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Resonance phenomena in viscous fluid configurations inside a spinning and coning cylinder

Selmi, Mohamed, Ph.D.
The Ohio State University, 1991

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Resonance Phenomena in Viscous Fluid Configurations inside a Spinning and Coning Cylinder

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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Abstract

The moments exerted by viscous fluids in cylindrical payloads can cause severe flight instabilities of liquid-filled projectiles. This study is concerned with the calculations of these moments for a spinning and coning payload having the following configurations: (1) A single fluid in a completely filled cylinder. (2) A single fluid in a cylinder containing a coaxial rod. (3) A single fluid in a partially filled cylinder. Small percentages of air inclusion during production essentially lead to this case. (4) Two immiscible fluids of different density and viscosity in a completely filled cylinder. Since a negative roll moment is caused by shear stresses at the cylinder wall, a heavy low-viscosity additive can reduce this moment. The viscous fluid motions, and consequently the moments, are analyzed with numerical methods based on spatial eigenfunction expansions (for the linearized problems) or spectral techniques (for the linear and nonlinear problems). For completely filled cylinders, it is found that the yaw and roll moments acquire maxima at critical aspect ratios (i.e. ratio of length to diameter of the cylinder) as the Reynolds number is increased. Similar results are found for the other configurations, but now at critical fill ratios (i.e. ratio of the volume of one fluid to the volume of the cylinder) for a given aspect ratio. These maxima are due to resonant inertial
waves. The inviscid fluid motions for the various configurations are also analyzed by eigenfunction expansions for the linearized equations and their solutions, in analytical form, provide criteria for the onset of resonance. Spectral techniques are used to investigate the effect of nonlinearities on the moments, and to calculate the pitch moment accurately since unlike the roll and yaw moments it is only partially predicted by linear theory. Comparisons of the linear and nonlinear results reveal that nonlinear effects are negligible for all practical applications.
To My parents

Magtouf and F'tima Selmi
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During this study I have used the computers and visualization equipments in the computational and visual fluid dynamics lab. Charlotte Hawley has kept these machines running and I called on her whenever I had a problem with them. Her help is greatly appreciated. Part of the computations was done on the Cray Y-MP8/864 of the Ohio Supercomputer Center.
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Finally I am in debt to my family in Tunisia for their love, understanding, and consistent encouragement. I have put them through a lot of suffering by being far away from them, and I am very appreciative for their forgiveness.
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List of Symbols

\( a \)  
Dimensional cylinder radius and reference length

\( a_0 \)  
Dimensional rod or fill radius

\( 2c \)  
Dimensional cylinder length

\( \omega \)  
Spinning rate and time reference

\( \Omega \)  
Coning rate

\( \eta \)  
Aspect ratio, \( \eta = c/a \)

\( r_0 \)  
Dimensionless rod or fill radius, \( r_0 = a_0/a \)

\( \theta \)  
Nutation angle

\( \tau \)  
Coning frequency, \( \tau = \Omega/\omega \)

\( \epsilon \)  
\( \epsilon = \tau \sin \theta \)

\( \tau_r \)  
\( \tau_r = -\epsilon \cos \phi \)

\( \tau_\phi \)  
\( \tau_\phi = \epsilon \sin \phi \)

\( \tau_z \)  
\( \tau_z = \tau \cos \theta \)

\( \tau_\theta \)  
\( \tau_\theta = 2(1 + \tau_z) \)

\( \rho_0 \)  
Density of inner fluid

\( \rho_1 \)  
Density of outer fluid

\( \nu \)  
Kinematic viscosity

\( \nu_0 \)  
Kinematic viscosity of inner fluid

\( \nu_1 \)  
Kinematic viscosity of outer fluid

\( \mu \)  
Fluid viscosity

\( \mu_0 \)  
Viscosity of inner fluid
\( \mu_1 \) Viscosity of outer fluid

\( Re \) Reynolds number, \( Re = \omega a^2/\nu \)

\( Re_0 \) Inner flow Reynolds number, \( Re_0 = \omega a^2/\nu_0 \)

\( Re_0 \) Outer flow Reynolds number, \( Re_1 = \omega a^2/\nu_1 \)

\( V \) Total cylinder volume

\( V_0 \) Volume of inner fluid

\( V_1 \) Volume of outer fluid

\( V_1/V \) Fill ratio

\( \rho_0/\rho_1 \) Density ratio

\( a_f \) Dimensional radial location of void-fluid or two-fluid interface

\( r_f \) Dimensionless radial location of void-fluid or two-fluid interface, \( r_f = a_f/a \)

\( \zeta \) Dimensionless radial deviation of interface from the surface \( r = r_0 \), \( r_f = r_0 + \zeta(\phi, z) \)

\((X, Y, Z)\) Inertial Cartesian coordinate system

\((i, j, k)\) Unit vectors associated with the \((X, Y, Z)\) system

\((x, y, z)\) Coning/Nutating Cartesian coordinate system

\((i, j, k)\) Unit vectors associated with the \((x, y, z)\) system

\((r, \phi, z)\) Coning/Nutating Cylindrical coordinate system

\((e_r, e_\phi, e_z)\) Unit vectors associated with the \((r, \phi, z)\) system

\( \Omega \) Angular coning velocity, \( \Omega = \Omega k = \Omega k \)

\( V \) Total velocity, \( V = v + v' \)

\( V_0 \) Inner fluid total velocity, \( V_0 = v_0 + v_0' \)

\( V_1 \) Outer fluid total velocity, \( V_1 = v_1 + v_1' \)
Dimensionless total pressure, \( P = -\frac{1}{8} \tau_0^2 r_0^2 + p^r + p^d \)

Inner fluid dimensionless total pressure, \( P_0 = -\frac{1}{8} \tau_0^2 r_0^2 + p_0^r + p_0^d \)

Outer fluid dimensionless total pressure, \( P_1 = -\frac{1}{8} \tau_0^2 r_0^2 + p_1^r + p_1^d \)

Perturbation pressure

Inner fluid perturbation pressure

Outer fluid perturbation pressure

Velocity deviation from solid body rotation, \( v = v_r e_r + v_\phi e_\phi + v_z e_z \)

Inner fluid velocity deviation from solid body rotation, \( v_0 = v_0^1 e_r + v_0^1 e_\phi + v_0^1 e_z \)

Outer fluid velocity deviation from solid body rotation, \( v_1 = v_1^1 e_r + v_1^1 e_\phi + v_1^1 e_z \)

Velocity due to solid body rotation, \( v^r = \omega e_\phi \)

Total moment

Moment caused by flow deviation from rigid-body motion, \( M = M_x i + M_y j + M_z k \)

Yaw moment caused by flow deviation from rigid-body motion

Pitch moment caused by flow deviation from rigid-body motion

Roll moment caused by flow deviation from rigid-body motion

\( \omega' = \omega + \Omega \cos \theta \)

\( \tau' = \frac{\Omega}{\omega'} = \tau/(1 + \tau_z) \)

\( Re' = \frac{\omega' a^2}{\nu} \)

Reference velocity

Reference pressure
$\rho \omega^2 a^5$  Reference moment
CHAPTER I

Introduction

1.1 Objective

In this era of increasing demand in space exploration and advancing aerospace technology, the study of rotating flows becomes increasingly important to gain insights into how to design aerospace vehicles or devices that involve such flow phenomena. Liquid-filled projectiles are examples of such devices. They are spun for the purposes of reducing aerodynamic drag forces on one hand and to assuring dynamical stability on the other. When released into the air after the initial spin-up motion, the projectile acquires a combined motion of spinning and nutating under the action of aerodynamic forces. The coning motion of the projectile about its trajectory is primarily responsible for the deviation of the motion of the liquid payload from solid-body rotation.

Liquid-filled projectiles are known to experience severe dynamical instabilities due to the motion of their liquid payloads. These instabilities are characterized by a rapid growth of the yaw angle and sometimes a substantial loss in the spin rate.
Two types of instabilities are currently understood for cylindrical containers with complete fill of a homogeneous fluid. Instability can be caused by resonance with inertial waves at critical coning frequencies (ratio of coning rate $\Omega$ to spinning rate $\omega$). This instability is most pronounced for low-viscosity fluids and is known to strongly depend on the aspect ratio (ratio of length to diameter) of the payload cylinder. For a given frequency, the aspect ratio is usually properly chosen to avoid resonance. The second type of instability is related to the viscous stresses exerted by the fluid on the walls of the container. This type of instability is associated with a rapid despin of the container and is most pronounced for high-viscosity fluids.

Our main goal in studying the above flow configurations is to gain insight and analytical capabilities for the design of stable configurations. While a central rod or partial fill may serve to intentionally change liquid moments and resonant frequencies, partial fills are also frequent off-design products. Fills with two immiscible fluids, and with a low-viscosity fluid in contact with the side wall might lessen the viscous shear stresses and thereby reduce the despin rate and ultimately eliminate the viscous type instability.

1.2 Literature Review

The analysis of liquid moments for engineering design is based throughout on the quasi-steady motion in the aeroballistic system where the coning and spin rates are constant. Theoretical and computational analyses of the flow in liquid-
filled cylinders in the large range of relevant Reynolds numbers require proper approximations. The boundary-layer approximation is the basis of the Stewartson-Wedemeyer theory (Stewartson 1959; Wedemeyer 1966). Since this approximation is only valid for flows at sufficiently large Reynolds numbers, the theory is primarily suited to predict the instability caused by inertial waves. Analysis based on the Navier-Stokes equations (Herbert & Li 1987, Herbert & Li 1990) shows, however, that resonance with inertial waves may severely influence the liquid moments at Reynolds numbers as low as $Re = 100$.

Another common approximation is the linearization of the Navier-Stokes equations when the nutation angle is sufficiently small. Linearization based on the parameter $\epsilon = \tau \sin \theta$, where $\tau$ is the coning frequency $\Omega/\omega$ and $\theta$ is the nutation angle, and the assumption of infinitely long cylinders are the basis of the analytical approach undertaken by Herbert (1985). Though valid for any Reynolds number, this theory can only hold for sufficiently large aspect ratios and it cannot be utilized to predict instability caused by resonance with inertial waves which depends on this length. However, estimates of the moments based on this theory for cylinders of large aspect ratios can be obtained at negligible cost.

An alternative approach to solving the linearized Navier-Stokes equations has been suggested by Hall, Sedney, and Gerber (1987, 1990). This approach expands velocity components and pressure in a series of products of trigonometric functions in axial direction and radial “eigenfunctions” that satisfy homogeneous boundary
conditions at the side wall. The expansion coefficients of the series can be found from the boundary conditions at the end walls by collocation or least squares method. While this method has the potential of treating both types of instabilities, its shortcoming is in the numerical determination of eigenvalues and eigenfunctions. Practical application is restricted to the range up to $Re = 1000$ and CPU times of $10 - 1800$ seconds per solution on a VAX 8600 computer.

For given parameters, the eigenvalues for the expansion of Hall et al. (1987) are obtained by iterative solution of a sixth-order complex system of ordinary differential equations. Good initial guesses are required for the iteration to converge. This problem is currently overcome by precalculating voluminous tables for interpolation of the initial estimates. The generation of these tables requires approximately 40 hours CPU time on a Cray. While this approach is successful for completely filled cylinders at moderate Reynolds numbers, the computational expense increases dramatically with the Reynolds number and for the configurations studied here.

We develop here an alternative approach to calculating the moments from the linearized Navier-Stokes equations. This approach is based on the observation that when using a control volume analysis to calculate the moments, these moments depend essentially on the axial velocity. We derive a single sixth-order partial differential equation for the axial velocity component. The solution to this equation is expanded in spatial eigenfunctions in the axial direction and Fourier series in
the azimuthal direction, while the radial structure is expressed in terms of Bessel functions. The eigenfunctions are given in closed form and the eigenvalues are determined by numerically solving a closed form characteristic system of equations.

The expansion coefficients are found by satisfying the boundary conditions at the side walls using either collocation or least-squares methods. This approach proves to be computationally efficient and flexible in treating other flows such as partial fill and two-fluid flows. The eigenfunctions (in the axial direction) need only be computed once to treat different configurations. The only difference is the change in boundary conditions in the radial directions.

Full Navier-Stokes solvers for completely filled cylinders have been developed by a number of investigators. Vaughn et al. (1985) employed finite difference techniques and Chorin’s method of artificial compressibility to solve for the steady flow quantities by integrating typically over $10^4$ to $8 \cdot 10^4$ time steps. Strikwerda & Nagel (1985) developed a code using finite differences in the radial and axial directions, and pseudospectral differencing in the azimuthal direction. The difference equations are solved by the method of successive over relaxation. Rosenblat et al. (1986) developed a finite element code to study the effect of viscoelasticity at small Reynolds numbers. Herbert & Li (1987, 1990) employed spectral techniques to solve the steady state Navier-Stokes equations. Each flow quantity is approximated by a triple series of Chebyshev polynomials in the radial and axial directions and Fourier functions in the azimuthal direction. The expansion coefficients are
determined by satisfying the governing equations and boundary conditions by collocation. This approach proved to be very powerful in studying the anatomy of the flow in completely filled cylinders and in predicting moments. We will use this approach here to study the effect of nonlinearity on the moments for the various configurations and for the purpose of flow visualizations.
CHAPTER II

Mathematical formulations

2.1 Governing equations

We consider the flows of incompressible viscous fluids in various configurations inside a cylinder of radius $a$ and length $2c$. The cylinder can be completely or partially filled with a single fluid, contain a central rod of radius $a_0 < a$ and one fluid, or contain two immiscible fluids of different densities and viscosities. The cylinder rotates about its axis at the spin rate $\omega$ and rotates at the coning rate $\Omega$ about a nutation axis that passes through its center. The angle between nutation axis and spin axis is denoted by $\theta$ as shown in figure 2.1. For this study, it is assumed that the coning rate $\Omega$, the spin rate $\omega$, and the nutation angle $\theta$ are constant.

Under the influence of centrifugal forces, the heavy fluid accumulates at the side walls while the lighter fluid or the void surrounds the cylinder axis. For convenience, we adopt the notion of an inner region, characterized by the index $0$, that contains the inner fluid, void, or the central rod and an outer region,
characterized by the index 1, that contains the outer fluid. In case of completely filled cylinders, there is only one region and it can be either the inner or outer region. While we need to solve for the inner solution in case of two-fluid flow, the inner solution for the flow in partially filled cylinders is meaningless and that in cylinders with a central rod is due to solid body rotation and therefore is known. The location of the void-fluid or two-fluid interface in cases of flows in partially
Figure 2.2: Nomenclature sketch and description of the various configurations.

filled cylinders or two-fluid flows needs to be determined. Figure 2.2 shows a
description of the various configurations and the nomenclature used in this study.

For the moments, we use Cartesian coordinates \( x, y, z \) where \( z \) is the spin axis
and \( x \) is normal to \( z \) and coplanar with both the spin and nutation axes. However, the flow quantities are more conveniently expressed in cylindrical coordinates
The governing equations for the various flows represent both conservation of mass and momentum balance. When written with respect to the nutating system \((x, y, z)\) which rotates about the \(Z\)-axis of the inertial system \((X, Y, Z)\) at the rate \(\Omega\), they take the form

\[
\nabla \cdot \mathbf{V} = 0, \tag{2.1}
\]

\[
\rho \left[ \frac{D\mathbf{V}}{Dt} + 2\Omega \times \mathbf{V} + \Omega \times (\Omega \times \mathbf{r}) \right] = -\nabla P + \mu \nabla^2 \mathbf{V}, \tag{2.2}
\]

where \(\mathbf{V}\) and \(P\) denote the velocity and pressure fields respectively, \(\mathbf{r}\) the position vector, \(\rho\) the fluid density, and \(\mu\) the fluid viscosity. Equations (2.1) and (2.2) are valid for both inner and outer flows with the appropriate fluid density and viscosity. They are also subject to the no-slip conditions at the walls and to the state of equal stresses at the interface of inner fluid and outer fluid.

The flow quantities are made dimensionless by using \(a\) to scale length, \(\omega\) to scale time, and \(\rho\) to scale mass. As a result, the various flows are found to depend on the aspect ratio \(\eta = c/a\), the coning frequency \(\tau = \Omega/\omega\), the nutation angle \(\theta\), the density ratio of both fluids \(\rho_0/\rho_1\), the fill ratio \(V_1/V\) (ratio of the volume of outer fluid to the volume of the cylinder), and the Reynolds number of each flow, i.e. \(Re_0 = \rho_0 \omega a^2/\mu_0\), and \(Re_1 = \rho_1 \omega a^2/\mu_1\), or the Reynolds number of one flow, say \(Re_1\), and the viscosity ratio of both fluids \(\mu_0/\mu_1\). Depending on the type of flow configuration, some of these parameters become degenerate.

The dimensionless radius of the central rod or the radial location of the void-fluid or two-fluid interface when the cylinder is only spinning is also a measure of
the fill ratio and we denote this quantity by $r_0$. We will refer to this quantity as the fill radius. When the cylinder is also coning, the radial interface location is no longer constant but takes the dimensionless form

$$r_f = r_0 + \zeta(\phi, z),$$

(2.3)

where $\zeta$ is the deviation of the interface radial location from the axisymmetric surface $r = r_0$.

Moreover, it is convenient to split the velocity and pressure fields according to

$$\mathbf{V} = \mathbf{v}^r + \mathbf{v},$$

(2.4)

$$P = -\frac{1}{2}(1 + \tau_z)^2 r_0^2 + p^r + p^d,$$

(2.5)

where $\mathbf{v}^r$ is the velocity due to solid body rotation, in cylindrical coordinates $\mathbf{v}^r = re_\phi$ where $e_\phi$ is a unit vector in the azimuthal direction, and $p^r$ is chosen so that the forcing terms in the governing equations appear only in the $(z)$-momentum equation,

$$p^r = \frac{1}{2}[r^2(1 + \tau_z)^2 + r^2 \tau_\phi^2 + z^2 \epsilon^2 - 2rz \tau_z \tau_r],$$

(2.6)

where $\tau_r = -\epsilon \cos \phi$, $\tau_\phi = \epsilon \sin \phi$, $\tau_z = \tau \cos \theta$, and $\epsilon = \tau \sin \theta$.

For convenience and in order to treat all flow configurations in a similar fashion, we define the flow quantities to span both inner and outer regions. In cylindrical coordinates $(r, \phi, z)$, we represent the velocity deviation from solid body rotation by

$$\mathbf{v} = (v_r, v_\phi, v_z) = \begin{cases} \mathbf{v}_0 = (v_r^0, v_\phi^0, v_z^0) & \text{if } 0 \leq r \leq r_f, \\ \mathbf{v}_1 = (v_r^1, v_\phi^1, v_z^1) & \text{if } r_f \leq r \leq 1, \end{cases}$$

(2.7)
and the pressure perturbation by

\[ p^d = \begin{cases} 
  p_0^d & \text{if } 0 \leq r \leq r_f, \\
  p_1^d & \text{if } r_f \leq r \leq 1.
\end{cases} \quad (2.8) \]

Similarly, we define the density and Reynolds number according to

\[ \rho = \begin{cases} 
  \rho_0 & \text{if } 0 \leq r \leq r_f, \\
  \rho_1 & \text{if } r_f \leq r \leq 1,
\end{cases} \quad (2.9) \]

\[ Re = \begin{cases} 
  Re_0 & \text{if } 0 \leq r \leq r_f, \\
  Re_1 & \text{if } r_f \leq r \leq 1.
\end{cases} \quad (2.10) \]

The equations governing the velocity components \( v_r, v_\phi, \) and \( v_z \) of the deviation from rigid-body motion and the perturbation pressure \( p^d \) take the form

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0, \quad (2.11) \]

\[ D'v_r - \frac{v_\phi^2}{r} - 2(1 + \tau_z)v_\phi + 2\tau_r v_z = -\frac{\partial p^d}{\partial r} + \frac{1}{Re} \left[D''v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi}\right], \quad (2.12) \]

\[ D'v_\phi + \frac{v_r v_\phi}{r} + 2(1 + \tau_z)v_r - 2\tau_r v_z = -\frac{1}{r} \frac{\partial p^d}{\partial \phi} + \frac{1}{Re} \left[D''v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi}\right], \quad (2.13) \]

\[ D'v_z + 2\tau_r v_\phi - 2\tau_\phi v_r = -\frac{\partial p^d}{\partial z} - 2r_\tau_r + \frac{1}{Re} D''v_z, \quad (2.14) \]

where

\[ D' = \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} + v_r \frac{\partial}{\partial \phi} + v_\phi \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}, \]

and

\[ D'' = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \]

The only forcing of the flow quantities comes from the term \(-2r_\tau_r = 2\varepsilon r \cos \phi\) present in the \( z \)-momentum equation and contains the primary effect of Coriolis
forces. When this term vanishes, \( \epsilon = 0 \), the governing equations admit the trivial solution \( v = 0, p^d = 0 \). Hence, the velocity deviation from solid body rotation is \( O(\epsilon) \) (Herbert 1985). Moreover, if \((v_r, v_\phi, v_z, p^d)\) is the solution at \((r, \phi, z)\), the solution at \((r, \phi + \pi, -z)\) is \((v_r, v_\phi, -v_z, p^d)\). These symmetries are exploited to save computational power.

2.2 Linearized governing equations

For ideal flights, projectiles are spun up to 6000 rpm and soon after they are fired, flight tests have shown that they acquire a coning rate of about 500 rpm and a coning angle of less than 20° (Miller 1981, 1982, 1989, 1991). Thus, for practical purposes the parameter \( \epsilon \) is small, i.e. \( \epsilon \leq 0.057 \) for \( \theta \leq 20^\circ \), \( \Omega \leq 500 \) rpm, and \( \omega \geq 3000 \) rpm. Since the flow quantities are \( O(\epsilon) \), then for sufficiently small \( \epsilon \), it is well justified (Herbert 1985) to use this parameter to linearize the governing equations. When this is done we obtain

\[
\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0, \tag{2.15}
\]

\[
D^* v_r - 2(1 + \tau_z) v_\phi = -\frac{\partial p^d}{\partial r} + \frac{1}{Re} \left[ D'' v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} \right], \tag{2.16}
\]

\[
D^* v_\phi + 2(1 + \tau_z) v_r = -\frac{1}{r} \frac{\partial p^d}{\partial \phi} + \frac{1}{Re} \left[ D'' v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} \right], \tag{2.17}
\]

\[
D^* v_z = -\frac{\partial p^d}{\partial z} - 2r \tau_r + \frac{1}{Re} D'' v_z, \tag{2.18}
\]

where

\[
D^* = \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi}.
\]
These equations support the additional symmetries:

\[ \mathbf{v}(r, \phi + \pi, z) = -\mathbf{v}(r, \phi, z), \quad (2.19) \]

\[ p^d(r, \phi + \pi, z) = -p^d(r, \phi, z). \quad (2.20) \]

### 2.3 Evaluation of moments

One method to calculate the moments exerted by the fluid on the walls of the cylindrical payload container is to evaluate the viscous stresses and the pressure at these walls. The evaluation of the viscous stresses requires knowledge of the gradients of all velocity components at the surface of the payload container. Consequently, this approach requires solving for all flow quantities—a task that can be both difficult and computationally expensive, especially for high-Reynolds-number flows. Moreover, as a result of the invalidity of the Navier-Stokes equations at the corners, the convergence of the pressure at the walls is very slow and this certainly affects the convergence of the moments when using this approach.

An alternative approach, based on control volume analysis, is very powerful in computing these moments more accurately. The control volume approach uses the concept of conservation of angular momentum,

\[
M = \frac{\partial}{\partial t} \int \int \int_R (\mathbf{r} \times \mathbf{V}) \rho dR + \int \int \int_R [\mathbf{r} \times (2\Omega \times \mathbf{V})] \rho dR \\
+ \int \int \int_R [\mathbf{r} \times \Omega \times (\Omega \times \mathbf{r})] \rho dR + \int \int_S (\mathbf{r} \times \mathbf{V})(\mathbf{V} \cdot \mathbf{n}) \rho dS \quad (2.21)
\]
where $S$ denotes the surface of the control volume $R$ (formed by the solid boundaries of the cylinder walls), $\mathbf{r}$ the position vector, $\mathbf{M}$ the resulting torque on the control volume, and $\mathbf{n}$ an outward unit vector normal to $S$. The surface integral on the right hand side of eq. (2.21) vanishes because the surface of the control volume is closed. Here, we are interested only in the moments caused by the flow deviation from rigid-body rotation. If we decompose the resulting torque on the control volume into $\mathbf{M} = \mathbf{M}^r + \mathbf{M}$, where $\mathbf{M}^r$ corresponds to pure rigid-body motion and $\mathbf{M}$ corresponds to the deviation velocity and pressure, and consider steady state conditions, then it can be shown (Herbert & Li 1990) that the moment calculation rests on the relation

$$M = \iint \int_{R} [\mathbf{r} \times \mathbf{(2\Omega \times v)}] \rho dR. \quad (2.22)$$

For convenience, we use the aeroballistic reference frame $x, y, z$ to express the moment $M$ in terms of Cartesian components ($M_x, M_y, M_z$). It can then be shown (Herbert & Li 1987, Herbert & Li 1990) that these components are related to the flow velocities by

$$M_x = 2\frac{\Omega}{\omega} (\rho \omega^2 a^5) \cos \theta \int_{-\pi}^{\pi} \int_{0}^{2\pi} \int_{0}^{1} v_z r^2 \cos \phi dr d\phi dz, \quad (2.23)$$

$$M_z = M_z \tan \theta, \quad (2.24)$$

$$M_y = 2\frac{\Omega}{\omega} (\rho \omega^2 a^5) \cos \theta \int_{-\pi}^{\pi} \int_{0}^{2\pi} \int_{0}^{1} v_z r^2 \sin \phi dr d\phi dz$$

$$+ \frac{\Omega}{\omega} (\rho \omega^2 a^5) \sin \theta \int_{-\pi}^{\pi} \int_{0}^{2\pi} \int_{0}^{1} v_\phi r^2 dr d\phi dz. \quad (2.25)$$
If we represent the velocity field by the Fourier series

\[ v(r, \phi, z) = \sum_{n=-\infty}^{\infty} v^n_n(r, z) e^{in\phi}, \quad v^n_n = (u^n_n, v^n_n, w^n_n), \quad i^2 = -1, \quad (2.26) \]

then it is evident from the expressions of the moments, upon performing the integrations over \( \phi \), that we need only consider the Fourier components \( w^1_n \) and \( v^0_n \). If we further expand the velocity components in a perturbation series in powers of \( \epsilon \), then it becomes obvious that \( w^1_\alpha \) is \( O(\epsilon) \) since the forcing term in the Navier-Stokes equations is simply periodic and is \( O(\epsilon) \), whereas the mean component \( v^0_\alpha \) is \( O(\epsilon^2) \). Hence, for the purpose of calculating the moments from the linearized Navier-Stokes equations, it is cheaper computationally to solve an equation governing only the fundamental of the axial velocity.

