure 5.50 shows the simulated response of the control system when these time differences are removed. The tracking performances in both figures are similar, indicating that no significant error is caused by the time differences between sampled inputs and sampled outputs in this case.

5.3.4.7 Sampling Interval

The sampling interval used in the discrete-time repetitive controller is 3msec, which is limited by the capability of the digital computer used here. It is expected that the repetitive control system can achieve better performance if higher sampling rate is available using faster computers. Figure 5.51 shows the responses of the same repetitive control system except that the sampling interval is 0.5msec. In this figure, the control system has better signal resolution, i.e. the signal frequency is exactly 10 Hz and there are 200 samples in a signal period. The signal frequency in Figure 5.49 is actually 10.101 Hz and there are only 33 samples in a signal period. The tracking accuracy is also seen to be better in Figure 5.51. Note however that the maximum error signal occurs at signal turnaround and is about the same in both cases.

5.3.4.8 Summary of Controller Investigation

The discrete-time repetitive control system can compensate effectively for the overlap/underlap characteristics of the servovalve and can significantly improve the system tracking performance. The accuracy of the control system is maintained at almost the same level even with large variations in specimen stiffness, fluid bulk modulus, and servovalve dynamics. Time differences between sampled inputs and output do not cause
noticeable problem in this case. Load sensor noise does not affect the test results significantly either. Higher sampling rates do improve the signal resolution. The repetitive control system designed here is very sensitive to the period of the reference signal. It is recommended therefore that the period of the reference signal be the same as the time delay in the repetitive controller.

5.4 Summary

The usefulness of the regeneration spectrum for analysis and synthesis of discrete-time repetitive control systems has been established. In particular, the stability, transient response and the stability robustness of such systems can be analyzed relatively simply using the regeneration spectrum, as long as the integer N is large. The requirement of a large N is not particularly restrictive since effective control dictates a need for large N as well. The limiting factor on the achievable value of N is the real-time computing power available. Current technology is adequate for achieving large values of N even for high bandwidth mechanical systems.

A design procedure based on the regeneration spectrum is outlined. This procedure is explicitly related to the accuracy, transient response, and stability robustness of the system and hence tradeoffs between these system properties can be rationally made. The controller design procedure is applied to an electrohydraulic material testing system, which is characterized by strong nonlinearities. The repetitive controller significantly enhances the controlled system performance. The controller form is shown to be effective for a wide range of specimen stiffnesses and for significant changes in system characteristics. The tuning of the controller is easy and may in fact be unnecessary for a wide range of test conditions.
Figure 5.29  Simulated Response of the Repetitive Control System
Servovalve Critically Centered
Figure 5.30  Simulated Response of the Repetitive Control System -
0.0002in Servovalve Overlap on One Side
Figure 5.31  Simulated Response of the Repetitive Control System - 0.0002in Servovalve Overlap on Both Sides
Figure 5.32  Simulated Response of the Repetitive Control System - 0.0005 in Servovalve Overlap on One Side
Figure 5.33  Simulated Response of the Repetitive Control System - 0.0002in Servovalve Underlap on One Side
Figure 5.34  Simulated Response of the Repetitive Control System - 0.0002 in Servovalve Underlap on Both Sides
Figure 5.35  Simulated Response of the Repetitive Control System - 0.002in Servovalve Underlap on One Side
Figure 5.36 Simulated Response of the Repetitive Control System -
Gaussian Load Sensor Noise With Standard Deviation of 3 lbf
Figure 5.37  Simulated Response of the Repetitive Control System - Gaussian Load Sensor Noise With Standard Deviation of 10 lbf
Figure S.38 Simulated Response of the Repetitive Control System - Fluid Bulk Modulus of 40,000 psi

Figure 5.39 Simulated Response of the Repetitive Control System - Fluid Bulk Modulus of 360,000 psi
Figure 5.40  Simulated Response of the Repetitive Control System - Servovalve Time Constant of 0.0006 sec

Figure 5.41  Simulated Response of the Repetitive Control System - Servovalve Time Constant of 0.0054 sec
Figure 5.42 Experimental Tests on Specimen #2
Figure 5.43 Experimental Tests on Specimen #3

(a) 3 Hz Triangular Signal, Steel Specimen, Diam.=0.175"

(b) 10 Hz Triangular Signal, Steel Specimen, Diam.=0.175"
Figure 5.44 Experimental Tests on Specimen #4

3 Hz Triangular Signal, Aluminum Specimen, Diam.=0.175"

10 Hz Triangular Signal, Aluminum Specimen, Diam.=0.175"
Figure 5.45  Simulated Response of the Repetitive Control System -
Equivalent System Stiffness of 30,000 lbf/in

Figure 5.46  Simulated Response of the Repetitive Control System -
Equivalent System Stiffness of 3,000,000 lbf/in
Figure 5.47  Experimental Response of the Repetitive Control System – Signal Period of 0.099 sec, Time Delay of 0.096 sec

Figure 5.48  Experimental Response of the Repetitive Control System – Signal Period of 0.099 sec, Time Delay of 0.102 sec
Figure 5.49  Details of the Steady State Response of the Repetitive Control System

Figure 5.50  Details of the Steady State Response of the Repetitive Control System - No Time Differences Between Inputs and Outputs
Figure 5.51 Detailed Steady-State Responses of the Repetitive Control System - With Sampling Interval of 0.5msec
6.1 Conclusions

Tuning of controller parameters is inevitable in most closed loop material testing applications because testing conditions and specimen characteristics are different from test to test. Manual controller tuning is obviously a trouble to users who do not have experience or expertise in designing feedback control systems. Consequently, automatic control of closed loop material testing would utilize certain types of self-tuning schemes to account for changes in testing conditions and specimen characteristics. Self-tuning schemes for electrohydraulic material testing machines (Lee, 1989), though feasible, may not have good performance due to the strong nonlinearities in the electrohydraulic systems. Repetitive control systems are effective in feedback control applications involving periodic reference inputs or disturbance inputs. Consequently, repetitive controllers are good alternatives for automatic control of material fatigue testing machines because of the periodic signals involved.

Repetitive controllers have a time delay in their controller structure. The regeneration spectrum for a class of time delayed systems, including repetitive control systems, has been introduced here along with the derivation of a quantitative criteria for a "large" time delay. An alternative interpretation of the regeneration spectrum has also been developed here to provide deeper insight into the transient response of time delayed systems. The usefulness and the validity of the regeneration spectrum as a relative
stability measure for continuous-time SISO repetitive control systems has been established by showing that the characteristic root distribution of the system is accurately inferred from the regeneration spectrum for such systems. It has been shown that the condition of a large time delay, needed for effective use of the regeneration spectrum, is usually satisfied for repetitive control systems. The regeneration spectrum combined with other frequency domain measures of control system performance, such as the sensitivity function and the complementary sensitivity function, can provide better understanding of the trade-offs inherent in repetitive control system design.

Multivariable repetitive control systems are also investigated because of their potential applications to multi-axial material testing and tire load tests on automobile structures. The regeneration spectrum concept has been extended to MIMO repetitive control systems. Application of the Generalized Nyquist Criterion leads naturally to the definition of the regeneration spectra for a MIMO system. The resulting sufficient stability condition is less conservative than that derived using the small gain theorem by Hara et al (1988). Simplifications in stability analysis that result for large values of the time delay for SISO repetitive control systems are shown to occur here as well. A MIMO repetitive control design procedure is outlined and illustrated by an example.

