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Array signal processing for polarized signals and signals with known waveforms

Li, Jian, Ph.D.
The Ohio State University, 1991
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ARRAY SIGNAL PROCESSING
FOR
POLARIZED SIGNALS
AND
SIGNALS WITH KNOWN WAVEFORMS

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University
by
Jian Li, B.S.E.E., M.S.

The Ohio State University
1991

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To my Parents
ACKNOWLEDGMENTS

I express my great appreciation to my advisor, Professor R. T. Compton, Jr. for his professional guidance, fatherly advice, encouragement, and support. I am grateful to Professor R. L. Moses for his help and advice throughout my graduate study. I thank Professor L. C. Potter for his comments and criticisms of this document. I also thank Professor A. K. Krishnamurthy for his support, and Professor Y. Bresler and Dr. J. Ward for their helpful discussions.

Finally I thank all wonderful teachers I have had throughout my education in China and in the US who, without their encouragement and care, I surely would have failed.
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CHAPTER I
INTRODUCTION

1.1 Background

Sensor array signal processing techniques are used in many fields including radar, sonar, communications, seismology, radio-astronomy, and ultrasonics. The objective of sensor array signal processing is to make efficient use of the information collected by a sensor array to detect the signals present in the field of view and to estimate their parameters.

An important problem in sensor array processing is the problem of estimating the directions of arrival (DOA) of multiple narrowband plane waves from the sensor array data. In this problem, the transmission medium is assumed isotropic and nondispersive, and the sources are assumed to be in the far-field of the array. When the incoming signals are narrowband, the signals received on different sensors differ only by a phase factor.

Many signal processing algorithms have been devised for this problem. These algorithms include optimal maximum likelihood (ML) approaches as well as suboptimal but computationally more efficient techniques.

Although ML estimates are optimal, direct maximization of the likelihood function, which is a highly nonlinear function of the signal parameters, involves a computationally expensive multidimensional search over parameter space [1, 2]. Many special ML algorithms have been devised to tackle this complicated optimization problem. Some well-known ones include Iterative Quadratic ML (IQML) [3, 4], sim-
ulated annealing [5], Alternating Projection (AP) [6], and the Estimate Maximize (EM) algorithm [7, 8].

The IQML algorithm [3] is designed for uniform linear arrays with equally spaced identical sensors. For such an array, the likelihood function may be expressed in terms of a linear prediction polynomial. The ML criterion may then be rearranged into a quadratic function, which the IQML algorithm solves iteratively to obtain the optimal estimates. The IQML algorithm avoids a multidimensional search, is computationally efficient, and converges in a small number of iterations.

Simulated annealing [5] refers to a class of numerical algorithms that imitate the slow cooling (annealing) of a substance to obtain a crystalline or glass-like state in choosing trial parameters for an optimization criterion. The current trial parameters depend on the previous trial parameters stochastically. Simulated annealing is a form of stochastic optimization. One of the most important features of simulated annealing is that the algorithm can escape from local stationary points of cost functions such as the likelihood function and proceed to find a globally optimal solution. Because of the imitation of the slow cooling process of a substance, simulated annealing algorithms have the disadvantage of being computationally expensive.

The AP algorithm [6] uses a simple “alternating maximization” idea for multidimensional maximization. At each iteration, the algorithm finds the maximum with respect to a single parameter while all the other parameters are held fixed. Thus, the multivariate nonlinear maximization problem is transformed into a sequence of one-dimensional maximization (or search) problems.

The EM algorithm [7, 8] decomposes the observed data into its individual signal components and then estimates the parameters of each signal separately. This algorithm is also iterative and involves a search over the parameter space for each signal.
Decomposing the observed data into separate signal components reduces the number of search dimensions and makes it possible to estimate the parameters of each signal in parallel. The use of the EM algorithm for DOA estimation was first described in [7], where the signals are assumed to be either deterministic signals with known waveforms and known gains, or zero-mean Gaussian processes with known covariance matrices. In [8], the EM algorithm is generalized to the case where the signals are deterministic with unknown waveforms. A special example involving signals with known waveforms but unknown complex gains is also considered in [8].

Since ML algorithms are, in general, computationally intensive, considerable attention has been paid to suboptimal but computationally more efficient algorithms. These include conventional beamforming techniques [9, p.17], Capon’s minimum variance method [10] [9, p.18], the linear prediction method [9, p.20], the maximum entropy method [11], and eigenstructure based algorithms [12, 13, 14, 9].

Beamforming techniques [9, p.17] use a weight vector to form a beam in a given look direction. The components of the weight vector consist of phase factors that steer the beam in the desired direction. The sum of the sensor outputs weighted by the phase factors forms the array output. The phase factors produce a maximum signal-to-noise ratio (SNR) at the array output for signals arriving from the look direction. The DOA estimates are taken to be those look directions corresponding to peaks in the array output power. However, the beamforming technique is known to have poor angular resolution.

Capon’s minimum variance method [10] [9, p.18] is similar to the beamforming technique. However, in Capon’s method, the weight vector is chosen to minimize the power at the array output while maintaining a constant gain in the look direction. The angle estimates are again obtained by scanning the look direction and deter-
mining the angles where the array output power has peaks. Capon’s method yields better resolution than the beamforming technique.

In the linear prediction method [9, p.20], one of the sensor outputs is predicted from a linear combination of the remaining sensor outputs. The linear prediction coefficients are obtained by minimizing the mean-square prediction error. In the maximum entropy method [11], on the other hand, a finite autocorrelation sequence is extrapolated in such a way that the entropy of the time series characterized by the extrapolated autocorrelation sequence is maximized. Both linear prediction and maximum entropy methods are used with uniform linear arrays. It can be shown that when the additive noise components at the sensor outputs are Gaussian random processes, the linear prediction method and the maximum entropy method result in the same system of linear equations [15].

Eigenstructure based algorithms [12, 13, 14, 9] utilize the eigenvectors of the array covariance matrix. Depending on the algorithm, the eigenvectors in either the signal subspace or in the noise subspace, which is orthogonal to the signal subspace, are used. These algorithms provide consistent (asymptotically accurate) DOA estimates. The most well-known eigenstructure based algorithms are the MUSIC (MUltiple Signal Classification) algorithm [12], which uses noise subspace eigenvectors, and the ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) algorithm [13], which uses signal subspace eigenvectors.

Compared with ML algorithms, the MUSIC algorithm is computationally more efficient. The MUSIC algorithm is not iterative but instead involves a search over parameter space. When each signal direction is specified by one spatial angle (e.g., the azimuth angle), the MUSIC algorithm requires a search over only one dimension, instead of a multidimensional search as in the ML algorithm. Nevertheless, this
search still requires extensive computations. Moreover, for two-dimensional angle estimation, when each signal direction is specified by two spatial angles (the azimuth and elevation angles), the MUSIC algorithm requires a two-dimensional search over parameter space. Such a search can be computationally prohibitive.

The ESPRIT algorithm dramatically reduces the amount of computation required, as compared to the MUSIC algorithm. In the ESPRIT algorithm, computational efficiency is achieved by requiring that the sensor array possess a displacement invariance: sensors must occur in matched pairs (in doublets) with identical displacement vectors. The ESPRIT algorithm exploits the invariance property of such an array. With ESPRIT, DOA estimates are computed directly, instead of being found with a search over parameter space.

The special structure of uniform linear arrays is also exploited by other eigenstructure based algorithms such as the KT (Kumaresan-Tufts) [14] method and the GEESE (GEneralized Eigenvalues utilizing Signal subspace Eigenvectors) [9] algorithm. In the KT method, the DOA estimates are found by solving a linear prediction polynomial whose coefficients are computed from the noise subspace eigenvectors of the array covariance matrix. The GEESE algorithm is similar to the ESPRIT algorithm but is used with uniform linear arrays where the displacement invariance is guaranteed automatically.

Other techniques may also be used with eigenstructure based algorithms to improve DOA estimates. For example, spatial filters may be used to increase the SNR of signals from the directions of interest [16]. Such spatial filters may be formed from discrete prolate spheroidal sequences through eigendecomposition [16].

Unlike ML algorithms, eigenstructure based algorithms are derived under the assumption that the arriving signals are uncorrelated or at most partially correlated.
The performance of these algorithms degrades rapidly as the incident signals become highly correlated. They fail to work at all when the signals are coherent (i.e., perfectly correlated).

Several approaches have been proposed to deal with the case of coherent signals. These techniques include forward-only and forward/backward spatial smoothing [17, 18, 19, 20, 21], both of which can be used with eigenstructure based algorithms. In this approach, the uniform linear array is partitioned into subarrays and the angle estimation is done by using the average of the covariance matrices for the subarrays. This procedure decorrelates the coherent signals, but at the cost of reducing the effective array aperture. Spatial smoothing techniques are known to provide consistent DOA estimates [19]. However, their performance deteriorates rapidly as the incident signals become closely-spaced [22, 23].

The Toeplitz approximation method (TAM) is another approach that has been proposed to deal with coherent signals [24]. In this approach, the array covariance matrix is approximated with a Toeplitz structure. Unlike spatial smoothing, TAM does not reduce the effective array aperture [24, 22]. Also, TAM can yield better resolution than spatial smoothing [22]. However, the DOA estimates obtained from TAM are biased [22]. TAM does not yield consistent DOA estimates due to the Toeplitz approximation.

The problem of DOA estimation is simplest for narrowband signals. However, techniques developed for narrowband signals may also be useful for wideband signals. For example, a wideband incident signal may be decomposed into several frequency bins by using the FFT (fast Fourier transform) [25]. Eigenstructure based techniques for narrowband signals may then be applied to each frequency bin. Angle estimates obtained from different frequency bins can be averaged to obtain an improved angle.
estimate [25]. Alternatively, narrowband signals in different frequency bins may be used *coherently* by forming a focused array covariance matrix, as described in [26]. The focused array covariance matrix is a composite of the array covariance matrices obtained from the different frequency bins [26, 27]. Improved angle estimates may then be obtained by using narrowband eigenstructure based techniques with the focused array covariance matrix [26, 27]. Yet another approach for wideband signals is to model each wideband signal as an AR (autoregressive) or ARMA (autoregressive moving average) process [28, 29]. Narrowband eigenstructure based techniques may then be used at each pole of the AR or ARMA model [28, 29].

### 1.2 Present Work

In this study, we investigate array signal processing techniques that address two new classes of angle estimation problems not handled by existing algorithms. First, we study techniques for incorporating known or partially known signal waveforms in the angle estimation process. Second, we consider angle and polarization estimation techniques for electromagnetic signals that arrive with unknown and arbitrary polarization.

Our first class of problems is motivated by the fact that algorithms are needed for estimating the arrival angles of signals incident on an array of sensors when one or all of the signals have known waveforms. For example, in some packet radio systems currently under study [30], a special acquisition preamble is added to the beginning of every packet [30] and an antenna array is used to receive the packets. This acquisition preamble consists of a known code, such as an integer number of periods of a pseudonoise (PN) code [31]. In unslotted packet radio systems, each packet arrives at the antenna array at an arbitrary time. For these systems, the
preamble is used to distinguish the newly arrived packet from other packets that arrive earlier or later. The newly arrived packet is the signal of interest and is the desired signal. Because of the acquisition code, the desired packet waveform is known, but not its exact amplitude or phase. Other packets that arrive around the same time are considered interfering signals. In unslotted packet radio systems, the waveforms of the interfering signals are best modeled as unknown. In slotted packet radio systems, however, all the incident signals have known waveforms. This difference is due to the timing of the packets. In slotted systems, all packets start at essentially the same time, at the beginning of a time slot. With the packets aligned in time, the presence of the known preambles means that each packet waveform is known (except for an unknown amplitude or phase). In either case, when only the desired signal waveform is known or when all the signal waveforms are known, these waveforms may be incorporated into the process of obtaining DOA estimates of the packets.

Few existing angle estimation techniques make any use of the signal waveforms. Techniques such as the IQML algorithm [3], the AP algorithm [6], the MUSIC algorithm [12] and the ESPRIT algorithm [13] are examples of what are called conditional and unconditional angle estimators by Stoica and Nehorai [32]. In these estimators, the incident signals are assumed to be either unknown deterministic signals or Gaussian random processes with unknown correlations. The EM (Estimate Maximize) algorithm presented by Feder and Weinstein [7] is an exception. The EM algorithm splits the search for the Maximum Likelihood (ML) estimate into a set of parallel searches. However, this algorithm is based on the assumption that both the signal waveforms and the signal gains are known exactly. If these gains are unknown, it cannot be used. Miller and Fuhrmann [8] also described a generalized EM algorithm
that is intended primarily for the case of unknown signal waveforms. They briefly consider signals with known waveforms and unknown gains, but they do not explore this problem fully.

In Chapter II, we consider maximum likelihood estimation of the arrival angle of a narrowband signal with a known waveform but an unknown amplitude or phase in the presence of interfering signals with unknown waveforms. We present a computationally efficient and rapidly converging algorithm that maximizes the likelihood function iteratively by searching over one-dimensional parameter space. We derive Cramer-Rao bounds for this estimator. We determine the conditions under which incorporating knowledge of the signal waveform in the estimator improves the accuracy of the angle estimates.

In Chapter III, we consider maximum likelihood estimation of the arrival angles of narrowband plane waves when all signal waveforms are known, except for the unknown amplitudes or phases. We present two computationally efficient and rapidly converging maximum likelihood algorithms that iteratively compute the maximum likelihood estimates. We also obtain Cramer-Rao bounds for these estimators. For this case we also describe the conditions under which incorporating the known signal waveforms in the estimators improves the accuracy of the angle estimates.

The second major thrust of this study is to consider methods for estimating the arrival angles and polarizations of electromagnetic signals with unknown and arbitrary polarizations. Specifically, we show how the ESPRIT algorithm may be used with polarization sensitive arrays to estimate both the directions and the polarizations of incoming narrowband plane waves. In Chapters IV and V, we consider angle and polarization estimation in one spatial coordinate. Chapter IV treats the case where the signals are uncorrelated. The techniques presented in Chapter IV can also
be used for partially correlated signals. Chapter V treats the case where they are completely correlated. In Chapter VI, we consider angle and polarization estimation in two spatial coordinates.

There appear to be no previous angle estimation techniques that address the problem of electromagnetic polarization of signals. However, in this work we show that if an array uses elements that respond to more than one polarization, the ESPRIT algorithm may be used with such an array to estimate both signal directions and signal polarizations.

Specifically, in Chapter IV we show how to use the ESPRIT algorithm with a uniform linear array of crossed dipole pairs to estimate both the signal directions in one spatial angle and the signal polarizations. Moreover, we also present an alternative approach that may be used with the same array to estimate arrival angles only. This alternative approach is useful when the signal polarizations are not of interest but it is necessary that the estimator work properly regardless of the signal polarizations. This method has the advantage that it requires far fewer computations than the first method. We compare the performance of these approaches with each other and with that of conventional ESPRIT estimators. The effects of signal direction and polarization on the estimator performance are investigated.

In Chapter IV, it is assumed that the incoming signals are uncorrelated and the techniques presented in the chapter can also be used for partially correlated signals. In Chapter V, we take up the same angle estimation problem when the signals are completely correlated (or coherent). We show how spatial smoothing techniques may be used with the same uniform linear array of crossed dipoles and the ESPRIT algorithm to estimate both arrival angles and polarizations for coherent signals. We also compare the performance of different spatial smoothing techniques.
In Chapter VI, we show how the ESPRIT algorithm may be used with a square array of crossed dipoles to estimate both the two-dimensional arrival angles and the polarizations of incoming signals. For this estimator we again determine the effects of signal direction and polarization on the estimator performance.

Finally, we summarize and conclude our study in Chapter VII.
CHAPTER II

MAXIMUM LIKELIHOOD ANGLE ESTIMATION FOR A SIGNAL WITH KNOWN WAVEFORM IN THE PRESENCE OF INTERFERING SIGNALS

2.1 Introduction

We start with the first major topic of this dissertation, angle estimation for signals with known or partially known waveforms. In this chapter, we consider the problem of estimating the arrival angle of a signal with a known waveform but an unknown gain when interfering signals with unknown waveforms are also present. In Chapter III, we shall consider the problem of estimating multiple signal arrival angles when all signals have known waveforms but unknown gains.

Specifically, in this chapter, we consider maximum likelihood angle estimation for a signal with a known waveform but an unknown gain. This signal is assumed to arrive in the presence of other interfering signals, which have unknown waveforms. We describe a computationally efficient maximum likelihood (ML) algorithm that may be used for this purpose. To avoid the need for a computationally expensive multidimensional search, which would be required to maximize the likelihood function directly when interference signals are present, we propose an iterative algorithm that combines the merits of the IQML (Iterative Quadratic ML) approach of Bresler and Macovski [4, 3] and the Alternating Maximization approach of Ziskind and Wax [6]. It transforms the multidimensional search problem into an iterative one-dimensional search problem. We compare the performance of this estimator with
that of an estimator that first uses the IQML estimator and then extracts the desired signal angle. We also present Cramer-Rao (CR) bounds for these estimators and compare the performance of the estimators to their CR-bounds.

The chapter is organized as follows. In Section 2.2, we formulate the problem. In Section 2.3, we describe the IQML approach, which does not utilize the known desired signal waveform, and the new ML approach that does. In Section 2.4, we present the CR-bounds for these two cases. In Section 2.5, we examine the performance of these algorithms. Finally, Section 2.6 contains our summary.

2.2 Formulation

Consider a uniform linear array with \( L \) isotropic sensors. The distance between two adjacent sensors is assumed to be a half wavelength at the signal frequency. Suppose \( K + 1 \) narrowband plane waves impinge on the array from angles \( \theta_k, k = 0, 1, \cdots, K \), relative to the array normal. The incident signals are assumed analytic signals. Assume that the signal from \( \theta_0 \) is the desired signal, whose waveform is

\[
s_0(t) = g w_0(t),
\]

where \( w_0(t) \) is a known waveform but the gain \( g \) may be unknown. The \( K \) remaining signals are interfering signals, with the number \( K \) assumed known. (If \( K \) is unknown, it may be estimated by the method described in [33].) The waveforms \( s_k(t) \) of the interfering signals from \( \theta_k, k = 1, 2, \cdots, K \), are assumed unknown. The desired signal is assumed to be uncorrelated with the interfering signals so that its waveform can be used to distinguish it from the interfering signals. However, the interfering signals may be correlated (or even perfectly correlated) with each other.
With \( K + 1 \) signals incident, the total signal \( x_l(t) \) received at the \( l \)th sensor is the sum of the \( K + 1 \) signals plus an additive noise component,

\[
x_l(t) = \sum_{k=0}^{K} s_k(t)e^{-j(l-1)\pi \sin \theta_k} + n_l(t),
\]

where \( n_l(t) \) is a zero-mean Gaussian noise process with variance \( \sigma^2 \). The \( n_l(t) \) are assumed independent of each other and the incident signals.

Let \( x(t) \), \( s(t) \), and \( n(t) \) be column vectors containing the received signals, incident signals, and noise respectively, i.e.,

\[
x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_L(t) \end{bmatrix}^T,
\]

\[
s(t) = \begin{bmatrix} s_0(t) & s_1(t) & \cdots & s_K(t) \end{bmatrix}^T,
\]

\[
n(t) = \begin{bmatrix} n_1(t) & n_2(t) & \cdots & n_L(t) \end{bmatrix}^T,
\]

where \((\cdot)^T\) denotes the transpose. The received signal vector has the form

\[
x(t) = A(\theta)s(t) + n(t),
\]

where \( A(\theta) \) (with \( \theta = \begin{bmatrix} \theta_0 & \theta_1 & \cdots & \theta_K \end{bmatrix}^T \)) is the direction matrix, whose columns are the direction vectors of the incident signals

\[
A(\theta) = \begin{bmatrix} a(\theta_0) & a(\theta_1) & \cdots & a(\theta_K) \end{bmatrix},
\]

with

\[
a(\theta_k) = \begin{bmatrix} 1 & q_k & \cdots & q_k^{L-1} \end{bmatrix}^T,
\]

and

\[
q_k = e^{-j\pi \sin \theta_k}.
\]
The phase factors $q_k$ are referred to as spatial phase factors.

The array output is sampled at $N$ distinct times $t_n$, $n = 1, 2, \cdots, N$. The random noise vectors $n(t_n)$ at different sample times are assumed independent of each other. The problem of interest is to determine the angle $\theta_0$ of the signal with known waveform from the measurements $x(t_n)$, $n = 1, 2, \cdots, N$. In the next section, we describe two approaches based on ML estimation that may be used to solve this problem.

2.3 Maximum Likelihood Angle Estimation Algorithms

One way of estimating the desired signal angle is to start with the IQML algorithm of Bresler and Macovski [3], a method originally discussed by Kumaresan, Scharf and Shaw [4]. This method estimates all the signal arrival angles, both desired and interference, but does not provide any way to distinguish the desired signal angle from the others. However, we can augment the IQML algorithm by using the known desired signal waveform to determine which angle corresponds to the desired signal. We consider this method in the first part below. The performance of this method is suboptimal because it does not incorporate knowledge of the desired signal waveform in the angle estimation process.

Then, in the second part, we present a new algorithm that properly incorporates the known desired signal waveform in the estimation process. This method, like the IQML method, is based on maximum likelihood estimation. We present an iterative procedure that converges in a few steps for solving the ML equation.

2.3.1 The IQML Algorithm with Signal Correlation

One method for estimating the desired signal angle, which we shall call Method II.A, is based on the IQML algorithm [3].
In this method, we start by using the IQML algorithm to estimate all the signal angles, desired and interference, without taking advantage of the known signal waveform. The maximum likelihood (ML) estimator for the case when all the incident signals (including $s_0(t)$) are unknown and the noise is Gaussian is derived in Appendix A. As shown in (A.7), the ML estimate of $\theta = [\theta_0 \ \theta_1 \ \theta_2 \ \cdots \ \theta_K]^T$ is given by the value of $\theta$ that minimizes

$$q = \text{tr} \left\{ P_{A^\perp(\theta)} \hat{R} \right\}, \quad (2.10)$$

where $P_{A^\perp(\theta)}$ is the projection operator onto the null space of $A(\theta)$,

$$P_{A^\perp(\theta)} = I - A(\theta) \left[ A^H(\theta)A(\theta) \right]^{-1} A^H(\theta), \quad (2.11)$$

and

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} x(t_n)x^H(t_n). \quad (2.12)$$

Minimizing $q$ directly would involve a computationally expensive $(K+1)$-dimensional search over the $\theta_k$. Instead, we take advantage of the IQML algorithm [3] to simplify the process.

In the IQML algorithm, the ML criterion is formulated in terms of a linear prediction (LP) polynomial $b(z) = b_0 z^{K+1} + b_1 z^K + \cdots + b_{K+1}$ [4, 3]. The $b(z)$ is the polynomial whose zeros are the $q_k$, $k = 0, 1, \cdots, K$, defined in (2.9). Given $b(z)$, $q$ in (2.10) can be expressed in terms of the coefficients of $b(z)$, as [3]

$$q = \frac{1}{N} b^H \left[ \sum_{n=1}^{N} X^H(t_n) \left( B^H B \right)^{-1} X(t_n) \right] b, \quad (2.13)$$

where $b = \begin{bmatrix} b_0 & b_1 & \cdots & b_{K+1} \end{bmatrix}^T$, $B$ is the $L \times (L - K - 1)$ Toeplitz matrix.
and $X(t_n)$ is the $(L - K - 1) \times (K + 2)$ Toeplitz matrix

$$X(t_n) = \begin{bmatrix} x_{K+2}(t_n) & x_{K+1}(t_n) & \cdots & x_1(t_n) \\ x_{K+3}(t_n) & x_{K+2}(t_n) & \cdots & x_2(t_n) \\ \vdots & \vdots & & \vdots \\ x_L(t_n) & x_{L-1}(t_n) & \cdots & x_{L-K-1}(t_n) \end{bmatrix}. \quad (2.15)$$

The IQML algorithm minimizes $q$ in (2.13) iteratively [4, 3]. This algorithm consists of the following steps:

1. Initialize: Let $i = 0$ and $b(0) = [1 \ 0 \ \cdots \ 0]^T$, where $b(0)$ is the initial value of $b(i)$, the $i$th estimate of $b$.

2. Compute

$$C(i) = \sum_{n=1}^{N} X^H(t_n) \left[ B(i)^H B(i) \right]^{-1} X(t_n). \quad (2.16)$$

where $B(i)$ is the matrix in (2.14) evaluated for the coefficients in $b(i)$.

3. Solve for $b(i+1)$ to minimize the quadratic form

$$b^H_{(i+1)} C(i) b_{(i+1)}. \quad (2.17)$$
4. Check convergence: If the Euclidean norm $\|b_{(i+1)} - b_{(i)}\| < \epsilon_1$, let $\hat{b} = b_{(i)}$ and continue; otherwise, let $i = i + 1$ and go to (2). ($\epsilon_1$ is a small constant chosen to give the desired precision.)

5. Find the roots $z_k$, $k = 0, 1, \cdots, K$, of $\hat{b}(z)$.

6. Compute the estimated signal angles $\tilde{\theta}_k$,
   \[
   \tilde{\theta}_k = -\sin^{-1}\left[\frac{\text{arg}(z_k)}{\pi}\right], \quad k = 0, 1, \cdots, K.
   \] (2.18)

This algorithm converges in a small number of iterations (fewer than 10 in the examples below). Like many other algorithms, it may converge to a local minimum instead of a global one.

In the IQML algorithm, constraints may be imposed on $\hat{b}$ to yield an appropriate $b$ for the problem at hand [3]. Of course, the first constraint is that $\hat{b}$ must be nontrivial, i.e., $\hat{b} \neq 0$. Also, for this problem the zeros $q_k$, $k = 0, 1, \cdots, K$, of $b(z)$ must lie on the unit circle, so we impose the conjugate symmetry constraint (see Appendix B)
   \[
   \hat{b}_k = \hat{b}_{K+1-k}^*, \quad k = 0, 1, \cdots, K + 1.
   \] (2.19)

This constraint guarantees that the zeros of $\hat{b}(z)$ are either on or at reciprocal radii inside and outside the unit circle [34]. The method of applying this constraint is discussed below. Finally, for those cases where the zeros of $\hat{b}(z)$ do not lie on the unit circle, we use the rule in (2.18) to evaluate $\tilde{\theta}_k$.

To satisfy the above constraints, Step 3 of the IQML algorithm is implemented as follows [3]:

- For $K$ even: Let
and let

\[ \mathbf{c}_{(i+1)} = \begin{bmatrix} \text{Re}(\mathbf{b}^T_{(i+1)}) & \text{Im}(\mathbf{b}^T_{(i+1)}) \end{bmatrix}^T. \]  

\[ \text{(c}_{(i+1)\text{ is a real vector.) In Step 3, we solve for the vector } \mathbf{c}_{(i+1)} \text{ such that } \begin{align*} \mathbf{c}^T_{(i+1)} \mathbf{C}_{(i)} \mathbf{c}_{(i+1)} &= \min, \\ \| \mathbf{c}_{(i+1)} \| &= 1, \end{align*} \]  

where

\[ \mathbf{C}_{(i)} = \sum_{n=1}^{N} \text{Re} \left\{ \mathbf{X}^H(t_n) \left[ \mathbf{B}_{(i)}^H \mathbf{B}_{(i)} \right]^{-1} \mathbf{X}(t_n) \right\}, \]  

\[ \mathbf{X}(t_n) = \begin{bmatrix} \mathbf{X}_1(t_n) + \mathbf{X}_2(t_n) \mathbf{J} & j \left[ \mathbf{X}_1(t_n) - \mathbf{X}_2(t_n) \mathbf{J} \right] \end{bmatrix}, \]  

where \( \mathbf{X}_1(t_n) \) and \( \mathbf{X}_2(t_n) \) are the submatrices of \( \mathbf{X}(t_n) \) consisting of the first and the last \( (K + 2)/2 \) columns of \( \mathbf{X}(t_n) \), respectively, and where \( \mathbf{J} \) is the exchange matrix

\[ \mathbf{J} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}. \]  

Then

\[ \mathbf{b}_{(i+1)} = \begin{bmatrix} \mathbf{b}^T_{(i+1)} & \mathbf{b}^H_{(i+1)} \mathbf{J} \end{bmatrix}^T. \]  

- For \( K \) odd: Let
\[ \mathbf{b}_{(i+1)} = [ b_0^{(i+1)} \ b_1^{(i+1)} \ \ldots \ b_{(K+1)/2}^{(i+1)} ]^T, \quad (2.27) \]

and

\[ \mathbf{c}_{(i+1)} = \begin{bmatrix} \text{Re}(\mathbf{b}_{(i+1)}^T) & b_{(K+1)/2}^{(i+1)} & \text{Im}(\mathbf{b}_{(i+1)}^T) \end{bmatrix}^T. \quad (2.28) \]

\((\mathbf{c}_{(i+1)}\text{ is real.})\) We solve for the \(\mathbf{c}_{(i+1)}\) such that

\[ \mathbf{c}_{(i+1)}^T \overline{\mathbf{C}}(i) \mathbf{c}_{(i+1)} = \min, \quad \|\mathbf{c}_{(i+1)}\| = 1, \quad (2.29) \]

with

\[ \overline{\mathbf{C}}(i) = \sum_{n=1}^{N} \text{Re} \left\{ \mathbf{X}^H(t_n) \left[ \mathbf{B}_{(i)}^H \mathbf{B}_{(i)} \right]^{-1} \mathbf{X}(t_n) \right\}, \quad (2.30) \]

where

\[ \mathbf{X}(t_n) = \begin{bmatrix} \mathbf{X}_1(t_n) + \mathbf{X}_2(t_n) \mathbf{J} & \mathbf{X}_3(t_n) & j[\mathbf{X}_1(t_n) - \mathbf{X}_2(t_n) \mathbf{J}] \end{bmatrix}, \quad (2.31) \]

\(\mathbf{X}_1(t_n)\) and \(\mathbf{X}_2(t_n)\) are defined as above, and \(\mathbf{X}_3(t_n)\) consists of the middle column of \(\mathbf{X}(t_n)\). Then

\[ \mathbf{b}_{(i+1)} = \begin{bmatrix} \mathbf{b}_{(i+1)}^T & b_{(K+1)/2}^{(i+1)} & \mathbf{b}_{(i+1)}^H \mathbf{J} \end{bmatrix}^T. \quad (2.32) \]

Note that the solution to either (2.22) or (2.29) is the normalized eigenvector that corresponds the smallest eigenvalue of the Hermitian matrix \(\overline{\mathbf{C}}(i)\).

After obtaining the ML estimates of \(\theta_0, \ldots, \theta_K\) from the IQML algorithm, we next use the known desired signal waveform to determine which angle in the set \(\{\hat{\theta}_0, \ldots, \hat{\theta}_K\}\) corresponds to the desired signal. The angle estimate obtained in this way is suboptimal, because the waveform is not utilized in forming the angle estimate. Nevertheless, this is one method of finding the desired signal angle.
To implement this method, we first estimate the waveform of each incident signal and then correlate these waveform estimates with the known desired signal waveform. The waveforms of the incident signals may be estimated from $\hat{\theta}_k$, $k = 0, 1, \cdots, K$, as follows [13]:

$$s(t_n) = \left[ A^H(\hat{\theta})A(\hat{\theta}) \right]^{-1} A^H(\hat{\theta})x(t_n), \quad n = 1, 2, \cdots, N,$$

(2.33)

where

$$A(\hat{\theta}) = \begin{bmatrix} a(\hat{\theta}_0) & a(\hat{\theta}_1) & \cdots & a(\hat{\theta}_K) \end{bmatrix},$$

(2.34)

$$\hat{s}(t_n) = \begin{bmatrix} \hat{s}_0(t_n) & \hat{s}_1(t_n) & \cdots & \hat{s}_K(t_n) \end{bmatrix}^T,$$

(2.35)

and where $\hat{s}_k(t_n)$ is the estimated value of $s_k(t_n)$.

Let $\hat{\varepsilon}_k$ be the cross-correlation between each $\hat{s}_k(t)$ and the desired signal waveform and be normalized by the power of $\hat{s}_k(t)$,

$$\hat{\varepsilon}_k = \frac{\sum_{n=1}^{N} w^*_n(t_n)\hat{s}_k(t_n)^2}{\sum_{n=1}^{N} |\hat{s}_k(t_n)|^2}, \quad k = 0, 1, \cdots, K.$$

(2.36)

The final estimate of the desired signal angle, $\hat{\theta}_0$, is then the element in the set $\{\hat{\theta}_0, \hat{\theta}_1, \cdots, \hat{\theta}_K\}$ that corresponds to the maximum $\{\hat{\varepsilon}_0, \hat{\varepsilon}_1, \cdots, \hat{\varepsilon}_K\}$. (The rest of the angles in the set $\{\hat{\theta}_0, \hat{\theta}_1, \cdots, \hat{\theta}_K\}$ are the interfering signal angles.)