### 2.4 Axial flow governing equations

To derive an equation governing the fundamental component of the axial velocity, we write the linearized momentum equations in vector form

\[
\frac{\partial}{\partial \phi} v + 2r \tau_r e_z + 2 \vec{\tau} \times v + \nabla p - \frac{1}{Re} \nabla^2 v = 0, \quad (2.27)
\]

where \( e_z \) is a unit vector in the \( z \)-direction and

\[
\vec{\tau} = (0, 0, 1 + \tau_z).
\]

We obtain the vorticity equation by taking the curl of the momentum equations

\[
\frac{\partial}{\partial \phi} (\nabla \times v) + \nabla \times (2r \tau_r e_z) + 2 \nabla \times (\vec{\tau} \times v) - \frac{1}{Re} \nabla \times (\nabla^2 v) = 0.
\]
With $\xi = \nabla \times \mathbf{v}$, the vorticity equation takes the vector form,

$$\frac{\partial \xi}{\partial \phi} - 2(1 + \tau_z)\frac{\partial \mathbf{v}}{\partial z} - \frac{1}{Re} \nabla^2 \xi = -2\tau_{\phi} \mathbf{e}_r + 2\tau_r \mathbf{e}_\phi,$$  

(2.28)

where $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are unit vectors in the $r, \phi$-direction respectively. Taking the curl of the vorticity equation, we obtain

$$\nabla \times [\nabla \times (\frac{\partial \mathbf{v}}{\partial \phi})] + \nabla \times [\nabla \times (2\tau_{\tau_r} \mathbf{e}_z)] + 2\nabla \times [\nabla \times (\tau \times \mathbf{v})] - \frac{1}{Re} \nabla \times [\nabla \times (\nabla^2 \mathbf{v})] = 0,$$

which can be written as

$$- \frac{\partial}{\partial \phi} \nabla^2 \mathbf{v} - 2(1 + \tau_z)\frac{\partial \mathbf{v}}{\partial z} + \frac{1}{Re} \nabla^4 \mathbf{v} = 0.$$  

(2.29)

Furthermore, the vorticity equation (2.28) can be rewritten as

$$(\frac{\partial}{\partial \phi} - \frac{1}{Re} \nabla^2)\xi = 2(1 + \tau_z)\frac{\partial \mathbf{v}}{\partial z} - 2\tau_{\phi} \mathbf{e}_r + 2\tau_r \mathbf{e}_\phi.$$

Now we apply $\frac{\partial}{\partial \phi} - (1/Re)\nabla^2$ to equation (2.29) and obtain

$$-\left(\frac{\partial}{\partial \phi} - \frac{1}{Re} \nabla^2\right)\frac{\partial}{\partial \phi} \nabla^2 \mathbf{v} - 2(1 + \tau_z)\frac{\partial}{\partial z} \left(\frac{\partial}{\partial \phi} - \frac{1}{Re} \nabla^2\right)\xi + \frac{1}{Re} \left(\frac{\partial}{\partial \phi} - \frac{1}{Re} \nabla^2\right) \nabla^4 \mathbf{v} = 0.$$

Note that $\tau_{\phi}$ and $\tau_r$ are functions of $\phi$ only, hence

$$2(1 + \tau_z)\frac{\partial}{\partial z}(2\tau_{\phi} \mathbf{e}_r - 2\tau_r \mathbf{e}_\phi) = 0,$$

and consequently

$$- \frac{\partial^2}{\partial \phi^2} \nabla^2 \mathbf{v} + \frac{2}{Re} \frac{\partial}{\partial \phi} \nabla^4 \mathbf{v} - \frac{1}{Re^2} \nabla^6 \mathbf{v} - 4(1 + \tau_z)^2 \frac{\partial^2}{\partial z^2} \mathbf{v} = 0.$$  

(2.30)
For the calculation of the moments, we only need the mean flow component of $v_\phi$ and the fundamental of $v_z$. For simplicity we drop the subscript $\alpha$ unless otherwise necessary to differentiate between inner and outer regions. We also drop the superscript 1 since from here on we will only be interested in the fundamental components. Suppose the fundamental of $v$ is expressed as

$$\begin{pmatrix} v_r \\ v_\phi(r, \phi, z) \\ v_z \end{pmatrix} = \begin{pmatrix} u \\ v(r, z) \\ w \end{pmatrix} e^{i\phi} + \begin{pmatrix} \bar{u} \\ \bar{v}(r, z) \\ \bar{w} \end{pmatrix} e^{-i\phi} \tag{2.31}$$

where the bar denotes the complex conjugate. Then, the continuity equation takes the form

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{iv}{r} + \frac{\partial w}{\partial z} = 0, \tag{2.32}$$

the $r$-component of the vorticity equation (2.28) takes the form

$$\begin{aligned} &i\left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) - 2(1 + \tau_z)\frac{\partial u}{\partial z} \\
&- \frac{1}{Re}\left[\nabla^2\left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) - \frac{1}{r^2}\left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) - \frac{2i}{r^2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = i\epsilon, \tag{2.33} \end{aligned}$$

the $\phi$-component of the vorticity equation (2.28) is

$$\begin{aligned} &i\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) - 2(1 + \tau_z)\frac{\partial v}{\partial z} \\
&- \frac{1}{Re}\left[\nabla^2\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) + \frac{2i}{r^2}\left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) - \frac{1}{r^2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = -\epsilon, \tag{2.34} \end{aligned}$$

and the $z$-component of the vorticity equation (2.28) takes the form

$$\begin{aligned} &i\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{i}{r}u\right) - 2(1 + \tau_z)\frac{\partial w}{\partial z} - \frac{1}{Re}\nabla^2\left(\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{i}{r}u\right) = 0 \tag{2.35} \end{aligned}$$
While the \( z \)-component of equation (2.29) is

\[
- i \nabla^2 w - 2(1 + \tau_z) \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{i}{r} u \right) + \frac{1}{Re} \nabla^4 w = 0, \tag{2.36}
\]

the \( z \)-component of equation (2.30) (Li & Herbert 1989) takes the form

\[
\nabla^2 w + \frac{2i}{Re} \nabla^4 w - \frac{1}{Re^2} \nabla^6 w - 4(1 + \tau_z)^2 \frac{\partial^2 w}{\partial z^2} = 0, \tag{2.37}
\]

where

\[
\nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}.
\]

Equation (37) is a Sixth Order Partial Differential Equation (SOPDE) for \( w \). The governing equations for \( u \) and \( v \) come from the \( r \)-component and \( \phi \)-component of equation (2.30). They are coupled and very complicated because in cylindrical coordinates

\[
\nabla^2 A_r \rightarrow \frac{A_r}{r^2} - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi}
\]

\[
\nabla^2 A_\phi \rightarrow - \frac{A_\phi}{r^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi}
\]

\[
\nabla^2 A_z
\]

for any vector \( A \) having cylindrical components \((A_r, A_\phi, A_z)\). We have not presented the other components of equation (2.29) because they are not used in deriving the boundary conditions associated with equation (2.37).
2.5 Axial flow boundary conditions

2.5.1 Conditions at the end walls

For the linearized equation, the flow field exhibits strong symmetries. Combination with the no-slip condition at the end walls \((z = \pm \eta)\) provides

\[
\frac{\partial^m}{\partial r^m} (u, v, w) = 0, \quad \frac{\partial^n}{\partial \phi^n} (u, v, w) = 0, \quad \frac{\partial^{m+n}}{\partial r^m \partial \phi^n} (u, v, w) = 0,
\]

for any integers \(m\) and \(n\), and the first two boundary conditions are

\[
w = 0, \quad (2.38)
\]

\[
\frac{\partial w}{\partial z} = 0, \quad (2.39)
\]

where the latter directly follows from the continuity equation. To obtain the third boundary condition, we consider \(\partial/\partial z\) eq. (2.36), namely,

\[-i \nabla^2 \frac{\partial w}{\partial z} - 2(1 + \tau_z) \frac{\partial^2}{\partial z^2} \left( \frac{\partial v}{\partial r} + \frac{v}{r - \frac{i}{u}} \right) + \frac{1}{Re} \nabla^2 \frac{\partial w}{\partial z} = 0,
\]

and rewrite eq. (2.35) in the form

\[
i \left( \frac{\partial v}{\partial r} + \frac{v}{r - \frac{i}{u}} \right) - 2(1 + \tau_z) \frac{\partial w}{\partial z} - \frac{1}{Re} \left[ \nabla^2_1 \left( \frac{\partial v}{\partial r} + \frac{v}{r - \frac{i}{u}} \right) + \frac{\partial^2}{\partial z^2} \left( \frac{\partial v}{\partial r} + \frac{v}{r - \frac{i}{u}} \right) \right] = 0,
\]

where

\[
\nabla^2_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}.
\]
We note from the above equation that
\[ \frac{\partial^2}{\partial z^2} \left( \frac{v}{r} + \frac{\partial v}{\partial r} - \frac{i}{r} u \right) = \frac{\partial^2}{\partial z^2} \xi_z = 0. \]

Hence, we obtain the third boundary condition:
\[ -i \frac{\partial^3 w}{\partial z^3} + \frac{1}{Re} \left( 2 \nabla^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^5 w}{\partial z^5} \right) = 0. \quad (2.40) \]

### 2.5.2 Conditions at the side walls

The boundary conditions at the side walls \((r = 1 \text{ or } r = r_0 \text{ for cylinders containing a central rod})\) are quite involved. The derivation is more difficult than that for the governing equation (2.37). At the side walls

\[ \frac{\partial^n}{\partial z^n} (u, v, w) = 0, \quad \frac{\partial^{m+n}}{\partial \phi^m \partial z^n} (u, v, w) = 0, \]

for any integers \(m\) and \(n\), and the first condition is

\[ w = 0. \quad (2.41) \]

For the second condition, consider eqs. (2.36) and (2.33). Note that

\[ \frac{\partial}{\partial r} \left( \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{i}{r} u \right) = \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) - i \left( \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) \]

\[ = \nabla^2 v - \frac{\partial^2 v}{\partial z^2} - i \left( \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right). \]

Further, we use \(\partial/\partial r\) eq. (2.36) which takes the form

\[ -i \frac{\partial}{\partial r} (\nabla^2 w) + \frac{1}{Re} \frac{\partial}{\partial r} (\nabla^4 w) - 2(1 + \tau_z) \frac{\partial}{\partial z} [\nabla^2 v - \frac{\partial^2 v}{\partial z^2} - i \left( \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right)] = 0, \]
and obtain from the continuity equation

\[
\frac{\partial}{\partial z}(\frac{\partial u}{\partial r}) = -\frac{\partial}{\partial z}(\frac{u}{r} + \frac{i}{r}v + \frac{\partial w}{\partial z}) = -\frac{1}{r}\frac{\partial u}{\partial z} - \frac{i}{r}\frac{\partial v}{\partial z} - \frac{\partial^2 w}{\partial z^2}.
\]

From eq. (2.33), we obtain

\[
\nabla^2 \frac{\partial v}{\partial z} = i \nabla^2 \left(\frac{w}{r}\right) - \frac{1}{r^2} \left(\frac{i}{r}w - \frac{\partial v}{\partial z} - \frac{2i}{r^2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right) + i\epsilon Re - iRe \left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) + 2(1 + \tau_z)Re \frac{\partial u}{\partial z}.
\]

Hence, \(\partial/\partial r\) eq. (2.36) takes the form

\[
-i \frac{\partial}{\partial r}(\nabla^2 w) - 2(1 + \tau_z) \left\{ i \nabla^2 \left(\frac{w}{r}\right) - \frac{1}{r^2} \left(\frac{i}{r}w - \frac{\partial v}{\partial z} - \frac{2i}{r^2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right) + i\epsilon Re - iRe \left(\frac{i}{r}w - \frac{\partial v}{\partial z}\right) + 2(1 + \tau_z)Re \frac{\partial u}{\partial z}\right\} + 2(1 + \tau_z) \frac{\partial^3 v}{\partial z^3}
\]

\[
+ 2(1 + \tau_z) \frac{i}{r} \left[ -\frac{1}{r} \frac{\partial u}{\partial z} - \frac{i}{r} \frac{\partial v}{\partial z} - \frac{\partial^2 w}{\partial z^2} \right] - 2(1 + \tau_z) \frac{i}{r^2} \frac{\partial u}{\partial z} + \frac{1}{Re} \frac{\partial}{\partial r}(\nabla^4 w) = 0 \quad (2.42)
\]

At the wall eq. (2.42) is reduced to

\[
-i \frac{\partial}{\partial r}(\nabla^2 w) - 2(1 + \tau_z) \left[ i \nabla^2 \left(\frac{w}{r}\right) + \frac{2i}{r^3} \frac{\partial w}{\partial r} + i\epsilon Re \right] + \frac{1}{Re} \frac{\partial}{\partial r}(\nabla^4 w) = 0,
\]

and if we note that

\[
\nabla^2_i \left(\frac{w}{r}\right) = i \nabla^2_i \left(\frac{w}{r}\right) = \frac{i}{r} \nabla^2_i w + \frac{i}{r^3} \frac{\partial w}{\partial r} - 2\frac{\partial w}{\partial r},
\]

we obtain

\[
-i \frac{\partial}{\partial r}(\nabla^2_i w) - 2(1 + \tau_z) \left[ \frac{i}{r} \nabla^2_i w + \frac{i}{r^3} \frac{\partial w}{\partial r} + \frac{2i}{r} \frac{\partial^2 w}{\partial z^2} + i\epsilon Re \right]
\]
\[-i \frac{\partial^3 w}{\partial r \partial z^2} + \frac{1}{Re} \frac{\partial}{\partial r} \left[ \nabla_1^2 w + 2 \nabla_1^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^4 w}{\partial z^4} \right] = 0,\]

and consequently the second condition
\[-i \frac{\partial}{\partial r} (\nabla_1^2 w) - i \frac{\partial^3 w}{\partial r \partial z^2} - 2(1 + \tau_z) \frac{i}{r} \nabla_1^2 w\]
\[+ \frac{1}{Re} \left[ \frac{\partial}{\partial r} (\nabla_4^2 w) + 2 \frac{\partial}{\partial r} \nabla_1^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^5 w}{\partial r \partial z^4} \right] = 2(1 + \tau_z)i \epsilon Re. \quad (2.43)\]

For the third boundary condition, we consider \((i/r)\) eq. (2.36),
\[\frac{1}{r} \nabla^2 w - 2(1 + \tau_z) \left( \frac{i}{r} \frac{\partial^2 v}{\partial r \partial z} + \frac{i}{r^2 \partial z} + \frac{1}{r^2 \partial z} \right) + \frac{i}{r} \frac{1}{Re} \nabla^4 w = 0,\]

and note that
\[\frac{\partial}{\partial z} (\nabla_1^2 u) = \frac{\partial}{\partial z} \frac{\partial u}{\partial r} \left( \frac{u}{r} \right) = \frac{\partial}{\partial z} \frac{\partial}{\partial r} \left( \frac{-i v}{r} - \frac{\partial w}{\partial z} \right)\]
\[= -i \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) - \frac{\partial^3 w}{\partial r \partial z^2}\]
\[= -i \frac{\partial^2 v}{r \partial r \partial z} + \frac{i \partial v}{r^2 \partial z} - \frac{\partial^3 w}{\partial r \partial z^2}.\]

From eq. (2.34) we have
\[i \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) - 2(1 + \tau_z) \frac{\partial v}{\partial z} - \frac{1}{Re} \left\{ - \frac{\partial}{\partial r} (\nabla^2 w) \right.\]
\[- \frac{1}{r^2} \frac{\partial w}{\partial r} + \frac{2}{r^3} w + \left( - \frac{i}{r} \frac{\partial^2 v}{\partial r \partial z} + \frac{i}{r^2} \frac{\partial v}{\partial z} - \frac{\partial^3 w}{\partial r \partial z^2} + \frac{\partial^3 u}{\partial z^3} \right)\]
\[+ \frac{2i}{r^2} \left( \frac{i}{r} - \frac{\partial u}{\partial z} \right) - \frac{1}{r^2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) \right\} = -\epsilon,\]

from which we obtain
\[i \frac{\partial^2 v}{r \partial r \partial z} = -i \text{Re} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) + 2(1 + \tau_z) \text{Re} \frac{\partial v}{\partial z} - \epsilon \text{Re}\]
\[- \frac{\partial}{\partial r} \nabla^2 w - \frac{1}{r^2} \frac{\partial w}{\partial r} + \frac{2}{r^3} w + \frac{i}{r^2} \frac{\partial v}{\partial z} - \frac{\partial^3 w}{\partial r \partial z^2}\]
\[+ \frac{\partial^3 u}{\partial z^3} + \frac{2i}{r^2} \left( \frac{i}{r} - \frac{\partial u}{\partial z} \right) - \frac{1}{r^2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right)\]
and consequently \( i/r \) eq. (2.36) takes the form

\[
\begin{align*}
+ \frac{1}{r} \nabla^2 w - 2(1 + \tau_z) \left\{ - i Re \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) + 2(1 + \tau_z) Re \frac{\partial v}{\partial z} 
- \varepsilon Re \left[ \frac{\partial}{\partial r} (\nabla^2 w) - \frac{\partial w}{\partial r} \right] + \frac{2}{r^2} \nabla^2 w + \frac{i}{r^2} \frac{\partial v}{\partial z} - \frac{\partial^2 w}{\partial r^2} + \frac{\partial^3 v}{\partial z^3}
+ \frac{2i}{r^2} (\frac{i}{r} w - \frac{\partial v}{\partial z}) - \frac{1}{r^2} (\frac{\partial w}{\partial r} - \frac{\partial u}{\partial r}) + \frac{i}{r^2} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial z} \right\}
+i \frac{1}{r} \frac{\partial^3 w}{\partial z^3} + \frac{1}{r^{4}} \frac{\partial w}{\partial z} = 0. \quad (2.44)
\end{align*}
\]

Hence, at the wall

\[
\begin{align*}
- 2(1 + \tau_z) \left[ i Re \frac{\partial w}{\partial r} - \frac{\partial}{\partial r} (\nabla^2 w) - 2 \frac{\partial^3 w}{\partial r \partial z^2} \right]
+ \frac{1}{r} \nabla^2 w + i \frac{1}{r} \frac{\partial^3 w}{\partial z^2} + \frac{1}{r Re} (\nabla^2 w + 2 \nabla^2 \frac{\partial^2 w}{\partial z^2}) = -2(1 + \tau_z) \varepsilon Re \quad (2.45)
\end{align*}
\]

### 2.5.3 Conditions at void-fluid interface

At the interface we require equality of the stress components and equality of velocity components due to the inner and outer fluid. For small nutation angles and/or nutation frequencies, the void-fluid interface is only a slight perturbation from the axisymmetric surface \( r = r_0 \).

When the cylinder is only spinning, \( \varepsilon = 0 \), the interface assumes the shape of the axisymmetric surface \( r = r_0 \). Hence, similar to the velocity deviation from solid body rotation, the deviation of the interface from this surface is \( O(\varepsilon) \). For sufficiently small \( \varepsilon \), it is well justified to linearize the conditions at the interface with \( \varepsilon \). As a result, we obtain for the no-shear stress conditions at a void-fluid interface,

\[
\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} = 0, \quad (2.46)
\]
\[ r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \phi} = 0. \]  

(2.47)

While the leading order term, balancing the normal stress with the void pressure, takes the form

\[-p^d + \frac{2}{Re} \frac{\partial v_r}{\partial r} = \frac{1}{2} \left[ 2 \tau_0 \zeta (1 + \tau_z)^2 + 2 \tau_0 \tau_z \epsilon \cos \phi \right], \]  

(2.48)

where \( \zeta \) denotes the radial distance from the surface \( r = r_0 \) to the interface location.

For sufficiently small \( \epsilon \) this quantity is related to the \( r \)-component of the velocity field by

\[ v_r(r_0, \phi, z) = \frac{\partial \zeta}{\partial \phi}. \]  

(2.49)

Suppose in addition to expressing the fundamental of \( v \) according to equation (2.31) we express the fundamental of the pressure and the \( \zeta \) according to

\[ p^d = p(r, z)e^{i\phi} + \bar{p}(r, z)e^{-i\phi}, \]  

(2.50)

\[ \zeta(z, \phi) = \zeta_1(z)e^{i\phi} + \bar{\zeta}_1(z)e^{-i\phi}. \]  

(2.51)

We now substitute these expressions into conditions (2.46) through (2.49) and obtain

\[ \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial r}, \]  

(2.52)

\[ \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{i}{r} u = 0, \]  

(2.53)

\[-p + \frac{2}{Re} \frac{\partial u}{\partial r} + i \tau_0 (1 + \tau_z)^2 u = \frac{1}{2} \tau_0 \tau_z \epsilon, \]  

(2.54)

\[ \zeta_1(z) = -iu(r_0, z). \]  

(2.55)
To derive the first condition, consider the z-momentum equation,

$$\frac{\partial v_z}{\partial \phi} = -\frac{\partial p^d}{\partial z} - 2r \tau_r + \frac{1}{Re} D'' v_z, \quad (2.56)$$

which takes the form for the fundamental components, evaluated at the interface,

$$iw = -\frac{\partial p}{\partial z} + \tau_0 \epsilon + \frac{1}{Re} \nabla^2 w. \quad (2.57)$$

We differentiate eq. (2.54) with respect to $z$,

$$-\frac{\partial p}{\partial z} + 2 \frac{\partial^2 u}{Re \partial r \partial z} + i\tau_0 (1 + \tau_z)^2 \frac{\partial u}{\partial z} = \frac{1}{2} \tau_0 \tau_z \epsilon. \quad (2.58)$$

Adding eq. (2.57) to eq. (2.58) and making use of eq. (2.52) leads to

$$iw + \frac{2}{Re} \frac{\partial^2 u}{\partial r \partial z} - i\tau_0 (1 + \tau_z)^2 \frac{\partial w}{\partial r} = \frac{1}{2} (2 + \tau_z) \tau_0 \epsilon + \frac{1}{Re} \nabla^2 w. \quad (2.59)$$

Note that we need an expression for $\partial^2 u / \partial r \partial z$ at $r = r_0$. Consider the continuity equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{i v}{r} + \frac{\partial w}{\partial z} = 0.$$  

Differentiating the continuity equation with respect to $z$ yields

$$\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} + \frac{i}{r} \frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial z^2} = 0, \quad (2.60)$$

and differentiating eq. (2.53) with respect to $z$ leads to

$$\frac{\partial^2 v}{\partial r \partial z} - \frac{1}{r} \frac{\partial v}{\partial z} + \frac{i}{r} \frac{\partial u}{\partial z} = 0. \quad (2.61)$$

Then, $-2i$ eq. (2.60) + eq. (2.61) is

$$\frac{\partial^2 v}{\partial r \partial z} + \frac{1}{r} \frac{\partial v}{\partial z} - \frac{i}{r} \frac{\partial u}{\partial z} = 2i \left( \frac{\partial^2 u}{\partial r \partial z} + \frac{\partial^2 w}{\partial z^2} \right). \quad (2.62)$$
Consider eq. (2.36),

$$-i\nabla^2 w - 2(1 + \tau_z)\left[ \frac{\partial^2 w}{\partial z \partial r} - \frac{1}{r} \frac{\partial w}{\partial r} - i \frac{\partial u}{\partial r} \right] + \frac{1}{Re} \nabla^4 w. \quad (2.63)$$

Substituting eq. (2.62) into eq. (2.63) we obtain

$$-i\nabla^2 w - 4(1 + \tau_z)i\left( \frac{\partial^2 w}{\partial r \partial z} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{1}{Re} \nabla^4 w = 0. \quad (2.64)$$

Multiplying eq. (2.59) by $2(1 + \tau_z)iRe$ yields

$$-2(1 + \tau_z)Re w + 4(1 + \tau_z)i\frac{\partial^2 u}{\partial r \partial z} + 2(1 + \tau_z)^3 r_0 Re \frac{\partial w}{\partial r} = (2 + \tau_z)(1 + \tau_z)r_0 \epsilon iRe + 2(1 + \tau_z)i\nabla^2 w. \quad (2.65)$$

Adding eq. (2.65) to eq. (2.64), to cancel out $\partial^2 u/\partial r \partial z$, yields the first condition

$$-i\nabla^2 w + \frac{1}{Re} \nabla^4 w - 2(1 + \tau_z)\left[ Re w + i\nabla^2 w + 2i\frac{\partial^2 w}{\partial z^2} \right]$$

$$+ 2(1 + \tau_z)^3 r_0 Re \frac{\partial w}{\partial r} = (2 + \tau_z)(1 + \tau_z)r_0 \epsilon iRe. \quad (2.66)$$

For the second condition, we consider eq. (2.44) found when we derived the boundary conditions at the side walls,

$$\frac{1}{r} \nabla^2 w - 2(1 + \tau_z)\left[ -iRe\left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) - \epsilon Re - \frac{\partial}{\partial r}(\nabla^2 w) \right]$$

$$- \frac{\partial^3 w}{\partial r \partial z \partial r} + \frac{\partial^3 u}{\partial r \partial z^2} \right] - 4(1 + \tau_z)^2 Re \frac{\partial v}{\partial z} + \frac{i}{r Re} \nabla^4 w = 0. \quad (2.67)$$

Combining eqs. (2.36) and (2.60) yields

$$-i\nabla^2 w - 2(1 + \tau_z)\left[ \frac{2\partial v}{r \partial z} + \frac{2i\partial w}{r \partial r} \right] + \frac{1}{Re} \nabla^4 w = 0 \quad (2.68)$$
and if multiplied by $r \Re(1 + \tau_z)$ this equation becomes

$$ir \Re(1 + \tau_z) \nabla^2 w + 4(1 + \tau_z)^2 \Re \left[ \frac{\partial v}{\partial z} + i \frac{\partial w}{\partial r} \right] - (1 + \tau_z) r \nabla^4 w = 0. \quad (2.69)$$

The addition of this equation to eq. (2.67) to cancel out $\partial v / \partial z$ and making use of eq. (2.52) produces the second condition

$$- \frac{1}{r} \nabla^2 w + \frac{1}{r} \frac{1}{i \Re} \nabla^4 w - 2(1 + \tau_z) \left[ \frac{\partial}{\partial r} (\nabla^2 w) + 2 \frac{\partial^3 w}{\partial r \partial z^2} - 2i \Re \frac{\partial w}{\partial r} \right]$$

$$+ \frac{1}{2} i r \Re \nabla^2 w - \frac{1}{2} r \nabla^4 w \right] - 4(1 + \tau_z)^2 i \Re \frac{\partial w}{\partial r} = 2(1 + \tau_z) \epsilon \Re. \quad (2.70)$$

For the third condition, we consider eq. (2.42), namely,

$$- i \frac{\partial}{\partial r} (\nabla^2 w) - 2(1 + \tau_z) \left[ i \nabla^2 \left( \frac{w}{r} \right) - \frac{i}{r^3} w + \frac{2i}{r^2} \frac{\partial w}{\partial r} \right]$$

$$+ i \epsilon \Re - \frac{\partial^3 v}{\partial z^3} + \frac{1}{r} \Re w + i \Re \frac{\partial v}{\partial z} + 2(1 + \tau_z) \Re \frac{\partial u}{\partial z} + \frac{i}{r} \frac{\partial^2 w}{\partial z^2} \right] + \frac{1}{\Re} \frac{\partial}{\partial r} (\nabla^4 w) = 0 \quad (2.71)$$

We need to eliminate $\partial v / \partial z, \partial^3 v / \partial z^3$. From eq. (2.68) we can write

$$- \frac{1}{2} r \Re \nabla^2 w + 2(1 + \tau_z) i \Re \frac{\partial v}{\partial z} - 2(1 + \tau_z) \Re \frac{\partial w}{\partial r} - \frac{i}{2} r \nabla^4 w = 0 \quad (2.72)$$

and if differentiated twice with respect to $z$, this equation becomes

$$- \frac{i}{2} r \nabla^2 \frac{\partial^2 w}{\partial z^2} - 2(1 + \tau_z) \frac{\partial^3 v}{\partial z^3} - 2(1 + \tau_z)i \frac{\partial^2 w}{\partial r \partial z^2} + \frac{1}{2} \frac{1}{\Re} r \nabla^4 \frac{\partial^2 w}{\partial z^2} = 0. \quad (2.73)$$

The addition of eqs. (2.71), (2.72), and (2.73), and the use of eq. (2.52) leads to the third condition

$$- 2(1 + \tau_z) \left[ \frac{i}{r} \nabla^2 w + \frac{i}{r} \frac{\partial^2 w}{\partial z^2} - \frac{i}{r^2} \frac{\partial w}{\partial r} + \Re \left( \frac{w}{r} + \frac{\partial w}{\partial r} \right) + i \frac{\partial^3 w}{\partial r \partial z^2} \right]$$
\[-i\frac{\partial}{\partial r}(\nabla^2 w) + \frac{1}{Re} \frac{\partial}{\partial r}(\nabla^4 w) - \frac{1}{2} Re \nabla^2 w - \frac{i}{2} r \nabla^2 \frac{\partial^2 w}{\partial z^2} - \frac{i}{2} r \nabla^4 w\]
\[\frac{1}{2} Re r \nabla^4 \frac{\partial^2 w}{\partial z^2} + 4(1 + \tau_z)^2 Re \frac{\partial w}{\partial r} = 2(1 + \tau_z)ieRe. \quad (2.74)\]

### 2.5.4 Conditions at two-fluid interface

For two-fluid flows, the velocities must be equal at the interface, namely,

\[(v_{r0}, v_{\phi 0}, v_z^0) = (v_{r1}, v_{\phi 1}, v_z^1), \quad (2.75)\]

the shear stresses must be equal,

\[\frac{\rho_0}{Re_0} \left( \frac{\partial v_r^0}{\partial r} + \frac{\partial v_{\phi 0}}{\partial z} \right) = \frac{\rho_1}{Re_1} \left( \frac{\partial v_r^1}{\partial r} + \frac{\partial v_{\phi 1}}{\partial z} \right), \quad (2.76)\]

\[\frac{\rho_0}{Re_0} \left( r \frac{\partial}{\partial r} \left( \frac{v_r^0}{r} \right) + \frac{1}{r} \frac{\partial v_{\phi 0}}{\partial \phi} \right) = \frac{\rho_1}{Re_1} \left( r \frac{\partial}{\partial r} \left( \frac{v_r^1}{r} \right) + \frac{1}{r} \frac{\partial v_{\phi 1}}{\partial \phi} \right), \quad (2.77)\]

and the normal stresses must also be equal,

\[\rho_0[-p_0^d + \frac{2}{Re} \frac{\partial v_r^0}{\partial r} - \frac{1}{2} \{(2\rho_0 \zeta)(1 + \tau_z)^2 + 2\rho_0 \tau_z \varepsilon \cos \phi\}] = \rho_1[-p_1^d + \frac{2}{Re} \frac{\partial v_r^1}{\partial r} - \frac{1}{2} \{(2\rho_0 \zeta)(1 + \tau_z)^2 + 2\rho_0 \tau_z \varepsilon \cos \phi\}], \quad (2.78)\]

From kinematics, we have

\[v^0_r = v^1_r = \frac{\partial \zeta}{\partial \phi}. \quad (2.79)\]

We can eliminate \(p_\alpha^d\) if we differentiate eq. (2.78) with respect to \(z\) and recognize,

from the \(z\)-momentum equations, that

\[-\frac{\partial p_\alpha^d}{\partial z} = \frac{\partial v_\alpha^0}{\partial \phi} + 2\rho_0 \tau_r - \frac{1}{Re_\alpha} D_{\alpha\alpha} v_{z\alpha}^0, \quad \alpha = 0, 1. \quad (2.80)\]
When eq. (2.80) is substituted into $\partial/\partial z$ eq. (2.78), we obtain

$$
\rho_0 \left[ \frac{\partial v_0}{\partial \phi} - \frac{1}{Re_0} D'' v_0 - r_0 (1 + \tau_z)^2 \frac{\partial \zeta}{\partial z} - r_0 (2 + \tau_z) \epsilon \cos \phi + \frac{2}{Re_0} \frac{\partial^2 v_0}{\partial r \partial z} \right] = \rho_1 \left[ \frac{\partial v_1}{\partial \phi} - \frac{1}{Re_1} D'' v_1 - r_0 (1 + \tau_z)^2 \frac{\partial \zeta}{\partial z} - r_0 (2 + \tau_z) \epsilon \cos \phi + \frac{2}{Re_1} \frac{\partial^2 v_1}{\partial r \partial z} \right]
$$

(2.81)