Discrete time versions of repetitive control systems are also studied since it is convenient and beneficial to implement repetitive controllers using digital computers. The relationship of the regeneration spectrum to the closed loop characteristic roots is established, and measures of stability robustness and performance robustness based on the regeneration spectrum determined. The repetitive controller is formulated as a plug-in module and can be easily added to the pre-existing control system. This is useful because controllers of material testing machines can be upgraded easily.
The electrohydraulic material testing machine studied here has significant servovalve overlap characteristics. This nonlinearity reduces reliability of system parameter identification procedures used in most self-tuning schemes and hence degrades the performance of most self-tuning controllers. Moreover, even with a well tuned lead-lag controller, the control system has significant error in experimental tests. For example, errors of 12% and 29% are seen in the gains associated with the first and the third harmonics of a 10 Hz triangular input signal. It can be expected that conventional self-tuning controllers such as those resulting from pole placement methods would have the same performance at the best. But when repetitive control is applied to a pre-existing proportional control system, the errors above are reduced to 0% and 2.6% respectively. Simulations also show that the repetitive control system can effectively adjust servovalve motion to overcome valve underlap/overlap characteristics to achieve high accuracy. This feature is important because manufacturing error can be tolerated in the fabrication of the servovalve.

The fact that the simple proportional control used in the pre-existing control system does not degrade the performance of the repetitive control system implies easy tuning of the control system. Extensive simulations and experimental results also show that the accuracy of the control system, with the fixed repetitive controller, is maintained at almost the same level even with large variations of specimen stiffness, fluid bulk modulus, and servovalve dynamics. The results show the effectiveness of the repetitive control system. The repetitive control system, being largely insensitive to system variations, would most likely make tuning of controller parameters unnecessary. Even if controller tuning is necessary, the dominant factor in the controller tuning is the amount of the time advance compensation in parameter b, which can be easily adjusted.
The contributions of the study are: 1) Measures of stability robustness, transient response, and performance robustness for SISO repetitive control systems have been established. 2) The development above for SISO repetitive control systems has also been extended to MIMO repetitive control systems. 3) The study leads to new interpretations of the regeneration spectrum, for large time delay systems. Also, a quantitative measure for the largeness of the time-delay has been established. 4) A repetitive control system has been successfully applied to an electrohydraulic material testing machine. The feasibility of the repetitive control of material testing is verified by the robust performance of the material testing machine.

6.2 Recommendations

As shown in this research, the tuning of repetitive controller parameters is unnecessary for a wide range of specimen stiffnesses. However, the tuning of repetitive controller parameters would become necessary if a much wider range of specimen stiffnesses is encountered or if a wider range of input frequencies were employed. The most dominant factor among the two controller parameters is the amount of the phase advance compensation used in the controller parameter b. Manual tuning of the time advance compensation is not difficult because the time advance is an integral multiple of the sampling interval and is not required to be exactly equal to the equivalent time lag of the pre-existing control system. For automatic tuning, it is essential for the controller to find the amount of time advance compensation that results in good stability and performance. To identify the time lag of the pre-existing system on-line is not especially difficult if no significant nonlinearity exists. Even though the pre-existing control system reduces the nonlinearity in the hydraulic system, the slight nonlinearity in the pre-existing control system may pose a problem for on-line identification of the pre-existing system.
Self-tuning of the time advance compensation, in the presence of slight nonlinearity, should be investigated to enable automatic tuning.

The stability criterion and the design procedure are established here assuming that the controlled plant is linear. In practice, the electrohydraulic system is nonlinear. However, the repetitive controller parameters are chosen based on the response of the pre-existing control system. The pre-existing control system can significantly reduce nonlinearity of the hydraulic system, resulting in a nearly linear response of the pre-existing control system. Successful application of the linear repetitive control system theory to the hydraulic material testing machine supports the above statements. Nevertheless, the stability of the repetitive control system in the presence of nonlinear effects needs to be investigated. If the nonlinear terms of the hydraulic system can be explicitly related to norm of $q(1-bG)$, for example, the stability of the control system could be analytically guaranteed through the use of the small gain theorem. Uncertainty in the nonlinear terms should also be considered in the formation of the norm of $q(1-bG)$.

The repetitive control system studied here is based on the internal model principle. Recently, an external model based repetitive control system has also been proposed (Tomizuka et al., 1990). The new controller predicts the periodic disturbance signal using an external model of the disturbance signal and rejects the disturbance by using counteracting control input. The external model allows on-line identification and hence the controller is inherently an adaptive control system. On the other hand, internal model based repetitive control systems may have problems if realized as adaptive controllers, especially when the error signal approaches low values. External model based repetitive controllers may be better controllers for automotive control of material testing machines because of their adaptive nature. However, external model based repetitive control systems need significant computational power to identify disturbance
signals, especially when the period of the signal is large. External model based repetitive control systems with some modifications may have potential for automatic control of closed loop material testing.

The successful application of repetitive control to single axis material testing certainly makes repetitive control of biaxial material testing very promising. Multivariable repetitive control systems has been investigated in Chapter 4. This work certainly pave the way for applying repetitive control systems to biaxial material testing. Theoretical results and design procedures for MIMO repetitive control systems discussed in Chapter 4 need to be investigated further to accommodate possible practical issues arising from the application of repetitive control to biaxial material testing.
REFERENCES


APPENDIX A.
SUPPLEMENTARY STATEMENTS AND PROOFS
A.1 Existence of the Inverse Laplace Transform of the Power Series $h(s)$

Given

(i) $R(s)$ is analytical in the right half $s$-plane

(ii) $|R(j\omega)| < 1 \ \forall \ \omega$ \hspace{1cm} (A.1-1)

the power series,

$$h(s) = 1 + e^{-T_d s} R(s) + e^{-2T_d s} R^2(s) + e^{-3T_d s} R^3(s) + \ldots$$ \hspace{1cm} (A.1-2)

is convergent for all $s$ on the imaginary axis of the $s$-plane and its inverse Laplace Transform exists

Proof:

(1) The regeneration spectrum is less than one (condition ii); therefore,

$$\rho = |e^{-T_d s} R(s)| < 1, \ s = j\omega$$ \hspace{1cm} (A.1-3)

Hence,

$$|h(s)| = |1 + e^{-T_d s} R(s) + e^{-2T_d s} R^2(s) + e^{-3T_d s} R^3(s) + \ldots|_{s=j\omega} \leq 1 + \rho + \rho^2 + \rho^3 + \ldots$$ \hspace{1cm} (A.1-4)

$h(s)$ is convergent for all $s$ on the imaginary axis.

(2) $R(s)$ is also analytical in the right half plane, i.e. it is $(L_2)$ stable.

$$\int_0^\infty |r_1(t)|^2 < \infty$$ \hspace{1cm} (A.1-5)

Hence the inverse Laplace transform of $R(s)$,

$$r_1(t) = \int_{-\infty}^{\infty} e^{c+js} R(s) ds, \quad c = 0$$ \hspace{1cm} (A.1-6)
exists, i.e. it can be evaluated along the imaginary axis. The same reasoning holds for higher order terms of \( R(s) \). Consequently, the inverse Laplace Transform of the power series, \( h(s) \), exists.

### A.2 Power Series Expansion of the Impulse Transfer Function

Consider the impulse transfer function of the repetitive control system,

\[
h = \frac{1}{1 + \left[1 + \frac{qe^{-T_0s}}{1 - qe^{-T_0s}b}\right]G_p}
\]

(A.2-1)

After some manipulations, it can be expressed as

\[
h = s_o (1 - qe^{-T_0s}) \cdot \frac{1}{1 - e^{-T_0s}q[1 - bG]}
\]

(A.2-2)

where

\[
s_o = \frac{1}{1 + G_p}
\]

(A.2-3)

and

\[
G = \frac{G_p}{1 + G_p}
\]

(A.2-4)

Letting \( R = q(1 - bG) \), the quotient on the right hand side of equation (A.2-2) can be expressed as the power series,

\[
h = s_o (1 - qe^{-T_0s}) \cdot [1 + e^{-T_0s}R + e^{-2T_0s}R^2 + e^{-3T_0s}R^3 + ...]
\]

\[
= s_o [1 + e^{-T_0s}(R - q) + e^{-2T_0s}(R - q) + e^{-3T_0s}(R - q) + ...]
\]

\[
= s_o + s_o (R - q)[e^{-T_0s}R + e^{-2T_0s}R + e^{-3T_0s}R^2 + ...]
\]

\[
= s_o - s_o (bGqe^{-T_0s})[1 + e^{-T_0s}R + e^{-2T_0s}R^2 + ...]
\]

(A.2-5)
In the above derivations, the complex arguments is omitted for simplicity. q,b, and G are stable (analytical in the right half plane) and hence R is stable (analytical in the right half plane).

