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Hypergroups and semiproper functions

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The Ohio State University, 1991
HYPERGROUPS AND SEMIPROPER FUNCTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

Robert Lincoln Craighead, Jr., B.S., M.S.

* * * * *

The Ohio State University
1991

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To My Father
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CHAPTER I
INTRODUCTION

Frederic Marty first introduced hypergroups in his 1936 paper Sur les Groups et Hypergroups Attachés à une Fraction Rationelle [10]. Kenneth Stephenson used Marty’s definition of hypergroup to study compositions of inner functions and presented the results in his 1982 paper Analytic Functions and Hypergroups of Function Pairs [14]. We will use some definitions and theorems from Stephenson pertaining to hypergroups of function pairs to develop a theory for hypergroups of semiproper functions. Chapter II develops the theory of semiproper functions and Chapter III is devoted to hypergroups. One of the main results is in Chapter III and asserts that every semiproper function with a finite number of branch points has a hypergroup. Chapter IV contains the examples. In particular, an example from Stephenson shows that there is a function with no hypergroup. Chapter IV also contains the second important result which establishes conditions for expressing a finite Blaschke product as a composition of two Blaschke products of lower order.
Proper functions are functions under which the inverse image of each compact set is compact. The only proper analytic functions defined from the unit disk $\mathcal{U}$ into the unit disk are finite Blaschke products. If we only require that each component of the inverse image of a compact set be compact, then we get a broader class of functions, which we call semiproper functions. As far as we are aware, semiproper functions have not been studied previously. New results in this chapter include Proposition 2.5 (Annular functions are semiproper), Theorem 2.8 (Blaschke products are the only possible semiproper functions from $\mathcal{U}$ onto $\mathcal{U}$), and Example 2.11 (There exist semiproper infinite Blaschke products).

**Definition 2.1.** Let $X$ and $Y$ be Hausdorff topological spaces and $f$ a continuous function from $X$ into $Y$.

(i) The function $f$ is said to be proper if for every compact $K$ in $Y$, $f^{-1}(K)$ is compact.

(ii) The function $f$ is said to be semiproper if for every compact $K$ in $Y$, each component of $f^{-1}(K)$ is compact.

When we study proper and semiproper functions in the context of complex functions, we shall implicitly assume that we are considering only analytic functions.
**Proposition 2.2.** If $f : X \to Y$ is proper, then $f$ is semiproper.

**Proof.** Let $K$ be compact in $Y$. If $C$ is a component of $f^{-1}(K)$, then $f^{-1}(K)$ compact implies $C \subseteq \text{cl}(C) \subseteq f^{-1}(K)$. Since $C$ is a component of $f^{-1}(K)$, $C = \text{cl}(C)$. Now $f^{-1}(K)$ compact implies $C$ is compact. ■

**Definition 2.3.** An analytic function from the unit disk $\mathcal{U}$ into the complex plane $\mathbb{C}$ is said to be annular if there exists a sequence of Jordan curves $\{J_n\}_{n=1}^{\infty}$ such that

(i) $J_n$ is contained in the interior of $J_{n+1}$,

(ii) For every $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that $n \geq M$ implies $J_n \subseteq \{z \mid 1 - \epsilon < |z| < 1\}$ and

(iii) $\min\{|f(z)| \mid z \in J_n\} = m(f, J_n) \to \infty$.

An excellent source for annular functions is [3]. We shall state a result from [3] that will be needed to show that every annular function is semiproper and not proper.

**Proposition 2.4.** [3, pr. 3.2, p. 30] If $f : \mathcal{U} \to \mathbb{C}$ is annular then $\{z \mid f(z) = a\}$ is countably infinite for every complex number $a \neq \infty$.

**Proposition 2.5.** If $f : \mathcal{U} \to \mathbb{C}$ is annular then $f$ is semiproper and $f$ is not proper.

**Proof.** Let $K$ be compact in $\mathbb{C}$. If $w_0 \in K$, then $f^{-1}\{w_0\}$ is countably infinite and contained in $\mathcal{U}$. Hence, there is a limit point $z_0$ of $f^{-1}(K)$ which
cannot be in $U$. Therefore $|z_0| = 1$, $f^{-1}(K)$ is not closed in $C$ and hence $f^{-1}(K)$ cannot be compact. Since $f^{-1}(K)$ is not compact, $f$ cannot be proper.

To show that $f$ is semiproper, let $C$ be a component of $f^{-1}(K)$. There is an $N \in \mathbb{N}$ such that $f^{-1}(K) \cap J_n = \emptyset$ for all $n \geq N$. Since $C$ is connected, $C \subseteq B(0,R)$ for some $R$, $0 < R < 1$. If $x_0$ is a limit point of $C$ then $x_0$ is in $U$ and $x_0 \in f^{-1}(K)$ since $f^{-1}(K)$ is closed in $U$. Therefore $C = \text{cl}(C) \subseteq f^{-1}(K)$. Hence, $C$ closed and bounded in $C$ implies $C$ is compact. ■

**Definition 2.6.** An analytic function $f : U \to U$ has a radial limit $\alpha$ with respect to $\theta$ if $\lim_{r \to 1^-} f(re^{i\theta}) = \alpha$.

When we want to emphasize the point $z = e^{i\theta}$ on the unit circle $C(0,1)$ rather than the angle, we say $f$ has a radial limit $\alpha$ at $z$.

**Definition 2.7.** An analytic function $f : U \to U$ is said to be an inner function if the radial limits $\lim_{r \to 1^-} f(re^{i\theta})$ have modulus one a.e. $\theta$, $0 \leq \theta < 2\pi$.

**Theorem 2.8.** If $f$ is a semiproper function from $U$ into $U$ then $f$ is a finite or infinite Blaschke product.

**Proof.** Case 1. The function $f$ is an inner function. Write $f$ as the product of a Blaschke product $B$ and a singular function $S$. Since $S$ is singular, $S$ has at least one radial limit $0$. See [9, p. 73] and [12, ex. 17, p. 383]. WLOG, let $\lim_{r \to 1^-} f(r) = 0$. Let $K = \{f(x) \mid 0 \leq x < 1\} \cup \{0\}$. Then $K$ is compact. Since $[0,1)$ is contained in one component $C$ of $f^{-1}(K)$ and $1 \not\in C$, $C$ is
not compact. This is a contradiction unless the singular factor $S$ is trivial. Therefore, $f$ is a Blaschke product.

Case 2. The function $f$ is not inner. Now $f \in H^\infty(\mathcal{U})$ implies by Fatou's theorem that $\lim_{r \to 1^-} f(re^{i\theta})$ exists a.e. Since $f$ is not inner, $\lim_{r \to 1^-} f(re^{i\theta}) = a$ with $|a| < 1$ exists on a set of measure greater than 0. Proceed as before. ■

This theorem has a partial converse in that every finite Blaschke product is semiproper. In fact, for $K$ compact and $f$ a finite Blaschke product, $f^{-1}(K) \subseteq B(0, R)$ for some $R$, $0 < R < 1$. This implies that $f^{-1}(K)$ is closed in $\mathbb{C}$ and hence compact. It follows that $f$ is proper and by proposition 2.2 $f$ is semiproper. If $f$ is an infinite Blaschke product then $f$ cannot be proper. The inverse image of the singleton $\{0\}$ is not compact. We have proved the following corollary to theorem 2.8.

**Corollary 2.9.** [13, p. 300] A function $f : \mathcal{U} \to \mathcal{U}$ is proper if and only if $f$ is a finite Blaschke product.

The converse of Theorem 2.8 is false. The class of infinite Blaschke products contains examples of both semiproper functions and examples of functions that are not semiproper. Frostman [8] gave the example $B(z) = \prod_{n=1}^{\infty} \frac{(1-1/n^2) - z}{1-(1/n^2)z}$ of an infinite Blaschke product that has radial limit zero at one. See also Rudin [12, ex 13, p 341]. A similar proof to that of theorem 2.8 can be used to conclude that $B$ cannot be semiproper.

The following proposition generalizes the property that causes the above function $B$ not to be semiproper.
Proposition 2.10. Let $f$ be a continuous function from the unit disk into the unit disk. If $E$ is connected, $\text{cl}(E) \cap C(0,1) \neq \emptyset$ and $\text{cl}(f(E)) \subseteq U$, then $f$ is not semiproper.

Proof. The set $\text{cl} f(E)$ is compact and let $C$ be the component of $f^{-1}(\text{cl} f(E))$ with $E \subseteq C$. Since $C \subseteq U$ and a limit point of $C$ is in $C(0,1)$, $C$ cannot be compact. □

The following is an example of a semiproper infinite Blaschke product.

Example 2.11. There exists a sequence $\{x_\nu\}$, $0 < x_1 < x_2 < \ldots < 1$ such that the infinite Blaschke product $B(z) = \prod_{\nu=1}^{\infty} \frac{x_{\nu} - z}{1 - x_{\nu} z}$ is semiproper. We will choose the sequences $\{x_\nu\}$ and $\{R_\nu\}$ inductively so that if $B_n(z) = \prod_{\nu=1}^{n} \frac{x_{\nu} - z}{1 - x_{\nu} z}$, then $|B_n(z)| > 1 - \frac{2}{\nu^2}$ on $|z| = R_\nu$, $\nu = 1, 2, \ldots, n$. Let $x_1 = \frac{1}{2}$, $B_1(z) = \frac{1 - z}{1 - z/2}$. Choose $R_1$ so that $|z| \geq R_1$ implies $|B_1(z)| > 3/4$.

Suppose $x_1, \ldots, x_n$; have been chosen so that

$$|B_n(z)| > 1 - \frac{2}{\nu^2}; \quad |z| = R_\nu, \quad \nu = 1, 2, \ldots, n.$$  

Consider the function $f_t(z) = \frac{t - z}{1 - t^2}$, where $t$ is a real parameter not yet specified, with $\max\{x_n, 1 - 2^{-n}\} < t$. As $t \to 1$, $\frac{t - z}{1 - t^2} \to 1$ on $U$. Hence $f_t(z)B_n(z) \to B_n(z)$ as $t \to 1$ a.u. on $U$. So for all $t$ close enough to 1 we have

$$|f_t(z)B_n(z)| > 1 - \frac{2}{\nu^2}, \quad |z| = R_\nu, \quad \nu = 1, 2, \ldots, n.$$  

Let $x_{n+1}$ be one of these $t$'s and define $B_{n+1}$ by

$$B_{n+1}(z) = f_{x_{n+1}}(z)B_n(z).$$
Then
\[ |B_{n+1}(z)| > 1 - \frac{2}{\nu^2}, \quad |z| = R_\nu, \quad \nu = 1, 2, \ldots, n. \]

We need \( |B_{n+1}(z)| > 1 - \frac{2}{(n+1)^2}, \quad |z| = R_{n+1} \), to complete the induction.

Since \( B_{n+1}(z) \) is a finite Blaschke product there is an \( R = R_{n+1} \) such that
\[ |z| \geq R \Rightarrow |B_{n+1}(z)| > 1 - \frac{2}{(n+1)^2}. \]
Hence \( |B_{n+1}(z)| > 1 - \frac{2}{\nu^2} \) for \( |z| = |R_\nu|, \quad \nu = 1, 2, \ldots, n+1 \). Define \( B(z) = \prod_{\nu=1}^{\infty} \frac{x_\nu - z}{1 - x_\nu z} \). Since \( x_{n+1} \) was chosen such that
\[ \max\{x_n, 1 - 2^{-n}\} < x_{n+1} \]
we have that \( \sum_{\nu=1}^{\infty} 1 - |x_\nu| < \infty \) and
\[ \prod_{\nu=1}^{\infty} \frac{x_\nu - z}{1 - x_\nu z} \]
is an analytic function in \( U \).

We assert that \( B \) is semiproper.

