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Topics in ergodic theory: Existence of invariant elements and ergodic decompositions of Banach lattices

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The Ohio State University, 1991
TOPICS IN ERGODIC THEORY: EXISTENCE OF INVARIANT ELEMENTS AND ERGODIC DECOMPOSITIONS OF BANACH LATTICES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

by

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* * * * *

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To My Parents
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The question of existence of invariant measure has been important since the dawn of ergodic theory, since in the presence of such a measure the ergodic theorem holds for a large class of functions.

The first asymptotic conditions for the existence of an equivalent invariant measure were given for a nonsingular point-transformation $\tau$ on a probability space $(X, F, p)$.

Let $M$ is the maximal value of the Banach limits on a bounded sequence of real numbers $\{x_n\}$. L. Sucheston [21] identified $M$ by

$$M[\{x_n\}] = \lim \sup_n 1/n \sum_{i=j}^{j+n-1} x_i.$$ 

One has the following result:

**Theorem 1.0.1** Let $(X, F, p)$ be a probability space. Let $\tau$ be a nonsingular measurable mapping from $X$ into $X$. Then the following conditions are equivalent:

(i) There exists an equivalent finite invariant measure,

(ii) $\lim \inf_n p(\tau^{-n}A) > 0$ if $p(A) > 0$,

(iii) $\lim \inf_n 1/n \sum_0^{n-1} p(\tau^{-i}A) > 0$ if $p(A) > 0$,

(iv) $\lim \sup_n 1/n \sum_0^{n-1} p(\tau^{-i}A) > 0$ if $p(A) > 0$,

(v) $M[p(T^{-n}A)] = \lim_n [\sup_j 1/n \sum_{i=j}^{j+n-1} p(\tau^{-i}A)] > 0$ if $p(A) > 0$. 

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First Y. N. Dowker [10] and, independently, A. Calderon [5] proved the equivalence of (i) and (ii); A. Calderon [5] also showed the equivalence of (i) and (iii). Again Y. N. Dowker [11] and, by a different method, Hajian and Kakutani [14] later proved the equivalence of (i) and (iv). L. Sucheston [21] proved the equivalence of (i) and (v).

E. Granirer [12] extends the equivalence of the conditions (i), (ii) and (v) in the above theorem to the case of equivalent finite measure invariant under (left) amenable semigroup of point-transformations. In that case $M$ in condition (v) generalizes to the maximal value of the (left) invariant means on the (left) amenable semigroup. In the same paper, E. Granirer also provided the identification of the maximal value $M$ of the (left) invariant means.

If $\tau$ is a nonsingular transformation on $(X, F, p)$ then $\tau$ generates a positive contraction $T$ on $L_1(X, F, p)$, defined by $Tf = g$ if $f = d\phi/dp$ and $g = d(\phi \circ \tau^{-1})/dp$. Thus the problem of existence of equivalent invariant measures generalizes to the problem of existence of strictly positive fixed points in $L_1(X, F, p)$.

U. Sachdeva [19] generalized the equivalence of the conditions (i), (ii) and (v) in the above theorem to the case of equivalent measure invariant under left amenable semigroup of positive linear contractions on $L_1(X, F, p)$.

Now we consider the generalization of the results of the above theorem in a different direction. Consider a weakly sequentially complete Banach lattice $E$ which has a weak unit, i.e., an element $e \in E_+$ such that $e \wedge |f| = 0$ for $f \in E$ implies $f = 0$.

We have the following results:
Theorem 1.0.2 Let $T$ be a bounded positive linear operator on $E$. The following conditions are equivalent:

(i) There exists an invariant weak unit $v$ in $E$, i.e., weak unit $v$ such that $Tv = v$.

(ii) $\inf_n H(T^n e) > 0$, $\forall H \in E_{++}^*$.

(iii) $\liminf_n H(A_n e) > 0$, $\forall H \in E_{++}^*$.

(iv) $\limsup_n H(A_n e) > 0$, $\forall H \in E_{++}^*$.

(v) $M[H(T^n e)] > 0$, $\forall H \in E_{++}^*$.

Here $M$ denotes the maximal value of the Banach Limits on $l_\infty$, the class of bounded real sequences.

P. Shields [20] proved the equivalence of (i) and (ii). The usual argument that uses construction of a countably additive measure in proving similar results in the case of a point transformation on a probability space or a contraction on $L_1(X, F, p)$ does not work in this case. P. Shields used the concept of countably additive functionals, instead.

A. Brunel and L. Sucheston [3] proved the equivalence of (ii), (iv) and (v) of the above theorem. In their work the condition of contraction is relaxed and $T$ is assumed to be either power bounded or mean bounded.

For all our results we assume the underlying space to be a weakly sequentially complete Banach lattice $E$ having a weak unit $e$. Let $\Sigma$ be a countably generated left amenable semigroup. For the operators on $E$, we consider a positive linear operator representation, $\{T_\sigma : \sigma \in \Sigma\}$ of $\Sigma$ with $\sup\{||T_\sigma|| : \sigma \in \Sigma\} = K < \infty$. For certain results we make a further assumption of commutativity on $\Sigma$, in which case we state
this assumption explicitly in the statement of our results (theorem or proposition).

In Chapter III, we first generalize the equivalence of (i) and (ii) to our case. In our proof we use the concept of countably additive functionals as P. Shields [20] did. But, instead of the Theorem 4 of [20] we use a weaker result from our main reference LT [16]. Once a sub-invariant weak unit is found, our proof differs entirely as the approach in [20] will not work in our case. The argument used in [20] is as follows: If \( u \) is a subinvariant under \( T \) then so are all the iterates of \( u \) and in the limiting case one obtains an invariant element, which can be shown to be a weak unit if \( u \) is.

In our case, for an element \( u \) subinvariant under all \( T_\sigma, \sigma \in \Sigma \), even any \( T_\sigma u \) may not be a subinvariant element, unless we assume commutativity. Even with the commutativity assumption, finding a limiting process is not straightforward as we do not have a countable sequence in our case.

Also in Chapter III, we generalize the equivalence of (ii), (iv) and (v) to our case under additional commutativity assumption on the operators. The scheme of these proofs is similar to that of A. Brunel and L. Sucheston [3]. A crucial part in our proof is the generalization of Cesàro averages \( A_n(T) \) in the above theorem to the case of a countably generated left amenable semigroup of operators in such a way that we have a generalization that corresponds to the following condition on the Cesàro averages of linear contractions:

\[
\|A_n(T) - TA_n(T)\| \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\]

U. Krengel ([15], pp 222-223) addresses the question of existence of such sequences of averages. Sufficient conditions for such convergence were given in terms of the the
topological structure of the underlying topological group. There was only one class of a semigroups satisfying the sufficient conditions was given in [15], namely $\sigma$-compact locally compact right amenable group. Recognizing the existence of ergodic nets, the author concluded writing "However we shall need averages over sets $I_n$ in the discussion of pointwise convergence." (Here $I_n$ refers to sequences of subsets of the semigroup.).

Chapter IV deals with the conditions for the existence of invariant elements which are not necessarily weak units. We make an additional monotonicity assumption (C) on a semi-norm defined on the Banach lattice $E$; this assumption was introduced in [1] and without this assumption (C) on the lattice norm the the TL (or stochastic) ergodic theorem fails. The results of Chapter IV are generalizations of part of the results of A. Millet and L. Sucheston [17], derived in the case of a semigroup generated by finitely many positive mean bounded commutative operators. Instead of this setting, for the results involving existence of $T_\sigma$ invariant elements we consider countably generated left amenable semigroup representation $\{T_\sigma : \sigma \in \Sigma\}$ of uniformly bounded positive operators and only in the latter part of chapter IV, for the results involving existence of $T_\sigma^*$ invariant elements and the decomposition theorem, we assume commutativity.

As in [17], we obtain a decomposition of the Banach lattice of the form $E = Y + Z = P + D + Z$, where the 'remaining part' $Y$ is the the largest support of $T_\sigma^*$ invariant element and $Z$ is the 'disappearing part'. $Y$ further decomposes into $P + D$ where 'positive part' $P$ is the maximal support of a $T$ invariant element element and
\[ N = D + Z \] is the 'null part'.
CHAPTER II
PRELIMINARY RESULTS

2.1 On Banach Lattices

Let $E$ be a Banach lattice. We denote the set all elements $f \in E$ such that $f \geq 0$ by $E_+$ and $E^{++} = E_+ - \{0\}$.

The dual of the Banach lattice $E$ is denoted by $E^*$.

We consider a Banach lattice $E$ satisfying the following conditions (A) and (B).
(A) $E$ has a weak unit $e$, i.e., there exists $e \in E_+$ such that for any $f \in E$, $e \wedge |f| = 0 \Rightarrow f = 0$.

(B) Any norm bounded increasing sequence in $E$ has a strong limit.

The condition (B) is equivalent to weak sequential completeness (LT [16], p. 34).

The condition (B) implies order continuity of the lattice norm (OCN):
(OCN) For every downwards directed net $(f_i, i \in I)$ with $\bigwedge_{i \in I} f_i = 0$, one has $\lim_{i} ||f_i|| = 0$.

Another condition that is equivalent to (OCN) is that every order interval $[f, g] = \{h : f \leq h \leq g\}$ in $E$ is weakly compact. (LT [16], p. 28).

For a Banach lattice satisfying conditions (A) and (OCN) there exists a strictly positive element $U$ in the positive cone of the dual, i.e., $U \in E^*_+ = E^{++}_+$ such that if $f \in E_+$
and $U(f) = 0$, then $f = 0$ (LT [16], p. 25)

Since $E$ is a order continuous Banach lattice with weak unit $E$ has a Köthe function space representation over a probability space $(\Omega, \mathcal{F}, \mu)$ (LT [16], p. 25); this means that $E$ is order isometric to an order ideal $X$ of $L^1(\Omega, \mathcal{F}, \mu)$ such that

(i) $X$ is dense in $L^1(\Omega, \mathcal{F}, \mu)$ and $L^\infty(\Omega, \mathcal{F}, \mu)$ is dense in $X$ and

(ii) The dual of the isometry between $E$ and $X$ maps $E^*$ onto the Banach lattice $X^*$ of all $\mu$ measurable functions $g$ for which

$$\|g\|_{X^*} = \sup\{ \int fg d\mu : \|f\|_X \leq 1 \} < \infty.$$  

Furthermore, one has $g(f) = \int fg d\mu$ for $f \in X$ and $g \in X^*$.

We denote strong convergence in $E$ by slim or $\longrightarrow$ and weak convergence in $E$ by wlim or $\rightarrow_w$. We denote order convergence of monotone nets by or $\uparrow$ or $\downarrow$.

**Definition 2.1.1** Let $\delta \in E_{++}$. A band projection $P_\delta : E_+ \to E_+$ is defined by,

$$P_\delta f = \lim_n f \wedge (n\delta), \text{ for every } f \in E.$$  

In the functional notation, $P_\delta f$ is the restriction of $f$ to the support of $\delta$.

**Definition 2.1.2** An ideal in a Banach lattice $E$ is a linear subspace $D$ for which $y \in D$ whenever $|y| \leq |x|$ for some $x \in D$.

We have the following result from Lindenstrauss (LT [16], p.28).

**Theorem 2.1.3** A Banach lattice $E$ is order continuous if and only if the canonical image of $E$ into its second dual $E^{**}$ is an ideal of $E^{**}$.  

Now consider the following theorem which is due to H. Nakano [18]. This result assumes only the condition (B).

**Theorem 2.1.4** If \( \{a_\alpha\}_{\alpha \in \Lambda} \) is an increasing net of bounded norm in \( E_+ \), then \( \mathcal{V}a_\alpha \) exists in \( E \). Furthermore there exists a sequence \( \{\alpha_n\} \in \Lambda \) such that

\[
\mathcal{V}a_\alpha = \mathcal{V}_n a_{\alpha_n}.
\]

**Proof:**
We have \( 0 \leq a_\alpha \uparrow \alpha \) and \( \sup_{\alpha} ||a_\alpha|| < \infty \). Therefore we can select a sequence \( \alpha_n \in \Lambda, n = 1, 2, \ldots \) such that \( a_{\alpha_n} \uparrow \) and for \( n = 1, 2, \ldots \),

\[
||a_{\alpha_{n+1}} - a_{\alpha_n}|| \geq \sup_{\alpha \geq a_{\alpha_n}} ||a_\alpha - a_{\alpha_n}|| - 1/2^n. \tag{2.1}
\]

By condition (B), there exists \( a \in E \) such that

\[
\lim_{n} a_{\alpha_n} = a. \tag{2.2}
\]

Let \( a, \gamma \in \Lambda \) be an arbitrary element of the net. For \( n = 1, 2, \ldots \) we have

\[
||((a_\gamma \vee a_{\alpha_n}) - a_{\alpha_n})|| \\
\leq \sup_{a_\alpha \geq a_{\alpha_n}} ||a_\alpha - a_{\alpha_n}|| \\
\leq ||a_{\alpha_{n+1}} - a_{\alpha_n}|| + 1/2^n \quad \text{(by relation (2.1))} \\
\leq ||a - a_{\alpha_n}|| + 1/2^n \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

But from equation (2.2) it follows that

\[
\lim_{n}((a_\gamma \vee a_{\alpha_n}) - a_{\alpha_n}) = (a_\gamma \vee a) - a.
\]
Therefore we have

\[||(a_\gamma \lor a) - a|| = 0\]

and hence

\[V_{\alpha_n}a_{\alpha_n} = a = (a_\gamma \lor a) \geq a_\gamma.\]

Since \(a_\gamma\) is arbitrary, it follows that \(V_{\gamma}a_{\gamma}\) exists and

\[V_{\alpha_n}a_{\alpha_n} = V_{\gamma}a_{\gamma}.\]

This completes the proof of the theorem.
2.2 On Truncated Limits

Now, let us recall the definitions and properties of truncated limits, a notion introduced by M. A. Akcoglu and L. Sucheston ([1] and [2]). For details on truncated limits, see [1] and [2]).