A.3 Reduction of the Plant Uncertainty by Feedback Control

Consider an uncompensated plant with multiplicative unstructured uncertainty, $\Delta(s)$,

$$G_{\infty}(s) = G_{\infty,0}(s)(1 + \Delta(s))$$  \hspace{1cm} (A.3-1)

The plant is compensated by a precenzosator $K(s)$. The closed loop system is therefore,

$$G(s) = \frac{G_{\infty}(s)K(s)}{1 + G_{\infty}(s)K(s)} = G_{0}(s)(1 + \Delta_{0}(s))$$  \hspace{1cm} (A.3-2)

where $\Delta_{0}(s)$ is the unstructured uncertainty in the closed loop system. Substituting (A.3-1) into (A.3-2) yields,

$$G(s) = \frac{G_{\infty}(s)(1 + \Delta(s))K(s)}{1 + G_{\infty}(s)(1 + \Delta(s))K(s)}$$

$$= \frac{G_{\infty}(s)K(s)}{1 + G_{\infty}(s)K(s)} \frac{(1 + \Delta(s))}{1 + \Delta(s)} \frac{G_{\infty}(s)K(s)}{1 + G_{\infty}(s)K(s)}$$

$$\Delta \ G_{0}(s)(1 + \Delta(s))(1 + \Delta(s)G_{0}(s))^{-1}$$

$$= G_{0}(s)(1 + \Delta(s))(1 - \Delta(s)G_{0}(s) + \frac{1}{2!}\Delta^{2}(s)G_{0}^{2}(s) + ...)$$

$$= G_{0}(s)(1 + \Delta(s))(1 - G_{0}(s) + \frac{1}{2!}\Delta^{2}(s)G_{0}^{2}(s) + ...)$$  \hspace{1cm} (A.3-3)
If $\Delta(s)$ is lower than one in magnitude in the low frequency range, the higher order terms of $\Delta(s)$ in the above equation can be neglected. Hence, we have

$$\Delta_0(s) = \Delta(s) \left| (1 - G_p(s)) \right| = \Delta(s)S_o(s) \quad (A.3-4)$$

Within the 3-dB bandwidth of $G_0(s)$,

$$\left(1 - G_0(s)\right) \ll 1 \quad (A.3-5)$$

Hence,

$$\Delta_0(s) \ll \Delta(s) \quad (A.3-6)$$

i.e., the uncertainty in the plant is reduced by the feedback control. The reduction factor of the uncertainty is approximately equal to the sensitivity function as shown in equation (A.3-4).

### A.4. Sensitivity Function of the MIMO Repetitive Control System

Form Figure 4.1,

$$E = V - G_p(I + B(I - Qe^{-T_t})^{-1}Qe^{-T_t})E \quad (A.4-1)$$

Algebraic manipulation yields

$$(I + T_d B(I - Qe^{-T_t})^{-1}Qe^{-T_t})E = (I + G_p)^{-1}V \quad (A.4-2)$$

Since the system is linear, replacing $V$ by $(I - Qe^{-T_t})V$ will result in $E$ being replaced by $(I - Qe^{-T_t})E$.

$$(I + T_d B(I - Qe^{-T_t})^{-1}Qe^{-T_t})(I - Qe^{-T_t})E = (I + G_p)^{-1}(I - Qe^{-T_t})V \quad (A.4-3)$$
Recognizing also that

\[ Qe^{-T_s}(I - Qe^{-T_s})^{-1} = (I - Qe^{-T_s})^{-1}Qe^{-T_s} \]  \( (A.4-4) \)

we get

\[ (I - (I - T_0B)Qe^{-T_s})E = (I + G_p)^{-1}(I - Qe^{-T_s})V \]  \( (A.4-5) \)

The equivalent system in Figure 4.2 is derived from equation (A.4-5), i.e.

\[ E = (I - T_0B)Qe^{-T_s}E + (I + G_p)^{-1}(I - Qe^{-T_s})V \]  \( (A.4-6) \)

From (A.4-5),

\[ E = (I - (I - T_0B)Qe^{-T_s})^{-1}(I + G_p)^{-1}(I - Qe^{-T_s})V \]  \( (A.4-7) \)

The expression for the sensitivity function S follows readily from here.

A.5 Complementary Sensitivity Function for the MIMO Repetitive Control System

From Figure 4.1,

\[ C = G_p(I + B(I - Qe^{-T_s})^{-1}Qe^{-T_s})(V - C) \]  \( (A.5-1) \)

Algebraic manipulation yields

\[ ((I + G_p) + G_pB(I - Qe^{-T_s})^{-1}Qe^{-T_s})C = G_p(I + B(I - Qe^{-T_s})^{-1}Qe^{-T_s})V \]  \( (A.5-2) \)

Premultiply by \((I + G_p)^{-1}\) and replace \((I + G_p)^{-1}G_p\) by \(T_0\)

\[ (I + T_0B(I - Qe^{-T_s})^{-1}Qe^{-T_s})C = T_0(I + B(I - Qe^{-T_s})^{-1}Qe^{-T_s})V \]  \( (A.5-3) \)
Since the system is linear, the relationship will hold if \( C \) and \( V \) are replaced by

\[(I - Qe^{-T_x})C \text{ and } (I - Qe^{-T_x})V\]

respectively in equation (A.5-3). Using (A.4-4), we get from equation (A.5-3)

\[(I - Qe^{-T_x} + T_0BQe^{-T_x})C = T_0(I - Qe^{-T_x} + BQe^{-T_x})V \quad (A.5-4)\]

This may be rewritten as

\[C = (I - (I - T_0B)Qe^{-T_x})^{-1}T_0(I - (I - B)Qe^{-T_x})V \quad (A.5-5)\]

The expression for the complementary sensitivity function follows.

A.6 Stability Condition of the Discrete-Time Repetitive Control System

Let the characteristic equation for a discrete-time repetitive control system be

\[P(z) + Q(z)z^{-N} = 0 \quad (A.6-1)\]

where \( P(z) \) and \( Q(z) \) are polynomials in \( z \). \( P(z) \) is the characteristic polynomial of the closed loop system in the absence of the repetitive control system. The characteristic equation (A.6-1) may be rewritten as

\[P(z)(1 + \frac{Q(z)}{P(z)}z^{-N}) = 0 \quad (A.6-2)\]

The definition of the regeneration spectrum is equivalent to

\[R(\omega) = \left|\frac{Q(z)}{P(z)}\right|_{z = e^{j\omega}} \quad (A.6-3)\]
If
\[ R(\omega) < 1 \] (A.6-4)

at all frequencies, it is clear that
\[
\left| \frac{Q(e^{j\omega T})}{P(e^{j\omega T})} (e^{j\omega T})^{-N} \right| < 1
\] (A.6-5)

and hence there is no enclosure of the point (-1, i0) by the curve of \( Q(z)z^{-N}/P(z) \) as \( z \) goes over the Nyquist path in the \( z \)-plane, namely, a unit circle suitably indented to include singularities of \( Q(z)z^{-N}/P(z) \) lying on the unit circle (Phillips and Nagle, 1984).

