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Stabilization and robust stability of discrete-time, time-varying systems

Dale, Wilbur Nolan, Ph.D.
The Ohio State University, 1991
Stabilization and Robust Stability of Discrete-Time, Time-Varying Systems

A Dissertation
Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

by

Wilbur Nolan Dale, M.S.

* * * * *

The Ohio State University
1991

Dissertation Committee:
Professor Malcolm C. Smith
Professor Ümit Özgüner
Professor Hitay Özbay

Approved by:

M. C. Smith
Adviser
Department of Electrical Engineering
To my parents
Julian E. and Alma H. Dale
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VITA

November 21, 1961 .......................................Born—Portsmouth, Virginia

1980–1984 summers only ...........................Department of Commerce,
National Oceanic and Atmospheric
Administration,
National Ocean Service

1984 ................................................................B.S.E.E.,
Old Dominion University,
Norfolk, Virginia

1988 ................................................................M.S.,
The Ohio State University,
Columbus, Ohio

Spring 1991 ....................................................Visitor, Gonville and Caius College,
Cambridge University
Cambridge, UK

FIELDS OF STUDY

Major Field: Electrical Engineering

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CHAPTER I

Introduction

Most of engineering consists of finding the appropriate translation of the engineering problem into the language of mathematics, and automatic control systems is no different. If fact, an argument could be made that, in control systems, the mathematics is usually the easy part of the problem. Finding an appropriate description of the physical system so that the engineering problem corresponds to a solvable problem in mathematics usually takes most of an engineers time. In this dissertation, we describe a framework that allows many questions concerning linear, discrete-time, time-varying systems to be translated into questions about the graph of an operator that models the plant.

From an operator-theoretic viewpoint, a physical system $G$ is described or modeled by a mapping of the input signals to the output signals. In other words, we have complete knowledge of the system and know that if a certain input signal is applied to the input of the system, we will observe a set of output signals that is a function only of the input signals. Often, a physical system does not behave as we desire. A common method of altering the behavior of the system is to connect another system (the compensator) as in Figure 1 to form a closed-loop system which we denote it by $\{G, F\}$. In Figure 1, $G$ represents the plant and $F$ the compensator. It is assumed that $G$ is a given physical system and we will construct a compensator $F$ so that the closed-loop system will meet certain design criteria.
In the first section of this chapter, we will discuss reasons that feedback is used to alter the behavior of a physical system. We will continue the chapter with a section describing the problems to be examined and the organization of the dissertation.

1.1 Ignorance and Feedback

One of the earliest discussions on why feedback is necessary can be found in Horowitz's book [23]. Horowitz suggested that there are three reasons for using feedback:

1. Use of feedback to contend with ignorance of a plant.

2. Use of feedback to contend with ignorance of the environment acting on the plant.

3. Use of feedback to exploit the fact that an element with feedback around it has static and dynamic characteristics which are different from those of the element by itself.
The justification of these categories is best demonstrated by some examples. Suppose the plant $G$ is unstable (i.e. the output can grow without bound for a bounded input). In theory, it is possible to construct an open-loop controller as in Figure 2 that would only apply control signals such that the output signals remain bounded. However, this is impractical for two reasons (related to 1. and 2. above). The first reason is that the design of the compensator would require exact knowledge of the plant; otherwise, the compensator might accidentally apply a signal that causes the output signal of the plant to grow without bound. Since all plant models are based on measurements that are corrupted by noise, there is always uncertainty in our model. Even without noise, the Heisenberg uncertainty principle would play a role in our measurement of the plants parameters. The second reason the open-loop compensator is impractical is that there are always uncontrollable inputs from the environment to a system (i.e. noise). If we could predict the noise, it would be possible to apply controllable input signals to cancel the effects of the noise on the plant. Unfortunately, one can not predict the noise signals and noise can cause a signal to be applied to the plant that causes the output signals to grow without bound.

![Figure 2: Open-loop Compensator](image)
If the plant $G$ is stable (i.e. bounded input signals yield bounded output signals), more subtle reasoning must be applied to show the need for feedback. However, it turns out that the same two reasons give the necessity of feedback. If there is plant uncertainty, there is uncertainty in the output. For example a common transistor (2N2222A) has a DC current gain ($\beta$) that varies from 50 to 500 depending on the manufacturer, operating temperature, and the individual device. A poorly designed amplifier with a 1 mV input could have an output between 50 mV and 500 mV. Both could have disastrous effects on the performance of a system because a signal that is too small might be lost in the ambient noise and a signal that is too large can saturate the system and the linear model would no longer apply. With feedback, it is possible to use the measured output signal to modify the input signal in such a way that there is less variation of the output signal with plant uncertainty. In fact, a properly designed amplifier uses feedback to reduce the variation of the output signal with respect to uncertainty of the parameters of the transistor. For the second reason, if there are uncontrollable inputs to a system, they also effect the output signal. For example in a radar system, the antenna must be accurately pointed for proper system operation. A random gust of wind will exert forces on the antenna to deviate the attitude of the antenna from the desired orientation. It may be that the effect of the uncontrollable input (the wind) is too great to meet the design criteria for the particular radar system. However, feedback allows measurement of the disturbance at the antenna output to alter the controllable input signals (the torque of the drive motor) such that the disturbance from the wind is reduced.

The third reason feedback is used is to exploit the fact that an element with feedback around it has static and dynamic characteristics which are different from those of the element by itself. In other words, a plant may exhibit undesirable
behavior such as large overshoot or a long settling time and the control engineer must change this. Horowitz shows that once the feedback system satisfies criteria caused by ignorance in the first two categories, the third category can (if desired) be satisfied using an open-loop compensator in front of the feedback system. Thus, the problems caused by plant uncertainty and rejection of unwanted signals decouple from the problem of changing the system response. Hence, it is not necessary to use feedback to change the system response, but feedback is often used anyway.

1.2 Problem Description and Organization

In Figure 1 the closed-loop system inputs $u_1$ and $u_2$ may represent either command inputs that we wish to track or disturbances that we wish to reject or attenuate. Different criteria are represented by selecting different input signals to be tracked or rejected. In this dissertation, we will examine three problems of general linear, discrete-time, possibly time-varying plants that are stabilized with linear, discrete-time, possibly time-varying compensators. The three general problems are stabilizability, parametrization of all stabilizing compensators, and robust compensators. The dissertation will continue by examining some plants that are not stabilizable with linear, possibly time-varying compensators and we conclude with an examination of eventually time-invariant systems.

The dissertation starts with an introduction to Banach and Hilbert spaces, operator theory, and nest algebras in Chapter II and an introduction to stable feedback systems in Chapter III. In Chapter IV, we will prove that the existence of a compensator that stabilizes the plant (stabilizability) is equivalent to the existence of a representation that completely characterizes the relationship between the input signals and the output signals of the plant. The result is an extension of
a similar result for linear, time-invariant systems that proves that stabilizability is equivalent to the existence of coprime fractions. We continue the chapter with a parametrization of all compensators that stabilize the plant $G$. We will show that representations of the compensator can be constructed from the representation of the plant and all stabilizing compensators have a representation of this form. This result is an extension of the Youla parametrization for linear, time-invariant systems where coprime fractions of the compensator are constructed from the coprime fraction of the plant and all stabilizing compensators have such coprime fractions.

The third general problem is robust stabilization and is examined in Chapter V. As stated in Horowitz's book [23], we need to contend with ignorance of the plant. Our controller must stabilize not only the plant $G$ for which the compensator was designed, but it must also stabilize all plants "close" to $G$ because the actual plant connected in the feedback loop may not have the exact properties that we attributed to its model $G$. In this chapter, we discuss several models of plant uncertainty with particular emphasis on the gap metric. We extend a result for time-invariant systems by Glover and McFarlane [20] that yields the size of the largest gap ball of uncertainty that a compensator can tolerate and still guarantee uniformly bounded closed-loop operators.

After the study of the three general problems of time-varying systems, we present in Chapter VI several examples of linear plants that are not stabilizable with linear, possibly time-varying compensators. We also give a theorem that states that a linear, discrete-time, time-invariant plant that is not stabilizable with a linear, time-invariant compensator is not stabilizable with a linear, time-varying compensator. The dissertation continues by examining eventually time-invariant systems in
Chapter VII. *Finally, the dissertation concludes with Chapter VIII summarizing the contributions of this dissertation and a description of future research.*
CHAPTER II
Mathematical Preliminaries

To understand control system theory, it is necessary to introduce certain mathematical concepts and to relate them to the physical world. In this chapter, we introduce notation and definitions used throughout the dissertation and use them to produce insight to physical systems. To this end, we will define continuous-time and discrete-time signals and relate them to the abstract vector spaces $L_p$, $\ell_p$, and $H_p$. From these spaces, we will develop a description of a physical system called the graph and relate the graph to familiar system concepts such as linear systems, time-invariant systems, and causal systems. We also will view systems as operators (possibly unbounded) and will introduce several bounded operators with special properties that are used in later chapters. We will continue the chapter with a section on nest algebras and inner/outer factorization which are used in the study of bounded, linear, causal or anti-causal systems. Finally, we conclude with a section on two standard problems: one is the time-invariant Nehari problem and the other is the time-varying Arveson distance problem.

2.1 Introduction to Vector Spaces

Let $\mathbb{Z}$ be the integers, $\mathbb{R}$ be the set of real numbers, and $\mathbb{C}$ be the complex numbers. Let $f_c$ be a mapping $f_c : \mathbb{R} \rightarrow \mathbb{C}$. Since time is a physical quantity that can be described with a real number, we can use such a mapping to describe the value of a quantity as it changes with time. Such a mapping is called a continuous-
time function because the dependent variable (time) can be any real value. These functions are useful in describing many physical quantities. For example, the function could represent capacitor voltage as a function of time, airplane altitude as a function of time, gas pressure as a function of time, and room temperature as a function of time. Some physical quantities are best described using another type of function because their time scale is not continuous. Let $f_d$ be a mapping $f_d : \mathbb{Z} \mapsto \mathbb{C}$, it is a discrete-time function because the dependent variable (time) can be only discrete values. These functions are useful in describing physical quantities that are sampled by a computer (i.e. the data is sampled every $t$ seconds) and arise whenever a computer is used to either monitor or to control a physical quantity.

Usually, a physical quantity has only real values. However, it is sometimes mathematically advantageous to consider real functions to be complex functions with a zero imaginary part. Sometimes it is computationally advantageous to combine two real functions into one complex function with one of the functions being the real part and the other function being the imaginary part. Since the proofs are the same regardless of whether the functions are real or complex, we will assume complex functions.

The above definitions allow the independent variable (time) to take on values from $-\infty$ to $+\infty$. Later, some of the mathematics is easier if time is allowed to take on only non-negative values. Some researchers do allow time to take on values from $-\infty$ to $+\infty$; however, we will only consider non-negative time ($\mathbb{R}+$ and $\mathbb{Z}+$). We feel that this is no handicap in our analysis since all physical systems have a definite beginning. The beginning may be when the system is turned on, it may be when the system is built, it may be when the system is first observed, or it may be when the universe was created (either the Big Bang or an act of God), but the beginning is
well-defined. Most of the following definitions are extendable in an obvious manner for $-\infty$ to $+\infty$ time.

The set of all possible functions is very large and many of the functions are "undesirable" within a physical system. To clarify what is meant by an "undesirable," functions are placed into classes that describe some property the functions within the class all have in common. One common classification is the $L_p[0, \infty)$ (continuous-time) and $\ell_p(0, \infty)$ (discrete-time) classes. A continuous-time function $f_c(t)$ is in $L_p[0, \infty)$ for some $0 < p \leq \infty$ if

$$\int_{0}^{\infty} |f_c(t)|^p \, dt < \infty \quad \text{for } 0 < p < \infty$$

$$\text{ess sup}_{t \in \mathbb{R}^+} |f_c(t)| < \infty \quad \text{for } p = \infty. \quad (2.1)$$

In a similar fashion, the discrete-time function $f_d(k)$ is in $\ell_p[0, \infty)$ for some $0 < p \leq \infty$ if

$$\sum_{k=0}^{\infty} |f_d(k)|^p < \infty \quad \text{for } 0 < p < \infty$$

$$\sup_{k \in \mathbb{Z}^+} |f_d(k)| < \infty \quad \text{for } p = \infty. \quad (2.2)$$

The $L_p[0, \infty)$ classes are not mutually exclusive nor collectively exhaustive. Many functions, such as the finite rectangular pulse, are in $L_p[0, \infty)$ for more than one $p$ and there are functions, such as $\exp(t)$, that are not in any $L_p[0, \infty)$ class. Similar statements are true for $\ell_p[0, \infty)$. When the range of the time variable is not important, it is often suppressed to give $L_p$ and $\ell_p$. Because $L_p$ and $\ell_p$ have so many properties in common, when we wish to refer to both we will denote this by $L_p$.

When $1 \leq p \leq \infty$, $L_p$ functions have the important mathematical properties of complete normed linear vector spaces or Banach spaces. We thus introduce the following definitions [35] and relate them to $L_p$ spaces.
Definition 1 A linear vector space is a set $V$, whose elements are called vectors and in which two operations, called addition and scalar multiplication are defined, with the following algebraic properties:

1. To every pair of vectors $x$ and $y$ there corresponds a vector $x + y$, in such a way that $x + y = y + x$ and $x + (y + z) = (x + y) + z$; $V$ contains a unique vector $0$ (the zero vector or origin of $V$) such that $x + 0 = x$ for all $x$ in $V$; and for each $x \in V$ there corresponds a unique vector $-x$ such that $x + (-x) = 0$.

2. To each pair $(\alpha, x)$, where $x \in V$ and $\alpha$ is a scalar, there is associated a vector $\alpha x \in V$, in such a way that $1x = x$, $\alpha(\beta x) = (\alpha \beta)x$, and such that the two distributive laws $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$ hold.

For our purposes, the vectors are functions in time (either in $L_p$ or $\ell_p$) and the scalars are the complex numbers. Addition and scalar multiplication are defined for the vectors by respectively adding the functions pointwise and multiplying the function with a scalar pointwise. The main property of linear vector spaces is we can add two vectors and multiply a scalar and a vector with the result remaining in the vector space.

Definition 2 A linear vector space $V$ is said to be a normed linear vector space if to each $x \in V$ there is an associated non-negative real number $\|x\|$, called the norm of $x$, such that

1. $\|x + y\| \leq \|x\| + \|y\|$ for all $x$ and $y \in V$,

2. $\|\alpha x\| = |\alpha|\|x\|$ if $x \in V$ and $\alpha$ is a scalar,

3. $\|x\| = 0$ implies $x = 0$. 
For $L_p$, the norm is defined
\[ \|f_c\|_p = \left\{ \int_0^\infty |f_c(t)|^p \, dt \right\}^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty \]
\[ \|f_c\|_\infty = \text{ess sup}_{t \in \mathbb{R}^+} |f_c(t)| \quad \text{for } p = \infty. \] (2.3)
and for $\ell_p$
\[ \|f_d\|_p = \left\{ \sum_{k=0}^\infty |f_d(k)|^p \right\}^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty \]
\[ \|f_d\|_\infty = \sup_{k \in \mathbb{Z}^+} |f_d(k)| \quad \text{for } p = \infty. \] (2.4)

**Definition 3** A sequence of vectors $\{x_i\}$ in a normed linear vector space $V$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists a $N(\epsilon) \in \mathbb{Z}^+$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > N(\epsilon)$.

**Definition 4** A normed linear vector space $V$ is said to be complete if for any Cauchy sequence $\{x_i\}$ there exists a vector $x \in V$ such that for any $\epsilon > 0$ there exists a $N(\epsilon) \in \mathbb{Z}^+$ in a way that $\|x_i - x\| < \epsilon$ for all $i \geq N(\epsilon)$. The sequence $\{x_i\}$ is said to converge to $x$ and complete normed linear vector spaces are called Banach spaces.

In simple terms, a Banach space has the property that a convergent sequence converges to an element in the Banach space.

While all of the $L_p$ and $\ell_p$ classes are Banach spaces for $1 \leq p \leq \infty$, not all are in common use in control systems literature. $p = \infty$ is popular because these signals are bounded and the norm is a measure of the maximum deviation from the zero level at any point in time. $p = 1$ is popular because the norm can be used as a time-weighted measure of variation around the zero level. In other words, large variations from the zero level are tolerable if they are for a short time. $p = 2$ is
popular because these signals are bounded in energy and \( \| \cdot \|^2 \) is the energy. \( p = 2 \) has the additional advantage of being a complete inner product space or a Hilbert space.

**Definition 5** A linear vector space \( V \) is an inner product space if to each ordered pair of vectors \( x \) and \( y \in V \) there is associated a complex number \( (x, y) \), the so called inner product of \( x \) and \( y \), such that the following hold:

1. \( (y, x) = \overline{(x, y)} \). (The bar denotes complex conjugation.)
2. \( (x + y, z) = (x, z) + (y, z) \) if \( x, y, \) and \( z \in V \).
3. \( (\alpha x, y) = \alpha (x, y) \) if \( x \) and \( y \in V \) and \( \alpha \) is a scalar.
4. \( (x, x) \geq 0 \) for all \( x \in V \).
5. \( (x, x) = 0 \) only if \( x = 0 \).

Immediate consequences of the definition of an inner product space are:

1. \( (0, y) = 0 \) for all \( y \in V \).
2. \( (x, \alpha y) = \overline{\alpha} (x, y) \).
3. A second distributive law \( (z, x + y) = (z, x) + (z, y) \).
4. We may define the norm of \( x \in V \) to be \( \|x\| = \sqrt{(x, x)} \) so all inner product spaces are normed linear vector spaces.

**Definition 6** An inner product space is a Hilbert space if it is complete.

We will often be dealing with only part of a vector space or subsets of these spaces. A Cauchy sequence in a subset converges to a limit point that may or may not be
in the subset. The union of a subset $S$ with all of its limit points is the closure $\overline{S}$ of the set. If the subset $S$ is a linear vector space, it is called a subspace. For inner product spaces, we define the following subspace. Let $S$ be a subspace within an inner product space, the subspace $S^\perp = \{ x \mid \langle x, y \rangle = 0 \text{ for all } y \in S \}$. $S^\perp$ is called the orthogonal complement of $S$.

As stated earlier, the only $L_p$ and $\ell_p$ spaces that are Hilbert spaces are for $p = 2$. Since Hilbert spaces have an inner product, they have more "structure" with which to work. This along with the physical relationship between the norm and energy is why we chose to work with $p = 2$.

Since we are using $L_p$ to study physical systems, we must extend these spaces to include multiple-input, multiple-output systems. We accomplish this by defining a new linear vector space $L_p^n$ with each element an $n$-tuple of $L_p$ functions. Also, the norm and inner product of the new vector space are defined in terms of the norm and inner product of each term of the $n$-tuple. For example, $x \in L_p^n$ if $x = \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right)$ and $x_1, x_2, \ldots, x_n \in L_p$. In a similar fashion, $x \in \ell_p^n$ if $x = \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right)$ and $x_1, x_2, \ldots, x_n \in \ell_p$. The $L_p^n$ and $\ell_p^n$ norms are defined as

$$
\|x\|_p = \left\{ \sum_{i=1}^{n} \|x_i\|_p^p \right\}^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty
$$

$$
\|x\|_\infty = \sup_{1 \leq i \leq n} \|x_i\|_\infty \quad \text{for } p = \infty.
$$

(2.5)
In addition, the $L^p_n$ and $\ell^p_n$ inner products are defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} (x_i, y_i).$$

Finally, all $L^p_n$ are Banach spaces for $1 \leq p \leq \infty$ and $L^2_2$ is a Hilbert space. As with the time scale, where the dimension of the space is unimportant, it is suppressed to give $L^p$ and $\ell^p$ or $L^p$ when we wish to denote both.

For time-invariant systems, it is usually better not to work directly with the time functions as the input and output signals. Instead, one usually works with the Laplace transform of the continuous-time signal defined as

$$x(s) = \int_{0}^{\infty} e^{-st} x(t) \, dt$$

or the $z$-transform of the discrete-time signal defined as

$$x(z) = \sum_{k=0}^{\infty} z^k x(k).$$

Note that the $z$-transform defined above is the standard one used by mathematicians and is not the standard one used by engineers. To convert from the mathematicians definition to the engineers definition, replace $z^k$ with $z^{-k}$. Because the transforms of the signals are complex analytic functions, we introduce the following linear vector spaces of complex analytic functions.

A function $f_c(s)$ that is analytic in the open right half-plane $\mathbb{C}+$ is in $H_p(\mathbb{C}+)$ for some $0 < p \leq \infty$ if

$$\sup_{\sigma > 0} \int_{0}^{\infty} |f_c(\sigma + j\omega)|^p \, d\omega < \infty \quad \text{for } 0 < p < \infty$$

$$\text{ess sup}_{s \in \mathbb{C}+} |f_c(s)| < \infty \quad \text{for } p = \infty.$$
In a similar manner, a function \( f_d(z) \) that is analytic in the open unit disk \( \mathbb{D} \) is in \( H_p(\mathbb{D}) \) for some \( 0 < p \leq \infty \) if

\[
\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_d(re^{i\theta})|^p \, d\theta < \infty \quad \text{for} \quad 0 < p < \infty
\]

\[
\operatorname{ess \ sup}_{z \in \mathbb{D}} |f_d(z)| < \infty \quad \text{for} \quad p = \infty.
\]  

(2.10)

Most of the properties of these two spaces are the same and in most cases we will suppress the domain and write \( H_p \). As with the \( L_p \) spaces, for \( 1 \leq p \leq \infty \), the \( H_p \) spaces are Banach spaces and for \( p = 2 \), they are Hilbert spaces.

