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Current near the vertex of a perfectly conducting angular sector

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The Ohio State University, 1991
Current Near the Vertex of a Perfectly Conducting Angular Sector

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Timothy J. Brinkley

* * * * *

The Ohio State University

1991

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Dedication

To my parents
Acknowledgements

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CHAPTER I
Introduction

Electromagnetic scattering from perfectly conducting, electrically large flat plates with straight edges and plane wave illumination has been under investigation for many years. For all practical purposes, the field scattered by the vertices of a plate is the leading term in the asymptotic series describing the total far field scattered by a plate with straight edges for plane wave illumination. In the far zone, the UTD reflected ray and diffracted rays of each of the edges do not contribute to the scattered field for most observation directions. In the specular direction or Keller directions where one or more of the UTD reflected or diffracted rays contribute, the corner diffracted fields alone will still yield the correct result, if they are evaluated carefully. All present high frequency solutions to the scattering from a vertex are based on some approximation to the current on an angular sector. Present high frequency techniques and current based approaches (physical optics and its extensions) fail to give accurate results, as compared with Moment Method or other techniques, for many nonspecular regions around a flat plate. An improved, but still approximate, expression for the current on a perfectly conducting angular sector for plane wave illumination is found here. It is illustrated how the improvement in the approximation of the current leads to scattered fields that compare better with Moment Method and measurements for most cases.
A great deal of research has been done on the scattering from a perfectly conducting angular sector. A dyadic Green's function for the scattering from a perfectly conducting angular sector has been found by Satterwhite [1,2] and recently verified by Hansen [3]. The current on a perfectly conducting angular sector for plane wave illumination is a special case of Satterwhite's solution. Unfortunately, Satterwhite's solution is in the form of an eigenfunction expansion which is slowly convergent. Therefore the calculation of Satterwhite's solution is only practical for points near the vertex of the angular sector. Attempts to derive a useful high frequency solution directly from Satterwhite's solution by Hansen [4] and Smyshlyaev [5] have been unsuccessful to date due to the complicated nature of the eigenfunction solution, involving Lamé polynomials in spheroidal coordinates. Attempts to characterize the behavior of the current near the vertex by Brinkley and Marhefka [6,7,8] and Hansen [4] have not led to useful solutions.

With these difficulties in mind an approximate, but analytic solution for the current on a perfectly conducting angular sector is derived in Chapter II. The derivation is done by separating the current into several components based on the Physical Theory of Diffraction (PTD) [9] and ray concepts. The known analytic currents are subtracted from the total current calculated using some reference solution, normally an eigenfunction or numerical solution. Fitting an approximate analytic expression to the result is much simpler than trying to fit an analytic expression to the total current. This approach was first used in [10] and [11] to find approximate expressions for the current on a wedge for plane wave illumination. Later, Hansen [12] used the same basic idea in developing a corner diffraction coefficient valid for the special case of a quarter plane and plane wave illumination from a limited angular region. In the same way the current on the angular sector is
separated into simpler components. Additional currents, which are corrections to the physical optics current, are associated with one or more discontinuities on the angular sector, specifically an edge or edges, or the vertex. Each current on the angular sector will be referred to by the specific ray path associated with it. For example, the current arising from each edge will be referred to as an edge diffracted current. The edge diffracted currents are conceptually the same as the currents referred to as the nonuniform currents in [10] and [11], the only difference being that the edge diffracted currents are valid for oblique incidence on a knife edge only, instead of a wedge. After the expression for the vertex current is derived, it will be verified by direct comparison with Satterwhite’s solution, and also integrated numerically over finite plates so the scattered field may be compared with Moment Method results.

In Chapter III the current on the angular sector derived in Chapter II will be evaluated asymptotically, resulting in a general expression for the far zone field scattered by a perfectly conducting angular sector for plane wave illumination of arbitrary polarization. This may be easily extended to scattering from finite plates in the context of the Geometrical Theory of Diffraction [13]. The first corner diffraction coefficient [14] is based on approximate equivalent edge currents resulting in some limitations in calculating the bistatic scattered field due to the basically two dimensional nature of the equivalent edge currents used. A later corner diffraction coefficient described in [15] and [16] corrected the problems associated with finding the bistatic scattered field, but as shown in [16] gave essentially the same results for backscatter. It has long been suspected that the backscattered fields found using either of the previous corner diffraction coefficients are inaccurate for certain regions. It is verified in Chapter III that the previous corner diffraction coefficients
are inaccurate in the region near the plane of the plate. This is accomplished by comparison with measurements where the field diffracted by each corner has been isolated using time domain techniques. This allows direct comparison of the corner diffracted field calculations to a reference solution, unlike comparisons with patterns on finite plates where the effect of higher order terms can make verification of the corner diffraction coefficient difficult. The new corner diffraction coefficient, which is an additive correction to the previous corner diffraction coefficient described in [15] and [16], plus the additional corrections described in [17], matches the measurements closely over the regions where measurements are available. Time harmonic fields with a time dependence of $e^{j\omega t}$ are assumed throughout.
CHAPTER II

Current

An approximate expression for the surface current on a perfectly conducting angular sector is developed for plane wave incidence. The total current is written as the sum of the physical optics current, the edge diffracted currents, the double diffracted currents, and the newly derived vertex diffracted current. Writing the total current as the sum of these currents allows the results to be applied directly to approximating the current on finite plates with straight edges. The vertex diffracted current will be derived by fitting a function to the difference current found by subtracting the physical optics current and edge diffracted currents from the total current on the angular sector. The total current on the angular sector is calculated using the eigenfunction solution in [1].

The magnitude and phase of the vertex diffracted current will be compared with the magnitude and phase of the difference current for some cases to verify that it is fairly accurate in the vicinity of the vertex. Far from the vertex the total current will be dominated by the physical optics current or the edge diffracted current (near each edge) and thus, as long as the vertex diffracted current approaches zero sufficiently fast, the current will be accurate far from the vertex. Further verification of the approximate expression for the vertex diffracted current will be provided by comparing field calculations for the far zone bistatic scattered fields.
from a flat plate with calculations made using the Method of Moments and the exact angular sector current.

2.1 Asymptotic Currents on an Angular Sector

The current on an angular sector is represented as the sum of several ray contributions to the current. The current will be expressed as the sum of the physical optics, edge diffracted, double diffracted and vertex diffracted currents. Interactions involving orders higher than the double diffracted current will be assumed to be negligible in comparison to the double diffracted and vertex diffracted currents. Comparing the phase variation of the total current on the angular sector minus the physical optics and edge diffracted currents with the expected phase variation of a double diffracted and vertex diffracted wave will verify this assumption for the problems illustrated here.

The various ray paths associated with the different currents on an angular sector are illustrated in Figure 1. As with any ray solution the ray path must not be shadowed for the current to contribute. The physical optics current is obviously ray optical and exists everywhere on the lit side of the angular sector. The edge diffracted currents are based on the exact solution of the current on a half plane which is valid everywhere on the half plane, including points arbitrarily close to the edges. The edge diffracted currents described here are not strictly associated with the usual UTD diffracted field [18] since the edge diffracted currents include an additional component parallel to the edge, which is important near the edge, but does not appear in the UTD solution. The vertex diffracted current will be found from the difference between the exact solution for the current on an angular sector and the sum of the physical optics and edge diffracted currents. The
Figure 1: Ray paths for different currents on an angular sector.
double diffracted current also remains in the difference between the exact solution for the current on an angular sector and the physical optics and edge diffracted currents. The phase variation of the difference current will be used to separate the cases where the difference current is dominated by the vertex diffracted current from those where it is dominated by the double diffracted current. In those cases where the phase variation of the vertex diffracted current and the double diffracted current is essentially the same, as illustrated in Figure 2, it will be assumed that the difference current is dominated by the vertex diffracted current.

2.2 Physical Optics and Edge Diffracted Currents

The classically used currents on a perfectly conducting angular sector are the physical optics current and the edge diffracted currents. The physical optics current is well known and given by

\[ j^{PO} = 2\hat{n} \times \vec{H}^i \] (2.1)
where \( \hat{n} \) is the surface normal. In this case, a knife edged angular sector, the optics field is never shadowed and the surface normal may be defined in either of two directions. The edge diffracted current on a half plane arising from a plane wave of arbitrary polarization and incidence angle may be easily derived from the closed form expressions for the total current given in [19]. The edge diffracted current is found by subtracting the physical optics current from the total current on a half plane. The edge diffracted currents, using the same form as the expressions given in [20], for cases when the diffraction point lies on the corresponding edge of the angular sector \((\phi_c^i < \beta_1^i)\) for \(J^{d1}\) and \(\phi_c^i > \alpha_v + \beta_2^i - \pi\) for \(J^{d2}\) where the geometry is shown in Figure 3, are

\[
\vec{J}^{\text{dn}} = 4 \left( \frac{-e^{-j\frac{\pi}{4}}}{2\sqrt{2\pi k}} \right) \frac{e^{-jks_n^d}}{\sqrt{s_n^d \sin \beta_n^i}} \left\{ H_n^1(Q_{En}) \delta_n^d \frac{F(\delta_n^e)}{\cos \frac{\psi_n^i}{2}} \right. \\
- \hat{x}_n \left[ H_n^1(Q_{En}) \cos \beta_n^i \cos \frac{\psi_n^i}{2} + H_n^1(Q_{En}) \sin \frac{\psi_n^i}{2} \right] 2 \left( 1 - F(\delta_n^e) \right) \right\} \tag{2.2}
\]

where

\[
\delta_n^e = 2ks_n^d \sin^2 \beta_n^i \cos^2 \frac{\psi_n^i}{2} \tag{2.3}
\]

\[
F(x) = 2j\sqrt{x}e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau, \tag{2.4}
\]

\( n=1 \) for edge 1, \( n=2 \) for edge 2, \( H_n^1 \) refers to the complex value of the \( \beta_n^i \) component of the incident magnetic field at \( Q_{En} \) with the phase referenced to the vertex, and the other incident field components are similarly defined. The angles, distances, and vector directions are defined with respect to the edge fixed coordinate system in Figure 3. The edge diffracted current from either edge is zero if the diffraction point does not lie on the angular sector. Therefore, the edge diffracted
current is

\[ \vec{j}^{d1} = 0; \phi_c' > \beta_1^1 \]

\[ \vec{j}^{d2} = 0; \phi_c' < \alpha_v + \beta_2^2 - \pi \]

for the regions where the diffraction point does not lie on the angular sector.

For large distances from the edge, the function \( F(\delta) \) in Equation (2.2) approaches one, and the edge diffracted currents reduce to the currents from the GTD diffracted fields. The condition on \( \phi_c' \) is the same as shadowing the field in a ray optical manner. This obviously creates discontinuities in the sum of the physical optics and edge diffracted currents on an angular sector with vertex angle \( \alpha_v \) for \( \phi_c' = \beta_1^1 \), if \( \beta_1^1 < \alpha_v \), and for \( \phi_c' = \alpha_v + \beta_2^2 - \pi \), if \( \beta_2^2 > \pi - \alpha_v \), where the edge diffracted component of the current from a given edge is suddenly terminated (Figure 4). The vertex diffracted current should have a corresponding discontinuity which makes the total current continuous at these boundaries.

### 2.3 Difference Current

The current associated with the vertex of the angular sector is determined by subtracting the physical optics and edge diffracted currents from the total current on an angular sector. This current,

\[ \vec{j}^{df} = \vec{j}^{AS} - (\vec{j}^{PO} + \vec{j}^{d1} + \vec{j}^{d2}) \]

will be referred to as the difference current throughout this chapter. In the ray interpretation of the current used here, the difference current must be associated with either vertex diffraction or multiple edge diffraction. To determine the relative magnitudes of the vertex and multiple edge interaction components of the
Figure 3: Edge fixed coordinates on an angular sector.
difference current, the phase of the difference current will be plotted for some cases and compared with the phase variation of a spherical wave emanating from the vertex and the double diffraction path for the same location on the plate. This illustrates that in most cases the phase varies as a spherical wave traveling from the vertex of the angular sector. Therefore, in most cases, the vertex diffracted current is much stronger than the higher order edge interaction currents. Approximate expressions will be derived for the difference current which may then be taken as the vertex diffracted current in most cases. The approximate expressions for the difference current will be derived by examining the radial dependence of the difference current and fitting a simple function to it. The difference current will be expressed in terms of $\hat{r}'$ and $\hat{\phi}_c$ components, corresponding to the natural coordinate system for the vertex diffracted current which is the dominant contribution to the difference current in most cases. The $\hat{r}'$ component of the difference current may be expected to be larger since it corresponds to the transverse component of the vertex diffracted field, while the $\hat{\phi}_c$ component of the current corresponds to
the radial component of the vertex diffracted field which must decay faster as the
distance from the vertex increases, although their magnitudes may be of the same
order near the vertex.

The coordinate system used throughout this chapter in connection with the
current on an angular sector is shown in Figure 3. The positive $x_c$ axis coincides
with one edge of the angular sector and the remaining coordinate directions are
defined such that $\hat{x}_c$, $\hat{y}_c$, and $\hat{z}_c$ form a right hand coordinate system and the
angular sector lies in the first (and second if $\alpha_y > 90^\circ$) quadrant of the $x_c$-$y_c$
plane. For all the numerical examples, the magnitude of the incident field is $1 V/\lambda$
and the phase of the incident field is $0^\circ$ at the origin of the coordinate system.
The magnitude of the physical optics current is listed in each of the figures as a
reference level. The total current on an angular sector for a $\phi_i$ polarized plane wave
incident from the direction $\theta_i = 45^\circ$, $\phi_i = 225^\circ$ is shown in Figure 5. The behavior
of the total current is fairly complicated, since the path differences between the
currents create amplitude and phase variations over the plate resulting in the
interference pattern illustrated in Figure 5. In contrast, the difference current
shown in Figure 6 is monotonically decreasing in the radial direction and smooth
in the angular direction with the exception of the directions corresponding to the
edge diffraction boundaries on the angular sector ($\phi'_c = 54.6^\circ$ for edge 1 and
$\phi'_c = 35.4^\circ$ for edge 2).

The behavior of the difference current for different angles of incidence is illus-
trated next. Several radial cuts on a quarter plane are taken for three angles of
incidence, corresponding to a front, side, and back corner of a finite rectangular
plate. The quarter plane lies in the first quadrant and the geometry is as de-
scribed previously. The current on the front corner is shown first. The magnitude
Figure 5: Components of the total current on a quarter plane for $x_c \leq 2\lambda$ and $y_c \leq 2\lambda$ with $\hat{E}_{\phi_1}$ incidence from $\theta = 45^\circ$, $\phi = 225^\circ$ ($|\hat{J}^PO| = 0.00375 A/\lambda$).
Figure 6: Components of $\hat{J}AS - \hat{J}PO - \hat{J}d1 - \hat{J}d2$ on a quarter plane for $x_c \leq 2\lambda$ and $y_c \leq 2\lambda$ with $\hat{E}_{\phi_1}$ incidence from $\theta = 45^\circ$, $\phi = 225^\circ$ ($|\hat{J}PO| = 0.00375$ $A/\lambda$).
and phase of the $\hat{r}'$ component of the difference current are shown in Figure 7 for both a $\hat{\phi}_i$ polarized field and a $\hat{\theta}_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$. Because of the symmetry of the incident direction with respect to the quarter plane only directions less than $45^\circ$ are shown. The results for directions, $\phi'_c$, greater than $45^\circ$ are the same as those for $90^\circ - \phi'_c$. Only three radial cuts are shown in the phase plot for clarity. In all cases the difference current decays monotonically as the the distance from the vertex increases. Comparing the curves for different directions ($\phi'_c$) on the plate shows that the functional dependence on the distance from the vertex is a function of the angular coordinate ($\phi'_c$) on the plate. The phase behaves essentially the same as the phase of a wave traveling away from the vertex. The results for the $\hat{\phi}'_c$ component of the difference current are very similar to those shown for the $\hat{r}'$ component of the difference current, except that the magnitudes are generally a factor of ten smaller.

Figure 8 illustrates the magnitude and phase of the $\hat{r}'$ component, for a few radial cuts on the quarter plane, of the difference current for a $\hat{\theta}_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = -45^\circ$ which corresponds to a side corner. The phase behavior again indicates that the difference current behaves essentially as a wave traveling away from the vertex of the angular sector. The difference current behaves similarly for the other component and the other incident polarization so the plots have been omitted.