Consequently,
\[ N_{p-z} \triangleq \text{No. of encirclements of the } (-1, i0) \text{ point by the Nyquist diagram} \]
\[ = 0 \] (A.6-6)

Now,
\[ N_p \triangleq \text{No. of poles of } Q(z)z^{-N}/P(z) \text{ outside the unit circle in the } z \text{-plane} \]
\[ = \text{No. of poles of } P(z) \text{ outside the unit circle} \] (A.6-7)

By inspection of equations (5.1) and (5.2) in Chapter 5, it is clear that the zeros of \( P(z) \) are simply the poles of \( q(z) \), \( b(z) \), the poles of the closed loop system in the absence of the repetitive control action and, in the case of Figure 5.2, the poles of \( G_h(s) \) and \( G_a(s) \) mapped to the \( z \)-plane. Since \( q(z) \), \( b(z) \), \( G_h(s) \) and \( G_a(s) \) will be chosen to be stable transfer functions, if the closed loop system is stable in the absence of repetitive control action, \( N_p \) in equation (A.6-7) will be zero. Consequently,
\[ N_z \triangleq \text{No. of roots of the equation (A.6-1) outside the unit circle} \]
\[ = N_z \cdot N_{p-z} \]
\[ = 0 \] (A.6-8)

and the discrete-time repetitive control system is stable.
A.7 Stability Robustness of the Discrete-Time Repetitive Control System

The stability robustness condition is derived for the system in Figure 5.2. For stability, equations (5.4) and (5.17) in Chapter 5 indicate that

\[ R(\omega) = |q(z)[1 - b(z)G(z)]|_{z = e^{j\omega T}} < 1 \]  \hspace{1cm} (A.7-1)

We omit the arguments for notational convenience below. Note that

\[ |q(1 - b(G_o + \Delta G))| \leq 1 \] \hspace{1cm} (A.7-2)

if

\[ |q(1 - bG_o)| + |qb\Delta G| < 1 \] \hspace{1cm} (A.7-3)

or

\[ R_o + |qb\Delta G| < 1 \] \hspace{1cm} (A.7-4)

where \( R_o \) is the nominal value of the regeneration spectrum. Alternatively, including the arguments omitted above,

\[ \left| \frac{\Delta G(z)}{G_o(z)} \right|_{z = e^{j\omega T}} \leq \frac{1 - R_o(\omega)}{\left| q(z)b(z)G_o(z) \right|_{z = e^{j\omega T}}} \] \hspace{1cm} (A.7-5)

An identical result can be obtained for the system in Figure 5.1, if \( G(z) \) is defined by equation (5.17) in Chapter 5. The result is a conservative one firstly because the requirement that \( R(\omega) \) be less than unity for stability is a conservative one, and secondly because of the simplification used in equation (A.7-3) which yields an upper bound for equation (A.7-2).
A.8 Performance Robustness of the Discrete-Time Repetitive Control System

The sensitivity function of hybrid control systems with both digital and analog components, is usually difficult to obtain. Moreover, the sensitivity function may not even have a closed form for some hybrid systems. In the discrete-time repetitive control system shown in Figure A-1, a fictitious sample and hold function is inserted before the disturbance input. If the sampling rate is fast enough, the original disturbance input is approximately equal to the signal after the sample and hold function.

Consider the error equation

\[ e(s) = -s_0(s)G_{dh}(s)d^*(s) - G(s)G_h(s)u^*_{rep}(s) + s_o(s)r(s) \]  

(A.8-1)

where \( s_o(s) \) and \( G(s) \) are defined by equation (A.2-3) and (A.2-4) respectively. The sampled function of \( c^*_s(s) \) is therefore
The input/output relationship of the discrete-time repetitive controller is given as

\[ u_{mp}^*(s) = G_v^*(s) e_\ast^*(s) \]  
(A.8-3)

Substituting equation (A.8-3) into equation (A.8-2), we have

\[ e_\ast^*(s) = -[G_v(s)s_o(s)G_{dr}(s)]^*d^*(s) - [G_v(s)G(s)G_h(s)]^*G_v^*(s)e^*_\ast^*(s) + [G_v(s)s_o(s)r(s)]^* \]

or

\[ e_\ast^*(s) = \frac{1}{1 + [G_v(s)G(s)G_h(s)]^*G_v^*(s)} \left\{ [G_v(s)s_o(s)G_{dr}(s)]^*d^*(s) - [G_v(s)s_o(s)r(s)]^* \right\} \]

(A.8-5)

The output is expressed as

\[ c(s) = r(s) - e(s) \]  
(A.8-6)

Substituting equation (A.8-1), (A.8-3), (A.8-5) into (A.8-6), we have

\[ c(s) = \\
\frac{s_o(s)G_{dr}(s)d^*(s) - G(s)G_h(s)G_v^*(s)}{1 + [G_v(s)G(s)G_h(s)]^*G_v^*(s)} \left[ G_v(s)s_o(s)G_{dr}(s) \right]^*d^*(s) + G(s)G_h(s)G_v^*(s) \frac{1}{1 + [G_v(s)G(s)G_h(s)]^*G_v^*(s)} \left[ G_v(s)s_o(s)r(s) \right]^* + G(s)r(s) \]

(A.8-7)

Performing the starred transform on both side of equation (A.8-7) and neglecting the reference input yields
where

$$W(s) = [s_o(s)G_{dh}(s)]^*[G^*(s)G(s)G_h(s)] G^*_r(s)$$

Note that $W(s)$ is nearly zero. The sensitivity function is the relationship between the disturbance signal and the output,

$$S^*(s) = \frac{c^*(s)}{d^*(s)} = \frac{1}{1 + [G^*(s)G(s)G_h(s)] G^*_r(s)} [s_o(s)G_{dh}(s)]^*$$

Taking z-transform of the above equation, we obtain the discrete-time sensitivity function after some manipulations.

$$S(z) = \frac{1}{1 + \frac{q(z)z^{-N}}{1 - q(z)z^{-N}} b(z)Z[G^*(s)G(s)G_h(s)]} Z[s_o(s)G_{dh}(s)]$$

$$= \frac{1 - q(z)z^{-N}}{1 - z^{-N}q(z) \{1 - b(z)Z[G^*(s)G(s)G_h(s)]\}} Z[s_o(s)G_{dh}(s)]$$

$$= \frac{1 - q(z)z^{-N}}{1 - z^{-N}R(z)} Z[s_o(s)G_{dh}(s)]$$
A.9 Magnitude Response of Zero Phase Discrete-Time Systems

Consider a discrete-time system with two real zeros,

\[ F(z) = (z + a)(z^{-1} + a) = a \frac{(z + a)(z + 1/a)}{z} \]  (A.9-1)

where \(-a\) is the unstable zero of the system, i.e. the magnitude being larger than unity. The system \(F(z)\) has zero phase frequency response and is typically a resulting system from zero phase compensation of a discrete-time system with one real unstable zero. The magnitude response of the \(F(z)\) can be obtain graphically in Figure A.2.

\[ |F(z)|_{\omega_0} = a \overline{AC} \cdot \overline{A'C} \]  (A.9-2)

where the point \(C\) is on the unit circle with center at point \(O\). Points \(A\) and \(A'\) indicate the locations of the zeros at \(-a\) and \(-1/a\) respectively.

