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Application of the Schwinger closed time-path method to relativistic quantum field theory

Davis, John Edward, Ph.D.
The Ohio State University, 1991
Application of the Schwinger Closed Time–Path Method to Relativistic Quantum Field Theory

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

John E. Davis, B.S.

*****

The Ohio State University

1991

Dissertation Committee

Prof. R. J. Perry
Prof. B. C. Clark
Prof R. L. Mills

Adviser’s Approval

[Signatures]

R. J. Perry
Co-Adviser
Dept. of Physics

B. C. Clark
Co-Adviser
Dept. of Physics
To my parents
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Vita

July 22, 1961 ................................................. Born Marion, Ohio
Spring 1983 ................................................. BS., The Ohio State University
Spring 1984–Fall 1986 ................................. Teaching Assistant
Winter 1986–Summer 1988 ......................... Research Assistant
                                                Experimental Nuclear Physics
Summer 1988–present ................................ Research Assistant
                                                Theoretical Nuclear Physics

List of Publications

"Relativistic Kinetic Equations with Mesonic Degrees of Freedom."

Fields of Study

Major Field: Physics
(Theoretical Nuclear Physics)
Advisor: Prof. R. J. Perry.
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CHAPTER I

Introduction

1.1 Introduction

Since the main emphasis of this work is on non-equilibrium relativistic quantum field theory, it is appropriate in this chapter to explain why I am interested in this problem.

The field operators form the fundamental objects in the usual presentation of quantum field theory where one works in the Heisenberg picture [1, 2, 3, 4, 5]. For bosons, the operators satisfy a commutator algebra while fermion operators satisfy an anti-commutator algebra. The temporal development of the operators is governed by certain partial differential equations— the so-called field equations. One seeks these operators as solutions to the field equations subject to the constraint that the operator solutions obey the proper algebra which characterize their fermionic or bosonic nature. In general, except for the simple case of non-interacting fields, very little is known about whether or not such solutions exist. However, assuming their existence, one also requires that the operators obey cer-
tain initial conditions for their complete determination. From the outset one is forced to consider the initial value problem as the most general problem of quantum field theory. Hence, from the pure mathematical standpoint, non-equilibrium quantum field theory is interesting in its own right.

Quantum Chromodynamics (QCD) is widely believed to be the correct theory of the strong interaction. At low temperatures and densities, quarks and gluons are confined in hadrons. As the temperature or density is increased, a deconfinement phase transition is believed to occur from the hadronic phase to a quark-gluon plasma phase (Fig. 1.1). There is considerable controversy as to the nature of such a phase transition as well as the experimental signature of the production of a quark-gluon plasma. If a quark-gluon plasma is produced in an energetic heavy ion collision, it is expected to be short lived before undergoing the confinement phase transition. Whether or not equilibrium is achieved in the quark-gluon plasma phase is an open question.

Hence, it is important that one works in a framework which is capable of encompassing non-equilibrium phenomena. One such formalism is the closed time-path method of Schwinger[6]. Although it is familiar from studies in non-relativistic transport theory[7], it has seen few applications in the study of relativistic quantum field theory. In the following chapters, we work within this formalism to study both equilibrium and non-equilibrium quantum fields. With an eye on potential applications to heavy ion collisions, through a series of approximations and
Figure 1.1: Temperature \( T \) vs density \( \rho \) for hadronic matter. At high densities and temperatures, hadronic matter undergoes a phase transition (shaded region) to a quark–gluon plasma.
Ansatze, a Boltzmann transport equation is derived for a simple meson-nucleon system from its underlying quantum field theory. Although the "approximations" made are quite common in physical applications, very little is known about their validity. In the last chapter, we investigate an exactly solvable model to gain some insight into the validity of such approximations.

1.2 Choice of Model

Motivated by the desire to understand matter under extreme conditions, the primary thrust of this work is to gain insight into the general initial value problem of relativistic quantum field theory. For this purpose, a model is needed. Ideally, we should work with the QCD lagrangian since QCD is considered to be the correct theory of the strong interaction. However, for reasons which will become clear, we propose to work with a simpler model which is free of many of the complications associated with QCD but nevertheless contains much of the physics of the problem. The lagrangian employed in much of this work is the Yukawa lagrangian which describes the interaction of spin-half Dirac particles with spin-zero scalar bosons. Besides being an excellent model for pedagogical purposes, the relativistic Yukawa model is also of intrinsic interest when viewed from the realm of the relativistic nuclear many body problem. Once the success of quantum electrodynamics was realized through the proper understanding of perturbative renormalization it was only natural to wonder if the same formalism could be applied to the strong
interaction. Hence, interacting meson–nucleon field theories were born. However, unlike the case of QED where a perturbative expansion in powers of the fine structure constant is valid, an expansion in powers of the strong coupling constant is not possible. Hence is forced either to abandon the idea in favor of a new approach or develop a non-perturbative means to do strong interaction physics. Even today, the latter alternative has not been realized and perturbation theory remains the only framework capable of obtaining unambiguous results in relativistic quantum field theories. The former approach has been a fruitful alternative where one resorts to doing phenomenology.

The most naive phenomenological idea is to imagine the nucleus as a non-relativistic fermi system interacting via two-body potentials. Typically, one uses phenomenological potentials fit to scattering data as well as the binding of the deuteron. Although one is able to do a reasonable job of explaining nuclear shell structure it has been impossible to reproduce the correct energy and saturation of nuclear matter. For a reasonable description of nuclear matter one is forced to consider three and four-body potentials. However, when one considers hot, dense nuclear systems such as those present in the core of a neutron star as well as in heavy ion experiments, one is forced to consider a relativistic formulation. In addition, Dirac phenomenology[8] suggests that the observed binding energy of a nucleon in a typical nucleus is the result of a delicate cancellation between a large attractive Lorentz scalar potential and a large repulsive vector potential. Thus,
if Dirac phenomenology is any indication, one must consider relativity from the outset when dealing with the nuclear many body problem.

The most successful model of the relativistic nuclear many body problem which reproduces the correct binding and saturation of nuclear matter as well being consistent with Dirac phenomenology is the phenomenological "σ–ω model" of Walecka[9]. Here one treats the nucleons as point like spin–half Dirac particles interacting through the exchange of the scalar σ and vector ω mesons. Walecka's original motivation for the model was a simple description of dense hadronic matter. It is assumed that in the limit of infinite density, a calculation performed at the mean field level is sufficient by replacing the boson operators by their groundstate expectation values. Hence the original philosophy of the Walecka model is that it be treated only phenomenologically where one ignores anti–particles and solve the theory at only the mean field level. Nevertheless, because of its phenomenological success, it has been treated in higher order approximations[9], e.g., Hartree–Fock, with and without anti–particles. In addition, it has been extended to the inclusion of other mesons as well. However, I believe it is important that one remember the phenomenological nature of the model and try not to push it too far.

For these reasons, we consider a different model ignoring the vector meson. Although this model cannot be taken seriously in describing nuclear systems, it does saturate (by a different mechanism) and allows for a description of creation and annihilation of dynamical mesons, thus providing a reasonable starting point
for the investigation of a relativistic quantum transport theory.

1.3 Organization and Notation

Non-equilibrium relativistic quantum field theory is a synthesis of two fields of physics—non-equilibrium many body theory and relativistic quantum field theory—both subjects notoriously difficult in their own right. It is for this reason that non-equilibrium relativistic quantum field theory has not yet matured, making it very difficult for those with little or no experience in the field to gain access to the relevant literature. One of the primary objectives of this work is to help bridge this gap. I have thought a great deal about the appropriate organization and presentation of the material which is reflected in the pedagogical style of this thesis. Although, the work is reasonably self-contained, I have had to make some general assumptions as to the background of the reader. I assume the reader to be familiar with relativistic quantum mechanics at the level of Vol I. of Bjorken and Drell[5] and is able to manipulate Dirac gamma matrices at a moment's notice.

Chapter II. is a review of classical relativistic field theory with a particular emphasis on the derivation of a symmetric energy–momentum tensor. Most texts concentrate upon the canonical stress tensor; however, the symmetric form is of utmost importance in relativistic many body theory. In Chapter III. quantization is introduced via the “Schwinger action principle”[10] which I believe is particularly elegant. The basic framework in which I work, namely the closed time–path
method of Schwinger, is described in Chapter IV. where a “sourceless” representation is introduced. Working in the sourceless representation facilitates the construction of Schwinger–Dyson equations for the Green functions of the theory. This technique is illustrated in this chapter for the Yukawa model. In Chapter V., the Wigner transform and gradient is described. In addition, the analytic properties of the Green functions are discussed as well as their sum rules. Chapter VI. is a review of many body systems with a description of the classical Boltzmann transport equation. The quantum analogue of the classical transport equation is derived as well as a discussion of the conditions necessary for this interpretation. Finite temperature systems are also briefly discussed. In Chapter VII., all the ingredients of formal apparatus developed up to this point are brought to bear against the Yukawa model where a transport equation is derived complete with collision terms as well as the transport of dynamical mesons. The assumptions and Ansätze are discussed in the context of an energetic relativistic heavy ion collision. Finally in the last chapter, we come back to some of the “approximations” made throughout this work and look at their validity in a simple model.

**Notation:** For the most part, we follow the conventions of Bjorken and Drell[5], setting $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $px \equiv p \cdot x = p_\mu x^\mu$. Also unless explicitly indicated, we work with natural units $\hbar = c = 1$. 
CHAPTER II

Classical Field Theory

2.1 Lorentz Invariance Properties of Fields

Let $\Phi^\alpha(x)$ denote the components of some geometrical object defined on the manifold of space-time and let $x$ denote a coordinate system on this manifold. Consider some group $L$ of coordinate transformations on the manifold and let $x'$ denote a new coordinate system obtained from the old by the action of some element of this group, i.e.,

$$x \rightarrow x' = Lx. \tag{2.1}$$

The objects are assumed to transform under some representation $D(L)$ of the group according to

$$\Phi^\alpha(x) \rightarrow \Phi'^\alpha(x') = D(L)^\alpha_\beta \Phi^\beta(x). \tag{2.2}$$

In general, the geometric objects may be characterized as being scalar, vector, spinor, and so on transforming under the corresponding representations of the group $L$. In the following, we confine ourselves to the Lorentz group of coordinate transformations.
Under an infinitesimal Lorentz transformation, the coordinate $x$ transforms as

$$x^\mu \rightarrow x'^\mu = x^\mu + \omega^\mu_\nu x^\nu,$$  \hspace{1cm} (2.3)

where $\omega^\mu_\nu = -\omega^\nu_\mu$ is the antisymmetric generator of the infinitesimal coordinate transformation. Similarly, $\Phi^\alpha$ will transform as

$$\Phi^\alpha(x) \rightarrow \Phi'^\alpha(x') = \Phi^\alpha(x) + \frac{1}{2} S^\alpha_{\beta\mu\nu} \omega^\mu_\nu \Phi^\beta(x),$$  \hspace{1cm} (2.4)

where $S^\alpha_{\beta\mu\nu}$ depends on the representation of the group and is antisymmetric in $\mu$ and $\nu$. For example, suppose that the geometric object is a vector field $A^\mu(x)$. Then under the infinitesimal Lorentz transformation, the vector field transforms as

$$A'^\mu(x') = A^\mu(x) + \omega^\mu_\nu A^\nu(x),$$  \hspace{1cm} (2.5)

which implies that

$$S^\alpha_{\beta\mu\nu} = g^{\alpha\beta} g^\mu_\nu - g^{\alpha\nu} g^\mu_\beta.$$  \hspace{1cm} (2.6)

Similarly, one can show that

$$S^\alpha_{\beta\mu\nu} = 0$$  \hspace{1cm} (2.7)

for a scalar field and

$$S^\alpha_{\beta\mu\nu} = -\frac{i}{2} (\sigma^\mu_\nu)^\beta$$  \hspace{1cm} (2.8)

for the Dirac field.

It is appropriate to make the following comment concerning a general gauge field $B_\mu(x)$. The transformation properties of gauge fields are complicated by the
fact that one has extra gauge degrees of freedom at his disposal. That is, one is really only concerned with an equivalence class of gauge fields which differ by a gauge transformation

\[ B_\mu(x) \rightarrow B'_\mu(x) = B_\mu(x) + \partial_\mu \Lambda(x). \]  

(2.9)

To fix this extra degree of freedom one must make a choice of gauge which is some constraint on the gauge field, i.e.,

\[ F[B] = 0. \]  

(2.10)

For example, if one works in the Lorentz gauge, one requires

\[ \partial_\mu B^\mu(x) = 0 \]  

(2.11)

whereas for a non-covariant gauge such as the radiation gauge one has

\[ \vec{\partial} \cdot \vec{B} = 0. \]  

(2.12)

In a covariant gauge, one maintains manifest Lorentz invariance with the gauge field transforming as a vector field

\[ B'_\mu(x') = B_\mu(x) + \omega_{\mu\nu} B^\nu(x). \]  

(2.13)

However, manifest Lorentz invariance is lost when one works in a non-covariant gauge. That is, under the transformation law above, one cannot maintain the choice of gauge. Lorentz invariance may be regained if one modifies the Lorentz
transformation law for the gauge field to

\[ B'_\mu(x') = B_\mu(x) + \omega_{\mu\nu}B''(x) + \partial_\mu \Lambda(x), \]  

(2.14)

where the function \( \Lambda(x) \) is chosen in such a way as to maintain the gauge.

### 2.2 Stress Tensor

Let \( \Phi(x) \) denote a set of fields of spinor or tensor character defined as in the previous section, and let \( \Phi_\mu = \partial_\mu \Phi \) denote the derivative of the set of fields with respect to \( x^\mu \). For notational simplicity, tensor indices will be suppressed; hence, the reader should also beware of implicit contractions of the indices. The action \( S[\Phi] \) is

\[ S[\Phi] = \int_V dx \mathcal{L}(\Phi, \Phi_\mu) \]  

(2.15)

where the integration is performed over some volume \( V \). Typically, the volume \( V \) will be the region bounded by two spacelike surfaces extending to spatial infinity. For example, if the spacelike surfaces consist of the two equal time planes \( x_0 = t_1 \) and \( x_0 = t_2 \) then \( \int_V dx = \int_{t_1}^{t_2} dt \int d^3x \).

We now consider the variation of the action with respect to arbitrary (but smooth) variations in both \( \Phi \) and \( x \). However, we do not allow ourselves the privilege of varying the functional form of the Lagrangian. This is not to say that the Lagrangian remains invariant under the variation; rather, a variation of it will
be induced by variations of $\Phi$ and $x$. We assume that under the variations

$$x \rightarrow x' = x + \delta x(x), \quad (2.16)$$

$$\Phi(x) \rightarrow \Phi'(x') = \Phi(x) + \delta \Phi(x). \quad (2.17)$$

It must be emphasized that $\delta x$ is not a constant but depends upon $x$, hence the notation $\delta x(x)$. For example, an infinitesimal Poincare transformation may consist of a translation as well as a rotation, i.e.,

$$x_{\mu} \rightarrow x'_{\mu} = x_{\mu} + \delta a_{\mu} + \delta \omega_{\mu\nu} x^\nu. \quad (2.18)$$

Also it is important to understand that $\delta \Phi(x)$ is not the variation of the functional form of $\Phi$ rather it is the total variation which consists of the variation with respect to the shape of $\Phi$ as well as a variation induced by the variation in $x$.

With the above cautionary statements understood, we find that under the variation,

$$\delta S = \int_{V'} dx' L(\Phi'(x'), \Phi'_{\mu}(x')) - \int_{V} dx L(\Phi(x), \Phi_{\mu}(x)). \quad (2.19)$$

where $V'$ is the image of $V$ under the mapping $x \rightarrow x'$. In the first integral, we make a change of variables from $x'$ to $x$ where $x' = x + \delta x$; this brings the volume of integration $V'$ back to $V$. Then the first integral becomes

$$\int_{V'} dx' L[\Phi'(x'), \Phi'_{\mu}(x')] = \int_{V} dx J(x) L[\Phi(x) + \delta \Phi(x), \Phi'_{\mu}(x + \delta x)]$$

$$= \int_{V} dx J(x) \left\{ L[\Phi(x), \Phi_{\mu}(x)] + \frac{\partial L}{\partial \Phi(x)} \delta \Phi(x) \right\}$$

$$+ \frac{\partial L}{\partial \Phi_{\mu}(x)} \left[ \Phi_{\mu}'(x') - \Phi_{\mu}(x) \right]. \quad (2.20)$$
where
\[ J(x) = 1 + \partial_\mu \delta x^\mu + \ldots \] (2.21)
is the Jacobian \( J(x) \) of the transformation \( x' \rightarrow x \) and higher order terms in the variation have been neglected.

Write
\[ \Phi'_\mu(x') = \Phi_\mu(x) + \partial_\mu \delta \Phi(x) \]
\[ = \Phi_\mu(x) \partial_\mu x^\nu + \partial_\nu \delta \Phi(x) \partial_\mu x^\nu \] (2.22)
and
\[ \partial_\mu x^\nu = \partial_\mu [x^\nu - \delta x^\nu(x)] \]
\[ = g^\nu_\mu - \partial_\lambda \delta x^\nu(x) \partial_\mu x^\lambda \]
\[ = g^\nu_\mu - \partial_\lambda \delta x^\nu(x) g^\lambda_\mu + \ldots \]
\[ = g^\nu_\mu - \partial_\mu \delta x^\nu(x), \] (2.23)
again neglecting higher order terms in the variation. So finally,
\[ \Phi'_\mu(x') = \Phi_\mu(x) - \Phi_\nu(x) \partial_\mu \delta x^\nu + \partial_\mu \delta \Phi(x) \] (2.24)
or
\[ \Phi'_\mu(x') - \Phi_\mu(x) = -\Phi_\nu(x) \partial_\mu \delta x^\nu + \partial_\mu \delta \Phi(x). \] (2.25)
Inserting the above expression into Eq. (2.20) and using Eq. (2.19) for \( \delta S \) yields
\[ \delta S = \int_v dx \left\{ \frac{\partial L}{\partial \Phi} \delta \Phi + \frac{\partial L}{\partial \Phi_\mu} (\partial_\mu \delta \Phi - \partial_\nu \Phi \partial_\mu \delta x^\nu) + L \partial_\mu \delta x^\mu \right\}, \] (2.26)
which after some rearrangement followed by integration by parts gives the final form

$$\delta S = \int_V dx \left[ \frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \Phi'} \right) \right] (\delta \Phi - \partial_v \Phi \delta x^v) + \int_{\partial V} d\sigma_{\mu} \left[ \frac{\partial L}{\partial \Phi_{\mu}} \delta \Phi - \delta x^v \left( \frac{\partial L}{\partial \Phi'} \partial_v \Phi - g_{\mu} \right) \right].$$

(2.27)

Here $\partial V$ denotes the boundary of the volume $V$ and care has been taken not to throw away any of the surface terms arising from the partial integration.

At this point we have not made any assumptions as to the nature of the variation and we have regarded the variation in $x$ to be independent of the variation of the field $\Phi$. We now consider some specific variations.

### 2.2.1 Field Equations

The equation for the field $\Phi(x)$ is obtained from Hamilton’s principle of least action. Demanding that the action be stationary under a variation of the field only ($\delta x = 0$) with the variation vanishing on the boundary $\partial V$ yields the celebrated Euler–Lagrange equations of motion for the system:

$$0 = \frac{\partial L}{\partial \Phi} - \partial_{\mu} \frac{\partial L}{\partial \Phi_{\mu}}.$$  

(2.28)

Hence, any physical field will satisfy the above equation.

### 2.2.2 Noether’s Theorem

Suppose that the variation $x \rightarrow x'$ and $\Phi(x) \rightarrow \Phi'(x')$ is a symmetry of the action. That is, $\delta S = 0$ under the transformation. Furthermore, assuming that
the fields obey the Euler–Lagrange equations given in the previous section we see that the first term of Eq. (2.27) vanishes and we can write

\[ 0 = \int_{\partial V} d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \dot{\Phi} - \partial^\nu \left( \frac{\partial \mathcal{L}}{\partial \Phi_{\nu}} \Phi_{\nu} - g_{\nu} \mathcal{L} \right) \right]. \] (2.29)

Assuming that the boundary \( \partial V \) consists of two space like surfaces \( \sigma_1 \) and \( \sigma_2 \) and that the fields vanish sufficiently rapidly at spatial infinity, we can write the above equation as

\[ 0 = G(\sigma_2) - G(\sigma_1) \] (2.30)

where

\[ G(\sigma) = \int_{\sigma} d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \dot{\Phi} - \partial^\nu \left( \frac{\partial \mathcal{L}}{\partial \Phi_{\nu}} \Phi_{\nu} - g_{\nu} \mathcal{L} \right) \right]. \] (2.31)

Hence, we see that \( G(\sigma) \) is a constant of motion of the system. This is the essential content of Noether's theorem which states that if the action is invariant under an \( n \) parameter continuous group, then the symmetry gives rise to \( n \) constants of motion.

We will use this result in the next section to show that Poincare invariance gives rise to conservation of energy–momentum and angular momentum of the field.

### 2.2.3 Stress–Energy Tensor

Under a Lorentz transformation \( x \rightarrow x + \delta x \) the field transforms in a well defined manner. We have seen in the previous section that under a Lorentz transformation

\[ x^\mu \rightarrow x'^\mu = x^\mu + \omega^{\mu\nu} x_\nu \] (2.32)
the field $\Phi^\alpha(x)$ transforms as

$$\Phi^\alpha(x) \longrightarrow \Phi'(x') = \Phi^\alpha(x) + \frac{1}{2} S^\alpha_{\beta \mu} \omega_{\mu \nu} \Phi^\beta(x)$$  \hspace{1cm} (2.33)

or suppressing internal indices (tensor or spinor)

$$\Phi(x) \longrightarrow \Phi'(x') = \Phi(x) + \frac{1}{2} S^{\mu \nu} \omega_{\mu \nu} \Phi(x).$$  \hspace{1cm} (2.34)

In the rest of this section, the internal indices will be implicit; hence, $S^{\mu \nu}$ and $\Phi(x)$ should be regarded as a matrix and vector, respectively, in the space of internal variables.

