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Small signal AC model for the velocity-saturated MODFET and the prediction of the microwave characteristics of MODFETs

Kang, Sung Choon, Ph.D.
The Ohio State University, 1991
SMALL SIGNAL AC MODEL FOR THE VELOCITY-SATURATED MODFET AND THE PREDICTION OF THE MICROWAVE CHARACTERISTICS OF MODFETS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University

1991

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ACKNOWLEDGEMENTS

I wish to express my sincere appreciation and gratitude to my advisor, Professor Patrick Roblin for his advice, guidance and financial support throughout my Master’s and Ph.D. research at The Ohio State University.

Furthermore, I wish to thank Professor Steve Biblyk and Professor Furrukh S. Khan for reading my dissertation and providing constructive criticism. I also express my gratitude to Dr. Hardis Morkoç in University of Illinois at Urbana-Champaign for providing the measured data for my research work.

Finally, I wish to express my gratitude to my wife, Kyungsook, my son, Sangwoo, and my daughter, Yousun, for their help and understanding. Especially, I owe to my parents and my relatives in Korea who have supported my Ph.D. studies.
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2. P. Roblin, S.C. Kang, and H. Morkoç, “Microwave Characteristics of the MOD-
FET and the Velocity-Saturated MOSFET Wave-Equation,” Proceedings of the
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May 1990

Microwave Characteristics,” IEEE Trans. Electron Devices, vol. ED-34, No.9,
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**LIST OF SYMBOLS**

- $C_g$: the gate capacitance per unit area
- $C_{GS}$: the fringe capacitors between gate and source
- $C_{GD}$: the fringe capacitors between gate and drain
- $d_s$: the channel width in the saturation region
- $f_{MAX}$: maximum oscillation frequency
- $E_c$: the critical electric field to attain the peak velocity
- $\varepsilon_1$: the dielectric constant for the channel material
- $\varepsilon_2$: the average dielectric constant for the high-bandgap region
- $I(x,t)$: total current in the channel
- $I_{dc}(x)$: the dc channel current
- $i(x)$: the ac channel current
- $q$: electron charge
- $L_g$: the gate length
- $l$: the length of the saturation region
- $n_s(x,t)$: two dimensional electron density in the channel
- $\mu$: the channel mobility
- $v_s$: saturation velocity of electron
- $\tau_s$: $\frac{l}{v_s}$ time delay due to saturation region
- $v_s$: the saturation velocity of electrons
- $V_T$: the threshold voltage
- $V_{CS}(x)$: DC channel to source voltage at the position $x$
- $v_{CS}(x)$: total channel to source voltage at the position $x$
- $V_{GC}(x)$: total gate to channel voltage at the position $x$
- $V_{GC}(x)$: DC gate to channel voltage at the position $x$
- $v_{gs}(x)$: AC gate to channel voltage at the position $x$
- $V_{GS}$: DC applied voltage between the gate and the source
- $v_{gs}$: AC applied voltage between the gate and the source
- $V_{DS}$: DC applied voltage between the drain and the source
\( v_{ds} \) : AC applied voltage between the drain and the source
\( k = \frac{V_{DS}}{V_{GS}} - V_T \) for unsaturated device
\( x_S \) : instantaneous position of the GCA/saturation boundary
\( X_S \) : dc position of the GCA/saturation boundary
\( x_a \) : ac motion of the GCA/saturation boundary
\( W_g \) : the gate width
\( \omega_o = \frac{\mu (V_{GS} - V_T)}{L_g^2} \)
CHAPTER I

INTRODUCTION

1.1 Background

The recently developed MODFET is a promising device for both microwave and millimeter-wave applications and high-speed digital circuits. It has demonstrated remarkable high-speed performances [1]. Propagation delays as small as 12 ps and below 10 ps have been obtained in ring oscillator measurements at 300°K and 77°K respectively [2], [3]. Very low noise figures have been measured at microwave frequencies (0.4 db with 14 db gain at 10GHz at 77°K [4]). Its low noise figure at high frequencies makes it attractive in microwave applications. MODFETs based on novel compound semiconductors are showing very promising results (see [5] for a review). Recently the GE Electronics Laboratory reported 0.15 μ gate length InAlAs/InGaAs/InP lattice-matched MODFETs with maximum frequency of oscillation as high as 405 GHz [6].

In support of the development of the MODFET technology, there has been a strong modeling effort reported in the literature. Indeed since the first dc model reported by Delagebeaudeuf [7] a large number of dc models for the MODFETs have been published. Some authors have in addition derived small-signal models for the MODFET using the quasi-static formula $C_{gs} = \frac{\partial Q}{\partial V_{gs}}$ and $C_{gd} = \frac{\partial Q}{\partial V_{gd}}$ where
$Q$ is the charge stored in the channel. So far, however, there have been fewer reported attempts to compare the microwave performance predicted from these models with the published microwave data available for the MODFETs. Yeager and Dutton [8] reported a large-signal model which they compared with reasonable success to scattering parameters measured at 4 GHz. A numerical microwave model based on a transmission line circuit model was reported [9]. The simulation results obtained with their numerical model show a good agreement with the scattering parameters of a 0.3 micron gate length MODFET measured at a single bias point.

An analytic microwave model including distributed effects for the unsaturated MODFET [10] was recently reported. It permitted us to reasonably reproduce simultaneously the dc characteristics and microwave performance of an unsaturated MODFET using a unique set of device parameters. This analytic ac model was however limited to the linear regime up to the edge of saturation.

Since the MODFET wave equation has the same form as the three terminal MOSFET's, it is useful to investigate the previous work on the MOSFET wave equation. The wave-equation for the unsaturated MOSFET was derived independently by Candler and Jordan [11], Geurst [12] and Hauser [13]. Geurst derived an exact solution of the three terminal MOSFET wave-equation in terms of Stokes' functions [12]. Burns [14] and Treleaven and Trofimenkoff [15] derived independently an exact solution in terms of Bessel functions. These exact solutions are not however analytic per se, as they involve Stokes or Bessel functions which much be numerically generated. Approximate equivalent circuits were derived by both Burns [14] and Treleaven and
Trofimenkoff [15] for the case of the MOSFET operated in pinch-off. For this mode of operation they reported approximated analytic expressions for $Y_{11}$ and $Y_{21}$ (in pinch-off $Y_{12} = Y_{22} = 0$). An alternative procedure based on an iterative scheme was introduced by Van Nielen [16] to obtain accurate approximate results of the MOSFET wave equation. This iterative solution was used by Bagheri and Tsividis [17] and Bagheri [18] for deriving the small-signal $Y$-parameters of the long-channel four terminal MOSFET and three-terminal MODFET, respectively. More recently using a frequency power series first introduced by Van der Ziel and Ero [19] for the junction FET, Van der Ziel and Wu [20] solved the MODFET wave-equation and calculated $Y_{11}$ in terms of a frequency power series for the unsaturated MODFET. Roblin and Kang [10] continued their calculation and derived the remaining $Y$ parameters $Y_{12}$, $Y_{21}$ and $Y_{22}$. The frequency power series has the advantage of being analytic and holding up to high frequencies.

To summarize there exist both an exact solution and an analytic frequency power series solution of the MODFET (and MOSFET) wave-equation. The MODFET (and MOSFET) wave-equation applies however only to the unsaturated MOSFET up to the edge of saturation or to long-channel devices operated in pinch-off. However saturation in a MOSFET or MODFET results from velocity saturation and not pinch-off for submicron gate length.

1.2 Problem Statement

In order to analyze the microwave characteristics of the velocity-saturated MODFET, an equivalent circuit is often used to fit the device performance measured at various
frequencies. A typical equivalent circuit is shown in Figure 1. This approach does not permit to determine the element values from the device parameters and to predict its bias dependence. The element values of the equivalent circuit are generally valid only over the frequency range for which the parameter extraction is performed, so that attempts to extrapolate the response of the circuit beyond this frequency range can produce misleading results.

The small-signal ac models quoted in previous section are directly derived from their dc models using the quasi-static approximation. Since these models cannot account directly for the propagation delay across the channel and for distributed effects such as the effective channel charging resistances of the device capacitances, these might not successfully simulate the observed frequency and bias dependence of the MODFET characteristics. A high frequency ac model should account for the
propagation delay and the distributed effects to predict the correct high frequency dependence of the MODFET characteristics.

Recently Rohdin [21] extended the unsaturated model [10] to the saturated MODFET with the aid of a drain resistor and capacitor in parallel to represent the characteristics of the saturation region. There is however no systematic way to predict the values of the drain conductance and capacitance from the device parameters or the bias conditions. Consequently, their values are fitted so as to obtain a good agreement with the measured data.

In order to account for these saturation effects in the microwave characteristics of the velocity-saturated MODFET, the wave equation in saturation region should be derived and solved. The equivalent circuit based on wave equation will give better representation of the velocity-saturated MODFET. It will give the basis for

Figure 2: Equivalent circuit for the velocity-saturated intrinsic MODFET proposed by Rohdin [21]
the development of a large signal model for short channel MODFETs.

1.3 Structure of Dissertation

In Chapter II the wave equation of velocity-saturated MODFET including both velocity-saturation and channel length modulation effects will be derived, using a simple transport picture. The intrinsic MODFET is divided into two regions separated by a floating boundary. The wave equation holding in these regions will be derived. Both an exact solution using Bessel function as reported in [14] and an approximate frequency power-series solution are derived. The comparison between two methods will be made to estimate the validity of the analytic frequency power-series solution.

In Chapter III an equivalent circuit representation of the velocity-saturated MODFET will be introduced. First a simple RC equivalent circuit for the GCA region (or long channel MODFET) will be derived based on the frequency power-series solution of the GCA wave equation. An optimal second-order equivalent circuit developed using fourth order frequency power-series solution will also be derived. An equivalent circuit for the saturation region based on the exact solution in the saturation region, will also be developed. The total equivalent circuit for the velocity saturated MODFET will be then constructed by combining the circuits for the GCA region and the saturation region.

In Chapter IV we will discuss the integration of this ac-model with a dc-model. We then compare the frequency and bias dependence of the scattering parameters calculated from this microwave model with the scattering parameters of a AlGaAs/GaAs
MODFET and GaAlAs/InGaAs/GaAs pseudomorphic MODFET measured for several bias conditions. In order to introduce some physical insights, a discussion on the significance of some of the physical parameters will be given.

In Chapter V the exact solution of the wave equation of velocity-saturated MODFET will be used to analyze the microwave characteristics of intrinsic MODFETs. The dependence of the unilateral gain, $f_T$ and $f_{MAX}$ dependence upon gate length, parasitics resistance and capacitance will then be discussed.

Chapter VI concludes this dissertation by discussing the future development required in order to improve the ac model and develop a large signal model.

The appendices give the detailed calculation of the frequency power-series solution (Appendix A), exact solution (Appendix B) and equivalent circuit for the saturated MODFET (Appendix D). Also included is Macsyma program to calculate the fourth-order frequency power-series solution of the wave equation.
CHAPTER II

THE VELOCITY-SATURATED MODFET WAVE EQUATION AND ITS SOLUTION

2.1 Derivation of the Wave-Equation

The distributed ac-model for the saturated MODFET is based on a simple but well founded transport picture which assumes that transport is taking place in the 2DEG channel and relies on the following electron velocity-field relation

\[ v_e = \mu E \quad \text{for} \quad E < E_c \]

\[ = v_s = \mu E_c \quad \text{for} \quad E > E_c \]

(2.1)

where \( v_s \) is an effective saturation velocity. \( v_s \) typically corresponds to the peak velocity of the stationary velocity-field relation of the material constituting the channel [22]. Velocity overshoot over the stationary velocity is partially implied in the assumption that the electron velocity remains the peak velocity \( v_s \) for channel fields beyond the critical field \( E_c \). Velocity overshoot above the effective saturation velocity is neglected since it has a minimal impact on the dc characteristics for moderately sub-micron gate-length MODFETs. Indeed it have recently been shown in [22] for a 0.5 \( \mu \) MODFET that a hydrodynamic model developed by Widiger and Hess [23], allowing the velocity to overshoot up to three times the peak stationary velocity,
predicted the same drain current and transconductance as a simple analytic model using the peak stationary velocity for effective saturation velocity. The hydrodynamic model predicts however a transport induced degradation in the drain conductance not accounted for by the analytic model. Therefore the simple transport model selected here is more appropriate than the stationary velocity-field relation used in [9] for micron and moderately sub-micron gate length FET's.

For the purpose of analysis the intrinsic MODFET is divided into two regions, the so-called gradual-channel approximation (GCA) and saturation regions, as was done in a previously reported dc model for the saturated MODFET [24]. As shown in Figure 3, the GCA and saturation regions are located between the gate and the 2DEG channel on the source and drain side respectively. In the GCA region the
gradual channel approximation holds and the 2DEG concentration $n_S$ is controlled by the gate to channel potential

$$q n_S(x,t) = C_g [v_{GS}(t) - v_{CS}(x,t) - V_T]$$

(2.2)

where $C_g$ is the 2DEG gate capacitance. In the saturation region two dimensional field effects dominate and the GCA approximation breaks down. The channel potential $v_{CS}$ can then be approximately obtained by solving the Poisson Equation along the 2DEG channel

$$\frac{d^2 v_{CS}(x,t)}{dx^2} = \frac{q n_S(x,t)}{d_s \varepsilon_1} = \beta I(x,t)$$

(2.3)

where $\beta = 1/\varepsilon_1 v_g W_g d_s$. This simple model has the advantage over the Grebene Ghandhi model [24] of predicting a larger drain conductance. However a detailed analysis of two-dimensional field effects in the saturation region [22] reveals that the charge distribution in the channel only partially account for the dc drain conductance $g_D$ (additional contributions to $g_D$ appear to be transport and traps related). As it will be seen in Chapter IV, it is preferable for the microwave model to use a physical value for the channel width $d_s$ even though an artificial closer fit of the measured dc drain conductance $g_D$ can be achieved with unphysically large channel openings $d_s$.

Following the pioneering work of Grebene and Ghandhi it is assumed as in dc model [24] that the boundary between the GCA and saturation regions occurs when the channel field $dv_{CS}/dx$ reaches the critical field $E_c$. Consequently the GCA region includes the portion of the channel where the electron velocity has not yet reached saturation and the saturation region includes the portion of the channel in which the velocity saturation is taking place.
Let us now establish the wave equation which applies in each region. The wave equation for the GCA region was derived in [13], [12] and, [11]. The relationship between the ac current and voltage in the GCA region is [20]

\[ i(x) = -\frac{d}{dx}[g(V_{GC}(x))v_{gc}(x)] \]  (2.4)

The wave equation obtained for the GCA region is [20]

\[ \frac{d^2}{dx^2}[g(V_{GC}(x))v_{gc}(x)] = j\omega C_g W_g v_{gc}(x) \]  (2.5)

using the function \( g(V_{GC}(x)) = \mu W_g C_g (V_{GC}(x) - V_T) \). The dc potential \( V_{GC}(x) - V_T \) is given by (see Appendix B)

\[ V_{GC}(x) - V_T = (V_{GS} - V_T)\sqrt{1 + (k_s^2 - 2k_s)} \frac{x}{X_S} \]  (2.6)

with \( k_s = V_{CS}(X_S)/(V_{GS} - V_T) \) and \( V_{CS}(X_S) \) the dc channel to source potential across the entire GCA region.

The channel current in the saturation region can be expressed by

\[ I(x,t) = I_{dc} + i(x)e^{j\omega t} = qW_g n_S(x,t)v_s \]  (2.7)

and the continuity equation in the channel

\[ \frac{\partial I(x,t)}{\partial x} = qW_g \frac{\partial v_s n_S(x,t)}{\partial x} = -qW_g \frac{\partial n_S(x,t)}{\partial t} = -\frac{1}{v_s} \frac{\partial I(x,t)}{\partial t} \]  (2.8)

Extracting the ac part from Equation (2.8) and retaining the first order terms yields

\[ \frac{di(x)}{dx} = -j\omega i(x) \]  (2.9)
In the saturation region the ac current is related to the ac voltage by the Poisson Equation (2.3). Decomposing Equation (2.3) into dc and ac parts yields the following relationship between the ac voltage $v_{gc}(x)$ and current $i$

$$\frac{d^2 v_{gc}(x)}{dx^2} = -\beta i(x)$$

Equations (2.9) and (2.10) make up the wave equation for the saturation region.

The solution of the wave equation across the entire channel requires a set of boundary conditions to be enforced at $x = 0$ and $x = L_g$ and at the boundary between the GCA and saturation region. The boundary conditions to be used at $x = 0$ and $x = L_g$ for the common source configuration are

$$v_{gc}(0) = v_{gs}$$

$$v_{gc}(L_g) = v_{gs} - v_{ds}$$

The continuity of the 2DEG carrier concentration, channel electric field and channel potential, electron velocity and current are enforced at the GCA/saturation boundary. These are naturally enforced by the continuity of the ac voltage $v_{gc}$ and ac current $i$ at the boundary.

Note that according to saturation picture, the channel electric field at the floating boundary between the GCA and saturation region is the dc (constant) critical field $E_c$. The ac channel field is therefore null at the boundary, and the GCA/saturation boundary must move when ac voltages are applied at the device terminals so as to maintain a zero ac channel field. In the small signal analysis the total (dc + ac) position of the GCA/saturation boundary is written
where $X_S$ is the dc position and $x_a$ the ac motion of the boundary. Let us now derive the relationship between the ac motion $x_a$ of the GCA/saturation boundary and the GCA ac field $v'_{gc}$. The total (dc + ac) channel field at the floating boundary $x_S$ is the spatial derivative of the total potential $v_{gc}$ at this boundary

$$v'_{GC}(x_S) = V'_{GC}(x_S) + v'_{gc}(x_S)e^{i\omega t}$$

The ac electric field at the floating boundary is then, neglecting second order terms

$$v'_{gc}(x_S) = V''_{GC}(X_S)x_a + v'_{gc}(X_S)$$

Setting the ac electric field at the floating boundary to zero yields the boundary motion $x_a$ as a function of $v'_{gc}$

$$x_a = -\frac{1}{V''_{GC}(X_S)}v'_{gc}(X_S)$$

where one can easily calculate $V''_{GC}(X_S)$ to be given by: (see Appendix A.2 for detail calculation)

$$V''_{GC}(X_S) = -\frac{k_s^2(1 - \frac{1}{2}k_s)^2}{(1 - k_s)^3} \frac{V_{GS} - V_T}{X_S^2}$$

The solution of the wave equation across the entire intrinsic MODFET relies on the continuity of the ac voltage and ac current at the floating boundary. It is therefore necessary to calculate the ac voltage at the floating boundary and account for the motion of the GCA/saturation boundary. Let us derive the modified GCA channel potential obtained at the floating boundary. The total (dc + ac) channel potential at the floating boundary is given by
\[ v_{GC}(x_s, t) = V_{GC}(x_s) + v_{gc}(x_s)e^{j\omega t} \]  
(2.18)

where \( v_{gc} \) is the ac potential obtained by solving Equation (2.5). Expanding Equation (2.18) with a Taylor series around the dc boundary position \( X_S \) for small variations \( x_s \) of the boundary position yields the ac voltage \( v_{gc}(x_s) \) at the floating boundary \( x_s \) (second order terms are neglected)

\[ v_{gc}(x_s) = V_{GC}'(X_S)x_s + v_{gc}(X_S) \]  
(2.19)
\[ = -E_c x_s + v_{gc}(X_S) \]  
(2.20)

The potential drop across the saturation region is also modified by the motion of the boundary which modulates the width of the saturation region. Integrating the Poisson equation

\[ \frac{d^2(v_{GC}(x))}{dx^2} = -\beta I(x, t) = -\beta (I_{dc} + i(x)e^{j\omega t}) \]  
(2.21)

across the time varying saturation region yields the ac potential at \( x = L_g \) (see Appendix A.1 for detail calculation)

\[ v_{gc}(L_g) = (E_c + \beta I_{dc} l) x_s + v_{gc}(x_s) + \Delta v_{gc}(l) \]  
(2.22)

where we introduced \( l = (L_g - X_S) \) the dc width of the saturation region and where \( \Delta v_{gc}(l) \) is the ac potential \( v_{gc}(x) \) obtained by solving the Poisson Equation (2.10) for a fixed saturation region width \( l \), and zero ac potential \( v_{gc}(X_S) = 0 \) and zero ac field \( v_{gc}'(X_S) = 0 \) at \( X_S \). Substituting Equation (2.20) into Equation (2.22) gives

\[ v_{gc}(L_g) = \beta I_{dc} l x_s + \Delta v_{gc}(l) + v_{gc}(X_S) \]  
(2.23)
One observes that the contribution of the motion $x_s$ of the GCA/saturation boundary is to add the ac potential term $\beta I_{dc} l x_s$.

Finally note that the ac current at the floating boundary $x_s$ is to first order equal to the ac current at the fixed boundary $X_s$. This originates in the fact that the dc current $I_{dc}$ is continuous (constant) along the channel.

2.2 Exact Solution of the Velocity-Saturated MODFET wave-equation

It has been shown by Burns [14] that the voltage-wave solution of the wave-equation (2.5) can be expressed in terms of the modified Bessel functions $I_{\pm 2/3}(Y)$ (see Appendix B for detail calculation)

$$v(x,\omega) = C_1 I_{2/3}(Y) + C_2 I_{-2/3}(Y)$$

(2.24)

The current-wave is then derived from Equation (2.4) to be

$$i(x,\omega) = G'_{d_{os}} \sqrt{S'}^{1/4} [C_1 I_{-1/3}(Y) + C_2 I_{1/3}(Y)]$$

(2.25)

where $Y$ is a variable defined by $Y = 4/3\sqrt{S'}(P)^{3/4}$, $P$ is a position variable defined by $P = 1 - (2k_s - k_3^2)x/X_s$, $S'$ is the normalized frequency $S' = j\omega/\omega_0 k$ with $\omega_0 k = \mu(V_G - V_T)(2k_s - k_3^2)X_s^2$, and $G'_{d_{os}} = (2k_s - k_3^2)G_{d0s}$ with $G_{d0s}$ in Section 2.3. The wave-equation in the saturation region can be readily derived. The current-wave is obtained by integrating Equation (2.9)

$$i(x) = i(X_s) e^{-j S'_{1/3}(x - X_s)}$$

(2.26)

and the voltage-wave by integrating Equation (2.10)
The unknown coefficients $C_1$ and $C_2$ are obtained from the boundary conditions (2.11) and (2.12),

\[
\begin{align*}
V_{gc}(L_g) &= A_{11}C_1 + A_{12}C_2 = v_{gs} - v_{ds} \\
V_{gc}(0) &= A_{21}C_1 + A_{22}C_2 = v_{gs}
\end{align*}
\]  

(2.28)  

(2.29)

where the coefficients $A_{ij}$ are evaluated using Equations (2.22) and (2.24)

\[
\begin{align*}
A_{11} &= I_{2/3}(Y_s) + G'_{dss}\sqrt{S'}P_s^{1/4} \left[ \beta \left( \frac{v_{gs}}{\omega} \right)^2 \left[ e^{-j\frac{\pi}{2}} - 1 \right] + j\beta \frac{v_{gs}}{\omega} \right] I_{1/3}(Y_s) \\
&\quad - \beta I_{dc} (1 - k_s)^3 X_s \sqrt{S'}P_s^{1/4} \frac{[I_{1/3}(Y_s) + I_{5/3}(Y_s)]}{k_s (1 - \frac{1}{2} k_s) V_{out}} \\
A_{12} &= I_{-2/3}(Y_s) + G'_{dss}\sqrt{S'}P_{s^{-1/4}} \left[ \beta \left( \frac{v_{gs}}{\omega} \right)^2 \left[ e^{-j\frac{\pi}{2}} - 1 \right] + j\beta \frac{v_{gs}}{\omega} \right] I_{1/3}(Y_s) \\
&\quad - \beta I_{dc} (1 - k_s)^3 X_s \sqrt{S'}P_{s^{-1/4}} \frac{[I_{1/3}(Y_s) + I_{5/3}(Y_s)]}{k_s (1 - \frac{1}{2} k_s) V_{out}} \\
A_{21} &= I_{2/3} \left( \frac{4}{3} \sqrt{S'} \right) \\
A_{22} &= I_{-2/3} \left( \frac{4}{3} \sqrt{S'} \right)
\end{align*}
\]

using $\Delta = A_{11}A_{22} - A_{12}A_{21}$, $Y_s = 4/3\sqrt{S'}(P_s)^{3/4}$, and $P_s = (1 - k_s)^2$. The unknown coefficients $C_1$ and $C_2$ are then derived from the system of equations (2.29) to be

\[
\begin{align*}
C_1 &= \frac{A_{22} - A_{12}}{\Delta} v_{gs} - \frac{A_{22}}{\Delta} v_{ds} \\
C_2 &= \frac{A_{11} - A_{21}}{\Delta} v_{gs} + \frac{A_{21}}{\Delta} v_{ds}
\end{align*}
\]  

(2.30)  

(2.31)

Finally the ac current flowing into the gate $i_g$ and the ac current flowing into the drain $i_d$ are given in terms of the applied gate to source voltage $v_{gs}$ and drain to
source voltage $v_{gs}$ by

\[ i_d = i(L_g) = i(X_s)e^{-j \frac{\pi}{2} l} \]  
\[ = G'_{dos} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{2} l}[C_1 I_{-1/3}(Y_s) + C_2 I_{1/3}(Y_s)] \]  
\[ i_g = i(0) - i(L_g) \]  
\[ = G'_{dos} \sqrt{S'} \left[ C_1 I_{-1/3} \left( \frac{4}{3} \sqrt{S'} \right) + C_2 I_{1/3} \left( \frac{4}{3} \sqrt{S'} \right) \right] \]  
\[ - G'_{dos} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{2} l}[C_1 I_{-1/3}(Y_s) + C_2 I_{1/3}(Y_s)] \]

These currents hold for arbitrary large frequencies. Note that the modified Bessel functions can be numerically calculated using the expansion [25]

\[ I_n(Y) = \left( \frac{Y}{2} \right)^n \sum_{j=0}^{\infty} \frac{\left( \frac{Y}{2} \right)^{2j}}{j! \Gamma(n + j + 1)} \]  

The drain and gate currents obtained for the saturated MODFET model reduce for $l = 0$, $X_S = L_g$ and $V_{CS}(X_S) = V_{DS}$ to the drain and gate currents of the unsaturated MODFETs.