Let \( K \) be a compact set in \( U \) with \( C \) a component of \( B^{-1}(K) \). If \( C \) is not compact, then there is a limit point \( a_0 \) of \( C \) on \( C(0,1) \). Since \( C \subset U \) is connected and has \( a_0 = e^{i\phi} \) in its closure, the sequence \( \{a_n \mid a_n = R_n e^{i\phi}\} \) satisfies \( \lim_{n \to \infty} a_n = a_0 \). Since \( |B_n(a_n)| \to 1, \ B_n(a_n) \not\in K \) if \( n \) is sufficiently large. The contradiction establishes the result. \( \blacksquare \)
CHAPTER III
HYPERGROUPS

The first part of this chapter introduces hypergroups of function pairs. It is not automatic that when given a function $f: U \rightarrow \mathcal{R}$, the function pairs will form a hypergroup. Example 4.4 in chapter IV demonstrates this failure and the property not satisfied is associativity. Analytic continuation is important in this example and plays a central role in the theory developed in the second part of this chapter. This theory is used to prove theorem 3.16. If $f$ is semiproper with a finite number of branch points, then the function pairs always form a hypergroup. The remaining part of the chapter contains results which show the interplay between the function, analytic continuation and the function pairs. We begin with a few definitions from Stephenson [14].

Definition 3.1. Let $H$ be a nonempty set and $m$ a function $m : H \times H \rightarrow P(H)$, the power set of $H$. $H$ is a hypergroup under $m$ if

(i) $m$ is associative in that for any three elements $a, b, c$ of $H$, 

$$\bigcup\{m(d, c) \mid d \in m(a, b)\} = \bigcup\{m(a, d) \mid d \in m(b, c)\}$$

(ii) There is an identity element $i \in H$ such that for every $a \in H$ $m(i, a) = m(a, i) = \{a\}$
(iii) For every $a \in H$, there is a unique element $a^{-1}$ in $H$ such that $(a^{-1})^{-1} = a$ and $i \in m(a, a^{-1}) \cap m(a^{-1}, a)$.

**Definition 3.2.** Let $\mathcal{U} = \{z \mid |z| < 1\}$ be the unit disk and $f$ and $g$ nonconstant analytic functions from $\mathcal{U}$ into a Riemann surface $\mathcal{R}$.

(i) A pair $(\phi, \psi)$ of analytic functions from $\mathcal{U}$ into $\mathcal{U}$ is said to be an $f$-$g$ pair if it satisfies the functional equation $f \circ \phi = g \circ \psi$. If $f = g$ then the pair $(\phi, \psi)$ is called an $f$ pair or a function pair for $f$.

(ii) A pair $(\phi, \psi)$ of analytic functions from $\mathcal{U}$ into $\mathcal{U}$ is said to match $w_1$ to $w_2$ if there is a $z$ in $\mathcal{U}$ with $\phi(z) = w_1$ and $\psi(z) = w_2$.

The following theorem establishes the existence of $f$-$g$ pairs. In particular, when $f = g$ we have the existence of $f$ pairs $(\phi, \psi)$, the elements needed to establish the hypergroup of $f$. We will state the theorem and briefly discuss how a pair is constructed. The construction is important since it shows the relationship between the functions $f$ and $g$, the intermediate surface created from continuations and the unit disk. A detailed proof of this theorem can be found in Stephenson [14, p. 850].

**Theorem 3.3 (Stephenson).** Let $f$ and $g$ be nonconstant analytic functions from $\mathcal{U}$ into a Riemann surface $\mathcal{R}$. If $z_1$ and $z_2$ are points of $\mathcal{U}$ with $f(z_1) = g(z_2)$, then there exists an $f$-$g$ pair $(\phi, \psi)$ which matches $z_1$ to $z_2$.

First, assume that $f$ and $g$ are smooth at $z_1$ and $z_2$ respectively. Choose domains $D_1$ and $D_2$ such that $z_i \in D_i$, $f$ is one-to-one from $D_1$ onto $\Delta$ and $g$ is $1-1$ from $D_2$ onto $\Delta$. Define $h = g^{-1} \circ f$ where $g^{-1}$ is the local inverse
of $g$ from $\Delta$ onto $D_2$. Taking all analytic continuations $(h_\alpha, D_\alpha)$ of $(h, D_1)$ with the restriction $|z| < 1$ and $|h_\alpha(z)| < 1$ for all $z \in D_\alpha$ leads to a Riemann surface we will denote by $W_h$. $W_h$ is a Riemann surface which serves as the domain for two functions $p^*$ and $H^*$ defined as follows:

$$p^*([h_\alpha, z_\alpha]) = z_\alpha \quad \text{projection}$$

$$H^*([h_\alpha, z_\alpha]) = h_\alpha(z_\alpha) \quad \text{evaluation.}$$

Since $p^*$ is bounded and nonconstant from $W_h$ into $\mathcal{U}$, $W_h$ is hyperbolic and has $\mathcal{U}$ as its universal cover [2, p 156].

Designate by $\rho$ the universal covering map (ucm) of $W_h$ and define the $f$-$g$ pair $(\phi, \psi)$ by the formulas $\phi = p^* \circ \rho$ and $\psi = H^* \circ \rho$. We have the following diagram:

![Diagram]

Figure 3.1: Relationship of $h$, $f$, $g$, $W_h$ and the pair $(\phi, \psi)$.
Notice that \( h \) may continue to a subdomain of \( U \) in one situation, and in another may continue to \( U - \{ \text{branch points} \} \) but be multivalued.

We check that \((\phi, \psi)\) satisfies \( f \circ \phi = f \circ \psi \) and that \((\phi, \psi)\) matches \( z_1 \) to \( z_2 \).

The domain \([h,D_1]\) in \( \mathcal{W}_h \) is conformal to \( D_1 \) in \( U \) when \( p^* \) is restricted to \([h,D_1]\). Since \( \rho \) is ucm, there is a \( \tilde{z} \) over \([h,z_1]\) in \([h,D_1]\) and a neighborhood \( V_\tilde{z} \) of \( \tilde{z} \) such that \( V_\tilde{z} \to \mathcal{W}_h \in [h,D'_1] \subseteq [h,D_1] \). The two computations
\[
\phi(\tilde{z}) = p^* \circ \rho(\tilde{z}) = p^*([h,z_1]) = z_1 \quad \text{and} \\
\psi(\tilde{z}) = H^* \circ \rho(\tilde{z}) = H^*([h,z_1]) = \psi(z_1) = z_2
\]
show that \((\phi, \psi)\) matches \( z_1 \) to \( z_2 \). If \( \tilde{z} \in V_\tilde{z} \), then
\[
\begin{align*}
\phi(z) &= f \circ \phi(z) = f \circ p^* \circ \rho(z) = f \circ p^*([h,z']) = f(z') \\
\psi(z) &= g \circ \psi(z) = g \circ H^* \circ \rho(z) = g \circ H^*([h,z']) = \\
g \circ \psi(z) &= g \circ (g^{-1} \circ f)(z') = f(z').
\end{align*}
\]
By the principle of analytic continuation,
\[
f \circ \phi = g \circ \psi \quad \text{on} \quad U.
\]
If \( f \) has a branch point at \( z_1 \) and/or \( g \) has a branch point at \( z_2 \), then we can choose small neighborhoods \( V_{z_1} \) and \( V_{z_2} \) about \( z_1 \) and \( z_2 \) where \( f \) is an \( n \) to \( 1 \) mapping and \( g \) is an \( m \) to \( 1 \) mapping and these neighborhoods contain no other branch points. Now choose \( z_1' \) and \( z_2' \) in \( V_{z_1} \) and \( V_{z_2} \) respectively where \( f \) and \( g \) are smooth. Repeat the above construction for these points and obtain through analytic continuation an \( f - g \) pair \((\phi, \psi)\) matching the original branch.
points $z_1$ to $z_2$. We remark that each pair $(\phi, \psi)$ constructed from smooth points is unique up to the choice of the universal covering map $\rho$. Changing this universal covering map leads to an equivalent pair as defined in 3.4 (iii) below. The pairs constructed from branch points are not unique and may lead to nonequivalent pairs.

We will need some other results from Stephenson [14] in order to begin proving our main theorem. These results will be terminology and the definition of the hypergroup $\mathcal{P}_f$ of a function $f$.

**Definition 3.4.** Let $f$ and $g$ be nonconstant analytic functions from the unit disk $U$ into a Riemann surface $\mathcal{R}$.

(i) An $f$-$g$ pair $(\phi, \psi)$ constructed as in theorem 3.3 is called a principal $f$-$g$ pair. When $f = g$, the pair $(\phi, \psi)$ is called a principal $f$ pair.

(ii) The pair $(\phi, \psi)$ of analytic functions from $U$ into $U$ is said to be subordinate to the pair $(\alpha, \beta)$ if there is an analytic function $\omega$ from $U$ into $U$ such that $\phi = \alpha \circ \omega$ and $\psi = \beta \circ \omega$. We denote this by $(\phi, \psi) \prec (\alpha, \beta)$.

(iii) The pair $(\phi, \psi)$ of analytic functions from $U$ into $U$ is said to be equivalent to $(\alpha, \beta)$ if there is a $\sigma \in \mathfrak{M}$, the group of Möbius transformations of $U$, such that $\phi = \alpha \circ \sigma$ and $\psi = \beta \circ \sigma$. We denote this by $(\alpha, \beta) \sim (\phi, \psi)$ and designate by $(\phi, \psi)$ the equivalence class containing $(\phi, \psi)$.

**Theorem 3.5.** [14, th. 4, p. 852] If $(\alpha, \beta)$ is an $f$-$g$ pair, then there is a principal $f$-$g$ pair $(\phi, \psi)$ such that $(\alpha, \beta) \prec (\phi, \psi)$. The pair $(\phi, \psi)$ is unique up to the universal covering map in Theorem 3.3.
To define multiplication let $(\phi_1, \psi_1)$ be a principal $f$ pair matching $z_1$ to $z_2$ and $(\phi_2, \psi_2)$ be a principal $f$ pair matching $z_2$ to $z_3$; say $\phi_1(w_1) = z_1$, $\psi_1(w_1) = z_2$, $\phi_2(w_2) = z_2$ and $\psi_2(w_2) = z_3$. See figure 3.2 below. From Theorem 3.3, there is a $\psi_1 \cdot \phi_2$ pair $(\alpha, \beta)$ matching $w_1$ to $w_2$.

We assert that $(\phi_1 \circ \alpha, \psi_2 \circ \beta)$ is an $f$ pair matching $z_1$ to $z_3$. The two computations

\[(\phi_1 \circ \alpha)(x) = \phi_1(w_1) = z_1 \quad \text{and} \quad (\psi_2 \circ \beta)(x) = \psi_2(w_2) = z_3\]

show that $(\phi_1 \circ \alpha, \psi_2 \circ \beta)$ does match $z_1$ to $z_3$. Since $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ are principal $f$ pairs,

\[f \circ \phi_1 \circ \alpha = f \circ \psi_1 \circ \alpha \quad \text{and} \quad f \circ \phi_2 \circ \beta = f \circ \psi_2 \circ \beta.\]

Now, $(\alpha, \beta)$ is a principal $\psi_1 \cdot \phi_2$ pair so that $f \circ \psi_1 \circ \alpha = f \circ \phi_2 \circ \beta$ and hence

\[f \circ \phi_1 \circ \alpha = f \circ \phi_2 \circ \beta.\]
By Theorem 3.5 there is a unique principal $f$ pair $(\phi_3, \psi_3)$ matching $z_1$ to $z_3$ such that $(\phi_1 \circ \alpha, \psi_2 \circ \beta) \prec (\phi_3, \psi_3)$. Define the product of $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ by

$$(\phi_1, \psi_1) \otimes (\phi_2, \psi_2) = (\phi_3, \psi_3).$$

**Definition 3.6.** Let $f : U \to \mathcal{R}$ be a nonconstant analytic function. Define the set $\mathcal{P}_f$ of equivalence classes by

$$\mathcal{P}_f = \{ (\phi, \psi) \mid (\phi, \psi) \text{ is a principal } f \text{ pair} \}.$$ 

If $(\phi_1, \psi)$ and $(\phi_2, \psi_2)$ belong to $\mathcal{P}_f$, define $(\phi_1, \psi_1) \otimes (\phi_2, \psi_2)$ by the set

$$\langle \phi_1, \psi_1 \rangle \otimes \langle \phi_2, \psi_2 \rangle = \{ (\phi, \psi) \mid (\phi, \psi) = (\alpha_1, \beta_1) \otimes (\alpha_2, \beta_2), \ (\alpha_i, \beta_i) \in \langle \phi_i, \psi_i \rangle \}.$$ 

Any element of this last set is called a determination of the product

$$\langle \phi_1, \psi_1 \rangle \otimes \langle \phi_2, \psi_2 \rangle.$$ 

When $\mathcal{P}_f$ exists, we will say that $\mathcal{P}_f$ is the hypergroup of $f$ under $\otimes$.