Figure A.2 Graphical Representation of System Gain, Equation (A.9-2)

Note that

\[ \overline{AC} = \sqrt{\overline{OA'}^2 + \overline{OC}^2 + 2 \overline{OA'} \cdot \overline{OC} \cos \theta} \]  (A.9-3)
\[ \overline{AC} = \sqrt{OA'^2 + OC'^2 + 2OA' \cdot OC \cos \theta} \]  

(A.9-4)

Substituting the above two equations into equation (A.9-3) yields

\[
f(\theta) \Delta |F(\theta)| = a\sqrt{\left( a^2 + 1 + 2a \cos \theta \right) \left( a^2 + 1 + 2 \cos \theta \right)} \]

\[
= a\sqrt{\left( a^2 + 1/a^2 + 2 \right) + 4(a + 1/a) \cos \theta + 4 \cos^2 \theta} \]

\[
= a\sqrt{2 \cos \theta + a + 1/a} \]  

(A.9-5)

Note that \( a \) is a positive real number,

\[
(a + 1/a) > 2 \geq -2 \cos \theta, \quad \forall \theta \]  

(A.9-6)

Hence,

\[
f(\theta) = a(2 \cos(\theta) + a + 1/a) \]  

(A.9-7)

Note that the gain response of the system, \( f(\theta) \), is a sinusoidal function offset by a positive value. The maximum of \( f(\theta) \) is \( a(2a+1/a) \) when \( \theta \) is equal to zero. In other words, the maximum gain of a zero phase system, with one negative real zero, occurs at a \( \theta \) of zero radians. Flipping Figure A.2 horizontally, we can show that the maximum gain of a zero phase system, with one positive real zero, occurs at a \( \theta \) of \( \pi \) radians.

Now, a zero phase system with two unstable zeros is shown in Figure A.3. The zero phase system discussed here is assumed to have only real coefficients in its transfer function, resulting in a pair of conjugate complex roots \(-z_1\) and \(-z_1^*\). Two zeros are added to the system to obtain a zero phase characteristic. Consider the system transfer function.
\[ F(z) = (z + z_1)(z + z_1^*)(z^{-1} + z_1)(z^{-1} + z_1^*) \]

\[ = |z_1|^2 \frac{(z + z_1)(z + z_1^*)(z + 1/z_1)(z + 1/z_1^*)}{z^2} \]  

(A.9-8)

The two zeros are denoted by points A and B and their reciprocals denoted by points A' and B' in the Figure A.3. Hence the system gain is

\[ |F(z)|_{k=1^*} = a^2 \overline{AC} \cdot \overline{BC} \cdot \overline{A'C} \cdot \overline{B'C} \]

\[ = (a \overline{AC} \cdot \overline{BC}) \cdot (a \overline{AC} \cdot \overline{B'C}) \]

\[ = f(\beta + \theta) \cdot f(\beta - \theta) \]  

(A.9-10)

where \( a \) is the magnitude of the unstable zeros and \( f() \) is the function described in equation (A.9-7).
\[ |F(z)|_{z=e^j\theta} = a(2\cos(\beta + \theta) + a + 1/a) \cdot a(2\cos(\beta - \theta) + a + 1/a) \]

\[ = a^2[4\cos(\beta + \theta)\cos(\beta - \theta) + 2(a + 1/a)(\cos(\beta + \theta) + \cos(\beta - \theta)) + (a + 1/a)^2] \]

\[ = a^2[2(\cos(2\beta) + \cos(2\theta)) + 4(a + 1/a)(\cos \beta \cos \theta) + (a + 1/a)^2] \]

\[ = a^2[4\cos^2 \beta + 4\cos^2 \theta - 4 + 4(a + 1/a)(\cos \beta \cos \theta) + (a + 1/a)^2] \]

\[ = a^2[(2\cos \theta + (a + 1/a)\cos \beta)^2 + 4\cos^2 \beta - 4(a + 1/a)^2 \cos^2 \beta + (a + 1/a)^2 - 4] \]

\[ = a^2[(2\cos \theta + (a + 1/a)\cos \beta)^2 - (a - 1/a)^2 \cos^2 \beta + (a - 1/a)^2] \]

\[ = a^2[(2\cos \theta + (a + 1/a)\cos \beta)^2 + ((a - 1/a)\sin \beta)^2] \quad \text{(A.9-12)} \]

The zeros are on the left half of the z-plane, and hence,

\[ -\frac{\pi}{2} < \beta < \frac{\pi}{2} \quad \text{and} \quad \cos \beta > 0 \quad \text{(A.9-13)} \]

Form equation (A.9-12), the maximum system gain is

\[ \max_{\theta}|F(\theta)| = a^2[(2 + (a + 1/a)\cos \beta)^2 + ((a - 1/a)\sin \beta)^2] \quad \text{(A.9-14)} \]

at a \( \theta \) of zero radian. In other words, the maximum gain of a zero phase system, having two complex conjugate zeros with negative real parts, occurs at zero radians. Flipping Figure A.3 horizontally, we can prove that the maximum gain of a zero phase system, having two complex conjugate roots with positive real parts, occurs at \( \pi \) radians.
APPENDIX B
COMPUTER PROGRAMS
B.1 Simulation Program of the Two-Variable Repetitive Control System

"CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC"
"C REPETITIVE CONTROL OF THE TWO-INPUT-TWO-OUTPUT SYSTEM  C"
"C                                  FENG-RONG SHAW MAY 1990  C"
"CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC"
INITIAL "$ " SIMULATION CONSTANTS 
CONSTANT FINITIM=20.0 PERI=0.5, ...
AMP=1.

" -- CONTROLLER PARAMETERS ...................................................... "
CONSTANT DDX=0.05,DDY=0.05

CONSTANT WLX=120.,CH2=1.44
PLX=1./WLX/WLY
QLX=1.4/WLY
DDCX=CH2/WLY
PERIAX=PERI-DDX-DDCX

CONSTANT WLY=120.
PLY=1./WLY/WLY
QLY=1.4/WLY
DDCY=CH2/WLY
PERIAY=PERI-DDY-DDCY

" -- NUMERICAL ALGORITHM FOR SIMULATIONS .................................... "
CINTERVAL CINT=0.01
ALGORITHM IALG=5
NSTEPS NSTP=4

END $ "OF INITIAL."

DYNAMIC
"**********************************************
"* PLANT DANAMICS AND CONTROL SYSTEMS  *
"**********************************************
DERIVATIVE

" -- REFERENCE SIGNAL ..................................................... "

" SINE WAVE "
YSI=SIN(T*6.283185/PERI)

" SQRARE WAVE "
YSQR=1-2*PULSE(PERI/4,PERI,PERI/2)

" TRIANGULAR "
YTRI=4/PERI*INTEG(YSQR,0)
"-- PLANT DYNAMICS -------------------------------------------------------------"

ARRAY P1(3), Q1(4), P2(3), P3(3), P4(3)
CONSTANT P1=2.1500, 16.0000, 32.6000
CONSTANT P2=1.1000, 4.0000, 9.4000
CONSTANT P3=1.0500, 4.0000, 6.2000
CONSTANT P4=1.0, 4.0, 3.0
CONSTANT Q1=1.0, 6.0, 11.0, 6.0
X1=TRAN(2,3,P1,Q1,UX)
X2=TRAN(2,3,P2,Q1,UY)
Y1=TRAN(2,3,P3,Q1,UX)
Y2=TRAN(2,3,P4,Q1,UY)
X=X1+X2
Y=Y1+Y2

"-- REPETITIVE CONTROLLER -----------------------------------------------------"

REFX=AMP*YSI

ERX=REFX-X
U1X=ERX
ER3X=DELAY(ER1X,0,PERIAX,5000)
ER2X=CMPXPL(PLX,Q1X,ER3X,0.0,0.0,0.0)
ER1DX=DELAY(ER2X,0,DDX,5000)
ER1X=ERX+ER1DX
UXI=U1X+ER2X

REFY=AMP*YSI

ERY=REFY-Y
U1Y=ERY
ER3Y=DELAY(ER1Y,0,PERIAY,5000)
ER2Y=CMPXPL(PLY,QLY,ER3Y,0.0,0.0,0.0)
ER1DY=DELAY(ER2Y,0,DDY,5000)
ER1Y=ERY+ER1DY
UYI=U1Y+ER2Y