### 2.3 Y-Parameters within the Frequency Power-Series approximation

The exact solution derived in the previous section was obtained in terms of modified Bessel functions. Since these functions must be generated numerically it is convenient to use instead a frequency power series solution which usually holds up to very high frequencies. This frequency power series solution have directly derived from the wave-equation using the method proposed by Ziel and Ero [19]. The detailed calculation is described in Appendix A.
The frequency power series yields the ac current flowing into the gate $i_g$ and the ac current flowing into the drain $i_d$ in terms of the applied gate to source voltage $v_{ds}$ and drain to source voltage $v_{gs}$.

$$i_g = Y_{11}v_{gs} + Y_{12}v_{ds} \quad (2.38)$$

$$i_d = Y_{21}v_{gs} + Y_{22}v_{ds} \quad (2.39)$$

The Y coefficients calculated are the common source Y-parameters of the intrinsic MODFET. Port 1 is defined between gate and source and port 2 between drain and source.

Before giving the calculated Y-parameters let us first define the following terms:

$$A(k_s) = \frac{1}{1 - \frac{1}{2}k_s}$$

$$B(k_s) = \frac{1}{6 \left(1 - \frac{1}{2}k_s\right)^2}$$

$$C(k_s) = \frac{\frac{1}{2} - \frac{1}{3}k_s}{\left(1 - \frac{1}{2}k_s\right)^2}$$

$$D(k_s) = \frac{\frac{1}{2} - \frac{1}{6}k_s}{\left(1 - \frac{1}{2}k_s\right)^2}$$

$$E(k_s) = \frac{\frac{1}{6} - \frac{1}{6}k_s + \frac{1}{30}k_s^2}{\left(1 - \frac{1}{2}k_s\right)^3}$$

$$F(k_s) = \frac{\frac{1}{4} - \frac{1}{30}k_s + \frac{1}{180}k_s^2}{\left(1 - \frac{1}{2}k_s\right)^4}$$

$$G(k_s) = \frac{\frac{1}{4} - \frac{1}{20}k_s + \frac{1}{72}k_s^2}{\left(1 - \frac{1}{2}k_s\right)^4}$$

$$H(k_s) = \frac{\frac{1}{120} - \frac{1}{80}k_s + \frac{1}{180}k_s^2 - \frac{1}{1440}k_s^3}{\left(1 - \frac{1}{2}k_s\right)^5}$$
\[ R_i = \frac{(1 - k_s)^{2}}{k_s^2(1 - \frac{1}{2} k_s)^{2}} \frac{I_{dc} \beta I_{\text{X}}}{V_{\text{out}}} \]

\[ R_{gc} = \frac{C_{0s}}{G_{\text{d0s}}} \]

\[ R_y = \frac{E_c X S}{(1 - k_s)V_{\text{out}}} \]

\[ R_A = (1 - k_s)R_i + \frac{\beta I^2}{2} G_{\text{d0s}} (1 - k_s) \]

\[ R_B = 1 + (1 - k_s)R_i R_y \]

\[ R_d = \frac{1}{R_A + R_B} = \frac{1}{1 + \frac{\beta I^2}{2} G_{\text{d0s}} (1 - k_s)} \]

\[ G_{\text{d0s}} = \frac{\mu C_{g} W_{g} (V_{GS} - V_{T})}{X_S} \]

\[ k_s = \frac{V_{CS}(X_S)}{V_{GS} - V_{T}} \]

\[ C_{0s} = C_{g} W_{g} X_S \]

The \( Y_{11} \) and \( Y_{21} \) parameters for the saturated MODFET are:

\[ Y_{11} = \omega^2 \left[ \frac{G_{gs} \tau_s^2}{2} - C_{gs} \tau_s + H_{gs} + F_{gs} \right] - j \omega \left[ -G_{gs} \tau_s + C_{gs} + E_{gs} \right] \]

\[ Y_{21} = G_{gs} - \omega^2 \left[ \frac{G_{gs} \tau_s^2}{2} - C_{gs} \tau_s + H_{gs} \right] + j \omega \left[ -G_{gs} \tau_s + C_{gs} \right] \]

where

\[ G_{gs} = G_{\text{d0s}} R_d (k_s + (1 - k_s)R_y R_i) \]

\[ C_{gs} = R_{gc} G_{gs} D - C_{0s} A - E_{gs} \]

\[ H_{gs} = R_{gc}^2 G_{gs} F - R_{gc} C_{0s} B - R_{gc} E_{gs} D - F_{gs} \]

\[ E_{gs} = R_{gc} C_{gs} R_d [R_A D + R_B E] - C_{0s} R_d [R_A A + R_B C] - \frac{\beta I^2 \tau_s^2}{6} G_{gs} G_{ds} \]

\[ F_{gs} = R_{gc} R_d [R_{gc} G_{gs} (R_A F + R_B H) - C_{0s} (R_A B + R_B G) - E_{gs} (R_A D + R_B E)] + \frac{\beta I^2 \tau_s^2}{24} G_{ds} G_{gs} - \frac{\beta I^2 \tau_s^2}{6} G_{ds} C_{gs} \]
\( G_{ds} \) is defined in the next section.

The \( Y_{12} \) and \( Y_{22} \) for the saturated MODFET are:

\[
Y_{12} = \omega^2 \left[ \frac{G_{ds}}{2} \tau_s^2 - C_{ds} \tau_s + H_{ds} + F_{ds} \right] - j\omega \left[ -G_{ds} \tau_s + C_{ds} + E_{ds} \right]
\]
\[
Y_{22} = G_{ds} - \omega^2 \left[ \frac{G_{ds}}{2} \tau_s^2 - C_{ds} \tau_s + H_{ds} \right] + j\omega \left[ -G_{ds} \tau_s + C_{ds} \right]
\]

where

\[
G_{ds} = G_{ds}R_d(1 - k_s)
\]
\[
C_{ds} = R_{gc}G_{ds}D - E_{ds}
\]
\[
H_{ds} = R_{gc}G_{ds}F - R_{gc}G_{ds}D - F_{ds}
\]
\[
E_{ds} = R_{gc}R_dG_{ds} [R_A D + R_B E] - \frac{\beta l^2 \tau_s^2 G_{ds}^2}{6}
\]
\[
F_{ds} = R_{gc}R_d [R_{gc} G_{ds} (R_A F + R_B H) - E_{ds} (R_A D + R_B E)] + \frac{\beta l^2 \tau_s^2 G_{ds}^2}{24} - \frac{\beta l^2 \tau_s^2 G_{ds} C_{ds}}{6}
\]

Note that the Y-parameters derived for the saturated MODFET reduce for \( l = 0 \), \( X_S = L_g \) and \( V_{CXS}(X_S) = V_{DS} \) to the Y-parameters reported in [10] for the unsaturated MODFET, except, however, for the term \( R_{22}(k) \), which as pointed out by [26] is incorrect. The correct term \( R_{22}(k) \) is

\[
R_{22}(k) = \frac{(1 - k)(1 - \frac{7}{180} k + \frac{17}{720} k^2 - \frac{1}{160} k^3 + \frac{9}{14400} k^4)}{(1 - \frac{2}{7} k)^6}
\]  \hspace{1cm} (2.40)

The correction introduced is small.

It is necessary to assess the frequency range of validity of the frequency power-expansion solution. For this purpose the frequency dependence of the Y parameters
Figure 4: Exact solution (solid lines) and frequency power-series solution (dotted lines) for magnitude of $Y_{11}$

Figure 5: Exact solution (solid lines) and frequency power-series solution (dotted lines) for phase of $Y_{11}$
Figure 6: Exact solution (solid lines) and frequency power-series solution (dotted lines) for magnitude of $Y_{12}$

Figure 7: Exact solution (solid lines) and frequency power-series solution (dotted lines) for phase of $Y_{12}$
Figure 8: Exact solution (solid lines) and frequency power-series solution (dotted lines) for magnitude of $Y_{21}$

Figure 9: Exact solution (solid lines) and frequency power-series solution (dotted lines) for phase of $Y_{21}$
Figure 10: Exact solution (solid lines) and frequency power-series solution (dotted lines) for magnitude of $Y_{22}$

Figure 11: Exact solution (solid lines) and frequency power-series solution (dotted lines) for phase of $Y_{22}$
Table 1: Device parameters for calculating the intrinsic Y-parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_g$ gate length ($\mu m$)</td>
<td>1</td>
</tr>
<tr>
<td>$W_g$ gate width ($\mu m$)</td>
<td>290</td>
</tr>
<tr>
<td>$\mu$ mobility ($cm^2/V.sec$)</td>
<td>4400</td>
</tr>
<tr>
<td>$v_s$ Saturation velocity ($m/sec$)</td>
<td>$3.45 \times 10^5$</td>
</tr>
<tr>
<td>$V_T$ threshold voltage ($V$)</td>
<td>-0.3</td>
</tr>
<tr>
<td>$d$ gate to channel spacing ($\AA$)</td>
<td>430</td>
</tr>
<tr>
<td>$d_s$ channel width in saturation ($\AA$)</td>
<td>1500</td>
</tr>
<tr>
<td>$\epsilon_1$ channel dielectric constant</td>
<td>$13.1 \epsilon_0$</td>
</tr>
<tr>
<td>$\epsilon_2$ gate dielectric constant</td>
<td>$12.2 \epsilon_0$</td>
</tr>
</tbody>
</table>

are compared for a fixed bias of both the analytic and exact solutions in Figure 4 - 11.

The device parameters used for this comparison are shown in Table 1. The intrinsic Y-parameters are calculated for $V_{GS} = 0V$ and $V_{DS} = 3V$. One observes that the analytic solution compare to the exact solution for frequencies up to 40 GHz. This frequency is near the extrinsic $f_{\text{max}}$ as will be found in Chapter IV.


CHAPTER III

EQUIVALENT CIRCUIT REPRESENTATION

3.1 Introduction

The MODFET wave-equation admits an exact small-signal solution in the frequency domain in terms of the modified Bessel functions. However Bessel functions are difficult to generate and do not permit the development of a large-signal model. The approximate analytic solutions are preferred for CAD applications since they are fast and permit the development of equivalent circuits useful for time domain analysis. The equivalent circuit for the velocity-saturated MODFET can be directly derived from the frequency power-series solution using a first-order RC topology for the entire MODFET. However this does not provide a physical representation of the circuit so that it is desirable to derive the equivalent circuits for each region and cascade them together to obtain a more physical equivalent circuit.

Two methods, the iterative and power-series methods, have been used to obtain approximate solutions of the MODFET wave equation in the GCA region. The small-signal Y parameters obtained by an iteration [18] of order two admit a frequency power-series expansion valid up to power two. These iterative Y-parameters hold for higher frequencies and have the advantage of providing a more graceful degradation
outside their frequency range of validity, compared to the Y-parameters obtained by the frequency power-series of order two [18]. The iterative procedure yields for MODFET up to first order, small-signal Y-parameters of the following form

\[ Y_{ij} = \frac{g_{ij} + j\omega a_{ij}}{1 + j\omega b_{ij}} \]  

(3.1)

This equation suggests the first-order RC topology for the equivalent circuit model in GCA region. However the circuit directly derived from the iterative method does not improve the total performance of the circuit, so that it is needed to derive the circuit from the second-order power-series solution using the first-order RC topology. Based on the first-order RC topology, the second-order RC topology can be derived from the fourth-order frequency power-series expansion.

In order to develop the equivalent circuit for the saturation region the Poisson’s equation and wave equation have to be solved. The solution will account for the drain delay and the potential drop in the drain region. As seen in Chapter II this equation can be solved exactly and this is used to develop the equivalent circuit of the saturation region. Combining the equivalent circuit of the GCA and saturation regions give the complete equivalent circuit.

### 3.2 The First Order Equivalent Circuit for the GCA region

A simple RC equivalent circuit model will be introduced to provide a graceful degradation of the Y-parameters for frequencies \( \omega \) larger than \( \omega_0 \). The RC model selected consists of the DC (\( \omega = 0 \)) small-signal parameters \( g_{ij} \) shunted by a capacitor \( C_{ij} \) in
series with a charging resistor $R_{ij}$. The resulting intrinsic $Y$-parameters are

\[
\begin{align*}
Y_{11} &= \frac{j\omega C_{11}}{1 + j\omega R_{11}C_{11}} \\
Y_{12} &= \frac{j\omega C_{12}}{1 + j\omega R_{12}C_{12}} \\
Y_{21} &= g_m + \frac{j\omega C_{21}}{1 + j\omega R_{21}C_{21}} \\
Y_{22} &= g_d + \frac{j\omega C_{22}}{1 + j\omega R_{22}C_{22}}
\end{align*}
\]

The associated equivalent-circuit for the intrinsic MODFET is shown in Figure 12. For frequencies $\omega << 1/(R_{ij}C_{ij})$ these $Y$-parameters admit the frequency power-series

\[
Y_{ij} = g_{ij} + j\omega C_{ij} + \omega^2 R_{ij}C_{ij}^2
\]

We can now readily identify the resistors and capacitors to be

\[
\begin{align*}
C_{11} &= \frac{g_o(V_{GS})I_{11}(k)}{\omega_o} \\
C_{12} &= -\frac{g_o(V_{GS})I_{12}(k)}{\omega_o} \\
R_{11} &= \frac{R_{11}}{g_o(V_{GS})I_{11}(k)} \\
R_{12} &= -\frac{R_{12}}{g_o(V_{GS})I_{12}(k)}
\end{align*}
\]
\[ C_{21} = \frac{g_o(V_{GS})I_{21}(k)}{\omega_o} \quad \quad R_{21} = -\frac{\mathcal{R}_{21}}{g_o(V_{GS})I_{21}^2(k)} \]
\[ C_{22} = \frac{g_o(V_{GS})I_{22}(k)}{\omega_o} \quad \quad R_{22} = \frac{\mathcal{R}_{22}}{g_o(V_{GS})I_{22}^2(k)} \]

where

\[ \mathcal{R}_{11}(k) = \frac{\frac{1}{12} - \frac{1}{6}k + \frac{9}{80}k^2 - \frac{7}{240}k^3 + \frac{1}{360}k^4}{(1 - \frac{1}{2}k)^6} \]
\[ \mathcal{I}_{11}(k) = \frac{1 - k + \frac{1}{2}k^2}{(1 - \frac{1}{2}k)^2} \]
\[ \mathcal{R}_{12}(k) = \frac{(1 - k)(\frac{1}{24} - \frac{41}{720}k + \frac{1}{49}k^2 - \frac{1}{360}k^3)}{(1 - \frac{1}{2}k)^5} \]
\[ \mathcal{I}_{12}(k) = \frac{(1 - k)(1 - \frac{1}{2}k)}{2(1 - \frac{1}{2}k)^2} \]
\[ \mathcal{R}_{21}(k) = \frac{\frac{1}{24} - \frac{1}{10}k + \frac{43}{480}k^2 - \frac{3}{40}k^3 + \frac{11}{1440}k^4 - \frac{1}{1600}k^5}{(1 - \frac{1}{2}k)^6} \]
\[ \mathcal{I}_{21}(k) = \frac{\frac{1}{2} - \frac{3}{4}k + \frac{1}{3}k^2 - \frac{1}{20}k^3}{(1 - \frac{1}{2}k)^3} \]
\[ \mathcal{R}_{22}(k) = \frac{(1 - k)(\frac{1}{45} - \frac{7}{180}k + \frac{17}{720}k^2 - \frac{1}{160}k^3 + \frac{9}{14400}k^4)}{(1 - \frac{1}{2}k)^6} \]
\[ \mathcal{I}_{22}(k) = \frac{(1 - k)(\frac{1}{3} - \frac{1}{4}k + \frac{1}{20}k^2)}{(1 - \frac{1}{2}k)^3} \]

The time-constants \( \tau_{ij} = R_{ij}C_{ij} \) appearing in the small-signal Y-parameters are then given by

\[ \tau_{11} = R_{11}C_{11} = \frac{1}{\omega_o} \frac{60 - 120k + 81k^2 - 21k^3 + 2k^4}{15(2 - k)^3(6 - 6k + k^2)} \]
\[ \tau_{12} = R_{12}C_{12} = \frac{1}{\omega_o} \frac{30 - 41k + 16k^2 - 2k^3}{15(2 - k)^3(3 - k)} \]
\[ \tau_{21} = R_{21}C_{21} = \frac{1}{\omega_o} \frac{600 - 1440k + 1290k^2 - 540k^3 + 110k^4 - 9k^5}{30(2 - k)^3(30 - 45k + 20k^2 - 3k^3)} \]
\[ \tau_{22} = R_{22}C_{22} = \frac{1}{\omega_o} \frac{320 - 560k + 340k^2 - 90k^3 + 9k^4}{30(2 - k)^3(20 - 15k + 3k^2)} \]
Figure 13: Comparison of the amplitude of $Y_{11}/g_0$.

Figure 14: Comparison of the phase of $Y_{11}/g_0$. 
Figure 15: Comparison of the amplitude of $Y_{12}/g_0$.

Figure 16: Comparison of the phase of $Y_{12}/g_0$. 
Figure 17: Comparison of the amplitude of $Y_{21}/g_0$.

Figure 18: Comparison of the phase of $Y_{21}/g_0$. 
Figure 19: Comparison of the amplitude of $Y_{22}/g_0$.

Figure 20: Comparison of the phase of $Y_{22}/g_0$. 
The magnitude and phase of each Y-parameters are shown in Figure 13 - 20 for \( k = 0.65 \), obtained with the RC equivalent-circuit (dashed-dotted line, EQ), the exact solution (plain line, EXACT), the frequency power-series (dashed line, POWER), and the second-order iterative Y-parameters derived in [18] (dashed line, B). As can be seen in figures, the first-order equivalent shows the graceful degradation.

In order to establish the range of validity of the RC circuit representation for all bias conditions, the frequency \( f_{\text{opt}}(Y_{ij}) \) for each parameter \( Y_{ij} \) is calculated. An error \( Err(Y_{ij}) \) of 5% is obtained between the exact Bessel solution and the approximate results for the frequency \( f_{\text{opt}}(Y_{ij}) \). The error \( Err(Y_{ij}) \) is

\[
Err(Y_{ij}) = \frac{|Y_{ij}(\text{exact}) - Y_{ij}(\text{approximate})|}{|Y_{ij}(\text{exact})|}
\]  

For the sake of comparison \( f_{\text{opt}}(Y_{ij})/f_o \) for each \( Y_{ij} \) parameter are plotted in Figure 21-24 as a function of the biasing parameter \( k \) for the frequency power-series model (dashed line, POWER), the second-order iterative results [18] (dashed line, B2), the first-order iterative results [18] (dashed line, B1) and the simple RC circuit representation of the frequency power-series model (dashed-dotted line, EQ). One observes that the simple RC representation of the frequency power series holds for all bias conditions up to a higher frequency than both the frequency power series and the iterative results. On the same curve we have also plotted the unity current gain cut-off frequency \( f_T/f_o \) (dashed line, FT) and the maximum frequency of oscillation \( f_{\text{max}}/f_o \) (plain line, FMAX), (frequency at which the unilateral gain is one [27]). Both \( f_T \) and \( f_{\text{max}} \) are calculated using the exact Bessel solution.

All approximate small-signal models except the first-order iterative model hold
Figure 21: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{11}$ as a function of the biasing parameter $k$.

Figure 22: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{12}$ as a function of the biasing parameter $k$. 
Figure 23: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{21}$ as a function of the biasing parameter $k$.

Figure 24: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{22}$ as a function of the biasing parameter $k$. 
for frequencies larger than the cut-off frequency $f_T$ for all bias conditions. The RC circuit representation holds for frequencies larger than the maximum frequency of oscillation $f_{\text{max}}$ for $k$ smaller than $\sim 0.9$. For $k$ larger than $\sim 0.9$, $f_{\text{s}}$ is however smaller than $f_{\text{max}}$. Note that both the exact and the approximate models predict an infinite maximum frequency of oscillation at $k = 1$. Obviously in the extrinsic device the unavoidable source, drain and gate resistances and drain output conductance will limit $f_{\text{max}}$ to a finite value. The infinite $f_{\text{max}}$ predicted for the intrinsic FET is nonetheless an indication of the limited validity of the long-channel model. Indeed even in long channel devices the drain current saturation ultimately results from velocity saturation and not pinch-off so that we always have $k < 1$ in the unsaturated part of the channel.

To conclude note that the normalization frequency $f_o$ is bias dependent. For gate voltages approaching the threshold voltage, the normalization frequency $f_o$ is small and none of these so-called high-frequency approximate models can account for the distributed effects arising even at low frequencies.

### 3.3 The Optimal Second-Order Equivalent Circuit

The simple RC equivalent circuit shown in Figure 12 is valid when the frequency considered is small enough so that the unsaturated MODFET behaves like a lumped device. At high-frequencies transmission line-effects become important and a second-order equivalent circuit becomes desirable.

The topology of the optimal second-order equivalent circuit will be based on the second-order RC topology obtained by rewriting the second-order iterative Y-
parameters under the form

\[ Y_{ij} = g_{ij} + \frac{j\omega a_{ij} + (j\omega)^2 b_{ij}}{1 + j\omega a_{ij} + (j\omega)^2 d_{ij}} \]  \hspace{1cm} (3.4)

Actually two different second-order RC equivalent circuits can be used to implement this equation, as is seen in Figure 25. (a) is preferable over (b) as its topology physically implements the distributed effects of the channel.

Let us now evaluate each element in the equivalent circuit (a). First Equation (3.4) is rewritten in terms of the time constants \( \tau_{1ij} \), \( \tau_{2ij} \), and \( \tau_{3ij} \).

\[ Y_{ij} = g_{ij} + j\omega C_{ij} \frac{1 + j\frac{\omega}{\omega_0} \tau_{1ij}}{1 + j\frac{\omega}{\omega_0} \tau_{2ij} + (j\frac{\omega}{\omega_0})^2 \tau_{3ij}^2} \]  \hspace{1cm} (3.5)
where

\[
C_{ij} = a_{ij} = C_{1ij} + C_{ij2}
\]

\[
\tau_{1ij} = \omega_0 b_{ij} = \omega_0 \frac{C_{1ij}C_{2ij}R_{2ij}}{C_{1ij} + C_{2ij}}
\]

\[
\tau_{2ij} = \omega_0 c_{ij} = \omega_0 (C_{2ij}R_{2ij} + C_{1ij}R_{1ij} + C_{2ij}R_{1ij})
\]

\[
\tau_{3ij}^2 = \omega_0^2 d_{ij} = \omega_0^2 C_{1ij}C_{2ij}R_{1ij}R_{2ij}
\]

\[
\omega_0 = \frac{2\pi f_0}{L_g} (V_{GS} - V_T)
\]

In order to extract the value of \( \tau_{1ij}, \tau_{2ij}, \) and \( \tau_{3ij}^2 \) Equation (3.5) is expanded in a fourth-order frequency power series. The denominator of Equation (3.5) is obtained

\[
\frac{1}{1 + j \omega_0 \tau_{2ij} + (j \omega_0)^2 \tau_{3ij}^2} = 1 - [j \frac{\omega}{\omega_0} \tau_{2ij} + (j \frac{\omega}{\omega_0})^2 \tau_{3ij}^2]
\]

\[
+[j \frac{\omega}{\omega_0} \tau_{2ij} + (j \frac{\omega}{\omega_0})^2 \tau_{3ij}^2]^2 - [j \frac{\omega}{\omega_0} \tau_{2ij} + (j \frac{\omega}{\omega_0})^2 \tau_{3ij}^2]^3
\]

(3.6)

Substituting Equation (3.6) in Equation (3.5) and neglecting the fifth- and higher order terms in \( \omega \) gives the fourth-order frequency-power series.

\[
Y_{ij} = g_{ij} + j \frac{\omega}{\omega_0} F_{ij} - (j \frac{\omega}{\omega_0})^2 S_{ij} + (j \frac{\omega}{\omega_0})^3 T_{ij} - (j \frac{\omega}{\omega_0})^4 D_{ij}
\]

(3.7)

where the coefficients, \( \tau_{1ij}, \tau_{2ij}, \) and \( \tau_{3ij}^2 \) are given in terms of \( F_{ij}, S_{ij}, T_{ij}, \) and \( D_{ij} \) by

\[
C_{ij} = F_{ij}
\]

\[
\tau_{1ij} = \frac{F_{ij}D_{ij} - S_{ij}T_{ij}}{F_{ij}T_{ij} - S_{ij}^2}
\]

\[
\tau_{2ij} = \frac{F_{ij}^2 D_{ij} - 2F_{ij}S_{ij}T_{ij} + S_{ij}^3}{F_{ij}T_{ij} - S_{ij}^2}
\]

\[
\tau_{3ij}^2 = \frac{S_{ij}D_{ij} - T_{ij}^2}{F_{ij}T_{ij} - S_{ij}^2}
\]

(3.8)
Figure 26: Comparison of the magnitude of $Y_{11}/g_0$ for $k = 0.65$

Figure 27: Comparison of the phase of $Y_{11}/g_0$ for $k = 0.65$
Figure 28: Comparison of the magnitude of $Y_{12}/g_0$ for $k = 0.65$

Figure 29: Comparison of the phase of $Y_{12}/g_0$ for $k = 0.65$
Figure 30: Comparison of the magnitude of $Y_{21}/g_0$ for $k = 0.65$

Figure 31: Comparison of the phase of $Y_{21}/g_0$ for $k = 0.65$
Figure 32: Comparison of the magnitude of $Y_{22}/g_0$ for $k = 0.65$

Figure 33: Comparison of the phase of $Y_{22}/g_0$ for $k = 0.65$
The coefficients $F_{ij}$, $S_{ij}$, $T_{ij}$ and $D_{ij}$ can be obtained from the MODFET wave-equation using the method developed by Ziel [20]. The procedure used and the obtained $F_{ij}$, $S_{ij}$, $T_{ij}$ and $D_{ij}$ coefficients for each $Y$ parameters $Y_{ij}$ are given in Appendix C. Note that these parameters are all dependent on the normalized bias parameter $k = \frac{V_{ds}}{V_{gs}-V_{T}}$. The elements of the optimal second-order RC circuit can now be obtained by inverting the system of Equations (3.5) so as to express $R_{1ij}$, $R_{2ij}$, $C_{1ij}$, and $C_{2ij}$ in terms of $\tau_{1ij}$, $\tau_{2ij}$, and $\tau_{3ij}$.

\[
R_{1ij} = \frac{\tau_{3ij}^2}{\tau_{1ij}} \\
R_{2ij} = -\frac{(C_{ij}\tau_{3ij}^2 - \tau_{1ij}\tau_{2ij})^2}{\tau_{1ij}(C_{ij}\tau_{3ij}^2 - C_{ij}\tau_{1ij}\tau_{2ij} + \tau_{1ij}^2)} \\
C_{1ij} = -\frac{\tau_{1ij}^2}{C_{ij}\tau_{3ij}^2 - \tau_{1ij}\tau_{2ij}} \\
C_{2ij} = \frac{C_{ij}\tau_{3ij}^2 - C_{ij}\tau_{1ij}\tau_{2ij} + \tau_{1ij}^2}{C_{ij}\tau_{3ij}^2 - \tau_{1ij}\tau_{2ij}}
\]

The magnitude and phase of each $Y$-parameters are plotted in Figure 26 - 33 to demonstrate the graceful degradation. The $Y$-parameters are obtained with the second-order equivalent circuit (dashed-dotted line, EQ), the exact solution (plain line, EXACT), the frequency power series (dashed line, POWER), and fourth order iterative solution (dashed dashed line, ITER). As can be seen in figures, the second-order equivalent circuit exhibits a more graceful degradation compared to the fourth-order iterative solution and power series solution. The frequency range of validity of this equivalent circuit can be evaluated by calculating the frequency $f_{5\%}(Y_{ij})$ for each parameter $Y_{ij}$ for which a 5% error is obtained. Figure 34 - 37 show $f_{5\%}(Y_{ij})$ for each parameter $Y_{ij}$. One observes that the circuit holds to a much higher frequency than
Figure 34: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{11}$ as a function of the biasing parameter $k$.