Comparing with Eq. (2.17) we see that

$$\delta x^\mu = \omega^{\mu \nu} x^\nu$$  \hspace{1cm} (2.35)

and

$$\delta \Phi(x) = \frac{1}{2} S_{\mu \nu} \Phi(x) \omega^{\mu \nu}$$

$$= \frac{1}{2} S_{\mu \nu} \Phi(x) \partial^\mu \delta x^\nu$$  \hspace{1cm} (2.36)

where in the last line I have used $\omega^{\mu \nu} = \partial^\nu \delta x^\mu = -\partial^\mu \delta x^\nu$. In accordance with Noether's theorem, this symmetry implies that the generator

$$G(\sigma) = \int_{\sigma} d\sigma \lambda \left[ -\frac{1}{2} \frac{\partial L}{\partial \Phi_{\lambda}} S_{\mu \nu} \Phi \partial^\nu \delta x^\nu - \delta x^\nu \left( \frac{\partial L}{\partial \Phi_{\lambda}} \partial_{\nu} \Phi - g^{\nu \lambda} L \right) \right]$$  \hspace{1cm} (2.37)

is a constant of motion. Write the first term in the above integral as

$$\frac{1}{2} \frac{\partial L}{\partial \Phi_{\lambda}} S^{\mu \nu} \Phi \partial_{\mu} \delta x_{\nu} = f^{\lambda \mu \nu}(x) \partial_{\mu} \delta x_{\nu},$$  \hspace{1cm} (2.38)
defining the tensor $f_{\lambda}^{\mu\nu}(x)$. Since $\partial_{\mu}\delta x_{\nu} = -\partial_{\nu}\delta x_{\mu}$ is antisymmetric in $\mu$ and $\nu$, $f_{\lambda}^{\mu\nu}(x)$ is determined up to a term symmetric in $\mu$ and $\nu$. This ambiguity may be exploited to write $f_{\lambda}^{\mu\nu}(x)$ in a form which is antisymmetric in $\mu$ and $\lambda$,

$$f_{\lambda}^{\mu\nu}(x) = \frac{1}{2} \left[ \frac{\partial L}{\partial \Phi_{\lambda}} S_{\mu
u}^{\lambda} - \frac{\partial L}{\partial \Phi_{\mu}} S_{\lambda\nu}^{\lambda} - \frac{\partial L}{\partial \Phi_{\nu}} S_{\lambda\mu}^{\lambda} \right] \Phi(x) = -f_{\mu}^{\lambda\nu}(x). \quad (2.39)$$

Hence, the generator $G(\sigma)$ becomes

$$G(\sigma) = \int_{\sigma} d\sigma_{\lambda} \left[ -f_{\lambda}^{\mu\nu}(x) \partial_{\mu}\delta x_{\nu} - \delta x_{\nu}\left( \frac{\partial L}{\partial \Phi_{\lambda}} \partial_{\nu} \Phi - g_{\nu}^{\lambda} L \right) \right]$$

$$= -\int_{\sigma} d\sigma_{\lambda} \partial_{\mu} \left[ f_{\lambda}^{\mu\nu}(x) \delta x_{\nu} \right] + \int_{\sigma} d\sigma_{\lambda} \delta x_{\nu} \left[ \partial_{\mu} f_{\lambda}^{\mu\nu}(x) - \left( \frac{\partial L}{\partial \Phi_{\lambda}} \partial_{\nu} \Phi - g_{\nu}^{\lambda} L \right) \right]. \quad (2.40)$$

It is easy to see that the first integral in the above expression is just a constant.

Indeed, we can use the antisymmetry of $f_{\lambda}^{\mu\nu}$ in $\lambda$ and $\mu$ to write

$$\int_{\sigma} d\sigma_{\lambda} \partial_{\mu} \left[ f_{\lambda}^{\mu\nu}(x) \delta x_{\nu} \right] = \frac{1}{2} \int_{\sigma} (d\sigma_{\lambda} \partial_{\mu} - d\sigma_{\mu} \partial_{\lambda}) \left[ f_{\lambda}^{\mu\nu}(x) \delta x_{\nu} \right] \quad (2.41)$$

which implies that the above expression is independent of the surface $\sigma$. Omitting this unimportant constant leads to the expression

$$G(\sigma) = \int_{\sigma} d\sigma_{\lambda} \left[ \partial_{\mu} f_{\lambda}^{\mu\nu}(x) \delta x_{\nu} - \delta x_{\nu}\left( \frac{\partial L}{\partial \Phi_{\lambda}} \partial_{\nu} \Phi - g_{\nu}^{\lambda} L \right) \right]$$

$$= -\int_{\sigma} d\sigma_{\lambda} \delta x_{\nu} T_{\mu\nu} \quad (2.42)$$

where the stress–energy tensor $T_{\mu\nu}(x)$ is defined as

$$T_{\mu\nu}(x) = -\partial_{\lambda} f^{\mu\lambda}(x) + \frac{\partial L}{\partial \Phi_{\mu}} \partial_{\nu} \Phi - g_{\mu\nu} L. \quad (2.43)$$
Using the fact that for an infinitesimal Poincare transformation

\[ \delta x_\mu = a_\mu + \omega_{\mu\nu} x^\nu, \]  

(2.44)

we may write

\[ G(\sigma) = - \int d\sigma_\mu \delta x_\nu T^{\mu\nu} \]

\[ = -a_\nu \int d\sigma_\mu T^{\mu\nu} - \omega_{\mu\nu} \int d\sigma x^\nu T^{\lambda\mu} \]

\[ = -a_\nu P^\nu + \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu}, \]

where

\[ P^\nu = \int d\sigma_\mu T^{\mu\nu} \]

(2.45)

is the conserved energy–momentum four vector of the system and

\[ J^{\mu\nu} = \int d\sigma_\lambda (x^\mu T^{\lambda\nu} - \omega^{\lambda\nu} x^\nu T^{\lambda\mu}) \]

(2.47)

is the angular momentum tensor of the field. Since both \( P^\nu \) and \( J^{\mu\nu} \) are constants, it follows from their defining equations that

\[ \partial_\mu T^{\mu\nu} = 0 \]

(2.48)

and

\[ \partial_\lambda (T^{\lambda\mu} x^\nu - T^{\lambda\nu} x^\mu) = 0. \]

(2.49)

From these two expressions, it is trivial to show that

\[ T_{\mu\nu}(x) = T_{\nu\mu}(x), \]

(2.50)
i.e., the stress tensor is symmetric.

One often sees another stress–tensor, the so–called canonical stress–tensor. It is defined as

$$T_{\mu\nu}^{\text{can}} = \frac{\partial L}{\partial \Phi_{\mu}} \partial_{\nu} \Phi - g_{\mu\nu} L$$  \hspace{1cm} (2.51)

and is related to the symmetric stress–tensor by

$$T_{\mu\nu}(x) = -\partial^\lambda f_{\mu\lambda\nu}(x) + T_{\mu\nu}^{\text{can}}.$$  \hspace{1cm} (2.52)

In general, the canonical stress–tensor is not symmetric. Note that the two tensors differ by a term which is a total divergence. This implies that the energy–momentum 4–vector computed using the canonical stress tensor will be conserved and differ by a constant from the corresponding vector computed from the symmetric tensor. However, the corresponding angular momentum tensor, $J^{\mu\nu}$ will not be conserved as is the case for the one calculated with the symmetric tensor. The reason is simple. In deriving the symmetric tensor, care was taken to ensure that the proper transformation of the field variables under Lorentz transformations was taken into account. In derivations of the canonical tensor, one ignores the internal transformation laws of the field variables. This is tantamount to ignoring spin degrees of freedom violating total (spin + orbital) angular momentum conservation. The two tensors agree only for scalar fields which have no intrinsic spin.

Up to this point we have used a matrix notation suppressing all internal indices. For future reference, the symmetric stress–energy tensor $T^{\mu\nu}$ is displayed in its full
glory with all internal indices in full view:

\[
T_{\mu\nu} = \frac{1}{2} \partial_\lambda \left[ \frac{\partial L}{\partial \partial_\lambda \Phi^\alpha} S_\beta^{\alpha\mu} \Phi^\beta - \frac{\partial L}{\partial \partial_\mu \Phi^\alpha} S_\beta^{\lambda\nu} \Phi^\beta - \frac{\partial L}{\partial \partial_\nu \Phi^\alpha} S_\beta^{\mu\lambda} \Phi^\beta \right] + \frac{\partial L}{\partial \partial_\mu \Phi^\alpha} \partial_\nu \Phi^\alpha - g_{\mu\nu} L.
\]

(2.53)

In order of increasing complexity, we consider the scalar, vector, and spinor fields.

**Scalar Field.** A Lagrangian appropriate for a free scalar field \( \phi(x) \) with mass \( m_s \) is

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_s^2 \phi^2.
\]

(2.54)

Under the Lorentz transformation \( x \rightarrow x' = x + \delta x \) a scalar field \( \phi(x) \) remains invariant, i.e.,

\[
\phi(x') = \phi(x)
\]

(2.55)

This implies that the quantities \( S_\beta^{\alpha\mu} \) are 0. Thus the stress energy tensor is

\[
T_{\mu\nu} = T_{\mu\nu}^{\text{can}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} L
\]

(2.56)

which is obviously symmetric.

**Vector Field.** We take the Lagrangian

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},
\]

(2.57)

which is suitable for a free electromagnetic field and where

\[
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu
\]

(2.58)
is the anti-symmetric field strength tensor. In addition to being Lorentz invariant, this Lagrangian is also gauge invariant, i.e., it is unchanged under the gauge transformation

\[ A^\mu \rightarrow A^\mu + \partial^\mu \Lambda. \] (2.59)

For the vector field \( A^\mu(x) \), the tensor \( S^\alpha_{\beta \nu} \) is given by

\[ S^\alpha_{\beta \nu} = g^\alpha_{\beta \nu} - g^\alpha_{\gamma \nu} g^\mu_{\beta}. \] (2.60)

We first calculate the canonical stress tensor. It is straightforward to show that

\[ T^\text{can}_{\mu \nu} = F_{\alpha \mu} \partial_\nu A^\alpha - g_{\mu \nu} \mathcal{L}. \] (2.61)

Obviously, in addition to not being symmetric, it is also not gauge invariant. We leave it as an exercise for the reader to show that the symmetric stress tensor is given by

\[ T^\mu_{\nu} = F^{\mu \alpha} F_{\alpha \nu} - g^{\mu \nu} \mathcal{L}. \] (2.62)

which is gauge invariant as one would expect it to be.

**Dirac Field.** As a final example, we consider a spin-half Dirac field \( \Psi(x) \) coupled to a spin zero boson field \( \phi(x) \) via a Yukawa coupling. The Lagrangian is

\[ \mathcal{L} = \bar{\Psi}(i\gamma_\mu \partial^\mu - m)\Psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_s^2 \phi^2 + g \bar{\Psi} \phi \Psi \] (2.63)

where \( m \) is the mass of the fermion field, \( m_s \) is the mass of the boson field and \( g \) is the strength with which the two fields couple. The Euler-Lagrange equations
yield
\[(\Box + m_b^2)\phi(x) - g\bar{\Psi}(x)\Psi(x) = 0 \quad (2.64)\]
for the boson field and
\[(i\gamma_\mu\partial^\mu - m)\Psi(x) + g\phi(x)\Psi(x) = 0 \quad (2.65)\]
for the fermion field. There is an analogous equation for the adjoint field \(\bar{\Psi}(x)\).

For the fermion field, \(S^{\alpha\mu\nu}_\beta\) is given by
\[S^{\alpha\mu\nu}_\beta = \frac{-i}{2}(\sigma^{\mu\nu})^\alpha_\beta \quad (2.66)\]
where
\[\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \quad (2.67)\]

It is straightforward to show that
\[T^{\mu\nu} = -\frac{1}{4}\partial_\lambda \cdot \bar{\Psi}(\gamma^\mu\sigma^\lambda\nu + \gamma^\lambda\sigma^\nu\mu - \gamma^\nu\sigma^\lambda\mu)\Psi + i\bar{\Psi}\gamma^\mu\partial^\nu\Psi + \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L}. \quad (2.68)\]

From the above equation, it follows (though somewhat long and tedious) from the field equations that
\[T^{\mu\nu} = T^{\mu\nu}_F + T^{\mu\nu}_B \quad (2.69)\]
where
\[T^{\mu\nu}_F = \frac{i}{4}\bar{\Psi}(\gamma_\mu \bar{\partial}_\nu + \gamma_\nu \bar{\partial}_\mu)\Psi \quad (2.70)\]
is the fermion contribution and
\[T^{\mu\nu}_B = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi - m^2_b\phi^2) \quad (2.71)\]
is the boson piece. Here we are using the notation \(A \bar{\partial}_\nu B = A \cdot \partial_\mu B - \partial_\mu A \cdot B\).
CHAPTER III

Quantization

3.1 Canonical Quantization

In the preceding chapter, the fields $\Phi^\sigma(x)$ are regarded as classical objects. The goal of the present chapter is to make the transition from a classical to a quantum system. Upon making this transition, the fields become operators in the Heisenberg representation acting upon some Hilbert space.

The most popular and well known method of quantization is the method of canonical quantization\[5\]. This technique consists of making the transition from a Lagrangian formalism to a Hamiltonian one and replacing classical Poisson brackets by commutation relations in accordance with the correspondence principle. We shall illustrate this technique with a simple example from classical mechanics. We start with a one dimensional classical Lagrangian, $L(q, \dot{q})$, which is a function of “position” $q$ and “velocity” $\dot{q}$. The canonical momentum $p$ conjugate to coordinate $q$ is defined as

$$p = \frac{\partial L}{\partial \dot{q}}. \qquad (3.1)$$
The transition from a Lagrangian formalism to a Hamiltonian one is accomplished through the Legendre transformation

\[ H(p, q) = p\dot{q} - L(q, \dot{q}), \]  

(3.2)

where \( \dot{q} = \dot{q}(q, p) \) is obtained by inverting Eq. (3.1). We emphasize that the transition to a Hamiltonian system is possible only if this inversion is possible. With this assumption, the equations of motion in the Hamiltonian formalism are

\[ \dot{p} = -\frac{\partial H}{\partial q} = [q, H]_{PB} \]  

(3.3)

and

\[ \dot{q} = \frac{\partial H}{\partial p} = [p, H]_{PB}, \]  

(3.4)

where the Poisson bracket of two functions \( F \) and \( G \) is defined as

\[ [F, G]_{PB} = \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q}. \]  

(3.5)

In particular,

\[ [q, p]_{PB} = 1 \]  

(3.6)

and

\[ [q, q]_{PB} = [p, p]_{PB} = 0. \]  

(3.7)

Canonical quantization is performed at this point by making the correspondence \([F, G]_{PB} \rightarrow i/\hbar [F, G]\) where now \( F \) and \( G \) are regarded as operators and
$[F, G]$ is the quantum mechanical commutator. In particular, this prescription yields

$$[q, p] = -i\hbar$$

(3.8)

and

$$[q, q] = [p, p] = 0,$$

(3.9)

which are the familiar commutation relations for the quantum mechanical operators $p$ and $q$.

We repeat, however, that this whole quantization recipe is based upon being able to make the transition from a Lagrangian to a Hamiltonian framework, i.e., one must be able to perform the above Legendre transformation. However it turns out that for most interesting physical systems, this is not possible. As an example consider the Dirac field which is described by the Lagrangian

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \gamma_\mu \partial^\mu \psi - \frac{i}{2} \partial^\mu \bar{\psi} \gamma_\mu \psi - m \bar{\psi} \psi.$$  (3.10)

The momentum conjugate to $\psi$ is

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \psi} = \frac{i}{2} \bar{\psi} \gamma_0$$

(3.11)

and that conjugate to $\bar{\psi}$ is

$$\Pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = -\frac{i}{2} \psi \gamma_0 \psi.$$  (3.12)

Evidently it is impossible to complete the Legendre transformation since one is
unable to invert the above equations for $\psi$ and $\dot{\psi}$. Hence, for this example canonical quantization fails and one is forced to resort to another method of quantization.

The quantization method described below is based on Schwinger's quantum action principle. Here one works entirely in a Lagrangian framework without the need to transform to a Hamiltonian formalism.

### 3.2 Schwinger's Action Principle

We have seen in the previous section that it is not always possible to work in a Hamiltonian formalism. For this reason, it is advantageous to quantize in the Lagrangian framework. The method described in this section is due to Schwinger[10]. However, I believe that the presentation given here to be more transparent, though perhaps less elegant, than Schwinger's. Nevertheless, the spirit is the same.

In the previous chapter, we have seen that the variations

$$x \rightarrow x' = x + \delta x \quad (3.13)$$

and

$$\Phi(x) \rightarrow \Phi'(x') = \Phi(x) + \delta \Phi(x) \quad (3.14)$$

induce the following variation of the action:

$$\delta S = \int_{\mathcal{V}} dx \left[ \frac{\partial L}{\partial \Phi} - \partial_\mu \frac{\partial L}{\partial \Phi_\mu} \right] (\delta \Phi - \partial_\nu \Phi \delta x^\nu + G(\sigma_2) - G(\sigma_2)) \quad (3.15)$$

where

$$G(\sigma) = \int_{\sigma} d\sigma_\mu \left[ \frac{\partial L}{\partial \Phi_\mu} \delta \Phi - \delta x^\nu \left( \frac{\partial L}{\partial \Phi_\mu} \partial_\nu \Phi - g_\nu^\mu L \right) \right]. \quad (3.16)$$
In particular, if the variation is a dynamical one, i.e., it is generated by the evolution of the system, then the volume integral in the above expression for $\delta S$ is zero and we can write

$$\delta S = G(\sigma_2) - G(\sigma_1). \quad (3.17)$$

The fundamental postulate of Schwinger’s is to regard $G(\sigma)$ as a generator of the canonical transformation and quantization is achieved by demanding

$$F'(x) - F(x) = i[G(\sigma), F(x)]. \quad (3.18)$$

for some arbitrary function $F$ of the operator $\Phi$ and where $x \in \sigma$. In addition to the equations of motion satisfied by the field operator $\Phi(x)$, the above equation places a constraint upon the field operators in the form of commutation relations. Hence, quantization is achieved by requiring the above equation to be consistent with the field equations. To see the power of this postulate, consider variations for which $\delta x = 0$. Then

$$G(\sigma) = \int_\sigma d\sigma_\mu \frac{\partial L}{\partial \Phi_\mu} \delta \Phi, \quad (3.19)$$

or by defining

$$\Pi_\mu(x) = \frac{\partial L}{\partial \Phi_\mu(x)} \quad (3.20)$$

we can write

$$G(\sigma) = \int_\sigma d\sigma_\mu \Pi_\mu \delta \Phi. \quad (3.21)$$

Hence, it follows from the fundamental postulate that

$$\delta \Phi(x) = i \int_{x' \in \sigma} d\sigma [n_\mu \Pi'(x') \delta \Phi(x'), \Phi(x)], \quad (3.22)$$
where $d\sigma_\mu = d\sigma n_\mu$. Upon using the operator identity

$$[AB, C] = [A, C]B + A[B, C], \quad (3.23)$$

it is easily established that

$$\delta \Phi(x) = i \int_{x' \in \sigma} d\sigma \left\{ [n_\mu \Pi^\mu(x'), \Phi(x)] \delta \Phi(x') + n_\mu \Pi^\mu(x') [\delta \Phi(x'), \Phi(x)] \right\}. \quad (3.24)$$

This equation may be satisfied if we demand that

$$[n_\mu \Pi^\mu(x'), \Phi(x)] = -i \delta_\sigma(x - x') \quad (3.25)$$

and

$$[\delta \Phi(x'), \Phi(x)] = 0, \quad (3.26)$$

where both $x$ and $x'$ belong to the surface $\sigma$ and $\delta_\sigma(x - x')$ is a delta function on $\sigma$. For example, if the surface $\sigma$ is an equal time surface then the surface delta function is just $\delta(x - x')$. Since the variation $\delta \Phi$ is arbitrary, it follows from the last expression that

$$[\Phi(x'), \Phi(x)] = 0. \quad (3.27)$$

Furthermore, putting $F(x) = n_\mu \Pi^\mu(x)$ in the dynamical principle followed by similar manipulations yields

$$[n^\mu \Pi_\mu(x'), n^\nu \Pi_\nu(x)] = 0. \quad (3.28)$$

Eq. (3.25), Eq. (3.27), and Eq. (3.28), are the familiar equal-time commutation relations for a boson field operator $\Phi$ generalized to an arbitrary space like surface.
What about fermion fields? Instead of using the identity, Eq. (3.23), we could have employed

\[ [AB, C] = A\{B, C\} - \{A, C\}B, \]

(3.29)

which would have yielded the familiar anti-commutation relations for the fermion field operators. This is left as a trivial exercise for the reader.

For a Poincare invariant Lagrangian, we saw in the previous chapter that the generator corresponding to the variations

\[ \delta x_\mu = a_\mu + \omega_{\mu\nu} x^\nu \]

(3.30)

and

\[ \delta \Phi(x) = \frac{1}{2} \omega_{\mu\nu} S^{\mu\nu} \Phi(x) \]

(3.31)

is

\[ G(\sigma) = -a_\mu F^\mu + \frac{1}{2} \omega_{\mu\nu} J^{\mu\nu} \]

(3.32)

Then the quantization postulate implies

\[ \frac{1}{2} \omega_{\mu\nu} S^{\mu\nu} F(x) - (a_\mu + \omega_{\mu\nu} x^\nu) \partial_\mu F(x) = -ia_\mu [P^\mu, F(x)] - \frac{i}{2} \omega_{\mu\nu} [J^{\mu\nu}, F(x)], \]

(3.33)

where the fact that \( F'(x) - F(x) = \delta F(x) - \partial_\mu F(x) \delta x^\mu \) was used. Since the above equation must hold for arbitrary infinitesimal \( a_\mu \) and \( \omega_{\mu\nu} \), it follows that

\[ i\partial_\mu F(x) = [F(x), P_\mu] \]

(3.34)

and

\[ -i S^{\mu\nu} F(x) - i(x^\mu \partial^\nu - x^\nu \partial^\mu) F(x) = [F(x), J^{\mu\nu}]. \]

(3.35)
Hence, Schwinger's principle has led directly to the Heisenberg equations of motion for the operator $F(x)$. Eq. (3.34) may be integrated immediately to yield

$$F(x) = e^{iP \cdot (x-x_0)} F(x_0) e^{-iP \cdot (x-x_0)},$$

(3.36)

showing that indeed $P^\mu$ is an operator generating translations in the $x^\mu$ direction.
CHAPTER IV

Closed Time–Path Quantum Field Theory

4.1 Emergence of the Closed Time–Path

In the previous chapters, we have set up the equations of quantum field theory as well as the operator properties of the fields. However, no attempt was made to discuss their solution. The reason for this is simple: except for simple non-interacting fields, we do not know how to obtain exact solutions to the equations. For this reason, various "approximation" schemes have been devised to obtain "approximate" solutions. Typically, these involve equations for the Green functions of the theory. The purpose of this chapter is to set up a formalism based upon the use of Green functions which not only encompasses equilibrium or static situations but allows for a description of very general time-dependent non-equilibrium phenomena as well. The formalism described here was pioneered by Schwinger[6] and is known in the literature[12] as "closed time-path" quantum field theory. In this section, we show how the closed time-path emerges naturally from the very structure of quantum mechanics.
The basic problem of quantum field theory is to solve the *Heisenberg* equations of motion

\[ \frac{d}{dt} Q(t) = i[H, Q(t)]. \] (4.1)

whose solutions are constrained to respect the correct (fermionic or bosonic) operator algebra. For our purposes, it is useful to couple an external \( c \)-number current \( j(x) \) to the field operator \( \Phi(x) \). The resulting equation for the operator \( Q(t) \) in the presence of the source \( j(x) \) is

\[ \frac{d}{dt} Q(t) = i[H - j(t) \cdot \Phi(t), Q(t)]. \] (4.2)

Here and in most of this section we have adopted the convention of suppressing the spatial dependence of \( \Phi \) and we use the shorthand notation \( j(t) \cdot \Phi(t) \) for \( \int d^3x j(t, x) \Phi(t, x) \), suppressing spatial integrations. The source \( j(x) \) is arbitrary except that it is assumed to transform under the Lorentz group in such a way that \( j(x) \Phi(x) \) transforms as a scalar under the group. In addition, we assume that \( j(x) \) has compact support on the space–time manifold.