"-- PRECOMPENSATOR OF THE CONTROLLER ------------------------------------------"

UX=5.8903*INTEG(UXI,0)-18.4534*INTEG(UYI,0)+...
5.5984*UXI-6.1474*UYI
UY=-12.1727*INTEG(UXI,0)+63.9969*INTEG(UYI,0)-...
5.8676*UXI+12.0252*UYI

END $ "OF DERIVATIVE"

TERMT(T .GE. FINTIM)

END $ "OF DYNAMIC"
END $ "OF PROGRAM"
B.2. Fortran Program for Solving Characteristic Equation of the Two-Variable Repetitive Control System

```fortran
REAL ERRABS, ERRREL
INTEGER INFO(3), ITMAX, NKGUESS, NKNOWN, NNEW
COMPLEX F, Z(3), ZINIT(3)
EXTERNAL F, ZANLY, WRCRN

C INITIAL GUESSES OF THE ZEROS
C
ZINIT(1) = (-0.56208, 13.59)
ZINIT(2) = (-0.6729, 13.58)
ZINIT(3) = (-0.6730, 13.55)

C PARAMETERS OF THE SUBROUTINE ZANLY
C
ERRABS = 0.00001
ERRREL = 0.00001
KNOWN = 0
NEW = 3
GUESS = 3
ITMAX = 400

C IMSL SUBROUTINE FOR FINDING ZEROS OF A COMPLEX FUNCTION
C
CALL ZANLY(F, ERRABS, ERRREL, KNOWN, NEW, GUESS,
N=Z, IMAX, INFO)
CALL WRCRN( 'THE ZEROS ARE', 1, 3, Z, 0)
WRITE(9, *) Z(I), I=1, 3
STOP
END

C DEFINITION OF THE COMPLEX FUNCTION
C
COMPLEX FUNCTION F(Z)
COMPLEX ZD, ZP, ZBX, ZBY, ZQ, ZT, ZN1, ZN2, ZN3, ZN4
REAL N1(9), N2(9), N3(9), N4(9), D(9)
N1(9) = 0.
N1(8) = 849052.241
N1(7) = 19767892.354
N1(6) = 208586233.597
N1(5) = 1240250950.204
N1(4) = 3974556316.514
N1(3) = 644614203.133
N1(2) = 4237249778.703
N1(1) = 915671670.436
```
\[ N_2(9) = 0. \\
N_2(8) = 1644.183 \\
N_2(7) = -2961517.8610 \\
N_2(6) = -37071305.458 \\
N_2(5) = -148243203.360 \\
N_2(4) = -229708090.45 \\
N_2(3) = -115580792.459 \\
N_2(2) = -7761.949 \\
N_2(1) = 0. \\
\]

\[ N_3(9) = 0. \\
N_3(8) = 1630.528 \\
N_3(7) = -1064758.980 \\
N_3(6) = -7649527.778 \\
N_3(5) = -19131191.344 \\
N_3(4) = -19868051.046 \\
N_3(3) = -7318478.691 \\
N_3(2) = 1519.279 \\
N_3(1) = 0. \\
\]

\[ N_4(9) = 0. \\
N_4(8) = 847261.980 \\
N_4(7) = 20176024.457 \\
N_4(6) = 212179422.685 \\
N_4(5) = 1251442600.153 \\
N_4(4) = 3989147246.863 \\
N_4(3) = 6452678399.2155 \\
N_4(2) = 4237231427.662 \\
N_4(1) = 915671670.456 \\
\]

\[ D(9) = 152099.928 \\
D(8) = 3521513.365 \\
D(7) = 4403644.365 \\
D(6) = 329384469.578 \\
D(5) = 1564873382.749 \\
D(4) = 4428995425.996 \\
D(3) = 6745780591.004 \\
D(2) = 4308049144.473 \\
D(1) = 915671670.456 \\
\]

C
C COMPELEX POLYNOMIALS
C
ZI1=0
DO 1 I=1,9
ZI1=ZI1+NI1*(Z**(I-1))
1 CONTINUE

ZI2=0
DO 2 J=1,9
ZI2=ZI2+NI2*(Z**(J-1))
2 CONTINUE
ZN3 = 0
DO 3 I = 1, 9
   ZN3 = ZN3 + N3(I) * (Z**I - 1))
3 CONTINUE

ZN4 = 0
DO 4 I = 1, 9
   ZN4 = ZN4 + N4(I) * (Z**I - 1))
4 CONTINUE

ZD = 0
DO 5 I = 1, 9
   ZD = ZD + D(I) * (Z**I - 1))
5 CONTINUE

C
C LOWPASS FILTER Q
C
   ZP = Z**Z/120. + 1.4*Z/120. + 1.
   ZQ = EXP(0.012*Z)

C
C TIME ADVANCE COMPENSATION
C
   ZBX = EXP(0.05*Z)
   ZBY = EXP(0.05*Z)

C
C TIME-DELAY OF THE SYSTEM
C
   ZT = EXP(-0.5*Z)

C
C DEFINITION OF THE COMPLEX FUNCTION
C
   F = (ZD*ZP - (ZD-ZN1*ZBX)*ZQ*ZT)* (ZD*ZP - (ZD-ZN4*ZBY)*ZQ*ZT)
      & (ZN2*ZN3*(ZBX*ZBY)*(ZQ)*ZQ)*ZT*ZT
C
RETURN
END
B.3 Simulation Program of the Material Testing Machine Control System

"REPEATED CONTROL OF THE MATERIAL TESTING MACHINE"
"OCT 10, 1990 FENG-RONG SHAW"

"INITIALIZATION BLOCK"

INITIAL $ " SIMULATION CONSTANTS "

CONSTANT FINTIM=2.0, PERI=0.1, ...
EBIAS=0.0, AMP=0.25

" -- PROCESS CONSTANTS "

CONSTANT KG=30.07, KK=0.2817, ...
MT=0.0363, BP=0.0, ...
CIP=2.5E-5, CEP=0.0E-6, ...
KSP=1.5E6, KSM=3.0E5, ...
FC=30., AP=1.17, ...
VA1=4.212, VA2=4.212, ...
CD=0.61, W=0.881, ...
K1=1.62, KSV=3.0E4, ...
DELAI=-0.000, Delta2=-0.000, ...
DELAI3=-0.000, DELA4=-0.000, ...
RHO=7.8E-5, BETA=1.2E5, ...
PS=1000., TAU=0.00187

KS=(KSP*KSM)/(KSP+KSM) $ " EQUIVALENT SYSTEM STIFFNESS"
VA1=VAI1
VA2=VAI2
KGA=KG+4.7

" -- SENSOR GAINS "

CONSTANT XPMK=2.5, FSMK=0.001

" -- COEFFICIENTS OF B FILTER AND Q FILTER "

CONSTANT WB=500, RQ=0.98
C1=4/DTSAMP/DTSAMP/WB/WB
C2=2/DTSAMP*1.4/WB

" COEFFICIENTS OF B "
D1=(C1+C2+1)
D2=2*(C1-1)
D3=(C1-C2+1)

" COEFFICIENTS OF Q "
QN2=2*RQ
QN3=RQ**2
QS=1+QN2+QN3
" -- INITIALIZATION OF STATES AND REFERENCE SIGNALS -- "

" IF THE SIMULATION IS TO BE CONTINUED FROM PREVIOUS SIMULATION, SET CON=1, SET FINTIM, AND TYPE REINIT "
" OTHERWISE, THE START COMMAND WILL START FROM TIME=0 "

CONSTANT CON=2
ARRAY VP(1000), ERP(1000), REF(1000)