The norm of the function \( f_c(s) \) in \( H_p(\mathbb{C}^+) \) is

\[
\|f_c(s)\|_p = \left\{ \sup_{\sigma > 0} \frac{1}{2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} |f_c(\sigma + j\omega)|^p \, d\omega \right\}^{\frac{1}{p}}
\]

for \( 1 \leq p < \infty \)

\[
\|f_c(s)\|_\infty = \operatorname{ess \ sup}_{s \in \mathbb{C}^+} |f_c(s)| \quad \text{for} \quad p = \infty.
\]

(2.11)

In a similar manner, the norm of a function \( f_d(z) \) in \( H_p(\mathbb{D}) \) is

\[
\|f_d(z)\|_p = \left\{ \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_d(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}
\]

for \( 1 \leq p < \infty \)

\[
\|f_d(z)\|_\infty = \operatorname{ess \ sup}_{z \in \mathbb{D}} |f_d(z)| \quad \text{for} \quad p = \infty.
\]

(2.12)

The \( H_p \) functions are not necessarily defined on the boundary of either \( \mathbb{C}^+ \) or \( \mathbb{D} \) respectively. However, the non-tangential limit exists almost everywhere and the limit is an \( L_p \) function. This fact is used to define an inner product for \( H_2 \). For \( H_2(\mathbb{C}^+) \) the inner product is defined as

\[
\langle x(s), y(s) \rangle = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} x(s)\overline{y(s)} \, ds
\]

(2.13)

and for \( H_2(\mathbb{D}) \) as

\[
\langle x(z), y(z) \rangle = \frac{1}{2\pi j} \oint_{\mathbb{D}} x(z)\overline{y(z)} \, dz
\]

(2.14)

where the overbar denotes complex conjugation.
As with $L_p$ functions, $H_p$ can be extended to $H_p^n$ by defining a new linear vector space with each element an $n$-tuple of $H_p$ functions. Also, the norm and inner product of the new vector space are defined in terms of the norm and inner product of each term of the $n$-tuple. For example, $x \in H_p^n$ if $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $x_1, x_2, \ldots,$ $x_n \in H_p$. The $H_p^n$ norms are defined as

$$
\|x\|_p = \left( \sum_{i=1}^{n} \|x_i\|_p^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty
$$

$$
\|x\|_\infty = \sup_{1 \leq i \leq n} \|x_i\|_\infty \quad \text{for } p = \infty.
$$

In addition, the $H_p^n$ inner product is defined by

$$
\langle x, y \rangle = \sum_{i=1}^{n} \langle x_i, y_i \rangle.
$$

Finally, where the dimension of the space is unimportant, it is suppressed to give $H_p$.

2.2 Introduction to Operators

An operator maps elements in a Banach space $V_1$ to elements of a Banach space $V_2$ (not necessarily the same Banach space) with the zero element of $V_1$ mapped to the zero element of $V_2$. The operator $G$ is defined only on its domain $D\{G\} \subset V_1$ and the domain need not be closed. The range is defined as $R\{G\} = GD\{G\}$ with $R\{G\} \subset V_2$. The kernel $K\{G\}$ of the operator is the subset of the domain that is
mapped to the zero element of $V_2$. The graph, $\mathcal{G} \{G\}$, of an operator $G : \mathcal{D} \{G\} \rightarrow \mathcal{R} \{G\}$ is defined to be:

$$\mathcal{G} \{G\} := \left[ \begin{array}{c} I \\ G \end{array} \right] \mathcal{D} \{G\}.$$  \hspace{1cm} (2.17)

The inverse graph, $\mathcal{G}^{-1} \{G\}$, of $G$ is:

$$\mathcal{G}^{-1} \{G\} := \left[ \begin{array}{c} G \\ I \end{array} \right] \mathcal{D} \{G\}.$$  \hspace{1cm} (2.18)

An operator is said to be linear if its graph is a subspace. An operator $G$ is bounded with norm $\|G\|$ if $\mathcal{D} \{G\} = V_1$ and

$$\|G\| = \sup_{x \in \mathcal{D} \{G\}} \frac{\|Gx\|}{\|x\|} < \infty.$$  \hspace{1cm} (2.19)

In this dissertation, the continuous-time operators will have $\mathcal{D} \{G\} \subset L^m_2$ and $\mathcal{R} \{G\} \subset L^n_2$ while the discrete-time operators will have $\mathcal{D} \{G\} \subset \ell^m_2$ and $\mathcal{R} \{G\} \subset \ell^n_2$. For time-invariant operators, we will often use the Laplace transform or the $x$-transform of the input and output signals to yield $\mathcal{D} \{G\} \subset H^m_2$ and $\mathcal{R} \{G\} \subset H^n_2$ as an alternate representation of the operator. The operators are multiple-input, multiple-output operators that represent multiple-input, multiple-output systems ($m$ inputs and $n$ outputs) and their graphs and inverse graphs are subsets of the Hilbert space $L^m_2 + n$ or $\ell^m_2 + n$ or $H^m_2 + n$.

In several proofs and definitions, we will use bounded, linear operators with special properties which we introduce in the following.

Let the shift operator or the delay operator for $L^m_2$ be denoted by $S^\tau_m$ and defined as follows:

$$y = S^\tau_m x = \begin{cases} 0 \text{ for all } t < \tau \\ x(t - \tau) \text{ for all } t \geq \tau \end{cases}.$$  \hspace{1cm} (2.20)
For discrete-time systems, \( \tau \) is usually assumed to be 1 and suppressed.

Let the truncation operator for \( L^2_{\mathbb{R}} \) be denoted by \( P_m(\tau) \) and defined as follows:

\[
y = P_m(\tau)x = \begin{cases} 
0 & \text{for all } t < \tau \\
x(t) & \text{for all } t \geq \tau 
\end{cases}
\]  

(2.21)

If \( V_1 \) and \( V_2 \) are Hilbert spaces, then the adjoint operator \( G^* \) of the bounded operator \( G \) has the property that \( \langle Gx, y \rangle = \langle x, G^*y \rangle \) for all \( x \in V_1 \) and \( y \in V_2 \). If \( G \) is linear, \( G^* \) exists and is unique. Also \( \|G\| = \|G^*\|, \mathcal{K}\{G\}^\perp = \overline{\mathcal{K}\{G^*\}}, \) and \( \overline{\mathcal{R}\{G\}} = \mathcal{K}\{G^*\} \). As an example, the adjoint of the shift operator \( S_m^T \) is the advance operator \( S_m^{T*} \) defined as follows:

\[
y = S_m^{T*}x = x(t + \tau) \text{ for all } t \geq \tau.
\]  

(2.22)

It should be noted that all information from time 0 to \( \tau \) is lost and set equal to zero. Thus, while \( S_m^{T*}S_m^T = I, S_m^T S_m^{T*} = P_m(\tau) \).

Any operator with the property \( \Pi^2 = \Pi \) is a projection operator. The projection operators are classified according to the relationship between their range and their kernel. For the special case where \( \mathcal{R}\{\Pi\} \perp \mathcal{K}\{\Pi\} \), the projection operator is called an orthogonal projection. Otherwise, the projection operator is called a parallel projection. Projection operators are said to project onto \( \mathcal{R}\{\Pi\} \) along \( \mathcal{K}\{\Pi\} \). The truncation operator is an example of an orthogonal projection.

If an operator has the property that \( G^{-1} = G^* \), it is said to be unitary. Operator norm is invariant under multiplication by a unitary operator and unitary operators are used in the derivation of formulae for the norms of operators. Unitary operators map one Hilbert space onto another Hilbert space so that the inner product is preserved. In other words if \( x \) and \( y \) are in a Hilbert space \( V_1 \) and the unitary operator \( G \) maps \( V_1 \) one-to-one and onto the Hilbert space \( V_2 \), then \( \langle x, y \rangle = \langle Gx, Gy \rangle \). Unitary
operators are also called Hilbert space isomorphisms. Two of the most important Hilbert space isomorphisms are given in the following theorem [2, 35, 45].

Theorem 7 The Laplace transform and the z-transform are Hilbert space isomorphisms from $L_2$ onto $H_2(\mathbb{C}^+)$ and from $\ell_2$ onto $H_2(\mathbb{D})$ respectively.

An operator $G : V_1 \rightarrow V_2$ is said to be an isometry if it maps $V_1$ into $V_2$ and preserves inner product. In other words, if $G$ is an isometry $\langle x, y \rangle = \langle Gx, Gy \rangle$ for all $x$ and $y \in V_1$. Isometries differ from unitary operators in that the range of an isometry need not be the entire space. $G^*G = I$ for isometries, but $GG^* \neq I$. Instead, $GG^*$ is an orthogonal projection onto $\overline{\mathcal{R}} \{G\} = \mathcal{K} \{G^*\}^\perp$.

Any closed subspace in a Hilbert space can be thought of as a Hilbert space contained in a larger Hilbert space. This allows us to describe operators that are an isometry on only part of a space. Thus, an operator $G$ is a partial isometry if it is an isometry on the orthogonal complement of its kernel. For partial isometries, $G^*G$ is an orthogonal projection onto $\overline{\mathcal{R}} \{G^*\} = \mathcal{K} \{G\}^\perp$ and $GG^*$ is an orthogonal projection onto $\mathcal{R} \{G\} = \mathcal{K} \{G^*\}^\perp$. Also, the adjoint of a partial isometry is a partial isometry.

An operator $G : V_1 \rightarrow V_2$ is said to be compact if for every bounded sequence $\{x_n\} \in V_1$, the sequence $\{Gx_n\}$ has a convergent subsequence. A compact operator must be bounded and, for $V_1$ and $V_2$ Hilbert spaces, the compact operators are the closure of the finite rank operators [45] where the rank of an operator is defined as the dimension of the range. The compact operators are a two-sided ideal of bounded operators. This means any bounded operator multiplied by a compact operator (either from the left or right) is a compact operator.
The spectrum of an operator $G$ is the set of all complex numbers $\lambda$ such that $\lambda I - G$ is not invertible. The spectrum of a compact operator is either a finite number of points or it is a countable number of points that tend to the origin of the complex plane. Also, if $G$ is compact and $\lambda \neq 0$, then $\lambda$ is in the spectrum if and only if $\lambda I - G$ has a nonzero kernel [21].

We are now ready to classify operators according to properties of their graphs. An operator $G$ is said to be linear if its graph $\{G\}$ is a linear vector space. It is said to be shift-invariant or time-invariant if $S^\tau_{m+n}G \{G\} \subset G \{G\}$ for all $\tau \geq 0$. For the special case of bounded operators, the definition of time-invariant system reduces to $S^\tau_n*GS^\tau_m = G$ where $S^\tau_n*$ is the adjoint of $S^\tau_n$. And finally, an operator $G$ is said to be causal if

\[
\begin{bmatrix}
1 & I \\
G & G
\end{bmatrix}
\{D \{G\} \cap L^m_2(\tau, \infty)\} \subset L^m_2(\tau, \infty)
\]

for all $\tau \geq 0$. In other words, an operator is causal if the future has no effect on the past. If $G$ is bounded, this condition reduces to $[I - P_n(\tau)]GP_m(\tau) = 0$ for all $\tau \geq 0$.

A bounded, linear operator $G$ is said to be eventually time-invariant if the sequence $S^\tau_n*GS^\tau_m$ converges in norm as $\tau \to \infty$. Since, $S^\tau_n*GS^\tau_m = G$ for a time-invariant plant, we see that an eventually time-invariant plant behaves more like a time-invariant plant as time approaches infinity because $S^\tau_n*GS^\tau_m$ is converging to an operator. In [9] it was proved that $G$ is eventually time-invariant if and only if $G = G_T + G_K$ where $G_T$ is time-invariant and $G_K$ is compact.

Under certain conditions, we have a characterization of the bounded, linear operators. For example, all bounded, linear operators on $\ell_2$ can be represented as a convolution sum

\[
y(k) = \sum_{i=0}^{\infty} g(k,i)x(i) = Gx
\]

(2.23)
where \( g(i, k) \) is a matrix for multiple-input, multiple-output systems. This is easily seen for the multiple-input, single-output case because if the operator is bounded, the energy at each time must be bounded. Thus for a fixed time \( k \), \( y(k) \) is a bounded linear functional. By [35], all bounded, linear functionals on a Hilbert space are characterized by inner product with a unique vector. Thus, \( y(k) = (g(i), x(i)) \) where \( g \) and \( x \in \ell_2 \). Since \( g(i) \) will be different for each time \( k \) and using the definition of the inner product for \( \ell_2 \) we obtain the convolution sum. To extend to the multiple-output case, use linearity. Another interpretation of the convolution sum is an infinite matrix representation

\[
y = \begin{bmatrix}
  y(0) \\
y(1) \\
y(2) \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
  g_{00} & g_{01} & g_{02} \\
g_{10} & g_{11} & g_{12} & \cdots \\
g_{20} & g_{21} & g_{22} \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
  x(0) \\
x(1) \\
x(2) \\
\vdots
\end{bmatrix}
= Gx.
\]

(2.24)

Again the entries are matrices for multiple-input, multiple-output systems. This representation is easier to visualize and is used in several places in this dissertation. The bounded, linear operator \( G \) is causal if and only if the matrix is block lower triangular i.e.

\[
G = \begin{bmatrix}
  g_{00} & 0 & 0 \\
g_{10} & g_{11} & 0 & \cdots \\
g_{20} & g_{21} & g_{22} \\
\vdots & \ddots & \ddots & \ddots
\end{bmatrix}
\]

(2.25)

In a similar fashion, the operator is \textit{anti-causal} if and only if the matrix representation is upper triangular and the operator is \textit{strictly causal} if it is causal and the diagonal blocks are all zero. In addition, the complex conjugate transpose of the matrix representation of the operator \( G \) is the matrix representation of its adjoint \( G^* \).
A similar convolution representation does not exist for the continuous-time case of $L_2$. If one assumes that the identity $I$ has a convolution representation, $x(t) = \int_0^\infty g(t, \tau)x(\tau)\,d\tau$ a contradiction arises that $g(t, \tau) = 0$ almost everywhere [44]. The only way around the contradiction is to work with distributions. Representation of operators on various vector spaces is discussed in Dunford and Schwartz [6]. The case for $L_2$ is an easy exercise. By the Fundamental Theorem of Integral Calculus,

$$y(t) = \frac{d}{dt} \int_0^t y(\tau)\,d\tau = \frac{d}{dt}(y(\tau), x[0,t](\tau))$$

(2.26)

where $x[0,t](\tau)$ is the characteristic function of the set $[0, t)$. Since $(y(\tau), x[0,t](\tau)) = (Gx(\tau), x[0,t](\tau))$ is a bounded linear functional for a fixed time $t$, we can use the same reasoning as in the $l_2$ case to represent it as a convolution. Hence,

$$y(t) = \frac{d}{dt} \int_0^\infty g(t, \tau)x(\tau)\,d\tau$$

(2.27)

As with the $l_2$ representation, if one views the function $g(t, \tau)$ as a function on a two dimensional plane, the operator $G$ is causal if and only if $g(t, \tau)$ is lower triangular. However, the anti-causal operators do not correspond to upper triangular functions. Also, there is no simple relationship between the representation of an operator and its adjoint [6]. The lack of a good representation for the linear, bounded operators on $L_2$ that easily characterizes the concepts of causality, anti-causality, and the adjoint is a hindrance in the study of continuous-time systems. If the system is bounded, causal, linear, and time-invariant, then a representation exists if we use the transform of the signals (Laplace transform for continuous-time signals and $z$-transform for discrete-time signals) [2, 43]. For a single-input, single-output operator with input $x$ and output $y$ and the corresponding transforms $x$ and $y$, then $y = Gx$ where $G \in H_\infty$. Thus, the operator can be represented as
a multiplication of the $H_2$ input signal with an $H_{\infty}$ function to give the $H_2$ output signal. By linearity, we can extend this to the multiple-input, multiple-output case where $y = Gx$ and $y \in H_2^n$, $x \in H_2^m$, and $G$ is an $m \times n$ matrix with $H_{\infty}$ functions as entries. In this dissertation, $H_{\infty}^{m \times n}$ refers to the symbol of an operator. Where the dimension is unimportant, we will suppress it to yield $H_{\infty}$.

Since $L_2$ and $H_2$ are Hilbert spaces and the linear, causal, time-invariant operators on $L_2$ have a representation in the frequency domain (on $H_2$), the operator has an adjoint in the frequency domain. For example, let $y = Gx$ be the time domain operator and $y = Gx$ be the corresponding frequency domain operator. Also, let $u \in L_2$ and $u \in H_2$ its corresponding transform. Then, $\langle Gx, u \rangle = \langle x, G^* u \rangle$ where the $*$ operator is the adjoint on the $H_2$ space. $G^*(s) = G(-\bar{s})$ for continuous-time operators and $G^*(z) = G(\bar{z}^{-1})$ for discrete-time operators where the overbar denotes complex conjugate transpose. We also have $\langle Gx, u \rangle = \langle x, G^* u \rangle$. Unfortunately, $G^*$ is not the symbol of $G^*$. In the frequency domain, $G^*$ would be represented as $\Pi_{H_2} G^*$ where $\Pi_{H_2}$ is an orthogonal projection onto $H_2$. $\Pi_{H_2}$ is not a shift-invariant operator; hence, it is not a multiplication operator in the frequency domain. In the time domain, the $*$ operator corresponds to the adjoint on the Hilbert space $L_2(-\infty, \infty)$ while $*$ is the adjoint on $L_2[0, \infty)$. Be forewarned $*$ and $*$ both denote adjoints, but they do not correspond to the same operator.

2.3 Nest Algebras and Inner/Outer Factorizations

The study of linear, causal systems is aided by some theorems in nest algebras. Nest algebras view the inputs and outputs of the system as a chain of subspaces. The only requirement is that the subspaces be linearly ordered (nested). For example, we can write $\ell_2^m[0, \infty) = \bigcup_{k=0}^{\infty} M_k$ where $M_k = P_m(k)\ell_2^m$ and $M_0 \supset M_1 \cdots \supset M_k \supset \cdots$.
\[\mathcal{M}_{k+1} \cdots \subseteq 0.\] An element in \(\mathcal{M}_k\) is zero up to time \(k - 1\) and is possibly nonzero from time \(k\) onward. A typical element in \(\mathcal{M}_k\) is shown in Figure 3. Each of these subspaces has an associated orthogonal projection operator \(\Pi_{\mathcal{M}_k}\) that projects onto the subspace \(\mathcal{M}_k\) and the set of projection operators determines an algebra of linear operators such that \([I - \Pi_{\mathcal{M}_k}]G \Pi_{\mathcal{M}_k} = 0\) for all \(k \in \mathbb{Z}\). In our language, this nest algebra is the set of all bounded, linear, causal operators \(G : \ell^m_2 \to \ell^m_2\).

A different choice of subspaces yields a different set of operators in the nest algebra. As a second example consider, \(\ell^m_2[0, \infty) = \cup_{k=0}^{\infty} \mathcal{M}^*_k\) where \(\mathcal{M}^*_k = [I - P_m(k)]\ell^m_2\) and \(0 \subseteq \mathcal{M}^*_0 \subseteq \mathcal{M}^*_1 \cdots \subseteq \mathcal{M}^*_k \subseteq \mathcal{M}^*_k \cdots\). An element in \(\mathcal{M}^*_k\) is possibly nonzero up to time \(k - 1\) and is zero from time \(k\) onward. A typical element in \(\mathcal{M}^*_k\) is shown in Figure 4. Each of these subspaces has an associated orthogonal projection operator \(\Pi_{\mathcal{M}^*_k}\) and the set of projection operators determines an algebra of linear operators such that \([I - \Pi_{\mathcal{M}^*_k}]G^* \Pi_{\mathcal{M}^*_k} = 0\) for all \(k \in \mathbb{Z}\). Note that this nest algebra is the set of all bounded, linear, anti-causal operators \(G^* : \ell^m_2 \to \ell^m_2\).
Also, the adjoint of any operator in the first nest algebra is an operator in the second
nest algebra.

Figure 4: Typical Element in $\mathcal{M}_k^*$

Similar constructions are possible for the continuous-time case of $L^2_0[0, \infty)$. In
this case, the subspaces are indexed by the real numbers and the nest is called a
continuous nest.

Arveson in [1] defined inner/outer factorizations for time-varying operators in
a nest algebra. The results of Arveson and others are brought together in a unified
fashion in [3]. Few of the properties that define inner and outer are used in this
dissertation. However, the existence of inner/outer factorizations with certain prop­
erties is crucial to our proofs. For completeness, we include the following definitions.
An operator $A$ is outer if the range projection $\Pi_{\mathcal{R}\{A\}}$ commutes with the subspace
projection $\Pi_{\mathcal{M}_k}$ for all $k$ and $A\mathcal{M}_k$ is dense in $\mathcal{M}_k \cap \mathcal{R}\{A\}$. An operator $U$ is inner
if $U$ is a partial isometry and $U^*U$ commutes with $\Pi_{\mathcal{M}_k}$ for all $k$. If the operators
are time-invariant, then the above definitions of inner and outer are equivalent to
the usual definitions for time-invariant operators [1]. The following theorem can be found in [1, Theorem 3.2, 3.3 and Corollary 1] and [3, Theorem 14.20 and 14.21].

**Theorem 8** Let $\mathcal{N}$ be a nest algebra. If every $\mathcal{M}_k$ has an immediate successor, then every operator $G \in \mathcal{N}$ has an inner/outer factorization $G = UA$ such that $U \in \mathcal{N}$ is inner, $A \in \mathcal{N}$ is outer, $\mathcal{R}\{G\} = \mathcal{R}\{U\}$, $\mathcal{R}\{A\} = \mathcal{K}\{U\}^\perp$, and $\mathcal{K}\{A\} = \mathcal{K}\{G\}$.

In addition if $G^*G = A^*A = B^*B$ with $B$ outer, then there exists a partial isometry $V \in \mathcal{N} \cap \mathcal{N}^*$ such that $V^*V = \Pi_{\mathcal{R}\{A\}}$, $VV^* = \Pi_{\mathcal{R}\{B\}}$, $VA = B$ and $G = WB$ is another inner/outer factorization with $W = UV^*$ inner and $B$ outer.

Since the subspace $\mathcal{M}_k$ has an immediate successor $\mathcal{M}_{k+1}$, the inner/outer factorizations exist for discrete-time systems. However, this is not the case for continuous-time systems and will be discussed in a later chapter.

The two examples of nest algebras and Theorem 8 assume the operators are “square” or equal number of inputs and outputs. We shall need a factorization result for “tall” operators with more outputs than inputs. The extension is accomplished by packing the operator with zero operators such that the new operator is square and using Theorem 8. We will also need the following result [3, Theorem 14.19].

**Theorem 9** If $\{\mathcal{M}_k\}$ is a well-ordered nest, then every positive operator $Q$ factors as $A^*A$ where $A$ is outer.