Figure 9 illustrates the magnitude and phase of the $\hat{r}'$ component, for a few radial cuts on the quarter plane, of the difference current for a $\hat{\theta}_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = 45$. As before, the phase indicates that the difference current behaves as a wave traveling away from the vertex. Only two radial cuts are shown in the phase plot for clarity. The phase of the $\hat{r}'$ component
Figure 7: Difference current for a $\phi_i$ polarized incident field (dash, $|JPO| = 0.00265 \ A/\lambda$) and a $\theta_i$ polarized incident field (solid, $|JPO| = 0.00531 \ A/\lambda$) incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$. 
Figure 8: Difference current for a $\dot{\theta}$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = -45^\circ$ ($|J^{PO}| = 0.00531 \, A/\lambda$).
Figure 9: Difference current for a \( \hat{\theta}_i \) polarized field incident from the direction \( \theta_i = 60^\circ, \phi_i = 45^\circ \) (\( |\mathbf{j}^{PO}| = 0.00531 \, A/\lambda \)).
of the difference current along the two radial cuts $\phi_c' = 25^\circ$ and $\phi_c' = 35^\circ$ is within 15° of the phase along the $\phi_c' = 45^\circ$ cut.

Observing the phase variation of the difference current shows that it may be associated with the vertex diffracted current, and therefore the component of the current associated with the multiple interaction of the edges, including the double diffracted current is small in comparison for the cases illustrated above. An example showing a case where the double diffraction current is the dominant component in the difference current is given later in this section.

Next, the behavior of the difference current as a function of the angle on the plate ($\phi_c'$) is investigated for the same directions of incidence shown for the radial plots. The magnitude and phase of the $\vec{r}'$ component of the difference current is shown in Figure 10 for both a $\hat{\phi}_i$ polarized field and a $\hat{\theta}_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$ for two different distances from the vertex. The difference current is discontinuous along the two edge diffraction boundaries, $\phi_c' = 52.2^\circ$ (edge 1) and $\phi_c' = 37.8^\circ$ (edge 2), since one of the edge diffracted currents is discontinuous along each edge diffraction boundary. The phase is also discontinuous at the diffraction boundaries. The difference current is also singular along the edges, but it is consistent with Meixner's edge condition [21] since the difference current is the difference between the exact solution for the fields scattered by an angular sector, which meets the edge condition [2] and the exact solution for the field scattered by a half plane which must satisfy the edge condition.

The magnitude of the $\vec{r}'$ component of the difference current is shown in Figure 11 for both a $\hat{\phi}_i$ polarized field and a $\hat{\theta}_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = -45^\circ$ for two different distances from the vertex. The difference current is only discontinuous at the diffraction boundary for edge 2,
Figure 10: Difference current for a \( \phi_i \) polarized field (dash, \( |\vec{J}^{PO}| = 0.00265 \, A/\lambda \)) and a \( \theta_i \) polarized field (solid, \( |\vec{J}^{PO}| = 0.00531 \, A/\lambda \)) incident from the direction \( \theta_i = 60^\circ, \phi_i = 225^\circ \).
The magnitude of the \( r' \) component of the difference current is shown in Figure 12 for both a \( \hat{\phi}_i \) polarized field and a \( \hat{\theta}_i \) polarized field incident from the direction \( \theta_i = 60^\circ, \phi_i = 45^\circ \) for two different distances from the vertex. For this angle of incidence, the difference current is continuous over the entire angular sector since both edge diffraction currents are continuous on the angular sector.

The magnitude of the \( \phi'_c \) component of the difference current is shown in Figure 13 for a \( \hat{\phi}_i \) polarized field incident from the three directions (\( \theta_i = 60^\circ, \phi_i = 225^\circ \), (\( \theta_i = 60^\circ, \phi_i = 45^\circ \)), and (\( \theta_i = 60^\circ, \phi_i = 45^\circ \)). At the edge diffraction boundaries, the angular dependence of the \( \phi'_c \) component of the difference current is similar to the \( r' \) component. Near the edges the \( \phi'_c \) component of the total current
Figure 12: Magnitude of the difference current for a $\hat{\phi}_i$ polarized field (dash, $|\hat{J}^{PO}| = 0.00265 \ A/\lambda$) and a $\hat{\theta}_i$ polarized field (solid, $|\hat{J}^{PO}| = 0.00531 \ A/\lambda$) incident from the direction $\theta_i = 60^\circ, \phi_i = 45^\circ$.

Figure 13: Magnitude of the difference current for a $\hat{\phi}_i$ polarized field for $r' = 0.4\lambda$ ($|\hat{J}^{PO}| = 0.00265 \ A/\lambda$).
must be zero, so in the cases where the edge diffraction current from one edge is nonzero at the other edge, the difference current must cancel the $\phi^l_e$ component of the edge diffracted current along the edge. Notice that the $\phi^l_e$ component of the physical optics current and the $\phi^l_e$ component of the edge diffracted current cancel at the edge where the edge diffracted current originates.

The total current on a finite plate with straight edges may now be approximated by summing the difference currents from each corner with the edge diffracted currents, shadowed for the finite edge not the semi-infinite edge, from each edge and the physical optics current. The resulting current may be integrated numerically to find the fields scattered by a finite plate. As an example, the bistatic scattering from a $2\lambda$ square plate lying in the $x - y$ plane and defined by edges parallel to the $x$ and $y$ axes is investigated. The complete scattering matrix for the bistatic scattering from the plate for a fixed source direction of $\theta_i = 60^\circ, \phi_i = 45^\circ$ and a spherical pattern, $\phi_s = 60^\circ$, is shown in Figures 14, 15, 16, and 17. The results show good agreement between the Method of Moments and the currents based on the quarter plane solution with the exception of the region $30^\circ < \theta_s < 150^\circ$ for the $\hat{\phi}_i$ polarized incident field where the ray approximation to the current is five to ten dB too low. It is speculated that the discrepancy is the result of not fully including the effects of the vertex diffracted-edge diffracted current. The comparison with the physical optics and edge diffracted currents indicates that the difference current only makes a significant contribution to the scattered field in the region $30^\circ < \theta_s < 150^\circ$ for this geometry. The physical optics and edge diffracted currents make the dominant contribution to the scattered field in the region near the specular directions ($\theta_s = 300^\circ, \phi_s = 60^\circ$ and $\theta_s = 240^\circ, \phi_s = 60^\circ$).
Figure 14: Bistatic cross section $\sigma_{\theta_i\theta_i}$ in $\text{dB}/\lambda^2$ of a $2\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$.

Figure 15: Bistatic cross section $\sigma_{\phi_s\phi_i}$ in $\text{dB}/\lambda^2$ of a $2\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$. 
Figure 16: Bistatic cross section $\sigma_{0,\phi_i}$ in $dB/\lambda^2$ of a $2\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$.

Figure 17: Bistatic cross section $\sigma_{\theta,\phi_i}$ in $dB/\lambda^2$ of a $2\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$. 
The difference current is not always dominated by the vertex diffracted current. In some cases, the double diffracted current becomes the dominant contribution to the difference current. An example illustrating a case where the double diffracted current dominates the difference current is shown next. Figure 18 illustrates the $\phi'_c$ component of the difference current for $\phi'_c = 1^\circ$ and a $\hat{\phi}_i$ polarized plane wave incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$ with the vertex angle as a parameter. As shown in the figure, the phase variation of the $\phi'_c$ component of the difference current is essentially the same as the phase variation of the double diffraction ray path. Notice that this is not the incident polarization normally associated with double diffraction. Previous work in the area of double diffraction (including [22], [23], [24], [25], [17], [26], and [27]) is based on the asymptotic evaluation of spectral integrals assuming that the two diffraction points are well separated and that the source and field points are not close to either of the diffraction points. These assumptions are obviously not valid in this case. In this case the $\phi'_c$ component of the difference current must cancel the $\phi'_e$ component of the edge diffracted current at the edge so that the $\phi'_e$ component of the total current is zero at the edge (see Figure 19 for the pertinent ray path geometry).

A significant contribution to the $\phi'_c$ component of the edge 2 diffracted current is the component given explicitly as the $\hat{z}_{cn}$ ($n=2$) component in Equation (2.2). This component of the edge diffracted current is only important near the diffracting edge and is usually neglected in double diffraction solutions. As a result there are no known double diffraction solutions available at this time which could properly account for the double diffraction current contribution to the difference current.

The magnitude of the $\phi'_c$ component of the difference current still decreases monotonically as the distance from the vertex increases, as may be expected from
Figure 18: The difference current (solid) and the double diffracted current (dashed, phase only) for a $\phi_i$ polarized field incident from the direction $\theta_i$. $\theta_i = 89^\circ$, $\phi_i = 91^\circ$ for $\phi_c = 1^\circ$ ($|J_{PO}| \approx 10^{-4} \ A/\lambda$).
Figure 19: Double diffraction ray paths for incidence in the plane of an angular sector.

the double diffraction ray paths shown in Figure 19. The double diffraction current also affects the behavior of the magnitude of the \( r' \) component of the difference current, but does not alter the phase from the expected vertex diffracted phase variation. This component will be looked at in greater detail in Sections 2.4 and 2.5.

Overall, the results show that the vertex diffracted current makes the dominant contribution, in many cases, to the difference current. With this in mind, an expression for the vertex diffracted current may be found using the difference current as a reference.

2.4 Least Squares Fit Current

The behavior of the vertex diffracted current far from the vertex will be used to hypothesize a simple, but approximate functional form for the difference current. Since the vertex diffracted current is a corner diffracted field the \( r' \) component of the current (corresponding to a transverse component of the corner diffracted
field) must have a radial dependence of \( \frac{e^{-jk\nu t}}{r'} \) far from the vertex and away from the discontinuities in the edge diffracted current. Similarly, the \( \phi' \) component of the vertex current (corresponding to the radial component of the corner diffracted field) must have a radial dependence of \( \frac{e^{-jk\nu t}}{(r')^{3/2}} \) far from the vertex and away from the discontinuities in the edge diffracted current. Also far from the vertex, it is known that the \( r' \) component of the current must have a radial dependence of \( e^{-jk\nu t} \) along the discontinuity of the edge diffracted current, if the two edge diffraction boundaries are far apart. Likewise the \( \phi' \) component of the vertex current has a radial dependence of \( \frac{e^{-jk\nu t}}{(r')^{3/2}} \) in the same situation. The regions of space near the edge diffraction boundaries, where the fields are not ray optical, are transition regions where the behavior of the current “transitions” from one form to the other. Unfortunately, the behavior of the vertex diffracted fields is not known in these transition regions, and in fact the extent of the transition regions is not known. The functional behavior may be partially described by the transition function derived in [20], however the correct arguments of the transition function and the angular spectrum of the corner diffracted field are still not known. A much simpler approach will be taken to avoid these problems.

A least squares fit on the components of the difference current for the different incident polarizations to a function of the form

\[
J_{r', \phi_c}^{f_s} (r', \phi_c) = A_{r', \phi_c} (\phi_c' (kr') - \nu_{r', \phi_c} (\phi_c')) e^{-jk\nu t}
\]

where the usual least squares fit parameters for a variable raised to an unknown power are

\[
\nu_{r', \phi_c} (\phi_c') = -\frac{\left( \sum_{n=1}^{N} \frac{Y_n}{X_n} \right) \sum_{n=1}^{N} X_n - N \left( \sum_{n=1}^{N} Y_n \right)}{\left( \sum_{n=1}^{N} X_n \right)^2 - N \left( \sum_{n=1}^{N} X_n^2 \right)}
\]
Equations (2.9) and (2.10) involve the calculation of the difference current over some range of distances from the vertex for a constant direction along the plate. This is done for several angles of incidence, calculating $\nu$ and $|A(\phi'_c)|$ for both the $\tau'$ and $\phi'_c$ components of the difference current over a range of $0.2\lambda < r' < 1.2\lambda$. The least squares fit current is found to be in fairly good agreement with the difference current in most cases. A comparison of the magnitude and phase of the $\tau'$ component, for several radial cuts on the quarter plane, of the current given by the least squares fit and the difference current is shown in Figure 20 for a polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$. Only two radial cuts are shown in the phase plot for clarity. The least squares fit current also agrees fairly well with the difference current for the incident directions $\theta_i = 60^\circ, \phi_i = -45^\circ$ and $\theta_i = 60^\circ, \phi_i = 45^\circ$, although the results are not shown here. In general, the $\phi'_c$ component of the difference and least squares fit currents do not agree as well, but as mentioned earlier the $\phi'_c$ component of the difference current is generally a factor of five to ten times smaller than the $\tau'$ component for the same incident polarization and sample point on the angular sector.

One case where the least squares fit current does not agree well with the difference current is mentioned at the end of Section 2.3. In this case the double
(a) Magnitude of the $r'$ component

(b) Phase of the $r'$ component

Figure 20: Difference current (solid) and the least squares fit (dash) for a $\theta_i$ polarized field incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ (|J_{PO}| = 0.00531 A/\lambda)$.
diffraction current makes a significant contribution to the difference current. Figure 21 illustrates the \( \hat{r} \) component of the difference current for \( \phi^c = 1^\circ \) and a \( \hat{p} \) polarized plane wave incident from the direction \( \theta_i = 89^\circ, \phi_i = 91^\circ \) for two different vertex angles. The \( \hat{\phi} \) component of the difference current is discussed further in Sections 2.3 and 2.5. Both components of the current indicate that the difference current is not dominated by the vertex diffracted current. The magnitude of the radial component of the difference current does not decrease monotonically as the distance from the vertex increases, instead the difference current oscillates around the least squares fit current indicating that two different currents are making significant contributions. The oscillations do not appear very significant, but it should be pointed out that the levels are ten to twenty times higher than those shown in Figure 20. The oscillations also lead to an increase in errors as further approximations are made (see Section 2.5). On the other hand the \( \hat{\phi} \) component of the current decreases monotonically as the distance from the vertex increases, but the phase does not vary in the manner of a vertex diffracted wave \( (e^{-jkr}) \).

As illustrated previously in Figure 18, the phase variation is essentially the same as the variation expected from a double diffracted wave.

In general, the least squares fit current agrees well with the difference current for most of the geometries investigated, with the exception of the case illustrated above. A further approximation is described in the next section which does not require the calculation of the exact solution for large distances from the vertex, therefore making the calculation of the current practical.
Figure 21: Difference current (solid) and the least squares fit (dash) for a polarized field incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$ for $\phi_c = 1^\circ$ with the vertex angle as a parameter.
2.5 Approximate Current

An approximate expression for the difference current is postulated in this section based on the least squares fit current plus an additional phase term to approximate the phase behavior of the double diffracted current. The approximate current is given by

\[
J_{app}^{r',\phi'}_{r',\phi'}(r',\phi',\phi_c) = \left\{ \begin{array}{ll}
J_{df}^{r',\phi'}_{r',\phi_c} &= J_{AS}^{r',\phi'}_{r',\phi_c} - \left( J_{PO}^{r',\phi'}_{r',\phi_c} - J_{d1}^{r',\phi'}_{r',\phi_c} - J_{d2}^{r',\phi'}_{r',\phi_c} \right) ; r' < r'_{1} \\
A_{r',\phi'}(\phi_c) \left( kr' \right)^{-\nu_{r',\phi'}(\phi_c)} e^{-j k B_{r',\phi'}^{r'} r'_{1}} &; r' > r'_{1}
\end{array} \right. \tag{2.14}
\]

\[
\nu_{r',\phi'}(\phi_c) = \ln \left| \frac{J_{d1}^{r',\phi'}_{r',\phi_c}(r'_{1},\phi_c)}{J_{d1}^{r',\phi'}_{r',\phi_c}(r'_{2},\phi_c)} \right| \ln \left( \frac{r'_{2}}{r'_{1}} \right) \tag{2.15}
\]

\[
B_{r',\phi'}^{r'} = \frac{\operatorname{Arg} \left( \frac{J_{d1}^{r',\phi'}_{r',\phi_c}(r'_{1},\phi_c)}{J_{d1}^{r',\phi'}_{r',\phi_c}(r'_{2},\phi_c)} \right)}{k \left( r'_{2} - r'_{1} \right)} \tag{2.16}
\]

\[
A_{r',\phi'}(\phi_c) = J_{d1}^{r',\phi'}_{r',\phi_c}(r'_{1},\phi_c) \left( kr'_{1} \right)^{\nu_{r',\phi'}(\phi_c)} e^{j k B_{r',\phi'}^{r'} r'_{1}} \tag{2.17}
\]

and the distances \( r'_{2} \) and \( r'_{1} \) are chosen so that they are small enough that \( J_{AS}^{r',\phi'} \) may be easily calculated using the eigenfunction solution (Appendix A) and such that \( |r'_{2} - r'_{1}| < \lambda \). The second condition is much less stringent than the first in practice and is only required so that the phase does not change more than \( 360^\circ \), causing a large error in the calculation of \( B_{r',\phi'}^{r'} \). The series used to calculate \( J_{AS}^{r',\phi'} \) has been found to have converged sufficiently with \( N_{ev} = 15 \) and \( N_{ef} = 15 \), for \( r'_{1} = 0.2 \lambda \) and \( N_{ev} = 25 \) and \( N_{ef} = 15 \) for \( r'_{2} = 0.3 \lambda \) in connection with calculating \( \nu \) in Equation (2.15). The current must be calculated very accurately in this case since \( \nu \) is a sensitive function of the difference current, especially when the difference current is small. Comparing Equation (2.14) with Equation (2.8) it is obvious
that the approximate current is the same as the least squares fit current except that the parameters $\nu, \nu'_{\phi_c}, A_{\nu'},$ and $A_{\phi_c}$ are calculated using only two points near the vertex instead of a fairly large number of points of varying distances from the vertex. Also, the parameter $B'_{r',\phi_c}$ has been introduced in the phase to approximate the phase variation for those cases when the double diffraction current is important. In order to prevent results which are obviously nonphysical the parameter $\nu$ is restricted to $\nu_{r',\phi_c} > 0$. This restriction ensures that the approximate current remains finite as $r' \to \infty$.