In the Heisenberg picture, observables are calculated by taking expectation values of the appropriate operators with respect to the initial state of the system. In general, the initial state is a *mixed* state and must be described by a density matrix \( \rho(t_0) \), \( t_0 \) being the initial time. The expectation value of a Heisenberg operator \( Q(t) \) is given by

\[ \langle Q(t) \rangle = \text{Tr}[\rho(t_0)Q(t)]. \] (4.3)
For example, for an equilibrium system at an inverse temperature $\beta$, $\rho(t_0)$ assumes the form

$$\rho_\beta = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}},$$

(4.4)

where the trace is taken over all eigenvectors of $H$.

To facilitate the construction of Green functions, it is convenient to work in a sourceless representation defined by

$$Q'(t) = U(t, t_0)Q(t)U(t, t_0),$$

(4.5)

where the unitary operator

$$U(t_1, t_2) \equiv \begin{cases} \mathcal{T}\exp[i \int_{t_2}^{t_1} dt(j(t) \cdot \Phi'(t))] & \text{for } t_1 \geq t_2 \\ \mathcal{\bar{T}}\exp[i \int_{t_1}^{t_2} dt(j(t) \cdot \Phi'(t))] & \text{for } t_1 \leq t_2 \end{cases}$$

(4.6)

satisfies the usual group property $U(t_1, t_3)U(t_3, t_2) = U(t_1, t_2)$. Here $\mathcal{T}$ is the usual time ordering operator while $\bar{T}$ is an anti-time ordering operator and $\Phi'$ is the field operator in the sourceless representation. It is important to note that for $t > t_0$, $U(t, t_0) = U(t_0, t)$ describes backward time evolution from $t$ to $t_0$. The sourceless representation should not be confused with the more familiar interaction picture representation which is commonly used for deriving Feynman rules. As will be illustrated later, the sourceless representation is ideally suited for deriving Schwinger–Dyson equations; these are non-perturbative equations satisfied by the Green functions whereas the Feynman rules are used in connection with perturbation theory.
The transformed operator $Q'(t)$ satisfies the *sourceless* field equation

$$\frac{d}{dt} Q'(t) = i[H', Q'(t)],$$  \hspace{1cm} (4.7)

where the transformed Hamiltonian $H' = U(t, t_0)H\Gamma(t, t_0)$ is time independent by virtue of $[H', H'] = 0$. To show this, we note that for $t \geq t_0$, $U(t, t_0)$ satisfies

$$\frac{\partial U(t, t_0)}{\partial t} = iJ(t) \cdot \Phi'(t)U(t, t_0)$$  \hspace{1cm} (4.8)

with a similar equation for $U\Gamma(t, t_0)$. From the above equation, it immediately follows that

$$i\frac{dQ'(t)}{dt} = i\frac{\partial U(t, t_0)}{\partial t} Q(t)U\Gamma(t, t_0)$$

$$+ iU(t, t_0)\frac{dQ'(t)}{dt} U\Gamma(t, t_0) + iU(t, t_0)Q(t)\frac{\partial U\Gamma(t, t_0)}{\partial t}$$

$$= U(t, t_0)[Q(t), H]U\Gamma(t, t_0)$$

$$= [Q'(t), H'].$$  \hspace{1cm} (4.9)

The correlation function, $\langle \Phi(t_1)\Phi(t_2) \rangle$, may be expressed in the sourceless representation as

$$\langle \Phi(t_1)\Phi(t_2) \rangle = \langle U\Gamma(t_1, t_0)\Phi'(t_1)U(t_1, t_2)\Phi'(t_2)U(t_2, t_0) \rangle,$$  \hspace{1cm} (4.10)

or by functional differentiation

$$\langle \Phi(t_1)\Phi(t_2) \rangle = \langle \frac{\delta U\Gamma(\infty, t_0)}{i\delta j(t_1)} \frac{\delta U(\infty, t_0)}{i\delta j(t_2)} \rangle.$$  \hspace{1cm} (4.11)
Using a trick originally due to Schwinger[6], we write

\[
\langle \Phi(t_1)\Phi(t_2) \rangle = \left. \frac{1}{i^2} \frac{\delta^2}{\delta j_-(t_1)\delta j_+(t_2)} \langle [\hat{T} \exp(i \int_{t_0}^{\infty} -j_\Phi')] [T \exp(i \int_{t_0}^{\infty} j_\Phi')] \rangle \right|_{j_+=j_-=j},
\]

(4.12)

where we have introduced two sources, \( j_+ \) and \(-j_-\), allowing the functional derivatives to be pulled out of the expectation value. We have explicitly indicated here that the sources \( j_+ \) and \( j_- \) are to be identified with the physical source \( j \) after functional differentiation. This expression may be further simplified by introducing a time contour, \( C \), which runs from \( t = t_0 \) to \( t = +\infty \) and back to \( t = t_0 \)[12, 13]. This contour, shown in Fig. 4.1, is known in the literature as the closed time-path[12]. In addition we define a contour source, \( j_C \), which takes on the value \( j_+(t) \) or \(-j_-(t)\) for \( t \) on the forward or backward branch of \( C \), respectively. Ordering along the contour will be denoted by \( T_C \). With these definitions we may write

\[
\langle \Phi(t_1)\Phi(t_2) \rangle = \frac{1}{i^2} \frac{\delta^2}{\delta j_-(t_1)\delta j_+(t_2)} Z[j_C],
\]

(4.13)

where

\[
Z[j_C] = \langle T_C \exp(i \int_C j_C \Phi') \rangle
\]

(4.14)

is a contour generating functional allowing for the construction of contour Green functions.
Figure 4.1: Closed Time-Path Contour
Contour Green functions are defined as expectation values of contour-ordered products of the field operators which may be obtained through functional differentiation of the generating functional, e.g.,

$$i\Delta_c(x, y) \equiv \langle T_c \Phi(x) \Phi(y) \rangle = \frac{1}{i^2} \frac{1}{Z[jc]} \frac{\delta^2}{\delta j_c(x) \delta j_c(y)} Z[jc],$$   \hspace{1cm} (4.15)

with higher order Green functions defined analogously. Here, functional differentiation has been extended to the contour by writing

$$\frac{\delta j_c(y)}{\delta j_c(x)} = \delta_c(x - y),$$   \hspace{1cm} (4.16)

where the contour delta function is a delta distribution on the contour.

We mention that it is possible to write a path integral representation of the contour generating functional. Indeed, Calzetta and Hu[14] have formulated closed time path QFT in this manner and derived a Boltzmann equation from a two-loop approximation to the path integral. Of course, collision terms do not occur at the one-loop level, so one is forced to go to at least two loops to see collisions. However, there is no justification for the use of the loop expansion in strongly coupled theories[15], so we do not take this approach here.

Given the field equations for $\Phi$, we can derive a Dyson equation for the contour Green function, $\Delta_c$, using functional techniques which are made particularly simple in the sourceless representation. This technique will be illustrated later when we consider the Yukawa model. Until then we will content ourselves with the remark that quantum field theory on the closed time-path contour is formally identical
to ordinary "vacuum" quantum field theory. The only new feature present in the contour Dyson equations is, as the reader may have guessed, that time integrations are replaced by contour integrations. Thus a prototypical Dyson equation is of the form

$$\Delta_c(x, y) = \Delta_c^0(x, y) + \int_c dz \int_c dz' \Delta_c^0(x, z) \Pi_c(z, z') \Delta_c(z', y). \quad (4.17)$$

Before leaving this section we show what role the backward flowing part of the contour plays in the ordinary vacuum theory. The vacuum theory results when we take the density matrix to be $\rho(t_0) = |0\rangle \langle 0|$. Here, $|0\rangle$ is the exact vacuum of the system at time $t_0$ and we will take $t_0 = -\infty$. Furthermore, in accordance with the usual postulates of ordinary quantum field theory, we make the assumptions that the vacuum, $|0\rangle$, is non-degenerate and the source, $j(x)$, is adiabatically switched on and off in the past and future, respectively. With these two assumptions, the conditions for the quantum mechanical adiabatic theorem[2] are fulfilled. Concentrating on the backward time piece, we may write

$$\langle 0|U(-\infty, t) = \sum_n \langle 0|U(-\infty, \infty)|n\rangle \langle n|U(\infty, t)$$

$$= \langle 0|U(-\infty, \infty)|0\rangle \langle 0|U(\infty, t), \quad (4.18)$$

where we have introduced a complete set of states, $|n\rangle$, and exploited the group property of the evolution operator, $U$. The second equality follows because the adiabatic theorem guarantees that only the vacuum state, $|0\rangle$, contributes to the sum since an adiabatic perturbation cannot induce transitions between different
states. In addition, the adiabatic theorem implies that \( \langle 0|U(-\infty, \infty)|0 \rangle \) is just a pure phase factor and so by complex conjugation we can write

\[
(0|U(-\infty, t) = \frac{\langle 0|U(\infty, t)\rangle}{\langle 0|U(\infty, -\infty)|0 \rangle}. \tag{4.19}
\]

In this form, we see that the backward flowing part of the contour manifests itself in the vacuum theory by canceling disconnected vacuum to vacuum graphs!

At this point, the reader should be convinced that the structure of non-equilibrium quantum field theory within the closed time-path framework is formally identical to the ordinary vacuum theory. This concludes our discussion of closed time-path quantum field theory.

### 4.2 Example: The Yukawa Model

To illustrate the ideas presented in the previous section, we will derive the contour Dyson equations for the Yukawa model. Later, we shall use the equations derived in this section to generate a transport equation satisfied by a system of fermions interacting via boson exchange.

The model we will concentrate on is composed of spin-half Dirac fermions interacting with a massive scalar boson field via a Yukawa coupling. The Lagrangian density for this system is

\[
\mathcal{L}(x) = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) - \frac{1}{2} \Phi(x)(\Box + m_\Phi^2)\Phi(x) + g\bar{\Psi}(x)\Phi(x)\Psi(x) + j(x)\Phi(x) + \bar{\zeta}(x)\Psi(x) + \bar{\zeta}(x)\zeta(x). \tag{4.20}
\]
In anticipation of transforming to the sourceless representation, we have coupled
the fields to the external sources \( j(x), \zeta(x), \) and \( \bar{\zeta}(x). \) Here \( j(x) \) is an ordi-
nary \( c \)-number function while \( \zeta(x) \) and \( \bar{\zeta}(x) \) are Grassmann functions (i.e., anti-
commuting \( c \)-number functions). The Euler–Lagrange equations for the above
Lagrangian yield
\[
(\Box + m^2)\Phi(x) - g\bar{\Psi}(x)\Psi(x) = j(x)
\]
(4.21)
for the boson field and
\[
(i\gamma_{\mu}\partial^{\mu} - m)\Psi(x) + g\Phi(x)\Psi(x) = -\zeta(x)
\]
(4.22)
for the fermion field. There is an analogous equation for the adjoint field \( \bar{\Psi}(x). \) The
algebra of the field operators is given by the usual equal-time (anti-)commutation
relations,
\[
\left\{ \Phi_{\alpha}(t, x), \bar{\Psi}_{\beta}(t, y) \right\} = \gamma^{0}_{\alpha\beta}\delta^{3}(x - y)
\]
(4.23)
and
\[
\left[ \frac{\partial}{\partial t} \Phi(t, x), \Phi(t, y) \right] = -i\delta^{3}(x - y),
\]
(4.24)
with all other equal-time (anti-)commutators vanishing.

The first step towards our goal of deriving the Dyson equations consists of
taking expectation values of the field equations and making the replacements
\[
\langle \Phi(x) \rangle = \frac{1}{iZc} \frac{\delta}{\delta jc(x)} Z[jc, \zeta, \bar{\zeta}]
\]
(4.25)
and
\[
\langle \Psi(x) \rangle = \frac{1}{iZ_c} \frac{\delta}{\delta \zeta_c(x)} Z[j_c, \zeta_c, \bar{\zeta}_c].
\] (4.26)

The contour generating functional is defined as
\[
Z_c \equiv Z[j_c, \zeta_c, \bar{\zeta}_c] \equiv \langle T_c \exp[i \int_c (j_c \bar{\Phi} + \bar{\Psi}' \zeta_c + \bar{\zeta}_c \Psi')] \rangle,
\] (4.27)

where \( \bar{\Phi}' \), \( \Psi' \), and \( \Phi' \) are the field operators expressed in the sourceless representation introduced earlier. As a result we arrive at a set of functional differential equations on the closed-time path, i.e.,
\[
[(i \gamma_\mu \partial^\mu - m)_{c} + g \frac{i}{i^2} \frac{\delta^2 \zeta_c(x)}{\delta \zeta_c(x) \delta \bar{\zeta}_c(x)}] Z[j_c, \zeta_c, \bar{\zeta}_c] = 0,
\] (4.28)

and
\[
[(\Box + m^2)_{c} + g \frac{i}{i^2} \frac{\delta^2 \zeta_c(x)}{\delta \zeta_c(x) \delta \bar{\zeta}_c(x)} - j_c(x)] Z[j_c, \zeta_c, \bar{\zeta}_c] = 0.
\] (4.29)

We define the two-point contour fermion Green function as
\[
iG_c(x, y) \equiv \langle T_c \Psi(x) \bar{\Psi}(y) \rangle = \frac{1}{Z_c} \frac{\delta^2 \zeta_c(x)}{\delta \zeta_c(x) \delta \bar{\zeta}_c(y)} Z_c.
\] (4.30)

Similarly, we define the boson Green function as
\[
i\Delta_c(x, y) \equiv \langle T_c \Phi(x) \Phi(y) \rangle \equiv i \Delta_c(x, y) + \phi(x) \phi(y) = \frac{1}{i Z_c} \frac{\delta^2 Z_c}{\delta j_c(x) \delta j_c(y)},
\] (4.31)

explicitly indicating the presence of the mean field
\[
\phi(x) \equiv \langle \Phi(x) \rangle = \frac{1}{i Z_c} \frac{\delta}{\delta j_c(x)} Z_c.
\] (4.32)
In addition, due to complications introduced by the background field, $\phi$, we have introduced a connected contour Green function $\Delta_c$ for which the Dyson equations simplify. Also, we should emphasize that all contour Green functions defined up to this point depend on the external sources although this fact has not been explicitly indicated.

By rewriting Eq. (4.29) using the definitions (4.30) and (4.32) we obtain

$$\nabla^2 + m^2 \phi_c(x) + ig \text{tr} G_c(x, x) = 0 \quad (4.33)$$

for the mean field, $\phi$, in the absence of an external source. Upon functional differentiation of equations (4.28) and (4.29) with respect to $\zeta_c(y)$ and $j_c(y)$, respectively, and using definitions (4.30)-(4.32), we obtain

$$\left( i\gamma_\mu \partial_\mu - m + g\phi \right) G_c(x, y) - ig \frac{\delta G_c(x, y)}{\delta j_c(x)} = \delta_c(x - y) \quad (4.34)$$

for the fermion Green function and

$$\left( \nabla^2 + m^2 \right) \Delta_c(x, y) - ig \text{tr} \frac{\delta G_c(x, x)}{\delta j_c(y)} = -\delta_c(x - y) \quad (4.35)$$

for the connected boson Green function. We note that in obtaining these equations, we have set the external sources to zero after functional differentiation and assume that $\langle \Psi(x) \rangle = 0$ in their absence. These equations may be put in a more familiar form by defining a contour self-energy

$$\Sigma_c(x, y) = g \int_c dz \, dz' G_c(x, z) \Gamma_c(z, y; z') \Delta_c(z', x) \quad (4.36)$$
for the fermion field and a contour polarization insertion

\[ \Pi_C(x, y) \equiv -g \int_C \; dz \; dz' \; \text{tr} \; G_C(x, z) \Gamma_C(z, z'; y) G_C(z', x) \]  

(4.37)

for the boson field. Here the three-point contour vertex function is defined as

\[ \Gamma_C(x, y; z) \equiv i \frac{\delta G^{-1}_C(x, y)}{\delta \phi_c(z)}, \]  

(4.38)

where the inverse Green function, \( G^{-1}_C(x, y) \), is regarded as a functional of the background field \( \phi \). In terms of these functions, we may write the functional equations (4.34) and (4.35) as

\[ (i \gamma_{\mu} \partial^\mu - m + g \phi) c G_C(x, y) - \int_C \; dz \; \Sigma_c(x, z) G_C(z, y) = \delta_C(x - y), \]  

(4.39)

\[ (\Box + m^2) \Delta_c(x, y) + \int_C \; dz \; \Pi_c(x, z) \Delta_c(z, y) = -\delta_c(x - y), \]  

(4.40)

which are a set of coupled Dyson equations for the contour Green functions. A schematic representation of these equations is shown in figures 4.2 and 4.3. Although it looks as though we have a closed set of equations for the Green functions, we do not because the vertex function hides all of our ignorance. The vertex function can be found in perturbation theory from its definition, Eq. (4.38). Later, we shall approximate the vertex by the bare vertex— the lowest order term in the perturbation expansion. We find

\[ \Gamma_c(x, y; z) = ig \delta_c(x - y) \delta_c(x - z) + O(g^2). \]  

(4.41)

It should be emphasized that there is no justification for such an approximation, particularly in a strongly interacting field theory. Any reasonable discussion of
this point at this time would lead us far astray. Later, we will come back to this point and try to determine if such an approximation is valid.

4.3 Green Functions

Although the contour functions introduced in the last section are particularly suited for formal manipulations, they are not convenient for practical calculations. Instead, we find it useful to use a matrix representation of the contour functions whose matrix elements are ordinary Green functions. In this section, we will see a proliferation of Green functions which might seem intimidating at first to the reader. However, simple algebraic relations among the various two-point functions give rise to a few simple rules which tremendously simplify calculations on the closed time-path.

Contour Green functions have a natural matrix representation defined by

\[ [i\Delta_c(x, y)]_{a,b} = \frac{1}{i^2} \frac{1}{Z[jc]} \frac{\delta^2}{\delta j_a(x) \delta j_b(y)} Z[jc] \quad a, b \in \{+, -\}. \] (4.42)

With the definitions

\[ i\Delta_F(x, y) \equiv \langle T \Phi(x) \Phi(y) \rangle, \] (4.43)
\[ i\Delta_F(x, y) \equiv \langle \Phi(x) \Phi(y) \rangle, \] (4.44)
\[ i\Delta^>(x, y) \equiv \langle \Phi(x) \Phi(y) \rangle, \] (4.45)
\[ i\Delta^<(x, y) \equiv \langle \Phi(y) \Phi(x) \rangle. \] (4.46)
Figure 4.2: Dyson Equation for Fermion Green Function
Figure 4.3: Dyson Equation for Boson Green Function.
we can write

$$\Delta^c(x, y) = \begin{pmatrix} \Delta_F(x, y) & \Delta^<(x, y) \\ \Delta^>(x, y) & \Delta_F(x, y) \end{pmatrix}. \tag{4.47}$$

It is easy to see why the matrix elements are as they are without any calculations. For example, the 1, 1 or +, + component corresponds to both $x$ and $y$ on the + or positive flowing part of the contour. On this section of the contour, the contour ordering operator reduces to the ordinary time ordering operator and hence it picks out the Feynman functions. Similar considerations apply to the other matrix elements.

Since any contour two-point function, $A_C$, will have a matrix representation, we define a canonical form for $A_C$ as

$$A_C(x, y) = \begin{pmatrix} A_F(x, y) & A^<(x, y) \\ A^>(x, y) & A_F(x, y) \end{pmatrix}. \tag{4.48}$$

It is important to realize that in the matrix representation, the matrix elements of the contour functions are defined on the real axis and not the contour.

We also find it convenient to define retarded and advanced two-point functions as

$$A_R(x, y) \equiv \theta(x_0 - y_0) [A^>(x, y) - A^<(x, y)] \tag{4.49}$$

and

$$A_A(x, y) \equiv -\theta(y_0 - x_0) [A^>(x, y) - A^<(x, y)], \tag{4.50}$$

respectively. It is important to note that the definitions of $A_R$ and $A_A$, Eqs. (4.49) and (4.50), hold for fermions as well as bosons. We also assume that the Feynman
functions, $A_F,F$ have the decomposition

$$A_F,F(x,y) = A^{>\langle}(x,y)\theta(x_0-y_0) + A^{\langle>}(x,y)\theta(y_0-x_0), \quad (4.51)$$

indicating that the elements of $A_C$, Eq. (4.48), are not independent. Although the two-point Green functions can always be decomposed in this way, the self-energy cannot if it includes a mean field contribution, since the mean field contributes a piece proportional to $\delta(x-y)$. Such a contribution only has support for $x_0 = y_0$ whereas Eq. (4.51) is undefined at $x_0 = y_0$. Thus we will always separate out the mean field from the self-energy by absorbing it into the free propagator. This will be illustrated in more detail later. For an alternative procedure, the reader is referred to [16].

By considering the contraction, $D_C$, of two contour functions, $A_C$ and $B_C$, i.e.,

$$D_C(x,y) = \int_C dz A_C(x,z)B_C(z,y), \quad (4.52)$$

we can derive a convenient set of rules, the so-called Langreth-Wilkins rules[7], for continuing from the contour to the real time axis. Using the canonical matrix form for contour two-point functions, Eq. (4.48), we find in the limit $t_0 \to -\infty$,

$$D^{>\langle}(x,y) = \int dz[A^{>\langle}(x,z)B_A(z,y) + A_R(x,z)B^{>\langle}(z,y)] \quad (4.53)$$

and

$$D_{R,A}(x,y) = \int dz A_{R,A}(x,z)B_{R,A}(z,y). \quad (4.54)$$
These are the first two Langreth–Wilkins rules. The other two rules are useful for reducing \textit{anti-parallel} products of contour functions

\begin{equation}
 iD_c(x, y) = iA_c(x, y) iB_c(y, x). \tag{4.55}
\end{equation}

Indeed, as the reader can readily show,

\begin{equation}
 iD^>\cdot< (x, y) = iA^>\cdot< (x, y) iB^<\cdot> (y, x) \tag{4.56}
\end{equation}

and

\begin{equation}
 D_{R,A}(x, y) = A_{R,A}(x, y) iB^<(y, x) + iA^<(x, y) B_{A,R}(y, x). \tag{4.57}
\end{equation}

The anti-parallel Langreth–Wilkins rules are particularly useful for continuing contour self-energies to the real axis. Here we mention that these rules are valid only in the limit as $t_0$ goes to $-\infty$. This limit corresponds to the neglect of initial correlations. We will see later in a simple model that the initial correlations for the general time dependent problem damp out on a time scale given by roughly $1/m$ where $m$ is the mass of the lightest boson field in the problem. For a pion mass of about 140MeV, this corresponds to roughly $10^{-23}$ seconds, i.e., the strong interaction time scale. These considerations do not apply to massless fields due to the long range nature of the Coulomb interaction. However, for most plasmas where photons are produced, the Coulomb interaction is screened and we anticipate that the time scale is on the order of the Debye screening length.
CHAPTER V

Green Functions: General Properties

5.1 Wigner Transform and Gradient Expansion

5.1.1 Definition

For translationally invariant systems, calculations are often simplified when working in a Fourier transform representation, i.e., in momentum space. However, for the general non-equilibrium situation, the two-point functions depend on the difference coordinate \( r = x - y \) as well as the sum coordinate \( R = \frac{1}{2}(x + y) \). The Wigner transform is a generalization of the Fourier transform and is realized by Fourier transforming over the difference coordinate \( r \). Hence, we define the Wigner transform of a generic two-point function, \( A(x, y) \), as

\[
A(R, p) \equiv \int d^nre^{ip\cdot r/\hbar}A(R + \frac{r}{2}, R - \frac{r}{2}), \quad (5.1)
\]

with the inverse transform

\[
A(x, y) = \int \frac{d^n p}{(2\pi \hbar)^n} e^{-ip\cdot r/\hbar}A(R, p), \quad (5.2)
\]
where $n$ is the number of dimensions. Obviously, the Wigner transform reduces to the Fourier transform for translationally invariant systems.