Applying Theorem 8 to the packed operator we obtain

$$G = \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = UA \quad (2.28)$$
with $U$ inner and $A$ outer. Since $\mathcal{K}\{G\} = \mathcal{K}\{A\}$, we have

$$ G = \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} = U A. \quad (2.29) $$

We also have that

$$ G^* G = A^* A = \begin{bmatrix} A_{11}^* A_{11} + A_{21}^* A_{21} & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.30) $$

Since $A_{11}^* A_{11} + A_{21}^* A_{21}$ is a positive, square operator, there exists an outer square operator $B_{11}$ such that

$$ G^* G = B^* B = \begin{bmatrix} B_{11}^* & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.31) $$

and furthermore $B$ is outer in the larger nest algebra.

By Theorem 8, there exists an inner $W$ such that $G = WB$. Hence,

$$ G = \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix} = WB. \quad (2.32) $$

Since $\mathcal{K}\{W\}^\perp = \overline{\mathcal{K}}\{B\}$, then

$$ G = \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} = \begin{bmatrix} W_{11} & 0 \\ W_{21} & 0 \end{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix} = WB. \quad (2.33) $$

By unpacking the above operators, we obtain the following.

**Corollary 10** Let $G : \ell^m_2 \rightarrow \ell^n_2$ ($n \geq m$) be a bounded causal (resp. anti-causal) operator. Then there exist bounded causal (resp. anti-causal) operators $U : \ell^m_2 \rightarrow \ell^n_2$ and $A : \ell^m_2 \rightarrow \ell^n_2$ such that $G = UA$, $U$ is a partial isometry, $\overline{\mathcal{K}}\{G\} = \overline{\mathcal{K}}\{U\}$, $\overline{\mathcal{K}}\{A\} = \mathcal{K}\{U\}^\perp$ and $\mathcal{K}\{A\} = \mathcal{K}\{G\}$. 
2.4 The Standard Problems

In this section, we introduce two problems from mathematics that in the control literature are referred to as standard problems. A standard technique in control engineering is to manipulate the particular control problem under study to show that it is equivalent to one of the two standard problems. Since the standard problems are well understood and have many mathematical tools available for solving them, once we know that another problem is equivalent to a standard problem we can use the same tools on the new problem.

The first problem we introduce is the Nehari problem [15, 45]. The Nehari problem is used in time-invariant systems in many optimization problems (i.e. optimal sensitivity reduction and optimal robustness). First, we need to introduce some notation. $L_\infty(-j\omega, j\omega)$ are functions on the complex plane that are bounded on the imaginary axis. $L_\infty(T)$ are functions on the complex plane that are bounded on the unit circle. An interpretation of these functions is that they represent bounded, linear, time-invariant operators that are not necessarily causal. Functions in $L_\infty(-j\omega, j\omega)$ represent continuous-time operators while functions in $L_\infty(T)$ represent discrete-time operators.

The Nehari problem: Given $G \in L_\infty(-j\omega, j\omega)$ (or $G \in L_\infty(T)$) find a function $Q \in H_\infty(\mathbb{C}^+)$ (or $Q \in H_\infty(\mathbb{D})$) such that $\|G - Q\|_\infty$ is minimized.

In most cases the solution to the Nehari problem is not unique but a solution does exist [15, 45] and there are methods of calculating $\inf_{Q \in H_\infty} \|G - Q\|_\infty$ as well as methods of calculating a $Q$ that achieves the infimum for $G$ belonging to various classes of operators [14, 15, 31, 36, 45]. An interpretation of the Nehari problem is that we are trying to approximate a linear, time-invariant, non-causal operator with
a linear, time-invariant, causal operator.

The infimum of the infinity norm in the Nehari problem is expressed in terms of the norm of a Hankel operator \([15, 45]\). Thus, 
\[
\inf_{Q \in H_\infty} \| G - Q \|_\infty = \| \Pi_{H_2^\perp} G \Pi_{H_2} \|
\]
where \( \Pi_{H_2} \) projects onto \( H_2 \) and \( \Pi_{H_2^\perp} \) projects onto \( H_2^\perp \). \( H_2^\perp \) functions are defined in a similar manner to \( H_2 \) functions except they are analytic in the left half-plane for continuous-time signals and are analytic outside the unit disk for discrete-time signals.

In a typical optimization problem that occurs in control theory, some set of linear operators is represented by \( M - N Q \) where \( Q \) is any \( H_\infty \) function. The optimization requires \( \| M - N Q \|_\infty \) to be minimized. An appropriate unitary (possibly non-causal) operator \( X \) is found and since a unitary operator does not change the norm, we have
\[
\inf_{Q \in H_\infty} \| M - N Q \|_\infty = \inf_{Q \in H_\infty} \| X M - X N Q \|_\infty
\]
with \( X M \) being non-causal and \( X N Q \) being causal. Ideally, we would like \( X N \) to be an invertible operator so that
\[
\inf_{Q \in H_\infty} \| X M - X N Q \|_\infty = \inf_{\widetilde{Q} \in H_\infty} \| X M - \widetilde{Q} \|_\infty
\]
and the optimization problem is mapped to the Nehari problem exactly. However, this may not occur. Often if \( Q \in H_\infty \) then \( \widetilde{Q} = X N Q \) will form a dense set within \( H_\infty \). If this occurs, the value of the infimum is the same as the Nehari problem, but a solution that achieves the infimum may not exist. Under these circumstances, a solution is calculated that is arbitrarily close to the infimum.

The second standard problem we introduce is the Arveson distance problem \([1, 3, 11]\). The Arveson distance problem is used in time-varying systems in the same way that the Nehari problem is used for time-invariant systems. In fact, the Arveson distance problem is an extension of the Nehari problem to time-varying operators.
The Arveson distance problem: Given $G$ a linear, bounded operator that is not necessarily causal, find a linear, bounded, causal operator $Q$ that minimizes $\|G - Q\|$.

Like the Nehari problem, there exists an operator that achieves the infimum [1, 3, 11] but it may not be unique. One interpretation of the Arveson distance problem is that we are trying to approximate a linear, bounded operator that is not necessarily causal with a linear, bounded, causal operator. This interpretation is similar to the interpretation of the Nehari problem.

The infimum of the operator norm in the Arveson distance problem is expressed in terms of the supremum of operator norms [1, 3, 11]. Thus, $\inf_{Q \text{ causal}} \|G - Q\| = \sup_{0 \leq t < \infty} \| (I - P_n(t)) G P_m(t) \|$ where $P_m(t)$ is the orthogonal projection onto $L^2[t, \infty)$. For now, there are no general methods for calculating either the Arveson distance or an operator that achieves the infimum for any large classes of time-varying operators.
CHAPTER III

Stable Feedback Systems

In the last chapter, we introduced the mathematical concept of an operator and used it to model a physical system. In this chapter, we connect two physical systems together as in Figure 1 to form a closed-loop system and denote it by \( \{ G, F \} \). In Figure 1, the operators \( G : \mathcal{D}\{ G \} \rightarrow \mathcal{R}\{ G \} \) and \( F : \mathcal{D}\{ F \} \rightarrow \mathcal{R}\{ F \} \) represent the plant and the compensator respectively. It is assumed that \( G \) is a given physical system and we will construct a compensator \( F \) so that the closed-loop system will meet certain design criteria. In this chapter, we will define a stable closed-loop system and examine some of the consequences of the definition on the properties of the plant and on the properties of the closed-loop system.

The closed-loop system equations that describe the system in Figure 1 are

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  I & F \\
  G & I
\end{bmatrix}
\begin{bmatrix}
  e_1 \\
  e_2
\end{bmatrix} =: Re
\]  

(3.1)

and stability is defined as follows.

**Definition 11** The closed-loop system \( \{ G, F \} \) is stable if

\[
\begin{bmatrix}
  I & F \\
  G & I
\end{bmatrix} : \mathcal{D}\{ G \} \times \mathcal{D}\{ F \} \rightarrow L_2^{m+n}
\]  

(3.2)

has a bounded inverse defined on \( L_2^{m+n} \).

As interpreted from a geometric point of view by Foias, Georgiou, and Smith in [13], this definition has several consequences that provide insight into stability and
the graphs of stabilizable operators. Note that \[
\begin{bmatrix}
I \\
G
\end{bmatrix} e_1 \in \mathcal{G}\{G\} \quad \text{and} \quad \begin{bmatrix}
F \\
I
\end{bmatrix} e_2 \in \mathcal{G}^{-1}\{F\}.
\]
Since the existence of an inverse implies the operator is one-to-one and onto, each element in the Hilbert space \(L_m^{m+n}\) is uniquely decomposed into two elements: one is on \(\mathcal{G}\{G\}\) and the other is on \(\mathcal{G}^{-1}\{F\}\). In addition, the uniqueness of the decomposition requires \(\mathcal{G}\{G\} \cap \mathcal{G}^{-1}\{F\} = 0\).

Many of the properties of the closed-loop stable system are inherited from the properties of the plant and the compensator. For example, if \(G\) and \(F\) are linear, their graphs are subspaces. Thus, \(R\) is linear and its graph \(\mathcal{G}\{R\}\) is a subspace. Since \(\mathcal{G}^{-1}\{R^{-1}\} = \mathcal{G}\{R\}\), \(R^{-1}\) is linear. Likewise, if \(G\) and \(F\) are time-invariant (not necessarily linear), then \(\mathcal{G}\{R\}\) is shift invariant. Since \(\mathcal{G}^{-1}\{R^{-1}\} = \mathcal{G}\{R\}\), \(R^{-1}\) is time-invariant. However, there is no general proof that if \(G\) and \(F\) are causal that \(R^{-1}\) is causal. For causality, we do have the following results.

**Theorem 12** If \(\{G, F\}\) is stable and \(G\) and \(F\) are time-invariant and causal (not necessarily linear), then \(R^{-1}\) is time-invariant and causal.

**Proof:** We already know that \(\mathcal{G}\{R\} = \mathcal{G}^{-1}\{R^{-1}\}\) is shift-invariant from above. Choose any \(u \in P^r_{m+n} L_2^{m+n} = L_2^{m+n}[\tau, \infty)\). Thus, there exists a uniquely defined \(e \in L_2^{m+n}\) so that \(\begin{bmatrix} e \\ u \end{bmatrix} \in \mathcal{G}\{R\}\). To show that \(R^{-1}\) is causal, one need show that \(e \in L_2[\tau, \infty)\). Because \(S^*_{m+n} u \in L_2^{m+n}\), there exist a unique \(e' \in L_2[0, \infty)\) such that \(\begin{bmatrix} e' \\ S^*_{m+n} u \end{bmatrix} \in \mathcal{G}\{R\}\). Since \(\mathcal{G}\{R\}\) is shift-invariant, we have

\[
S^*_{2(m+n)} \begin{bmatrix} e' \\ S^*_{m+n} u \end{bmatrix} = \begin{bmatrix} S^*_{m+n} e' \\ S^*_{m+n} \end{bmatrix} = \begin{bmatrix} S^*_{m+n} e' \\ u \end{bmatrix} \in \mathcal{G}\{R\} \quad (3.3)
\]
and \( e = S_{m+n}^r e' \in L_2^{m+n}[r, \infty) \). Therefore, \( R^{-1} \) is causal.

Additionally, the following result was proved in [18] for the case of linear systems defined over \( \ell_2[0, \infty) \).

**Theorem 13** Suppose the closed-loop system \( \{G, F\} \) is stable and \( G \) and \( F \) are discrete-time, linear, causal operators. Then the operator

\[
\begin{bmatrix}
I & F \\
G & I
\end{bmatrix}^{-1} : \ell_2^m \times \ell_2^m \rightarrow D\{G\} \times D\{F\}
\]

is causal.

**Proof:** Since the mapping \( K := \begin{bmatrix} I & F \\ G & I \end{bmatrix} \) is causal, it induces a well-defined linear map from a subspace of \( \ell_2^{m+n}[0, k] \) onto \( \ell_2^{m+n}[0, k] \) for all \( k \). This mapping must have a matrix representation and the representation must be square and non-singular, otherwise it can't be onto. Now suppose that \( H := \begin{bmatrix} I & F \\ G & I \end{bmatrix}^{-1} \) is not causal. Then there exists \( x, y \in \ell_2^{m+n} \) such that \( y = Hx \), with \( (I - P_{m+n}(k+1))x = 0 \) and \( (I - P_{m+n}(k+1))y \neq 0 \) for some \( k \). But \( x = Ky \), so the restriction of \( K \) to \([0, k]\) must have a kernel. This contradicts the fact that the matrix representation of this restriction is non-singular.

Throughout the rest of this dissertation, we will assume that \( G \) and \( F \) are linear and causal, though possibly time-varying and unbounded unless stated otherwise. With this assumption, we now prove some facts about the closed-loop system operator \( R^{-1} \) and the graphs of stabilizable plants.
Theorem 14 Let $G$ and $F$ be linear and suppose that the closed-loop system $\{G, F\}$ is stable. Then
\[
\begin{bmatrix}
I & F \\
G & I \\
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - FG)^{-1} & -F(I - GF)^{-1} \\
-G(I - FG)^{-1} & (I - GF)^{-1} \\
\end{bmatrix} =: R^{-1}
\]
and all four elements of $R^{-1}$ are bounded.

Proof: Since $\{G, F\}$ is stable, we can set $u_2 = 0$ in Figure 1 to show that $D \{ (I - FG)^{-1} \} = L^m_2$, $R \{ (I - FG)^{-1} \} \subset D \{ G \}$ and $(I - FG)^{-1}$ is bounded. It also follows that $D \{ -G(I - FG)^{-1} \} = L^n_2$, $R \{ -G(I - FG)^{-1} \} \subset D \{ F \}$, and $-G(I - FG)^{-1}$ is bounded. In a similar fashion, we can set $u_1 = 0$ to show that $D \{ (I - GF)^{-1} \} = L^m_2$, $R \{ (I - GF)^{-1} \} \subset D \{ F \}$, and $(I - GF)^{-1}$ is bounded. It also follows that $D \{ -F(I - GF)^{-1} \} = L^n_2$, $R \{ -F(I - GF)^{-1} \} \subset D \{ G \}$, and $-F(I - GF)^{-1}$ is bounded. Thus,
\[
D \left\{ \begin{bmatrix}
(I - FG)^{-1} & -F(I - GF)^{-1} \\
-G(I - FG)^{-1} & (I - GF)^{-1} \\
\end{bmatrix} \right\} = L^m_2 \times L^n_2
\]
and the following is well-defined:
\[
\begin{bmatrix}
I & F \\
G & I \\
\end{bmatrix} \begin{bmatrix}
(I - FG)^{-1} & -F(I - GF)^{-1} \\
-G(I - FG)^{-1} & (I - GF)^{-1} \\
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I \\
\end{bmatrix}.
\]
Because the inverse is unique, we have
\[
\begin{bmatrix}
I & F \\
G & I \\
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - FG)^{-1} & -F(I - GF)^{-1} \\
-G(I - FG)^{-1} & (I - GF)^{-1} \\
\end{bmatrix}
\]
and the proof is complete.
Suppose that the closed-loop system \( \{G, F\} \) is causal and bounded. Then the following two operators:

\[
P_1 = \begin{bmatrix} I \\ G \end{bmatrix} \begin{bmatrix} (I - FG)^{-1} & -F(I - GF)^{-1} \\ (I - G F)^{-1} & (I - GF)^{-1} \end{bmatrix}
\]

(3.10)

\[
P_2 = \begin{bmatrix} F \\ I \end{bmatrix} \begin{bmatrix} -G(I - FG)^{-1} & (I - GF)^{-1} \end{bmatrix}
\]

(3.11)

are causal and bounded. It was pointed out in Foias, Georgiou and Smith [13] that \( P_1 \) is the parallel projection operator onto \( \mathcal{G} \{G\} \) along \( \mathcal{G}^{-1} \{F\} \). In particular \( P_1^2 = P_1 \).

Also, \( P_2 \) is the parallel projection operator onto \( \mathcal{G}^{-1} \{F\} \) along \( \mathcal{G} \{G\} \) and \( P_2^2 = P_2 \) with \( P_1 + P_2 = I \).

The following is a necessary condition for stabilizability in both the continuous-time and discrete-time cases.

**Theorem 15** If a continuous-time or discrete-time plant \( G \) is linear, possibly time-varying, and stabilizable with a linear, possibly time-varying compensator \( F \), then it has a closed graph.

**Proof:** Since \( G \) is stabilizable, choose a compensator \( F \) such that the closed-loop system \( \{G, F\} \) is stable. From Equation (3.10), the parallel projection \( P_1 \) onto \( \mathcal{G} \{G\} \) along \( \mathcal{G}^{-1} \{F\} \) is defined in terms of the closed-loop operators. Thus,

\[
\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = P_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

where \( P_1 \) is a bounded, linear operator that maps \( L_2^{m+n} \) onto \( \mathcal{G} \{G\} \) and \( P_1^2 = P_1 \). If \( e_1 \in \mathcal{D} \{G\} \) and \( y_1 = Ge_1 \) then,

\[
\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = P_1 \begin{pmatrix} e_1 \\ y_1 \end{pmatrix}
\]

Now
consider a Cauchy sequence \( \left( \begin{array}{c} e_{1i} \\ y_{1i} \end{array} \right) \) on \( G \{ G \} \) with a limit point \( \left( \begin{array}{c} e'_1 \\ y'_1 \end{array} \right) \). Since \( P_1 \) is a bounded, linear operator, it has a closed graph. Thus,

\[
\left( \begin{array}{c} e_{1i} \\ y_{1i} \\
\\
\end{array} \right) = \left( \begin{array}{c} e_{1i} \\ y_{1i} \\
\\
\end{array} \right) \rightarrow \left( \begin{array}{c} e'_1 \\ y'_1 \end{array} \right) \in G \{ P_1 \}.
\]

Hence, \( \left( \begin{array}{c} e'_1 \\ y'_1 \end{array} \right) = P_1 \left( \begin{array}{c} e'_1 \\ y'_1 \end{array} \right) \) and \( \left( \begin{array}{c} e'_1 \\ y'_1 \end{array} \right) \in G \{ G \} \) and the graph of \( G \) is closed.  

Notice how the plant inherits the property of having a closed graph from the closed graph of the closed-loop system. This is not true for non-linear systems since there exist stable, non-linear plants that do not have a closed graph.
CHAPTER IV

Stabilization of Linear, Discrete-time, Time-varying Systems

In the last chapter, we examined the definition of closed-loop stability of feedback systems and some of the consequences of the definition. In this chapter, we will examine a method of representing an operator (coprime factorizations) that was developed in the study of linear, time-invariant systems and we will redefine some of the terms to extend the result to linear, time-varying, discrete-time systems. The results of this chapter will be necessary and sufficient conditions for a linear, discrete-time, possibly time-varying plant to be stabilizable. We will also characterize all of the linear, discrete-time, possibly time-varying compensators that stabilize the plant. This is an important starting point in the analysis of time-varying systems because we will usually have several design criteria to satisfy. If all the stabilizing compensators are parametrized, then we can optimize the other criteria using only the stabilizing compensators.

4.1 Coprime Fractions

For linear, time-invariant systems with a transfer function $G$, we define the following. An element $x \in H_{\infty}^{1 \times 1}$ is a common divisor of a finite set $\{y \mid y_i \in H_{\infty}^{1 \times 1}\}$ if there exist a set $\{\alpha \mid \alpha_i \in H_{\infty}^{1 \times 1}\}$ such that $y_i = \alpha_i x$ for all $i$. A greatest common divisor is a common divisor which is a multiple of any other common divisor. A greatest common divisor is unique up to multiplication by an invertible element in $H_{\infty}^{1 \times 1}$. As was proved by Smith [37], $H_{\infty}^{1 \times 1}$ is a greatest common divisor domain so
a greatest common divisor always exists. A matrix $M \in H_{\infty}^{m \times n}$ is irreducible if 1 is a greatest common divisor of all highest order minors of $M$.

**Definition 16** A plant $G$ has a weakly coprime right fraction factorization if $G = NM^{-1}$ where $M$ and $N \in H_{\infty}$ and $\begin{bmatrix} M \\ N \end{bmatrix}$ is irreducible. If there exist $X$ and $Y \in H_{\infty}$ so that $YM + XN = I$, then the coprime right fraction is strongly coprime. If the coprime right fraction has the property that $M^*M + N^*N = I$, then the coprime right fraction is said to be $*$-normalized.

**Definition 17** A plant $G$ has a weakly coprime left fraction factorization if $G = \overline{M}^{-1}\overline{N}$ where $\overline{M}$ and $\overline{N} \in H_{\infty}$ and $\begin{bmatrix} \overline{N} \\ \overline{M} \end{bmatrix}$ is irreducible. If there exist $\overline{X}$ and $\overline{Y} \in H_{\infty}$ so that $\overline{M}\overline{Y} + \overline{N}\overline{X} = I$, then the coprime left fraction is strongly coprime. If the coprime left fraction has the property that $\overline{M}\overline{M}^* + \overline{N}\overline{N}^* = I$, then the coprime left fraction is said to be $*$-normalized.

Coprime factorizations of linear, time-invariant systems have been extensively studied in recent years. The following theorem summarizes several results which have been obtained for the case of linear systems on $L_2[0, \infty)$ or $\ell_2[0, \infty)$. See [5, 37, 40, 42] and the references therein.

**Theorem 18** A linear, time-invariant causal plant $G$ (either continuous-time or discrete-time) is stabilizable if and only if there exists functions $M, N, X, Y, \overline{M}, \overline{N}, \overline{X}, \overline{Y} \in H_{\infty}$ with

$$G = NM^{-1} = \overline{M}^{-1}\overline{N}$$

(4.1)
such that the following double Bezout identity holds

\[
\begin{bmatrix}
  Y & X \\
-\bar{N} & \bar{M}
\end{bmatrix}
\begin{bmatrix}
  M & -\bar{X} \\
N & \bar{Y}
\end{bmatrix}
= \begin{bmatrix}
  I & 0 \\
0 & I
\end{bmatrix}
\]

(4.2)

Also, a linear, time-invariant compensator \( F \) stabilizes \( G \) if and only if

\[
F = (-\bar{X} - MQ) (\bar{Y} - NQ)^{-1}
= (Y - Q\bar{N})^{-1} (-X - Q\bar{M})
\]

(4.3)

for some \( Q \in H_{\infty} \).