Figure 35: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{12}$ as a function of the biasing parameter $k$. 
Figure 36: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{21}$ as a function of the biasing parameter $k$.

Figure 37: Plot of $f_{5\%}(Y_{ij})/f_0$ for $Y_{22}$ as a function of the biasing parameter $k$. 
any other model for all $k$ values except for $Y_{21}$. $f_{35}\%(Y_{21})/f_0$ for $Y_{21}$ is the same as that of first-order RC equivalent circuits at $k = 0.9$.

As mentioned above two different topologies for the second-order equivalent circuit are possible (see Figure 25). The transfer functions of both circuits for small-signal analysis in the frequency domain are the same, even though different RC elements are used. For the same bias condition, the values of $R_{1i j}$ and $C_{1i j}$ in circuit (b) are the same as that of $R_{ij}$ and $C_{ij}$ in the optimal first-order RC circuit. This means that circuit (b) extends the frequency range by adding the $R_{2ij}$, $C_{2ij}$ circuit to the initial RC circuit. However in circuit (a) the sum of capacitors, $C_{1ij} + C_{2ij}$, give the capacitor $C_{ij}$ of the optimal first-order RC circuit. The small-signal analysis does not differentiate between (a) and (b), however circuit (a) will be preferable to circuit (b) since circuit (a) is a more physical representation of the distributed channel for large-signal analysis. The resulting equivalent circuit for long gate length device is
shown in Figure 38.

3.4 The Velocity-Saturated MODFET Equivalent Circuit

The small-signal model presented above for the intrinsic MODFET holds only for the region of the channel for which the gradual channel approximation (GCA) holds. However in saturation it becomes necessary to account for the contribution of the built-in potential. A more complex equivalent circuit results in which the equivalent circuit introduced for the MODFET wave-equation is now just a subcircuit.

Let us demonstrate this approach for the velocity-saturated MODFET wave-equation. In this conventional MODFET model the FET channel is divided into the GCA and saturation regions of length \( X_s = L_g - \ell \) and \( \ell \) respectively. In the saturation region the electron velocity is assumed to saturate (to a value \( v_s \)) while the GCA is failing. The channel potential in the saturation region is then assumed to be supported uniquely by the electron distribution in the channel.

The derivation of the equivalent circuit for saturation region is given in Appendix D, which is based on the exact solution. By combining two equivalent circuit, one of GCA region and the other of saturation region, one can obtain Figure 39.

The total Y-parameters \( Y_{ij} (s) \) in terms of the Y-parameters of the GCA region \( Y_{ij} (g) \) of reduced gate length \( X_s = L_g - \ell \), are obtained as follows

\[
Y_{11}^{(sat)} = \frac{Y_{11}(g) + Y_{12}(g)\gamma_s}{1 + Y_{22}(g)Z_s(\omega)} (1 - e^{-j\omega \tau_s} - Z_s(\omega)Y_{12}(g))
\]

\[
Y_{12}^{(sat)} = Y_{12}(g)\gamma_s + \frac{Y_{22}(g)\gamma_s}{1 + Y_{22}(g)Z_s(\omega)} (1 - e^{-j\omega \tau_s} - Z_s(\omega)Y_{12}(g))
\]

\[
Y_{21}^{(sat)} = \frac{Y_{21}(g) + Y_{22}(g)\gamma_s}{1 + Y_{22}(g)Z_s(\omega)} e^{-j\omega \tau_s}
\]
Figure 39: First-order non-quasi-static equivalent circuit for the velocity-saturated MODFET wave-equation

\[ Y_{22}^{(sat)} = \frac{Y_{22}(g)\delta_s}{1 + Y_{22}(g)Z_s(\omega)}e^{-j\omega\tau_s} \]

where \( \tau_s = \frac{v_s}{\ell} \) is the transit time of the saturation region, \( Z_s(\omega) \) an impedance specified below and \( \delta_s \) and \( \gamma_s \) two constants given by

\[ \gamma_s = 1 - \delta_s = \frac{1}{1 + \beta I_D C_l A} \]

with

\[ A = \frac{2X_S(1 - k_s)}{(2k_s - k_s^2)(V_{GS} - V_T)} \]
\[ B = \frac{4X_S(1 - k_s)^2}{G_{d0s}(2k_s - k_s^2)^2(V_{GS} - V_T)} \]

Note that \( k_s = \frac{V_{CS}(X_S)}{(V_{GS} - V_T)} \) and \( G_{d0s} = \mu C_g W_g (V_{GS} - V_T)/X_S \) are the values used for \( k \) and the drain conductance \( g_d \) respectively, in the GCA Y-Parameters \( Y_{ij}(g) \) given in section 3.2.
Figure 40: Comparison of the amplitude of $Y_{11}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$.

Figure 41: Comparison of the phase of $Y_{11}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$. 
Figure 42: Comparison of the amplitude of $Y_{12}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$.

Figure 43: Comparison of the phase of $Y_{12}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$. 
Figure 44: Comparison of the amplitude of $Y_{21}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$.

Figure 45: Comparison of the phase of $Y_{21}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$. 
Figure 46: Comparison of the amplitude of $Y_{22}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$.

Figure 47: Comparison of the phase of $Y_{22}$ for $V_{DS} = 3V$ and $V_{GS} = 0V$. 
The impedance $Z_a(\omega)$ is approximated by a first order RC network providing the correct second-order frequency power-series expansion

$$Z_a = R_{s1} + \frac{R_{s2}}{1 + j\omega C_s R_{s2}}$$  \hspace{1cm} (3.9)$$

with

$$R_{s1} = \frac{\beta I_{DC} \ell B - \frac{1}{6} \beta \ell^2}{1 + \beta I_{DC} \ell A}$$
$$R_{s2} = \frac{2 \beta \ell^2}{3(1 + \beta I_{DC} \ell A)}$$
$$C_s = \frac{3}{8} \tau_s \left(1 + \beta I_{DC} \ell A\right) \frac{1}{\beta \ell^2}$$

using $\beta = 1/\epsilon_1 v_s W_d d_s$.

The resulting equivalent-circuit provides an optimal first-order non-quasi-static equivalent-circuit admitting the correct second-order frequency power expansion as well as a graceful degradation. This is demonstrated in Figure 40 - 47 for an intrinsic MODFET with the parameters given in Table 1 in Chapter II and for an intrinsic bias of $V_{DS} = 3V$ and $V_{GS} = 0V$. The phase and amplitude of Y-parameters versus frequency calculated using this first-order RC equivalent-circuit (dashed-dotted line, EQUI), the exact solution (plain line, EXACT), and the frequency power-series approximation (dashed line, POWER) are compared in figures. The optimal first-order RC model (EQUI) is seen to hold to a much higher frequency than the frequency power-series approximation (POWER).
CHAPTER IV

PREDICTION OF THE MICROWAVE CHARACTERISTICS OF MODFET'S

4.1 Extraction of parameters for the ac model

The intrinsic ac model developed relies on the material, device and bias parameters \( \epsilon_1, L_g, W_g, \mu, v_s, C_g, V_T, V_{GC}(X_S), I \) or \( X_S, V_G, I_{dc} \), which should all be obtained directly from the dc-model. As was mentioned in Chapter II, it is necessary to use an accurate dc model to model the microwave performance. The dc model used is the dc model recently reported [24], except for the saturation voltage \( V_h = V_{DS} - V_{CS}(X_S) \) which in accordance with the proposed saturation picture is derived from Equation (2.3) to be

\[
V_h = I_{dc}X_S^2/(2\epsilon_1d_sW_gv_s) - E_cX_S \tag{4.1}
\]

This dc model provides two features: a field dependent mobility (see Equation (4.3) below) and a non-linear charge control

\[
n_S = n_SO \left[ (1 - \alpha) + \alpha \tanh \frac{V_{GC} - V_{GM}}{V_1} \right] \tag{4.2}
\]

which permits one to obtain an improved fit of the dc-characteristics \( (n_{SO}, V_{GM}, \alpha, \) and \( V_1 \) are used as fitting parameters). Since the ac-model introduced in Chapter II
relies on a constant mobility $\mu$ (in the GCA region), constant gate capacitance $C_g$, and constant threshold voltage $V_T$ an extraction theory is required. Following the approach described in reference [10] the mobility is given by,

$$\mu = \frac{\mu_0}{1 + E(0)/E_1}$$  \hspace{1cm} (4.3)

where $E_1 = E_c/(\mu E_c/v_s - 1)$. Note that the mobility used by the ac-model is the chordal mobility and not the differential mobility (see Gunn [28]).

In reference [10] the threshold voltage was calculated using

$$V_T = V_{GC}(X_S) - V_1 \left[ \frac{\alpha}{(1 - \alpha)} + \tanh \left( \frac{V_{GC}(X_S) - V_{GM}}{V_1} \right) \right]$$

$$\times \cosh^2 \left( \frac{V_{GC}(X_S) - V_{GM}}{V_1} \right)$$  \hspace{1cm} (4.4)

and the 2DEG capacitance using

$$C_g = \frac{qn_0(1 - \alpha)}{V_1} \text{sech}^2 \left( \frac{V_{GC}(0) - V_{GM}}{V_1} \right)$$  \hspace{1cm} (4.5)

These expressions are only valid for small gate-to-channel voltages. At large gate-to-channel voltages the high-bandgap material between the gate and the 2DEG channel is no longer fully depleted. This leads to the saturation of the 2DEG concentration in Equation (4.2), an effect reflected in turn in Equations (4.4) and (4.5) by the reduction of both the threshold voltage and the 2DEG capacitance. However due to the large RC constant of the depleted parasitic MESFET channel, the charge distribution (electrons and ionized donors) in the high-bandgap material does have time to respond at high-frequencies. The 2DEG capacitance at large gate-to-channel voltages is then limited by the maximum 2DEG capacitance $C_{g_{\text{max}}} = \epsilon_2/(d + \Delta d)$,
where \( d \) is the gate to channel spacing and \( \Delta d \) is a constant arising solely from the variation of the Fermi level with the 2DEG concentration [29]. The gate-to-channel voltage \( V_{GC0} \) at which this takes place is simply obtained from \( C_g(V_{GC0}) = C_{g_{max}} \). For gate-to-channel voltage larger than \( V_{GC0} \) the threshold voltage selected is then given by \( V_{T_{max}} = V_T(V_{GC0}) \). The use of the maximum 2DEG capacitance and maximum threshold voltage at microwave frequencies for large gate-to-channel voltages supports the notion that the RF transconductance \( g_M(RF) \) can be larger than the dc transconductance \( g_M(dc) \). A greatly improved fit of the scattering parameters results from this choice.

4.2 Comparison of the measured and calculated data

In order to test this microwave model, the theory is applied to two one-micron gate length devices; an \( nAl_{0.25}Ga_{0.75}As/iAl_{0.25}Ga_{0.75}As/GaAs \) (350/30/10,000 Å) MODFET (device 2045) and an \( nAl_{1.15}Ga_{0.85}As/iAl_{0.15}Ga_{0.85}As/In_{0.3}Ga_{0.6}As/GaAs \) (350/30/150/10,000 Å) pseudomorphic MODFET (device 2379). The data are obtained from University of Illinois at Urbana-Champaign.

The device parameters used for fitting the IV characteristics of the devices are shown in Table 2. The comparison of the calculated and measured IV characteristics are shown in Figures 48 and 49. The device parameters for device 2045 are the same as reported in [10] except for \( d_s, R_s, \) and \( R_d \). A smaller value for \( d_s \) (closer to the equilibrium channel width, see [22]) is used. The nonlinear source and drain resistance model reported in [30] is used for \( R_s \) and \( R_d \).

The equivalent circuit used for the extrinsic model is shown in Figure 50. Dis-
Table 2: Device parameters for the dc characteristics of the AlGaAs/GaAs MODFET (2045) and the GaAlAs/InGaAs/GaAs pseudomorphic MODFET (2379)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>2045</th>
<th>2379</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_g$ gate length ($\mu m$)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$W_g$ gate width ($\mu m$)</td>
<td>290</td>
<td>290</td>
</tr>
<tr>
<td>$\mu$ mobility ($cm^2/V$.sec)</td>
<td>4400</td>
<td>5600</td>
</tr>
<tr>
<td>$v_s$ Saturation velocity ($m/sec$)</td>
<td>$3.45 \times 10^6$</td>
<td>$3.5 \times 10^5$</td>
</tr>
<tr>
<td>$E_c$ Critical Field ($KV/cm$)</td>
<td>10.9</td>
<td>31.2</td>
</tr>
<tr>
<td>$n_{so}$ ($cm^{-2}$)</td>
<td>$1.02 \times 10^{12}$</td>
<td>$1.0 \times 10^{12}$</td>
</tr>
<tr>
<td>$V_1$ (V)</td>
<td>0.3</td>
<td>0.19</td>
</tr>
<tr>
<td>$V_{GM}$ (V)</td>
<td>0.15</td>
<td>-0.12</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$d$ gate to channel spacing ($Å$)</td>
<td>380</td>
<td>380</td>
</tr>
<tr>
<td>$\Delta d$ see [29] (Å)</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$d_s$ channel width in saturation (Å)</td>
<td>1500</td>
<td>500</td>
</tr>
<tr>
<td>$R_s$ low field source resistance ($Ω$)</td>
<td>3</td>
<td>2.7</td>
</tr>
<tr>
<td>$R_d$ low field drain resistance ($Ω$)</td>
<td>6</td>
<td>2.7</td>
</tr>
<tr>
<td>$\epsilon_1$ channel dielectric constant</td>
<td>$13.1 \epsilon_0$</td>
<td>$12.44 \epsilon_0$</td>
</tr>
<tr>
<td>$\epsilon_2$ channel dielectric constant</td>
<td>$12.2 \epsilon_0$</td>
<td>$12.65 \epsilon_0$</td>
</tr>
</tbody>
</table>
Figure 48: Measured (solid lines) and calculated (dotted lines) IV characteristics of the AlGaAs/GaAs MODFET 2045

Figure 49: Measured (solid lines) and calculated (dotted lines) IV characteristics of the pseudomorphic GaAlAs/InGaAs/GaAs MODFET 2379
tributed effects along the gate width are not included, as they were found to be small for the gate width and frequencies considered. The value of the parasitics used for calculating the extrinsic Y parameters are shown in Table 3. The gate to source $C_{GS}$ and gate to drain $C_{GD}$ fringe capacitances were estimated to be 50-70 fF (see [21]). The bond inductance at the gate, source and drain terminals and gate resistance $R_G$ are obtained from deembedding. The source and drain resistances used are obtained from the dc model.

The scattering parameters were measured from 2 GHz to 18.4 GHz for 5 bias conditions for device 2045 and 13 bias conditions for device 2379.

The extrinsic Y parameters, including parasitics, were calculated from the ac-
Figure 51: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{GS} = .08$ V and $V_{DS} = 0.5$ V for the AlGaAs/GaAs MODFET 2045.

Figure 52: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{GS} = .08$ V and $V_{DS} = 0.5$ V for the AlGaAs/GaAs MODFET 2045.
Figure 53: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{GS} = 0.08\,\text{V}$ and $V_{DS} = 0.75\,\text{V}$ for the AlGaAs/GaAs MODFET 2045.

Figure 54: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{GS} = .08\,\text{V}$ and $V_{DS} = 0.75\,\text{V}$ for the AlGaAs/GaAs MODFET 2045.
Figure 55: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{GS} = 0.08$ V and $V_{DS} = 1.0$ V for the AlGaAs/GaAs MODFET 2045.

Figure 56: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{GS} = 0.08$ V and $V_{DS} = 1.0$ V for the AlGaAs/GaAs MODFET 2045.
Figure 57: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{GS} = .08$ V and $V_{DS} = 3.0$ V for the AlGaAs/GaAs MODFET 2045.

Figure 58: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{GS} = .08$ V and $V_{DS} = 3.0$ V for the AlGaAs/GaAs MODFET 2045.
Figure 59: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{DS} = 3$ V and $V_{GS} = -0.15$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.

Figure 60: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{DS} = 3$ V and $V_{GS} = -0.15$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.
Figure 61: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{DS} = 3$ V and $V_{GS} = -0.08$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.

Figure 62: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{DS} = 3$ V and $V_{GS} = -0.08$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.
Figure 63: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{DS} = 3$ V and $V_{GS} = 0.25$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.

Figure 64: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{DS} = 3$ V and $V_{GS} = 0.25$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.
Figure 65: Measured (solid lines) and calculated (dotted lines) scattering parameters for $V_{DS} = 3$ V and $V_{GS} = 0.56$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.

Figure 66: Measured (solid lines) and calculated (dotted lines) unilateral power gain for $V_{DS} = 3$ V and $V_{GS} = 0.56$ V for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET 2379.
Table 3: Microwave parasitics for the AlGaAs/GaAs MODFET (2045) and the GaAlAs/InGaAs/GaAs pseudomorphic MODFET (2379)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>2045</th>
<th>2379</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{GS}$ Gate to source parasitic capacitance (fF)</td>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>$C_{GD}$ Gate to drain parasitic capacitance (fF)</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>$R_G$ Gate resistance (Ω)</td>
<td>4.16</td>
<td>5.4</td>
</tr>
<tr>
<td>$L_G$ Gate bond inductance (nH)</td>
<td>0.35</td>
<td>0.3</td>
</tr>
<tr>
<td>$L_S$ Source bond inductance (nH)</td>
<td>0.08</td>
<td>0.07</td>
</tr>
<tr>
<td>$L_D$ Drain bond inductance (nH)</td>
<td>0.33</td>
<td>0.36</td>
</tr>
</tbody>
</table>

model for the same bias conditions and then converted to scattering parameters for comparison with the measured data.

The scattering parameters and unilateral power gain versus frequency for four bias conditions are shown in Figures 51 - 58 for device 2045 and Figures 59 - 66 for device 2379. The scattering parameters for $V_{GS} = 0.08$ V and $V_{DS} = 0.5$, 0.75, 1 V, and 3 V are shown in Figures 51, 53, 55, 57 respectively for the AlGaAs/GaAs MODFET. The scattering parameters for $V_{DS} = 3$ V and $V_{GS} = -0.15$ V, -0.08 V, 0.25 V, and 0.56 V are shown in Figures 59, 61, 63, 65 respectively for the GaAlAs/InGaAs/GaAs pseudomorphic MODFET. In the polar and Smith chart plots, the solid line is used to represent the measured data and the dotted line is used for the calculated data.

Figures 52, 54, 56, 58, 60, 62, 64, and 66 show the unilateral power gain given by [31]
Table 4: Deviation of calculated S parameters from the measured data for all bias conditions for the device 2045 (d_s = 1500 Å, C_GD = 50 fF, and C_GS = 50 fF)

<table>
<thead>
<tr>
<th>V_g</th>
<th>V_d</th>
<th>ΔS_{11}</th>
<th>ΔS_{21}</th>
<th>ΔS_{12}</th>
<th>ΔS_{22}</th>
<th>ΔU(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>0.25</td>
<td>0.09709</td>
<td>0.12828</td>
<td>0.04286</td>
<td>0.08892</td>
<td>2.41553</td>
</tr>
<tr>
<td>0.08</td>
<td>0.5</td>
<td>0.12094</td>
<td>0.41329</td>
<td>0.03103</td>
<td>0.12960</td>
<td>0.60579</td>
</tr>
<tr>
<td>0.08</td>
<td>0.75</td>
<td>0.14125</td>
<td>0.61521</td>
<td>0.03263</td>
<td>0.21576</td>
<td>0.61067</td>
</tr>
<tr>
<td>0.08</td>
<td>1.0</td>
<td>0.12278</td>
<td>0.61916</td>
<td>0.04603</td>
<td>0.25717</td>
<td>1.45705</td>
</tr>
<tr>
<td>0.08</td>
<td>3.0</td>
<td>0.30447</td>
<td>1.32780</td>
<td>0.07536</td>
<td>0.28605</td>
<td>1.16160</td>
</tr>
</tbody>
</table>

\[ U = \frac{|Y_{21} - Y_{12}|^2}{4(\text{Re}(Y_{11})\text{Re}(Y_{22}) - \text{Re}(Y_{12})\text{Re}(Y_{21}))} \quad (4.6) \]

In these figures, the solid line represents the value of \(U\) obtained from the measured data and the dotted line the value of \(U\) calculated from the microwave model.

To estimate the performance of the model the error obtained for all measured bias conditions is summarized in Table 4 and Table 5 for devices 2045 and 2379 respectively. Since scattering parameters are normalized quantities, the error used is

\[ \Delta S_{ij} = \frac{1}{N} \sum_{i=1}^{N} |S_{ijdata}(\omega_i) - S_{ijcalculated}(\omega_i)| \quad (4.7) \]

As can be seen in Figures 52, 54, 56, 58, 60, 62, 64, and 66 the calculated and measured unilateral gain plotted on a log scale differ by a few dB over the entire frequency range. The error used for the unilateral power gain is therefore

\[ \Delta U_{dB} = \frac{1}{N} \sum_{i}^{N} |10\log(U_{data}(\omega_i)) - 10\log(U_{calculated}(\omega_i))| \quad (4.8) \]
Table 5: Deviation of calculated S parameters from the measured data for all bias conditions for the device 2379 (\( d_s = 500 \text{ Å} \), \( C_{GD} = 70 \text{ fF} \), and \( C_{GS} = 50 \text{ fF} \))

<table>
<thead>
<tr>
<th>( V_{gs} )</th>
<th>( V_{ds} )</th>
<th>( \Delta S_{11} )</th>
<th>( \Delta S_{21} )</th>
<th>( \Delta S_{12} )</th>
<th>( \Delta S_{22} )</th>
<th>( \Delta U(\text{dB}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.07</td>
<td>0.25</td>
<td>0.10488</td>
<td>0.09403</td>
<td>0.06810</td>
<td>0.12519</td>
<td>3.25594</td>
</tr>
<tr>
<td>0.07</td>
<td>0.5</td>
<td>0.13381</td>
<td>1.01989</td>
<td>0.06220</td>
<td>0.54647</td>
<td>3.10049</td>
</tr>
<tr>
<td>0.07</td>
<td>0.75</td>
<td>0.14001</td>
<td>0.93923</td>
<td>0.03592</td>
<td>0.43403</td>
<td>1.10848</td>
</tr>
<tr>
<td>0.07</td>
<td>1.0</td>
<td>0.14074</td>
<td>0.67644</td>
<td>0.03820</td>
<td>0.32584</td>
<td>2.06435</td>
</tr>
<tr>
<td>0.07</td>
<td>2.0</td>
<td>0.15359</td>
<td>0.62508</td>
<td>0.05199</td>
<td>0.26670</td>
<td>3.08550</td>
</tr>
<tr>
<td>-0.15</td>
<td>0.5</td>
<td>0.14178</td>
<td>0.70650</td>
<td>0.03433</td>
<td>0.34632</td>
<td>1.24863</td>
</tr>
<tr>
<td>-0.08</td>
<td>0.5</td>
<td>0.12351</td>
<td>0.83907</td>
<td>0.04749</td>
<td>0.43257</td>
<td>0.99085</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5</td>
<td>0.11836</td>
<td>0.16675</td>
<td>0.05722</td>
<td>0.10181</td>
<td>4.08520</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.13181</td>
<td>0.06170</td>
<td>0.06089</td>
<td>0.15007</td>
<td>0.87395</td>
</tr>
<tr>
<td>-0.08</td>
<td>3.0</td>
<td>0.14849</td>
<td>0.67419</td>
<td>0.04942</td>
<td>0.26523</td>
<td>2.96939</td>
</tr>
<tr>
<td>-0.15</td>
<td>3.0</td>
<td>0.15725</td>
<td>0.72022</td>
<td>0.04232</td>
<td>0.27461</td>
<td>2.60278</td>
</tr>
<tr>
<td>0.25</td>
<td>3.0</td>
<td>0.12164</td>
<td>0.89009</td>
<td>0.06550</td>
<td>0.24644</td>
<td>1.71173</td>
</tr>
<tr>
<td>0.56</td>
<td>3.0</td>
<td>0.09435</td>
<td>1.42184</td>
<td>0.07672</td>
<td>0.27800</td>
<td>1.19325</td>
</tr>
</tbody>
</table>
The error averaged over all bias conditions is given in Tables 4 and 5 for devices 2045 and 2379, respectively. A minimum average error on the unilateral gain of 1.25 dB and 2.17 dB is obtained for all bias conditions for devices 2045 and 2379 respectively. Note that an error of +2dB in the unilateral gain corresponds to an error in the calculated $f_{\text{max}}$ given by $f_{\text{max}}(\text{data})/f_{\text{max}}(\text{theory}) = 1.26$. This error is reasonable considering that apart from $d_s$ and $C_{GS}$ and $C_{GD}$, all of the model parameters are extracted from the dc measurement or the microwave deembedding. A main source of error originates from the dc drain conductance and transconductance, which are only approximately fitted with the present extraction techniques. This emphasizes the importance of an accurate dc model. It is believed that the error obtained could be reduced with an improved dc parameter extraction technique. Additional substantial error might also originate from calibration and deembedding.