The preliminary definitions and theorems are in place to begin proving that a semiproper function with a finite number of branch points has a hypergroup.

**Definition 3.7.** The function $h = f^{-1} \circ f$ is said to generate the principal pair $(\phi, \psi)$ if $(\phi, \psi)$ results from the construction stemming from $h$ as outlined in the discussion following theorem 3.3.

**Proposition 3.8.** Let $h_i$ generate $(\phi_i, \psi_i)$, $i = 1, 2$. If $h_2 \circ h_1$ is defined, then $h_2 \circ h_1$ generates $(\phi_1, \psi_1) \otimes (\phi_2, \psi_2)$. 
Proof. WLOG, we can assume that the domain of $h_2$ is equal to the range of $h_1$. Let $D_1$, $D_2$ and $D_3$ be the domain of $h_1$, the domain of $h_2$ and the range of $h_2$ respectively. See figure 3.2. Let $z_i \in D_i$, $f$ smooth on $D_i$, $h_1(z_1) = z_2$ and $h_2(z_2) = z_3$. We have that $h_1(z_1) = f^{-1}(f(z_1)) = z_2$ which implies $f(z_1) = f(z_2)$. Also $h_2(z_2) = f^{-1}(f(z_2)) = z_3$ implies $f(z_2) = f(z_3)$.

From the definition of $\otimes$, $(\phi_1, \psi_1) \otimes (\phi_2, \psi_2)$ was chosen as the principal pair matching $z_1$ to $z_3$. When $f$ is smooth at $z_1$, $z_2$ and $z_3$, the pair is unique up to the universal covering map chosen. For our case, $h_2 \circ h_1$ is of the form $f^{-1} \circ f$ and $f(z_1) = f(z_3)$. Since $f$ is smooth at $z_1$ and $z_3$, the principal pair generated by $h_2 \circ h_1$ is equivalent to $(\phi_1, \psi_1) \otimes (\phi_2, \psi_2)$. ■

Proposition 3.9. For $i = 1, 2$ let $h_i$ generate the principal pair $(\phi_i, \psi_i)$ and let $(h_i, D_i)$ be function elements with $z_i \in D_i$. The following are equivalent:

(i) $h_1$ and $h_2$ generate the same equivalence class $(\phi, \psi)$ of principal $f$ pairs.

(ii) $\mathcal{W}_{h_1} = \mathcal{W}_{h_2}$.

(iii) $(h_1, D_1)$ and $(h_2, D_2)$ are analytic continuations via $(h_t, D_t)$, $t \in [1, 2]$ along $\gamma$ connecting $z_1 \in D_1$ and $z_2 \in D_2$ such that for every $z \in D_t$, $|z| < 1$ and $|h_t(z)| < 1$.

Proof (i) ⇒ (ii). The functions $h_1$ and $h_2$ generate the same equivalence class $(\phi, \psi)$ iff there is $\omega \in \mathcal{M}$ s.t.

$$\phi_1 = \phi_2 \circ \omega \quad \text{and} \quad \psi_1 = \psi_2 \circ \omega. \quad (1)$$
Moreover, from the construction in Stephenson [14] we have the following com-
mutative diagrams relating the equivalence of \((\phi_1, \psi_1)\) and \((\phi_2, \psi_2)\).

Figure 3.3: The pair \((\phi_1, \psi_1)\) generated by \(h_1\).

Figure 3.4: Equivalence of \((\phi_1, \psi_1)\) and \((\phi_2, \psi_2)\).

The equations involving the pairs, the evaluation and projection maps are

\[
\begin{align*}
\phi_1 &= p_1^* \circ \rho, & \psi_1 &= H_1^* \circ \rho, & \phi_2 &= p_2^* \circ \sigma, & \psi_2 &= H_2^* \circ \sigma
\end{align*}
\]

where \(\rho\) and \(\sigma\) are universal covering maps.

We show that \(W_{h_1} \subseteq W_{h_2}\). The idea is to start with \([h_\alpha, \bar{z}]\) in \(W_{h_1}\) and show that this germ is also in \(W_{h_2}\). First, we have to locate a germ \(\sigma \circ \omega(\bar{z})\) in \(W_{h_2}\) and establish that this germ is at \(z\). Then we must show that the function defining the germ \(\sigma \circ \omega(\bar{z})\) agrees with \(h_\alpha\) near \(z\). After some preliminary arranging, the appropriate domain will be \(D_\alpha\) as defined below.

Let \([h_\alpha, z] \in W_{h_1}\). Since \(\rho\) and \(\sigma\) are universal and \(\omega\) conformal there is a \(\bar{z} \in \mathcal{U}\) and a neighborhood \(N_{\bar{z}}\) of \(\bar{z}\) such that
The element $\sigma \circ \omega(\tilde{z})$ of $W_{h_2}$ is at $z$ since
\[ p_2^* \circ \sigma \circ \omega(\tilde{z}) = \phi_2 \circ \omega(\tilde{z}) = \phi_1(\tilde{z}) = p_1^* \circ \rho(\tilde{z}) = p_1^*([h_\alpha, z]) = z. \]

Hence we may write $[h_\beta, z]$ for $\sigma \circ \omega(\tilde{z})$. Since sets of the form $[g, D]$ with $D \subseteq U$ form a base for the topology in $W_{h_2}$, [2, p 99], we may extend figure 3.6 to

\[
\begin{align*}
\tilde{z} &\in N_{\tilde{z}} \\
\omega \downarrow &\quad \omega \downarrow \\
[\alpha_1, z] &\in N[\alpha_1, z] \\
\sigma &\downarrow \quad \sigma \downarrow \\
\sigma \circ \omega(\tilde{z}) &\in \sigma \circ \omega(N_{\tilde{z}})
\end{align*}
\]

**Figure 3.5:** The germ $[h_\alpha, z]$ covered by $\tilde{z}$.

**Figure 3.6:** The germ $\sigma \circ \omega(\tilde{z})$ covered by $\tilde{z}$.

**Figure 3.7:** The extension of figure 3.6 to include the germ $[h_\beta, z] = \sigma \circ \omega(\tilde{z})$. 
where $\sigma \circ \omega^{-1}([h_\beta, D_\beta]) \cap N_{\tilde{x}}$ is a neighborhood of $\tilde{x}$. Hence

$$
\rho(\sigma \circ \omega^{-1}([h_\beta, D_\beta]) \cap N_{\tilde{x}})
$$

is a neighborhood of $[h_\alpha, z]$, and there exists a domain $[h_\alpha, D_\alpha]$ such that figure 3.5 extends to

$$
\tilde{x} \in \rho^{-1}([h_\alpha, D_\alpha]) \cap (\sigma \circ \omega^{-1}([h_\beta, D_\beta]) \cap N_{\tilde{x}}) \subseteq (\sigma \circ \omega^{-1}([h_\beta, D_\beta]) \cap N_{\tilde{x}}) \subseteq N_{\tilde{x}}
$$

Let $\tilde{V} = \rho^{-1}([h_\alpha, D_\alpha]) \cap (\sigma \circ \omega^{-1}([h_\beta, D_\beta]) \cap N_{\tilde{x}})$. We assert that $h_\alpha = h_\beta$ on $D_\alpha$.

If $x \in D_\alpha$ then $[h_\alpha, x] \in [h_\alpha, D_\alpha] \Rightarrow \exists \tilde{x} \in \tilde{V}$ such that $\rho(\tilde{x}) = x$. But $\tilde{x} \in \tilde{V} \Rightarrow \sigma \circ \omega(\tilde{x}) \in [h_\beta, D_\beta]$ and $\sigma \circ \omega(\tilde{x})$ is at $x$ since $p_2^* \circ \sigma \circ \omega(\tilde{x}) = \phi_2 \circ \omega(\tilde{x}) = \phi_1(\tilde{x}) = x$. Hence $\sigma \circ \omega(\tilde{x}) \in [h_\beta, D_\beta]$ at $x$ means $\sigma \circ \omega(\tilde{x}) = [h_\beta, x]$ by definition of $[h_\beta, D_\beta]$. Also $\sigma \circ \omega(\tilde{x}) = [h_\beta, x] \in [h_\beta, D_\beta] \Rightarrow x \in D_\beta$. Also, $h_\beta(x) = H_2^*([h_\beta, x]) = H_2^* \circ \sigma \circ \omega(\tilde{x}) = \psi_2 \circ \omega(\tilde{x}) = \psi_1(\tilde{x}) = H_1^* \circ \rho(\tilde{x}) = H_1^*([h_\alpha, \chi]) = h_\alpha(x)$. Therefore, $D_\alpha \subseteq D_\beta$ and $h_\alpha = h_\beta$ on $D_\alpha \Rightarrow h_\alpha$ agrees with $h_\beta$ near $z$. Hence $[h_\alpha, z] = [h_\beta, z]$ and $W_{h_1} \subseteq W_{h_2}$.

If $W_{h_1} \subseteq W_{h_2}$, then $h_1$ is a continuation of $h_2$ and so $W_{h_1} = W_{h_2}$.
(ii) $\Rightarrow$ (i). This is clear since if the universal map $\rho$ is fixed, then the same pair will be constructed.

(ii) and (iii) are equivalent since the component of $p^*(-1)(U) \cap H^*{-1}(U)$ containing the germ $[h_1, z_1]$ is $\mathcal{W}_{h_1}$. Since $(h_2, D_2)$ is a continuation of $(h_1, D_1)$ with the restriction $|z| < 1$ and $|h_t(z)| < 1$, $[h_2, z_2] \in \mathcal{W}_{h_1}$ and so $\mathcal{W}_{h_1} = \mathcal{W}_{h_2}$. See also [14, p. 850].

The following definition and proposition are from Conway [6, p. 241] with some minor changes. $\mathbb{C}$ has been replaced by $\mathcal{R}$, a Riemann surface and disks in $\mathbb{C}$ are replaced by parametric disks. The proof of the proposition will not be given.

**Definition 3.10.** Let $G$ be a region in $\mathbb{C}$ and $G \xrightarrow{f} \mathcal{R}$, an analytic smooth function. If $a \in G$, and $f(a) = \alpha$, let $(g, D)$ be a function element such that $\alpha \in D$ and $f(g(z)) = z$ for $z$ in $D$. The complete analytic function $\mathcal{F}$ obtained from $(g, D)$ will be called the complete analytic function of local inverses of $f$. We will deliberately abuse notation and let $\mathcal{W}_{f^{-1}}$ denote the Riemann surface over $f(G) \subseteq \mathcal{R}$ as well as the set of germs $[f^{-1}, w]$ in $\mathcal{F}$ with $w \in f(G)$.