We now express the fundamentals of the velocities as

$$
v_r^0(r, \phi, z) = u_\alpha(r, z)e^{i\phi} + \bar{u}_\alpha(r, z)e^{-i\phi}, \quad (2.82)
$$

$$
v_\phi^0(r, \phi, z) = v_\alpha(r, z)e^{i\phi} + \bar{v}_\alpha(r, z)e^{-i\phi}, \quad (2.83)
$$

$$
v_z^0(r, \phi, z) = w_\alpha(r, z)e^{i\phi} + \bar{w}_\alpha(r, z)e^{-i\phi}, \quad (2.84)
$$

$$
\zeta(\phi, z) = \zeta_1(z)e^{i\phi} + \bar{\zeta}_1(z)e^{-i\phi}, \quad (2.85)
$$

$$
p_\alpha^d(r, \phi, z) = p_\alpha(r, z)e^{i\phi} + \bar{p}_\alpha(r, z)e^{-i\phi}, \quad (2.86)
$$

where $\alpha$ takes the values of 0 and 1. The substitution of these expressions into the boundary conditions leads to

$$
(u_0, v_0, w_0) = (u_1, v_1, w_1) \quad (2.87)
$$

$$
u_0(r_0, z) = i\zeta_1(z), \quad (2.88)
$$

$$
\left( \frac{\rho_0}{Re_0} \left( \frac{\partial w_0}{\partial r} + \frac{\partial v_0}{\partial z} \right) + \frac{\rho_1}{Re_1} \left( \frac{\partial w_1}{\partial r} + \frac{\partial v_1}{\partial z} \right) \right) = \frac{\rho_0}{Re_0} \left( \frac{\partial u_0}{\partial r} + \frac{\partial u_1}{\partial z} \right), \quad (2.89)
$$

$$
\left( \frac{\rho_0}{Re_0} \left( \frac{\partial v_0}{\partial r} - \frac{v_0}{r} + \frac{i}{r} u_0 \right) + \frac{\rho_1}{Re_1} \left( \frac{\partial v_1}{\partial r} - \frac{v_1}{r} + \frac{i}{r} u_1 \right) \right) = \frac{\rho_0}{Re_0} \left( \frac{\partial u_0}{\partial r} - \frac{u_0}{r} + \frac{i}{r} u_1 \right), \quad (2.90)
$$

$$
\rho_0 [i w_0 - \frac{1}{Re_0} \nabla^2 w_0 + i r_0 (1 + \tau_z)^2 \frac{\partial w_0}{\partial z} - \frac{1}{2} (2 + \tau_z) r_0 \epsilon + \frac{2}{Re_0} \frac{\partial^2 u_0}{\partial r \partial z}] = \rho_1 [i w_1 - \frac{1}{Re_1} \nabla^2 w_1 + i r_1 (1 + \tau_z)^2 \frac{\partial w_1}{\partial z} - \frac{1}{2} (2 + \tau_z) r_1 \epsilon + \frac{2}{Re_1} \frac{\partial^2 u_1}{\partial r \partial z}] \quad (2.91)
$$
\[ \rho_1 [i w_1 - \frac{1}{Re_1} \nabla^2 w_1 + i r_0 (1 + \tau_z)^2 \frac{\partial u_1}{\partial z} - \frac{1}{2} (2 + \tau_z) r_0 \epsilon + \frac{2}{Re_1} \frac{\partial^2 u_1}{\partial r \partial z}] \]

For the fundamental components \( w_0 \) and \( w_1 \), the first condition follows from eq. (2.87), namely,

\[ w_0 = w_1 \] (2.92)

The second condition is not as easy as the first one. Consider the derivative of the continuity equation with respect to \( z \),

\[ \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} + \frac{i}{r} \frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial z^2} = 0. \] (2.93)

Consider also eq. (2.36),

\[ -i \nabla^2 w - 2(1 + \tau_z) \left[ \frac{\partial^2 v}{\partial r \partial z} + \frac{1}{r} \frac{\partial v}{\partial z} - \frac{i}{r} \frac{\partial u}{\partial z} \right] + \nabla^4 w = 0, \] (2.94)

from which we obtain

\[ -i \frac{\rho_0}{Re_0} \nabla^2 w_0 - 2(1 + \tau_z) \frac{\rho_0}{Re_0} \left[ \frac{\partial^2 v_0}{\partial r \partial z} + \frac{1}{r} \frac{\partial v_0}{\partial z} - \frac{i}{r} \frac{\partial u_0}{\partial z} \right] + \frac{\rho_0}{Re_0} \nabla^4 w_0 \]

\[ = \]

\[ -i \frac{\rho_1}{Re_1} \nabla^2 w_1 - 2(1 + \tau_z) \frac{\rho_1}{Re_1} \left[ \frac{\partial^2 v_1}{\partial r \partial z} + \frac{1}{r} \frac{\partial v_1}{\partial z} - \frac{i}{r} \frac{\partial u_1}{\partial z} \right] + \frac{\rho_1}{Re_1} \nabla^4 w_1. \] (2.95)

The addition of eq. (2.95) to \( 2(1 + \tau_z) \partial / \partial z \) eq. (2.90) leads to

\[ -i \frac{\rho_0}{Re_0} \nabla^2 w_0 - 2(1 + \tau_z) \frac{\rho_0}{Re_0} \left[ \frac{2 \partial v_0}{r \partial z} - \frac{2i \partial u_0}{r \partial z} \right] + \frac{\rho_0}{Re_0} \nabla^4 w_0 \]

\[ = \]

\[ -i \frac{\rho_1}{Re_1} \nabla^2 w_1 - 2(1 + \tau_z) \frac{\rho_1}{Re_1} \left[ \frac{2 \partial v_1}{r \partial z} - \frac{2i \partial u_1}{r \partial z} \right] + \frac{\rho_1}{Re_1} \nabla^4 w_1 \] (2.96)
From eq. (2.89) we can write

$$- 2(1 + \tau_z) \frac{\rho_0}{Re_0} \left[ \frac{2i \partial w_0}{r \partial r} + \frac{2i \partial u_0}{r \partial z} \right] = -2(1 + \tau_z) \frac{\rho_1}{Re_1} \left[ \frac{2i \partial w_1}{r \partial r} + \frac{2i \partial u_1}{r \partial z} \right]. \quad (2.97)$$

Adding eqs. (2.97) and (2.96), we obtain

$$-i \frac{\rho_0}{Re_0} \nabla^2 w_0 - 2(1 + \tau_z) \frac{\rho_0}{Re_0} \left[ \frac{2 \partial w_0}{r \partial z} + \frac{2i \partial w_0}{r \partial z} \right] + \frac{\rho_0}{Re_0} \nabla^4 w_0$$

$$= -i \frac{\rho_1}{Re_1} \nabla^2 w_1 - 2(1 + \tau_z) \frac{\rho_1}{Re_1} \left[ \frac{2 \partial w_1}{r \partial z} + \frac{2i \partial w_1}{r \partial z} \right] + \frac{\rho_1}{Re_1} \nabla^4 w_1. \quad (2.98)$$

From the continuity equation, we get

$$\frac{1}{r} \frac{\partial v_\alpha}{\partial z} = i \frac{\partial^2 u_\alpha}{\partial r \partial z} + \frac{i \partial u_\alpha}{r \partial z} + i \frac{\partial^2 w_\alpha}{\partial z^2}. \quad (2.99)$$

Substituting eq. (2.99) into eq. (2.96) to eliminate $\partial v_\alpha/\partial z$, we obtain

$$-i \frac{\rho_0}{Re_0} \nabla^2 w_0 - 2(1 + \tau_z) \frac{\rho_0}{Re_0} \left[ \frac{2 \partial^2 u_0}{r \partial r \partial z} + \frac{2i \partial^2 w_0}{r \partial z^2} \right] + \frac{\rho_0}{Re_0} \nabla^4 w_0$$

$$= -i \frac{\rho_1}{Re_1} \nabla^2 w_1 - 2(1 + \tau_z) \frac{\rho_1}{Re_1} \left[ \frac{2 \partial^2 u_1}{r \partial r \partial z} + \frac{2i \partial^2 w_1}{r \partial z^2} \right] + \frac{\rho_1}{Re_1} \nabla^4 w_1. \quad (2.100)$$

From eq. (2.87), we have

$$\frac{\partial u_0}{\partial z} = \frac{\partial u_1}{\partial z}. \quad (2.101)$$

Using eqs. (2.101) and (2.89) to solve for $\partial u_\alpha/\partial z$, we obtain

$$\left( \frac{\rho_0}{Re_0} - \frac{\rho_1}{Re_1} \right) \frac{\partial u_0}{\partial z} = \frac{\rho_1}{Re_1} \frac{\partial u_1}{\partial r} - \frac{\rho_0}{Re_0} \frac{\partial w_0}{\partial r}, \quad (2.102)$$

$$\left( \frac{\rho_0}{Re_0} - \frac{\rho_1}{Re_1} \right) \frac{\partial u_1}{\partial z} = \frac{\rho_1}{Re_1} \frac{\partial u_1}{\partial r} - \frac{\rho_0}{Re_0} \frac{\partial w_0}{\partial r}. \quad (2.103)$$

Multiplying eq. (2.91) by $2(1 + \tau_z)i$ and adding the resulting equation to eq. (2.100) yield the second condition

$$2(1 + \tau_z) \rho_0 [w_0 - \frac{1}{i \rho_0} \nabla^2 w_0 - \frac{2}{i \rho_0} \frac{\partial^2 w_0}{\partial z^2}] - \frac{\rho_0}{Re_0} \nabla^2 w_0 - \frac{\rho_0}{Re_0} \nabla^4 w_0$$
\[-2(1 + \tau_z)\rho_1[w_1 - \frac{1}{i\text{Re}_1} \nabla^2 w_1 - \frac{2}{i\text{Re}_1} \frac{\partial^2 w_1}{\partial z^2}] - \rho_1 \nabla^2 w_1 - \frac{\rho_1}{\text{Re}_1^2} \nabla^4 w_1 \]
\[+2(1 + \tau_z)^3 \tau_0 (\rho_0 - \rho_1) \left[ \frac{\rho_1 \text{Re}_1}{(\rho_0 \text{Re}_1 - \rho_1 \text{Re}_0)} \frac{\partial w_1}{\partial r} - \frac{\rho_0 \text{Re}_1}{(\rho_0 \text{Re}_1 - \rho_1 \text{Re}_0)} \frac{\partial w_0}{\partial r} \right] \]
\[= (2 + \tau_z)(1 + \tau_z)(\rho_1 - \rho_0) i \epsilon \tau_0. \quad (2.104)\]

For the third conditions, we use some results of the derivation of the boundary conditions at the side walls. From eq. (2.44) we have

\[
\frac{1}{\text{Re}} \frac{1}{r} \nabla^2 w - 2(1 + \tau_z) \left\{ -i \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) + 2(1 + \tau_z) \frac{\partial v}{\partial z} 
- \epsilon - \frac{1}{\text{Re}} \frac{\partial}{\partial r} \left( \nabla^2 w \right) - \frac{1}{\text{Re}} \frac{\partial^2 w}{\partial r \partial z^2} + \frac{1}{\text{Re}} \frac{\partial^3 u}{\partial z^3} \right\} + \frac{i}{r} \frac{1}{\text{Re}} \nabla^4 w = 0 \quad (2.105)\]

Writing this equation for both inner and outer flows and equating the resulting equations to eliminate \(\partial v_0 / \partial \phi\) lead to

\[
\frac{1}{\text{Re}_0} \frac{1}{r} \nabla^2 w_0 - 2(1 + \tau_z) \left\{ i \frac{\partial w_0}{\partial r} - \frac{1}{\text{Re}_0} \frac{\partial}{\partial r} \left( \nabla^2 w_0 \right) - \frac{1}{\text{Re}_0} \frac{\partial^3 w_0}{\partial r \partial z^2} \right\} + \frac{i}{r} \frac{1}{\text{Re}_0^2} \nabla^4 w_0 
- \frac{1}{\text{Re}_1} \frac{1}{r} \nabla^2 w_1 + 2(1 + \tau_z) \left\{ i \frac{\partial w_1}{\partial r} - \frac{1}{\text{Re}_1} \frac{\partial}{\partial r} \left( \nabla^2 w_1 \right) - \frac{1}{\text{Re}_1} \frac{\partial^3 w_1}{\partial r \partial z^2} \right\} - \frac{i}{r} \frac{1}{\text{Re}_1^2} \nabla^4 w_1 
- 2(1 + \tau_z) \left( \frac{1}{\text{Re}_0} - \frac{1}{\text{Re}_1} \right) \left[ \frac{1}{(\rho_0 \text{Re}_1 - \rho_1 \text{Re}_0)} \frac{\partial^3 w_1}{\partial r \partial z^2} - \rho_0 \text{Re}_1 \frac{\partial^3 w_0}{\partial r \partial z^2} \right] 
= 0 \quad (2.106)\]

We obtain the fourth condition by combining eqs. (2.105) and (2.98). We first rewrite eq. (2.98) as

\[-i \frac{\rho_0}{\text{Re}_0} (1 + \tau_z) \nabla^2 w_0 - 4(1 + \tau_z)^2 \frac{\rho_0}{\text{Re}_0} \frac{1}{r} \frac{\partial w_0}{\partial z} - 4(1 + \tau_z)^2 \frac{\rho_0}{\text{Re}_0} \frac{i}{r} \frac{\partial v_0}{\partial r} + \frac{\rho_0}{\text{Re}_0^2} \nabla^4 w_0 \]
\[= 0 \quad (2.107)\]
From eq. (2.105) we have

\[-4(1 + \tau_z)^2 \frac{\partial v_\alpha}{\partial z} = -\frac{1}{r \operatorname{Re}_\alpha} \frac{1}{r^2} \nabla^2 w_\alpha - \frac{i}{r} \frac{1}{\operatorname{Re}_\alpha} \nabla^4 w_\alpha + \frac{1}{\operatorname{Re}_\alpha} \frac{1}{r^2} \frac{\partial^3 w_\alpha}{\partial z^2} \]

Substituting eq. (2.108) into eq. (2.107) and making use of eqs. (2.102) and (2.103), we obtain

\[-i \frac{\partial}{\partial r} (\nabla^2 w_\alpha) - 2(1 + \tau_z) i \frac{\partial w_\alpha}{\partial r} + \frac{1}{r^2} \frac{\partial^3 w_\alpha}{\partial z^2} \]

To derive the fifth condition, we use eq. (2.42) found when deriving the boundary conditions at the side walls,
Multiplying eq. (2.110) by 2(1 + \tau_z) yields

\[-i2(1 + \tau_z)\frac{\partial}{\partial r}(\nabla^2 w_\alpha) - 4(1 + \tau_z)^2\left[\frac{i}{r}\nabla^2 w_\alpha + i\epsilon Re_\alpha + \frac{w_\alpha}{r} Re_\alpha + \frac{i}{r} \frac{\partial^2 w_\alpha}{\partial z^2}\right]
\]

\[-4(1 + \tau_z)^2 i Re_\alpha \frac{\partial v_\alpha}{\partial z} - 8(1 + \tau_z)^3 Re_\alpha \frac{\partial u_\alpha}{\partial z} + 4(1 + \tau_z)^2 \frac{\partial^3 v_\alpha}{\partial z^3} + \frac{1}{Re_\alpha} \frac{\partial}{\partial r}(\nabla^4 w_\alpha)\]

\[= 0. \quad (2.111)\]

Using eq. (2.108) we can write

\[4(1 + \tau_z)^2 i Re_\alpha \frac{\partial v_\alpha}{\partial z} - \frac{i}{r} \nabla^2 w_\alpha + \frac{1}{Re_\alpha} \nabla^4 w_\alpha\]

\[+ 2(1 + \tau_z)[Re_\alpha(\frac{\partial u_\alpha}{\partial z} - \frac{\partial w_\alpha}{\partial r}) - i Re_\alpha \epsilon - i \frac{\partial}{\partial r}(\nabla^2 w_\alpha) - i \frac{\partial^3 w_\alpha}{\partial r \partial z^2} + i \frac{\partial^3 u_\alpha}{\partial z^3}]\]

\[= 0. \quad (2.112)\]

Adding eq. (2.111) to eq. (2.112) yields

\[-4(1 + \tau_z) i \frac{\partial}{\partial r}(\nabla^2 w_\alpha) + 2(1 + \tau_z)\frac{1}{Re_\alpha} \frac{\partial}{\partial r}(\nabla^4 w_\alpha) - \frac{i}{r} \nabla^2 w_\alpha + \frac{1}{Re_\alpha} \frac{1}{r} \nabla^4 w_\alpha\]

\[+ 2(1 + \tau_z)[- Re_\alpha \frac{\partial w_\alpha}{\partial r} - i Re_\alpha \epsilon - i \frac{\partial^3 w_\alpha}{\partial r \partial z^2} + i \frac{\partial^3 u_\alpha}{\partial z^3} + Re_\alpha \frac{\partial u_\alpha}{\partial r}]\]

\[-4(1 + \tau_z)^2\left[\frac{i}{r} \nabla^2 w_\alpha + i\epsilon Re_\alpha + \frac{w_\alpha}{r} Re_\alpha + \frac{i}{r} \frac{\partial^2 w_\alpha}{\partial z^2}\right]\]

\[+ 4(1 + \tau_z)^2 \frac{\partial^3 v_\alpha}{\partial z^3} - 8(1 + \tau_z)^3 Re_\alpha \frac{\partial u_\alpha}{\partial z} = 0. \quad (2.113)\]

Equating the above equations and making use of eq. (2.102) and (2.103) yield

\[+2(1 + \tau_z)[\frac{1}{Re_0} \frac{\partial}{\partial r}(\nabla^4 w_0) - 2i \frac{\partial}{\partial r}(\nabla^2 w_0) - i \frac{\partial^3 w_0}{\partial r \partial z^2} - Re_0 \frac{\partial w_0}{\partial r}]\]
\[ +2(1 + \tau_z)(\frac{1}{Re_1}\frac{\partial}{\partial r}(\nabla^4 w_1) - 2i\frac{\partial}{\partial r}(\nabla^2 w_1) - i\frac{\partial^3 w_1}{\partial r \partial z^2} - Re_1\frac{\partial w_1}{\partial r}) \]

\[ -4(1 + \tau_z)^2[\frac{i}{r}\nabla^2 w_0 + Re_0\frac{w_0}{r}] + 4(1 + \tau_z)^2[\frac{i}{r}\nabla^2 w_1 + Re_1\frac{w_1}{r}] \]

\[ + \frac{i}{r}\nabla^2 w_1 - \frac{1}{Re_1}\frac{1}{r}\nabla^4 w_1 - \frac{i}{r}\nabla^2 w_0 + \frac{1}{Re_0}\frac{1}{r}\nabla^4 w_0 \]

\[ + (2(1 + \tau_z) - 8(1 + \tau_z)^3)(\frac{Re_0 - Re_1}{\rho_0 Re_1 - \rho_1 Re_0})(\rho_1 Re_0\frac{\partial w_1}{\partial r} - \rho_0 Re_1\frac{\partial w_0}{\partial r}) \]

\[ = [2(1 + \tau_z) + 4(1 + \tau_z)^2](Re_0 - Re_1)i\epsilon. \quad (2.114) \]

For the sixth condition, multiplying eq. (2.113) by \( \rho_\alpha/rRe_\alpha \) gives

\[ 2(1 + \tau_z)\rho_\alpha[-\frac{2i}{rRe_\alpha}\frac{\partial}{\partial r}(\nabla^2 w_\alpha) + \frac{1}{rRe_\alpha}\frac{\partial}{\partial r}(\nabla^4 w_\alpha) - \frac{1}{r}\frac{\partial w_\alpha}{\partial r} - \frac{i}{rRe_\alpha}\frac{\partial^3 w_\alpha}{\partial z^2}] \]

\[ -4(1 + \tau_z)^2\rho_\alpha[\frac{i}{r^2Re_\alpha}\nabla^2 w_\alpha + \frac{w_\alpha}{r^2} + \frac{i}{r^2Re_\alpha}\frac{\partial^2 w_\alpha}{\partial z^2}] - \frac{i\rho_\alpha}{r^2Re_\alpha}\nabla^2 w_\alpha + \frac{\rho_\alpha}{r^2Re_\alpha}\nabla^4 w_\alpha \]

\[ + 2(1 + \tau_z)\frac{i\rho_\alpha}{rRe_\alpha}\frac{\partial^3 w_\alpha}{\partial z^3} + (2(1 + \tau_z) - 8(1 + \tau_z)^3)\frac{\rho_\alpha}{r}\frac{\partial u_\alpha}{\partial z} + 4(1 + \tau_z)^2\frac{\rho_\alpha}{rRe_\alpha}\frac{\partial^3 v_\alpha}{\partial z^3} \]

\[ = 2(1 + \tau_z)(1 + 2(1 + \tau_z))\frac{i}{r}\rho_\alpha. \quad (2.115) \]

Multiplying eq. (2.98) by \((1 + \tau_z)\) and differentiating with respect to \( z \) twice, and making use of eq. (2.115) we obtain

\[ 2(1 + \tau_z)\rho_0[-\frac{i}{2Re_0}\nabla^2 \frac{\partial^2 w_0}{\partial z^2} + \frac{1}{2Re_0}\nabla^4 \frac{\partial^4 w_0}{\partial z^2} - \frac{2i}{rRe_0}\frac{\partial}{\partial r}(\nabla^2 \frac{\partial^2 w_0}{\partial z^2})] \]

\[ -2(1 + \tau_z)\rho_0[-\frac{i}{2Re_1}\nabla^2 \frac{\partial^2 w_1}{\partial z^2} + \frac{1}{2Re_1}\nabla^4 \frac{\partial^4 w_1}{\partial z^2} - \frac{2i}{rRe_1}\frac{\partial}{\partial r}(\nabla^2 \frac{\partial^2 w_1}{\partial z^2})] \]

\[ + 2(1 + \tau_z)\rho_0[\frac{1}{rRe_0}\frac{\partial}{\partial r}(\nabla^4 \frac{\partial^2 w_0}{\partial z^2}) - \frac{1}{r}\frac{\partial w_0}{\partial r} - \frac{i}{rRe_0}\frac{\partial^3 w_0}{\partial r \partial z^2}] \]

\[ -2(1 + \tau_z)\rho_1[\frac{1}{rRe_1}\frac{\partial}{\partial r}(\nabla^4 \frac{\partial^2 w_1}{\partial z^2}) - \frac{1}{r}\frac{\partial w_1}{\partial r} - \frac{i}{rRe_1}\frac{\partial^3 w_1}{\partial r \partial z^2}] \]
\[-4(1 + \tau_z)^2 \rho_0 \left[ \frac{i}{\bar{R}e_0} \frac{1}{r^2} \nabla^2 w_0 + \frac{w_0}{r^2} \frac{i}{\bar{R}e_0} \frac{\partial^2 w_0}{\partial z^2} + \frac{i}{r \bar{R}e_0} \frac{\partial^3 w_0}{\partial r \partial z^2} \right] \]
\[+ 4(1 + \tau_z)^2 \rho_1 \left[ \frac{i}{\bar{R}e_1} \frac{1}{r^2} \nabla^2 w_1 + \frac{w_1}{r^2} \frac{i}{\bar{R}e_1} \frac{\partial^2 w_1}{\partial z^2} + \frac{i}{r \bar{R}e_1} \frac{\partial^3 w_1}{\partial r \partial z^2} \right] \]
\[- \frac{\rho_0}{\bar{R}e_0} \frac{i}{r^2} \nabla^2 w_0 + \frac{\rho_0}{\bar{R}e_0} \frac{1}{r^2} \nabla^4 w_0 + \frac{\rho_1}{\bar{R}e_1} \frac{i}{r^2} \nabla^2 w_1 - \frac{\rho_1}{\bar{R}e_1} \frac{1}{r^2} \nabla^4 w_1 \]
\[+ 2(1 + \tau_z) \frac{i}{r} \left( \frac{\rho_1}{\bar{R}e_1} \frac{\partial^3 w_1}{\partial r \partial z^2} - \frac{\rho_0}{\bar{R}e_0} \frac{\partial^3 w_0}{\partial r \partial z^2} \right) \]
\[+ (2(1 + \tau_z) - 8(1 + \tau_z)^3) \left( \frac{\rho_0 - \rho_1}{r(\rho_0 \bar{R}e_1 - \rho_1 \bar{R}e_0)} \right) \left( \rho_1 \bar{R}e_0 \frac{\partial w_1}{\partial r} - \rho_0 \bar{R}e_1 \frac{\partial w_0}{\partial r} \right) \]
\[= [2(1 + \tau_z) + 4(1 + \tau_z)^2](\rho_0 - \rho_1) \frac{i}{r} \epsilon \quad (2.116) \]

which is the sixth condition.
CHAPTER III
Linear Viscous Analysis

3.1 Introduction

This chapter is concerned with the solutions of the linearized viscous equations describing the fundamentals of the axial velocities for the various configurations. In section 2, we present the solutions obtained by eigenfunction expansions. We first describe the method, then present how to calculate the moments, discuss the effects of the various configurations on the roll, pitch, and yaw moments, and finally present how to calculate the location of the void-fluid or two-fluid interface.

The method requires solving a sixth order eigenvalue problem. For given flight conditions, i.e. $\tau$ and $\theta$, and aspect ratio $\eta$, the eigenfunctions are only functions of the Reynolds number. Hence, all configurations require solving similar eigenvalue problems. For instance, in case of cylinders containing two immiscible fluids, the eigenvalue problem is solved once for $Re = Re_0$ to solve for the eigenfunctions in the inner region and once more for $Re = Re_1$ to solve for the eigenfunctions in the outer region.
Finding the eigenvalues is, however, a difficult task. To check the results obtained by eigenfunction expansions, we have also solved the same linear viscous equations by spectral techniques. The procedure and results are presented in section 3. The method proved very useful in providing an alternative way to compute the moments inexpensively and with comparable accuracy to those obtained by eigenfunction expansions. We have only done this for completely filled cylinders. The extension of the technique to compute the moments for the rest of the configurations is straightforward.

3.2 Solutions by eigenfunction expansions

3.2.1 Solutions procedure

For convenience, we introduce the scaled variables

\[ w_\alpha^1 = \epsilon w_\alpha, \]

\[ q_\alpha = (1 + i)\sqrt{Re_\alpha / 2}, \]

where \( \alpha \) is 0 in the inner region, 1 in the outer region, and

\[ (\tilde{r}, \tilde{z}) = \begin{cases} (r q_0, z q_0) & \text{if } 0 \leq r \leq r_f, \\ (r q_1, z q_1) & \text{if } r_f \leq r \leq 1. \end{cases} \]

We obtain the scaled form of the governing equations for the fundamental components of the axial velocities when eqs. (3.1) through (3.3) are substituted into eq. (2.37), namely,

\[ \tilde{\nabla}^2 w_\alpha - 2 \tilde{\nabla}^4 w_\alpha + \tilde{\nabla}^6 w_\alpha - \tau_0^2 \frac{\partial^2 w_\alpha}{\partial \tilde{z}^2} = 0, \]
where \( \tau_0 = 2(1 + \tau_z) \) and

\[
\hat{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}.
\]

Similarly, we obtain for the boundary conditions at the end walls \((z = \pm \eta)\),

\[
w_\alpha = 0, \quad \frac{\partial w_\alpha}{\partial z} = 0, \quad - \frac{\partial^3 w_\alpha}{\partial z^3} + 2 \hat{\nabla}_1^2 \frac{\partial^3 w_\alpha}{\partial z^3} + \frac{\partial^5 w_\alpha}{\partial z^5} = 0, \quad (3.5)
\]

where

\[
\hat{\nabla}_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}.
\]

Note that the form of the governing equations and the boundary conditions at the end walls are the same for both inner and outer regions. Hence, we can treat both inner and outer solutions in a similar fashion and consequently we drop the index \( \alpha \) unless otherwise necessary to distinguish between both solutions.