Let \( \{f_n\} \) in \( E_+ \). Then \( TL f_n = \phi \) means that for a weak unit \( u \), \( slim_n(f_n \wedge ku) = \phi_k \) exists for each \( k \) and \( \phi_k \uparrow \phi \).

For a sequence \( \{f_n\} \) in \( E \), \( TL f_n \) is defined by \( TL f_n = TL f_n^+ - TL f_n^- \), provided that both the truncated limits to the right exist.

Weak truncated limit of \( \{f_n\} \), denoted \( WTL f_n \), is defined analogously, requiring only \( wlim_n(f_n \wedge ku) = \phi_k \) exists for each \( k \) and \( \phi_k \uparrow \phi \).

These definitions are independent of the choice of the weak unit \( u \).

A sequence \( \{f_n\} \) is called \( TL \) null if \( TL |f_n| = 0 \).

From the definition, it follows that \( TL |f_n| = 0 \) if and only if \( |f_n \wedge u| \rightarrow 0 \).

If, in the above definitions, the role of weak unit \( u \) is played by an arbitrary \( \delta \in E_{++} \), one writes \( TL_\delta f_n = \phi \) or \( WTL_\delta f_n = \phi \). First, let us consider the following lemma.

**Lemma 2.2.1** Let \( \{f_n\} \) in \( E_+ \) be such that \( sup f_n \in E \).

If \( f_n \xrightarrow{w} 0 \), then \( f_n \rightarrow 0 \).

By the above lemma, \( \{f_n\} \) is \( TL \) null if and only if \( |f_n \wedge u| \rightarrow 0 \).

**Proposition 2.2.2** (Compactness for \( WTL \))

Let \( \{f_n\} \) in \( E_+ \) and \( sup ||f_n|| = M < \infty \).

Then there is a subsequence \( \{f_{n_i}\} \) of \( \{f_n\} \) such that \( WTL f_{n_i} = \phi \) exists.
Furthermore, if \( \{f_n\} \) is not TL null, then this subsequence can be chosen so that \( \phi \neq 0 \).

**Proposition 2.2.3 (Additivity and Fatou property for operators)**

Let \( \{f_n\} \) and \( \{g_n\} \) be in \( E_+ \).

Let \( WTL f_n = \phi \) and \( WTL g_n = \psi \).

(a) If \( WTL (f_n + g_n) = \gamma \) exists, then \( \gamma = \phi + \psi \).

(b) If \( T : E \rightarrow E \) is a positive linear operator and \( T f_n = g_n \), then \( T \phi \leq \psi \)

**Proposition 2.2.4** Let \( f_n \in E_+ \) and \( \sup ||f_n|| = M < \infty \).

If \( WTL f_n = \phi \), then \( WTL \phi f_n = \phi \).

**Definition 2.2.5** A seminorm \( N \) is a map \( N : E_+ \rightarrow R \) such that

(i) \( N(f + g) \leq N(f) + N(g) \) for every \( f, g \in E_+ \).

(ii) \( N(\alpha f) = \alpha Nf \) for every \( \alpha \geq 0 \) and \( f \in E_+ \).

(iii) \( 0 \leq f \leq g \) implies \( 0 = N(0) \leq N(f) \leq N(g) \).

If furthermore we have

(iv) If \( f_n \downarrow 0 \) then \( \lim N(f_n) = 0 \),

then \( N \) is called order continuous (OC) seminorm.

**Lemma 2.2.6** Let \( N \) be an OC seminorm and \( \{f_n\} \) be a sequence in \( E_+ \).

If \( slim f_n = f \) then \( \lim N(f_n) = N(f) \).

In some of the results, we make assumption that the converse of the above result holds for the OC seminorm \( N \); more precisely we have the following:
Definition 2.2.7 Let $N$ be an OC seminorm. The Banach lattice norm is said to be continuous with respect to $N$ on $E_+$ if for each sequence $\{f_n\}$ in $E_+$ with $\sup_n f_n \in E$ and $\lim_n N(f_n) = 0$, one has $\lim f_n = 0$.

Lemma 2.2.8 Let $U$ be a strictly positive element in $E_*^{++}$, and let $\{f_n\}$ be a sequence in $E_+$ such that $\lim U(f_n) = 0$

(a) Then $TL f_n = 0$.

(b) If furthermore $\sup f_n \in E$, then $\lim f_n = 0$.

In some of the results, we impose the additional condition $(C)$ or $(C_1)$ on $N$:

$(C)$ For every $f \in E_+$ and for every $\alpha > 0$, there exists a number $\beta = \beta(f, \alpha)$ such that if $g \in E_+$, $N(g) \leq 1$, $0 \leq h \leq f$ and $N(h) \geq \alpha$, then $N(g + h) \geq N(g) + \beta$.

$(C_1)$ For any $f, g$ in $E_+$, if $N(f) > 0$, then $N(f + g) > N(g)$.

The condition $(C)$ is stronger than the condition $(C_1)$:

For a seminorm $N$ defined on an Orlicz space with non-atomic measure, condition $(C)$ is equivalent to condition $(C_1)$ and both are equivalent to the classical $\Delta_2$ condition.

Also note that for a given $F \in E_*^+$, if we define $N$ is by $N(f) = F(f)$ for every $f \in E_+$, then $N$ satisfies condition $(C)$. 
2.3 On Countably Additive Functionals

In this section we consider some results on countably additive functionals. These results are based on the work of P. C. Shields [20] and these results generalize the corresponding results on countably additive measures by L. Sucheston [21].

Let $L$ be a Banach lattice satisfying the following condition:

(SC) Suppose that $\{x_n\}$ is an increasing sequence in $L$ and for some $x \in L$ one has $x_n \leq x$, $n = 1, 2, \ldots$. Then $\vee x_n$ exists.

Note that the condition (SC) follows from condition (B).

Let $F$ and $G$ be in $L^*$. Then $F \wedge G$ and $F \vee G$ are given by

$$(F \wedge G)(b) = \inf_{b = b_1 + b_2, b_1 \geq 0} [F(b_1) + G(b_2)], \ \forall b \in L, b \geq 0$$

and

$$(F \vee G)(b) = \sup_{b = b_1 + b_2, b_1 \geq 0} [F(b_1) + G(b_2)], \ \forall b \in L, b \geq 0.$$ 

and also it follows that for any Banach lattice $L$, $L^*$ is a Banach lattice satisfying the condition (SC) (LT [16], p. 3).

Definition 2.3.1 We say that a positive linear functional $G$ on $L$ is countably additive if any sequence $\{y_n\}$ in $L$ is such that $y_n \downarrow 0$ then $G(y_n) \downarrow 0$.

Lemma 2.3.2 Suppose $F$ and $G$ are positive linear functionals on $L$ and that $G$ is countably additive. Suppose $(F \wedge G)(b) = 0$ for some $b \geq 0$. Let $\epsilon > 0$. Then we can find $b_1 \in [0, b]$ such that

$$G(b_1) < \epsilon \text{ and } F(b_1) = F(b).$$
Proof: Since \((F \wedge G)(b) = 0\), from the discussion in the beginning of this section, it follows that
\[
\inf_{b = b_1 + b_2, b_1 \geq 0} [F(b_1) + G(b_2)] = 0.
\]
Set \(a_0 = 0\). We select \(a_1, a_2 \cdots\) recursively as follows. Once \(a_1, a_2 \cdots, a_{n-1}\) are selected, choose
\[
a_n \in [0, b - \sum_{k=0}^{n-1} a_k]
\]
such that
\[
G(a_n) < \epsilon/2^n \quad \text{and} \quad F(b - \sum_{k=0}^{n} a_k) < \epsilon/2^n.
\]
Set
\[
b_1 = \sum_{k=1}^{\infty} a_n.
\]
Then \(b_1 \in [0, b]\) and
\[
G(b_1) = \sum_{k=1}^{\infty} G(a_k) < \epsilon.
\]
Furthermore, for each \(n\) we have
\[
F(b) \geq F(b_1) \geq F(\sum_{k=1}^{n} a_k) \geq F(b) - \epsilon/2^n.
\]
and hence
\[
F(b_1) = F(b).
\]
This completes the proof of lemma.

**Lemma 2.3.3** Suppose \(F\) is a positive linear functional on \(L\). Suppose further that \(x_n \geq 0\) and \(F(x_n) = 0, n = 1, 2, \cdots\) and \(x = \liminf x_n\) exists in \(L\). Then there exists \(a_n \geq 0\) such that \(x = \sum_{k=1}^{\infty} a_n\) and \(F(a_n) = 0, n = 1, 2, \cdots\).
Proof:

Set

\[ z_k = \land_{n \geq k} x_n \]
\[ a_1 = z_1 \]
\[ a_{n+1} = z_{n+1} - z_n, \quad n = 1, 2, \ldots. \]

We have

\[ 0 \leq a_k \leq z_k \leq x_k. \]

Since \( F(x_k) = 0 \), from the above relations it follows that

\[ F(a_k) = 0. \]

Furthermore, since

\[ \sum_{n=1}^{k} a_n = z_k \uparrow x \quad \text{as} \quad n \to \infty, \]

it follows that

\[ x = \sum_{n=1}^{\infty} a_n. \]

Lemma 2.3.4 Let \( F, G \) be positive linear functionals on \( L \), and let \( G \) be countably additive.

If \( b \geq 0 \) and \( (F \land G)(b) = 0 \), then we can find \( a_k \geq 0, \; k = 0, 1, 2, \ldots \) such that

\[ b = \sum_{k=1}^{\infty} a_k \quad \text{and} \]
\[ 0 = G(a_0) = F(a_1) = F(a_2) = \cdots. \]
Proof:

By Lemma 2.3.2, we can select \( b_n \in [0, b] \) such that

\[
G(b_n) < 1/2^n \quad \text{and} \quad F(b_n) = F(b).
\]

Set

\[
a_0 = \limsup b_n = \lim \sup_{n} \nu_{k \geq n} b_k.
\]

Then we have

\[
G(\nu_{k \geq n} b_k) \leq \sum_{k \geq n} G(b_k) = \sum_{k \geq n} 1/2^{k-1} \leq 1/2^{n-1}.
\]

Hence we have

\[
G(a_0) = 0. \tag{2.3}
\]

Let

\[
x_n := b - b_n.
\]

Then

\[
F(x_n) = F(b) - F(b_n) = 0.
\]

Also we have

\[
\liminf x_n
\]

\[
= \liminf(b - b_n)
\]

\[
= b - \limsup b_n
\]

\[
= b - a_0.
\]
Thus $x := \lim \inf x_n$ exists. Therefore by lemma 2.3.3 we can find $a_n, n = 1, 2, \ldots$ such that
\[ x = b - a_0 = \sum_{n=1}^{\infty} a_n \text{ and } F(a_n) = 0. \] (2.4)
Combining equation 2.3 and equation 2.4, we get
\[ b = \sum_{k=1}^{\infty} a_k \text{ and } 0 = G(a_0) = F(a_1) = F(a_2) = \cdots. \]
This completes the proof of the lemma.

If, in the above lemma, both $F$ and $G$ are both countably additive functionals we have the following consequence:

**Corollary 2.3.5** Suppose $F, G$ are positive countably additive linear functionals on $L$. If $b \geq 0$ in $L$ is such that $(F \wedge G)(b) = 0$, then we can find $b_1, b_2 \geq 0$ in $L$ so that $b = b_1 + b_2$ and $F(b_1) = 0 = G(b_2)$.

**Proof:**
In the above lemma, set
\[ b = b_1 \text{ and } a_0 = b_2. \]
Then, by using countable additivity of $F$, we obtain
\[ G(b_2) = 0 \text{ and } F(b_1) = \sum_{k=1}^{\infty} F(a_k) = 0. \]
2.4 On Amenable Semigroups

Let $\Sigma$ be a semigroup and $l_\infty(\Sigma)$ denote the Banach space of bounded real-valued functions on $\Sigma$, with supremum norm.

A linear functional $\phi$ on $l_\infty(\Sigma)$ is called a mean if

$$\inf_\sigma h(\sigma) \leq \phi(h) \leq \sup_\sigma h(\sigma)$$

for any $h \in l_\infty(\Sigma)$.

Let $1_\sigma, \sigma \in \Sigma$ be the evaluation functional given by $1_\sigma h = h(\sigma)$.

With each linear functional $\phi$ on $l_\infty(\Sigma)$ of the form $\phi = \sum_{k=1}^{m} \beta_k 1_{\sigma_k}$, where $\sum_{k=1}^{m} |\beta_k| < \infty$, we associate the $l_1$-norm $||\phi||_1 = \sum_{k=1}^{m} |\beta_k|$.

A mean $\phi$ is called a finite mean if

$$\phi = \sum_{k=1}^{m} \beta_k 1_{\sigma_k}$$

for some $\beta_k \geq 0$ such that $\sum_{k=1}^{m} \beta_k = 1$ and $\sigma_k \in \Sigma$. Let us denote the set of all finite means by $FM$.