2.3.2 A Maximum Likelihood Algorithm that Incorporates the Known Waveform

Now we present an ML algorithm that incorporates the known desired signal waveform from the beginning. We shall refer to this method as Method II.B. As shown in (A.16) and (A.18) of Appendix A, in order to include the known signal waveform in the estimation process, we should minimize the quantity

$$q = \frac{1}{N} \sum_{n=1}^{N} x^H(t_n)P_{A(\theta_I)}x(t_n)$$

(2.37)
where
\begin{align}
x_I(t_n) &= x(t_n) - a(\theta_0)gw_0(t_n), \\
P_{\theta_I} &= I - A(\theta_I)[A^H(\theta_I)A(\theta_I)]^{-1}A^H(\theta_I), \\
\hat{\theta}_I &= \frac{1}{N} \sum_{n=1}^{N} x_I(t_n)x_I^H(t_n).
\end{align}

The algorithm we propose for minimizing \( q \) is an iterative approach similar to the alternating maximization approach of Ziskind and Wax \[6\]. At each iteration, a minimization is performed first with respect to \( g \) (if \( g \) is unknown), then with respect to \( \theta_I = [\theta_1 \theta_2 \ldots \theta_K]^T \), and finally with respect to \( \theta_0 \).

As the first step in the \((i+1)\)st iteration, we fix \( \theta_0^{(i)} \) and \( \theta_0 \). If \( g \) is known, we simply set
\begin{equation}
g^{(i+1)} = g.
\end{equation}

However, if \( g \) is unknown, we minimize \( q \) with respect to \( g^{(i+1)} \). By substituting (2.39) in (2.37) and setting derivatives with respect to the appropriate variables to zero, it is easy to show that if \( g \) is unknown and real,
\begin{equation}
g^{(i+1)} = \text{Re} \left\{ \frac{a^H(\theta_0^{(i)})P_{\theta_I}a(\theta_0^{(i)})}{a^H(\theta_0^{(i)})P_{\theta_I}a(\theta_0^{(i)})} \sum_{n=1}^{N} w_0^*(t_n)x(t_n) \right\}.
\end{equation}

If \( g \) is unknown and complex,
\begin{equation}
g^{(i+1)} = \frac{a^H(\theta_0^{(i)})P_{\theta_I}a(\theta_0^{(i)})}{a^H(\theta_0^{(i)})P_{\theta_I}a(\theta_0^{(i)})} \sum_{n=1}^{N} w_0^*(t_n)x(t_n)
\end{equation}
and if \( g = e^{j\nu} \) with \( \nu \) unknown and real,
\[
g^{(i+1)} = e^{j\nu^{(i+1)}},
\]
where
\[
\nu^{(i+1)} = \arg \left\{ a^H \left( \theta^{(i)}_0 \right) P_{A^\perp} \left( \theta^{(i)}_I \right) \sum_{n=1}^N w_0^*(t_n)x(t_n) \right\}.
\]

For the second step in the \((i+1)\)st iteration, we solve for \( \theta^{(i+1)}_I = [\theta^{(i+1)}_1, \theta^{(i+1)}_2, \ldots, \theta^{(i+1)}_K]^T \) by fixing \( g^{(i+1)} \) and \( \theta^{(i)}_0 \). For this step, we minimize \( g \) by applying the IQML algorithm as described in Steps 1 to 5 above, with \( X(t_n) \) in (2.16) derived from \( x^{(i+1)}_I(t_n) \), where
\[
x^{(i+1)}_I(t_n) = x(t_n) - a^H \left( \theta^{(i)}_0 \right) g^{(i+1)} w_0(t_n),
\]
instead of from \( x(t_n) \) as in (2.15). In this case, the initial polynomial for the IQML algorithm is chosen so that its zeros are
\[
e^{-j\pi \sin \theta^{(i)}_k}, \quad k = 1, 2, \ldots, K.
\]

For the last step of the \((i+1)\)st iteration, we fix \( g^{(i+1)} \) and \( \theta^{(i+1)}_I \) and then find \( \theta^{(i+1)}_0 \). The \( \theta^{(i+1)}_0 \) is the value of \( \theta_0 \) that minimizes
\[
g^{(i+1)}(\theta_0) = \text{tr} \left\{ P_{A^\perp} \left( \theta^{(i+1)}_I \right) R^{(i+1)}_I(\theta_0) \right\},
\]
where
\[
R^{(i+1)}_I(\theta_0) = R - 2 \text{Re} \left\{ g^{(i+1)} a(\theta_0) y^H \right\} + \left| g^{(i+1)} \right|^2 P_0 a(\theta_0) a^H(\theta_0),
\]
with
\[
y = \frac{1}{N} \sum_{n=1}^N w_0^*(t_n)x(t_n),
\]
and
\[ P_0 = \frac{1}{N} \sum_{n=1}^{N} |w_0(t_n)|^2. \]  \hspace{1cm} (2.52)

The required solution for \( \theta_0^{(i+1)} \) is obtained with a one-dimensional search. We limit the search domain to \( \left[ \theta_0^{(i)} - \Delta \theta_0^{(i)}, \quad \theta_0^{(i)} + \Delta \theta_0^{(i)} \right] \), where \( \Delta \theta_0^{(i)} \) is chosen according to our confidence in \( \theta_0^{(i)} \).

To summarize, the ML algorithm that utilizes the known desired signal waveform consists of the following steps:

1. Initialize: Let \( i = 0 \) and obtain \( \theta_0^{(0)} \) and \( \theta_I^{(0)} \) using the IQML algorithm, as described in Method II.A.

2. Update \( g^{(i+1)} \) from (2.42) if \( g \) is known, (2.43) if \( g \) is unknown and real, (2.44) if \( g \) is unknown and complex, or (2.45) if \( g = e^{j\nu} \) and \( \nu \) is unknown and real.

3. Compute \( x_I^{(i+1)}(t_n) \) using (2.47).

4. Compute \( \theta_I^{(i+1)} \) by applying the IQML algorithm to \( x_I^{(i+1)}(t_n) \).

5. Find \( \theta_0^{(i+1)} \) that minimizes \( q^{(i+1)}(\theta_0) \) in (2.49).

6. Check convergence: if \( \left| \theta_0^{(i+1)} - \theta_0^{(i)} \right| < \epsilon_2 \), let \( \hat{\theta}_0 = \theta_0^{(i)} \); otherwise, let \( i = i + 1 \) and go to (2). (\( \epsilon_2 \) is a suitable small constant. \( \hat{\theta}_0 \) is the final estimate of \( \theta_0 \).)

This algorithm is bound to converge to a local minimum. Since the minimization of \( q \) is performed at every step, the value of \( q \) will never increase. However, as usual, there is no guarantee that it will converge to the global minimum. Nevertheless, in the examples we have tried, it always converges to the proper result in a small number of iterations. In general, the number of iterations required drops as the angle between
the desired and the interfering signals becomes larger. In our simulation examples, fewer than 10 iterations were needed. The number of iterations required by the IQML algorithm in Step 4 was fewer than 5.

2.4 Cramer-Rao Bounds

Using the results in [35], we may obtain Cramer-Rao bounds (CRB) for any unbiased estimator of $\theta_0$ for the cases of both known and unknown desired signal waveform.

2.4.1 Unknown Desired Signal Waveform

If all signal waveforms are unknown, it has been shown in [35] that the mean-square error (MSE) of any unbiased estimate of $\theta = [\theta_0 \ \theta_1 \ \cdots \ \theta_K]^T$ is bounded below by the CRB,

$$CRB(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)D^H(\theta)P_{A^\perp(\theta)}D(\theta)S(t_n) \right] \right\}^{-1},$$

where $P_{A^\perp(\theta)}$ is defined in (2.11),

$$S(t_n) = \text{diag} \{s_0(t_n), s_1(t_n), \cdots, s_K(t_n)\},$$

and

$$D(\theta) = \begin{bmatrix} d(\theta_0) & d(\theta_1) & \cdots & d(\theta_K) \end{bmatrix}^T,$$

with

$$d(\theta_k) = \begin{bmatrix} 0 & -j\pi \cos(\theta_k)q_k & \cdots & -j(L-1)\pi \cos(\theta_k)q_k^{L-1} \end{bmatrix}^T.$$
2.4.2 Known Desired Signal Waveform

The CRBs for the cases of known $g$, unknown real $g$, unknown complex $g$, and $g = e^{j\nu}$ with $\nu$ unknown and real are considered below. The CRBs given below have been derived by modifying the results in [35] under the assumption that the desired signal waveform is known and the interfering signal waveforms are unknown. Detailed derivations of the results below are given in Appendix C.

If $g$ is known, it is shown in (C.30) of Appendix C that the MSE of any unbiased estimate of $\theta$ is bounded below by the CRB,

$$
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) \right\}^{-1},
$$

(2.57)

where

$$
\Gamma(t_n) = \Re \left[ S^H(t_n) D^H(\theta) P_{A \perp (\theta_j)} D(\theta) S(t_n) \right].
$$

(2.58)

If $g$ is unknown and complex, the MSE of any unbiased estimate of $\theta$ is bounded below by (see (C.70) of Appendix C)

$$
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\Re \left( \left[ \sum_{n=1}^{N} f(t_n) \right] \left[ \sum_{n=1}^{N} f^H(t_n) \right] \right)}{\sum_{n=1}^{N} \eta(t_n)} \right\}^{-1},
$$

(2.59)

where

$$
f(t_n) = S^H(t_n) D^H(\theta) P_{A \perp (\theta_j)} a(\theta_0) s_0(t_n),
$$

(2.60)

and

$$
\eta(t_n) = |s_0(t_n)|^2 a^H(\theta_0) P_{A \perp (\theta_j)} a(\theta_0).
$$

(2.61)

If $g$ is unknown and real, it is shown in (C.76) of Appendix C that the MSE of any unbiased estimate of $\theta$ is bounded below by
Finally, if $g = e^{j\nu}$ with $\nu$ unknown and real, the MSE of any unbiased estimate of $\theta$ is bounded below by (see (C.89) of Appendix C)

$$\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\text{Re} \left[ \sum_{n=1}^{N} f(t_n) \right] \text{Re} \left[ \sum_{n=1}^{N} f(t_n) \right]}{\sum_{n=1}^{N} \eta(t_n)} \right\}^{-1}. \quad (2.62)$$

Comparing the bounds in (2.53), (2.57), (2.62), (2.59), and (2.63), we note that the ratio between any two of the bounds is independent of the variance of the additive noise or the signal-to-noise ratio. Also, the bounds in (2.57), (2.62), (2.59), and (2.63) are independent of $g$ as long as $s_0(t)$ is fixed so that the SNR of the desired signal is fixed.

Also, it can be shown that, in the absence of interfering signals, the bounds in (2.53), (2.59), and (2.63) are the same, and all are equal to

$$\text{CRB}(\theta) = \frac{6\sigma^2}{NL(L^2 - 1)P_0\pi^2 \cos^2 \theta}, \quad (2.64)$$

where $P_0$ is defined in (2.52). Therefore, with no interfering signals, the CR-bound is the same in three cases: (1) for unknown desired signal, (2) for known desired signal waveform but unknown complex $g$, and (3) for known desired signal waveform but unknown real $\nu$ with $g = e^{j\nu}$. In the absence of interference, the bounds in (2.57) and (2.62) are also the same and are

$$\text{CRB}(\theta) = \frac{3\sigma^2}{NL(L - 1)(2L - 1)P_0\pi^2 \cos^2 \theta}. \quad (2.65)$$

Thus, with no interference and a known desired signal waveform, the CR-bound is the same for both known $g$ and unknown real $g$. Comparison of (2.64) and (2.65) shows that these two bounds differ by a factor $2(2L - 1)/(L + 1)$. For large $L$,
$2(2L - 1)/(L + 1)$ is about 6 dB. Note also that the ratio between the bounds in (2.64) and (2.65) depends only on the number of sensors $L$. The smaller bound (2.65) occurs because the angle of the desired signal depends only on the spatial phase factor $q_0$.

2.5 Typical Results

In this section, we show some typical performance curves for Methods II.A and II.B. The results below were obtained by using fifty Monte Carlo simulations with independent trials. In all cases, the desired signal arrives from $\theta_0 = 30^\circ$. The waveform of the desired signal is a BPSK (binary phase-shift keyed) signal modulated by one period of a 31-bit PN (pseudonoise) sequence [31]. The interfering signals have BPSK modulations derived from binary bit streams that are independent of the desired signal PN sequence. All incident signals have unit power at each sensor and are sampled at a rate of one sample per bit. The signal-to-noise ratio (SNR) at each sensor output, defined as $-10 \log_{10} \sigma^2$ dB, is assumed to be 20 dB. For the case of known $g$ and unknown complex $g$, we assumed $g = 2e^{j\pi/4}$. For the case of unknown real $g$, we assumed $g = 2$. For the case $g = e^{j\nu}$ with unknown real $\nu$, we assumed $\nu = 2$. The convergence constants used in the iterative algorithms were chosen to be $\epsilon_1 = 0.01$ and $\epsilon_2 = 0.005^\circ$.

We first present a case with one interfering signal. The interfering signal is assumed to arrive from $\theta_1 = (30 - \Delta \theta)^\circ$, so $\Delta \theta$ is the angle separation between the desired and the interfering signals. Figure 1(a) shows the MSE of $\hat{\theta}_0$ as a function of $\Delta \theta$ when the number of data samples is $N = 31$ and the number of sensors is $L = 10$. Figure 1(b) shows the corresponding Cramer-Rao bounds. Note that the estimates from Method II.B are very close to the best unbiased estimates one can
get. (Because of the limited number of Monte Carlo simulations, the MSE curves may occasionally fall below the CRBs.)

In Figure 1, we see that Method II.B is not better than Method II.A when $g$ is unknown and complex. However, for known $g$ and for $g$ unknown and real, Method II.B has significantly better performance than Method II.A. When $g = e^{j\nu}$ with $\nu$ unknown and real, the estimates obtained with Method II.B are better for small $\Delta \theta$, but not for large $\Delta \theta$.

Figure 2(a) shows the MSE of $\hat{\theta}_0$ as a function of the number of samples $N$ when $\Delta \theta = 5^\circ$ and for $L = 10$ sensors. Figure 2(b) shows the corresponding Cramer-Rao bounds. Note that the differences among the curves change little as $N$ ($N > 1$) changes.

Figure 3(a) shows the MSE of $\hat{\theta}_0$ as a function of the number of sensors $L$ when $\Delta \theta = 5^\circ$ and for $N = 31$ samples. Figure 3(b) shows the corresponding Cramer-Rao bounds. Note that when $g = e^{j\nu}$ with $\nu$ unknown and real, the improvement that results from using the desired signal waveform decreases as the number of sensors increases. For known $g$ or unknown real $g$, the improvement decreases slightly as $L$ increases.

Finally, we show a case with three interfering signals. The interfering signals are assumed to arrive from $\theta_1 = (30 - \Delta \theta)^\circ$, $\theta_2 = -30^\circ$, and $\theta_3 = 50^\circ$, so $\Delta \theta$ is the angle separation between the desired and the first interfering signals. All three interference signals are assumed to have identical waveforms, so they are perfectly correlated with each other. Figure 4(a) shows the MSE of $\hat{\theta}_0$ as a function of $\Delta \theta$ for $N = 31$ samples and $L = 10$ sensors. Figure 4(b) shows the corresponding Cramer-Rao bounds. It is interesting that the accuracy of the estimates is not significantly reduced by the extra interfering signals.
Figure 1: MSE and CRB vs. $\Delta \theta$ for $N = 31$, $L = 10$. Solid curves (which coincide visually): Method II.A and Method II.B for unknown complex $g$; dashed curve: Method II.B for $g = e^{j\nu}$, unknown real $\nu$; dotted curve: Method II.B for unknown real $g$; dashdot curve: Method II.B for known $g$. 
Figure 2: MSE and CRB vs. $N$ for $\Delta \theta = 5^\circ$, $L = 10$. Solid curves (which coincide visually): Method II.A and Method II.B for unknown complex $g$; dashed curve: Method II.B for $g = e^{j\nu}$, unknown real $\nu$; dotted curve: Method II.B for unknown real $g$; dashdot curve: Method II.B for known $g$. 
Figure 3: MSE and CRB vs. $L$ for $\Delta \theta = 5^\circ$, $N = 31$. Solid curve (which coincide visually): Method II.A and Method II.B for unknown complex $g$; dashed curve: Method II.B for $g = e^{j\nu}$, unknown real $\nu$; dotted curve: Method II.B for unknown real $g$; dashdot curve: Method II.B for known $g$. 

(a) MSE curves.

(b) CR-bounds.
Figure 4: MSE and CRB vs. $\Delta \theta$ for 3 interfering signals, $N = 31$, $L = 10$. Solid curve (which coincide visually): Method II.A and Method II.B for unknown complex $g$; dashed curve: Method II.B for $g = e^{j\nu}$, unknown real $\nu$; dotted curve: Method II.B lor unknown real $g$; dashdot curve: Method II.B for known $g$. 
Our simulations also show that the results observed above for the case of coherent interfering signals also hold when the interfering signals are uncorrelated with each other.

2.6 Summary

We have presented a maximum likelihood algorithm (Method II.B) that incorporates a knowledge of the desired signal waveform into the process of estimating the desired signal angle. The algorithm solves the ML solution iteratively and converges in a few iterations, so it avoids the complexity of a multi-dimensional search. Since the algorithm is based on maximum likelihood estimation, it is applicable even when the interfering signals are perfectly correlated with each other. We have presented curves that compare the performance of this algorithm with that of the IQML algorithm [3] when augmented to extract the desired signal angle (Method II.A). We have also described the conditions under which incorporating knowledge of the desired signal waveform in the estimator improves the accuracy of the angle estimates. We have shown that the improvement is most significant when the gain of the desired signal is known or unknown and real. When the gain is of unit magnitude and has an unknown and real phase, the improvement occurs only when the desired and interfering signals are closely spaced. Little improvement is obtained when the gain is unknown and complex.
CHAPTER III

MAXIMUM LIKELIHOOD ESTIMATION OF THE ARRIVAL DIRECTIONS OF SIGNALS WITH KNOWN WAVEFORMS

3.1 Introduction

In the previous chapter, we have considered the problem of angle estimation of a single signal with known waveform but an unknown gain in the presence of interfering signals with unknown waveforms. In this chapter, we turn to the problem of estimating the angles of all the signals when all signals have known waveforms but unknown gains.

As in Chapter II, we again use maximum likelihood angle estimation techniques. We describe two computationally efficient ML algorithms that may be used for this purpose. To avoid the need for a multidimensional search, as is required to maximize the likelihood function directly when multiple signals are present, we present two iterative algorithms for computing the angle estimates. One approach is based on the Alternating Maximization (AM) approach of Ziskind and Wax [6] and the other is based on the EM approach of Feder and Weinstein [7]. Our approaches differ from the AM approach and the EM algorithm, however, in that we consider a uniform linear array of sensors and we obtain the angle estimates by finding polynomial roots rather than by searching over parameter space. We compare the performance of these two estimators with each other and with that of a suboptimal estimator that first estimates the angles without using the signal waveforms (the IQML estimator [3]) and then determines which angle estimate corresponds to which waveform. We also
present Cramer-Rao (CR) bounds for these estimators and compare the performance of the estimators to their CR-bounds.

This chapter is organized as follows. In Section 3.2, we formulate the problem. In Section 3.3, we describe the IQML approach, which does not utilize the known signal waveforms, and the two new ML approaches, which do. In Section 3.4, we present the CR-bounds that apply when the known signal waveforms are either incorporated or not incorporated. In Section 3.5, we show numerical results and examine the performance of these algorithms. Finally, Section 3.6 contains our summary.

3.2 Problem Formulation

We again consider a uniform linear array with $L$ isotropic sensors. The distance between two adjacent sensors is a half wavelength at the signal frequency. Suppose $K$ narrowband plane waves with known waveforms impinge on the array from distinct angles $\theta_k$, $k = 1, \cdots, K$, relative to the array normal. The number of incident signals $K$ is assumed known. (If the number of signals is unknown, it may be estimated as described in [33].) Suppose that the signal from $\theta_k$ has waveform $s_k(t) = g_k w_k(t)$, where $w_k(t)$ is a known waveform but the gain $g_k$ may be unknown. The incident signals may be correlated (or even perfectly correlated) with each other.

To simplify the problem, we shall assume that when two waveforms $w_k(t)$ are perfectly correlated, they are actually identical. In other words, we exclude from consideration the case where two $w_k(t)$ differ only by a phase factor. Such a phase difference between two $w_k(t)$ can be incorporated into the definitions of the gains $g_k$, so there is no loss of generality with this assumption.

With $K$ signals incident, the total signal $x_l(t)$ received at the $l$th sensor is the sum of the $K$ signals plus an additive noise component,
where $n_l(t)$ is a zero-mean Gaussian noise process with variance $\sigma^2$. The $n_l(t)$ are independent of each other and the incident signals.

Let $x(t)$, $W(t)$, $g$, and $n(t)$ be

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_L(t) \end{bmatrix}^T,$$

$$W(t) = \text{diag}\{w_1(t), w_2(t), \cdots, w_K(t)\},$$

$$g = \begin{bmatrix} g_1 & g_2 & \cdots & g_K \end{bmatrix}^T,$$

$$n(t) = \begin{bmatrix} n_1(t) & n_2(t) & \cdots & n_L(t) \end{bmatrix}^T,$$

where $(\cdot)^T$ denotes the transpose. The received signal vector has the form

$$x(t) = A(\theta)W(t)g + n(t),$$

where $A(\theta)$ (with $\theta = [\theta_1 \ \theta_2 \ \cdots \ \theta_K]^T$) is the direction matrix, whose columns are the direction vectors of the incident signals

$$A(\theta) = \begin{bmatrix} a(\theta_1) & a(\theta_2) & \cdots & a(\theta_K) \end{bmatrix},$$

with

$$a(\theta_k) = \begin{bmatrix} 1 & q_k & \cdots & q_k^{L-1} \end{bmatrix}^T,$$

and

$$q_k = e^{-j\pi\sin\theta_k}.$$
each other. The problem of interest is to determine the angles \( \theta_k, k = 1, 2, \ldots, K \), from the measurements \( x(n), n = 1, 2, \ldots, N \). In the next section, we describe three approaches based on ML estimation that may be used to solve this problem.

3.3 Maximum Likelihood Angle Estimation Algorithms

One way of estimating the signal angles is to start with the IQML algorithm of Bresler and Macovski [3], a maximum likelihood method originally discussed by Kumaresan, Scharf and Shaw [4]. This method does not tell us which angle estimate corresponds to which known signal waveform. However, we can augment the IQML algorithm by estimating the received signal waveforms and then correlating them with the known waveforms to determine which waveform corresponds to each angle. We consider this approach in the first part below. Note that the performance of this method is suboptimal because it does not incorporate the known signal waveforms in the angle estimation process.

Then, in the second part, we present two algorithms that do incorporate the signal waveforms. Both of these methods are based on maximum likelihood estimation. The two methods involve two different iterative procedures for computing the ML estimates.

3.3.1 The IQML Algorithm with Signal Correlation

One method for estimating the signal angles, which we shall call Method III.A, is based on the IQML algorithm [3]. This method is similar to Method II.A which has been described in Section 2.3.1. In this method, the first step is to use the IQML algorithm to estimate signal angles without taking advantage of the known signal waveforms. The IQML algorithm has been described in Section 2.3.1.
After obtaining the ML estimates $\hat{\theta}_1, \cdots, \hat{\theta}_k$ of $\theta_1, \cdots, \theta_K$ from the IQML algorithm, we next use the known signal waveforms to determine which waveform corresponds to which angle in the set \{\hat{\theta}_1, \cdots, \hat{\theta}_k\}. To do this, we first estimate the waveform of each incident signal and then correlate these waveform estimates with the known waveforms. The estimated waveforms of the incident signals are obtained from $\hat{\theta}_k$, $k = 1, 2, \cdots, K$, as follows [13]:

$$\bar{s}(t_n) = \left[ A^H(\hat{\theta})A(\bar{\theta}) \right]^{-1} A^H(\hat{\theta})x(t_n), \quad n = 1, 2, \cdots, N,$$

where

$$A(\bar{\theta}) = \begin{bmatrix} a(\bar{\theta}_1) & a(\bar{\theta}_2) & \cdots & a(\bar{\theta}_K) \end{bmatrix}, \quad (3.11)$$

and

$$\bar{s}(t_n) = \left[ \bar{s}_1(t_n) \quad \bar{s}_2(t_n) \quad \cdots \quad \bar{s}_K(t_n) \right]^T. \quad (3.12)$$

Next, we correlate each waveform in $\bar{s}(t)$ against all of the known waveforms $w_k(t)$. To determine which $w_k(t)$ corresponds to a given $\hat{\theta}_k$, the cross-correlation between the corresponding signal estimate $\bar{s}_k(t)$ and each $w_k(t)$, $k = 1, 2, \cdots, K$, is computed and normalized with respect to the power of each $w_k(t)$. Note that this cross-correlation is different from the cross-correlation given in (2.36). In (2.36), the cross-correlation between $w_0(t)$ and each of the signal estimates is computed and normalized with respect to the power of the corresponding signal estimate since $w_0(t)$ is the only known waveform for the case considered in Chapter II.

For $k' = 1, 2, \cdots, K$, let $\tilde{c}_{k,k'}$ be the cross-correlation between the signal estimate $\bar{s}_{k'}(t)$ and the $k$th waveform $w_k(t)$ normalized with respect to $P_k$, the power of $w_k(t)$,

$$\tilde{c}_{k,k'} = \frac{1}{NP_k} \left| \sum_{n=1}^{N} w^*_k(t_n)\bar{s}_{k'}(t_n) \right|^2, \quad 1 \leq k, k' \leq 1, \quad (3.13)$$
where
\[ P_k = \frac{1}{N} \sum_{n=1}^{N} |w_k(t_n)|^2. \] (3.14)

To determine which \( w_k(t) \) corresponds to a given \( \tilde{\theta}_{k'} \), for each \( \tilde{\theta}_{k'}(t) \) we choose the \( w_k(t) \) for which \( \tilde{\tilde{e}}_{k,k'} \) is maximum.

The angle estimates obtained in this way are suboptimal, because the waveforms are not utilized in forming the estimates. Nevertheless, this is one method of determining the arrival angle for each signal waveform.

### 3.3.2 Maximum Likelihood Algorithms that Incorporate the Known Signal Waveforms

Now we present two ML algorithms that incorporate the known signal waveforms in the estimation.

When all signal waveforms are known, the conditional multivariate Gaussian probability density function for the vector \( x(t) \) of \( L \) complex random variables in (3.6) is [36, p.114]
\[
p [x(t)|g, \theta, W(t)] = \left[ \pi \sigma^2 \right]^{-L} \exp \left\{ -\frac{1}{2} [x(t) - A(\theta)W(t)g]^H [x(t) - A(\theta)W(t)g] \right\},
\] (3.15)

where \((\cdot)^H\) denotes the complex conjugate transpose. When \( N \) independent samples \( x(t_n) \) of \( x(t) \) are taken, the density function for the \( x(t_n) \) is
\[
p [x(t_1), \cdots, x(t_N)|g, \theta, W(t)] = \left[ \pi \sigma^2 \right]^{-LN} \exp \left\{ -N\sigma^{-2} q \right\},
\] (3.16)

where
\[
q = \frac{1}{N} \sum_{n=1}^{N} [x(t_n) - A(\theta)W(t_n)g]^H [x(t_n) - A(\theta)W(t_n)g].
\] (3.17)
To maximize \( p[\mathbf{x}(t_1), \cdots, \mathbf{x}(t_N) | \mathbf{g}, \theta, \mathbf{W}(t)] \), one must minimize \( q \). The minimization of \( q \) is done over \( \theta \) if \( \mathbf{g} \) is known or over \( \{\mathbf{g}, \theta\} \) if \( \mathbf{g} \) is unknown.

To minimize \( q \) in (3.17) directly would involve either (1) a \( K \)-dimensional search if \( \mathbf{g} \) is known, (2) a \( 3K \)-dimensional search if \( \mathbf{g} \) is unknown and complex, or (3) a \( 2K \)-dimensional search if \( \mathbf{g} \) is unknown and real, or if \( g_k = e^{i\nu_k}, k = 1, 2, \cdots, K \), with \( \nu_k \) unknown and real. We describe below two iterative approaches that may be used instead to avoid the multidimensional search. These approaches are based on the Alternating Maximization (AM) approach of Ziskind and Wax [6] and the Estimate Maximize (EM) approach of Feder and Weinstein [7].

These two approaches are given in the two sections below. Both methods involve iteration on both \( \mathbf{g} \) and \( \theta \). To begin either algorithm, it is necessary to have initial estimates for \( \mathbf{g} \) and \( \theta \). The initial estimate for \( \theta \) may be obtained by using the angle estimates resulting from Method III.A above or from other computationally efficient techniques such as MUSIC, ESPRIT, or others [12, 13, 9]. We let \( \theta^{(0)} = \left[ \theta_1^{(0)}, \theta_2^{(0)}, \cdots, \theta_K^{(0)} \right]^T \) be the initial set of angle estimates obtained with one of these methods. The initial estimate \( \mathbf{g}^{(0)} = \left[ g_1^{(0)}, g_2^{(0)}, \cdots, g_K^{(0)} \right]^T \) for \( \mathbf{g} \) is obtained as follows. Of course, if \( \mathbf{g} \) is known, we simply set

\[
\mathbf{g}^{(0)} = \mathbf{g}.
\]  

But if \( \mathbf{g} \) is unknown, a least squares method is used to determine \( \mathbf{g}^{(0)} \). Let

\[
\mathbf{e} = \left[ \sum_{n=1}^{N} \mathbf{W}^H(t_n) \mathbf{A}^H \left( \theta^{(0)} \right) \mathbf{A} \left( \theta^{(0)} \right) \mathbf{W}(t_n) \right]^{-1} \left[ \sum_{n=1}^{N} \mathbf{W}^H(t_n) \mathbf{A}^H \left( \theta^{(0)} \right) \mathbf{x}(t_n) \right].
\]  

Then, if \( \mathbf{g} \) is unknown and complex, we choose
\[ g^{(0)} = 0. \]  

(3.20)

If \( g \) is unknown and real, we let

\[ g^{(0)} = \text{Re}(g). \]  

(3.21)

Or, if \( g_k = e^{j\nu_k} \) for each \( k \), with \( \nu_k \) unknown and real, we choose

\[ g_k^{(0)} = e^{j\nu_k^{(0)}}, \]  

(3.22)

where \( \nu^{(0)} = [\nu_1^{(0)} \nu_2^{(0)} \ldots \nu_K^{(0)}]^T \) is

\[ \nu^{(0)} = \text{arg}(g). \]  

(3.23)

Now we present the two methods.

**A Maximum Likelihood Method Based on Alternating Maximization**

The first approach, which we call Method III.B.1, is based on the AM algorithm [3]. At each iteration, a minimization is performed first with respect to \( \theta_1 \), then with respect to \( g_1 \), then with respect to \( \theta_2 \), then with respect respect to \( g_2 \), and so forth.

At the \( k \)th step of the \( (i+1) \)st iteration, for \( k = 1, 2, \ldots, K \), we first fix

\[ \theta_1^{(i+1)}, g_1^{(i+1)}, \ldots, \theta_k^{(i+1)}, g_k^{(i+1)}, \theta_{k+1}^{(i)}, g_{k+1}^{(i)}, \ldots, \theta_K^{(i)}, g_K^{(i)}, \]  

and compute \( \theta_k^{(i+1)} \). Dropping all terms in \( q \) in (3.17) not involving \( \theta_k \), \( k = 1, 2, \ldots, K \), yields

\[ \arg \min_{\theta_k} \left\{ \sum_{l=1}^{L} \text{Re} \left[ \left( g_k^{(i)} \right)^* e^{j(l-1)\pi \sin \theta_k y_k^{(i+1)}} \right] \right\} = \theta_k^{(i+1)}, \quad k = 1, \ldots, K, \]  

(3.24)
where we have defined

\[ y_{k,l}^{(i+1)} = \frac{1}{N} \sum_{n=1}^{N} \left[ x_l(t_n) - \sum_{k'=1}^{k-1} g_{k'}^{(i+1)} w_{k'}(t) e^{-j(l-1)\pi \sin \theta_{k'}^{(i+1)}} - \sum_{k'=k+1}^{K} g_{k'}^{(i)} w_{k'}(t) e^{-j(l-1)\pi \sin \theta_{k'}^{(i)}} \right] w_k^*(t_n). \] 

(3.25)

The problem of finding \( \theta_k^{(i+1)} \) from (3.24) is a problem of solving for the roots of a polynomial of order \( 2(L-1) \) (see Appendix D). The polynomial has the form

\[ b^{(i+1)}(z) = g_k^{(i)} \sum_{i=1}^{L} (l-1) \left( y_{k,l}^{(i+1)} \right)^* z^{l-1} - \left( g_k^{(i)} \right)^* \sum_{l=1}^{L} (l-1) y_{k,l}^{(i+1)} z^{-(l-1)}, \] 

(3.26)

where

\[ z = e^{-j\pi \sin \theta_k}. \] 

(3.27)

Let the zeros of the polynomial be denoted by \( z_{k,i'}^{(i+1)} \), \( i' = 1, 2, \cdots, 2(L-1) \). Then \( \theta_k^{(i+1)} \) is the element in the set \( \{ \theta_k^{(i+1)} , \theta_{k,2}^{(i+1)} , \cdots , \theta_{k,2(L-1)}^{(i+1)} \} \) that minimizes the left side of (3.24), where

\[ \theta_k^{(i+1)} = -\sin^{-1} \left[ \frac{1}{\pi} \arg \left( z_{k,i'}^{(i+1)} \right) \right], \quad i' = 1, 2, \cdots, 2(L-1). \] 

(3.28)

After finding \( \theta_k^{(i+1)} \), we next fix the quantities

\[ \theta_1^{(i+1)}, g_1^{(i+1)} \cdots , \theta_{k-1}^{(i+1)}, g_{k-1}^{(i+1)}, \theta_k^{(i+1)}, g_k^{(i)}, \theta_{k+1}^{(i)}, g_{k+1}^{(i)} \cdots , \theta_K^{(i)}, g_K^{(i)}, \] 

and compute \( g_k^{(i+1)} \). Dropping all terms in \( q \) not involving \( g_k \) yields

\[ \arg \min_{g_k} \left\{ \sum_{l=1}^{L} -2\text{Re} \left[ g_k^* e^{j(l-1)\pi \sin \theta_k^{(i+1)} y_{k,l}^{(i+1)}} \right] + |g_k|^2 P_k \right\} = g_k^{(i+1)}. \] 

(3.29)

The solution for \( g_k^{(i+1)} \) in (3.29) depends on the assumed form of \( g_k \) (see Appendix D). If \( g_k \) is known,
\( g_{k}^{(i+1)} = g_{k} \).  