Because of its close relationship with the Fourier transform, the Wigner transform enjoys similar properties. In particular, one can show that

$$
\int d^n z \ A(x, z)B(z, y) \rightarrow \exp\left[\frac{\imath \hbar}{2} (\partial_R^A \cdot \partial_R^B - \partial_R^A \cdot \partial_R^B)\right] A(R, p)B(R, p)
$$

and

$$
A(x, y)B(y, x) \rightarrow \int \frac{d^n q}{(2\pi\hbar)^n} A(R, q)B(R, q - p),
$$

which are useful in transforming products of two point functions. In addition, one often needs the equality

$$
\int d^n x \ d^n y A(x, y)B(y, x) = \int \frac{d^n R d^n p}{(2\pi\hbar)^n} A(R, p)B(R, p).
$$

Finally, we note that if the functions $A(x, y)$ and $B(x, y)$ have no matrix structure, the transform of the commutator of $A$ and $B$ is

$$
\int dz \ [A(x, z)B(z, y) - B(x, z)A(z, y)] \rightarrow \imath \hbar [A, B]_{PB}(R, p) + O(\hbar^3),
$$

where the generalized Poisson bracket is defined

$$
[A, B]_{PB} = [\partial_p A \cdot \partial_R B - \partial_R A \cdot \partial_p B](R, p).
$$

If the two-point functions have a matrix structure, then there are extra terms which involve commutators of the functions. Usually when this situation arises, e.g., for spin systems, one expands the two-point functions in a matrix basis and
deals directly with the scalar coefficients. We shall see an example of this technique when we derive a transport equation for a Yukawa model of spin one-half fermions interacting with spin zero bosons.

### 5.1.2 Gradient Expansion

For the above expansion to be meaningful, the variation with respect to the sum coordinate, $R$, must be small on a scale set by the variations of the relative coordinate $r$. If this condition is fulfilled then the resulting expansion is known as a gradient expansion in derivatives with respect to $R$.

It is also obvious that the gradient expansion is in some sense an expansion in powers of $\hbar$. This is because the gradient expansion is valid if $\Delta R \Delta p \gg \hbar$ where $\Delta R$ is on the order of the scale set by a typical variation with respect to $R$ and $\Delta p \sim \hbar/\Delta r$ sets the scale at which the function varies with $p$. For this reason, it is also a semi-classical expansion. Hence, one expects such an approximation to hold only for systems which are close to equilibrium.

### 5.1.3 Examples

As hinted above, Wigner transforms are ideal for semi-classical expansions. Later we will use the Wigner transform to make a semi-classical expansion of Schwinger–Dyson equations resulting in a Boltzmann transport equation. For the moment, we confine ourselves to some simple examples from elementary quantum mechanics which illustrate this remarkable property.
Consider the Schrodinger equation familiar from ordinary quantum mechanics

\[ H \psi(t) = i\hbar \psi(t), \]  

which we write in the position representation

\[ \int dy \langle x|H|y \rangle \langle y|\psi(t) \rangle = i\hbar \frac{\partial}{\partial t} \langle x|\psi(t) \rangle. \]  

Defining the wavefunction

\[ \psi(x, t) = \langle x|\psi(t) \rangle, \]

and Hamiltonian

\[ H(x, y) = \langle x|H|y \rangle, \]

we can write

\[ \int dy H(x, y) \psi(y, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t). \]

Expectation values of an arbitrary operator \( A \) may be written

\[ \langle A \rangle = \langle \psi(t)|A|\psi(t) \rangle \]

\[ = \int dx dy \psi^*(x, t) A(x, y) \psi(x, t) \]

\[ = \int dx dy A(x, y) \rho(y, x; t) \equiv \text{Tr} A \rho(t), \]  

where we have defined

\[ \rho(y, x; t) = \psi^*(x, t) \psi(y, t). \]

Now consider the same equations when expressed in the Wigner representation.

The average of some observable \( A(x, y) \) is

\[ \int dx dy A(x, y) \rho(y, x; t) = \int \frac{dR dp}{2\pi \hbar} A(R, p) \rho(R, p; t), \]
which may be interpreted as the familiar “ensemble average” from statistical mechanics. This interpretation requires that $\rho(R, p; t)$ be regarded as some sort of a generalized phase space density. Indeed, one may show that $\rho(x, y; t)$ satisfies

$$\int dz \left[ H(x, z) \rho(z, y) - \rho(x, z) H(z, y) \right] = i\hbar \frac{\partial}{\partial t} \rho(x, y; t),$$

which is just the continuity equation expressed in the coordinate representation.

Wigner transformation followed by a gradient expansion results in

$$\frac{\partial}{\partial t} \rho(R, p; t) + [H(R, p), \rho(R, p; t)]_{PB} + O(\hbar^2) = 0$$

which, in the classical limit, is just the familiar Liouville equation for a classical distribution function $\rho(R, p; t)$ defined in the classical phase space. Hence, it appears as though the description of a single quantum mechanical particle is equivalent to the description an infinite number of classical particles. Indeed this equivalence is embodied in the Feynman path integral formulation of quantum mechanics where one calculates transition amplitudes by summing over an infinite number of classical trajectories. Further discussion of the path integral is outside the scope of this work.

We now consider a few concrete examples.

**Free Particle:** A free non-relativistic quantum mechanical particle has a Hamiltonian $H = \frac{p^2}{2m}$ which when expressed in the Wigner representation becomes

$$H(R, p) = \frac{p^2}{2m}.$$
The wave functions consist of plane waves which in the x-representation are given by \( e^{ipx} \). The corresponding phase space density is easily seen to be

\[
\rho(R, p; t) = 2\pi \hbar \delta(p)
\]  

which represents a "smeared out" point in phase space of "area" \( 2\pi\hbar \). We see that unlike a classical particle with momentum \( p \) which can be localized at a given \( R \), the quantum mechanical particle is spread over all \( R \). This is just a manifestation of the uncertainty principle \( \Delta R \Delta p \geq \hbar \).

**Harmonic Oscillator.** As the reader can easily show, the Hamiltonian for the harmonic oscillator expressed in the Wigner representation is just

\[
H(R, p) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 R^2.
\]  

The stationary state wave functions are

\[
\psi_n(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{\alpha}x) e^{-\frac{\alpha x^2}{2}}
\]  

where \( \alpha = m\omega/\hbar \) and \( H_n(x) \) are the well known Hermite polynomials; in particular \( H_0(x) = 1 \) and \( H_1(x) = x \). The corresponding phase space densities for the ground state and first excited state are given by

\[
\rho_0(R, p) = \sqrt{2} e^{-\frac{R^2}{2\hbar \omega}} H(R, p)
\]  

and

\[
\rho_1(R, p) = \left[ \frac{H(R, p)}{\hbar \omega} - \frac{1}{4} \right] \rho_0(R, p),
\]  

\[
\rho_1(R, p) = [H(R, p) - \hbar \omega] \rho_0(R, p).
\]
respectively. We note that the ground state may be described by an ensemble of classical oscillators distributed in phase space with a temperature $kT = \frac{1}{2} \hbar \omega$. The distribution function for the first excited state illustrates another important feature of generalized phase space densities; namely, they are not always positive definite. This feature distinguishes the classical and quantum distribution functions and much of the richness of quantum mechanics stems from this fact. For example, quantum mechanical interference phenomena are manifestations of the non-positive definite character of the quantum distribution functions since it is impossible to describe the wave nature of quantum systems solely in terms of classical distribution functions.

5.2 Analytic Properties

The Green functions defined above have a number of interesting analytic properties which often simplify calculations. These properties may be exploited to remove infinities from quantum field theory calculations as well as removing unwanted non-physical behavior introduced through bad approximations to the equations satisfied by the Green functions.

For translationally invariant systems, the Wigner transform depends only upon the Lorentz scalar $p^2$. However, for the general problem, the Wigner transform may depend upon all 4 components of $p_\mu$. In the following, we consider the continuation of $p_0$ into the complex plane. We first show that only the retarded and advanced
functions may be continued into the complex energy plane. To this end, consider
the transform of $A_R(t) = \theta(t)A(t)$:

$$A_R(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t}\theta(t)A(t)$$
$$= \int_{0}^{\infty} dt\ e^{i\omega t}A(t). \quad (5.24)$$

Analytically continuing $\omega$ to complex values, we can write

$$A_R(\Re \omega + i \Im \omega) = \int_{0}^{\infty} dt\ \exp(i\Re\omega t - \Im\omega t)A(t). \quad (5.25)$$

Hence we see that the retarded expression can only be continued into the upper
half complex plane of $\omega$. The reason for this is that the above expression fails to
exist for $\Im\omega < 0$; this fact is entirely due to the presence of the step function
in the definition of $A_R(t)$. Similar considerations show that the transform of the
advanced function $A_A(t) = -\theta(-t)A(t)$ has an analytic continuation into the lower
half plane of $\omega$. However, the Feynman and anti-Feynman functions have no
analytic continuation off the real axis since they are the sum of an advanced and
a retarded function.

Since the retarded function is analytic in the upper half complex energy plane,
it follows that one may write a dispersion relation between its real and imaginary
parts. Consider some point $z$ in the upper half complex plane. Using Cauchy’s
integral theorem, we can write

$$A_R(z) = \int_{\gamma} \frac{dz'}{2\pi i} \frac{A_R(z')}{z' - z}, \quad (5.26)$$
where $\gamma$ is some path in the upper half plane enclosing $z$. It is possible to deform the path $\gamma$ into a large semi-circle in the upper half plane with its diameter along the real axis and with the radius tending to infinity. However, one must be careful here. In general $A_R(z)$ is a tempered distribution on the boundary of its domain of analyticity. This implies that $A_R(z)$ may have a polynomial growth as $|z| \to \infty$, which means that one cannot naively take the radius to infinity. However, for the case of a polynomial growth of order $n$, we can perform $n$ subtractions of $A_R(z)$ and then take the infinite radius limit. Thus, assuming this is the case, we write

$$A_R(z) = \sum_{k=0}^{n-1} \frac{A_R^{(k)}(z_0)}{k!}(z-z_0)^k + \tilde{A}_R(z), \quad (5.27)$$

where $\tilde{A}_R(z)$ will tend at most to a constant as $|z| \to \infty$. The point $z_0$ is known as the subtraction point. Hence we can write

$$A_R(z) = \sum_{k=0}^{n-1} \frac{A_R^{(k)}(z_0)}{k!}(z-z_0)^k + \int_\gamma \frac{dz'}{2\pi i} \frac{\tilde{A}_R(z')}{z'-z}, \quad (5.28)$$

which has a well defined limit as the radius of the semi-circle $\gamma$ tends to infinity. In the rest of this section, we assume enough subtractions have been made to render the above expression finite in this limit and with this understanding we will simply write

$$A_R(z) = \int_\gamma \frac{dz'}{2\pi i} \frac{A_R(z')}{z'-z}. \quad (5.29)$$

Alternatively, we could just focus attention on $\tilde{A}_R(z)$.

In the limit of an infinite radius, the contribution from the circular arc vanishes
and we are left with

$$A_R(z) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{A_R(\omega')}{\omega' - z},$$

(5.30)

or

$$A_R(\omega + i\epsilon) = -\int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{A_R(\omega')}{\omega - \omega' + i\epsilon}.$$

(5.31)

Using the identity

$$\frac{1}{\omega - \omega' + i\epsilon} = \mathcal{P} \frac{1}{\omega - \omega'} - i\pi \delta(\omega - \omega'),$$

(5.32)

valid for infinitesimal $\epsilon$ we can write

$$A_R(\omega) = -\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi i} \frac{A_R(\omega')}{\omega - \omega'} + \frac{1}{2} A_R(\omega).$$

(5.33)

Taking the real part of the above expression yields

$$\text{Re} \ A_R(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im} A_R(\omega')}{\omega - \omega'},$$

(5.34)

or

$$A_R(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im} A_R(\omega')}{\omega - \omega' + i\epsilon}.$$

(5.35)

The above dispersion relation holds for any retarded function and is a consequence of the retarded function being analytic in the upper half plane.

Similar manipulations show that the advanced function satisfies

$$A_A(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im} A_A(\omega')}{\omega - \omega' - i\epsilon},$$

(5.36)

which indicates that the retarded and advanced functions are complex conjugates of one another.
From the dispersion relations, it follows that the retarded functions are completely determined once their imaginary parts are known. This is a consequence of analyticity. This fact can be used to simplify certain calculations for the Green functions. The advantage of requiring only knowledge of the imaginary parts of Green functions is that the imaginary parts are directly related to observable quantities. In fact, the earliest use of dispersion relations in physics was by Kramers and Kronig.

5.3 Sum Rules

In the previous section, we studied the analytic properties of a generic retarded function. In this section, we concentrate on a particular retarded two-point function, namely the retarded propagator. We will see that the commutation relations for the field operators give rise to sum rules.

Consider the two correlation functions $G_{AB}^>(x, y)$ defined by

$$G_{AB}^>(x, y) = \langle A(x)B(y) \rangle$$ (5.37)

and

$$G_{AB}^<(x, y) = \pm \langle B(x)A(y) \rangle,$$ (5.38)

where the "plus" sign is for bosons and the "minus" sign is for fermions. The "minus" sign arises for fermions because the fermion field operators anti-commute
under the contour ordering operator. The retarded function is defined as

\[ G_R(x, y) = \theta(x_0 - y_0)[G^{\gamma}_{AB}(x, y) - G^\gamma_{AB}(x, y)] \]

\[ = \theta(x_0 - y_0)[A(x), B(y)]_\pm \]

\[ = \theta(x_0 - y_0)G_{AB}(x, y), \quad (5.39) \]

where the spectral function \( G_{AB}(x, y) \) has been defined

\[ G_{AB}(x, y) = ([A(x), B(y)]_\pm). \quad (5.40) \]

In most literature on relativistic quantum field theory, \( G_{AB}(x, y) \) is known as the "Pauli–Jordan" function. However, the terminology used in this work is that of many-body theory, and "spectral function" is a more descriptive term in any case.

From the above equation, it follows from Wigner transforming over the time variable that

\[ G_{AB}(p_0, R_0; x, y) = \int_{-\infty}^{\infty} dr_0 e^{ip_0 r_0} ([A(x), B(y)]_\pm), \quad (5.41) \]

where \( R \) and \( r \) are the sum and difference variables defined earlier. Integrating the above expression over all \( p_0 \) and noting that

\[ \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} = \delta(x), \quad (5.42) \]

we find

\[ \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} G_{AB}(p_0, R_0; x, y) = \int_{-\infty}^{\infty} dr_0 \delta(r_0) ([A(x), B(y)]_\pm) \]

\[ = ([A(x_0, x), B(x_0, y)]_\pm). \quad (5.43) \]
The right hand side of the above equation is just the equal time commutator of the field operators; hence, the right hand side is a known quantity. To be more specific, it is assumed that the commutator has the form

$$[A(x_0,x), B(x_0,y)]_{\pm} = D_{AB}\delta(x-y),$$  \hspace{1cm} (5.44)

where the matrix $D$ depends upon the nature of the field operators. For example, if $A = B = \Phi$ represents a scalar field, then $D = 0$. Inserting this in the sum rule derived above and completing the Wigner transform over the spatial variables yields the final result

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} G_{AB}(p,R) = D_{AB}. \hspace{1cm} (5.45)$$

The above sum rule is an exact result depending only upon the commutation relations of the operators $A$ and $B$. It is rather amazing that it is independent of the detailed dynamics of the system, i.e., upon whether or not the system is in or out of equilibrium. By considering other commutators in the manner presented above, one can derive other sum rules; this is left to the reader.

Before leaving this section, we will show that the spectral function $G(R,p)$ is just the imaginary part of the retarded Green function. It follows from the definition that

$$G(R,p) = G^>(R,p) - G^<(R,p)$$

$$= G_R(R,p) - G_A(R,p)$$

$$= 2i \text{ Im} G_R(R,p), \hspace{1cm} (5.46)$$
where in the last line we used the fact that the retarded and advanced functions are complex conjugates of one another.

We now show that the sum rules are related to the asymptotic behavior of the theory. Here it is assumed that $\omega G_R(\omega)$ is bounded as $\omega \to \infty$. For the more general case, one must use a subtracted dispersion relation. With this assumption, we may write

\[
G_R(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im} G_R(\omega')}{\omega - \omega' + i\epsilon} = \sum_{k=0}^{\infty} \frac{M_k}{(\omega + i\epsilon)^{k+1}},
\]

(5.47)

where

\[
M_k = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \omega'^k \text{Im} G_R(\omega')
\]

(5.48)

is the $k$th moment of the spectral function $\text{Im} G_R(\omega)$. It then follows that

\[
M_0 = \lim_{\omega \to \infty} \omega G_R(\omega)
\]

(5.49)

\[
M_1 = \lim_{\omega \to \infty} \omega[G_R(\omega) - M_0],
\]

(5.50)

with similar expressions for the higher order moments. For $G_R$ of the form

\[
G_R(\omega) = \frac{1}{\omega - \Sigma(\omega)},
\]

(5.51)

then it follows

\[
M_0 = 1
\]

(5.52)

and

\[
M_1 = \Sigma(\infty),
\]

(5.53)
where we have assumed that $\Sigma(\omega)/\omega$ goes to zero for large $\omega$. Since most observables depend upon the first few moments of the spectral function, it follows that if one could come up with an approximation to the self energy with the correct asymptotic behavior, then perhaps it is possible to calculate the observables. This exciting possibility is realized in the last chapter where, although an approximation to $\Sigma$ fails to produce the correct spectral function, it is found that observables which depend only upon the first two moments are calculated correctly. Of course, the above results apply only if it permissible to use unsubtracted dispersion relations.
6.1 Ergodic Systems

In this section, the dynamics of a classical $N$ particle system is reviewed. For simplicity, we consider the case of only one spatial dimension. It is easy to extend the ideas expressed here to three spatial dimensions; however, no further insight will be gained.

At any given time, a system of $N$ interacting classical particles can be characterized by a point $(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N)$ in a $2N$ dimensional space where the momentum and position of the $i$th particle is $p_i$ and $q_i$, respectively. As the system evolves in time, the point will trace out a trajectory in the space. Actually for systems which have constraints due to conservation laws (e.g., energy, etc...) the entire space is not accessible to the system point. Instead, for systems with $f$ constants of motion, the point is constrained to move on a $2N - f$ dimensional manifold.
The system is said to be *ergodic* if the trajectory traced out by the system passes arbitrarily close to every point on the manifold accessible to the system.

It is not hard to find systems which are not ergodic. Consider for example a pendulum in a uniform vertical gravitational field. It is free to swing about any vertical plane containing its point of suspension. More generally, it is free to move with the pendulum bob constrained to a sphere of radius equal to the length of the pendulum. However, it is not ergodic. If it is initially swinging in a vertical plane, it will continue to swing in the same vertical plane with its trajectory never entering other parts of the two dimensional manifold accessible to it.

If the system is ergodic, then it is possible to introduce a density function \( \rho(p_1, \ldots, q_N) \) on the manifold such that the time average of some dynamical quantity \( f(q, p) \) may be written

\[
\langle f(p, q) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, f(q(t), p(t)) = \int_{\mathcal{M}} dp \, dq \, \rho(p, q) f(p, q)
\] (6.1)

where the shorthand notation \((p, q) = (p_1, \ldots, q_N)\) has been introduced. Usually in statistical mechanics, one speaks of an ensemble of *macroscopically* identical systems distributed on the manifold with density \( \rho(p, q) \). Evidently, by macroscopically identical, it is meant that each system is equivalent to a single system viewed at different times in its course of evolution. Hence for an ergodic system, one may replace time averages by ensemble averages circumventing the impossible task of calculating the exact trajectory of the system. This idea, based upon the
assumption that the system is ergodic, forms one of the cornerstones of statistical mechanics.

### 6.2 Classical Boltzmann Equation

Above, we characterized the system by a point in a $2N$ dimensional space. In the following, we consider the 2 dimensional space labelled by $(p, q)$ known as the phase space of the system. The $N$ particle system will now be represented by $N$ points $(p_1, q_1), \ldots, (p_N, q_N)$ in the phase space. As the system evolves in time, the points trace out $N$ trajectories reminiscent of a swarm of bees.

For concreteness, let us assume the particles are in an external field $V(q)$ which does not depend upon the momentum of the particles. For the time being, we neglect any direct interaction (e.g. two-body forces) between the particles. The Hamiltonian of the system is

$$H(\vec{p}, \vec{q}) = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + V(q_i) \right), \tag{6.2}$$

which implies

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \tag{6.3}$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}. \tag{6.4}$$

If the number of particles in the system is very large, it is convenient to speak of a time dependent density $\rho(p, q, t)$ on the phase space such that $\rho(p, q, t)dp\, dq$
is the number of particles with momentum between $p$ and $p + dp$ and position between $q$ and $q + dq$ at time $t$. Obviously,

$$N = \int dp\ dq\ \rho(p,q,t)$$

(6.5)

since we assume particle number is conserved. Consider the time rate of change of $\rho$. That is, concentrate on a subset of particles in a small volume $dp\ dq$ centered about $(p,q)$ at time $t$. In a time $\delta t$ later, the particles in this volume will occupy a new volume $dp'\ dq'$ about the point $(p',q')$. From conservation of particle number, we may write

$$\rho(p,q,t)dp\ dq = \rho(p',q',t + \delta t)dp'\ dq'$$

(6.6)

where from the equations of motion

$$p' = p - \frac{\partial V}{\partial q} \delta t$$

(6.7)

and

$$q' = q + \frac{p}{m} \delta t.$$

(6.8)

Under the transformation $(p,q) \rightarrow (p',q')$ we have

$$dp' \wedge dq' = (dp - \delta t \frac{\partial^2 V}{\partial q^2} dq) \wedge (dq + \frac{dp}{m} \delta t)$$

(6.9)

$$= dp \wedge dq + O(\delta t^2),$$

(6.10)

implying that the volume is invariant under the flow. Hence it follows that

$$\rho(p,q,t) = \rho(p',q',t + \delta t)$$

(6.11)
In other words, with respect to a frame of reference attached to a particle in the flow, the density is invariant.

The above analysis was specific to a system interacting with an external field, but ignored the possibility of direct interparticle interactions. This is appropriate for sufficiently dilute or weakly interacting systems. However, for dense strongly interacting systems direct interparticle collisions cannot be ignored and the above analysis breaks down. We can still define a phase space density for the system, but it will no longer be invariant under the flow. That is, Eq. (6.12) must be supplemented by a term \((\partial \rho / \partial t)_{\text{collision}}\) due to collisions:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial p} \dot{p} + \frac{\partial \rho}{\partial q} \dot{q} = (\frac{\partial \rho}{\partial t})_{\text{collision}}. \tag{6.13}
\]

The interpretation is simple. The left hand side corresponds to the change in the phase space density due to particles drifting in and out through the boundary of an element of phase space. The right hand side acts like a source or sink of particles which appear or disappear in the element without actually passing continuously through the boundary of the element. For this reason, the left hand side is known as the "drift" term while the right hand side is known as the "collision" term. For this interpretation to hold, the timescale associated with the collision has to be much smaller than the timescale associated with the drift. As an example consider
a two particle collision $p_1 + p_2 \rightarrow p'_1 + p'_2$ which takes the particles from different parts of the phase space and throws them into two entirely different parts of the phase space. If the time scale of the actual collision is short compared to the drift time scale, it will appear as though the particles are created and destroyed in disjoint parts of phase space.