Let $L_\sigma, \sigma \in \Sigma$ be the left shift defined on $l_\infty(\Sigma)$ by

$$L_\sigma h(\rho) = h(\sigma \rho), \forall h \in l_\infty(\Sigma).$$

For $\psi = \sum_{k=1}^{n} \beta_k 1_{\sigma_k} \in l_\infty^*(\Sigma)$, we define

$$L_\psi h = \sum_{k=1}^{m} \beta_k L_{\sigma_k} h.$$

Furthermore, if we choose $\psi$ to be a finite mean then $L_\psi$ will be a contraction on $l_\infty$. Indeed, if $\psi = \sum_{k=1}^{n} \beta_k 1_{\sigma_k}$ with $\sum_{k=1}^{n} \beta_k = 1$ and $\beta_k \geq 0$ then for any $h \in l_\infty(\Sigma)$
we have

\[ \|L_\psi h\|_\infty = \| \sum_{k=1}^{m} \beta_k L_{\sigma_k} h \|_\infty \leq \sum_{k=1}^{m} \beta_k \cdot \|L_{\sigma_k} h\|_\infty \leq \sum_{k=1}^{m} \beta_k \cdot \|h\|_\infty = \|h\|_\infty. \]

A mean \( \phi \) is said to be left invariant if

\[ \phi(L_\sigma h) = \phi(h), \quad \forall h \in l_\infty(\Sigma), \quad \forall \sigma \in \Sigma. \]

Note that for every left invariant mean \( \phi \) and for every finite mean \( \psi \), we have \( \phi(L_\psi h) = \phi(h) \).

Similarly we define right invariant means. A mean which is left invariant as well as right invariant is called an invariant mean.

Let us denote the set of all left invariant means by \( \text{LIM} \), the set of all right invariant means by \( \text{RIM} \) and the set of all invariant means by \( \text{IM} \).

A semigroup \( \Sigma \) is called left amenable if \( \text{LIM} \neq \emptyset \), i.e., there exists a left invariant mean. A semigroup \( \Sigma \) is called right amenable if \( \text{RIM} \neq \emptyset \), i.e., there exists a right invariant mean. A semigroup \( \Sigma \) is called amenable if \( \text{IM} \neq \emptyset \), i.e., there exists an invariant mean.

First we prove some results under the assumption that \( \Sigma \) is a left amenable semigroup. For a part in our main theorem we assume \( \Sigma \) to be an abelian semigroup. Obviously under the abelian assumption on the semigroup \( \Sigma \), the concepts of left amenability, right amenability and amenability all coincide.

Consider a left amenable semigroup \( \Sigma \). We define the maximal value of the left invariant means by

\[ M(h) := \sup_{\phi \in \text{LIM}} [\phi(h)], \quad h \in l_\infty(\Sigma) \]
Let us start here with the statement of a theorem of Day [8].

**Theorem 2.4.1 (Day)**

Let $\Sigma$ be a left amenable semigroup. Then there exists a net $\psi_{\alpha} \in \mathcal{F}M$ such that $\psi_{\alpha}$ converges in norm to left invariance, i.e.,

$$\lim_{\alpha} \|\psi_{\alpha} L_\sigma - \psi_\sigma\|_1 = 0, \forall \sigma \in \Sigma$$

In the case of countably generated left amenable semigroups, we have the following consequence, which is due to Sachdeva [19].

**Theorem 2.4.2** Let $\Sigma$ be a countably generated left amenable semigroup. Then there exists a sequence $\psi_n \in \mathcal{F}M$ such that $\psi_n$ converges in norm to left invariance, i.e.,

$$\lim_{n} \|\psi_n L_\sigma - \psi_\sigma\|_1 = 0, \forall \sigma \in \Sigma$$

Now, let us consider the following theorem on identification of maximal left invariant mean by Granirer [12].

**Theorem 2.4.3 (Granirer)**

Let $\Sigma$ be a left amenable semigroup and $\psi_{\alpha} \in \mathcal{F}M$ be a net converging to left invariance. Then

$$M(h) = \inf_{\psi \in \mathcal{F}} \sup_{\sigma \in \Sigma} (L_\psi h)(\sigma) = \lim_{\alpha} \sup_{\sigma \in \Sigma} (L_{\psi_{\alpha}} h)(\sigma), \forall h \in l_\infty(\Sigma)$$

From theorem 2.4.2 and theorem 2.4.3, it follows that in the case of countably generated amenable group we have

$$M(h) = \limsup_{n} \sup_{\sigma \in \Sigma} (L_{\psi_n} h)(\sigma), \forall h \in l_\infty(\Sigma) \quad (2.5)$$

where $\psi_n$ is defined in theorem 2.4.2.
CHAPTER III

ON EXISTENCE OF INARIANT WEAK UNITS

3.1 Results on Existence of Invariant Weak Units

Let $E$ be a Banach lattice satisfying the following conditions (A) and (B). Let $\Sigma$ be a countably generated left amenable semigroup. For the operators on $E$, we consider a positive linear operator representation, $\{T_\sigma : \sigma \in \Sigma\}$ of $\Sigma$ with $\sup\{||T_\sigma|| : \sigma \in \Sigma\} = K < \infty$.

Consider any finite mean of the form

$$\psi = \sum_{k=1}^{m} \beta_k 1_{\sigma_k},$$

where $\sum_{k=1}^{m} \beta_k = 1$ and $\beta_k \geq 0$. \hfill (3.1)

With this finite mean we associate the operator averages $A_\psi$ given by

$$A_\psi = \sum_{k=1}^{m} \beta_k T_{\sigma_k}.$$ \hfill (3.2)

Also, it follows that for any $f \in E$ and for any $F \in E_{++}^*$, if we define $h \in l_\infty(\Sigma)$ by

$$h(\sigma) = F(T_\sigma f)$$

then we have

$$F[A_\psi(f)] = \psi(h).$$ \hfill (3.3)

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Indeed, both expressions above are equal to $\sum_{k=1}^{m} \beta_{k} F(T_{\sigma_{k}}f)$.

We construct our sequence of operator averages $\{A_{\psi_{n}}\}$ from the sequence of finite means $\{\psi_{n}\}$, which converges in mean to left invariance. Note that the theorem 2.4.2 guarantees the existence of such sequences of finite means. Therefore, if

$$\psi_{n} = \sum_{k=1}^{m} \beta_{n_{k}} \sigma_{n_{k}} \text{ with } \sum_{k=1}^{m} \beta_{n_{k}} = 1 \text{ and } \beta_{n_{k}} \geq 0$$

then

$$A_{\psi_{n}} = \sum_{k=1}^{m} \beta_{n_{k}} T_{\sigma_{n_{k}}}.$$ 

First we shall show that for any $f \in E$ the averages $A_{\psi_{n}}f$ asymptotically satisfy left invariant property.

**Definition 3.1.1** Consider the operator averages $A_{\psi_{n}}, n = 1,2,\ldots$, of the form $A_{\psi_{n}} = \sum_{k=1}^{m} \beta_{n_{k}} T_{\sigma_{n_{k}}}$, where $\sum_{k=1}^{m} \beta_{n_{k}} = 1$. We say that the operator averages converge in norm to the left invariance if

$$||T_{\sigma}A_{\psi_{n}} - A_{\psi_{n}}|| \rightarrow 0, \text{ as } n \rightarrow \infty \text{ for each } \sigma \in \Sigma.$$ 

**Proposition 3.1.2** Let $\Sigma$ be a countably generated amenable semigroup, with $\{\psi_{n}\}$ being a sequence of finite means which converges in norm to left invariance. Let $A_{\psi_{n}}$ be the operator averages associated with $\psi_{n}$, as described in the beginning of this section. Assume $\sup\{||T_{\sigma}|| : \sigma \in \Sigma\} = K < \infty$. Then operator averages $A_{\psi_{n}}$ converge in norm to the left invariance.

**Proof:** Let $F \in E^{\ast}$ and $f \in E$.

Let $\tau \in \Sigma$ be fixed. We have

$$|F(T_{\tau}A_{\psi_{n}}f - A_{\psi_{n}}f)|$$
Define \( h \in l_{\infty}(\Sigma) \) by

\[
h(\sigma) = F(T_\sigma f)
\]

Then it follows that

\[
L_T h(\sigma) = F(T_\tau \sigma f)
\]

and therefore

\[
\psi_n(h) = \sum_{k=1}^{m_n} \beta_{n_k} F(T_{\sigma_{n_k}} f)
\]

\[
= F[\sum_{k=1}^{m_n} \beta_{n_k} T_{\sigma_{n_k}} f]
\]

\[
= F[A_{\psi_n} f]
\]

\[
\psi_n(L_T h) = \sum_{k=1}^{m_n} \beta_{n_k} F(T_\tau \sigma_{n_k} f)
\]

\[
= (T_\tau^* F)(\sum_{k=1}^{m_n} \beta_{n_k} T_{\sigma_{n_k}} f)
\]

\[
= (T_\tau^* F)(A_{\psi_n} f).
\]

Therefore we can write

\[
|F(T_\tau A_{\psi_n} f - A_{\psi_n} f)|
\]

\[
= |\psi_n L_T (h) - \psi_n(h)|
\]

\[
\leq ||\psi_n L_T - \psi_n||_{L_1} \cdot ||h||_{\infty}
\]

\[
\leq ||\psi_n L_T - \psi_n||_{L_1} \cdot sup_{\sigma \in \Sigma} |F(T_\sigma f)|
\]

\[
\leq ||\psi_n L_T - \psi_n||_{L_1} \cdot ||F|| \cdot K \cdot ||f||
\]

\[
\leq ||\psi_n L_T - \psi_n||_{L_1} \cdot K \cdot ||f||, \text{ if } ||F|| \leq 1.
\]
Since $F \in E^*$ is arbitrary, we have

$$\sup_{\|F\| \leq 1} |F(T_{rA} A_{\psi_n} f - A_{\psi_n} f)| \leq \|\psi_n L_r - \psi_n\|_{1} \cdot K \cdot \|f\|.$$ 

Therefore we have

$$||T_{rA} A_{\psi_n} f - A_{\psi_n} f|| = ||T_{rA} A_{\psi_n} f - A_{\psi_n} f|| \leq ||\psi_n L_r - \psi_n\|_{1} \cdot K \cdot \|f\||.$$ 

The above result being true for each $f \in E$, we obtain

$$||T_{rA} A_{\psi_n} f - A_{\psi_n} f|| \leq \|\psi_n L_r - \psi_n\|_{1} \cdot K.$$ 

But, by assumption

$$||\psi_n L_r - \psi_n\|_{1} \xrightarrow{n \to \infty} 0.$$ 

Therefore we have

$$||T_{rA} A_{\psi_n} f - A_{\psi_n} f|| \xrightarrow{n \to \infty} 0.$$ 

This concludes the proof of the proposition.

Hereafter, by $A_{\psi_n}, n = 1, 2, \ldots$, we always mean a fixed sequence of operator averages converging in norm to the left invariance. The existence of such a sequence is guaranteed by the above proposition.

**Theorem 3.1.3** Let $E$ be a Banach lattice satisfying conditions (A) and (B). Let $\{T_\sigma : \sigma \in \Sigma\}$ be a representation of a countably generated left amenable semigroup as a semigroup of uniformly bounded positive linear operators on $E$. Then the following conditions are equivalent:

(i) There exists an invariant weak unit $v$ in $E$, i.e., $\exists$ a weak unit $v \in E_+$ such that $T_\sigma v = v, \forall \sigma \in \Sigma$.
(ii) \( \inf_{\sigma} \in \Sigma H(T_{\sigma}e) > 0, \ \forall H \in E_{++}^* \)

Furthermore, if we assume \( \Sigma \) is commutative then the following conditions are equivalent and they are equivalent to each of the conditions (i) and (ii) above.

(iii) \( \liminf_n H(A_{\phi_n}e) > 0, \ \forall H \in E_{++}^* \)

(iv) \( \limsup_n H(A_{\phi_n}e) > 0, \ \forall H \in E_{++}^* \)

(v) \( M[H(T_{\sigma}e)] > 0, \ \forall H \in E_{++}^* \).

Here \( M \) denotes the maximal value of the left invariant means on \( l_\infty(\Sigma) \), the class of bounded functionals on \( \Sigma \), i.e., \( M(h) = \sup_{\phi \in LIM} \phi(h), \ \forall \ h \in l_\infty(\Sigma) \).

The proof of equivalence of (i) and (ii) is given in the following section.

The equivalence of (iii) with the remaining conditions is obvious, once we show the equivalence of the remaining.

In section 3.4 we show the equivalence of (ii), (iv) and (v) under the additional commutativity assumption. For the proofs of (iv) \( \Rightarrow \) (ii) and (v) \( \Rightarrow \) (ii) we follow the scheme of Brunel and Sucheston [3] in the case of \( \Sigma = N \).
3.2 Examples.

Example 3.2.1 Let $\Sigma = \{1,2,3,\cdots\}$ with addition. Then any $x \in l_\infty(\Sigma)$ is a bounded sequence of the form $(x_1,x_2,\cdots)$ with $x_n = x(n)$ and $||x|| = \sup |x_n| < \infty$. Given $x = (x_1,x_2,x_3,\cdots)$, for any positive integer $k$ define the sequence obtained by shifting the elements of the sequence $x$ to the left by $k$ places by $x^{(k)} = (x_{(k+1)},x_{(k+2)},\cdots)$. Then left shift operators $L_k$ on $l_\infty(\Sigma)$ are given by $L_k(x) = x^{(k)}$.

Consider the finite means on $l_\infty(\Sigma)$ defined by

$$\psi_n = \frac{1}{n} \sum_{i=1}^{n} 1_i \in l_\infty^*(\Sigma).$$

We have

$$\psi_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_i(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$(\psi_n L_k)(x) = \frac{1}{n} \sum_{i=1}^{n} 1_i(x^{(k)}) = \frac{1}{n} \sum_{i=1}^{n} x^{(k)}(i) = \frac{1}{n} \sum_{i=1}^{n} x_{i+k}$$

Therefore for any $k$, $1 \leq k \leq n$ we have

$$|(\psi_n - \psi_n L_k)(x)| = \left| \frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n} \sum_{i=1}^{n} x_{i+k} \right|$$

$$= \frac{1}{n} \left| x_1 + x_2 + \cdots + x_k - x_{n+1} - x_{n+2} - \cdots x_{n+k} \right|$$

$$\leq \frac{1}{n} \cdot 2k \cdot ||x||$$

Thus we have

$$||\psi_n - \psi_n L_k||_1 \leq 2k/n \to 0, \text{ as } n \to \infty.$$
By our definition, the operator averages associated with \( \psi_n = \frac{1}{n} \sum_{i=1}^{\infty} 1_i \) are given by

\[ A_n = A_{\psi_n} = \frac{1}{n} \sum_{i=1}^{n} T^i \]

which are the Cesaro Averages of the operator \( T \). Thus in the case of cyclic group, the results of theorem 3.1.3 reduces to the results in [3], in the case of power bounded operators.