(3.30)

If \( g_{k} \) is unknown and complex,
\[
g_{k}^{(i+1)} = \frac{1}{LP_{k}} \sum_{l=1}^{L} y_{k,l}^{(i+1)} e^{j(l-1)\pi \sin \theta_{k}^{(i+1)}}.
\]

(3.31)

If \( g_{k} \) is unknown and real,
\[
g_{k}^{(i+1)} = \text{Re} \left\{ \frac{1}{LP_{k}} \sum_{l=1}^{L} y_{k,l}^{(i+1)} e^{j(l-1)\pi \sin \theta_{k}^{(i+1)}} \right\}.
\]

(3.32)

Finally, if \( g_{k} = e^{j\nu_{k}} \) with \( \nu_{k} \) unknown and real,
\[
g_{k}^{(i+1)} = \exp \left\{ j \arg \left[ \sum_{l=1}^{L} y_{k,l}^{(i+1)} e^{j(l-1)\pi \sin \theta_{k}^{(i+1)}} \right] \right\}.
\]

(3.33)

Thus, Method III.B.1 for incorporating the known signal waveforms in the estimator consists of the following steps:

1. Initialize: Let \( i = 0 \) and obtain \( \theta^{(0)} \) using the IQML algorithm (or some other algorithm).

2. Compute \( g^{(0)} \) from (3.18), (3.20), (3.21), or (3.22).

3. For \( k = 1, 2, \ldots, K \), obtain \( \theta_{k}^{(i+1)} \) from (3.24), and \( g_{k}^{(i+1)} \) from (3.29).

4. Check convergence: If \( \max_{k} \left| \theta_{k}^{(i+1)} - \theta_{k}^{(i)} \right| < \epsilon_{2} \), let \( \hat{\theta} = \theta^{(i)} \); otherwise, let \( i = i + 1 \) and go to (3). (\( \epsilon_{2} \) is a suitable convergence constant. \( \hat{\theta} \) is the final estimate of \( \theta \).)

This method is bound to converge to at least a local minimum of \( q \). Since \( q \) is minimized at every step, the value of \( q \) will never increase. However, it is possible that the algorithm may converge to a local minimum instead of the global minimum.

Nevertheless, in the examples we have tried, it has always converged to the proper result in a small number of iterations.
A Maximum Likelihood Method Based on the Estimate Maximize Algorithm

The second approach, Method III.B.2, is based on the EM algorithm [7] and works as follows. At each iteration, the observed signals \( x(t) \) are decomposed into their signal components plus noise [7], and the angle estimates are updated individually from these separate components.

As the first step in the \((i+1)\)st iteration, we decompose the observed signals into their signal components plus noise. The purpose of this decomposition is to decouple the complicated multidimensional minimization of \( q \) in (3.17) into \( K \) separate minimizations. For \( k = 1, 2, \ldots, K \), we let

\[
x_k^{(i+1)}(t_n) = a \left( \theta_k^{(i)} \right) g_k^{(i)} w_k(t_n) + \frac{1}{K} \left[ x(t_n) - \sum_{k'=1}^{K} a \left( \theta_{k'}^{(i)} \right) g_{k'}^{(i)} w_{k'}(t_n) \right],
\]

where

\[
x_k^{(i+1)}(t_n) = \begin{bmatrix} x_{k,1}^{(i+1)}(t_n) & x_{k,2}^{(i+1)}(t_n) & \cdots & x_{k,L}^{(i+1)}(t_n) \end{bmatrix}^T.
\]

Note that the first term of the right side of (3.34) is the \( k \)th signal component and the second term the noise component.

For the second step of the \((i+1)\)st iteration, we calculate \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) from \( x_k^{(i+1)}(t_n) \). We solve

\[
\arg \min_{g_k, \theta_k} \sum_{n=1}^{N} \left[ x_k^{(i+1)}(t_n) - a(\theta_k) g_k w_k(t_n) \right]^H \left[ x_k^{(i+1)}(t_n) - a(\theta_k) g_k w_k(t_n) \right] = g_k^{(i+1)}, \theta_k^{(i+1)}.
\]

By dropping all terms not involving \( g_k \) and \( \theta_k \), (3.36) can be rewritten as

\[
\arg \min_{g_k, \theta_k} \left\{ \sum_{l=1}^{L} -2\text{Re} \left[ g_k^* e^{j(l-1)\pi \sin \theta_k g_k^{(i+1)}} \right] + |g_k|^2 P_k \right\} = g_k^{(i+1)}, \theta_k^{(i+1)}.
\]
where
\[ y_k^{(i+1)} = \frac{1}{N} \sum_{n=1}^{N} x_{k,l}^{(i+1)}(t_n)w_k^*(t_n). \] (3.38)

As shown in Appendix D, solutions to (3.37) may be found as the roots of a polynomial. The particular polynomial depends on the assumed form of \( g \). For the case where \( g \) is known,
\[ g_k^{(i+1)} = g_k, \] (3.39)
and \( \theta_k^{(i+1)} \) is found by solving for the roots of the following polynomial of order \( 2(L-1) \),
\[ g^{(i+1)}(z) = g_k \sum_{l=1}^{L} (z - l)^{-1} - (g_k)^* \sum_{l=1}^{L} (l-1) y_k^{(i+1)} z^{-(l-1)}, \] (3.40)
where
\[ z = e^{-j\pi \sin \theta_k}. \] (3.41)

Let the zeros of this polynomial be denoted by \( z_{k,i'}^{(i+1)} \), \( i' = 1, 2, \cdots, 2(L-1) \). Then \( \theta_k^{(i+1)} \) is the element in the set \( \{ \theta_k^{(i+1)} \}_{i' = 1}^{2(L-1)} \) that minimizes the left side of (3.37), where
\[ \theta_k^{(i+1)} = -\sin^{-1} \left[ \frac{1}{\pi} \text{arg} \left( z_{k,i'}^{(i+1)} \right) \right], \quad i' = 1, 2, \cdots, 2(L-1). \] (3.42)

For the case when \( g \) is unknown and complex, finding \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) requires solving for the roots of the polynomial (see Appendix D)
\[ g^{(i+1)}(z) = \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} (l_1 - l_2) \left( y_{k,l_1}^{(i+1)} \right)^* y_{k,l_2}^{(i+1)} z^{l_1-l_2}, \] (3.43)
where \( z \) is defined in (3.41). If the zeros of this polynomial are denoted by \( z_{k,i'}^{(i+1)} \), \( i' = 1, 2, \cdots, 2(L-1) \), then \( (g_k^{(i+1)}, \theta_k^{(i+1)}) \) is the pair in the set.
that minimizes the left side of (3.37), where \( \theta_k^{(i+1)} \) is computed from (3.42), and

\[
\frac{y_{k-1}^{(i+1)}}{2} = \sum_{l=1}^{L} \left( y_{k,l}^{(i+1)} \right) \left( z_{k,l}^{(i+1)} \right)^{-(l-1)} , \quad i' = 1, 2, \ldots, 2(L-1).
\]

For the case \( g \) unknown and real, \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) are found from the roots of the polynomial (see Appendix D)

\[
b^{(i+1)}(z) = \sum_{l=1}^{L} (l - 1) \left( y_{k,l}^{(i+1)} \right)^{*} z^{l-1} - \sum_{l=1}^{L} (l - 1) y_{k,l}^{(i+1)} z^{-(l-1)}, \quad (3.45)
\]

If the zeros of this polynomial are \( z_{k,i'}^{(i+1)} = 1, 2, \ldots, 2(L-1), \) then \( \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right) \) is the pair in the set

\[
\left\{ \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right), \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right), \ldots, \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right) \right\}
\]

that minimizes the left side of (3.37), where \( \theta_k^{(i+1)} \) is found from (3.42), and

\[
g_k^{(i+1)} = \text{Re} \left\{ \frac{1}{L P_k} \sum_{l=1}^{L} y_{k,l}^{(i+1)} \left( z_{k,i'}^{(i+1)} \right)^{-(l-1)} \right\} , \quad i' = 1, 2, \ldots, 2(L-1). \quad (3.46)
\]

Finally, for \( g_k = e^{j \nu_k} \) with \( \nu_k \) unknown and real, \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) are again obtained from roots of (3.43). If the zeros of (3.43) are denoted by \( z_{k,i'}^{(i+1)} = 1, 2, \ldots, 2(L-1), \) \( \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right) \) is the pair in the set

\[
\left\{ \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right), \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right), \ldots, \left( g_k^{(i+1)}, \theta_k^{(i+1)} \right) \right\}
\]

that minimizes the left side of (3.37), where \( \theta_k^{(i+1)} \) is computed from (3.42), and

\[
g_k^{(i+1)} = e^{j \nu_k^{(i+1)}}
\]

with
\[ \nu_{k,i'}^{(i+1)} = \arg \left\{ \frac{1}{L} \sum_{l=1}^{L} y_{k,i'}^{(i+1)} \left[ z_{k,i'}^{(i+1)} \right]^{-l-1} \right\}, \quad i' = 1, 2, \ldots, 2(L-1). \] (3.48)

Thus, Method III.B.2 consists of the following steps:

1. Initialize: Let \( i = 0 \) and obtain \( \theta^{(0)} \) using the IQML algorithm (or some other algorithm).

2. Compute \( g^{(0)} \) from (3.18), (3.20), (3.21), or (3.22).

3. For \( k = 1, 2, \ldots, K \), obtain \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) from (3.37).

4. Check convergence: If \( \max_k |\theta_k^{(i+1)} - \theta_k^{(i)}| < \epsilon_3 \), let \( \hat{\theta} = \theta^{(i)} \); otherwise, let \( i = i + 1 \) and go to (3). (\( \epsilon_3 \) is a suitable convergence constant. \( \hat{\theta} \) is the final estimate of \( \theta \)).

This method always converges at least to a local minimum of \( q \) [7]. But as usual there is no guarantee that it will converge to the global minimum. Nevertheless, in our examples, it has always converged to the proper result in a small number of iterations.

Methods III.B.1 and III.B.2 both minimize \( q \) in (3.17) iteratively. Within each iteration, however, Method III.B.2 computes \( g_k^{(i+1)} \) and \( \theta_k^{(i+1)} \) in parallel while Method III.B.1 computes them serially.

Both Methods III.B.1 and III.B.2 involve finding polynomial roots within each iteration. To reduce the root solving computations, the roots obtained in a previous iteration may be used as starting points in the current iteration. The roots may be updated by root-polishing techniques such as the fast convergent Newton-Raphson method [37]. Furthermore, we may only polish those roots from which the angle estimates are updated.
3.4 Cramér-Rao Bounds

Using the results in [35], we may obtain the Cramér-Rao bounds (CRB) of any unbiased estimator of $\theta$ for both cases of known and unknown signal waveforms.

3.4.1 Unknown Signal Waveforms

If all signal waveforms are unknown, the CR-bound of any unbiased angle estimate is the same as in (2.53) except that $\theta = [ \theta_0 \ \theta_1 \ \cdots \ \theta_K ]^T$ in (2.53) is replaced by $\theta = [ \theta_1 \ \theta_2 \ \cdots \ \theta_K ]^T$ for this case. I.e., the mean-square error (MSE) of any unbiased estimate of $\theta = [ \theta_1 \ \theta_2 \ \cdots \ \theta_K ]^T$ is bounded below by the CRB,

$$
\text{CRB}(\theta) = \sigma^2 \left\{ \sum_{n=1}^{N} \text{Re} \left[ S(t_n) D^H(\theta) P_{A\perp}(\theta) D(\theta) S(t_n) \right] \right\}^{-1},
$$

(3.49)

where

$$
P_{A\perp}(\theta) = I - A(\theta) A^H(\theta A(\theta))^{-1} A^H(\theta),
$$

(3.50)

$$
S(t_n) = \text{diag} \{ s_1(t_n), s_2(t_n), \cdots, s_K(t_n) \},
$$

(3.51)

and

$$
D(\theta) = \begin{bmatrix} d(\theta_1) & d(\theta_2) & \cdots & d(\theta_K) \end{bmatrix},
$$

(3.52)

with

$$
d(\theta_k) = \begin{bmatrix} 0 -j\pi \cos(\theta_k) q_k \cdots -j(L-1)\pi \cos(\theta_k) q_k^{L-1} \end{bmatrix}^T.
$$

(3.53)
3.4.2 Known Signal Waveforms

We present the CRBs for the four cases of known $g$, unknown real $g$, unknown complex $g$, and $g_k = e^{j\nu_k}$, $k = 1, 2, \ldots, K$, with $\nu_k$ unknown and real. These CRBs have been obtained by modifying the results in [35] under the assumption that the signal waveforms are known. Detailed derivations of the results below may be found in Appendix E.

For $g$ known, it is shown in (E.11) of Appendix E that the MSE of any unbiased estimate of $\theta$ is bounded below by the CRB

$$\text{CRB}(\theta) = \frac{\sigma^2}{2} \{\Gamma\}^{-1},$$

(3.54)

where

$$\Gamma = \text{Re} \left[ \sum_{n=1}^{N} S^H(t_n)D^H(\theta)D(\theta)S(t_n) \right].$$

(3.55)

For $g$ unknown and complex, the MSE of any unbiased estimate of $\theta$ is bounded below by the CRB (see (E.36) of Appendix E)

$$\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{\Gamma - \text{Re} \left( \Delta^H \Lambda^{-1} \Delta \right) \right\}^{-1},$$

(3.56)

where

$$\Delta = \sum_{n=1}^{N} S^H(t_n)A^H(\theta)D(\theta)S(t_n),$$

(3.57)

$$\Lambda = \sum_{n=1}^{N} S^H(t_n)A^H(\theta)A(\theta)S(t_n).$$

(3.58)

For $g$ unknown and real, it is shown in (E.39) of Appendix E that the MSE of any unbiased estimate of $\theta$ is bounded below by the CRB

$$\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{\Gamma - \text{Re}(\Delta)^T \left[\text{Re}(\Lambda)^{-1} \text{Re}(\Delta)\right] \right\}^{-1}.$$

(3.59)
For \( g_k = e^{j\nu_k}, \ k = 1, 2, \ldots, K \), with \( \nu_k \) unknown and real, the MSE of any unbiased estimate of \( \theta \) is bounded below by the CRB (see (E.50) of Appendix E)

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \Gamma - \text{Im}(\Delta)^T [\text{Re}(\Delta)]^{-1} \text{Im}(\Delta) \right\}^{-1}.
\]  \hspace{1cm} (3.60)

Comparing the bounds in (3.49), (3.54), (3.56), (3.59), and (3.60), we note that the ratio between any two of the bounds is independent of the variance of the additive noise or the signal-to-noise ratio (SNR). Also, the bounds in (3.54), (3.56), (3.59), and (3.60) are independent of \( g \) as long as \( S(t) \) is fixed so that the SNRs of the incident signals \( s(t) \) are fixed.

It can be shown that the bounds in (3.49) and (3.56) are the same when the number of samples \( N = 1 \), as one would expect. (For \( N = 1 \), not knowing the value of complex \( g \) is the same as not knowing the only data sample of the signals.)

3.5 Typical Results

In this section, we show typical performance curves that result when the signal waveforms are either incorporated or not incorporated. For the curves below, it is assumed that two signals arrive from \( \theta_1 = 30^\circ \) and \( \theta_2 = (30 - \Delta \theta)^\circ \), so \( \Delta \theta \) is the angle separation between the signals. Each incident signal is a BPSK (binary phase-shift keyed) signal modulated by one period of a 31-bit PN (pseudonoise) sequence [31]. Both incident signals \( s_k(t), k = 1, 2, \) have unit power at each sensor and are sampled at a rate of one sample per bit. The incident signals \( s_k(t) \) are defined differently under the different assumptions for \( g \). For the case of unknown complex \( g \), we assumed \( g = [2e^{j\pi/4} \quad 2e^{-j\pi/3}]^T \). For the case of unknown real \( g \), we assumed \( g = [2 \quad 2]^T \). For the case of \( g_k = e^{j\nu_k} \) with unknown real \( \nu_k \), we
assumed \( \nu = [ \frac{x}{4}, \frac{-x}{3} ]^T \). The SNR at each sensor output, defined as \(-10 \log_{10} \sigma^2\) dB, is assumed to be 20 dB.

First, we show typical curves of the CR-bounds to illustrate the conditions under which incorporating knowledge of the known signal waveforms in the ML estimator can improve the accuracy of the angle estimates.

For the first series of curves, the two signal waveforms are from two different PN sequences with low cross-correlation. Thus the incident signals are almost uncorrelated (or non-coherent). Figure 5 shows the CR-bounds (in dB with respect to degree squared) of \( \hat{\theta}_1 \) as a function of \( \Delta \delta \) when the number of data samples is \( N = 31 \) and the number of sensors is \( L = 10 \). Figures 5(a) and 5(c) show that when \( g \) is known or when \( g \) is unknown and real, incorporating the known signal waveforms significantly improves the CR-bounds for \( \hat{\theta}_1 \), especially for small \( \Delta \theta \). Moreover, the amount of improvement is similar for these two cases of \( g \). However, Figures 5(b) and 5(d) show that when \( g \) is unknown and complex, or when \( g_k = e^{i\nu_k}, k = 1,2 \), with the \( \nu_k \) unknown and real, incorporating the known waveforms again improves the CR-bounds for \( \hat{\theta}_1 \), but the improvement for this case is limited to small \( \Delta \theta \). The amount of improvement is similar for these last two types of \( g \).

Figure 6 shows the CR-bounds for \( \hat{\theta}_1 \) as a function of the number of data samples \( N \) when \( \Delta \theta = 5^\circ \) and for \( L = 10 \) sensors for the same assumptions on \( g \) as in Figure 5. Note that the differences between the curves change little as \( N (N > 1) \) changes.

Figure 7 shows the CR-bounds for \( \hat{\theta}_1 \) as a function of the number of sensors \( L \) when \( \Delta \theta = 5^\circ \) and for \( N = 31 \) samples. The improvement in the CR-bounds decreases slightly as \( L \) increases.

For the second series of curves, the two signal waveforms are from two identical PN sequences, so the incident signals are coherent (perfectly correlated). Figures 8,
Figure 5: CR-bounds of $\hat{\theta}_1$ vs. $\Delta \theta$ for non-coherent signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 6: CR-bounds of $\hat{\theta}_1$ vs. $N$ for non-coherent signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 7: CR-bounds of $\hat{\theta}_1$ vs. $L$ for \textit{non-coherent} signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
9, and 10 show the resulting CR-bounds vs. $\Delta \theta$, $N$ and $L$, respectively. Except for the signal coherence, all parameters used in Figures 8, 9, and 10 are same as those for Figures 5, 6, and 7, respectively.

Figure 8(a) shows that when $g$ is known, incorporating the known signal waveforms in the estimator can significantly lower the CR-bound for $\hat{\theta}_1$, especially for small $\Delta \theta$. On the other hand, when $g$ is unknown and complex, utilizing the signal waveforms yields little improvement in the CR-bound for $\hat{\theta}_1$, as may be seen in Figure 8(b). For $g$ unknown and real, use of the signal waveforms lowers the CR-bound for large $\Delta \theta$ (see Figure 8(c)), whereas for $g_k = e^{j\nu_k}$ with $\nu_k$ unknown and real, use of the signal waveforms lowers the CR-bound for small $\Delta \theta$ (see Figure 8(d)). Figure 9 shows that for coherent signals the differences between the curves change little as $N$ ($N > 1$) changes. Figure 10 shows that the improvement in the CR-bounds changes only slightly as $L$ increases.

Next, we show some typical performance curves for Methods III.A, III.B.1, and III.B.2. The results below were obtained by using fifty Monte Carlo simulations with independent trials. The convergence constants used in the iterative algorithms were chosen to be $\epsilon_1 = 0.01$ and $\epsilon_2 = \epsilon_3 = 0.005^\circ$.

We first present a case where the two signal waveforms are from two different PN sequences with low cross-correlation. Figure 11 shows the MSE (in dB with respect to degrees squared) of $\hat{\theta}_1$ as a function of the $\Delta \theta$ for $N = 31$ samples and $L = 10$ sensors. Comparing Figures 11 and 5 shows that the ML estimates from Methods III.B.1 and III.B.2 are very close to the best unbiased estimates one can get. (Because of the limited number of Monte Carlo simulations, the MSE curves may occasionally fall below the CRBs.) Note also that comparing the performance of Method III.A with that of Methods III.B.1 and III.B.2 is similar to comparing the
Figure 8: CR-bounds of $\hat{\theta}_1$ vs. $\Delta \theta$ for coherent signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 9: CR-bounds of $\hat{\theta}_1$ vs. $N$ for coherent signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 10: CR-bounds of $\hat{\theta}_1$ vs. $L$ for coherent signals. Solid curve: known signal waveforms; dashed curve: unknown signal waveforms. For known signal waveforms, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}, k = 1, 2$, with $\nu_k$ unknown real.
Figure 12 shows the average number of iterations needed to obtain convergence in Methods III.B.1 and III.B.2. Note that Method III.B.1 requires fewer iterations than Method III.B.2 when \( g \) is known or when \( g \) is unknown and real. On the other hand, Method III.B.2 requires fewer iterations than Method III.B.1 when \( g \) is unknown and complex or when \( g_k = e^{j\nu_k}, k = 1, 2 \), with \( \nu_k \) unknown and real, especially for small \( \Delta \theta \). For the later two types of \( g \), Method III.B.2 is more attractive since it can be used to compute angle estimates in parallel during each iteration.

Figures 13 and 14 show simulation results when the two signal waveforms are from two identical PN sequences. The other parameters used in Figures 13 and 14 are same as those in Figures 11 and 12, respectively. Comparing Figures 13 and 14 again shows that the ML estimates from Methods III.B.1 and III.B.2 are very close to the best unbiased estimates one can get. In Figure 14, we note that the average number of iterations required by Methods III.B.1 or III.B.2 is almost the same for the case of coherent signals.

3.6 Summary

We have presented two maximum likelihood algorithms (Methods III.B.1 and III.B.2) that incorporate knowledge of known signal waveforms with unknown gains into the process of estimating the signal angles. Both algorithms compute the ML estimates iteratively and converge in a few iterations, so the complexity of a multidimensional search is avoided. For comparison, we have also used the IQML algorithm augmented to determine which signal waveform corresponds to which estimated angle. Curves were presented that compare the performance of the two ML estimators with each other and also with the IQML algorithm. Both the actual per-
Figure 11: MSE of $\hat{\theta}$ vs. $\Delta \theta$ for non-coherent signals. Dashed curve: Method III.A; solid curve with symbol $\times$: Method III.B.1; solid curve with symbol $+$: Method III.B.2; For Methods III.B.1 and III.B.2, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 12: The number of iterations vs. $\Delta \theta$ for non-coherent signals. Solid curve with symbol $\times$: Method III.B.1; solid curve with symbol $+$: Method III.B.2. For Methods III.B.1 and III.B.2, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 13: MSE of $\hat{\theta}_1$ vs. $\Delta \theta$ for coherent signals. Dashed curve: Method III.A; solid curve with symbol $\times$: Method III.B.1; solid curve with symbol $+ $: Method III.B.2; For Methods III.B.1 and III.B.2, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
Figure 14: The number of iterations vs. $\Delta \theta$ for coherent signals. Solid curve with symbol x: Method III.B.1; solid curve with symbol +: Method III.B.2. For Methods III.B.1 and III.B.2, we assume (a) $g$ known, (b) $g$ unknown complex, (c) $g$ unknown real, and (d) $g_k = e^{j\nu_k}$, $k = 1, 2$, with $\nu_k$ unknown real.
formance and the Cramer-Rao bounds were shown under several assumptions for the unknown signal gains. These curves show that the ML angle estimates of the desired signal obtained by the estimators are very close to the best unbiased estimates one can get. These curves also show the conditions under which incorporating the known signal waveforms in the estimator improves the angle estimates.
CHAPTER IV

ANGLE AND POLARIZATION ESTIMATION USING THE ESPRIT ALGORITHM WITH A POLARIZATION SENSITIVE ARRAY

4.1 Introduction

Now we take up the second major topic of this dissertation, the estimation of signal arrival angles and polarizations for electromagnetic signals with arbitrary polarizations. For this purpose, we turn to a different algorithm, the ESPRIT algorithm [13]. The ESPRIT algorithm is a signal subspace method derived for angle estimation and does not require a search over a multidimensional parameter space, as the MUSIC algorithm [12] does. The ESPRIT algorithm computes angle estimates directly and achieves this computational efficiency by exploiting the spatial invariance properties of arrays possessing displacement invariance properties. We shall extend the use of the ESPRIT algorithm and show that both angle and polarization estimates may be computed directly by using the ESPRIT algorithm with polarization-sensitive arrays.

The purpose of this chapter is to show how the ESPRIT algorithm may be used with a one-dimensional crossed dipole array to estimate signal directions and polarizations. We present two types of estimators, one that estimates both angles and polarizations and another simplified one that estimates only angles but takes polarization into account. We also study the effects of signal direction and polarization on the performance of these estimators. The approaches we describe below assume that the incident signals are uncorrelated. They can also be used for partially correlated

66
signals. The case where the signals are perfectly correlated will be treated in Chapter V. In both Chapters IV and V, we consider the case where each arrival direction is specified by one spatial angle. The problem of estimating arrival directions in two spatial angles is taken up in Chapter VI.

In Section 4.2, we define the array geometry and formulate the problem of interest. In Section 4.3, we describe how to use ESPRIT to estimate the signal directions and polarizations. In Section 4.4, we describe the simplified approach that may be used when polarization estimates are not needed. In Section 4.5, we present simulation results and describe and compare the estimation performance. Finally, Section 4.6 contains our summary.

4.2 Problem Formulation

Consider a 2L-element array consisting of L pairs of crossed dipoles, as shown in Figure 15. The signal from each dipole is to be processed separately in the direction and polarization estimation. The \( l \)th dipole pair, \( l = 1, 2, \ldots, L \), has its center on the y-axis at \( y = (l - 1)\delta \). For the \( l \)th dipole pair, let \( x_l(t) \) be the signal received from the \( x \)-axis dipole, or the dipole parallel to the \( x \)-axis, and \( y_l(t) \) the signal received from the \( y \)-axis dipole, or the dipole parallel to the \( y \)-axis. The distance between two adjacent dipole pairs \( \delta \) is assumed to be less than or equal to a half wavelength to avoid angle ambiguity problems.

Suppose \( K \) (with \( K \leq L \)) continuous wave (CW) signals impinge on the array from angular directions \( \theta_k \) in the \( y - z \) plane, \( k = 1, 2, \ldots, K \), where \( r \) and \( \theta \) denote standard polar coordinates in the \( y - z \) plane, as shown in Figure 15. Furthermore, assume each signal has an arbitrary elliptical electromagnetic polarization [38].

To specify the polarization of the signals, we use the following definitions. Given
Figure 15: A uniform linear array of crossed dipoles.
a transverse electromagnetic (TEM) wave propagating into the array, we consider
the polarization ellipse produced by its electric field as the incoming wave is viewed
from the coordinate origin. Note that unit vectors, $-\mathbf{e}_x,\mathbf{e}_\theta, -\mathbf{e}_r$, in that order, form
a right-handed coordinate system for an incoming signal. Suppose the electric field
has transverse components

$$
\mathbf{E} = -E_x\mathbf{e}_x + E_\theta\mathbf{e}_\theta.
$$

(E$_x$ will be called the horizontal component and E$_\theta$ the vertical component of the
field.) In general, as time progresses, E$_x$ and E$_\theta$ will describe a polarization ellipse
as shown in Figure 16. Given this ellipse, we define $\beta$ to be the orientation angle of
the major axis of the ellipse with respect to E$_x$, as shown in Figure 16. To eliminate
ambiguities, we define $\beta$ to be in the range $0 < \beta < \pi$. We also define the ellipticity
angle $\alpha$ to have a magnitude given by

$$
\alpha = \tan^{-1} \rho,
$$

where $\rho$ is the axial ratio

$$
\rho = \frac{\text{minor axis}}{\text{major axis}}.
$$

In addition $\alpha$ is defined positive when the electric vector rotates clockwise and neg­
ative when it rotates counterclockwise (when the incoming wave is viewed from the
coordinate origin, as in Figure 16). $\alpha$ is always in the range $-\pi/4 \leq \alpha \leq \pi/4$. Figure
16 depicts a situation in which $\alpha$ is positive.

For a given signal polarization, specified by $\alpha$ and $\beta$, the electric field components
are given by (aside from a common phase factor)

$$
E_x = E \cos \gamma, \quad \text{(4.4)}
$$

$$
E_\theta = E \sin \gamma e^{i\eta}, \quad \text{(4.5)}
$$
where $\gamma$ and $\eta$ are related to $\alpha$ and $\beta$ by [39]

$$\cos 2\gamma = \cos 2\alpha \cos 2\beta, \quad (4.6)$$
$$\tan \eta = \tan 2\alpha \csc 2\beta. \quad (4.7)$$

$\gamma$ is always in the range $0 \leq \gamma \leq \pi/2$, and $\eta$ is in the range $-\pi \leq \eta < \pi$. The inverse relations are [39]

$$\tan 2\beta = \tan 2\gamma \cos \eta, \quad (4.8)$$
$$\sin 2\alpha = \sin 2\gamma \sin \eta. \quad (4.9)$$

The relationship among the four angular variables $\alpha$, $\beta$, $\gamma$, and $\eta$ is most easily visualized by making use of the Poincaré sphere concept [39]. This technique represents the state of polarization by a point on a sphere, such as point $M$ in Figure 17. For a given $M$, $2\gamma$, $2\beta$, and $2\alpha$ form the sides a right spherical triangle, as shown. $2\gamma$ is the side of the triangle between $M$ and a point labeled $H$ in the figure; $H$ is the
Figure 17: Poincaré sphere and signal polarization.  

point representing horizontal linear polarization. Side $2\beta$ extends along the equator and side $2\alpha$ is vertical, i.e., perpendicular to side $2\beta$. The angle $\eta$ in (4.4)-(4.9) is the angle between sides $2\gamma$ and $2\beta$. The special case when $\alpha = 0$ in (4.2) and Figure 16 corresponds to linear polarization; in this case the point $M$ lies on the equator. If in addition, $\beta = 0$, only $E_x$ is nonzero and the wave is horizontally polarized. This case defines the point $H$ in Figure 17. If instead $\beta = \pi/2$, only $E_\theta$ is nonzero and the wave is vertically polarized. Point $M$ then lies on the equator diametrically behind $H$. The poles of the sphere correspond to circular polarization ($\alpha = \pm 45^\circ$), with clockwise circular polarization ($\alpha = +45^\circ$) at the upper pole.