**Proposition 3.11.** Let $G$ be a region in $\mathbb{C}$ and $G \xrightarrow{f} \mathcal{R}$ an analytic smooth function. Let $a, b \in G$, $f(a) = \alpha$, $f(b) = \beta$ and let $\Delta_0$ and $\Delta_1$ be parametric disks about $\alpha$ and $\beta$ respectively such that there are analytic functions $g_0 : \Delta_0 \rightarrow G$, $g_1 : \Delta_1 \rightarrow G$ with $g_0(\alpha) = a$, $g_1(\beta) = b$, $f(g_0(\zeta)) = \zeta$ for $\zeta$ in $\Delta_0$ and $f(g_1(\zeta)) = \zeta$ for $\zeta$ in $\Delta_1$. Then there is a curve $\sigma$ in $f(G) \subseteq \mathcal{R}$ from...
\( \alpha \) to \( \beta \) such that \((g_1, \Delta_1)\) is a continuation of \((g_0, \Delta_0)\) along \( \sigma \). The curve \( \sigma \) is \( f \circ \gamma \) where \( \gamma \) is any curve in \( G \) with initial point \( \alpha \) and terminal point \( b \).

Also, there is a curve \( \tilde{\sigma} \) in \( \mathcal{W}_{f^{-1}} \) over \( \sigma \) with initial point \([g_0, \alpha]\) and terminal point \([g_1, \beta]\).

**Proposition 3.12.** Let \( G \) be a region in \( \mathcal{U} \) and \( G \xrightarrow{f} \mathcal{R} \) a smooth analytic function. Let the function element \((h_0, D_0) = (f_0^{-1} \circ f, D_0)\) generate \((\phi_0, \psi_0)\) and the function element \((h_1, D_1) = (f_1^{-1} \circ f, D_1)\) generate \((\phi_1, \psi_1)\).

If \((\mathcal{W}_{f^{-1}}, p^*)\) has the curve lifting property then \((\phi_0, \psi_0) \otimes (\phi_1, \psi_1)\) is defined.

**Proof.** Proposition 3.8 says the product will exist if \( h_1 \circ h_0 \) is defined. Since we are not given the existence of \( h_1 \circ h_0 \), we have to find an analytic continuation \((h, \Omega_0)\) of \((h_1, D_1)\) such that \( h \circ h_0 \) is defined. We can use proposition 3.9 to conclude that the product \((\phi_0, \psi_0) \otimes (\phi_1, \psi_1)\) exists.

The following diagram shows the relationship between the functions \( h_0, h_1, h = f^{-1}_r \circ f_0 \) and \( \mathcal{W}_{f^{-1}} \). We will show that \((h_1, D_1)\) has an analytic continuation to a function element \((h, \Omega_0)\) where \( \Omega_0 = h_0(D_0) \). We remark that we have subscripted all \( f \)'s to distinguish among the various germs.
We are to find $f^{-1}_\tau$. Let $\gamma$ be any curve in $G$ with initial point $z_1 \in D_1$ and terminal point $\zeta_0 \in \Omega_0$. The curve $f \circ \gamma$ is in $\mathcal{R}$ and by hypothesis lifts to the curve $\tilde{f} \circ \gamma$ in $(\mathcal{W}_{f^{-1}})_2$ with initial point $[f^{-1}_\beta, w_1]$ and some terminal point $[f^{-1}_\tau, w_0]$. Hence, we have a continuation of $(f^{-1}_1, \Delta_1)$ to $(f^{-1}_\tau, \Delta_0)$ along $f \circ \gamma$.

We assert that $(f^{-1}_\tau \circ f_0, \Omega_0)$ is an analytic continuation of $(f^{-1}_1 \circ f_\beta, D_1)$ along $\gamma$.

Let $(f^{-1}_\beta, \Lambda_1)$ be the continuation of $(f^{-1}_\beta, \Delta_1)$ to $(f^{-1}_0, \Delta_0)$ along $f \circ \gamma$ and $(f^{-1}_\tau, \Gamma_1)$ be the continuation from $(f^{-1}_1, \Delta_1)$ to $(f^{-1}_\tau, \Delta_0)$ along $f \circ \gamma$. 

Figure 3.9: Construction of $h$ as related to given $f_0$ and $f_\tau$ to be found and to $\mathcal{W}_{f^{-1}}$. 

We assert that $(f^{-1}_\tau \circ f_0, \Omega_0)$ is an analytic continuation of $(f^{-1}_1 \circ f_\beta, D_1)$ along $\gamma$.
We will show that \((f^{-1}_{tt} \circ f_\beta, f^{-1}_{\beta t}(\Lambda_t \cap \Gamma_t))\) is an analytic continuation from 
\((f^{-1}_{1t} \circ f_\beta, f^{-1}_{\beta 0}(\Lambda_0 \cap \Gamma_0))\) to 
\((f^{-1}_{r} \circ f_0, f^{-1}_{\beta 1}(\Lambda_1 \cap \Gamma_1))\). See [6, p 216]. We remark that the above functions \(f^{-1}_{tt}\) are appropriate local inverses and are \(1-1\).

Let \(t \in [0,1]\). Then \(f \circ \gamma(t) \in \Lambda_t\) and \(f \circ \gamma(t) \in \Gamma_t\). Hence \(f(\gamma(t)) \in \Lambda_t \cap \Gamma_t\), and \(\gamma(t) \in f^{-1}_{\beta t}(\Lambda_t \cap \Gamma_t)\). Also, there are \(\delta_1\) and \(\delta_2\) such that:

(i) \(|s - t| < \delta_1 \Rightarrow f \circ \gamma(s) \in \Lambda_t\) and \(f^{-1}_{\beta s} = f^{-1}_{\beta t}\) near \(f(\gamma(s))\), say on \(B(f(\gamma(s)), \epsilon_1)\).

(ii) \(|s - t| < \delta_2 \Rightarrow f \circ \gamma(s) \in \Gamma_t\) and \(f^{-1}_{ts} = f^{-1}_{tt}\) near \(f(\gamma(s))\), say on \(B(f(\gamma(s)), \epsilon_2)\).

For \(|s - t| < \delta = \min\{\delta_1, \delta_2\}\) we have

\[ f \circ \gamma(s) \in \Lambda_t \cap \Gamma_t, \quad \text{and so} \quad \gamma(s) \in f^{-1}_{\beta t}(\Lambda_t \cap \Gamma_t). \]

And for

\[ z \in f^{-1}_{\beta t}(B(f(\gamma(s)), \epsilon_1)) \cap B(f(\gamma(s)), \epsilon_2) \]

we have

\[ f^{-1}_{1t} \circ f_{\beta t}(z) = f^{-1}_{1t} \circ f_{\beta s}(z) = f^{-1}_{1s} \circ f_{\beta s}(z). \]

Hence the definition for analytic continuation is satisfied. Define the function \(h\) by \(h(z) = f^{-1}_r \circ f_0(z)\). Now \(h \circ h_0\) is defined and by proposition 3.9 the product \((\phi_0, \psi_0) \otimes (\phi_1, \psi_1)\) is defined. ■
Proposition 3.13. Let $G$ be a domain in $U$ and $G \xrightarrow{f} \mathcal{R}$ be analytic and smooth with $D_0$ and $D_1$ domains of $G$ such that $(h_0, D_0)$ analytically continues along $\gamma \subseteq G$ to $(h_1, D_1)$. If $h_0$ is of the form $f_0^{-1} \circ f$ and $h_t(\gamma(t)) \in G$ for the function elements $(h_t, D_t)$ defining the analytic continuation, then $h_1$ is of the form $f_1^{-1} \circ f$ and the curve $f \circ \gamma$ lifts to $\mathcal{W}_{f^{-1}}$ with initial point $[f_0^{-1}, w_0]$ where $h_0(\gamma(0)) = f_0^{-1}(w_0)$.

Proof. Let $h_0$ generate the principal $f$ pair $(\phi, \psi)$. Since $(h_0, D_0)$ continues to $(h_1, D_1)$, $\gamma$ lifts to $\mathcal{W}_{h_0}$ with initial point $[h_0, \gamma(0)]$ and terminal point $[h_1, \gamma(1)]$. Designate this lift by $\tilde{\gamma}$. If $\rho$ is the ucm of $\mathcal{W}_{h_0}$, designate by $\Gamma$ the lift of $\tilde{\gamma}$ in $(U, \rho)$. See figure 3.10.

![Figure 3.10](image-url)
In completing the proof, it would be notationally convenient to distinguish between the restrictions of $f$ as they pertain to $h_0$ and $h_1$.

Let $h_0 = f_{0}^{-1} \circ f_{\alpha}$. We want to find functions $f_{\beta}$ and $f_{1}$ such that $f_{1}^{-1} \circ f_{\beta} = h_{1}$. See figure 3.11.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3_11.png}
\caption{Required continuation and equality of curves $f \circ \gamma$ and $f \circ H^* \circ \tilde{\gamma}$.}
\end{figure}

It is enough to show that $f \circ \gamma = f(H^*(\tilde{\gamma}))$. We can define $f_\beta$ using $f$ near $\gamma(1)$. Say that $f_\beta$ is defined from $D_1$, $1-1$ and onto $\Omega_1$. By proposition 3.11 we have a continuation of $(f_{\alpha}^{-1}, \Omega_0)$ to $(f_{\beta}^{-1}, \Omega_1)$ along $f \circ \gamma$. Since $h_t(\gamma(t)) \in G$ for each $h_t$ in the function elements $(h_t, D_t)$, $H^*(\tilde{\gamma}) \subseteq G$. Therefore, for the curve $f(H^*(\tilde{\gamma}))$, there is a continuation from $(f_0^{-1}, \Omega_0)$ to $(f_1^{-1}, \Omega_1')$ along $f(H^*(\tilde{\gamma}))$. If $f \circ \gamma = f(H^*(\tilde{\gamma}))$, then we can just as well say $\Omega_1 = \Omega_1'$. Using the same ideas as in proposition 3.12, $(f_1^{-1} \circ f_{\beta}, D_1)$ is an analytic continuation of $(f_0^{-1} \circ f_{\alpha}, \Omega_0)$ along $\gamma$. Note that the continuation
from \((f^{-1}_0, \Omega_0)\) to \((f^{-1}_1, \Omega'_1)\) and the assumption that \(f \circ \gamma = f(H^*(\tilde{\gamma}))\) say that \(f \circ \gamma\) lifts to \(W_{f^{-1}}\) with initial point \([f^{-1}_0, w_0]\) where \(f^{-1}_0(w_0) = h_0(\gamma(0))\). In order to show that \(f \circ \gamma = f \circ H^*(\tilde{\gamma})\) we have to verify that \(f \circ \phi = f \circ \psi\) on a domain containing \(\Gamma\). Since \(H^*(\tilde{\gamma}) \subseteq G\) and \(\gamma \subseteq G\), \(\tilde{\gamma}\) is contained in some component \(C\) of the open set \(p^*\Gamma \cap H^*(G)\). Since \(C\) is open, \(\rho^{-1}(C)\) is open and \(\Gamma\) is contained in a component \(\tilde{C}\) of \(\rho^{-1}(C)\). Just as in the discussion following theorem 3.3, \(f \circ \phi = f \circ \psi\) on a small disk containing \(\Gamma(0)\), so by the principle of analytic continuation \(f \circ \phi = f \circ \psi\) on \(\tilde{C}\). We complete the proof with

\[
f \circ \gamma = f \circ p^* (\tilde{\gamma}) = f \circ p^* \rho (\Gamma) = f \circ \phi (\Gamma) = f \circ \psi (\Gamma) = f \circ H^* \circ \rho (\Gamma) = f \circ H^*(\gamma).
\]

The above proposition shows that we can use \(W_{f^{-1}}\) to compute the analytic continuation of any \(h\) of the form \(f^{-1} \circ f\).