To validate the use of Equation (2.15) to determine the parameter $\nu$, calculations using Equation (2.15) with $r'_1 = 0.2\lambda$ and $r'_2 = 0.3\lambda$ are compared with values found using the least squares fit, Equation (2.9), on the difference current data over the range $0.2\lambda < r' < 1.2\lambda$. The different methods of determining the parameter $\nu$ as a function of $\phi_c$ are compared in Figure 22 for the case of a $\theta_i$ polarized plane wave incident from the direction $\theta_i = 60^\circ$ and three different values of $\phi_i$. The fairly good agreement between the approximate values of the parameter $\nu$ and the values found using a least squares fit on a larger range of the data indicates that the approximate current is essentially the same as the least squares fit current for most of the cases illustrated here. The only exception being the region between the edge diffraction boundaries, $37.8^\circ < \phi_c < 52.2^\circ$, when the source is in the direction $\theta_i = 60^\circ$ and $\phi_i = 225^\circ$. In this case, the parameter, $\nu_{\phi}$ is underestimated which will result in the approximate current being too large. Since the least squares fit current has been shown to be a good approximation to the difference current (see Section 2.4), it follows that the approximate current should match the difference current fairly well.
Figure 22: Approximate (solid) and least squares (dash) values of the parameter $\nu_{ij}$ for a $\hat{\theta}_i$ polarized field incident from three different directions with a fixed angle $\theta_i = 60^\circ$.

The results shown here do not support the theory in [28] and [4] that there is a cylindrical wave guided down each edge. If this were the case, the parameter $\nu_{ij}$ should approach $\frac{1}{2}$ at each edge. Instead of being related to the distance from the edges, the parameter $\nu_{ij}$ is more closely related to the distance from the edge diffraction boundaries. Although only results for a $\hat{\theta}_i$ polarized incident field are shown, none of the results obtained to this point, for either incident polarization indicate the presence of an edge guided wave.

The approximate current agrees well with the difference current for the geometries described in Section 2.3, except for the case of a $\hat{\phi}_i$ polarized field incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$. Figures 23 and 24 show the parameters $\nu_{ij}$ and $\nu_{ij}'$ as a function of $\phi_c'$ on a quarter plane for a $\hat{\phi}_i$ polarized plane wave incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$. Figure 23 shows that the param-
Figure 23: Approximate (solid) and least squares (dash) values of the parameter
$\nu_\tau$ for a $\hat{\phi}_i$ polarized field incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$.

Figure 24: Approximate (solid) and least squares (dash) values of the parameter
$\nu_\phi$ for a $\hat{\phi}_i$ polarized field incident from the direction $\theta_i = 89^\circ, \phi_i = 91^\circ$.  

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eter \( \nu_r \) is not determined very accurately due to the oscillations in the difference current resulting from the significant contributions to the difference current from the double diffraction current. Figure 25 illustrates that the error in calculating \( \nu_r \) results in the magnitude of the \( r^l \) component of the approximate current being significantly different from the magnitude of the \( r^l \) component of difference current. The other significant difference is in the phase of the \( \phi_c \) component of the current, shown in Figure 26, which matches the difference current much more closely than the least squares current since the additional parameter \( B_{\phi_c}^l \) allows for a phase variation other than a wave emanating from the vertex. As shown in Figure 25, the approximate current may be inaccurate in cases where the vertex diffracted current and the double diffraction current have magnitudes on the same order. This may cause the calculation of the parameter \( \nu \) to be inaccurate. The discontinuities in the approximate calculations of the parameter \( \nu \) occur at the edge diffraction boundaries. As stated before, the assumed functional behavior of the current is not accurate in the regions near the edge diffraction boundaries and is the most likely cause of this problem. This does not lead to any problems in the calculation of the scattered field.

Replacing the difference current with the approximate current and repeating the calculation of the bistatic scattered field of Section 2.3, the approximate current gives essentially the same results (within a few dB over the entire pattern) for \( \sigma_{\theta_s \theta_i} \) and \( \sigma_{\theta_s \phi_i} \), and the reader is thus referred to Figures 14 and 17. The results for \( \sigma_{\phi_s \theta_i} \) and \( \sigma_{\phi_s \phi_i} \) are different in the region of \( \theta \approx 90^\circ \). These results are shown and discussed in Section 2.6.
Figure 25: Approximate current (dash) and difference current (solid) for \( \phi_i \) polarized field incident from the direction \( \theta_i = 89^\circ, \phi_i = 91^\circ \) for \( \phi_c' = 1^\circ \).
Figure 26: Approximate current (dash) and difference current (solid) for \( \phi \) polarized field incident from the direction \( \theta_i = 89^\circ, \phi_i = 91^\circ \) for \( \phi_c = 1^\circ \).
2.6 Vertex Diffracted Current

Combining the results of the previous sections, an expression for the the vertex diffracted current is postulated to be

\[
J_{r', \phi'_c}^{v} (r'_1, \phi'_c) = \begin{cases} J_{r', \phi'_c}^{d} = J_{r', \phi'_c}^{AS} - \left( J_{r', \phi'_c}^{PO} - J_{r', \phi'_c}^{d1} - J_{r', \phi'_c}^{d2} \right); & r'_1 < r'_1' \\
A_{r', \phi'_c} (\phi'_c) (kr'_1) e^{-jkr'}; & r'_1 > r'_1'
\end{cases}
\]

(2.18)

where

\[
\nu_{r', \phi'_c} = \frac{\ln \left( \frac{J_{r', \phi'_c}^{d} (r'_1, \phi'_c)}{J_{r', \phi'_c}^{d} (r'_2, \phi'_c)} \right)}{\ln \left( \frac{r'_2}{r'_1} \right)}
\]

(2.19)

\[
B_{r', \phi'_c} = \frac{\arg \left( \frac{J_{r', \phi'_c}^{d} (r'_1, \phi'_c)}{J_{r', \phi'_c}^{d} (r'_2, \phi'_c)} \right) k (r'_2 - r'_1)}
\]

(2.20)

\[
A_{r', \phi'_c} (\phi'_c) = \begin{cases} J_{r', \phi'_c}^{d} (kr'_1) e^{jkr'_1}; & B_{r', \phi'_c}^{d1} < B_{r', \phi'_c}^{d1} < B_{r', \phi'_c}^{d1} \max \\
0; & B_{r', \phi'_c}^{d1} < B_{r', \phi'_c}^{d1} \min \text{ or } B_{r', \phi'_c}^{d1} > B_{r', \phi'_c}^{d1} \max
\end{cases}
\]

(2.21)

and the distances \(r'_2\) and \(r'_1\) are chosen so that they are small enough that the total current on the angular sector, \(J^{AS}\), may be easily calculated and such that \(|r'_2 - r'_1| < \lambda\). This is the same expression as the approximate current in Equation (2.14) with two minor, but important changes. The phase variation has been assumed to be the same as a spherical wave traveling away from the vertex for reasons that will be explained in Section 3.1. The parameter \(B_{r', \phi'_c}^{d1}\) is still calculated, however, and it must be between some parameters \(B_{r', \phi'_c}^{d1} \min\) and \(B_{r', \phi'_c}^{d1} \max\) chosen to minimize the effect of the double diffraction current. For the calculations, \(B_{r', \phi'_c}^{d1} \min\) has been chosen to be 0.5 and \(B_{r', \phi'_c}^{d1} \max\) has been chosen to be 1.5. In those instances where the ray path of the double diffraction current is essentially the same as the vertex diffracted current ray path (as illustrated in Figure 2) the expression in Equation (2.18) for the vertex diffracted current will obviously be in error, since
it will include the effects of the double diffraction current. In other regions of
the pattern where the double diffraction current is fairly large compared with the
vertex diffracted current, but not large enough to dominate the phase variation of
the difference current (e.g. Figure 25), Equation (2.18) will also be in error. At
the present time, it is not practical to remove the double diffracted current from
the difference current for the reason given in Section 2.3. The vertex diffracted
current may also be in error for cases when the two edge diffraction boundaries are
close together. The two edge diffraction boundaries are fairly close together when
the plane wave is incident from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$. The magnitude
and phase of the total current on the angular sector, as a function of the angular
position, is shown in Figure 27 along with the various combinations of the physi­
cal optics, edge diffracted and vertex diffracted currents. The vertex diffracted
current should sum with the physical optics and edge diffracted currents to give a
continuous result. In this case, the assumed functional form of the vertex diffracted
current between the edge diffracted boundaries is not sufficiently accurate to prop­
erly correct the discontinuities. In cases where the edge diffraction boundaries are
well separated, the vertex diffracted current does properly balance the discontinu­
ity so that the sum of the physical optics, edge diffracted, and vertex diffracted
currents is essentially continuous across the diffraction boundary (see Figure 28).

The bistatic scattered field from a square plate for the case described in Sec­
tion 2.3 is repeated using the vertex diffracted current instead of the difference
current. The vertex diffracted current is added to the physical optics and edge
diffracted currents and integrated numerically to find the bistatic scattered fields
from a two wavelength square plate. Similar calculations are made using the ap­
proximate current (Section 2.5) instead of the vertex diffracted current. Both the
Figure 27: The $r'$ component of $J^{AS}$ (solid), $J^{PO} + J^{d1} + J^{d2} + J^{u}$ (dash), and $J^{PO} + J^{d1} + J^{d2}$ (other) for $E_{\theta_i}$ incidence from $\theta_i = 60^\circ, \phi_i = 225^\circ$. 

(a) Magnitude of the $r'$ component 

(b) Phase of the $r'$ component
Figure 28: The $r'$ component of $J^{AS}$ (solid), $J^{PO} + J^{d1} + J^{d2} + J^{v}$ (dash), and $J^{PO} + J^{d1} + J^{d2}$ (other) for $E_{2i}$ incidence from the direction $\theta_i = 60^\circ, \phi_i = 225^\circ$.

Approximate current and the vertex diffracted current give results that are essentially the same as those given in Figures 14 and 17 for the $\hat{\phi}_s$ component of the scattered field calculated using the difference current. The $\hat{\phi}_s$ component of the scattered field is shown in Figures 29 and 30. The most likely cause of the inaccuracy in the approximate current, which should give essentially the same results as the difference current, is in the assumed functional form of the approximate current near the diffraction boundaries which may result in an overestimate of the current far from the vertex. Although this overestimate of the current should be small, near the plane of the plate where the contribution from the physical optics and edge diffracted currents is small, this error in the approximate current may lead to noticeable errors in the calculated scattered field. The differences between the approximate current and the vertex diffracted current result from the limited removal of the double diffracted current.
Figure 29: Bistatic cross section $\sigma_{\phi_s \theta_i}$ in dB/$\lambda^2$ of a 2\$\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$.

Figure 30: Bistatic cross section $\sigma_{\phi_s \theta_i}$ in dB/$\lambda^2$ of a 2\$\lambda$ square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ for $\phi_s = 60^\circ$. 

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2.7 Conclusion

An analytic, but approximate, expression for the current on a perfectly conducting angular sector for a plane wave incident field has been found and is given by

\[ j^{AS} \approx j^{PO} + j^{d1} + j^{d2} + j^v + j^{d1,d2} + j^{d2,d1} \]  (2.22)

where the separate currents are given in Equations (2.1), (2.2), (2.18), and the double diffraction currents are assumed to be zero here, but will be used in the next chapter. The total current is written as the sum of the physical optics, edge diffracted, and vertex diffracted currents so that it may be easily extended to finite plates. The analytic expression for the total current, given in Equation (2.22) approximately meets several important conditions on the total current on an angular sector. The vertex diffracted current decreases the magnitude of the discontinuity in the total current where one or both of the edge diffracted currents are discontinuous, but does not completely remove these discontinuities. The component of the total current normal to the edge \((J_{\phi_c}^{AS})\) should be zero at the edges of the angular sector. The double diffracted current is needed to meet this condition for angles of incidence where \(\beta_1^1 > \alpha_v\), \(\beta_2^1 < \pi - \alpha_v\), or both. As mentioned before, a double diffraction solution applicable to this case is not available at this time.

The vertex diffracted current has been verified for regions near the vertex for a few cases by comparing it with the exact solution after the physical optics and edge diffracted currents have been removed. For regions of the angular sector far from the vertex, the solution reduces to the physical optics current away from the edges or to the sum of the physical optics and an edge diffracted current near an edge since the vertex diffracted current decreases monotonically as the distance to the vertex increases. The ray based current has been further verified by comparison...
with Method of Moment calculations for the bistatic scattered field from a square plate.

Although almost all of the results shown are for a quarter plane, the equations for the ray based currents are valid for an angular sector with an arbitrary vertex angle. As the vertex angle decreases it seems reasonable that the higher order edge interactions will be more important, and the ray based current will not be as accurate as it is for the quarter plane. However, results shown in Chapter III indicate that the ray based current is fairly accurate for vertex angles as small as 30°.
CHAPTER III

Corner Diffraction

The field diffracted by a corner may now be found by asymptotically evaluating the radiation integral over the total current on the angular sector. Writing the current as the sum of currents as described in the previous chapter will simplify the problem in several ways. In fact, the radiation integrals involving all of the ray based currents discussed in Chapter II with the exception of the vertex diffracted current have already been evaluated asymptotically by one or more authors. Specifically, the asymptotic contribution from the physical optics current is given in [29], [15], [16], [4], and possibly others for a field point in the far zone. Recently, Hill [20] has done a uniform asymptotic evaluation of the radiation integral using the physical optics current on an angular sector extending the results to a near zone field point and reducing to the previous results for a far zone field point. The contribution from the edge diffracted currents for a far zone field point is given in [15], [16], and [4] based on work by Michaeli [30,31], although the spectral representation of the current is used instead of the analytic expression given in Equation (2.2). Hill [20] also obtained a uniform solution for the contribution from the edge diffracted currents on an angular sector valid for a near zone field point, once again reducing to the previous work for the special case of a far zone field point. The contribution from the double diffracted current for a far zone field point is given in [17]. As explained in Section 2.3, the double diffraction current used
in [17] is based on the transverse component of the fields only, and is therefore incomplete in the vicinity of the vertex. The contribution from the double diffracted current is therefore still incomplete, but the results indicate that the contribution from the missing component of the double diffracted current is insignificant for the cases shown. The contribution from the vertex diffracted current is obtained here. After finding the contribution from the vertex diffracted current, calculations of the field diffracted by a corner will be compared with backscatter measurements processed to isolate a single corner. Comparisons with Method of Moment results and pattern measurements for the scattering from finite plates will also be made.