### 6.3 Generalized Boltzmann Equation

In the previous section, the dynamics of a classical many body system was reviewed within the framework of the Boltzmann equation. It is of interest to see what the corresponding quantum equation is. This equation is known as the “generalized Boltzmann equation” or the “Kadanoff-Baym equation”\[7, 17\]. Later we will show that with certain assumptions as to the nature of the underlying dynamics, this equation may be reduced to the classical Boltzmann equation in the classical limit.

In a previous chapter, we discussed the Schwinger-Dyson equations appropriate for the closed time-path. They have the generic form

$$\Delta_c(x, y) = \Delta_c^0(x, y) + \int_c dz \int_c dz' \Delta_c^0(x, z) \Pi_c(z, z') \Delta_c(z', y), \quad (6.14)$$

where $\Pi_c(x, y)$ is the contour self energy correction to the contour Green function $G_c(x, y)$. Using the Langreth-Wilkins rules, Eq. (4.53) and Eq. (4.54) for continuing the contraction of contour functions to the real axis, the above contour Dyson equation may be expressed as

$$\Delta_0^{-1} \Delta^{>_<} = \Pi^{>_< \Delta} + \Pi_R \Delta^{>_<}, \quad (6.15)$$
\[ \Delta_{R,A} = \Delta^0_{R,A} + \Delta^0_{R,A} \Pi_{R,A} \Delta_{R,A}. \] 

(6.16)

Here we have used the fact that the greater and lesser components of the contour delta function \( \delta_c \) are zero, i.e., \( \delta^{>\!\!\langle}(x-y) = 0 \). Subtracting Eq. (6.15) and its adjoint we obtain the \textit{generalized Boltzmann equation}:

\[ [\Delta^{>\!\!\langle}, \Delta^{-1}_0\{\Pi_R + \Pi_A\}] + [\Pi^{>\!\!\langle}, \frac{1}{2}(\Delta_R + \Delta_A)] = \frac{1}{2}\{\Pi^{\!\langle}, \Delta^{>\!\!\langle}\} + \frac{1}{2}\{\Pi^{>\!\!\langle}, \Delta^{\!\langle}\}. \] 

(6.17)

We shall see that, upon using a semi-classical expansion of the generalized Boltzmann equation, the first term on the left-hand side corresponds to the familiar drift term of a classical Boltzmann equation while the second term is a quantum-mechanical correction to the drift. The drift term contains contributions from other particles in the medium through the self-consistent Hartree-Fock potential set up by the particles in the background. The right-hand-side corresponds to the collision term—the first term provides scattering into a phase space element and the second term provides scattering out. However, it must be emphasized that this interpretation is valid only in a semi-classical limit. Furthermore, the generalized Boltzmann equation above does not have the single time structure that one expects a Boltzmann equation to have. Hence, in general it is \textit{non-Markovian}, that is, knowledge of the \textit{complete} past is necessary to predict the future of the system. In this respect, it more closely resembles a generalized master equation. Later, in this chapter, we describe an \textit{Ansatz}, due to Kadanoff and Baym[17], which relates the double-time function \( \Delta^{\!\langle}(x,y) \) to a single time function which
has the interpretation of a phase-space density in a semi-classical approximation. Thus, the Kadanoff-Baym Ansatz gives the generalized Boltzmann equation the Markovian behavior of the classical Boltzmann equation.

Of course one must question the validity of such an Ansatz. After all, the generalized Boltzmann equation is inherently non-Markovian. However, there may exist interesting quantum systems where the non-Markovian behavior is not important. In fact, in recent years the phenomenological Boltzmann equation has been used quite successfully in the study of heavy ion collisions[18]. In a later chapter, we will derive a Boltzmann equation for the Yukawa model and examine the validity of these assumptions within the context of a relativistic heavy ion collision.

### 6.4 Equilibrium Systems

In an earlier chapter, we saw that a continuation from the contour to the real axis necessitated the introduction of several Green functions which are required to describe the dynamics of a quantum system. A linearly independent set consists of $G_R(x, y)$, $G_A(x, y)$, and $G^<(x, y)$. Alternatively, in place of the retarded and advanced functions, one can use the Feynman and anti-Feynman functions. However, for reasons that will become clear later, we choose to work with the former. Of course, due to the infinite tower of equations which make up the Schwinger-Dyson equations, in practice one requires an infinite number of Green functions to characterize a quantum field theory. Hence, one is always always forced to make
some sort of truncation to close this set. We will come back to this issue later; in the mean time we will concentrate on the two-point functions.

As stated above, in general the three two-point functions \( G_R(x,y) \), \( G_A(x,y) \), and \( G^<(x,y) \) are linearly independent. However, we will show in this section that the complete dynamics of an equilibrium system may be described solely in terms of the retarded Green function. In a finite temperature system, the generic two-point function \( A(x,y) \) does not depend upon the temporal variables \( x_0 \) and \( y_0 \) separately but only upon their difference, i.e., \( A(x,y) = A(x_0 - y_0, x,y) \). Alternatively, one may take this as a definition of equilibrium. For infinite systems, only the difference \( x - y \) enters as well; however, we allow the possibility of a finite system, e.g., a \(^{40}\text{Ca} \) nucleus.

Let us define the two functions

\[
G^>_{AB}(t,x,y) = \langle A(t,x)B(0,y) \rangle
\]

(6.18)

and

\[
G^<_{AB}(t,x,y) = \pm \langle B(0,y)A(t,x) \rangle,
\]

(6.19)

where the "±" sign in the above equation is "positive" if the operators \( A \) and \( B \) are bosonic and "minus" if the operators represent fermions. In addition, we imagine the system to be in contact with a heat bath of inverse temperature \( \beta \). In this case we write

\[
\langle \cdot \rangle = \frac{1}{Z} \text{Tr} [\cdot e^{-\beta H}],
\]

(6.20)
where $Z$ is the partition function $\text{Tr} \, e^{-\beta H}$. Concentrating on $G_{AB}^{\gamma}(t)$ we have

$$G_{AB}^{\gamma}(t, x, y) = \text{Tr} \left[ e^{-\beta H} A(t, x) B(0, y) \right]$$

$$= \text{Tr} \left[ e^{-\beta H} e^{iHt} A(0, x) e^{-iHt} B(0, y) \right]. \quad (6.21)$$

Similarly,

$$G_{AB}^{\prec}(t, x, y) = \pm \text{Tr} \left[ e^{-\beta H} B(0, y) A(t, x) \right]$$

$$= \pm \text{Tr} \left[ A(t, x) e^{-\beta H} B(0, y) \right]$$

$$= \pm \text{Tr} \left[ e^{iHt} A(0, x) e^{-iHt} e^{-\beta H} B(0, y) \right], \quad (6.22)$$

where in the second step we have exploited the cyclic property of the trace. Comparing the two expressions, we immediately find that

$$G_{AB}^{\gamma}(t, x, y) = \pm G_{AB}^{\prec}(t + i\beta, x, y), \quad (6.23)$$

which is the well known Kubo–Martin–Schwinger[19] (KMS) boundary condition. Though it is a simple consequence of the cyclic property of the trace, it forms one of the cornerstones of finite temperature quantum field theory. It is useful to express the above equation in the Wigner representation. It follows from Wigner transforming both sides of the above equation that

$$G^{\gamma}(p, R) = \pm \int dt e^{i\rho t} G^{\prec}(t + i\beta; p, R)$$

$$= \pm e^{\rho_0} G^{\prec}(p, R), \quad (6.24)$$

where we have dropped the subscript "AB". Using the identity $G_R - G_A = G^{\gamma} - G^{\prec}$ together with the fact that in the Wigner representation the retarded and advanced
functions are complex conjugate pairs, we find immediately that

$$ GR(p, R) - GA(p, R) = (\pm e^{\beta p_0} - 1)G^<(p, R) $$

(6.25)

or

$$ iG^<(p, R) = \mp 2\frac{1}{e^{\beta p_0} - 1} \text{Im} G_R(p, R) $$

$$ = -2f_\beta^{FE}(p_0) \text{Im} G_R(p, R). $$

(6.26)

where we have defined

$$ f_\beta^{FE}(p_0) = \pm \frac{1}{e^{\beta p_0} - 1}. $$

(6.27)

The functions $f_\beta^{FE}(p_0)$ are related to the Bose-Einstein and Fermi-Dirac functions; i.e.,

$$ f_\beta^{BE}(p_0) = f_\beta(p_0) = \frac{1}{e^{\beta p_0} - 1} $$

(6.28)

and

$$ f_\beta^{FD}(p_0) = f_\beta^{FD}(p_0) = \frac{1}{e^{\beta p_0} + 1}. $$

(6.29)

Although, through the use of the KMS boundary condition, we have related the correlation function $G^<$ to the imaginary part of the retarded Green function, the reader should be warned that this simple result does not hold for all equilibrium two-point functions. It only holds for two point functions which are constructed from the thermal average of two operators.

So far we have only considered the retarded propagator. What about the Feynman propagator? Pick up almost any book on quantum field theory and you will
inevitably find Feynman propagators all over the place with the retarded propagator almost entirely neglected. The reason for this is clear. Unlike retarded propagators, Feynman propagators are ideally suited for perturbative calculations. Indeed one can write down a set of rules, the so-called Feynman rules, for calculating quantum corrections to any order in perturbation theory. Here lies the utility and widespread use of the Feynman propagator.

Suffice it to say that most literature on finite temperature relativistic quantum field theory deal almost exclusively with Feynman propagators for much the same reasons. There exist three finite temperature formalisms utilizing the Feynman propagator (see [20]). The oldest and most familiar is the Matsubara formalism. Here one formulates the theory in imaginary time so that the evolution operator $e^{iHt}$ is formally identical to $e^{-\beta H}$. The time contour consists of a line from $t = 0$ in the complex time plane to $t = -i\beta$, with time-ordering of field operators along this contour. Since the "time variable" runs along a contour of finite length $\beta$, upon Fourier transformation, the frequency $\omega$ is no longer a continuous variable; instead it takes on discrete values, the so-called Matsubara frequencies. Despite the simplicity of the Matsubara approach, it is inconvenient to use because one must analytically continue back from imaginary time to real time. Here one is presented with a problem since one cannot analytically continue in a unique way from a finite set to an infinite one without additional assumptions.

In their famous paper, Dolan and Jackiw[21] tried to formulate a finite tempera-
ture relativistic quantum field theory in Minkowski space. They reported that such a theory is not well defined since they ran into squares of delta functions which are meaningless. However, they failed to appreciate the matrix structure of the contour Green functions working with only the 1,1 or Feynman matrix element. With the matrix structure of the finite temperature Green function fully taken into account, one finds that the ill-defined distributions encountered by Dolan and Jackiw are canceled by similar contributions from the other matrix elements. The lesson to be learned is that one has no alternative but to use matrix propagators when doing finite temperature field theory in Minkowski space. What about the zero temperature limit? The theory of Dolan and Jackiw has the correct limit. This is because one can show that the other matrix elements contributing to a general Feynman diagram decouple and one is left with only Feynman propagators, i.e., the 1,1 element of the contour propagator.

Recently, Niemi and Semenoff[22] made another attempt to formulate a finite temperature theory in Minkowski space. Their attempt was successful. Basically their approach is just a relativistic path integral representation of the work in non-relativistic finite temperature theory associated with the names Keldysh[23], Craig[24], and Mills[13]. In the Neimi–Semenoff theory, one introduces two fields, type one and type two. These fields are a result of the matrix structure of the thermal Green function and are associated with the 1,1 and 2,2 matrix elements, respectively. The type two field plays the role of a ghost field and occurs only as
an internal line of a general Feynman diagram. The sole purpose of this field is to cancel the unwanted ill-defined products of distributions encountered by Dolan and Jackiw. The result is a well-defined set of Feynman rules allowing the calculation to any order in perturbation theory. The only shortcoming of their work is that one cannot extend it to non-equilibrium systems. This is because a finite temperature equilibrium contour is built in at the very outset. For more details, the reader is referred to their paper.

The above three approaches dealt exclusively with Feynman propagators, which are well suited for perturbative calculations. However, for non-perturbative calculations I feel that the retarded propagator has more to offer; particularly for finite temperature systems. As shown in this section, one only requires the imaginary part of the retarded propagator. One of the major advantages of the retarded propagator is the fact that it is analytic in the complex energy plane. Hence one has dispersion relations at one's disposal. In a later chapter, we exhibit these features by doing a non-perturbative calculation which employs the retarded propagator.

6.5 Kadanoff–Baym Ansatz

In the last section, we were able to relate the correlation function $G^<$ to the imaginary part of the retarded Green function for an equilibrium system. One of the most salient features of this relationship is the appearance of the Fermi–Dirac and Bose–Einstein distribution functions. One might ask if such an equation exists
for the general non-equilibrium case and with the replacement of the equilibrium distribution functions by a non-equilibrium distribution function. The answer is no.

Kadanoff and Baym[17] assumed such an equation to hold locally for the non-equilibrium case, i.e.,

$$G^<(R,p) = f(R,p)2i \text{Im} G_R(R,p),$$  \hspace{1cm} (6.30)

where $f(R,p)$ plays the role of a non-equilibrium quasi-particle distribution function. Note that unlike the equilibrium case, it depends upon $R$ and $p$ as well as $p_0$.

The spectral function, $2i \text{Im} G_R$, has the formal solution

$$G_R - G_A = \frac{2i\text{Im} \Sigma_R}{(G_0^{-1} - \text{Re} \Sigma_R)^2 + (\text{Im} \Sigma_R)^2} + O(h^2),$$  \hspace{1cm} (6.31)

which follows from a gradient expansion and Wigner transform of the Dyson equations (6.15) and (6.16) and from the fact that $G_R(R,p) = G_A^*(R,p)$. In the limit of weak scattering, $\text{Im} \Sigma_R$ is small and the spectral function is sharply peaked about some value of $p_0 = E_p(R)$. If the width of the peak is sufficiently narrow, the excitations of the system may be described in terms of quasi-particles and the distribution function becomes a distribution function for the quasi-particles. In the limit of vanishing width, the spectral function has a delta function–like behavior and we may write

$$G^<(R,p) = 2if(R,p,E_p) \text{Im} G_R(R,p),$$  \hspace{1cm} (6.32)
replacing $p_0$ in the argument of the distribution function, $f$, by $E_p$. Thus $G^<$ can be eliminated from the Boltzmann equation in favor of the time-dependent distribution function $f(R,p,E_p)$ defined on the phase-space $(R,p)$. This is the essence of the Kadanoff-Baym ansatz\cite{Kadanoff-Baym}. We want to emphasize that this is an ansatz, and not an approximation; it is strictly valid only for equilibrium. As such, it can only be justified a posteriori. Throughout this work, the word "ansatz" is used in this fashion.

Finally, we would like to mention that the Kadanoff-Baym ansatz is not the only ansatz on the market. Lipavski et al\cite{Lipavski} have constructed an ansatz which overcomes some of the limitations of the Kadanoff-Baym ansatz. Namely, the Kadanoff-Baym ansatz violates causality\cite{Lipavski} and in addition it is not clear how to improve it. The ansatz of Lipavski et al is causal and one can derive correction terms for it. Moreover, unlike the Kadanoff-Baym ansatz, it is not limited to the gradient expansion. However, unlike the Kadanoff-Baym ansatz, it leads to a non-Markovian Boltzmann equation. For more details the reader is referred to their paper. For a discussion of the Markovian ansatz that we employ here as it pertains to heavy ion collisions, the reader is referred to the next chapter.

To summarize, under certain conditions, the generalized Boltzmann equation can be reduced to a Boltzmann equation for a quasi-particle distribution function. The essential ingredients for this interpretation are: i) validity of the gradient expansion, ii) long lived quasi-particle excitations, and iii) validity of the Kadanoff-
Baym ansatz. We emphasize that the three assumptions are not independent of each other. Indeed, in motivating the ansatz, the gradient expansion was used and the quasi-particle pole was assumed to dominate.
CHAPTER VII

A Transport Theory with Dynamical Mesons

In the preceding chapters we have described a formalism capable of handling quantum fields which are in equilibrium as well those which are out of equilibrium. In this chapter, we bring together all this formal apparatus and apply it to the relativistic Yukawa model.

7.1 Introduction

In recent years, non-relativistic Boltzmann-Uehling-Uhlenbeck (BUU) equations have been employed in the study of heavy ion collisions with some success[18]. As one goes to increasingly higher energies, relativistic kinematics become necessary. To meet this need, relativistic Vlasov equations have been developed[26]. In addition, these equations have been extended through the ad-hoc addition of collision terms using free-space nucleon-nucleon cross sections[27, 28, 29]. However, with few exceptions[30, 31], little work has been done to justify this naive extension of the collisionless Boltzmann equation. It is the purpose of this chapter to investi-
gate whether or not such a picture is justified through the systematic construction of collision terms with medium dependence.

It has been emphasized that phenomenological hadronic field theories[9] provide a framework for incorporating the needs of special relativity and quantum mechanics. In addition, because they are based on quantum field theory, they allow a discussion of creation and annihilation processes. As a starting point we have adopted a quantum field theory consisting of spin-half fermions (nucleons) interacting with spin-zero scalar bosons (σ mesons) via a Yukawa coupling. This model has also been studied by Li et al[30]. However, in addition to ignoring retardation effects, their description does not allow for a discussion of boson creation and annihilation processes. Botermans and Malfliet[32] have investigated a similar model using a Dirac-Brueckner T-matrix approach but also failed to include retardation effects. In this chapter, we go beyond their investigations by including retardation effects and deriving a kinetic equation describing meson production. We want to emphasize that in this chapter we propose to study what elements of a complete relativistic transport theory may be important, and leave the more ambitious goal of using a "realistic" theory for later. At this point it is not even clear how one should calculate ground state properties of nuclei with a realistic hadronic theory[15].
7.1.1 Distribution Function for the Scalar Field

In a previous chapter, we have derived the generalized Boltzmann equation from the contour Dyson equations. We have also obtained the contour Dyson equations for the Yukawa Lagrangian, Eqs. (4.39) and (4.40). To complete the transport theory for the Yukawa model, we must take the generalized Boltzmann equation specialized to this model and perform a semi-classical expansion of it. In addition, we will use the Kadanoff-Baym Ansatz to construct the non-equilibrium distribution functions as discussed earlier. In this section, we construct a quasi-particle distribution function for the scalar field. The analogous construction for the fermions is more difficult due to the Dirac matrix structure. We postpone this problem until the next section.

The equation for the connected contour boson propagator, Eq. (4.40), can be put in the form

\[(\Delta_0^0)^{-1}\Delta_c = 1 + \Pi_c \Delta_c, \quad (7.1)\]

where

\[(\Delta_0^0)^{-1}(x, y) = -[\Box + m_s^2]\delta_c(x - y), \quad (7.2)\]

with integration over the contour implied. The generalized Boltzmann equation, Eq. (6.17), becomes, upon Wigner transformation and gradient expansion,

\[i\hbar[\Delta^{>\cdot<}, p^2 - m_s^2 - \text{Re} \Pi_R]_{PB}(R, p) + i\hbar[\Pi^{>\cdot<}, \text{Re} \Delta_R]_{PB}
\]

\[= [\Delta^{>\cdot<} - \Delta^{<\cdot>}](R, p) + O(\hbar^2), \quad (7.3)\]
where we have used the properties of the Wigner transform listed in the Appendix.

We also keep the \( \hbar \) from the gradient expansion for bookkeeping purposes.

In accordance with the general recipe outlined in previous sections, we now implement the Kadanoff-Baym Ansatz. To this end, we find from Wigner transforming and gradient expanding Eq. (7.3) for the retarded function \( \Delta_R \),

\[
[p^2 - m_s^2 - \Pi_R(R, p)]\Delta_R(R, p) = 1 - i\hbar [p^2 - m_s^2 - \Pi_R, \Delta_R]_{PB}(R, p) + O(\hbar^2). \tag{7.4}
\]

In addition, one can show that the retarded function, \( \Delta_R(R, p) \), satisfies the dispersion relation

\[
\Delta_R(R, p) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dq_0 \frac{\text{Im} \Delta_R(R, p, q_0)}{p_0 - q_0 + i\epsilon}. \tag{7.5}
\]

In fact, such a relation holds for any retarded function defined analogously. Here, we have assumed that \( \text{Im} \Delta_R(R, p) \) vanishes sufficiently rapidly as \( p_0 \) goes to infinity. In the general case of a polynomial growth for large \( p_0 \), one must use a subtracted dispersion relation and we leave it as a simple exercise for the reader to show that the argument still goes through. Using the Poisson bracket property, \( [A, f(A)]_{PB} = 0 \) for \( A = A(R, p) \), we can write the solution to Eq. (7.4) as

\[
\Delta_R = \frac{1}{p^2 - m_s^2 - \text{Re} \Pi_R - i \text{Im} \Pi_R} + O(\hbar^2). \tag{7.6}
\]

Since \( \Delta_R \) is a retarded function, it must be analytic in the upper half complex plane of \( p_0 \), so poles must occur in the lower half plane. Suppose \( z_p = \omega_p(R) - i\gamma_p(R) \) for \( \gamma_p > 0 \) is a pole in the lower half plane of \( p_0 \). Then

\[
z^2 - p^2 - m_s^2 - \Pi_R(p, z) = 0, \tag{7.7}
\]
where for clarity we have not explicitly indicated the $R$ dependence. Using the property $\Delta^>(R, p) = \Delta^<(R, -p)$ and the identity $\Delta^> - \Delta^< = \Delta_R - \Delta_A$, together with the dispersion relation Eq. (7.5), it is easily verified that

$$\Delta_R(R, p, z) = [\Delta_R(R, -p, -z^*')]^*$$

(7.8)

for every complex $z$. Then, as the reader can readily show, if $z_p$ is a pole then $-z^*_p$ is also a pole. This method of constructing one pole from the other cannot return the same pole since for a non-interacting system this would imply $\omega_p = -\omega_{-p}$ which is incorrect. Hence, we conclude that the poles are distinct. Assuming $\gamma_{\pm p}$ are small, an approximate solution to Eq. (7.7) is given by

$$\gamma_{\pm p} = -Z_B(p, p_0)|_{p_0 = \pm \omega_{\pm p}} \text{ Im } \Pi_R(\pm \omega_{\pm p}),$$

(7.9)

with $\omega_{\pm p}$ satisfying the transcendental equation

$$\omega_{\pm p}^2 - p^2 - m^2 - \text{ Re } \Pi_R(p, \pm \omega_{\pm p}) = 0.$$  

(7.10)

The residue, $Z_B$, is defined by

$$Z_B^{-1}(p, p_0) = [2p_0 - \frac{\partial \text{ Re } \Pi_R(p, p_0)}{\partial p_0}],$$

(7.11)

with the property

$$Z_B(p, p_0) = -Z_B(-p, -p_0).$$

(7.12)

Assuming there are only two poles and neglecting off-pole contributions we can write the retarded function, Eq. (7.6), as

$$\Delta_R(R, p) = \frac{Z_B(p, \omega_p)}{p_0 - \omega_p + i\gamma_p} + \frac{Z_B(p, -\omega_{-p})}{p_0 + \omega_{-p} + i\gamma_{-p}},$$

(7.13)
where for clarity, the $R$ dependence has been omitted on the right hand side of the equation. The spectral function may be found by taking the imaginary part of Eq. (7.13). In the limit of vanishing width, we can write

$$\text{Im} \Delta_R(R, p) = -\pi \text{sgn}[Z_B(p, p_0)] \delta(p^2 - m^2 - \text{Re} \Pi_R), \quad (7.14)$$

where for meaningful results, all distributions are considered to be regulated as in the finite temperature quantum field theory[20].