The governing equations (3.4) are homogeneous and have homogeneous boundary conditions in the \( \tilde{z} \) direction. This suggests the use of separation of variables to solve the linearized axial equations. We start by assuming the product solution

\[
w(\tilde{r}, \tilde{z}) = R(\tilde{r}) Z(\tilde{z})
\]

and we let

\[
\hat{\nabla}_1^2 R = B R, \quad (3.7)
\]

where \( B \) is a complex parameter (i.e. separation constant). When eqs. (3.6) and (3.7) are substituted into eq. (3.4), we obtain

\[
Z^{(6)} + (3 B - 2) Z^{(4)} + (3 B^2 - 4 B + 1 - \tau_0^2) Z'' + (B^3 - 2 B^2 + B) Z = 0, \quad (3.8)
\]
where prime denotes differentiation with respect to \( \tilde{z} \). A similar substitution into the boundary conditions at the end walls \((\tilde{z} = \pm \eta)\) provides the conditions

\[
Z = 0, \quad Z' = 0, \quad Z^{(5)} + (2B - 1)Z^{(3)} = 0 \quad \text{at} \quad \tilde{z} = \pm \eta. \tag{3.9}
\]

Eqs. (3.8) and (3.9) constitute an eigenvalue problem for the eigenvalue \( B \) and the eigenfunction \( Z \). This eigenvalue problem is converted into a system of nonlinear algebraic equations by assuming a solution to equation (3.8) in the form

\[
Z = \cos(a \tilde{z}), \tag{3.10}
\]

where \( a \) is a complex constant that depends on \( B \). The substitution of eq. (3.10) into eq. (3.8) yields the sixth-order algebraic equation for the constant \( a \),

\[
a^6 - (3B - 2)a^4 - (3B^2 - 4B + 1 - \tau^2_\theta)a^2 - (B^3 - 2B^2 + B) = 0. \tag{3.11}
\]

If \( \pm a_1, \pm a_2, \) and \( \pm a_3 \) denote six distinct solutions to eq. (3.11), then the general solution to eq. (3.8) can be written as

\[
Z = c_1 \frac{\cos(a_1 \tilde{z})}{\cos(a_1 \eta \eta)} + c_2 \frac{\cos(a_2 \tilde{z})}{\cos(a_2 \eta \eta)} + c_3 \frac{\cos(a_3 \tilde{z})}{\cos(a_3 \eta \eta)}, \tag{3.12}
\]

where \( c_1, c_2, \) and \( c_3 \) are constants of integration and eq. (3.11) can be written as

\[
a_1^2 + a_2^2 + a_3^2 - 3B + 2 = 0, \tag{3.13}
\]

\[
a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2 - 3B^2 + 4B - 1 + \tau^2_\theta = 0, \tag{3.14}
\]

\[
a_1^2 a_2^2 a_3^2 - B^3 + 2B^2 - B = 0. \tag{3.15}
\]
The application of boundary conditions (3.9) provides a linear homogeneous system of algebraic equations for the determination of the constants \( c_1, c_2, \) and \( c_3 \), namely,

\[
c_1 + c_2 + c_3 = 0, \tag{3.16}
\]

\[
a_1 c_1 \tan(a_1 q \eta) + a_2 c_2 \tan(a_2 q \eta) + a_3 c_3 \tan(a_3 q \eta) = 0, \tag{3.17}
\]

\[
c_1 \left[a_1^5 - (2B - 1) a_1^2 \right] \tan(a_1 q \eta) + c_2 \left[a_2^5 - (2B - 1) a_2^2 \right] \tan(a_2 q \eta)
\]
\[+ c_3 \left[a_3^5 - (2B - 1) a_3^2 \right] \tan(a_3 q \eta) = 0. \tag{3.18}
\]

In order for system (3.16) through (3.18) to have nontrivial solutions, its determinant must vanish, namely,

\[
(a_1^2 - a_2^2) (2B - 1 - a_1^2 - a_2^2) a_1 \tan(a_1 q \eta) a_2 \tan(a_2 q \eta)
\]
\[+ (a_2^2 - a_3^2) (2B - 1 - a_2^2 - a_3^2) a_2 \tan(a_2 q \eta) a_3 \tan(a_3 q \eta)
\]
\[+ (a_3^2 - a_1^2) (2B - 1 - a_3^2 - a_1^2) a_3 \tan(a_3 q \eta) a_1 \tan(a_1 q \eta) = 0. \tag{3.19}
\]

Equations (3.13) through (3.15) and (3.19) represent four nonlinear algebraic equations for the unknown quadruples \((B; a_1, a_2, a_3)\). Finding these quadruples is a nontrivial task, and details are described in Appendix B. The relatively simple form of equations (3.11) and (3.19), however, permits application of analytical tools unavailable to the numerical approach of Hall et al. (1987). Equation (3.11) has four isolated roots with real \( B \). Only the double root \( B = 1 \) (with \( a_1 = 0 \)) satisfies equation (3.19) and provides a nonoscillatory eigenfunction.
Figure 3.1: Plot of the first 60 eigenvalues on branch 1 (\(\triangle\)), branch 2 (\(\diamondsuit\)), and branch 3 (\(\bigcirc\)) for \(Re = 20, \tau = 0.1111, \eta = 3, \text{ and } \theta = 2^\circ\).

For sufficiently large Reynolds numbers (and \(q\)), it can be shown that one of the \(a_i\)'s, \(a_1\) say, must be located near the diagonal \(\arg(a_1) \approx -\pi/4\). We observe that equation (3.11) can be interpreted as determining either the \(a_i\) for given \(B\) or three solutions \(B_j, j = 1, 2, 3\) for any given \(a\). Varying \(a\) along the diagonal \(-\pi/4\), provides three branches as the locus of the \(B_j\), two of them \(B_1\) and \(B_2\), originating at \(B = 1\), the third \(B_3\), at \(B = 0\) as can be seen from figure 3.1. A small correction to these branches and the position of the eigenvalues along the branches is provided by equation (3.19) when it is solved along with equations (3.13) through (3.15) by the Newton-Raphson iteration. The process takes negligible CPU time when using today's computer workstations, i.e. Sun 3/180 or apollo DN3500.

Given a triplet \(B_j\), we obtain for each \(B_j\) the associated values \(a_{ji}, i = 1, 2, 3\),
Figure 3.2: Plot of the first 60 values of $a_i$ for $Re = 20, \tau = 0.1111, \eta = 3, \text{ and } \theta = 2^\circ$.

where the $a_j$ are located in a small neighborhood near the diagonal $-\pi/4$ and similar to the results for $B_j$, the values $a_j$ for each $a_i$ lie on three branches in the $a_i$-complex plane. Figures 3.2 through 3.4 show the first sixty—twenty on each branch—values (from the origin) for $a_1, a_2,$ and $a_4$ respectively at $Re = 20, \tau = 0.08333, \theta = 2^\circ, \text{ and } \eta = 3$. When the Reynolds number is increased, the branches maintain the same shape. However, the eigenvalues become more closely spaced and more difficult to find. We have, nonetheless, obtained results for Reynolds number up to $Re = 10^6$.

These eigenvalues are infinitely many and their corresponding eigenfunctions, except for the one corresponding to $B = B_0 = 1$, become more oscillatory as $|B| \to \infty$, which indicates that such a system of eigenfunctions is complete and
Figure 3.3: Plot of the first 60 values of $a_2$ for $Re = 20, \tau = 0.1111, \eta = 3$, and $\theta = 2^\circ$.

Figure 3.4: Plot of the first 60 values of $a_3$ for $Re = 20, \tau = 0.1111, \eta = 3$, and $\theta = 2^\circ$. 
can be used as a basis for spanning the solution. This system can be ordered in any desirable fashion. However, we prefer that eigenfunctions be ordered in the order of increasing number of oscillations or zeros. Since the number of zeros of eigenfunctions corresponding to eigenvalues on one branch varies in a similar fashion as that of eigenfunctions corresponding to eigenvalues on the other two branches, we must include in the solution as many eigenfunctions corresponding to eigenvalues on one branch as those corresponding to eigenvalues on other branches.

For convenience, we denote by \( \{B_m\}, \{a_{1m}\}, \{a_{2m}\}, \{a_{3m}\}, m = 0, 1, 2, \ldots, \) the solution sets to the nonlinear system (3.13) through (3.15) and (3.19) ordered from the neighborhood of \( B = 0 \) and \( B = 1 \) to \( B \to \infty \) simultaneously along the three branches. For example \( B_0 \) is on branch 1, \( B_1 \) is on Branch 2, \( B_2 \) is on branch 3, \( B_3 \) is on branch 1, and so on. With this in mind, the closed form of the eigenfunctions associated with the eigenvalues \( \{B_m\} \) are found by expressing \( c_2 \) and \( c_3 \) in terms of \( c_1 \) according to (3.16) and (3.17). If we normalize them by choosing \( c_1 = (1 + i) \cos(a_1 q \eta) \), we obtain the infinite set of eigenfunctions

\[
Z_m = (1 + i) [\cos(a_{1m} \tilde{z}) + c_{2m} \cos(a_{2m} \tilde{z}) + c_{3m} \cos(a_{3m} \tilde{z})], \quad m \neq 0, \quad (3.20)
\]

where

\[
c_{2m} = \frac{a_{1m} \sin(a_{1m} q \eta) \cos(a_{3m} q \eta) - a_{3m} \sin(a_{3m} q \eta) \cos(a_{1m} q \eta)}{a_{3m} \sin(a_{3m} q \eta) \cos(a_{2m} q \eta) - a_{2m} \sin(a_{2m} q \eta) \cos(a_{3m} q \eta)},
\]

\[
c_{3m} = \frac{a_{2m} \sin(a_{2m} q \eta) \cos(a_{1m} q \eta) - a_{1m} \sin(a_{1m} q \eta) \cos(a_{2m} q \eta)}{a_{3m} \sin(a_{3m} q \eta) \cos(a_{2m} q \eta) - a_{2m} \sin(a_{2m} q \eta) \cos(a_{3m} q \eta)}.
\]
and for the special case corresponding to

\[ B_0 = 1, \ a_{10} = 0, \ a_{20} = \frac{\sqrt{(4 \tau_0^2 + 1)^{1/2} + 1}}{\sqrt{2}}, \ a_{30} = \frac{\sqrt{1 - (4 \tau_0^2 + 1)^{1/2}}}{\sqrt{2}}, \]  

(3.21)

we have

\[ Z_0 = (1 + i)[1 - \frac{a_{30}}{\Delta} \sin(a_{30} q \eta) \cos(a_{20} \bar{z}) + \frac{a_{20}}{\Delta} \sin(a_{20} q \eta) \cos(a_{30} \bar{z})], \]  

(3.22)

where

\[ \Delta = a_{30} \cos(a_{20} q \eta) \sin(a_{30} q \eta) - a_{20} \cos(a_{30} q \eta) \sin(a_{20} q \eta). \]

For large Reynolds numbers, plots of the eigenfunctions reveal the presence of a small thin layer near the end walls (Ekman layer) where the greatest changes occur.

We now turn to the solution of equation (3.7). Since \( B = 0 \) is not an eigenvalue, the solutions to eq. (3.7) corresponding to the infinitely many eigenvalues are simply

\[ R_m(\bar{r}) = I_1(\sqrt{B_m} \bar{r}), \ m = 0, 1, 2, 3, \ldots \]  

(3.23)

Let \( Z_m^0, Z_m^1 \) be the eigenfunctions that are solutions to the above eigenvalue problem for \( q = q_0, q_1 \) respectively and their corresponding eigenvalues be \( B_m^0 \) and \( B_m^1 \), then the inner solution is given by

\[ w_0 = \sum_{m=0}^{\infty} F_m^0 I_1(\sqrt{B_m^0} \bar{r}) Z_m^0(\bar{z}), \]  

(3.24)

while the outer solution takes the form

\[ w_1 = \sum_{m=0}^{\infty} \left[ F_m^1 I_1(\sqrt{B_m^1} \bar{r}) + G_m^1 K_1(\sqrt{B_m^1} \bar{r}) \right] Z_m^1(\bar{z}), \]  

(3.25)
where \( I_1 \) and \( K_1 \) are the modified Bessel functions of the first and second kind of order 1 respectively. The expansion coefficients \( F_m^\alpha \) and \( G_m^1 \) are found by forcing the solutions to satisfy the boundary conditions at the side walls and at the interface depending on the configuration considered. For completely filled cylinders \((r_0 = 1)\), we have at \( r = 1 \)

\[
\sum_{m=0}^{\infty} F_m^0 \mathcal{L}_j^w I_1(\sqrt{B_m^0 \hat{r}}) Z_m^0(\hat{z}) = C_j^w, \quad j = 1, 2, 3, \quad (3.26)
\]

where \( \mathcal{L}_j^w \) are side wall scaled boundary operators. For convenience, they are listed in Appendix A along with the forcing constants \( C_j^w \). For cylinders containing a central rod, we have at \( r = 1 \) and at \( r = r_0 \)

\[
\sum_{m=0}^{\infty} F_m^1 \mathcal{L}_j^w I_1(\sqrt{B_m^1 \hat{r}}) Z_m^1(\hat{z}) + G_m^1 \mathcal{L}_j^w K_1(\sqrt{B_m^1 \hat{r}}) Z_m^1(\hat{z}) = C_j^w, \quad j = 1, 2, 3, \quad (3.27)
\]

while for partially filled cylinders, in addition to condition (3.27) applied at \( r = 1 \), we have at \( r = r_0 \)

\[
\sum_{m=0}^{\infty} F_m^1 \mathcal{L}_j^v I_1(\sqrt{B_m^1 \hat{r}}) Z_m^1(\hat{z}) + G_m^1 \mathcal{L}_j^v K_1(\sqrt{B_m^1 \hat{r}}) Z_m^1(\hat{z}) = C_j^v, \quad j = 1, 2, 3, \quad (3.28)
\]

where \( \mathcal{L}_j^v \) are scaled void-fluid interface operators and \( C_j^v \) are forcing constants, and are listed in Appendix A. In addition to condition (3.27) applied at \( r = 1 \), for two-fluid flows, we have

\[
\sum_{m=0}^{\infty} F_m^0 \mathcal{L}_j^0 I_1(\sqrt{B_m^0 \hat{r}}) Z_m^0(\hat{z}) - \sum_{m=0}^{\infty} F_m^1 \mathcal{L}_j^1 I_1(\sqrt{B_m^1 \hat{r}}) Z_m^1(\hat{z})
\]
\[- \sum_{m=0}^{\infty} G_m^1 \mathcal{L}_j^1 K_1(\sqrt{B_m^1 \tilde{r}}) Z_m^1(\tilde{z}) = C_j, \quad j = 1, \ldots, 6, \] (3.29)

where \( \mathcal{L}_j^0 \) and \( \mathcal{L}_j^1 \) are scaled two-fluid-interface operators, and are listed in Appendix A along with the forcing constants \( C_j \).

These conditions are too difficult to satisfy analytically and the infinite series are truncated to \( M \) terms. The appropriate conditions for each configuration are converted into a system of linear algebraic equations for the coefficients \( F_m^0, G_m^1 \), and/or \( F_m^1 \) by collocation or least squares method. When using the collocation method, each condition is satisfied exactly at \( M/3 \) equidistant points along only half of the cylinder length since the solution is symmetric about its center plane, \( \tilde{z} = 0 \). Since the number of conditions and coefficients vary with the type of configuration (3 for completely filled cylinders, 6 for either cylinders containing a central rod or partially filled cylinders, and 9 for cylinders containing two fluids), the size of the algebraic system changes from configuration to configuration for fixed collocation points. For completely filled cylinders, we generate a \( M \times M \) system for determining \( M \) coefficients, \( \{F_m^0; m = 0, \ldots, M - 1\} \). While for cylinders containing a partial fill or a central rod, we generate a \( 2M \times 2M \) system for the determination of 2\( M \) coefficients, \( \{F_m^1; m = 0, \ldots, M - 1\} \) and \( \{G_m^1; m = 0, \ldots, M - 1\} \). Finally for cylinders containing two immiscible fluids, a \( 3M \times 3M \) system is generated for determining 3\( M \) coefficients, \( \{F_m^0; m = 0, \ldots, M - 1\} \), \( \{F_m^1; m = 0, \ldots, M - 1\} \) and \( \{G_m^1; m = 0, \ldots, M - 1\} \). The resulting system of equations is solved by Gauss
Figure 3.5: Convergence of the value of $M_x/\left(\rho \omega^2 a^5\right)$ for completely filled cylinders at $Re = 20, \tau = 0.16667, \eta = 4.368,$ and $\theta = 20^\circ$ obtained by collocation method (Δ) and the least squares method (○).

elimination. For completely filled cylinders, the whole process including evaluation of moments with 120 eigenfunctions ($M = 120$) takes about 0.24 second CPU time on a Cray Y-MP8/864 for the case of $Re = 20, \eta = 4.368, \tau = 0.16667,$ and $\theta = 20^\circ$. The same process for cylinders containing a central rod or a void takes about 0.86 second, and for cylinders containing two immiscible fluids it takes about 1.84 second on a Cray Y-MP8/864.

When using the discrete least squares method we over-determined the system by satisfying the boundary conditions at $M$ equidistant points rather than $M/3$ points. We have only done this for completely filled cylinders to compare convergence properties. Figure 3.5 and table 3.1 compare the convergence of the yaw
Figure 3.6: Convergence of the expansion coefficients for completely filled cylinders at $Re = 20, \tau = 0.16667, \eta = 4.368$, and $\theta = 20^0$ obtained by collocation.

moment for $Re = 20, \eta = 4.368, \tau = 0.16667$, and $\theta = 20^0$ obtained by both methods. As expected, the yaw moment must converge to the same value regardless of the method used. The CPU time for this method is of the same order as that for the collocation method. We see no reason why we should favor one over the other. Hence, from here on, all results are obtained by collocation if they are obtained by eigenfunction expansions. Figure 3.6 shows the convergence of the expansion coefficients for completely filled cylinders at the same parameters as in figure 3.5. While the convergence of the coefficients is spectral, the convergence of the solution at any location may not be spectral since in the expansion, the coefficients are also multiplied by Bessel functions that may increase exponentially depending on the values of their arguments $q\sqrt{B_m r}$. Table 3.2 through 3.5 present the con-
Table 3.1: Convergence of the value of $M_x/(\rho_0^2 a^5)$ for completely filled cylinders at $Re = 20, \tau = 0.16667, \eta = 4.36842$, and $\theta = 20^\circ$.

<table>
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<tr>
<th>No of Eigenfunctions</th>
<th>Collocation</th>
<th>Least squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>0.081914829</td>
<td>0.082767458</td>
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<td>45</td>
<td>0.082209134</td>
<td>0.082589925</td>
</tr>
<tr>
<td>54</td>
<td>0.082348282</td>
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<td>63</td>
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<td>0.082498105</td>
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<tr>
<td>72</td>
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<td>0.082477601</td>
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<td>0.082467566</td>
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<tr>
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</tr>
<tr>
<td>117</td>
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<td>0.082466807</td>
</tr>
</tbody>
</table>

Convergence of the yaw, pitch, and roll moments for the various configurations for the same parameters as in table 3.1 and a fill radii of 0.2 for partially filled cylinders and 0.5 for cylinders containing a central rod and cylinders containing two fluids of density ratio $\rho_0/\rho_1 = 0.9$. As can be noted from the tables, the number of eigenfunctions needed for a given accuracy increases with the degree of complexity of the configuration—completely filled cylinders being the least complex and cylinders containing two fluids being the highest.

### 3.2.2 Calculation of Moments

We use the volume integral approach expressions described in chapter 2. If we let

$$S_1^0(r_0) = \int_{-\eta}^{\eta} \int_{0}^{2\pi} \int_{0}^{r_0} v_2^0(r, \phi, z)r^2 \cos(\phi) dr d\phi dz,$$  \hspace{1cm} (3.30)

$$S_1^1(r_0) = \int_{-\eta}^{\eta} \int_{0}^{2\pi} \int_{0}^{r_0} v_z^1(r, \phi, z)r^2 \cos(\phi) dr d\phi dz,$$ \hspace{1cm} (3.31)
Table 3.2: Convergence of the moments for completely filled cylinders at $Re = 20$, $\eta = 4.36842$, $\tau = 0.16667$, and $\theta = 20^\circ$.

<table>
<thead>
<tr>
<th>No of Eigenfunctions</th>
<th>$M_x/(\rho \omega^2 a^5)$</th>
<th>$M_y/(\rho \omega^2 a^5)$</th>
<th>$M_z/(\rho \omega^2 a^5)$</th>
</tr>
</thead>
<tbody>
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Table 3.3: Convergence of the moments for cylinders containing a central rod at \(Re = 20, r_0 = 0.5, \eta = 4.368, \tau = 0.16667,\) and \(\theta = 20^\circ\).

<table>
<thead>
<tr>
<th>No of Eigenfunctions</th>
<th>(M_x/(\rho \omega^2 a^5))</th>
<th>(M_y/(\rho \omega^2 a^5))</th>
<th>(M_z/(\rho \omega^2 a^5))</th>
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Table 3.4: Convergence of the moments for partially filled cylinders at $Re = 20$, $r_0 = 0.2$, $\eta = 4.368$, $\tau = 0.16667$, and $\theta = 20^\circ$.

<table>
<thead>
<tr>
<th>No of Eigenfunctions</th>
<th>$M_x/(\rho \omega^2 a^5)$</th>
<th>$M_y/(\rho \omega^2 a^5)$</th>
<th>$M_z/(\rho \omega^2 a^5)$</th>
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Table 3.5: Convergence of the moments for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 20$, $\rho_0/\rho_1 = 0.9$, $r_0 = 0.5$, $\eta = 4.368$, $\tau = 0.16667$, and $\theta = 20^\circ$.

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\[ S^0_2(r_0) = \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^{r_0} v^0_2(r, \phi, z) r^2 \sin(\phi) dr d\phi dz, \quad (3.32) \]

and

\[ S^1_2(r_0) = \int_{-\pi}^{\pi} \int_0^{2\pi} \int_0^{r_0} v^1_2(r, \phi, z) r^2 \sin(\phi) dr d\phi dz, \quad (3.33) \]

then for completely filled cylinders we have

\[ M_x = 2 \cos \theta (\rho_0 \omega^2 a^5) S^0_1(r_0 = 1), \quad (3.34) \]

\[ M_y = 2 \cos \theta (\rho_0 \omega^2 a^5) S^1_2(r_0 = 1), \quad (3.35) \]

for cylinders containing a central rod or a void, we have

\[ M_x = 2 \cos \theta (\rho_1 \omega^2 a^5) S^1_1(r_0), \quad (3.36) \]

\[ M_y = 2 \cos \theta (\rho_1 \omega^2 a^5) S^1_2(r_0), \quad (3.37) \]

and for cylinders containing two immiscible fluids, we have

\[ M_x = 2 \cos \theta (\rho_0 \omega^2 a^5) S^0_1(r_0) + 2 \cos \theta (\rho_1 \omega^2 a^5) S^1_1(r_0), \quad (3.38) \]

\[ M_y = 2 \cos \theta (\rho_0 \omega^2 a^5) S^0_2(r_0) + 2 \cos \theta (\rho_1 \omega^2 a^5) S^1_2(r_0), \quad (3.39) \]

The substitution of the fundamental of the axial velocities,

\[ v^\omega_z = \epsilon[w_\alpha(r, z)e^{i\phi} + \bar{w}_\alpha(r, z)e^{-i\phi}] \quad (3.40) \]

into \( S^\alpha = S^1_1 - i S^2_2 \) yields

\[ S^0 = 4 \pi \epsilon \int_{-\pi}^{\pi} \int_0^{r_0} w_0(r, z) r^2 dr dz, \quad (3.41) \]
and

\[ S^1 = 4 \pi \epsilon \int_0^\eta \int_{r_0}^1 w_1(r,z)r^2 dr dz. \]  \hfill (3.42)

We now substitute the expansion of \( w_\alpha(r,z) \) into eqs. (3.41) and (3.42) to get for the inner region,

\[ S^0 = 4 \pi \epsilon \sum_{m=0}^{M-1} F^0_m \int_0^1 r^2 I_1(\sqrt{B^0_m q_0} r) dr \int_0^n Z^0_m(z) dz, \]  \hfill (3.43)

and for the outer region we get

\[
S^1 = 4 \pi \epsilon \sum_{m=0}^{M-1} \{ F^1_m \int_{r_0}^1 r^2 I_1(\sqrt{B^1_m q_1} r) dr \\
+ G^1_m \int_{r_0}^1 r^2 K_1(\sqrt{B^1_m q_1} r) dr \} \int_0^n Z^1_m(z) dz, \]

where one can easily show that

\[ \int_0^1 r^2 I_1(\sqrt{B_m q_1} r) dr = \frac{1}{q \sqrt{B_m}} [I_2(q \sqrt{B_m})], \]  \hfill (3.45)

\[ \int_{r_0}^1 r^2 I_1(\sqrt{B_m q_1} r) dr = \frac{1}{q \sqrt{B_m}} [I_2(q \sqrt{B_m}) - r_0^2 I_2(r_0 q \sqrt{B_m})], \]  \hfill (3.46)

\[ \int_{r_0}^1 r^2 K_1(\sqrt{B_m q_1} r) dr = \frac{1}{q \sqrt{B_m}} [r_0^2 K_2(r_0 q \sqrt{B_m}) - K_2(q \sqrt{B_m})], \]  \hfill (3.47)

and if \( a_{1m} = 0 \), we have

\[ \int_0^n Z_m(z) dz = (1 + i) \frac{1}{q} [\eta q - \frac{a_{30}^2 - a_{20}^2}{a_{20} a_{30} \Delta} \sin(a_{20} q \eta) \sin(a_{30} q \eta)], \]  \hfill (3.48)
whereas when \( a_{1m} \neq 0 \), we have

\[
\int_0^\eta Z_m(z)dz = (1 + i)^{-1} q \sin(a_{1m} q \eta) \left[ \frac{1}{a_{1m}} - \frac{a_{3m} \Delta_{2m} + a_{2m} \Delta_{3m}}{a_{2m} a_{3m} \Delta_m} \right], \quad (3.49)
\]

where

\[
\Delta_{2m} = a_{1m} \delta_{3m} - a_{3m} \delta_{1m}, \quad \Delta_{3m} = a_{2m} \delta_{1m} - a_{1m} \delta_{2m},
\]

\[
\Delta_m = a_{2m} \delta_{3m} - a_{3m} \delta_{2m},
\]

with

\[
\delta_{jm} = a_{jm}^5 - (2 B_m - 1) a_{jm}^3; \quad j = 1, 2, 3.
\]

We have first calculated these moments for completely filled cylinders and the results are shown in figures 3.7 through 3.11. Figures 3.7 through 3.9 show plots of the roll, pitch, and yaw moments versus the Reynolds number at \( \tau = 0.1667 \), \( \eta = 4.368 \), and \( \theta = 20^\circ \) respectively. The results compare quite well with those obtained by the 3D-Spectral code of Herbert & Li (1987) for both yaw and roll moments, i.e. at \( Re = 900 \) the difference between the yaw moment predicted by the spectral code and that predicted by our method is about 1.4%. However, figure 3.8 shows a relatively significant difference between the pitch moments predicted by the two methods. This significance is attributed to the fact that the linear theory does not predict the mean flow distortion (\( v^0 \)) needed for accurate prediction of the pitch moment. This difference increases in magnitude as the Reynolds number increases. Nevertheless, it is negligible for engineering applications, i.e. 4% for \( Re = 900 \).
Figure 3.7: Roll moment versus Reynolds number for completely filled cylinders at $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. Comparison with the results of Herbert & Li (1987) (+), Rosenblat et al. (1986) ($\triangle$), and Experimental data of Miller (1982) (o).

Figure 3.8: Pitch moment versus Reynolds number for completely filled cylinders at $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. Comparison with the results of Herbert & Li (1987) (+) and Rosenblat et al. (1986) ($\triangle$).
We have also compared our results to those of Rosenblat et al. (1987) obtained by a finite element code. Figure 3.8 shows that for the pitch moment, the discrepancy with the finite element code is slightly more severe than that with the spectral code, i.e. the difference is about 10.7 % for $Re = 500$, and we believe it is attributed to the low accuracy of the finite element code since it also does not predict the yaw and roll moments accurately as can be seen from figures 3.7 and 3.9. For flows at $Re = 500$, the difference between the yaw moment obtained by the finite element code and that obtained by our method is about 14 % as opposed to 0.6 % for the difference between yaw moment obtained by the spectral code and that obtained by our method. The discrepancies for the yaw and roll moments...
are also more severe in the neighborhood of the Reynolds number at which both
moments acquire maxima. These maxima can be responsible for flight failure of
the projectile. Since this phenomena occur at relatively low Reynolds number,
$Re \approx 30$, the flight instability resulting from it is referred to as viscous instability.

With our method, we can obtain results for flows at high Reynolds numbers.
We have calculated the yaw, roll, and pitch moments for cylinders of different
aspect ratios at different Reynolds numbers for a given coning frequency. Figure
3.10 shows a plot of the yaw moment versus the aspect ratio for $\tau = 0.08333$ and
$\theta = 2^\circ$. The figure clearly shows that the yaw moment, and consequently the roll
moment since $M_x = M_{x \tan \theta}$, acquires maxima at critical aspect ratios. As the
Reynolds number increases, these moments increase dramatically at these critical
aspect ratios. We believe this phenomena is due to resonance with inertial waves
and can be responsible for flight failure of liquid-filled projectiles. The locus of the
critical aspect ratios that lead to resonance will be discussed in the next chapter
where the inviscid equations are solved for the various configurations.

Similar results are obtained for the pitch moment and they are presented in
figure 3.11. We note, however, the qualitative differences between the behavior of
the roll moment and the pitch moment in the neighborhood of the critical aspect
ratios. The pitch moment approaches $\infty$ as the critical aspect ratio is approached
from the right and it approaches $-\infty$ as the critical aspect ratio is approached
from the left while the yaw and roll moments approach $\infty$ as the critical aspect
Figure 3.10: Yaw moment versus aspect ratio for completely filled cylinders at \( \tau = 0.08333, \theta = 2^0, Re = 10^3 \) (---), and \( Re = 10^4 \) (—).

Figure 3.11: Pitch moment versus aspect ratio for completely filled cylinders at \( \tau = 0.08333, \theta = 2^0, Re = 10^3 \) (---), and \( Re = 10^4 \) (—).
ratio is approached from either the left or the right. If the coning frequency is increased the critical aspect ratios shift to the right and vice versa. We recognize that in real life this frequency is not constant but rather changing, and its slight variations can cause this moment to oscillate between a large negative value and a large positive value. This of course can cause an increase in the nutation angle and consequently can lead to flight failure of projectiles.

One way to avoid resonance for a given coning frequency, is to avoid these critical aspect ratios. The versatility of our method makes this possible since it does not require lots of computing time on one hand and it can handle flows at high Reynolds numbers, unaccessible to other approaches, on the other hand. Moreover, it is quite efficient for flight simulators that are routinely used to predict flight instabilities of projectiles. Rather than reading the moments for different flight conditions from massive tables generated apriori to flight simulation, they can be found interactively at negligible computing costs.

An alternative way to avoiding resonance if the aspect ratio cannot be modified due to other design constraints is to design the cylinder to contain a central rod or have the cylinder partially filled during production. Figure 3.12 shows plots of the roll moments versus the fill ratio obtained for both configurations at $\tau = 0.1111$, $\eta = 3.35$, $\theta = 2^0$, and $Re = 10^4$. We have chosen these parameters because they lead to resonance for completely filled cylinders. The figure clearly indicates that the moments can be cut by as much as 96% when using cylinders with a central rod
Figure 3.12: Roll moment versus fill ratio $V_1/V$ for partially filled cylinders (○) and cylinders containing a central rod (△) at $Re = 10^4$, $\eta = 3.35$, $\tau = 0.1111$, and $\theta = 20^\circ$.

of $r_0 = .15$ ($V_1/V \approx 98\%$) and by as much as 95% when the cylinder is partially filled with a fill ratio of 75% ($r_0 = 0.5$). While the central rod configuration seems more effective, the partial fill configuration allows room for expansion and might be preferred for certain applications.