The contribution to the corner diffracted field from the vertex diffracted current is found using a straightforward approach. The expression for the vertex diffracted current is placed in the radiation integral. This expression is only valid for plane wave incidence, since the expression for the vertex diffracted current is only valid for plane wave incidence. The surface integral over the angular sector is then separated into two regions. The first region is the small region near the vertex where the vertex diffracted current is given by the difference current \( r' < r_1 \). The second region consists of the remainder of the angular sector, where the second expression for the vertex diffracted current in Equation (2.18) is valid \( r' > r_1 \). The surface integral over the first region (near the vertex) is done numerically. The surface integral over the second region is reduced to a one-dimensional integral by finding the asymptotic endpoint contribution in the radial direction. Once again the remaining one-dimensional integral in the angular direction is done numerically. Several results are shown to illustrate how the vertex diffracted current improves
the corner diffraction calculations for the field scattered from flat perfectly con­
ducting plates. Both measurements and Method of Moments calculations are used as reference solutions.

3.1 Asymptotic Evaluation of the Radiation Integral

The far zone scattered field from the vertex diffracted current is found here. The field radiated by the vertex diffracted current (Section 2.6) is found by substi-
tuting the expression for the vertex diffracted current into the radiation integral

$$\vec{E}(\vec{r}) = \frac{j k Z_0}{4 \pi} \int \int \hat{R} \times \hat{R} \times \vec{J}(\vec{r}') \frac{e^{-jkR}}{R} d\sigma'.$$

Several approximations are now made. The field point is assumed to be in the far zone so that the usual parallel ray approximations, $R \approx r - \hat{r} \cdot \hat{r}'$ in the phase term and $R \approx r$ in the magnitude terms, may be used. Using the coordinates shown in
Figure 31 and writing the integral appropriately results in

$$E_{c,J_y}(\vec{r}) = \frac{jkZ_0 e^{-jk\vec{r}}}{4\pi} \int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c'$$

$$+ \int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c'. \quad (3.2)$$

The first integral in Equation (3.2) is done numerically. The second integral may be simplified further. First separating the vertex diffracted current into \( r' \) and \( \phi_c' \) components and noting that the resulting cross products are not a function of \( r' \) results in

$$\int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c' =$$

$$\int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c' =$$

$$+ \int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c'. \quad (3.3)$$

Using the expression for the vertex diffracted current in Equation (2.18) valid for \( r' > r_1' \) in Equation (3.3) results in

$$\int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c' =$$

$$\int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c' =$$

$$+ \int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c'. \quad (3.4)$$

$$\int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c' =$$

$$+ \int_0^{\alpha_y} \int_0^{\alpha_y r_1'} \hat{\vec{r}} \times \hat{\vec{r}} \times \hat{\vec{j}^y}(\vec{r}') e^{i k \hat{\vec{r}} \cdot \hat{\vec{r}}'} \vec{r}' \, d\vec{r}' \, d\phi_c'. \quad (3.5)$$

where

$$B(\phi_c') = \sin \theta_s \cos (\phi_d - \phi_c'). \quad (3.6)$$

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Both radial integrals have the same form and may be evaluated using the method of stationary phase. Since there are no stationary points within the range of integration, the leading term in the asymptotic evaluation of the integral is the endpoint contribution at \( r' = r'_1 \). Evaluating the integral using the method of stationary phase \([32]\) gives

\[
\int_{r'_1}^{r'} \left( r' \right)^{1-\nu_p(\phi'_c)} e^{-jkr'(1-\sin \theta_s \cos(\phi_s-\phi'_c))} dr' \sim
\]

\[
\frac{\left( r'_1 \right)^{1-\nu_p(\phi'_c)} e^{-jk(1-\sin \theta_s \cos(\phi_s-\phi'_c))}}{jk(1-\sin \theta_s \cos(\phi_s-\phi'_c))}
\]  

(3.7)

where \( p \) represents either \( r' \) or \( \phi'_c \). The integration over the second region has now been reduced to

\[
\int_{\phi'_c}^{\phi'_{c0}} \mathbf{j}_v \cdot \mathbf{E} \left( \mathbf{r}^r \right) e^{jkr' \mathbf{r}'} dr' d\phi'_c =
\]

\[
\int_0^{\phi'_{c0}} \mathbf{j}_v \cdot \mathbf{E} \left( \mathbf{r}^r \right) e^{jkr' \mathbf{r}'} dr' d\phi'_c
\]

(3.8)

which may be evaluated numerically. Using Equation (3.8) in Equation (3.2) gives the field radiated by the vertex diffracted current as

\[
\begin{align*}
\mathbf{E}_v = & \frac{jkZ_0}{4\pi r} \left[ \int_0^{\phi'_{c0}} \mathbf{j}_v \cdot \mathbf{E} \left( \mathbf{r}^r \right) e^{jkr' \mathbf{r}'} dr' d\phi'_c \\
+ \int_0^{\phi'_{c0}} \mathbf{j}_v \cdot \mathbf{E} \left( \mathbf{r}^r \right) e^{jkr' \mathbf{r}'} dr' d\phi'_c \\
+ \int_0^{\phi'_{c0}} \mathbf{j}_v \cdot \mathbf{E} \left( \mathbf{r}^r \right) e^{jkr' \mathbf{r}'} dr' d\phi'_c \right]
\end{align*}

(3.9)

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where \( J_1, A_{\nu}, A_{\kappa}, \nu_1, \) and \( \nu_\phi \) are given in Section 2.6. The integrand in the last two integrals is singular for the observation direction \( \hat{r} = \hat{r}' \). This direction corresponds to an observation point located on the angular sector. The cause of the singularity in this direction is the fact that the parallel ray approximation in the asymptotic evaluation of the integral in the radial direction is no longer valid. The field point cannot be in the far field of the surface since it lies on the surface of the angular sector. In practice, this problem may be easily overcome by changing the last integral in Equation (3.9) to

\[
\int_{0}^{\alpha} \mathbf{r} \times \frac{A_1}{k \nu_{\phi}} \left( \mathbf{r}', \phi_1 \right) \left( r_1 \right)^{1-\nu} e^{-jk(1-\sin \theta \cos(\phi - \phi'))} \frac{d\phi'}{jk(1 - B(\phi'))}
\]

(3.10)

where

\[
B(\phi') = \begin{cases} 
\sin \theta \cos(\phi - \phi') ; \sin \theta \cos(\phi - \phi') < 0.9 \\
0.9 ; \sin \theta \cos(\phi - \phi') > 0.9 
\end{cases}
\]

(3.11)

which has been found to give satisfactory results for the examples shown here.

The entire corner diffracted field, including the contribution from the physical optics, edge diffracted, vertex diffracted, and double diffracted currents, is given in Appendix B.

3.2 Results

Since the calculation of the corner diffracted field is based on an approximation of the current on an angular sector, it is compared with other methods to determine the accuracy of the calculations. The results labeled the new corner throughout this chapter are calculated using the contributions from the physical optics current, edge diffracted currents, the double diffracted current, and the vertex diffracted currents. The complete expression for the new corner is given in Appendix B. The parameters used in calculating the contribution from the vertex
diffracted current are $N_{ev} = 15$ and $N_{ef} = 15$ to find $\tilde{J}^{AS}(r'_1), N_{ev} = 25$ and $N_{ef} = 15$ to find $\tilde{J}^{AS}(r'_2), r'_1 = 0.2\lambda, r'_2 = 0.3\lambda$, 20 equally spaced rectangular approximations for the numerical evaluation of the linear integral, $N_{ev} = 7$ and $N_{ef} = 9$ to calculate $\tilde{J}^{AS}$ for the surface integral, and 4 equally spaced rectangular approximations in the $r'$ part of the surface integral, and 20 equally spaced rectangular approximations in the $\phi'_c$ part of the surface integral. The results are compared with a reference solution and the previous corner diffraction coefficient based on the contributions from the physical optics current [29,15,16], edge diffracted currents [15,16], and double diffracted currents [17] to show where the contribution from the vertex diffracted currents is important.

3.2.1 Triangle

Measurement data is processed using the following procedure to provide a reference solution for the field diffracted by the corner of an angular sector. Frequency swept measurements over a frequency range of 2 to 18 GHz are made on the triangle shown in Figure 32 for both the case $\alpha = 32''$, $\alpha_v = 30^\circ$ and the case $\alpha = 32''$, $\alpha_v = 45^\circ$ at 5° degree intervals in the $\theta = 90^\circ$ pattern. The frequency swept data at each pattern point is transformed to the time domain to obtain the band limited impulse response. The time response, typical of those obtained for pattern points in the region $95^\circ < \phi < 175^\circ$, for the pattern angle $\phi \approx 145^\circ$ is shown in Figure 33. The two first order mechanisms are labeled E1, which corresponds to the return from the front edge, and C3 which corresponds to the return from corner 3. The higher order mechanisms labeled CE and CC correspond to corner-edge interactions and corner-corner interactions. The path length to corner 3 is sufficiently different from the path lengths of the other mechanisms that the
Figure 32: Vertical triangle.

Figure 33: Impulse response ($\phi$ polarized co-pol response) of the triangle in Figure 32 with $a=32^\circ$ and $\alpha_v = 30^\circ$ at $\theta = 90^\circ$ and $\phi = 145^\circ$. 
field diffracted by corner 3 may be gated out in the time domain over most of the pattern. The response from corner 3 is isolated in the time domain and then transformed back to the frequency domain to obtain the contribution from this corner over the entire frequency band. Further details of the time gating process may be found in [33] and [34]. The contribution from corner 3 as a function of frequency is shown in Figure 34.

The measurements may then be used to determine the scattered field from an angular sector as a function of the pattern angle. From the frequency dependence of all of the components of the corner diffracted field in Appendix B and Figure 34 (which gives the cross section in dBsm) it is seen that the corner diffracted field has a frequency response that varies as one over the frequency. The corner diffracted

Figure 34: Frequency response ($\phi$ polarized co-pol response) of corner 3 of the vertical triangle with $a=32^\circ$ and $\alpha_v = 30^\circ$ at $\theta = 90^\circ$ and $\phi = 145^\circ$. 

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field is therefore constant for a given look angle when it is expressed in terms of dBSW. The roll off at the upper and lower limits of the frequency response in Figure 34 is a result of the time gating process, so only the data in the center of the frequency range is used. The measurements are averaged (the contribution from the corner should be a constant when it is expressed in dBSW) over 50 evenly spaced samples in the region of 6 to 14 GHz to obtain the measured value of the corner diffracted field. Repeating this process for all of the points in the pattern where the individual corner contribution may be gated out in the time domain results in the measured curves in Figure 35. The results show that the additional correction to the corner diffracted field improves the high frequency calculations considerably for this pattern.

The backscatter from a scaled version \((a = 3\lambda)\) of the triangle in Figure 32 is shown next. Calculations made using the new corner diffracted field added to the corner-edge and corner-corner higher order terms given in [17] are compared with calculations made using the previous corner diffracted field summed with the same higher order terms and Moment Method calculations in Figures 36 and 37. The results show fairly good agreement between the new corner calculations and the Moment Method calculations over most of the patterns. The most likely cause of the remaining differences in the new high frequency calculations and the Moment Method calculations is that the corner term in the edge-corner interactions in the new high frequency calculations is based on the physical optics and edge diffracted currents. It is speculated that the addition of the vertex diffracted current to the corner term in the edge-corner interactions would bring the new high frequency solution into fairly close agreement with the Moment Method calculations over the pattern, although even higher order interactions may also be important. In
Figure 35: Calculated and measured contribution from corner 3 to the RCS ($\sigma_{\phi\phi}$) of the vertical triangle for a $\theta = 90^\circ$ pattern.
Figure 36: Backscatter ($\sigma_{\phi\phi}$) from the vertical triangle with $a = 3\lambda$ and $\alpha_y = 30^\circ$ for a $\theta = 90^\circ$ pattern cut.

Figure 37: Backscatter ($\sigma_{\theta\theta}$) from the vertical triangle with $a = 3\lambda$ and $\alpha_y = 30^\circ$ for a $\theta = 90^\circ$ pattern cut.
a manner similar to the addition of double diffraction from a strip, the addition of the more complete edge-corner interaction should produce the expected null in the $\sigma_{\phi\phi}$ pattern located at $\phi = 180^\circ$ and shift the energy to create a lobe in the vicinity of $\phi = 150^\circ$ which matches the Moment Method calculations fairly well. Whether improving the edge-corner interactions would also improve the high frequency results for $\sigma_{\theta\theta}$ is not clear at this time.

Measurements are also made in the plane of the triangle shown in Figure 38. The contributions from each corner are isolated where possible using the technique described previously. The measurements and calculations for this geometry are shown in Figures 39-41. The curves illustrate how the addition of the vertex diffracted current improves the results in the regions away from normal incidence to an edge (i.e. the Keller cone, for backscatter), while it makes only a negligible contribution in the region near the Keller cone where the previous corner is known to give good results. Unlike the scattering from an edge, where it is well known that the front edge dominates the return from a strip when the incident electric field is
Figure 39: Calculated and measured contribution to the RCS from corner 1 ($\sigma_{\phi\phi}$) of the triangle ($\alpha_v = 45^\circ$) at $\theta = 90^\circ$.

Figure 40: Calculated and measured contribution to the RCS from corner 2 ($\sigma_{\phi\phi}$) of the triangle ($\alpha_v = 67.5^\circ$) at $\theta = 90^\circ$. 
parallel to the edge (or more generally $\beta_i$ polarized in the case of oblique incidence), in the region $95^\circ < \phi < 175^\circ$, the field scattered by either back corner (Figures 39 and 40) is greater than the field scattered by the front corner (Figure 41). A general rule is surmised at the end of this chapter which is consistent with the results shown here.

### 3.2.2 Square Plate

Similar measurements [35] are made on a two foot square plate tilted $44^\circ$ with respect to the $x-y$ plane as illustrated in Figure 42. The frequency swept measurements are taken every $5^\circ$ along a $\theta = 90^\circ$ pattern. The measured results are obtained using the same time domain extraction procedure that is used in Section 3.2.1. Both the new corner and the previous corner diffraction coefficient are compared with the measurements for the co-polarized, backscattered fields. Only
the new corner diffraction calculations are compared with the measurements for the cross polarized, backscattered field. Both $\sigma_{\theta\phi}$ and $\sigma_{\phi\theta}$ are shown for the corner diffracted fields to illustrate how close the calculated fields are to being reciprocal. The complete scattering matrix for all four corners is shown in Figures 43-53. It is not possible to separate the individual contribution from all of the corners over the entire pattern so there are some gaps in the measured results. The results for the $\sigma_{\phi\phi}$ contribution from corner 4 have been omitted since the measured levels are below the noise floor. Overall the new corner diffraction coefficient matches the measurements more closely than the previous corner diffraction coefficient. The previous corner diffraction coefficient does match the measured results more closely than the new corner diffraction coefficient for corner 1. The vertex diffracted current is not as accurate on the front vertex for reasons stated in Section 2.6, so in this case the calculated corner diffracted field may be less accurate when the field radiated by the vertex diffracted current is included. Including the field radiated by the vertex diffracted current does improve the calculations for the corner.
Figure 43: Calculated and measured backscatter ($\sigma_{\theta \theta}$) from corner 1 of the square plate, tilted 44° with respect to the $x - y$ plane, for a $\theta = 90°$ pattern.

Figure 44: Calculated and measured backscatter ($\sigma_{\phi \phi}$) from corner 1 of the square plate, tilted 44° with respect to the $x - y$ plane, for a $\theta = 90°$ pattern.
Figure 45: Calculated and measured cross polarized backscatter ($\sigma_{\phi\theta}$ or $\sigma_{\theta\phi}$) from corner 1 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90°$ pattern.

Figure 46: Calculated and measured backscatter ($\sigma_{\theta\theta}$) from corner 2 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90°$ pattern.
Figure 47: Calculated and measured backscatter ($\sigma_{\phi\phi}$) from corner 2 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90°$ pattern.

Figure 48: Calculated and measured cross-polarized backscatter ($\sigma_{\phi\theta}$ or $\sigma_{\theta\phi}$) from corner 2 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90°$ pattern.
Figure 49: Calculated and measured backscatter (σ_θθ) from corner 3 of the square plate, tilted 44° with respect to the x – y plane, for a θ = 90° pattern.