PROCEDURAL
  IF (CON .EQ. 1) GO TO 10
  PA1=0.
  PA2=0.
  DXP=0.
  XP=0.
  UZ=0.
  ER=0.
  ERD=0.
  ERD1=0.
  IX=0.
  UP=0.
  UP2=0.
  UREP=0.
  REFOUT=0.
  DO 111*I=1,1000
     VP(I)=0.
     ERP(I)=0.
  11.. CONTINUE
  10.. CONTINUE

" -- REFERENCE SIGNAL FROM THE DIGITAL COMPUTER -- "

" NSIG=1, FOR SINUSOIDAL SIGNAL "
" NSIG=2, FOR TRIANGULAR SIGNAL "

CONSTANT NSIG=2

  NN=IFIX(PERI/DTSAMP) $ " NUMBER OF SAMPLES PER PERIOD "

" SINUSOIDAL SIGNAL "
  DO 20 I=1,NN
     REF(I)=AMP*SIN(FLOAT((I-1))/NN*6.2832)
  20.. CONTINUE

" TRIANGULAR SIGNAL "
  IF (NSIG .EQ. 1) GO TO 25
  PP4=PERI/4.
  PP43=PP4*3.
  PP4K=AMP/PP4.
  DO 51 I=1,NN
     TT=I*DTSAMP
     VAL=PP4K*TT
     VREF=VAL
     IF (TT .GE. PP4) VREF=2*AMP-VAL
     IF (TT .GE. PP43) VREF=VAL-4*AMP
     REF(I)=VREF
  25.. CONTINUE

END $ " OF PROCEDURAL "
" -- NUMERICAL ALGORITHM FOR SIMULATIONS ------------------------------------------ "
CONSTATE DTSAMP=0.003 $ " COMPUTER SAMPLING TIME "
CINTERVAL CINT=0.0001
ALGORITHM IALG=5
NSTEPS NSTP=1

END $ "OF INITIAL"

"* DYNAMIC SIMULATION BLOCK
*  

DYNAMIC
DERIVATIVE

" -- CONTROL VOLTAGE TO THE 406.11 CONTROLLER ------------------------------- "
VV=(UZ+EBIAS)*KG*KK

" HYDRAULIC MACHINE DYNAMICS

"  

" -- SATURATION OF SERVOAMPLIFIER -------------------------------------------- "
PROCEDURAL(V=VV)
V=VV
IF(VV .GE. 15.45) V=15.45
IF(VV .LT. -15.45) V=-15.45
END

" -- SERVOVALVE DYNAMICS ------------------------------------------------------- "
IA=K*I*V
XVV=KSV*IA
XV=REALPL(TAU,XVV,0.0)

" -- ORIFICE FLOW EQUATIONS --------------------------------------------------- "
PROCEDURAL (QA1,QA2=XV,W,CD,PA1,PA2,PS,RHO)
IF(PA1 .LT. 0.)PA1=0.
IF(PA2 .LT. 0.)PA2=0.
PSPA1=PS-PA1
PSPA2=PS-PA2
IF(PSPA1 .LT. 0.)PSPA1=0.
IF(PSPA2 .LT. 0.)PSPA2=0.
IF(XV .GE. -DELA1)QA1=CD*W*(DELA1+XV)*SQRT(2./RHO*(PSPA1))
IF(XV .LT. -DELA1)QA1=0.
IF(XV .GE. -DELA2)QA2=CD*W*(DELA2+XV)*SQRT(2./RHO*(PA2))
IF(XV .LT. -DELA2)QA2=0.
IF(XV .LE. DELA3)Q3=CD*W*(DELA3-XV)*SQRT(2./RHO*(PSPA2))
IF(XV .GT. DELA3)Q3=0.
IF(XV .LE. DELA4)Q4=CD*W*(DELA4-XV)*SQRT(2./RHO*(PA1))
IF(XV .GT. DELA4)Q4=0.
QA1=Q1-Q4
QA2=Q2-Q3

END $ "OF PROCEDURAL"
"-- FORCE BALANCE AND CONTINUITY EQUATIONS-------------------------------------" 
VA1=VA1I+XP*AP 
VA2=VA2I-XP*AP 
DPA1=(QA1-AP)*DP+CIP*(PA1-PA2)*CEP*PA1*BETA/VA1 
DPA2=(AP-PA2)*DP+QA2+CIP*(PA1-PA2)*CEP*PA2*BETA/VA2 
CONSTANT PA1I=1000, PA2I=1000 
PA1=INTEG(DPA1,PA1I) 
PA2=INTEG(DPA2,PA2I) 
DELP=PA1-PA2 
FP=DELP*AP 
FB=FC*SDXP 
FS=KS*XP 
FIT=FP-FB-FS 
D2XP=FIT/MT 
DXP=INTEG(D2XP,0) 
XPA=INTEG(DXP,0) 

"-- DRY FRICTION MODEL-------------------------------------------------------" 
PROCEDURAL(XP,SDXP=XPA,DXP) 
IF(DXP LE 0.) SDXP=1. 
IF(DXP NE 0.) SDXP=DXP/ABS(DXP) 
XP=XPA 
IF(XPA GT 3.6) XP=3.6 
IF(XPA LT -3.6) XP=-3.6 
END $ "OF PROCEDURAL"

" ANALOG FEEDBACK SIGNALS " 
"=========================================================================

" POSITION MEASUREMENT " 
XPM=XP*XPMK 

" LOAD MEASUREMENT WITH SENSOR NOISE " 
CONSTANT NST=0.0, NTAU=0.001, ... 
SDEV=0.003, NMEAN=0 
PROCEDURAL 
NOISE=0 
IF (T GT NST) NOISE=OU(NTAU,NMEAN,SDEV) 
FSM=FS*FSMK 
FSM=FSMN+NOISE 
END 

"-- ANALOG ERROR SIGNAL------------------------------------------------------" 
ER=REFOUT-FSM 

"-- SUM OF ERROR SIGNAL AND REPETITIVE CONTROL SIGNAL " 
USUM=ER+UREP
" -- ANALOG LEAD-LAG CONTROLLER ------------------------------------------"
ARRAY ZZ(3),PP(3)
CONSTANT ZZ=1,300,20000, SPAIN=0.5, ...
PP=1,602,1200,
UU=TRAN(2.2,ZZ,PP,USUM)*3
UZ=USUM*SPAIN
END $ "OF DERIVATIVE"

"==-----------------------------------------------------------------------"
" DIGITAL COMPUTER SIGNALS "
"==-----------------------------------------------------------------------"

DISCRETE OUTREF
INTERVAL DTSAMP=0.003

" -- REFERENCE SIGNAL OUTPUT ------------------------------------------"
PROCEDURAL
IX=IX+1
IF (IX .LE. NN) GO TO 30
IX=1
30.. CONTINUE
REFOUT=REF(IX)

" --- TIME DELAY OF ERROR INPUT AND CONTROL OUTPUT ---------------------- "
CONSTANT DTIN=0.0005, DTOUT=0.0025
SCHEDULE INERR .AT. T+DTIN
SCHEDULE OUTCON .AT. T+DTOUT
END $ "PROCEDURAL "
END $ "OF OUTREF"

" -- ERROR SIGNAL INPUT TO THE DIGITAL COMPUTER ------------------------ "
DISCRETE INERR
PROCEDURAL
ERD=ER
END

" -- REPEETITIVE CONTROL OUTPUT FROM THE DIGITAL COMPUTER "
DISCRETE OUTCON
PROCEDURAL
" TIME TO SWITCH TO REPEETITIVE CONTROL "
CONSTANT TSW=0.1
IF (T .LT. TSW) GO TO 60
" SIGNAL PROCESSING OF THE DIGITAL REPETITIVE CONTROLLER "
" PROCESSING OF TIME DELAY AND FILTER B "

CONSTANT ID=5 $ " TIME ADVANCE IN B, ID*DSAMP "