4.3 Discussion

It is now relevant to elaborate on the physical insights gained from the ac-model reported for the saturated MODFET. First let us justify the need for an ac-model for the saturated MODFET.

As mentioned the previous solutions for the wave-equation were only valid up to the edge of saturation which occurs when the channel field is the critical field $E_c$. It is important to note that one cannot infer the performance of the device in saturation from the performance of the device at the edge of saturation. Indeed once the device is driven deeper in saturation, one observes a substantial increase of the transconductance, a strong reduction of the drain conductance and a reduction of
the effective channel length. These effects drastically modify, and in fact improve the device performance.

In a recent paper [21], Rohdin reported an ac-model for the saturated MODFET (shown in Figure 2 in Chapter I). In his model the GCA region is described by an equivalent circuit based on the Y-parameters derived for the unsaturated MODFET [10]. To account for the saturated region, Rohdin introduces in series in the drain terminal, a conductance $g_{dd'}$ limiting the output conductance. A capacitance $C_{dd'}$ in parallel with $g_{dd'}$ is introduced to account for the charge modulation in the saturation region. Using his model Rohdin develops a method permitting one to extract from the measured Y-parameters the source resistance, fringe capacitances, gate length and effective saturation velocity. The physical equivalent circuit model proposed by Rohdin emphasizes the importance of the saturation region.

Interestingly the GCA region in a saturated MODFET does not behave, at high-frequencies, as the GCA region of the unsaturated MODFET. This is verified by developing a model relying on the Y-parameters derived for the unsaturated MODFET (using for gate length: $X_S$) and the exact solution of the wave-equation in the saturation region. The continuity of the total current and voltage was enforced at the floating boundary. This approach failed completely, and the S-parameters calculated were uncorrelated to the measured ones, for one cannot use the unsaturated Y-parameters in the GCA region of the saturated MODFET. Indeed these unsaturated Y-parameters are derived with the boundary conditions

$$v_{gco}(X_S-) = v_{gco}(X_S+, t)$$

(4.9)
Table 6: Deviation of calculated S parameters from the measured data for device 2045

<table>
<thead>
<tr>
<th>$d_s$ (Å)</th>
<th>$\langle \Delta S_{11} \rangle$</th>
<th>$\langle \Delta S_{21} \rangle$</th>
<th>$\langle \Delta S_{12} \rangle$</th>
<th>$\langle \Delta S_{22} \rangle$</th>
<th>$\langle \Delta U \rangle$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500 Å</td>
<td>0.1522</td>
<td>0.7418</td>
<td>0.04848</td>
<td>0.2574</td>
<td>1.8898</td>
</tr>
<tr>
<td>1500 Å</td>
<td>0.1574</td>
<td>0.6207</td>
<td>0.04558</td>
<td>0.1955</td>
<td>1.2501</td>
</tr>
<tr>
<td>2500 Å</td>
<td>0.1673</td>
<td>0.5809</td>
<td>0.04429</td>
<td>0.1897</td>
<td>1.5260</td>
</tr>
<tr>
<td>$C_{gs} = C_{gd} = 0$</td>
<td>0.3549</td>
<td>1.2738</td>
<td>0.06199</td>
<td>0.3783</td>
<td>3.0426</td>
</tr>
</tbody>
</table>

$v_{gc1}(X_S-) = v_{gc2}(X_S-) = 0$  \hspace{1cm} (4.10)

which are incorrect for the saturated MODFET. Instead it must be assumed, as is done in Appendix A, that each voltage component is continuous: $v_{gc0,1,2}(X_S-) = v_{gc0,1,2}(X_S+)$. Note that these correct boundary conditions are naturally enforced by a circuit implementation such as the one proposed by Rohdin [21].

Let us now consider in more detail the impact of $d_s$ upon the scattering parameters. $d_s$ is intended to represent the effective channel width in the saturation region. The best fit of the IV characteristics at all bias conditions was obtained for $d_s = 2500\text{Å}$ and $d_s = 1500\text{Å}$ on devices 2045 and 2379 respectively. In Tables 6 and 7 it is compared the error on the scattering parameters and unilateral gain averaged over all bias points obtained for different values of $d_s$. It is noted from Tables 6 and 7 that a reduced error results from using a value of $d_s = 1500\text{Å}$ and $d_s = 500\text{Å}$ which is closer to the equilibrium channel width (see [22]). This supports the concept that the RF extrinsic drain conductance $g_D(RF)$ can be smaller than the dc extrinsic drain.
Table 7: Deviation of calculated S parameters from the measured data for device 2379

<table>
<thead>
<tr>
<th>$d_s$ (Å)</th>
<th>$\langle \Delta S_{11} \rangle$</th>
<th>$\langle \Delta S_{21} \rangle$</th>
<th>$\langle \Delta S_{12} \rangle$</th>
<th>$\langle \Delta S_{22} \rangle$</th>
<th>$\langle \Delta U \rangle$(dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>0.1323</td>
<td>0.7153</td>
<td>0.05459</td>
<td>0.3133</td>
<td>2.3031</td>
</tr>
<tr>
<td>500</td>
<td>0.1316</td>
<td>0.6796</td>
<td>0.05310</td>
<td>0.2918</td>
<td>2.1762</td>
</tr>
<tr>
<td>1000</td>
<td>0.1315</td>
<td>0.6114</td>
<td>0.05080</td>
<td>0.2577</td>
<td>2.206</td>
</tr>
<tr>
<td>1500</td>
<td>0.1324</td>
<td>0.5654</td>
<td>0.04975</td>
<td>0.2390</td>
<td>2.3746</td>
</tr>
<tr>
<td>$C_{gs} = C_{gd} = 0$</td>
<td>0.2531</td>
<td>1.8118</td>
<td>0.09147</td>
<td>0.6197</td>
<td>5.2966</td>
</tr>
</tbody>
</table>

conductance $g_D (dc)$. The difference between the dc drain conductance $g_D (dc)$ and RF drain conductance $g_D (RF)$ is thought to originate in some devices from traps present in the buffer [32].

It is pointed out in Section 4.1 that the RF transconductance $g_M (RF)$ was larger in this device than the dc transconductance $g_M (dc)$. This ac-model does not account in itself for either the large charging time constant of the buffer traps or the large RC time constant in the wide bandgap material. These effects must therefore be reflected in the choice made for the parameters of the ac-model. For example for a device with negligible gate leakage (perfect insulated gate) the intrinsic drain conductance $G_{ds}$ and the intrinsic transconductance $G_{gs}$ in the ac-model should be related to the extrinsic RF drain conductance $g_D (RF)$ and extrinsic RF transconductance $g_M (RF)$ by Chou.
and Antoniadis relationships inverted [33]

\[
\frac{G_{ds}}{1 + G_{ds}(R_s + R_d) + G_{gs}R_s} = g_D(R_F) \\
\frac{G_{gs}}{1 + G_{ds}(R_s + R_d) + G_{gs}R_s} = g_M(R_F)
\]

(4.11) (4.12)

Finally let us note the impact of the parasitics on the microwave characteristics. Tables 6 and 7 show that the error in the unilateral power gain increases by 2-3 dB when we set \(C_{GS} = C_{GD} = 0\). Therefore parasitics play an important role in the prediction of the device performance.
CHAPTER V

UNILATERAL POWER GAIN RESONANCES AND $f_T-f_{MAX}$ ORDERING

5.1 Introduction

In the velocity-saturated model the FET is divided into two regions [34], a gradual-channel region (GCA) and a saturation region. In the gradual-channel region [35] the carrier concentration is determined by

$$qn_a(V_{GC}) = C_g(V_{GC} - V_T) \quad \text{for } V_T < V_{GC} < V_{GS_{MAX}}$$

$$qn_{SMAX} \quad \text{for } V_{GS_{MAX}} \leq V_{GC}$$

Constant mobility $\mu$, gate capacitance $C_g$ and threshold voltage $V_T$ are assumed in this region. In the saturation region, the Poisson equation is solved in the direction parallel to the channel, assuming a constant channel width $d_s$. The velocity of the electron is assumed to have saturated to $v_s$ in this region. The boundary between the GCA region and the saturation region is the position in the channel where the channel electric field reaches the critical field $E_c = v_s/\mu$. This simple model can handle both long and short channel FETs. In the long-channel mode the transconductance varies linearly with gate voltage. In the short-channel mode the transconductance saturates.
to a constant value for large gate voltages. The switch from the long channel mode to the short channel mode is determined by the ratio $\alpha = (E_c L_g)/(V_{GS} - V_T)$. Indeed one can easily verify for the velocity-saturated MODFET model that the ratio of the drain current at the onset of saturation $I_{dc}(sat)$ (when the channel field at the drain is equal to the critical field $E_c$) by the drain current in pinch-off $I_{dc}(pinch)$ for the same $V_{GS}$ voltage is given by

$$\frac{I_{dc}(sat)}{I_{dc}(pinch)} = 2\alpha^2 \left[ \frac{1}{\alpha^2} - \frac{1}{\alpha^2} \right] = w_1(\alpha)$$

(5.1)

Similarly the ratio of the transconductance $g_m(sat)$ at the onset of saturation by the maximum transconductance $g_{m,MAX}(sat)$ is

$$\frac{g_m(sat)}{g_{m,MAX}(sat)} = \frac{\alpha^{-1}}{\sqrt{1 + (\alpha^{-1})^2}} = w_2(\alpha^{-1})$$

(5.2)

The weight functions $w_1(\alpha)$ (plain line) and $w_2(\alpha^{-1})$ (dashed line) are plotted in Figure 67 versus $\alpha$ and $\alpha^{-1}$ respectively. From the tangential dashed-dotted lines shown in Figure 67 it is seen that the transition from long- to short-channel mode occurs for $\alpha = 1/2$ for the saturation current ratio $w_1$ and $\alpha^{-1} = 1$ for the saturation transconductance ratio $w_2$. Note that this criterion is only applicable in the range of validity of the GCA approximation. A GCA region is always expected in the case of high-aspect ratio $(L_g/d)$ FET where $d$ is the gate to channel spacing. Small aspect ratio FETs, where two-dimensional field effects are important over the entire gated channel are not considered here. The GCA approximation will also fail in conventional high-aspect ratio FET when the $V_{GS}$ voltage reaches $V_{GSMAX}$ and the channel charge $n_s$ saturates to $n_{sMAX}$. In a MODFET this occurs when the parasitic MESFET turns on [36]. The ratio $\alpha$ is therefore more correctly defined by
Figure 67: Current and transconductance weight functions $w_1(\alpha)$ (plain line) and $w_2(\alpha^{-1})$ (dashed line) plotted versus $\alpha$ and $\alpha^{-1}$ respectively.

\[
\alpha = \frac{E_c L_g}{\min[V_{GMAX}, V_{GS}] - V_T}.
\]  

(5.3)

As a consequence it is not possible in practice to turn on the short-channel mode in a long gate-length FET (e.g., 10 $\mu$).

5.2 Long and Short Channel Mode and the ac-current Gain

The high-frequency small-signal characteristics of the velocity-saturated MODFET model will be studied in both the short and long channel mode. Obviously the transport picture upon which this ac-model is based will in practice break down for frequencies corresponding to the energy relaxation time ($\sim 1$ THz) and the momentum relaxation time ($\sim 10$ THz).
The exact solution derived in Chapter II is used to calculate the unity current-gain cutoff frequency $f_T$ of the intrinsic MODFET versus gate length $L_g$ for the gate to source voltages $V_{GS} = 0, 0.1,$ and $0.2$ V and a drain to source voltage of $V_{DS} = 1$ V. The MODFET parameters used are given in Table 8. These unilateral current-gain cutoff frequencies $f_T(int)$ are plotted versus $\alpha$ in log scale in Figure 68. $f_T$ is defined here as the frequency at which we have

$$h_{21}(\omega_T) = \frac{|y_{21}(\omega_T)|}{|y_{11}(\omega_T)|} = \frac{|z_{21}(\omega_T)|}{|z_{22}(\omega_T)|} = 1$$

(5.4)

$f_T(int)$ increases with shrinking gate length ($\alpha$). The increase of $f_T(int)$ switches from the $1/L_g^2$ law of the long channel FET to the $1/L_g$ law expected for the short channel FET. This results from the saturation of the transconductance due to the

Figure 68: Variation of the unity current gain cutoff frequency $f_T$ versus gate length $L_g$ plotted versus $\alpha = E_cL_g/(V_{GS} - V_T)$ in log scale for an intrinsic MODFET with $V_{GS} = 0, 0.1,$ and $0.2$ V and $V_{DS} = 1$ V.
Table 8: Device parameters for the intrinsic short-channel MODFET

<table>
<thead>
<tr>
<th>Parameters</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_g )</td>
<td>gate width (( \mu m ))</td>
</tr>
<tr>
<td>( V_T )</td>
<td>threshold voltage (V)</td>
</tr>
<tr>
<td>( \mu )</td>
<td>mobility (( cm^2/V .sec ))</td>
</tr>
<tr>
<td>( v_s )</td>
<td>saturation velocity (( m/sec ))</td>
</tr>
<tr>
<td>( d )</td>
<td>gate to channel spacing (( \AA ))</td>
</tr>
<tr>
<td>( d_s )</td>
<td>channel width in saturation (( \AA ))</td>
</tr>
<tr>
<td>( \epsilon_1 )</td>
<td>channel dielectric constant</td>
</tr>
<tr>
<td>( \epsilon_2 )</td>
<td>gate dielectric constant</td>
</tr>
</tbody>
</table>

velocity saturation in the FET channel.

Note that the switch from the long to short channel mode occurs for \( \alpha \) between 1 and 2 as predicted in the previous section. For the saturation velocity and mobility of Table 1 and \( V_{DS} = 1 \text{ V} \) the corner point \( \alpha = 1 \) corresponds to a gate length of 1 and 1.66 \( \mu \) for \( V_{GS} = 0 \) and 0.2 V respectively.

If the effective saturation velocity \( v_s \) were to increase with decreasing gate length \( L_g \) one would have in the sub-micron regime an \( 1/L_g^\gamma \) law with \( 1 \leq \gamma \leq 2 \).

Rohdin [21] has demonstrated that despite the expected occurrence of velocity overshoot the effective saturation velocity is essentially independent of gate length for MODFETs with gate length varying from 0.9 to 0.3 \( \mu \). His analysis is based on the systematic reverse modeling of large number of FETs on different wafers. A
constant saturation velocity of $1.85 \times 10^5$ m/sec is used in this analysis.

### 5.3 Unilateral power gain of the wave-equation model

The analysis of the high-frequency performance of a device is typically done using the unilateral power gain $U$ derived by Mason [27].

$$
U = \frac{|y_{21} - y_{12}|^2}{4[\Re(y_{11})\Re(y_{22}) - \Re(y_{12})\Re(y_{21})]} \\
= \frac{|z_{21} - z_{12}|^2}{4[\Re(z_{11})\Re(z_{22}) - \Re(z_{12})\Re(z_{21})]}
$$

(5.5)

$U$ is the maximum available power gain (MAG) of a device once it has been unilaterized ($y_{12} = z_{12} = 0$) using feedback techniques. Figure 69 shows a possible feedback circuit (proposed by Mason himself) to unilaterize a three-terminal device. The maximum frequency of oscillation $f_{MAX}$ is defined as the frequency at which $U$ is unity. $f_{MAX}$ is often referred to as the frequency at which a three-port device switches from active to passive. The importance of $U$ and $f_{MAX}$ for characterizing a device hinges on their invariance upon loss-less coupling (feedback and loading).
Figure 70: Magnitude of the unilateral power gain versus frequency for an intrinsic MODFET ($V_{GS} = 0$ V and $V_{DS} = 1$ V) with a gate length of 3 $\mu$m (dashed-dotted line), 1 $\mu$m (dashed line), and 0.3 $\mu$m (plain line).

However because the feedback network required to unilaterize a device could only be achieved at a single frequency with loss-less passive components, $f_{MAX}$ is a narrow-band figure of merit. A narrow-band figure of merit is useful in classifying transistors for the design of tuned amplifiers and oscillators. $f_{MAX}$ is therefore used as a RF or microwave figure of merit. This is in contrast with $f_T$ which is a broad-band figure of merit and is therefore more relevant for classifying transistors for the design of broad-band and large-signal circuits.

Figure 70 shows the magnitude of the intrinsic unilateral gain calculated using the solution of the velocity-saturated wave-equation for a 3, 1 and 0.3 $\mu$m gate length FET with $V_{GS} = 0V$ and $V_{DS} = 1V$. For small frequencies one observes the usual 20 dB per
decade decrease of the intrinsic Unilateral power gain. In all these FETs, one observes at large-frequencies a periodic divergence of the intrinsic $|U|$ and alternate regions of positive $U$ and negative $U$. The resonant frequencies are approximately given by $f_n = (n + \frac{1}{2}) \frac{1}{2\tau_s}$ for positive integer $n$ where $\tau_s(V_{GS}, V_{DS}) = \ell/v_s$ is the transit-time through the saturation region of bias-dependent length $\ell(V_{GS}, V_{DS})$. Clearly these resonances are associated with the saturation region and occur for frequencies for which the phase of the drain current phaser $exp(-j\omega \tau_s)$ is approximately $\pm \pi$. The appearance of the negative $U$ regions are also correlated with the periodic development of a negative output ac-resistance.

The unilateral power gain resonances could be an artifact of the FET model which assumes the existence of a constant velocity saturation region indirectly controlled by the gate. The possibility of steady-state gain at frequencies above the 20 dB/decade extrapolated $f_{MAX}(int)$ is not however in contradiction with the principle of operation of an FET. An FET is a transit time device, and its switching speed is therefore limited by the length of its gate. However the amplification of a steady-state signal does not convey any information. For such a steady-state application, an FET is therefore not necessarily transit-time limited. Note that for the 0.3 $\mu$ intrinsic MODFET, the first resonant frequency $f_o$ occurs before the 20 dB/decade extrapolated $f_{MAX}(int)$. A resonance can also be predicted by the approximate solution of the wave-equation based upon the frequency-power series.

In practical devices, lossy parasitics (see Figure 71) will prevent the observation of the unilateral power gain resonances. Figure 72 show the impact of a source, gate
Figure 71: Equivalent circuit for the extrinsic MODFET. $C_{GS}$ and $C_{GD}$ are the fringe capacitors of the gate.

Figure 72: Unilateral power gain versus frequency for a 0.3 $\mu$m extrinsic MODFET with parasitics resistances $R_S = R_G = R_D = 0.01 \, \Omega$ (plain line), 0.1 $\Omega$ (dashed line), 1 $\Omega$ (dotted dashed line), and 5 $\Omega$ (dashed dashed line).
and drain resistances \( R_S = R_G = R_D = 0.01, 0.1, 1 \) and \( 5 \) \( \Omega \) upon the Unilateral power gain of the 0.3 \( \mu \) MODFET in the presence of the parasitics capacitors \( C_{GD} = C_{GS} = 50 \) fF (see Figure 71) and with bias \( V_{GS} = 0 \) V and \( V_{DS} = 1 \) V. For large enough resistances the unilateral power gain exhibits a switch from the 20 dB to the 40 dB drop per decade approximately at the resonant frequency \( f_o \). This is not without resemblance with the performance of the equivalent circuit reported by Steer and Trew [37]. Note however that here the corner frequency \( f_o \) is above the extrinsic \( f_{MAX}(ext) \). Up to now a 40 dB per decade decrease of the unilateral power gain has not been experimentally observed/reported in MODFETs (or MOSFETs).

It is noted that \( f_o \) can be smaller than the intrinsic \( f_{MAX}(int) \) (extrapolated with a 20 dB per decade slope) in a 0.3 \( \mu \) MODFET. However for realistic lossy parasitics the extrinsic \( f_{MAX}(ext) \) is much smaller than \( f_o \) (see Figure 72) and the unilateral power gain calculated from the extrinsic velocity-saturated MODFET wave-equation will exhibit a 20 dB per decade decrease in the entire active range.

Let us now address the issue of the ordering of \( f_T \) and \( f_{MAX} \). It is shown in Figure 73 that by the use of a sufficiently large gate resistance in presence of the parasitics resistance \( R_S = R_D = 2 \) \( \Omega \) and parasitics capacitance \( C_{GS} = C_{GD} = 50 \) fF with bias \( V_{GS} = 0 \) V and \( V_{DS} = 1 \) V, it is possible to reduce the extrinsic unilateral power gain \( U \) while maintaining a constant extrinsic \( f_T(ext) \) until \( f_{MAX}(ext) \) is smaller than \( f_T(ext) \). Large gate resistances are indeed a problem in submicron gate FET's and mushroom T- and L- shape gates are used to circumvent it.

As it was seen, lossy parasitics (gate, source and drain resistances combined with
Figure 73: Unilateral power gain and short circuit current gain (plain line) versus frequency for a 0.3 $\mu$ extrinsic MODFET using two different gate resistances $R_G = 5 \Omega$ (dashed line) and 25 $\Omega$ (dotted dashed line).

The parasitics capacitors $C_{GS}$ and $C_{GD}$) play a dominant role in shaping the high-frequency characteristics of a short-channel device (e.g., 12 dB drop of $U$ per octave).
CHAPTER VI

CONCLUSION

6.1 Conclusion

The velocity-saturated MODFET wave-equation was derived. This ideal model is based on a piece-wise linear charge-control model and velocity field relation and accounts for velocity saturation, and channel length narrowing. The exact solution was obtained in terms of Bessel functions and analytic expressions for the Y parameters in terms of a frequency power series.

A simple RC equivalent circuit developed from the frequency power-series Y-parameters was presented for the unsaturated intrinsic MODFET. This first-order RC equivalent circuit was found to hold to higher frequencies than the frequency power-series from which it is derived or the more complicated second-order iterative Y-parameters reported by [18]. Like the iterative Y-parameters this RC equivalent circuit features a graceful degradation of the small-signal parameters at high frequencies. Although quite simple the RC equivalent circuit selected departs from conventional equivalent circuit models which usually rely on a transmission line or RC delay for the drain transconductance and a C or series RC feedback element between the drain and gate and an inductor in series with the drain conductance.
In order to increase the frequency range of validity an optimal second-order RC equivalent circuit which admits a fourth-order frequency power-series solution was developed. It was demonstrated that this equivalent circuit exhibits a graceful degradation and holds to much higher frequencies than the first-order RC equivalent circuit, and even fourth-order iterative $Y$ parameters of the MODFET wave equation.

This non-quasi-static small-signal equivalent circuit model was then extended to the short-channel velocity-saturated MODFET wave-equation. The resulting equivalent circuit provided a graceful degradation of the small-signal $Y$-parameters at high frequencies.

To apply this ideal MODFET ac-model to real MODFET devices a parameter extraction technique was proposed. The resulting microwave model permitted one to reasonably predict the microwave characteristics (scattering parameters and unilateral power gain versus frequency for different bias) of a one micron gate length AlGaAs/GaAs MODFET and GaAlAs/InGaAs/GaAs MODFET. The parameters used by the intrinsic ac-model were all obtained from the fit of the IV characteristics alone. The microwave parasitic elements were either measured ($R_g, L_G, L_S, L_D$) or estimated ($C_{GS}, C_{GD}$). The merit of this analytic ac model is therefore its capacity to predict the microwave performance from the dc characteristics.

Using the exact solution of MODFET wave equation, the high-frequency characteristics of the saturated-velocity MODFET wave-equation was studied. This non-quasi-static small-signal model presents some novel features (e.g., unilateral power gain resonances). The observation or use of these non-quasi-static effects in the ex-
trinsic MODFET seems however to require unrealistically small source and drain resistances assuming this canonic ac-model is applicable to real devices.