**Corollary 3.14.** If \(W_{h_0}\) is the surface generated from the continuation in \(G\) then \(W_{h_0}\) is conformal to \(W_{h_0^{-1}}\).

**Proof.** Define \(\Lambda : W_{h_0} \to W_{h_0^{-1}}\) by \(\Lambda([h_\alpha, z]) = [h_\alpha^{-1}, h_\alpha(z)]\). Use figure 3.11 to observe that \(h_1 = f_1^{-1} \circ f_\beta\) is an analytic continuation of \(h_0 = f_0^{-1} \circ f_\alpha\) along \(\gamma\) if and only if \(h_0^{-1}\) continues analytically to \(h_1^{-1}\) along the curve \(H^*(\tilde{\gamma})\). This says that if \(h_\alpha\) is a continuation of \(h_0\), then \(h_\alpha^{-1}\) is a continuation of \(h_0^{-1}\) and hence \(\Lambda\) is well defined. It also says that \(\Lambda\) is \(1-1\). To show that \(\Lambda\) is onto, let \([g_\tau, x] \in W_{h_0^{-1}}\). Since \(\gamma\) and \(H^*(\tilde{\gamma})\) are contained in \(G\), by proposition 3.13, \(g_\tau\) is of the form \(f^{-1} \circ f\). Applying the observation again, \(g_\tau^{-1}\) is an analytic continuation of \(h_0\). Hence \(\Lambda([g_\tau^{-1}, g_\tau(x)]) = [g_\tau, x]\).
Finally, we assert that $A$ is analytic. The atlas of the Riemann surface $\mathcal{W}_{h_0}$ is the set $\{p^* \uparrow [h_\alpha, D_\alpha]\}$. See [2, p. 99]. We show that the function $\Lambda_{\beta \alpha} = p^* \circ A \circ p^{-1}$ defined on $p^*([h_\alpha, D_\alpha] \cap \Lambda^{-1}([h_\beta^{-1}, D_\beta])) \subseteq \mathcal{U}$ is holomorphic. Since $h_\alpha = h_\beta = h$ on $D_\alpha \cap h_\beta^{-1}(D_\beta)$, for every $z \in D_\alpha \cap h_\beta^{-1}(D_\beta)$ we have

$$\Lambda_{\beta \alpha}(z) = \Lambda_{\beta \alpha}(p^*([h,z])) = p^* \circ A \circ p^{-1}((p^*([h,z]))) = p^*([h^{-1}, h(z)]) = h(z).$$

Hence, each $\Lambda_{\beta \alpha}$ is holomorphic and so $A$ is analytic. See [2, p. 57]. ■

The notation for corollary 3.14 will be $\mathcal{W}_{h_0} \simeq \mathcal{W}_{h_0^{-1}}$.

**Corollary 3.15.** If $h_0$ generates $(\phi, \psi)$ then $h_0^{-1}$ generates $(\psi, \phi)$.

**Proof.** Let $D_0 \xrightarrow{h_0} \Delta$ and $\rho$ the ucm for the surface $\mathcal{W}_{h_0}$. Since $\mathcal{W}_{h_0} \simeq \mathcal{W}_{h_0^{-1}}$, $\Lambda \circ \rho$ is ucm for $\mathcal{W}_{h_0^{-1}}$. Let $(\alpha, \beta)$ be the principal pair associated with $(\mathcal{W}_{h_0^{-1}}, \Lambda \circ \rho)$ and $\mathcal{D}'$ in $\mathcal{U}$ over a subset $[h_0, D_0']$ of $[h_0, D_0]$ where $\rho$ is 1–1. If $\tilde{z} \in \mathcal{D}'$ then $\alpha(\tilde{z}) = p^* \circ \Lambda \circ \rho(\tilde{z}) = p^* \circ \Lambda([h_0, z]) = p^*([h_0^{-1}, h_0(z)]) = h_0(z)$ and $\psi(\tilde{z}) = H^* \circ \rho(\tilde{z}) = H^*([h_0, z]) = h_0(z)$. By the principle of analytic continuation $\alpha = \psi$. Similarly, $\beta(\tilde{z}) = H^* \circ \Lambda \circ \rho(\tilde{z}) = H^* \circ \Lambda([h_0, z]) = H^*([h_0^{-1}, h_0(z)]) = z$ and $\phi(\tilde{z}) = p^* \circ \rho(\tilde{z}) = p^*([h_0, z]) = z$ show that $\beta = \phi$. That is, $(\alpha, \beta) = (\psi, \phi)$. ■

We now state and prove our main result.

**Theorem 3.16.** If $\mathcal{U} \xrightarrow{f} \mathcal{R}$ is a nonconstant semiproper analytic function with a finite number of branch points, then $f$ has a hypergroup $\mathcal{P}_f$ under $\otimes$. 
Proof. The idea is to replace \( f \) by a function \( g \) such that \( g \) is semiproper with no branch points. We can then show that \( g \) lifts curves and has the same hypergroup as \( f \). Let

\[ V = \{ z \in \mathcal{U} | z \text{ is not the preimage of the image of a branch point of } f \} . \]

If we define \( g = f \upharpoonright V \), then \( V \to \mathcal{R} \) is smooth, \( \mathcal{U} - V \) is countable and if \( z_0 \in \mathcal{U} - V \) then there is \( \delta \) such that \( B(z_0, \delta) \cap \mathcal{U} - V = \{ z_0 \} \). Furthermore, \( \mathcal{U} - V \) is closed in \( \mathcal{U} \), \( V \) is open in \( \mathcal{C} \) and \( g(V) \) is open in \( \mathcal{R} \). Hence \( g(V) \) is a Riemann surface.

We assert that \( V \to \mathcal{R}_g = g(V) \) is semiproper. Let \( K \) be compact in \( \mathcal{R}_g \). Then \( K \) is compact in \( \mathcal{R} \) and \( K \cap f(\{ z \in \mathcal{U} | z \text{ is a branch point of } f \}) = \emptyset \). If \( z \in f^{-1}(K) \) then \( f(z) \in K \subseteq \mathcal{R}_g \) and \( f(z) \) is not the image of a branch point. Therefore \( z \) is not the preimage of the image of a branch point which implies \( z \in V \). That is, \( f^{-1}(K) \subseteq g^{-1}(K) \). Hence, \( g^{-1}(K) = f^{-1}(K) \). Since the components of \( f^{-1}(K) \) are compact because \( f \) is semiproper, we have that \( g \) is semiproper.

We now assert that \( V \to \mathcal{R}_g \) lifts curves. Let \( w \in \mathcal{R}_g \). Since \( \mathcal{R}_g \) is a Riemann surface, there is a parametric disk \( Q_w \) in \( \mathcal{R}_g \) such that:

(i) \( w \in \cl(Q_w) \),

(ii) \( \cl(Q_w) \) does not contain the image of any branch point of \( f \) and

(iii) \( \cl Q_w \) is compact.

Since \( g \) is semiproper, the components of \( g^{-1}(\cl(Q_w)) \) are compact. By [2, th. 7.4.5], \( V \to \mathcal{R}_g \) lifts curves. But if \( V \to \mathcal{R}_g \) lifts curves, and \( G^{-1} \) is \( 1 - 1 \) in the diagram
we must have that \((W_{g^{-1}}, p^*)\) lifts curves. Since \((W_{g^{-1}}, p^*)\) lifts curves, we can apply proposition 3.12 to any two pairs \((\phi_1, \psi_1)\), and \((\phi_2, \psi_2)\) to conclude that the product \((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)\) exists. If we designate by \(\mathcal{P}_g\) the equivalence classes from \(g\), then we have immediately that \(\mathcal{P}_g \subseteq \mathcal{P}_f\). Now if \(\langle \phi, \psi \rangle \in \mathcal{P}_f\), then \((\phi, \psi)\) was generated from some \(h = f^{-1} \circ f\) defined on smooth points of \(f\). Hence \(h = g^{-1} \circ g\) and generates a principal \(g\) pair \((\alpha, \beta)\) that is equivalent to \((\phi, \psi)\). Therefore \(\langle \phi, \psi \rangle = \langle \alpha, \beta \rangle \in \mathcal{P}_g\) and we conclude that \(\mathcal{P}_f = \mathcal{P}_g\).

We now show that \(\mathcal{P}_g\) is a hypergroup under \(\otimes\). Let \(\langle \phi_i, \psi_i \rangle \in \mathcal{P}_g\), \(i = 1, 2, 3\). There is \(h_1\) which generates \((\phi_1, \psi_1)\) by theorem 3.3 and the discussion following theorem 3.3. By proposition 3.12 there is an \(h_2\) such that \(h_2 \circ h_1\) is defined and by proposition 3.8. \(h_2 \circ h_1\) generates \((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)\). Similarly there is an \(h_3\) such that \(h_3 \circ (h_2 \circ h_1)\) is defined and \(h_3 \circ (h_2 \circ h_1)\) generates \(((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)) \otimes (\phi_3, \psi_3)\). Since \((\phi_1, \psi_1) \otimes (\phi_2, \psi_2) \in (\phi_1, \psi_1) \otimes (\phi_2, \psi_2)\) by definition, we have \(((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)) \otimes (\phi_3, \psi_3) \subseteq ((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)) \otimes (\phi_3, \psi_3)\). Since \(\circ\) is associative and \((h_3 \circ h_2) \circ h_1\) is defined we may repeat the argument above to conclude that \((h_3 \circ h_2) \circ h_1\) generates equivalent pairs, we

\[\begin{align*}
W_{g^{-1}} \xrightarrow{G^{-1}} p^* & \xrightarrow{g} \mathcal{P}_g \\
V & \xrightarrow{g} R_g
\end{align*}\]

**Figure 3.12:** The function \(g\) as related to the surface \(W_{g^{-1}}\) and the maps \(p^*\) and \(G^*\).
have that \( (\phi_1, \psi_1) \otimes ((\phi_2, \psi_2) \otimes (\phi_3, \psi_3)) = ((\phi_1, \psi_1) \otimes (\phi_2, \psi_2)) \otimes (\phi_3, \psi_3) \). That is, \( \otimes \) is associative.

We assert that \( (\chi, \chi) \) is the identity and the only \( h \) that can generate \( (\alpha, \beta) \in (\chi, \chi) \) is the identity. The principal pair \( (\alpha, \beta) \sim (\chi, \chi) \) iff there is \( \sigma \in \mathcal{M} \) such that \( \alpha = \chi \circ \sigma \) and \( \beta = \chi \circ \sigma \). Hence \( \alpha = \beta = \sigma \). By construction, we have \( \sigma = p^* \circ \rho = H^* \circ \rho \). Since \( \sigma \) is \( 1-1 \), \( \rho \) is \( 1-1 \) and \( p^* \) is onto. Similarly \( H^* \) is onto. Since \( \rho \) is ucm, \( \rho \) is onto and hence \( \sigma \) being \( 1-1 \) implies \( p^* \) and \( H^* \) are \( 1-1 \). If \( (h,D) \) is any generator of \( (\alpha, \beta) = (\sigma, \sigma) \) then on \( D \subseteq \mathcal{U} \) we have

\[
h(z) = H^*([h, z]) = H^* \circ \rho(\tilde{z}) = p^* \circ \rho(\tilde{z}) = p^*([h, z]) = z.
\]

If \( (\sigma, \sigma) \in (\chi, \chi) \) and \( (\mu, \nu) \in (\phi, \psi) \) then \( (\sigma, \sigma) \otimes (\mu, \nu) \) is equivalent to \( (\phi, \psi) \) since the only \( h \) that can generate \( (\sigma, \sigma) \) is the identity. Hence \( (\chi, \chi) \otimes (\phi, \psi) = \{(\phi, \psi)\} \). Similarly \( (\phi, \psi) \otimes (\chi, \chi) = \{(\phi, \psi)\} \).