Figure 50: Calculated and measured backscatter (σ_φφ) from corner 3 of the square plate, tilted 44° with respect to the x – y plane, for a θ = 90° pattern.
Figure 51: Calculated and measured cross-polarized backscatter ($\sigma_{\phi\theta}$ or $\sigma_{\theta\phi}$) from corner 3 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90^\circ$ pattern.

Figure 52: Calculated and measured backscatter ($\sigma_{\theta\theta}$) from corner 4 of the square plate, tilted 44° with respect to the $x-y$ plane, for a $\theta = 90^\circ$ pattern.
Figure 53: Calculated and measured cross-polarized backscatter ($\sigma_{\phi\theta}$ or $\sigma_{\theta\phi}$) from corner 4 of the square plate, tilted 44° with respect to the $x-y$ plane, for $\theta = 90°$ pattern.
3.2.3 Bistatic Results for a Two Wavelength Square

Calculations made using the new corner diffracted field and the additional higher order terms mentioned in Section 3.2.1 are repeated for the bistatic scattered field from a two wavelength square plate located in the $x-y$ plane, as described in Section 2.3. The entire scattering matrix for a fixed source located at $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ over the pattern $\phi_s = 60^\circ$ is shown in Figures 54, 55, 56, and 57. As with the numerical integration of the currents in Section 2.3, the new corner agrees with the Moment Method results fairly well throughout the pattern with the exception of the region $5^\circ < \theta < 175^\circ$ for a $\hat{\theta}_i$ polarized incident field. In this region the corner diffraction calculations are around 5 to 10 dB lower than
Figure 55: Bistatic cross section $\sigma_{\phi_i \theta_i}$ in $dB/\lambda^2$ of a 2\lambda square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ at $\phi_s = 60^\circ$.

Figure 56: Bistatic cross section $\sigma_{\phi_s \phi_i}$ in $dB/\lambda^2$ of a 2\lambda square plate for $\theta_i = 60^\circ$ and $\phi_i = 45^\circ$ at $\phi_s = 60^\circ$. 72
the Moment Method, but agree fairly well with the numerical integration of the currents explained in Section 2.3. As mentioned previously in Section 2.3, the lack of a complete description of the higher order terms involving the corner-edge interactions seems the most likely cause of the differences in this region of the pattern.

3.2.4 Large Square Plate

The improvement in the calculation of the backscatter from large objects in regions away from specular is illustrated here. The measured backscatter [36] in the plane of the plate, at 10 GHz, from the 11.875" square plate shown in Figure 58 is shown in Figure 59. Calculations made using the new corner and additional higher order terms (corner to edge interactions, double diffraction, and corner to corner interactions) described in [17] are shown in Figure 60. Calculations made
Figure 58: 11.875" square plate in the $x - y$ plane.

Figure 59: Measured backscatter ($\sigma_{\phi\phi}$) of the 11.875" square plate.
Figure 60: Calculated backscatter ($\sigma_{\Phi\Phi}$) of the 11.875" square plate using the new corner and higher order terms (see text).
Figure 61: Calculated backscatter ($\sigma_{\phi \phi}$) of the 11.875" square plate using the previous corner and higher order terms (see text).

using the previous corner and the same additional higher order terms (corner to edge interactions, double diffraction, and corner to corner interactions) described in [17] are shown in Figure 61. The calculations using the new corner do not match the measurements well in the regions of $\theta = 0^\circ$ and $\theta = 90^\circ$ where the corner diffracted field is singular for edge on incidence. Aside from this problem the calculations including the new corner show good agreement with the measurements and significant improvement over calculations using the previous corner.
3.2.5 Large Triangular Plate

The measured backscatter in the plane of the plate, at 4 GHz, from the triangular plate shown in Figure 38 is shown in Figure 62. Calculations made using the new corner and additional higher order terms (corner to edge interactions, double diffraction, and corner to corner interactions) described in [17] are shown in Figure 63. Calculations made using the previous corner and the same additional higher order terms (corner to edge interactions, double diffraction, and corner to corner interactions) described in [17] are shown in Figure 64. Both the previous corner diffracted field and the new corner diffracted field are singular in some regions of this pattern. The contribution from the edge diffracted current to the corner diffracted field is singular for $\phi = 0^\circ$, and the contributions from both the edge diffracted current and the double diffracted current are singular for $\phi = \pm 112.5^\circ$. The singularities are discussed in a little more detail in Appendix B.
Figure 63: Calculated backscatter ($\sigma_{b*}$), using the new corner and higher order terms (see text), from the horizontal triangle at 4 GHz for $\theta = 89^\circ$.

Figure 64: Calculated backscatter ($\sigma_{\phi\phi}$), using the previous corner and higher order terms (see text), from the horizontal triangle at 4 GHz for $\theta = 89^\circ$. 

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Figure 65: Calculated backscatter ($\sigma_{\phi\phi}$), using the new corner and higher order terms (see text), from the horizontal triangle at 4 GHz for $\theta = 80^\circ$.

Compared with the calculations using the previous corner diffracted field, the calculations made using the new corner diffracted field agree more closely with the measurements. In addition to the singularities already present in the previous corner diffracted field the new corner diffracted field is singular for edge on incidence ($\phi = \pm 22.5^\circ$, $\phi = \pm 90^\circ$, and $\phi = \pm 157.5^\circ$). As explained in Appendix B, a more complete solution for the corner-corner diffracted field should cancel this singularity and result in calculations that match the measurements fairly closely. Until further work is done in this area, Figure 65 illustrates that taking the pattern out of the plane of the plate still results in calculations that come fairly close to matching the measured results in the plane of the plate. Calculations made using the previous corner diffracted field are essentially the same for both the $\theta = 89^\circ$ pattern shown in Figure 64 and the $\theta = 80^\circ$ pattern, so the second pattern has been omitted.
3.3 Conclusion

A new expression for the field diffracted by a corner has been obtained which represents a correction to previous work on the field diffracted by the corner. Several comparisons with measurements show that this additional term contributes significantly to the corner diffracted field in many regions of space for the special case of backscatter. In fact, in some cases, particularly for corners other than the front tip, it has been shown that the vertex diffracted current makes the dominant contribution to the corner diffracted field. In general, it appears that the co-polarized backscatter from a "back" corner is stronger for a $\hat{\phi}$ polarized field than a $\hat{\theta}$ polarized field, while the co-polarized backscatter from a "front" corner is stronger for a $\hat{\phi}$ polarized field than for a $\hat{\theta}$ polarized field. The unit vectors, $\hat{\theta}$ and $\hat{\phi}$, are the sphero-conal unit vectors described in Equations (A.35) and (A.36), and the pertinent geometry is shown in Figures 66 and 67.
CHAPTER IV

Conclusion

An analytic, but approximate, expression for the current on a perfectly conducting angular sector for a plane wave incident field has been found. The expression for the current on the angular sector is valid for an arbitrary vertex angle and the computation time required to calculate the current is independent of the distance from the vertex of the angular sector. The total current is written as the sum of the physical optics, edge diffracted, and vertex diffracted currents so that it may be easily extended to finite plates. The approximate expression for the current has been shown to be in good agreement with the exact solution for the special case of a quarter plane.

The approximate expression for the current on an angular sector is also used to find a new expression for the field diffracted by the vertex. The new expression for the field diffracted by a corner has been obtained which is an additive correction to previous work on the field diffracted by the corner. Several comparisons with measurements show that this additional term improves the calculations for backscatter from a flat plate.
APPENDIX A

Current on a Perfectly Conducting Angular Sector

A complete description of the exact solution for the electromagnetic field scattered by a perfectly conducting angular sector is given in [1]. The solution is expressed in the form of an eigenfunction expansion of the dyadic Green's function. The solution is specialized to the case of the current on a perfectly conducting angular sector assuming plane wave illumination. The solution is expressed in the sphero-conal coordinate system which is described briefly in the first section. The second section describes the implementation of the exact solution. For further information on the theoretical aspects of the solution, the sphero-conal coordinate system, or the more general problem of determining the field off of the angular sector the reader is referred to [1].

In this appendix only, the symbol \( \kappa \) will be used for the wave number and the symbol \( k \) will be used for the parameter related to the vertex angle in the sphero-conal coordinate system as in [1].

A.1 Sphero-conal Coordinate System

The sphero-conal coordinate system \((r, \theta, \phi)\) is related to the usual Cartesian coordinates shown in Figure 66 by

\[
\begin{align*}
x &= r \cos \theta \sqrt{1 - k'^2 \cos^2 \phi} \\
y &= r \sin \theta \sin \phi
\end{align*}
\] (A.1) (A.2)
Figure 66: Sphero-conal coordinate system.

\[ z = r \cos \phi \sqrt{1 - k^2 \cos^2 \theta} \]  \hspace{1cm} (A.3)

where

\[ k'^2 = \sqrt{1 - k^2}, 0 \leq k \leq 1 \]  \hspace{1cm} (A.4)

\[ 0 \leq \theta \leq \pi \]  \hspace{1cm} (A.5)

\[ 0 \leq \phi \leq 2\pi \]  \hspace{1cm} (A.6)

\[ r \geq 0 \]  \hspace{1cm} (A.7)

and the branches of the square root in Equation (A.1) and (A.3) must be evaluated appropriately to determine the correct sign of \( x \) and \( z \). Notice that in this case the parameter \( k \) is related to the vertex angle of the angular sector through

\[ k^2 = \cos^2 \frac{\alpha_v}{2}, 0 \leq k \leq 1 \]  \hspace{1cm} (A.8)

where the vertex angle, \( \alpha_v \), is illustrated in Figure 67.

Converting from Cartesian coordinates to sphero-conal coordinates is more complicated. Satterwhite [1] and other authors express this inverse transformation in terms of several other parameters. In order to avoid the elliptic functions
Figure 67: Angular sector geometry.

involved in those expressions, the inverse transform is expressed as

\[ r = \sqrt{x^2 + y^2 + z^2} \]  

(A.9)

and the solutions to

\[ (x \sin \theta)^2 - (y \cos \theta)^2 - \frac{(kz \sin \theta \cos \theta)^2}{1 - k^2 \cos^2 \theta} = 0 \]  

(A.10)

\[ (r k' \cos \theta \sin \phi \cos \phi)^2 + (y \cos \phi)^2 - (z \sin \phi)^2 = 0 \]  

(A.11)

which are solved numerically combined with the conditions

\[ 0 \leq \theta \leq \frac{\pi}{2} \quad \text{for} \quad x > 0 \]  

(A.12)

\[ \frac{\pi}{2} \leq \theta \leq \pi \quad \text{for} \quad x < 0 \]  

(A.13)

\[ 0 \leq \phi \leq \frac{\pi}{2} \quad \text{for} \quad y > 0, z > 0 \]  

(A.14)

\[ \frac{\pi}{2} \leq \phi \leq \pi \quad \text{for} \quad y > 0, z < 0 \]  

(A.15)

\[ \pi \leq \phi \leq \frac{3\pi}{2} \quad \text{for} \quad y < 0, z < 0 \]  

(A.16)

\[ \frac{3\pi}{2} \leq \phi \leq 2\pi \quad \text{for} \quad y < 0, z < 0 \]  

(A.17)
which make the spherico-conal coordinates of a point in space unique. The restrictions on the angles follow from Figure 66. Equations (A.9) to (A.11) give insight into the physical interpretation of the spherico-conal coordinate system. A constant \( r \) surface corresponds to a spherical surface centered at the origin and having a radius \( r \). A constant \( \theta \) surface corresponds to an elliptical cone with axis coinciding with the \( z \) axis and with the tip at the origin. The angle \( \theta \) is the angle in the \( z = 0 \) plane from the positive \( x \) axis to the surface of the elliptic cone. Similarly a constant \( \phi \) surface corresponds to half an elliptic cone with axis coinciding with the \( z \) axis and with the tip at the origin. The angle \( \phi \) is the angle in the \( x = 0 \) plane from the positive \( z \) axis to the surface of the elliptic cone. The two elliptic surfaces are shown in Figure 66. Numerical problems in solving Equations (A.10) and (A.11) may be avoided by using the several different approximations for points near the \( x \) axis, \( y \) axis, \( z \) axis, \( x = 0 \) plane, \( y = 0 \) plane, or \( z = 0 \) plane. Near the \( z \) axis, \( |y|/r \approx 1 \),

\[
\phi \approx \begin{cases} 
\frac{y}{r} & ; z \geq 0 \\
\frac{\pi}{2} - \frac{y}{r} & ; z < 0
\end{cases} \tag{A.18}
\]

\[
\theta \approx \frac{\pi}{2} - \frac{x}{r} \frac{1}{\sqrt{1 - k'^2 \cos^2 \phi}} \tag{A.19}
\]

Near the \( y \) axis, \( |x|/r \approx 1 \),

\[
\phi \approx \begin{cases} 
\frac{\pi}{2} - \epsilon & ; y \geq 0 \\
\frac{3\pi}{2} + \epsilon & ; y < 0
\end{cases} \tag{A.20}
\]

\[
\theta \approx \frac{\pi}{2} - \frac{x}{r} \frac{1}{\sqrt{1 - k'^2 \cos^2 \phi}} \tag{A.21}
\]

where

\[
\epsilon = \frac{\left[1 + k'^2 \left(\frac{z}{r}\right)^2\right] \pm \sqrt{\left[1 + k'^2 \left(\frac{z}{r}\right)^2\right]^2 - 4k'^2 \left[\left(\frac{z}{r}\right)^2 + k^2 \left(\frac{z}{r}\right)^2\right]}}{2k'^2} \tag{A.22}
\]
choosing the sign which best solves

\[ \frac{y}{r} = \sin \theta \sin \phi. \]  
(A.23)

Near the \( y - z \) plane, \( \frac{|y|}{r} \ll 1, \frac{|y|}{r} > > \frac{|x|}{r}, \) and \( \frac{|y|}{r} > > \frac{|z|}{r}, \)

\[ \phi \approx \sin^{-1}\left(\frac{y}{r}\right) \]  
(A.24)

\[ \theta \approx \frac{\pi}{2} - \frac{x}{r} \frac{1}{\sqrt{1 - k^2 \cos^2 \phi}}. \]  
(A.25)

Near the \( x \) axis, \( \frac{|z|}{r} \approx 1, \)

\[ \phi \approx \begin{cases} \frac{\pi}{2} - \frac{z}{r} \frac{1}{k} & ; y \geq 0 \\ \frac{3\pi}{2} + \frac{z}{r} \frac{1}{k} & ; y < 0 \end{cases} \]  
(A.26)

\[ \theta \approx \begin{cases} \frac{y}{r} & ; x \geq 0 \\ \pi - \frac{y}{r} \frac{1}{\sin \phi} & ; x < 0 \end{cases} \]  
(A.27)

Near the constant \( \theta \) part of the \( x - z \) plane, \( \frac{|y|}{r} \ll 1, \frac{|y|}{r} > > \frac{|x|}{r}, \frac{|z|}{r} > > \frac{|y|}{r}, \) and

\[ \tan^{-1}\left|\frac{z}{x}\right| < \frac{\alpha_r}{2}, \]

\[ \phi \approx \cos^{-1}\left(\frac{\text{sgn}(z)}{k} \sqrt{1 - \left(\frac{z}{r}\right)^2}\right) \]  
(A.28)

\[ \theta \approx \begin{cases} \frac{y}{r} \frac{1}{\sin \phi} & ; x \geq 0 \\ \pi - \frac{y}{r} \frac{1}{\sin \phi} & ; x < 0 \end{cases} \]  
(A.29)

Near the constant \( \phi \) part of the \( x - z \) plane, \( \frac{|y|}{r} \ll 1, \frac{|y|}{r} > > \frac{|x|}{r}, \frac{|z|}{r} > > \frac{|y|}{r}, \) and

\[ \tan^{-1}\left|\frac{z}{x}\right| > \frac{\alpha_r}{2}, \]

\[ \theta \approx \cos^{-1}\left(\frac{\text{sgn}(z)}{k} \sqrt{1 - \left(\frac{z}{r}\right)^2}\right) \]  
(A.30)

\[ \phi \approx \begin{cases} \frac{y}{r} \frac{1}{\sin \phi} & ; z \geq 0 \\ \pi - \frac{y}{r} \frac{1}{\sin \phi} & ; z < 0 \end{cases} \]  
(A.31)

Near the \( x - y \) plane, \( \frac{|z|}{r} \ll 1, \frac{|z|}{r} > > \frac{|x|}{r}, \frac{|y|}{r} > > \frac{|z|}{r}, \)

\[ \theta \approx \sin^{-1}\left(\frac{y}{r}\right) \]  
(A.32)

\[ \phi \approx \begin{cases} \frac{\pi}{2} - \frac{z}{r} \frac{1}{\sqrt{1 - k^2 \cos^2 \theta}} & ; y \geq 0 \\ \frac{3\pi}{2} + \frac{z}{r} \frac{1}{\sqrt{1 - k^2 \cos^2 \theta}} & ; y < 0 \end{cases} \]  
(A.33)
The unit vectors in spherico-conal coordinates are related to the usual Cartesian coordinate unit vectors in Figure 66 by

\[
\mathbf{\hat{r}} = \frac{x}{r} \mathbf{\hat{x}} + \frac{y}{r} \mathbf{\hat{y}} + \frac{z}{r} \mathbf{\hat{z}} \tag{A.34}
\]

\[
\mathbf{\hat{\theta}} = \frac{-\sin \theta \sqrt{1 - k^2 \cos^2 \phi} \sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{x}}
+ \frac{\cos \theta \sin \phi \sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{y}}
+ \frac{k^2 \cos \theta \sin \theta \cos \phi}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{z}} \tag{A.35}
\]

\[
\mathbf{\hat{\phi}} = \frac{k'^2 \cos \theta \cos \phi \sin \phi \sqrt{1 - k^2 \cos^2 \theta}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{x}}
+ \frac{\sin \theta \cos \phi \sqrt{1 - k'^2 \cos^2 \phi}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{y}}
- \frac{\sin \phi \sqrt{1 - k^2 \cos^2 \theta \sqrt{1 - k'^2 \cos^2 \phi}}}{\sqrt{k^2 \sin^2 \theta + k'^2 \sin^2 \phi}} \mathbf{\hat{z}} \tag{A.36}
\]

The spherico-conal unit vector directions are also indicated in Figure 66. Notice that the vectors \(\mathbf{\hat{r}}, \mathbf{\hat{\theta}}, \mathbf{\hat{\phi}}\) form the right hand coordinate system even though the coordinates are given in the other order, \((r, \theta, \phi)\). Also, there is obviously a typographical error in the expression for \(\mathbf{\hat{\phi}}\) given in [1].