6.2 Future Work

New models are required to evaluate the device parasitics and improve the extraction techniques for the gate capacitance $C_g$ and the threshold voltage $V_T$. Finally it might be possible to develop an extraction techniques for the mobility $\mu$ and the effective saturation velocity $v_s$, permitting to apply this simple ac model to sub-micron MODFETs for which velocity overshoot and undershoot have a strong effect on device performance.

The large-signal analysis is beyond the purpose of the dissertation. However a large-signal model based on the proposed first-order equivalent circuit was reported [38] recently for the long channel MODFET. However a large-signal model for the velocity-saturated MODFET has not yet been developed. Even though the analysis for the saturation region is not as simple as for the GCA region, it should be possible to develop a large signal model from the small signal equivalent circuit model developed in this dissertation for the velocity-saturated MODFET.
Appendix A

Frequency Power-Series Solution for the Velocity-Saturated MODFET wave equation

A.1 Calculation of $V_{GC}(x = L_g)$

The Poisson’s equation in the channel is given in (2.3). For convenience the equation is repeated below

$$\frac{d^2V_{GC}}{dx^2} = -\beta I(x, t) = -\beta (I_{dc} + ie^{j\omega t})$$  \hspace{1cm} (A.1)

For simplicity a new variable $x' = x - X_S$ is introduced, which make (A.1) as

$$\frac{d^2V_{GC}}{dx'^2} = -\beta I(x', t) = -\beta (I_{dc} + ie^{j\omega t})$$  \hspace{1cm} (A.2)

The boundary conditions are

$$\begin{align*}
    v_{GC}(x' = 0) &= v_{GC}(x_S) \\
    v_{GC}(x' = 0) &= -E_c
\end{align*}$$

Let’s start with integrating (A.2) regarding to $x'$.

$$\begin{align*}
    \frac{dV_{GC}}{dx'} &= -\beta \int_0^{x'} (I_{dc} + i)dx + \frac{dV_{GC}}{dx'} \\
    &= -\beta I_{dc}x' - E_c - \beta \int_0^{x'} i(x)dx
\end{align*}$$  \hspace{1cm} (A.3)
\( v_{GC}(L_g) \) is obtained by integration from 0 to \( \ell \)

\[
v_{GC}(L_g) = v_{GC}(x_S) - E_c(L_g - x_S) - \beta \int_0^{L_g-x_S} I_{dc} x' dx' - \beta \int_0^{L_g-x_S} \int_0^{x'} i(x) \, dx
\]

\[
= v_{GC}(x_S) - E_c(L_g - x_S) - \beta \frac{1}{2} I_{dc}(L_g - x_S)^2
- \beta \int_0^{L_g-x_S} \int_0^{x'} i(x) dx' dx'.
\] (A.4)

Retaining the first order terms gives

\[
v_{GC}(L_g) \approx v_{GC}(x_S) - E_c \ell - \beta \frac{1}{2} I_{dc} \ell^2
+ E_c x_s e^{jut} + v_{gc}(x_S) e^{jut} + \Delta v_{gc}(\ell) e^{jut} + \beta I_{dc} x_s e^{jut}
\] (A.5)

where \( \ell = L_g - X_S \) and

\[
\Delta v_{gc}(\ell) = -\beta \int_0^{\ell} \int_0^{x'} i(x) dx' dx'
\]

The ac potential at \( x = L_g \) is then:

\[
v_{gc}(L_g) = E_c x_s e^{jut} + v_{gc}(x_S) e^{jut} + \Delta v_{gc}(\ell) e^{jut} + \beta I_{dc} x_s e^{jut}
\] (A.6)

### A.2 Calculation of \( V_{GC}'(X_S) \)

The dc potential \( V_{GC} - V_T \) is given in Equation (2.6). For convenience the equation is rewritten as below

\[
V_{GC}(x) - V_T = (V_{GS} - V_T) \sqrt{1 + \left( k_s^2 - 2k_s \right) \frac{x}{X_S}}
\] (A.7)

Differentiating both side of the above equation gives

\[
V_{GC}'(X_S) = V_{cut} \frac{d^2}{dx^2} \left( \sqrt{1 - \frac{x}{X_S} + (1 - k_s)^2 \frac{x}{X_S}} \right) \bigg|_{x=X_S}
\]
\[ \begin{align*}
&= V_{\text{out}} \frac{d^2}{dx^2} \left( \sqrt{1 + \left( k_s^2 - 2k_s \right) \frac{x}{X_s}} \right) \bigg|_{x=x_s} \\
&= V_{\text{out}} \frac{d}{dx} \left[ \frac{1}{2} \frac{k_s^2 - 2k_s}{X_s} \left( 1 + \left( k_s^2 - 2k_s \right) \frac{x}{X_s} \right)^{-1/2} \right] \bigg|_{x=x_s} \\
&= V_{\text{out}} \frac{d}{dx} \left[ - \frac{k_s(1 - \frac{1}{2}k_s)}{X_s} \left( 1 + \left( k_s^2 - 2k_s \right) \frac{x}{X_s} \right)^{-1/2} \right] \bigg|_{x=x_s} \\
&= - \frac{k_s(1 - \frac{1}{2}k_s)}{X_s} V_{\text{out}} \left[ \frac{1}{2} \frac{k_s^2 - 2k_s}{X_s} \left[ 1 + \left( k_s^2 - 2k_s \right) \frac{x}{X_s} \right]^{-3/2} \right]_{x=x_s} \\
&= - \frac{k_s^2(1 - \frac{1}{2}k_s)^2}{X_s^3(1 - k_s)} V_{\text{out}}[1 + k_s^2 - 2k_s]^{-3/2} \\
&= - \frac{V_{\text{out}} k_s^2(1 - \frac{1}{2}k_s)^2}{X_s^3(1 - k_s)^3} \\
\end{align*} \]

where we use \( V_{\text{out}} = V_{GS} - V_T \).

\[ V''_{\text{GC}}(X_s) = - \frac{V_{\text{out}} k_s^2(1 - \frac{1}{2}k_s)^2}{X_s^3(1 - k_s)^3} \] (A.9)

### A.3 Power-Series Solution of Wave-Equation

The wave equation in the GCA and saturation regions can be solved using the frequency power series analysis which was first introduced by Ziel and Wu [20]. Since we are interested in the range of frequency up to \( f_{\text{max}} \) we will use an expansion up to the second order terms. The frequency power-series solution up to the fourth order will be shown in Appendix C without detailed calculation. The ac voltage and current, \( v_{ac}(x) \), \( i(x) \), and \( x \), are expanded in Taylor series in powers of \( j\omega \) up to the second order

\[ i(x) = i_0 + j\omega i_1 + (j\omega)^2 i_2 \] (A.10a)
\[ x_s = x_{s0} + j \omega x_{s1} + (j \omega)^2 x_{s2} \]  

(A.10c)

Substituting \( v_{gc}(x) \) and \( i(x) \) into Equations (2.5), (2.9) and (2.10) and equating the power of \( j \omega \) yields for the GCA region

\[ \frac{d^2}{dx^2} [g(V_{GC}(x))v_{gsc0}(x)] = 0 \]  
\[ \frac{d^2}{dx^2} [g(V_{GC}(x))v_{gsc1}(x)] = W_g C_g v_{gsc0}(x) \]  
\[ \frac{d^2}{dx^2} [g(V_{GC}(x))v_{gsc2}(x)] = W_g C_g v_{gsc1}(x) \]

(A.11a)  
(A.11b)  
(A.11c)

and for the saturation region,

\[ \frac{d i_0(x)}{d x} = 0 \]  
\[ \frac{d i_1(x)}{d x} = -\frac{1}{v_s} i_0 \]  
\[ \frac{d i_2(x)}{d x} = -\frac{1}{v_s} i_1 \]

(A.12)

and

\[ \frac{d^2 \Delta v_{gsc0}(x)}{dx^2} = -\beta i_0(x) \]  
\[ \frac{d^2 \Delta v_{gsc1}(x)}{dx^2} = -\beta i_1(x) \]  
\[ \frac{d^2 \Delta v_{gsc2}(x)}{dx^2} = -\beta i_2(x). \]

(A.13)

Introducing the new variable \( x' = x - X_S \) in Equation (A.12) gives

\[ \frac{d i_0}{d x'} = 0, \]  
\[ \frac{d i_1}{d x'} = -\frac{i_0}{v_s}, \]  
\[ \frac{d i_2}{d x'} = -\frac{i_1}{v_2}. \]  

(A.14)
and

\[ \frac{d^2 \Delta v_{g0}}{dx'^2} = -\beta i_0 \]
\[ \frac{d^2 \Delta v_{g1}}{dx'^2} = -\beta i_1 \]
\[ \frac{d^2 \Delta v_{g2}}{dx'^2} = -\beta i_2 \]  

(A.15)

Solving the Equation (A.14) yields

\[ i_0 = C_1 \]
\[ i_1 = -\frac{C_1}{v_s} x' + C_2 \]  
\[ i_2 = \frac{1}{2} \frac{C_1}{v_s^2} x'^2 - \frac{C_2}{v_s} x' + C_3 \]  

(A.16)

where \( C_1 \) and \( C_2 \) are arbitrary constants. Substituting Equation (A.16) into Equation (A.15) and integrating from 0 to \( \ell \) gives the potential across the saturation boundary.

\[ \Delta v_{g0} = \frac{\beta C_1}{2} x'^2 \]  
\[ \Delta v_{g1} = \frac{\beta C_1}{6v_s} x'^3 - \frac{\beta C_2}{2} x'^2 \]  
\[ \Delta v_{g2} = -\frac{\beta C_1}{24v_s^2} x'^4 + \frac{\beta C_2}{6v_s} x'^3 - \frac{\beta C_3}{2} x'^2 \]  

(A.17c)

where we used the boundary conditions

\[ \Delta v_{g0,1,2}(X_S) = 0 \]
\[ \Delta v_{g0,1,2}'(X_S) = 0 \]

Now the boundary conditions are needed in order to solve the above wave equation.

The boundary conditions used at \( x = 0 \) and \( x = L_g \) are

\[ v_{g0}(0) = v_{gs} \]  

(A.18a)
\[ v_{gc0}(L_g) = v_{gs} - v_{ds} \quad \text{(A.18b)} \]
\[ v_{gc1}(0) = v_{gc2}(0) = v_{gc1}(L_g) = v_{gc2}(L_g) = 0 \quad \text{(A.18c)} \]

And the following boundary conditions are used at the GCA/saturation boundary:

- Each component of the ac current is continuous at the boundary
  \[ i_{0,1,2}(X_S-) = -\frac{d}{dx} \left[ g(VGC(X_S))v_{gc0,1,2}(X_S) \right] = i_{0,1,2}(X_S+) \quad \text{(A.19)} \]

- Each component of the ac channel electric field in the saturation region is set to zero at the boundary \( \Delta v'_{gc0,1,2} = 0 \).

- Each component of the boundary motion is calculated using
  \[ x_{0,1,2} = -\frac{1}{\nu_{GC}(X_S)}v_{gc0,1,2}(X_S) \quad \text{(A.20)} \]

- Each component of the ac voltage is continuous at the boundary so that we have
  \[ v_{gc0,1,2}(L_g) = \beta I_{dc}x_{0,1,2} + \Delta v_{gc0,1,2}(l) + v_{gc0,1,2}(X_S) \quad \text{(A.21)} \]

Before the calculation of the channel voltage for entire region, let us define the following simpler notation.

\[ V_{out} = V_{GS} - V_T \]
\[ L = L_g \]
\[ V = V_{GC}(x) \]
\[ v = v_{GC}(x) \]
\[ V_s = V_{GC}(X_S) \]
And it will be helpful for understanding the calculation that the following terms are calculated first.

\[ g(V) = G_{dos}(1 - y)X_S \]  
(A.22)

\[ g(V_s) = G_{dos}(1 - k_s)X_S \]  
(A.23)

\[ g'(V_s) = -G_{dos} \frac{E_cX_S}{V_{out}} \]  
(A.24)

We are interested in the ac current in terms of \( v_{gs} \) and \( v_{ds} \) in order to calculate the intrinsic \( Y \) parameters. However the boundary conditions are given in form of ac voltages so that the ac channel voltages have to be calculated before the ac current can be obtained.

### A.3.1 Calculation of \( v_0 \)

The first order ac voltage \( v_0 \) for the entire channel can be obtained by solving the Equation (A.11a) for the GCA region and Equation (A.17a) for the saturation region with the boundary conditions (A.18a), (A.18b) and GCA/saturation boundary conditions. Integrating Equation (A.11a) gives the channel voltage in the GCA region for \( v_0 \) in terms of \( x \)

\[ g(V) v_0(x) = d_1 x + d_2 \]  
(A.25)

where \( d_1 \) and \( d_2 \) are arbitrary constant. \( d_2 \) can be obtained from the boundary condition (A.18a) with Equation (A.25)

\[ d_2 = v_{gs}G_{dos}X_S \]  
(A.26)

Substituting Equation (A.26) into Equation (A.25) yields
\[ v_0(x) = \frac{d_1 x + v_{gs} G_{dos} X_S}{g(V)} \]  

(A.27)

d_1 will be determined later on when the wave equation is solved in the saturation region.

The calculation of \( \Delta v_0 \) in the saturation region (Equation (A.17a)) is obtained by solving the Poisson equation (A.13) for a fixed saturation region width \( \ell \), and zero ac potential \( v_{gc}(X_S) = 0 \) and zero ac field \( v'_{gc}(X_S) = 0 \). The relation between \( d_1 \) and \( C_1 \) can be obtained by setting the ac current continuous at GCA/saturation boundary.

The ac current at the GCA/saturation boundary is

\[ \dot{i}_0(X_S) = -\frac{d}{dx}[g(V)v_0(x)]|_{x=X_S} = -d_1 \]

and the ac current in the saturation region is \( C_1 \) so that we can find \( C_1 = -d_1 \).

\( v_0(x = L_g) \) can be found in Equation (A.21) from the zero order term which is

\[ v_0(L_g) = \beta I_{dc} l x_0 + \Delta v_0(l) + v_0(X_S) \]  

(A.28)

The value of \( C_1 (= -d_1) \) can be found from this equation using the boundary condition (A.18b).

In order to obtain \( C_1 \), we have to express the above equation in terms of \( C_1 \). Let us start with the calculation \( v_0(X_S-) \) at the GCA/saturation boundary

\[ v_0(X_S) = \frac{d_1 X_S + v_{gs} G_{dos} X_S}{G_{dos}(1 - k_s) X_S} \]

\[ = \frac{-C_1 X_S + v_{gs} G_{dos} X_S}{G_{dos}(1 - k_s) X_S} \]  

(A.29)

\( x_0 \) can be obtained by substituting Equation (A.9) into Equation (A.20) and extracting the zero order component.
\[ x_0 = \frac{X_S^2(1 - k_s)^3}{V_{out}k_s^2(1 - \frac{1}{2}k_s)^2} v_0'(X_S) \]  

(A.30)

\( v_0'(X_S) \) is obtained by differentiating Equation (A.27) at GCA/saturation boundary and expressing it in terms of \( d_x \).

\[
 v_0'(X_S) = \left. \frac{d}{dx} \left[ \frac{d_x x + v_g g_{dos} X_S}{g(V)} \right] \right|_{x=X_S}
 = d_1 \left. \frac{d}{dx} \left[ \frac{x}{g(V)} \right] \right|_{x=X_S} + v_g g_{dos} X_S \left. \frac{1}{g(V)} \right|_{x=X_S}
 = d_1 \left. \frac{d}{dx} \left[ \frac{g(V) - xg'(V)}{g^2(V)} \right] \right|_{x=X_S} + v_g g_{dos} X_S \left. \frac{g'(V)}{g^2(V)} \right|_{x=X_S}
\]

(A.31)

Substituting Equation (A.24) into Equation (A.31) yields

\[
 v_0'(X_S) = \frac{C_1}{G_{dos}(1 - k_s)X_S} \left[ 1 + \frac{E_c X_S}{(1 - k_s)V_{out}} \right] + v_g g_{dos} X_S \left[ \frac{E_c X_S}{(1 - k_s)^2 X_S V_{out}} \right]
\]

(A.32)

From Equation (A.30) and Equation (A.32) we obtain

\[
 x_0 = -\frac{(1 - k_s)^2 C_1}{(\frac{V_{out}}{X_S})G_{dos}k_s^2(1 - \frac{1}{2}k_s)^2} \left[ 1 + \frac{E_c X_S}{(1 - k_s)V_{out}} \right]
 + \frac{(1 - k_s)^2 v_g}{(\frac{V_{out}}{X_S})k_s^2(1 - \frac{1}{2}k_s)^2} \left[ \frac{E_c X_S}{V_{out}(1 - k_s)} \right]
\]

(A.33)

The next step is to calculate \( \Delta v_0(\ell) \) in Equation (A.17a).

\[
 \Delta v_0(\ell) = -\frac{\beta C_1}{2} \ell^2
\]

(A.34)

Substituting Equation (A.33), (A.34), (A.29) into Equation (A.28) gives

\[
 v_0(L_g) = -\frac{(1 - k_s)^2 C_1 \beta I_{dc} \ell}{(\frac{V_{out}}{X_S})G_{dos}k_s^2(1 - \frac{1}{2}k_s)^2} \left[ 1 + \frac{E_c X_S}{(1 - k_s)V_{out}} \right]
 + \frac{(1 - k_s)^2 v_g \beta I_{dc} \ell}{(\frac{V_{out}}{X_S})k_s^2(1 - \frac{1}{2}k_s)^2} \left[ \frac{E_c X_S}{V_{out}(1 - k_s)} \right]
 - \frac{C_1}{G_{dos}(1 - k_s)} + \frac{v_g}{(1 - k_s)} = v_{gs} - v_{ds}
\]

(A.35)
By solving Equation (A.35) \( C_1 \) will be known in terms of \( v_{gs} \) and \( v_{ds} \). For simplicity we will introduce the new symbols \( R_i \) and \( R_y \) which are defined in Section 2.3.

\[
\begin{align*}
\nu_0(L_y) &= -\frac{R_i C_1}{G_{dos}} [1 + R_y] + R_i R_y v_{gs} - \frac{\beta \ell^2}{2} C_1 \\
&= -\frac{C_1}{G_{dos}(1 - k_s)} + \frac{v_{gs}}{(1 - k_s)} = v_{gs} - v_{ds}
\end{align*}
\] (A.36)

Manipulating the above equation gives

\[
C_1 = \frac{G_{dos} [(k_s + R_i R_y(1 - k_s))v_{gs} + (1 - k_s)v_{ds}]}{1 + \frac{\beta \ell^2}{2} G_{dos}(1 - k_s) + (1 - k_s)R_i(1 + R_y)}
\]

\[
= \frac{G_{dos} R_d [(k_s + R_i R_y(1 - k_s))v_{gs} + (1 - k_s)v_{ds}]}{(1 - k_s)(1 + R_y)}
\]

where

\[
R_d = 1 + \frac{\beta \ell^2}{2} G_{dos}(1 - k_s) + (1 - k_s)R_i(1 + R_y)
\]

**A.3.2 Calculation of \( v_1 \)**

We shall now integrate the wave equation (A.11b) in order to get \( v_1 \) in the GCA region

\[
\frac{d^2}{dx^2} [g(V)v_1] = C_g W_g v_0(x)
\]

\[
= C_g W_g \frac{-C_1 x + v_{gs} G_{dos} X_S}{g(V)}
\] (A.37)

Before to do so let us introduce some new variables which will simplify the integration.

The channel current is given by

\[
I_{dc} = \mu C_g W_g [V_{GS} - V_T - V_{CS}] \frac{dV_{CS}(x)}{dx}
\] (A.38)

Using Equation (A.38) \( dx \) can be expressed in terms of \( V_{CS}(x) \) as follows
\[ dx = \frac{\mu C_s W_g}{I_d} [V_{GS} - V_T - V_{CS}(x)] dV_{CS}(x) \] (A.39)

Integrating Equation (A.39) from 0 to \( x \) yields

\[
x = \frac{\mu C_s W_g}{I_d} \left[ (V_{GS} - V_T) V_{CS}(x) - \frac{1}{2} V_{CS}^2(x) \right]
= \frac{\mu C_s W_g (V_{GS} - V_T)^2}{I_d} \left[ V_{CS}(x) \frac{V_{GS} - V_T}{V_{GS} - V_T} - \frac{1}{2} \left( \frac{V_{CS}(x)}{V_{GS} - V_T} \right)^2 \right]
\]

Let us introduce the new variable \( y \)

\[
y = \frac{V_{CS}(x)}{V_{GS} - V_T}
\]

so that we can write

\[
x = \frac{\mu C_s W_g (V_{GS} - V_T)^2}{I_d} \left( y - \frac{1}{2} y^2 \right) \] (A.40)

Integrating Equation (A.38) from 0 to \( X_s \) gives

\[
I_d = \frac{\mu C_s W_g}{X_s} \left[ (V_{GS} - V_T) V_{CS}(X_s) - \frac{1}{2} V_{CS}^2(X_s) \right]
= \frac{\mu C_s W_g (V_{GS} - V_T)^2}{X_s} \left[ V_{CS}(X_s) \frac{V_{GS} - V_T}{V_{GS} - V_T} - \frac{1}{2} \left( \frac{V_{CS}(X_s)}{V_{GS} - V_T} \right)^2 \right]
= \frac{\mu C_s W_g (V_{GS} - V_T)^2}{X_s} \left( k_s - \frac{1}{2} k_s^2 \right) \] (A.41)

where we introduced the new variable

\[
k_s = \frac{V_{CS}(X_s)}{V_{GS} - V_T}
\]

Replacing Equation (A.41) into Equation (A.40) we obtain an expression related to \( y \)

\[
x = \frac{X_s}{k_s \left( 1 - \frac{1}{2} k_s \right)} \left( y - \frac{1}{2} y^2 \right)
\]
Let us differentiate the above equation by $y$ in order to obtain the relation between $dx$ and $dy$

\[
\frac{dx}{dy} = \frac{X_S}{k_s \left(1 - \frac{1}{2}k_s\right)} (1 - y)
\]

so that $dx$ can be expressed in terms of $dy$

\[
dx = \frac{X_S(1 - y)}{k_s \left(1 - \frac{1}{2}k_s\right)} dy
\]  \hspace{1cm} (A.42)

Let's start integration of the wave equation (A.37) using these new variables

\[
\frac{d}{dx} [g(V)v_1(x)]
= C_gW_g \int_0^x -\frac{C_1 x + v_g G_{d0s} X_S}{g(V)} dx

= -C_gW_g \int_0^y C_1 X_S (y - \frac{1}{2}y^2) \frac{1}{k_s \left(1 - \frac{1}{2}k_s\right) G_{d0s} (1 - y) X_S k_s \left(1 - \frac{1}{2}k_s\right)} dy

+ C_gW_g \int_0^y \frac{G_{d0s} X_S v_g}{G_{d0s} (1 - y) X_S} dy + e_1

= -\frac{C_0 v_g}{G_{d0s} k_s^2 \left(1 - \frac{1}{2}k_s\right)^2} \int_0^y \left(y - \frac{1}{2}y^2\right) dy + \frac{C_0 v_g}{G_{d0s} k_s \left(1 - \frac{1}{2}k_s\right)} \int_0^y dy + e_1

= -\frac{C_0 v_g}{G_{d0s} k_s^2 \left(1 - \frac{1}{2}k_s\right)^2} \left(\frac{1}{2}y^2 - \frac{1}{6}y^3\right) + \frac{C_0 v_g}{G_{d0s} k_s \left(1 - \frac{1}{2}k_s\right)} y + e_1
\]

Integrating once more yields

\[
g(V)v_1(x)
= -\frac{C_0 v_g}{G_{d0s} k_s^2 \left(1 - \frac{1}{2}k_s\right)^2} \int_0^x \left(\frac{1}{2}y^2 - \frac{1}{6}y^3\right) dx + \frac{C_0 v_g}{k_s \left(1 - \frac{1}{2}k_s\right)} \int_0^x y dx + e_1 x + e_2

= -\frac{C_0 v_g}{G_{d0s} k_s^2 \left(1 - \frac{1}{2}k_s\right)^2} \int_0^y \left(\frac{1}{2}y^2 - \frac{1}{6}y^3\right) \frac{X_S (1 - y)}{k_s \left(1 - \frac{1}{2}k_s\right)} dy

+ \frac{C_0 v_g}{k_s \left(1 - \frac{1}{2}k_s\right)} \int_0^y \frac{X_S (1 - y)}{k_s \left(1 - \frac{1}{2}k_s\right)} dy + e_1 x + e_2

= -\frac{C_0 C_1 X_S}{G_{d0s} k_s^2 \left(1 - \frac{1}{2}k_s\right)^3} \int_0^y \left(\frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{6}y^4\right) dy
\]
\[ u(x) = \frac{C_{os}v_{gs}X_S}{k_s^2(1 - \frac{1}{2}k_s)^2} \int_0^y (y - y^2) dy + e_1x + e_2 \]
\[ = -\frac{C_{os}X_SC_1}{G_{dos}k_s^3(1 - \frac{1}{2}k_s)^3} \left[ \frac{1}{6}y^3 - \frac{1}{6}y^4 + \frac{1}{30}y^5 \right] \]
\[ + \frac{v_{gs}C_{os}X_S}{k_s^2(1 - \frac{1}{2}k_s)^2} \left[ \frac{1}{2}y^2 - \frac{1}{3}y^3 \right] + e_1x + e_2 \]

\( e_2 = 0 \) because \( v_0(0) = 0 \) from boundary condition (A.18c).