To use (4.7) or (4.8) to solve for $\eta$ or $\beta$, the question of which solution to take for the arctangent may be resolved by considering the Poincaré sphere. For example, when $\alpha \neq 0$, Equation (4.7) may be solved correctly for $\eta$ by noting that the sign of $\eta$ must be the same as the sign of $\alpha$. When $\alpha = 0$, we have $\eta = 0$ if $0 < \beta < \pi/2$.  

and \( \eta = -\pi \) if \( \pi/2 < \beta < \pi \). To solve (4.8) for \( \beta \), we note that if \( \gamma \neq 0 \) or \( \pi/2 \), 
\[ 0 \leq \beta < \pi/2 \] 
when \( |\eta| < \pi/2 \), and \( \pi/2 < \beta < \pi \) when \( |\eta| > \pi/2 \). When \( |\eta| = \pi/2 \), 
we note that \( \beta = 0 \) if \( 0 < \gamma < \pi/4 \), and \( \beta = \pi/2 \) if \( \pi/4 < \gamma < \pi/2 \). Also, note that if \( \gamma = 0 \), then \( \beta = 0 \), and if \( \gamma = \pi/2 \), then \( \beta = \pi/2 \). These relationships are easily seen from the Poincaré sphere in Figure 17.

An arbitrary plane wave coming into the array may be characterized by three angular parameters and an amplitude. For example, the \( k \)th signal, \( k = 1, 2, \ldots, K \), will be characterized by its arrival angle \( \theta_k \), its polarization ellipticity angle \( \alpha_k \) and orientation angle \( \beta_k \), and its amplitude \( E_k \) (i.e., \( E_k \) is the value of \( E \) in (4.4) and (4.5) for the \( k \)th signal). We will say the \( k \)th signal is defined by \((\theta_k, \alpha_k, \beta_k, E_k)\).

We assume that each dipole in the array is a short dipole, i.e., the output voltage from each dipole is proportional to the electric field component along the dipole. Therefore, the outputs of the dipoles parallel to the \( x \)- and \( y \)-axis will be proportional to the \( x \)- and \( y \)-components, respectively, of the electric field. An incoming signal, with arbitrary electric field components \( E_x \) and \( E_\theta \), has \( x \), \( y \), \( z \) components:

\[
E = -E_x e_x + E_\theta e_\theta \\
= (-E_x)e_x + (E_\theta \cos \theta)e_y - (E_\theta \sin \theta)e_z.
\]

(4.10)

When \( E_x \) and \( E_\theta \) are expressed in terms of \( E \), \( \gamma \), and \( \eta \) as in (4.4) and (4.5), the electric field components become

\[
E = E \left[ (-\cos \gamma)e_x + (\sin \gamma \cos \theta e^{i\eta})e_y - (\sin \gamma \sin \theta e^{i\eta})e_z \right].
\]

(4.11)

Let us define the space phase factor

\[
q = e^{i2\pi \theta / \lambda} \sin \theta,
\]

(4.12)
where $\lambda$ is the wavelength of the signal. Including the time and space phase factors in (4.11), we find that an incoming signal characterized by $(\theta, \alpha, \beta, E)$ produces a signal vector in the crossed dipole pair centered at $y = (l - 1)\delta$ as follows:

\[
z_l = \begin{bmatrix} x_l(t) \\ y_l(t) \end{bmatrix} = u s(t) q^{l-1}, \tag{4.13}
\]

where

\[
u = \begin{bmatrix} -\cos \gamma \\ \sin \gamma \cos \theta e^{i\eta} \end{bmatrix}, \tag{4.14}
\]

and

\[s(t) = E e^{i(\omega t + \psi)}, \tag{4.15}\]

with $\omega$ the frequency of the signal and $\psi$ the carrier phase of the signal at the coordinate origin at $t = 0$.

We assume that $K$ such signals, specified by $\theta_k, k = 1, 2, \cdots, K$, are incident on the array. In addition we assume a thermal noise voltage vector $n_l(t) = \begin{bmatrix} n_{x_l}(t) \\ n_{y_l}(t) \end{bmatrix}^T$ is present on each signal vector $z_l(t)$. The $n_l(t)$, where

\[
n_l(t) = \begin{bmatrix} n_{x_l}(t) \\ n_{y_l}(t) \end{bmatrix}, \tag{4.16}
\]

are assumed to be zero mean, complex Gaussian processes statistically independent of each other, and to have covariance $\sigma^2 I$, where $I$ denotes the identity matrix.

Under these assumptions, the total signal vector received by the crossed dipole pair centered at $y = (l - 1)\delta$ is given by

\[z_l(t) = \sum_{k=1}^{K} u_k s_k(t) q_k^{l-1} + n_l(t), \quad l = 1, 2, \cdots, L, \tag{4.17}\]
where $u_k$ and $q_k$ are given by (4.14) and (4.12), respectively, with subscript $k$ added to each angular quantity, and $s_k(t)$ given by (4.15) with subscript $k$ added to the amplitude and carrier phase quantities. The $\psi_k$ are assumed to be random phase angles, each uniformly distributed on $[0,2\pi)$ and statistically independent of each other.

Let $z(t)$, $s(t)$, and $n(t)$ be column vectors containing the received signals, incident signals, and noise, respectively, i.e.,

\[
\begin{align*}
  z(t) &= \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_L(t) \end{bmatrix}, \\
  s(t) &= \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_K(t) \end{bmatrix}, \\
  n(t) &= \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_L(t) \end{bmatrix}.
\end{align*}
\]  

The received signal vector has the form

\[ z(t) = As(t) + n(t), \]  

where $A$ is a $2L \times K$ matrix

\[ A = [ \begin{array}{cccc} a_1 & a_2 & \cdots & a_K \end{array} ], \]  

with $2L \times 1$ columns

\[
  a_k = \begin{bmatrix} u_k \\ u_kq_k \\ \vdots \\ u_kq_k^{L-1} \end{bmatrix}. \]

The columns $a_k$ are assumed linearly independent. They define a $K$-dimensional signal subspace in a $2L$-dimensional space.
By assuming the columns in \( \mathbf{A} \) are linearly independent, we are excluding from consideration degenerate cases, such as when two signals arrive from the same direction with the same polarization, when more than two signals of arbitrary polarization arrive from the same direction, or when a signal causes zero output at the \( x \)- and \( y \)-axis dipoles at the same time.

We assume that the element signals are sampled at \( N \) distinct times \( t_n, n = 1, 2, \ldots, N \). The random noise vectors \( \mathbf{n}(t_n) \) at different sample times are assumed independent of each other. The problem of interest is to determine the quantities \( (\theta_k, \alpha_k, \beta_k), k = 1, 2, \ldots, K \), from the measurements \( \mathbf{z}(t_n), n = 1, 2, \ldots, N \).

### 4.3 Direction and Polarization Estimation Using ESPRIT

The array geometry described in the previous section possesses different invariance properties that may be exploited so that the ESPRIT algorithm may be used to estimate both signal directions and polarizations. We shall refer to this approach as the \textit{full polarization method}. We shall first illustrate the method for the case where the array covariance matrix and the number of incident signals are known. We then consider the practical situation where only a finite number of data samples is available and the number of incident signals is unknown.

The array covariance matrix has the form

\[
\mathbf{R} = E\{\mathbf{z}(t)\mathbf{z}^H(t)\} = \mathbf{R}_0 + \sigma^2 \mathbf{I}, \tag{4.22}
\]

where

\[
\mathbf{R}_0 = \mathbf{A} \mathbf{R}_s \mathbf{A}^H, \tag{4.23}
\]

with \((\cdot)^H\) denoting the complex conjugate transpose and \( \mathbf{R}_s = E\{\mathbf{s}(t)\mathbf{s}^H(t)\} \) representing the source covariance matrix.
From the array covariance matrix, the signal directions and polarizations may be calculated by using the ESPRIT algorithm [13, 9]. Let $\lambda_1 + \sigma^2 \geq \lambda_2 + \sigma^2 \geq \cdots \geq \lambda_K + \sigma^2 > \sigma^2 = \cdots = \sigma^2$ be the eigenvalues of $R$, and $e_1, e_2, \cdots, e_K, e_{K+1}, \cdots, e_{2L}$ be the corresponding orthonormal eigenvectors. Since the noise contribution to $R$ for this case is simply the term $\sigma^2 I$, the eigenvectors of $R$ are also the eigenvectors of $R_0$. It can be shown that the columns in $E_s = [e_1 \ e_2 \ \cdots \ e_K]$ span the same signal subspace as the column vectors in $A$ [12]. Therefore, there must exist a unique nonsingular $T$ such that

$$E_s = AT.$$  \hspace{1cm} (4.24)

The columns in $E_s$ are the *signal subspace eigenvectors*. The signal directions and polarizations are computed from them.

Consider first the calculation of the space factors $q_k$, $k = 1, 2, \cdots, K$. From Figure 15, we note that the overlapping subarrays consisting of the left and the right $L - 1$ pairs of the crossed dipoles are the same except for a displacement $\delta$ parallel to the $y$-axis. For the $k$th incident signal, the displacement $\delta$ results in the space factor $q_k$. This can also be seen from $a_k$ in (4.21). The subvectors of $a_k$ consisting of the first and the last $2(L - 1)$ elements of $a_k$ differ by the factor $q_k$. Let $A_{q1}$ and $A_{q2}$ be the $2(L - 1) \times K$ submatrices of $A$ consisting of the first and the last $2(L - 1)$ rows of $A$, respectively. Then $A_{q2} = A_{q1} \Phi_q$, where $\Phi_q$ is the diagonal matrix

$$\Phi_q = \text{diag}\{ q_1 \ q_2 \ \cdots \ q_K \}.$$  \hspace{1cm} (4.25)

Thus the subspaces spanned by the columns in $A_{q1}$ and $A_{q2}$ are the same except for the phase rotation caused by the diagonal matrix $\Phi_q$. Let $E_{q1}$ and $E_{q2}$ be the $2(L - 1) \times K$ submatrices formed from $E_s$ in the same way that $A_{q1}$ and $A_{q2}$ are
formed from $A$. Then from (4.24), we have

$$E_{q1} = A_{q1} T,$$

$$E_{q2} = A_{q2} T = A_{q1} \Phi_q T,$$

(4.26)  

(4.27)

Under the assumption that $A_{q1}$ is of full column rank, therefore, the columns of $E_{q1}$ and $E_{q2}$ span the same $K$-dimensional subspaces as the columns of $A_{q1}$ and $A_{q1} \Phi_q$, respectively. As shown in [13], the diagonal elements $q_k$, $k = 1, 2, \cdots, K$, of $\Phi_q$ are the eigenvalues of the unique matrix $\Psi_q = T^{-1} \Phi_q T$ that satisfies

$$E_{q2} = E_{q1} \Psi_q.$$  

(4.28)

Consider next the calculation of the ratios $r_k$ from which the polarization angles may be calculated, where $r_k$ is the ratio between the first and the second elements of $u_k$, i.e.,

$$r_k = \frac{-\cos \gamma_k}{\sin \gamma_k \cos \theta_k e^{j\eta_k}}, \quad k = 1, 2, \cdots, K.$$  

(4.29)

From Figure 15, we note that the space factors for the two dipoles in each of the crossed dipole pairs are the same. In a crossed dipole pair, the $k$th signal output at the $y$-axis dipole is related to the corresponding output at the $x$-axis dipole by a factor $r_k$. This can also be seen from $a_k$ in (4.21). The subvectors of $a_k$ consisting of the even and the odd numbered elements of $a_k$ differ by the factor $r_k$. Let $A_{r1}$ and $A_{r2}$ be the $L \times K$ submatrices of $A$ consisting of the even and the odd numbered rows of $A$, respectively. Then $A_{r2} = A_{r1} \Phi_r$, where $\Phi_r$ is the diagonal matrix

$$\Phi_r = \text{diag} \{ r_1, r_2, \cdots, r_K \}.$$  

(4.30)
Let $E_{r1}$ and $E_{r2}$ be the $L \times K$ submatrices formed from $E_\delta$ in the same way that $A_{r1}$ and $A_{r2}$ are formed from $A$. Then

$$E_{r1} = A_{r1}^T,$$  \hspace{1cm}  (4.31)  

$$E_{r2} = A_{r2}^T = A_{r1}^T \Phi_r^T,$$  \hspace{1cm}  (4.32)  

Under the assumption that $A_{r1}$ is of full column rank, therefore, the columns of $E_{r1}$ and $E_{r2}$ span the same $K$-dimensional subspaces as the columns of $A_{r1}$ and $A_{r1} \Phi_r$, respectively. The diagonal elements $\bar{r}_k, k = 1, 2, \cdots, K$, of $\Phi_r$ are the eigenvalues of the unique matrix $\Psi_r = T^{-1} \Phi_r T$ that satisfies

$$E_{r2} = E_{r1} \Psi_r.$$  \hspace{1cm}  (4.33)  

In the case of multiple incident signals, we must determine the pairing of the eigenvalues $\Psi_q$ and $\Psi_r$. I.e., we must determine which eigenvalue of $\Psi_q$ corresponds to which eigenvalue of $\Psi_r$. Note that

$$\Psi_{qr} = \Psi_r^{-1} \Psi_q = T^{-1} \Phi_r^{-1} \Phi_q T.$$  \hspace{1cm}  (4.34)  

I.e., the eigenvalues of $\Psi_{qr}, \bar{r}_k, k = 1, 2, \cdots, K$, are the ratios between the eigenvalues of $\Psi_q$ and their corresponding eigenvalues of $\Psi_r$. Thus, for $k_1 = 1, 2, \cdots, K$, the eigenvalue of $\Psi_r$ that corresponds to the eigenvalue $q_{k_1}$ of $\Psi_q$ is the element in the set $\{r_{k_2}, k_2 = 1, 2, \cdots, K\}$ that corresponds to the minimum of

$$\left\{ \left\| \frac{q_{k_1}}{r_{k_2}} - \bar{r}_{k_3} \right\|, k_2, k_3 = 1, 2, \cdots, K \right\}.$$  \hspace{1cm}  (4.35)  

From the paired sets of eigenvalues of $\Psi_q$ and $\Psi_r$, we can determine the pairing $(q_k, r_k), k = 1, 2, \cdots, K$. 

The arrival angles, ellipticity angles, and orientation angles can be computed from the sets \((q_k, r_k), k = 1, 2, \cdots, K\). The arrival angles \(\theta_k, k = 1, 2, \cdots, K\), are calculated from \(q_k\) as
\[
\theta_k = \sin^{-1} \left\{ \arg(q_k) \frac{|\lambda|}{2\pi} \right\}.
\] (4.36)

To determine the ellipticity angles \(\alpha_k\) and the orientation angles \(\beta_k\), we must first find \(\gamma_k\) and \(\eta_k\) from \(r_k\). The angles \(\gamma_k \in [0, \pi/2]\) and \(\eta_k \in [-\pi, \pi]\) can be determined uniquely from
\[
\gamma_k = \tan^{-1}(|\xi_k|),
\] (4.37)
\[
\eta_k = \arg(\xi_k),
\] (4.38)
where
\[
\xi_k = -\frac{1}{r_k \cos \theta_k}.
\] (4.39)
From \(\gamma_k\) and \(\eta_k\), \(\alpha_k \in [-\pi/4, \pi/4]\) and \(\beta_k \in [0, \pi]\) can then be determined as described in Section 4.2.

The full polarization method may be used to estimate the signal directions and polarizations as long as the matrices \(A, A_q,\) and \(A_r\) are of full column rank. This rank condition is satisfied in most cases. When the signals are from distinct directions and the number of signals \(K\) satisfies \(K < L\), for example, it is easy to show that the matrices \(A, A_q,\) and \(A_r\) are guaranteed to be of full column rank. Depending on the \(q_k\) and \(r_k, k = 1, 2, \cdots, K\), of the incident signals, the full polarization method can deal with a maximum of \(L\) signals.

However, the full polarization method may fail to estimate the signal directions and polarizations for some special cases. This is because even when \(A\) is assumed
to have full column rank, $A_{q1}$ or $A_{r1}$ may not have full rank. For example, when two signals with distinct polarizations arrive from the same direction, $A$ is of full rank, but $A_{r1}$ is not. For such cases, the signal polarizations cannot be obtained by solving (4.33), although the signal directions can still be found by solving (4.28).

Thus whether the full polarization method can be used or not depends on the rank conditions of $A_{q1}$ and $A_{r1}$. The rank conditions of $A_{q1}$ and $A_{r1}$ can be determined from the rank conditions of $E_{q1}$ and $E_{r1}$, respectively, as can be seen from (4.26) and (4.31).

For the cases where the full polarization method fails, the computationally expensive MUSIC algorithm [12] may be used to determine the signal directions and polarizations as long as $A$ has full column rank and $K < 2L$.

In the following, we shall restrict our consideration to the cases where the full polarization method is applicable. For these cases, when the array covariance matrix is known exactly, the signal directions and polarizations can be found exactly by using the full polarization method.

In practical situations, we have only a finite number of noisy measurements made at the dipole outputs. The estimates of the signal directions and polarizations must be made from the available measurements. Also, the number of incident signals is unknown and must be estimated from the measurements. We shall use the minimum description length (MDL) criterion described by Wax and Kailath [33] to estimate the number of incident signals. The total least squares (TLS) algorithm [13, 40] is used to estimate $\Psi_q$ and $\Psi_r$. The estimation steps in the full polarization method are as follows:

1) Compute the maximum likelihood estimate of the array covariance matrix $R$
\[
\hat{R} = \frac{1}{N} \sum_{n=1}^{N} z(t_n)z^H(t_n),
\]
(4.40)

where \( N \) denotes the number of measurements.

2) Compute the eigenvalues \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{2L} \) of \( \hat{R} \).

3) Estimate the number of incident signals \( \hat{K} \) using the MDL criterion. The MDL estimate \( \hat{K} \) of the number of signals is given by the value of \( K \in \{0, 1, \cdots, 2L-1\} \) that minimizes the following MDL function [33]:

\[
\text{MDL}(K) = -\log \left\{ \frac{\prod_{i=K+1}^{2L} \hat{\lambda}_i^{1/(2L-K)}}{\frac{1}{2L-K} \sum_{i=K+1}^{2L} \hat{\lambda}_i} \right\}^{(2L-K)N} + \frac{1}{2} K(4L-K) \log N.
\]
(4.41)

4) Obtain \( \hat{E}_s \) whose columns are the eigenvectors of \( \hat{R} \) that correspond to the \( \hat{K} \) largest eigenvalues of \( \hat{R} \).

5) Form \( \hat{E}_q1, \hat{E}_q2, \hat{E}_r1, \) and \( \hat{E}_r2 \) from \( \hat{E}_s \) in the same way that \( E_q1, E_q2, E_r1, \) and \( E_r2 \) are formed from \( E_s \).

6) Calculate the TLS (Total Least Squares) solution \( \hat{\Psi}_q \) [13, 40] from

\[
\hat{E}_q2 = \hat{E}_q1 \hat{\Psi}_q.
\]
(4.42)

To calculate \( \hat{\Psi}_q \), let \( V \) be the \( 2\hat{K} \times \hat{K} \) matrix containing the \( \hat{K} \) right singular vectors of \( [\hat{E}_q1: \hat{E}_q2] \) that correspond to the \( \hat{K} \) largest singular values of \( [\hat{E}_q1: \hat{E}_q2] \). Let \( V_1 \) and \( V_2 \) be the \( \hat{K} \times \hat{K} \) submatrices of \( V \) consisting of the first and the last \( \hat{K} \) rows of \( V \). Then the TLS solution \( \hat{\Psi}_q \) is

\[
\hat{\Psi}_q = [V_1^H]^{-1} V_2^H.
\]
(4.43)

7) Calculate the TLS solution \( \hat{\Psi}_r \) from

\[
\hat{E}_r2 = \hat{E}_r1 \hat{\Psi}_r.
\]
(4.44)
8) Compute \( \hat{q}_k, \hat{r}_k, \) and \( \hat{r}_k \), \( k_1, k_2, k_3 = 1, 2, \cdots, K \), by determining the eigenvalues of \( \hat{\Psi}_q, \hat{\Psi}_r, \) and \( \hat{\Psi}_{qr} = \hat{\Psi}_r^{-1}\hat{\Psi}_q \), respectively.

9) Determine the pairing scheme \((\hat{q}_k, \hat{r}_k), k = 1, 2, \cdots, K, \) as discussed in (4.34)-(4.35).

10) Calculate the direction and polarization estimates \((\hat{\theta}_k, \hat{\alpha}_k, \hat{\beta}_k), k = 1, 2, \cdots, K, \) from \((\hat{q}_k, \hat{r}_k) \) as discussed in (4.36)-(4.39).

4.4 The Simplified Approach

In the previous section, we have described how the ESPRIT algorithm may be applied to \( z(t_n), n = 1, 2, \cdots, N \), to estimate both the arrival directions and the polarizations of the incoming plane waves. In the present section, we consider the case where we are interested in estimating the signal directions only. We describe below a simple approach that may be used when polarization estimates are not needed. The simplified approach described here will be called the angle-only method.

The central idea of the angle-only method is to consider the \( x \)-axis dipoles and the \( y \)-axis dipoles as separate subarrays. The average of the covariance matrices for the \( x \)- and \( y \)-axis subarrays is then used in the ESPRIT algorithm instead of the full covariance matrix of the crossed dipole array. The signal directions are estimated from the averaged \( x \)- and \( y \)-axis covariance matrices.

Specifically, let \( x(t) \) denote the column vector of signals received on the \( x \)-axis dipoles, i.e.,

\[
x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_L(t) \end{bmatrix}^T.
\]

(4.45)

Note that \( x(t) \) is the subvector of \( z(t) \) consisting of the odd-numbered elements of \( z(t) \). Let \( n_x(t) \) be a vector containing the corresponding noise voltages,
\[ n_x(t) = \begin{bmatrix} n_{x_1}(t) & n_{x_2}(t) & \cdots & n_{x_L}(t) \end{bmatrix}^T. \] (4.46)

\( n_x(t) \) is a zero-mean complex Gaussian process with covariance \( \sigma^2 \mathbf{I} \). Then \( x(t) \) can be written

\[ x(t) = \overline{A}_L \Phi x s(t) + n_x(t), \] (4.47)

where \( \overline{A}_L \) is the direction matrix

\[ \overline{A}_L = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_K \\ \vdots & \vdots & \cdots & \vdots \\ q_1^{L-1} & q_2^{L-1} & \cdots & q_K^{L-1} \end{bmatrix}, \] (4.48)

and \( \Phi x \) is defined as

\[ \Phi x = \text{diag} \{ -\cos \gamma_1, -\cos \gamma_2, \cdots, -\cos \gamma_K \}. \] (4.49)

The covariance matrix of \( x(t) \) is given by

\[ R_x = E \{ x(t)x^H(t) \} = \overline{A}_L \Phi x R_s \Phi^H x \overline{A}_L^H + \sigma^2 \mathbf{I}. \] (4.50)

Similarly, let \( y(t) \) denote the column vector of signals received on the \( y \)-axis dipoles, i.e.,

\[ y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_L(t) \end{bmatrix}^T. \] (4.51)

\( y(t) \) is the subvector of \( z(t) \) consisting of the even-numbered elements of \( z(t) \). \( y(t) \) can be written

\[ y(t) = \overline{A}_L \Phi y s(t) + n_y(t), \] (4.52)
where

$$\Phi_y = \text{diag}\left\{-\sin \gamma_1 \cos \theta_1 e^{i\eta_1}, -\sin \gamma_2 \cos \theta_2 e^{i\eta_2}, \ldots, -\sin \gamma_K \cos \theta_K e^{i\eta_K}\right\},$$

and

$$n_y(t) = \begin{bmatrix} n_{y_1}(t) & n_{y_2}(t) & \cdots & n_{y_L}(t) \end{bmatrix}^T.$$  \hfill (4.54)

\(n_y(t)\) is also a zero-mean complex Gaussian process with covariance \(\sigma^2 I\). The covariance matrix of \(y(t)\) is

$$R_y = E\left\{y(t)y^H(t)\right\} = \overline{A}_L \Phi_y R_s \Phi_y^H \overline{A}_L^H + \sigma^2 I.$$  \hfill (4.55)

Next let \(\overline{R}\) be the average of \(R_x\) and \(R_y\),

$$\overline{R} = \frac{1}{2} (R_x + R_y).$$  \hfill (4.56)

\(\overline{R}\) can be written

$$\overline{R} = \overline{A}_L \overline{R}_s \overline{A}_L^H + \sigma^2 I,$$  \hfill (4.57)

where \(\overline{R}_s\) is

$$\overline{R}_s = \frac{1}{2} \left( \Phi_x R_s \Phi_x^H + \Phi_y R_s \Phi_y^H \right),$$ \hfill (4.58)

$$= \frac{1}{2} \begin{bmatrix} \Phi_x & \Phi_y \end{bmatrix} \begin{bmatrix} R_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} \Phi_x^H \\ \Phi_y^H \end{bmatrix}. \hfill (4.59)$$

Since the incident signals are uncorrelated, \(R_s\) is nonsingular. As long as none of the incident signals produces a zero output on both the \(x\)- and \(y\)-axis dipoles at the same time, \([\Phi_x \mid \Phi_y]\) is of rank \(K\). Then \(\overline{R}_s\) is also nonsingular.
To apply the ESPRIT algorithm [13, 9] to $\mathbf{R}$, the direction matrix $\mathbf{A}_L$ must be of full column rank. Therefore the angles of arrival must be distinct so that the columns of $\mathbf{A}_L$ define a $K$-dimensional signal subspace in an $L$-dimensional space.

From $\mathbf{R}$, the signal directions may be calculated as follows [13, 9]. Let $\lambda_1 + \sigma^2 \geq \lambda_2 + \sigma^2 \geq \cdots \geq \lambda_K + \sigma^2 > \cdots = \sigma^2$ be the eigenvalues of $\mathbf{R}$, and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_K, \mathbf{e}_{K+1}, \cdots, \mathbf{e}_L$ be the corresponding orthonormal eigenvectors. The columns in $\mathbf{E}_s = [ \mathbf{e}_1 \mathbf{e}_2 \cdots \mathbf{e}_K ]$ are referred to as *signal subspace eigenvectors*. They span the same signal subspace as the direction vectors in $\mathbf{A}_L$.

Let $\mathbf{E}_{q1}$ and $\mathbf{E}_{q2}$ be the $(L-1) \times K$ submatrices of $\mathbf{E}_s$ consisting of the first and the last $L-1$ rows of $\mathbf{E}_s$, respectively. Then the columns in $\mathbf{E}_{q1}$ and $\mathbf{E}_{q2}$ span the same subspaces as the columns in $\mathbf{A}_{L-1}$ and $\mathbf{A}_{L-1}\Phi_q$, respectively, where

$$\Phi_q = \text{diag}\{q_1, q_2, \cdots, q_K\}.$$  \hfill (4.60)

As shown in [13], the diagonal elements of $\Phi_q$ are the eigenvalues of the unique matrix $\mathbf{V}_q$ that satisfies

$$\mathbf{E}_{q2} = \mathbf{E}_{q1} \mathbf{V}_q.$$  \hfill (4.61)

From the diagonal elements of $\Phi_q$, the signal directions $\theta_k$ can be computed from (4.36).

The approach described above is computationally simpler than the full polarization method in Section 4.3, because it requires the eigendecomposition of only the $L \times L$ matrix $\mathbf{R}$ in (4.56). The full polarization method requires the eigendecomposition of $\mathbf{R} = E \{ z(t)z^H(t) \}$, which is $2L \times 2L$. Hence the new approach requires approximately one eighth as many computations [41].

If the ideal array covariance matrix in (4.56) were known, the signal directions could be calculated exactly with ESPRIT. In practical situations, however, only a
finite number of noisy measurements are taken at the dipole outputs, and the estimates of the signal directions must be made from the available measurements. Also, the number of incident signals is unknown and must be estimated. The minimum description length (MDL) criterion described by Wax and Kailath [33] can be used to estimate the number of incident signals, and the total least squares (TLS) ESPRIT algorithm [13, 40] can be used to estimate \( \mathbf{\Psi}_q \). The steps in this process are as follows:

1) Compute

\[
\hat{\mathbf{R}} = \frac{1}{2N} \sum_{n=1}^{N} \left[ \mathbf{x}(t_n)\mathbf{x}^H(t_n) + \mathbf{y}(t_n)\mathbf{y}^H(t_n) \right], \tag{4.62}
\]

where \( N \) denotes the number of measurements.

2) Compute the eigenvalues \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_L \) of \( \hat{\mathbf{R}} \).

3) Estimate the number of incident signals \( \hat{K} \) using the MDL criterion. The MDL estimate \( \hat{K} \) of the number of signals is the value of \( K \in \{0, 1, \cdots, L-1\} \) that minimizes the following MDL function [33]:

\[
\text{MDL}(K) = -\log \left\{ \prod_{i=K+1}^{L} \frac{\hat{\lambda}_i^{1/(L-K)}}{\hat{\lambda}_i} \right\}^{(L-K)^N} + \frac{1}{2} K(2L-K) \log N. \tag{4.63}
\]

4) Obtain \( \hat{\mathbf{E}}_s \) whose columns are the eigenvectors of \( \hat{\mathbf{R}} \) that correspond to the \( \hat{K} \) largest eigenvalues of \( \hat{\mathbf{R}} \).

5) Form \( \hat{\mathbf{E}}_{q1} \) and \( \hat{\mathbf{E}}_{q2} \) from \( \hat{\mathbf{E}}_s \) in the same way that \( \mathbf{E}_{q1} \) and \( \mathbf{E}_{q2} \) are formed from \( \mathbf{E}_s \).

6) Calculate the TLS (Total Least Squares) solution \( \hat{\mathbf{\Psi}}_q \) from \( \hat{\mathbf{E}}_{q2} = \hat{\mathbf{E}}_{q1} \hat{\mathbf{\Psi}} \) [40].

7) Compute \( \hat{\vartheta}_k, k = 1, 2, \cdots, \hat{K} \), by determining the eigenvalues of \( \hat{\mathbf{\Psi}}_q \).

8) Calculate the direction estimates \( \hat{\vartheta}_k \) from
\[ \hat{\theta}_k = -\sin^{-1}\left\{ \frac{\lambda}{2\pi\delta} \arg(\hat{q}_k) \right\}, \quad k = 1, 2, \ldots, \hat{K}. \] (4.64)

4.5 Simulation Results

We show below several examples illustrating the use of the estimators described in the previous two sections. We also compare these estimators with ESPRIT estimators using only the x- or y-axis dipoles (i.e., a conventional ESPRIT array).

The results below were obtained by using fifty Monte Carlo simulations. The array used in the examples consists of \( L = 5 \) pairs of crossed dipoles and a spacing \( \delta \) between adjacent dipole pairs of a half wavelength. All incident signals are assumed to have the same unit amplitude \( E_k \). The signal-to-noise ratio (SNR) used in the simulations \((-10\log_{10}\sigma^2 \text{ dB})\) is assumed to be 20 dB. In the examples, the number of data samples taken at each dipole output is \( N = 31 \).

Before we present simulation examples, however, we first describe the method we shall use to describe the accuracy of the polarization estimates.

Of course, the performance of the polarization estimation could be presented just by showing the variance of each of the polarization estimates \( \hat{\alpha} \) and \( \hat{\beta} \). However, the importance of an error in one of the two parameters usually depends on what the other parameter is. For example, an error in \( \hat{\beta} \) is significant when \( \alpha = 0 \) but of no significance when \( \alpha = \pi/4 \). Moreover, since the value of \( \beta \) is indeterminate when \( \alpha = \pi/4 \), the variance of \( \hat{\beta} \) tends to be very high for \( \alpha = \pi/4 \). The high variance of \( \hat{\beta} \) near \( \alpha = \pi/4 \) is misleading, because errors in \( \hat{\beta} \) are not important in this case. For this reason, we shall use a different measure of polarization estimation performance.

For polarization estimates, we define the estimation error to be the spherical distance between the two points \( \hat{M} \) and \( \hat{M} \) on the Poincaré sphere that represent the actual signal polarization \( (\alpha, \beta) \) and the estimated polarization \( (\hat{\alpha}, \hat{\beta}) \), respectively.
Figure 18: Great circle arc between points $M$ and $\hat{M}$.

This definition is reasonable, because it is well-known that the response of a receiving antenna of one polarization to a signal of another polarization depends only on the distance between the two polarization points on the Poincaré sphere [39].

The distance between two points $M$ and $\hat{M}$ is computed as follows. Let $M$ and $\hat{M}$ be the two points corresponding to $(\gamma, \eta)$ and $(\hat{\gamma}, \hat{\eta})$, respectively, and let $\zeta$ be the angular distance between $M$ and $\hat{M}$. Then $2\gamma$, $2\hat{\gamma}$, and $\zeta$ form the sides of a spherical triangle, as shown in Figure 18. The angle $\eta - \hat{\eta}$ is the angle opposite side $\zeta$. Using the well-known law of cosines in spherical trigonometry [42], we have

$$
\cos \zeta = \cos 2\gamma \cos 2\hat{\gamma} + \sin 2\gamma \sin 2\hat{\gamma} \cos(\eta - \hat{\eta}).
$$

(4.65)

where $\zeta$ is always in the range $0 \leq \zeta \leq \pi$. The variances of the polarization estimates plotted below are the mean-squared values of $\zeta$ for all the Monte Carlo simulations.
We first present below several examples showing the use of the full polarization method described in Section 4.3 for direction and polarization estimation.