The two cartesian coordinate systems shown in Figure 70 are related through

\[
x_c = -x \cos \frac{\alpha_x}{2} - z \sin \frac{\alpha_y}{2} \tag{A.37}
\]

\[
y_c = z \cos \frac{\alpha_x}{2} - x \sin \frac{\alpha_y}{2} \tag{A.38}
\]

\[
z_c = y \tag{A.39}
\]

and

\[
x = -x_c \cos \frac{\alpha_y}{2} - y_c \sin \frac{\alpha_y}{2} \tag{A.40}
\]

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\begin{align}
    y &= z_c \quad \text{(A.41)} \\
    z &= -x_c \sin \frac{\alpha_y}{2} + y_c \cos \frac{\alpha_y}{2} \quad \text{(A.42)}
\end{align}

### A.2 Current on a Perfectly Conducting Angular Sector

The current on a perfectly conducting angular sector for plane wave incidence may be derived from the expression for the current in Reference [1] by replacing the spherical Hankel functions involving the distance to the source $R_0$ by their asymptotic representation. After some simplification and the identification of the incident field the resulting expression is

\begin{align}
    j'_{AS} (r) &= \frac{E_0}{15} \sum_{l=1}^{\infty} \left\{ \frac{j^{\nu_{ol2}+1} \left[ \bar{m}_{ol2} (\theta_0, \phi_0) \cdot \hat{a} \right]}{\Lambda_{ol2}} \Theta_{ol2} (\pi) \right. \\
    &\quad \left. \left[ \frac{\frac{d}{dr} (r j_{ol2} (\kappa r))}{\kappa r \sin \phi} \sqrt{\left( \frac{1}{k'} \right)^2 - \cos^2 \phi \Phi_{ol2} (\phi) \hat{r}} \\
    &\quad \quad - \nu_{ol2} (\nu_{ol2} + 1) \frac{j_{ol2} (\kappa r) \Phi_{ol2} (\phi) \hat{\phi}}{\kappa r} \right] \\
    &\quad \left. + \frac{j^{\nu_{el1}} \left[ \bar{\tilde{f}}_0 \times \bar{m}_{el1} (\theta_0, \phi_0) \cdot \hat{a} \right]}{\Lambda_{el1}} \sin \phi \Phi_{el1} (\pi) \Phi_{el1} (\phi) \hat{r} \right\} \quad \text{(A.43)}
\end{align}

where the sum will be terminated after $N_{ev}$ terms and

\begin{align}
    \bar{m}_{ol2} (\theta_0, \phi_0) &= \frac{\sqrt{1 - k'^2 \cos^2 \theta_0}}{\sqrt{k^2 \sin^2 \theta_0 + k'^2 \sin^2 \phi_0}} \Theta_{ol2} (\theta_0) \Phi_{ol2} (\phi_0) \hat{\phi}_0 \quad \text{(A.44)} \\
    \bar{\tilde{f}}_0 \times \bar{m}_{el1} (\theta_0, \phi_0) &= \frac{\sqrt{1 - k'^2 \cos^2 \phi_0}}{\sqrt{k^2 \sin^2 \theta_0 + k'^2 \sin^2 \phi_0}} \Theta_{el1} (\theta_0) \Phi_{el1} (\phi_0) \hat{\phi}_0 \\
    &\quad + \frac{\sqrt{1 - k^2 \cos^2 \theta_0}}{\sqrt{k^2 \sin^2 \theta_0 + k'^2 \sin^2 \phi_0}} \Theta_{el1} (\theta_0) \Phi_{el1} (\phi_0) \hat{\phi}_0 \quad \text{(A.45)}
\end{align}

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\[ \Lambda_q = \left\{ \int_{-\pi}^{\pi} \frac{[\Phi_q(\phi)]^2}{\sqrt{1 - k'^2 \cos^2 \phi}} d\phi \right\} \left\{ \int_0^\pi \left[ \Theta_q'(\theta) \right]^2 \frac{1}{\sqrt{1 - k'^2 \cos^2 \theta}} d\theta \right\} \\
+ \left\{ \int_{-\pi}^{\pi} \frac{[\Phi_q'(\phi)]^2}{\sqrt{1 - k'^2 \cos^2 \phi}} d\phi \right\} \left\{ \int_0^\pi \frac{[\Theta_q(\theta)]^2}{\sqrt{1 - k'^2 \cos^2 \theta}} d\theta \right\} \right. \]

\( q \) denotes either \( ol2 \) or \( el1 \), the angles of incidence \( \theta_0 \) and \( \phi_0 \) are in sphero-conal coordinates, the observation angle on the sector, \( \phi \), is in sphero-conal coordinates, and the incident field, polarized in the arbitrary direction \( \hat{\alpha} \), is \( \vec{E}_i = \hat{\alpha} \vec{E}_0^i \) with the phase referenced to the origin. The functions \( \Theta_{ol2}(\theta_0) \) and \( \Phi_{ol2}(\phi_0) \) are the odd eigenfunctions meeting the Neumann boundary conditions corresponding to the \( l^{th} \) eigenvalue \( \nu_{ol2} \) and the functions \( \Theta_{el1}(\theta_0) \) and \( \Phi_{el1}(\phi_0) \) are the even eigenfunctions meeting the Dirichlet boundary conditions and corresponding to the \( l^{th} \) eigenvalue \( \nu_{el1} \). The prime denotes differentiation with respect to the argument of the function. The eigenfunctions are given by

\[ \Theta_{ol2}(\theta) = \sum_{m=-\infty}^{\infty} A_m \sin \left[ (2m - \frac{1}{2}) \theta \right] \] (A.47)

\[ \Phi_{ol2}(\phi) = \sum_{m=1}^{\infty} B_{2m} \sin (2m\phi) \] (A.48)

\[ \Phi_{ol2}(\phi) = \sum_{m=0}^{\infty} B_{2m+1} \sin [(2m + 1)\phi] \] (A.49)

\[ \Theta_{el1}(\theta) = \sum_{m=-\infty}^{\infty} A_m \cos \left[ (2m - \frac{1}{2}) \theta \right] \] (A.50)

\[ \Phi_{el1}(\phi) = \sum_{m=0}^{\infty} B_{2m} \cos (2m\phi) \] (A.51)

\[ \Phi_{el1}(\phi) = \sum_{m=0}^{\infty} B_{2m+1} \cos [(2m + 1)\phi] \]. (A.52)

In practice the sums will be terminated after summing \( N_{ef} \) terms with nonzero coefficients. The two expressions shown for \( \Phi_{ol2}(\phi) \) and \( \Phi_{el1}(\phi) \) have the same form, but are given separately to show explicitly that only even or odd terms are
nonzero in the infinite summations. Which of the forms is appropriate depends
on the corresponding eigenvalue. The coefficients \( A_m \) are determined using the
recurrence relations

\[
\frac{A_{m-1}}{A_m} = -\frac{b_m}{a_m} - \frac{c_m/a_m}{A_m/A_{m+1}} \tag{A.53}
\]

\[
\frac{A_{m+1}}{A_m} = -\frac{b_m}{c_m} - \frac{a_m/c_m}{A_m/A_{m-1}} \tag{A.54}
\]

where

\[
a_m = \frac{k^2}{4} \left[ \frac{(4m - 3)(4m - 5)}{4} - \nu_q \left( \nu_q + 1 \right) \right] \tag{A.55}
\]

\[
b_m = \frac{(4m - 1)^2}{4} \left( \frac{k^2}{2} - 1 \right) + \frac{\nu_q \left( \nu_q + 1 \right) k^2}{2} + \mu_q \tag{A.56}
\]

\[
c_m = \frac{k^2}{4} \left[ \frac{(4m + 1)(4m + 3)}{4} - \nu_q \left( \nu_q + 1 \right) \right] \tag{A.57}
\]

and \( \nu_q, \mu_q \) is the appropriate eigenvalue pair and \( q \) is either 012 or 11. The
recurrence relationship used to find \( B_{2m} \) in Equations (A.48) and (A.51) is

\[
\frac{B_{2m-2}}{B_{2m}} = -\frac{b_{2m}}{a_{2m}} - \frac{c_{2m}/a_{2m}}{B_{2m}/B_{2m+2}} \tag{A.58}
\]

where

\[
a_2 = -\frac{\nu_q \left( \nu_q + 1 \right) k'^2}{2} \tag{A.59}
\]

\[
b_2 = \frac{(k'^2 - 1)}{4} + \frac{\nu_q \left( \nu_q + 1 \right) k'^2}{2} - \mu_q \tag{A.60}
\]

\[
c_2 = \frac{k'^2}{4} \left[ 12 - \nu_q \left( \nu_q + 1 \right) \right] \tag{A.61}
\]

and

\[
a_{2m} = \frac{k'^2}{4} \left[ (2m - 2)(2m - 1) - \nu_q \left( \nu_q + 1 \right) \right] \tag{A.62}
\]

\[
b_{2m} = \frac{(2m)^2}{4} \left( \frac{k'^2}{2} - 1 \right) + \frac{\nu_q \left( \nu_q + 1 \right) k'^2}{2} - \mu_q \tag{A.63}
\]

\[
c_{2m} = \frac{k'^2}{4} \left[ (2m + 2)(2m + 1) - \nu_q \left( \nu_q + 1 \right) \right] \tag{A.64}
\]
for $m \geq 2$ and $\nu_q$, $\mu_q$ and $q$ have the same meaning as above. The recurrence relationship used to find $B_{2m+1}$ in Equations (A.49) and (A.52) is

$$\frac{B_{2m-1}}{B_{2m+1}} = \frac{b_{2m+1}}{a_{2m+1}} \frac{c_{2m+1}/a_{2m+1}}{B_{2m+1}/B_{2m+3}}$$  \hspace{1cm} (A.65)$$

where

$$a_{2m+1} = \frac{k'^2}{4} \left[ (2m - 1) 2m - \nu_q (\nu_q + 1) \right]$$  \hspace{1cm} (A.66)$$

$$b_{2m+1} = \left[ (2m + 1)^2 \left( \frac{k'^2}{2} - 1 \right) + \nu_q \left( \nu_q + 1 \right) \frac{k'^2}{2} - \mu_q \right]$$  \hspace{1cm} (A.67)$$

$$c_{2m+1} = \frac{k'^2}{4} \left[ (2m + 3) (2m + 2) - \nu_q (\nu_q + 1) \right]$$  \hspace{1cm} (A.68)$$

for $m \geq 1$ and $\nu_q$, $\mu_q$, and $q$ have the same meaning as above. The eigenvalue pairs $(\nu_q, \mu_q)$ are the simultaneous solutions of the eigenvalues of two matrices of infinite order. The eigenvalue pairs corresponding to the eigenfunctions satisfying the Neumann boundary conditions are the simultaneous solutions of

$$\begin{bmatrix} . & . & . & . & . & . & . & . & . & . & . \\ -b_2 & c_{-2} & 0 & 0 & 0 \\ b_{-1} & c_{-1} & 0 & 0 & 0 \\ a_0 & b_0 & c_0 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 \\ 0 & 0 & 0 & a_3 & c_3 \end{bmatrix} = 0$$  \hspace{1cm} (A.69)$$

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where $a_m, b_m,$ and $c_m$ are given for all $m$ by Equations (A.55) to (A.57), and for the $\Phi$ eigenfunctions with only nonzero even terms

$$
\begin{vmatrix}
  b_2 & c_2 & 0 & 0 & 0 \\
  a_4 & b_4 & c_4 & 0 & 0 \\
  0 & a_6 & b_6 & c_6 & 0 \\
  0 & 0 & a_8 & b_8 & c_8 \\
  \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0
$$

(A.70)

where $a_{2m}, b_{2m},$ and $c_{2m}$ are given for all $m$ by Equations (A.62) to (A.64). The eigenvalues corresponding to the $\Phi$ eigenfunctions with only nonzero odd terms are the simultaneous solutions of Equation (A.69) and

$$
\begin{vmatrix}
  b_1 & c_1 & 0 & 0 & 0 \\
  a_3 & b_3 & c_3 & 0 & 0 \\
  0 & a_5 & b_5 & c_5 & 0 \\
  0 & 0 & a_7 & b_7 & c_7 \\
  \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0
$$

(A.71)

where

$$
b_1 = \left( \frac{k'^2}{2} - 1 \right) + \frac{3\nu_q (\nu_q + 1)}{4} k'^2 - \mu_q
$$

(A.72)

$$
c_1 = \frac{k'^2}{4} \left[ 6 - \nu_q (\nu_q + 1) \right]
$$

(A.73)

and $a_{2m+1}, b_{2m+1},$ and $c_{2m+1}$ are given for $m \geq 1$ by Equations (A.66) to (A.68). The eigenvalue pairs corresponding to the eigenfunctions satisfying the Dirichlet boundary conditions and corresponding to the $\Phi$ eigenfunctions with only nonzero
even terms are the simultaneous solutions of Equation (A.69) and

\[
\begin{bmatrix}
  b_0 & c_0 & 0 & 0 & 0 \\
  a_2 & b_2 & c_2 & 0 & 0 \\
  0 & a_4 & b_4 & c_4 & 0 \\
  0 & 0 & a_6 & b_6 & c_6 \\
  & & & & \\
\end{bmatrix} = 0
\]  

(A.74)

where

\[
b_0 = \frac{\nu_q (\nu_q + 1) k'^2}{2} - \mu_q \tag{A.75}
\]

and

\[
c_0 = \frac{k'^2}{4} \left[ 2 - \nu_q (\nu_q + 1) \right] \tag{A.76}
\]

and \(a_{2m}, b_{2m},\) and \(c_{2m}\) are given for \(m \geq 1\) by Equations (A.59) to (A.64). The eigenvalues satisfying the Dirichlet boundary conditions and corresponding to the \(\Phi\) eigenfunctions with only nonzero odd terms are the simultaneous solutions of Equation (A.69) and Equation (A.71) with

\[
b_1 = \left( \frac{k'^2}{2} - 1 \right) + \frac{\nu_q (\nu_q + 1) k'^2}{4} - \mu_q \tag{A.77}
\]

\[
c_1 = \frac{k'^2}{4} \left[ 6 - \nu_q (\nu_q + 1) \right] \tag{A.78}
\]

and \(a_{2m+1}, b_{2m+1},\) and \(c_{2m+1}\) are given for \(m \geq 1\) by Equations (A.66) to (A.68).