Now let \( (\phi, \psi) \in \mathcal{P}_g \). We will show that the hypothesis of proposition 3.13 is satisfied. We can then apply corollary 3.15 to conclude \( (\psi, \phi) \) is the inverse of \( (\phi, \psi) \). We have to prove that if \( (h, D) = (g_{\alpha_0}^{-1} \circ g_0, D_0) \) has a continuation along \( \gamma \subseteq V \), then \( H^*(\tilde{\gamma}) \subseteq V \) where \( \tilde{\gamma} \) is the lift of \( \gamma \) in \( \mathcal{W}_h \). Since \( (\mathcal{W}_{g^{-1}}, p^*) \) lifts curves, \( g \circ \gamma \) lifts to \( \Gamma_1 \) and \( \Gamma_2 \) in \( \mathcal{W}_{g^{-1}} \) with initial points \([g_0^{-1}, g_0(\gamma(0))] \) and \([g_{\alpha_0}^{-1}, g_0(\gamma(0))] \) respectively. This leads to a continuation of \( (h, D) \) expressed in terms of \( g \). If the analytic continuation is defined by \( (g_{\alpha t}^{-1} \circ g_t, D_t) \) then

\[
H^*(\tilde{\gamma}) = H^*([g_{\alpha t}^{-1} \circ g_t, \gamma(t)]) = \{g_{\alpha t}^{-1}(g_t(\gamma(t)))\} = G^{-1*}(\{[g_{\alpha t}^{-1}, g_t(\gamma(t))]\}) = G^{-1*}(\Gamma_2) \subseteq V.
\]
Hence by corollary 3.15, $h^{-1}$ generates $\langle \psi, \phi \rangle$. We can conclude that

$$\langle \chi, \chi \rangle \in \langle \phi, \psi \rangle \otimes \langle \psi, \phi \rangle \cap \langle \psi, \phi \rangle \otimes \langle \phi, \psi \rangle.$$ 

To show uniqueness of $\langle \psi, \phi \rangle$, let $(\chi, \chi) \in \langle \phi, \psi \rangle \otimes \langle \alpha, \beta \rangle$. WLOG there exist $h_1$ and $h_2$ such that $h_1$ generates $(\phi, \psi)$, $h_2$ generates $(\alpha, \beta)$ and $h_2 \circ h_1$ generates $(\phi, \psi) \otimes (\alpha, \beta) \sim (\chi, \chi)$. From above, the only $g^{-1} \circ g$ that can generate an element in $(\chi, \chi)$ is the identity. Hence, $h_2 \circ h_1 = \chi$ which implies $h_2 = h_1^{-1}$. That is, $(\alpha, \beta) = (\psi, \phi)$.

Since $\mathcal{P}_g = \mathcal{P}_f$ we have that $\mathcal{P}_f$ is a hypergroup under $\otimes$. ■

Stephenson [14] gets the following corollary through inner functions. We get the result using semiproper functions.

**Corollary 3.17.** [14, th. 10, p. 870] If $B$ is a finite Blaschke product, then $\mathcal{P}_B$ is a hypergroup under $\otimes$.

**Proof.** By corollary 2.6 $B$ is proper. ■

The following two results along with proposition 3.13 will be used to study the examples.

**Proposition 3.18.** Let $U \xrightarrow{f} \mathcal{R}$ and $f(z_1) = f(z_2)$ with $f$ smooth at $z_1$ and $z_2$. Let $h = f^{-1} \circ f$ such that $h(z_1) = z_2$ with $D_1 \xrightarrow{h} D_2$, $z_i \in D_i$. Suppose that the continuations of $h$ are single-valued and there is $D(\subseteq U) \xrightarrow{g} U$ with the following properties:

(i) $D_1 \subseteq D$ and $g \upharpoonright D_1 = h$, 


that the continuations of \( h \) are single-valued and there is \( D(\subseteq U)^{\rightarrow U} \) with the following properties:

(i) \( D_1 \subseteq D \) and \( g \upharpoonright D_1 = h \),

(ii) \( z \in D \Rightarrow |g(z)| < 1 \) and

(iii) Either \((g, D)\) cannot be analytically continued across the boundary of \( D \) or if it can be analytically continued across the boundary of \( D \) to \((\bar{g}, \Delta)\) then for every \( z \in \Delta - D \) either \( |z| \geq 1 \) or \( |\bar{g}(z)| \geq 1 \).

Then \( W_h \) is conformal to \( D \). In this case the pair \((\phi, \psi)\) generated by \( h \) is \((\rho, g \circ \rho)\) where \( \rho \) is a universal covering map of \( D \).

**Proof.** By hypothesis, the function pair \((g, D)\) is an analytic continuation of \((h, D_1)\). Hence, \( D \subseteq U \) and \( z \in D \Rightarrow |g(z)| < 1 \) means that \([g,D] \subseteq W_h\).

To show that \( W_h \subseteq [g,D] \) let \([h_\alpha, z_\alpha] \in W_h\). If \((h_\alpha, D_\alpha) \in [h_\alpha, z_\alpha]\), then \((h_\alpha, D_\alpha)\) is an analytic continuation of \((h, D_1)\) and hence a continuation of \((g, D)\). We want to show that \([h_\alpha, z_\alpha] \in [g,D]\). This amounts to showing that \( z_\alpha \in D \) and \( h_\alpha = g \) near \( z_\alpha \). If the boundary of \( D \) is the natural boundary of \( g \) and \((h_\alpha, D_\alpha)\) is an analytic continuation of \((g, D)\) then \( D_\alpha \subseteq D \). Since the continuations of \((h, D_1)\) are single valued, \( h_\alpha = g \) near \( z_\alpha \). Hence \([h_\alpha, z_\alpha] \in [g,D]\).

Suppose that \((h_\alpha, D_\alpha)\) is a continuation across the boundary of \( D \). If \( z_\alpha \in D_\alpha - D \) then either \( |z_\alpha| \geq 1 \) or \( |h_\alpha(z_\alpha)| \geq 1 \). Either one would contradict \([h_\alpha, z_\alpha] \in W_h\). Hence \( z_\alpha \in D \). Again \((h, D_1)\) has only single valued continuations means that \( h_\alpha = g \) near \( z_\alpha \). Hence \([h_\alpha, z_\alpha] \in [g,D]\) and \( W_h \subseteq [g,D]\).
Therefore, the function pair \((\phi, \psi) = (p^* \circ p^{*-1} \circ \rho, H^* \circ p^{*-1} \circ \rho)\) reduces to \((\rho, H^* \circ p^{*-1} \circ \rho)\). Finally, \(g \circ p^* = H^*\) implies \(g = H^* \circ p^{*-1}\) since the diagram commutes. Hence \((\phi, \psi) = (\rho, g \circ \rho)\). ■

**Figure 3.13:** Relationship between \(g\) and \(W_h\) when \(h\) continues to domain in \(U\).

**Corollary 3.19.** The function \(h\) generates the principal \(f\) pair \((\chi, \psi)\) iff \(h\) can be continued analytically to \(\psi\) defined on all of \(U\) with \(|\psi(z)| < 1\). In this case \(U\) is conformally equivalent to \(W_h\).

**Proof.** Let \((\chi, \psi)\) be a principal \(f\) pair. We have the following diagram:
Figure 3.14: Relationship between \( g \) and \( \mathcal{U} \) when \( h \) continues to all of \( \mathcal{U} \).

Since \( \chi = p^* \circ \rho \), \( \rho \) is 1 - 1. Since \( \rho \) is onto, \( \mathcal{W}_h \simeq \mathcal{U} \). Also \( \rho \) onto \( \Rightarrow p^* \) is 1 - 1. Since \( p^* \) is onto by \( \chi = p^* \circ \rho \), we have that \( \rho = p^{*-1} \). We have \( H^* \circ p^{*-1} = \psi \) by construction and \( (H^* \circ p^{*-1}) \upharpoonright D = h \) by the commutative diagram. Hence \( (\psi, \mathcal{U}) \) is an analytic continuation of \( (h, D) \).

Conversely, let \( (h, D) \) continue to \( (\psi, \mathcal{U}) \) with \( |\psi(z)| < 1 \). We check the hypothesis of 3.18. Since \( (h, D) \) continues to all of \( \mathcal{U} \) and \( \mathcal{U} \) is simply connected, the analytic continuations of \( (h, D) \) are single valued. Proposition 3.18 (i), (ii) and (iii) are satisfied so that \( \mathcal{U} \) is conformal to \( \mathcal{W}_h \). Since \( \chi \) is a universal map of \( \mathcal{U} \) we have \( (\chi, \psi \circ \chi) = (\chi, \psi) \) a principal \( f \) pair. \( \blacksquare \)
CHAPTER IV
EXAMPLES AND FURTHER RESULTS

The finite Blaschke products are good examples of semiproper functions. We will discuss the nature of the hypergroup of some finite Blaschke products using the theorems developed in Chapter III. Also, we will find the actual elements of the hypergroups when the formula of the Blaschke product lends itself to such computation.

The Riemann surface $W_{f^{-1}}$ of a function $f$ will be needed as the theoretical tool for studying some of these examples. In conjunction with proposition 3.13, the surface $W_{f^{-1}}$ provides insight into the nature of the surface $W_h$; that is, the number of sheets and the location and number of branch points. The surface $W_{f^{-1}}$ appears in another example besides that of a finite Blaschke product. This is Stephenson's [14] counterexample to the existence of a hypergroup of a function $f$ and is presented in terms of $W_{f^{-1}}$ alone.

We begin with our approach to constructing the Riemann surface of a finite Blaschke product.

Theorem 4.1. (The Riemann Surface of a Finite Blaschke Product). Let $B$ be a finite Blaschke product of order $n$. Then $U$ can be decomposed into domains $D_1, \ldots, D_n$ with boundaries $\beta_1, \ldots, \beta_n$ such that:
(i) the $D_i$'s are mutually disjoint

(ii) each $D_i$ is simply connected

(iii) $B$ is $1 - 1$ on each $D_i$

(iv) each $\beta_i$ has an arc belonging to $C(0,1)$.

(v) each subarc of $\beta_i$ with initial point and terminal point the only points on $C(0,1)$ contains exactly one branch point, and all of the branch points are on the $\beta_i$.

Proof. Counting multiplicities, $B'$ has $n - 1$ zeros in $\mathcal{U}$ [11, pr. 192, p. 142]. Let $z_1, \ldots, z_k$ be the distinct zeros of $B'$ with multiplicities $m_1, \ldots, m_k$ respectively. The set $\{f(z_i)\}_{i=1}^k$ contains at least 1 but at most $k$ points. Let these points be $w_1, \ldots, w_j$. Arrange the notation so that

$$f(z_i) = w_1, \quad 1 \leq i \leq J_1;$$

$$f(z_i) = w_2, \quad 1 + J_1 \leq i \leq J_2;$$

$$\vdots$$

$$f(z_i) = w_j, \quad 1 + J_{k-1} \leq i \leq k$$

Let $\gamma_1, \ldots, \gamma_j$ be curves in the range of $B$ in $\mathcal{U}$ such that

(i) the initial point of $\gamma_i$ is $w_i$,

(ii) the $\gamma_i$'s are nonintersecting,

(iii) the $\gamma_i$'s are simple and

(iv) the terminal point of each $\gamma_i$ is on $C(0,1)$ and the terminal points are distinct.
Now $z_1$ is over $w_1$ and $B$ is an $m_1 + 1$ mapping in a neighborhood $N_{z_1}$ of $z_1$. Hence, for a point $\zeta_{11}$ close to $w_1$ there are $m_1 + 1$ preimages in $N_{z_1}$ where curves can initiate over the portion of $\gamma_1$ beginning at $\zeta_{11}$. $B$ is a semiproper function so we may obtain unique lifts as in the proof of theorem 3.16. Allow $\zeta_{11}$ to approach $w_1$ and the result is $m_1 + 1$ curves with initial point $z_1$ and terminal points on $C(0,1)$. Since curve lifting is unique, the only point common to the curves lying over $\gamma_1$ is $z_1$.