Furthermore by proposition 3.1.2 we obtain that if \( T \) is power bounded then

\[ \|T^n A_n - A_n\| \rightarrow 0, \text{ as } n \rightarrow \infty \]

This is a well-known result; in fact this result holds under a weaker condition, namely \( T \) is mean bounded and \( \|T^n\|/n \) converges to zero.

**Example 3.2.2** Consider an abelian semigroup \( \Sigma \) generated by finitely many elements \( \sigma_1, \sigma_2, \ldots, \sigma_d \).

Define \( \psi_n^k \in L^\infty(\Sigma) \) for \( k = 1, 2, \ldots, d \) by

\[ \psi_n^k = \frac{1}{n} \sum_{i=1}^{n} 1_{\sigma_i^k}. \]

We have \( \|\psi_n^k\|_{l_1} = 1 \). Define the multiplication \( \psi_n^k \psi_n^l \) by

\[ \psi_n^k \psi_n^l = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1_{\sigma_i^k \sigma_j^l}. \]

Clearly the multiplication defined above is commutative. Let \( \psi_n = \psi_n^1 \psi_n^2 \ldots \psi_n^d \). For any given \( \sigma_k \) in \( \Sigma \) we have

\[ \|(\psi_n - \psi_n L_{\sigma_k})\|_{l_1} \]
\[\begin{align*}
&= \|\psi_n^1 \cdot \psi_n^2 \cdot \psi_n^{k-1} \cdot \psi_n^{k+1} \cdot \psi_n^{d} \cdot (\psi_n^k - \psi_n^{k} L_{\sigma_k})\|_{l_1} \\
&\leq \|\psi_n^1\|_{l_1} \cdot \|\psi_n^2\|_{l_1} \cdot \cdots \cdot \|\psi_n^{k-1}\|_{l_1} \cdot \|\psi_n^{k+1}\|_{l_1} \cdot \|\psi_n^{d}\|_{l_1} \cdot \|\psi_n^k - \psi_n^{k} L_{\sigma_k}\|_{l_1} \\
&= \|(\psi_n^k - \psi_n^{k} L_{\sigma_k})\|_{l_1} \\
&\to 0, \text{ as } n \to 0.
\end{align*}\]

Note that the last step above follows by the same approach as in the previous example.

Now consider two elements \(\rho_1, \rho_2\) such that for \(k = 1, 2\) one has

\[\lim_n \|(\psi_n - \psi_n L_{\rho_k})\|_{l_1} = 0\]

Consider the product \(\rho = \rho_1 \cdot \rho_2\). We have

\[\begin{align*}
\|(\psi_n - \psi_n L_{\rho})\|_{l_1} &= \|(\psi_n - \psi_n L_{\rho_1} + \psi_n L_{\rho_1} - \psi_n L_{\rho_1 \cdot \rho_2})\|_{l_1} \\
&\leq \|(\psi_n - \psi_n L_{\rho_1})\|_{l_1} + \|\psi_n L_{\rho_1} - \psi_n L_{\rho_1 \cdot \rho_2}\|_{l_1} \\
&\leq \|(\psi_n - \psi_n L_{\rho_1})\|_{l_1} + \|\psi_n - \psi_n L_{\rho_2}\|_{l_1} \cdot \|L_{\rho_1}\|_{l_1} \\
&\to 0, \text{ as } n \to 0.
\end{align*}\]

Now consider any arbitrary element \(\sigma \in \Sigma\). Then \(\sigma\) is given by a finite product of \(\sigma_k\), \(k = 1, 2, \ldots d\). Therefore we have,

\[\lim_n \|(\psi_n - \psi_n L_{\sigma})\|_{l_1} = 0\]

Hence

\[\begin{align*}
\psi_n &= \psi_n^1 \cdot \psi_n^2 \cdot \cdots \cdot \psi_n^d \\
&= \frac{1}{n^d} \sum_{i_1, i_2, \ldots, i_k=1}^{n} 1_{\sigma_1}^{i_1} \cdot 1_{\sigma_2}^{i_2} \cdot \cdots \cdot 1_{\sigma_d}^{i_d}.
\end{align*}\]
are sequences of finite means converging in norm to left invariance.

By our definition, the corresponding operator averages are given by

$$A_{\psi_n} = \frac{1}{n^d} \sum_{i_1, i_2, \ldots, i_k=1}^{n} T_{i_1}^{i_1} T_{i_2}^{i_2} \cdots T_{i_d}^{i_d}.$$  

By proposition 3.1.2, if the operators are uniformly bounded, then the above averages satisfy the condition

$$\lim_{k \to \infty} ||A_{\psi_n} - T_{\sigma_k} A_{\psi_n}|| = 0, \ k = 1, 2, \ldots, d.$$  

For example, in the case of operators generated by two commutative operators $S$ and $T$ the operator averages are given by

$$A_{\psi_n} = \frac{1}{n^2} \sum_{i,j=1}^{n} S^i T^j.$$  

Also in this case theorem 3.1.3 takes the following form:

**Theorem 3.2.3** Let $E$ be a Banach lattice satisfying conditions (A) and (B). Let $S, T$ be power bounded commutative operators. Then the following conditions are equivalent:

(i) There exists an invariant weak unit $v$ in $E$ such that $Tv = v = Sv$.

(ii) $\inf_{i,j} H(S^i T^j e) > 0, \ \forall \ H \in E_{++}^*$

(iii) $\lim \inf_n H[A_n(S)A_n(T)(e)] > 0, \ \forall \ H \in E_{++}^*$

(iv) $\lim \sup_n H[A_n(S)A_n(T)(e)] > 0, \ \forall \ H \in E_{++}^*$

(v) $M[H(S^i T^j e)] > 0, \ \forall \ H \in E_{++}^*$.

Here $M$ denotes the maximal value of the (left) invariant means on $l_\infty(\Sigma)$, the class of bounded functionals on $\Sigma = N^2$.  

Now let us consider an example of an amenable group which is not abelian.

**Example 3.2.4** Consider a group $\Sigma$ generated by two elements $\sigma_1$ and $\sigma_2$ such that $\sigma_i = \sigma_i^{-1}, i = 1, 2$. Dixmier [9] proved that such a group is amenable. Therefore from theorem 3.1.3, the following result follows:

**Proposition 3.2.5** Let $E$ be a Banach lattice satisfying conditions (A) and (B). Let $T_1, T_2$ be positive linear contractions on $E$ such that $T_i^2 = I, i = 1, 2$. Then the following conditions are equivalent:

(i) There exists a weak unit $v$ in $E$ such that $T_1v = v = T_2v$.

(ii) $\inf_T H(Te) > 0, \forall H \in E^*_+.$

Here the infimum is taken over all $T$, where $T$ denotes a finite product of the operators $T_1$ and $T_2$.

In the case of point transformations acting on a probability space, a similar result was proved by Blum and Friedman [4].

Groups generated by more than two elements are not amenable (see Dixmier [9]). Therefore, the results of the above proposition cannot be extended to cases involving more than two elements.
3.3 Equivalence of (i) and (ii).

First we prove the following lemma.

**Lemma 3.3.1** Let $e$ be any weak unit and $H \in E^*_+$. Suppose we have uniform bounded operators, $T_{\rho_n}, n = 1, 2, \ldots$, such that $\sup_n ||T_{\rho_n}|| \leq K < \infty$ Assume $\lim_n H(T_{\rho_n}e) = 0$. Then $\lim_n H(T_{\rho_n}u) = 0$ for any $u$ in $E_+$.

Proof:

Let $u$ be any element in $E_+$. Given any $\epsilon > 0$, there exists an integer $k$ and $w \in E_+$ such that $u = u \wedge ke + w$ and $||w|| < \epsilon/2K||H||$.

Since $\lim_n H(T_{\rho_n}e) = 0$, we can choose a positive integer $N$ such that for $n > N$

$$H(T_{\rho_n}e) < \epsilon/2k.$$

Thus for $n > N$ we get

$$H(T_{\rho_n}u) \leq k \cdot H(T_{\rho_n}e) + H(T_{\rho_n}w)$$

$$\leq k \cdot H(T_{\rho_n}e) + ||H|| \cdot \sup_{\sigma} ||T_{\sigma}|| \cdot ||w||$$

$$\leq k \cdot H(T_{\rho_n}e) + ||H|| \cdot K \cdot ||w||$$

$$\leq k \cdot \epsilon/2k + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows.

**Proof of (i) $\Rightarrow$ (ii)**
Assume that there exists an invariant weak unit \( v \) in \( E_+ \). Suppose (ii) does not hold. Then there exists \( H_0 \in E_{++}^* \) such that

\[
\inf_{\sigma} H_0(T_\sigma e) = 0.
\]

By lemma 3.3.1, it follows

\[
\inf_{\sigma} H_0(T_\sigma v) = 0.
\]

But \( T_\sigma v = v \) for any \( \sigma \in \Sigma \); this implies

\[
0 = \inf_{\sigma} H_0(T_\sigma v) = \inf_{\sigma} H_0(v) = H_0(v).
\]

Since \( v \) is a weak unit this leads to \( H_0 = 0 \), a contradiction.

\textbf{Proof of (ii) \( \Rightarrow \) (i)}

Assume \( \inf_{\sigma} H(T_\sigma e) > 0 \), \( \forall H \in E_{++}^* \). Let \( \phi \) be a left invariant mean on \( l_\infty(\Sigma) \).

Define \( \lambda \in E_{++}^{**} \) by

\[
\lambda(H) = \phi[H(T_\sigma e)], \ H \in E_{++}^*.
\] (3.4)

Then we have

\[
\lambda(H) > 0, \ \forall H \in E_{++}^*. \tag{3.5}
\]

For any \( H \in E^* \) and \( \tau \in \Sigma \), we have

\[
(T^*_{\tau**} \lambda)H = \lambda(T^*_\tau H)
\]

\[
= \phi[T^*_\tau H(T_\sigma e)]
\]

\[
= \phi[H(T_\tau T_\sigma e)]
\]

\[
= \phi[H(T_\sigma e)]
\]

\[
= \lambda(H).
\]
Therefore for any $\sigma \in \Sigma$

$$T^{**}_\sigma \lambda = \lambda. \quad (3.6)$$

Now, define

$$u = \sup \{ w \in E \mid 0 \leq w \leq \lambda \}. \quad (3.7)$$

Theorem 2.1.4 guarantees that $u \in E$. Also we have

$$0 \leq T_\sigma u = T^{**}_\sigma u \leq T^{**}_\sigma \lambda = \lambda.$$

Therefore, by equation (3.7) it follows that

$$T_\sigma u \leq u \quad (3.8)$$

Now, we shall show that $u$ is a weak unit. Consider $(\lambda - u) \wedge e$. By theorem 2.1.3, we get

$$(\lambda - u) \wedge e \in E_+.$$

Thus, from equation (3.7) it follows that

$$(\lambda - u) \wedge e = 0. \quad (3.9)$$

Indeed, since $u + (\lambda - u) \wedge e \leq u + (\lambda - u) = \lambda$, the above result follows from the maximality of $u$ as given by equation (3.7).

Now, suppose $H(u) = 0$ for some $H \in E^*_+$. Equation (3.9) implies that

$$< (\lambda - u) \wedge e, H > = 0.$$

Therefore, by lemma 2.3.4 applied with $L = E^*$, there exists a sequence $\{ H_k \}$ in $E^*_+$ such that

$$H = \sum_{k=0}^{\infty} H_n,$$
\(< (\lambda - u), H_k > = 0, \text{ for } k = 0, 1, 2, \ldots \) and
\(< e, H_0 > = 0.\)

But, \(H(u) = 0\) and \(0 \leq H_k \leq H\) implies
\[H_k(u) = 0, \text{ for } k = 0, 1, 2, \ldots;\]
this result, together with \(< (\lambda - u), H_k >= 0, k = 0, 1, 2, \ldots\) implies
\[\lambda(H_k) = 0, \text{ for } k = 1, 2, \ldots.\]

Therefore, by equation (3.5), we get
\[H_k = 0, \text{ for } k = 1, 2, \ldots.\]

Also, \(< e, H_0 >= 0\) implies \(H_0 = 0.\) Therefore we have
\[H = \sum_{k=0}^{\infty} H_k = 0.\]

Thus we have shown for an arbitrary \(H \in E_+^*\) that \(H(u) = 0\) implies \(H = 0;\) hence \(u\)
is a weak unit. Thus, we have a weak unit \(u\) such that \(T_\sigma u \leq u, \text{ for any } \sigma \in \Sigma.\)

To complete the proof, it remains to find a weak unit \(v\) such that \(T_\sigma v = v\) for all \(\sigma \in \Sigma.\)
Consider a sequence \(\{A_{\psi_n}\}\) of operator averages which converge in norm to left invariance. Since \(0 \leq T_\sigma u \leq u,\) we have
\[0 \leq A_{\psi_n} u \leq u.\]

Recall that order continuity (hence a fortiori condition (B)) implies that each order interval is weakly compact (LT [16], p.28), and hence we have a subsequence \(\{\psi_{n_k}\}\)
of \{\psi_n\} such that \{A_{\psi_{n_k}} u\} converges weakly to some element \(v\) in \(E_+\).

\textbf{claim} \(v\) is a weak unit.