\[ v_1(x) = \frac{1}{g(V)} \left[ -\frac{C_{os}X_SC_1}{G_{dos}k_s^3(1 - \frac{1}{2}k_s)^3} \left( \frac{1}{6}y^3 - \frac{1}{6}y^4 + \frac{1}{30}y^5 \right) \right. \]
\[ + \left. \frac{v_{gs}C_{os}X_S}{k_s^2(1 - \frac{1}{2}k_s)^2} \left( \frac{1}{2}y^2 - \frac{1}{3}y^3 \right) + e_1x \right] \]

(A.43)

AC potential at \( X_S \), where \( y = k_s \), is

\[ v_1(X_S) = \frac{1}{G_{dos}(1 - k_s)} \left[ -\frac{C_{os}C_1}{G_{dos}}E + v_{gs}C_{os}C + e_1 \right] \]

(A.44)

Unknown variable \( e_1 \) will be obtain by applying the remaining boundary condition to Equation (A.21) for the first order term. For convenience Equation (A.21) will be rewritten for the first order term.

\[ v_1(L_g) = \beta I_{dc}x_1 + \Delta v_1(\ell) + v_1(X_S) \]

(A.45)

The other boundary condition \( v_1(L_g) = 0 \) will be used to obtain \( e_1 \) with Equation (A.46). Equation (A.46) must be rewritten in terms of \( e_1 \) through the following steps. First the relation between \( e_1 \) and \( C_2 \), which is used in saturation region, will be established through GCA/saturation boundary condition, ac current continuity, the ac current in GCA region at \( X_S \) is

\[ i_1(X_S) = -\frac{d}{dx} [g(V)v_1(x)]_{x=X_S} \]
\[ = \frac{C_{os}}{G_{dos}} DC_1 - v_{gs}C_{os}A - e_1 \]
And ac current in saturation region at $X_S(x' = 0)$ is

$$i_1(x' = 0) = C_2$$

$C_2$ can be expressed in terms of $e_1$ from the above equations

$$C_2 = \frac{C_{os}}{G_{dos}} DC_1 - v_{gs} C_{os} A - e_1$$ (A.47)

where $A$ and $D$ are defined in Chapter II section 3. Next step is to rewrite $\Delta v_1(\ell)$ in terms of $e_1$. From Equation (A.17b)

$$\Delta v_1(\ell) = \frac{\beta C_1}{6v_s} \ell^3 - \frac{\beta C_2}{2} \ell^2$$ (A.48)

Substituting Equation (A.47) into Equation (A.48) yields

$$\Delta v_1(\ell) = \frac{\beta C_1}{6v_s} \ell^3 - \frac{\beta C_2}{2} \ell^2 \left[ \frac{C_{os}}{G_{dos}} DC_1 - v_{gs} C_{os} A - e_1 \right]$$ (A.49)

Last step is calculation $x_1$ in terms of $e_1$. Substituting Equation (A.9) into Equation (A.20) and extracting the first order term yields

$$x_1 = \frac{X_S^2(1 - k_x)^3}{V_{out} k_x^2 (1 - \frac{1}{2} k_x)^2} v_1'(X_S)$$ (A.50)

In order to get $x_1$ in terms of $e_1$, we must calculate ac electric field, $v_1'(X_S)$, in terms of $e_1$. Ac electric field in GCA region is differentiating Equation (A.44) by $x$,

$$v_1'(X_S) = \frac{1}{g(V_s)} \left[ \frac{C_{os} C_1}{G_{dos}} D + v_{gs} C_{os} A + e_1 \right]$$

$$- \frac{g'(V_s)}{g^2(V_s)} \left[ - \frac{C_{os} X_S C_1}{G_{dos}} E + v_{gs} C_{os} X_S C + e_1 X_S \right]$$

$$= \frac{1}{G_{dos} X_S (1 - k_x)} \left[ - \frac{C_{os} C_1}{G_{dos}} D + v_{gs} C_{os} A + e_1 \right]$$

$$+ \frac{G_{dos} E x_x}{G_{dos} X_S^2 (1 - k_x)^2} \left[ - \frac{C_{os} X_S C_1}{G_{dos}} E + v_{gs} C_{os} X_S C + e_1 X_S \right]$$

$$= \frac{1}{G_{dos} X_S (1 - k_x)} \left[ - \frac{C_{os} C_1}{G_{dos}} (D + E R_y) \right]$$

$$+ \frac{v_{gs} C_{os} (A + C R_y) + e_1 (1 + R_y)}{G_{dos} X_S (1 - k_x)}$$
where $A$, $C$, $D$, and $E$ are defined in Chapter II. Now $x_1$ can be expressed in terms of $e_1$

\[
x_1 = \frac{(1 - k_s)^2}{G_{dos} \left( \frac{V_{out}}{X_s} \right) k_s^2 (1 - \frac{1}{2} k_s)^2} \left[ \frac{C_{os} C_1}{G_{dos}} (D + E R_y) + v_{gs} C_{os} (A + C R_y) + e_1 (1 + R_y) \right]
\]

(A.51)

Replacing Equation (A.51), (A.49), (A.45) into Equation (A.46) and applying boundary condition (A.18c), $v_1(L_y) = 0$.

\[
0 = -\frac{C_1}{G_{dos} (1 - k_s)} \left[ \frac{C_{os} R_i}{G_{dos}} (D + E R_y)(1 - k_s) + \frac{C_{os}}{G_{dos}} E \right] + v_{gs} C_{os} \left[ R_i (1 - k_s) (A + C R_y) + C \right]
\]

\[
+ e_1 \left[ \frac{C_{os}}{G_{dos}} \left[ 1 + R_i (1 + R_y) (1 - k_s) \right] + \beta C_1 \ell^2 \frac{G_{dos} (1 - k_s)}{6 v_y} \right]
\]

\[
= -\frac{C_1 C_{os}}{G_{dos} (1 - k_s)} \left[ R_i (D + E R_y)(1 - k_s) + E + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) D \right]
\]

\[
+ v_{gs} C_{os} \left[ R_i (1 - k_s) (A + C R_y) + C + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) A \right]
\]

\[
+ e_1 \left[ 1 + R_i (1 + R_y) (1 - k_s) + \frac{\beta \ell^2}{2} G_{dos} (1 + k_s) \right] + \beta C_1 \ell^2 \frac{G_{dos}}{6 \tau_s}
\]

where $R_i$ and $R_y$ are defined in Chapter II. Multiplying $G_{dos} (1 - k_s)$ to both sides to make calculation easy gives

\[
0 = -\frac{C_1 C_{os}}{G_{dos}} \left[ R_i (1 - k_s) D + (1 + R_i R_y (1 - k_s)) E + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) D \right]
\]

\[
+ v_{gs} C_{os} \left[ R_i (1 - k_s) A + (1 + R_i R_y (1 - k_s)) C + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) A \right]
\]

\[
+ e_1 R_d + \frac{\beta C_1 \ell^2}{6 \tau_s G_{dos} (1 - k_s)}
\]
where \( R_A \) and \( R_B \) are given in Chapter II. Equation (A.47) and (A.52) gives

\[
C_2 = C_{os} \left[ \frac{C_1}{G_{dos}} D - v_{gs}A \right] - e_1
\]

(A.53)

Now \( v_1 \) is known for entire gate length.

**A.3.3 Calculation of \( v_2 \)**

The wave equation for the second order ac potential in GCA region is

\[
\frac{d^2}{dx^2} [g(V)v_2] = C_g W_g v_1
\]

(A.54)

Substituting Equation (A.44) into Equation (A.54) and integrating both side gives
Integrating again gives

\[ g(V)v_2(x) \]

\[ = - \frac{C_{0a}^2C_1}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \frac{1}{g(V)} \left( \frac{1}{2}y^2 + \frac{1}{3}y^3 \right) dy + C_{0a}Wg \int_0^\infty \frac{e_1x}{g(V)} dx + f_1 \]

\[ = - \frac{C_{0a}^2C_1}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \frac{1}{G_{da}X_S(1 - y)} \left( \frac{1}{6}y^3 + \frac{1}{180}y^6 \right) X_S(1 - y) dy + f_1 \]

\[ + C_{0a}Wg \int_0^\infty \frac{1}{G_{da}X_S(1 - y)} \left( \frac{1}{12}y^4 \right) X_S(1 - y) dy + f_1 \]

\[ = - \frac{C_{0a}^2C_1}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \frac{1}{6}y^3 \left( \frac{1}{12}y^4 \right) X_S(1 - y) \frac{1}{k_s(1 - \frac{1}{2}k_s)} dy \]

\[ + \frac{C_{0a}^2v_{gs}}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{2}y^2 - \frac{1}{6}y^3 \right) X_S(1 - y) \frac{1}{k_s(1 - \frac{1}{2}k_s)} dy + f_1 + f_2 \]

\[ = - \frac{C_{0a}^2C_1}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{24}y^4 - \frac{1}{30}y^5 + \frac{1}{180}y^6 \right) X_S(1 - y) \frac{1}{k_s(1 - \frac{1}{2}k_s)} dy \]

\[ + \frac{C_{0a}^2v_{gs}}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{2}y^2 - \frac{1}{12}y^4 \right) X_S(1 - y) \frac{1}{k_s(1 - \frac{1}{2}k_s)} dy \]

\[ + \frac{C_{0a}^2C_1X_S}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{24}y^4 - \frac{1}{4}y^5 + \frac{7}{180}y^6 - \frac{1}{180}y^7 \right) dy \]

\[ + \frac{C_{0a}^2v_{gs}X_S}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{6}y^3 - \frac{1}{4}y^4 \right) dy + f_1 + f_2 \]

\[ + \frac{C_{0a}^2v_{gs}X_S}{G_{da}k_2^2(1 - \frac{1}{2}k_s)^3} \int_0^\infty \left( \frac{1}{2}y^2 - \frac{2}{3}y^3 + \frac{1}{6}y^4 \right) dy + f_1 + f_2 \]
\[ v_2(x) = \frac{1}{g(V)} \left[ \frac{C_{os}^2 X_S C_1}{G_{dos}^2 k_s^5 (1 - \frac{1}{2} k_s)^5} \left[ \frac{1}{120} y^5 - \frac{1}{80} y^6 + \frac{1}{180} y^7 - \frac{1}{1440} y^8 \right] \\
+ \frac{v_{gs} C_{os}^2 X_S}{G_{dos} k_s^4 (1 - \frac{1}{2} k_s)^4} \left[ \frac{1}{24} y^4 - \frac{1}{20} y^5 + \frac{1}{72} y^6 \right] \\
+ \frac{C_{os} X_S e_1}{G_{dos} k_s^2 (1 + \frac{1}{2} k_s)^3} \left( \frac{1}{6} y^3 - \frac{1}{6} y^4 + \frac{1}{30} y^5 \right) \right] + f_1 x \] 

Since boundary condition (A.18c) gives \( v_2(x = 0) = 0 \), \( f_2 \) is zero.

\[ v_2(x) = \frac{1}{g(V)} \left[ \frac{C_{os}^2 X_S d_1}{G_{dos}^2 k_s^5 (1 - \frac{1}{2} k_s)^5} \left[ \frac{1}{120} y^5 - \frac{1}{80} y^6 + \frac{1}{180} y^7 - \frac{1}{1440} y^8 \right] \\
+ \frac{v_{gs} C_{os}^2 X_S}{G_{dos} k_s^4 (1 - \frac{1}{2} k_s)^4} \left[ \frac{1}{24} y^4 - \frac{1}{20} y^5 + \frac{1}{72} y^6 \right] \\
+ \frac{C_{os} X_S e_1}{G_{dos} k_s^2 (1 + \frac{1}{2} k_s)^3} \left( \frac{1}{6} y^3 - \frac{1}{6} y^4 + \frac{1}{30} y^5 \right) \right] + f_1 x \] 

The second order ac potential at \( x = X_s \), where \( y = k_s \), is given as

\[ v_2(X_s) = \frac{1}{G_{dos}(1 - k_s) X_s} \left[ -\frac{C_{os}^2 X_S C_1}{G_{dos}^2} H + \frac{v_{gs} C_{os}^2 X_S}{G_{dos}} G \\
+ \frac{C_{os} X_S e_1}{G_{dos}} E + f_1 X_s \right] \\
= \frac{1}{G_{dos}(1 - k_s)} \left[ -\frac{C_{os}^2 C_1}{G_{dos}^2} H + \frac{v_{gs} C_{os}^2}{G_{dos}} G + \frac{C_{os} e_1}{G_{dos}} E + f_1 \right] \quad (A.55) \]

where \( H, G, \) and \( E \) are given in Chapter II. The other boundary condition gives ac potential at drain side zero, so that we can calculate the \( f_1 \). The second order ac potential at drain side can be obtained from Equation (A.21) by extracting second order terms

\[ v_2(L_y) = \beta I_{dl} x_2 + \Delta v_2(\ell) + v_2(X_s) \quad (A.56) \]

\( \Delta v_2(\ell) \) is given in Equation (A.17c)

\[ \Delta v_2(\ell) = -\frac{\beta C_1}{24 v_s^2} \ell^4 + \frac{\beta C_2}{6 v_s} \ell^3 - \frac{\beta C_3}{2} \ell^2 \quad (A.57) \]
ac current is continuous at GCA/saturation boundary so that

\[
i_2(x_s) = \frac{d}{dz}[g(V)v_2(x)]|_{x=x_s}
= -\frac{C_{os}d_1}{G_{dos}^2}F - \frac{v_{gs}C_{os}^2}{G_{dos}}B + \frac{C_{os}e_1}{G_{dos}}D - f_1
= C_3 = i_2(x' = 0)
\]  

(A.58)

where \(F, B,\) and \(D\) are given in Chapter II. From Equation (2.16) and Equation (A.9)

\[
x_2 = \frac{X_2^2(1-k_s)^3}{V_{out}k_s^2(1-\frac{1}{2}k_s)^2}v_2'(x_s)
\]  

(A.59)

\[
v_2'(x_s) = \frac{1}{g(V_s)} \left[ \frac{C_{os}^2d_1}{G_{dos}^2}F + \frac{v_{gs}C_{os}^2}{G_{dos}}B + \frac{C_{os}e_1}{G_{dos}}D + f_1 \right]
- \frac{g'(V_s)}{g(V_s)} \left[ \frac{C_{os}d_1X_s}{G_{dos}^2}H + \frac{X_sv_{gs}C_{os}^2}{G_{dos}}G + \frac{X_scos e_1}{G_{dos}}E + f_1X_s \right]
= \frac{1}{G_{dos}(1-k_s)x_s} \left[ \frac{C_{os}d_1}{G_{dos}^2}F + \frac{v_{gs}C_{os}^2}{G_{dos}}B + \frac{C_{os}e_1}{G_{dos}}D + f_1 \right]
+ \frac{G_{dos}E_{os}x_s}{G_{dos}^2(1-k_s)^2} \left[ \frac{C_{os}d_1X_s}{G_{dos}}H + \frac{v_{gs}C_{os}^2X_s}{G_{dos}}G \right]
+ \frac{C_{os}e_1X_s}{G_{dos}}E + f_1X_s
= \frac{1}{G_{dos}(1-k_s)x_s} \left[ \frac{C_{os}d_1}{G_{dos}^2}(F + R_yH) + \frac{v_{gs}C_{os}^2}{G_{dos}}(B + R_yG) \right]
+ \frac{C_{os}e_1}{G_{dos}}(D + R_yE) + f_1(1 + R_y)
\]  

(A.60)

Substituting Equation (A.60) into Equation (A.59) yields

\[
x_2 = \frac{(1-k_s)^2}{G_{dos}(V_{out}/X_s)k_s^2(1-\frac{1}{2}k_s)^2} \left[ \frac{C_{os}d_1}{G_{dos}^2}(F + R_yH) \right]
+ \frac{v_{gs}C_{os}^2}{G_{dos}}(B + R_yG) + \frac{C_{os}e_1}{G_{dos}}(D + R_yE) + f_1(1 + R_y)
\]  

(A.61)
Substituting Equation (A.55), (A.57), and (A.61) into Equation (A.56)

\[
0 = v_2(L_g) = \frac{R_i}{G_{dos}} \left[ -\frac{C_{os}^2 C_1 G_{dos}^2}{G_{dos}^2} (F + R_y H) + \frac{v_{gs} C_{os}^2}{G_{dos}} (B + R_y G) \right. \\
+ \left. \frac{C_{os} e_1}{G_{dos}} (D + R_y E) + f_1 (1 + R_y) \right] - \frac{\beta C_1^3}{24 v_s^2} + \frac{\beta C_1^2 C_2}{6 v_s} - \frac{\beta C_1^3}{2} \\
+ \frac{1}{G_{dos} (1 - k_s)} \left[ -\frac{C_{os}^2 C_1}{G_{dos}^2} H + \frac{v_{gs} C_{os}^2}{G_{dos}} G + \frac{C_{os} e_1}{G_{dos}} E + f_1 \right] \tag{A.62}
\]

From Equation (A.58) we can obtain \( C_2 \) and introduce it in Equation (A.62)

\[
0 = v_2(L_g) = \frac{R_i}{G_{dos}} \left[ -\frac{C_{os}^2 C_1 G_{dos}^2}{G_{dos}^2} (F + R_y H) + \frac{v_{gs} C_{os}^2}{G_{dos}} (B + R_y G) \right. \\
+ \left. \frac{C_{os} e_1}{G_{dos}} (D + R_y E) + f_1 (1 + R_y) \right] - \frac{\beta \ell^2 \tau_s^2 C_1}{24} \\
+ \frac{\beta \ell^2 C_2 \tau_s}{6} + \frac{\beta \ell^2}{2} \left[ -\frac{C_{os}^2 C_1}{G_{dos}^2} F + \frac{v_{gs} C_{os}^2}{G_{dos}} B + \frac{C_{os} e_1}{G_{dos}} D + f_1 \right] \\
+ \frac{1}{G_{dos} (1 - k_s)} \left[ -\frac{C_{os}^2 d_1}{G_{dos}^2} H + \frac{v_{gs} C_{os}^2}{G_{dos}} G + \frac{C_{os} e_1}{G_{dos}} E + f_1 \right] \\
= -\frac{\beta \ell^2 \tau_s^2 C_1}{24} + \frac{\beta \ell^2 C_2 \tau_s}{6} \\
+ \frac{1}{G_{dos} (1 - k_s)} \left[ C_{os}^2 \left[ R_i (1 - k_s) (F + R_y H) + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) F + H \right] \\
+ \frac{v_{gs} C_{os}^2}{G_{dos}} \left[ R_i (1 - k_s) (B + R_y G) + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) B + G \right] \\
+ \frac{C_{os} e_1}{G_{dos}} \left[ R_i (1 - k_s) (D + R_y E) + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) D + E \right] \\
+ f_1 \left[ R_i (1 - k_s) (1 + R_y) + \frac{\beta \ell^2}{2} G_{dos} (1 - k_s) + 1 \right] \right] = 0
\]

\[
f_1 = \frac{R_d}{G_{dos} (1 - k_s)} \left[ \frac{\beta \ell^2 \tau_s^2 C_1}{24} - \frac{\beta \ell^2 C_2 \tau_s}{6} \right]
\]
\[
- \frac{C_2^2 d_1}{G_{dos}^2} [R_A F + R_B H] - \frac{v_{gs} C_2^2}{G_{dos}} [R_A B + R_B G] - \frac{C_{ox} e_1}{G_{dos}} [R_A D + R_B E]
\]
\[
= \frac{\beta \ell^2 \tau_s C_1}{24} G_{dos} (1 - k_s) R_d - \frac{\beta \ell^2 \tau_s C_1}{6} R_d G_{dos} (1 - k_s) + \frac{C_{ox} R_d}{G_{dos}}
\]
\[
\left[ \frac{C_{ox} C_1}{G_{dos}} [R_A F + R_B H] - v_{gs} C_{ox} [R_A B + R_B G] - E_1 [R_A D + R_B E] \right]
\]
\[
C_3 = \frac{C_2^2 C_1}{G_{ dos}^2} F - \frac{v_{gs} C_2^2}{G_{ dos}} B - \frac{C_{ox} e_1}{G_{ dos}} D - f_1
\]

### A.3.4 Calculation of \( i_g \) and \( i_d \)

The ac current flowing into the gate can be obtained by subtracting \( i(x = L_g) \) from \( i(x = 0) \). We assume that the ac channel current is flowing from drain to source. The ac current flowing into the drain is \( i(x = L_g) \). \( i(x = 0) \) can be obtained from the current equation of the GCA region and \( i(x = L_g) \) from the current equation of the saturation region. The relation between ac current and ac potential in GCA region is given by Equation (2.4). Rewriting this equation in terms of the frequency components at \( x = 0 \) yields

\[
i_0(0) = - \frac{d}{dx} [g(V) v_0]_{x=0} = - d_1 = C_1 \tag{A.63}
\]
\[
i_1(0) = - \frac{d}{dx} [g(V) v_1]_{x=0} = - e_1 \tag{A.64}
\]
\[
i_2(0) = - \frac{d}{dx} [g(V) v_2]_{x=0} = - f_1 \tag{A.65}
\]

The ac current in the saturation region is given by Equation (A.16) so that \( i(x = L_g) \) can be obtained by substituting \( x' = l \) into Equation (A.16).

\[
i_0(L_g) = C_1 \tag{A.66}
\]
\[
i_1(L_g) = - C_1 \tau_s + C_2 \tag{A.67}
\]
\[ i_2(L_g) = \frac{1}{2} C_1 \tau_s^2 - C_2 \tau_s + C_3 \]  

(A.68)

where \( \tau_s = l/v_s \).

As shown in Equation (A.10a) the ac current consists of the zero, first and second order terms. In order to get the total gate and drain current we have to combine them using to Equation (A.10a). The ac current will be calculated up to second order terms

\[
i(0) = C_1 - j \omega e_1 - (j \omega)^2 f_1
\]  

(A.69)

\[
i(L_g) = C_1 - j \omega (C_1 \tau_s - C_2) + (j \omega)^2 \left( \frac{1}{2} C_1 \tau_s^2 - C_2 \tau_s + C_3 \right)
\]  

(A.70)

The gate current is then given by

\[
i_g = j \omega [-e_1 + C_1 \tau_s - C_2] + (j \omega)^2 \left[ -f_1 - \frac{C_1}{2} \tau_s^2 + C_2 \tau_s - C_3 \right]
\]  

(A.71)

and the drain current by Equation (A.70).