Several articles have appeared in the literature exploring similar factorizations for time-varying systems or non-linear systems. Feintuch [8] found a necessary condition for the existence of coprime factorizations for discrete-time systems. Also, an example was presented of a plant that has no coprime factorization. Hammer [22] developed a framework for coprime factorizations for discrete-time, non-linear systems on \( \ell_{\infty} \). Verma [38] examined fractional representations for non-linear, time-varying systems in a general setting which included continuous-time systems. In both [22, 38], the existence of strong coprime factorizations for both the plant and the compensator was assumed. Thus far, the question of whether all stabilizable plants have strong coprime factorizations has remained unresolved. However, for the set of plants that can be realized by a finite set of state equations, such results do exist. Poolla and Khargonekar [32] examined linear, discrete-time, time-varying, finite-dimensional plants and proved the existence of coprime factorizations for stabilizable plants. Rotea and Khargonekar in [34] proved the existence of coprime factorizations for continuous-time, finite-dimensional, stabilizable plants. Desoer and Kabuli in [4] derive a right coprime factorization for finite dimensional, continuous-time, non-linear, time-varying plants which are uniformly completely controllable.
However, there are many infinite-dimensional (distributed parameter) plants that do not satisfy this criteria.

We now develop an alternate, but equivalent framework to the coprime factorization in previous work. In an operator theoretic approach to coprime factorization, one works with products of operators in the form $NM^{-1}$ and $\tilde{M}^{-1}\tilde{N}$. The terms $M^{-1}$ and $\tilde{M}^{-1}$ can be unbounded and one must ensure that the domains and ranges are properly aligned so that the products are defined. Although this is not an insurmountable difficulty, we prefer to work with graphs of unstable systems and their representations.

**Definition 19** A plant $G$ has a **right representation**
\[
\begin{bmatrix}
M \\
N
\end{bmatrix}
\]
if $M$ and $N$ are causal, bounded operators such that
\[
\mathcal{G}\{G\} = \mathcal{R}\left\{\begin{bmatrix}
M \\
N
\end{bmatrix}\right\}.
\tag{4.4}
\]
The right representation is a **strong right representation** if it has a causal, bounded left inverse. If the right representation has the property that $M^*M + N^*N = I$, then the representation is said to be $\ast$-normalized.

**Definition 20** A plant $G$ has a **left representation**
\[
\begin{bmatrix}
-\tilde{N} & \tilde{M}
\end{bmatrix}
\]
if $\tilde{M}$ and $\tilde{N}$ are causal, bounded operators such that
\[
\mathcal{G}\{G\} = \mathcal{K}\left\{\begin{bmatrix}
-\tilde{N} & \tilde{M}
\end{bmatrix}\right\}.
\tag{4.5}
\]
The left representation is a **strong left representation** if it has a causal, bounded right inverse. If the left representation has the property that $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$, then the representation is said to be $\ast$-normalized.
The main goal in this chapter is to show that the existence of strong right and left representations is equivalent to stabilizability for linear, causal, discrete-time systems. The proof of existence will make use of certain results of Arveson on inner/outer factorizations in nest algebras. The key technical step involves the factorization of the adjoint of a certain causal, bounded operator, which can be viewed as belonging to a nest algebra of anti-causal operators.

4.2 Existence of Strong Representations

We now present our main theorem which shows that stabilizability implies the existence of strong representations. In the proof of this theorem, we assume that the plant $G$ is stabilizable; hence, a stabilizing compensator $F$ exists. We then use the bounded, causal parallel projection operators $P_1$ and $P_2$ from Equations (3.10) and (3.11) to construct the necessary bounded, causal operators.

**Theorem 21** Let $G$ and $F$ be causal, discrete-time operators and suppose that the closed-loop system $\{G, F\}$ is stable (and causal). Then there exist bounded, causal operators $M, N, X, Y, \overline{M}, \overline{N}, \overline{X},$ and $\overline{Y}$ such that $P_1 = \left[ \begin{array}{c} M \\ N \end{array} \right] \left[ \begin{array}{cc} Y & X \end{array} \right]$ and $P_2 = \left[ \begin{array}{c} -X \\ \overline{Y} \end{array} \right] \left[ \begin{array}{cc} -\overline{N} & \overline{M} \end{array} \right]$. For any such factorizations, $\left[ \begin{array}{c} M \\ N \end{array} \right]$ is a strong right representation of $G$, $\left[ \begin{array}{cc} -\overline{N} & \overline{M} \end{array} \right]$ is a strong left representation of $G$, and the double Bezout identity
is satisfied. Further, both right and left representations of $G$ can be taken to be $*$-normalized.

**Proof:** Assume $G$ and $F$ are causal, discrete-time operators and suppose that the closed-loop system $\{G, F\}$ is stable (and causal). We thus have bounded, causal operators

$$P_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} F \\ I \end{bmatrix} \begin{bmatrix} -G(I - FG)^{-1} & (I - GF)^{-1} \\ (I - FG)^{-1} & -G(I - FG)^{-1} \end{bmatrix}$$

with $P_1 + P_2 = I$. To prove the second part of the theorem, consider any bounded, causal $M, N, X, Y, \bar{M}, \bar{N}, \bar{X}, \bar{Y}$ such that

$$P_1 = \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} -\bar{X} \\ -\bar{Y} \end{bmatrix} \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}.$$
Then

\[
\begin{bmatrix}
M & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix}
\begin{bmatrix}
Y & X \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
= \begin{bmatrix}
M \\
N
\end{bmatrix}
\begin{bmatrix}
Y & X \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
+ \begin{bmatrix}
-\tilde{X} \\
\tilde{Y}
\end{bmatrix}
\begin{bmatrix}
-\tilde{N} & \tilde{M}
\end{bmatrix}
= P_1 + P_2 = I. \tag{4.11}
\]

Both the matrix operators on the left hand side of the above equation are causal, bounded operators on \(\ell_2^{m+n}\). They therefore have block lower triangular matrix representations. Since their product is equal to the identity, the diagonal blocks must all be non-singular. Hence, neither operator has a kernel and so they are inverses of each other. Thus,

\[
\begin{bmatrix}
Y & X \\
-\tilde{N} & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
M & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix} \tag{4.12}
\]

and the double Bezout identity is satisfied.

Since \(\begin{bmatrix}
-\tilde{X} \\
\tilde{Y}
\end{bmatrix}\) is left invertible, it has no kernel and \(G\{G\} = K\{P_2\} = K\{\begin{bmatrix}
-\tilde{N} & \tilde{M}
\end{bmatrix}\}\). \tag{4.13}

Hence, \(\begin{bmatrix}
-\tilde{N} & \tilde{M}
\end{bmatrix}\) is a strong left representation of \(G\). Because \(\begin{bmatrix}
M & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix}\) is invertible, it has full range and

\[
\left[
\begin{array}{cc}
-\tilde{N} & \tilde{M}
\end{array}
\right]
\left[
\begin{array}{c}
M & -\tilde{X} \\
N & \tilde{Y}
\end{array}
\right]
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\iff
\begin{bmatrix}
0 & I
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}. \tag{4.14}
\]}
Therefore,

$$\mathcal{G}\{G\} = \mathcal{K}\left\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\right\} = \mathcal{R}\left\{\begin{bmatrix} M \\ N \end{bmatrix}\right\}$$  \hspace{1cm} (4.15)

and \(\begin{bmatrix} M \\ N \end{bmatrix}\) is a strong right representation of \(G\).

It remains to be shown that such factorizations exist. Since \(\{G, F\}\) is stable, then \(P_2\) from Equation (3.11) is bounded and causal with \(\mathcal{R}\{P_2\} = G^{-1}\{F\}\) and \(\mathcal{K}\{P_2\} = \mathcal{G}\{G\}\). Write

$$P_2 = \begin{bmatrix} F \\ I \end{bmatrix}\begin{bmatrix} A_1 & A_2 \end{bmatrix}$$  \hspace{1cm} (4.16)

where

$$A := \begin{bmatrix} A_1 & A_2 \end{bmatrix} := \begin{bmatrix} -G(I - FG)^{-1} & (I - GF)^{-1} \end{bmatrix}.$$  \hspace{1cm} (4.17)

Since \(A\) is bounded and causal, its adjoint \(A^*\) is bounded and anti-causal. Applying Corollary 10 to \(A^*\) we have a factorization \(A^* = \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}\bar{Y}^*\) where \(\begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}\) is a partial isometry and \(\mathcal{K}\left\{\begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}\right\} = \mathcal{R}\\{\bar{Y}^*\}\). Taking the adjoint we have

\(A = \bar{Y}^*\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\) with \(\bar{Y}\) and \(\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\) being bounded and causal. Since \(\mathcal{R}\left\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\right\} \perp \mathcal{K}\\{\bar{Y}\}\), then \(\mathcal{K}\left\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\right\} = \mathcal{G}\{G\}\) and \(\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\) is a left representation of the plant. It is also a partial isometry because the adjoint of a partial isometry is a partial isometry. We now write

$$P_2 = \begin{bmatrix} F\bar{Y} \\ \bar{Y} \end{bmatrix}\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} = \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix}\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}.$$  \hspace{1cm} (4.18)
Observe that \( \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} \) is causal since both \( \bar{Y} \) and \( F \) are causal operators. We now wish to show that \( \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} \) is bounded. Since \( \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} \) is a partial isometry, \( I - \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} \) is an orthogonal projection onto

\[
\mathcal{R}\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\} \perp = \mathcal{K}\begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}
= \mathcal{R}\{\bar{Y}^*\} \perp = \mathcal{K}\{\bar{Y}\} = \mathcal{K}\begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix}.
\]

(4.19)

Hence,

\[
\begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}
+ \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} \left\{ I - \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} \right\} = P_2 \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}
\]

(4.20)

and so \( \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix} \) is bounded because it is the product of two bounded operators. We also deduce from Equation (4.20) and the fact that \( \mathcal{R}\{\begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix}\} \perp \mathcal{K}\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\} \) that

\[
\mathcal{K}\{\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\} = \mathcal{R}\{P_2\} = G^{-1}\{F\}.
\]

(4.21)
Now we apply the same reasoning to $P_1$ from Equation (3.10) where $\mathcal{R}\{P_1\} = \mathcal{G}\{G\}$ and $\mathcal{K}\{P_1\} = G^{-1}\{F\}$. Write

$$P_1 = \begin{bmatrix} I \\ G \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

where

$$B := \begin{bmatrix} B_1 & B_2 \end{bmatrix} := \begin{bmatrix} (I - FG)^{-1} & -F(I - GF)^{-1} \end{bmatrix}.$$  \hspace{1cm} (4.23)

We inner/outer factorize $B^*$ and take the adjoint of the factorization. We then obtain

$$P_1 = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$ \hspace{1cm} (4.24)

where the operators $U_1, U_2, V_1,$ and $V_2$ are causal and bounded with $\mathcal{G}\{G\} = \mathcal{R}\left\{ \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \right\}$. We inner/outer factorize $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ to give

$$P_1 = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} W \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$$=: \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} Y & X \end{bmatrix}$$ \hspace{1cm} (4.25)

where $\mathcal{R}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \right\} = \mathcal{G}\{G\}$ and $\begin{bmatrix} M \\ N \end{bmatrix}$ is a right representation of the plant that is also partial isometry. We thus have a factorization of the parallel projections $P_1$ and $P_2$ and we know that the representations are strong and the double Bezout identity is satisfied.
To finish the proof, we now show that the representations are *-normalized. Since \[
\begin{bmatrix}
M \\
N
\end{bmatrix}
\] has a left inverse, it has no kernel. Because it is a partial isometry \[
\begin{bmatrix}
M^* & N^*
\end{bmatrix}
\begin{bmatrix}
M \\
N
\end{bmatrix}
\]
is an orthogonal projection onto \( \mathcal{R}\{\begin{bmatrix}
M^* & N^*
\end{bmatrix}\} = K \{\begin{bmatrix}
M \\
N
\end{bmatrix}\} \perp = l_2 \). Hence,
\[
\begin{bmatrix}
M^* & N^*
\end{bmatrix}
\begin{bmatrix}
M \\
N
\end{bmatrix} = I. \tag{4.26}
\]
Likewise, \[
\begin{bmatrix}
-N & \tilde{M}
\end{bmatrix}
\]
has a right inverse and thus has full range. Because it is a partial isometry, \[
\begin{bmatrix}
-N & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
-N^* \\
\tilde{M}^*
\end{bmatrix}
\]
is an orthogonal projection onto \( \mathcal{R}\{\begin{bmatrix}
-N & \tilde{M}
\end{bmatrix}\} = l_2 \). Therefore,
\[
\begin{bmatrix}
-N & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
-N^* \\
\tilde{M}^*
\end{bmatrix} = I. \tag{4.27}
\]

In [37], in the time-invariant case, reduction of fractional representations to coprime representations was achieved by \textit{two} inner/outer factorizations (one on the \( H_\infty \) matrix \[
\begin{bmatrix}
M \\
N
\end{bmatrix}
\] and the other on the transpose of this matrix). It is easy to check that both these factorizations are necessary in general. It is interesting to note that a similar reduction was achieved in the proof of the above theorem by means of just \textit{one} inner/outer factorization of the adjoint operator in the star algebra.
4.3 Youla Parametrization

In this section, we will use the strong representations of the plant to construct strong representations of the compensator. We will prove that the Youla parametrization from the time-invariant case extends to the discrete-time, time-varying case and gives all of the stabilizing compensators.

**Theorem 22 (Youla Parametrization)** Let $G$ be a discrete-time, causal, possibly time-varying plant $G$ which is stabilizable (i.e. $\{G, F\}$ is stable and causal for some $F$). Consider any bounded, causal operators $M, N, X, Y, M, N, \bar{N}, \bar{X}$, and $\bar{Y}$ with

\[
\begin{bmatrix}
M \\
N
\end{bmatrix}
\]

a strong right representation of $G$ and

\[
\begin{bmatrix}
-\bar{N} \\
\bar{M}
\end{bmatrix}
\]

a strong left representation of $G$ such that the following double Bezout identity holds

\[
\begin{bmatrix}
Y & X \\
-\bar{N} & \bar{M}
\end{bmatrix}
\begin{bmatrix}
M & -\bar{X} \\
N & \bar{Y}
\end{bmatrix}
= 
\begin{bmatrix}
M & -\bar{X} \\
N & \bar{Y}
\end{bmatrix}
\begin{bmatrix}
Y & X \\
-\bar{N} & \bar{M}
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

Then, a compensator $F$ stabilizes $G$ (i.e. $\{G, F\}$ is stable and causal for $F$) if and only if $F$ has a strong right representation

\[
\begin{bmatrix}
\bar{Y} - NQ \\
-\bar{X} - MQ
\end{bmatrix}
\]

and a strong left representation

\[
\begin{bmatrix}
X + Q\bar{M} & Y - Q\bar{N}
\end{bmatrix}
\]

for some causal, bounded $Q$. 
Proof: Assume that $F$ stabilizes $G$. Then $F$ also has a strong left and right representation that satisfy the double Bezout identity. Thus, there exist bounded, causal operators such that

$$\mathcal{G}\{F\} = \mathcal{R}\left\{ \begin{bmatrix} M_F \\ N_F \end{bmatrix} \right\} = \mathcal{K}\left\{ \begin{bmatrix} -\tilde{N}_F \\ \tilde{M}_F \end{bmatrix} \right\}$$

(4.31)

and

$$\begin{bmatrix} Y_F & X_F \\ -\tilde{N}_F & \tilde{M}_F \end{bmatrix} \begin{bmatrix} M_F & -\tilde{X}_F \\ N_F & \tilde{Y}_F \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$  

(4.32)

Define

$$H := \begin{bmatrix} M_F & -\tilde{N}_F \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}.$$  

(4.33)

Since the left representation of the compensator is right invertible and the right representation of the plant is left invertible, $\mathcal{R}\left\{ \begin{bmatrix} M_F & -\tilde{N}_F \end{bmatrix} \right\} = \ell_2^n$, $\mathcal{K}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \right\} = 0$,

$$\mathcal{R}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \right\} = \mathcal{G}\{G\}, \text{ and } \mathcal{K}\left\{ \begin{bmatrix} M_F & -\tilde{N}_F \end{bmatrix} \right\} = G^{-1}\{F\}.$$  

Because $\{G, F\}$ is stable, we must have $\mathcal{G}\{G\} + G^{-1}\{F\} = \ell_2^{m+n}$ and $\mathcal{G}\{G\} \cap G^{-1}\{F\} = 0$. Therefore, $\mathcal{R}\{H\} = \ell_2^n$ and $\mathcal{K}\{H\} = 0$ which means that $H$ has an inverse $H^{-1}$. We will now establish that $H^{-1}$ is bounded and causal. Define

$$\Phi := \begin{bmatrix} M \\ N \end{bmatrix} H^{-1} \begin{bmatrix} \tilde{M}_F & -\tilde{N}_F \end{bmatrix}.$$  

(4.34)

Therefore, $\mathcal{K}\{\Phi\} = G^{-1}\{F\}$, $\mathcal{R}\{\Phi\} = \mathcal{G}\{G\}$, and $\Phi^2 = \Phi$. Thus, $\Phi$ is a parallel projection onto $\mathcal{G}\{G\}$ along $G^{-1}\{F\}$ (which is the bounded, causal operator $P_1$.
defined in terms of the closed-loop operators in Equation (3.10)). We now write

\[
\begin{bmatrix}
Y \\ X
\end{bmatrix} \Phi \begin{bmatrix}
\bar{Y}_F \\ -\bar{X}_F
\end{bmatrix} =
\begin{bmatrix}
Y \\ X
\end{bmatrix} \begin{bmatrix}
M \\ N
\end{bmatrix} H^{-1} \begin{bmatrix}
\bar{M}_F & -\bar{N}_F \\
-H^{-1}\bar{N}_F & H^{-1}\bar{M}_F
\end{bmatrix} \begin{bmatrix}
\bar{Y}_F \\ -\bar{X}_F
\end{bmatrix} = H^{-1}
\] (4.35)

where the last equality follows from the double Bezout identities. Then \( H^{-1} \) is bounded and causal since it is the product of three bounded, causal operators. Hence,

\[
G \{ F \} = \mathcal{K} \left\{ \begin{bmatrix}
-H^{-1}\bar{N}_F & H^{-1}\bar{M}_F
\end{bmatrix} \right\},
\] (4.36)

\[
\begin{bmatrix}
Y_F & X_F \\
-H^{-1}\bar{N}_F & H^{-1}\bar{M}_F
\end{bmatrix} \begin{bmatrix}
M_F & -\bar{X}_FH \\
N_F & \bar{Y}_FH
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix},
\] (4.37)

and

\[
\begin{bmatrix}
H^{-1}\bar{M}_F & -H^{-1}\bar{N}_F
\end{bmatrix} \begin{bmatrix}
M \\ N
\end{bmatrix} = I.
\] (4.38)

Thus, without loss of generality, we can assume

\[
\begin{bmatrix}
\bar{M}_F & -\bar{N}_F
\end{bmatrix} \begin{bmatrix}
M \\ N
\end{bmatrix} = I.
\] (4.39)

Since we also have

\[
\begin{bmatrix}
Y \\ X
\end{bmatrix} \begin{bmatrix}
M \\ N
\end{bmatrix} = I,
\] (4.40)

then

\[
\begin{bmatrix}
\bar{M}_F - Y & -\bar{N}_F - X
\end{bmatrix} \begin{bmatrix}
M \\ N
\end{bmatrix} = 0.
\] (4.41)
Define the causal, bounded operator

\[ Q := \begin{bmatrix} \tilde{M}_F - Y & -\tilde{N}_F - X \\ \tilde{\mathcal{X}} & \tilde{\mathcal{Y}} \end{bmatrix}. \]  

(4.42)

Then

\[
\begin{align*}
\begin{bmatrix} Y - Q \tilde{N} & X + Q \tilde{M} \end{bmatrix} &= \begin{bmatrix} Y & X \end{bmatrix} + Q \begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix} \\
&= \begin{bmatrix} Y & X \end{bmatrix} + \begin{bmatrix} \tilde{M}_F - Y & -\tilde{N}_F - X \end{bmatrix} \begin{bmatrix} -\tilde{X} \\
&\quad \tilde{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix} \\
&= \begin{bmatrix} Y & X \end{bmatrix} + \begin{bmatrix} \tilde{M}_F - Y & -\tilde{N}_F - X \end{bmatrix} \begin{bmatrix} I & 0 \\
&\quad 0 & I \end{bmatrix} - \begin{bmatrix} M \\
&\quad N \end{bmatrix} \begin{bmatrix} Y & X \end{bmatrix} \\
&= \begin{bmatrix} \tilde{M}_F & -\tilde{N}_F \end{bmatrix} - \left( \begin{bmatrix} \tilde{M}_F - Y & -\tilde{N}_F - X \end{bmatrix} \begin{bmatrix} M \\
&\quad N \end{bmatrix} \right) \begin{bmatrix} Y & X \end{bmatrix} \\
&= \begin{bmatrix} \tilde{M}_F & -\tilde{N}_F \end{bmatrix}.
\end{align*}
\]

(4.43)

Since \[ \begin{bmatrix} X + Q \tilde{M} & Y - Q \tilde{N} \end{bmatrix} = \begin{bmatrix} -\tilde{N}_F & \tilde{M}_F \end{bmatrix}, \] all stabilizing compensators have a strong left representation in the form of the Youla parametrization.

Conversely, choose a bounded, causal operator for \( Q \). This yields a compensator \( F \) with a right representation

\[ \mathcal{G} \{ F \} = \mathcal{R} \left\{ \begin{bmatrix} \tilde{Y} - NQ \\
&\quad -\tilde{X} - MQ \end{bmatrix} \right\} \]  

(4.44)

or

\[ \mathcal{G}^{-1} \{ F \} = \mathcal{R} \left\{ \begin{bmatrix} -\tilde{X} - MQ \\
&\quad \tilde{Y} - NQ \end{bmatrix} \right\}. \]  

(4.45)
Choose $e_1 \in \mathcal{D}\{G\}$ and $e_2 \in \mathcal{D}\{F\}$ and calculate the closed-loop system inputs as follows:

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix} = \begin{bmatrix}
  I \\
  G
\end{bmatrix} e_1 + \begin{bmatrix}
  F \\
  I
\end{bmatrix} e_2 = \begin{bmatrix}
  M \\
  N
\end{bmatrix} w_1 + \begin{bmatrix}
  -\bar{X} - MQ \\
  \bar{Y} - NQ
\end{bmatrix} w_2 \\
\]

\[
= \begin{bmatrix}
  M & -\bar{X} \\
  N & \bar{Y}
\end{bmatrix} \begin{bmatrix}
  I & -Q \\
  0 & I
\end{bmatrix} \begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix}.
\] (4.46)

Because the range of the right representation is the graph of the operator, we are guaranteed that $w_1$ and $w_2$ exist.