\textbf{proof} Suppose the opposite. Then there exists an \(H \in E_{++}^*\) such that \(H v = 0\). Thus we obtain
\[
H(A_{\psi_{n_k}} u) \to H(v) = 0.
\]

Let
\[
\psi_{n_k} = \sum_{k=1}^{m_n} \beta_{n_k} 1_{\sigma_{n_k}}, \text{where } \beta_{n_k} \geq 0 \text{ and } \sum_{k=1}^{m_n} \beta_{n_k} = 1.
\]

Then we have
\[
\lim_{n_k} \sum_{k=1}^{m_n} \beta_{n_k} H(T_{\sigma_{n_k}} u) = 0.
\]

This implies that
\[
\inf_{\sigma} H(T_\sigma u) = 0.
\]

Since \(u\) is a weak unit, from lemma 3.3.1 it follows that \(\inf_{\sigma} H(T_\sigma e) = 0\). This contradicts our assumption.

\textbf{claim} \(v\) is such that \(T_\sigma v = v, \forall \sigma \in \Sigma\).

\textbf{proof} Take any \(\sigma \in \Sigma\). Since \(A_{\psi_{n_k}} u \xrightarrow{w} v\), for any \(H \in E_{++}^*\) we have,
\[
H(A_{\psi_{n_k}} u) \to Hv.
\]

Also for each \(H \in E_{++}^*\) we have,
\[
(T_\sigma^* H)(A_{\psi_{n_k}} u) \to (T_\sigma^* H)v.
\]

and hence,
\[
H(T_\sigma A_{\psi_{n_k}} u) \to H(T_\sigma v).
\]

(3.11)
But,

\[ |H(T_\sigma A_{\psi n_k} u) - H(A_{\psi n_k} u)| \to 0, \text{ as } n_k \to \infty. \tag{3.12} \]

Indeed, by proposition (3.1.2), we have

\[ |H(T_\sigma A_{\psi n_k} u) - H(A_{\psi n_k} u)| \leq ||H|| \cdot ||T_\sigma A_{\psi n_k} - A_{\psi n_k}|| |u| \to 0, \text{ as } n_k \to \infty. \]

Equations (3.10), (3.11) and (3.12) imply

\[ H(T_\sigma v) = Hv. \]

Therefore \( H(T_\sigma v - v) = 0 \); this result being true for any \( H \in E^*_+ \), we get

\[ T_\sigma v = v. \]

This completes the proof of (i) \( \Leftrightarrow \) (ii) of theorem 3.1.3.
3.4 Equivalence of (ii), (iv) and (v).

In our proofs in this section, we use the concepts of weakly wandering element and m-weakly wandering element. These concepts generalize the concept of weakly wandering set.

**Definition 3.4.1** An element $W \in E_{++}^*$ is said to be weakly wandering in $E^*$, if there is a sequence $\{\sigma_k\}$ in $\Sigma$ and an element $G \in E_{++}^*$ such that $\sum_{k=1}^{\infty} T_{\sigma_k}^* W \leq G$.

**Definition 3.4.2** An element $W \in E_{++}^*$ is said to be m-weakly wandering in $E_{++}^*$, if there is a sequence $\{\sigma_k\}$ in $\Sigma$ and an element $G \in E_{++}^*$ such that for any positive integer $m$, one has $\sum_{k=m}^{\infty} T_{\sigma_m}^* \cdots T_{\sigma_{m+k}}^* W \leq G$.

From the definitions, it follows that the existence of m-weakly wandering element in $E_{++}^*$ implies the existence of weakly wandering element in $E^*$.

**Definition 3.4.3** Let $Z$ be a functional on $E_{++}^*$. We say $Z$ is additive on the transforms if for any $G$ in $E_{++}^*$ and for any finite collection of elements $\sigma_k$, $k = 1, 2, \ldots, m$, in $\Sigma$ we have $Z[(\sum_{k=1}^{m} T_{\sigma_k}^*) G] = m \cdot Z[G]$.

We call $Z$ order preserving if for any $F$ and $F'$ in $E^*$, we have $F \leq F'$ in $E^*$, we have $F \leq F'$ implies $Z(F) \leq Z(F')$. We have the following lemma:

**Lemma 3.4.4** Let $Z$ be a positive order preserving functional on $E_{++}^*$ which is additive on the transforms. Assume $Z(H) > 0$ for all $H \in E_{++}^*$. Then there exists no weakly wandering element in $E^*$.
Proof: Suppose the result of the lemma is not true. Then there exists $W$ and $G$ in $E^*_+$ and a sequence $\{\sigma_k\}$ in $\Sigma$ such that $\sum_{k=1}^{\infty} T_{\sigma_k}^* W \leq G$.

Let $m$ be any positive integer. We have

$$m \cdot Z(W) = Z(\sum_{k=1}^{m} T_{\sigma_k}^* W) \leq Z(G) < \infty.$$ 

This result is true for any arbitrary $m$; therefore $Z(W) = 0$. This contradicts our assumption and hence there exists no weakly wandering element in $E^*$.

**Lemma 3.4.5** Let $f \in E_{++}$ be fixed. Then

(a) 

$$Z(F) = \limsup_{n} F[(A_{\psi_n} f)], \ F \in E^*_+$$

defines a positive order preserving functional on $E^*_+$, which is additive on the transforms.

(b) 

$$M[F(T_\sigma f)], \ F \in E^*_+$$

defines a positive order preserving functional on $E^*_+$, which is additive on the transforms.

Proof: (a) Consider the transforms $T_{\sigma_1}^* F$, $T_{\sigma_2}^* F$, $\cdots$, $T_{\sigma_m}^* F$ of $F \in E^*$. We have

$$|m \cdot F[A_{\psi_n} f] - (\sum_{k=1}^{m} T_{\sigma_k}^* F)[A_{\psi_n} f]|$$

$$= |m \cdot F[A_{\psi_n} f] - F[\sum_{k=1}^{m} T_{\sigma_k} A_{\psi_n} f]|$$

$$= |\sum_{k=1}^{m} \{F[A_{\psi_n} f] - F[T_{\sigma_k} A_{\psi_n} f]\}|$$

$$\leq \sum_{k=1}^{m} |F[A_{\psi_n} f] - F[T_{\sigma_k} A_{\psi_n} f]|$$
\[\begin{align*}
&= \sum_{k=1}^{\infty} |F[(A^n_{\psi_{k}} - T_{\sigma_k}A^n_{\psi_{k}})]| \\
&\leq \sum_{k=1}^{\infty} ||F|| \cdot ||A^n_{\psi_{k}} - T_{\sigma_k}A^n_{\psi_{k}}|| \cdot ||f|| \\
&\rightarrow 0, \text{ as } n \rightarrow \infty \text{ (by proposition 3.1.2)}. 
\end{align*}\]

Therefore we have,

\[
\limsup_n (\sum_{k=1}^{m} T^*_{\sigma_k} F)[A_{\psi_n} f] = m \cdot \limsup_n F[A_{\psi_n} f],
\]

as required.

(b) Consider the transforms \( T^*_{\sigma_1} F, T^*_{\sigma_2} F, \ldots, T^*_{\sigma_m} F \) of \( F \in E^* \). We have

\[
M[(\sum_{k=1}^{m} T^*_{\sigma_k} F)T_\sigma f]
\]
\[
= M[F((\sum_{k=1}^{m} T_{\sigma_k})T_\sigma f)] \\
= \sup_{\phi \in LIM} \phi[F((\sum_{k=1}^{m} T_{\sigma_k})T_\sigma f)] \\
= \sup_{\phi \in LIM} \sum_{k=1}^{m} \phi[F(T_{\sigma_k}f)] \\
= \sup_{\phi \in LIM} \sum_{k=1}^{m} \phi[F(T_\sigma f)] \quad \text{(using } \phi \in LIM) \\
= m \cdot M[F(T_\sigma f)]
\]

Hence the proof of lemma.

From lemma 3.4.5 and lemma 3.4.4, we have the following theorem:

Theorem 3.4.6 Let \( f \in E_{++} \) be fixed.

(a) If \( \limsup_n H(A^n_{\psi_n}f) > 0 \) for all \( H \in E_{++}^* \) then there is no weakly wandering element in \( E^* \).

(b) If \( M[H(T_\sigma f)] > 0 \) for all \( H \in E_{++}^* \) then there is no weakly wandering element in \( E^* \).
Lemma 3.4.7 Let $F$ be a given element in $E^*_+$. 

Assume either 

(a) $\limsup_{n} F(A_{\psi_n} e) > 0$ 

or 

(b) $M[F(T_\sigma e)] > 0$. 

Then there exists $G$ and $H$ in $E^*_+$ such that $T_\sigma^* G = G$ for any $\sigma \in \Sigma$, $H \leq G$ and $H \leq T_{\alpha^*}^* F$ for some $\alpha^* \in \Sigma$.

Proof: 

(a) Let $R := \limsup_{n} F(A_{\psi_n} e) > 0$. Then $\{\psi_n\}$ has a subsequence, still denoted $\{\psi_n\}$, such that 

$$ F(A_{\psi_n} e) \to R > 0. $$

Consider

$$ F_n := A_{\psi_n}^* F = \sum_{k=1}^{m_n} \beta_{n_k} T_{\sigma_{n_k}}^* F \in E^*_+. $$

We have

$$ \|F_n\| \leq \|A_{\psi_n}^*\| \cdot \|F\| \leq \|K\| \cdot \|F\|. $$

Therefore by Alaoglu's theorem, $\{F_n\}$ has a subsequence which converges in the $\sigma(E^*, E)$ topology (weak* convergence). Denoting this subsequence by the same notation as the original sequence, for some $G \in E^*_+$ we have

$$ F_n(f) \to G(f), \ \forall f \in E. $$

In particular, we have $F_n(e) \to G(e)$. But $F_n(e) \to R > 0$. Therefore, we have that

$$ G(e) = R > 0. $$
Thus we have $G \in E_{*+}^+$. 

**Proof of $T_\sigma^* G = G$, $\forall \sigma \in \Sigma$**

Consider any $\sigma \in \Sigma$ and $f \in E$. Since $F_n \to G$ (weak*), we have

$$F_n(f) \to G(f)$$

and

$$(T_\sigma^* F_n)(f) \to (T_\sigma^* G)(f).$$

To show $T_\sigma^* G = G$ it is enough to show that $|T_\sigma^* F_n(f) - F_n(f)| \to 0$. We have

$$|T_\sigma^* F_n(f) - F_n(f)|$$

$$= |T_\sigma^* A_{\psi_n} F(f) - A_{\psi_n}^* F(f)|$$

$$= |F[A_{\psi_n} T_\sigma(f)] - F[A_{\psi_n}(f)]|$$

$$= |F[T_\sigma A_{\psi_n}(f)] - F[A_{\psi_n}(f)]| \quad \text{(using commutativity)}$$

$$\leq ||F|| \cdot ||T_\sigma A_{\psi_n} - A_{\psi_n}|| \cdot ||f||$$

$$\to 0, \text{ as } n \to \infty \text{ (by proposition 3.1.2).}$$

Thus we have $T_\sigma^* G = G$.

We have

$$F_n = \sum_{k=1}^m \beta_{n_k} (T_{\sigma_{n_k}}^* F) \xrightarrow{w^*} G, \text{ as } n \to \infty.$$ 

This implies that $T_{\sigma_F}^* F \wedge G \neq 0$ for some $\sigma_F \in \Sigma$. Let

$$H = T_{\sigma_F}^* F \wedge G.$$ 

Then

$$H \leq G \text{ and } H \leq T_{\sigma_F}^* F.$$
This completes the proof of part(a) of lemma 3.4.7.

proof of part(b):

Let $R := M[F(T_\sigma e)] > 0$. Define $h(\sigma) = F(T_\sigma e)$. By using the identification of $M$ as given by theorem 2.4.3, we have

$$ R = \limsup_n [(L_{\psi_n} h)(\sigma)]. $$

claim We can find a subsequence $\{\psi_{n_i}\}$ of $\{\psi_n\}$ and a sequence $\{\sigma_i\}$ in $\Sigma$ such that

$$ \lim_i (L_{\psi_{n_i}} h)(\sigma_i) = R. $$

proof of claim Let $\epsilon > 0$ be given. Let $i > 0$ be a positive integer. Since $R = \lim_n \sup_\sigma [(L_{\psi_n} h)(\sigma)]$, there exists a positive integer $n_i$ such that for any $n \geq n_i$

$$ R - \epsilon/2i \leq \sup_\sigma (L_{\psi_n} h)(\sigma) \leq R + \epsilon/2i. $$

In particular, we have

$$ R - \epsilon/2i \leq \sup_\sigma (L_{\psi_{n_i}} h)(\sigma) \leq R + \epsilon/2i. $$

Now, we can find $\sigma_i \in \Sigma$ such that

$$ R - \epsilon/i < (L_{\psi_{n_i}} h)(\sigma_i) < R + \epsilon/i. $$

Thus the claim follows.

By denoting above subsequence $\{\psi_{n_i}\}$ by $\{\psi_i\}$ and then replacing the index $i$ by $n$, we can write

$$ \lim_n (L_{\psi_n} h)(\sigma_n) = R. \tag{3.13} $$
Now, write

\[ \psi_n = \sum_{k=1}^{m_n} \beta_n \sigma_n, \] where \( \beta_n \geq 0 \), and \( \sum_{k=1}^{m_n} \beta_n = 1. \)

Then we have

\[
(L_{\psi_n} h)(\sigma) = \sum_{k=1}^{m_n} \beta_n [(L_{\sigma_n} h)(\sigma)]
= \sum_{k=1}^{m_n} \beta_n [(h)(\sigma_n \sigma)]
= \sum_{k=1}^{m_n} \beta_n \sigma_n P(T_{\sigma_n} \sigma e)
= \sum_{k=1}^{m_n} \beta_n (T_{\sigma_n} \sigma F)(e)
\]

Define

\[ F_n := \sum_{k=1}^{m_n} \beta_n T_{\sigma_n} \sigma_n F \in E^*. \]

Since \( \lim_n F_n(e) = \lim_n (L_{\psi_n} h)(\sigma_n) \), by equation (3.13), we have

\[
\lim_n F_n(e) = R \tag{3.14}
\]

Since

\[
||F_n|| \leq \sum_{k=1}^{m_n} \beta_n ||T_{\sigma_n} \sigma_n F|| \cdot ||F|| \leq \sum_{k=1}^{m_n} \beta_n \cdot K \cdot ||F|| = K \cdot ||F|| < \infty,
\]

by Alaoglu's theorem \( \{F_n\} \) has a subsequence which converges in the \( \sigma(E^*, E) \) topology (weak* convergence). Denoting this subsequence still by \( \{F_n\} \), we obtain for some \( G \in E_{++} \)

\[ F_n(f) \to G(f), \forall f \in E. \]

But by equation (3.14),

\[ F_n(e) \to R > 0. \]
Therefore $G(e) = R > 0$; thus we have $G \in E^*_+$. 