The eigenvalue pairs are found in basically the same way as described by Satterwhite [1], with one minor change. Satterwhite has shown that all of the eigenvalue pairs consist of a pair of real numbers \((\nu_q, \mu_q)\) with the additional restriction that all the unique eigenvalue pairs lie in the half plane \(\nu_q > 0\). Since all of the eigenvalues are real and \(\nu_q\) is known to be positive, \(\mu_q\) is calculated starting with \(\nu_q = 0\) and then at a constant interval \(\Delta \nu\) until some predetermined value of \(\nu_q\) is reached. Instead of implementing the procedure described in [1],
the individual eigenvalue problems described by Equations (A.69), (A.70), (A.71), and (A.74) are reduced, when \( \nu_q \) is fixed and the matrix is truncated to some finite size, to finding the eigenvalues of real Hessenberg matrices. A real Hessenberg matrix has zero entries in the lower triangular portion of the matrix, with the exception of the first diagonal below the main diagonal which may have nonzero entries, and real valued entries in the nonzero portion of the matrix. Thus, in general, \( \mu_q \) will have multiple values for each value of \( \nu_q \). Notice that \( \mu_q \) is the negative of the eigenvalues of the matrix associated with Equation (A.69) as may be seen by Equation (A.56). In all other cases, \( \mu_q \) is the same as the eigenvalue of the associated matrix. The real eigenvalues of a real Hessenberg matrix may be found numerically using widely available computer subroutines such as the one described in [37]. The eigenvalue pairs, \( \nu_{ol2}, \mu_{ol2} (\nu_{el1}, \mu_{el1}) \), corresponding to eigenfunctions satisfying the Neumann (Dirichlet) boundary conditions are found using a general purpose root finder to find the simultaneous solutions of both Equation (A.69) (Equation (A.69)) and Equation (A.70) (Equation (A.74)) and also Equation (A.69) (Equation (A.69)) and Equation (A.71) (Equation (A.71)). The previously determined values of \( \mu_q \) are used along with linear interpolation to describe \( \mu_q \) as a function of \( \nu_q \) for any \( \nu_q \). The incremental value \( \Delta \nu \) described above therefore determines the accuracy with which the eigenvalue pairs may be calculated and also the computer time required to make the calculations. Finding the eigenvalues of a large matrix is very time consuming and may not be practical on some computers. In those cases, the values of \( \mu_q \) as a function of \( \nu_q \) may be found using the method described by Satterwhite [1] which is much faster, but not very stable, especially for larger values of \( \nu_q \). Curves giving the first few eigenvalues as a function of vertex angle are shown in [38]. Since there are multiple
values of $\mu_q$ for every value of $\nu_q$, there are actually several curves associated with each equation. Examples of these curves for a vertex angle of $30^\circ$ are shown in Figure 68 and Figure 69.

The intersections of the curves in the figure are the eigenvalue pairs for this vertex angle. Notice that in general there are a total of $2N + 1$ eigenvalue pairs contained in the interval $N - 1/2 < \nu < N + 1/2$. The coefficients for the eigenfunctions are then calculated using the appropriate recurrence relations given in Equations (A.54) and (A.53) after assuming $A_{\pm n} = 0$, $n > M$ where $M$ is a large integer ($M=14$ has been used for the calculation of the coefficients of the eigenfunctions here). The recurrence relationship (A.54) gives all the terms with nonpositive indices and the recurrence relationship (A.53) gives all the terms with nonnegative indices. The terms with negative indices are then renormalized so that the value of $A_0$ calculated starting with the term with the most negative index is the same as

Figure 68: Portion of the curves used to find the eigenvalue pairs meeting the Neumann boundary conditions for a vertex angle of $\alpha_v = 30^\circ$.  

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Figure 6.9: Portion of the curves used to find the eigenvalue pairs meeting the Dirichlet boundary conditions for a vertex angle of \( \alpha_v = 30^\circ \).

The coefficients for the \( \Phi \) eigenfunctions are calculated using the appropriate recurrence relations given in Equations (A.58) or \( \text{capb}(2m+1) \) assuming the sum may be terminated after some finite number of terms. The recurrence relationship gives all previous terms in terms of the term with the largest index. For large values of the eigenvalue \( \nu_q \), the recurrence relationships must be applied carefully to avoid division by zero. Specifically for the \( \Theta \) eigenfunctions, if \( \nu_q = 2N - 5/2 \), where \( N \) is a positive integer, then \( A_n = 0 \) for \( n \geq N \) and \( A_{N-1} \) should be used to begin the recurrence relationship. Also for the \( \Theta \) eigenfunctions, if \( \nu_q = -(2N - 3/2) \), where \( N \) is a non-positive integer, then \( A_n = 0 \) for \( n \leq N \) and \( A_{N+1} \) should be used to begin the recurrence relationship. For the \( \Phi \) eigenfunctions with even indices, if \( \nu_q = 2N - 2 \), where \( N \) is a positive integer, then \( B_{2n} = 0 \) for \( n \geq N \) and \( B_{2N-2} \) should be used to begin the recurrence relationship. For the \( \Phi \) eigenfunctions with odd indices, if \( \nu_q = 2N - 1 \), where \( N \) is a positive integer, then \( B_{2n+1} = 0 \)
for $n \geq N$ and $B_{2N-1}$ should be used to begin the recurrence relationship. Calculating the coefficients in the manner described above minimizes the round off error introduced at each step in the recurrence relationship. To prevent overflow problems, the coefficients may be renormalized throughout the calculations to the maximum coefficient. To compare with Satterwhite's results [1] the coefficients are renormalized so that $\Theta_{el1}(0) = 1$, $\Theta_{el2}'(0) = 1$, $\Phi_{el1}(0) = 1$ and $\Phi_{el2}'(0) = 1$. Further checks for round off errors in calculating the eigenvalues and coefficients of the eigenfunctions are described by Satterwhite [1]. Finally the normalization constants are found by numerically evaluating the integrals in Equation (A.46).

The first three eigenvalue pairs, normalization constants, and the coefficients for the eigenfunctions for each boundary condition are listed in Tables 1 and 2 for an angular sector with a vertex angle of 30°. Eigenfunction coefficients smaller than 0.001 have been omitted. In this case, and in all other results given, the matrices are terminated after 100 terms, the eigenfunctions are terminated after 29 terms, and the increment used in finding $\mu_q$ as a function of $\nu_q$ ($\Delta \nu$) is 0.01. These are the same parameters as Satterwhite [1] used, and seem to give satisfactory results for the calculations made here.

A.3 Conclusion

The calculation of the exact solution for the current near the vertex of a perfectly conducting angular sector using the solution in [1] has been described. A more numerically stable, but less efficient, method of determining the eigenvalues has been proposed. The new method has been used to find several eigenvalues for different vertex angles. The exact solution has been evaluated for distances of up to 2.8$\lambda$ from the vertex and used in the approximation of the current on a finite
Table 1: Coefficients for the odd eigenfunctions satisfying the Neumann boundary conditions.

<table>
<thead>
<tr>
<th>$\nu_{011}$</th>
<th>$\mu_{011}$</th>
<th>$\lambda_{011}$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_{-1}$</th>
<th>$A_{-2}$</th>
<th>$B_1$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.9816</td>
<td>-0.8687</td>
<td>8.7268</td>
<td>-0.93090</td>
<td>0.4623</td>
<td>0.0968</td>
<td>0.0381</td>
<td>0.9996</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\nu_{021}$</td>
<td>$\mu_{021}$</td>
<td>$\lambda_{021}$</td>
<td>$A_0$</td>
<td>$A_1$</td>
<td>$A_{-1}$</td>
<td>$A_{-2}$</td>
<td>$B_1$</td>
<td>$B_3$</td>
</tr>
<tr>
<td>1.9402</td>
<td>-0.6799</td>
<td>5.8082</td>
<td>0.2207</td>
<td>0.4437</td>
<td>-0.2299</td>
<td>-0.0239</td>
<td>1.0246</td>
<td>-0.0081</td>
</tr>
<tr>
<td>$\nu_{031}$</td>
<td>$\mu_{031}$</td>
<td>$\lambda_{031}$</td>
<td>$A_0$</td>
<td>$A_1$</td>
<td>$A_{-1}$</td>
<td>$A_{-2}$</td>
<td>$B_1$</td>
<td>$B_2$</td>
</tr>
<tr>
<td>1.9996</td>
<td>-3.6651</td>
<td>75.1091</td>
<td>-2.7044</td>
<td>1.8610</td>
<td>0.8160</td>
<td>-0.1456</td>
<td>0.0000</td>
<td>0.5000</td>
</tr>
</tbody>
</table>
Table 2: Coefficients for the even eigenfunctions satisfying the Dirichlet boundary conditions.

<table>
<thead>
<tr>
<th>( \nu_{e11} = 0.1810 )</th>
<th>( A_0 = 1.1216 )</th>
<th>( B_0 = 1.0018 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{e11} = 0.0071 )</td>
<td>( A_1 = -0.1635 )</td>
<td>( B_2 = -0.0019 )</td>
</tr>
<tr>
<td>( \Lambda_{e11} = 1.8356 )</td>
<td>( A_{-1} = 0.0609 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_2 = -0.0289 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_{-2} = 0.0155 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \nu_{e21} = 1.2500 )</th>
<th>( A_0 = 0.3110 )</th>
<th>( B_0 = 1.0251 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{e21} = 0.0945 )</td>
<td>( A_1 = 0.7715 )</td>
<td>( B_2 = -0.0250 )</td>
</tr>
<tr>
<td>( \Lambda_{e21} = 10.505 )</td>
<td>( A_{-1} = -0.1133 )</td>
<td>( B_4 = -0.0001 )</td>
</tr>
<tr>
<td></td>
<td>( A_2 = 0.0455 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_{-2} = -0.0228 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_3 = 0.0127 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \nu_{e31} = 1.0163 )</th>
<th>( A_0 = 3.4078 )</th>
<th>( B_1 = 1.0001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_{e31} = -0.9322 )</td>
<td>( A_1 = -1.7304 )</td>
<td>( B_3 = -0.0001 )</td>
</tr>
<tr>
<td>( \Lambda_{e31} = 120.868 )</td>
<td>( A_{-1} = -0.3807 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_2 = -0.1336 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_{-2} = -0.0713 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_3 = -0.0346 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_{-3} = -0.0220 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( A_4 = -0.0120 )</td>
<td></td>
</tr>
</tbody>
</table>
plate (Section 2.3). The number of terms required in the summation is on the order of $N_{ev} \approx 280$, and requires the resources of a very large computer, such as a Cray, to find the eigenvalues and evaluate the sum to find the current. On the other hand, the small number of eigenvalues required to find the solution near the vertex may be found in a reasonable amount of time on a much smaller computer, if the method described by Satterwhite [1] is used. Once a sufficient number of the eigenvalues have been found, calculating the current near the vertex is possible on most modern computers in a reasonable amount of time.
APPENDIX B
Corner Diffracted Field Expression

B.1 Corner Diffracted Field

All of the numerical results are expressed as the radar cross section in dB which is related to the scattered field by

$$\sigma = 10 \log \left( \frac{4\pi r^2 |E^s|^2}{|E^i|^2} \right)$$  \hspace{1cm} (B.1)

where $|E^s|$ is the magnitude of the scattered field in the far zone, $|E^i|$ is the magnitude of the incident field at the scatterer, and $r$ is the far zone distance.

The field diffracted by the corner of an angular sector is written as the sum of the contribution to the scattered field from the different currents on the angular sector. The far zone field scattered by an angular sector is

$$\vec{E}^c = \vec{E}^c,jpo + \vec{E}^c,jd1 + \vec{E}^c,jd2 + \vec{E}^c,jv$$

$$+ \vec{E}^c,jd1t2 + \vec{E}^c,jd2t1 + \vec{E}^c,jd1d2 + \vec{E}^c,jd2d1$$  \hspace{1cm} (B.2)

The contribution from the physical optics current is

$$\vec{E}^{c,jpo} = \sum_{m=1}^{2} \vec{E}^{c,jpo}_m$$  \hspace{1cm} (B.3)

where

$$\begin{bmatrix} \vec{E}^{c,jpo}_m \\ \beta_m \\ E^c_{\psi_m} \\ E^c_{\phi_m} \end{bmatrix} = \begin{bmatrix} D^i_{zi} & D^i_{zi,jpom} \\ 0 & D^i_{zi,jpom} \end{bmatrix} \begin{bmatrix} \vec{E}^{i}_{\beta_m} (Q_c) \\ \vec{E}^{i}_{\psi_m} (Q_c) \end{bmatrix} e^{-jkr} r$$  \hspace{1cm} (B.4)
\[
D_{s,i pom}^{c,j \beta m} = (-1)^{n-1} \left( \frac{1}{8 \pi k} \right) \left( \frac{1}{\cos \beta_m - \cos \beta_m^i} \right) \begin{cases}
\sin \beta_m \\
-\sin \psi_m \\
\sin \beta_m
\end{cases}
\]
\[
\cdot \sum_{n=1}^{2} \left[ S^i C_{n1}^{s,h,2} (\gamma_m, \psi_m^i) \right]
\]

\[
C_{n1}^s (\eta, \eta^i) = (-1)^{n-1} \left[ \cot \left( \frac{\pi - (\eta - (-1)^n \eta^i)}{4} \right) \right] + (-1)^n \cot \left( \frac{\pi + (\eta - (-1)^n \eta^i)}{4} \right)
\]  

(B.6)

\[
C_{n1}^h (\eta, \eta^i) = \frac{1}{\sin \eta} \left[ \cot \left( \frac{\pi - (\eta - (-1)^n \eta^i)}{4} \right) \right] + (-1)^n \cot \left( \frac{\pi + (\eta - (-1)^n \eta^i)}{4} \right)
\]  

(B.7)

\[
C_{n1}^{2} (\eta, \eta^i) = (\cot \beta_m \cos \psi_m + \cot \beta_m^i \cos \eta^i) C_{n1}^h (\eta, \eta^i)
\]  

(B.8)

\[
\cos \gamma_m = \frac{\sin \beta_m \cos \psi_m}{\sin \beta_m^i}
\]
\[
+ \frac{(\cos \beta_m - \cos \beta_m^i)^2}{\sin \beta_m \sin \beta_m^i \cos \psi_m + \sin \beta_m^i \cos \psi_m}
\]  

(B.9)

\[
\cos^{-1} \mu = -j \ln \left( \mu + \sqrt{\mu^2 - 1} \right)
\]  

(B.10)

\[
\sqrt{\mu^2 - 1} = \begin{cases}
-j \sqrt{\mu^2 - 1} & \mu < 1 \\
\sqrt{\mu^2 - 1} & -1 \leq \mu \leq 1 \\
\sqrt{\mu^2 - 1} & \mu > 1
\end{cases}
\]  

(B.11)

\[
S^i = \begin{cases}
-1, \pi - \psi^i < 0 \\
1, \pi - \psi^i > 0
\end{cases}
\]  

(B.12)

The various angles are shown in Figure 70. The details of the derivation may be found in [29], [15], or [16].
Figure 70: Edge fixed coordinates on an angular sector.
The contribution from the edge diffracted current of edge \( m \) \((m = 1,2)\) terminated by edge \( m \) \((m = 1,2)\) is

\[
\begin{align*}
E_{c,jdm}^e & = \beta_m^i E_{c,jdm}^{\psi_m} + \psi_m E_{c,jdm}^m
\end{align*}
\]

\[
\begin{align*}
E_{c,jdm}^{\psi_m} & = \begin{bmatrix} D_{s,jdm}^c & D_{2,jdm}^c \end{bmatrix} \begin{bmatrix} E_{\beta_m^i}^{i} (Q_c) \end{bmatrix} e^{-jkr} r
\end{align*}
\]

\[
D_{s,jdm}^c = (-1)^{m-1} \frac{j}{8\pi k} \left( \frac{1}{\cos \beta_m^i - \cos \beta_m^i} \right)
\begin{bmatrix}
\sin \beta_m^i & -\sin \psi_m & \sin \beta_m^i
\end{bmatrix}
\]

\[
C_{n_2}^2 (\eta, \eta^i) = \frac{\left( \cot \beta_m^i \cos \psi_m - \cot \beta_m^i \cos \alpha_m \right) C_{n_2}^h (\eta, \eta^i)}{\sin \beta_m^i + \left( \cot \beta_m^i \cos \alpha_m \right) \cos \beta_m^i}.
\]

The functions \( C_{np}^s, C_{np}^h, C_{n_1}^2, S^i, \) and \( \cos^{-1} \) have been defined previously in Equations (B.7), (B.8), (B.9), (B.13), and (B.11), and the angles are defined in Figure 70. The details of the derivation may be found in [15] or [16].