Finally, we define distribution functions, $N^{>\cdot<}(R, p)$, through

$$\Delta^{>\cdot<}(R, p) = 2i N^{>\cdot<}(R, p) \text{Im} \Delta_R(R, p) \quad (7.15)$$

$$= -2\pi i \text{sgn}[Z_B(p, p_0)] N^{>\cdot<}(R, p) \delta(p^2 - m^2 - \text{Re} \Pi_R), \quad (7.16)$$

where

$$N^{>}(R, p) = N^{<}(R, p) + 1, \quad (7.17)$$

completing the implementation of the Kadanoff–Baym Ansatz.

Inserting the Ansatz, Eq. (7.16) into the Boltzmann equation, Eq. (7.3), we can write

$$\text{sgn}[Z_B(p, p_0)] \delta(p^2 - \omega_p^2)[N^{>\cdot<}(R, p), p^2 - m^2 - \text{Re} \Pi_R]_{PB} =$$

$$-i \text{sgn}[Z_B(p, p_0)] \delta(p_0^2 - \omega_p^2)[N^>(R, p) \Pi^<(R, p) - N^<(R, p) \Pi^>(R, p)], \quad (7.18)$$

where by construction, the delta function commutes with the Poisson bracket. In writing this expression, we note that the correction to the drift, the term involving
Re $\Delta$ in Eq. (7.3), does not contribute in the pole approximation[16, 17]. Using the easily derived identity

$$\text{sgn} \left( \frac{\partial f}{\partial p_0} \right)^{-1} \delta(f)[N(R, p), f(R, p)]_{PB} =$$

$$\sum_r \left[ \frac{\partial n_r}{\partial T} + \frac{\partial n_r}{\partial p} \cdot \frac{\partial \omega_r}{\partial R} - \frac{\partial n_r}{\partial R} \cdot \frac{\partial \omega_r}{\partial p} \right] \delta(p_0 - \omega_r),$$

where

$$f(R, p, \omega_r) = 0,$$

$$n_r(R, p) = N(R, p, \omega_r),$$

and all roots, $\omega_r$, are summed over, we can write Eq. (7.18) in the form

$$\left[ \frac{\partial}{\partial T} - \frac{\partial \omega_p}{\partial R} \cdot \frac{\partial}{\partial p} + \frac{\partial \omega_p}{\partial p} \cdot \frac{\partial}{\partial R} \right] n(p) =$$

$$iZ_B(p, \omega_p) \left\{ [1 + n(p)] \Pi^<(p, \omega_p) - n(p) \Pi^>(p, \omega_p) \right\}. \quad (7.19)$$

Here we have introduced a boson quasi-particle distribution function

$$n(p) = N^<(p, \omega_p) = N^>(p, \omega_p) - 1, \quad (7.20)$$

and exploited the fact that $\Delta^>(R, p) = \Delta^<(R, -p)$ to write

$$N^<(p, \omega_p) = -N^>(-p, -\omega_p). \quad (7.21)$$

Before construction of the distribution function for the fermions, we discuss the physical interpretation of the boson function, $n(R, p)$, by expressing the energy-momentum stress tensor, Eq. (2.71), in terms of $n(R, p)$. To this end, we introduce
the two-point symmetric tensor

\[ T_{\mu\nu}^B(x, y) \equiv \frac{1}{2} \left[ \partial_\mu \partial_\nu + \partial_\nu \partial_\mu - g_{\mu\nu} (\partial^x \cdot \partial^y - m^2_*) \right] (\Phi(y)\Phi(x)), \]  

(7.22)

which upon setting \( x = y \) becomes the stress tensor Eq. (2.71). By Wigner transforming this expression and integrating over all \( p \) we can write a local average

\[ T_{\mu\nu}^B(R) \equiv \int \frac{d^4p}{(2\pi)^4} T_{\mu\nu}^B(R, p) = \]

\[ \partial_\mu \phi(R) \partial_\nu \phi(R) - \frac{1}{2} g_{\mu\nu} (\partial^x \phi(R)\partial^y \phi(R) - m^2_\phi^2(R)) \]

\[ + \frac{1}{4} (\partial_\mu R \partial_\nu R - \frac{1}{2} g_{\mu\nu} \Box R) \int \frac{d^3p}{(2\pi)^3} 2Z_B(p, \omega_p)(n_p + \frac{1}{2}) \]

\[ + \int \frac{d^3p}{(2\pi)^3} 2Z_B(p, \omega_p)[p_\mu p_\nu - \frac{1}{2} g_{\mu\nu} (p^2 - m^2_\star)] \bigg|_{p_0 = \omega_p} \left( n_p + \frac{1}{2} \right). \]  

(7.23)

In deriving this expression, we have used the "lesser" component, \( \Delta^c(x, y) \), of the contour green function \( \Delta(x, y) \), Eq. (4.31), to rewrite \( (\Phi(y)\Phi(x)) \) and we have employed the Ansatz for \( \Delta_c(R, p) \), Eq. (7.16), together with Eq. (7.20) for \( n(p) \). Finally, we have exploited the symmetry of \( Z_B(p) \), Eq. (7.12).

Each of these terms has a straightforward interpretation. The first term provides the familiar mean field energy and momentum. The remaining terms are easily understood by realizing that \( \frac{d^3p}{(2\pi)^3} 2Z_B(p, \omega_p) \) is the Lorentz invariant momentum space volume element at \( R \), while \( n_p \) is the boson number density at space–time point \( R \). The second term gives the energy–momentum associated with space–time inhomogeneities of the boson number density, while the third term corresponds to energy–momentum of the quasi–particles at \( R \).
Note, that in the vacuum limit, the energy–momentum tensor does not vanish due to the additional sums of $\frac{1}{2}$ in the integrands of Eq. (7.23), which are the familiar zero-point vacuum fluctuation contributions. In ordinary vacuum quantum field theory these terms are made to vanish by normal ordering with respect to the vacuum. However, in more general situations as the ones we consider here, such terms must persist. The best one can do is to subtract the vacuum expectation value of $T_{\mu\nu}$ and then, with the appropriate counterterms, $T_{\mu\nu}$ can be made finite[20]. We emphasize that this is not the same as simply ignoring the extra terms of $\frac{1}{2}$ in the integrands of Eq. (7.23); only by properly dealing with the vacuum and through proper renormalization of the theory can the energy–momentum tensor be made finite.

Finally, we make the remark that the local averaging procedure employed above for a composite operator is equivalent to the point-splitting technique of Schwinger[10]. To see this, it is sufficient to consider the composite operator $\Phi^2(x)$ in one dimension with an obvious generalization to higher dimensions. The point-splitting technique consists of defining

$$
\langle \Phi^2(x) \rangle \equiv \lim_{\varepsilon \to 0^+} \langle \Phi(x - \varepsilon)\Phi(x + \varepsilon) \rangle. \tag{7.24}
$$

We will prove that the right-hand side of this equation is equivalent to

$$
\int \frac{dp}{2\pi} \Delta'(R,p), \tag{7.25}
$$
where $\Delta'(R,p)$ is the Wigner transform of

$$\Delta'(x,y) \equiv \langle \Phi(y)\Phi(x) \rangle. \tag{7.26}$$

Indeed, from the inverse Wigner transform we find

$$\int \frac{dp}{2\pi} \int dr e^{ipr} \langle \Phi(R - \frac{r}{2})\Phi(R + \frac{r}{2}) \rangle = \lim_{\varepsilon \to 0^+} \int dr \delta_\varepsilon(r) \langle \Phi(R - \frac{r}{2})\Phi(R + \frac{r}{2}) \rangle, \tag{7.27}$$

where we have regularized the delta function according to the prescription

$$\delta_\varepsilon(r) = \begin{cases} (2\varepsilon)^{-1} & \text{for } |r| \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}. \tag{7.28}$$

Using the mean–value theorem of integral calculus, we can write the right–hand side of Eq. (7.27) as

$$\lim_{\varepsilon \to 0^+} (2\varepsilon)^{-1} \cdot 2\varepsilon \langle \Phi(R - \frac{\eta_\varepsilon}{2})\Phi(R + \frac{\eta_\varepsilon}{2}) \rangle, \tag{7.29}$$

where the mean value theorem guarantees that this expression holds for at least one $\eta_\varepsilon \in [-\varepsilon, \varepsilon]$. From this, the result follows trivially.

### 7.1.2 Fermion Distribution Function

In this section we extend the Kadanoff–Baym Ansatz to the Dirac field. Because of the matrix structure of the Green functions this is not as straightforward as in the boson case. In the latter case, upon Wigner transformation and gradient expansion, commutators were replaced by Poisson brackets with ordinary products replacing anti–commutators. In this case, due to lack of commutativity, this prescription fails.
For simplicity, in the following we adopt the spin-symmetric Ansatz of Bezzerides and Dubois[31]. This Ansatz consists of decomposing fermion Green functions as

$$G(R, p) = G(R, p) + G_\mu(R, p)\gamma^\mu,$$  \hspace{1cm} (7.30)

neglecting tensor, pseudo-scalar, and pseudo-vector contributions. It is obvious from Eq. (6.16) that the retarded Green function, $G_R$, satisfies the anticommutation relation

$$\frac{1}{2}\{G_0^{-1} - \Sigma_R, G_R\}(x, y) = \delta(x - y).$$  \hspace{1cm} (7.31)

Upon Wigner transformation and gradient expansion, this equation decomposes as

$$1 = [p^\mu - \sigma_R^\mu(R, p)]G_{R\mu}(R, p) - [m - g\phi(R) + \sigma_R(R, p)]G_R(R, p),$$  \hspace{1cm} (7.32)

and

$$0 = [p^\mu - \sigma_R^\mu(R, p)]G_R(R, p) - [m - g\phi(R) + \sigma_R(R, p)]G_R^\mu(R, p),$$  \hspace{1cm} (7.33)

which may be solved immediately to yield

$$G_R(R, p) = \frac{m - g\phi(R) + \sigma_R(R, p)}{[p^\mu - \sigma_R^\mu(R, p)][p_\mu - \sigma_R\mu(R, p)] - [m - g\phi(R) + \sigma_R(R, p)]^2},$$  \hspace{1cm} (7.34)

$$G_R^\mu(R, p) = \frac{p^\mu - \sigma_R^\mu(R, p)}{[p^\mu - \sigma_R^\mu(R, p)][p_\mu - \sigma_R\mu(R, p)] - [m - g\phi(R) + \sigma_R(R, p)]^2}.\hspace{1cm} (7.35)$$

Here, $\Sigma_R$ has the form

$$\Sigma_R(R, p) = \sigma_R(R, p) + \gamma_\mu\sigma_R^\mu(R, p).$$  \hspace{1cm} (7.36)
To find the dispersion relation for the quasi-particles, we must look for poles in the retarded propagator. We assume there are only two poles at \( p_0 = \epsilon_\pm^+ - i \Gamma_\pm^+ \) and \( p_0 = -\epsilon_\pm^- - i \Gamma_\pm^- \), or more concisely, at \( p_0 = \pm \epsilon_\pm^\pm - i \Gamma_\pm^\pm \) where \( \Gamma_\pm^\pm > 0 \). Note that in the boson case, the two poles are related. This is due to the charge conjugation symmetry of the scalar field. Here, this is not the case and we must introduce two distribution functions— one for particles and one for anti-particles.

For small \( \Gamma_\pm^\pm \) we find

\[
\Gamma_\pm^\pm = -2Z_F(p, \pm \epsilon_\pm^\pm) \left[ p^\mu(p, \pm \epsilon_\pm^\pm) \text{Im} \sigma^\mu(p, \pm \epsilon_\pm^\pm) + m^*(p, \pm \epsilon_\pm^\pm) \text{Im} \sigma(p, \pm \epsilon_\pm^\pm) \right],
\]

(7.37)

where \( \pm \epsilon_\pm^\pm \) satisfies the mass-shell condition

\[
p^\mu(p, \pm \epsilon_\pm^\pm)^2 - m^*(p, \pm \epsilon_\pm^\pm)^2 = 0.
\]

(7.38)

Here we have introduced the momentum variable

\[
p^\mu_\mu(R, p, p_0) = p_\mu - \text{Re} \sigma_\mu(R, p, p_0),
\]

(7.39)

and the effective mass

\[
m^*(R, p, p_0) = m - g\phi(R) + \text{Re} \sigma(R, p, p_0).
\]

(7.40)

The value of the residue at the pole, \( Z_F \), is given by

\[
Z_F^{-1}(p, p_0) = \frac{\partial}{\partial p_0} (p^\mu p^\mu - m^* m^*).
\]

(7.41)
Hence, neglecting off pole contributions, we can write the imaginary part of the retarded function as

\[ \text{Im} G_R(R, p) = -\pi \text{sgn} Z_F(R, p) \left[ p^*\gamma^\mu + m^* \right] \delta(p^*^2 - m^*^2), \]  

(7.42)

where we have taken the width, \( \Gamma_{\pm}^\pm \), to be vanishingly small. We point out that when taking complex conjugates, it is understood that Dirac gamma matrices are not conjugated, e.g., \( \text{Im} G_R = \text{Im} \mathcal{G}_R + \gamma_\mu \text{Im} \mathcal{G}_R^\mu \).

In accordance with the Kadanoff-Baym Ansatz, Eq. (6.30), we define quasiparticle distribution functions, \( \pm F^{>}^{<}(R, p) \), through

\[ G^{>}^{<}(R, p) = \pm 2i F^{>}^{<}(R, p) \text{Im} G_R(R, p) \]

\[ = \mp 2\pi i F^{>}^{<}(R, p) \text{sgn} Z_F(R, p) \left[ p^*\gamma^\mu + m^* \right] \delta(p^*^2 - m^*^2). \]

(7.43)

Since \( G^> - G^< = G_R - G_A = 2i \text{Im} G_R \), it is easy to see that

\[ F^>(R, p) + F^<(R, p) = 1. \]  

(7.44)

After Wigner transformation, gradient expansion, and standard manipulations with the Dirac algebra, we can write the scalar piece of the generalized Boltzmann equation, Eq. (6.17), in the form

\[ i[G^{>}^{<}, -m^*]_{PB}(R, p) + i[G^\mu^{>}^{<}, p^*^\mu]_{PB}(R, p) = \]

\[ [\sigma^< G^> + \sigma_\mu^< G^>^\mu](R, p) - [\sigma^> G^< + \sigma_\mu^> G^<^\mu](R, p). \]

(7.45)
Upon substituting the Ansatz, Eq. (7.43), into the previous equation and after making the observation that \( \text{Im} \, G_R \) commutes with the Poisson bracket we find

\[
\text{sgn} Z_F(R, p) \delta(p^* - m^*) \left[ \pm F^{\geq}(R, p), p^* - m^* \right]_{PB} =
\text{sgn} Z_F(R, p) \delta(p^* - m^*) \left\{ \left[ \sigma^m + \sigma^- p^\mu \right] F^{\geq}(R, p) \right\} + \left[ \sigma^m + \sigma^- p^\mu \right] F^{\leq}(R, p). \tag{7.46}
\]

Using the identity, Eq. (7.19), we see that the particle distribution function \( f(p) \) defined by

\[
f(R, p) = F^{\leq}(R, p, \epsilon_p^+) \tag{7.47}
\]
satisfies

\[
\left[ \frac{\partial}{\partial T} - \frac{\partial \epsilon_p^+}{\partial R} \cdot \frac{\partial}{\partial p} + \frac{\partial \epsilon_p^+}{\partial p} \cdot \frac{\partial}{\partial R} \right] f(p) =
-2iZ_F(p, \epsilon_p^+) \left\{ [1 - f(p)] \left[ \sigma^m(p, \epsilon_p^+) m^*(p, \epsilon_p^+) + \sigma^- p^\mu(p, \epsilon_p^+) \right] \right\} + f(p) \left[ \sigma^m(p, \epsilon_p^+) m^*(p, \epsilon_p^+) + \sigma^- p^\mu(p, \epsilon_p^+) \right]. \tag{7.48}
\]

Similarly, the anti-particle distribution function, \( f(p) \), given by

\[
f(R, -p) = F^{\geq}(R, p, -\epsilon_p^-) \tag{7.49}
\]
satisfies the equation

\[
\left[ \frac{\partial}{\partial T} - \frac{\partial \epsilon_p^-}{\partial R} \cdot \frac{\partial}{\partial p} + \frac{\partial \epsilon_p^-}{\partial p} \cdot \frac{\partial}{\partial R} \right] f(p) = -2iZ_F(-p, -\epsilon_p^-) \left\{ [1 - f(p)] \left[ \sigma^m(-p, -\epsilon_p^-) m^*(-p, -\epsilon_p^-) \right] \right\}
+ \sigma^- (-p, -\epsilon_p^-) p^\mu(-p, -\epsilon_p^-) \right\} + f(p) \left[ \sigma^m(-p, -\epsilon_p^-) m^*(-p, -\epsilon_p^-) + \sigma^- (-p, -\epsilon_p^-) p^\mu(-p, -\epsilon_p^-) \right]. \tag{7.50}
\]
We can get some insight into the meaning of the distribution functions, \( f(p) \) and \( \tilde{f}(p) \), by writing the fermion contribution to the energy–momentum stress tensor, Eq. (2.70). We proceed as in the boson case by defining the two-point function

\[
T_{\mu\nu}^F(x, y) \equiv \frac{i}{4} (\partial_\mu - \partial_\nu) \bar{\Psi}(y) \gamma_\nu \Psi(x) + \frac{i}{4} (\partial_\mu - \partial_\nu) \bar{\Psi}(y) \gamma_\nu \Psi(x) \\
= \frac{1}{4} (\partial_\mu - \partial_\nu) \text{tr} \gamma_\nu G^<(x, y) + \frac{1}{4} (\partial_\mu - \partial_\nu) \text{tr} \gamma_\nu G^<(x, y), \tag{7.51}
\]

which reduces to the ordinary stress tensor upon setting \( x = y \). Upon Wigner transforming with the help of Eq. (7.43) for \( G^<(R, p) \) along with the distribution functions, Eq. (7.47) and Eq. (7.49), we find

\[
T_{\mu\nu}^F(R) \equiv \int \frac{d^4p}{(2\pi)^4} T_{\mu\nu}^F(R, p) = \\
\gamma \int \frac{d^3p}{(2\pi)^3} 2Z_F(p, \epsilon_p^+)[p_\mu p_\nu^*(p, \epsilon_p^+ + p_\nu p_\mu^*(p, \epsilon_p^+)] \bigg|_{p_0 = \epsilon_p} f_p \\
+ \gamma \int \frac{d^3p}{(2\pi)^3} 2Z_F(-p, -\epsilon_p^-)[p_\mu p_\nu^*(-p, -\epsilon_p^-) + p_\nu p_\mu^*(-p, -\epsilon_p^-)] \bigg|_{p_0 = \epsilon_p} \tilde{f}_p \\
+ \text{vacuum contribution}, \tag{7.52}
\]

where we have not explicitly written down the divergent zero–point vacuum fluctuation contribution. As in the boson case, some caution has to be exercised here because the vacuum contribution cannot be eliminated by normal ordering the field operators. Also, \( \gamma \), the isospin degeneracy parameter arising from the trace operation, takes on a value of \( n \) for a system with \( n \) flavors of fermions. So for nuclear matter, with only neutrons and protons, \( \gamma = 2 \).
To provide a more transparent interpretation of the distribution functions, we consider the energy momentum tensor in the mean field or collisionless approximation. This approximation is realized by the substitutions \( p_\mu^* \rightarrow p_\mu \) and \( m^* \rightarrow m - g\phi \). We find

\[
T_{MF}^{\mu\nu} = 2\gamma \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E^*(p)} (f_p + \bar{f}_p),
\]

where

\[
E^*(p) = +\sqrt{p^2 + (m - g\phi)^2}.
\]

We see that, at least in the mean field approximation, \( f_p \) and \( \bar{f}_p \) represent the number of fermions and anti-fermions, respectively, with momentum \( p \). We emphasize that \( f \) and \( \bar{f} \) also depend upon the space-time point \( R \) but this dependence has been suppressed for clarity.