Unfortunately, it is not always the case that these configurations reduce the moments for any given flight conditions and geometric characteristics with any fill ratio. Certain fill ratios can be responsible for the onset of resonance. In fact, completely filled cylinders are only a special case of both configurations since as $r_0 \to 0$ both configurations approach the complete fill configuration. We, thus, need to study the effect of the fill ratio on the onset of resonance. Figures 3.13
and 3.14 show plots of the roll moment versus \( r_0 \) at different Reynolds numbers for cylinders containing a central rod at \( \tau = 0.08674, \eta = 4.5, \) and \( \theta = 20^\circ \). Similar results for partially filled cylinders are presented in figures 3.15 and 3.16. The figures show that for Reynolds numbers up to about 30, the roll moment increases with the Reynolds number for any given fill ratio. It does level out at \( Re \approx 30 \), then it starts descending, but not for any fill ratio. For certain fill ratios, this moment starts to ascend again as the Reynolds number increases. It continues to increase as \( Re \) increases. This phenomena is, thus, inviscid in nature and is due to resonance with inertial waves.

The critical fill ratio at which the roll moment is most amplified at large Reynolds numbers depends on the configuration considered. For the parameters
Figure 3.14: Roll moment versus fill radius for cylinders containing a central rod at \( \eta = 4.5 \), \( \tau = 0.08674 \), and \( \theta = 20^\circ \).

Figure 3.15: Roll moment versus fill radius for partially filled cylinders at \( \eta = 4.5 \), \( \tau = 0.08674 \), and \( \theta = 20^\circ \).
Figure 3.16: Roll moment versus fill radius for partially filled cylinders at $\eta = 4.5$, $\tau = 0.08674$, and $\theta = 20^\circ$.

depicted in figures 3.13 through 3.16, we see that the critical fill radius for cylinders containing a central rod is much smaller than that for partially filled cylinders. These critical values also depend on the aspect ratio and the coning frequency as can be seen from figures 3.17 through 3.19. For certain aspect ratios and coning frequencies there exist more than one critical radius as in figure 3.19. We will discuss the locus of these critical radii in the next chapter. We should mention here that resonance also affects the pitch moment as can be seen from figures 3.20 and 3.21. Like the results obtained for completely filled cylinders, the pitch moment is decreased dramatically as the critical radius is approached from the left while it increases dramatically when the critical radius is approached from the right as the Reynolds numbers is increased. Far from these critical radii, the moment increases
Figure 3.17: Roll moment versus fill radius for cylinders containing a central rod (---) and partially filled cylinders (- - -) at \( Re = 10^4, \eta = 3.0, \tau = 0.1111, \) and \( \theta = 2^\circ. \)

A quick glance at figure 3.15 reveals that the roll moment is almost constant for up to \( r_0 \approx 0.4, \) these results are very useful since in practice cylinders are often designed to carry a partial fill for safety reasons for that liquids might expand. Consequently, this property allow for no alternative design criteria from that for completely filled cylinders. However, we caution the reader that the critical radius move to the left or right as the coning frequency is altered and we recommend consulting with the results presented in the next chapter to know these critical radii for a wide range of aspect ratios and coning frequencies.

While the partial fill and central rod configurations can eliminate the insta-
Figure 3.18: Roll moment versus fill radius for cylinders containing a central rod (—) and partially filled cylinders (-----) at $Re = 10^4$, $\eta = 2.0$, $\tau = 0.1111$, and $\theta = 2^\circ$.

Figure 3.19: Roll moment versus fill radius for cylinders containing a central rod (—) and partially filled cylinders (-----) at $Re = 10^4$, $\eta = 1.5$, $\tau = 0.1111$, and $\theta = 2^\circ$. 
Figure 3.20: Pitch moment versus fill radius for cylinders containing a central rod at $\eta = 4.5$, $\tau = 0.08674$, and $\theta = 20^0$.

Figure 3.21: Pitch moment versus fill radius for partially filled cylinders at $\eta = 4.5$, $\tau = 0.08674$, and $\theta = 20^0$. 
bility due to inertial waves, they certainly fail to eliminate the viscous instability. Figures 3.12 through 3.15 show that the roll moments for each configuration peak at $Re \approx 30$ for fill ratio not leading to resonance. It has been suggested that having a low viscosity high density additive in contact with the walls to lubricate the high viscosity core fluid might lessen this peak and thereby eliminate the viscous instability. It is for this reason that we are studying the dynamics of two-fluid flows inside spinning and coning cylinders.

We have calculated the roll moment caused by the flow in a completely filled cylinder at $\tau = 0.008$, $\eta = 4.5$, and $\theta = 1^\circ$ for different Reynolds numbers and found out that it acquires a maximum at $Re \approx 25$. We have made sure that $\tau$ and $\eta$ picked do not lead to resonance at high Reynolds numbers. We used these parameters to study the effect of the density and fill ratio on the roll moment for cylinders containing two immiscible fluids. We used an inner fluid of $Re_0 = 25$ and an outer fluid of $Re_1 = 10^4$. The reason for choosing $\tau$ so small is to make sure that the outer fluid stays in contact with the side walls for wide range of density ratios. Figure 3.22 shows plots of the ratio of the roll moment for cylinders containing two fluids to that for cylinders filled with the inner fluid versus the ratio of the volume of the additive to the volume of the cylinder ($V_1/V$). When $V_1/V = 0$, the cylinder is completely filled with the high-viscosity fluid and when $V_1/V = 1$, the cylinder is completely filled with the low-viscosity fluid.

Figure 3.22 clearly shows that for a density ratio $\rho_0/\rho_1 = 0.98$, the roll moment
Figure 3.22: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

can be cut by as much as 30% with a fill ratio of only 5%. This value decreases as the density ratio is reduced. As the fill ratio is varied, we see the roll moment acquires maxima that increase as the density ratio is reduced. The largest of these maxima occurs at $V_1/V \approx 0.65$. This maximum becomes larger than the value for completely filled cylinders at $\rho_0/\rho_1 \approx 0.75$. The smaller maxima are due to resonance with inertial waves of shorter wavelengths than that corresponding to the largest maximum.

For the purpose of only reducing the roll moment and not actually transporting two fluids, we are interested in small fill ratios. We concentrate on those for a moment. Figure 3.23 shows plots of the roll moment ratio versus the fill ratio
Figure 3.23: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

for up to $V_1/V = 20\%$ for different density ratios. We can clearly see that the roll moment is no longer reduced for density ratios below about 40\% but rather increased. This demonstrate the effect of the density ratio on the roll moment: It is not always the case that the roll moment is reduced by having a lower viscosity additive in contact with the wall.

For the purpose of transporting two fluids, we would like to consider all possible values for the fill ratio. Figure 3.24 shows plots of roll moment ratio for density ratios less than those depicted in figure 3.21. We can see clearly from this figure, the development of a secondary peak at $V_1/V \approx 0.07$ and the increase of the roll moment at the location of the primary peak which occurs at $V_1/V \approx 0.65$ as the
Figure 3.24: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

density ratio is further reduced. The secondary peaks are due to resonance with inertial waves of wave numbers higher than that corresponding to $V_1/V \approx 0.65$. Further reduction of the density ratio below a sufficiently small value, i.e. 0.1 say, causes the roll moment to approach that for partially filled cylinders since as $\rho_0/\rho_1 \to 0$, the inner fluid approaches the status of a void. Figure 3.25 shows a comparison of the results obtained for two-fluid flows at $\rho_0/\rho_1 = 0.001$ with those obtained for flows in partially filled cylinders. The comparison is quite good and serves us very well in checking our two-fluid flow solutions.

The results presented in figures 3.22 through 3.25 were obtained for fixed outer-flow and inner-flow Reynolds numbers. With $Re_0 = 25$ and $Re = 10^4$, when the
density ratio $\rho_0/\rho_1$ varies from 0.98 to 0.001, the viscosity ratio $\mu_0/\mu_1$ varies from 392 to 0.4 since $\mu_0/\mu_1 = (Re_1/Re_0)(\rho_0/\rho_1)$. This could lead us to believe that the inner fluid behaves like a void if its viscosity is comparable to that of the outer fluid and the density ratio is very small. To investigate this, we have let the inner flow Reynolds number be 25 as before, but now we have fixed the viscosity ratio and calculated the roll moment for various density ratios. Figures 3.26 and 3.27 show plots of the roll moment versus the fill ratio for $\mu_0/\mu_1 = 100$. The results are comparable qualitatively to those in figures 3.22 through 3.25—as the density ratio is lowered, we see the development of maxima at the same locations as before and for sufficiently small density ratios the inner fluid behaves like a void. We
Figure 3.26: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $\mu_0/\mu_1 = 100$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

conclude, thus, that even if the outer fluid is much less viscous than the core fluid, this latter cannot be lubricated if the density ratio is sufficiently small and behaves like a void.

The above results for two fluid flows have shown that for the purpose of lubricating the core fluid, the density ratio must be little less than one for best results. In figure 3.28, we show the roll moment versus the fill ratio of the additive for $Re_1 = 10^3$ and $Re_1 = 10^4$. These results show that having an outer-fluid of $Re_1 > Re_0$ does not necessarily reduce the roll moment if the fill ratio is small. In fact for $Re = 1000$, the roll moment starts to fall below that for completely filled cylinders only if $V_1/V > 3\%$ unlike the case for $Re = 10^4$ where we can see the
Figure 3.27: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $\rho_0/\rho_1 = 0.25$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

Figure 3.28: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $\rho_0/\rho_1 = 0.98$, $\eta = 4.5$, $\tau = 0.008$, $\theta = 1^\circ$, $Re_1 = 10^3$ ($\Diamond$), and $Re_1 = 10^4$ ($\Delta$).
Figure 3.29: Ratio of roll moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the outer fluid Reynolds number at $R_e_0 = 25$, $\rho_0/\rho_1 = 0.98$, $V_1/V = 4\%$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

roll moment drop with a fill ratio as little as 1 %. The figure also shows that the higher the Reynolds number, the higher the drop in the roll moment. However, for a given volume ratio this drop is not monotonic with the outer flow Reynolds number $R_e_1$ as can be seen from figure 3.29. The figure shows a plot of the roll moment ratio versus the outer-flow Reynolds number, $R_e_1$, for a volume ratio of $V_1/V = 4\%$.

So far we have only considered the roll and yaw moments. Figure 3.30 shows plots of the pitch moment versus the fill ratio for $R_e_0 = 25$, $R_e_1 = 10^4$, $\tau = 0.008$, and $\theta = 1^\circ$. Those are the same parameters used in figures 3.21 through 3.25 for the roll moment. Figure 3.30 shows that the pitch moment more than doubles
Figure 3.30: Ratio of pitch moment for cylinders containing two fluids to that for cylinders completely filled with inner fluid versus the fill ratio at $Re_0 = 25$, $Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

with a volume ratio about 4%. In fact for density ratios close to unity, i.e. 98%, a value that proved useful in reducing the roll moment, the pitch moment approach the value of the low viscosity fluid with a fill ratio as little as 4%. This could be a drawback in eliminating the viscous instability. While the roll and yaw moments are reduced, the pitch moment is increased. From the results obtained for completely filled cylinders, we know that the pitch moment increases as the Reynolds number increases, it follows then from the results presented in figure 3.30, that the pitch moment increases for cylinders containing two immiscible fluids as the viscosity of the outer fluid is reduced for purposes of lubricating the core fluid as can be seen from figure 3.31. Whether this increases the instability of the
projectile or not cannot be determined except by flight simulation. As the density ratio is reduced below that of 0.98, the pitch moment increases for a given fill ratio and for sufficiently small density ratios we can see resonance developing and obtain the same results obtained for the void configuration.

The above results for two-fluid flows have been obtained for a fixed inner-fluid Reynolds number, i.e. $Re = 25$. In practice, however, the inner-fluid Reynolds number changes since in general the spin rate decreases during the flight of the projectile and according to the viscosity of the liquid payload. The above results show that the most reduction in the roll moment is achieved when the density ratio is close to unity, the fill ratio is small, and the outer-fluid Reynolds number
Figure 3.32: Roll moment versus inner-fluid Reynolds number for cylinders containing two fluids at \( \rho_0/\rho_1 = 0.98 \), \( \eta = 4.5 \), \( \tau = 0.16667 \), and \( \theta = 20^0 \). The curve describing the roll moment as a function of the Reynolds number is shifted to the left because of the additive. Similar results are found for the pitch moment as can be seen from figure 3.33 and for fill ratio of \( V_1/V = 10\% \) as presented in figures...
Figure 3.33: Pitch moment versus inner-fluid Reynolds number for cylinders containing two fluids at $\rho_0/\rho_1 = 0.98$, $\eta = 4.5$, $\tau = 0.16667$, and $\theta = 20^\circ$.

3.34 and 3.35. The figures show that it is not always the case that the roll moment is reduced for any inner-fluid Reynolds number. For certain $Re_0$, the roll moment actually increases. The results for the roll moment are, however, interesting. Since in actual flight the Reynolds number is initially about $Re = 1000$ and decreases as the spin rate decreases, the effect of the additive in lowering the critical Reynolds number at which the roll moment acquires a maximum helps stabilize the otherwise unstable projectile as confirmed by flight tests (Miller 1991).

Other interesting results for cylinders containing two immiscible fluids can easily be obtained by our method. Figures 3.36 through 3.44 present plots of roll and pitch moments for flows at the same inner and outer flow Reynolds numbers, $Re_0 = Re_1 = 10^4$. We have used the same flight parameters and aspect ratio.
Figure 3.34: Roll moment versus inner-fluid Reynolds number for cylinders containing two fluids at $\rho_0/\rho_1 = 0.98$, $\eta = 4.5$, $\tau = 0.16667$, and $\theta = 20^\circ$.

Figure 3.35: Pitch moment versus inner-fluid Reynolds number for cylinders containing two fluids at $\rho_0/\rho_1 = 0.98$, $\eta = 4.5$, $\tau = 0.16667$, and $\theta = 20^\circ$. 
Figure 3.36: Roll moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

Figure 3.37: Roll moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$. 
Figure 3.38: Roll moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

Figure 3.39: Pitch moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$. 
Figure 3.40: Pitch moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

Figure 3.41: Pitch moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$. 
Figure 3.42: Pitch moment versus fill radius for cylinders containing two immiscible fluids at $Re_0 = Re_1 = 10^4$, $\eta = 4.5$, $\tau = 0.008$, and $\theta = 1^\circ$.

as before. Figure 3.36 shows that for density ratios close to unity, i.e. 0.98, the variation with $r_0$ is almost constant and we expect it that way. As the density ratio is reduced further, we see the development of a single significant peak in the roll moment for $1 < \rho_0/\rho_1 < 0.4$. We do not believe that this peak is due to resonance because we have seen this phenomena even at very low Reynolds numbers. Also the variation of the yaw moment around where the peak occurs is not sharp enough to lead us to believe that it is due to resonance. This will become clear when we solve the inviscid equations. We believe that this phenomena is similar to that corresponding to the viscous instability and it comes as a penalty for carrying two fluids. For density ratios lower than 0.4 we see clearly the development of two peaks instead of one as can be seen from figure 3.37. Figure 3.38 demonstrates once
again the fact that for sufficiently small density ratios the moments approach that for the void-fluid configuration as can also be seen from figure 3.42 for the Pitch moment. Figures 3.39 through 3.42 show similar results for the Pitch moments for a variety of density ratios.

We can also generate results for a fixed density ratio and various inner flow and outer flow Reynolds numbers. Figures 3.43 and 3.44 show plots of the roll and pitch moments respectively for a density ratio of 0.20. The figures clearly show the development of peaks at critical values of \( r_0 \) for the roll moments as the Reynolds numbers are increased. The value of the roll moment at those peaks increases as the Reynolds numbers increase and as before we believe this is due to resonance with inertial waves. Similar results are found for the Pitch moment and as before it tends to different values for \( r_0 \) below and above the critical radius. In the next chapter we will investigate the case where \( Re_0 = Re_1 \rightarrow \infty \) or in other words the flow of two inviscid fluids and compare results of both approaches.

Finally, Figure 3.45 and 3.46 present our results for the nondimensional yaw moment as a function of the fill radius in comparison with data of Murphy et al. (1989) obtained with the KGS method. Murphy et al. (1989) present results of different approaches. In all cases, we find the best agreement with the results of the KGS method. The origin of the slight discrepancy shown in figures 3.40 and 3.41 is unclear. We have spent some efforts to convert correctly from their to our parameters. Therefore, we suspect the deviation is caused by using the KGS
Figure 3.43: Roll moment versus fill radius for cylinders containing two immiscible fluids at \( \rho_0/\rho_1 = 0.2, \eta = 4.5, \tau = 0.008, \) and \( \theta = 1^\circ. \)

Figure 3.44: Pitch moment versus fill radius for cylinders containing two immiscible fluids at \( \rho_0/\rho_1 = 0.2, \eta = 4.5, \tau = 0.008, \) and \( \theta = 1^\circ. \)
Figure 3.45: Yaw moment versus fill radius for cylinders containing a central rod at $\eta = 3$, $\tau = 0.1111$, and $\theta = 2^\circ$. Comparison with the results of Murphy et al. (1989) obtained by the KGS method (o).

Figure 3.46: Yaw moment versus fill radius for partially filled cylinders at $\eta = 3$, $\tau = 0.1111$, and $\theta = 2^\circ$. Comparison with the results of Murphy et al. (1989) obtained by the KGS method (o).
boundary-layer approach at relatively low Reynolds numbers.

### 3.2.3 Interfacial shape

It is important to calculate the radial location of the void-fluid or two-fluid interface as a function of \( z \) and \( \phi \) and learn when its deviation from the axisymmetric surface \( r = r_0 \) is small since our results are based on this assumption. From the relation

\[
\zeta_1(z) = -i u_1(r_0, z),
\]

we obtain

\[
\frac{\partial \zeta_1}{\partial z} = -i \frac{\partial u_1}{\partial z}(r_0, z).
\]

From the shear stress relations at the interface, we have for partially filled cylinders,

\[
\frac{\partial u_1}{\partial z}(r_0, z) = -\frac{\partial w_1}{\partial r}(r_0, z),
\]

and for cylinders containing two immiscible fluids,

\[
\left( \frac{\rho_0}{Re_0} - \frac{\rho_1}{Re_1} \right) \frac{\partial u_1}{\partial z} = \frac{\rho_1}{Re_1} \frac{\partial w_1}{\partial r} - \frac{\rho_0}{Re_0} \frac{\partial w_0}{\partial r}.
\]

Consequently, for a void-fluid interface, we have

\[
\frac{\partial \zeta_1}{\partial z} = i \frac{\partial w_1}{\partial r}(r_0, z),
\]

while for two-fluid interface, we have

\[
\left( \frac{\rho_1}{Re_1} - \frac{\rho_0}{Re_0} \right) \frac{\partial \zeta_1}{\partial z} = i \left( \frac{\rho_1}{Re_1} \frac{\partial w_1}{\partial r} - \frac{\rho_0}{Re_0} \frac{\partial w_0}{\partial r} \right).
\]
Since the interface is untisymmetric about the center plane, $\zeta_1(0) = 0$. Thus, integrating eq. (3.54), we obtain

$$\zeta_1(z) = i \int_0^z \frac{\partial w_1(r_0, \lambda)}{\partial r} (r_0, \lambda) d\lambda. \quad (3.56)$$

Similarly the integration of eq. (3.55) yields

$$(\frac{\rho_1}{Re_1} - \frac{\rho_0}{Re_0}) \zeta_1(z) = i \int_0^z \left[ \frac{\rho_1}{Re_1} \frac{\partial w_1(r_0, \lambda)}{\partial r} (r_0, \lambda) - \frac{\rho_0}{Re_0} \frac{\partial w_0(r_0, \lambda)}{\partial r} \right] d\lambda. \quad (3.57)$$

Hence, if we substitute for $w_1$,

$$w_1 = \sum_{m=0}^{M-1} \left[ F_m^1 R_{1m}(r) + G_m^1 R_{2m}(r) \right] Z_m^1(z), \quad (3.58)$$

and for $w_0$,

$$w_0 = \sum_{m=0}^{M-1} F_m^0 R_{0m}(r) Z_m^0(z), \quad (3.59)$$

where

$$R_{0m}(r) = I_1(q_0 \sqrt{B_m^0 r}), \quad R_{1m}(r) = I_1(q_1 \sqrt{B_m^1 r}), \quad R_{2m}(r) = K_1(q_1 \sqrt{B_m^1 r}), \quad (3.60)$$

we obtain, for a void-fluid interface

$$\zeta_1(z) = i \sum_{m=0}^{M-1} \left[ F_m^1 R_{1m}(r_0) + G_m^1 R_{2m}(r_0) \right] \int_0^z Z_m^1(\lambda) d\lambda, \quad (3.61)$$

and for two-fluid interface

$$\left(\frac{\rho_1}{Re_1} - \frac{\rho_0}{Re_0}\right) \zeta_1(z) = i \left(\frac{\rho_1}{Re_1}\right) \sum_{m=0}^{M-1} \left[ F_m^1 R_{1m}(r_0) + G_m^1 R_{2m}(r_0) \right] \int_0^z Z_m^1(\lambda) d\lambda$$

$$- i \left(\frac{\rho_0}{Re_0}\right) \sum_{m=0}^{M-1} [F_m^0 R_{0m}(r_0)] \int_0^z Z_m^0(\lambda) d\lambda, \quad (3.62)$$
Figure 3.47: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 20$, $r = 0.16667$, $\eta = 4.368$, $\theta = 20^\circ$, $r_0 = 0.2$, and $r_0 = 0.8$.

where prime denotes differentiation with respect to $r$ and for $a_{1m} = 0$

$$
\int_0^2 Z_m^\alpha(\lambda)d\lambda = \frac{(1 + i)}{q_\alpha}[q_\alpha z + \frac{c_{2m}}{a_{2m}} \sin(a_{2m}q_\alpha z) + \frac{c_{3m}}{a_{3m}} \sin(a_{3m}q_\alpha z)], \quad (3.63)
$$

while for $a_{1m} \neq 0$

$$
\int_0^2 Z_m^\alpha(\lambda)d\lambda = \frac{(1 + i)}{q_\alpha}[-\frac{1}{a_{1m}} \sin(a_{1m}q_\alpha z) + \frac{c_{2m}}{a_{2m}} \sin(a_{2m}q_\alpha z) + \frac{c_{3m}}{a_{3m}} \sin(a_{3m}q_\alpha z)]. \quad (3.64)
$$

For a cylinder with partial fill, we show in figure 3.47 the shape of the interface for relatively small Reynolds number, $Re = 20$, and relatively large nutation angle, $\theta = 20^\circ$. The figure shows that the interface distortion from the axisymmetric surface $r = r_0$ is negligible. This result has also been found for different aspect ratios and different planes not shown here.
Figure 3.48: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 453$, $r = 0.11037$, $\eta = 3$, $\theta = 20$, $r_0 = 0.2$, and $r_0 = 0.8$.

However, as the Reynolds number increases, the distortion increases as can be seen from figures 3.48 through 3.51. The figures show interface shapes for different aspect ratios, different Reynolds numbers that range from $Re = 453$ to $Re = 10^4$, and fill radii of $0.2$ and $0.8$. The nutation angle has been kept constant, $\theta = 20^\circ$, while the coning frequency was varied a little. It is important to see from these figures the changes of waviness of the interface as the aspect and fill ratios change.

We also see from these figures that the fill ratio has a significant effect on the void distortion since the radial pressure gradient increases with $r$. For small fill ratios or large ‘fill radius’ $r_0$, a given normal force can be balanced by a small displacement of the interface. However, figures 3.52 and 3.53 show that this is not the case. Figure 3.52 show the shape of the interface at fill radii of 0.7 and
Figure 3.49: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 1000$, $r = 0.16667$, $\eta = 4.368$, $\theta = 20^0$, $r_0 = 0.2$, and $r_0 = 0.8$.

Figure 3.50: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $r = 0.08333$, $\eta = 3$, $\theta = 20^0$, $r_0 = 0.2$, and $r_0 = 0.8$. 
Figure 3.51: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 4.5$, $\theta = 20^0$, $r_0 = 0.2$, and $r_0 = 0.8$. 0.9. Their distortion is relatively small compared to that with fill radius of 0.8 shown in figure 3.53. It follows that this is not contradicting to our above results, and found out that $r_0 = 0.8$ is close to the fill radius that causes resonance. The fill radius that causes resonance can be found by the results of the next chapter and is approximately 0.83. At resonance, it is expected that the interface can be significantly distorted. This result is also found for different aspect ratios and fill radii, since resonance is greatly affected by these two parameters.

In figures 3.54 and 3.55, we show the interface shape for an aspect ratio of 3 and a fill radius of 0.5 that is close to that leading to resonance with inertial waves, $r_0 = 0.51$, in the $\phi = 90$ and $\phi = 0$ respectively. We note from these figures that the distortion of the interface is quite significant. In fact in the $y - z$
Figure 3.52: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 2$, $\theta = 20^\circ$, $r_0 = 0.7$, and $r_0 = 0.9$.

Figure 3.53: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 2$, $\theta = 20^\circ$, and $r_0 = 0.8$. 
Figure 3.54: Interface shape in the plane $\phi = 90$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 3$, $\theta = 20^\circ$, and $r_0 = 0.5$.

Figure 3.55: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 3$, $\theta = 20^\circ$, and $r_0 = 0.5$. 
plane the distortion is so significant that the void partially touches the side wall. Note also from figures 3.53 and 3.55 that the distortion of the interface at fill radii in the neighborhood of a large resonant fill radius is relatively smaller than that corresponding to a small resonant fill radius. These distortions, however, become insignificant for small nutation angles as can be seen from figures 3.56 and 3.57 which show the interface shape in the neighborhood of a resonant fill radius at a nutation angle of $2^\circ$. With small nutation angles, less significant distortions are obtained for fill ratios not leading to resonance as seen in figures 3.58 and 3.59.

For cylinders with two fluids, the interface shape is similar to that for partially filled cylinders. For small nutation angles, the interface distortion is insignificant as can be seen from figures 3.60 and 3.61. When the nutation angle is not small, i.e.
Figure 3.57: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 3$, $\theta = 2^o$, and $r_0 = 0.5$.

Figure 3.58: Interface shape in the plane $\phi = 0$ for partially filled cylinders at $Re = 10^4$, $\tau = 0.08333$, $\eta = 3$, $\theta = 2^o$, $r_0 = 0.2$, and $r_0 = 0.8$. 
Figure 3.59: Interface shape in the plane $\phi = 90$ for partially filled cylinders at $Re = 10^4$, $r = 0.08333$, $\eta = 3$, $\theta = 2^\circ$, $r_0 = 0.2$, and $r_0 = 0.8$.

Figure 3.60: Interface shapes in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.98$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 2^\circ$, $r_0 = 0.2$, and $r_0 = 0.8$. 
Figure 3.61: Interface shapes in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.10$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, $r_0 = 0.2$, and $r_0 = 0.8$.

$20^\circ$, and the flow parameters do not lead to resonance, the distortion is no longer insignificant, and similar to the case for partially filled cylinders the distortion increases as the fill radius decreases for any density ratio. Figures 3.62 through 3.65 illustrate this property. Thus the density has little if no effect at all on the distortion of the interface.

However, figures 3.66 and 3.67 show that the density ratio affects the distortion of the interface even at large fill radius, i.e. $r_0 = 0.84$. It turns out that this fill radius leads to resonance at small density ratios and that is why the distortion in figure 3.67 is more severe than that in figure 3.66 since the density ratio used in the results of figure 3.67 is much smaller than that used to generate the results of figure 3.66. The distortion of the interface shown in figure 3.67 is, nevertheless, small.
Figure 3.62: Interface shape in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.98$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.2$.

Figure 3.63: Interface shape in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.10$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.2$. 
Figure 3.64: Interface shape in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.90$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.4$.

Figure 3.65: Interface shape in the plane $\phi = 0$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.01$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.4$. 
Figure 3.66: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.98$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.84$.

Figure 3.67: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.01$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.84$. 
Figure 3.68: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.98$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$.

and this is because the fill radius is large on one hand and it leads to secondary resonance on the other hand. Resonance at this fill radius is due to secondary waves—waves of shorter wavelengths than those causing the most amplification in the moments that we call primary waves or the most dangerous waves.

When the fill radius is selected so that we have resonance with primary waves, the distortion of the interface can be quite significant. For the parameters $\tau = .1$, $\eta = 4.5$, $\theta = 20^\circ$, $Re_0 = 30$, $Re_1 = 10^4$, we have found that the roll moments is most amplified at a fill radius of $r_0 \approx 0.66$. We calculated the interface at this fill ratio for different density ratios and the results are shown in figures 3.68 through 3.74. For density ratios not far from unity as in figure 3.68, the distortion is relatively small. As the density ratio is reduced further as in figures 3.69, 3.70,
Figure 3.69: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.50$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$.

Figure 3.70: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.30$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$. 
Figure 3.71: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.10$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.66$.

Figure 3.72: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.01$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^0$, and $r_0 = 0.66$. 
Figure 3.73: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.001$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$.

and 3.71, the distortions become more significant, and for sufficiently small density ratios as in figure 3.72 and 3.73, the distortions become large and the inner fluid almost touches the wall. Note that the interface shape in figure 3.72 is identical to that in figure 3.73 which support the fact that below sufficiently small density ratio, the distortion no longer changes and the inner fluid acts like a void. Note also the difference in the wavelength of the interface at primary resonance and that at secondary resonance.

Resonance can also be seen for different inner and outer flow Reynolds number at sufficiently small density ratio. We obtain the same results for a fill radius of $r_0 = 0.66$ discussed above with $Re_0 = Re_1 = 10^4$. The results are shown in figures 3.74 for $\rho_0/\rho_1 = 0.98$ where it shows that the interface distortion is relatively small.
Figure 3.74: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 10^4$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.98$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$.

and in figure 3.75 for $\rho_0/\rho_1 = 0.001$ where it shows that the interface is the same as that in figure 3.73. The distortion at resonance is, however, quite small when the nutation angle is small as can be seen from figures 3.76 and 3.77.