**Proof of** $T^* \tau G = G$, $\forall \tau \in \Sigma$

Consider any $\tau \in \Sigma$. Then for any $f \in E$, we have

$$|T^*_\tau F_n(f) - F_n(f)|$$

$$= |T^*_\tau \sum_{k=1}^n \beta_{n_k} (T^*_\sigma_{n_k} \sigma_n F)(f) - \sum_{k=1}^n \beta_{n_k} (T^*_\sigma_{n_k} \sigma_n F)(f)|$$

$$= |\sum_{k=1}^n \beta_{n_k} F(T\sigma_{n_k} \sigma_n \tau f) - \sum_{k=1}^n \beta_{n_k} F(T\sigma_{n_k} F)|$$

$$= |\sum_{k=1}^n \beta_{n_k} F(T\sigma_{n_k} \sigma_n f) - \sum_{k=1}^n \beta_{n_k} F(T\sigma_{n_k} \sigma_n f)| \quad (\text{using commutativity of } \Sigma).$$

Define $h_n \in l_\infty(\Sigma)$ by

$$h_n(\sigma) = F(T\sigma_{n_k} F).$$

Then it follows that $(L\tau h_n)(\sigma) = F(T\tau \sigma_{n_k} F)$ and thus we have

$$\psi_n(h_m) = \sum_{k=1}^n \beta_{n_k} F(T\sigma_{n_k} \sigma_n F) \quad \text{and}$$

$$\psi_n(L\tau h_n) = \sum_{k=1}^n \beta_{n_k} F(T\tau \sigma_{n_k} \sigma_n F).$$

Thus we can write

$$|T^*_\tau F_n(f) - F_n(f)|$$

$$= |\psi_n L\tau h_n - \psi_n(h_n)|$$

$$\leq ||\psi_n L\tau - \psi_n|| \cdot ||h_n||_\infty$$

$$\leq ||\psi_n L\tau - \psi_n|| \cdot ||F|| \cdot \sup_{\sigma \in \Sigma} ||T\sigma|| \cdot ||f||$$

$$\leq ||\psi_n L\tau - \psi_n|| \cdot ||F|| \cdot K \cdot ||f||$$

$$\rightarrow 0 \quad \text{as } n \rightarrow \infty.$$
But we have
\[ F_n(f) \to G(f) \]
and
\[ (T_{n^*}F_n)(f) \to (T_{n^*}G)(f). \]

Therefore we obtain \( T_{n^*}G = G \).

Furthermore, we have \( G \in E_{++}^* \) and
\[ \sum_{k=1}^{m^n} \beta_{n_k} (T_{\sigma_n \sigma_n} F)(f) \to G(f), \forall f \in E. \]

This implies \( T_{\sigma_F} F \wedge G \neq 0 \) for some \( \sigma_F \in \Sigma \). Set
\[ H := T_{\sigma_F} F \wedge G. \]

Then we have \( H \leq G \) and \( H \leq T_{\sigma_F} F \). This completes the proof of lemma 3.4.7.

**Theorem 3.4.8** Assume there exists a sequence \( \{\rho_n\} \) in \( \Sigma \) and \( H \in E_{++}^* \) such that
\[ \lim_{n} H(T_{\rho_n} e) = 0 \quad (3.15) \]
Further assume that there exists \( G \in E_{++}^* \) such that \( T_{\sigma_F} G \leq G \) and \( H \leq G \). Then there exists a \( m \)-weakly wandering element in \( E^* \).

**Proof:**
Assume equation (3.15).

Let \( 0 < \epsilon < 1 \) be given. Choose \( \epsilon_k > 0 \) such that \( \prod_{k=1}^{\infty} (1 + \epsilon_k) = 1 + \epsilon \).

Claim There exists a subsequence \( \{\rho_{n_k}\} \) of the sequence \( \{\rho_n\} \), denoted briefly by \( \{\sigma_k\} \), such that for any integer \( m > 0 \) we have
\[ H[(I + T_{\sigma_m})(I + T_{\sigma_{m-1}}) \ldots (I + T_{\sigma_1}) e] \leq \prod_{k=1}^{m} (1 + \epsilon_k) H(e) < (1 + \epsilon) H(e) \quad (3.16) \]
Proof of claim (by induction on \( m \)) Equation (3.15) implies that there exists an element \( \sigma_1 = \rho_{n_1} \in \{\rho_n\} \) such that

\[
H(T_{\sigma_1} e) < \epsilon_1 H(e)
\]

and therefore

\[
H[(I + T_{\sigma_1})e] < (1 + \epsilon_1)H(e).
\]

Now assume the result for \( m > 0 \). Then we have

\[
H(u) \leq \prod_{k=1}^{m} (1 + \epsilon_k)H(e) < (1 + \epsilon)H(e), \text{ where}
\]

\[
u = (I + T_{\sigma_m})(I + T_{\sigma_{m-1}}) \cdots (I + T_{\sigma_1})e > \epsilon.
\]

By lemma 3.3.1, it follows from equation (3.15) that \( \lim_n H(T_{\rho_n} u) = 0 \). Therefore there exists \( \sigma_{m+1} = \rho_{n_{m+1}} \in \{\rho_n\} \) such that

\[
H(T_{\sigma_{m+1}}u) < \epsilon_{m+1}H(u).
\]

Therefore we have

\[
H[(I + T_{\sigma_{m+1}})(I + T_{\sigma_m}) \cdots (I + T_{\sigma_1})e] = H[(I + T_{\sigma_{m+1}})u] \\
< (1 + \epsilon_{m+1})H(u) \\
\leq \prod_{k=1}^{m+1} (1 + \epsilon_k)H(e) \\
< (1 + \epsilon)H(e).
\]

This completes the proof of our claim.
Now let

\[ B_k := (I + T_{\sigma_1}^*)(I + T_{\sigma_2}^*) \cdots (I + T_{\sigma_k}^*)H \] and

\[ W_k := (2H - B_k)^+. \]

Since \( B_k \uparrow \), therefore \( W_k \downarrow \). Let \( W_k \downarrow W \).

Proof of \( W \neq 0 \)

We have

\[ W_k(e) > (2H - B_k)(e) \]

\[ = 2H(e) - B_k(e) \]

\[ > 2H(e) - (1 + \epsilon) H(e) \quad \text{(by equation (3.16))} \]

\[ = (1 - \epsilon) H(e) > 0. \quad \text{(since } H \in E_{++}^*) \]

Therefore \( W(e) > 0 \).

Proof of \( W \leq H \)

On \( \{B_k \geq 2H\} \), we have \( W_k = 0 \). On \( \{B_k \leq 2H\} \), we have

\[ W_k = 2H - B_k = H + (H - B_k) \leq H. \]

Since \( W_k \downarrow W \), in both cases it follows that \( W \leq H \).

Now we shall show that for any integer \( k > 0 \),

\[ W + T_{\sigma_k}^* W \leq G. \quad (3.17) \]

On \( \{W = 0\} \), we have

\[ W + T_{\sigma_k}^* W = T_{\sigma_k}^* W \leq T_{\sigma_k}^* H \leq T_{\sigma_k}^* G \leq G. \]
On \( \{ W > 0 \} \), we have \( W_k \geq W > 0 \) and hence \( W_k = 2H - B_k \).

But one has \( W \leq H \) and \( B_k \leq H \); therefore

\[
W + (I + T_{\sigma_k}^*)H \leq W + B_k \leq 2H,
\]

yielding

\[
W + T_{\sigma_k}^* W \leq W + T_{\sigma_k}^* H \leq H \leq G.
\]

This proves equation (3.17).

Now, we shall show for any integer \( m \leq n \), one has

\[
V_m := W + T_{\sigma_m}^* W + T_{\sigma_m}^* T_{\sigma_{m+1}}^* \cdots + T_{\sigma_{m-1}}^* \cdots T_{\sigma_n}^* W \leq G. 
\tag{3.18}
\]

Proof (by induction):

Fix \( n \). By equation (3.17), the result is true for \( m = n \). Assume the result for some integer \( m, 1 < m \leq n \). We shall show

\[
V_{m-1} = W + T_{\sigma_{m-1}}^* \cdots T_{\sigma_n}^* W \leq G.
\]

On \( W = 0 \) we have

\[
V_{m-1} = T_{\sigma_{m-1}}^* V_m \leq T_{\sigma_{m-1}}^* G \leq G.
\]

On \( W > 0 \) we have \( W_n > W > 0 \) and hence \( W_n + B_n = 2H \). Therefore

\[
W + (I + T_{\sigma_{m-1}}^*)(I + T_{\sigma_m}^*) \cdots (I + T_{\sigma_n}^*)H \leq 2H,
\]

which implies

\[
W + T_{\sigma_{m-1}}^* H + T_{\sigma_{m-1}}^* T_{\sigma_m}^* H + \cdots + T_{\sigma_{m-1}}^* \cdots T_{\sigma_n}^* H \leq H \leq G.
\]
But $W \leq H$; therefore we obtain

$$V_{m-1} = W + T^{*}_{\sigma_{m-1}} W + T^{*}_{\sigma_{m-1}} T^{*}_{\sigma_{m}} W + \ldots + T^{*}_{\sigma_{m-1}} \cdots T^{*}_{\sigma_{n}} W \leq H \leq G$$

Therefore the result of equation (3.18) follows by induction.

This result is true for any $n$; therefore we have

$$W + \sum_{k=m}^{\infty} T_{\sigma_{m}}^{*} \cdot T_{\sigma_{m+1}}^{*} \cdot T_{\sigma_{k}}^{*} W \leq G.$$ 

Thus we have a weakly wandering element $W$ in $E^{*}$. Hence the proof of theorem 3.4.8.

Now we prove the remaining equivalences of our main theorem.

Proof of theorem 3.1.3:

Now we prove the conditions (ii), (iv) and (v) in theorem 3.1.3 are equivalent. Proof of (ii) $\Rightarrow$ (iv) is immediate.

Proof of (ii) $\Rightarrow$ (v)

Let $\phi \in LIM \neq \emptyset$. We have

$$M[H(T_{\sigma}e)] \geq \phi[H(T_{\sigma}e)] \geq \inf_{\bar{\sigma}} H(T_{\sigma}e)$$

and hence the result follows.

Proof of (iv) $\Rightarrow$ (ii)

Assume condition (iv) of the theorem. Suppose $\inf_{\sigma} F(T_{\sigma}e) = 0$, for some $F \in E_{++}^{*}$.

By condition (iv) we have $\lim sup_{n} F(A_{\psi_{n}}e) > 0$ and therefore by part(a) of lemma 3.4.7, there exists $H$ and $G$ in $E_{++}^{*}$ and $\sigma_{F}$ in $\Sigma$ such that

$$H \leq G, \quad H \leq T_{\sigma_{F}}^{*} F \quad \text{and} \quad T_{\sigma}^{*} G = G, \quad \forall \sigma \in \Sigma.$$
Since \( \inf_{\sigma} \in \Sigma F(T_{\sigma}f) = 0 \), it follows that there exists a sequence \( \{\sigma_n\} \) in \( \Sigma \) such that

\[
\lim_{n} F(T_{\sigma_n}e) = 0.
\]

Applying the lemma 3.3.1 with \( H = F \) and \( u = T_{\sigma_F}e \), we obtain

\[
\lim_{n} F(T_{\sigma_n}T_{\sigma_F}e) = 0.
\]

By using commutativity assumption, we get

\[
\lim_{n} F(T_{\sigma_F}T_{\sigma_n}e) = 0.
\]

Since \( H \leq T_{\sigma_F}F \), it follows that

\[
0 = \lim_{n} F(T_{\sigma_F}T_{\sigma_n}e) = \lim_{n} (T_{\sigma_F}^{*}F)(T_{\sigma_n}e) \geq \lim_{n} H(T_{\sigma_n}e) \geq 0
\]

and hence

\[
\lim_{n} H(T_{\sigma_n}e) = 0.
\]

Since \( H \leq G \) and \( T_{\sigma}G = G \) for all \( \sigma \in \Sigma \), by theorem 3.4.8, the above equation implies the existence of a \( m \)-weakly wandering set in \( E^{*} \). This is a contradiction, since by part(a) of theorem 3.4.6, \( \lim \sup_{n} F(A_{\psi_n}F(e)) > 0 \) implies that there is no weakly wandering set; this completes the proof of \((iv) \Rightarrow (ii)\).

Proofs of \((v) \Rightarrow (ii)\)

The proof is similar to the proof of \((iv) \Rightarrow (ii)\) given above.
CHAPTER IV

ON EXISTENCE OF INVARIANT ELEMENTS

In this chapter we give certain sufficient conditions for existence of invariant elements which are not necessarily weak units.

The results of this chapter generalize part of the results obtained by A. Millet and L. Sucheston [17] in the case operators generated by finitely many positive, mean-bounded commuting operators $T_1, T_2, \ldots T_d$ to the case of positive uniformly norm bounded operators $T_\sigma$ representing a countably generated left amenable semigroup.