### 7.1.3 Boltzmann Equations

Now that we have constructed distribution functions for the particles we can complete the task of finding Boltzmann equations for the interacting fermion-boson system. As previously mentioned we approximate the self-energies by using the bare vertex, Eq. (4.41). In this lowest order approximation, the contour self-energies, Eqs. (4.36) and (4.37), are given by

\[
\Sigma_c(x, y) = ig^2 G_c(x, y) \Delta_c(y, x),
\]
and

$$\Pi_c(x, y) = -ig^2 \text{tr} G_c(x, y)G_c(y, x). \quad (7.56)$$

Using the Langreth–Wilkins rule for anti-parallel products, Eq. (4.56), we find

$$\Sigma^{>\!<}(x, y) = ig^2 G^{>\!<}(x, y)\Delta^{<\!>}(y, x) \quad (7.57)$$

and

$$\Pi^{>\!<}(x, y) = -ig^2 \text{tr} G^{>\!<}(x, y)G^{<\!>}(y, x). \quad (7.58)$$

Upon Wigner transforming with the help of Eq. (A.5) in the Appendix and inserting Eqs. (7.16) and (7.43), we obtain, after tedious algebra,

$$\Sigma^>(p) = -(2\pi)^4 ig^2 \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} [\gamma^a p^*_{\mu}(q_\mu, \epsilon^+_q) + m^*(q, \epsilon^+_q)]Z_F(q, \epsilon^+_q)Z_B(k, \omega_k)$$

$$+ (n_0 - f_q)\delta(p_0 - \epsilon^+_q + \omega_k)\delta(p - q + k)$$

$$+ (1 + n_0)(1 - f_q)\delta(p_0 - \epsilon^+_q - \omega_k)\delta(p - q - k)$$

$$- (2\pi)^4 ig^2 \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} [\gamma^a p^*_{\mu}(-q_\mu, -\epsilon^-_q) + m^*(-q, -\epsilon^-_q)]Z_F(-q, -\epsilon^-_q)Z_B(k, \omega_k)$$

$$+ (n_k f_q)\delta(p_0 + \epsilon^-_q + \omega_k)\delta(p + q + k) + (1 + n_k)f_q\delta(p_0 + \epsilon^-_q - \omega_k)\delta(p + q - k),$$

$$\Sigma^<(p) = (2\pi)^4 ig^2 \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} [\gamma^a p^*_{\mu}(q_\mu, \epsilon^+_q) + m^*(q, \epsilon^+_q)]Z_F(q, \epsilon^+_q)Z_B(k, \omega_k)$$

$$+ (n_0 f_q)\delta(p_0 - \epsilon^+_q - \omega_k)\delta(p - q - k) + (1 + n_0)f_q\delta(p_0 - \epsilon^+_q + \omega_k)\delta(p - q + k)$$

$$+ (2\pi)^4 ig^2 \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} [\gamma^a p^*_{\mu}(-q_\mu, -\epsilon^-_q) + m^*(-q, -\epsilon^-_q)]Z_F(-q, -\epsilon^-_q)Z_B(k, \omega_k)$$

$$+ (n_k(1 - f_q))\delta(p_0 + \epsilon^-_q - \omega_k)\delta(p + q - k)$$

$$+ (1 + n_k)(1 - f_q)\delta(p_0 + \epsilon^-_q + \omega_k)\delta(p + q + k), \quad (7.60)$$
and

\[
\Pi^<(p) = \Pi^>(-p) = -4ig^2\gamma (2\pi)^4 \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \left\{ \Theta(-q,-\epsilon_q^-,-k,-\epsilon_k^-)ZF(-k,-\epsilon_k^-)ZF(-q,-\epsilon_q^-) \right. \\
\times f_q(1-f_k)\delta(p+q-k)\delta(p_0+\epsilon_q^-+\epsilon_k^-) \right. \\
+\Theta(-q,-\epsilon_q^-,-k,\epsilon_k^+)ZF(k,\epsilon_k^+)ZF(-q,-\epsilon_q^-) \\
\left. \times (1-f_q)(1-f_k)\delta(p+q+k)\delta(p_0+\epsilon_q^++\epsilon_k^+) \right. \\
+\Theta(q,\epsilon_q^+,k,-\epsilon_k^-)ZF(-k,-\epsilon_k^-)ZF(q,\epsilon_q^+)f_qf_k\delta(p-q-k)\delta(p_0-\epsilon_q^+-\epsilon_k^-) \right. \\
+\Theta(q,\epsilon_q^+,k,\epsilon_k^+)ZF(k,\epsilon_k^+)ZF(q,\epsilon_q^+)f_q(1-f_k)\delta(p-q+k)\delta(p_0-\epsilon_q^++\epsilon_k^+) \right\},
\]

(7.61)

where, for convenience, we have defined the function

\[
\Theta(p,p_0,q,q_0) = p^*_{\mu}(p,p_0)p^*_{\mu}(q,q_0) + m^*(p,p_0)m^*(q,q_0).
\]

(7.62)

Inserting expressions (7.59) and (7.60) into Eq. (7.48) for the particle distribution function yields the Boltzmann equation

\[
\begin{aligned}
&\left[ \frac{\partial}{\partial T} - \frac{\partial \epsilon_p^+}{\partial \mathbf{R}} \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{\partial \epsilon_p^+}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{R}} \right] f(p) = \\
&2g^2 \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \Theta(q,\epsilon_q^+,p,\epsilon_p^+)ZF(p,\epsilon_p^+)ZF(q,\epsilon_q^+)Z_B(k,\omega_k) \\
&\{ [n_kf_q(1-f_p)(2\pi)^4\delta(p-q-k)\delta(\epsilon_p^+-\epsilon_q^-+\omega_k) \\
&+ (1+n_k)(1-f_p)f_q(2\pi)^4\delta(p-q+k)\delta(\epsilon_p^+-\epsilon_q^-+\omega_k)] \\
&\} - [n_kf_p(1-f_q)(2\pi)^4\delta(p-q+k)\delta(\epsilon_p^+-\epsilon_q^++\omega_k)]
\end{aligned}
\]
\[(1 + n_k)f_p(1 - f_q)\frac{d}{d\tau} \delta(p - q - k)\delta(\epsilon_p^+ - \epsilon_q^+ - \omega_k)\]  
+ \text{anti-particle terms} \} . \quad (7.63)

Similarly, upon substitution of Eq. (7.61) for the polarization insertion into Eq. (7.19) for the boson distribution function, \(n\), we find

\[
\left[ \frac{\partial}{\partial T} - \frac{\partial \omega_p}{\partial R} \cdot \frac{\partial}{\partial p} + \frac{\partial \omega_p}{\partial p} \cdot \frac{\partial}{\partial R} \right] n(p) =
\]

\[
4g^2\gamma \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \Theta(q, \epsilon_q^+, k, \epsilon_k^+) Z_F(k, \epsilon_k^+) Z_B(q, \epsilon_q^+) Z_B(p, \omega_p)
\]

\[
\{ [(1 + n_p)f_q(1 - f_k) - n_p f_k(1 - f_q)] (2\pi)^4 \delta(p - q + k)\delta(\omega_p - \epsilon_q^+ + \epsilon_k^+) + \text{anti-particle terms} \} . \quad (7.64)
\]

For simplicity, in deriving Eqs. (7.63) and (7.64), we neglect any coupling to anti-particle states. It is easy to include them, \textit{formally}, but it is not clear what to do with them due to the \textit{non-perturbative} nature of the equations. Such an inclusion would imply renormalization but, at this point, an adequate non-perturbative renormalization scheme does not exist. Moreover, since the theory is not asymptotically free, it is questionable whether their inclusion makes any sense[33, 34, 35]. This omission is also consistent with our view of the \textit{phenomenological} nature of hadronic field theories.

Equations (7.63) and (7.64) are coupled Boltzmann equations for the fermion distribution function \(f(R, p)\) and the boson distribution function \(n(R, p)\), respectively. The left hand sides have the familiar form of a drift term while the right hand sides represent collision terms. The collision terms, illustrated in Fig. 7.1,
include Pauli-blocking as is evident from the presence of the $1 - f$ factors; in addition, they contain the familiar terms proportional to $1 + n$ indicating stimulated emission of bosons.

While the coupled Boltzmann equations, (7.63) and (7.64) present a formidable numerical challenge, it is possible under certain conditions\[31\] to “integrate” out the boson distribution function $n(R,p)$, reducing the task to that of solving a single Boltzmann equation. To see this, note that Eq. (6.15) for the boson function $\Delta^{>\!<}(x,y)$ may be put in the form

$$\Delta^{-1}_R \Delta^{>\!<} = \Pi^{>\!<} \Delta_A,$$ \hspace{1cm} (7.65)

or upon Wigner transformation and gradient expansion,

$$\Delta^{>\!<}(R,p) = \Delta_R(R,p)\Pi^{>\!<}(R,p)\Delta_A(R,p) + O(\hbar).$$ \hspace{1cm} (7.66)

The $O(\hbar)$ term may be neglected if the meson distribution functions vary slowly enough that local equilibrium is maintained. It is easy to see that the approximation, Eq. (7.66), guarantees that the collision terms of the boson Boltzmann equation, Eq. (7.3), vanish provided one neglects terms of $O(\hbar)$. This defines local equilibrium. Indeed, inserting Eq. (7.66) into the RHS of Eq. (7.3) and neglecting terms of $O(\hbar)$,

$$0 = \Pi^{<}(R,p)\Delta_R(R,p)\Pi^{>}(R,p)\Delta_A(R,p) - \Pi^{>}(R,p)\Delta_R(R,p)\Pi^{<}(R,p)\Delta_A(R,p).$$ \hspace{1cm} (7.67)
Figure 7.1: Collision Terms
The right hand side of this equation vanishes and hence the Boltzmann equation for the boson field is trivially satisfied in this limit. With the approximation, Eq. (7.66), we can write

$$\Sigma^>(R,p) = -4i\gamma g^2 \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left| \Delta_R(q-p) \right|^2 \Theta(q,q') Z_F(p') Z_F(q') Z_F(q)$$

$$[\gamma^\mu p^\mu(p') + m^*(p')] f_q (1 - f_{p'}) (1 - f_q') (2\pi)^4 \delta^4(p + q - p' - q') \quad (7.68)$$

and

$$\Sigma^<(R,p) = 4i\gamma g^2 \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left| \Delta_R(p' - p) \right|^2 \Theta(q,q') Z_F(p') Z_F(q') Z_F(q)$$

$$[\gamma^\mu p^\mu(p') + m^*(p')] f_p f_q (1 - f_q) (2\pi)^4 \delta^4(p + q - p' - q'), \quad (7.69)$$

where we have neglected anti-particle contributions. Furthermore it is understood that all four-momenta are on shell, i.e. $p_0 = \epsilon^+(p)$. Substitution into Eq. (7.48) yields the single Boltzmann equation

$$\left[ \frac{\partial}{\partial T} - \frac{\partial \epsilon_\perp^+(p)}{\partial p} \cdot \frac{\partial}{\partial p} + \frac{\partial \epsilon_\perp^+(p)}{\partial p} \cdot \frac{\partial}{\partial p} \right] f(p) =$$

$$8\gamma g^2 \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left| \Delta_R(p' - p) \right|^2 \Theta(q,q') \Theta(p,p') Z_F(p) Z_F(p') Z_F(q) Z_F(q')$$

$$\times \left[ f_p f_q (1 - f_p) (1 - f_q) - f_p f_q (1 - f_q) (1 - f_{p'}) \right] (2\pi)^4 \delta^4(p + q - p' - q'). \quad (7.70)$$

This equation has an easy interpretation. The left-hand side is the drift or streaming term. This term describes the measure preserving evolution of a particle with momentum $p$ and velocity $\partial \epsilon_\perp^+(p)/\partial p$ "drifting" through phase space under the influence of a "force" $-\partial \epsilon_\perp^+(p)/\partial R$. This "force" is due to the direct and exchange
contributions of the self-consistent Hartree-Fock potential generated by the other particles in the medium. In the absence of collisions, the right hand side is zero and we recover the classical Liouville equation. The terms in square brackets on the right hand side describe collisions in and out of a phase space element with Pauli-blocking; the delta function ensures that energy–momentum conservation is maintained. The $Z_F(p)$ factors, or, more precisely $Z_F(p) = 2p_0Z_F(p)$ are renormalizations for the outgoing fermion lines due to the momentum dependence of the self-energy; this feature is absent in previous investigations\[30, 32\]. The remaining factors represent the effective cross section for the collisions which include dynamical screening effects due to medium modifications. *We emphasize that no matter how physically intuitive and appealing Eq. (7.70) is, it is only valid when the boson fields are in local equilibrium— an unlikely scenario in an energetic heavy ion collision.*

### 7.2 Discussion

In the previous section, we present two sets of Boltzmann equations. The first set consists of two coupled equations, (7.63) and (7.64), describing the self-consistent transport of mutually interacting fermions and bosons. These two equations are derived under the assumption that the pole value of the propagators dominate the dynamics (Kadanoff–Baym Ansatz) thus leading to the on-shell propagation of particles. Hence, the collision terms reflect this on-shell constraint through the presence of energy–momentum conserving delta functions and as a result, the
bosons are created on-shell. Such production processes cannot take place in the vacuum because energy and momentum cannot be simultaneously conserved. Creation via this process can only take place in the presence of a medium through the modification of effective masses, etc. For this reason, the production process that we describe here is analogous to Cerenkov radiation. Although it is easy to produce photons via Cerenkov radiation, it may be very difficult to produce mesons in this manner since the mesons are quite massive. Because the mesons are so massive, the only way for them to be produced by this process is for the medium to significantly modify the masses of the particles in such a way as to maintain energy–momentum conservation. Clearly, this could be a problem for the heavy mesons, and the utility of these equations is problematic.

The second consists of a single equation, Eq. (7.70), for the fermion distribution function. It was derived under the assumption that the mesons remain in equilibrium during transport. Stated in another way, the mesons interacting with the nucleons are virtual particles. This picture is expected to be valid at low energies below or just above pion production threshold.

Clearly the most physical scenario lies somewhere between the two extreme pictures described above. In a reasonable picture, it should be possible to describe the creation and destruction of real as well as virtual particles and not just one or the other. In this picture, the production mechanism for mesons is bremsstrahlung, where two nucleons exchange virtual mesons with space–like momenta producing
an off-shell nucleon which then decays to an on-shell nucleon and an on-shell meson (Fig. 7.2). This mechanism is more physically appealing and much more likely for meson production than the Cerenkov process described above. It must be emphasized that any transport theory which employs a mass-shell constraint will inevitably lead to meson production via a Cerenkov-like process. For this reason, it is necessary to go beyond the quasi-particle approximation if a consistent transport theory is to be constructed.

In this work, through a series of approximations and Ansätze, we derive semi-classical transport equations from the underlying quantum field theory. Now one must ask whether or not these approximations or Ansätze are valid and if so under what conditions.

In an attempt to describe the dynamics classically, we make the assumption that the gradient expansion is a useful expansion. The validity of this expansion hinges upon the assumption that the space–time inhomogeneities of the system vary much slower than the average space–time distance between successive collisions. Equivalently, the average distance between collisions is assumed to be much larger than the distance scale set by the collision time. In a typical NN collision, the mean free path between collisions, \( \lambda \), is on the order of \( \lambda \sim (\rho \sigma)^{-1} \sim (1 - 2) \text{ fm}^{-1} \) for energies around 1 Gev per nucleon. However, the nuclear force is of approximately the same range, hence, quantum effects are expected to be important. The argument presented here is for an individual nucleon–nucleon collision. How-
Figure 7.2: An Off Shell Boson Production Mechanism
ever, in a heavy ion collision, many such collisions take place and it is hoped that the quantum interference effects will tend to cancel out. Perhaps this is the reason for the success of semi-classical kinetic equations in the description of heavy ion collisions; however, it has yet to be demonstrated.

The most crucial Ansatz which dictates a Markovian behavior of the resulting transport equations is the Kadanoff-Baym Ansatz. This Ansatz is equivalent to putting the particles on mass-shell between collisions. However, from the energy-time uncertainty principle, nucleons are expected to be off-shell on the order of $\hbar/\tau \sim 100$ Mev or more where $\tau = (\rho \sigma v)^{-1} \sim 2 \text{ fm/c}$ is the time between collisions. Since this is a substantial fraction of the nucleon mass it is not clear whether the assumption of putting the nucleons on their mass-shell between collisions is valid. This point was first emphasized by Danielewicz [16]. Moreover, as eluded to earlier, putting the mesons on mass-shell results in the production of mesons via Cerenkov radiation. Any other production mechanism is outside the domain of the pole approximation.

In this thesis an attempt is made to present a systematic program for constructing relativistic Boltzmann equations from first principles by generalizing the same techniques that have been successful in non-relativistic transport theory. We present the closed time-path method of Schwinger using a contour-functional approach which parallels the approach used in the ordinary vacuum theory. Using the Langreth-Wilkins rules to continue from the contour to the real time axis we
derive the generalized Boltzmann equation from the contour Dyson equation, showing that the generalized Boltzmann equation is nothing more than a re-statement of the Dyson equation. At this point it is necessary to introduce an Ansatz to relate the “off-time-diagonal components” of the Green function, $G^{<}(t, t')$, to its “time-diagonal” components $G^{<}(t, t)$ since the generalized Boltzmann equation has a two-time structure while a Boltzmann equation only depends upon a single time. The Ansatz fulfilling these needs used here is the Kadanoff-Baym Ansatz. This Ansatz consists of describing the system in terms of weakly interacting quasi-particles with a “delta function–like” width for the energy spread. This is found to be equivalent to the so-called “mass-shell” constraint used by other authors[26, 30, 32].

To illustrate these ideas we work with a relativistic system of spin-half particles interacting with spin-zero quanta via a Yukawa coupling. Although the program outlined above is straightforward to implement for the bosons we find it convenient to use the spin-symmetric Ansatz, Eq. (7.30), of Bezzerides and Dubois to deal with the matrix structure of the fermion Green functions. Having dealt with the discrete variables we are then able to derive a set of coupled transport equations for the quasi-particle distribution functions. Finally, assuming local equilibrium for the boson quasi-particles we eliminate the boson function in favor of one Boltzmann equation for the fermion distribution function. We have demonstrated that above meson production threshold one cannot describe the dynamics of a system...
in terms of the fermion distribution function alone, ignoring the dynamics of the meson fields, regardless of whether or not medium-dependent cross sections are used, if the boson fields are not in equilibrium. This condition should be violated in heavy ion collisions of sufficiently high energy, invalidating any attempt which ignores dynamical mesons, relativistic or nonrelativistic. This claim should be intuitively obvious. At high energies, mesons must carry a significant fraction of the energy and momentum after a collision. This means any description of the equilibration process should refer to the self-consistent transport in this sector. In other words, one should be forced to consider coupled Boltzmann equations for the particle distribution functions. It is possible that an energy regime exists within which only pions need be explicitly treated.

In this work, although the Lagrangian used is renormalizable, we neglect vacuum effects such as creation and annihilation of anti-particles. Even if one were dealing with a fundamental rather than an effective theory, the nonperturbative approximations we employ would require nonperturbative renormalizations. Thus, the consideration of vacuum effects is premature at this time. Moreover, this work is aimed at phenomenological hadronic field theories in which one usually ignores the vacuum anyway. We feel that a more immediate problem is the choice of the parameters, e.g.,, meson masses and coupling constants. Unlike QED, in which the renormalized parameters are measured experimentally, hadronic field theories such as the Walecka model[9] have the peculiarity that parameters are adjusted to fit
the bulk properties of nuclei; hence, the masses and coupling constants are chosen differently at each level of approximation. The Boltzmann equations derived in this paper come from a "self-consistent Hartree-Fock plus RPA" approximation to the Dyson equation— an approximation that goes far beyond what has been done for the ground state in the Walecka model. So before attempting a numerical solution of the Boltzmann equations presented here, one needs to study the ground state at this same level of sophistication (neglecting Dirac sea effects).
8.1 Introduction

Since their inception, Schwinger–Dyson equations have been employed as a means of going beyond the framework of perturbation theory. As we have seen, these equations form an infinite hierarchy of equations for the Green functions of the theory. At present, their solution requires that the hierarchy be truncated at some point by making an ansatz relating higher order Green functions to lower order ones, thus closing the set of equations. Because of the tremendous complexity of the resulting Schwinger–Dyson equations, the truncation is usually performed at a very low level, often at the one body level leaving a set of equations for the two-point functions of the theory. Truncation at this level is usually carried out by neglecting the dressing of the three point functions of the theory, i.e. replacing the full vertex by its bare value. Making this ansatz in QED leads to an approximation known as the "ladder" approximation. If, in addition, one neglects
the contribution from fermion loops then the resulting truncation is known as the “quenched-ladder” approximation. These two approximations are used quite often in field theory. Studies of chirally invariant gauge theories have been carried out within the quenched-ladder approximation and have led to the speculation of a possible ultraviolet fixed point in QED. The same approximations have been used in the study of high density nuclear matter within the context of phenomenological hadronic field theories where the quenched ladder approximation is commonly called the Hartree-Fock approximation. Such approximations are also made in trying to motivate the form of transport equations that have been successful in the study of heavy ion reactions.

Used in this sense, the term “approximation” should be regarded with some suspicion; perhaps “Ansatz” would be more appropriate. This is because, despite the wide use of these “approximations”, very little is known about their validity. One should not find this too surprising due to the enormous difficulties inherent in relativistic quantum field theories. In this work, an attempt is made to examine the validity of the ladder approximation. We present an exactly soluble quantum field theory in 0+1 dimensions. This model describes fermion and boson fields interacting via a Yukawa coupling. A similar theory has been studied by Brooks and Frautschi[36]; however, unlike their model, there are no anti-particles in the present theory. Although, the model might at first seem too simplistic, the associated Schwinger-Dyson equations are by no means trivial. However, due to the lack
of anti-fermions the quenched approximation is exact, but the ladder approximation is not, due to the presence of a nontrivial vertex function. Hence this model is ideal for studying the ladder approximation since one is not forced to make other approximations which undermine the analysis, casting doubt upon the entire study.

This chapter is organized as follows. In the next section, the model is introduced and the energy spectrum of the theory is derived. The exact two-point functions of the theory are presented in the third section. In the fourth section, we set up the Schwinger-Dyson equations followed by their numerical solution in section five. The conclusion follows in the last section.

8.2 A Model Field Theory

The work in this chapter is based on the 0+1 dimensional Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\frac{d}{dt} - m)\Psi - \frac{1}{2} \Phi \left(\frac{d^2}{dt^2} + m^2_\Phi\right)\Phi + g\Phi\bar{\Psi}\Psi$$

(8.1)

which describes a fermion field operator $\Psi$ interacting with a hermitian boson field operator $\Phi$ via a Yukawa coupling with a dimensionful coupling constant $g$. The hermitian conjugate to $\Psi$ is denoted as $\bar{\Psi}$.

The operator field equations which follow from the Lagrangian in the usual fashion are

$$(i\frac{d}{dt} - m)\Psi + g\Phi\Psi = 0$$

(8.2)
for the fermion field and
\[
\left( \frac{d^2}{dt^2} + m_f^2 \right) \Phi - g \bar{\Psi} \Psi = 0 \tag{8.3}
\]
for the boson field.

It is easily seen that the fermion number operator
\[
N = \bar{\Psi} \Psi \tag{8.4}
\]
and the energy operator or Hamiltonian
\[
H = (m - g \Phi) \bar{\Psi} \Psi + \frac{1}{2} \left( \frac{d \Phi}{dt} \right)^2 + \frac{1}{2} m_f^2 \Phi^2 \tag{8.5}
\]
are constants of motion for the system.

Quantization is achieved by demanding that the Hamiltonian be the generator of time translations according to the rule
\[
i \frac{d}{dt} Q(t) = [Q(t), H] \tag{8.6}
\]
for any operator \( Q(t) \). In particular, replacing \( Q(t) \) by the field operators \( \Psi(t) \), \( \Phi(t) \), and \( \dot{\Phi}(t) \) in the above equation and demanding that the resulting equations be consistent with the field equations yields the usual equal-time algebra for the field operators
\[
0 = [\Phi(t), \Phi(t)] = [\dot{\Phi}(t), \dot{\Phi}(t)] \tag{8.7}
\]
\[
0 = \{\Psi(t), \Psi(t)\} = \{\bar{\Psi}(t), \bar{\Psi}(t)\} \tag{8.8}
\]
\[
i = [\Phi(t), \dot{\Phi}(t)] \tag{8.9}
\]
\[
1 = \{\Psi(t), \bar{\Psi}(t)\}. \tag{8.10}
\]
The anti-commutator algebra of the fermion field operators follow from the physical requirement that the spectrum of the Hamiltonian have a finite lower bound.

From the operator algebra, it is easily seen that the number operator \( N \) commutes with the Hamiltonian, i.e.,

\[
[N, H] = 0. \tag{8.11}
\]

This fact permits the Hamiltonian to be easily diagonalized by the unitary transformation

\[
U = \exp(i \frac{gN}{m^2} \hat{\Phi}) \tag{8.12}
\]

with

\[
H \rightarrow H' = U H U^\dagger
\]

\[
= N(m - \frac{1}{2} \frac{g^2}{m^2}) + \frac{1}{2} (\frac{d}{dt} \Phi)^2 + \frac{1}{2} m^2 \Phi^2. \tag{8.13}
\]

Here we have used the fact that \( N \) is idempotent, i.e., \( N^2 = N \). The unitary transformation has effectively decoupled the Fermi and Bose fields with the interaction simply renormalizing the mass of the fermion. The energy of the system is trivially seen to be

\[
E_{N,n} = (m - \frac{1}{2} \frac{g^2}{m^2})N + m_s(n + \frac{1}{2}) \begin{cases} N = 0, 1 \\ n = 0, 1, 2, \ldots \end{cases} \tag{8.14}
\]

We see that there exists a critical coupling constant

\[
g_c^2 \equiv 2mm_s^2, \tag{8.15}
\]
such that for \( g^2 < g_c^2 \), the ground state consists of the fermion number zero sector, while for \( g^2 > g_c^2 \), the ground state consists of the fermion number one sector. Stated another way, for coupling below the critical coupling, the ground state is just the bare vacuum; however, above the critical coupling the ground state consists of a single fermion and a coherent boson state which dresses the fermion.