### A.3.5 Calculation of Y_{12} and Y_{22}

\( Y_{12} \) and \( Y_{22} \) are obtained from \( i_g/v_{ds} \) and \( i_d/v_{ds} \) respectively with \( v_{gs} = 0 \). For \( v_{gs} = 0 \) we must introduce the new constants \( C_1', C_2', C_3', e_1', \) and \( f_1' \) calculated in Equation (A.69), (A.70), and (A.71).

\[
C_1' = G_{dos}(1 - k_s) R_d v_{ds}
\]

\[
C_2' = \frac{C_{os} C_1}{G_{dos}} D - e_1
\]

\[
C_3' = \frac{C_{os}^2 C_1}{G_{dos}^2} F - \frac{C_{os} E_1}{G_{dos}} D - f_1
\]

\[
e_1' = \frac{C_{os} R_d C_1}{G_{dos}} \left[ R_A D + R_B E \right] - \frac{\beta C_1 \ell^2}{6} \tau_s R_d G_{dos} (1 - k_s)
\]
\[ f'_1 = \frac{\beta \ell^2 \tau_s^2 C_1}{24} G_{dos} (1 - k_s) R_d - \frac{\beta \ell^2 C_2 \tau_s}{6} R_d G_{dos} (1 - k_s) \]

\[ + \frac{C_{os} R_d}{G_{dos}} \left[ C_{os} C_1 \left[ R_A F + R_B H \right] - E_{ds} \left[ R_A D + R_B E \right] \right] \]

Substituting the above equations into Equation (A.71) and Equation (A.70) and dividing it by \( v_{ds} \) gives \( Y_{12} \) and \( Y_{22} \)

\[ Y_{12} = \frac{i_g}{v_{ds}} = \frac{j \omega \left[ -C_1^' + C_2^' \tau_s - C_3^' \right] + (j \omega)^2 [-f'_1 - \frac{1}{2} C_1^' \tau_s^2 + C_2^' \tau_s - C_3^']}{v_{ds}} \]

\[ = j \omega \left[ -E_{ds} + G_{ds} \tau_s - C_{ds} \right] + (j \omega)^2 \left[ -F_{ds} - G_{ds}/2 \tau_s^2 + C_{ds} \tau_s - H_{ds} \right] \]

\[ = \omega^2 \left[ F_{ds} + \frac{G_{ds}}{2} \tau_s^2 - C_{ds} \tau_s + H_{ds} \right] - j \omega \left[ E_{ds} - G_{ds} \tau_s + C_{ds} \right] \]

\[ Y_{22} = \frac{i_d}{v_{ds}} = \frac{C_1^' + j \omega \left[ -C_1^' \tau_s + C_2^' \tau_s \right] + (j \omega)^2 \left[ \frac{1}{2} C_1^' \tau_s^2 - C_2^' \tau_s + C_3^' \right]}{v_{ds}} \]

\[ = G_{ds} + j \omega \left[ -G_{ds} \tau_s + C_{ds} \right] + (j \omega)^2 \left[ G_{ds}/2 \tau_s^2 - C_{ds} \tau_s + H_{ds} \right] \]

\[ = G_{ds} - \omega^2 \left[ \frac{G_{ds}}{2} \tau_s^2 - C_{ds} \tau_s + H_{ds} \right] - j \omega \left[ G_{ds} \tau_s - C_{ds} \right] \]

where

\[ G_{ds} = \frac{C_1^'}{v_{ds}} = G_{dos} (1 - k_s) R_d \]

\[ C_{ds} = \frac{C_2^'}{v_{ds}} = \frac{C_{os} G_{ds} - E_{ds}}{G_{dos}} \]

\[ E_{ds} = \frac{C_1^'}{v_{ds}} = \frac{C_{os} R_d G_{ds} \left[ R_A D + R_B E \right]}{G_{dos}} - \frac{\beta \ell^2}{6} \tau_s G_{ds}^2 \]

\[ H_{ds} = \frac{C_2^'}{v_{ds}} = \frac{C_{os} E_{ds} D - F_{ds}}{G_{dos}} \]

\[ F_{ds} = \frac{f'_1}{v_{ds}} = \frac{\beta \ell^2}{24} \tau_s^2 G_{ds}^2 - \frac{\beta \ell^2}{6} \tau_s C_{ds} G_{ds} \]

\[ + \frac{C_{os} R_d}{G_{dos}} \left[ C_{os} G_{ds} \left( R_A F + R_B H \right) - E_{ds} \left( R_A D + R_B E \right) \right] \]
A.3.6 Calculation of $Y_{11}$ and $Y_{21}$

$Y_{11}$ and $Y_{21}$ is the ac gate and drain current divided by $v_{gs}$ when $v_{ds} = 0$.

\[
\begin{align*}
C_1'' &= G_{dos}R_d(k_s + R_i R_y(1 - k_s))v_{gs} \\
C_2'' &= C_{os} \left[ \frac{C_1}{G_{dos}} D - v_{gs} A \right] - e_1 \\
C_3'' &= \frac{C_{os} C_1}{G_{dos}^2} F - \frac{v_{gs} C_{os}^2 B}{G_{dos}} - \frac{C_{os} e_1}{G_{dos}} D - f_1 \\
e_1'' &= C_{os} R_d \left[ \frac{C_1}{G_{dos}} [R_A D + R_B E] - v_{gs} [R_A A + R_B C] \right] \\
&- \frac{\beta \ell^2}{6} C_1 \tau_s R_d G_{dos} (1 - k_s) \\
f_1'' &= \frac{\beta \ell^2 \tau_s^2 C_1}{24} C_{os} R_d G_{dos} (1 - k_s) R_d - \frac{\beta \ell^2 C_2}{6} \tau_s R_d G_{dos} (1 - k_s) \\
&+ \frac{C_{os} R_d}{G_{dos}} \left[ \frac{C_{os} C_1}{G_{dos}} [R_A F + R_B H] - v_{gs} C_{os} [R_A B + R_B G] \right] \\
&- E_1 [R_A D + R_B E]
\end{align*}
\]

Substituting the above equations into Equation (A.71) and Equation (A.70) and dividing it by $v_{gs}$ gives $Y_{11}$ and $Y_{21}$.

\[
\begin{align*}
Y_{11} &= \frac{i_g}{v_{gs}} = \frac{j \omega \left[ -e_1'' + C_1''(\tau_s - C_2'') + (j \omega)^2 \left[ -f_1'' - \frac{1}{2} C_1'' \tau_s^2 + C_2'' \tau_s - C_3'' \right] \right]}{v_{gs}} \\
&= \left[ -E_{gs} + G_{gs} \tau_s - C_{gs} \right] + (j \omega)^2 \left[ -F_{gs} - \frac{G_{gs}}{2} \tau_s^2 + C_{gs} \tau_s - H_{gs} \right] \\
&= \omega^2 \left[ F_{gs} + \frac{G_{gs}}{2} \tau_s^2 - C_{gs} \tau_s + H_{gs} \right] - j \omega [E_{gs} - G_{gs} \tau_s + C_{gs}] \\
Y_{21} &= \frac{i_d}{v_{gs}} = \frac{C_1'' + j \omega (-C_1'' \tau_s + C_2'') + (j \omega)^2 \left[ \frac{1}{2} C_1'' \tau_s^2 - C_2'' \tau_s + C_3'' \right]}{v_{gs}} \\
&= G_{gs} + j \omega (-G_{gs} \tau_s + C_{gs}) + (j \omega)^2 \left[ \frac{1}{2} G_{gs} \tau_s^2 - C_{gs} \tau_s + H_{gs} \right] \\
&= G_{gs} - \omega^2 \left[ \frac{G_{gs}}{2} \tau_s^2 - C_{gs} \tau_s + H_{gs} \right] - j \omega [G_{gs} \tau_s - C_{gs}]
\end{align*}
\]
where

\[
G_{gs} = \frac{C''}{v_{gs}} = G_{dos} R_d (k_s + R_t R_y (1 - k_s))
\]

\[
C_{gs} = \frac{C''}{v_{gs}} = \frac{C_{os} G_{gs}}{G_{dos}} D - C_{os} A - E_{gs}
\]

\[
E_{gs} = \frac{e_1''}{v_{gs}} = \frac{C_{os} G_{gs}}{G_{dos}} R_d [R_A D + R_B E] - C_{os} R_d [R_A A + R_B C] - \frac{\beta \ell^2}{6} \tau_s G_{gs} G_{ds}
\]

\[
H_{gs} = \frac{C_3''}{v_{gs}} = \frac{C_{os} G_{gs}}{G_{dos}} F - \frac{C_{os} E_{gs}}{G_{dos}} D - F_{gs}
\]

\[
F_{gs} = \frac{f_1''}{v_{gs}} = \frac{\beta \ell^2 \tau_s^2}{2 A} G_{gs} G_{ds} - \frac{\beta \ell^2 \tau_s}{6} C_{gs} G_{ds}
\]

\[
+ \frac{C_{os} G_{gs}}{G_{dos}^2} R_d [R_A F + R_B H] - \frac{C_{os}^2}{G_{dos}} R_d [R_A B + R_B G]
\]

\[
- \frac{C_{os} R_d E_{gs}}{G_{dos}} [R_A D + R_B E]
\]
Appendix B

Exact Solution for Velocity-Saturated MODFET Wave Equation

The exact solution of the MODFET wave equation is based on the original calculation of Burn [14] for the MOSFET in pinch off \((k = 1)\). His wave equation can be modified to hold for the unsaturated regime.

Before proceeding, we need to derive the channel voltage. The dc channel potential \(V_c(x)\) in the GCA region is obtained from the current equation. The dc current in GCA region is

\[
I_{DC}(x) = \mu C(V_{GS} - V_T - V_{CS}(x)) \frac{dV_{CS}(x)}{dx}
\]

Since the dc current \(I_{dc}\) is independent of \(x\), we have

\[
(V_{GS} - V_T - V_{CS}(x)) \frac{dV_{CS}(x)}{dx} = \text{constant}
\]

Integrating both sides from 0 to \(x\) and manipulating with boundary conditions, \(V_{CS}(0) = 0\) and \(V_{CS}(X_S) = V'_D\), yields

\[
V_{CS}(x) = (V_{GS} - V_T) \left[ 1 - \sqrt{1 - \frac{x}{X_s}} + \left(1 - \frac{V'_D}{V_{GS} - V_T}\right)^2 \frac{x}{X_s} \right]
\]
It is convenient to introduce the dc gate to channel voltage, \( V_{GC}(x) \), defined as

\[
V_{GC}(x) = V_{GS} - V_T - V_{CS}(x)
\]

\[
= (V_{GS} - V_T) \sqrt{1 - \frac{(2k_s - k_s^2) x}{X_s}}
\]

where \( k_s = \frac{V_s}{V_{GS} - V_T} \).

The equation is derived for total voltage from continuity equation and current equation. The time dependent current equation can then be rewritten

\[
I(x, t) = -\mu C v_{GC}(x, t) \frac{\partial v_{GC}(x, t)}{\partial x}
\]

where

\[
v_{GC}(x, t) = V_{GC}(x) + v_{gc}(x, t)
\]

The continuity equation becomes

\[
\frac{\partial I(x, t)}{\partial x} = -C \frac{\partial v_{GC}(x, t)}{\partial t}
\]

Differentiating Equation (B.3) on both sides with respect to \( x \) and substituting in Equation (B.5) yields an equation for \( v_{GC}(x, t) \):

\[
\frac{\partial^2}{\partial x^2}(v_{GC}^2(x, t)) = \frac{2}{\mu} \frac{\partial v_{GC}(x, t)}{\partial t}
\]

For small signal analysis, the above equation will be decomposed into a dc part and a small-signal ac part. It is assumed that the second order term (such as \( v_{gc}^2(x, t) \)) of small-signal ac part is negligible. Using this assumption \( v_{GC}^2(x, t) \) can be rewritten approximately

\[
v_{GC}^2(x, t) \approx V_{GC}^2(x) + 2V_{GC}(x)v_{gc}(x, t)
\]
Substituting Equation (B.2) in the above equation and simplifying yields

\[
v_{GC}(x,t) \approx (V_{GS} - V_T)^2 \left(1 - (2k_s - k_s^2) \frac{x}{X_s} \right)
+ 2(V_{GS} - V_T) \sqrt{1 - (2k_s - k_s^2) \frac{x}{X_s}} v_{GC}(x,t)
= (V_{GS} - V_T)^2 P + 2(V_{GS} - V_T) \sqrt{P} v_{GC}(x,t)
\]

(B.7)

where

\[
P = 1 - (2k_s - k_s^2) \frac{x}{X_s}
\]

The new variable \(P\) is introduced to simplify calculation. The relation between \(dx\) and \(dP\) has to be calculated before differentiating Equation (B.7).

\[
dx = -\frac{X_s}{2k_s - k_s^2} dP
\]

The procedure to differentiate Equation (B.7) is as follows

\[
\frac{\partial^2}{\partial x^2} v_{GC}^2(x,t) = \left[-\frac{(2k_s - k_s^2)}{X_s} \right]^2 \frac{\partial^2}{\partial P^2} [v_{GC}^2(x,t)]
\]

\[
= 2 \left(\frac{2k_s - k_s^2}{X_s^2} (V_{GS} - V_T) \frac{\partial}{\partial P} \left[ \frac{1}{2\sqrt{P}v + \sqrt{P}} \frac{\partial v_{GC}}{\partial P} \right] \right)
\]

\[
= 2 \left(\frac{2k_s - k_s^2}{X_s^2} (V_{GS} - V_T) \frac{\partial}{\partial P} \left[ \frac{1}{4P^{3/2}} v + \frac{1}{2\sqrt{P}} \frac{dv_{GC}}{dP} + \frac{1}{2\sqrt{P}} \frac{dv_{GC}}{dP} + \sqrt{P} \frac{d^2v_{GC}}{dP^2} \right] \right)
\]

\[
= 2 \left(\frac{2k_s - k_s^2}{X_s^2} (V_{GS} - V_T) \frac{\partial}{\partial P} \left[ \frac{1}{4P^{3/2}} v + \frac{1}{2\sqrt{P}} \frac{dv_{GC}}{dP} - \frac{1}{4P^{3/2}} v_{GC} \right] \right)
\]

(B.8)

Replacing Equation (B.8) in Equation (B.6) yields

\[
\frac{\partial v_{GC}(x,t)}{\partial t} = \frac{\mu (V_{GS} - V_T)(2k_s - k_s^2)}{X_s^2} \left[ \sqrt{P} \frac{d^2v_{GC}(x,t)}{dP^2} + \frac{1}{\sqrt{P}} \frac{dv_{GC}(x,t)}{dP} - \frac{1}{4P^{3/2}} v_{GC} \right]
\]

\[
= \omega_{ok} \left[ \sqrt{P} \frac{d^2v_{GC}(x,t)}{dP^2} + \frac{1}{\sqrt{P}} \frac{dv_{GC}(x,t)}{dP} - \frac{1}{4P^{3/2}} v_{GC}(x,t) \right]
\]

(B.9)
The Laplace transform of Equation (B.9) is

\[ P^2 \frac{\partial^2 v_{gc}(S, t)}{\partial P^2} + P \frac{dv_{gc}(S, t)}{dP} - \left( \frac{1}{4} + S'P^{3/2} \right)v_{gc}(S, t) = 0 \]  

(B.10)

where \( S' = S/\omega_{0k} \).

Equation (B.10) is the s space representation of the MODFET wave equation and can be verified to be equivalent to Equation (2.5). Equation (B.10) can be derived from Equation (2.5) by substituting \( g(V_{GC}(x)) = \mu W s C_s (V_{GC}(x) - V_T) \) into Equation (2.5) and differentiating. This is a modified Bessel’s differential equation so that one can find an analytic solution [25]. The complete solution is written as

\[ v_{gc}(P, S) = C_1 I_{2/3} \left( \frac{4}{3} \sqrt{S'}(P)^{3/4} \right) + C_2 I_{-2/3} \left( \frac{4}{3} \sqrt{S'}(P)^{3/4} \right) \]  

(B.11)

where \( C_1 \) and \( C_2 \) are arbitrary constants. The boundary conditions will define \( C_1 \) and \( C_2 \).

The ac current will be obtained decomposing Equation (B.3) into its dc and ac components while neglecting second order terms:

\[ i(P, S) = -\mu C \frac{d}{dx}(v_{gc}(P, S)V_{GC}(P)) \]

\[ = -\mu C \left( \frac{(2k_s - k_s^2)}{X_s} \right) (V_{GS} - V_T) \frac{d}{dP}(v_{gc}\sqrt{P}) = G'_{dos} \frac{d}{dP}(\sqrt{P}) \]

\[ = G'_{dos} \left[ \frac{1}{2\sqrt{P}} v_{gc} + \frac{v_{gc} dv_{gc}}{dP} \right] \]  

(B.12)

where
\[ G'_{d_{os}} = \frac{\mu C_g W_g (V_{GS} - V_T)(2k_x - k_x^2)}{x_s} \]

Equation (B.12) has to be expanded in terms of \( C_1 \) and \( C_2 \) in order to apply the boundary conditions. It simplifies the calculation to introduce the new variable \( Y \)

\[ Y = \frac{4}{3} \sqrt{S}(P)^{3/4} \]
\[ \frac{dY}{dP} = \sqrt{S} P^{-1/4} \]

Note that the modified Bessel function has the following properties

\[ \frac{dI_n(x)}{dx} = \frac{1}{2}(I_{n+1}(x) + I_{n-1}(x)) \]
\[ I_n(Y) = \frac{Y}{2n}[I_{n-1}(Y) - I_{n+1}(Y)] \]

First \( dv_{gc}/dP \) will be expanded in terms of \( C_1 \) and \( C_2 \)

\[ \frac{dv_{gc}}{dP} = \frac{dY}{dP} \frac{dv_{gc}}{dY} = \sqrt{S} P^{-1/4} \frac{d}{dY}[C_1 I_{2/3}(Y) + C_2 I_{2/3}(Y)] \]
\[ = \sqrt{S} P^{-1/4} \left[ \frac{C_1}{2}(I_{5/3}(Y) + I_{-1/3}(Y)) + \frac{C_2}{2}(I_{1/3}(Y) + I_{-5/3}(Y)) \right] \quad (B.13) \]

Substituting Equation (B.13) and (B.11) in Equation (B.12) gives

\[ i(P, S) = G'_{d_{os}} \left[ \frac{1}{2} P^{-1/2}(C_1 I_{2/3}(Y) + C_2 I_{-2/3}(Y)) \right] \]
\[ + \sqrt{S} P^{1/4} \left[ \frac{C_1}{2}(I_{5/3}(Y) + I_{-1/3}(Y)) + \frac{C_2}{2}(I_{1/3}(Y) + I_{-5/3}(Y)) \right] \]
\[ = \left[ \frac{C_1}{2} P^{1/2} \frac{4 \sqrt{S}(P)^{3/4}}{2 \times \frac{3}{2}} [I_{-1/3}(Y) - I_{5/3}(Y)] \right] \]
\[ + \frac{C_2}{2} P^{-1/2} \frac{4 \sqrt{S}(P)^{3/4}}{2 \times (-\frac{5}{3})} [I_{-5/3}(Y) - I_{1/3}(Y)] \]
\[ + \sqrt{S} P^{1/4} \left[ \frac{C_1}{2}(I_{5/3}(Y) + I_{-1/3}(Y)) + \frac{C_2}{2}(I_{1/3}(Y) + I_{-5/3}(Y)) \right] \]
\[ = \frac{G'_{d_{os}}}{2} \sqrt{S} P^{1/4}[C_1 I_{-1/3}(Y) - C_1 I_{5/3}(Y) - C_2 I_{-5/3}(Y) + C_2 I_{1/3}(Y)] \]
\[ + C_1 I_{5/3}(Y) + C_1 I_{-1/3}(Y) + C_2 I_{1/3}(Y) + C_2 I_{-5/3}(Y) \]

\[ = G'_{dos} \sqrt{S'P^{1/4}} [C_1 I_{1/3}(Y) + C_2 I_{1/3}(Y)] \]

Now the ac voltage and current in the GCA region can be obtained in terms of \( C_1 \) and \( C_2 \)

\[ v_{gc}(P, S) = C_1 I_{2/3}(Y) + C_1 I_{-2/3}(Y) \] (B.14)

\[ i(P, S) = G'_{dos} \sqrt{S'P^{1/4}} [C_1 I_{-1/3}(Y) + C_2 I_{1/3}(Y)] \] (B.15)

Solving the continuity and Poisson’s equations in the saturation region gives the ac voltage and the ac current. Integrating the continuity equation (2.9) in chapter II gives the ac current

\[ i(x') = i_0 e^{-\frac{x'}{Y_s}} \] (B.16)

Since the ac current is continuous at the GCA/saturation boundary, we have

\[ i_0 = i(P, S)|_{x=x_s} \]

\[ = G'_{dos} \sqrt{S'P^{1/4}} [C_1 I_{-1/3}(Y_s) + C_2 I_{1/3}(Y_s)] \] (B.17)

where

\[ P_s = (1 - k_s)^2 \]

\[ Y_s = \frac{4}{3} \sqrt{S'P^{3/4}} \]

The channel potential can be approximately obtained by solving the Poisson equation along the 2DEG channel. Substituting Equation (B.16) in Poisson equation (2.10) and integrating gives
\[ v(x') = -\beta i_0 \left( \frac{v_s}{j\omega} \right)^2 e^{-j\frac{\omega}{v_s} x'} + ax' + b \]

Since the ac electric field is zero at the GCA/saturation boundary, \( \frac{dv_{ig}}{dx'} |_{x=x_s} = 0 \), the unknown constant \( a \) is

\[ a = \beta i_0 \left( \frac{v_s}{j\omega} \right)^2 \left(-j\frac{\omega}{v_s}\right) = j\beta i_0 \frac{v_s}{\omega} \] (B.18)

The channel voltage in the saturation region is

\[ v_{gc}(x') = \beta i_0 \left( \frac{v_s}{j\omega} \right)^2 e^{-j\frac{\omega}{v_s} x'} + j\beta i_0 \frac{v_s}{\omega} x' + b \] (B.19)

Now the ac potential and ac current for the entire region are known if \( C_1 \) and \( C_2 \) are known. As mentioned in Chapter II, the boundary conditions are \( v_{gc}(0) = v_{gs} \) and \( v_{gc}(L_g) = v_{gs} - v_{ds} \). The channel potential at the drain end is shown in Equation (2.23) in Chapter II. \( v_{gc}(0) \) can be directly obtained from the Equation (B.14) by setting \( x = 0 \), \( P = 1 \) and \( Y = 4\sqrt{3}/3 \).

\[ v_{gc}(0) = C_1 I_{2/3} \left( \frac{4}{3} \sqrt{3}/3 \right) + C_2 I_{-2/3} \left( \frac{4}{3} \sqrt{3}/3 \right) = v_{gs} \] (B.20)

Let us calculate \( v_{gc}(L_g) \) in terms of \( C_1 \) and \( C_2 \). First the voltage drop, \( \Delta v(\ell) \), in the saturation region is the difference of \( v(x' = \ell) \) and \( v_{gc}(x' = 0) \).

\[ \Delta v_{gc}(\ell) = \beta i_0 \left( \frac{v_s}{\omega} \right)^2 e^{-j\frac{\omega}{v_s} \ell} + j\beta i_0 \frac{v_s}{\omega} \ell - \beta i_0 \left( \frac{v_s}{\omega} \right)^2 \]

\[ = \beta i_0 \left( \frac{v_s}{\omega} \right)^2 \left[ e^{-j\frac{\omega}{v_s} \ell} - 1 \right] + j\beta i_0 \frac{v_s}{\omega} \ell \] (B.21)

\( x_s \) was given in Equation (2.16) in Chapter II,

\[ x_s = -\frac{1}{V_{gc}'(X_s)} v_{gc}'(X_s) \]
$V_{GC}$ is calculated in Appendix A.2:

$$V_{GC}'' = -\frac{k_s^2 \left( 1 - \frac{1}{2} k_s \right)^2}{(1 - k_s)^3} \frac{V_{GS} - V_T}{X_s^2}$$

The calculation of $v'_{gc}(X_S)$ in terms of $C_1$ and $C_2$ will give us

$$v'_{gc}(X_S) = \frac{dv_{gc}}{dx} \frac{dP}{dY} \frac{dv_{gc}}{dy}$$

$$= -\frac{2k_s - k_s^2}{X_s} \sqrt{S'P_s^{-1/4}} \left[ \frac{C_1}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} + \frac{C_2}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} \right]$$

so that

$$x_s = -\frac{1}{k_s^2 \left( 1 - k_s \right)^2} \frac{2k_s^2}{(1 - X_s)^3} \frac{\sqrt{S'P_s^{-1/4}}}{X_s}$$

$$\times \left[ \frac{C_1}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} + \frac{C_2}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} \right]$$

$$= -\frac{\left( 1 - k_s \right)^3 X_s \sqrt{S'P_s^{-1/4}}}{k_s \left( 1 - \frac{1}{2} k_s \right) \left( V_{GS} - V_T \right)} \left[ C_1 \{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \} \right]$$

$$+ \frac{C_2}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} \right]$$

(B.22)

Substituting Equation (B.22), (B.21), and (B.14) into Equation (2.23) in Chapter II and manipulating yields

$$v_{gc}(I_s) = \beta I_{dc} \ell x_s + \Delta v_{gc}(\ell) + v_{gc}(X_s)$$

$$= \beta I_{dc} \ell \left( \frac{(1 - k_s)^3 X_s \sqrt{S'P_s^{-1/4}}}{k_s \left( 1 - \frac{1}{2} k_s \right) V_{out}} \right) \left[ C_1 \{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \} \right]$$

$$+ \frac{C_2}{2} \left\{ I_{1/3}(Y_s) + I_{5/3}(Y_s) \right\} \right] + \left[ \beta \left( \frac{v_s}{\omega} \right)^2 \left[ e^{-j \omega \ell} - 1 \right]$$

$$+ j \beta \frac{v_s}{\omega} \left[ G'_{dss} \sqrt{S'P_s^{1/4}} \{ C_1 I_{1/3}(Y_s) + C_2 I_{1/3}(Y_s) \} \right]$$

$$+ C_1 I_{2/3}(Y_s) + C_2 I_{2/3}(Y_s) \right\}$$

(B.23)
Two equations (B.20) and (B.23) are used to obtain \( C_1 \) and \( C_2 \). For simplicity two equations may be rewritten as

\[
\begin{align*}
v_{gc}(L_g) &= A_{11}C_1 + A_{12}C_2 = v_{gs} - v_{ds} \\
v_{gc}(0) &= A_{21}C_1 + A_{22}C_2 = v_{gs}
\end{align*}
\]

where

\[
A_{11} = \frac{A_{22}}{\Delta} = \frac{A_{22}v_{gs} - A_{12}v_{ds}}{A_{22} - A_{12}}v_{gs} - \frac{A_{22}}{\Delta}v_{ds}
\]

\[
A_{21} = \frac{A_{22}}{\Delta} = \frac{A_{22}v_{gs} - A_{21}v_{gs} + A_{21}v_{ds}}{A_{11} - A_{21}}v_{gs}
\]

where \( \Delta = A_{11}A_{22} - A_{12}A_{21} \).