TSUM=0
TSUM=TSUM+(VP(NN)+ERP(NN-ID))*0.25+...
(VP(NN+1)+ERP(NN-ID+1))*0.5+(VP(NN+2)+ERP(NN-ID+2))*0.25
VR=TSUM*QS-VP(1)*QN2-VP(2)*QN3

" STATE VARIABLE UPDATE, VP "
NN1=NN+2
NN3=NN1-2
DO 40 J=0,NN3
VP(NN1-J)=VP(NN1-1-J)
40.. CONTINUE
VP(1)=VR

" STATE VARIABLE UPDATE, ERP "
DO 50 J=0,NN3
ERP(NN1-J)=ERP(NN1-1-J)
50.. CONTINUE
ERP(1)=ERD

" SIGNAL PROCESSING OF FILTER B "
UU=((VR+2*VP(2)+VP(3))-D2*UP-D3*UP2)/D1

" STATE VARIABLE UPDATE, FILTER B "
UP2=UP
UP=UU

" END OF DIGITAL REPETITIVE CONTROLLER "
" END OF PROCEDURAL "$

IF(TLT.TSW) UUF=0
IF(TGE.TSW) UUF=UU

" SATURATION OF THE DAC CONVERTER "
UZZ=UUF
IF(UZZ .GE. 5.) UZZ=5.
IF(UZZ .LE. -5.) UZZ=-5.

" QUANTIZATION OF THE REPETITIVE CONTROL SIGNAL"
UREP=QNTZR(0.00244,UZZ)

END $ "OF PROCEDURAL"
B.4 Repetitive Control Program Used For Experimental Tests

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C 409.6 = 1 VOLT

C REFERENC PERIODICAL SIGNAL FORMS
C
WRITE(6,*)'ENTER 1 FOR SINUSOIDAL SIGNAL'
WRITE(6,*)'ENTER 2 FOR TRIANGULAR SIGNAL'
READ(5,*)NU

C SINUSOIDAL
ENP=PERI/SAMP*1000.
NP=IFIX(ENP)
DO 5 I=1,NP
   REF(I)=AMP*SIN(FLOAT(I-1)/ENP*6.2832)
5 CONTINUE
IF (NU.EQ. 1) GO TO 51

C TRIANGULAR
P4=PERI/4.
P43=P4*3.
P4K=AMP/P4
DO 51 I=1,NP
   TT=FLOAT(I)*TSAMP
   VAL=P4K*TT
   V=VAL
   IF (TT .GE. P4) V=2.*AMP-VAL
   IF (TT .GE. P43) V=VAL-4.*AMP
   REF(I)=V
51 CONTINUE

C INPUT THE PHASE ADVANCE COMPENSATION, (5 SAMPLES ADVANCED)
WRITE(6,*)'TIME ADVANCE? (5, SAMPLING INTERVALS)'
READ(5,*)IA

C COEFFICIENTS OF FILTER Q
WRITE(6,*)' POLE LOCATION OF FILTER B , 0.96'
READ(5,*)RQ
Q1=RQ*2.
Q2=RQ*RQ
QS=1.+Q1+Q2
ZV(1)=0.25*QS
ZV(2)=0.5*QS
ZV(3)=0.25*QS
C COEFFICIENTS FOR LOWPASS FILTER B
C
WRITE(6,*)'BANDWIDTH OF B? (500) RAD/SEC'
READ(5,*)WB
C1=4./TSAMP/TSAMP/WB/WB
C2=2./TSAMP**1.4/WB
D1=(C1+C2+1.)
D2=-2.*(C1-1.)
D3=(C1-C2+1.)

C PARAMETERS NEEDED IN THE CIRCULAR QUEUES
C
NH=1
NH1=NH+1
NE=NP+NH+NH
N=2*NH+1
NT=NP-NH-2

C INITIALIZE VP,ERP
C
DO 30 I=1,NE
   ERP(I)=0.
30   VP(I)=0.

C GET THE SETUP READY AND THEN START THE CONTROL SYSTEM
C
WRITE(6,*)'WHEN READY HIT ANY NUMBER TO START'
READ(5,*)NU

C INITIALIZE THE INTERRUPT SERVICE ROUTINE
C
ICOUNT = IFIX(SAMP*153.67)
CALL PLMSAM(ICOUNT)
CALL INITINTERRUPT

C HALT THE MAIN PROGRAM UNTIL THE DESIRED NO. OF CYCLES IS REACHED
C
IFIN10=IFIN+100
40 CONTINUE
   IF (IT.LE. IFIN10) GO TO 40

C ZERO THE CONTROL OUTPUT WHEN THE TEST IS OVER
C
IZ0=0
CALL DA1(IZ0)
CALL DA2(IZ0)

C END OF MAIN PROGRAM
C
STOP 
END
SUBROUTINE USERCON

DIMENSION VP(1000), REF(1000), ERP(1000), ZV(10)
COMMON /DATA/ VP, REF, ERP, ZV
COMMON /NUMBERS/ NH, NP, N, NT, ION, IT, IFIN, IOFF, IA
COMMON D1, D2, D3, Q1, Q2
DATA IT, IP/1, 0, 1/
DATA UP, UP2, VRP, VRP2/0.0, 0.0, 0.0, 0.0/

CYCLE COUNTER

I = I + 1
IF (I .GT. NP) THEN
  I = 1
  IT = IT + 1
ENDIF

CHECK IF THE TEST IS OVER

IF (IT .LE. IFIN) GO TO 1
IZ0 = 0
CALL DA1(IZ0)
CALL DA2(IZ0)
GO TO 200

OUTPUT REFERENCE SIGNAL TO OUTPUT CHANNEL ONE

IREF = IFIX(REF(I))
CALL DA1(IREF)

INPUT ERROR SIGNAL FROM INPUT CHANNEL ONE

CALL AD1(IER)
ER = FLOAT(IER)

CHECK IF IT IS IN REPETITIVE CONTROL MODE

IF ((IT .LE. ION) .OR. (IT .GE. IOFF)) GO TO 100

PROCESSING OF THE REPETITIVE CONTROLLER

The error data of the previous cycle are retrieved, precessed, and stored in circular queues. The time shift is done by shifting the index pointer, IP.
VR = 0
DO 10 J = 1, N
M = NT + NH + IP + J
ME = M - IA
IF (M .GT. NE) M = M - NE
IF (ME .GT. NE) ME = ME - NE
10 VR = VR + (ERP(ME) + VP(M)) * ZV(J)
VR = VR - VRP * Q1 - VRP2 * Q2
C
C STORE THE DATA IN CIRCULAR QUEUES
C
  IP=IP-1
  IF (IP .EQ. 0) IP=NE
  VP(IP)=VR
  ERP(IP)=ER

C
C PROCESSING OF FILTER B
C
  U= ((VR+2.*VRP+VRP2)-D2*UP-D3*UP2)/D1

C
C UPDATE STATES OF THE FILTER B
C
  UP2=UP
  UP=U
  VRP2=VRP
  VRP=VR

C
UUF=U
GO TO 102

C
C NO REPEETITIVE CONTROL OUTPUT
C
100  UUF=0

C
C LIMITS OF OUTPUT VALUES: + -2047 i.e, + -5 Volts
C
102  IF (UUF .GE. 2047.) UUF=2047.
    IF (UUF .LE. -2047.) UUF=-2047.

C
C OUTPUT REPEETITIVE CONTROL SIGNAL TO OUTPUT CHANNEL TWO
C
  IUZ=IFIX(UUF)
  CALL DA2(IUZ)

C
C OUTPUT SAMPLED ERROR SIGNAL TO OUTPUT CHANNEL THREE
C
  CALL DA3(IER)

C
C END OF UERCON SERVICE ROUTINE
C
200  CONTINUE
    RETURN
END