### 8.3 Exact Results

Since the object of this study is to compare the solution of a truncated set of Schwinger–Dyson equations to the exact solution of this model, in this section we calculate the exact two–point functions of the theory. As elsewhere in this work, we work in the closed time–path formalism. The ground state of the system may be obtained by taking the zero temperature limit \( (\beta \to \infty) \) for which the distribution functions have the limiting forms

\[
\begin{align*}
n_\beta(p) & \longrightarrow -\theta(-p), \\
f_\beta(p) & \longrightarrow \theta(-p).
\end{align*}
\]

We leave it to the reader to show that the expectation value of the Hamiltonian in an equilibrium state is given by

\[
\langle H \rangle = -\frac{1}{\pi} \int_{-\infty}^{\infty} dp \, p f_\beta(p) \text{Im} R(p) + \frac{1}{2} m^2_s \langle \Phi \rangle - \int_{-\infty}^{\infty} \frac{dp}{2\pi} n_\beta(p)(p^2 + m^2_s) \text{Im} \Delta_R(p)
\]

\[\text{(8.18)}\]
and that the mean field ($\langle \Phi \rangle$) is given by

$$\langle \Phi \rangle = \frac{2g}{m_\phi^2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} n_\rho(p) \text{Im} G_R(p).$$

(8.19)

It is instructive to consider a free theory. For a free theory one can show that

$$G_R(p) = \frac{1}{p - m + i\varepsilon}$$

(8.20)

and

$$\Delta_R(p) = \frac{1}{p^2 - m_\phi^2 + i\varepsilon \text{sgn}(p)},$$

(8.21)

from which it follows that

$$\langle H \rangle = m f_\rho(m) + m_\phi [n_\rho(m_\phi) + \frac{1}{2}],$$

(8.22)

which has an obvious interpretation.

Our immediate goal in this section is to calculate the exact retarded two-point functions of the theory. To do so requires an exact solution of the operator field equations. To this end, we note that since $N$ is conserved, it immediately follows from the field equation for $\Phi$, Eq. (8.3), that

$$\Phi(t) = (\Phi(0) - \frac{gN}{m_\phi^2}) \cos(m_\phi t) + \frac{1}{m_\phi} \dot{\Phi}(0) \sin(m_\phi t) + \frac{gN}{m_\phi^2},$$

(8.23)

from which as the reader can readily show

$$\Delta_R(p) = \frac{1}{p^2 - m_\phi^2 + i\varepsilon \text{sgn}(p)}.$$  

(8.24)

This simple result should be no surprise since, with the exception of the boson condensate, the boson field is a free field.
An explicit solution for the fermion field is not so trivial. We start with

\[ \Psi(t) = e^{iHt}\Psi(0)e^{-iHt}, \quad (8.25) \]

from which one can show that

\[ \Psi(t) = e^{-im^*t}R(t)R^{-1}(0)\Psi(0) \quad (8.26) \]

where

\[ R(t) = \exp(-i\frac{g}{m^*_s}\Phi(t)) \quad (8.27) \]

and

\[ R(0)\Psi(0) = U\Psi(0)U^\dagger, \quad (8.28) \]

with the definition of \( U \) given by Eq. (8.12). Note that we have also introduced an effective mass

\[ m^* = m - \frac{1}{2}\frac{g^2}{m^*_s}, \quad (8.29) \]

and unless explicitly indicated, all operators are evaluated at \( t = 0 \). Defining

\[ R'(t) \equiv UR(t)U^\dagger, \quad (8.30) \]

with the important property that \([R'(t), \Psi(0)] = 0\), we can write

\[ \Psi(t) = U^\dagger R'(t)R^{-1}(0)e^{-im^*t}\Psi(0). \quad (8.31) \]

It then follows that

\[ \Psi(t)\bar{\Psi}(0) = R'(t)R^{-1}(0)e^{-im^*t}\Psi(0)\bar{\Psi}(0) \quad (8.32) \]
and

\[ \Phi(0)\Psi(t) = R'(t)R^{-1}(0)e^{-im\tau(t)}\Phi(0)\Psi(0). \]  

(8.33)

At this point, we transform to a new set of canonical variables \(a\) and \(a^\dagger\) defined through the relations

\[ \Phi(0) = \frac{i}{\sqrt{2m_\tau}}(a - a^\dagger), \]  

(8.34)

\[ \Phi(0) = \sqrt{\frac{m_\tau}{2}}(a + a^\dagger), \]  

(8.35)

where

\[ [a, a^\dagger] = 1. \]  

(8.36)

From the repeated application of famous the Baker–Campbell–Hausdorf formula

\[ e^{A+B} = e^A e^B e^{-\frac{1}{2} [A,B]}, \]  

(8.37)

which holds if \([A, B]\) is a c-number, one can easily show that

\[ R'(t)R^{-1}(0) = \exp(-\tilde{g}^2(1 - e^{-im\tau(t)}))\exp(i\tilde{g}a^\dagger(1 - e^{im\tau(t)}))\exp(i\tilde{g}a(1 - e^{-im\tau(t)})). \]  

(8.38)

Here we have introduced a dimensionless coupling constant \(\tilde{g}\) defined by

\[ \tilde{g}^2 = \frac{g^2}{2m_\tau^2}. \]  

(8.39)

For our purposes, it is convenient to transform from the bare boson field operators to the dressed operators defined by

\[ \alpha = U^\dagger aU = a + i\tilde{g}N \]  

(8.40)
\[ \alpha^\dagger = U^\dagger a^\dagger U = a^\dagger - igN. \]  

(8.41)

In the \( N = 0 \) sector, these operators are equivalent; however, for the \( N = 1 \) sector, the new operators create boson excitations on top of the coherent state. Under this transformation, it is straightforward to show that

\[
\langle \Phi(0)\Phi(t) \rangle = e^{-\im \omega^* t} \exp(-\tilde{g}^2(1 - e^{\im \omega^* t})) \sum_{j,k=0}^{\infty} \frac{(-i\tilde{g})^{j+k}}{j!k!}(e^{\im \omega^* t} - 1)^j(e^{-\im \omega^* t} - 1)^k(\alpha^\dagger j\alpha^k\Phi(0)\Phi(0))
\]

and

\[
\langle \Psi(t)\Phi(0) \rangle = e^{-\im \omega^* t} \exp(-\tilde{g}^2(1 - e^{\im \omega^* t})) \sum_{j,k=0}^{\infty} \frac{(-i\tilde{g})^{j+k}}{j!k!}(e^{\im \omega^* t} - 1)^j(e^{-\im \omega^* t} - 1)^k(\alpha^\dagger j\alpha^k\Phi(0)\Phi(0)).
\]

(8.42)  

We see that the general initial value problem for the correlation functions requires the knowledge of an infinite number of initial correlations. However, in this work we are interested only in equilibrium configurations of the system, in particular, the ground state. We defer the more general initial value problem to the future.

Not all initial correlations of the system are consistent with equilibrium configurations. To single out equilibrium configurations we appeal to thermodynamic arguments. We can ensure thermodynamic consistency by demanding that the equilibrium condition, Eq. (6.26), between \( \text{Im} \, G_R(p) \) and the correlation function \( G^<(p) \) hold. Doing so and noting from the previous section that there are no
dressed bosons in the ground state we find in the zero temperature limit

\[ G_R(p) = e^{-g^2} \sum_{k=0}^{\infty} \frac{\theta(m^*)}{k!} \left\{ \frac{\theta(m^*)}{p - m^* - k m_s + i \epsilon} + \frac{\theta(-m^*)}{p - m^* + k m_s + i \epsilon} \right\}. \quad (8.44) \]

We see that for \( m^* > 0 \), \( G_R(p) \) has discrete poles at positive values of \( p \) with the spacing between poles given by the meson mass \( m_s \). The fact that adjacent poles are separated by the meson mass is a consequence of the free nature of the boson field— the only effect of the Yukawa interaction is through the presence of a boson condensate which simply renormalizes the fermion mass. For \( m^* \) less than zero, the poles occur only at negative values of \( p \). This should also be contrasted with the situation for more "realistic" field theories in which the propagator acquires branch cuts at each meson production threshold. The discrete poles observed in this model are due to the requirement that only energy be conserved in the production process since there is no spatial momentum to be conserved.

Upon inserting the expression for \( G_R(p) \) derived above into the expression for \( \langle \Phi \rangle \), Eq. (8.19), and taking the zero temperature limit and we find

\[ \langle \Phi \rangle = \frac{g}{m_s^2} \theta(-m^*), \quad (8.45) \]

explicitly demonstrating the existence of a boson condensate when \( m^* \) becomes negative. It then follows from Eq. (8.18) that the ground state energy is

\[ \langle H \rangle = \frac{1}{2} m_s + m^* \theta(-m^*), \quad (8.46) \]

which agrees with the result obtained by conventional means in the previous section.
In this section, we solve the Schwinger-Dyson equations in the ladder approximation for the present model. Due to the absence of anti-fermions there are no fermion loops; hence, the boson retarded function is given by its free field expression whose imaginary part is

\[ \text{Im} \Delta_R(p) = -\pi \text{sgn}(p) \delta(p^2 - m^2_\phi). \]  

(8.47)

However, the fermion two-point function is not so trivial. It satisfies the Schwinger-Dyson equation

\[ G_R^{-1}(p) = p - m + g \phi - \Sigma_R(p) \]  

subject to the retarded boundary condition. Here \( \phi = \langle \Phi \rangle \) is the mean field given by Eq. (8.19) and \( \Sigma_R(p) \), the retarded self energy[37], is a functional of \( G_R(p) \).

The self-energy is given by a sum of all "rainbow" and "crossed-rainbow" diagrams. The crossed-rainbow diagrams are essentially a dressing of the vertex function which we ignore in this work, that is, we make the ladder approximation. In the ladder approximation, the imaginary part of the retarded self energy is given by

\[ \text{Im} \Sigma_R(p) = -2g^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \text{Im} G_R(q) \text{Im} \Delta_R(q - p) \left[ n_\beta(q - p) + f_\beta(q) \right]. \]  

(8.49)

Since \( \Sigma_R(p) \) is a retarded function, its real part may be determined from the

8.4 Schwinger–Dyson Equations
dispersion relation

\[ \Sigma_R(p) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dq \frac{\text{Im} \Sigma_R(q)}{p - q + i\varepsilon}. \quad (8.50) \]

The above expression assumes of course that the self energy goes to zero sufficiently fast as \( p \to \infty \); for the more general case of a polynomial growth at infinity, one must use a subtracted dispersion relation. However, this is an unnecessary complication for the present model.

It is instructive to first consider the Hartree or mean field solution of the Schwinger–Dyson equations. In this approximation we ignore \( \Sigma_R(p) \) obtaining

\[ G_R(p) = \frac{1}{p - m + g\phi + i\varepsilon}. \quad (8.51) \]

Inserting the above equation into the expression for \( \langle \phi \rangle \), Eq. (8.19), we obtain in the zero temperature limit

\[ \langle \phi \rangle = \frac{g}{m_\phi^2} \theta(g\phi - m) \]

\[ = \frac{g}{m_\phi^2} \theta\left( \frac{g^2}{m_\phi^2} - m \right) \quad (8.52) \]

Comparing this expression with the exact result, Eq. (8.45), we see that the mean field theory gives the correct magnitude for \( \langle \phi \rangle \) but it underestimates the critical coupling constant. The ground state energy of the system in this approximation is given by

\[ \langle H \rangle_{MF} = \frac{1}{2}m_\phi + m_\phi^* \theta(g^2 - mm_\phi^2). \quad (8.53) \]

We note that for small coupling, \( g^2 < mm_\phi^2 \), as well as for large coupling, \( g^2 > 2mm_\phi^2 \), the mean field theory gives the correct value for the ground state energy
of the system. However, for intermediate coupling mean field theory fails; this is a consequence of the failure to predict the correct value of the critical coupling constant in the ladder approximation.

We now consider the effect of self-energy corrections on the propagator. Inserting the expression for \( \text{Im} \Delta_R(p) \), Eq. (8.47), into Eq. (8.49) and taking the zero temperature limit we obtain

\[
\text{Im} \Sigma_R(p) = \frac{g^2}{2m_*} \left[ \theta(-p - m_s) \text{Im} G_R(p + m_s) + \theta(p - m_s) \text{Im} G_R(p - m_s) \right],
\]

from which the dispersion relation Eq. (8.50) implies

\[
\Sigma_R(p) = -\frac{g^2}{2\pi m_*} \int_{-\infty}^{\infty} dq \left[ \frac{\theta(-q) \text{Im} G_R(q)}{p - q + m_s + i\varepsilon} + \frac{\theta(q) \text{Im} G_R(q)}{p - q - m_s + i\varepsilon} \right].
\]

Note the presence of the theta functions in the expression for the imaginary part of the self energy. The theta functions signify that the imaginary part of the self energy is zero below meson production threshold— a result familiar from other field theories. However, there is a major difference here. Due to the lack of fermion loops on the boson lines which contribute to the self-energy there is no dissipation and hence the imaginary part of the self energy has zero width. This implies that \( \text{Im} G_R(p) \) also has zero width, i.e.,

\[
\text{Im} G_R(p) = -\pi \delta(p - m + g\phi - \text{Re} \Sigma_R(p))
\]

which yields

\[
\Sigma_R(p) = \frac{g^2}{2m_*} \int_{-\infty}^{\infty} dq \left[ \theta(-q) \delta(q - m + g\phi - \text{Re} \Sigma_R(q)) \right]
\]

\[
\frac{p - q + m_s + i\varepsilon}{p - q - m_s + i\varepsilon}
\]
Equations (4.21), (8.48), and (8.57) form a closed set of integral equations for \( G_R(p) \) in the ladder approximation. Due to the complexity of these equations, they cannot be solved analytically and one must turn to a numerical solution.

### 8.5 Numerical Solutions

By introducing the function

\[
    f(p) = p - m + g\phi - \text{Re} \Sigma_R(p) \tag{8.58}
\]

one may uncouple the integral equations (4.21), (8.48), and (8.57) yielding the single integral equation

\[
    f(p) = p - m - \frac{g^2 m_s}{\text{Re} R(p)} \int_{-\infty}^{\infty} dq \, \delta[f(q)] \left\{ \theta(q) \frac{m_s}{p - q - m_s} + \theta(-q) \frac{2(q - p) - m_s}{p - q + m_s} \right\} \tag{8.59}
\]

for the unknown function \( f(p) \). Writing

\[
    \delta[f(q)] = \sum_r Z_r \delta(q - p_r), \tag{8.60}
\]

where

\[
    f(p_r) = 0 \tag{8.61}
\]

and

\[
    Z_r^{-1} = \left| \frac{df(q)}{dq} \right|_{q=p_r}, \tag{8.62}
\]
we find
\[ f(p) = p - m - g^2 m_s \sum_r Z_r \left\{ \theta(p_r) \frac{m_s}{p - p_r - m_s} + \theta(-p_r) \frac{2(p_r - p) - m_s}{p - p_r + m_s} \right\}. \] (8.63)

Before proceeding to the numerical solution of Eq. (8.63) we make the following observation. First it is easy to convince oneself that \( f(p) \) has an infinite number of roots. With this in mind, consider the behavior of \( f(p) \) for large \( |p| \). In this limit, \( f(p) \) behaves like \( p \) except near the poles \( p_r + m_s \) where \( f(p) \) goes like \( 1/(p - p_r \pm m_s). \) Hence it follows that the roots of \( f(p) \) approach the poles of \( f(p) \) for large values of \( |p| \). More precisely, if \( p_r \) and \( p_{r+1} \) are two adjacent roots of \( f(p) \) then
\[ |p_{r+1} - p_r| \to m_s \quad \text{as} \quad p_r \to \infty. \] (8.64)

This simple result is a consequence of the fact that \( \Sigma_R(p) \) goes to zero as \( p \) goes to infinity, i.e., the theory approaches the free theory in the asymptotic limit.

Eq. (8.63) may be solved by iteration by choosing an initial set of values \( \{p_r, Z_r\} \) and iterating until convergence is achieved. In Fig. 8.1, we plot the exact values of \( \{p_r, Z_r\} \) along those determined numerically. We arbitrarily chose \( m_s = 0.5 \) on a scale in which \( m = 1 \). For small coupling, the Schwinger-Dyson results compare favorably with the exact. However, as the coupling is increased, major differences emerge. We see that for moderate values of the coupling the Schwinger-Dyson equation fails to produce even qualitatively the correct locations and residues of the poles of the spectral function. However, as noted above, for large \( p \) the spacing of the poles approaches the correct value of \( m_s \).
Figure 8.1: Spectral Function.
Figure 8.1 (continued)
Figure 8.1 (continued)

(i) $\bar{g}^2 = 2.0$  
$m_j = 0.5 \text{ m}$

(j) $\bar{g}^2 = 2.3$  
$m_j = 0.5 \text{ m}$

(k) $\bar{g}^2 = 3.0$  
$m_j = 0.5 \text{ m}$

(l) $\bar{g}^2 = 4.0$  
$m_j = 0.5 \text{ m}$
As in the mean field solution, the ladder approximation also underestimates the correct value of the critical coupling constant for which the nature of the ground state changes. For the value of $m_s/m = 0.5$ used in the numerical calculations, mean field theory gives $\tilde{g}_c^2 = 1$ while the ladder approximation predicts a value of $\tilde{g}_c^2 \approx 1.52$ in closer agreement with the exact value of 2.

Although the ladder approximation appears to have failed except in the small coupling regime it is very surprising that the ladder approximation reproduces the sum rules

$$ -\frac{1}{\pi} \int_{-\infty}^{\infty} dp \, \text{Im} G_R(p) = \sum_r Z_r = 1 $$ \hspace{1cm} (8.65)

and

$$ -\frac{1}{\pi} \int_{-\infty}^{\infty} dp \, p \, \text{Im} G_R(p) = \sum_r p_r Z_r = m - g(\Phi) $$ \hspace{1cm} (8.66)

for both weak and strong coupling to within the numerical precision of the calculation. We have performed the calculation for various values of $m_s/m$ and in each case the sum rules as predicted by the ladder approximation have agreed with the exact values. The only region where the sum rules fail is for the region of coupling between the critical value as predicted by the ladder approximation and the exact critical coupling. Again, this failure is due to the inability of the ladder approximation to give the exact critical coupling constant.

In the range of coupling where the sum rules are given exactly, the ground state energy as predicted by the Schwinger–Dyson analysis is also exact since it depends only upon these sum rules. In Fig. 8.2 the exact ground state energy is plotted
against the ground state energy as given in the ladder approximation.

8.6 Discussion

In this chapter we present an exactly soluble model quantum field theory of a fermi field interacting with a Bose field in 0+1 dimensions. We construct the exact retarded Green function for the fermion field and compare it to the one obtained from the ladder approximation to the Schwinger–Dyson equations. We find that they agree only in the limit of weak coupling. For intermediate and large couplings the ladder approximation fails to even give a qualitative picture of the spectral function.

Although the ladder approximation failed to produce a qualitative picture of the spectral function, it does reproduce the first two moments defined in Eqs. (5.48), (5.49), and (5.50). Hence, the ladder approximation reproduces the exact ground state energy in both the strong and weak coupling limits. For intermediate coupling, the approximation fails because it underestimates the value of the critical coupling constant for which the nature of the ground state changes from the fermion number zero sector to the fermion number one sector. The reason for this agreement is that the moments of the spectral function depend upon the asymptotic behavior of the theory. For this model, the theory approaches a free one in the asymptotic limit. The reader is referred to the discussion following Eq. (5.50).

We now turn to the relevance of the results of this investigation for more
Figure 8.2: Ground State Energy as Function of $\tilde{g}^2$.
“realistic” quantum field theories. The model employed here contains no anti­
particles and no ultraviolet divergences as are present in other relativistic quan­
tum field theories. Hence, one might believe that this work says nothing about
the ladder approximation used in the more realistic field theories. This we can­
not refute. However, we strongly feel that since the ladder approximation fails
in this simple model it must surely fail for the other field theories where one
must also worry about infinite renormalizations, etc.. This not to say that we
believe that Schwinger–Dyson equations cannot be profitably used in extracting
non–perturbative physics. Rather we believe that one must look beyond the ladder
approximation to the use of a dressed vertex function. We leave the investigation
of vertex dressing in the present model to the future.
Appendix A

Collection of Useful Formulae Involving Wigner Transforms

In this appendix, we gather together a number of useful formulae involving the Wigner transform. The proof of these formulae is left to the reader. The Wigner transform of a two-point function, $A(x, y)$ (this should not be confused with a contour two-point function), is defined

$$A(R, p) \equiv \int d^4 r \ e^{i r \cdot p / \hbar} A(R + \frac{r}{2}, R - \frac{r}{2}), \quad \text{(A.1)}$$

with the inverse Wigner Transform

$$A(x, y) = \int \frac{d^4 p}{(2\pi \hbar)^4} e^{-i r \cdot p / \hbar} A(R, p). \quad \text{(A.2)}$$

Here $R = \frac{1}{2}(x + y)$ and $r = x - y$.

From the definition of the Wigner transform it follows that:

$$A(x, y) \longrightarrow A(R, p) \quad \text{(A.3)}$$

$$\int d^4 z \ A(x, z) B(z, y) \longrightarrow \exp\left[\frac{i\hbar}{2} (\delta^A_R \cdot \delta^B_R - \delta^A_p \cdot \delta^B_p)\right] A(R, p) B(R, p) \quad \text{(A.4)}$$

$$A(x, y) B(y, x) \longrightarrow \int \frac{d^4 q}{(2\pi \hbar)^4} A(R, q) B(R, q - p) \quad \text{(A.5)}$$
where we use a right arrow to indicate a Wigner transformation has been performed. For the special case where the two-point functions have no discrete indices (i.e. no matrix structure), we find from Eq. (A.4) the useful result

\[ \int d^4z [A(x, z)B(z, y) - B(x, z)A(z, y)] \longrightarrow i\hbar [\partial_x A \cdot \partial_R B - \partial_R A \cdot \partial_x B](R, p) + O(\hbar^2), \]  

(A.6)

or in a more compact form

\[ [A, B](x, y) \longrightarrow i\hbar [A, B]_{PB}(R, p) + O(\hbar^2), \]  

(A.7)

which defines the generalized Poisson bracket. Similarly, when discrete indices are absent, we have

\[ \{A, B\}(x, y) \longrightarrow 2A(R, p)B(R, p) + O(\hbar^2). \]  

(A.8)

In addition, one can show that

\[ \int d^4x d^4y A(x, y)B(y, x) = \int \frac{d^4R d^4p}{(2\pi\hbar)^3} A(R, p)B(R, p), \]  

(A.9)

which is useful for calculating ensemble averages.

Finally we list a few useful transforms that are frequently encountered:

\[ \delta(x - y) \longrightarrow 1, \]  

(A.10)

\[ \phi(x)\delta(x - y) \longrightarrow \phi(R), \]  

(A.11)

\[ (i\hbar \partial_\mu)^n \delta(x - y) \longrightarrow (p_\mu)^n, \]  

(A.12)

\[ \theta(x_0 - y_0)\delta(x - y) \longrightarrow \frac{-i\hbar}{p_0 + i\epsilon}, \]  

(A.13)

\[ \theta(x_0 - y_0) \longrightarrow \frac{i\hbar}{p_0 + i\epsilon} (2\pi\hbar)^3 \delta(p). \]  

(A.14)
Bibliography