From Equation (B.24) it is obvious that

\[
C_1 = \frac{A_{22}(v_{gs} - v_{ds}) - A_{12}v_{gs}}{\Delta}
\]

\[
C_2 = \frac{A_{11}v_{gs} - A_{21}(v_{gs} - v_{ds})}{\Delta}
\]

For the calculation of \( Y_{11} \) and \( Y_{21} \) \( v_{ds} \) have to be set to zero so that \( C_1 \) and \( C_2 \) will
be modified as follows

\[ C_1' = \frac{(A_{22} - A_{12}) \Delta}{v_{gs}} \]
\[ C_2' = \frac{(A_{11} - A_{21}) \Delta}{v_{gs}} \]

The gate and drain current will be expressed in terms of \( C_1' \) and \( C_2' \) as

\[ i_d = i(x' = \ell) = i_0 e^{-j \frac{\pi}{2} \ell} \]
\[ = G'_{dos} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{2} \ell} [C_1' I_{-1/3}(Y_s) + C_2' I_{1/3}(Y_s)] \]
\[ i_g = i(x = 0) - i(x' = \ell) \]
\[ = G'_{dos} \sqrt{S'} \left[ C_1' I_{-1/3} \left( \frac{4}{3} \sqrt{S'} \right) + C_2' I_{1/3} \left( \frac{4}{3} \sqrt{S'} \right) \right] \]
\[ - G'_{dos} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{2} \ell} [C_1' I_{-1/3}(Y_s) + C_2' I_{1/3}(Y_s)] \]

The \( Y_{11} \) and \( Y_{21} \) parameters for the saturated MODFET are

\[ Y_{21} = \frac{i_d}{v_{gs}} \]
\[ = G'_{dos} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{2} \ell} [C_{1gs} I_{-1/3}(Y_s) + C_{2gs} I_{1/3}(Y_s)] \]
\[ Y_{11} = \frac{i_g}{v_{gs}} \]
\[ = G'_{dos} \sqrt{S'} \left[ C_{1gs} I_{-1/3} \left( \frac{4}{3} \sqrt{S'} \right) + C_{2gs} I_{1/3} \left( \frac{4}{3} \sqrt{S'} \right) \right] \]
\[ - P_s^{1/4} e^{-j \frac{\pi}{2} \ell} (C_{1gs} I_{-1/3}(Y_s) + C_{2gs} I_{1/3}(Y_s)) \]

where

\[ C_{1gs} = \frac{(A_{22} - A_{12}) \Delta}{\Delta} \]
\[ C_{2gs} = \frac{(A_{11} - A_{21}) \Delta}{\Delta} \]
$v_{gs}$ is set to zero to calculate $Y_{12}$ and $Y_{22}$ and we must introduce the new constant $C'_1$ and $C''_1$

\[ C'_1 = \frac{A_{22}}{\Delta} v_{ds} \]
\[ C''_1 = \frac{A_{21}}{\Delta} v_{ds} \]

The gate current and drain current are then

\[ i_d = i(x' = \ell) = i_0 e^{-j \frac{\pi}{2} \ell} \]
\[ = G'_{ds} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{6} \ell} [C''_1 I_{1/3}(Y_s) + C''_2 I_{1/3}(Y_s)] \]
\[ i_g = i(x = 0) - i(x' = \ell) \]
\[ = G'_{ds} \sqrt{S'} \left[ C'_1 I_{1/3} \left( \frac{f}{3} \sqrt{S'} \right) + C''_1 I_{1/3} \left( \frac{f}{3} \sqrt{S'} \right) + \right] \]
\[ - G'_{ds} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{6} \ell} [C''_1 I_{1/3}(Y_s) + C''_2 I_{1/3}(Y_s)] \]

The $Y_{12}$ and $Y_{22}$ parameters for the saturated MODFET are

\[ Y_{22} = \frac{i_d}{v_{ds}} \]
\[ = G'_{ds} \sqrt{S'} P_s^{1/4} e^{-j \frac{\pi}{6} \ell} [C_{1ds} I_{1/3}(Y_s) + C_{2ds} I_{1/3}(Y_s)] \]
\[ Y_{12} = \frac{i_g}{v_{ds}} \]
\[ = G'_{ds} \sqrt{S'} \left[ C_{1ds} I_{1/3} \left( \frac{4}{3} \sqrt{S'} \right) + C_{2ds} I_{1/3} \left( \frac{4}{3} \sqrt{S'} \right) + \right] \]
\[ - P_s^{1/4} e^{-j \frac{\pi}{6} \ell} [C_{1ds} I_{1/3}(Y_s) + C_{2ds} I_{1/3}(Y_s)] \]

where

\[ C_{1ds} = -\frac{A_{22}}{\Delta} \]
\[ C_{2ds} = -\frac{A_{21}}{\Delta} \]
Appendix C

The Fourth Order Frequency Power-Series Solution in the GCA Region

The fourth-order power-frequency solution of the wave equation can be obtained by expanding the Equation (6a),(6b),(6c) in [10] up to \( v_4(x) \) terms

\[
\frac{d^2}{dx^2} \left[ \mu(V_{GC}(x) - V_T)v_0(x) \right] = 0 \quad (C.1a)
\]
\[
\frac{d^2}{dx^2} \left[ \mu(V_{GC}(x) - V_T)v_1(x) \right] = v_0(x) \quad (C.1b)
\]
\[
\frac{d^2}{dx^2} \left[ \mu(V_{GC}(x) - V_T)v_2(x) \right] = v_1(x) \quad (C.1c)
\]
\[
\frac{d^2}{dx^2} \left[ \mu(V_{GC}(x) - V_T)v_3(x) \right] = v_2(x) \quad (C.1d)
\]
\[
\frac{d^2}{dx^2} \left[ \mu(V_{GC}(x) - V_T)v_4(x) \right] = v_3(x) \quad (C.1e)
\]

The current in the channel is obtained from

\[
i = -\mu C_g W_g \frac{d}{dx} \left[ v(x)(V_{GC}(x) - V_T) \right] \quad (C.2)
\]

with

\[
v(x) = v_0(x) + j \omega v_1(x) + (j \omega)^2 v_2(x) + (j \omega)^3 v_3(x) + (j \omega)^4 v_4(x) \quad (C.3a)
\]
\[
i(x) = i_0(x) + j \omega i_1(x) + (j \omega)^2 i_2(x) + (j \omega)^3 i_3(x) + (j \omega)^4 i_4(x) \quad (C.3b)
\]

\( Y_{11} \) and \( Y_{21} \) are determined by calculating \( i_g \) and \( i_d \) for \( v_{ds} = 0 \) with the boundary conditions \( v_0(0) = v_0(L_g) = v_{gs}; v_n(0) = v_n(L_g) = 0 \) for \( n \neq 0 \). \( Y_{gd} \) and \( Y_{dd} \) are
calvn (fx, vdsn) := block([i,m,n,c,gdos,k,tmp,c1,c2],
  fx: integrate(fx*C*2*(1-y)/k/(2-k),y),
  fx: integrate(fx*L*2*(1-y)/k/(2-k),y) + Cl*L*(2*y-y**2)/(2*k-k**2) + C2,
  fx:fx/gdos/L/(1-y),
  m:last(first(solve(ev(fx,y=0),c2))),
  fx:ev(fx,c2=m),
  n:last(first(solve(ev(fx,y=k)+vdsn,c1))),
  fx:ev(fx,c1=n),
  return(factor(fx)))$

Figure 74: List of calvn program

determined by calculating $i_g$ and $i_d$ for $v_{gs} = 0$ with the boundary conditions $v_0(0) = 0, v_0(L_g) = -v_{ds}, v_n(0) = v_n(L_g) = 0$ for $n \neq 0$. The gate and drain currents ($i_g$ and $i_d$) are obtained from the channel current $i(x)$ using

\[ i_d = i(x = L_g) \quad (C.4) \]
\[ i_g = i(x = L_g) - i(x = 0) \quad (C.5) \]

Due to the tedious nature of this calculation the symbolic manipulator MACSYMA was used. Two programs are used to calculate the ac voltage and current, which are calvn and calix function. calvn function solves the wave equation and calix function calculates the ac current from the ac voltage for each component. The programs are shown in Figure 74 and 75. In calvn function vdsn is a boundary condition for ac voltage.

The calculations gives:

\[ Y_{11} = j\omega F_{gg} - (j\omega)^2 S_{gg} + (j\omega)^3 T_{gg} - (j\omega)^4 D_{gg} \]
calix (fx) := block([tmp,i,j,k,m,n],
    ix: -k*(2-k)*diff(fx*g*gdos,L*(1-y),y)/2/L/(1-y),
    return(factor(ix)))$

Figure 75: List of calix program

with

\[
F_{gs} = 2C_0 \frac{6 - 6k + k^2}{3(2 - k)^2}
\]
\[
S_{gs} = 2C_0 \frac{60 - 12k + 81k^2 - 21k^3 + 2k^4}{45(2 - k)^5}
\]
\[
T_{gs} = C_0 \frac{7560 - 22680k + 26730k^2 - 15660k^3 + 4807k^4 - 757k^5 + 49k^6}{14175(2 - k)^8}
\]
\[
D_{gs} = 2C_0 \left( \frac{(2019600 - 8078400k + 13573560k^2 - 12446280k^3 + 6793245k^4}{233875(2 - k)^{11}}
    - \frac{2267490k^5 - 4574111k^6 + 51646k^7 - 2527k^8}{233875(2 - k)^{11}} \right)
\]

\[Y_{12} = j\omega F_{gd} - (j\omega)^2 S_{gd} + (j\omega)^3 T_{gd} - (j\omega)^4 D_{gd}\]

with

\[
F_{gd} = -2C_0 \frac{3 - 4k + k^2}{3(2 - k)^2}
\]
\[
S_{gd} = -2C_0 \frac{(1 - k)(30 - 41k + 16k^2 - 2k^3)}{45(2 - k)^5}
\]
\[
T_{gd} = -C_0 \frac{(1 - k)(3780 - 8970k + 7920k^2 - 3247k^3 + 637k^4 - 49k^5 + 49k^6)}{14175(2 - k)^8}
\]
\[
D_{gd} = -2C_0 \left( \frac{(1 - k)(1009800 - 3407580k + 4702830k^2 - 3430765k^3}{233875(2 - k)^{11}}
    + \frac{1432920k^4 - 346531k^5 + 45486k^6 - 2527k^7}{233875(2 - k)^{11}} \right)
\]
\[ Y_{21} = j\omega F_{dg} - (j\omega)^2 S_{dg} + (j\omega)^3 T_{dg} - (j\omega)^4 D_{dg} \text{ with} \]

\[ F_{dg} = -2C_0 \frac{30 - 45k + 20k^2 - 3k^3}{15(2 - k)^3} \]

\[ S_{dg} = -C_0 \frac{600 - 1440k + 1290k^2 - 540k^3 + 110k^4 - 9k^5}{225(2 - k)^6} \]

\[ T_{dg} = -4C_0 \left( \frac{415800 - 1405800k + 1945350k^2 - 1424610k^3}{779625(2 - k)^9} + \frac{599005k^4 - 146755k^5 + 19705k^6 - 1134k^7}{779625(2 - k)^9} \right) \]

\[ D_{dg} = -2C_0 \left( \frac{20196000 - 88387200k + 166148400k^2 - 175707600k^3}{11694375(2 - k)^{12}} + \frac{11522050k^4 - 48760380k^5 + 13410670k^6 - 2330920k^7}{11694375(2 - k)^{12}} + \frac{234250k^8 - 10449k^9}{11694375(2 - k)^{12}} \right) \]

\[ Y_{22} = j\omega F_{dd} - (j\omega)^2 S_{dd} + (j\omega)^3 T_{dd} - (j\omega)^4 D_{dd} \text{ with} \]

\[ F_{dd} = 2C_0 \frac{(1 - k)(20 - 15k + 3k^2)}{15(2 - k)^2} \]

\[ S_{dd} = C_0 \frac{(1 - k)(320 - 560k + 340k^2 - 90k^3 + 9k^4)}{225(2 - k)^6} \]

\[ T_{dd} = 8C_0(1 - k) \left( \frac{10560 - 290400k + 315920k^2 - 175175k^3}{779625(2 - k)^9} + \frac{53165k^4 - 8505k^5 + 567k^6}{779625(2 - k)^9} \right) \]

\[ D_{dd} = C_0(1 - k) \left( \frac{10137600 - 38016000k + 60064000k^2 - 52256000k^3}{11694375(2 - k)^{11}} + \frac{27500560k^4 - 9035640k^5 + 1825520k^6 - 208980k^7 + 10449k^8}{11694375(2 - k)^{11}} \right) \]
Appendix D

Development of Equivalent Circuit for the velocity-saturated MODFET

D.1 Equivalent circuit for the saturation region

The equivalent circuit for saturation region is based on the exact solution of the velocity-saturated MODFET wave equation so that the procedure for solving the wave equation will be repeated. The exact solution for the GCA wave equation is given in Appendix B.

\[ v_{gs}(Y, S) = C_1 I_{2/3}(Y) + C_2 I_{-2/3}(Y) \]  \hspace{1cm} (D.1a)

\[ i(Y, S) = G_{dos} \frac{1}{\sqrt{Y}} (C_1 I_{-1/3}(Y) + C_2 I_{1/3}(Y)) \]  \hspace{1cm} (D.1b)

where

\[ Y = \frac{4}{3} \sqrt{\frac{1}{\sqrt{Y}} P^{3/4}} \]

\[ P = 1 - (2k_s - k_s^2) \frac{x}{X_s} \]

It is assumed that \( v_{gs}(Y, S) = v_{gs} - v'_d \) and \( i = i'_d \) at boundary. Now the development of the equivalent circuit for the saturation region requires to derive the relationship between \( v'_d \) and \( i'_d \). Let us start with the voltage and current in the GCA
region. At the boundary \( x = X_s \) we have

\[
v(X_s) = C_1 I_{2/3}(Y_s) + C_2 I_{-2/3}(Y_s) = v_{gs} - v'_{ds}
\]  

\[
i(X_s) = C'_{ds} P_s^{1/4} \sqrt{S'}(C_1 I_{-1/3}(Y_s) + C_2 I_{1/3}(Y_s)) = i_d'
\]

The ac current and ac voltage in the saturation region is given by Equation (B.16) and (B.19) in Appendix B. In these equations \( i_0 \) can be replaced \( i_d' \) directly. For convenience these equations are rewritten

\[
i(x') = i_d' e^{-i \frac{\pi}{4} x'}
\]

\[
v(x') = \beta i_d' \left( \frac{u_1}{\omega} \right)^2 e^{-i \frac{\pi}{4} x'} + j \beta i_d' \frac{u_0}{\omega} x' + b
\]

The boundary condition at the drain side is \( v_{gc}(\ell) = v_{gs} - V_{ds} \). The gate to channel voltage \( v_{gc} \) at drain side is given by Equation (2.23)(see Chapter II).

\[
v_{gc}(L_d) = \beta I_{dc} \ell x_s + \Delta v_{gc}(\ell) + v_{gc}(X_s)
\]

In order to derive the equivalent circuit Equation (D.4) has to be expressed in terms of \( v'_{ds} \) and \( i_d' \). \( x_s \) is given in Equation (2.16) in Chapter II

\[
x_s = -\frac{1}{V''_{GC}(X_s)} v'_{gc}(X_s)
\]

where \( V''_{GC}(X_s) \) is given in Equation (2.17) in Chapter II. Note that the derivative of the Bessel functions verify

\[
\frac{dI_n(x)}{dx} = I_{n+1}(x) + \frac{n}{x} I_n(x)
\]

\[
\frac{dI_n(x)}{dx} = I_{n-1}(x) - \frac{n}{x} I_n(x)
\]
Using these properties one can obtain $v'_g(X_s)$

$$v'_g(X_s) = \left. \frac{dv_g}{dx} \right|_{x=X_s} = \frac{dP}{dx} \left. \frac{dY}{dP} \frac{dv_g}{dY} \right|_{x=X_s}$$

$$= -\frac{2k_s - k^2_s}{X_s} \sqrt{S} P_s^{-1/4} \left[ C_1 \left\{ I_{1/3}(Y_s) - \frac{2/3}{Y_s} I_{2/3}(Y_s) \right\} + C_2 \left\{ I_{1/3}(Y_s) + \frac{-2/3}{Y_s} I_{2/3}(Y_s) \right\} \right]$$

$$= -\frac{2k_s - k^2_s}{X_s} \sqrt{S} P_s^{-1/4} \left[ \{C_1 I_{1/3}(Y_s) + C_2 I_{1/3}(Y_s)\} - \frac{2}{3Y_s} \{C_1 I_{2/3}(Y_s) + C_2 I_{2/3}(Y_s)\} \right] \quad (D.8)$$

Substituting Equation (D.2a) and (D.2b) into Equation (D.8) yields

$$v'_g(X_s) = -\frac{2k_s - k^2_s}{X_s} \sqrt{S} P_s^{-1/4} \frac{i'_d}{G'_d X_s P'_s} + \frac{2k_s - k^2_s}{2X_s P_s} (v_{gs} - v'_{ds})$$

$$= -\frac{i'_d}{G'_d X_s (1 - k_s)} + \frac{2k_s - k^2_s}{2X_s (1 - k_s)^2} (v_{gs} - v'_{ds}) \quad (D.9)$$

$V'_G(X_s)$ was given in Equation (2.17) in Chapter II so that $x_s$ can be obtained in terms of $i'_d$ and $v'_{ds}$.

$$x_s = \frac{4X_S^2 (1 - k_s)^3}{(2k_s - k^2_s)^2 V_{out}} \left[ -\frac{i'_d}{G'_d X_s (1 - k_s)} + \frac{2k_s - k^2_s}{2X_s (1 - k_s)^2} (v_{gs} - v'_{ds}) \right]$$

$$= -\frac{4X_S^2 (1 - k_s)^2 i'_d}{G'_d (2k_s - k^2_s)^2 V_{out}} + \frac{2X_S^2 (1 - k_s)}{(2k_s - k^2_s)^2 V_{out}} (v_{gs} - v'_{ds}) \quad (D.10)$$

where $V_{out} = V_{GS} - V_T$. For convenience the new constant $A$ and $B$ are introduced

$$A = \frac{2X_S (1 - k_s)}{(2k_s - k^2_s) V_{out}}$$

$$B = \frac{4X_S^2 (1 - k_s)^2}{G'_d (2k_s - k^2_s)^2 V_{out}}$$
\( \Delta_S v_{gc}(\ell) \) is given in Equation (2.27) in Chapter II. Now Equation (D.4) can be rewritten in terms of \( v'_{ds} \) and \( i'_d \).

\[
v_{gc}(L_g) = \beta I_{dc} \ell x_s + \Delta_S v_{gc}(\ell) + v(X_s) = v_{gs} - v_{ds}
\]

\[
= \beta I_{dc} \ell [-B i'_d + A(v_{gs} - v'_{ds})] + \beta i'_d \left( \frac{v_s}{\omega} \right)^2 \left[ e^{-j \frac{\omega}{\omega}} - 1 \right]

+ j \beta i'_d \left( \frac{v_s}{\omega} \right) \ell + v_{gs} - v'_{ds}
\] (D.11)

From the above equation one can easily derive the relation between \( i'_d \) and \( v'_{ds} \).

\[
v'_{ds} = \frac{v_{ds}}{1 + \beta I_{dc} \ell A} + \frac{\beta I_{dc} \ell A v_{gs}}{1 + \beta I_{dc} \ell A} - \frac{\beta i'_d}{1 + \beta I_{dc} \ell A} \left[ I_{dc} \ell B - \left( \frac{v_s}{\omega} \right)^2 \left\{ e^{-j \frac{\omega}{\omega}} - 1 \right\} - j \left( \frac{v_s}{\omega} \right) \ell \right]
\] (D.12)

It is difficult to develop the equivalent circuit for the saturation region from the above equation so that the exponential term is expanded in power-series up to fourth order term.

\[
v'_{ds} \approx \frac{v_{ds}}{1 + \beta I_{dc} \ell A} + \frac{\beta I_{dc} \ell A v_{gs}}{1 + \beta I_{dc} \ell A} - \frac{\beta i'_d}{1 + \beta I_{dc} \ell A} \left[ I_{dc} \ell B - \left( \frac{v_s}{\omega} \right)^2 \left\{ 1 - j \frac{\omega}{v_s} \ell \right\}

+ \frac{1}{2} \left( -j \frac{\omega}{v_s} \ell \right)^2 + \frac{1}{6} \left( -j \frac{\omega}{v_s} \ell \right)^3 + \frac{1}{24} \left( -j \frac{\omega}{v_s} \ell \right)^4 - 1 \right] - j \left( \frac{v_s}{\omega} \right) \ell \]

= \frac{v_{ds}}{1 + \beta I_{dc} \ell A} + \frac{\beta I_{dc} \ell A v_{gs}}{1 + \beta I_{dc} \ell A}

- \frac{i'_d}{1 + \beta I_{dc} \ell A} \left[ \beta I_{dc} \ell B + \frac{1}{2} \beta \ell^2 - j \omega \tau_s \frac{\beta \ell^2}{6} - (\omega \tau_s)^2 \frac{\beta \ell^2}{24} \right]
\] (D.13)

where \( \tau_s = \ell / v_s \). The drain current is obtained from Equation (D.3a) by setting \( x' = \ell \).

\[
i_d = i'_d e^{-j \omega \tau_s}
\] (D.14)

One can easily obtained the gate current in the saturation region from the difference of the channel current at drain side and the GCA/saturation boundary.
The above three equations gives the equivalent circuit for the saturation region which is shown in Figure 76. Comparing Figure 76 with Equation (D.13) one can easily find $\gamma_s$ and $\delta_s$

$$\gamma_s = \frac{1}{1 + \beta I_{dc}\ell A} \quad \text{(D.16)}$$

$$\delta_s = \frac{\beta I_{dc}\ell A}{1 + \beta I_{dc}\ell A} \quad \text{(D.17)}$$

Let us calculate the value of each element in the impedance part. The following equation can be found from equivalent circuit

$$v_{ds}' = \gamma_s v_{ds} + \delta_s v_{gs} - \left( R_{s1} + \frac{1}{R_{s2} + j\omega C_s} \right) i_d' \quad \text{(D.18)}$$

In order to compare Equation (D.18) with Equation (D.13) we should expand the denominator in power-series up to the second order using the assumption that
\( C_s R_{s2} \ll \omega. \)

\[
v'_d \approx \gamma_s v_{ds} + \delta_s v_{gs} - \left[ R_{s1} + R_{s2} - j\omega C_s R_{s2}^2 - \omega^2 C_s^2 R_{s2}^3 \right] i'_d \tag{D.19}
\]

Comparing Equation (D.13) and (D.19) and equating the term of same order gives

\[
R_{s1} + R_{s2} = \frac{\beta I_{dc} B + \frac{1}{2} \beta \ell^2}{1 + \beta I_{dc} \ell A} \tag{D.20}
\]

\[
C_s R_{s2}^2 = \frac{\tau_s \beta \ell^2}{6(1 + \beta I_{dc} \ell A)} \tag{D.21}
\]

\[
C_s^2 R_{s2}^3 = \frac{\tau_s^2 \beta \ell^2}{24(1 + \beta I_{dc} \ell A)} \tag{D.22}
\]

Solving the above equations yields

\[
R_{s1} = \frac{\beta I_{dc} B - \frac{1}{6} \beta \ell^2}{1 + \beta I_{dc} \ell A} \tag{D.23}
\]

\[
R_{s2} = \frac{2 \beta \ell^2}{3(1 + \beta I_{dc} \ell A)} \tag{D.24}
\]

\[
C_s = \frac{3}{8} \frac{1 + \beta I_{dc} \ell A}{\beta \ell^2} \tag{D.25}
\]

Now \( v'_d \) is

\[
v'_d = \gamma_s v_{ds} + \delta_s v_{gs} - Z_s i'_d \tag{D.26}
\]

where

\[
Z_s = R_{s1} + \frac{1}{R_{s2} + j\omega C_s}
\]

**D.2 Calculation of the Y-parameters for the two region model**

The Y-parameters of the velocity saturated MODFET, \( Y_{ij}(sat) \), will be obtained from the equivalent circuit in terms of the Y-parameters of the GCA region \( Y_{ij}(g) \) of
reduced gate length \(X_S = L_g - \ell\). The gate and drain currents are easily obtained by inspection from the equivalent circuit (see Figure 39)

\[i_g = Y_{gs}(g)v_{gs} + Y_{gd}(g)v'_{ds} + \gamma_s \delta_s v_{gs} + \iota'_d (1 - e^{-j\omega t})\]  
\[i_d = i'_d e^{-j\omega t}\]  
\[i'_d = Y_{dg}(g)v_{gs} + Y_{dd}(g)v'_{ds}\]  

where \(v'_{ds}\) is given by Equation (D.26). Substituting Equation (D.26) into (D.29) yields

\[i'_d = Y_{dg}(g)v_{gs} + Y_{dd}(g) \gamma_s v_{ds} + Y_{dd}(g) \delta_s v_{gs} - Y_{dd}(g) Z_s v'_{ds}\]  

From Equation (D.30) one can easily obtain \(i'_d\) in terms of \(v_{gs}\) and \(v_{ds}\) as follows

\[i'_d = \frac{Y_{dg}(g) + Y_{dd}(g) \delta_s v_{gs} + \frac{Y_{dd}(g) \gamma_s}{1 + Y_{dd}(g) Z_s} v_{ds}}{1 + Y_{dd}(g) Z_s}\]  

Replacing Equation (D.31) in (D.28) gives

\[i_d = \left(\frac{Y_{dg}(g) + Y_{dd}(g) \delta_s}{1 + Y_{dd}(g) Z_s}\right) e^{-j\omega t} v_{gs} + \left(\frac{Y_{dd}(g) \gamma_s}{1 + Y_{dd}(g) Z_s}\right) e^{-j\omega t} v_{ds}\]  

Substituting Equation (D.26) into (D.27) yields

\[i_g = Y_{gs}(g)v_{gs} + Y_{gd}(g) \gamma_s v_{ds} + \delta_s v_{gs} - Z_s v'_{ds} + i'_d (1 - e^{-j\omega t})\]  
\[= (Y_{gs}(g) + Y_{gd}(g) \delta_s) v_{gs} + Y_{gd}(g) \gamma_s v_{ds} + \iota'_d (1 - e^{-j\omega t} - Z_s Y_{gd}(g))\]  

Substituting Equation (D.31) into (D.33) gives

\[i_g = (Y_{gs}(g) + Y_{gd}(g) \delta_s) v_{gs} + Y_{gd}(g) \gamma_s v_{ds} + \left(1 - e^{-j\omega t} - Z_s Y_{gd}(g)\right)\]
\[ \begin{aligned}
&\times \left( \frac{Y_{dg}(g) + Y_{dd}(g)\delta_s}{1 + Y_{dd}(g)Z_s} v_{gs} + \frac{Y_{dd}(g)\gamma_s}{1 + Y_{dd}(g)Z_s} v_{ds} \right) \\
&= \left[ Y_{gg}(g) + Y_{gd}(g)\delta_s + \frac{Y_{dg}(g) + Y_{dd}(g)\delta_s}{1 + Y_{dd}(g)Z_s} \left( 1 - e^{-j\omega_{rs}} - Z_s Y_{gd}(g) \right) \right] v_{gs} \\
&+ \left[ Y_{gd}(g)\gamma_s + \frac{Y_{dd}(g)\gamma_s}{1 + Y_{dd}(g)Z_s} \left( 1 - e^{-j\omega_{rs}} - Z_s Y_{gd}(g) \right) \right] v_{ds} \\
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
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