By direct multiplication of the operators and simplification using the double Bezout identities, we obtain

\[
\begin{align*}
\begin{bmatrix}
  I & Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  Y & X \\
  -\bar{N} & \bar{M}
\end{bmatrix} & = \begin{bmatrix}
  M & -\bar{X} \\
  N & \bar{Y}
\end{bmatrix} \begin{bmatrix}
  I & -Q \\
  0 & I
\end{bmatrix} \\
\end{align*}
\]

\[
= \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix}.
\] (4.47)

Therefore, \( \begin{bmatrix}
  M & -\bar{X} \\
  N & \bar{Y}
\end{bmatrix} \begin{bmatrix}
  I & -Q \\
  0 & I
\end{bmatrix} \) is a bounded, causal operator with a bounded, causal inverse. Hence,

\[
\begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix} = \begin{bmatrix}
  I & Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  Y & X \\
  -\bar{N} & \bar{M}
\end{bmatrix} \begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}.
\] (4.48)
Thus, is bounded for all bounded \( Q \) and the system is closed-loop stable. Hence, all strong right representations from the Youla parametrization stabilize the plant.

To complete the proof, we will show that, for the same \( Q \), the strong right and strong left representations correspond to the same operator \( F \). Select an arbitrary, causal, bounded \( Q \). Then the controller defined by

\[
G^{-1}\{F\} = R\left\{ \begin{bmatrix} -\bar{X} - MQ \\ \bar{Y} - NQ \end{bmatrix} \right\}
\]  

(4.50)

stabilizes \( G \). Furthermore,

\[
\begin{bmatrix} M & -\bar{X} - MQ \\ N & \bar{Y} - NQ \end{bmatrix}
\]  

(4.51)

is invertible and has range \( \ell^m_2 + n \). Hence,

\[
\begin{bmatrix} Y - Q\bar{N} & X + Q\bar{M} \end{bmatrix} \begin{bmatrix} M & -\bar{X} - MQ \\ N & \bar{Y} - NQ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

\[\iff \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]  

(4.52)
if and only if \( w_1 = 0 \). Thus
\[
\mathcal{K} \left\{ \begin{bmatrix} Y - Q \tilde{N} & X + Q \tilde{M} \end{bmatrix} \right\} = \mathcal{R} \left\{ \begin{bmatrix} -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\}.
\] (4.53)

This establishes the required equality and the proof is complete. \( \blacksquare \)

For future reference, we will derive formulae for the parallel projections in terms of the Youla parametrization.

**Theorem 23** If a plant \( G \) has strong right and left representations as in Theorem 22, then the parallel projections for a compensator \( F \) from the Youla parametrization are
\[
P_1 = \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix}
\]
\[
P_2 = \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} \begin{bmatrix} 0 & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix}
\] (4.54) (4.55)

where \( Q \) is the same bounded operator that parametrizes the compensator \( F \).

**Proof:** Using formulae from the proof of the Youla parametrization and the Bezout identity we get
\[
\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
\]
\[
= \begin{bmatrix} M & 0 \\ N & 0 \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
\[
= \begin{bmatrix} M & -\tilde{X} \\ N & \tilde{Y} \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = P_1 u
\] (4.56)
Similarly for $P_2$ we get

\[
\begin{bmatrix}
\begin{array}{c}
y_2 \\
e_2
\end{array}
\end{bmatrix} = \begin{bmatrix}
0 & -\bar{X} - MQ \\
0 & \bar{Y} - NQ
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -\bar{X} - MQ \\
0 & \bar{Y} - NQ
\end{bmatrix} \begin{bmatrix}
I & Q \\
0 & I
\end{bmatrix} \begin{bmatrix}
Y & X \\
-\bar{N} & \bar{M}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -\bar{X} - MQ \\
0 & \bar{Y} - NQ
\end{bmatrix} \begin{bmatrix}
Y & X \\
-\bar{N} & \bar{M}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = P_2u
\]  

(4.57)

4.4 Relationships Between Two Strong Representations

Theorem 21 does not say that the representation is unique. We will conclude this chapter with a derivation of the relationship between any two representations for a stabilizable plant. First, we will prove the following lemma.

Lemma 24 If a square, discrete-time operator $J$ is bounded and causal with a bounded inverse $K$, then $K$ is also causal.

Proof: Because $J$ and $K$ are bounded, each has a matrix representation. Because $J$ is causal, its matrix representation is lower triangular. Hence,

\[
J = \begin{bmatrix}
J_{00} & 0 & 0 \\
J_{10} & J_{11} & 0 & \ldots \\
J_{20} & J_{21} & J_{22} \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]  

(4.58)
and
\[
J^{-1} = K = \begin{bmatrix}
K_{00} & K_{01} & K_{02} \\
K_{10} & K_{11} & K_{12} \\
K_{20} & K_{21} & K_{22} \\
\vdots & \ddots & \ddots
\end{bmatrix}.
\] (4.59)

Because \(JK = I\) and all the diagonal blocks are square, we must have \(J_{00}K_{00} = I\) and \(J_{0i}K_{0i} = 0\) for \(i > 0\). Thus, \(K_{0i} = 0\) for \(i > 0\) and we proceed inductively for each row of \(K\) to conclude that \(K\) is lower triangular and thus, causal.

We now present the theorem that shows the relationship between any two representations of a stabilizable plant.

**Theorem 25** If a plant \(G\) is stabilizable, then any right (resp. left) representation that has no kernel (resp. has full range) is a strong right (resp. strong left) representation. Furthermore, two strong right (resp. strong left) representations are related to one another by multiplication on the right (resp. left) by a bounded, causal, square, invertible operator.

**Proof:** Define the right representation of the plant as \(\begin{bmatrix} M \\ N \end{bmatrix}\) and the left representation of the plant as \(\begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\). Since the plant is stabilizable, there exists a strong right representation, \(\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}\), and a strong left representation, \(\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix}\) such that the following double Bezout identity holds:
Choose a stabilizing compensator from the Youla parametrization by setting \( Q = 0 \). Since \( \{G, F\} \) is stable, we must have \( G \{G\} + G^{-1} \{F\} = \mathcal{E}_2^{m+n} \) and \( G \{G\} \cap G^{-1} \{F\} = 0 \). We also have that \( \begin{bmatrix} Y_1 & X_1 \end{bmatrix} \) is right invertible and thus, has full range. Therefore,

\[
\mathcal{R}\left\{ \begin{bmatrix} M & -X_1 \\ N & Y_1 \end{bmatrix} \right\} = \mathcal{E}_2^{m+n},
\]

\[
\mathcal{K}\left\{ \begin{bmatrix} M & -X_1 \\ N & Y_1 \end{bmatrix} \right\} = 0,
\]

\[
\mathcal{K}\left\{ \begin{bmatrix} Y_1 & X_1 \\ -N & M \end{bmatrix} \right\} = 0,
\]

and we can write

\[
\begin{bmatrix} Y_1 & X_1 \\ -N & M \end{bmatrix}\begin{bmatrix} M & -X_1 \\ N & Y_1 \end{bmatrix} = \begin{bmatrix} Y_1M + X_1N & 0 \\ 0 & N\bar{X}_1 + M\bar{Y}_1 \end{bmatrix} =: J.
\]

Because \( \begin{bmatrix} M & -X_1 \\ N & Y_1 \end{bmatrix} \) has full range, we have
\[ \mathcal{R} \{ J_1 \} = \mathcal{R} \{ Y_1 M + X_1 N \} = \mathcal{R} \left\{ \begin{bmatrix} Y_1 & X_1 \\ N & \bar{Y}_1 \end{bmatrix} \begin{bmatrix} M & -\bar{X}_1 \\ N & \bar{Y}_1 \end{bmatrix} \right\} = \mathcal{R} \left\{ \begin{bmatrix} Y_1 \\ X_1 \end{bmatrix} \right\} = \ell_2^m. \] (4.65)

Similarly \( \mathcal{R} \{ J_2 \} = \ell_2^n \), and so \( \mathcal{R} \{ J \} = \ell_2^{m+n} \). We also have that \( \mathcal{K} \{ J \} = 0 \), so \( J \) is one-to-one and onto. A consequence of the Open Mapping Theorem is [35, Theorem 5.10], a bounded operator that is one-to-one and maps a Banach space onto a Banach space has a bounded inverse. Therefore, \( J^{-1} \) exists and is bounded. By the lemma, \( J^{-1} \) is also causal as are \( J_1^{-1} \) and \( J_2^{-1} \).

To prove the first part of the theorem, we note that
\[
\begin{bmatrix} J_1^{-1} Y_1 \\ J_1^{-1} X_1 \end{bmatrix} \begin{bmatrix} M & -\bar{X}_1 J_2^{-1} \\ N & \bar{Y}_1 J_2^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\] (4.66)
and define
\[
\Psi_1 := \begin{bmatrix} M & -\bar{X}_1 J_2^{-1} \\ N & \bar{Y}_1 J_2^{-1} \end{bmatrix} \begin{bmatrix} J_1^{-1} Y_1 & J_1^{-1} X_1 \\ -\bar{N} & \bar{M} \end{bmatrix} = \begin{bmatrix} M & -\bar{X}_1 \\ N & \bar{Y}_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J_2^{-1} \end{bmatrix} \begin{bmatrix} J_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Y_1 & X_1 \\ -\bar{N} & \bar{M} \end{bmatrix}. \] (4.67)

Since \( \mathcal{K} \{ \Psi_1 \} = 0 \) and \( \Psi_1^2 = \Psi_1 \), then \( \Psi_1 = I \) and the first part is proved.

To prove the second part of the theorem, we note that
\[
\begin{bmatrix} Y_1 & X_1 \\ -J_2^{-1} \bar{N} & J_2^{-1} \bar{M} \end{bmatrix} \begin{bmatrix} M J_1^{-1} & -\bar{X}_1 \\ N J_1^{-1} & \bar{Y}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\] (4.68)
and define
\[
\Psi_2 := \begin{bmatrix} M J_1^{-1} & -\bar{X}_1 \\ N J_1^{-1} & \bar{Y}_1 \end{bmatrix} \begin{bmatrix} Y_1 & X_1 \\ -J_2^{-1} \bar{N} & J_2^{-1} \bar{M} \end{bmatrix} \\
= \begin{bmatrix} M & -\bar{X}_1 \\ N & \bar{Y}_1 \end{bmatrix} \begin{bmatrix} J_1^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J_2^{-1} \end{bmatrix} \begin{bmatrix} Y_1 & X_1 \\ -\bar{N} & \bar{M} \end{bmatrix}.
\]

(4.69)

Since \( K \{\Psi_2\} = 0 \) and \( \Psi_2^2 = \Psi_2 \), then \( \Psi_2 = I \). Finally, note that

\[
\begin{bmatrix} Y_1 & X_1 \\ -J_2^{-1} \bar{N} & J_2^{-1} \bar{M} \end{bmatrix} \begin{bmatrix} M_1 & -\bar{X}_1 \\ N_1 & \bar{Y}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

(4.70)

Since inverses are unique, we have

\[
\begin{bmatrix} -J_2^{-1} \bar{N} & J_2^{-1} \bar{M} \end{bmatrix} = \begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix}
\]

(4.71)

and

\[
\begin{bmatrix} M J_1^{-1} \\ N J_1^{-1} \end{bmatrix} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.
\]

(4.72)
CHAPTER V

Robust Stabilization of Discrete-time, Time-varying Systems

In the last chapter, we looked at necessary and sufficient conditions for a linear, discrete-time, possibly time-varying plant to be stabilizable. While the most important property a closed-loop system must possess is stability, it is not the only one. We still need to allow for ignorance about the plant and ignorance about the environment. In this chapter, we will examine the problem of ignorance about the plant by designing a compensator that will stabilize the plants “close” to the nominal plant (robustness). Because normalized fraction and normalized representations occur so often in this chapter, the first section uses an example and a theorem to highlight the difference between $\cdot$-normalized and $\ast$-normalized left representations for time-invariant systems. In the second section, we will examine a definition for the “distance” (the gap metric) between two plants and use it in the examination of the problem of robustness. In other words, for a given stable closed-loop system $\{G, F\}$, what is the largest perturbation allowed between $G$ and another plant $G_1$ which guarantees the closed-loop system $\{G_1, F\}$ stable. In the third section, we will examine how to design a compensator such that the size of the allowable perturbation is as large as possible (optimal robustness).

5.1 $\ast$-Normalized and $\cdot$-Normalized Left Representations

As was stated earlier, $\ast$ and $\cdot$ are not the same operation. This becomes apparent when the left $\ast$-normalized and the left $\cdot$-normalized operators are calculated for a
discrete-time, time-invariant system. For simplicity, we will denote time domain operators in non-bold type and the corresponding frequency domain operator in bold type. As an example, let $G = z$ which is the delay or shift operator. A $\ast$-normalized left coprime factorization is

$$\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix} = \begin{bmatrix} -\frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

with the corresponding matrix representation

$$\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix} = \begin{bmatrix} \left( \begin{array}{ccc} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) & \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) & \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \end{bmatrix} \cdots$$

A calculation of $\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix} \begin{bmatrix} -\bar{N}_1^* \\ \bar{M}_1^* \end{bmatrix}$ shows that $\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix}$ is not $\ast$-normalized. A slight modification to the $(1,1)$ term of $\begin{bmatrix} -\bar{N}_1 & \bar{M}_1 \end{bmatrix}$ will yield a time-varying $\ast$-normalized left representation

$$\begin{bmatrix} -\bar{N}_2 & \bar{M}_2 \end{bmatrix} = \begin{bmatrix} \left( \begin{array}{ccc} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) & \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) & \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \end{bmatrix} \cdots$$

Since the above $\ast$-normalized left representation is time-varying, a natural question is whether there exists a time-invariant $\ast$-normalized left representation to the time-invariant plant. This question is answered in the following theorem.
Theorem 26 If a linear, causal, discrete-time plant $G$ has a *-normalized left representation $\begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix}$ that is shift-invariant, then the plant is a constant plant.

Proof: If $\begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix}$ is shift-invariant, then the operator has a frequency domain representation that is analytic on the unit disk $\mathbb{D}$. Thus,

$$\begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} = \sum_{k=0}^{\infty} c_k z^k$$  \hspace{1cm} (5.4)

where $c_k$ are constant matrices.

The frequency domain representation of $\begin{bmatrix} -\overline{N}^* & \overline{M}^* \end{bmatrix}$ is $\Pi_{H_2} \begin{bmatrix} -\overline{N}^* & \overline{M}^* \end{bmatrix}$ where $\Pi_{H_2}$ is the orthogonal projection onto $H_2$. In other words, it truncates negative powers of $z$ and passes non-negative powers of $z$. We now apply the input signal represented by $I z^n$. Therefore,

$$I z^n = \begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \Pi_{H_2} \begin{bmatrix} -\overline{N}^* \\ \overline{M}^* \end{bmatrix} z^n$$

$$= \begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \Pi_{H_2} \sum_{k=0}^{\infty} \overline{c}_k z^{n-k}$$

$$= \begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \sum_{k=0}^{n} \overline{c}_k z^{n-k}$$  \hspace{1cm} (5.5)

where the overbar denotes complex conjugate transpose. Let $n = 0$ to obtain $\begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \overline{c}_0 = I$. Thus, for all $n \geq 1$

$$I z^n = \begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \overline{c}_0 z^n + \begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \sum_{k=1}^{n} \overline{c}_k z^{n-k}$$  \hspace{1cm} (5.6)

and

$$\begin{bmatrix} -\overline{N} & \overline{M} \end{bmatrix} \sum_{k=1}^{n} \overline{c}_k z^{n-k} = 0.$$  \hspace{1cm} (5.7)
Induction on Equation (5.7) over \(1 \leq n < \infty\) yields \(\begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix} \check{e}_n = 0\) for all \(n \geq 1\). Hence,

\[
0 = \begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix} \check{e}_n = \sum_{k=0}^{\infty} c_k z^k \check{e}_n = \sum_{k=0}^{\infty} c_k \check{e}_n z^k
\]

for all \(n \geq 1\). Because \(z^k\) forms an orthonormal basis on the unit disk, we know that \(c_k \check{e}_n = 0\) for all \(n \geq 1\). In particular, \(c_n \check{e}_n = 0\) so \(c_n = 0\) for all \(n \geq 1\). Thus,

\[
\begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix} = c_0.
\]

It is an interesting and unexpected result that most time-invariant systems do not possess a time-invariant \(\ast\)-normalized left representation.

The \(\ast\)-normalized representations of a system can be used to form the following unitary operators that are used several times in this chapter.

**Theorem 27** If a plant \(G\) has \(\ast\)-normalized strong right and left representations

\[
\begin{bmatrix} M \\ N \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix},
\]

then \(\begin{bmatrix} M^* & N^* \\ -\mathcal{N} & \mathcal{M} \end{bmatrix}\) and \(\begin{bmatrix} M & -\mathcal{N}^* \\ N & \mathcal{M}^* \end{bmatrix}\) are unitary.

**Proof:** Since the representations are strong (with left and right inverses) we know that \(\mathcal{K}\left\{\begin{bmatrix} M \\ N \end{bmatrix}\right\} = 0\), \(\mathcal{R}\left\{\begin{bmatrix} M \\ N \end{bmatrix}\right\} = \mathcal{G}\{G\}\), \(\mathcal{K}\left\{\begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix}\right\} = \mathcal{G}\{G\}\), and \(\mathcal{R}\left\{\begin{bmatrix} -\mathcal{N} & \mathcal{M} \end{bmatrix}\right\} = \ell_2\). From the definition of the adjoint of an operator, we obtain \(\mathcal{R}\left\{\begin{bmatrix} M^* & N^* \end{bmatrix}\right\} = \ell_2\), \(\mathcal{K}\left\{\begin{bmatrix} M^* & N^* \end{bmatrix}\right\} = \mathcal{G}\{G\}^\perp\), \(\mathcal{R}\left\{\begin{bmatrix} -\mathcal{N}^* \\ \mathcal{M}^* \end{bmatrix}\right\} = \mathcal{G}\{G\}^\perp\), and \(\mathcal{K}\left\{\begin{bmatrix} -\mathcal{N}^* \\ \mathcal{M}^* \end{bmatrix}\right\} = 0\). Thus,

\[
\begin{bmatrix} M^* & N^* \\ -\mathcal{N} & \mathcal{M} \end{bmatrix}\begin{bmatrix} M & -\mathcal{N}^* \\ N & \mathcal{M}^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]
Hence, $\mathcal{K}\left(\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}\right) = 0$. Because $\mathcal{G}\{G\} + \mathcal{G}\{G\}^\perp = \ell_2$, we also have $\mathcal{R}\left(\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}\right) = \ell_2$. Therefore, $\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}$ is invertible. Since the inverse is unique, we conclude $\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}^{-1} = \begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}^*$ and the theorem is proved. 

\[ \blacksquare \]

5.2 Robustness in the Gap Metric

Before one can start analyzing and comparing the robustness of controllers in any concrete terms, some sort of metric must be introduced to define the "distance" between two plants $G_1$ and $G_2$. Most of the research in the time-invariant case has been with additive perturbations where $G_2 = G_1 + \Delta$ and with multiplicative perturbations where $G_2 = G_1(I + \Delta)$. Perturbations of these types inherently fix the number of unstable modes. The idea is to find restrictions on $\Delta$ and the compensator $F$, such that the closed-loop system remains stable with the nominal plant $G_1$ replaced with the plant $G_2$. See [15, 28, 41] and the references therein.

These techniques do not handle cases where there are poles on the imaginary axis for continuous-time systems or poles on the unit circle for discrete-time systems. Also, the restriction concerning the number of unstable modes is often unnatural. For example, an airplane that flies at both supersonic and subsonic speeds has different numbers of unstable modes depending on which side of the sound barrier it is flying. We would prefer to use one controller to stabilize both sets of conditions and neither additive nor multiplicative perturbations will allow the analysis of this situation.
Two metrics were introduced to overcome these drawbacks—the graph metric and the gap metric. In [39], Vidyasagar introduced the graph metric which defined the “distance” between two plants in terms of *-normalized coprime factorizations of the plants.

Definition 28 The graph metric between two operators $G_1$ and $G_2$ is denoted by $d(G_1, G_2)$ and $d(G_1, G_2) = \max\{a, b\}$ where $G_1 = N_1M_1^{-1}$ and $G_2 = N_2M_2^{-1}$ are *-normalized coprime fractions,

$$a = \inf_{\|Q\|_{\infty} \leq 1} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} + \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|,$$ \hspace{1cm} (5.10)

and

$$b = \inf_{\|Q\|_{\infty} \leq 1} \left\| \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} + \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q \right\|.$$ \hspace{1cm} (5.11)

The gap metric has been in use in operator theory [25, 29] and was introduced into control literature by Zames and El-Sakkary [7, 46].

Definition 29 The gap between two operators $G_1$ and $G_2$ is denoted by $\delta(G_1, G_2)$ and

$$\delta(G_1, G_2) = \|\Pi_{G\{G_1\}} - \Pi_{G\{G_2\}}\|,$$ \hspace{1cm} (5.12)

where $\Pi_{G\{G_i\}}$ is the orthogonal projection onto $G\{G_i\}$.

Both metrics induce the same topology (the graph topology). The graph topology is the weakest topology in which feedback stability is a robust property. The last statement means that every open set in any topology in which feedback stability is a robust property is an open set in the graph topology. Thus, of all topologies in which
feedback stability is a robust property, the graph topology has the "most" open sets. The graph metric is extendable to time-varying operators by using $*$-normalized right representations and restricting $Q$ to causal operators with norm less than or equal to 1. The gap metric is already defined for time-varying operators.

A third approach of modeling plant uncertainty was introduced by Vidyasagar and Kimura in [41]. This approach is a modification of the bounded additive perturbation method and consist of adding a bounded perturbation to a coprime factorization of the nominal plant. It was proved by Georgiou and Smith in [19] that for the shift-invariant case with $*$-normalized coprime fractions, this approach is equivalent to the gap metric.