This give us $m_1 + 1$ curves and $m_1 + 1$ domains. One of the domains $\Delta$ contains $z_2$. We repeat the process and obtain $m_2 + 1$ nonintersecting curves in $\Delta$ ending at distinct points on $C(0,1)$. These curves do not intersect themselves and they do not intersect any of the $m_1 + 1$ curves obtained above. All terminal
points on $C(0,1)$ are distinct. We now have $m_1 + 1 + m_2 + 1$ curves and $m_1 + (m_2 + 1)$ domains. If we continue this process we will have $(\sum_{i=1}^{k} m_i) + 1 = n$ domains and $(\sum_{i=1}^{j} m_i) + j$ nonintersecting curves that terminate at distinct points on $C(0,1)$. The domains are all simply connected and the boundary of $D_i$ is $\beta_i$. By construction, the domains $D_i$ are mutually disjoint. Figure 4.2 above has 7 domains with domain number 2 outlined by arrows.

We claim that $B$ is $1-1$ on each domain $D_i$. The image of $D_i$ will be contained in a simply connected domain whose boundary consists of $C(0,1)$ and the image curves $\gamma_j$ of the curve $\beta_i$ containing the branch points. Figure 4.3 shows domain $D_i$ and a domain $V$ containing the image.

![Figure 4.3: Example of lifted curves showing domains $D_i$ and $V$.](image)

We will prove that $D_i$ is mapped onto $V$ and that $B \upharpoonright D_i$ lifts curves. Since $V$ is simply connected $B \upharpoonright D_i$ will be one to one. Now, $B(D_i) \subseteq V$ and $D_i$ is contained in some component of $B^{-1}(V)$. But if the component contained points not in $D_i$ then any path from inside $D_i$ to this point would have to intersect the boundary $\beta_i$. Hence $D_i$ is a component of $B^{-1}(V)$. Since
\(D_i = \text{cl}(D_i) \cap B^{-1}(V)\) we have \(B(D_i) = B(\text{cl}(D_i)) \cap V = \text{cl}B(D_i) \cap V\) and so \(B(D_i)\) is closed in \(V\). But \(D_i\) open \(\Rightarrow B(D_i)\) is open in \(V\) and so \(B(D_i) = V\).

There are no branch points in \(D_i\) so that \(B\) is smooth on \(D_i\). For each point \(y\) in \(V\), there are a finite number of points \(x_1, \ldots, x_p\) such that \(B(x_i) = y\). Therefore we can find a neighborhood \(N_y\) about \(y\) such that the components of \(B^{-1}(N_y)\) are compact. Hence by Beardon [2, th. 7.4.5, p. 103] \(B \upharpoonright D_i\) lifts curves. Since \(V\) is simply connected, \(B\) is \(1-1\) on \(D_i\).

By making the proper cuts and identifications along the curves constructed, we can build the Riemann surface of each finite Blaschke product.

The next theorem is due to Stephenson [14, cor, p. 873]. In this corollary, Stephenson also proves that the hypergroup of a finite Blaschke product \(B\) is cyclic. We will not use this result and the following proof of \(|\mathcal{P}_B| \leq n\) follows the theme of semiproper functions.

**Theorem 4.2.** If \(B\) is a finite Blaschke product of order \(n\), then \(|\mathcal{P}_B| \leq n\).

**Proof.** Let \(D_1, D_2, \ldots, D_n\) be as in theorem 4.1 and \(\Delta_1\) a small disk in \(D_1\). There are exactly \(n\) simply connected domains \(\Delta_i \subseteq D_i, \ i = 1, 2, \ldots, n\) such that \(B(\Delta_i) = B(\Delta_1)\) and \(B \upharpoonright \Delta_i\) is \(1-1\) for every \(i\). If we define \(h_i = (B \upharpoonright \Delta_i)^{-1} \circ B\) from \(\Delta_1\) onto \(\Delta_i\), then \(h_i\) generates an element \(\langle \phi_i, \psi_i \rangle\) in \(\mathcal{P}_B\). Hence \(|\langle \phi_i, \psi_i \rangle | i = 1, 2, \ldots, n\| \subseteq \mathcal{P}_B\). Let \((h, \Delta)\) be any function element such that \(h\) is of the form \(B^{-1} \circ B\). By the proof of theorem 3.12 \((h, \Delta)\) continues to some function element \(\langle \tilde{h}, \Delta_1 \rangle\). By proposition 3.13, \(\tilde{h}\) is of the form \(B^{-1} \circ B\) and so must agree with \((B \upharpoonright \Delta_i)^{-1} \circ B = h_k\) for some
For this $k$, $h_k$ and $h$ generate the same element $(\phi_k, \psi_k)$ in $\mathcal{P}_B$. Hence $|\mathcal{P}_B| \leq n$.

Example 4.3. Let $f(z) = \frac{z^{0.1} - 1}{1 - z^{0.1}}$. We want to compute $\mathcal{P}_f$ and interpret $\otimes$ in $\mathcal{P}_f$. The branch point of $f$ is $2 - \sqrt{3}$ and $B(2 - \sqrt{3}) = -.0718$. The function $f$ is two to one near $(2 - \sqrt{3})$ so according to theorem 4.1 we have two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ with initial point $2 - \sqrt{3}$, over the curve $\gamma$ from $-0.0718$ to $-1$. The terminal points of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ respectively.

Our surface is

\begin{figure}
  \centering
  \includegraphics[width=\textwidth]{example4.4.png}
  \caption{Riemann surface of example 4.4.}
\end{figure}
Let $h$ be defined such that $h : D_1 \rightarrow D_2$ where $D_1$ and $D_2$ are shown in figure 4.4. The function $h$ is of the form $f^{-1} \circ f$ where $D_1 \xrightarrow{f^{-1}} \Delta = p^*(\Delta_i) \xrightarrow{f^{-1}} D_2$. Figure 4.5 shows the structure of every $h$ of the form $f^{-1} \circ f$. The subscripts 1 and 2 distinguish between the portions of the figure dealing with $f$ and $f^{-1}$ respectively. The function $h = F_2^{-1} \circ p_{*}^{-1} \circ p^* \circ F_1^{-1}$ with appropriate restrictions on $p^*$ and $F^{-1}$. 

![Diagram](image)

**Figure 4.5:** The structure of every $h = f^{-1} \circ f$.

We assert that $h$ continues to all of $\mathcal{U}$ single valuedly. The only point that could possibly be a branch point resulting in $h$ being multivalued is $2 - \sqrt{3}$. For computational purposes we can delete $2 - \sqrt{3}$ and its image from the surface and use proposition 3.13 to test $2 - \sqrt{3}$ as a multivalued type branch point for $h$. Figure 4.6 is a version of figure 4.5 which shows the computation of the analytic continuation of $h : D_1 \rightarrow D_2$ along $\gamma$. 
The curve $\gamma$ begins at a point in $D_1$, circuits $2 - \sqrt{3}$ and returns to the same point in $D_1$. Now apply proposition 3.13 to lift $f \circ \gamma$ to a curve $\Gamma_2$ in $(W_{f^{-1}})_2$ with initial point $[f_2^{-1}, f(\gamma(0))]$. Notice that the curve $f \circ \gamma$ also lifts to a curve $\Gamma_1$ in $(W_{f^{-1}})_1$, with initial point $[f_1^{-1}, f(\gamma(0))]$. Now $F_2^{-1*}(\Gamma_2)$ begins and ends at the same point in $D_2$. Hence $2 - \sqrt{3}$ is not a branch point, and $h$ continues to $U - \{2 - \sqrt{3}\}$ single valuedly. But in this case $2 - \sqrt{3}$ becomes a removable singularity and $h$ continues to all of $U$. By corollary 3.19 $(\chi, h) \in \mathcal{P}_f$ where $h$
is the continuation. By theorem 4.2 \( P_f = \{(\chi, \chi), (\chi, h)\} \). The pair \( (h, \chi) \) is the inverse of \( (\chi, h) \) and so \( (\chi, h) \) and \( (h, \chi) \) must be equal. That is, there exists an \( \omega \in \mathcal{M} \) such that \( \chi = h \circ \omega \) and \( h = \chi \circ \omega \). Hence \( h = \omega = h^{-1} \in \mathcal{M} \).

Since \( (\chi, h) \in P_f \), we have \( f = f \circ h \). Hence
\[
\frac{h(z)(h(z) - 1/2)}{(1 - h(z)/2)} = \frac{z(z - 1/2)}{(1 - z/2)}.
\]
Simplifying we have
\[
h(z) = \frac{1 - z^2 \pm \sqrt{z^4 - 8z^3 + 18z^2 - 8z + 1}}{4(1 - 1/2)}.
\]
Assume one of the branches will give use \( h(z) = z \) and knowing \( h \in \mathcal{M} \) we assume
\[
\frac{1 - z^2 + \sqrt{z^4 - 8z^3 + 18z^2 - 8z + 1}}{4(1 - 1/2)} = z.
\]
Solving for \( \sqrt{z^4 - 8z^3 + 18z^2 - 8z + 1} \) we find
\[
\sqrt{z^4 - 8z^3 + 18z^2 - 8z + 1} = -z^2 + 4z - 1.
\]
Checking, we see that \( (z^2 - 4z + 1)^2 \) does equal \( z^4 - 8z^3 + 18z^2 - 8z + 1 \).

We can now calculate a formula for \( h(z) \):
\[
h(z) = \frac{(1 - z^2) \pm (z^2 - 4z + 1)}{4(1 - 1/2z)}.
\]
The plus sign yields \( h(z) = \frac{1 - 2z}{2z} \) and the minus sign gives \( h(z) = z \).

Finally \( (\chi, \chi) \otimes (\chi, h) = (\chi, h) \) since \( h \circ \chi = h \) so that \( \otimes \) is composition.

The next example is a every elegant one from Stephenson which shows that not all functions have hypergroups. The property that fails is associativity.
Example 4.4. [14, ex. 5, p. 863]. In figure 4.7 the surface $S$ is over $C$ and projects to $C$ via $p^*$. The universal cover of this surface is $U$ since the analytic function $p^*$ is bounded and nonconstant. Let $f = p^* \circ \rho$. The surface for $f$ is $S$ and so we can label this surface $W_{f^{-1}}$. Since $W_{f^{-1}}$ is simply connected, $\rho$ is $1 - 1$. Figure 4.8 shows one representative cover for $W_{f^{-1}}$.

Region 1 ($R_1$) maps $1 - 1$ and onto the upper square over $0$ ($U_0$) and region 2 ($R_2$) maps $1 - 1$ and onto the lower square over $0$ ($L_0$). Regions 3 and 4 ($R_3$ and $R_4$) map $1 - 1$ and onto the lower and upper square, respectively, over $1$ ($L_1$ and $U_1$). Define $h_0$ by

$$R_1 \xrightarrow{\rho^{-1}} U_0 \xrightarrow{p^*} C \xrightarrow{p^*^{-1}} L_0 \xrightarrow{\rho^{-1}} R_2$$

and let $(\phi, \psi)$ be the principal pair generated by $h_0$.