We begin with the case of a linearly polarized signal ($\alpha = 0^\circ$). For this case, we have

\[
 u = \begin{cases} 
 -\cos \gamma \\
 \sin \gamma \cos \theta 
\end{cases}, \quad \text{if } 0^\circ \leq \beta < 90^\circ, \\
\begin{cases} 
 -\cos \gamma \\
 -\sin \gamma \cos \theta 
\end{cases}, \quad \text{if } 90^\circ < \beta < 180^\circ.
\] (4.66)

Note that the dipole outputs depend only on the polarization angle $\gamma$ and the arrival angle $\theta$. For example, if the signal is horizontally polarized ($\beta = 0^\circ$), then $\gamma = 0^\circ$. For this case, the signal output from a $y$-axis dipole is zero while the signal output from an $x$-axis dipole has maximum power. If the signal is vertically polarized ($\beta = 90^\circ$), then $\gamma = 90^\circ$, and the signal from a $y$-axis dipole has power proportional to $\cos^2 \theta$ while the signal from an $x$-axis dipole is zero.

For the linearly polarized signal, Figure 19 shows the variance (in dB with respect to degrees squared) of the direction and polarization estimates as a function of $\beta$ for several elevation angles $\theta$. (The discontinuity of the curves at $\beta = 90^\circ$ is caused by the discontinuity of $u$ at $\beta = 90^\circ$.) Note that the polarization has little effect on either the direction or the polarization estimates when $\theta$ is small. This is because for small $\theta$, $\cos^2 \theta$ is close to 1. In this case, no matter what the signal polarization, the outputs of the $x$- and $y$-axis dipoles are not close to zero at the same time. For large $\theta$, however, $\beta$ has more effect on the error variances. When $\theta$ is large, $\cos^2 \theta$ is close to zero. For a vertically polarized signal from a large $\theta$, for example, the signal on a $y$-axis dipole is close to zero while the signal on an $x$-axis
dipole is zero. Hence, for large $\theta$, the direction and polarization estimates are greatly affected by the signal polarization, as seen in Figure 19.

Figure 20 shows another example of a single signal with $\beta = 0^\circ$. For this case, we have

$$u = \begin{cases} 
  \begin{bmatrix} -\cos \gamma \\
  -j \sin \gamma \cos \theta \\
  -\cos \gamma \\
  j \sin \gamma \cos \theta \end{bmatrix}, & \text{if } -45^\circ \leq \alpha < 0^\circ, \\
  \begin{bmatrix} -\cos \gamma \\
  -j \sin \gamma \cos \theta \\
  -\cos \gamma \\
  j \sin \gamma \cos \theta \end{bmatrix}, & \text{if } 0^\circ < \beta \leq 45^\circ.
\end{cases}$$ (4.67)

Note that the dipole outputs again depend only on the polarization angle $\gamma$ and the arrival angle $\theta$. For example, if the signal is horizontally polarized ($\alpha = 0^\circ$), then $\gamma = 0^\circ$. For this case, the signal output from a $y$-axis dipole is zero while the signal output from an $x$-axis dipole has maximum power. If the signal is circularly polarized ($\alpha = \pm 45^\circ$), then $\gamma = \pm 45^\circ$. For this case, the signal from a $y$-axis dipole has power proportional to $0.5 \cos^2 \theta$ while the signal from an $x$-axis dipole is half of the maximum power possible.

Figure 20 shows the variance of the direction and polarization estimates as a function of $\alpha$ for several $\theta$. Note that the accuracy of both the direction and polarization estimates is better for small $\theta$. Note also that as in the previous example, the polarization has little effect on either the direction or the polarization estimates when $\theta$ is small. For large $\theta$, $\alpha$ has more effect on the error variances. The polarization effect, however, is within 3 dB and is not as significant as in the previous example. This is because for this case, $-45^\circ \leq \gamma \leq 45^\circ$. Although the signal on a $y$-axis dipole is close to zero, the signal on an $x$-axis dipole is not close to zero for any $\alpha$. The signal on an $x$-axis dipole has at least half of the maximum power possible.
Figure 19: Variance of estimates versus $\beta$ for a linearly polarized signal.
Figure 20: Variance of estimates versus $\alpha$ for a signal with $\beta = 0^\circ$. 

(a) Variance of direction estimates.

(b) Variance of polarization estimates.
Now we present three examples that illustrate how separation in direction and polarization between two incident signals affect the performance of the full polarization method. Consider first a case where two incident signals have closely-spaced arrival angles and identical polarization. Suppose the two signals have identical circular polarization, \( \alpha_1 = \alpha_2 = 45^\circ \), (so the polarization separation is \( \zeta_p = 0^\circ \)), and arrive from \( \theta_1 = 20^\circ \) and \( \theta_2 = 24^\circ \) (so the angle difference \( \Delta \theta = \theta_2 - \theta_1 = 4^\circ \)). Figure 21 shows typical results obtained with the estimator described above. In Figure 21(a), each arrival angle estimate obtained with one of the 50 independent Monte Carlo trials is plotted on a unit circle at that angle from the center of the circle. The 50 pairs of estimates are superimposed on the same plot, so spread in the estimates can be seen. In Figure 21(b), the estimates of ellipticity and orientation angles obtained with each of the 50 independent trials are plotted on a polar plot with \( \cos 2\alpha \) as the radius and \( 2\beta \) as the angle.

Next consider a case where the polarizations of the incident signals are widely separated. Suppose the signals have opposite circular polarizations, i.e., \( \alpha_1 = 45^\circ \) and \( \alpha_2 = -45^\circ \). The signal arrival angles are assumed the same as above. For this case, \( \Delta \theta = 4^\circ \) and \( \zeta_p = 180^\circ \). Figure 22 shows the results for this case.

Figures 22(a) and 21(a) show that the direction resolution is improved significantly by when the polarization separation is large. This result occurs because when the two signals arrive from closely-spaced directions with identical polarizations, the columns of matrix \( \mathbf{A} \) in (4.20) become almost identical. Thus \( \mathbf{A} \), \( \mathbf{A}_{q_1} \), and \( \mathbf{A}_{r_1} \) become ill-conditioned, which makes the ESPRIT algorithm more sensitive to noise. Specifically, an ill-conditioned \( \mathbf{A} \) results in an ill-conditioned \( \mathbf{R}_0 \) in (4.23). Since the noise contribution to \( \hat{\mathbf{R}} \) is not just \( \sigma^2 \mathbf{I} \) when the number of data samples is finite, the signal subspace eigenvectors of \( \hat{\mathbf{R}} \) are perturbed by the noise from the true eigenvec-
Figure 21: Polar plot of estimates when $\Delta \theta = 4^\circ$ and $\zeta_p = 0^\circ$ ($\alpha_1 = \alpha_2 = 45^\circ$).
(a) Direction estimates.

(b) Polarization estimates.

Figure 22: Polar plot of estimates when $\Delta \theta = 4^\circ$ and $\zeta_p = 180^\circ$ ($\alpha_1 = 45^\circ$, $\alpha_2 = -45^\circ$).
tors of \( \mathbf{R}_0 \). The ill-conditioning of \( \mathbf{R}_0 \) then makes the signal subspace eigenvectors of \( \mathbf{R}_0 \) more sensitive to this perturbation [43]. Increasing the polarization separation reduces the ill-conditioning of \( \mathbf{A} \) and \( \mathbf{A}_{q1} \) and thus results in improved direction estimation.

Figures 22(b) and 21(b) show that the polarization estimates are also improved when \( \zeta_p = 180^\circ \), but not as much as the direction estimates. This is because the ill-conditioning of \( \mathbf{A}_{r1} \) is not reduced by increasing the polarization separation.

Finally, we consider an example in which the directions of the incident signals are more widely separated. Suppose the two signals have identical circular polarization (\( \alpha = 45^\circ \)) and arrive from 20° and 40°. For this case, \( \Delta \theta = 20^\circ \) and \( \zeta_p = 0^\circ \). Figure 23 shows the results for this case. By comparing Figures 23(b) and 21(b), it is seen that the polarizations are more easily resolved and the direction estimates are improved by increasing the direction separation. Increasing the direction separation of the two signals reduces the ill-conditioning of \( \mathbf{A} \), \( \mathbf{A}_{q1} \), and \( \mathbf{A}_{r1} \).

We next show below several examples that compare the performance of this estimator with that of the full polarization method. We also compare both estimators with ESPRIT estimators using only the x- or y-axis dipoles (i.e., a conventional ESPRIT array).

We begin with the case of a single linearly polarized signal (\( \alpha = 0^\circ \)). Figure 24 shows the variance (in dB with respect to degrees squared) of the direction estimate \( \hat{\theta} \) as a function of \( \beta \) when \( \theta = 20^\circ \). Four curves are shown, one for the full polarization method, one for the angle-only method discussed above, and two for conventional ESPRIT estimators using only the x-dipoles or y-dipoles.

Note that the performance of the angle-only method is almost the same as that of the full polarization method and is not sensitive to \( \beta \). The performance of
(a) Direction estimates.

(b) Polarization estimates.

Figure 23: Polar plot of estimates when $\Delta \theta = 20^\circ$ and $\zeta_p = 0^\circ$ ($\alpha_1 = \alpha_2 = 45^\circ$).
Figure 24: Variance of $\hat{\theta}$ versus $\beta$ for a linearly polarized signal ($\alpha = 0^\circ$), $\theta = 20^\circ$. Solid curve: full polarization method; Dashed curve: angle-only method (the solid and dashed curves overlap for this case); Dotted curve: $x$-axis dipoles only; Dashdot curve: $y$-axis dipoles only.
the conventional ESPRIT estimators using only the $x$- or $y$-axis dipoles, however, is sensitive to $\beta$. For example, when only the $x$-axis dipoles are used, the angle estimates deteriorate rapidly as $\beta$ approaches $90^\circ$ (as the signal becomes vertically polarized). Of course, the reason is that the signals on the $x$-axis dipoles approach zero as $\beta$ approaches $90^\circ$. It is thus an advantage to use an array with elements responding to more than one polarization.

Figure 25 shows another example of a single signal with $\beta = 0^\circ$ and $\theta = 20^\circ$. Figure 25 shows the variance of the direction estimates as a function of the ellipticity angle $\alpha$. Note again that the performance of the angle-only method is almost the same as that of the full polarization method and is not sensitive to $\alpha$. The performance of a conventional ESPRIT estimator using the $x$-axis dipoles is also not very sensitive to $\alpha$, because the signals on the $x$-axis dipoles are never close to zero for any $\alpha$. For an ESPRIT estimator using the $y$-axis dipoles, however, the variance blows up when the polarization approaches linear (near $\alpha = 0^\circ$).

Next we consider a case where two signals arrive from $\theta_1 = 20^\circ$ and $\theta_2 = 20^\circ + \Delta \theta$, so $\Delta \theta$ is the angular separation between the two signals. We assume the corresponding ellipticity angles are $\alpha_1 = 45^\circ$ and $\alpha_2 = 45^\circ - \Delta \alpha$ and the orientation angles are $\beta_1 = \beta_2 = 0^\circ$, so $\Delta \alpha$ is the only difference in polarization between the two signals. Figure 26 shows the variance of $\hat{\theta}_1$ as a function of $\Delta \alpha$ for two values of $\Delta \theta$. The variance is obtained by taking the smaller of the two angle estimates as $\hat{\theta}_1$.

Note that the performance of the angle-only approach is not sensitive to $\Delta \alpha$, but the performance of the full polarization approach is. For small $\Delta \theta$ and large $\Delta \alpha$, the full polarization method yields better performance than the angle-only method. For large $\Delta \theta$ or small $\Delta \alpha$, the two methods yield similar performance. The problem when $\Delta \theta$ is small and $\Delta \alpha$ is large occurs because $\overline{A}_L$ in (4.48) is ill-conditioned but
Figure 25: Variance of $\hat{\theta}$ versus ellipticity $\alpha$ for $\beta = 0^\circ$, $\theta = 20^\circ$. Solid curve: full polarization method; Dashed curve: angle-only method (the solid and dashed curves overlap for this case); Dotted curve: $x$-axis dipoles only; Dashdot curve: $y$-axis dipoles only.
Figure 26: Variance of $\hat{\theta}$ versus $\Delta \alpha$. Solid curve: full polarization method; Dashed curve: angle-only method; Dotted curve: $x$-axis dipoles only; Dashdot curve: $y$-axis dipoles only.

(a) $\Delta \theta = 4^\circ$.

(b) $\Delta \theta = 20^\circ$. 
A in (4.20) is not. The ill-conditioned $\mathbf{A}_L$ makes it hard to resolve the columns of $\mathbf{A}_L$ and thus to resolve the two closely-spaced directions.

Note also from Figure 26 that for a conventional ESPRIT estimator using the $y$-axis dipoles, the variance of $\hat{\theta}_1$ increases as $\Delta \alpha$ approaches $45^\circ$ (linear polarization) even though $\Delta \alpha$ affects only the polarization of the signal from $\theta_2$. The reason is that as $\Delta \alpha$ approaches $45^\circ$, the signals on the $y$-axis dipoles due to signal 2 go to zero, so the estimates of $\theta_2$ become very poor. Because the smaller of the two estimated angles take to be $\hat{\theta}_1$, a large error in the angle estimate for signal 2 can cause the two estimated angles to be assigned incorrectly to $\hat{\theta}_1$ and $\hat{\theta}_2$. The result is a large increase in the computed variance of $\hat{\theta}_1$.

We remark that the above results were obtained by assuming that the number of incident signals $K$ is known. We found, however, that the MDL criterion provided accurate estimates of $K$ for both the full polarization method and the angle-only method in all simulations. For a conventional ESPRIT estimator using only the $x$- or $y$-axis dipoles, however, estimates of $K$ can be wrong for certain signal polarizations.

4.6 Summary

We have described a full polarization method that uses the ESPRIT algorithm to estimate both the arrival directions and the polarizations of incoming plane waves with a uniform linear array of crossed dipoles. We have also described a simpler angle-only method that can be used to estimate only the arrival directions. The angle-only method requires approximately one eighth as many computations as the full polarization method. Typical examples showing the use of the estimators have been presented.
CHAPTER V

ANGLE AND POLARIZATION ESTIMATION IN A COHERENT SIGNAL ENVIRONMENT

5.1 Introduction

In the previous chapter, we described how the ESPRIT algorithm may be used to estimate the signal directions and the polarizations of incoming plane waves with a uniform linear array of crossed dipoles. Both the full polarization method and the angle-only method given in Chapter IV assume that the incident signals are uncorrelated. They can also be used for partially correlated signals. However, the performance of these methods will degrade rapidly as the incident signals become highly correlated. They fail to work properly when the signals are coherent (i.e., perfectly correlated).

The purpose of this chapter is to show how the ESPRIT algorithm can be combined with spatial smoothing techniques [17, 18, 19, 20, 21] and used with the same uniform linear array of crossed dipoles to estimate signal directions and polarizations for coherent signals. We present one method of spatial smoothing that can be used when it is necessary to estimate both the arrival angles and the polarizations of signals. We also present two additional methods that can be used when only the signal arrival angles are of interest, not the polarizations, but it is still necessary that the estimator work properly with arbitrarily polarized signals.

In Section 5.2, we formulate the problem. In Section 5.3, we show how forward-only spatial smoothing [18, 19] can be combined with ESPRIT and used with a
polarization-sensitive array to estimate both signal directions and polarizations for coherent signals. In Section 5.4, we describe an alternative procedure based on the angle-only method given in Chapter IV that may be used with both forward-only and forward/backward [20, 21] spatial smoothing to estimate signal directions only. In Section 5.5, we present some typical examples and compare the performance of the different techniques. Finally, Section 5.6 contains our summary.

5.2 Problem Formulation

The problem considered in this chapter is the same as the one described in Section 4.2 except that the incident signals $s_k(t)$ are now assumed perfectly correlated. We use the same array as in Chapter IV, shown in Figure 15. The carrier phase angle $\psi_1$ of $s_1(t)$ at the coordinate origin at $t = 0$ is assumed to be a random variable uniformly distributed on $[0, 2\pi)$. The other $\psi_k$ of $s_k(t)$, for $k = 2, 3, \cdots, K$, at the coordinate origin at $t = 0$ are also random variables but it is assumed that each of these differs from $\psi_1$ by a fixed amount. Thus, the $\psi_k$ are all rigidly tied to one another, and the incident signals are coherent (i.e., perfectly correlated). In this case, each signal $s_k(t)$ can be written as a scaled replica of some signal $w_0(t)$ for which $E \{|w_0(t)|^2\} = 1$, i.e.,

$$s_k(t) = g_k w_0(t), \quad k = 1, 2, \cdots, K,$$

(5.1)

where $g_1$ is a real constant, and $g_2, g_3, \ldots, g_K$ are complex constants. The problem of interest is again to determine the quantities $(\theta_k, \alpha_k, \beta_k)$, $k = 1, 2, \cdots, K$, from the measurements $z(t_n)$, $n = 1, 2, \cdots, N$.

In Section 5.3, we consider how to estimate all three parameters $\theta_k, \alpha_k$ and $\beta_k$ from the $z(t_n)$. In Section 5.4, we consider what can be done if we need to estimate
only the $\theta_k$.

### 5.3 Estimating Both Direction and Polarization for Coherent Signals

Consider the array covariance matrix of $z(t)$ which has the form

$$ R = E \{z(t)z^H(t)\} = R_0 + \sigma^2 I, \quad (5.2) $$

where

$$ R_0 = AR_sA^H, \quad (5.3) $$

with $(\cdot)^H$ denoting the complex conjugate transpose and $R_s = E\{s(t)s^H(t)\}$ representing the source covariance matrix.

In general, if the signals $s_1(t), \ldots, s_K(t)$ are uncorrelated, $R_s$ is diagonal. If the signals are partially correlated, $R_s$ is nondiagonal but nonsingular. For the case considered here, the signals are completely correlated, so $R_s$ is nondiagonal and singular. As long as the $s_k(t)$ are not completely correlated, the eigenvectors of $R_0$ (or $R$) that correspond to the $K$ largest eigenvalues of $R_0$ (or $R$) span the same signal subspace as the column vectors in $A$ [12]. This fact is used in the original ESPRIT algorithm [44, 13] for estimating signal direction and in the previous chapter for estimating both direction and polarization.

When the incident signals are coherent, however, using $E\{|w_0(t)|^2\} = 1$ and (5.1) yields

$$ R_s = gg^H, \quad (5.4) $$

where $g$ is a $K \times 1$ column vector

$$ g = \begin{bmatrix} g_1 & g_2 & \cdots & g_K \end{bmatrix}^T, \quad (5.5) $$
where $(\cdot)^T$ denotes the transpose. For this case, the matrix $R_s$ is of rank 1 and the signal subspace based approach will not work. To use signal subspace methods with coherent signals, the signals must first be "decorrelated".

One method of overcoming the singularity of $R_s$ for coherent signals is the forward-only (FO) spatial smoothing technique of Shan, Wax and Kailath [18, 19]. The idea of this technique is to divide a total array of $L$ elements into $L_0$ overlapping subarrays, as shown in Figure 27. Each subarray then has $L_s = L - L_0 + 1$ elements. As shown in [18, 19], averaging the covariance matrices associated with the subarrays restores the rank of $R_s$ and makes it possible to use signal subspace methods.

The FO method can be applied here. Let $\mathbf{z}_l(t)$ denote the column vector received from the $l$th subarray, i.e.,

$$
\mathbf{z}_l(t) = \begin{bmatrix} z_l^T(t) & z_{l+1}^T(t) & \cdots & z_{L_s-1}^T(t) \end{bmatrix}^T, \quad 1 \leq l \leq L_0.
$$

The covariance matrix of the $l$th subarray is [19]

$$
R_l^f = \mathbb{E} \left\{ \mathbf{z}_l(t) [\mathbf{z}_l(t)]^H \right\} = A \Phi_q^{l-1} R_s (\Phi_q^{l-1})^H A^H + \sigma^2 I,
$$

where

$$
\Phi_q = \text{diag} \{ q_1, q_2, \cdots, q_K \},
$$

and where $A$ is now the matrix in (6.16) and (6.17) with $L$ replaced by $L_s$.

We define the forward-only spatially smoothed array covariance matrix $R^f$ to be the average of the matrices $R_l^f$, i.e.,

$$
R^f = \frac{1}{L_0} \sum_{l=1}^{L_0} R_l^f.
$$
Figure 27: A spatial smoothing scheme.
The matrix $R^f$ can be written as

$$R^f = R_0^f + \sigma^2 I,$$  \hspace{1cm} (5.10)

where

$$R_0^f = AR_\phi A^H,$$  \hspace{1cm} (5.11)

with $R_\phi^f$ denoting the forward-only spatially smoothed source covariance matrix

$$R_\phi^f = \frac{1}{L_0} \sum_{l=1}^{L_0} \Phi_q^{-1} R_s \left( \Phi_q^{-1} \right)^H. $$  \hspace{1cm} (5.12)

By replacing $R_s$ with $gg^H$, (5.12) can be written

$$R_s^f = \frac{1}{L_0} CC^H,$$  \hspace{1cm} (5.13)

where

$$C = [ \ g \ \Phi_q g \ \cdots \ \Phi_q^{L_0-1}g \ ] = G\overline{A}_{L_0}^T, $$  \hspace{1cm} (5.14)

with

$$G = \text{diag}\{g_1, g_2, \cdots, g_K\}, $$  \hspace{1cm} (5.15)

and

$$\overline{A}_{L_0} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_K \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{L_0-1} & q_2^{L_0-1} & \cdots & q_K^{L_0-1} \end{bmatrix}. $$  \hspace{1cm} (5.16)
Note that as long as the signals arrive from distinct directions, the Vandermonde matrix $\mathbf{A}_L$ is nonsingular. If in addition $L_0 > K$, then $\mathbf{C}$ is nonsingular so the rank of $R_s^f$ is $K$.

A second potential method for overcoming the singularity of $R_s$ for coherent signals is the forward/backward spatial smoothing technique of [20, 21]. However, this method cannot be used in the present problem when we want to estimate $\alpha_k, \beta_k$ as well as $\theta_k$. The reason is that the amplitudes of the elements of $\mathbf{A}$ given in (6.16) are not all unity, as can be seen from (6.11). However, if we want to estimate only the $\theta_k$, but not the polarization, this technique can be used. We consider this case in the next section.

5.4 Estimating Direction Only for Coherent Signals

In Chapter IV, we have also described a method for using the array of Figure 15 to estimate signal directions only, but to do so in such a way that the estimator works properly regardless of signal polarization. This approach treated the $x$-axis dipoles and the $y$-axis dipoles as separate subarrays. The arithmetic average of the covariance matrices for the $x$- and $y$-axis dipoles was used as the total covariance matrix in the ESPRIT algorithm. We now show that, with coherent signals, this same approach can be used in combination with both the forward-only [18, 19] and forward/backward [20, 21] spatial smoothing techniques.

Let $\mathbf{x}(t)$ denote the column vector received from the $x$-axis dipoles, i.e.,

$$
\mathbf{x}(t) = \left[ x_1(t) \ x_2(t) \ \cdots \ x_{L_x}(t) \right]^T. \tag{5.17}
$$

$\mathbf{x}(t)$ is the subvector of $\mathbf{z}(t)$ consisting of the odd-numbered elements of $\mathbf{z}(t)$. $\mathbf{x}(t)$ can be written
\[ x(t) = \overline{A}_{L_s} \Phi_x s(t) + n_x(t), \]  

where \( \overline{A}_{L_s} \) is defined as in (5.16) with \( L_0 \) replaced by \( L_s \), \( \Phi_x \) is defined as

\[ \Phi_x = \text{diag} \{ -\cos \gamma_1, -\cos \gamma_2, \ldots, -\cos \gamma_K \}, \]

and \( n_x(t) \) is a zero-mean complex Gaussian process with covariance \( \sigma^2 I \). The matrix \( \overline{A}_{L_s} \) is the direction matrix for this approach. The angles of arrival are assumed to be distinct, so that the columns of \( \overline{A}_{L_s} \) define a \( K \)-dimensional signal subspace in an \( L_s \)-dimensional space. The covariance matrix of \( x(t) \) is

\[ R_x = E \{ x(t)x^H(t) \} = \overline{A}_{L_s} \Phi_x R_s \Phi_x^H \overline{A}_{L_s}^H + \sigma^2 I. \]  

Similarly, let \( y(t) \) denote the column vector received from the \( y \)-axis dipoles, i.e.,

\[ y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_{L_s}(t) \end{bmatrix}^T. \]

\( y(t) \) is the subvector of \( z(t) \) consisting of the even-numbered elements of \( z(t) \). \( y(t) \) can be written

\[ y(t) = \overline{A}_{L_s} \Phi_y s(t) + n_y(t), \]

where

\[ \Phi_y = \text{diag} \{ -\sin \gamma_1 \cos \theta_1 e^{j\eta_1}, -\sin \gamma_2 \cos \theta_2 e^{j\eta_2}, \ldots, -\sin \gamma_K \cos \theta_K e^{j\eta_K} \}, \]
and \( n_y(t) \) is a zero-mean complex Gaussian process with covariance \( \sigma^2 I \). The covariance matrix of \( y(t) \) is

\[
R_y = E \{ y(t)y^H(t) \} = \overline{A}_{L_s} \Phi_y R_s \Phi^H_y \overline{A}^H_{L_s} + \sigma^2 I .
\]  

(5.24)

Consider \( \overline{R} \), the average of \( R_x \) and \( R_y \),

\[
\overline{R} = \frac{1}{2} (R_x + R_y) .
\]  

(5.25)

\( \overline{R} \) can be written

\[
\overline{R} = \overline{R}_0 + \sigma^2 I ,
\]  

(5.26)

where

\[
\overline{R}_0 = \overline{A}_{L_s} \overline{R}_s \overline{A}^H_{L_s} ,
\]  

(5.27)

with \( \overline{R}_s \) defined as

\[
\overline{R}_s = \frac{1}{2} \left( \Phi_x R_x \Phi^H_x + \Phi_y R_y \Phi^H_y \right) = \frac{1}{2} \begin{bmatrix} \Phi_x & \Phi_y \end{bmatrix} \begin{bmatrix} R_s & 0 \\ 0 & R_s \end{bmatrix} \begin{bmatrix} \Phi^H_x \\ \Phi^H_y \end{bmatrix} .
\]  

(5.28)

(5.29)

As long as the incident signals are at most partially correlated, \( R_s \) is nonsingular.

Since by assumption none of the incident signals produces zero output on both the \( x \)- and \( y \)-axis dipoles at the same time, \( [\Phi_x \ | \ \Phi_y] \) is of rank \( K \). Thus \( \overline{R}_s \) is nonsingular and the eigenvectors of \( \overline{R}_0 \) that correspond to the \( K \) largest eigenvalues of \( \overline{R}_0 \) span the same signal subspace as the column vectors in \( \overline{A}_{L_s} \). For this case, the ESPRIT algorithm can be applied to \( \overline{R} \) for direction estimation.

For coherent signals, however, (5.29) becomes
in which case the rank of \( \mathbf{R}_s \) is at most 2. Thus the coherent signals must first be "decorrelated" before the ESPRIT algorithm can be applied.

The first method of decorrelating the signals is the FO spatial smoothing technique of [18, 19]. We shall apply this method to the averaged covariance matrix \( \mathbf{R} \) in (5.25). In the curves below, we call this composite method the alternative forward-only (AFO) method.

In this method, the total array of \( L \) elements is divided into \( L_0 \) overlapping subarrays with \( L_s = L - L_0 + 1 \) elements in each subarray. Let \( x_f^f(t) \) denote the column vector of the \( x \)-axis dipole signals in the \( l \)th subarray,

\[
x_f^f(t) = \begin{bmatrix} x_l(t) & x_{l+1}(t) & \cdots & x_{l+L_s-1}(t) \end{bmatrix}^T, \quad 1 \leq l \leq L_0.
\]

and let \( y_f^f(t) \) denote the column vector of the \( y \)-axis dipole signals in the \( l \)th subarray,

\[
y_f^f(t) = \begin{bmatrix} y_l(t) & y_{l+1}(t) & \cdots & y_{l+L_s-1}(t) \end{bmatrix}^T, \quad 1 \leq l \leq L_0.
\]

Then let \( \mathbf{R}_f^f \) be the average of the covariance matrices of \( x_f^f(t) \) and \( y_f^f(t) \),

\[
\mathbf{R}_f^f = \frac{1}{2} \mathbf{E} \left\{ x_f^f(t) [x_f^f(t)]^H + y_f^f(t) [y_f^f(t)]^H \right\}.
\]

The alternative forward-only spatially smoothed covariance matrix \( \mathbf{R}_f^f \) is the average of \( \mathbf{R}_f^f \):

\[
\mathbf{R}_f^f = \frac{1}{L_0} \sum_{l=1}^{L_0} \mathbf{R}_f^f.
\]

\( \mathbf{R}_f^f \) can be written

\[
\mathbf{R}_f^f = \mathbf{R}_0^f + \sigma^2 \mathbf{I},
\]
where

\[
\mathbf{R}_0^f = \mathbf{A}_{L_s} \mathbf{R}_s^f \mathbf{A}_{L_s}^H,
\]  

(5.36)

with \( \mathbf{R}_s^f \) denoting the alternative forward-only spatially smoothed source covariance matrix

\[
\mathbf{R}_s^f = \frac{1}{2L_0} \sum_{l=1}^{L_0} \left[ \Phi_z \Phi_q^{l-1} \mathbf{R}_s (\Phi_q^{l-1})^H \Phi_x^H + \Phi_y \Phi_q^{l-1} \mathbf{R}_s (\Phi_q^{l-1})^H \Phi_y^H \right].
\]  

(5.37)

By using (5.12)-(5.13), \( \mathbf{R}_s^f \) can be written

\[
\mathbf{R}_s^f = \frac{1}{2L_0} \left[ \Phi_z \mathbf{C} \Phi_x^H \Phi_x^H + \Phi_y \mathbf{C} \Phi_y^H \Phi_y^H \right]
\]

\[
= \frac{1}{2L_0} \mathbf{C}_1^H \mathbf{C}_1^H,
\]  

(5.38)

where (see (5.14))

\[
\mathbf{C}_1 = \begin{bmatrix} \Phi_z \mathbf{C} & \Phi_y \mathbf{C} \\ \Phi_z \mathbf{G} \mathbf{A}_{L_0}^T & \Phi_y \mathbf{G} \mathbf{A}_{L_0}^T \\ \end{bmatrix}
\]

(5.39)

The last equality holds because \( \mathbf{G} \), \( \Phi_z \), and \( \Phi_y \) are diagonal matrices.

The minimum number of subarrays \( L_0 \) required to make \( \mathbf{R}_s^f \) nonsingular is between \( K/2 \) and \( K \) depending on the signal scenario. On the one hand, suppose all incident signals are linearly polarized so there is no output on the \( y \)-axis dipoles. Since by assumption none of the signals produces zero output on both the \( x \)- and \( y \)-axis dipoles at the same time, the diagonal elements of \( \Phi_z \) are then nonzero. For \( \mathbf{R}_s^f \) to be nonsingular in this case, we must have \( L_0 \geq K \). On the other hand, assume \( K \) is even and suppose \( K/2 \) signals produce zero outputs on the \( x \)-axis dipoles and
the other $K/2$ signals produce zero outputs on the $y$-axis dipoles. In this case we must have $L_0 \geq K/2$ for $\mathbf{R}^f_s$ to be nonsingular.

The second method for decorrelating the signals is the forward/backward spatial smoothing technique of [20, 21]. In the curves below, we label this method, when used with the averaged covariance matrix $\overline{\mathbf{R}}$ in (5.25), the alternative forward/backward (AFB) method.