3.3 Solutions by spectral techniques

3.3.1 Introduction

At the beginning of the development of the solutions by eigenfunction expansions, we encountered some difficulties in finding the eigenvalues, especially in the neighborhood of the origin of the complex plane. This has lead us to not including some eigenfunctions in the expansion for the fundamental component of the axial velocity and consequently the method did not converge. When we accounted for all eigen-
Figure 3.75: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 10^4$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.001$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$.

Figure 3.76: Interface shape in the plane $\phi = 90$ for a cylinder containing two fluids at $Re_0 = 30$, $Re_1 = 10^4$, $\rho_0/\rho_1 = 0.001$, $\tau = 0.1$, $\eta = 4.5$, $\theta = 20^\circ$, and $r_0 = 0.66$. 
values that are associated with eigenfunctions, that are vital to the convergence of the solutions, we discovered a slight disagreement—due to the difference in resolution not known then—of the results with those generated by a three-dimensional spectral code developed by Herbert & Li (1987). This slight disagreement has prompted us to solve the SOPDE by a spectral collocation method.

Before presenting the details of this approach, however, I should mention that we have encountered many problems when solving the SOPDE by spectral techniques some of which are related to convergence and the singularity of the governing equations at the corners. In particular and more importantly than others, we have found that the convergence of the solution when expanded in a double Chebyshev series in \( r \) and \( z \) was poor.
To remedy this problem, we have found that it is best to split the solution into two parts, one which corresponds to the infinitely long cylinder that is readily expressed in terms of analytical functions and another solution which we seek. The resulting boundary conditions for the newly sought solution are homogeneous at the corners. In addition an asymptotic analysis near the corners revealed that such a solution behaves smoothly and much simpler than the one of the original formulation. This lead us to believe that we can achieve better convergence when we adopt the new formulation. Since the solution corresponding to the infinitely long cylinder satisfy the SOPDE, the newly sought solution is also governed by this equation.

We have also found that it is critical to satisfy the governing equation at the boundaries. We suspect this is due to the rapid change of the solution and its derivatives near the walls especially at high Reynolds numbers. The complexity of the boundary conditions prevents us from knowing whether a spectral convergence could be achieved. Because we had to satisfy the equation on the boundaries and some of the conditions at the corners become degenerate, we were not able to cluster the collocation points in such a way as to obtain the same number of algebraic equations as the number of unknown coefficients. Therefore, we have chosen the collocation points in such a way the number of equations exceeds the number of unknowns and have used the discrete least squares approach to solve such a system.
3.3.2 Spectral approximation

We now present this approach for completely filled cylinders. The extension to the other configurations is straightforward. We assume a solution to equation (2.37) of the form

\[ w = -i[r - \frac{I_1(qr)}{I_1(q)}] + \sum_{m=1}^{M+2} \sum_{n=1}^{N+2} c_{mn} \phi_m(r) \psi_n(z) \tag{3.65} \]

where \( I_1 \) is the modified Bessel function of order 1 and \( q^2 = iRe \). In equation (3.65), the first term on the right hand side is the solution corresponding to the infinitely long cylinder and for the second term, the basis functions \( \phi_m \) and \( \psi_n \) are linear combinations of Chebyshev polynomials, namely

\[
\phi_m(r) = T_{2m+1}(r) - T_{2m-1}(r), \tag{3.66}
\]

\[
\psi_n(z) = [(n - 1)^2 T_{2n}(z/\eta) - n^2 T_{2n-2}(z/\eta)]/n^2. \tag{3.67}
\]

The above choice permits us to satisfy the no-slip condition at the side walls and the no-gradient condition at the end walls implicitly. Note that we have exploited symmetry with respect to \( z = 0 \) by choosing even polynomials in \( z \), and for single valuedness at the center axis (\( r = 0 \)), we have chosen odd polynomials in \( r \).

We substitute the assumed solution into the governing equations and the rest of the boundary conditions that were not satisfied implicitly—two at one end wall and two at the side wall. The resulting equations are satisfied in the least squares sense at the Gauss – Radau collocation points:

\[
r_j = \cos(\frac{j \pi}{2M + 1}), \quad j = 0, \ldots, M, \tag{3.68}
\]
\[ z_k = \eta \cos\left(\frac{k\pi}{2N+1}\right), \quad k = 0,\ldots, N, \tag{3.69} \]

to solve for the Chebyshev coefficients \( c_{nm} \).

### 3.3.3 Calculation of Moments

We use the volume integral approach presented in chapter 2 and find

\[
M_x = 8\pi \eta r^2 \cos \theta \sin \theta (\rho a^5 \omega^2) \text{Real}\{\tilde{S}\}, \tag{3.70}
\]

\[
M_y = 8\pi \eta r^2 \cos \theta \sin \theta (\rho a^5 \omega^2) \text{Real}\{i \tilde{S}\}, \tag{3.71}
\]

where

\[
\tilde{S} = -i\left(\frac{1}{4} - \frac{I_2(q)}{q I_1(q)}\right) + \sum_{m=1}^{M+2} \sum_{n=1}^{N+2} c_{mn} \int_0^1 r^2 \phi_m(r) dr \int_0^1 \psi_n(\eta z) dz, \tag{3.72}
\]

and \( I_2 \) is the modified Bessel function of order 2. Note that the moments can be made dimensionless by scaling them using \( (\rho a^5 \omega^2) \).

We have calculated these moments for \( \tau = 0.16667, \eta = 4.368, \theta = 20 \), and a wide range of Reynolds number. The results for the roll, pitch, and yaw moments are shown as function of Reynolds number in figures 3.78 through 3.80 respectively. The results were obtained with a Chebyshev series of \( 36 \times 36 \) polynomials and the computation takes about 157 seconds on a Cray Y-MP8/864. In the same figure we plotted the moments obtained by the eigenfunction expansion and as can been seen the results compare very well. For detailed comparisons, we have tabulated these moments obtained by the Chebyshev expansion in table 3.6 and those obtained by
Figure 3.78: Dimensionless roll moment versus Reynolds number for completely filled cylinders at $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. Comparisons of the results obtained by eigenfunction expansions (—) and Chebyshev expansions ($\Delta$).

eigenfunction expansion in table 3.7. A quick glance at both tables reveals that the values agree to within 4 digits.

We also calculated these moments at a constant Reynolds number but with different number of polynomials in the expansion. The results for the error between the moments computed by the Chebyshev expansion and high resolution values obtained by eigenfunction expansions are shown in figures 3.81 and 3.82 for $Re = 20$. At this relatively low Reynolds number, the moments converge to within 4 digits of accuracy with at least 20 polynomials in each direction. This value, however, increases as the Reynolds number increases. Figure 3.83 and 3.84 show that this values must be in excess of 35 polynomials. Note also that with relatively
Figure 3.79: Dimensionless pitch moment versus Reynolds number for completely filled cylinders at $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. Comparisons of the results obtained by eigenfunction expansions (---) and Chebyshev expansions (△).

Figure 3.80: Dimensionless yaw moment versus Reynolds number for completely filled cylinders at $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. Comparisons of the results obtained by eigenfunction expansions (---) and Chebyshev expansions (△).
Table 3.6: Dimensionless moments versus Reynolds number obtained by 38 x 38 Chebyshev expansions for completely filled cylinders at $Re = 20$, $\eta = 4.368$, $\tau = 0.16667$, and $\theta = 20^\circ$.

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Table 3.7: Dimensionless moments versus Reynolds number obtained by eigenfunction expansions for completely filled cylinders at \( Re = 20, \eta = 4.368, \tau = 0.16667, \) and \( \theta = 20^\circ \).

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low number of polynomials, i.e. 15 to 25 in figures 3.83 and 3.84, the values of
the moments do not change as the number of polynomial is varied, yet they did
not converge to the values given by the eigenfunction expansion. We believe this
is due to the singularity at the corners, and only a big change in the number
of polynomials produce a significant change in the moments. Nevertheless, the
method is quite competitive compared to the 3D-spectral code since we are only
solving for one component of one flow quantity and not the total of 4 quantities.
The CPU time and memory allocation is incomparable. Note also that certain
components of the moments converge faster than others. This has to do with the
degree of complexity of the velocity at the plane the moments are heavily dependent
on. We have also shown these moments as functions of the number of polynomials
in each direction in table 3.8 and 3.9 for purposes of detailed comparisons.
Figure 3.81: Error in dimensionless yaw moment versus number of polynomials for completely filled cylinders at $Re = 20$, $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$.

Figure 3.82: Error in dimensionless pitch moment versus number of polynomials for completely filled cylinders at $Re = 20$, $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. 
Figure 3.83: Error in dimensionless yaw moment versus number of polynomials for completely filled cylinders at $Re = 1000$, $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. 

Figure 3.84: Error in dimensionless pitch moment versus number of polynomials for completely filled cylinders at $Re = 1000$, $\tau = 0.16667$, $\eta = 4.368$, and $\theta = 20^\circ$. 
Table 3.8: Convergence of the values of moments obtained by Chebyshev expansions for completely filled cylinders at \( Re = 20, \eta = 4.368, \tau = 0.16667, \) and \( \theta = 20^\circ \).

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Table 3.9: Convergence of the values of moments obtained by Chebyshev expansions for completely filled cylinders at $Re = 1000$, $\eta = 4.368$, $\tau = 0.16667$, and $\theta = 20^\circ$.

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CHAPTER IV

Linear Inviscid Analysis and Resonance Phenomena

4.1 Introduction

The results of the linear viscous analysis considered in chapter 3 have shown that the moments acquire maxima at specific fill ratios for given coning frequency and aspect ratio. These maxima increase as the Reynolds number increases and the variation of the moments in the neighborhood of the critical fill ratios is enormous. Since for a given spinning rate, increasing the Reynolds number means lowering the kinematic viscosity of the fluid, we believe that this phenomenon is inviscid in nature and the fact that it occurs at specific fill and aspect ratios lead us to believe it is due to resonance with inertial waves.

The aspect ratio and fill radius combinations that lead to resonance for a given coning frequency is of great importance to the designer in general and to us in particular since to understand the global picture of this phenomenon, we need to generate an enormous number of figures from the viscous analysis which may make
the size of this dissertation larger than what it should be. Consequently, in this chapter we undertake the inviscid analysis of the various flow configurations to solve for the critical parameters that cause these maxima and consequently lead to resonance.

4.2 Governing equations

If we let the Reynolds number \( Re \to \infty \) in the linearized Navier-Stokes equations, we obtain for steady state conditions

\[
\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0, \tag{4.1}
\]

\[
\frac{\partial v_r}{\partial \phi} - 2(1 + \tau_z) v_\phi = -\frac{\partial p^d}{\partial r}, \tag{4.2}
\]

\[
\frac{\partial v_\phi}{\partial \phi} + 2(1 + \tau_z) v_r = -\frac{1}{r} \frac{\partial p^d}{\partial \phi}, \tag{4.3}
\]

\[
\frac{\partial v_z}{\partial \phi} = -\frac{\partial p^d}{\partial z} - 2r \tau_r. \tag{4.4}
\]

The boundary conditions on \( \mathbf{v} = (v_r, v_\phi, v_z) \) associated with these equations depend on the configuration studied and we shall consider each case separately.

4.3 Boundary conditions

4.3.1 Completely filled cylinders

For a cylinder, completely filled with a single fluid, we let \( r_f = 0 \) and hence the inner solution is meaningless since there is no longer an inner region. However, the
outer flow is required to satisfy the impermeability condition at the walls, namely,

\[ v^1_z = 0 \text{ at } z = \pm \eta, \quad v^1_r = 0 \text{ at } r = 1, \]  

(4.5)

where as before \( \eta \) denotes the aspect ratio \( c/a \) of the cylinder.

### 4.3.2 Cylinders with a central rod

The motion of a coaxial rod inside the cylinder is only due to solid body rotation and hence the inner solution is trivial, \( \nu_0 \equiv 0 \), and \( r_f \) is reduced to the radius of the rod \( r_0 \). For the outer flow, we require no-fluid-penetration through all walls including the surface of the central rod. As a result, in addition to condition (4.5), we have

\[ v^1_r = 0 \text{ at } r = r_0 \]  

(4.6)

### 4.3.3 Partially filled cylinders

When the cylinder is partially filled, the fluid accumulates near the side walls under the influence of centrifugal forces. As a result, a void surrounds the cylinder axis and its motion is meaningless if it is taken to be a vacuum. However, in practice this is impossible and the void could be filled with air and since the density of air is far smaller than most liquids, its contribution to the moments is negligible as was confirmed by the viscous analysis of two fluids undertaken in chapter 3. For the outer flow, in addition to the impermeability condition represented by equation (4.5), the pressure of the outer flow is required to assume the value of that of the
void at the fluid-void interface, namely,

\[ p^d_1 = -\frac{1}{2} \left[ \frac{\tau^2_0}{4} (2 r_0 \zeta) + 2 r_0 z \tau_z \cos \phi \right] \text{ at } r = r_0. \tag{4.7} \]

### 4.3.4 Cylinders with two fluids

Similar to the void-fluid configuration, under the action of centrifugal forces the heavier fluid occupies the outer region while the less heavier fluid occupies the inner one, and hence we need to solve for both inner and outer flows. Thus, in addition to (4.5), we require continuity of the total pressure across the inner-fluid outer-fluid interface, namely,

\[ \rho_0 P_0 = \rho_1 P_1 \text{ at } r = r_0 \tag{4.8} \]

When expressed in terms of \( p^d_\alpha \), this condition becomes

\[ \rho_0 p^d_0 - \rho_1 p^d_1 = \frac{1}{4} (\rho_1 - \rho_0) \tau_\theta^2 r_0 \zeta + (\rho_1 - \rho_0) r_0 z \tau_z \cos \phi. \tag{4.9} \]

Moreover, from kinematics, we have

\[ v^0_r = v^1_r = \frac{\partial \zeta}{\partial \phi} \text{ at } r = r_0. \tag{4.10} \]

### 4.4 Solution

We represent the flow velocities, the pressure, and the interface by the Fourier series:

\[ (v_r^\alpha, v_\phi^\alpha, v_z^\alpha) = \sum_{n=-\infty}^{\infty} (u_n^\alpha, v_n^\alpha, w_n^\alpha) e^{i n \phi} \tag{4.11} \]
\[ p_{\alpha}^{d} = 2r \varepsilon z \cos \phi + \sum_{n=\infty}^{\infty} p_{\alpha}^{n} \epsilon^{n\phi} \quad (4.12) \]

\[ \zeta(z, \phi) = \sum_{n=\infty}^{\infty} \zeta_{n}(z) \epsilon^{n\phi} \quad (4.13) \]

Substituting these expressions into the governing equations (4.1) through (4.4) and realizing that for linear analysis only the fundamental components are relevant, we obtain

\[ (1 - \tau_{\theta}^{2}) u_{\alpha} = \frac{i}{r} \tau_{\theta} p_{\alpha} + i \frac{\partial p_{\alpha}}{\partial r} + i (1 + \tau_{\theta}) \varepsilon z, \quad (4.14) \]

\[ (1 - \tau_{\theta}^{2}) v_{\alpha} = -\frac{1}{r} p_{\alpha} - \tau_{\theta} \frac{\partial p_{\alpha}}{\partial r} - (1 + \tau_{\theta}) \varepsilon z, \quad (4.15) \]

\[ w_{\alpha} = \frac{1}{r} \frac{\partial p_{\alpha}}{\partial z}, \quad (4.16) \]

\[ \frac{\partial^{2} p_{\alpha}}{\partial r^{2}} + \frac{1}{r} \frac{\partial p_{\alpha}}{\partial r} - \frac{p_{\alpha}}{r^{2}} + (1 - \tau_{\theta}^{2}) \frac{\partial^{2} p_{\alpha}}{\partial z^{2}} = 0, \quad (4.17) \]

where we have let

\[ (u_{\alpha}, v_{\alpha}, w_{\alpha}, p_{\alpha}) = (u_{\alpha}^{1}, v_{\alpha}^{1}, w_{\alpha}^{1}, p_{\alpha}^{1}). \]

The boundary conditions associated with the pressure equation (4.17) at the end walls are independent of the flow configuration and they are

\[ \frac{\partial p_{\alpha}}{\partial z} = 0 \text{ at } z = \pm \eta. \quad (4.18) \]

Likewise, the boundary condition at the side wall is independent of the flow configuration since it involves only the outer fluid. It takes the form

\[ \frac{\tau_{\theta}}{r} p_{1} + \frac{\partial p_{1}}{\partial r} = -(1 + \tau_{\theta}) \varepsilon z \text{ at } r = 1. \quad (4.19) \]
For a cylinder containing a central rod, the boundary condition at the surface of
the rod takes the same form as (4.19) but evaluated at \( r = r_0 \), while the condition
at a void-fluid interface takes the form

\[
(1 - \tau_0^2 + \frac{\tau_0^3}{4}) \frac{p_1}{r} + \frac{\tau_0^2}{4} \frac{\partial p_1}{\partial r} = - \left( \frac{1}{2} + \frac{\tau_0}{4} - \frac{\tau_0^2}{4} \right) z \epsilon \text{ at } r = r_0. \tag{4.20}
\]

Finally, for two fluid interface, we have

\[
[p_0 (1 - \tau_0^2) + \frac{\tau_0^2}{4} (\rho_0 - \rho_1)] \frac{p_0}{r} + \frac{\tau_0^2}{4} (\rho_0 - \rho_1) \frac{\partial p_0}{\partial r}
- \left( \frac{1}{2} + \frac{\tau_0}{4} - \frac{\tau_0^2}{4} \right) (\rho_0 - \rho_1) z \epsilon \text{ at } r = r_0 \tag{4.21}
\]

and the kinematics condition (4.10) provides

\[
\zeta_1 = -i u_0 = -i u_1 \text{ at } r = r_0. \tag{4.22}
\]

Note when \( \rho_0 = 0 \), eq. (4.21) reduces to eq. (4.20) and when \( \rho_0 = \rho_1 \), eq. (4.21)
reduces to \( p_0 = p_1 \).

The pressure equation is elliptic, homogeneous, and possesses homogeneous
boundary conditions in the axial direction. By means of separation of variables, it
can be shown, that it supports solutions in the form of products of sine waves in \( z \)
and Bessel functions \( J_1 \) and \( Y_1 \) in \( r \). Hence, its general solution can be written as

\[
p_\alpha = \sum_{k=0}^{\infty} [A_k^\alpha J_1(\beta_k r) + B_k^\alpha Y_1(\beta_k r)] \sin(\gamma_k z), \tag{4.23}
\]

where

\[
\gamma_k = (2k + 1) \frac{\pi}{2 \eta}, \quad \beta_k = \gamma_k \sqrt{\tau_0^2 - 1}. \tag{4.24}
\]
The other flow quantities can be found from equations (4.14) through (4.16),

\[
\begin{align*}
\mathbf{u}_\alpha &= i \frac{\varepsilon z}{1 - \tau_\theta} \\
+ i \sum_{k=0}^{\infty} \frac{A'_k}{(1 - \tau_\theta)^2} \frac{\beta_k}{2} \left[ (1 + \tau_\theta) J_0(\beta_k r) + (\tau_\theta - 1) J_2(\beta_k r) \right] \sin(\gamma_k z) \\
+ i \sum_{k=0}^{\infty} \frac{B''_k}{(1 - \tau_\theta)^2} \frac{\beta_k}{2} \left[ (1 + \tau_\theta) Y_0(\beta_k r) + (\tau_\theta - 1) Y_2(\beta_k r) \right] \sin(\gamma_k z),
\end{align*}
\]

where \( J_l \) and \( Y_l \) are Bessel and Neumann functions of order \( l \) (\( l = 0, 1, 2 \)) respectively. The amplitude coefficients \( A'_k \) and \( B''_k \) are determined by satisfying the boundary conditions at the side walls and/or the interface. Most importantly, they provide us with criteria for the onset of resonance.

### 4.5 Criteria for resonance

#### 4.5.1 Completely filled cylinders

When the cylinder is completely filled, we require the solution to be finite at the center \( (r = 0) \) and as a result the coefficients \( B''_k \) are zeros. The coefficients \( A'_k \) are found by satisfying (4.19), namely,

\[
\sum_{k=0}^{\infty} A'_k \frac{\beta_k}{2} \left[ (1 + \tau_\theta) J_0(\beta_k) - (1 - \tau_\theta) J_2(\beta_k) \right] \sin(\gamma_k z) = -(1 + \tau_\theta) \varepsilon z.
\]
Figure 4.1: Criteria for resonance in completely filled cylinders.

But,

\[-(1 + \tau_0) \epsilon z = \sum_{k=0}^{\infty} \frac{2(1 + \tau_0)\epsilon}{\eta \gamma_k^2} (-1)^{k+1} \sin(\gamma_k z).\]  

(4.29)

Hence,

\[A_k^1 = \frac{4(1 + \tau_0) (-1)^{k+1} \epsilon}{\eta \gamma_k^2 \beta_k [(1 + \tau_0) J_0(\beta_k) - (1 - \tau_0) J_2(\beta_k)]},\]  

(4.30)

and consequently, the motion is in resonance if

\[[(1 + \tau_0) J_0(\beta_k) - (1 - \tau_0) J_2(\beta_k)] = 0.\]  

(4.31)

This is a transcendental equation and its roots determine the parameters \((\tau, \eta)\) that cause resonance. These roots depend on \(k\) and consequently there can be many critical values of \((\tau, \eta)\) corresponding to different values of \(k\). However, roots associated with the lowest values of \(k\) are more severe than others in causing
resonance in a specific region in the $\tau - \eta$ plane. These critical values are shown in figure 4.1 which indicates that an increase in coning frequency necessitates an increase in the aspect ratio, almost in a linear fashion, to cause resonance.

### 4.5.2 Cylinders with a central rod

For this case, the no-penetration conditions at the side wall and at the surface of the rod require

\[
\sum_{k=0}^{\infty} \left\{ A_k^1 [E_1 J_0(\beta_k) + E_2 J_2(\beta_k)] + B_k^1 [E_1 Y_0(\beta_k) + E_2 Y_2(\beta_k)] \right\} \frac{\beta_k}{2} \sin(\gamma_k z) = -E_1 \epsilon z, \tag{4.32}
\]

\[
\sum_{k=0}^{\infty} \left\{ A_k^1 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] + B_k^1 [E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)] \right\} \frac{\beta_k}{2} \sin(\gamma_k z) = -E_1 \epsilon z, \tag{4.33}
\]

where $E_1 = (1 + \tau_0)$ and $E_2 = (\tau_0 - 1)$. Hence, the coefficients $A_k^1$ and $B_k^1$ are related by

\[
A_k^1 [E_1 J_0(\beta_k) + E_2 J_2(\beta_k)] + B_k^1 [E_1 Y_0(\beta_k) + E_2 Y_2(\beta_k)] = \frac{4 E_1 (-1)^{k+1} \epsilon}{\eta \gamma_k^2 \beta_k}, \tag{4.34}
\]

\[
A_k^1 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] + B_k^1 [E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)] = \frac{4 E_1 (-1)^{k+1} \epsilon}{\eta \gamma_k^2 \beta_k}. \tag{4.35}
\]

The motion becomes resonant if the determinant of this system vanishes, namely,

\[
[E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)] [E_1 J_0(\beta_k) + E_2 J_2(\beta_k)] -
\]

\[
\]
Figure 4.2: Criteria for resonance in cylinders containing a central rod at $r = 0.08674$ and $\theta = 20^\circ$.

\[ [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] [E_1 Y_0(\beta_k) + E_2 Y_2(\beta_k)] = 0 \quad (4.36) \]

This is the appropriate transcendental equation to be solved for the determination of the parameters $(r, \eta, r_0)$ that cause resonance. Similar to the case of completely filled cylinders, the roots of equation (4.36) depend on $k$. Given a frequency $r$, the roots of equation (4.36) represent continuous functions $\eta = \eta(r_0)$ in the $r_0$-$\eta$ plane. The absolute maximum of these functions is important. Note that for certain $k$, there might be more than one function and in this case the maximum is taken over all maxima of each function. Suppose we denote by $\eta_{\text{max}}(k)$, the absolute maximum of these functions. For each $k$, we have found that the parameters $\eta, r_0$ that lead to the most amplified moments are in the interval $\eta_{\text{max}}(k - 1) \leq \eta \leq \eta_{\text{max}}(k)$. These critical values are presented in figures.
Figure 4.3: Criteria for resonance in cylinders containing a central rod at $\tau = 0.05$ and $\theta = 2^\circ$.

Figure 4.4: Criteria for resonance in cylinders containing a central rod at $\tau = 0.1111$ and $\theta = 2^\circ$. 

Figure 4.5: Criteria for resonance in cylinders containing a central rod at $\tau = 0.15$ and $\theta = 2^\circ$.

For this case, in addition to the condition at the side wall represented by equation (4.32), the interface condition (4.20) provides

$$
\sum_{k=0}^{\infty} \left\{ A_k \left[ E_3 J_0(\beta_k r_0) + E_4 J_2(\beta_k r_0) \right] 
+ B_k \left[ E_3 Y_0(\beta_k r_0) + E_4 Y_2(\beta_k r_0) \right] \right\} \frac{\beta_k}{2} \sin(\gamma_k z) = -E_5 \epsilon z,
$$

where we have let

$$
E_3 = (1 - \frac{3}{4} \tau_0^2 + \frac{1}{4} \tau_0^3), \quad E_4 = (1 - \frac{5}{4} \tau_0^2 + \frac{1}{4} \tau_0^3), \quad \text{and} \quad E_5 = -\left(\frac{1}{2} + \frac{\tau_0}{4} - \tau_0^2\right).
$$
Thus, in addition to equation (4.34), \( A_k^1 \) and \( B_k^1 \) are related by

\[
A_k^1 [E_3 J_0(\beta_k r_0) + E_4 J_2(\beta_k r_0)] + B_k^1 [E_3 Y_0(\beta_k r_0) + E_4 Y_2(\beta_k r_0)] = 4 \frac{E_5 \epsilon}{\eta \gamma_k^2 \beta_k},
\]

and the motion is in resonance when

\[
[E_3 Y_0(\beta_k r_0) + E_3 Y_2(\beta_k r_0)] [E_1 J_0(\beta_k) + E_2 J_2(\beta_k)] - [E_3 J_0(\beta_k r_0) + E_3 J_2(\beta_k r_0)] [E_1 Y_0(\beta_k) + E_2 Y_2(\beta_k)] = 0.
\]

This equation is similar to that for resonance in cylinders containing a central rod. Figures 4.6 through 4.9 present the values of \((r_0, \eta)\) that lead to the most amplifications in the moments for different values of \(\tau\). Note that the curves \(\eta(r_0)\)
Figure 4.7: Criteria for resonance in partially filled cylinders at $\tau = 0.05$ and $\theta = 2^\circ$.

Figure 4.8: Criteria for resonance in partially filled cylinders at $\tau = 0.1111$ and $\theta = 2^\circ$. 
Figure 4.9: Criteria for resonance in partially filled cylinders at $\tau = 0.15$ and $\theta = 2^0$.

giving the most dangerous aspect ratio for a given fill radius are almost constant for fill radii up to $r_0 \approx 0.2$ unlike those for cylinders containing a central rod. This property emphasizes the fact that a central rod is more effective in eliminating resonance than a void.

### 4.5.4 Cylinders containing two immiscible fluids

Similar to the partial fill configuration, the no-penetration condition at the side wall is represented by equation (4.32). While the continuity of the pressure across the interface provides

$$
\sum_{k=0}^{\infty} A_k^0 [E_0 J_0(\beta_k r_0) + E_7 J_2(\beta_k r_0)] - A_k^1 E_0 [J_0(\beta_k r_0) - J_2(\beta_k r_0)]
$$

$$
- B_k^1 E_0 [Y_0(\beta_k r_0) - Y_2(\beta_k r_0)] \frac{\beta_k}{2} \sin(\gamma_k z) = E_5 (\rho_0 - \rho_1) z \epsilon \quad (4.40)
$$
where

\[
E_6 = \rho_0 (1 - \tau_0^2) + \frac{1}{4} (\tau_0^2 + \tau_0^3) (\rho_0 - \rho_1),
\]

\[
E_7 = \rho_0 (1 - \tau_0^2) + \frac{1}{4} (\tau_0^3 - \tau_0^2) (\rho_0 - \rho_1),
\]

and the continuity of the radial velocity across the interface provides

\[
\sum_{k=0}^{\infty} A_k^0 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] - A_k^1 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] - B_k^1 [E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)] \frac{\beta_k}{2} \sin(\gamma_k z) = 0. \tag{4.41}
\]

Thus, in addition to equation (4.34), \(A_k^0, A_k^1,\) and \(B_k^1\) are related by

\[
A_k^0 [E_6 J_0(\beta_k r_0) + E_7 J_2(\beta_k r_0)] - A_k^1 [E_1 J_0(\beta_k r_0) - J_2(\beta_k r_0)]
\]

\[
- B_k^1 E_0 [Y_0(\beta_k r_0) - Y_2(\beta_k r_0)] = 4 E_5 \frac{(-1)^{k+1} e}{\eta \gamma_k^2 \beta_k}, \tag{4.42}
\]

\[
A_k^0 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] - A_k^1 [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)] - B_k^1 [E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)] = 0. \tag{4.43}
\]

Consequently the motion resonates if

\[
[E_1 J_0(\beta_k) + E_2 J_2(\beta_k)] [E_1 Y_0(\beta_k r_0) + E_2 Y_2(\beta_k r_0)]
\]

\[
[E_6 J_0(\beta_k r_0) + E_7 J_2(\beta_k r_0)] - [E_1 J_0(\beta_k) + E_2 J_2(\beta_k)]
\]

\[
[E_0 J_0(\beta_k r_0) - E_0 J_2(\beta_k r_0)] [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)]
\]

\[
- [E_1 Y_0(\beta_k) + E_2 Y_2(\beta_k)] [E_1 J_0(\beta_k r_0) + E_2 J_2(\beta_k r_0)]
\]

\[
[E_6 J_0(\beta_k r_0) + E_7 J_2(\beta_k r_0)] = 0, \tag{4.44}
\]
Figure 4.10: Plot of the roots of equation (4.44) for $k = 0$, $\tau = 0.008$, $\theta = 1^0$, and $\rho_0/\rho_1 = 0.2$.

where

$$E_0 = (1 - \tau_0^2) \rho_1, \ E_8 = E_6 - E_0, \ \text{and} \ E_0 = E_7 + E_0.$$ 

This equation is more complicated than the ones we have seen so far, since in addition to its dependence on $\tau$, $\theta$, $\eta$, and $k$, it also depends on the density ratio $\rho_0/\rho_1$. Given $\tau$, $\theta$, $\rho_0/\rho_1$, and $k$, the solution to equation (4.44) provides the pairs $(r_0, \eta)$ that lead to resonance. As before these pairs constitute continuous functions $\eta = \eta(r_0)$ and for a given interval, $\eta_{\text{min}} \leq \eta \leq \eta_{\text{max}}$, there may be more than one function and the number of functions increases with $k$. We have plotted these functions for $k = 0, 1, 2, 3, 4,$ and $5$ in figures 4.10 through 4.15 respectively. If we plot the same results in one figure, it would be filled with small circles indicating
Figure 4.11: Plot of the roots of equation (4.44) for $k = 1$, $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$.