In the study of the gap metric, it is useful to introduce the following definition and theorems from [17, 29, 30].

**Definition 30** The directed gap between two operators $G_1$ and $G_2$ is denoted by $\delta(G_1, G_2)$ and

$$\delta(G_1, G_2) = \left\| \left( I - \Pi_{G\{G_2\}} \right) \Pi_{G\{G_1\}} \right\|$$

where $\Pi_{G\{G_1\}}$ is the orthogonal projection onto $G\{G_1\}$.

**Theorem 31** For two plants $G_1$ and $G_2$,

$$\delta(G_1, G_2) = \max \left\{ \delta(G_1, G_2), \delta(G_2, G_1) \right\}.$$  

*Proof:* The proof presented here uses the fact that orthogonal projections are self-adjoint and that

$$\begin{bmatrix}
\Pi_{G\{G_i\}} & I - \Pi_{G\{G_i\}} \\
I - \Pi_{G\{G_i\}} & \Pi_{G\{G_i\}}
\end{bmatrix}$$

is unitary for $i = 1, 2$. Thus,
\[ \delta(G_1, G_2) = \| \Pi_{G_1} - \Pi_{G_2} \| = \begin{bmatrix} \Pi_{G_1} & I - \Pi_{G_1} \\ I - \Pi_{G_1} & \Pi_{G_2} \end{bmatrix} \begin{bmatrix} \Pi_{G_1} & 0 \\ 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} \Pi_{G_1} (I - \Pi_{G_2}) & 0 \\ -(I - \Pi_{G_1}) \Pi_{G_2} & 0 \end{bmatrix} \begin{bmatrix} \Pi_{G_2} & I - \Pi_{G_2} \\ I - \Pi_{G_2} & \Pi_{G_2} \end{bmatrix} \]

\[ = \max \{ \| \Pi_{G_1} (I - \Pi_{G_2}) \|, \| (I - \Pi_{G_1}) \Pi_{G_2} \| \} \]

\[ = \max \{ \delta(G_1, G_2), \delta(G_2, G_1) \}. \tag{5.15} \]

The last equality follows from the fact that an operator and its adjoint have the same norm.

For time-invariant operators, Georgiou in [17] proved that the directed gaps are computable by the following formula.

\[ \bar{\delta}(G_1, G_2) = \inf_{Q \in \mathcal{H}_\infty} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} + \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} \]

where \( G_1 = N_1 M_1^{-1} \) and \( G_2 = N_2 M_2^{-1} \) are \(*\)-normalized coprime fractions.

Solving this infimum is equivalent to a Nehari problem [15]. Notice that the formula for the directed gap is similar to the formula in the definition of the graph metric. The difference is that the norm of the operator \( Q \) is unconstrained in the calculation of the directed gap and makes calculation of the infimum easier.
Feintuch in [10], introduced a new metric for time-varying systems that reduced to the gap metric for time-invariant systems. Feintuch defined a directed gap $\tilde{\delta}_t$ between two subspaces of the graphs of the operators

$$\tilde{\delta}_t(G_1, G_2) = \left\| \left( I - \Pi \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} \ell_2[t,\infty) \right) \Pi \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \ell_2[t,\infty) \right\|$$

(5.17)

where $\begin{bmatrix} M_i \\ N_i \end{bmatrix}$ is a $*$-normalized representation of the plant $G_i$. Feintuch continues by defining a gap $\delta_t = \max\{\tilde{\delta}(G_1, G_2), \tilde{\delta}(G_2, G_1)\}$. For $t = 0$ this corresponds to the gap metric. Feintuch then defines a new metric $\alpha(G_1, G_2) = \sup_{t \geq \infty} \delta_t(G_1, G_2)$. Feintuch shows that the calculation of the metric $\alpha$ is equivalent to an Arveson distance problem to obtain

$$\alpha(G_1, G_2) = \max \left\{ \inf_{Q_{\text{causal}}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|, \right. \right.$$

$$\left. \inf_{Q_{\text{causal}}} \left\| \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} - \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} Q \right\| \right\}$$

(5.18)

which is similar to Georgiou's formula in Equation (5.16).

However, for operators in the time domain, we can develop a formula for the directed gap using $*$-normalized strong representations and unitary operators.
Theorem 32 For two plants $G_1$ and $G_2$ with $\ast$-normalized strong left and right representations $[-\overline{N}_i \overline{M}_i]$ and $[M_i \overline{N}_i]$ for $i = 1, 2$ then

$$\bar{\delta}(G_1, G_2) = \left\| \left[ \begin{array}{c} \overline{N}_2 \\ \overline{M}_2 \end{array} \right] \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \right\|.$$  \hspace{1cm} (5.19)

Proof: Since the strong representations are normalized, the orthogonal projections defining the directed gap are given by the following equations.

$$I - \Pi_{G_2} = \left[ \begin{array}{c} -\overline{N}_2^* \\ \overline{M}_2^* \end{array} \right] \left[ \begin{array}{c} -\overline{N}_2 \\ \overline{M}_2 \end{array} \right]$$  \hspace{1cm} (5.20)

and

$$\Pi_{G_1} = \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \left[ \begin{array}{cc} M_1^* & N_1^* \end{array} \right].$$  \hspace{1cm} (5.21)

Therefore,

$$\bar{\delta}(G_1, G_2) = \left\| (I - \Pi_{G_2}) \Pi_{G_1} \right\|$$

$$= \left\| \left[ \begin{array}{c} -\overline{N}_2^* \\ \overline{M}_2^* \end{array} \right] \left[ \begin{array}{c} -\overline{N}_2 \\ \overline{M}_2 \end{array} \right] \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \left[ \begin{array}{cc} M_1^* & N_1^* \end{array} \right] \right\|$$

$$= \left\| \left[ \begin{array}{c} M_2^* \\ N_2 \end{array} \right] \left[ \begin{array}{c} -\overline{N}_2^* \\ \overline{M}_2^* \end{array} \right] \left[ \begin{array}{c} -\overline{N}_2 \\ \overline{M}_2 \end{array} \right] \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \left[ \begin{array}{cc} M_1^* & N_1^* \end{array} \right] \right\|$$

$$= \left\| \left[ \frac{0}{I} \right] \left[ \begin{array}{c} -\overline{N}_2 \\ \overline{M}_2 \end{array} \right] \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \left[ \begin{array}{cc} M_1^* & N_1^* \end{array} \right] \left[ \begin{array}{cc} M_1 & -\overline{N}_1^* \\ N_1 & \overline{M}_1^* \end{array} \right] \right\|$$

$$= \left\| \left[ \frac{0}{I} \right] \left[ \begin{array}{c} -\overline{N}_2 \\ \overline{M}_2 \end{array} \right] \left[ \begin{array}{c} M_1 \\ N_1 \end{array} \right] \left[ \begin{array}{c} I \\ 0 \end{array} \right] \right\|. \hspace{1cm} (5.22)
Hence,

\[ \tilde{\delta}(G_1, G_2) = \left\| \begin{bmatrix} -\bar{N}_2 & \bar{M}_2 \\ -M_1 & N_1 \end{bmatrix} \right\|. \]  

(5.23)

While the above formula is very simple, it is important to note that it does not hold in the frequency domain because the operators are \( \ast \)-normalized. \( \ast \)-normalized left fractions and \( \ast \)-normalized left representations are not the same thing. \( \ast \)-normalized left fractions are \( H_\infty \) analytic functions that represent time-invariant systems. \( \ast \)-normalized left representations are usually time-varying operators for a time-invariant system.

For the rest of this chapter, we will concentrate on the gap as the metric to measure the "distance" between two plants and we present the following theorem from Foias, Georgiou, and Smith [13] which gives a bound on the gap that the plant must satisfy to guarantee that the system will be closed-loop stable and that the closed-loop operators are uniformly bounded.

**Theorem 33** For linear, possibly time-varying operators \( G_1, G_2, \) and \( F \), the following are equivalent:

1. \( \{G_1, F\} \) is closed-loop stable and \( b < \|P_1(G_1, F)\|^{-1} \) where \( P_1(G_1, F) \) is the parallel projection defined in Equation (3.10) using the plant \( G_1 \) and the compensator \( F \).

2. \( \{G_2, F\} \) is closed-loop stable and the parallel projection \( P_1(G_2, F) \) is uniformly bounded in norm for all \( G_2 \) where \( \delta(G_1, G_2) \leq b \). \( P_1(G_2, F) \) is the parallel projection defined in Equation (3.10) using the plant \( G_2 \) and the compensator \( F \).
Proof: See [13]. It is important to note that while the above theorem states that all plants inside the gap ball are stabilized, it does not state that all plants outside the ball are not stabilized by the compensator $F$. There exist plants outside the gap ball that are stabilized by the compensator $F$. In [13], Foias, Georgiou, and Smith also prove that for the same compensator $F$, $\|P_1\| = \|P_2\|$. These results are extensions of results for the time-invariant case found in [19, 20, 41] for either the gap metric or normalized coprime fraction perturbation.

Since the norm of the parallel projections are going to play a large role in the study of robustness, we present the following theorem which relates the norm of the parallel projection to the compensator from the Youla parametrization for the special case of $*$-normalized representations of the plant. See [41].

**Theorem 34** If a closed-loop system $\{G, F\}$ is stable with $*$-normalized strong representations $\begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} -\tilde{N} & \tilde{M} \end{bmatrix}$ that satisfy the double Bezout identity and $F$ is parametrized by $Q$ from the Youla parametrization, then

$$\|P_1\| = \left\| \begin{bmatrix} Y - Q\tilde{N} & X + Q\tilde{M} \end{bmatrix} \right\|$$

and

$$\|P_2\| = \left\| \begin{bmatrix} -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\|. \quad (5.25)$$

Proof: Since the norm of an operator is unchanged by an unitary operator and we have formulae from Theorem 23 for the norms of $P_1$ and $P_2$, we can write
Similarly, 

\[
\|P_2\| = \left\| \begin{bmatrix} M^* & N^* \\ \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} 0 & -Q \\ I & 0 \end{bmatrix} \begin{bmatrix} Y & X \\ -\tilde{N} & -\tilde{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{N}^* \\ -\tilde{M} & N \end{bmatrix} \right\|
\]

\[
= \left\| \begin{bmatrix} M -\tilde{X} \\ N \end{bmatrix} \begin{bmatrix} 0 & -Q \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -Y\tilde{N}^* + X\tilde{M}^* \\ -\tilde{X} - MQ \end{bmatrix} \right\|
\]

\[
= \left\| \begin{bmatrix} 0 & -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\|
\]

\[
= \left\| \begin{bmatrix} -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\|
\]  

\[\text{(5.27)}\]

It is interesting that the operators above are the strong left and the strong right representations of the compensator.
5.3 Optimal Robust Stabilization in the Gap Metric

In the last section, we calculated the size of the gap ball of uncertainty that a given compensator will allow. In this section, we will look at the problem of choosing a compensator that allows the largest gap ball of uncertainty: the optimal robust stabilization problem. In the time-invariant case, this problem was studied in the gap metric or the normalized coprime fraction perturbation by [19, 20, 41]. We will extend a result from Glover and McFarlane [20] to the time-varying case.

Since the size of the gap ball of allowable uncertainty is the inverse of the norm of the parallel projection and

\[ \|P_1\| = \left\| \begin{bmatrix} Y - Q\bar{N} & X + Q\bar{M} \end{bmatrix} \right\| \] (5.28)

and

\[ \|P_2\| = \left\| \begin{bmatrix} -\bar{X} - MQ \\ \bar{Y} - NQ \end{bmatrix} \right\|, \] (5.29)

with \( \|P_1\| = \|P_2\| \), the obvious approach is to directly minimize either of the operators above. The optimization problem is equivalent to an Arveson distance problem and an expression for the size of the optimal (largest) gap ball \( b_{opt} \) is easily obtained.

\[
\|P_1\|^2 = \left\| \begin{bmatrix} Y - Q\bar{N} & X + Q\bar{M} \end{bmatrix} \right\|^2 \\
= \left\| \begin{bmatrix} Y - Q\bar{N} & X + Q\bar{M} \end{bmatrix} \begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix} \right\|^2 \\
= \left\| \begin{bmatrix} I, & -Y\bar{N}^* + X\bar{M}^* + Q \end{bmatrix} \right\|^2 \\
= 1 + \left\| -Y\bar{N}^* + X\bar{M}^* + Q \right\|^2 (5.30)
\]
Thus,

\[ \frac{1}{b_{\text{opt}}} = \inf_{F_{\text{stabilizing}}} \| P_1 \|^2 = 1 + \inf_{Q_{\text{causal}}} \| -Y \tilde{N}^* + X \tilde{M}^* + Q \|^2 \]

\[ = 1 + \sup_{0 \leq t < \infty} \| (I - P(t)) \left[ -Y \tilde{N}^* + X \tilde{M}^* \right] P(t) \|^2 \]  

(5.31)

The dual of the above formula is obtained as follows.

\[ \| P_2 \|^2 = \left\| \begin{bmatrix} -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} -\tilde{X} - MQ \\ \tilde{Y} - NQ \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} -M^* \tilde{X} + N^* \tilde{Y} - Q \\ I \end{bmatrix} \right\|^2 \]

\[ = 1 + \| -M^* \tilde{X} + N^* \tilde{Y} - Q \|^2 \]  

(5.32)

Thus,

\[ \frac{1}{b_{\text{opt}}} = \inf_{F_{\text{stabilizing}}} \| P_2 \|^2 = 1 + \inf_{Q_{\text{causal}}} \| -M^* \tilde{X} + N^* \tilde{Y} - Q \|^2 \]

\[ = 1 + \sup_{0 \leq t < \infty} \| (I - P(t)) \left[ -M^* \tilde{X} + N^* \tilde{Y} \right] P(t) \|^2 . \]  

(5.33)

An alternative characterization of the optimal robustness problem was given in [20]. To extend the result to time varying systems, we need to prove the following lemmata for the time-varying case. [19, 20, 33].

**Lemma 35** \( E_1 \), and \( E_2 \) are bounded operators such that \[ \left\| \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\| \leq \alpha < 1, \] then

\[ \| E_1 (I - E_2)^{-1} \| \leq \frac{\alpha}{\sqrt{1 - \alpha^2}} . \]
Proof: By the hypothesis, $\|E_2\| < 1$. Therefore, $(I - E_2)^{-1}$ exists and is bounded. In addition, we have

$$\frac{1}{1 - \alpha^2}(\alpha^2 I - E_2)^* (\alpha^2 I - E_2) \geq 0 \quad (5.34)$$

$$\frac{\alpha^4}{1 - \alpha^2} I - \frac{\alpha^2}{1 - \alpha^2} (E_2 + E_2^*) + \frac{1}{1 - \alpha^2} E_2^* E_2 \geq 0 \quad (5.35)$$

$$\frac{\alpha^2}{1 - \alpha^2} [I - (E_2 + E_2^*) + E_2^* E_2] - (\alpha^2 I - E_1^* E_1 - E_2^* E_2) - E_1^* E_1 \geq 0 \quad (5.36)$$

$$\frac{\alpha^2}{1 - \alpha^2} (I - E_2)^* (I - E_2) - E_1^* E_1 \geq (\alpha^2 I - E_1^* E_1 - E_2^* E_2) \geq 0 \quad (5.37)$$

$$(I - E_2)^* \left\{ \frac{\alpha^2}{1 - \alpha^2} I - [E_1(I - E_2)^{-1}]^* [E_1(I - E_2)^{-1}] \right\} (I - E_2) \geq 0 \quad (5.38)$$

$$\left\{ \frac{\alpha^2}{1 - \alpha^2} I - [E_1(I - E_2)^{-1}]^* [E_1(I - E_2)^{-1}] \right\} \geq 0 \quad (5.39)$$

Therefore, $\|E_1(I - E_2)^{-1}\| \leq \frac{\alpha}{\sqrt{1 - \alpha^2}}$. 

Lemma 36 $E_1$ and $E_2$ are bounded operators such that $\left\| \begin{bmatrix} E_1 & E_2 \end{bmatrix} \right\| \leq \alpha < 1$, then $\|(I - E_2)^{-1}E_1\| \leq \frac{\alpha}{\sqrt{1 - \alpha^2}}$.

Proof: $\left\| \begin{bmatrix} E_1 & E_2 \end{bmatrix} \right\| \leq \alpha < 1$ implies that $\left\| \begin{bmatrix} E_1^* \\ E_2^* \end{bmatrix} \right\| \leq \alpha < 1$ and by the previous lemma we know $\|E_1^*(I - E_2^*)^{-1}\| \leq \frac{\alpha}{\sqrt{1 - \alpha^2}}$. Therefore, $\|(I - E_2)^{-1}E_1\| \leq \frac{\alpha}{\sqrt{1 - \alpha^2}}$. 

We now extend the theorem by Glover and McFarlane [20].
Theorem 37 A compensator \( F \) stabilizes a plant \( G \) and \( \|P_2\| \leq \gamma \) if and only if there exist operators \( U \) and \( V \) such that \[
\begin{bmatrix}
V \\
U
\end{bmatrix}
\] is a strong right representation of \( F \) and
\[
\left\| \begin{bmatrix}
-N^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\| \leq \sqrt{1 - \frac{1}{\gamma^2}} \tag{5.40}
\]
where \[
\begin{bmatrix}
-N \\
\bar{M}
\end{bmatrix}
\] is a *-normalized strong left representation of \( G \).

Proof: Assume \( F \) stabilizes \( G \), then \( \|P_2\| \) is bounded and *-normalized strong representations exist for \( G \). Thus, \[
\begin{bmatrix}
M^* \\
-N
\end{bmatrix}
\begin{bmatrix}
N^* \\
\bar{M}
\end{bmatrix}
\]
is unitary. Suppose \( \|P_2\| \leq \gamma \), then from the Youla parametrization in Theorem 22 there exists a \( Q \) so that \( G^{-1} \{F\} = \mathcal{R}\left\{ \begin{bmatrix}
-X \\
\bar{Y}
\end{bmatrix} - \begin{bmatrix}
M \\
N
\end{bmatrix} Q \right\} \). For this \( Q \) (and this stabilizing compensator),
\[
\|P_2\| = \left\| \begin{bmatrix}
-X \\
\bar{Y}
\end{bmatrix} - \begin{bmatrix}
M \\
N
\end{bmatrix} Q \right\|. \tag{5.41}
\]
Hence,
\[
\|P_2\|^2 = \left\| \begin{bmatrix}
M^* & N^*
\end{bmatrix} \begin{bmatrix}
M & -X \\
-N & \bar{M}
\end{bmatrix} \begin{bmatrix}
-Q \\
I
\end{bmatrix} \right\|^2
= \left\| \begin{bmatrix}
I & (-M^*X + N^*\bar{Y}) \\
0 & I
\end{bmatrix} \begin{bmatrix}
-Q \\
I
\end{bmatrix} \right\|^2
= \left\| (-M^*X + N^*\bar{Y} - Q) \right\|^2
\leq \gamma^2. \tag{5.42}
\]
Therefore,
\[
\|-M^*X + N^*\bar{Y} - Q\|^2 \leq \gamma^2 - 1. \tag{5.43}
\]
Define \[
\begin{bmatrix}
U \\
V
\end{bmatrix} = -\frac{1}{\gamma^2} \begin{bmatrix}
-\bar{X} - MQ \\
\bar{Y} - NQ
\end{bmatrix}.
\]
Thus \[
\begin{bmatrix}
V \\
U
\end{bmatrix}
\]
is a strong right representation of the compensator \( F \). Therefore,

\[
\left\| \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\|^2
\]

\[
= \left\| \begin{bmatrix}
M^* & N^*
\end{bmatrix} \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} - \frac{1}{\gamma^2} \begin{bmatrix}
-\bar{X} - MQ \\
\bar{Y} - NQ
\end{bmatrix} \right\|^2
\]

\[
= \left\| \begin{bmatrix}
-\frac{1}{\gamma^2} (-M^*\bar{X} + N^*\bar{Y} - Q)
\end{bmatrix} \right\|^2
\]

\[
= \left(1 - \frac{1}{\gamma^2}\right)^2 + \frac{1}{\gamma^4} \left\| -M^*\bar{X} + N^*\bar{Y} - Q \right\|^2.
\]

(5.44)

Hence, from Equation (5.43) we conclude that

\[
\left\| \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\|^2 \leq 1 - \frac{2}{\gamma^2} + \frac{1}{\gamma^4} + \frac{1}{\gamma^4} (\gamma^2 - 1) = 1 - \frac{1}{\gamma^2}.
\]

(5.45)

Conversely, assume

\[
\left\| \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\| \leq \sqrt{1 - \frac{1}{\gamma^2}}.
\]

(5.46)

Therefore,

\[
\left\| \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\|^2 = \left\| \begin{bmatrix}
M^* & N^*
\end{bmatrix} \begin{bmatrix}
-\bar{N}^* \\
\bar{M}^*
\end{bmatrix} + \begin{bmatrix}
U \\
V
\end{bmatrix} \right\|^2
\]

\[
= \left\| \begin{bmatrix}
M^*U + N^*V \\
I - \bar{N}U + \bar{M}V
\end{bmatrix} \right\|^2 \leq 1 - \frac{1}{\gamma^2}.
\]

(5.47)
By the assumption, $\|I - \hat{\mathbf{N}}\mathbf{U} + \hat{\mathbf{M}}\mathbf{V}\| < 1$ and $(\hat{\mathbf{N}}\mathbf{U} + \hat{\mathbf{M}}\mathbf{V})^{-1}$ exists and is bounded. Since the inverse can be expressed as a norm convergent series, we can also show that it is causal (see [12, 35]). Define $\mathbf{R} := -\hat{\mathbf{N}}\mathbf{U} + \hat{\mathbf{M}}\mathbf{V}$. From Lemma 35, we know that

$$\|(M^*\mathbf{U} + N^*\mathbf{V})\mathbf{R}^{-1}\|^2 \leq \frac{1 - \frac{1}{\gamma^2}}{1 - \left(1 - \frac{1}{\gamma^2}\right)} = \gamma^2 - 1.$$  \hspace{1cm} (5.48)

From the Youla parametrization, we know that

$$\begin{bmatrix}
\hat{\mathbf{Y}} \\
-\hat{\mathbf{X}}
\end{bmatrix} - \begin{bmatrix}
\mathbf{N} \\
\mathbf{M}
\end{bmatrix} \mathbf{Q}$$

is a strong right representation of an operator $\mathbf{F}$ that stabilizes $\mathbf{G}$. Choose

$$\mathbf{Q} := - \begin{bmatrix}
\mathbf{Y} & \mathbf{X}
\end{bmatrix} \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1}. \hspace{1cm} (5.49)$$

Hence,

$$\begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} - \begin{bmatrix}
\mathbf{M} \\
\mathbf{N}
\end{bmatrix} \mathbf{Q} = \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{M} \\
\mathbf{N}
\end{bmatrix} \begin{bmatrix}
\mathbf{Y} & \mathbf{X}
\end{bmatrix} \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1}$$

$$= \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} + \left\{ \begin{bmatrix}
\mathbf{I} & \mathbf{0}
\end{bmatrix} \begin{bmatrix}
\mathbf{Y} \\
-\hat{\mathbf{X}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{0} & \mathbf{I}
\end{bmatrix} \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} \right\} \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1}$$

$$= \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1} - \begin{bmatrix}
-\hat{\mathbf{X}} \\
-\hat{\mathbf{Y}}
\end{bmatrix} \begin{bmatrix}
\mathbf{N} \\
\mathbf{M}
\end{bmatrix} \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1}$$

$$= \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} + \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1} - \begin{bmatrix}
-\hat{\mathbf{X}} \\
\hat{\mathbf{Y}}
\end{bmatrix} \mathbf{R} \mathbf{R}^{-1} = \begin{bmatrix}
\mathbf{U} \\
\mathbf{V}
\end{bmatrix} \mathbf{R}^{-1}. \hspace{1cm} (5.50)$$

Therefore, $\begin{bmatrix}
\mathbf{V} \\
\mathbf{U}
\end{bmatrix}$ is a strong right representation of a compensator that stabilizes $\mathbf{G}$. 