The alternative forward/backward spatially smoothed array covariance matrix $\overline{\mathbf{R}}^{f/b}$ is defined as [20, 21]

$$\overline{\mathbf{R}}^{f/b} = \frac{1}{2} \left[ \mathbf{R}^f + \mathbf{J} \left( \mathbf{R}^f \right)^* \right], \quad (5.40)$$

where $\mathbf{R}^f$ is defined in (5.34), superscript $*$ denotes complex conjugate, and $\mathbf{J}$ is the exchange matrix

$$\mathbf{J} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \quad (5.41)$$

From (5.35), we have

$$\mathbf{J} \left( \mathbf{R}^f \right)^* \mathbf{J} = \mathbf{J} \left( \mathbf{R}_0^f + \sigma^2 \mathbf{I} \right)^* \mathbf{J} = \mathbf{J} \left( \mathbf{R}_0^f \right)^* \mathbf{J} + \sigma^2 \mathbf{I}. \quad (5.42)$$

Then using the relation

$$\overline{\mathbf{A}}_{L_s}^T \mathbf{J} = \Phi_q^{-1} \overline{\mathbf{A}}_{L_s}^H, \quad (5.43)$$

we have from (5.36)

$$\mathbf{J} \left( \mathbf{R}^f \right)^* \mathbf{J} = \mathbf{J} \left( \overline{\mathbf{A}}_{L_s} \overline{\mathbf{R}}^f_s \overline{\mathbf{A}}_{L_s}^H \right)^* \mathbf{J} \quad (5.44)$$

$$= \overline{\mathbf{A}}_{L_s} \Phi_q^{-(L_s-1)} \left( \overline{\mathbf{R}}^f_s \right)^* \left[ \Phi_q^{-(L_s-1)} \right]^H \overline{\mathbf{A}}_{L_s}^H. \quad (5.45)$$
Therefore $\bar{R}^{f/b}$ can be written as

$$\bar{R}^{f/b} = R_0^{f/b} + \sigma^2 I,$$  \hfill (5.46)

where

$$\bar{R}_0^{f/b} = A_{L_s} \bar{R}_s^{f/b} A_{L_s}^T,$$  \hfill (5.47)

and $\bar{R}_s^{f/b}$ denotes the alternative forward/backward spatially smoothed source covariance matrix

$$R_s^{f/b} = \frac{1}{2} \left[ R_s^f + \Phi_q^{-1} (L_s - 1) \left( \Phi_q^H \right)^f \right] \left[ \Phi_q^{-1} (L_s - 1) \right]^H.$$

Furthermore, from (5.38) we have

$$\begin{align*}
\bar{R}_s^{f/b} &= \frac{1}{4L_0} \left[ C_1 C_1^H + \Phi_q^{-1} (L_s - 1) C_1^T C_1^T \left[ \Phi_q^{-1} (L_s - 1) \right]^H \right] \\
&= \frac{1}{4L_0} \left[ C_1 C_1^H + \Phi_q^{-1} (L_s - 1) C_1^T J J^H C_1^T \left[ \Phi_q^{-1} (L_s - 1) \right]^H \right] \\
&= \frac{1}{4L_0} C_2 C_2^H,
\end{align*}$$

(5.49)

where we have used $J J^H = I$ and defined

$$C_2 = \left[ C_1 \mid \Phi_q^{-1} (L_s - 1) C_1^T J \right].$$

(5.50)

Next, using (5.39), we have

$$\begin{align*}
C_2 &= \left[ G \Phi_x A_{L_0}^T \mid G \Phi_y A_{L_0}^T \mid \Phi_q^{-1} (L_s - 1) G \Phi_x^* A_{L_0}^H \mid \Phi_q^{-1} (L_s - 1) G \Phi_y^* A_{L_0}^H \right] \\
&= \left[ G \Phi_x A_{L_0}^T \mid G \Phi_y A_{L_0}^T \mid G \Phi_x^* \Phi_q^{-1} (L_s - 1) A_{L_0}^T \mid G \Phi_y^* \Phi_q^{-1} (L_s - 1) A_{L_0}^T \right],
\end{align*}$$

(5.51)
where we have used (5.43) with $L_s$ replaced by $L_0$ and $L = L_s + L_0 - 1$. $C_2$ may then be written as

$$C_2 = G \begin{bmatrix} \Phi_x \mathbf{A}_{L_0}^T \mid \Phi_y \mathbf{A}_{L_0}^T \mid \Phi_x^* G^* G^{-1} \Phi_q^{-(L-1)} \mathbf{A}_{L_0}^T \mid \\
\Phi_y^* G^* G^{-1} \Phi_q^{-(L-1)} \mathbf{A}_{L_0}^T \end{bmatrix}$$

$$= G \begin{bmatrix} \Phi_x \mathbf{A}_{L_0}^T \mid \Phi_y \mathbf{A}_{L_0}^T \mid \Phi_x^* \mathbf{H} \mathbf{A}_{L_0}^T \mid \Phi_y^* \mathbf{H} \mathbf{A}_{L_0}^T \end{bmatrix},$$

(5.52)

where $H$ is the diagonal matrix

$$H = G^* G^{-1} \Phi_q^{-(L-1)}$$

$$= \text{diag}\{h_1, h_2, \ldots, h_K\},$$

(5.53)

(5.54)

with

$$h_k = \frac{g_k^{*}}{g_k} q_k^{-(L-1)}, \quad k = 1, 2, \ldots, K.$$

(5.55)

For $\mathbf{H}_s^{J/b}$ to be nonsingular, the minimum number of subarrays $L_0$ required is between $K/4$ and $K$, depending on the signal scenario. Each of the submatrices

$\Phi_x \mathbf{A}_{L_0}^T$, $\Phi_y \mathbf{A}_{L_0}^T$, $\Phi_x^* \mathbf{H} \mathbf{A}_{L_0}^T$, and $\Phi_y^* \mathbf{H} \mathbf{A}_{L_0}^T$, is a $K \times L_0$ matrix. To make $C_2$ have rank $K$, there must be $K$ independent column vectors somewhere in the set of these four submatrices. For most combinations of signals, $C_2$ will be full rank if $L_0 = K/4$, i.e., so there are $K/4$ columns in each of the submatrices in (5.52). Usually these $K/4$ columns in each submatrix will be linearly independent of each other, so $C_2$ will have rank $K$. However, for certain specific choices of signal parameters, some of these columns will be linearly dependent, in which case $L_0$ will need to be larger than $K/4$. Choosing $L_0 = K$, however, is always sufficient to make $C_2$ full rank, because then each of the four submatrices in (5.52) has rank $K$, regardless of the signal parameters.
5.5 Simulation Results

We now show several examples illustrating the use of these techniques with coherent signals. The results below were obtained by using fifty Monte Carlo trials. The array consisted of \( L = 10 \) pairs of crossed dipoles. All incident signals were assumed to have the same unit amplitude \( E_k \). The signal-to-noise ratio (SNR) shown in the figures is defined as \(-10\log_{10} \sigma^2\) dB. A finite number of data samples \( N \) was taken at each dipole output. The subarray covariance matrices were estimated from the available data samples, as described in the previous chapter. The spatially smoothed covariance matrices were obtained from the subarray covariance matrix estimates.

We first show an example that illustrates how the results for spatial smoothing with coherent signals compare with those using the method in Chapter IV with partially correlated signals and no spatial smoothing. We consider an example with 7 coherent signals. The SNR for each signal is 20 dB and the number of data samples is \( N = 200 \). The number of incident signals is assumed known in the estimator. The direction of arrival estimates are computed by using the AFB technique. Two subarrays \( (L_0 = 2) \) of 9 elements \( (L_s = 9) \) are used. Figure 28 shows the direction estimates obtained with each of the 50 independent trials plotted on a unit circle at those angles from the center of the circle. The 50 estimates of the angles are superimposed on the same plot, so the spread in angles can be seen. Figure 28(a) shows the results when the signals arrive from equally-spaced angles every \( 20^\circ \) between \(-55^\circ \) and \( 65^\circ \). The corresponding ellipticity angles are also equally spaced between \(-45^\circ \) and \( 45^\circ \) and the orientation angles are zero. Figure 28(b) shows the results for a smaller separation between signals, every \( 11.5^\circ \) between \(-29.5^\circ \) and \( 39.5^\circ \) and
for the same polarizations. As may be seen, the estimation accuracy is poor when the signals are spaced every 11.5°. The reason is that when the angles are closer the direction matrix $\vec{A}_L$ is becoming ill-conditioned. This example illustrates the resolution limits for this technique as the 7 arrival angles approach one another.

Next, for comparison, Figure 29 shows the corresponding results when 7 partially correlated signals are incident and the angle-only method described in the previous chapter is used, i.e., there is no spatial smoothing. The other parameters of the signals are the same as for Figure 28. In this case, an array of 9 elements is used, to make the results comparable to those in Figure 28. Figure 29(a) shows the results when the signals are evenly spaced from $-\pi/2$ to $\pi/2$, and Figure 29(b) shows them when they are spaced from $-\pi/4$ to $\pi/4$. Note that the results in Figures 28(a) and 29(a) are similar, but with the signals more closely spaced, the results in Figure 29(b) are much better than those in Figure 28(b).

This example illustrates that the resolution of the AFB technique for coherent arrivals is poorer than what can be achieved with partially correlated signals. This drop in performance occurs because the full rank spatially smoothed matrix $\bar{R}_s^{f/b}$ for coherent sources is different from the full rank source covariance matrix $R_s$ for non-coherent sources. The numerical condition of $R_s$ for non-coherent sources is determined by how strongly the sources are correlated, but not by the arrival directions. The numerical condition of $\bar{R}_s^{f/b}$, on the other hand, is determined by how closely the arrival angles are spaced, as can be seen from (5.49) and (5.52). Note that $\bar{R}_s^{f/b}$ depends on $A_{L_0}$, which is a function of the arrival angles. As the arrival angles become more closely-spaced, the columns of $A_{L_0}$ become more nearly linearly dependent, and $\bar{R}_s^{f/b}$ becomes more ill-conditioned. Moreover, as the coherent arrivals become more closely-spaced, $\bar{R}_0^{f/b}$ in (5.47) becomes more ill-conditioned.
Figure 28: Direction estimates of coherent signals obtained by using the AFB method.
Figure 29: Direction estimates of non-coherent signals obtained by using the angle-only method.
because both $\overline{A}_L$ and $\overline{R}_f^f/b$ become ill-conditioned. For partially correlated signals, the ill-conditioning is less serious because only $\overline{A}_L$ in $\overline{R}_0$ of (5.3) becomes ill-conditioned. The more ill-conditioned $\overline{R}_0^f/b$ or $\overline{R}_0$, the more sensitive the matrix is to noise perturbation [43]. (These comments apply to the FO and AFO spatial smoothing techniques as well.)

We next show the error performance of these spatial smoothing techniques with coherent signals. For the next three examples, we used $N = 31$ data samples.

First, we consider a case where three identical signals arrive from $20^\circ$, $25^\circ$, and $30^\circ$. The corresponding ellipticity angles are $45^\circ$, $40^\circ$, and $35^\circ$, respectively, and the orientation angles are zero. For this case, the signal directions are closely-spaced and the signal polarizations are similar. Figure 30(a) shows the direction estimate error variances for the first signal and for the FO, AFO, and AFB approaches as functions of the SNR when $L_0 = 3$. Figures 30(b) shows the variance of the polarization estimates for the FO approach for the first signal. These curves were obtained by assuming that the number of incident signals is known. The variance of the polarization estimates in Figure 30(b) is defined as the mean-squared value of the angular distance on the Poincaré sphere between $M$ and $\hat{M}$, the points representing the actual and estimated polarizations $(\gamma, \eta)$ and $(\hat{\gamma}, \hat{\eta})$, respectively, as described in Chapter IV.

When the number of incident signals is unknown, we also used the minimum description length (MDL) criterion [33, 45] with the spatially smoothed covariance matrix to estimate the number of incident signals. Figure 30(c) shows the probability that the correct number was obtained as a function of the SNR for each of the three methods.

Next, we consider an example where three identical signals arrive from $25^\circ$ -
Figure 30: Performance of the FO, AFO, AFB approaches as a function of SNR.

(a) Variance of direction estimates.  
(b) Variance of polarization estimates.  
(c) Probability of correct detection.
\(\Delta \theta\)^\circ, 25^\circ, \text{ and } (25 + \Delta \theta)^\circ\), so \(\Delta \theta\) is the angle separation between two adjacent angles. The corresponding ellipticity angles are 45\(^\circ\), 40\(^\circ\), \text{ and } 35\(^\circ\), respectively, \text{ and } the orientation angles are zero. The SNR per signal is 20 dB. Figure 31 shows the error variances and the probability of estimating the number of signals correctly for the FO, AFO, and AFB approaches as a function of \(\Delta \theta\) when \(L_0 = 3\).

Finally, we consider an example where three identical signals arrive from 13\(^\circ\), 25\(^\circ\), \text{ and } 37\(^\circ\). The corresponding ellipticity angles are 45\(^\circ\), \((45 - \Delta \alpha)^\circ\), \text{ and } \((45 - 2\Delta \alpha)^\circ\) \text{ and } the orientation angles are zero, so \(\Delta \alpha\) is the polarization separation between adjacent signals. The SNR per signal is again 20 dB. Figure 32 shows the performance of the FO, AFO, and AFB approaches as a function of \(\Delta \alpha\) when \(L_0 = 3\).

From Figures 30, 31, and 32, we note that using the AFB approach yields much better performance in estimating both the signal directions and the number of incident signals than the forward-only approaches do. The reason for this may be seen by comparing the AFO and AFB approaches. Using the forward/backward approach results in two extra submatrices in \(C_2\), as may be seen by comparing (5.52) with (5.39). The extra submatrices reduce the ill-conditioning of \(R_{s}^{f/b}\) when the signal directions are closely-spaced.

5.6 Summary

We have described how a uniform linear array of crossed dipoles may be used with the ESPRIT algorithm and spatial smoothing to estimate the directions and polarizations of arbitrarily polarized coherent signals. It is found that spatial smoothing yields poorer resolution for coherent signals than would be obtained with uncorrelated or partially correlated signals, \text{ but of course without spatial smoothing ESPRIT cannot be used at all with coherent sources. Some examples showing typical results}
Figure 31: Performance of the FO, AFO, AFB approaches as a function of $\Delta \theta$. 

(a) Variance of direction estimates. 

(b) Variance of polarization estimates. 

(c) Probability of correct detection.
Figure 32: Performance of the FO, AFO, AFB approaches as a function of $\Delta \alpha$. 

(a) Variance of direction estimates.  
(b) Variance of polarization estimates.  
(c) Probability of correct detection.
were presented.
CHAPTER VI

TWO DIMENSIONAL ANGLE AND POLARIZATION ESTIMATION USING THE ESPRIT ALGORITHM

6.1 Introduction

In Chapter IV, we have described how to use the ESPRIT algorithm with a uniform linear array of crossed dipoles to estimate signal directions in one spatial angle, along with the polarizations. In this chapter, we generalize these results to two spatial angles. The arriving signals are again assumed to be narrowband, so the signals received on different array elements differ only by a phase factor.

In this chapter, we consider a square array consisting of \( L^2 \) pairs of crossed short dipoles. We show how the ESPRIT algorithm may be used to estimate two dimensional signal directions and polarizations with this array. We also study the effects of signal direction and polarization on the performance of the estimator.

In Section 6.2, we define the array geometry and formulate the problem of interest. In Section 6.3, we show how to adapt the ESPRIT algorithm to the problem of estimating signal directions in two angles and signal polarizations. In Section 6.4, we present simulation results and describe the estimator performance. Finally, Section 6.5 contains our summary.

6.2 Problem Formulation

We consider the array shown in Figure 33. The array consists of \( L^2 \) pairs of crossed dipoles, or a total of \( 2L^2 \) elements. The signal from each dipole is to be
processed separately. The \( i \)th dipole pair, where \( i, l = 1, 2, \cdots, L \), has its center at \((x, y) = ((i - 1)\delta, (l - 1)\delta)\). The distance \( \delta \) between adjacent dipoles is assumed to be a half wavelength. At the \( i \)th dipole pair, \( x_{il}(t) \) denotes the signal received on the \( x \)-axis dipole and \( y_{il}(t) \) the signal received on the \( y \)-axis dipole.

Suppose \( K \) (with \( K \leq L^2 \)) continuous wave (CW) signals impinge on the array from angular directions \((\theta_k, \phi_k), k = 1, 2, \cdots, K\), where \( r, \theta \) and \( \phi \) denote standard polar coordinates, as shown in Figure 33. Assume each signal has an arbitrary elliptical electromagnetic polarization [38].

To specify the signal polarizations, we use the following definitions. Given a transverse electromagnetic (TEM) wave propagating into the array, we again consider the polarization ellipse produced by the electric field as the incoming wave is viewed from the coordinate origin. Note that in this case the unit vectors \( e_r, e_\phi, -e_r \), in that order, form a right-handed coordinate system for an incoming signal. The electric field now has transverse components \( E_\phi \) and \( E_\theta \),

\[
E = E_\phi e_\phi + E_\theta e_\theta. \tag{6.1}
\]

(We call \( E_\phi \) the horizontal component and \( E_\theta \) the vertical component of the field.) In the same manner as for \( E_x \) and \( E_\theta \) in Chapter IV, \( E_\phi \) and \( E_\theta \) describe a polarization ellipse as shown in Figure 34. In this ellipse, \( \beta \) is the orientation angle of the major axis of the ellipse with respect to \( E_\phi \). As before, \( \beta \) is in the range \( 0 \leq \beta < \pi \). The ellipticity angle \( \alpha \) has magnitude

\[
|\alpha| = \tan^{-1}\rho, \tag{6.2}
\]

where \( \rho \) is the axial ratio

\[
\rho = \frac{\text{minor axis}}{\text{major axis}}. \tag{6.3}
\]
Figure 33: Crossed dipole array.
$\alpha$ is positive when the electric vector rotates clockwise and negative when it rotates counterclockwise (when the incoming wave is viewed from the coordinate origin, as in Figure 34). $\alpha$ is always in the range $-\pi/4 \leq \alpha \leq \pi/4$. Figure 34 depicts a situation in which $\alpha$ is positive.

For a given signal polarization, specified by $\alpha$ and $\beta$, the electric field components are given by (aside from a common phase factor)

$$E_\phi = E \cos \gamma,$$

$$E_\theta = E \sin \gamma e^{i\eta},$$  \hspace{1cm} (6.4) (6.5)

where the relationship among the four angular variables $\alpha$, $\beta$, $\gamma$, and $\eta$ has been described by Equations (4.6)-(4.9).

Thus an arbitrary plane wave coming into the array may be characterized by four angular parameters and an amplitude. For example, the $k$th signal, $k = 1, 2, \cdots, K$, 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{polarization_ellipse.png}
\caption{Polarization ellipse.}
\end{figure}
will be characterized by its arrival angles \((\theta_k, \phi_k)\), its polarization ellipticity angle \(\alpha_k\) and orientation angle \(\beta_k\), and its amplitude \(E_k\) (i.e., \(E_k\) is the value of \(E\) in (6.4) and (6.5) for the \(k\)th signal). We will say the \(k\)th signal is defined by \((\theta_k, \phi_k, \alpha_k, \beta_k, E_k)\).

We assume that each dipole in the array is a short dipole, i.e., the output voltage from each dipole is proportional to the electric field component along the dipole. Therefore, the outputs of the dipoles parallel to the \(x\)- and \(y\)-axes will be proportional to the \(x\)- and \(y\)-components, respectively, of the electric field. An incoming signal, with arbitrary electric field components \(E^x\) and \(E^y\), has \(x\), \(y\), \(z\) components:

\[
E = E^x e_x + E^y e_y = (E_\theta \cos \theta \cos \phi - E_\phi \sin \phi)e_x \\
+ (E_\theta \cos \theta \sin \phi + E_\phi \cos \phi)e_y \\
- (E_\theta \sin \theta)e_z. \tag{6.6}
\]

When \(E^x\) and \(E^y\) are expressed in terms of \(E, \gamma, \) and \(\eta\) as in (6.4) and (6.5), the electric field components become

\[
E = E \left[ (\sin \gamma \cos \theta \cos \phi e^{j\eta} - \cos \gamma \sin \phi) e_x \\
+ (\sin \gamma \cos \theta \sin \phi e^{j\eta} + \cos \gamma \cos \phi) e_y \\
- (\sin \gamma \sin \theta e^{j\eta}) e_z \right]. \tag{6.7}
\]

Let us define the space phase factors

\[
p = e^{j \frac{2\pi \phi}{\lambda} \sin \theta \cos \phi}, \tag{6.8}
\]
\[
q = e^{j \frac{2\pi \phi}{\lambda} \sin \theta \sin \phi}. \tag{6.9}
\]
where \( \lambda \) is the wavelength of the signal. Including the time and space phase factors in (6.7), we find that an incoming signal characterized by \((\theta, \phi, \alpha, \beta, E)\) produces a signal vector in the crossed dipole pair centered at \((x, y) = ((i - 1)\delta, (l - 1)\delta)\) as follows:

\[
\mathbf{z}_{il} = \begin{bmatrix}
x_{il}(t) \\
y_{il}(t)
\end{bmatrix} = u_s(t)p_{i-1}q_{l-1}, \tag{6.10}
\]

where

\[
u = \begin{bmatrix}
\sin \gamma \cos \theta \cos \phi e^{j\eta} - \cos \gamma \sin \phi \\
\sin \gamma \cos \theta \sin \phi e^{j\eta} + \cos \gamma \cos \phi
\end{bmatrix}, \tag{6.11}
\]

and

\[
s(t) = E e^{j(\omega t + \psi)}, \tag{6.12}
\]

with \( \omega \) the frequency of the signal and \( \psi \) the carrier phase of the signal at the coordinate origin at \( t = 0 \).

We assume that \( K \) such signals, specified by \((\theta_k, \phi_k, \alpha_k, \beta_k, E_k)\), \( k = 1, 2, \cdots, K \), are incident on the array. The carrier phase angles \( \psi_k \) are assumed to be random variables, each uniformly distributed on \([0, 2\pi]\) and all statistically independent of each other. In addition we assume a thermal noise voltage vector \( \mathbf{n}_{il}(t) \) is present on each signal vector \( \mathbf{z}_{il}(t) \). The \( \mathbf{n}_{il}(t) \) are assumed to be zero mean, complex Gaussian processes statistically independent of each other, with covariance \( \sigma^2 \mathbf{I} \), where \( \mathbf{I} \) denotes the identity matrix.

Under these assumptions, the total signal vector received by the crossed dipole pair centered at \((x, y) = ((i - 1)\delta, (l - 1)\delta)\) is given by

\[
\mathbf{z}_{il}(t) = \sum_{k=1}^{K} u_{k}\mathbf{s}_k(t)p_{i-1}q_{l-1} + \mathbf{n}_{il}(t), \quad i, l = 1, 2, \cdots, L, \tag{6.13}
\]
where $u_k, p_k, q_k,$ and $s_k(t)$ are given by (6.11), (6.8), (6.9), and (6.12), respectively, with subscript $k$ added to the amplitude $E$ and to each angular quantity.

Let $z(t), s(t),$ and $n(t)$ be column vectors containing the received signals, incident signals, and noise, respectively, i.e.,

$$
\begin{align*}
\mathbf{z}(t) &= \begin{bmatrix}
    z_{11}(t) \\
    z_{12}(t) \\
    \vdots \\
    z_{1L}(t) \\
    z_{21}(t) \\
    z_{22}(t) \\
    \vdots \\
    z_{2L}(t) \\
    \vdots \\
    z_{L1}(t) \\
    z_{L2}(t) \\
    \vdots \\
    z_{LL}(t)
\end{bmatrix}, \\
\mathbf{s}(t) &= \begin{bmatrix}
    s_1(t) \\
    s_2(t) \\
    \vdots \\
    s_K(t)
\end{bmatrix}, \\
\mathbf{n}(t) &= \begin{bmatrix}
    n_{11}(t) \\
    n_{12}(t) \\
    \vdots \\
    n_{1L}(t) \\
    n_{21}(t) \\
    n_{22}(t) \\
    \vdots \\
    n_{2L}(t) \\
    \vdots \\
    n_{L1}(t) \\
    n_{L2}(t) \\
    \vdots \\
    n_{LL}(t)
\end{bmatrix}.
\end{align*}
$$

The received signal vector is then

$$
\mathbf{z}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t),
$$

where $\mathbf{A}$ is a $2L^2 \times K$ matrix

$$
\mathbf{A} = \begin{bmatrix}
    a_1 & a_2 & \cdots & a_K
\end{bmatrix},
$$

with each $2L^2 \times 1$ column given by
The columns $a_k$ are assumed linearly independent. They define a $K$-dimensional signal subspace in a $2L^2$-dimensional space.

By assuming linearly independent columns in $A$, we are excluding from consideration degenerate cases, such as when two signals arrive from the same direction with the same polarization, when more than two signals of arbitrary polarization arrive from the same direction, or when a signal yields zero output at the $x$- and $y$-axis dipoles at the same time.

We assume that the element signals are sampled at $N$ distinct times $t_n$, $n = 1, 2, \cdots, N$. The problem of interest is to determine the quantities $(\theta_k, \phi_k, \alpha_k, \beta_k)$, $k = 1, 2, \cdots, K$, from the measurements $z(t_n)$, $n = 1, 2, \cdots, N$. 

\[
a_k = \begin{bmatrix}
    u_k \\
    u_kq_k \\
    \vdots \\
    u_kq_k^{L-1} \\
    u_kp_k \\
    u_kp_kq_k \\
    \vdots \\
    u_kp_kq_k^{L-1} \\
    u_kp_k^{L-1}q_k \\
    \vdots \\
    u_kp_k^{L-1}q_k^{L-1}
\end{bmatrix}
\] (6.17)
6.3 Application of the ESPRIT Algorithm

The array geometry described above possesses several invariance properties that may be exploited by the ESPRIT algorithm. We shall first illustrate how the ESPRIT algorithm can be used for the case where the array covariance matrix and the number of incident signals are known. We then consider the practical situation where only a finite number of data samples is available and the number of incident signals is unknown.

The array covariance matrix has the form

\[ R = E\{z(t)z^H(t)\} = R_0 + \sigma^2 I, \]  

(6.18)

where

\[ R_0 = AR_sA^H, \]  

(6.19)

\((\cdot)^H\) denotes the complex conjugate transpose, \(R_s = E\{s(t)s^H(t)\}\) is the source covariance matrix, and \(E(\cdot)\) denotes expectation.

From the array covariance matrix, the signal directions and polarizations may be calculated as follows [13, 9]. Let \(\lambda_1 + \sigma^2 \geq \lambda_2 + \sigma^2 \geq \cdots \geq \lambda_K + \sigma^2 > \cdots = \sigma^2\) be the eigenvalues of \(R\), and \(e_1, e_2, \ldots, e_K, e_{K+1}, \ldots, e_{2L}\) be the corresponding orthonormal eigenvectors. Since the noise contribution to \(R\) for this ideal case is simply \(\sigma^2 I\), the eigenvectors of \(R\) are also the eigenvectors of \(R_0\). It can be shown that the columns in \(E_s = [e_1 \ e_2 \ \cdots \ e_K]\) span the same signal subspace as the column vectors in \(A\) [12]. Therefore, there exists a unique nonsingular \(T\) such that

\[ E_s = AT. \]  

(6.20)
The columns in $E_s$ are the \textit{signal subspace eigenvectors}. The signal directions and polarizations are computed from them.

Consider first the calculation of the space factors $p_k$, $k = 1, 2, \cdots, K$. From Figure 33, we note that the overlapping subarrays consisting of the first and the last $L - 1$ y-axis rows of the crossed dipoles are the same except for the displacement $\delta$ parallel to the z-axis. For the $k$th incident signal, the displacement $\delta$ results in the space factor $p_k$. The subvectors of $a_k$ consisting of the first and the last $2L(L - 1)$ elements of $a_k$ differ by the factor $p_k$, as can also be seen from (6.17). (Note that $u_k$ is a two-element column vector, as in (6.11).) Let $A_{p1}$ and $A_{p2}$ be the $2L(L - 1) \times K$ submatrices of $A$ consisting of the first and the last $2L(L - 1)$ rows of $A$, respectively. Then $A_{p2} = A_{p1} \Phi_p$, where $\Phi_p$ is the diagonal matrix

$$
\Phi_p = \text{diag}\{ p_1, p_2, \ldots, p_K \}. \quad (6.21)
$$

Thus the subspaces spanned by the columns in $A_{p1}$ and $A_{p2}$ are the same except for the phase rotation caused by the diagonal matrix $\Phi_p$. Let $E_{p1}$ and $E_{p2}$ be the $2L(L - 1) \times K$ submatrices formed from $E_s$ in the same way that $A_{p1}$ and $A_{p2}$ are formed from $A$. Then from (6.20), we have

$$
E_{p1} = A_{p1} T, \quad (6.22)
$$

$$
E_{p2} = A_{p2} T = A_{p1} \Phi_p T. \quad (6.23)
$$

As long as $A_{p1}$ is of full column rank, the columns of $E_{p1}$ and $E_{p2}$ span the same $K$-dimensional subspaces as the columns of $A_{p1}$ and $A_{p1} \Phi_p$, respectively. As shown in [13], the diagonal elements $p_k$, $k = 1, 2, \cdots, K$, of $\Phi_p$ are the eigenvalues of the unique matrix $\Psi_p = T^{-1} \Phi_p T$ that satisfies

$$
E_{p2} = E_{p1} \Psi_p. \quad (6.24)
$$
Consider next the calculation of the space factors $q_k$, $k = 1, 2, \cdots, K$. From Figure 33, we note that the overlapping subarrays consisting of the left and the right $L - 1$ $x$-axis columns of the crossed dipoles are the same except for a displacement $\delta$ parallel to the $y$-axis. For the $k$th incident signal, the displacement $\delta$ results in the space factor $q_k$, as can also be seen from (6.17). The subvectors of $a_k$ consisting of the elements of $a_k$ numbered $2L(i - 1) + l$ and $2L(i - 1) + l + 2$, respectively, for $l = 1, 2, \cdots, 2(L - 1)$ and $i = 1, 2, \cdots, L$, differ by the factor $q_k$. Let $A_{q1}$ and $A_{q2}$ be the $2L(L - 1) \times K$ submatrices of $A$ consisting of the rows of $A$ numbered $2L(i - 1) + l$ and $2L(i - 1) + l + 2$, respectively, $l = 1, 2, \cdots, 2(L - 1)$, $i = 1, 2, \cdots, L$.

Then $A_{q2} = A_{q1} \Phi_q$, where $\Phi_q$ is the diagonal matrix

$$
\Phi_q = \text{diag}\{ q_1, q_2, \cdots, q_K \}.
$$

Let $E_{q1}$ and $E_{q2}$ be the $2L(L - 1) \times K$ submatrices formed from $E$ in the same way that $A_{q1}$ and $A_{q2}$ are formed from $A$. Then

$$
E_{q1} = A_{q1}T,
$$

$$
E_{q2} = A_{q2}T = A_{q1} \Phi_q T.
$$

As long as $A_{q1}$ is of full column rank, the columns of $E_{q1}$ and $E_{q2}$ span the same $K$-dimensional subspaces as the columns of $A_{q1}$ and $A_{q1} \Phi_q$, respectively. Then the diagonal elements $q_k$, $k = 1, 2, \cdots, K$, of $\Phi_q$ are the eigenvalues of the unique matrix $\Psi_q = T^{-1} \Phi_q T$ that satisfies

$$
E_{q2} = E_{q1} \Psi_q.
$$

Finally, consider the calculation of the ratios $r_k$ from which the polarization angles may be calculated, where $r_k$ is the ratio between the first and the second elements of $u_k$, i.e.,
\[
r_k = \frac{\sin \gamma_k \cos \theta_k \cos \phi_k e^{i\eta_k} - \cos \gamma_k \sin \phi_k}{\sin \gamma_k \cos \theta_k \sin \phi_k e^{i\eta_k} + \cos \gamma_k \cos \phi_k}, \quad k = 1, 2, \ldots, K. \tag{6.29}
\]

From Figure 33, we note that both dipoles in a given crossed dipole pair have the same space factors \( p_k \) or \( q_k \). Moreover, in any dipole pair, the \( y \)-axis dipole output is related to the \( x \)-axis dipole output by a factor \( r_k \), as seen in (6.10) and (6.11). Because of this, the subvectors of \( a_k \) consisting of the even and the odd numbered elements of \( a_k \) differ by the factor \( r_k \). Let \( A_{r1} \) and \( A_{r2} \) be the \( L^2 \times K \) submatrices of \( A \) consisting of the even and the odd numbered rows of \( A \), respectively. Then

\[
A_{r2} = A_{r1} \Phi_r, \text{ where } \Phi_r \text{ is the diagonal matrix}
\]

\[
\Phi_r = \text{diag}\{ r_1, r_2, \ldots, r_K \}. \tag{6.30}
\]

Let \( E_{r1} \) and \( E_{r2} \) be the \( L^2 \times K \) submatrices formed from \( E_s \) in the same way that \( A_{r1} \) and \( A_{r2} \) are formed from \( A \). Then

\[
E_{r1} = A_{r1} T, \tag{6.31}
\]

\[
E_{r2} = A_{r2} T = A_{r1} \Phi_r T, \tag{6.32}
\]

As long as \( A_{r1} \) is of full column rank, the columns of \( E_{r1} \) and \( E_{r2} \) span the same \( K \)-dimensional subspaces as the columns of \( A_{r1} \) and \( A_{r1} \Phi_r \), respectively. The diagonal elements \( r_k, k = 1, 2, \ldots, K \) of \( \Phi_r \) are the eigenvalues of the unique matrix \( \Psi_r = T^{-1} \Phi_r T \) that satisfies

\[
E_{r2} = E_{r1} \Psi_r. \tag{6.33}
\]

With multiple incident signals, it is necessary to determine the grouping of the eigenvalues of \( \Psi_p \), \( \Psi_q \), and \( \Psi_r \). I.e., we must determine which eigenvalues of \( \Psi_p \) correspond to which eigenvalues of \( \Psi_q \) and to which eigenvalues of \( \Psi_r \). One way could be to use the MUSIC spectrum [12] to determine the grouping [46]. This
method, however, requires finding the extra eigenvectors \( \mathbf{e}_{K+1}, \mathbf{e}_{K+2}, \ldots, \mathbf{e}_{2L} \) and searching over \( K^3 \) possibilities. Instead, we introduce a different approach below.