The contribution from the vertex diffracted current is

\[
\begin{align*}
E_{c,jv}^e & = \frac{jk Z_0 e^{-jkr}}{4\pi} \frac{1}{r} \int_0^{\alpha_v} \int_0^{r_1} \hat{r} \times \hat{r} \times \tilde{j}_{dv} (\hat{r}_v) e^{jk \hat{r}_v \cdot \hat{r}} d\phi_c^i d\phi_c^j
\end{align*}
\]

where

\[
\tilde{j}_{dv} (\hat{r}_v) = \tilde{j}_{AS} (\hat{r}_v) - \tilde{j}_{PO} (\hat{r}_v) - \tilde{j}_{d_1} (\hat{r}_v) - \tilde{j}_{d_2} (\hat{r}_v)
\]
The angles are defined in Figure 31. Expressions for the currents used in Equation (B.20) are given in Equations (A.43), (2.1), and (2.2). The number of eigenvalues and the number of terms used in calculating the eigenfunctions are \( N_{ev} = 7 \) and \( N_{ef} = 9 \) to calculate \( J^{A_S}(r') \) for the surface integral in Equation (B.19), \( N_{ev} = 15 \) and \( N_{ef} = 15 \) to calculate \( J^{A_S}(r'_1) \) for the line integral in Equation (B.19), and \( N_{ev} = 25 \) and \( N_{ef} = 15 \) to calculate \( J^{A_S}(r'_2) \) for the line integral in Equation (B.19). The values of the parameters used in all of the calculations are \( r_1' = 0.2 \lambda, r_2' = 0.3 \lambda, B_{min}' = 0.5, \) and \( B_{max}' = 1.5 \) which have been found to give fairly accurate results, while keeping the computational time required reasonable.

The integrations in Equation (B.19) must be done numerically. A simple Riemann sum was used to do the integrations using 4 steps for the radial part of the surface integral, 20 steps for the angular part of the surface integral, and 20 steps for the line integration.

The contribution from the edge diffracted current of edge \( m \) (\( m = 1, 2 \)) terminated by the other edge (\( m = 2, 1 \)) is

\[
\vec{E}_{c,j} dt' = \vec{E}_{1,j} dt' = \vec{E}_{B1}^C dt' + \vec{E}_{\psi_1}^C dt'
\]

\[(B.25)\]
\[ E^c_{\psi m} = E^c_{\psi m} = g \beta_2 E^c_{\psi m} + \psi^i E^c_{\psi m} \]

where

\[ E^c_{\psi m} = -V(m) \frac{j \sin \left( \beta^i_m - \alpha^i \right)}{2 \pi k \sin^3 \beta^i_m} \left\{ E^i_m(Q_c) \frac{\sin \beta^i_m}{\sin \beta^i_m} \omega(2) \left( \beta^i_m, \psi^i_m; \beta^i_m, \psi^i_m \right) \right. \\
+ Z_0 H^i_m(Q_c) \left( \cos \beta^i_m \cos \psi^i_m \omega(0) \left( \beta^i_m, \psi^i_m; \beta^i_m, \psi^i_m \right) \right) \\
+ \sin \beta^i_m \cot \beta^i_m \omega(1) \left( \beta^i_m, \psi^i_m; \beta^i_m, \psi^i_m \right) \left\} e^{-jkx} \]

(B.27)

\[ E^c_{\psi m} = V(m) \frac{j Z_0 H^i_m(Q_c) \sin \left( \beta^i_m - \alpha^i \right) \sin \psi^i_m}{2 \pi k \sin^3 \beta^i_m} \omega(0) \left( \beta^i_m, \psi^i_m; \beta^i_m, \psi^i_m \right) \frac{e^{-jkx}}{r} \]

(B.28)

where

\[ V(m) = \begin{cases} U \left( \beta^i_1 - \alpha^i \right) ; m = 1 \\ -U \left( \beta^i_2 + \alpha^i - \pi \right) ; m = 2 \end{cases} \]

(B.29)

\[ E^i_m(Q_c) = \dot{x}_{cm} \cdot E^i(Q_c) \]

(B.30)

\[ H^i_m(Q_c) = \dot{x}_{cm} \cdot H^i(Q_c) \]

(B.31)

\[ \omega(0) \left( \beta, \psi; \beta^i, \psi^i \right) = \frac{U \left( \pi - \psi^i \right)}{\sin \alpha^i \left( \cos \alpha + \cos \psi^i \right) \left( \cos \delta + \cos \psi^i \right)} \]

\[- \frac{\sin^2 \beta^i}{2 \sin \left( \beta^i - \alpha^i \right) \left( \cos \beta - \cos \beta^i \right)} \cdot \left[ \frac{\csc \alpha \sin \left( \frac{\pi - \alpha}{2} \right)}{\cos \frac{\psi^i}{2} - \cos \left( \frac{\pi - \alpha}{2} \right)} - \frac{\csc \delta \sin \left( \frac{\pi - \delta}{2} \right)}{\cos \frac{\psi^i}{2} - \cos \left( \frac{\pi - \delta}{2} \right)} \right] \]

(B.32)

\[ \omega(1) \left( \beta, \psi; \beta^i, \psi^i \right) = \frac{U \left( \pi - \psi^i \right) \cos \psi^i}{\sin \alpha^i \left( \cos \alpha + \cos \psi^i \right) \left( \cos \delta + \cos \psi^i \right)} \]

\[ + \frac{\sin^2 \beta^i}{2 \sin \left( \beta^i - \alpha^i \right) \left( \cos \beta - \cos \beta^i \right)} \cdot \left[ \frac{\cot \alpha \sin \left( \frac{\pi - \alpha}{2} \right)}{\cos \frac{\psi^i}{2} - \cos \left( \frac{\pi - \alpha}{2} \right)} - \frac{\cot \delta \sin \left( \frac{\pi - \delta}{2} \right)}{\cos \frac{\psi^i}{2} - \cos \left( \frac{\pi - \delta}{2} \right)} \right] \]

(B.33)
\[
\varpi^{(2)} (\beta, \psi; \beta', \psi') = \frac{-U \left( \pi - \psi \right) \sin \psi}{\sin \alpha \left( \cos \alpha + \cos \psi \right) \left( \cos \delta + \cos \psi \right)} \\
+ \frac{2 \sin \beta \sin \beta'}{2 \sin \left( \beta - \alpha \right) \left( \cos \beta - \cos \beta' \right)} \cdot \left[ \frac{\sin \left( \frac{\pi - \alpha}{2} \right)}{\cos \beta - \cos \left( \frac{\pi - \alpha}{2} \right)} - \frac{\sin \left( \frac{\pi - \alpha}{2} \right)}{\cos \psi - \cos \left( \frac{\pi - \alpha}{2} \right)} \right] \\
\cos \alpha = \frac{\sin \beta \sin \beta' \left( \cos \beta - \cos \beta' \right)}{\sin \alpha \sin \beta'} \\
\cos \delta = \frac{\sin \alpha \sin \beta \cos \psi + \cos \alpha \left( \cos \beta - \cos \beta' \right)}{\sin \alpha \sin \beta'}
\]

The details of the derivation may be found in [17].

The contribution from the double diffracted current terminated by the last diffracting edge is

\[
\tilde{E}_{c,jd1d2} = \tilde{E}_{m}^{c,jd} \\
\tilde{E}_{c,jd1d2} = \tilde{E}_{m}^{c,jd}
\]

where

\[
\tilde{E}_{m}^{c,jd} = E_{m}^{i} (Q_c) V (m) \frac{i \pi}{4 (2\pi k)^{3/2}} \frac{\sin \beta'}{\sin \beta} \sin \beta_{m} \sin \beta'_{m} \left( \sin \beta_{m} \sin \beta'_{m} \right) \\
\cdot \left[ \beta_{m}^{i} I_{h} \left( \beta_{m}, \psi_{m}; \beta' \right) - \psi_{m} M_{h} \left( \beta_{m}, \psi_{m}; \beta' \right) \right] \\
\cdot \sum_{p=1}^{2} \cot \left( \frac{\pi + (-1)^{p} \psi_{m}^{i}}{4} \right) F_{c} \left( \alpha_{p} \left( \psi_{m}^{i}, 1 - v \right) e^{-jkr} \right)
\]

where

\[
I_{h} \left( \beta, \psi; \beta' \right) = \frac{1}{\sin \beta} \left\{ \frac{\cos \beta \cos \psi + \cot \beta'}{\cos \alpha + 1} \right\} \\
+ \frac{\cos \alpha \cot \beta' - \cos \beta \cos \psi}{\sin \alpha} \cot \left( \frac{\pi - \alpha}{4} \right) \\
M_{h} \left( \beta, \psi; \beta' \right) = \frac{\sin \psi}{\sin \beta \sin \beta'} \left\{ \frac{1}{\cos \alpha + 1} - \frac{1}{\sin \alpha} \cot \left( \frac{\pi - \alpha}{4} \right) \right\}
\]
\[
\cos \alpha = \frac{\sin \beta \cos \psi + \cos \beta' (\cos \beta - \cos \beta')}{\sin \beta' \sin \beta'} \quad (B.42)
\]

\[
F^c (a, 1 - v) = \frac{\sqrt{a} \sqrt{1 - v}}{\sqrt{a} + \sqrt{1 - v}} \quad (B.43)
\]

\[
\sqrt{1 - v} = \begin{cases} 
|\sqrt{1 - v}| ; & 1 - v > 0 \\
-j|\sqrt{1 - v}| ; & 1 - v < 0 
\end{cases} \quad (B.44)
\]

\[
v = \frac{\sin \alpha_v \sin \beta_m \cos \psi_m + \cos \alpha_v (\cos \beta_m - \cos \beta')}{\sin \alpha_v \sin \beta'} \quad (B.45)
\]

\[
\beta' = (-1)^m \left( \pi - \beta_m^i \right) - \alpha_v \quad (B.46)
\]

\[
a_p (\psi^i) = 1 + \cos \left[ 4N_p \pi - (-1)^p \psi_m^i \right] \quad (B.47)
\]

and \(N_p\) is the integer most nearly satisfying

\[
4N_p \pi - (-1)^p \psi_m^i = \pi. \quad (B.48)
\]

The angles are defined in Figure 70, and the details of the derivation may be found in [17].

### B.2 Properties

#### B.2.1 Singularities

The corner diffracted field described here becomes singular in several different directions. In most cases, these singularities are canceled by corresponding singularities from other corners on a finite plate. Other singularities correspond to caustics which are not presently accounted for and may cause problems over small regions of a pattern. The singularities in the corner diffracted field are:

- The so-called Ufimtsev singularity. The components \(\tilde{E}^{c,jp_0}\), at least one of \(\tilde{E}^{c,j_{d1}}\) or \(\tilde{E}^{c,j_{d2}}\), \(\tilde{E}^{c,j_{v}}\), and in some cases \(\tilde{E}^{c,j_{d1t2}}\) and/or \(\tilde{E}^{c,j_{d2t1}}\) become singular when the diffracted ray is in the same direction as the incident ray.
and the incident ray is in the plane of the angular sector, but not on the angular sector (i.e. forward scatter for glancing incidence).

- The components $\vec{E}^{c,jd_1}$ and in some cases $\vec{E}^{c,jd_1t_2}$ are singular when the observation direction is on the intersection of the half plane associated with edge 1 and the Keller cone from edge 1 ($\beta_1 = \beta^i_1$ and $\psi_1 = 0$). Similarly, the components $\vec{E}^{c,jd_2}$ and in some cases $\vec{E}^{c,jd_2t_1}$ are singular when the observation direction is on the intersection of the half plane associated with edge 2 and the Keller cone from edge 2 ($\beta_2 = \beta^i_2$ and $\psi_2 = 0$). Further details involving this singularity are illustrated in [16].

- The components $\vec{E}^{c,jpo_1}$, $\vec{E}^{c,jd_1}$, and in some cases $\vec{E}^{c,jd_1t_2}$ are singular when the observation direction is on the Keller cone associated with edge 1 ($\beta_1 = \beta^i_1$). Similarly, the components $\vec{E}^{c,jpo_2}$, $\vec{E}^{c,jd_2}$, and in some cases $\vec{E}^{c,jd_2t_1}$ are singular when the observation direction is on the Keller cone associated with edge 2 ($\beta_2 = \beta^i_2$). For observation points off of the plate, neither of these singularities is a problem in practice since the singularities are cancelled by corresponding singularities in the corner diffracted field at the other corner along the same edge, in the case of a finite plate with straight edges (see [16] for an explanation). For observation points on the plate the singularity does cause some minor problems as described in [16].

- The component of $\vec{E}^{c,jp_1}$ in the $\hat{\phi}_s$ direction is in error for field points on the angular sector. This may be seen by observing that the integrand in the third integral in Equation (B.19) would be singular for $\theta_s = 90^\circ$ and $\phi^i_c = \phi_s$ if the parameter $B(\phi^i_c)$ had not been introduced. The integrand involving the $\hat{r}'$ component of the current is not singular since there is a
corresponding zero in the term \( \hat{r} \times \hat{r} \times \hat{r}' \). Notice that the stationary phase evaluation of the integral is not valid here, since the phase term \( e^{-jkr'(1-B)} \) is one \( B = 1 \). In the important special case of backscatter this is not a problem for a \( \phi_i \) polarized source since the factor \( A(\phi'_c) \) in Equation is zero. The factor \( A(\phi'_c) \) is zero since \( J^{AS}, J^{PO}, J^{d1}, \) and \( J^{d2} \) are all zero for a \( \phi_i \) polarized source on the angular sector. The cross polarized field term for a \( \theta_i \) polarized source and a \( \phi_s \) polarized receiver will be nonzero resulting in an erroneous calculation of the cross polarized backscatter.

- The component \( \vec{E}_{c,jd2d1} \) is singular when the observation direction corresponds to the intersection of the Keller cone associated with the double diffraction from edge 1 and the half plane associated with edge 1 \( (\beta_1 = \pi - (\alpha_v + \beta_1^i), \psi_1 = 0) \). The component \( \vec{E}_{c,jd1d2} \) is singular when the observation direction corresponds to the intersection of the Keller cone associated with the double diffraction from edge 2 and the half plane associated with edge 2 \( (\beta_2 = \pi + \alpha_v - \beta_1^i, \psi_2 = 0) \).

- All of the terms in the corner diffracted field become singular for edge on illumination or observation. For finite plates, it seems reasonable that if the complete corner-corner interactions could be included, the corner diffracted and corner-corner diffracted fields, both infinite, would sum to a finite result. It is believed that this will occur in a manner similar to the case when the two corner diffracted fields along a single edge sum to give a finite result when the observation point lies on the Keller cone even though the individual corner diffracted fields are infinite.
B.2.2 Boundary Conditions

The corner diffracted field meets the boundary conditions on the angular sector for a source on the angular sector, but does not meet the boundary conditions for a receiver on the angular sector. The corner diffracted field is zero for a source polarized tangent to the angular sector and lying on the angular sector. The corner diffracted field on the angular sector and tangent to the angular sector is finite, but nonzero for an arbitrarily polarized incident field. The two previous statements imply that the boundary conditions are met on the angular sector for the important special case of backscatter.

B.2.3 Reciprocity

The corner diffracted field should be reciprocal. Several examples showing how close the calculations come to being reciprocal for the special case of backscatter are shown in Section 3.2.2. Although they are not in Section 3.2.2, calculations made using the previous corner diffraction coefficient are not any closer to giving a reciprocal answer and are worse in some cases.
BIBLIOGRAPHY


[4] T. B. Hansen, "An attempt to obtain a useful representation for the field scattered by a plane angular sector when both source and field points are far from the corner," Technical Report R 426, Technical University of Denmark, Electromagnetics Institute, March 1990.