In the latter part of this chapter, as in [17], with the additional commutativity assumption, we obtain a decomposition of the Banach lattice.

Let $E$ be a Banach lattice satisfying conditions (A) and (B). Let $\Sigma$ be a countably generated left amenable semigroup. For the operators on $E$, we consider a positive linear operator representation $\{T_\sigma : \sigma \in \Sigma\}$ of $\Sigma$ with $\sup\{|T_\sigma| : \sigma \in \Sigma\} = K < \infty$.

Let $\psi_n$ be a sequence of finite means on $l_\infty(\Sigma)$ converging in norm to left invariance. The operator averages $A_{\psi_n}$ are defined as described in section 3.1.

4.1 Results on Existence of Invariant Elements

First consider the following proposition on the existence of a subinvariant element.

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Proposition 4.1.1 Let \( f \in E_{++} \) and \( \eta \in E_{++} \) be such that \( \{P_n A_{\psi_n} f\} \) is not TL null. Then there exists a subsequence \( \{A_{\psi_n}\} \) of \( \{A_{\psi_n}\} \) such that \( \text{WTL} (P_\eta A_{\psi_n, k} f) = \delta_0 \), for some \( \delta_0 \in E_{++} \) and there exists a \( \delta \in E_{++} \) such that \( \delta \geq \delta_0 \) and \( T_{\sigma} \delta \leq \delta \) for any \( \sigma \in \Sigma \).

Proof:

Since we have \( ||P_\eta A_{\psi_n} f|| \leq ||A_{\psi_n}|| \cdot ||f|| \leq K ||f|| < \infty \) and \( \{P_n A_{\psi_n} f\} \) is not TL null, by proposition 2.2.2 there exists a subsequence \( \{A_{\psi_n}\} \) such that for some \( \delta_0 \in E_{++} \), we have

\[
\text{WTL} (P_\eta A_{\psi_n, k} f) = \delta_0.
\]

But

\[
\text{TL} (P_\eta A_{\psi_n, k} f) \neq 0 \Rightarrow \text{TL} (A_{\psi_n, k} f) \neq 0.
\]

Therefore, by proposition 2.2.2, for a further subsequence of \( \{A_{\psi_n}\} \), still denoted \( \{A_{\psi_n}\} \), there exists \( \delta \in E_{++} \) such that

\[
\text{WTL} (P_\eta A_{\psi_n, k} f) = \delta_0 \quad \text{and} \quad \text{WTL} (A_{\psi_n, k} f) = \delta.
\]

Clearly \( \delta_0 \leq \delta \).

Now, let \( \sigma \in \Sigma \) be fixed. We have

\[
||T_{\sigma} A_{\psi_n} f|| \leq ||T_{\sigma}|| \cdot ||A_{\psi_n}|| \cdot ||f|| \leq ||T_{\sigma}|| \cdot K \cdot ||f|| < \infty,
\]

Therefore by proposition 2.2.2, for a further subsequence of \( \{A_{\psi_n}\} \), still denoted \( \{A_{\psi_n}\} \), there exists \( \eta \) such that

\[
\text{WTL} (T_{\sigma} A_{\psi_n, k} f) = \eta \quad \text{and}
\]
\[ WTL (A_{\psi_{n_k}} f) = \delta. \]

By Fatou property of \( WTL \) (proposition 2.2.3) we have,
\[ T_\sigma \delta = T_\sigma (WTL A_{\psi_{n_k}} f) \leq WTL (T_\sigma A_{\psi_{n_k}} f) = \eta. \]

To prove the theorem, it remains to show \( \eta = \delta. \)

Since \( WTL (A_{\psi_{n_k}} f) = \eta \) and \( WTL (T_\sigma A_{\psi_{n_k}} f) = \eta \), we can write
\[ w\lim (A_{\psi_{n_k}} f \wedge mu) = \delta_m \uparrow \delta \quad \text{and} \]
\[ w\lim (T_\sigma A_{\psi_{n_k}} f \wedge mu) = \eta_m \uparrow \eta. \]

Let \( k \) be fixed. For any \( F \in E_+^* \) we have
\[
|F(T_\sigma A_{\psi_{n_k}} f \wedge mu) - F(A_{\psi_{n_k}} f \wedge mu)|
\]
\[
= |F(T_\sigma A_{\psi_{n_k}} f \wedge mu - A_{\psi_{n_k}} f \wedge mu)|
\]
\[
\leq ||F|| \cdot ||(T_\sigma A_{\psi_{n_k}} f \wedge mu - A_{\psi_{n_k}} f \wedge mu)||
\]
\[
\leq ||F|| \cdot ||T_\sigma A_{\psi_{n_k}} f - A_{\psi_{n_k}} f|| \wedge mu)||
\]
\[
\leq ||F|| \cdot ||T_\sigma A_{\psi_{n_k}} f - A_{\psi_{n_k}} f||
\]
\[
\to 0, \text{ as } n_k \to \infty. \]

The convergence above follows, since we have \( ||T_\sigma A_{\psi_{n_k}} f - A_{\psi_{n_k}} f|| \to 0 \), by proposition 3.1.2. Thus we have, \( F(\delta_m) = F(\eta_m) \). Since this holds for any arbitrary \( F \in E_+^* \), hence \( \delta_m = \eta_m \). This result being true for any \( m \) we obtain \( \delta = \eta \), completing the proof of the proposition.

**Theorem 4.1.2** Let \( N \) be an order continuous seminorm satisfying the following conditions:
(i) The Banach lattice norm is continuous with respect to $N$ on $E_+$. 

(ii) $N$ satisfies condition (C). 

(iii) For every $\sigma$, the operator $T_\sigma$ contracts the seminorm $N$, i.e., $N(T_\sigma f) \leq N(f)$ for every $f$ in $E_+$. 

Let $f \in E_{++}$ and $\eta \in E_{++}$ be such that $\{P_\eta A_{\psi_n}f\}$ is not TL null. Then there exists a subsequence $\{A_{\psi_{n_k}}f\}$ of $\{A_{\psi_n}f\}$ such that $WTL (P_\eta A_{\psi_{n_k}}f) = \delta_0$ for some $\delta_0 \in E_{++}$ and there is a $\delta \in E_{++}$ such that $\delta \geq \delta_0$ and $T_\sigma \delta = \delta$ for every $\sigma \in \Sigma$. 

Proof: We have $\sup_n ||A_{\psi_n}|| \leq K < \infty$. Assume, without loss of generality, $||f|| \leq 1/K$. 

By proposition 4.1.1, there exists a subsequence $\{A_{\psi_{n_k}}f\}$ of $\{A_{\psi_n}f\}$ such that $WTL (P_\eta A_{\psi_{n_k}}f) = \delta_0 \in E_{++}$ and there is a $\delta \geq \delta_0$ such that $T_\sigma \delta \leq \delta$, $\forall \sigma \in \Sigma$. 

Suppose $T_\tau \delta = \delta$ does not hold for some $\tau$ in $\Sigma$. Then 

$$\delta' = \delta - T_\tau \delta > 0$$

and by condition (i), we obtain $N(\delta') > 0$. 

Since $WTL (A_{\psi_{n_k}}f) = \delta$, by proposition 2.2.4, we have $WTL_{\delta}(A_{\psi_{n_k}}f) = \delta$. Therefore, we have $\eta_m, m = 1, 2, \ldots$, in $E_+$ such that 

$\text{wlim}_{n_k} (A_{\psi_{n_k}}f \wedge m\delta) = \eta_m \uparrow \delta$. 

By order continuity of seminorm $N$, it follows that $N(\delta - \eta_m) \to 0$. Therefore, we can find a large enough $m$ so that 

$$N(\delta - \eta_m) < N(\delta')/4.$$
Let $m$ be fixed. Define

\[ r_k = A_{\psi_{n_k}} f \wedge m\delta, \]
\[ s_k = A_{\psi_{n_k}} f - r_k, \]
\[ r'_k = (T_{\tau}A_{\psi_{n_k}} f) \wedge m\delta \quad \text{and} \]
\[ s'_k = T_{\tau}A_{\psi_{n_k}} f - r'_k. \]

Since, by proposition 3.1.2, $||A_{\psi_{n_k}} f - T_{\tau}A_{\psi_{n_k}} f|| \rightarrow 0$ we obtain

\[ ||r_k - r'_k|| \rightarrow 0 \quad \text{and} \quad ||s_k - s'_k|| \rightarrow 0. \]

Since $r_k \xrightarrow{w} \eta_m$, it follows that $r'_k \xrightarrow{w} \eta_m$ and $T_{\tau}r_k \xrightarrow{w} T_{\tau}\eta_m$.

From the definitions of $s_k$ and $s'_k$ we have,

\[ r'_k + s'_k = T_{\tau}A_{\psi_{n_k}} f = T_{\tau}r_k + T_{\tau}s_k. \]

Also since $T_{\tau}\delta \leq \delta$, from the definition of $r'_k$ it follows that $0 \leq T_{\tau}r_k \leq r'_k$. Therefore we have

\[ 0 \leq r'_k - T_{\tau}r_k \xrightarrow{w} \eta_m - T_{\tau}\eta_m. \]

Since $\eta_m - T_{\tau}\eta_m \geq 0$, we have

\[ (\eta_m - T_{\tau}\eta_m) + (\delta - \eta_m) + (T_{\tau}(\delta - \eta_m)) \]
\[ \geq (\eta_m - T_{\tau}\eta_m) + (\delta - \eta_m) - (T_{\tau}(\delta - \eta_m)) \]
\[ = \delta - T_{\tau}\delta \]
\[ \geq \delta'. \]
giving
\[ N(\eta_m - T_r\eta_m) + N(\delta - \eta_m) + N(T_r(\delta - \eta_m)) \geq N(\delta') \]

But \( T_r \) contracts seminorm \( N \); therefore
\[ N(\eta_m - T_r\eta_m) + 2N(\delta - \eta_m) \geq N(\delta') \]

and thus we get,
\[ N(\eta_m - T_r\eta_m) \geq N(\delta') - 2N(\delta - \eta_m) > N(\delta') - 2N(\delta')/4 = N(\delta')/2 > 0. \]

Now we shall show \( \lim \inf \limits_k N(r'_k - T_r r_k) > \epsilon \) for some \( \epsilon > 0 \). Suppose not; then we have
\[ \lim \inf \limits_k N(r'_k - T_r r_k) = 0. \]

Therefore for a subsequence of the sequence \( \{r'_k - T_r r_k\} \), still denoted the same way as the original sequence, we have
\[ \lim \limits_k N(r'_k - T_r r_k) = 0 \]

Therefore, by condition (i), it follows that \( \| (r'_k - T_r r_k) \| \to 0 \) for the subsequence, implying \( \eta_m - T_r \eta_m = 0 \), which is a contradiction. Therefore for some \( \epsilon > 0 \) we have,
\[ \lim \inf \limits_k N(r'_k - T_r r_k) > \epsilon > 0 \]

Now we choose a large enough \( k_0 \) so that \( N(r'_k - T_r r_k) > \epsilon/2 \) for all \( k > k_0 \). Choosing \( \beta = \beta(m\delta, \epsilon/2) \), we get from condition (C),
\[ N(T_r s_k) = N(s'_k + (r'_k - T_r r_k)) > N(s'_k) + \beta \text{ for all } k > k_0. \]
But, by condition (iii) we have

\[ N(T_\sigma s_k) \leq N(s_k). \]

Therefore

\[ N(s_k) > N(s_k') + \beta \text{ for all } k > k_0, \]

which implies

\[ N(|s_k - s_k'|) \geq |N(s_k) - N(s_k')| > \beta \text{ for all } k > k_0. \]

This is a contradiction, since \(|s_k - s_k'| \rightarrow 0\) implies \(N(|s_k - s_k'|) \rightarrow 0\); thus we have \(T_\sigma \delta = \delta\) for any \(\sigma\) in \(\Sigma\).

Corollary 4.1.3 Let \(f \in E_{++}\) be such that \(\{A_{\psi_n} f\}\) is not TL null. Assume that the Banach lattice norm satisfies condition (C) and the operators \(T_\sigma\) are contractions for every \(\sigma \in \Sigma\). Then there exists a \(\delta \in E_{++}\) invariant under \(\Sigma\).

Proof: The result follows by taking \(N(\cdot) = ||\cdot||\) in the above theorem.

Corollary 4.1.4 Let \(f \in E_{++}\) be such that \(\{A_{\psi_n} f\}\) is not TL null. If there is an \(H \in E_{++}^*\) such that \(T_\sigma^* H \leq H\) for any \(\sigma \in \Sigma\) and on \(E_+\) the Banach lattice norm is continuous with respect to the seminorm defined by \(H\), then there exists a \(\delta \in E_{++}\) invariant under \(\Sigma\).

In particular, if there exists a strictly positive element \(U \in E_{++}^*\) which is \(T_\sigma^*\) subinvariant for all \(\sigma \in \Sigma\), then \(\delta\) is invariant under \(Z\).

Proof: Let \(N(f) = H(f)\) in the above theorem. Since \(N\) is a linear functional, the condition (C) is automatically satisfied; the other conditions follow from the assumptions.
In the particular case of strictly positive element \( U \in E^*_{++} \), by lemma 2.2.8 it follows that on \( E^+ \) the Banach lattice norm is continuous with respect to the seminorm defined by \( U \).

By standard measure-theoretic argument, we now show that there exists an invariant element \( \delta \) with the maximal support.

**Lemma 4.1.5** There exists a \( \delta \in E^+ \), called maximal invariant element, such that

(i) \( T_\sigma \delta = \delta \) for each \( \sigma \in \Sigma \).

(ii) If \( \gamma \in E^+ \) is such that \( T_\sigma \gamma = \gamma \) for each \( \sigma \in \Sigma \), then \( P_\delta \gamma = \gamma \).