We assert that $(\phi, \psi) \otimes (\phi, \psi)$ does not exist. Since the range of $h_0$ is $R_2$ we need to find an $h_1$ defined on $R_2$ such that $h_1$ is an analytic continuation of $h_0$ and the defining function elements $(h_t, D_t)$ satisfy $|z| < 1$ and $|h_t(z)| < 1$ for every $t \in [0, 1]$. This last condition must be satisfied if $h_1$ is to generate a pair equivalent to $(\phi, \psi)$. Let $\gamma$ be a curve in $U$ along which $h_0$ may continue to $R_2$. See figure 4.8. Since $\gamma \subseteq U$ and we have assumed $|h_t(z)| < 1$, we can apply proposition 3.13 to obtain that $f \circ \gamma$ lifts to $\Gamma$ in $W_{f^{-1}}$ with initial point $[f^{-1}, f \circ \gamma(0)]$. This germ corresponds to $L_0$ not $U_0$ and the curve $\Gamma$ falls off the edge of $W_{f^{-1}}$. See figure 4.7. Since $f \circ \gamma$ does not lift, $h_0$ cannot be continued into region $R_2$ and at the same time satisfy $|h_t(z)| < 1$. Hence $h_1 \circ h_0$ cannot be defined and $(\phi, \psi) \otimes (\phi, \psi)$ does not exist. If $h$ is defined in some region other than the regions $R_1, R_2, R_3$, or $R_4$, then this $h$ can only
Figure 4.7: Surface for example 4.4.

$W_{f-1} = S$

Figure 4.8: Example of a cover for the surface in example 4.4.
generate \((\chi, \chi)\). Now, \(h_0^{-1}\) generates \((\psi, \phi)\) and \(((\phi, \psi) \otimes (\phi, \psi)) \otimes (\psi, \phi)\) is not defined while \((\phi, \psi) \otimes ((\phi, \psi) \otimes (\psi, \phi)) = (\phi, \psi)\).

**Example 4.5.** Let \(f(z) = \left(\frac{z - \alpha}{1 - \overline{\alpha} z}\right)^n, |\alpha| < 1\).

We will find \(P_f\) and interpret \(\otimes\) in \(P_f\). For every \(h\), \(h = f^{-1} \circ f\) or \(f = f \circ h\) for an appropriate restriction of \(f\). This gives

\[
\left(\frac{z - \alpha}{1 - \overline{\alpha} z}\right)^n = \left(\frac{h(z) - \alpha}{1 - \overline{\alpha} h(z)}\right)^n
\]

which implies

\[
\frac{h(z) - \alpha}{1 - \overline{\alpha} h(z)} = \epsilon \frac{z - \alpha}{1 - \overline{\alpha} z}, \quad \epsilon = n^{th} \text{ root of unity.}
\]

Solving for \(h(z)\) we have

\[
h(z) = \frac{\alpha(z - 1) - (\epsilon - |\alpha|^2)z}{(\epsilon|\alpha|^2 - 1) - \overline{\alpha}(\epsilon - 1)z} = \frac{\epsilon|\alpha|^2 - 1}{\epsilon - |\alpha|^2} \left[\frac{\overline{\alpha}(\epsilon - 1)}{\epsilon|\alpha|^2 - 1} - \frac{z}{1 - \overline{\alpha}(\epsilon - 1)/\epsilon|\alpha|^2 z}\right].
\]

Now, \(\epsilon|\alpha|^2 - 1\) and \(\epsilon - |\alpha|^2\) have the same modulus and the conjugate of \(\frac{\overline{\alpha}(\epsilon - 1)}{\epsilon|\alpha|^2 - 1}\) is \(\frac{\alpha(\epsilon - 1)}{\epsilon - |\alpha|^2}\). Hence \(h(z) = e^{i\theta} \left[\frac{\beta - z}{1 - \beta z}\right]\) with \(\beta = \frac{\alpha(\epsilon - 1)}{\epsilon|\alpha|^2 - 1}\). That \(|\beta| < 1\) can be checked by showing that

\[|\alpha(\epsilon - 1)|^2 < |\epsilon| |\alpha|^2 - 1|^2.\]

The left side reduces to \(2|\alpha|^2\) and the right side reduces to \(|\alpha|^4 + 1\). But \((|\alpha|^2 - 1)^2 > 0\).

Now \(h\) analytically continues to all of \(U\) single valuedly and so \((\chi, e^{i\theta} \left[\frac{\beta - z}{1 - \beta z}\right]) \in P_f\). The \(h\) that solves equation (4.1) on some small neighborhood with a fixed \(n^{th}\) root of unity will not solve the equation with another \(n^{th}\)
root of unity. Hence, there must be $n$ such principal pairs generated. Multiplication $\otimes$ is composition since $\sigma_2 \circ \sigma_1$ generates $(\chi, \sigma_1) \otimes (\chi, \sigma_2)$ by proposition 3.8. ■

Example 4.6 is from Stephenson. It shows that a finite Blaschke product of order $n$ may not have a hypergroup with $n$ elements.

Example 4.6. [14, ex. 7, p. 864]. Let $f(z) = z^{2(z-1/2)}$. The branch points of $f$ are 0 and .344 and their images are 0 and -.0223 respectively. The preimages of the images of the branch points are 0,.5,.344 and -.188. These are the only possible branch points of any $h$ of the form $f^{-1} \circ f$. By theorem 4.2 we know that $|\mathcal{P}_f| \leq 3$. The points 0 and .344 are not branch points of $h : D_1 \to D_2$ while .5 and -.188 are. We show the details of 0 and .5 in figures 4.9 and 4.10 respectively. In both figures, we have deleted the surface $(\mathcal{W}_{f^{-1}})_1$ since the continuations are computed on $(\mathcal{W}_{f^{-1}})_2$. In figure 4.9, $\gamma$ is a curve initiating in $D_1$ and circuits once around 0 and returns to the same point in $D_1$. The curve $f \circ \gamma$ is lifted to $\Gamma$ in $(\mathcal{W}_{f^{-1}})_2$ with initial point $[f^{-1}, f(\gamma(0))]$ where $f^{-1}(f(\gamma(0)))$ does not equal $\gamma(0)$ and is an element of $D_2$. Justification for this is proposition 3.13. The continuation $F^{-1} \circ \Gamma$ is computed in $\mathcal{U}$ terminating at its original point in $D_2$. Hence 0 is not a branch point.

In Figure 4.10 $\gamma$ is a curve initiating in $D_1$, circuiting about .5 and returning to the same point in $D_1$. The curve $f \circ \gamma$ is lifted to $\Gamma$ in $(\mathcal{W}_{f^{-1}})_2$ and the continuation $F^{-1} \circ \Gamma$ is computed in $\mathcal{U}$. In this situation the continuation did not return to its original position so that .5 is a branch point. Hence the original $h$ defined from region I to region II continues to an $\tilde{h}$ defined from region I to region III and so must generate the same pair $(\phi, \psi)$. The other pair is $(\chi, \chi)$. Therefore $\mathcal{P}_f = \{(\chi, \chi), (\phi, \psi)\}$. ■
Figure 4.9: Computation for 0 in example 4.6.
Figure 4.10: Computation for .5 in example 4.6.
The final result, theorem 4.9, is an application of two theorems in Stephenson [14, Lem 1, p. 847] and [14, th 5, p 854] which we have not yet stated. The first theorem gives necessary and sufficient conditions for the composition of two functions to be inner and the second theorem relates subhypergroups of \( \mathcal{P}_f \) with the decompositions of \( f \).

**Theorem 4.7 (Stephenson).** Let \( f, g \) and \( h \) be analytic from \( U \) into \( U \) with \( h = f \circ g \). Then \( h \) is inner if and only if \( f \) and \( g \) are inner.

**Theorem 4.8 (Stephenson).** Suppose that \( F : U \to \mathcal{R} \) and \( f_1 : U \to S \) are analytic from the unit disk into the Riemann surfaces \( \mathcal{R} \) and \( S \) respectively. Then \( \mathcal{P}_{f_1} \) is a subhypergroup of \( \mathcal{P}_F \) if and only if there is a function \( f_2 \) from the range of \( f_1 \) into the range of \( F \) such that \( F = f_2 \circ f_1 \).

**Theorem 4.9.** Let \( B \) be a Blaschke product of order \( km, \ k \geq 2 \) and \( m \geq 2 \). Suppose that \( h \in \mathcal{M} \) such that

(i) \( (\chi, h) \in \mathcal{P}_B \),

(ii) \( h(\gamma) = \gamma \), for some \( \gamma \), \( |\gamma| < 1 \), and

(iii) \( h^k = h \circ h \circ h \cdots \circ h \ (k \text{ times}) = \chi, \ h^\nu \neq \chi \ \nu = 1, 2, \ldots, k - 1 \).

Then there exist Blaschke products \( B_1 \) and \( B_2 \) of orders \( k \) and \( m \) respectively such that \( B = B_2 \circ B_1 \). Moreover we may choose \( B_1(z) = \left( \frac{z-\gamma}{1-\gamma \zeta} \right)^k \) and then \( (\chi, h) \in \mathcal{P}_{B_1} \).
Proof. Since \( \gamma \) is a fixed point of \( h \) we have

\[
\frac{h(z) - \gamma}{1 - \overline{\gamma}h(z)} = c \left( \frac{z - \gamma}{1 - \overline{\gamma}z} \right), \quad \text{for some } c, \quad |c| = 1.
\]

See [9, p 4].

Hence

\[
\frac{h(h(z)) - \gamma}{1 - \overline{\gamma}h(h(z))} = c \left( \frac{h(z) - \gamma}{1 - \overline{\gamma}h(z)} \right) = c^2 \left( \frac{z - \gamma}{1 - \overline{\gamma}z} \right)
\]

and inductively we have

\[
\frac{h^k(z) - \gamma}{1 - \overline{\gamma}h^k(z)} = c^k \left( \frac{z - \gamma}{1 - \overline{\gamma}z} \right).
\]

Since \( h^k(z) = z \), this means that \( c^k = 1 \). If we raise each side of equation 4.2) to the \( k \)th power, we have

\[
\left( \frac{h(z) - \gamma}{1 - \overline{\gamma}h(z)} \right)^k = c^k \left( \frac{z - \gamma}{1 - \overline{\gamma}z} \right)^k = \left( \frac{z - \gamma}{1 - \overline{\gamma}z} \right)^k.
\]

That is, \( B_1(z) = B_1(h(z)) \) and hence \( \langle x, h \rangle \in \mathcal{P}_{B_1} \). Now \( |\mathcal{P}_{B_1}| = k \) since \( \langle x, h \rangle^\nu = \langle x, h^\nu \rangle \neq \langle x, x \rangle \) if \( 1 \leq \nu < k \). Furthermore, \( \mathcal{P}_{B_1} \) is a subhypergroup of \( \mathcal{P}_B \). By theorem 4.8 there is an \( f_2 \) such that \( f_2 \) is analytic from the unit disk into the range of \( B_1 \) and \( B = f_2 \circ B_1 \). Since \( B \) is inner, \( f_2 \) is inner by theorem 4.7. We assert that \( f_2 \) is a finite Blaschke product. If it is not a Blaschke product, then there is a path \( \sigma \) in \( \mathcal{U} \) going to \( \text{br}(\mathcal{U}) \) such that

\[
\lim_{w \to \text{br}(\mathcal{U})} f_2(w) = \kappa, \quad |\kappa| < 1 \quad \text{[11, ex. 17, p. 383]}.\]

WLOG, \( \sigma \) does not intersect the images of the branch points of \( B_1 \). Since \( B_1 \) is semiproper, we may lift \( \sigma \) to \( \tilde{\sigma} \) in \( \mathcal{U} \). Since \( B_1 \) is a finite Blaschke product,
going toward the boundary means \( \sigma \) must also approach the boundary of \( \mathcal{U} \).

Hence \( \lim_{z \to \sigma \in \partial \mathcal{U}} B(z) = \kappa \). This is a contradiction since \( B \) is a finite Blaschke product. It follows that \( f_2 \) is a Blaschke product which we will now denote by \( B_2 \). Since \( B \) is a Blaschke product of order \( km \) and \( B_1 \) is a Blaschke product of order \( k \), it follows that \( B_2 \) is a Blaschke product of order \( m \). \( \blacksquare \)
LIST OF REFERENCES