The proper grouping of the eigenvalues of \( \Psi_p, \Psi_q, \) and \( \Psi_r \) may be determined in two steps: 1) by pairing the eigenvalues of \( \Psi_p \) and \( \Psi_q \), and 2) by pairing the eigenvalues of \( \Psi_p \) and \( \Psi_r \). To pair the eigenvalues of \( \Psi_p \) and \( \Psi_q \), note that

\[
\Psi_{pq} = \Psi_q^{-1}\Psi_p = T^{-1}\Phi_q^{-1}\Phi_p T.
\]

Thus the eigenvalues \( \tilde{q}_k \) of \( \Psi_{pq} \), \( k = 1, 2, \ldots, K \), are the ratios between the eigenvalues of \( \Psi_p \) and their corresponding eigenvalues of \( \Psi_q \). Therefore, for \( k_1 = 1, 2, \ldots, K \), the eigenvalue of \( \Psi_q \) that corresponds to the eigenvalue \( p_{k_1} \) of \( \Psi_p \) is the element in the set \( \{ q_{k_2}, k_2 = 1, 2, \ldots, K \} \) that corresponds to the minimum of

\[
\left\{ \left| \frac{p_{k_1}}{q_{k_2}} - \tilde{q}_{k_3} \right|, k_2, k_3 = 1, 2, \ldots, K \right\}.
\]

Similarly, the eigenvalues of \( \Psi_{pr} = \Psi_r^{-1}\Psi_p \), \( \tilde{r}_k \), \( k = 1, 2, \ldots, K \), are also the ratios between the eigenvalues of \( \Psi_p \) and their corresponding eigenvalues of \( \Psi_r \). Thus the eigenvalues of \( \Psi_p \) and \( \Psi_r \) can be paired in the same way as those of \( \Psi_p \) and \( \Psi_q \). From the paired sets of eigenvalues of \( \Psi_p \) and \( \Psi_q \) and of \( \Psi_p \) and \( \Psi_r \), we can determine the grouping \( (p_k, q_k, r_k) \), \( k = 1, 2, \ldots, K \).

The arrival angles, ellipticity angles, and orientation angles can be computed from the sets \( (p_k, q_k, r_k) \), \( k = 1, 2, \ldots, K \). The arrival angles \( (\theta_k, \phi_k) \), \( k = 1, 2, \ldots, K \), are calculated from \( p_k \) and \( q_k \) as

\[
\begin{align*}
\theta_k &= \sin^{-1} \left\{ \frac{\lambda}{2\pi\delta} \left[ \arctan \left( \frac{\arg(p_k)}{\arg(q_k)} \right) \right] \right\}, \\
\phi_k &= \tan^{-1} \left( \frac{\arg(p_k)}{\arg(q_k)} \right).
\end{align*}
\]

(6.36)
To determine the ellipticity angles $\alpha_k$ and the orientation angles $\beta_k$, we must first find $\gamma_k$ and $\eta_k$ from $r_k$. $\gamma_k \in [0, \pi/2]$ and $\eta_k \in [-\pi, \pi)$ can be determined from

$$\gamma_k = \tan^{-1}(|\xi_k|),$$  
$$\eta_k = \arg(\xi_k),$$

where

$$\xi_k = \frac{r_k \cos \phi_k + \sin \phi_k}{\cos \theta_k (-r_k \sin \phi_k + \cos \phi_k)}.$$  

(6.40)

From $\gamma_k$ and $\eta_k$, $\alpha_k \in [-\pi/4, \pi/4]$ and $\beta_k \in [0, \pi)$ can then be determined as described in Section 4.2.

This approach may be used to estimate the signal directions and polarizations as long as the matrices $A$, $A_{p1}$, $A_{q1}$, and $A_{r1}$ are all of full column rank, a condition that is satisfied in most cases. When the signals are from distinct directions and the number of signals $K$ satisfies $K < L$, for example, it is easy to show that the matrices $A$, $A_{p1}$, $A_{q1}$, and $A_{r1}$ are guaranteed to be of full column rank. In general, this approach can deal with a maximum of $L^2$ signals.

However, our approach may fail to estimate the signal directions and polarizations for some special cases. These cases occur when $A$ has full column rank, but $A_{p1}$, $A_{q1}$, or $A_{r1}$ do not. For example, when two signals with distinct polarizations arrive from the same direction, $A$ is of full rank, but $A_{r1}$ is not. For such cases, the signal polarizations cannot be obtained by solving (6.33), although the signal directions can still be found by solving (6.24) and (6.28).

Thus whether this approach can be used or not depends on the rank conditions of $A_{p1}$, $A_{q1}$, and $A_{r1}$. The rank conditions of $A_{p1}$, $A_{q1}$, and $A_{r1}$ can be determined
from the rank conditions of $E_p$, $E_q$, and $E_r$, respectively, as can be seen from (6.22), (6.26), and (6.31).

For the cases where the above approach fails, the computationally more expensive MUSIC algorithm [12] may be used to determine the signal directions and polarizations as long as $A$ has full column rank and $K < 2L^2$.

In the following, we shall restrict our consideration to the cases where the above approach is applicable. For these cases, when the array covariance matrix is known exactly, the signal directions and polarizations can be found \textit{exactly}.

In the discussion above, it was assumed that the array covariance matrix was known exactly. In practical situations, however, only a finite number of noisy measurements are made at the dipole outputs. The estimates of the signal directions and polarizations must then be made from the available measurements. Also, the number of incident signals is unknown and must be estimated from the measurements. We shall use the minimum description length (MDL) criterion described by Wax and Kailath [33] to estimate the number of incident signals. The total least squares (TLS) algorithm [13, 40] is then used to estimate $\Psi_p$, $\Psi_q$, and $\Psi_r$. The estimation steps in the signal subspace approach are as follows:

1) Compute the maximum likelihood estimate of the array covariance matrix $R$

$$
\hat{R} = \frac{1}{N} \sum_{n=1}^{N} z(t_n)z^H(t_n), \quad (6.41)
$$

where $N$ denotes the number of measurements.

2) Compute the eigenvalues $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{2L^2}$ of $\hat{R}$.

3) Estimate the number of incident signals $\hat{K}$ using the MDL criterion. The MDL estimate $\hat{K}$ of the number of signals is given by the value of $K \in \{0, 1, \cdots, 2L^2\}$—
that minimizes the following MDL function [33]:

$$\text{MDL}(K) = -\log \left\{ \frac{\prod_{i=K+1}^{2L^2} \lambda_i^{1/(2L^2-K)}}{\frac{1}{2L^2-K} \sum_{i=K+1}^{2L^2} \lambda_i} \right\}^{(2L^2-K)N} + \frac{1}{2} K(4L^2 - K) \log N. \quad (6.42)$$

4) Obtain $\hat{E}_s$ whose columns are the eigenvectors of $\hat{R}$ that correspond to the $K$ largest eigenvalues of $\hat{R}$.

5) Form $\hat{E}_{p1}$, $\hat{E}_{p2}$, $\hat{E}_{q1}$, $\hat{E}_{q2}$, $\hat{E}_{r1}$, and $\hat{E}_{r2}$ from $\hat{E}_s$ in the same way that $E_{p1}$, $E_{p2}$, $E_{q1}$, $E_{q2}$, $E_{r1}$, and $E_{r2}$ are formed from $E_s$.

6) Calculate the TLS (Total Least Squares) solution $\hat{\Psi}_p$ [13, 40] from

$$\hat{E}_{p2} = \hat{E}_{p1} \hat{\Psi}_p. \quad (6.43)$$

To calculate $\hat{\Psi}_p$, let $V$ be the $2K \times K$ matrix containing the $K$ right singular vectors of $[\hat{E}_{p1} \hat{E}_{p2}]$ that correspond to the $K$ largest singular values of $[\hat{E}_{p1} \hat{E}_{p2}]$. Let $V_1$ and $V_2$ be the $K \times K$ submatrices of $V$ consisting of the first and the last $K$ rows of $V$. Then the TLS solution $\hat{\Psi}_p$ is

$$\hat{\Psi}_p = [V_1^H]^{-1} V_2^H. \quad (6.44)$$

7) Calculate the TLS solution $\hat{\Psi}_q$ from

$$\hat{E}_{q2} = \hat{E}_{q1} \hat{\Psi}_q. \quad (6.45)$$

8) Calculate the TLS solution $\hat{\Psi}_r$ from

$$\hat{E}_{r2} = \hat{E}_{r1} \hat{\Psi}_r. \quad (6.46)$$
9) Compute $\hat{p}_k, \hat{q}_k, \hat{r}_k$, and $\hat{\gamma}_k$, $k = 1, 2, \ldots, \hat{K}$, by determining the eigenvalues of $\hat{\Psi}_p, \hat{\Psi}_q, \hat{\Psi}_{pq} = \hat{\Psi}_q^{-1}\hat{\Psi}_p$, and $\hat{\Psi}_{pr} = \hat{\Psi}_r^{-1}\hat{\Psi}_p$, respectively.

10) Determine the grouping scheme $(\hat{p}_k, \hat{q}_k, \hat{r}_k)$, $k = 1, 2, \ldots, \hat{K}$, as discussed in (6.34)-(6.35).

11) Calculate the direction and polarization estimates $(\hat{\theta}_k, \hat{\phi}_k, \hat{\alpha}_k, \hat{\beta}_k)$, $k = 1, 2, \ldots, \hat{K}$, from $(\hat{p}_k, \hat{q}_k, \hat{r}_k)$ as discussed in (6.36)-(6.40).

6.4 Simulation Results

We show below several examples of the use of this technique for direction and polarization estimation. These results were obtained by using fifty Monte Carlo simulations. The array used in the examples consists of four ($L = 2$) pairs of crossed dipoles with a spacing $\delta$ between adjacent dipole pairs of a half wavelength. All incident signals are assumed to have the same unit amplitude $E_k$. The signal-to-noise ratio (SNR) used in the simulations is $-10\log_{10}\sigma^2$ dB.

Before presenting these simulation examples, however, we first describe the method we shall use to describe the accuracy of the estimates.

Of course, the performance of the estimator could be presented just by showing the variance of each of the estimates $\hat{\theta}$, $\hat{\phi}$, $\hat{\gamma}$, and $\hat{\eta}$. However, the importance of an error in one of these parameters usually depends on what the other parameters are. For example, an error in $\hat{\phi}$ is most significant when $\theta = \pi/2$ but of no significance when $\theta = 0$. Moreover, since the value of $\phi$ is indeterminate when $\theta = 0$, the variance of $\hat{\phi}$ tends to be very high for $\theta = 0$. The high variance of $\hat{\phi}$ near $\theta = 0$ is misleading, because errors in $\hat{\phi}$ are not important in this case. Similar comments apply to the
polarization estimates. For this reason, we shall use the following different measures of estimator performance.

We define the error $\zeta_d$ of a direction estimate $(\hat{\theta}, \hat{\phi})$ when the actual arrival angle is $(\theta, \phi)$ to be the angular separation between the two directions as measured at the coordinate origin in Figure 33. The direction estimate variances plotted below are the mean-squared values of this angular error.

Similarly, for polarization estimates, the estimation error is the spherical distance between the two points $M$ and $\hat{M}$ on the Poincaré sphere that represent the actual signal polarization $(\alpha, \beta)$ and the estimated polarization $(\hat{\alpha}, \hat{\beta})$, as defined in Chapter IV.

There is one other feature that must be dealt with. For the special case where $\theta = 0$, the angle $\phi$ has the same effect on the signal as the angle $\beta$. When $\theta = 0$, changing either $\phi$ or $\beta$ simply rotates the principal axis of the polarization ellipse. (The principal axis is defined relative to $\bar{\phi}$.) To eliminate this ambiguity between $\phi$ and $\beta$, we always define $\beta = 0$ when $\theta = 0$.

We now present a few examples. We begin with the case of a linearly polarized signal $(\alpha = 0^\circ)$. Suppose a single linearly polarized signal arrives from azimuth angle $\phi = 0^\circ$ and elevation angle $\theta$. (For the case of one single incident signal, we drop the subscript $k$.) For this case, we have

$$ u = \begin{cases} \begin{bmatrix} \sin \gamma \cos \theta \\ \cos \gamma \\ -\sin \gamma \cos \theta \end{bmatrix}, & \text{if } 0^\circ \leq \beta < 90^\circ, \\ \begin{bmatrix} \cos \gamma \\ -\sin \gamma \cos \theta \end{bmatrix}, & \text{if } 90^\circ < \beta < 180^\circ. \end{cases} \quad (6.47) $$

Note that the signal powers in the dipole outputs depend only on the polarization angle $\gamma$ and the angle $\theta$. For example, if the signal is horizontally polarized $(\beta = 0^\circ)$,
then $\gamma = 0^\circ$. For this case, the output from an $x$-axis dipole is zero while the output from a $y$-axis dipole is maximum. If the signal is vertically polarized ($\beta = 90^\circ$), then $\gamma = 90^\circ$, and the signal from an $x$-axis dipole varies as $\cos \theta$ while the signal from a $y$-axis dipole is zero.

For such a signal, Figure 35(a) shows the variance (in dB with respect to degree squared) of the direction estimates as a function of $\beta$ for several elevation angles $\theta$. For these curves, the SNR was assumed to be 20 dB. The number of data samples taken at each dipole output was $N = 31$.

To understand why the curves in Figure 35(a) have the behavior shown, we show the variance of $\hat{\theta}$ and $\hat{\phi}$ as functions of $\beta$ in Figure 36. Note first that $\beta$ has little effect on either of the direction estimates $\hat{\theta}$ and $\hat{\phi}$ when $\theta$ is small. The reason is that for small $\theta$, the outputs of the $x$- and $y$-axis dipoles are not close to zero at the same time. For large $\theta$, however, $\beta$ has more effect. When $\theta$ is large, the total power received by the $x$- and $y$-axis dipoles becomes small as $\beta$ approaches $90^\circ$. This causes the performance of the angle estimates to deteriorate.

Also, note from Figure 36(a) that the accuracy of $\hat{\theta}$ becomes worse as $\theta$ increases. This behavior is due to the arcsin($\cdot$) in (6.36). Figure 37 shows a plot of $\sin^{-1}(w)$ versus $w$. Note that as $\sin^{-1}(w)$ approaches $90^\circ$, the slope of $\sin^{-1}(w)$ approaches infinity. When $\theta$ is near $90^\circ$, a small perturbation in the argument of the arcsin in (6.36) causes a large error in $\hat{\theta}$.

In addition, note from Figure 36(b) that when $\beta$ is close to $0^\circ$ or $180^\circ$, the accuracy of $\hat{\phi}$ improves as $\theta$ increases. (For $\beta$ near $0^\circ$ or $180^\circ$, the total power received by the $x$- and $y$-axis dipoles depends little on $\theta$.) The accuracy of $\hat{\phi}$ improves with $\theta$ because in (6.37) $\arg(q) = 0$ and $\arg(p)$ increases with $\theta$, as can been seen from (6.9) and (6.8), respectively. As $\arg(\hat{p}_k)$ in (6.37) increases, the error in in $\hat{\phi}$ decreases.
Figure 35: Variance of estimates versus $\beta$ for a linearly polarized signal ($\alpha = 0^\circ$, $\phi = 0^\circ$, SNR= 20 dB, $N = 31$).
Figure 36: Variance of $\hat{\theta}$ and $\hat{\phi}$ versus $\beta$ for a linearly polarized signal ($\alpha = 0^\circ$, $\phi = 0^\circ$, SNR = 20 dB, $N = 31$).
Figure 37: $\sin^{-1}(w)$ versus $w$. 
On the other hand, when $\beta$ is close to $90^\circ$, the accuracy of $\hat{\phi}$ deteriorates for large $\theta$. This behavior is due to the large errors in $\hat{p}$ and $\hat{q}$ that are caused by the small total power received by the $x$- and $y$-axis dipoles for $\beta \approx 90^\circ$ and large $\theta$.

Figure 35(b) shows the variance (in dB with respect to degree squared) of the polarization estimates as a function of $\beta$ for several values of $\theta$. In general, if we start with $\theta$ near zero and then increase $\theta$, the accuracy of the polarization estimates at first improves with $\theta$ but then finally becomes worse as $\theta$ nears $90^\circ$. The improvement with $\theta$ continues up to a higher value of $\theta$ when $\beta$ is near $90^\circ$ than for other values of $\beta$. The initial improvement with $\theta$ occurs because the accuracy of $\hat{\phi}$ improves with $\theta$, as noted above. When the accuracy of $\hat{\phi}$ improves, the accuracy of $\hat{\xi}$ in (6.40) improves, and hence the polarization estimates become better. However, as $\theta$ approaches $90^\circ$, errors in the term $\cos \hat{\theta}$ in the denominator of (6.40) and errors in $\hat{r}$ in (6.40) begin to dominate $\hat{\xi}$, so the polarization estimates become worse again.

Figure 38 shows another example, for a single circularly polarized signal ($\alpha = 45^\circ$). It is assumed again that SNR= 20 dB and $N = 31$. Figure 38 shows the variance of the direction and polarization estimates as a function of $\phi$ for several $\theta$. Note that the accuracy of both the direction and polarization estimates depends little on $\phi$ but is better for small $\theta$.

For this single circularly polarized signal, Figure 39 shows the variance of the direction and polarization estimates as a function of the SNR for several $\theta$ when $\phi = 90^\circ$ and $N = 31$. Figure 40 shows the variance of the estimates as a function of the number of data samples $N$ when $\phi = 90^\circ$ and SNR= 20 dB. As expected, the variances of both the direction and polarization estimates decrease as the SNR or $N$ increases. We note again that the accuracy of both the direction and polarization estimates is better for small $\theta$. 
Figure 38: Variance of estimates versus $\phi$ for a circularly polarized signal ($\alpha = 45^\circ$, SNR= 20 dB, $N = 31$).
Figure 39: Variance of estimates versus SNR for a circularly polarized signal ($\alpha = 45^\circ$, $\phi = 90^\circ$, $N = 31$).
Figure 40: Variance of estimates versus the number of data samples $N$ for a circularly polarized signal ($\alpha = 45^\circ$, $\phi = 90^\circ$, SNR= 20 dB).
Now we present three examples that illustrate how the separation in direction and polarization between two incident signals affect the performance of the estimator. We assume again that SNR = 20 dB and $N = 31$. Consider first a case where the two incident signals have closely-spaced arrival angles and similar polarizations. Suppose the two signals are characterized by

$$(\theta_1, \phi_1, \alpha_1, \beta_1, E_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ, 1),$$

and

$$(\theta_2, \phi_2, \alpha_2, \beta_2, E_2) = (25^\circ, 25^\circ, 30^\circ, 75^\circ, 1).$$

For these signals, the angle between the two signal directions is $\zeta_d = 10.53^\circ$, and the polarization separation is $\zeta_p = 23.57^\circ$. Figure 41 shows typical results obtained with the estimator. In Figure 41(a), the arrival angle estimates obtained with each of the 50 independent Monte Carlo trials are plotted on a polar plot with $\sin \theta$ as the radius and $\phi$ as the angle. (Each signal direction corresponds to a unique point on the upper half of a sphere centered at the coordinate origin. The plot shows the estimates projected from this sphere onto the $x$-$y$ plane.) The 50 pairs of estimates are superimposed on the same plot, so that one can see the spread in the estimates. Similarly, in Figure 41(b) the estimates of ellipticity and orientation angles obtained with each of the 50 independent trials are plotted on a polar plot with $\cos 2\alpha$ as the radius and $2\beta$ and the angle. (The plot shows the estimates projected from the Poincaré sphere onto the plane that passes through the equator of the sphere.) As may be seen in Figure 41(b), the polarization estimates for the two signals are hard to separate in this case.

Next consider a case where the polarizations of the incident signals are more widely separated. Suppose the signals are characterized by
Figure 41: Polar plot of estimates when \((\theta_1, \phi_1, \alpha_1, \beta_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ)\) and \((\theta_2, \phi_2, \alpha_2, \beta_2) = (25^\circ, 25^\circ, 30^\circ, 75^\circ)\).
\((\theta_1, \phi_1, \alpha_1, \beta_1, E_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ, 1),\)

and

\((\theta_2, \phi_2, \alpha_2, \beta_2, E_2) = (25^\circ, 25^\circ, -30^\circ, 105^\circ, 1).\)

For this case, \(\zeta_d = 10.53^\circ\) and \(\zeta_p = 119.35^\circ\). Figure 42 shows the results for this case. The polarization estimates are now more clearly separated.

Figures 41(a) and 42(a) illustrate that the direction resolution is improved significantly by increasing the polarization separation. The reason is that when the two signals arrive from closely-spaced directions with similar polarizations, the columns of matrix \(A\) in (6.16) become almost identical. \(A, A_{p1}, A_{q1}\), and \(A_{r1}\) then become ill-conditioned. This ill-conditioning makes the signal subspace approach more sensitive to noise. In other words, an ill-conditioned \(A\) results in an ill-conditioned \(\mathbf{R}_0\) in (6.19). Since the noise contribution to \(\hat{\mathbf{R}}\) is different from \(\sigma^2 \mathbf{I}\) when the number of data samples is finite, the signal subspace eigenvectors of \(\hat{\mathbf{R}}\) are perturbed by the noise from the true eigenvectors of \(\mathbf{R}_0\). The ill-conditioning of \(\mathbf{R}_0\) then makes the signal subspace eigenvectors of \(\mathbf{R}_0\) more sensitive to this perturbation [43].

The polarization estimates in Figure 42(b) are also improved over those in Figure 41(b), but not by as much as the direction estimates. This difference results because the ill-conditioning of \(A_{r1}\) is not reduced by increasing the polarization separation.

Finally, we consider an example in which the directions of the incident signals are more widely separated. The signals are characterized by

\((\theta_1, \phi_1, \alpha_1, \beta_1, E_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ, 1),\)

and

\((\theta_2, \phi_2, \alpha_2, \beta_2, E_2) = (55^\circ, -25^\circ, 30^\circ, 75^\circ, 1).\)
Figure 42: Polar plot of estimates when \((\theta_1, \phi_1, \alpha_1, \beta_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ)\) and \((\theta_2, \phi_2, \alpha_2, \beta_2) = (25^\circ, 25^\circ, -30^\circ, 105^\circ)\).
For this case, $\zeta_d = 44.24^\circ$ and $\zeta_p = 23.57^\circ$. Figure 43 shows the results for this case. By comparing Figures 41 and 43, it is seen that both the polarization estimates and the direction estimates have been improved by increasing the direction separation. Increasing the direction separation of the two signals reduces the ill-conditioning of $A$, $A_{p1}$, $A_{q1}$, and $A_{r1}$.

We remark that the MDL criterion provided accurate estimates of the number of incident signals $K$ for all simulations conducted.

6.5 Summary

We have described the use of the ESPRIT algorithm for estimating two dimensional arrival angles and polarizations of arbitrarily polarized signals with a square array of crossed dipoles. The ESPRIT algorithm exploits the invariance properties of such an array so that both angle and polarization estimates may be computed. Some typical examples showing the use of this approach have been presented.

We remark that when only the two-dimensional arrival angles are to be estimated, we may use a simplified approach similar to the angle-only method described in Chapter IV for this purpose. Also, when the incident signals are perfectly correlated, the spatial smoothing techniques described in Chapter V may again be applied to the square array for estimating two dimensional arrival angles and polarizations.
Figure 43: Polar plot of estimates when \((\theta_1, \phi_1, \alpha_1, \beta_1) = (15^\circ, 15^\circ, 20^\circ, 65^\circ)\) and \((\theta_2, \phi_2, \alpha_2, \beta_2) = (55^\circ, -25^\circ, 30^\circ, 75^\circ)\).
CHAPTER VII
SUMMARY AND CONCLUSIONS

In this dissertation, we have investigated sensor array signal processing techniques of two types: (1) techniques for incorporating a known desired signal waveform or multiple known signal waveforms in the estimation process, and (2) techniques for estimating the electromagnetic polarization of signals as well as their arrival angles. These techniques have been developed for narrowband plane waves.

First, we have presented a maximum likelihood (ML) algorithm that incorporates knowledge of a desired signal waveform into the angle estimation process with a uniform linear array. The desired signal is assumed to be corrupted by noise and also by interfering signals with unknown waveforms. The interfering signals are assumed uncorrelated with the desired signal. Although the desired signal waveform is assumed known, its gain may be unknown. The algorithm solves the ML solution iteratively and converges in a few iterations. It requires only a one-dimensional search over parameter space, so it avoids the complexity of a multi-dimensional search. Since the algorithm is based on ML estimation, it is applicable even when the interfering signals are perfectly correlated with each other. We have presented curves comparing the performance of this algorithm with that of the iterative quadratic ML (IQML) algorithm when augmented to extract the desired signal angle. We have also described the conditions under which incorporating the known signal waveform in the estimator improves the accuracy of the estimates. Finally, we have derived CR-bounds for both the new estimator and the augmented IQML estimator and have
compared the performance of these estimators with their CR-bounds. Our numerical results show that the ML angle estimates for the desired signal are very close to the best unbiased estimates one can get.

We have also presented two ML algorithms that incorporate knowledge of multiple known signal waveforms with unknown gains into the angle estimation process. Both algorithms compute the ML estimates iteratively and converge in a few iterations, so the complexity of a multi-dimensional search is again avoided. Since the algorithms are based on ML estimation, they are applicable even when the incident signals are perfectly correlated with each other. For comparison, we have also considered the IQML algorithm augmented to determine which signal waveform corresponds to which estimated angle. Curves were presented comparing the performance of the two ML estimators with each other and also with the IQML algorithm. Both the actual performance and the Cramer-Rao bounds were shown under several assumptions for the unknown signal gains. The curves show that the ML angle estimates are very close to the best possible unbiased estimates. They also show the conditions under which incorporating the known signal waveforms in the estimator improves the angle estimates.

In the second major part of this work, we have considered the problem of estimating the arrival angles and polarizations of electromagnetic signals that arrive with unknown polarization. We have shown that if an array uses elements responding to more than one polarization, both signal directions and polarizations may be estimated. In particular, we applied the ESPRIT algorithm to uniform arrays of crossed dipoles. The ESPRIT algorithm exploits multiple invariance properties of such arrays to allow both angle and polarization estimation. We considered angle estimation in one spatial coordinate with a uniform linear array and also angle es-
imation in two spatial coordinates with a square array. We have also presented a simpler alternative approach that can be used with a uniform linear array to estimate arrival directions only. This alternative approach has the advantage that it requires approximately one eighth as many computations as the original method. It also has the advantage over conventional angle estimation techniques that it works properly regardless of signal polarization. Numerical results show that this alternative method yields poorer direction estimates than the original approach when both the directions and the polarizations of two signals are nearly identical. Otherwise, the two approaches have similar performance. We have shown that the ESPRIT algorithm may be used with forward-only spatial smoothing techniques to estimate both signal directions and polarizations when the incident signals are coherent. (Without spatial smoothing, the ESPRIT algorithm fails to work properly in a coherent signal environment.) The simpler alternative approach used to estimate signal directions only can be used with both forward-only and forward/backward spatial smoothing techniques in a coherent signal environment. Spatial smoothing was found to yield poorer resolution for coherent signals than would be obtained with uncorrelated or partially correlated signals, but of course without spatial smoothing ESPRIT cannot be used at all with coherent sources. Typical examples showing the use of these approaches have been presented.
APPENDIX A
THE MAXIMUM LIKELIHOOD ESTIMATOR

To determine the maximum likelihood estimator in Chapter II, suppose first that all the signal waveforms, including $s_0(t)$, are unknown. The conditional multivariate Gaussian probability density function for the vector $x(t)$ of $L$ complex random variables in (2.6) is then [36, page 114]

$$p[x(t)|\theta, s(t)] = [\pi \sigma^2]^{-L} \exp \left\{ -\sigma^{-2} [x(t) - A(\theta)s(t)]^H [x(t) - A(\theta)s(t)] \right\},$$

(A.1)

where $(\cdot)^H$ denotes the complex conjugate transpose. If $N$ independent samples $x(t_n)$ of $x(t)$ are taken, the density function for the $x(t_n)$ is

$$p[x(t_1), \cdots, x(t_N)|\theta, s(t_1), \cdots, s(t_N)] = [\pi \sigma^2]^{-LN} \exp \left\{ -N \sigma^{-2} q \right\}$$

(A.2)

where

$$q = \frac{1}{N} \sum_{n=1}^{N} [x(t_n) - A(\theta)s(t_n)]^H [x(t_n) - A(\theta)s(t_n)].$$

(A.3)

To maximize $p[x(t_1), \cdots, x(t_N)|\theta, s(t_1), \cdots, s(t_N)]$, one must minimize $q$.

As shown in [47], minimizing $q$ can be carried out in two steps. For a given $\theta$, minimizing $q$ with respect to $s(t)$ is a standard least squares problem, with the result

$$\bar{s}(t_n) = \left[ A^H(\theta)A(\theta) \right]^{-1} A^H(\theta)x(t_n).$$

(A.4)

Replacing $s(t_n)$ in (A.3) with $\bar{s}(t_n)$ gives
\[ q = \frac{1}{N} \sum_{n=1}^{N} \left[ P_{A^\perp(\theta)} x(t_n) \right]^H \left[ P_{A^\perp(\theta)} x(t_n) \right], \]  

(A.5)

where

\[ P_{A^\perp(\theta)} = I - A(\theta) \left[ A^H(\theta) A(\theta) \right]^{-1} A^H(\theta). \]  

(A.6)

Then using the cyclic property of the trace operator and the fact that \( P_{A^\perp(\theta)} \) is Hermitian and idempotent gives

\[ q = \text{tr} \left\{ P_{A^\perp(\theta)} \hat{R} \right\}, \]  

(A.7)

where

\[ \hat{R} = \frac{1}{N} \sum_{n=1}^{N} x(t_n)x^H(t_n). \]  

(A.8)

Second, suppose the desired signal waveform is given by (2.1) with \( g \) unknown and \( w_0(t) \) known (and the interference signal waveforms \( s_k(t), k = 1, 2, \cdots, K, \) are assumed unknown). Equation (A.2) is then replaced by

\[ p[x(t_1), \cdots, x(t_N)|g, \theta, s_1(t_1), \cdots, s_1(t_N)] = [\pi \sigma^2]^{-LN} \exp\{-N\sigma^{-2}q\}, \]  

(A.9)

where

\[ q = \frac{1}{N} \sum_{n=1}^{N} [x_I(t_n) - A(\theta_I)s_I(t_n)]^H [x_I(t_n) - A(\theta_I)s_I(t_n)], \]  

(A.10)

with

\[ x_I(t_n) = x(t_n) - a(\theta_0)gw_0(t_n), \]  

(A.11)

\[ s_I(t_n) = [s_1(t_n) s_2(t_n) \cdots s_K(t_n)]^T, \]  

(A.12)

and

\[ A(\theta_I) = [a(\theta_1) a(\theta_2) \cdots a(\theta_K)]. \]  

(A.13)
where \( \theta_I = [\theta_1 \theta_2 \cdots \theta_K]^T \). Now the minimization of \( q \) is over \( \{\theta, s_I(t_1), s_I(t_2), \ldots, s_I(t_N)\} \) if \( g \) is known or over \( \{g, \theta, s_I(t_1), s_I(t_2), \ldots, s_I(t_N)\} \) if \( g \) is unknown.

We can again minimize \( q \) in two steps as described in [47]. With \( g \) and \( \theta \) fixed, minimizing \( q \) with respect to \( s_I(t) \) gives the result

\[
\mathbf{s}_I(t_n) = [\mathbf{A}^H(\theta_I)\mathbf{A}(\theta_I)]^{-1}\mathbf{A}^H(\theta_I)x_I(t_n). \tag{A.14}
\]

Replacing \( s_I(t_n) \) in (A.10) with \( \mathbf{s}_I(t_n) \) in (A.14) gives

\[
q = \frac{1}{N} \sum_{n=1}^{N} [\mathbf{P}_A(\theta_I)x_I(t_n)]^H[\mathbf{P}_A(\theta_I)x_I(t_n)], \tag{A.15}
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} x_I^H(t_n)\mathbf{P}_A(\theta_I)x_I(t_n) \tag{A.16}
\]

where

\[
\mathbf{P}_A(\theta_I) = \mathbf{I} - \mathbf{A}(\theta_I)[\mathbf{A}^H(\theta_I)\mathbf{A}(\theta_I)]^{-1}\mathbf{A}^H(\theta_I). \tag{A.17}
\]

Finally, from the cyclic property of the trace operator, we have

\[
q = \text{tr}\{\mathbf{P}_A(\theta_I)\hat{\mathbf{R}}_I\}, \tag{A.18}
\]

where

\[
\hat{\mathbf{R}}_I = \frac{1}{N} \sum_{n=1}^{N} x_I(t_n)x_I^H(t_n). \tag{A.19}
\]

Thus, maximizing the likelihood function for the case of known signal waveform is equivalent to minimizing the right side of (A.18). This minimization involves either

1. a \((K+1)\)-dimensional search if \( g \) is known,
2. a \((K+2)\)-dimensional search if \( g \) is unknown and real or if \( g = e^{j\nu} \) with \( \nu \) unknown and real, or
3. a \((K+3)\)-dimensional search if \( g \) is unknown and complex.
APPENDIX B

THE CONJUGATE SYMMETRY CONSTRAINT ON b

Let $a(z) = a_0z^{K+1} + a_1z^K + \cdots + a_{K+1}$ be a polynomial whose roots are the $q_k, \ k = 0, 1, \cdots, K$, in (2.9). It is shown by Marden [34] that such a polynomial, whose zeros are all on the unit circle, must be a self-inverse polynomial, i.e., the coefficients of $a(z)$ must satisfy

$$a_k = ua_{K+1-k}^*, \ |u| = 1, \ k = 0, 1, \cdots, K + 1,$$

where $u$ is a complex constant. Although $q$ in (2.13) could be expressed in terms of the coefficients of $a(z)$, which satisfy the above property, we instead express $q$ in terms of a new polynomial that satisfies a simpler property.