(iii) For every \( f \in E \), \( TL (I - P_\delta) A_\psi f = 0 \).

**Proof:**

Since \( E \) is an order continuous Banach lattice with weak unit, \( E \) has a Köthe function space representation over a probability space \( (\Omega, \mathcal{F}, \mu) \).

Let \( C \) be the class of functions in \( E^+ \) invariant under \( T_\sigma \) for each \( \sigma \in \Sigma \). Set

\[
\alpha = \sup \{ \mu(f > 0), f \in C \}
\]

Let \( \{ f_k \} \) be a sequence in \( C \) such that \( ||f|| = 1 \), and \( \mu(f_k > 0) \to \alpha \). Set

\[
\delta = \sum_{k=1}^{\infty} 2^{-k} f_k.
\]

Consider any \( \sigma \in \Sigma \). Since \( f_k \in C \) for \( k = 1, 2, \ldots \) we have,

\[
T_\sigma (\sum_{k=1}^{m} 2^{-k} f_k) = \sum_{k=1}^{m} 2^{-k} f_k.
\]

Therefore, by Lebesgue monotone convergence theorem, we get

\[
T_\sigma (\sum_{k=1}^{\infty} 2^{-k} f_k) = \sum_{k=1}^{\infty} 2^{-k} f_k.
\]
and thus we have

\[ T_\sigma \delta = \delta. \]

Hence (i) holds.

Let \( \gamma \in C \), then \( \delta + \gamma \in C \) and therefore we have \( \mu \{ \delta > 0 \} = \alpha \geq \mu \{ \delta + \gamma > 0 \} \).

But \( \mu \{ \delta > 0 \} \leq \mu \{ \delta + \gamma > 0 \} \); therefore \( \mu \{ \delta > 0 \} = \mu \{ \delta + \gamma > 0 \} \). Thus we obtain

\[ P_\delta \gamma = \gamma. \]

To prove (iii), suppose \( TL (I - P_\delta)A_{\psi_n}f = 0 \) fails for some \( f \in E_+ \). Let \( P_\eta = I - P_\delta \); then \( TL (P_\eta A_{\psi_n}f) = 0 \) fails. Therefore by theorem 4.1.2, there exists a subsequence \( \{ A_{\psi_{n_k}} \} \) of \( \{ A_{\psi_n} \} \) and \( \gamma_0 \in E \) such that

\[ WTL (P_\eta A_{\psi_{n_k}}f) = \gamma_0 \neq 0 \quad (4.1) \]

and \( \gamma \geq \gamma_0 \) such that \( \gamma \in C \).

Since \( \gamma \in C \), therefore by condition (ii), we get

\[ P_\delta \gamma = \gamma. \]

Since \( \gamma \geq \gamma_0 \), it follows that \( P_\delta \gamma_0 = \gamma_0 \), which implies

\[ P_\eta \gamma_0 = 0. \]

But from equation 4.1 it follows

\[ P_\eta \gamma_0 = \gamma_0, \]

giving \( \gamma_0 = 0 \), a contradiction. Therefore \( TL (I - P_\delta)A_{\psi_n}f = 0 \) holds for any \( f \in E_+ \) and hence for any \( f \in E \).
4.2 Results on Decompositions of Banach Lattice.

Lemma 4.2.1 Let $N$ be an order continuous seminorm satisfying the following conditions:

(i) The Banach lattice norm is continuous with respect to $N$ on $E_+$.  
(ii) $N$ satisfies condition $(C_1)$.  
(iii) For every $\sigma$, the operator $T_\sigma$ contracts the seminorm $N$, i.e., $N(T_\sigma f) \leq N(f)$ for every $f$ in $E_+$.  

Let $H \in E_+^*$ be such that $T_\sigma^* H \leq H$ for each $\sigma \in \Sigma$ and such that $TL A_{\psi_n} g = 0$ if $H(|g|) = 0$. Then for every element $\delta \in E_+$ invariant under each $T_\sigma$, we have $P_H \delta = \delta$, i.e., $\{\delta > 0\} \subseteq \{H > 0\}$.

Proof: Let $\delta \in E_+$ be such that $T_\sigma \delta = \delta$ for each $\sigma \in \Sigma$.  

Set

$$f = P_H \delta \text{ and } g = (I - P_H) \delta.$$  

Then $\delta = f + g$ and

$$\delta = A_{\psi_n} \delta = A_{\psi_n} f + A_{\psi_n} g. \quad (4.2)$$  

Also, since $H(|g|) = H((I - P_H) \delta) = 0$, by the assumption it follows that

$$TL A_{\psi_n} g = 0.$$  

Furthermore, since $g \leq \delta$ it follows that $0 \leq A_{\psi_n} g \leq \delta$, and therefore we have by lemma 2.2.8, $\lim A_{\psi_n} g = 0$ and therefore by order continuity of $N$ we have

$$\lim N(A_{\psi_n} g) = 0. \quad (4.3)$$
Using condition (iii), from equation (4.2) we get

\[ N(\delta) = N(A_{\psi_n} f + A_{\psi_n} g) \]
\[ \leq N(A_{\psi_n} f) + N(A_{\psi_n} g) \]
\[ \leq N(f) + N(A_{\psi_n} g). \]

Therefore \( N(\delta) \leq N(f) + N(A_{\psi_n} g) \) and by taking the limit as \( n \to \infty \), we get from equation (4.3)

\[ N(\delta) \leq N(f). \]

Suppose \( N(g) > 0 \); then by the assumption (ii), we get \( N(\delta) = N(f + g) > N(f) \), contradicting \( N(\delta) \leq N(f) \). Therefore we have \( N(g) = 0 \), yielding \( g = 0 \) by condition (i). Since \( (I - P_H)\delta = g = 0 \), we have \( P_H \delta = \delta \), i.e., \( \{\delta > 0\} \subseteq \{H > 0\} \).

**Lemma 4.2.2** Assume \( \Sigma \) is abelian. Then there exists an \( G \in E_+^* \) such that

(i) \( T^* G = G \) for any \( \sigma \in \Sigma \).

(ii) For each \( f \in E \) with \( G(|f|) = 0 \), \( T L A_{\psi_n} f = 0 \).

Proof:

Let \( U \) be a strictly positive element in \( E_+^* \). Consider \( A_{\psi_n} U \). Since

\[ ||A_{\psi_n} U|| \leq K \cdot ||U|| < \infty, \]

by Alaoglu's theorem \( \{A_{\psi_n} U\} \) has a subsequence \( \{A_{\psi_{n_k}} U\} \) such that \( \{A_{\psi_{n_k}} U\} \) is weak* convergent to some element \( H \in E_+^* \).

By using the similar argument as in the proof of part (a) of lemma 3.4.7, we can show

\[ T^* \sigma H = H, \text{ for each } \sigma \in \Sigma. \]
Now, let $G^*$ be the set of all $H \in E_+^*$, invariant under each $T_\alpha$. Consider the Köthe space representation of $E$ over a probability space $(\Omega, \mathcal{F}, \mu)$. Let

$$\alpha = \sup\{\mu(H > 0); \, H \in G^*\}$$

First consider the case when $\alpha \neq 0$.

Choose a sequence $H_k \in G^*$ such that $\lim \mu(H_k > 0) = \alpha$. Set

$$G = \sum_{k=1}^{\infty} 2^{-k}||H_k||^{-1} H_k \in E_+^*.$$  

Then $G \in G^*$ and $G$ has the maximal support.

To prove (ii), suppose $f \in E_+$ be such that $G(f) = 0$, and $TL A_{\psi_n}f \neq 0$. Then, by lemma 2.2.8,

$$\lim_n U(A_{\psi_n}f) \neq 0.$$  

Therefore there exists a subsequence of the sequence $\{A_{\psi_n}f\}$, still denoted as $\{A_{\psi_n}f\}$, such that for some $\gamma$,

$$\lim U(A_{\psi_n}f) = \gamma > 0.$$  

Since $\|A_{\psi_n}^* U\| \leq K \cdot ||U|| < \infty$, by Alaoglu's theorem the sequence $\{A_{\psi_n}^* U\}$ has a further subsequence, still denoted $\{A_{\psi_n}^* U\}$, such that for some $H_0$ in $E_+^*$ one has

$$\text{weak}^* \lim A_{\psi_n}^* U = H_0.$$  

Also by the same arguments as in the beginning of this proof it follows that $H_0 \in G^*$. Since $G(f) = 0$ it follows that $H_0(f) = 0$. But we have

$$H_0(f) = \lim[A_{\psi_n}^* U](f) = \lim U(A_{\psi_n}^* f) = \lim U(A_{\psi_n} f) = \gamma > 0,$$
which is a contradiction.

In the case when \( \alpha = 0 \) it follows that the equation \( TL A_{\psi_n}f = 0 \) holds for every \( f \in E \). Therefore the lemma holds trivially with \( G = 0 \); this completes the proof of the lemma.

**Theorem 4.2.3** Assume \( \Sigma \) is abelian. Let \( N \) be an order continuous seminorm satisfying the following conditions:

(i) The Banach lattice norm is continuous with respect to \( N \) on \( E_+ \).

(ii) \( N \) satisfies condition \( (C) \).

(iii) For every \( \sigma \), the operator \( T_\sigma \) contracts the seminorm \( N \), i.e., \( N(T_\sigma f) \leq N(f) \) for every \( f \) in \( E_+ \).

If \( E \) is represented as a function space over \( (\Omega, \mathcal{F}, \mu) \), then \( \Omega \) admits a decomposition \( \Omega = Y + Z = P + D + Z \) such that

(a) There exists \( H \in E_+^* \) with \( T_\sigma^* H = H \) for each \( \sigma \in \Sigma \), and \( Y = \{ H > 0 \} \).

(b) There exists \( \delta \in E_+ \) with \( T_\sigma \delta = \delta \) for each \( \sigma \in \Sigma \) and \( P = \{ \delta > 0 \} \).

(c) For \( f \in E_+ \) with \( \{ f > 0 \} \subseteq Z \), we have \( TL A_{\psi_n}f = 0 \)

(d) For \( f \in E_+ \) with \( \{ f > 0 \} \subseteq Y \), we have \( TL (1_{D+Z} A_{\psi_n}f) = 0 \)

**Proof:**

By lemma 4.1.5 there exists a maximal element \( \delta \in E_+ \) invariant under \( T_\sigma \) for each \( \sigma \in \Sigma \) such that for every \( f \in E \) one has \( TL (I - P_{\delta})A_{\psi_n}f = 0 \). Set \( P = \{ \delta > 0 \} \); then the results of part(b) follows and for every \( f \in E \) one has \( TL (1_{D+Z} A_{\psi_n}f) = 0 \).

By lemma 4.2.2, there exists \( H \in E_+^* \) such that \( T_\sigma^* H = H \) for any \( \sigma \in \Sigma \) such that for every \( f \in E \) with \( H(|f|) = 0 \), one has \( TL A_{\psi_n}f = 0 \). Set \( Y = \{ H > 0 \} \); Then the
results of part(a) and part(c) follow. This completes the proof of the theorem.

**Corollary 4.2.4** Assume $\Sigma$ is abelian. Assume that the Banach lattice norm satisfies condition (C) and the operators $T_\sigma$ are contractions for every $\sigma \in \Sigma$.

If $E$ is represented as a function space over $(\Omega, \mathcal{F}, \mu)$, then $\Omega$ admits a decomposition $\Omega = Y + Z = P + D + Z$ such that

(a) There exists $H \in E_+^*$ with $T_\sigma^* H = H$ for each $\sigma \in \Sigma$, and $Y = \{H > 0\}$.

(b) There exists $\delta \in E_+$ with $T_\sigma \delta = \delta$ for each $\sigma \in \Sigma$ and $P = \{\delta > 0\}$.

(c) For $f \in E_+$ with $\{f > 0\} \subseteq Z$, we have $TL A_{\psi_n} f = 0$

(d) For $f \in E_+$ with $\{f > 0\} \subseteq Y$, we have $TL (1_{D+Z} A_{\psi_n} f) = 0$

Proof: The result follows by taking $N(\cdot) = || \cdot ||$ in the above theorem.

**Corollary 4.2.5** Assume $\Sigma$ is abelian. If there is an $H \in E_{++}^*$ such that $T_\sigma^* H \leq H$ for any $\sigma \in \Sigma$ and on $E_+$ the Banach lattice norm is continuous with respect to the seminorm defined by $H$.

If $E$ is represented as a function space over $(\Omega, \mathcal{F}, \mu)$, then $\Omega$ admits a decomposition $\Omega = Y + Z = P + D + Z$ such that

(a) There exists $H \in E_+^*$ with $T_\sigma^* H = H$ for each $\sigma \in \Sigma$, and $Y = \{H > 0\}$.

(b) There exists $\delta \in E_+$ with $T_\sigma \delta = \delta$ for each $\sigma \in \Sigma$ and $P = \{\delta > 0\}$.

(c) For $f \in E_+$ with $\{f > 0\} \subseteq Z$, we have $TL A_{\psi_n} f = 0$

(d) For $f \in E_+$ with $\{f > 0\} \subseteq Y$, we have $TL (1_{D+Z} A_{\psi_n} f) = 0$

In particular, if there exists a strictly positive element $U \in E_{++}^*$ which is $T_\sigma^*$ subinvariant for all $\sigma \in \Sigma$, then the above result holds.
Proof: Let \( N(f) = H(f) \) in the above theorem. Since \( N \) is a linear functional, the condition (C) is automatically satisfied; the other conditions follow from the assumptions.

In the particular case of strictly positive element \( U \in E^{**}_+ \), by lemma 2.2.8, it follows that on \( E_+ \) the Banach lattice norm is continuous with respect to the seminorm defined by \( U \). Hence the result follows.
BIBLIOGRAPHY