Figure 4.12: Plot of the roots of equation (4.44) for $k = 2$, $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$. 
Figure 4.13: Plot of the roots of equation (4.44) for $k = 3$, $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$.

Figure 4.14: Plot of the roots of equation (4.44) for $k = 4$, $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$. 
Figure 4.15: Plot of the roots of equation (4.44) for $k = 5$, $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$.

Figure 4.16: Plot of the most critical roots of equation (4.44) for $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$. 
resonance of some value $k$. However, not every critical pair $(r_0, \eta)$ lead to the most amplifications in moments. For a given region in the $r_0 - \eta$ plane some are more critical than others. We could choose to plot the critical values associated with the lowest values of $k$ as in figure 4.16 or plot only the critical values leading to the most amplifications in the moments as in figure 4.17. We prefer the latter than the former and we can obtain plots or diagrams of this sorts for different values of density ratios. These are shown in figures 4.18 through 4.22. Note that as $\rho_0 \to \rho_1$ we retrieve the results of the complete fill configuration. However, some critical values become degenerate since they would cancel out with terms in the numerators of the expansion coefficients in the limit as $\rho_0 \to \rho_1$. Degenerate modes are also found in the limit as $r_0 \to 0$ in this configuration as well as the partial fill and rod configurations. They are also found in the two fluid flow configuration as $\rho_0/\rho_1 \to 0$. Some of the most critical values approach those of the partial fill configuration and some are degenerate. We recommend that the appropriate limiting configuration be used to compute the critical values.

4.6 Comparisons with viscous results

In this section we have computed the parameters that lead to resonance for some results of the viscous analysis. We have indicated the results of the inviscid theory by vertical dashed lines. In all cases, the comparison is quite good as can be seen from figures 4.23 through 4.34.
Figure 4.17: Criteria for resonance in cylinders containing two immiscible fluids at $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.2$.

Figure 4.18: Criteria for resonance in cylinders containing two immiscible fluids at $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.3$. 
Figure 4.19: Criteria for resonance in cylinders containing two immiscible fluids at \( \tau = 0.008, \theta = 1^\circ, \) and \( \rho_0/\rho_1 = 0.4. \)

Figure 4.20: Criteria for resonance in cylinders containing two immiscible fluids at \( \tau = 0.008, \theta = 1^\circ, \) and \( \rho_0/\rho_1 = 0.5. \)
Figure 4.21: Criteria for resonance in cylinders containing two immiscible fluids at $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.7$.

Figure 4.22: Criteria for resonance in cylinders containing two immiscible fluids at $\tau = 0.008$, $\theta = 1^\circ$, and $\rho_0/\rho_1 = 0.98$. 
Figure 4.23: Yaw moment versus aspect ratio for completely filled cylinders at $\tau = 0.08333$, $\theta = 2^0$, $Re = 10^3$ (- - -), and $Re = 10^4$ (—). Comparison of viscous and inviscid (vertical dashed lines) results.

Figure 4.24: Pitch moment versus aspect ratio for completely filled cylinders at $\tau = 0.08333$, $\theta = 2^0$, $Re = 10^3$ (- - -), and $Re = 10^4$ (—). Comparison of viscous and inviscid (vertical dashed lines) results.
Figure 4.25: Roll moment versus fill radius for cylinders containing a central rod at $\tau = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$. Comparison of viscous results (solid lines) and inviscid results (vertical dashed lines).

Figure 4.26: Roll moment versus fill radius for partially filled cylinders at $\tau = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$. Comparison of viscous results (solid lines) and inviscid results (vertical dashed lines).
Figure 4.27: Roll moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 3$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).

Figure 4.28: Roll moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 2$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).
Figure 4.29: Roll moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 1.5$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).

Figure 4.30: Pitch moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 3$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).
Figure 4.31: Pitch moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 2$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).

Figure 4.32: Pitch moment versus fill radius for partially filled cylinders (---) and for cylinders containing a central rod (—) at $\tau = 0.1111$, $\eta = 1.5$, and $\theta = 2^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).
Figure 4.33: Roll moment versus fill radius for cylinders containing two immiscible fluids at $\rho_0/\rho_1 = 0.2$, $\tau = 0.008$, $\eta = 4.5$, and $\theta = 1^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).

Figure 4.34: Pitch moment versus fill radius for cylinders containing two immiscible fluids at $\rho_0/\rho_1 = 0.2$, $\tau = 0.008$, $\eta = 4.5$, and $\theta = 1^\circ$. Comparison of viscous results (dashed and solid lines) and inviscid results (vertical dashed lines).
CHAPTER V

Nonlinear Analysis and Pseudo-Spectral Methods

5.1 Introduction

So far we have only considered linear analysis, which was based on the observation that in practice the parameter \( \epsilon \) is small. It is important, however, to investigate the extent of this assumption and learn the effects of nonlinearities on the moments.

Moreover, the linearized Navier-Stokes equations in \( O(\epsilon) \) can only be solved for the fundamental velocities. Knowing the fundamental of the axial velocity is enough to accurately predict the yaw and roll moments exerted by viscous fluids on the walls. The pitch moment, however, can only be estimated from these components and the nonlinear problem needs be solved for accurate prediction. In addition to the fundamental axial velocities, the pitch moment is dependent on the mean flow distortion of the azimuthal velocity component.

In this chapter, a pseudo-spectral method is devised to solve the three-dimensional linearized and nonlinear flow equations for partially filled cylinders and cylinders
with a central rod. For completely filled cylinders, the reader is to consult the paper by Herbert & Li (1987) and the extension of the method to two fluid flows is straightforward. The method is based on representing the flow quantities by Chebyshev series in the axial and radial directions and a Fourier series in the azimuthal direction.

5.2 Spectral approximation

For convenience, we treat both configurations simultaneously by introducing the parameter $\alpha$ that assumes the value of $\alpha = 0$ in the case of a cylinder with a central rod and $\alpha = 1$ for the partially filled cylinder. We approximate the flow quantities by

$$ (v_r, v_\phi, v_z, p) = \sum_{k=1}^{K+\alpha} \sum_{l=1}^{L} \sum_{m=1}^{M} (u_{klm}, v_{klm}, w_{klm}, p_{klm}) R_k(\tilde{r}) F_l(\phi) Z_m(z/\eta) $$

where $R_k(\tilde{r})$ and $Z_m(z/\eta)$ are linear combinations of Chebyshev polynomials suitably chosen for each flow quantity and flow configuration,

$$ \tilde{r} = (2r - 1 - r_0)/(1 - r_0) $$

is an algebraic mapping that maps the radial domain $[r_0, 1]$ into the interval $[-1, 1]$ where the Chebyshev polynomials are defined, and

$$ F_l = \left\{ \begin{array}{ll}
\cos \frac{l-1}{2} \phi, & l \text{ odd,} \\
\sin \frac{l}{2} \phi, & l \text{ even,}
\end{array} \right. $$

are the azimuthal expansion functions. The linear combinations of Chebyshev polynomials for the radial and axial expansion functions are selected such that...
each function satisfies the no-slip conditions at the walls and yields the proper permutation with the azimuthal functions $F_i$ to maintain the symmetries with respect to the center point of the cylinder. The interface conditions need be satisfied explicitly, and consequently the number of radial expansion functions for the case of partially filled cylinder is one higher than the number required for the cylinder with a central rod as shown in the expansion above.

The governing differential equations are converted into a system of algebraic equations for the $4 \cdot (K + \alpha) \cdot L \cdot M$ unknown expansion coefficients by spectral collocation at the points

$$(\tilde{r}_k, \phi_l, \frac{Z_m}{\eta}) = (\cos \frac{2k - 1}{2K}\pi, 2\pi \frac{l - 1}{L}, \sin \frac{2m - 1}{4M}\pi), \quad (5.4)$$

where $k = 1, 2, ..., K$, $l = 1, 2, ..., L$, and $m = 1, 2, ..., M$. In the first step, the linear algebraic system resulting from linearization in $\epsilon$ is solved. Subsequently, the non-linear problem is solved iteratively. For each iteration step, the nonlinear terms are evaluated using the solution at the previous step. This approach is computationally less expensive than Newton-Raphson iteration where each step requires solving a new algebraic system of relatively large size to obtain the corrections to the previous approximations. In our approach, the system is solved only once, and in subsequent steps, only the right hand side of the algebraic system is modified.
5.3 Results

We have used the expressions presented in chapter 2 for the control volume approach to calculate the moments exerted by the payload on the cylinder walls. Figures 5.1 through 5.8 present the yaw and pitch moments as functions of the fill radius for different Reynolds numbers for both configurations at typical aspect ratio and coning frequency, but small nutation angle, i.e. $\theta = 2^\circ$. The figures compare the nonlinear results shown as circles with the linear ones shown by solid lines. As can be seen there is no difference in both results even for the pitch moment that requires in addition to the fundamental of the axial velocity, the mean of the azimuthal velocity. We conclude, thus, that for relatively small nutation angles the nonlinear effects are negligible.

For high nutation angles, i.e. $\theta = 20^\circ$, computations of the moments for the same parameters as above and for $\theta = 20^\circ$ show no nonlinear effects on the yaw and roll moments as can be seen from figures 5.9 and 5.10. However, there is a slight nonlinear effect on the Pitch moment as can be seen from figures 5.11 and 5.12. Nevertheless, this effect is insignificant for engineering practice. The difference between linear and nonlinear results does not surpass 2% for parameters not leading to resonance.

In case of resonance, this difference can be more significant as can be seen from figures 5.13 and 5.14. Note that the difference is higher for a partially filled
Figure 5.1: Yaw moment versus fill radius for a cylinder containing a central rod at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 2^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.2: Yaw moment versus fill radius for a cylinder containing a central rod at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 2^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.3: Pitch moment versus fill radius for a cylinder containing a central rod at \( \tau = 0.1, \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.4: Pitch moment versus fill radius for a cylinder containing a central rod at \( \tau = 0.1, \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.5: Yaw moment versus fill radius for a partially filled cylinder at \( r = 0.1, \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.6: Yaw moment versus fill radius for a partially filled cylinder at \( r = 0.1, \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.7: Pitch moment versus fill radius for a partially filled cylinder at \( \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.8: Pitch moment versus fill radius for a partially filled cylinder at \( \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.9: Yaw moment versus fill radius for a cylinder containing a central rod at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.10: Yaw moment versus fill radius for a partially filled cylinder at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.11: Pitch moment versus fill radius for a cylinder containing a central rod at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.12: Pitch moment versus fill radius for a partially filled cylinder at $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
cylinder than that for a cylinder containing a central rod. We should also mention here that in case of fill radii leading to resonance we have seen small nonlinear effects on both yaw and roll moments as can be seen from figures 5.15 and 5.16. The figure present the yaw moments as functions of the fill radius for \( Re = 500 \). Similar results for \( Re = 1000 \) are presented in figures 5.17 and 5.18. Note that we have not shown the nonlinear results for fill ratios leading to resonance. The reason for this is that with a triple series expansion for each flow quantity composed of 11 Chebyshev polynomials in \( r \), 11 Chebyshev polynomials in \( z \), and 5 Fourier functions in \( \phi \), the nonlinear results did not converge as the fill radius approach that leading to resonance. We think by increasing the resolution, the results would converge.

We have compared the linear results of the 3D-Spectral codes to those of the eigenfunction expansions. The comparisons of the moments as function of the fill radius for \( Re = 30 \) are shown for small nutation angle, i.e. \( \theta = 2^\circ \), in figures 5.19 through 5.22 and for large nutation angle, i.e. \( \theta = 20^\circ \), in figures 5.23 through 5.26. As can be seen from these figures regardless of the nutation angle there is a small discrepancy between both results. We believe it is due to the high resolution of the method of eigenfunction expansions. By comparison with the results for completely filled cylinders, obtained by solving the SOPDE by spectral techniques, we believe to achieve the accuracy of the eigenfunction expansion, we must include around 50 Chebyshev polynomials in each direction in the triple series expansion used in
Figure 5.13: Pitch moment versus fill radius for a cylinder containing a central rod at \( Re = 500, \tau = 0.1, \eta = 4.5 \) and \( \theta = 20^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.14: Pitch moment versus fill radius for a partially filled cylinder at \( Re = 500, \tau = 0.1, \eta = 4.5 \) and \( \theta = 20^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.15: Yaw moment versus fill radius for a cylinder containing a central rod at $Re = 500$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.16: Yaw moment versus fill radius for a partially filled cylinder at $Re = 500$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
Figure 5.17: Yaw moment versus fill radius for a cylinder containing a central rod at \( Re = 1000, \tau = 0.1, \eta = 4.5 \) and \( \theta = 20^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.

Figure 5.18: Yaw moment versus fill radius for a partially filled cylinder at \( Re = 1000, \tau = 0.1, \eta = 4.5 \) and \( \theta = 20^\circ \). Comparison of the linear (solid line) and nonlinear (circles) results obtained by the 3D-Spectral code.
the 3D-Spectral code. Doing so is quite costly at this time in terms of computer memory. We wait for the next generation of computers to be able to study for example the stability of such flows since we need high accuracy in calculating the basic flow. For higher Reynolds number, the comparison is comparable to that for $Re = 30$, except at resonance the discrepancies are slightly more significant as seen from figures 5.27 through 5.30.

The 3D spectral code does not only provide us with means to calculate the moments, it also provides solutions of the 3D flow for the various configurations that can be visualized on a graphic workstation to study the detail of the fluid motion. Figures 5.31 through 5.35 show some of the details of the flow of the
Figure 5.20: Pitch moment versus fill radius for a cylinder containing a central rod at $Re = 30$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 2^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

Figure 5.21: Yaw moment versus fill radius for a partially filled cylinder at $Re = 30$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 2^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).
Figure 5.22: Pitch moment versus fill radius for a partially filled cylinder at \( Re = 30, \tau = 0.1, \eta = 4.5 \) and \( \theta = 2^\circ \). Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

Figure 5.23: Yaw moment versus fill radius for a cylinder containing a central rod at \( Re = 30, \tau = 0.1, \eta = 4.5 \) and \( \theta = 20^\circ \). Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).
Figure 5.24: Pitch moment versus fill radius for a cylinder containing a central rod at $Re = 30$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

Figure 5.25: Yaw moment versus fill radius for a partially filled cylinder at $Re = 30$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).
Figure 5.26: Pitch moment versus fill radius for a partially filled cylinder at $Re = 30$, $r = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

Figure 5.27: Yaw moment versus fill radius for a cylinder containing a central rod at $Re = 500$, $r = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).
Figure 5.28: Pitch moment versus fill radius for a cylinder containing a central rod at $Re = 500$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^0$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

Figure 5.29: Yaw moment versus fill radius for a partially filled cylinder at $Re = 500$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^0$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).
Figure 5.30: Pitch moment versus fill radius for a partially filled cylinder at $Re = 500$, $\tau = 0.1$, $\eta = 4.5$ and $\theta = 20^\circ$. Comparison of the linear results of the eigenfunction expansion (solid line) and nonlinear results of the 3D-Spectral code (dashed lines).

various configurations obtained by the program Plot3D.
Figure 5.31: Vector plot of the velocity in the plane $x = 0$ for the flow in a cylinder containing a central rod of radius $r_0 = 0.2$ at $Re = 30$, $r = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$.

Figure 5.32: Vector plot of the velocity in the plane $x = 0$ for the flow in a cylinder containing a void of mean radius $r_0 = 0.2$ at $Re = 30$, $r = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$. 
Figure 5.33: Contour plot of the pressure field in the plane $y = 0$ for the flow in a cylinder containing a central rod of radius $r_0 = 0.2$ at $Re = 500$, $\tau = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$.

Figure 5.34: Contour plot of the pressure field in the plane $y = 0$ for the flow in a cylinder containing a void of mean radius $r_0 = 0.2$ at $Re = 500$, $\tau = 0.08674$, $\eta = 4.5$, and $\theta = 20^\circ$. 
Figure 5.35: Contour plot of the pressure field in the plane $y = 0$ for the flow in a cylinder containing a void of mean radius $r_0 = 0.2$ at $Re = 500$, $\tau = 0.08674$, $\eta = 4.5$, and $\theta = 2^\circ$. 
CHAPTER VI

Summary and Conclusions

We have calculated the moments caused by various viscous flows in a spinning and coning cylinder. The cylinder can be completely or partially filled with a single fluid, contain one fluid and a central rod, or be filled with two immiscible fluids.

The calculations are based on the assumption of laminar flow, the solution of either the linearized or full Navier-Stokes equations, and the control volume approach in calculating these moments. When using this approach, it was found that the moments depend essentially on the axial velocity deviation from solid body rotation. In fact, when expressing the velocity in a Fourier series in $\phi$—the angle in the azimuthal direction—only the fundamental component is needed for accurate calculations of both yaw and roll moments and good estimates of the pitch moment.

For this reason, we have derived a single Sixth Order Partial Differential Equation (SOPDE) in $r$, the radial distance from the spin axis, and $z$, the axial distance from the center of the cylinder, governing this component for the various configu-
rations. This equation is based on the linearization of the Navier-Stokes equations in \( e = \tau \sin \theta \), where \( \tau \) is the coning frequency and \( \theta \) is the nutation angle, and therefore is linear. A nonlinear equation of this sort is not possible, though, a systematic perturbation in \( e \) can be devised to include higher order terms or the full Navier-Stokes equations need be solved for all flow quantities to study the effect of nonlinearities on the moments.

The reason for this linearization is purely practical. Considering constant spinning and coning rotations is only a simple model to the more complicated motion of real life projectiles. Nonetheless, field tests have shown that soon after the firing of the projectile and its initial spin-up, it acquires a nearly steady coning motion around its trajectory with a coning rate that is far smaller than that of its spinning rate and a small nutation angle before its flight fails. Consequently, in practice, the parameter \( e \) is small and since the forcing term in the Navier-Stokes equations is order \( e \) and since as \( e \to 0 \), these equations admit the trivial solution, it is well justified to use \( e \) to linearize these equations. We, however, wanted to make sure of this fact and employed spectral techniques to solve the full Navier Stokes equations to investigate the effect of nonlinearities on the moments. It was found that for all practical applications, nonlinear effects are negligible.

Although the derivation of the boundary conditions that accompany the SOPDE are quite complex, the relatively simple conditions at the end walls made it possible to investigate the effect of various configuration on the moments.
The three conditions at each end wall are homogeneous and since the governing equation is also homogeneous, the application of separation of variables has lead to solving a sixth order eigenvalue problem in the axial direction.

Considered that for a given frequency, nutation angle, and aspect ratio, these conditions are the same for all configurations, only one eigenvalue problem need be solved to compute the moments for all configurations. In the case of two-fluid flows of different inner and outer flow Reynolds numbers, the same eigenvalue problem is solved twice—once corresponding to the inner-flow Reynolds number $Re_0$ and once corresponding to the outer-flow Reynolds number $Re_1$.

This approach is in itself more advantageous than that of solving a radial eigenvalue problem for that when doing so, a different eigenvalue problem need be solved for each configuration since the boundary conditions in the radial direction vary from configuration to the next. A total of four eigenvalue problems need be solved for the configurations considered here and their solutions is a nontrivial task. This approach has been undertaken by Hall et al. (1987) to solve the low order linearized Navier-Stokes equations and in addition to being restricted to Reynolds numbers up to 1000, it can be quite expensive computationally especially as the degree of complexity of the configuration is increased. The expense comes through the necessity to iterate for the eigenfunctions and to solve for all flow quantities.

In our approach, even though the eigenvalues are found numerically and finding them is a nontrivial task, the eigenfunctions are given in closed form and their
integrability makes it possible to achieve high accuracy in calculating the moments through performance of the integrations in their expressions given by the control volume approach analytically.

We have obtained these eigenvalues and eigenfunctions for moderate as well as high Reynolds numbers and although at high Reynolds number, the Bessel functions which describe the radial structure of the flow field become difficult to evaluate, their asymptotic expansions made it possible to compute the moments at Reynolds numbers as high as 25,000. This enabled us to see the remarkable maxima acquired by the moments at critical aspect ratios, coning frequencies, and fill ratios.

In addition to its ability to predict both types of instability—viscous and inviscid—it's low requirement of CPU memory and execution time makes the solution by eigenfunction expansions more appealing at least for purposes of calculating the moments than the solution of the full Navier-Stokes equations by spectral techniques.

Although full or linearized Navier-Stokes equations must be solved for all flow quantities for purposes of flow visualizations, the invalidity of these equations at the corner makes their solutions converge slowly no matter what method of solution is used. This singularity is really a big hurdle in achieving high accuracy with few polynomials in approximating the solution by spectral techniques. The flow near the corner is quite complicated and a large number of polynomials is needed in
each direction to fully resolve such a flow.

In all moment calculations we have done by solving the full Navier-Stokes equations, we have approximated each of the four flow quantities, namely, $v_r$, $v_\phi$, $v_z$, and $p^d$, by a triple series composed of 11 Chebyshev polynomials in $r$, 11 Chebyshev polynomials in $z$, and 5 Fourier functions in $\phi$. The collocation discretization of the flow equations with this approximation yields a $2420 \times 2420$ system of nonlinear algebraic equations that require about 16 Megawords of Memory on a Cray Y-MP8/864. The CPU time on the same machine is typically of the order of 78 seconds.

By comparison with the Chebyshev expansion used to solve the SOPDE, we need close to 50 polynomials to approximate each flow quantity, at least in each of the $r$ and $z$ directions, to achieve 5 digits of accuracy. Suppose we further approximate the flow variation in the $\phi$ direction by 5 harmonic functions—a conservative number—then the collocation discretization of the flow equations with this approximation yields a $50,000 \times 50,000$ system of nonlinear algebraic equations that requires large size of Memory which prevented us from doing a stability analysis of the solutions computed here, even though it was not our intention to do so. However, we recommend such a study in the future now that we know a great deal about the basic flow.

The solution of the SOPDE by Chebyshev expansions offers an alternative way to evaluate the moments as accurately as those obtained by the eigenfunction
expansions at affordable CPU times and memory. Typical runs with $36 \times 36$ polynomials take about 116 seconds on a Cray Y-MP8/864. Since we only solve for one harmonic component, we can employ as many as $46 \times 46$ polynomials to achieve the 16 Megaword mark on a Cray Y-MP8/864.

We conclude that for steady state conditions, the moments are best calculated by solving the SOPDE by eigenfunction expansions because it is inexpensive computationally. The solution of the same equation by spectral techniques, though require more memory and time, is also competitive by today's standards. Although we have presented the spectral solution of the SOPDE for only completely filled cylinders, the extension of the method to other flow configurations is straightforward. One must deal with singularities at the corners for such a technique to converge. We believe that subtracting the solution corresponding to the infinitely long cylinder for each configuration improves the convergence dramatically like it did for the case of completely filled cylinders.

With the eigenfunction expansions, we have calculated the moments for the various configurations at a wide range of flow parameters. For completely filled cylinders, it was found that for a given coning frequency, nutation angle, and aspect ratio, the roll and yaw moments acquire maxima at critical Reynolds number, say $Re_c$. The pitch moment, however, increases monotonically as the Reynolds number increases. These maxima, primarily that of the roll moment, can be responsible for the flight failure of the projectile. The flight instability caused by
this phenomenon is referred to as *viscous* instability since it occurs at relatively low Reynolds numbers, i.e. \( Re_c \approx 30 \). Furthermore, if the frequency and nutation angle are kept fixed while the aspect ratio is varied, we observed, for certain aspect ratios, dramatic increases in both yaw and roll moments as the Reynolds number becomes large. These critical aspect ratios are best seen when looking at a plot of either moment versus the aspect ratio. Such a plot locate the aspect ratios at which both yaw and roll moments acquire maxima. Since the moments increase at these critical aspect ratios as the Reynolds number increases, we believe that this phenomenon is inviscid in nature and is due to resonance with inertial waves. The flight instability caused by this phenomena is referred to as *inviscid* instability. Our method, thus, can handle both types of flight instability and therefore works very well for flight simulations.

We have presented criteria for the onset of resonance by solving the linearized inviscid equations in analytical form. The solutions for the flow quantities are expanded in series of products of Bessel functions in \( r \) and trigonometric functions in \( z \) that satisfy the no-penetration condition at both end walls and having wave numbers dependent on the aspect ratio, the coning frequency, and the nutation angle. The expansion coefficients are found by satisfying the no-penetration condition at the side wall. It was found that these coefficients acquire poles at critical aspect ratios, frequencies, and nutation angles.

The characteristic equations describing these poles represent criteria for the on-
set of resonance. These equations are transcendental, composed of Bessel functions, and for given frequency and nutation angle, they are solved by Newton-Raphson method for the aspect ratio that causes resonance. Since the number of coefficients is infinite, the number of these equations is infinite, and consequently, there can be infinite critical values for the onset of resonance. However, in viscous flows, while all modes are damped to a certain extent because of the effect of viscosity, modes with largest wavelengths are the most dangerous—they lead to the most amplifications in the moments.

We have obtained loci of the parameters in the $\tau$-$\eta$ plane that lead to these dangerous modes. For stable configurations, these loci need be avoided. However since in real life the coning frequency might vary a little, these loci could be crossed and resonance can occur. Also design criteria can force the designer to choose an aspect ratio that causes resonance. It was suggested that a partial fill or a central rod inside the cylinder can eliminate resonance. It was not, however, known that the fill ratio is crucial to the onset of resonance and consequently cylinders were routinely left partially filled for safety purposes without knowing that this could be the cause of flight instability of projectiles carrying fluids of low viscosity.

We have calculated the moments for partially filled cylinders and cylinders containing a central rod. It was found, that for low Reynolds numbers the moments decrease monotonically as the fill ratio is decreased. However, as the Reynolds number is increased, we saw the development of remarkable peaks in both yaw
and roll moments at critical fill radii. The moments at these peaks increase as the Reynolds number increases, and hence, they are due to resonance with inertial waves.

Similar to completely filled cylinders, we have solved the inviscid equations for both configurations in analytical form. The same method of solution has been used, but now in addition to satisfying the no-penetration conditions at the side wall, the coefficients are solved for by satisfying the no-penetration condition at the surface of the rod or the vanishing of the stresses at the interface. The characteristic equations describing the poles of these coefficients represent the criteria for resonance. We have solved these equations and provided loci of the fill radii and aspect ratios that lead to the most amplification in the moments for given frequency and nutation angle. We can retrieve the results for completely filled cylinders when we let the fill radius in either configuration approach zero. Thus in the case when the aspect ratio leads to resonance for completely cylinders, increasing the fill radius can eliminate resonance. This can be easily seen when examining curves of the loci of the parameters leading to resonance $\eta(r_0)$—increasing the fill radius $r_0$ slightly, avoids this curve, and therefore avoids resonance. We have found that the central rod configuration is more effective in eliminating resonance—with a central rod resonance is eliminated with smaller rod radii than that corresponding to partially filled cylinders. This also can be seen and from the curves $\eta(r_0)$. Since for partially filled cylinders the curves $\eta(r_0)$ are almost constant for fill radii
up to \( r_0 \approx 0.2 \) unlike that for cylinders with a central rod.

Finally we have calculated the moments for cylinders containing two immiscible fluids. It was suggested that having a low viscosity fluid in contact with the wall can lubricate the core fluid and thereby reduce the roll moments, and consequently eliminate the viscous instability. We investigated this case for a small nutation angle, i.e. \( \theta = 1^\circ \), and a small (though not necessary) coning frequency, i.e. \( \tau = 0.008 \), to make sure that the interface distortion is insignificant for a wide range of density ratios, and be able to present a comprehensive study of the effect of the density ratio of inner-fluid to outer-fluid on the moments.

For these conditions, we have found the Reynolds number at which the maximum roll moment of the core fluid occurs if the cylinder is completely filled with this fluid, and we have used it in our study as the Reynolds number of the inner fluid. For the outer fluid, we have selected a Reynolds number of \( Re_1 = 10^4 \). We have then calculated the moments for the two-fluid configuration for a wide number of fill radii \( 0 \leq r_0 \leq 1 \). It was found that for density ratios close to but less than unity, i.e. \( \rho_0/\rho_1 = 0.98 \), the roll moment starts to decrease monotonically as the fill ratio is increased. The fill ratio here is defined as the ratio of the volume of low viscosity additive to the volume of the cylinder. The biggest drops in the roll moment occur for small fill ratios which is desirable since we do not want to fill the whole cylinder with a lot of additive. It is mainly used for lubrication and it is desirable to use the smallest quantity that lead to a substantial decrease in
the roll moment.