To finish the proof, we compute the norm of the parallel projection.

\[
\|P_2\|^2 = \left\| \begin{bmatrix} -X - MQ & \bar{Y} - NQ \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} U & V \end{bmatrix} R^{-1} \right\|^2 = \left\| \begin{bmatrix} M^* & N^* \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix} R^{-1} \right\|^2
\]

\[
= \left\| \begin{bmatrix} (M^*U + N^*V)R^{-1} \\ RR^{-1} \end{bmatrix} \right\|^2 = 1 + \|(M^*U + N^*V)R^{-1}\|^2.
\]

Thus, from Equation (5.48) we conclude

\[
\|P_2\|^2 \leq 1 + \gamma^2 - 1 = \gamma^2.
\]

\[\text{(5.52)}\]

The following is the dual to the previous theorem.

**Theorem 38** A compensator \(F\) stabilizes a plant \(G\) and \(\|P_1\| \leq \gamma\) if and only if there exist operators \(\bar{U}\) and \(\bar{V}\) such that \(\begin{bmatrix} \bar{V} & \bar{U} \end{bmatrix}\) is a strong left representation of \(F\) and

\[
\left\| \begin{bmatrix} M^* & N^* \end{bmatrix} + \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix} \right\| \leq \sqrt{1 - \frac{1}{\gamma^2}}
\]

\[\text{(5.53)}\]

where \(\begin{bmatrix} M \\ N \end{bmatrix}\) is a *-normalized strong right representation of \(G\).

**Proof:** Assume \(F\) stabilizes \(G\), then \(\|P_1\|\) is bounded and *-normalized strong representations exist for \(G\). Thus, \(\begin{bmatrix} M & -\bar{N}^* \\ N & \bar{M}^* \end{bmatrix}\) is unitary. Suppose \(\|P_1\| \leq \gamma\), then from the Youla parametrization in Theorem 22 there exists a \(Q\) so that \(G^{-1}\{F\} = \mathcal{K}\{\begin{bmatrix} Y & X \end{bmatrix} + Q \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix}\}\). For this \(Q\) (and this stabilizing compensator),

\[
\|P_1\| = \left\| \begin{bmatrix} Y & X \end{bmatrix} + Q \begin{bmatrix} -\bar{N} & \bar{M} \end{bmatrix} \right\|.
\]

\[\text{(5.54)}\]
Hence,

\[ \|P_1\|^2 = \left\| \begin{bmatrix} I & Q \\ \bar{M} & \bar{N} \end{bmatrix} \begin{bmatrix} Y & X \\ -\bar{N} & \bar{M} \end{bmatrix} \begin{bmatrix} M & -\tilde{N}^* \\ \tilde{N} & \tilde{M}^* \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} I & (-Y\tilde{N}^* + X\tilde{M}^*) \\ 0 & I \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} I & (-Y\tilde{N}^* + X\tilde{M}^* + Q) \end{bmatrix} \right\|^2 \]

\[ = 1 + \| -Y\tilde{N}^* + X\tilde{M}^* + Q \|^2 \leq \gamma^2. \quad (5.55) \]

Therefore,

\[ \| -Y\tilde{N}^* + X\tilde{M}^* + Q \|^2 \leq \gamma^2 - 1. \quad (5.56) \]

Define \[ \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = -\frac{1}{\gamma^2} \begin{bmatrix} Y & Q\bar{N} \\ X & Q\bar{M} \end{bmatrix} \]. Thus \[ \begin{bmatrix} \tilde{V} & \tilde{U} \end{bmatrix} \] is a strong left representation of the compensator \( F \). Therefore,

\[ \left\| \begin{bmatrix} M^* & N^* \\ \tilde{U} & \tilde{V} \end{bmatrix} \right\|^2 \]

\[ = \left\| \left\{ \begin{bmatrix} M^* & N^* \end{bmatrix} - \frac{1}{\gamma^2} \begin{bmatrix} Y & Q\bar{N} \\ X & Q\bar{M} \end{bmatrix} \right\} \begin{bmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{bmatrix} \right\|^2 \]

\[ = \left\| \begin{bmatrix} \left( 1 - \frac{1}{\gamma^2} \right) I & -\frac{1}{\gamma^2} (-Y\tilde{N}^* + X\tilde{M}^* + Q) \end{bmatrix} \right\|^2 \]

\[ = \left( 1 - \frac{1}{\gamma^2} \right)^2 + \frac{1}{\gamma^4} \left\| -Y\tilde{N}^* + X\tilde{M}^* + Q \right\|^2. \quad (5.57) \]

Hence, from Equation (5.56) we conclude that

\[ \left\| \begin{bmatrix} M^* & N^* \end{bmatrix} + \begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} \right\|^2 \leq 1 - \frac{2}{\gamma^2} + \frac{1}{\gamma^4} + \frac{1}{\gamma^4} (\gamma^2 - 1) = 1 - \frac{1}{\gamma^2}. \quad (5.58) \]

Conversely, assume

\[ \left\| \begin{bmatrix} M^* & N^* \end{bmatrix} + \begin{bmatrix} \tilde{U} & \tilde{V} \end{bmatrix} \right\| \leq \sqrt{1 - \frac{1}{\gamma^2}}. \quad (5.59) \]
Therefore,

\[
\left\| \begin{bmatrix} M^* & N^* \\ \bar{U} & \bar{V} \end{bmatrix} \right\|^2 
= \left\| \left\{ \begin{bmatrix} M^* & N^* \\ \bar{U} & \bar{V} \end{bmatrix} \right\} \begin{bmatrix} M & -\bar{N}^* \\ N & M^* \end{bmatrix} \right\|^2
= \left\| \begin{bmatrix} (I + \bar{U}M + \bar{V}N) & -\bar{U}\bar{N}^* + \bar{V}\bar{M}^* \end{bmatrix} \right\|^2 \leq 1 - \frac{1}{\gamma^2}.
\] (5.60)

By the assumption, \( \|I + \bar{U}M + \bar{V}N\| < 1 \) and \((\bar{U}M + \bar{V}N)^{-1}\) exists and is bounded. Since the inverse can be expressed as a norm convergent series, we can also show that it is causal (see [12, 35]). Define \( \bar{R} := \bar{U}M + \bar{V}N \). From Lemma 36, we know that

\[
\left\| \bar{R}^{-1}(-\bar{U}\bar{N}^* + \bar{V}\bar{M}^*) \right\|^2 \leq \frac{1 - \frac{1}{\gamma^2}}{1 - \left(1 - \frac{1}{\gamma^2}\right)} = \gamma^2 - 1.
\] (5.61)

From the Youla parametrization, we know that \( \begin{bmatrix} X & Y \end{bmatrix} + Q \begin{bmatrix} \bar{M} & -\bar{N} \end{bmatrix} \) is a strong left representation of an operator \( F \) that stabilizes \( G \). Choose

\[
Q := \bar{R}^{-1} \begin{bmatrix} \bar{U} & \bar{V} \end{bmatrix} \begin{bmatrix} -\bar{X} \\ \bar{Y} \end{bmatrix}
\] (5.62)

Hence,
\[
\begin{bmatrix}
Y & X
\end{bmatrix} + Q \begin{bmatrix}
-N & M
\end{bmatrix}
\]
\[
= \begin{bmatrix}
Y & X
\end{bmatrix} + \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} \begin{bmatrix}
-X
\tilde{Y}
\end{bmatrix} \begin{bmatrix}
-N & M
\end{bmatrix}
\]
\[
= \begin{bmatrix}
Y & X
\end{bmatrix} + \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} Y & X \end{bmatrix} \right\}
\]
\[
= \begin{bmatrix}
Y & X
\end{bmatrix} + \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} - \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} Y & X \end{bmatrix}
\]
\[
= \begin{bmatrix}
Y & X
\end{bmatrix} + \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} - \tilde{R}^{-1} \tilde{R} \begin{bmatrix} Y & X \end{bmatrix}
\]
\[
= \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix}.
\] (5.63)

Therefore, \( \begin{bmatrix}
\tilde{V} & \tilde{U}
\end{bmatrix} \) is a strong left representation of a compensator that stabilizes \( G \).

To finish the proof, we compute the norm of the parallel projection.

\[
\|P_1\|^2 = \left\| \begin{bmatrix}
Y - Q\tilde{N} & M + Q\tilde{Y}
\end{bmatrix} \right\|^2 = \left\| \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} \right\|^2
\]
\[
= \left\| \tilde{R}^{-1} \begin{bmatrix}
\tilde{U} & \tilde{V}
\end{bmatrix} \begin{bmatrix} M & -\tilde{N}^* \\ N & \tilde{M}^* \end{bmatrix} \right\|^2 = \left\| \tilde{R}^{-1} \tilde{R} \begin{bmatrix} Y & X \end{bmatrix} \right\|^2
\]
\[
= 1 + \left\| \tilde{R}^{-1}(-\tilde{U}\tilde{N}^* + \tilde{V}\tilde{M}^*) \right\|^2.
\] (5.64)

Thus, from Equation (5.61) we conclude

\[
\|P_1\|^2 \leq 1 + \gamma^2 - 1 = \gamma^2.
\] (5.65)

If equality holds in one half of the above theorems, then equality must hold in the other half or a contradiction results. Thus, if \( \|P_2\| = \gamma \) for a compensator \( F \),
then
\[ \left\| \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\| = \sqrt{1 - \frac{1}{\gamma^2}} = \alpha \] (5.66)

and minimization of \( \gamma \) corresponds to minimization of \( \alpha \). Hence, the optimal robustness problem is an Arveson distance problem. Therefore,

\[ \sqrt{1 - \frac{1}{b_{opt}^2}} = \inf Q \left\| \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\| 
= \sup_{0 \leq t < \infty} \left\| (I - P(t)) \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} P(t) \right\|. \] (5.67)

Thus,

\[ \frac{1}{b_{opt}^2} = 1 - \sup_{0 \leq t < \infty} \left\| (I - P(t)) \begin{bmatrix} -\bar{N}^* \\ \bar{M}^* \end{bmatrix} P(t) \right\|^2. \] (5.68)

In a similar fashion, if \( \| P \| = \gamma \), then

\[ \left\| \begin{bmatrix} M^* & N^* \\ \bar{U} & \bar{V} \end{bmatrix} \right\| = \sqrt{1 - \frac{1}{\gamma^2}} = \alpha. \] (5.69)

Since minimization of \( \gamma \) is equivalent to minimization of \( \alpha \), we obtain

\[ \sqrt{1 - \frac{1}{b_{opt}^2}} = \inf Q_{\text{causal}} \left\| \begin{bmatrix} M^* & N^* \\ \bar{U} & \bar{V} \end{bmatrix} \right\| 
= \sup_{0 \leq t < \infty} \left\| (I - P(t)) \begin{bmatrix} M^* & N^* \end{bmatrix} P(t) \right\|. \] (5.70)

or

\[ \frac{1}{b_{opt}^2} = 1 - \sup_{0 \leq t < \infty} \left\| (I - P(t)) \begin{bmatrix} M^* & N^* \end{bmatrix} P(t) \right\|^2. \] (5.71)

At present, there are no methods available to solve these equations for time-varying systems except for pathological examples. However, these formulae from
Glover and McFarlane [20] have more structure to exploit in the calculation of the norm. The standard Arveson distance problem tries to approximate a non-causal operator with a causal operator. In these formulae, we are trying to approximate an anti-causal operator with a causal operator. Hopefully, it will be possible to exploit the anti-causality of the $*$-normalized representations to solve these Arveson distance problems which will yield the size of the optimal gap ball of allowable uncertainty as well as the representation of an optimal compensator.
CHAPTER VI

Nonstabilizable Systems

Thus far in this dissertation, we have considered only general questions about stabilizable, linear, discrete-time, time-varying systems. In this chapter, we will apply some of the results of the previous chapters to examine nonstabilizable systems. In the first section, we will examine two linear, time-invariant systems and we will prove one theorem about discrete-time, time-invariant systems. The first example is a continuous-time example and the second is a discrete-time example. Next, we will prove that any linear, discrete-time, time-invariant plant that is not stabilizable with a linear, time-invariant compensator is not stabilizable with a linear, time-varying compensator. Finally, in the second section, we will examine a linear, time-varying plant of Feintuch [8] and show that it is not stabilizable with any linear, possibly time-varying compensator.

6.1 Time-Invariant Plants

In Kailath's book [24], a continuous-time example by Shefi was introduced in Example 1.1-1 to illustrate some subtle points about linearity and time-invariance. Consider a single-input, single-output, continuous-time plant $G$ defined on all piecewise continuous functions over $[0, \infty)$ having only a finite number of simple jump discontinuities in a finite time. For an input $x$, the output $Gx$ at any time $t$ is the algebraic sum of the jumps of the input $x$ from 0 up to the present time $t$. Since we are interested only in $L_2$ signals, the $D \{G\}$ is the intersection of
$L_2$ with the above piecewise continuous functions that also yield $L_2$ output functions.

It is easy to show that the plant $G$ is linear, time-invariant, and unbounded. Surprisingly, the plant does not possess a transfer function in the frequency domain. The proof of this proceeds by contradiction. Assume the plant $G$ has a transfer function $G$. If the input signal $x_1$ is a finite amplitude, finite time duration rectangular pulse, then the output $y_1 = Gx_1$ will be a rectangular pulse of the same amplitude and same duration as the input pulse. Since $y_1 = x_1$, then the Laplace transforms are also equal ($y_1 = x_1$) and $G = 1$. If the input signal $x_2$ is a finite amplitude, finite time duration triangular pulse that is continuous, then the output $y_1 = Gx_1$ will be zero. Since $y_1 = 0$, then its Laplace transform will be zero and $G = 0$. Thus, we have a contradiction so $G$ does not have a transfer function. It is important to remember in Chapter II that we guaranteed existence of a transfer function only if the plant is linear, time-invariant, and bounded. For unbounded, time-invariant plants we must have a closed graph so we may invoke the Beurling-Lax theorem to yield a transfer function.

We will show that the graph of the above continuous-time plant is not closed. First note that if a pulse of finite amplitude and duration is applied to the input, the output is a pulse of the same amplitude and duration. Since the continuous functions are dense in $L_2$ and the output to any continuous $L_2$ function is 0, we can construct a Cauchy sequence of continuous functions on $\mathcal{G} \{G\}$ such that the input sequence converges to a pulse and the output sequence is always 0. Thus, the graph is not closed nor is $\mathcal{G} \{G\}$ the graph of any operator because $\mathcal{G} \{G\}$ has multiple possible output functions for one input function. Since the graph is not closed, the plant is not stabilizable with any linear, possibly time-varying compensator by Theorem 15.
For the second example, consider the following unbounded, discrete-time, linear operator

\[
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{4}} & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{5}} & 0 & 0 & 0 & 0 \\
\vdots & \ddots & & & \\
0 & 0 & \ddots & & \\
0 & 0 & \ddots & \ddots & \\
0 & 0 & \ddots & \ddots & \ddots
\end{bmatrix}.
\] (6.1)

The domain of this operator is \( \ell_2[1, \infty) \) and the range is 0. It is interesting that even though the operator's matrix representation is not constant along diagonals, the operator is time-invariant by our definition \((SG \{G\} \subset G \{G\})\). The operator is not stabilizable because no strong right representation exists. To see this, assume there exists a strong right representation \( \begin{bmatrix} M \\ N \end{bmatrix} \) with a bounded, causal left inverse \( \begin{bmatrix} Y & X \end{bmatrix} \). Then,

\[
\ell_2[0, \infty) = \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \ell_2[0, \infty) = \begin{bmatrix} Y & X \end{bmatrix} G \{G\}. \] (6.2)

Because \( R \{G\} = 0 \) and \( D \{G\} = \ell_2[1, \infty) \), we have

\[
\ell_2[0, \infty) = Y \ell_2[1, \infty) + X0 = Y \ell_2[1, \infty) \] (6.3)

which is a contradiction since \( Y \) is causal. This example has several consequences in the analysis of slowly time-varying systems. In this analysis, the frozen-time plant at time \( k \) is defined as the time-invariant plant in which the \( k \)-th row is equal to the \( k \)-th row of the original time-varying plant. The research in slowly time-varying
plants tries to relate the behavior of the time-varying plant to the behavior of the set of frozen-time plants assuming the rate of change is slow enough. The plant $G$ has frozen-time plants uniformly bounded in norm and the norm of the frozen-time plants are monotonically decreasing to zero. Yet, the plant is unstable and nonstabilizable. Thus, the behavior of this time-invariant plant has no relation to the behavior of its frozen-time plants.

Finally, we prove the following theorem about discrete-time, time-invariant systems.

**Theorem 39** Let $G$ be a discrete-time, time-invariant, causal plant. If $G$ is not stabilizable with a time-invariant, causal compensator $F$, then it is not stabilizable with a time-varying, causal $F$.

**Proof:** In this proof, we will be changing our viewpoint (time domain vs. frequency domain) several times. Thus, we present the following notation. Let $M$ be a shift-invariant operator in the time domain and $M(z)$ be the corresponding $H_\infty$ matrix multiplication operator. Also, let $y(t)$ be a signal of finite energy in the time domain and $y(z)$ be the corresponding $H_2$ vector function.

In the first part of the proof, we construct a sequence of inputs to show that the right representation is not left invertible. The argument used is closely related to a proof in [19, Proposition 7].

From the Beurling-Lax Theorem, we can write (in the frequency domain)

$$G \{G(z)\} = \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} H_2^m$$

(6.4)

where $M(z)$ and $N(z)$ are matrices over $H_\infty$ such that

$$M(z)^*M(z) + N(z)^*N(z) = I$$

(6.5)
and $H^m_p$ is the Hardy $p$-space of vector valued functions on the disk. Since $G$ is not stabilizable by a time-invariant, causal $F$, \[
abla F \\begin{bmatrix} M(z) \\ N(z) \end{bmatrix} \] is not left invertible as a matrix over $H_\infty$. This means that $\inf_{|z|<1} \sigma_{\text{min}} \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} = 0$ by the Matrix Valued Corona (Fuhrmann) Theorem. Thus, we can find a sequence $z_i$ with $|z_i| < 1$ and complex $m$-vectors $x_i$ of unit norm such that \[
abla \begin{bmatrix} M(z_i) \\ N(z_i) \end{bmatrix} x_i \rightarrow 0 \text{ as } i \rightarrow \infty.\]

Construct the $(H^m_2)^\perp$ vectors (analytic outside the disk $D$)

$$y_i''(z) = \left( \frac{c_i}{z - z_i} \right) x_i \quad (6.6)$$

where $c_i$ are complex constants chosen so that $\|y_i''(z)\|_2 = 1$. Then

$$w_i''(z) := \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} y_i''(z) = \begin{bmatrix} M(z_i) \\ N(z_i) \end{bmatrix} y_i''(z) + \begin{bmatrix} M(z) - M(z_i) \\ N(z) - N(z_i) \end{bmatrix} y_i''(z) = (w_i'')^-(z) + (w_i'')^+(z) \quad (6.7)$$

where $\|w_i''(z)\|_2 = 1$, $(w_i'')^-(z) \in (H^m_2)^\perp$, and $(w_i'')^+(z) \in H^m_2$. Furthermore, $\|(w_i'')^-(t)\|_2 \rightarrow 0$ as $i \rightarrow \infty$.

Consider the $\ell^m_2(\infty, -1)$ vector corresponding to $y_i''(z)$. There exist normalized truncations $y_i'(t) \in \ell^m_2[-k_i, -1]$ such that $\|y_i'(t)\|_2 = 1$ and

$$w_i'(z) := \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} y_i'(z) = (w_i')^+(z) + (w_i')^-(z) \quad (6.8)$$
where \((w_i')^-(z) \in (H^m_2)\), \((w_i')^+(z) \in H^m_2\), and \(\|(w_i')^-(z)\|_2 \to 0\). This follows since 
\[
\begin{bmatrix}
M(z) \\
N(z)
\end{bmatrix}
\]
is a bounded operator on \(L^m_2\) of the unit circle.

Shifting \(y_i'(t)\) yields \(y_i(t) \in \ell^m_2[0, k_i - 1]\) with \(\|y_i(t)\|_2 = 1\) and

\[
w_i(t) := \begin{bmatrix} M \\ N \end{bmatrix} y_i(t) =: w_i^+(t) + w_i^-(t)
\]

where \(w_i^+(t) \in \ell^m_2[0, k_i - 1]\), \(w_i^-(t) \in \ell^m_2[k_i, \infty)\), and \(\|w_i^-(t)\|_2 \to 0\).