We have found that a fill ratio of 2% can reduce the roll moment, and consequently the yaw moment, by about 30%. However, the pitch moment increases dramatically to that corresponding to the values for the outer fluid in completely filled cylinders. With a 2% fill ratio, this component can be as much as two and half times that for 0% fill ratio. Consequently, we conclude that while having a low viscosity additive reduces the roll moment, and therefore reduces the despin of the cylinder, it might still lead to flight failure of the projectile. The only definite way to know this as a matter of fact is to do the flight simulation which is not our objective here. As the outer-flow Reynolds number increases, the reduction of the roll moment for a given fill ratio increases but not monotonically. For a fill ratio of 2%, extrapolation yields a 34% reduction for $Re_1 \approx 100,000$.

Further more as we lowered the density ratio we found out that below a certain density ratio, i.e. $\rho_0/\rho_1 \approx 0.3$, the roll moment no longer decreases, but rather increases. We have seen also the development of a significant hump at a "critical" fill ratio at which the roll moment increases when the density ratio decreases. This significant hump is due to resonance with inertial waves. There are other smaller humps that appear as "wiggles" in the plot of the roll moment versus the fill ratio and we believe they are due to resonance with inertial secondary waves—waves of shorter wavelengths than that corresponding to the most significant hump or humps. Further reduction of the density ratio beyond a sufficiently small value
causes the development of sharper peaks in both yaw and roll moments that are characteristics of resonance in partially filled cylinders. In fact for sufficiently low density ratios, the solution corresponding to two immiscible fluids tends to that corresponding to partially filled cylinders. We, thus, conclude that for sufficiently low density ratios, the inner fluid behaves like a void and the void resonance criteria be used for predicting resonance for such a case.

For best reduction of the roll moment, we have found that the density ratio must be close to unity and the additive must be much less viscous than the core fluid. For small quantities of the low viscosity additive, we have investigated the behavior of the roll moment as a function of the core fluid Reynolds number. We have found that the effect of the additive is to shift the curve describing the roll moment as a function of the core fluid Reynolds number to the left and to lower the critical Reynolds number at which the roll moment acquires a maximum. Similar results are found for the other moment components. The results for the roll moment indicate that this component cannot be reduced with lubrication for any core fluid Reynolds number. For some Reynolds numbers below the critical Reynolds number the roll moment increases rather than decreases with the presence of a low viscosity additive. Although the roll moment at the location of the hump for the core fluid with the additive is a little larger than that for the core fluid without the additive, the fact that the critical Reynolds number is lower can improve the stability of an otherwise unstable projectile.
For completeness and if it is desirable to carry two fluids, we have calculated the moments for two fluids of comparable Reynolds number. We have presented results for high Reynolds numbers because we wanted to see resonance. We have calculated these for $Re_0 = Re_1 = 10,000$ and for the same parameters as before. We varied the density ratio and presented results for the moments as functions of fill radius. We have found that resonance depends on the density ratio. For certain density ratios, we have seen the developments of humps that are not due to resonance because we have seen those at low Reynolds number on one hand and when the inviscid equations were solved there were no corresponding roots at the location of the maximum of the hump on the other hand. However, for certain others, we have clearly seen resonance at critical fill ratios, at which the roll and yaw moments amplify sharply. For certain values of the density ratio, we saw more than one critical fill radius and for certain others we have seen some nonsignificant amplifications in the moments. We know that these nonsignificant amplifications are due to resonance with inertial waves of longer wavelengths than the most critical ones.

Therefore similar to the partial fill and central rod configurations, we have solved the inviscid equations for this configuration and developed plots for the loci of the critical parameters leading to resonance. In the $r_0-\eta$ plane, we have presented for a given density ratio, the loci of the most critical parameters for each region or the most dangerous. We prefer the plot of the most dangerous
parameters and we have done this for quite few density ratios. These plots guided us to the understanding of why for certain interval of the density we see significant amplifications in the way and roll moments and in others we do not. They also provided a comprehensive study of the effect of the density ratio on resonance for wide range of aspect ratios. These plots are quite complicated and we believe that for fluids of low viscosity and large density difference, projectiles carrying this type of configuration might never fly.
Appendix A

Scaled form of boundary operators

A.1 Side walls boundary operators

These boundary operators are either applied at the side wall \( r = 1 \) or at the surface of a central rod \( r = r_0 \). They take the form

\[
\mathcal{L}^w_1 = \mathcal{I}, \quad \text{ (A.1)}
\]

\[
\mathcal{L}^w_2 = -\frac{\partial}{\partial \tau} (\tilde{\nabla}^2_1) - \frac{\partial^3}{\partial \tau^2 \partial z_2} - 2(1 + \tau_z) \frac{1}{\tau} \tilde{\nabla}^2_1 + \frac{\partial}{\partial \tau} (\tilde{\nabla}^4_1) + 2 \frac{\partial}{\partial \tau} \tilde{\nabla}^2_1 \frac{\partial^2}{\partial z^2} + \frac{\partial^5}{\partial \tau^2 \partial z^2}, \quad \text{ (A.2)}
\]

\[
\mathcal{L}^w_3 = -\frac{1}{\tau} \tilde{\nabla}^2_1 + 2(1 + \tau_z) \frac{\partial}{\partial \tau} - 2(1 + \tau_z) \frac{\partial}{\partial \tau} (\tilde{\nabla}^2_1) - 4(1 + \tau_z) \frac{\partial^3}{\partial \tau^3 \partial z^2} + \frac{1}{\tau} [\tilde{\nabla}^4_1 + 2 \tilde{\nabla}^2_1 \frac{\partial^2}{\partial z^2}], \quad \text{ (A.3)}
\]

where \( \mathcal{I} \) is the identity operator, i.e. \( \mathcal{I} w = w \), and

\[
\tilde{\nabla}^2_1 = \frac{\partial^2}{\partial \tau^2} + \frac{1}{\tau} \frac{\partial}{\partial \tau} - \frac{1}{\tau^2}, \quad \text{ (A.4)}
\]
The forcing constants associated with the above operators are:

\[ C_1^w = 0, \]  
\[ C_2^w = \frac{2(1 + \tau_z)}{iq_1}, \]  
\[ C_3^w = \frac{2(1 + \tau_z)}{iq_1}. \]  

(A.5) 
(A.6) 
(A.7)

A.2 Boundary operators at void-fluid interface

These interface operators are applied at \( r = r_0 \). They are

\[ \mathcal{L}_1^u = -(1 + \tau_0) \frac{1}{r} \hat{\nabla}^2 + \frac{1}{r} \hat{\nabla}^4 + \frac{\tau_0}{r} - \frac{2}{}\frac{\tau_0}{r} \frac{\partial^2}{\partial r^2} - \frac{\tau_0}{4} \frac{\partial}{\partial r}, \]  
\[ (A.8) \]

\[ \mathcal{L}_2^u = -\tau_0 \frac{\partial}{\partial r} \hat{\nabla}^2 - \frac{2}{\tau_0} \frac{\partial^3}{\partial r^3} + \frac{2}{\tau_0} \frac{\partial^2}{\partial r^2} - \frac{\tau_0}{2} \hat{r} \hat{\nabla}^2 
+ \frac{\tau_0}{2} \hat{r} \hat{\nabla}^4 - \frac{1}{r} \hat{\nabla}^4 \frac{1}{r} \hat{\nabla}^4 - \tau_0^2 \frac{\partial}{\partial r}, \]  
\[ (A.9) \]

\[ \mathcal{L}_3^u = -\tau_0 \left( \frac{1}{r} \hat{\nabla}^2 + \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{1}{r} - \frac{\partial}{\partial r} + \frac{\partial^3}{\partial r^3} \right) 
- \frac{\partial}{\partial r} \hat{\nabla}^2 + \frac{\partial}{\partial r} \hat{\nabla}^4 + \frac{\hat{r}}{2} \hat{\nabla}^2 - \frac{\hat{r}}{2} \hat{\nabla}^4 \frac{\partial^2}{\partial r^2} 
- \frac{\hat{r}}{2} \hat{\nabla}^4 + \frac{\hat{r}}{2} \hat{\nabla}^4 \frac{\partial^2}{\partial r^2} - \tau_0^2 \frac{\partial}{\partial r}, \]  
\[ (A.10) \]

and the forcing constants associated with them are:

\[ C_1^u = \frac{(2 + \tau_0)\tau_0}{4i q_1}, \]  
\[ C_2^u = \frac{\tau_0}{iq_1}, \]  
\[ C_3^u = \frac{\tau_0}{iq_1}. \]  

(A.11) 
(A.12) 
(A.13)
A.3 Boundary operators at two-fluid interface

These interface operators are applied at $r = r_0$. They take the form

\[ \mathcal{L}_1^\alpha = \mathcal{I}, \]  

(A.14)

\[ \mathcal{L}_2^\alpha = \rho_\alpha \left[ \frac{1}{r} \hat{\nabla}_1^2 - \frac{1}{r} \hat{\nabla}_4^4 - \tau_0 \frac{\partial}{\partial \hat{r}} - \frac{2}{r} \frac{\partial^2}{\partial \hat{z}^2} \right] - \hat{\nabla}_2^4 + \hat{\nabla}_4^4 + t_2^\alpha \frac{\partial}{\partial \hat{r}} \right], \]  

(A.15)

\[ \mathcal{L}_3^\alpha = q_\alpha \left[ \frac{1}{r} \hat{\nabla}_1^2 - \frac{1}{r} \hat{\nabla}_4^4 - \tau_0 \frac{\partial}{\partial \hat{r}} + \tau_0 \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 + \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} \right], \]  

(A.16)

\[ \mathcal{L}_4^\alpha = \rho_\alpha \left[ \frac{1}{r} \hat{\nabla}_1^2 - \frac{1}{r} \hat{\nabla}_4^4 + \frac{1}{r} \hat{\nabla}_2^4 + \tau_0 \frac{\partial}{\partial \hat{r}} + \frac{1}{2} \frac{\partial}{\partial \hat{r}} \right] \]  

\[ + 2 \tau_0 \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 - \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} + \tau_0 \frac{3}{2} \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 + \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} \right], \]  

(A.17)

\[ \mathcal{L}_5^\alpha = q_\alpha \left[ - (1 + \tau_0^2) \frac{1}{r} \hat{\nabla}_1^2 + \frac{1}{r} \hat{\nabla}_4^4 + \frac{1}{r} \hat{\nabla}_2^4 + \frac{1}{r} \hat{\nabla}_4^4 \right] 

\[ - 2 \tau_0 \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 - \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} + \tau_0 \frac{3}{2} \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 + \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} \right], \]  

(A.18)

\[ \mathcal{L}_6^\alpha = \rho_\alpha q_\alpha \left[ (1 + \tau_0^2) \frac{1}{r} \hat{\nabla}_1^2 - \frac{1}{r} \hat{\nabla}_4^4 + \frac{1}{r} \hat{\nabla}_2^4 + \frac{1}{r} \hat{\nabla}_4^4 \right] 

\[ - \frac{2}{r} \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 - \frac{2}{r} \frac{\partial}{\partial \hat{r}} \hat{\nabla}_4^4 - \frac{2}{r} \frac{\partial}{\partial \hat{r}} \hat{\nabla}_4^4 \]  

\[ + (t_6^\alpha - \tau_0) \frac{\partial}{\partial \hat{r}} + (\tau_0^2 + 2 \tau_0) \frac{\partial}{\partial \hat{r}} \hat{\nabla}_2^4 + \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} + \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} - \tau_0 \frac{3}{2} \frac{\partial^3}{\partial \hat{r} \partial \hat{z}^2} \right]. \]  

(A.19)
The forcing constants associated with these operators are

\begin{align*}
C_1 &= 0, \quad (A.20) \\
C_2 &= \frac{\tau_0}{4} (\tau_0 + 2) i (\rho_1 - \rho_0) r_0, \quad (A.21) \\
C_3 &= 0, \quad (A.22) \\
C_4 &= \tau_0 \frac{i}{r_0} \left( \frac{\rho_0}{q_0^2} - \frac{\rho_1}{q_1^2} \right), \quad (A.23) \\
C_5 &= (\tau_0 + \tau_0^2) i (q_1^2 - q_0^2), \quad (A.24) \\
C_6 &= (\tau_0 + \tau_0^2) \frac{i}{r_0} (\rho_0 - \rho_1), \quad (A.25)
\end{align*}

and

\begin{align*}
\tau_2^0 &= \frac{\tau_0^3}{4} q_1^2 \frac{\rho_1 - \rho_0}{\rho_0 q_1^2 - \rho_1 q_0^2}, \quad (A.26) \\
\tau_2^1 &= \frac{q_0^2}{q_1^2} \tau_2^0, \quad (A.27) \\
\tau_3^0 &= \rho_0 \tau_0 \frac{q_0^2 - q_0^2}{\rho_0 q_1^2 - \rho_1 q_0^2}, \quad (A.28) \\
\tau_4^0 &= \frac{\tau_0}{q_1^2} \frac{\rho_0 q_1^4 - \rho_1 q_0^4}{\rho_0 q_1^2 - \rho_1 q_0^2}, \quad (A.29) \\
\tau_4^1 &= \frac{q_1^2}{q_0^2} \tau_4^0, \quad (A.30) \\
\tau_5^0 &= (\tau_0 - \tau_0^3) \frac{\rho_0 q_0^2 q_0^2 - q_1^4}{\rho_0 q_1^2 - \rho_1 q_0^2}, \quad (A.31) \\
\tau_5^1 &= \frac{\rho_1 q_0^4}{\rho_0 q_1^2} \tau_5^0, \quad (A.32) \\
\tau_6^0 &= (\tau_0 - \tau_0^3) \frac{\rho_1 - \rho_0}{\rho_0 q_1^2 - \rho_1 q_0^2}, \quad (A.33) \\
\tau_6^1 &= \frac{q_0^2}{q_1^2} \tau_6^0, \quad (A.34)
\end{align*}
Appendix B

Roots of the characteristic equations

B.1 General procedure

Given are the parameters of the problem:

\[ Re, \ \eta, \ \tau, \ \theta \]  \hspace{1cm} (B.1)

which enter the linearized equations. The effect of nutation appears only in the combination

\[ \tau_z = \tau \cos \theta \]  \hspace{1cm} (B.2)

The expansion functions take the form

\[ w = R(\tilde{r})Z(\tilde{z}) \]  \hspace{1cm} (B.3)

where

\[ \tilde{r} = qr, \ \tilde{z} = qz, \ q^2 = iRe \]  \hspace{1cm} (B.4)

\[ \tilde{\nabla}^2 R(\tilde{r}) - BR(\tilde{r}) = 0, \ \tilde{\nabla}^2 = \frac{\partial^2}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} - \frac{1}{\tilde{r}^2} \]  \hspace{1cm} (B.5)
and

\[ Z^{vi} + Z^{iv}(3B - 2) + \int [3B^2 - 4B + 1 - \tau_\theta^2] + Z(B^3 - 2B^2 + B) = 0 \quad (B.6) \]

where \( \tau_\theta = 2(1 + \tau_z) \). The solution of equation (6) can be written in the form

\[ Z(\hat{z}) = \sum_{i=1}^{3} C_i \frac{\cos a_i \hat{z}}{\cos a_i \eta} \quad (B.7) \]

where \( a_i = b_i^{1/2} \) are the roots of

\[ b^3 - b^2(3B - 2) + b[3B^2 - 4B + 1 - \tau_\theta^2] - (B^3 - 2B^2 + B) = 0 \quad (B.8) \]

and the boundary conditions require

\[
\begin{align*}
(a_1^2 - a_2^2)(1 + 2B - a_1^2 - a_2^2)a_1 \tan(a_1 \eta) & a_2 \tan(a_2 \eta) \\
+ (a_2^2 - a_3^2)(1 + 2B - a_2^2 - a_3^2)a_2 \tan(a_2 \eta) & a_3 \tan(a_3 \eta) \\
+ (a_3^2 - a_1^2)(1 + 2B - a_3^2 - a_1^2)a_3 \tan(a_3 \eta) & a_1 \tan(a_1 \eta) = 0 \quad (B.9)
\end{align*}
\]

Equations (8) and (9) represent a transcendental nonlinear system that provides the four unknown quantities \( B \) and \( a_i \).

We attempt a solution of this system by Newton’s method and rewrite equation (8) in the form

\[
\begin{align*}
a_1^2 + a_2^2 + a_3^2 - 3B + 2 &= 0 \quad (B.10) \\
a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_4^2 - 3B^2 + 4B - 1 + \tau_\theta^2 &= 0 \quad (B.11) \\
a_1^2 a_2^2 a_3^2 - B^3 + 2B^2 - B &= 0 \quad (B.12)
\end{align*}
\]
B.2 An alternative approach to the roots

For arbitrary \( B \), we can write

\[
a_i = \frac{b_i^{1/2}}{b_i} \quad b_i = B + c_i
\]  

(B.13)

and obtain by substitution into (10), (11), and (12)

\[
c_1 + c_2 + c_3 = -2
\]  

(B.14)

\[
c_1(c_2 + c_3) + c_2c_3 = 1 - \tau_\theta^2
\]  

(B.15)

\[
c_1c_2c_3 = \tau_\theta^2B
\]  

(B.16)

Elimination of \( c_2, c_3 \) by use of (14), (16) provides a single equation for \( c_1 \):

\[
c_1^2 + 2c_1^2 + (1 - \tau_\theta^2)c_1 - \tau_\theta^2B = 0
\]  

(B.17)

The remaining \( c_2 \) and \( c_3 \) appear symmetrically and are governed by the quadratic equation

\[
c^2 + (2 + c_1)c + \frac{\tau_\theta^2B}{c_1} = 0, \quad c_1 \neq 0
\]  

(B.18)

with the solutions

\[
c_{2,3} = -1 - \frac{c_1}{2} \pm \left[ (1 + \frac{c_1}{2})^2 - \frac{\tau_\theta^2B}{c_1} \right]^{1/2}
\]  

(B.19)

This procedure is applied to find the roots \( a_i \) for given \( B \) before the simultaneous solution of equations (9), (10), (11), and (12). For \( c_1 = 0 \), equation (17) can only
be satisfied if $\tau_0 B = 0$. Since $\tau_0 \neq 0$ in the cases of interest, $c_1 = 0$ implies $B = 0$.

From (14) and (15) we obtain

$$c_2^2 + 2c_2 + (1 - \tau_0^2) = 0$$

with the solutions

$$c_{2,3} = -1 \pm \sqrt{\tau_0^2}, \quad c_1 = 0$$

(B.20)

(B.21)

B.3 Solutions with $a_1 = 0$

For given $b = a^2$, equation (8) can be rewritten in the form

$$B^3 - (2 + 3b)B^2 + (1 + 4b + 3b^2)B - [b^3 + 2b^2 + (1 - \tau_0^2)b] = 0$$

(B.22)

For $a_1 = 0$, this equation reduces to

$$B^3 - 2B^2 + B = 0$$

(B.23)

with the three roots

$$B_1 = 0, \quad B_{2,3} = 1$$

(B.24)

From equations (10) and (11) we obtain with $a_1 = 0$, $B_1 = 0$,

$$b^2 + 2b + 1 - \tau_0^2 = 0$$

(B.25)

with the roots

$$b_{2,3} = -1 \pm \sqrt{\tau_0^2}, \quad b_1 = 0.$$
while for $a_1 = 0$, $B_2 = 1$

\[ b^2 - b - \tau^2 = 0 \]  \hspace{1cm} (B.27)

with the roots

\[ b_{2,3} = \frac{1}{2}(1 \pm \sqrt{1 + 4\tau^2}), \quad b_1 = 0 \]  \hspace{1cm} (B.28)

The solution corresponding to $B = 0$ do not satisfy the boundary conditions and hence they are excluded from the expansion. While the solutions with $B = 1$ do satisfy the boundary conditions and therefore must be included in the expansion.

**B.4 Two isolated solutions**

There are two isolated solutions with real $B$ and real $a_2^2 = a_3^2$, hence real $c_2 = c_3$ according to equation (15). Eliminating $c_1$ from equations (14) and (15) provides

\[ c^2 + \frac{4}{3}c + \frac{1 - \tau^2}{3} = 0 \]  \hspace{1cm} (B.29)

and

\[ c_{2,3} = \frac{1}{3}(-2 \pm \sqrt{1 + 3\tau^2}), \quad c_1 = -2(1 + c_2), \quad B = \frac{1}{\tau^2}c_1c_2^2 \]  \hspace{1cm} (B.30)

These solutions do not satisfy the boundary conditions and therefore must be excluded from the expansion.

**B.5 Solution at Large Reynolds Numbers**

Starting from the observation that the eigenvalues $B$ tend to one, $B \to 1$ as $\eta \to \infty$, we let $B = 1 + \epsilon$. From (8) we obtain

\[ b^3 - b^2(3\epsilon + 1) + b[3\epsilon^2 + 2\epsilon - \tau^2] - (\epsilon^3 + \epsilon^2) = 0 \]  \hspace{1cm} (B.31)
Equation (9) can be written in the form

\[ a_1 \tan(a_1\eta)[a_2(a_1^2-a_2^2)(3 + 2\epsilon - a_1^2-a_2^2) + a_3(a_1^2-a_3^2)(3 + 2\epsilon - a_1^2-a_3^2)] \]  
(B.32)

\[ = a_2a_3(a_2^2-a_3^2)(1 + 2B - a_2^2 - a_3^2) \]  
(B.33)

where the term in [...] on the left hand side and the right hand side are of order \(O(1)\).

For discrete finite arguments \(a_1\eta\) of the tangent it is necessary that \(|a_1| \to 0\) as \(Im(\eta) \to \infty\). Then, however, equation (31) can only be satisfied if \(|\tan(a_1\eta)| \to \infty\). Therefore,

\[ a_1 \to (2n + 1)\frac{\pi}{2\eta} = (2n + 1)\frac{(1 - i)\pi}{2\eta(2Re)^{1/2}} \text{ as } Re \to \infty \]  
(B.34)

where \(n\) is an integer.

**B.6 Solutions for given \(a_1\) with \(arg(a_1) = -\pi/4\)**

As discussed earlier, solutions with \(arg(a_1) \approx -\pi/4\) have particular importance.

For given \(b = a_1^2\) along the negative imaginary axis, the three branches for the solutions \(B_i(b)\) can be found from equation (22) by using Cardani's formula

\[ B_1 = -\frac{f}{2} (\sqrt{3}i + 1) + \frac{1}{18f} (\sqrt{3}i - 1) + b + \frac{2}{3} \]  
(B.35)

\[ B_1 = \frac{f}{2} (\sqrt{3}i - 1) + \frac{1}{18f} (\sqrt{3}i + 1) + b + \frac{2}{3} \]  
(B.36)

\[ B_3 = f + \frac{1}{9f} + b + \frac{2}{3} \]  
(B.37)

where

\[ f = \sqrt[3]{\frac{1}{18}(3br_0^2(27br_0^2 + 4) - \frac{1}{54}(27br_0^2 + 2))} \]  
(B.38)
Appendix C

Solutions for flows at small Reynolds numbers

The procedure for solving the SOPDE by eigenfunction expansion at finite Reynolds numbers can be quite difficult to understand. We find it useful to discuss the case of flows at small Reynolds numbers, i.e. high viscosity fluids. This limiting case offers great insights into how the SOPDE is solved by the classical method of separation of variables. We will only present the solution for completely filled cylinders. The extension of the method to the other configurations is straightforward. For small Reynolds numbers, $Re \ll 1$, we assume a solution to the SOPDE in the form of a perturbation expansion in powers of $Re$, namely,

$$w(r,z) = \tilde{w}_0(r,z) + \tilde{w}_1(r,z) Re + \tilde{w}_2(r,z) Re^2 + ...$$  \hspace{1cm} (C.1)

It can be easily shown, upon substituting expansion (1) into the SOPDE and the wall boundary conditions, and equating terms of equal powers of $Re$, that the equations governing the two lowest order perturbation velocities admit the trivial solutions, $\tilde{w}_0 = \tilde{w}_1 = 0$, and the perturbation velocity $\tilde{w}_2$ is governed by

$$\nabla^6 \tilde{w}_2 = 0$$  \hspace{1cm} (C.2)
and must satisfy the conditions:

\[
\dot{w}_2 = 0, \quad \frac{\partial \dot{w}_2}{\partial z} = 0, \quad 2\nabla_1^2 \frac{\partial^3 \dot{w}_2}{\partial z^3} + \frac{\partial^5 \dot{w}_2}{\partial z^5} = 0 \text{ at } z = \pm \eta. \tag{C.3}
\]

The equation governing \( \dot{w}_2 \) is homogeneous and so are the boundary conditions at the end walls. We can then use the method of separation of variables to solve for \( \dot{w}_2 \). We assume a product solution to (2) of the form

\[
\dot{w}_2(r,z) = \tilde{R}(r) \tilde{Z}(z), \tag{C.4}
\]

and we let

\[
\nabla_1^2 \tilde{R} = \tilde{B} \tilde{R}, \tag{C.5}
\]

where \( \tilde{B} \) is a complex separation constant. Upon substituting (4) and (5) into (2) and (3), and separating variables, we obtain the eigenvalue problem

\[
\tilde{Z}^{(6)} + 3 \tilde{B} \tilde{Z}^{(4)} + 3 \tilde{B}^2 \tilde{Z}'' + \tilde{B}^3 \ddot{Z} = 0, \tag{C.6}
\]

\[
\tilde{Z} = 0, \quad \tilde{Z}' = 0, \quad 2 \tilde{B} \tilde{Z}''' + \tilde{Z}^{(5)} = 0 \text{ at } z = \pm \eta, \tag{C.7}
\]

where prime denotes differentiation with respect to \( z \). We solve this eigenvalue problem by assuming a solution to (6) having the form

\[
\tilde{Z} = \cos(\tilde{a} z), \tag{C.8}
\]

which converts (6) into the algebraic equation

\[
(\tilde{a}^2 - \tilde{B})^3 = 0. \tag{C.9}
\]
Equation (9) has the solutions:

\[ \tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \pm \sqrt{\tilde{B}}. \]  \hspace{1cm} (C.10)

When symmetries about \( z = 0 \) are considered, the general solution to (6) takes the form

\[ \tilde{Z} = \tilde{c}_1 \cos(\sqrt{\tilde{B}} z) + \tilde{c}_2 z \sin(\sqrt{\tilde{B}} z) + \tilde{c}_3 z^2 \cos(\sqrt{\tilde{B}} z), \]  \hspace{1cm} (C.11)

where \( \tilde{c}_1, \tilde{c}_2, \) and \( \tilde{c}_3 \) are constants of integration. Applying boundary conditions (7) provides 3 linear homogeneous equations for the constants \( \tilde{c}_1, \tilde{c}_2, \) and \( \tilde{c}_3 \). For nontrivial solutions, the determinant of such system must vanish, and hence we obtain

\[ \sin \eta \sqrt{\tilde{B}} = 0 \]  \hspace{1cm} (C.12)

and

\[ \sin 2\eta \sqrt{\tilde{B}} + 2\sqrt{\eta} \tilde{B} = 0. \]  \hspace{1cm} (C.13)

The roots of equations (12) and (13) constitute the spectrum \( \{\tilde{B}_m\}_{m=0}^\infty \) of the eigenvalue problem. In particular, the roots of (12) take the form

\[ \tilde{B}_l^1 = \frac{l^2 \pi^2}{\eta^2}, \quad l = 0, 1, 2, 3, \ldots \]  \hspace{1cm} (C.14)

They lie on the positive part of the real axis which is one of the branches on which the eigenvalues are located in the complex plane. The other two branches on which the rest of the eigenvalues, i.e. \( \tilde{B}_l^2 \) and \( \tilde{B}_l^3 \), are located, are obtained when equation
Figure C.1: Spectrum for small Reynolds numbers.

(13) is solved numerically. The two branches are symmetric about the real axis and they originate from the origin of the plane that we refer to by the index 0, i.e. \( \tilde{B}_0 = 0 \). These eigenvalues, scaled by the aspect ratio, are found in figure 1. Note that they are independent of \( \tau \) and \( \theta \). The eigenfunctions associated with real eigenvalues are

\[
\tilde{Z}_m = \cos(\sqrt{\tilde{B}_m} z) + \frac{2}{\eta^2 \sqrt{\tilde{B}_m}} z \sin(\sqrt{\tilde{B}_m} z) - \frac{z^2}{\eta^2} \cos(\sqrt{\tilde{B}_m} z), \quad \text{if } \tilde{B}_m \neq 0, \tag{C.15}
\]

\[
\tilde{Z}_m = 1 - 2 \left( \frac{\tilde{z}}{\eta} \right)^2 + \left( \frac{\tilde{z}}{\eta} \right)^4, \quad \text{if } \tilde{B}_m = 0, \tag{C.16}
\]

while those associated with complex eigenvalues take the form

\[
\tilde{Z}_m = \cos(\sqrt{\tilde{B}_m} z) - \frac{z}{\eta} \sin(\sqrt{\tilde{B}_m} z) \tan \eta \sqrt{\tilde{B}_m}. \tag{C.17}
\]
As can be seen from figure 1, the branches on which the eigenvalues lie originate from the origin and extend to \( \infty \). The eigenfunction associated with the origin is nonoscillatory. However, as we depart from the origin on a certain branch, the eigenfunctions become oscillatory. Their number of zeros increases as the eigenvalues approach \( \infty \). Hence, they must constitute a complete system. We use this system to expand the axial velocity as

\[
\tilde{w}_2(r,z) = \sum_{m=0}^{\infty} A_m \tilde{R}_m(r) \tilde{Z}_m(z) \tag{32}
\]

where

\[
\tilde{R}_0 = r, \quad \tilde{R}_m = I_1(\sqrt{B_m}r), \quad m \neq 0, \tag{33}
\]

is a solution to equation (5) and describes the radial structure of such flow. \( I_1 \) is the modified Bessel function of order 1. The expansion coefficients \( A_m \) are easily found by satisfying the boundary conditions at the side wall by means of collocation or least squares method.
Bibliography