The proof now proceeds by contradiction. Suppose \(G\) is stabilizable with a time-varying, causal \(F\). Then 
\[
\begin{bmatrix} M \\ N \end{bmatrix}
\]
is a strong right representation of \(G\) by Theorem 25.

Hence, there exists bounded, causal operators \(Y\) and \(X\) such that

\[
\begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I.
\]

(6.10)

Let \(\|\begin{bmatrix} Y & X \end{bmatrix}\| = c\). Then

\[
1 = \|y_i(t)\|_2 = \|[I - P_m(k_i)]y_i(t)\|_2
\]

\[
= \left\| [I - P_m(k_i)] \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} y_i(t) \right\|_2
\]

\[
= \left\| [I - P_m(k_i)] \begin{bmatrix} Y & X \end{bmatrix} [I - P_{m+n}(k_i)] \begin{bmatrix} M \\ N \end{bmatrix} y_i(t) \right\|_2
\]

\[
\leq c \left\| [I - P_{m+n}(k_i)] \begin{bmatrix} M \\ N \end{bmatrix} y_i(t) \right\|_2 = c\|w_i^-(t)\|_2 \to 0
\]

(6.11)

which is a contradiction and the theorem is proved.
6.2 A Discrete-Time, Time-Varying Plant

In this section, we now prove that an example presented by Feintuch in [8] is not stabilizable. Consider the following single-input, single-output, unbounded operator

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 \\
\vdots & \ddots
\end{bmatrix}
\]

\[G = (6.12)\]

Let

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 0 \\
0 & 0 & 0 & 0 & 1/5 \\
\vdots & \ddots
\end{bmatrix}
\]

\[M = (6.13)\]

and

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\vdots & \ddots
\end{bmatrix}
\]

\[N = (6.14)\]
It can be seen that \( \begin{bmatrix} M \\ N \end{bmatrix} \) is a right representation of \( G \). In addition, \( \mathcal{K} \left\{ \begin{bmatrix} M \\ N \end{bmatrix} \right\} = 0 \). Following Feintuch [8], assume the existence of a bounded, causal left inverse \( \begin{bmatrix} Y & X \end{bmatrix} \). Since \( N \) has zeros on its diagonal, \( XN \) will also have zeros on its diagonal. Therefore, \( YM \) must have ones on its diagonal. This implies that the diagonal of \( Y \) is \( \{1, 2, 3, 4, \cdots \} \) which implies that \( Y \) is unbounded. Thus, \( \begin{bmatrix} M \\ N \end{bmatrix} \) has no left inverse. By Theorem 25, if \( G \) is stabilizable, then \( \begin{bmatrix} M \\ N \end{bmatrix} \) must have a left inverse. Thus, \( G \) is not stabilizable.
In a paper of Feintuch [9], the notion of an eventually time-invariant system was defined. Several interesting results were obtained. In particular, it was shown that any eventually time-invariant system can be written as the sum of a time-invariant operator and a compact operator. It was also shown that any linear time-invariant compensator which stabilizes the time-invariant part also stabilizes the original system. We will extend this result by showing that a compensator (possibly time-varying) stabilizes an eventually time-invariant system if and only if it stabilizes the time-invariant part.

Feintuch [9] also introduced a notion of minimal weighted sensitivity over the set of stabilizing, time-invariant compensators. It was asserted that the minimal sensitivity for the eventually time-invariant system is no greater than that of its time-invariant part. We will present a counterexample to this claim and also prove the reverse inequality. Moreover, we will see that both the inequality and the counterexample are still valid if one defines minimal sensitivity over the set of stabilizing time-varying compensators.

7.1 Stabilizing Compensators of Eventually Time-Invariant Systems

Eventually time-invariant plants are a useful description of several physical systems. An example is a flexible robot arm that moves from one position to another. This system is highly non-linear but can be modeled as a linear time-varying system.
After the arm moves, vibrations tend to die out slowly and a compensator is added to reduce the level and the duration of the vibrations. The robot arm model has a time-invariant part that describes the arm in the final position (for small vibrations) after the motion is complete and a compact part that describes the difference between the time-invariant part of model and the behavior of the arm during the motion from one position to another. Since the time-invariant part is a function only of the final position and the compact part is a function of the initial position, the path, and the final position, a compensator must possess stability robustness of two types. The first type is robustness with respect to the final position (time-invariant part of the plant). The second type is robustness with respect to the transient (compact part of the plant). Our theorem will show that for a set of conditions, if the compensator stabilizes the time-invariant part, it will also stabilize the eventually time-invariant plant (the complete model), but performance (sensitivity) will always be worse than the performance calculated using only the time-invariant part of the model.

In [9] it was proved that $G$ is **eventually time-invariant** if and only if $G = G_T + G_K$ where $G_T$ is time-invariant and $G_K$ is compact. Throughout this chapter we will use subscripts $T$ and $K$ to denote the time-invariant and the compact parts of an eventually time-invariant operator.

First, we prove the following theorem.

**Theorem 40** For all bounded, discrete-time, causal operators $G_b$ and all compact, discrete-time, strictly causal operators $G_K$, a compensator $F$ stabilizes $G_b$ if and only if $F$ stabilizes $G = G_b + G_K$.

**Proof:** The proof closely follows the proof of the Youla parametrization. Since $G_b$ is bounded and causal, a strong representation for the operator is $M = \bar{M} =$
Therefore, from the Youla parametrization, we have that $F$ stabilizes $G_b$ if and only if

$$g^{-1}\{F\} = \mathcal{R}\left\{ \begin{bmatrix} \bar{X} - MQ \\ \bar{Y} - NQ \end{bmatrix} \right\} = \mathcal{R}\left\{ \begin{bmatrix} -Q \\ I - G_bQ \end{bmatrix} \right\}.$$  \hspace{1cm} (7.1)

We need to show that such an $F$ will stabilize $G$. Choose $e_1 \in \mathcal{D}\{G\}$ and $e_2 \in \mathcal{D}\{F\}$ and calculate the closed-loop system inputs as follows:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I \\ G \end{bmatrix} e_1 + \begin{bmatrix} F \\ I \end{bmatrix} e_2 = \begin{bmatrix} I \\ G_b + G_K \end{bmatrix} w_1 + \begin{bmatrix} -Q \\ I - G_bQ \end{bmatrix} w_2$$

$$= \begin{bmatrix} I & -Q \\ G_b + G_K & I - G_bQ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left\{ \begin{bmatrix} I & -Q \\ G_b & I - G_bQ \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_K & 0 \end{bmatrix} \right\} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} I & 0 \\ G_b & I \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_K & 0 \end{bmatrix} \right\} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \hspace{1cm} (7.2)$$

Since $\begin{bmatrix} I & 0 \\ G_b & I \end{bmatrix}$ and $\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}$ are invertible, we can factor the equation to yield
\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  G_b & I
\end{bmatrix} \begin{bmatrix}
  I & -Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  -G_b & I
\end{bmatrix} \begin{bmatrix}
  G_K & 0 \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I - Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  G_K & 0 \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  I & 0 \\
  G_b & I
\end{bmatrix} \begin{bmatrix}
  I - Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  QG_K & 0 \\
  G_K & 0
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  I & 0 \\
  G_b & I
\end{bmatrix} \begin{bmatrix}
  I - Q \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  G_K & I
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]

(7.3)

Since \( G_K \) is strictly causal and compact, \( QG_K \) is strictly causal and compact. Thus, \[
\begin{bmatrix}
  I + QG_K & 0 \\
  G_K & I
\end{bmatrix}
\]
has \( I \) on the main block diagonal and it has no kernel. Because it has no kernel, it is invertible because it is in the form of \( I + K \) where \( K \) is compact.

If it was not invertible, \(-1\) would be in the spectrum of the compact operator \( K \) and \( I + K \) would have a kernel. Since the other two operators are invertible, and the inverses are all causal by Lemma 24, then

\[
\begin{bmatrix}
  e_1 \\
  e_2
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  0 & I - G_b Q
\end{bmatrix} \begin{bmatrix}
  I + QG_K & 0 \\
  G_K & I
\end{bmatrix}^{-1} \begin{bmatrix}
  I & 0 \\
  0 & I
\end{bmatrix} \begin{bmatrix}
  I & 0 \\
  -G_b & I
\end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix}
\]

(7.4)

and \( \{G, F\} \) is closed-loop stable. The other direction is proved by the same argument using \( G \) as the bounded operator and using the fact that \( G_b = G - G_K \). \( \blacksquare \)
For the special case of $G = G_T + G_K$ where $G_T$ is time-invariant and $G_K$ is compact and strictly causal we obtain the following.

**Corollary 41** For all bounded discrete-time operators $G = G_T + G_K$ where $G_T$ is time-invariant and $G_K$ is compact and strictly causal, a compensator $F$ (possibly time-varying) stabilizes $G$ if and only if $F$ stabilizes $G_T$.

### 7.2 Minimal Weighted Sensitivity

Consider the eventually time-invariant plant $G = G_T + G_K$ with given stable time-invariant operators $W_1, W_2$ with stable inverses (i.e. $W_1, W_2, W_1^{-1},$ and $W_2^{-1}$ are all bounded, causal, and time-invariant). The objective is to minimize a cost function

$$J = \|W_1(I - FG)^{-1}W_2\|$$

(7.5)

over all compensators that stabilize $G$. This cost function corresponds to the weighted sensitivity function of the system. The sensitivity function describes how much the closed-loop operator varies with variation of the plant [16, 23]. It is often weighted to penalize some bands of frequencies more heavily than others. By minimizing the cost function, we are minimizing variation of the closed-loop operators to variation of plant parameters much like the transistor amplifier example mentioned in Chapter I. In some cases, the compensator $F$ may be restricted to some class such as the time-invariant compensators.

We now define two optimal weighted sensitivities. The first, $\mu$, is the minimal weighted sensitivity of a plant over all time-varying compensators or
The second, $\hat{\mu}$, is the minimal weighted sensitivity of a plant over all time-invariant compensators that stabilize the plant or

$$\hat{\mu}(G) = \inf \left\{ \|W_1(I - FG)^{-1}W_2\| : F \text{ stabilizes } G \right\} \tag{7.6}$$

Clearly $\mu(G) \leq \hat{\mu}(G)$. From [26] and [27], we know that $\mu(G_f) = \hat{\mu}(G_f)$.

In [9], the claim was made that for $G$ eventually time-invariant, $\hat{\mu}(G) \leq \hat{\mu}(G_f)$. This claim is false as is shown by the single-input, single-output counterexample below. We remark that an error occurs in [9] by splitting the infimum in the final equation of p.113.

Let

$$G = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \ddots & \ddots
\end{bmatrix}, \tag{7.8}
$$

$W_1 = 1/(1 - 0.5z)$, and $W_2 = 1$. The time-invariant part of the operator $G$ is the shift operator $S$ and can be described with the transfer function $G_T = z$. The sensitivity is $\hat{S}(G_T, F) = (I - FG_T)^{-1} = I - QG_T$ where $Q$ is time-invariant and causal. Thus, $\hat{\mu}(G_T) = \inf_{Q \in H\infty} \|1/(1 - 0.5z)(1 - Qz)\|_{\infty}$. Since $\lim_{z \to 0} \|1/(1 - 0.5z)(1 - Qz)\|_{\infty} = 1$ for all $Q$, $\hat{\mu}(G_T) \geq 1$. However, if $Q = 0.5$ then $\|1/(1 - 0.5z)(1 - Qz)\|_{\infty} = 1$ so $\hat{\mu}(G_T) = 1$. 

$$\mu(G) = \inf \left\{ \|W_1(I - FG)^{-1}W_2\| : F \text{ stabilizes } G \right\}$$
For $Q$ causal,

$$Q = \begin{bmatrix}
q_{00} & 0 & 0 & 0 & 0 \\
q_{10} & q_{11} & 0 & 0 & 0 \\
q_{20} & q_{21} & q_{22} & 0 & 0 \\
q_{30} & q_{31} & q_{32} & q_{33} & 0 \\
q_{40} & q_{41} & q_{42} & q_{43} & q_{44} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}$$

(7.9)

and the sensitivity is

$$\hat{S}(G, F) = I - QG = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -q_{22} & 1 & 0 & 0 \\
0 & -q_{32} & -q_{33} & 1 & 0 \\
0 & -q_{42} & -q_{43} & -q_{44} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}$$

(7.10)

For the input sequence $\{1, 0, 0, 0, \ldots\}$, the output sequence is $\{1, 0, 0, 0, \ldots\}$ for any $Q$. Thus, the output sequence of the weighted sensitivity operator $W_1 \hat{S}(G, F)W_2$ is $\{1, 1/2, 1/4, 1/8, \ldots\}$. Since the above computation is valid for any $Q$ (not necessarily time-invariant), we conclude that $\hat{\mu}(G_T) = 1 < \frac{2}{3} \sqrt{3} \leq \mu(G) \leq \hat{\mu}(G)$.

We now claim the that the inequality $\hat{\mu}(G_T) \leq \hat{\mu}(G)$ is true in general. Moreover, the inequality remains valid if we minimize over time-varying compensators.

**Theorem 42** If $G = G_T + G_K$, with $G$ bounded, causal, and eventually time-invariant such that $G_T$ is its time-invariant part and it compact part $G_K$ is also strictly causal, then $\hat{\mu}(G_T) = \mu(G_T) \leq \mu(G) \leq \hat{\mu}(G)$. 
Proof: The first equality is from [26] and [27]. The last inequality is obvious. We note that \( W_1(I - FG_T)^{-1}W_2 = W_1(I - QG_T)W_2 \) for some causal \( Q \). Also, if \( Q \)
is causal then \( S^nQ^n \) is causal for all \( n \). Thus,

\[
\mu(G_T) = \inf_{Q \text{ causal}} \|W_1(I - QG_T)W_2\|
\leq \inf_{Q \text{ causal}} \|W_1(I - S^nQ^nG_T)W_2\|
= \inf_{Q \text{ causal}} \|S^nW_1(I - QG_T)W_2S^n\|
\leq \inf_{Q \text{ causal}} \|W_1(I - QG_T)W_2S^n\|
= \inf_{Q \text{ causal}} \{\|W_1(I - QG)W_2S^n + W_1QGKW_2S^n\|\}
\leq \inf_{Q \text{ causal}} \{\|W_1(I - QG)W_2S^n\| + \|W_1QGKW_2S^n\|\}
\leq \inf_{Q \text{ causal}} \{\|W_1(I - QG)W_2\| + \|W_1QGKW_2S^n\|\}.
\tag{7.11}
\]

Therefore, \( \mu(G_T) \leq \|W_1(I - QG)W_2\| + \|W_1QGKW_2S^n\| \) for all \( n \) and \( Q \). Since \( W_1QGKW_2 \) is compact, then \( \lim_{n \to \infty} \|W_1QGKW_2S^n\| = 0 \). Hence \( \mu(G_T) \leq \|W_1(I - QG)W_2\| \) for any causal \( Q \). Thus, \( \mu(G_T) \leq \mu(G) \).

We have thus established that the minimal weighted sensitivity of an eventually time-invariant system is no smaller than the sensitivity of its time-invariant part no matter what class of compensator we minimize over. Furthermore, the inequality may be strict.
CHAPTER VIII

Conclusions

In this dissertation, we have examined some general problems concerning feedback control of discrete-time, time-varying systems. In this chapter, we summarize the main contributions of this dissertation and discuss topics needing further research.

In trying to extend coprime factorizations to non-linear or time-varying systems (from an operator point of view), it was realized by Hammer [22] and Verma [38] that the meaning of the factorization needed clarification. Coprime factorizations of linear, time-invariant systems were defined as $NM^{-1}$ and $M^{-1}N$ and if the definition was extended directly to time-varying systems one would obtain $NM^{-1}$ and $M^{-1}N$ where the operators $M^{-1}$ and $M^{-1}$ may be unbounded. Such a definition is difficult to work with because of the unboundedness of the operators. The meaning that was attributed to the right representation (coprime factorization) was that $\mathcal{R}\left\{ \begin{bmatrix} M \\ N \end{bmatrix} \right\} = \mathcal{G}\{G\}$ and Hammer and Verma were able to produce some results.

Just as the world is often prejudiced against the left handed, so control systems research has been prejudiced against the left coprime factorization (left representation) because it was less understood. One of the main contributions of this dissertation is the extension of the left coprime factorization to the left representation by defining $\mathcal{K}\left\{ \begin{bmatrix} -N & M \end{bmatrix} \right\} = \mathcal{G}\{G\}$ thus allowing questions about right and left representations to be translated into questions about the graph of the operator. Careful
attention to the proofs of the theorems in this dissertation will reveal that often the left and right representation were used simultaneously in key steps and our better understanding of left representations allowed us to proceed to obtain our results.

In Chapter IV, we were able to obtain necessary and sufficient conditions for a linear, discrete-time, possibly time-varying system to be stabilizable with a linear, possibly time-varying compensator. We were also able to extend the Youla parametrization to linear, discrete-time, time-varying plants.

The conditions for stabilizability and the Youla parametrization assumed exact knowledge of the plant and this is an unrealistic assumption. Thus, in Chapter V we examined the problem of designing a compensator that stabilizes not only a plant $G$, but also the plants “close” to $G$ (the robustness problem). We described several methods of measuring the “distance” between two plants, but concentrated on the gap metric. We extended a result of Glover and McFarlane [20] to the discrete-time, time-varying case which yields a concise formula that relates the $\ast$-normalized representation of the plant to the representation of the optimal robust controller and the size of the largest gap ball of uncertainty that the optimal compensator will tolerate and guarantee stability and uniform boundedness of the closed-loop operators. Unfortunately, there are no methods for evaluating the formula for most time-varying plants. Because $\ast$-normalized representations played such a large role in this chapter, we examined the $\ast$-normalized left representation and highlighted a major characteristic of the $\ast$-normalized left representation of time-invariant plants. The $\ast$-normalized left representation of a time-invariant plant must be time-varying unless the plant is a constant gain.

In Chapter VI, we used our results from the previous chapters to examine some examples of nonstabilizable systems. We proved that a continuous-time example by
Shefi does not have a transfer function and is not stabilizable. We also proved that any discrete-time, time-invariant plant that is not stabilizable with a time-invariant compensator is not stabilizable with a time-varying compensator. In addition, we presented an example of a time-invariant plant that had uniformly bounded frozen-time plants that was not stabilizable. Finally, we proved that a time-varying example of Feintuch was not stabilizable.

In Chapter VII, we studied a class of bounded, time-varying operators in which the variation decreases as time increases to infinity (eventually time-invariant). Feintuch showed that such operators are equal to a time-invariant operator plus a compact operator. We showed that a compensator stabilizes an eventually time-invariant operator if and only if it stabilizes the time-invariant part of the operator. We were also able to show that the performance (as measured by the sensitivity) of an eventually time-invariant plant with a compensator is no better than the time-invariant part of the plant with a compensator. In addition, an example was presented that showed that the performance can be strictly worse.

Almost all areas investigated in this dissertation need further research. We proved the existence of right and left representations only for linear, discrete-time plants. We need to investigate the continuous-time case to see if the theorem is true. Unfortunately, our proof does not carry over to the continuous-time case.

Our proof of the equivalence of stabilizability and the existence of strong right and strong left system representations fails for continuous-time plants in two important steps. The first step where the proof fails is that inner/outer factorizations do not exist for continuous-time operators as stated in the following theorem found in [3, Theorem 14.20].
Theorem 43 Let $\mathcal{N}$ be a nest algebra. If there exists $\mathcal{M}_k$ that has no immediate successor, then there exists an operator $G \in \mathcal{N}$ that does not have an inner/outer factorization.

For a continuous-time nest algebra, the subspaces have no immediate successor and the inner/outer factorizations may not exist. Thus, our proof can not be used in the continuous-time case. However, the properties of $U$ that we use in our proof are that $U$ is a partial isometry with $\overline{\mathcal{R}}\{G\} = \overline{\mathcal{R}}\{U\}$ and that $\overline{\mathcal{K}}\{A\} = \overline{\mathcal{K}}\{U\}^\perp$ and $\overline{\mathcal{K}}\{A\} = \overline{\mathcal{K}}\{G\}$. Thus, if a factorization can be found that possesses these properties, this step in the proof would carry over to the continuous-time case.

The second step where our proof fails is in proving
\[
\begin{bmatrix}
Y & X \\
-N & M
\end{bmatrix}
\begin{bmatrix}
M & -\bar{X} \\
N & \bar{Y}
\end{bmatrix} = I.
\]

In this step, we used the fact that all bounded, causal operators on $\ell_2^n$ have a lower triangular matrix representation and no similar representation exists for the bounded, continuous-time operators.

The greatest need in the optimal robustness problem is a method of calculating a solution to the Arveson distance problem. However, even if such a method is found, the optimal robustness problem is far from completely solved. We are presently assuming that we have a nominal time-varying plant $G$ and we are optimizing for plants close to $G$. Implicit in this assumption is that we have knowledge of the time-variation of the plant before it occurs. For example, if an airplane is modeled as a linear, time-varying plant then we would need to know the flight path, the times when the pilot banks, and the pilot's maneuvers during take off and landing. Except for a few eventually time-invariant cases (i.e. chemical plant startup), this assumption
of non-causal knowledge of the time-variation of the plant is unreasonable. One method of attacking this problem would be to assume that the variation is slow and we know the set of plant parameters over which the plant can vary. Thus, we would be taking a slowly time-varying approach to the problem. As pointed out in Chapter VI, this method presently may not work because the characteristics of the plant may be totally different from the characteristics of the frozen-time plants. A better understanding of the example of the nonstabilizable time-invariant plant with uniformly bounded frozen-time plants is needed in order to appropriately redefine what is meant by the term frozen-time plant and the term slowly time-varying. We mention that it may be necessary to assume knowledge of past values of the plant parameters and to use this knowledge to calculate the present parameters of a time-varying controller. This approach would yield an adaptive controller for the time-varying plant.

Despite the pessimism in the foregoing paragraph about being unable to calculate the controllers of a time-varying system, our results have immediate applications in the design of time-invariant systems. For a time-invariant system, we have techniques to calculate time-invariant controllers that are optimally robust and the maximum gap of uncertainty the compensator will tolerate. The time-invariant compensator that is computed by these techniques will satisfy our formulae with the same maximum gap of uncertainty. Thus, the compensator not only guarantees robustness with respect to time-invariant uncertainty in the plant, but also guarantees robustness with respect to time-varying uncertainty in the plant with the same maximum gap of uncertainty.
BIBLIOGRAPHY