Given $u$, let $v$ be a square root of $u$ in (B.1), i.e., $u = v^2$. Since $|u| = 1$, then $|v| = 1$ so $v^{-1} = v^*$. The polynomial $a(z)$ can then be rewritten

$$a(z) = v[v^*a_0z^{K+1} + v^*a_1z^K + \cdots + v^*a_{K+1}]$$

$$= v[b_0z^{K+1} + b_1z^K + \cdots + b_{K+1}]$$

$$= vb(z),$$

where

$$b_k = v^*a_k, \ k = 0, 1, \cdots, K + 1,$$

and

$$b(z) = b_0z^{K+1} + b_1z^K + \cdots + b_{K+1}.$$
The new polynomial $b(z)$ also has the same roots as $a(z)$, since $b(z) = v^* a(z)$.

However, by writing (B.1) in the form

$$v^* a_k = v a_{K+1-k}^*,$$  \hspace{1cm} (B.7)

and noting (B.5), we see that the coefficients $b_k$ of $b(z)$ satisfy the simpler conjugate symmetry property,

$$b_k = b_{K+1-k}^*. \hspace{1cm} (B.8)$$

We use the coefficients of this $b(z)$ in (2.13), instead of those of $a(z)$. Note that the coefficients of $b(z)$ yield the same $q$ in (2.13) as the coefficients of $a(z)$ would. In (2.19), we impose relation (B.8) on $\hat{b}(z)$. 
APPENDIX C

DERIVATION OF THE CR-BOUNDS IN CHAPTER II

To derive the CR-bounds in Chapter II, we follow the derivation in Appendix E of [35] and use the results proven there.

Using (A.9), we find the following log-likelihood function:

\[
\ln \mathcal{L} = \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} [\mathbf{x}(t_n) - \mathbf{A}(\theta)\mathbf{s}(t_n)]^H \\
[\mathbf{x}(t_n) - \mathbf{A}(\theta)\mathbf{s}(t_n)]
\]

(C.1)

\[
= \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} [\mathbf{x}(t_n) - \mathbf{A}(\theta_I)\mathbf{s}_I(t_n) - \mathbf{a}(\theta_0) \mathbf{g} w_0(t_n)]^H \\
[\mathbf{x}(t_n) - \mathbf{A}(\theta_I)\mathbf{s}_I(t_n) - \mathbf{a}(\theta_0) \mathbf{g} w_0(t_n)]
\]

(C.2)

\[
= \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} \mathbf{e}^H(t_n)\mathbf{e}(t_n),
\]

(C.3)

where \( \mathbf{A}(\theta_I) \) and \( \mathbf{s}_I(t_n) \) have been defined in (A.13) and (A.12), respectively, and we have defined

\[
\mathbf{e}(t_n) = \mathbf{x}(t_n) - \mathbf{A}(\theta_I)\mathbf{s}_I(t_n) - \mathbf{a}(\theta_0) \mathbf{g} w_0(t_n).
\]

(C.4)

If \( g \) is known, using the results in Appendix E of [35], we find

\[
\frac{\partial \ln \mathcal{L}}{\partial \sigma^{-2}} = -\frac{LN}{\sigma^2} + \frac{1}{\sigma^2} \sum_{n=1}^{N} [\mathbf{e}^H(t_n)\mathbf{e}(t_n)],
\]

(C.5)

\[
\frac{\partial \ln \mathcal{L}}{\partial \text{Re}[\mathbf{s}_I(t_n)]} = \frac{2}{\sigma^2} \text{Re} [\mathbf{A}^H(\theta_I)\mathbf{e}(t_n)], \quad n = 1, 2, \ldots, N,
\]

(C.6)

\[
\frac{\partial \ln \mathcal{L}}{\partial \text{Im}[\mathbf{s}_I(t_n)]} = \frac{2}{\sigma^2} \text{Im} [\mathbf{A}^H(\theta_I)\mathbf{e}(t_n)], \quad n = 1, 2, \ldots, N,
\]

(C.7)
\[ \frac{\partial \ln \mathcal{L}}{\partial \theta} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ \mathbf{S}^H(t_n)\mathbf{D}^H(\theta)\mathbf{e}(t_n) \right], \quad (C.8) \]

where \( \mathbf{S}^H(t_n) \) and \( \mathbf{D}^H(\theta) \) have been defined in (2.54) and (2.55), respectively.

Let
\[ \Gamma_1 = \sum_{n=1}^{N} \text{Re} \left[ \mathbf{S}^H(t_n)\mathbf{D}^H(\theta)\mathbf{D}(\theta)\mathbf{S}(t_n) \right], \quad (C.9) \]
\[ \mathbf{H}_I = \mathbf{A}^H(\theta_I)\mathbf{A}(\theta_I), \quad (C.10) \]

and
\[ \Delta(t_n) = \mathbf{A}^H(\theta_I)\mathbf{D}(\theta)\mathbf{S}(t_n), \quad n = 1, 2, \ldots, N. \quad (C.11) \]

Then
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right]^2 \right\} = \frac{LN}{\sigma^4}, \quad (C.12) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re} \left[ \mathbf{s}_I(t_n) \right]} \right]^T \right\} = 0, \quad (C.13) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im} \left[ \mathbf{s}_I(t_n) \right]} \right]^T \right\} = 0, \quad (C.14) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = 0, \quad (C.15) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re} \left[ \mathbf{s}_I(t_{n_1}) \right]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re} \left[ \mathbf{s}_I(t_{n_2}) \right]} \right]^T \right\} = \frac{2}{\sigma^2} \text{Re} \left[ \mathbf{A}^H(\theta_I)\mathbf{A}(\theta_I) \right] \delta_{n_1,n_2} \]
\[ = \frac{2}{\sigma^2} \text{Re} \left( \mathbf{H}_I \right) \delta_{n_1,n_2}, \quad (C.16) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re} \left[ s_I(t_{n_1}) \right]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im} \left[ s_I(t_{n_2}) \right]} \right]^T \right\} = \frac{2}{\sigma^2} \text{Im} \left[ A^H(\theta_I)A(\theta_I) \right] \delta_{n_1,n_2} \]
\[ = \frac{2}{\sigma^2} \text{Im} (H_I) \delta_{n_1,n_2}, \quad (C.17) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re} \left[ s_I(t_n) \right]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \text{Re} \left[ A^H(\theta_I)D(\theta)S(t_n) \right] \quad (C.18) \]
\[ = \frac{2}{\sigma^2} \text{Re} [\Delta(t_n)], \quad (C.19) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im} \left[ s_I(t_{n_1}) \right]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im} \left[ s_I(t_{n_2}) \right]} \right]^T \right\} = \frac{2}{\sigma^2} \text{Re} \left[ A^H(\theta_I)A(\theta_I) \right] \delta_{n_1,n_2} \]
\[ = \frac{2}{\sigma^2} \text{Re} (H_I) \delta_{n_1,n_2}, \quad (C.20) \]
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im} \left[ s_I(t_n) \right]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \text{Im} \left[ A^H(\theta_I)D(\theta)S(t_n) \right] \quad (C.21) \]
\[ = \frac{2}{\sigma^2} \text{Im} [\Delta(t_n)], \quad (C.22) \]

and
\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)D^H(\theta)D(\theta)S(t_n) \right] \quad (C.23) \]
\[ = \frac{2}{\sigma^2} \Gamma_1, \quad (C.24) \]

where
\[ \delta_{n_1,n_2} = \begin{cases} 
1 & \text{if } n_1 = n_2, \\
0 & \text{if } n_1 \neq n_2.
\end{cases} \quad (C.25) \]
Thus for \[ \begin{bmatrix} \sigma^2 & \text{Re}^T[s_f(t_n)] & \text{Im}^T[s_f(t_n)] & \theta^T \end{bmatrix}^T, \]
Fisher's information matrix is
\[
F = \frac{2}{\sigma^2} \begin{bmatrix}
\frac{LN}{2\sigma^2} & 0 & \text{Re}(H_I) & -\text{Im}(H_I) \\
0 & 0 & \text{Im}(H_I) & \text{Re}(H_I) \\
0 & \text{Re}^T(\Delta(t_1)) & \text{Im}^T(\Delta(t_1)) & \cdots \\
0 & \text{Re}^T(\Delta(t_N)) & \text{Im}^T(\Delta(t_N)) & \Gamma_1
\end{bmatrix},
\]
where we have used
\[
-\text{Im}^T(H_I) = \text{Im}(H_I)
\]
(C.27)
since \(H_I\) is Hermitian and hence its imaginary part must be skew-symmetric. Using
the standard matrix inversion lemma, we have
\[
\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \{\Gamma_1 - \\
\begin{bmatrix}
\text{Re}^T(\Delta(t_1)) & \text{Im}^T(\Delta(t_1)) & \cdots \\
\text{Re}^T(\Delta(t_N)) & \text{Im}^T(\Delta(t_N))
\end{bmatrix}
\]
\[
\begin{bmatrix}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & 0 \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) & 0 \\
0 & \text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) \\
0 & \text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1})
\end{bmatrix}
\]

\[\text{Re}[\Delta(t_1)]
\]
\[\text{Im}[\Delta(t_1)]
\]
\[\vdots
\]
\[\text{Re}[\Delta(t_N)]
\]
\[\text{Im}[\Delta(t_N)]
\]

\[= \frac{2}{\sigma^2} \left\{ \Gamma_1 - \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)D^H(\theta)A(\theta_I) \right] \right\}
\]

\[= \frac{2}{\sigma^2} \left[ \sum_{n=1}^{N} \Gamma(t_n) \right]
\]

where \(t_n\) has been defined in (2.58), and we have used (2.40) and (see Appendix E of [35])

\[\begin{bmatrix}
\text{Re}(H_I) & -\text{Im}(H_I) \\
\text{Im}(H_I) & \text{Re}(H_I)
\end{bmatrix}
\]

\[-1
\]=

\[\begin{bmatrix}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1})
\end{bmatrix}
\]

We have now proven Equation (2.57).

*If g is unknown and complex,* we calculate \(\partial \ln \mathcal{L}/\partial \text{Re}(g)\) and \(\partial \ln \mathcal{L}/\partial \text{Im}(g)\).

Using the results in Appendix E of [35], we get

\[\frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ a^H(\theta_0)w^*_0(t_n)e(t_n) \right],
\]

and
\[ \frac{\partial \ln L}{\partial \text{Im}(g)} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Im} \left[ a^H(\theta_0)w_0^*(t_n)e(t_n) \right]. \]  

(C.33)

Let

\[ f_1(t_n) = A^H(\theta_I) \ a(\theta_0)w_0(t_n), \]  

(C.34)

\[ f_2 = \sum_{n=1}^{N} \left[ S^H(t_n)D^H(\theta)a(\theta_0)w_0(t_n) \right], \]  

(C.35)

and

\[ \eta_1 = \sum_{n=1}^{N} a^H(\theta_0)w_0^*(t_n)w_0(t_n)a(\theta_0). \]  

(C.36)

Then

\[ E \left\{ \left[ \frac{\partial \ln L}{\partial \sigma^2} \right] \left[ \frac{\partial \ln L}{\partial \text{Re}(g)} \right] \right\} = 0, \]  

(C.37)

\[ E \left\{ \left[ \frac{\partial \ln L}{\partial \text{Re}[S_I(t_n)]} \right] \left[ \frac{\partial \ln L}{\partial \text{Re}(g)} \right] \right\} = \frac{2}{\sigma^2} \text{Re} \left[ A^H(\theta_I) a(\theta_0)w_0(t_n) \right] \]  

\[ = \frac{2}{\sigma^2} \text{Re} \left[ f_1(t_n) \right], \]  

(C.38)

\[ E \left\{ \left[ \frac{\partial \ln L}{\partial \text{Im}[S_I(t_n)]} \right] \left[ \frac{\partial \ln L}{\partial \text{Re}(g)} \right] \right\} = \frac{2}{\sigma^2} \text{Im} \left[ A^H(\theta_I) a(\theta_0)w_0(t_n) \right] \]  

\[ = \frac{2}{\sigma^2} \text{Im} \left[ f_1(t_n) \right], \]  

(C.39)

\[ E \left\{ \left[ \frac{\partial \ln L}{\partial \theta} \right] \left[ \frac{\partial \ln L}{\partial \text{Re}(g)} \right] \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)D^H(\theta)a(\theta_0)w_0(t_n) \right] \]  

\[ = \frac{2}{\sigma^2} \text{Re} \left( f_2 \right), \]  

(C.40)
\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ a^H(\theta_0) w_0^*(t_n) w_0(t_n) a(\theta_0) \right]
\]
\[
= \frac{2}{\sigma^2} \text{Re}(\eta_1)
\]
\[
= \frac{2}{\sigma^2} \eta_1,
\]
(C.41)

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right] \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} -\text{Im} \left[ a^H(\theta_0) w_0^*(t_n) w_0(t_n) a(\theta_0) \right]
\]
\[
= -\frac{2}{\sigma^2} \text{Im}(\eta_1)
\]
\[
= 0,
\]
(C.42)

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right] \right\} = 0,
\]
(C.43)

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}[s_I(t_n)]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right] \right\} = -\frac{2}{\sigma^2} \text{Im} \left[ A^H(\theta_I) a(\theta_0) w_0(t_n) \right]
\]
\[
= -\frac{2}{\sigma^2} \text{Im}[f_1(t_n)],
\]
(C.44)

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}[s_I(t_n)]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right] \right\} = \frac{2}{\sigma^2} \text{Re} \left[ A^H(\theta_I) a(\theta_0) w_0(t_n) \right]
\]
\[
= \frac{2}{\sigma^2} \text{Re}[f_1(t_n)],
\]
(C.45)

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right] \right\} = -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Im} \left[ S^H(t_n) D^H(\theta) a(\theta_0) w_0(t_n) \right]
\]
\[
= -\frac{2}{\sigma^2} \text{Im}(f_2),
\]
(C.46)

and

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right]^2 \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ a^H(\theta_0) w_0^*(t_n) w_0(t_n) a(\theta_0) \right]
\]
\[
= \frac{2}{\sigma^2} \text{Re}(\eta_1)
\]
\[
= \frac{2}{\sigma^2} \eta_1.
\]
(C.47)
Let

\[ \varphi_c = [\theta^T \quad \text{Re}(g) \quad \text{Im}(g)]^T. \] (C.48)

Thus for

\[ \left[ \sigma^2 \quad \text{Re}^T[s_I(t_n)] \quad \text{Im}^T[s_I(t_n)] \quad \varphi_c^T \right]^T, \]

the Fisher's information matrix is

\[
F = \frac{2}{\sigma^2} \left[ \begin{array}{cccc}
\frac{LN}{2\sigma^2} & 0 & 0 & 0 \\
0 & \text{Re}(H_I) & -\text{Im}(H_I) & 0 \\
0 & \text{Im}(H_I) & \text{Re}(H_I) & 0 \\
0 & 0 & 0 & \text{Re}(H_I) \\
0 & \text{Re}^T[\Delta(t_1)] & \text{Im}^T[\Delta(t_1)] & \cdots & \text{Re}^T[\Delta(t_N)] \\
0 & \text{Re}^T[f_1(t_1)] & \text{Im}^T[f_1(t_1)] & \cdots & \text{Re}^T[f_1(t_N)] \\
0 & -\text{Im}^T[f_1(t_1)] & \text{Re}^T[f_1(t_1)] & \cdots & -\text{Im}^T[f_1(t_N)] \\
0 & \text{Re}^T[\Delta(t_N)] & \text{Im}^T[\Delta(t_N)] & \cdots & \text{Re}^T[\Delta(t_N)] \\
0 & \text{Re}(H_I) & \text{Im}[\Delta(t_N)] & \text{Im}[f_1(t_N)] & \text{Re}[f_1(t_1)] \\
0 & -\text{Im}(H_I) & \text{Re}[\Delta(t_1)] & \text{Re}[f_1(t_1)] & -\text{Im}[f_1(t_1)] \\
0 & \text{Im}[\Delta(t_1)] & \text{Im}[f_1(t_1)] & \text{Re}[f_1(t_1)] & 0 \\
0 & \cdots & \cdots & \cdots & \cdots \\
-\text{Im}(H_I) & \text{Re}[\Delta(t_N)] & \text{Re}[f_1(t_N)] & -\text{Im}[f_1(t_N)] & 0 \\
\text{Re}(H_I) & \text{Im}[\Delta(t_N)] & \text{Im}[f_1(t_N)] & \text{Re}[f_1(t_N)] & 0 \\
\text{Im}^T[\Delta(t_N)] & \text{Re}^T(f_2) & \eta_1 & 0 & \text{Re}^T(f_2) \\
\text{Im}^T[f_1(t_N)] & \text{Re}^T(f_2) & \eta_1 & 0 & \eta_1 \\
\text{Re}^T[f_1(t_N)] & -\text{Im}^T(f_2) & 0 & \eta_1 & 0 \\
\end{array} \right]. \] (C.49)
Using the standard matrix inversion lemma, we have

\[
\text{CRB}^{-1}(\varphi_c) = \frac{2}{\sigma^2} \left[ \begin{array}{ccc}
\Gamma_1 & \text{Re}(f_2) & \text{Im}(f_2) \\
\text{Re}^T(f_2) & \eta_1 & 0 \\
\text{Im}^T(f_2) & 0 & \eta_1 \\
\end{array} \right]
\]

\[
- \left[ \begin{array}{ccc}
\text{Re}^T[\Delta(t_1)] & \text{Im}^T[\Delta(t_1)] & \cdots \\
\text{Re}^T[f_1(t_1)] & \text{Im}^T[f_1(t_1)] & \cdots \\
-\text{Im}^T[f_1(t_1)] & \text{Re}^T[f_1(t_1)] & \cdots \\
\end{array} \right]
\]

\[
\left[ \begin{array}{cccc}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & 0 & \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) & 0 & \\
0 & \text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & \\
\end{array} \right]
\]

\[
\left[ \begin{array}{cccc}
\text{Re}[\Delta(t_1)] & \text{Re}[f_1(t_1)] & -\text{Im}[f_1(t_1)] \\
\text{Im}[\Delta(t_1)] & \text{Im}[f_1(t_1)] & \text{Re}[f_1(t_1)] \\
\vdots & \vdots & \vdots \\
\text{Re}[\Delta(t_N)] & \text{Re}[f_1(t_N)] & -\text{Im}[f_1(t_N)] \\
\text{Im}[\Delta(t_N)] & \text{Im}[f_1(t_N)] & \text{Re}[f_1(t_N)] \\
\end{array} \right].
\]

This equation may be simplified as follows. For \( n = 1, 2, \cdots, N \), the results given below can be seen easily:

\[
\left[ \begin{array}{ccc}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) & \\
\end{array} \right]
\left[ \begin{array}{c}
\text{Re}[\Delta(t_n)] \\
\text{Im}[\Delta(t_n)] \\
\end{array} \right]
= \left[ \begin{array}{c}
\text{Re}[H_I^{-1}\Delta(t_n)] \\
\text{Im}[H_I^{-1}\Delta(t_n)] \\
\end{array} \right],
\]
\[
\begin{bmatrix}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1})
\end{bmatrix}
\begin{bmatrix}
\text{Re}[f_1(t_n)] \\
\text{Im}[f_1(t_n)]
\end{bmatrix}
= 
\begin{bmatrix}
\text{Re}[H_I^{-1}f_1(t_n)] \\
\text{Im}[H_I^{-1}f_1(t_n)]
\end{bmatrix}, 
\quad (C.52)
\]

and
\[
\begin{bmatrix}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1})
\end{bmatrix}
\begin{bmatrix}
-\text{Im}[f_1(t_n)] \\
\text{Re}[f_1(t_n)]
\end{bmatrix}
= 
\begin{bmatrix}
-\text{Im}[H_I^{-1}f_1(t_n)] \\
\text{Re}[H_I^{-1}f_1(t_n)]
\end{bmatrix}. 
\quad (C.53)
\]

For \( n = 1, 2, \ldots, N \), we also have
\[
\begin{bmatrix}
\text{Re}[\Delta(t_n)] \\
\text{Im}[\Delta(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
\text{Re}[H_I^{-1}\Delta(t_n)] \\
\text{Im}[H_I^{-1}\Delta(t_n)]
\end{bmatrix}
= 
\text{Re}\left[\Delta^H(t_n)H_I^{-1}\Delta(t_n)\right], 
\quad (C.54)
\]
\[
\begin{bmatrix}
\text{Re}[\Delta(t_n)] \\
\text{Im}[\Delta(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
\text{Re}[H_I^{-1}f_1(t_n)] \\
\text{Im}[H_I^{-1}f_1(t_n)]
\end{bmatrix}
= 
\text{Re}\left[\Delta^H(t_n)H_I^{-1}f_1(t_n)\right], 
\quad (C.55)
\]
\[
\begin{bmatrix}
\text{Re}[\Delta(t_n)] \\
\text{Im}[\Delta(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
-\text{Im}[H_I^{-1}f_1(t_n)] \\
\text{Re}[H_I^{-1}f_1(t_n)]
\end{bmatrix}
= -\text{Im}\left[\Delta^H(t_n)H_I^{-1}f_1(t_n)\right], 
\quad (C.56)
\]
\[
\begin{bmatrix}
\text{Re}[f_1(t_n)] \\
\text{Im}[f_1(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
\text{Re}[H_I^{-1}f_1(t_n)] \\
\text{Im}[H_I^{-1}f_1(t_n)]
\end{bmatrix}
= 
\text{Re}\left[f_1^H(t_n)H_I^{-1}f_1(t_n)\right], 
\quad (C.57)
\]
\[
\begin{bmatrix}
\text{Re}[f_1(t_n)] \\
\text{Im}[f_1(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
-\text{Im}[H_I^{-1}f_1(t_n)] \\
\text{Re}[H_I^{-1}f_1(t_n)]
\end{bmatrix}
= -\text{Im}\left[f_1^H(t_n)H_I^{-1}f_1(t_n)\right], 
\quad (C.58)
\]
\[
= 0, 
\quad (C.59)
\]

and
\[
\begin{bmatrix}
-\text{Im}[f_1(t_n)] \\
\text{Re}[f_1(t_n)]
\end{bmatrix}^T
\begin{bmatrix}
-\text{Im}[H_I^{-1}f_1(t_n)] \\
\text{Re}[H_I^{-1}f_1(t_n)]
\end{bmatrix}
= \text{Re}\left[f_1^H(t_n)H_I^{-1}f_1(t_n)\right]. 
\quad (C.60)
\]
Applying the above results to the right side of (C.50), we get

\[
\text{CRB}^{-1}(\theta_c) = \frac{2}{\sigma^2} \left\{ \begin{array}{ccc}
\Gamma_1 & \text{Re}(f_2) & \text{Im}(f_2) \\
\text{Re}^T(f_2) & \eta_1 & 0 \\
\text{Im}^T(f_2) & 0 & \eta_1
\end{array} \right\} - \sum_{n=1}^{N} \left[ \text{Re} \left[ \Delta^H(t_n)H_I^{-1}\Delta(t_n) \right] \right.
\left. \begin{array}{ccc}
\text{Re}^T \left[ \Delta^H(t_n)H_I^{-1}f_1(t_n) \right] \\
-\text{Im}^T \left[ \Delta^H(t_n)H_I^{-1}f_1(t_n) \right] \\
\text{Im} \left[ \Delta^H(t_n)H_I^{-1}f_1(t_n) \right]
\end{array} \right] \] (C.61)

\[
= \frac{2}{\sigma^2} \sum_{n=1}^{N} \left[ \begin{array}{ccc}
\Gamma(t_n) & \text{Re}[f_d(t_n)] & -\text{Im}[f_d(t_n)] \\
\text{Re}^T[f_d(t_n)] & \eta_d(t_n) & 0 \\
-\text{Im}^T[f_d(t_n)] & 0 & \eta_d(t_n)
\end{array} \right], \quad (C.62)
\]

where \( \Gamma(t_n) \) has been defined in (2.58), and we have defined

\[
f_d(t_n) = S^H(t_n)D^H(\theta)a(\theta_0)w_0(t_n) - \Delta^H(t_n)H_I^{-1}f_1(t_n) \quad (C.63)
\]

\[
f_d(t_n) = S^H(t_n)D^H(\theta)P_{A^\perp(\theta_f)}a(\theta_0)w_0(t_n), \quad (C.64)
\]

and

\[
\eta_d(t_n) = a^H(\theta_0)w_0(t_n)w_0(t_n)a(\theta_0) - f_1^H(t_n)H_I^{-1}f_1(t_n) \quad (C.65)
\]

\[
= |w_0(t_n)|^2 a^H(\theta_0)P_{A^\perp(\theta_f)}a(\theta_0), \quad (C.66)
\]

where \( P_{A^\perp(\theta_f)} \) has been defined in (A.17). Applying the partitioned matrix inversion lemma to the right side of (C.62) yields

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\text{Re} \left[ \sum_{n=1}^{N} f_d(t_n) \right] \left[ \sum_{n=1}^{N} f_d^H(t_n) \right]}{\sum_{n=1}^{N} \eta_d(t_n)} \right\}^{-1}. \quad (C.67)
\]
Since \( g \omega_0(t_n) = s_0(t_n) \), we have
\[
f_d(t_n)g = f(t_n),
\]
and
\[
\eta_d(t_n)|g|^2 = \eta(t_n),
\]
where \( f(t_n) \) and \( \eta(t_n) \) have been defined in (2.60) and (2.61), respectively. Using the above two equations, we note that (C.67) can be rewritten as
\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\text{Re} \left( \left[ \sum_{n=1}^{N} f(t_n) \right] \left[ \sum_{n=1}^{N} f^H(t_n) \right] \right)}{\sum_{n=1}^{N} \eta(t_n)} \right\}^{-1}.
\]

Now we have proven Equation (2.59).

*If \( g \) is unknown and real*, using the above results, we can find the Fisher's information matrix for this case. Let
\[
\vartheta_r = \begin{bmatrix} \theta^T & g \end{bmatrix}^T.
\]
Then for \( \left[ \sigma^2 \begin{bmatrix} \text{Re}^T[s_f(t_n)] & \text{Im}^T[s_f(t_n)] & \vartheta_r^T \end{bmatrix} \right]^T \), the Fisher's information matrix is a submatrix of the matrix at the right side of (C.49) consisting of all except the last row and column of the matrix at the right side of (C.49). Applying the partitioned matrix inversion lemma to the Fisher's information matrix to the submatrix yields
\[
\text{CRB}^{-1}(\vartheta_r) = \frac{2}{\sigma^2} \begin{bmatrix} \Gamma_1 & \text{Re}(f_2) \\ \text{Re}^T(f_2) & \eta_1 \end{bmatrix} - 
\begin{bmatrix} \text{Re}^T[\Delta(t_1)] & \text{Im}^T[\Delta(t_1)] & \cdots \\ \text{Re}^T[f_1(t_1)] & \text{Im}^T[f_1(t_1)] & \cdots \end{bmatrix}
\]
\[
\begin{bmatrix}
\Re^T[\Delta(t_N)] & \Im^T[\Delta(t_N)] \\
\Re^T[f_1(t_N)] & \Im^T[f_1(t_N)] \\
\end{bmatrix}
\begin{bmatrix}
\Re(H_I^{-1}) & -\Im(H_I^{-1}) & 0 \\
\Im(H_I^{-1}) & \Re(H_I^{-1}) & 0 \\
\vdots & & \ddots \\
0 & \Re(H_I^{-1}) & -\Im(H_I^{-1}) \\
0 & \Im(H_I^{-1}) & \Re(H_I^{-1}) \\
\end{bmatrix}
\begin{bmatrix}
\Re[\Delta(t_1)] & \Re[f_1(t_1)] \\
\Im[\Delta(t_1)] & \Im[f_1(t_1)] \\
\vdots & \vdots \\
\Re[\Delta(t_N)] & \Re[f_1(t_N)] \\
\Im[\Delta(t_N)] & \Im[f_1(t_N)] \\
\end{bmatrix}
\]

\[
= \frac{2}{\sigma^2} \left\{ \frac{1}{\sigma^2} \sum_{n=1}^{N} \left[ \begin{array}{c}
\Re^2(t_n) \\
\end{array} \right] \Re\left[ \Delta^H(t_n)H_I^{-1}\Delta(t_n) \right] \\
- \sum_{n=1}^{N} \left[ \begin{array}{c}
\Re^T\left[ \Delta^H(t_n)H_I^{-1}\Delta(t_n) \right] \\
\end{array} \right] \right\}
\]

\[
= \frac{2}{\sigma^2} \sum_{n=1}^{N} \left[ \begin{array}{c}
\Re[\Gamma(t_n)] \\
\Re^T[\Gamma(t_n)] \\
\end{array} \right] \left[ \begin{array}{c}
\Re[f_d(t_n)] \\
\eta_d(t_n) \\
\end{array} \right].
\]

Applying the partitioned matrix inversion lemma to the right side of (C.74) and using (C.68) and (C.69) yield

\[
\text{CRB}(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\Re\left[ \sum_{n=1}^{N} f_d(t_n) \right]}{\sum_{n=1}^{N} \eta_d(t_n)} \right\}^{-1}
\]

(C.75)
Now we have proven Equation (2.62).

Finally, if \( g = e^{i\nu} \) with \( \nu \) unknown and real, we calculate \( \delta \ln \mathcal{L} / \partial \nu \). We have

\[
\frac{\partial \ln \mathcal{L}}{\partial \nu} = -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ j a^H(\theta_0) g^* w_0^*(t_n) e(t_n) \right]
\]

\[= -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ j a^H(\theta_0) s_0^*(t_n) e(t_n) \right], \tag{C.77}
\]

where we have used \( g w_0(t_n) = s_0(t_n) \). We also have

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right] \right\} = 0, \tag{C.78}
\]

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}[s_f(t_n)]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right] \right\} = \frac{2}{\sigma^2} \text{Re} \left[ j A^H(\theta) a(\theta_0) s_0(t_n) \right]
\]

\[= -\frac{2}{\sigma^2} \text{Im} \left[ g f_1(t_n) \right], \tag{C.79}
\]

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}[s_f(t_n)]} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right] \right\} = \frac{2}{\sigma^2} \text{Im} \left[ j A^H(\theta) a(\theta_0) s_0(t_n) \right]
\]

\[= \frac{2}{\sigma^2} \text{Re} \left[ g f_1(t_n) \right], \tag{C.80}
\]

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right] \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ j S^H(t_n) D^H(\theta) a(\theta_0) s_0(t_n) \right]
\]

\[= -\frac{2}{\sigma^2} \text{Im} \left( g f_2 \right), \tag{C.81}
\]

and

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right]^2 \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ a^H(\theta_0) s_0^*(t_n) s_0(t_n) a(\theta_0) \right]
\]

\[= \frac{2}{\sigma^2} \text{Re} \left[ |g|^2(\eta_1) \right]
\]
where we have used \(|g|^2 = 1\). Let

\[ \vartheta_a = \begin{bmatrix} \theta^T & \nu \end{bmatrix}^T, \]  

(C.83)

Thus for \( \left[ \sigma^2 \quad \text{Re}^T [s_I(t_n)] \quad \text{Im}^T [s_I(t_n)] \quad \vartheta_a^T \right]^T \), the Fisher's information matrix is

\[
F = \frac{2}{\sigma^2} \begin{bmatrix}
\frac{LN}{2\sigma^2} & 0 & 0 & 0 \\
0 & \text{Re}(H_I) & -\text{Im}(H_I) & 0 \\
0 & \text{Im}(H_I) & \text{Re}(H_I) & 0 \\
0 & 0 & 0 & \cdots \\
\text{Re}^T [\Delta(t_1)] & \text{Im}^T [\Delta(t_1)] & \cdots & \text{Re}^T [\Delta(t_N)] \\
0 & -\text{Im}^T [g \Gamma_1(t_1)] & \text{Re}^T [g \Gamma_1(t_N)] & \cdots & -\text{Im}^T [g \Gamma_1(t_N)] \\
0 & 0 & \text{Re}[\Delta(t_1)] & -\text{Im}[g \Gamma_1(t_1)] \\
0 & \text{Im}[\Delta(t_1)] & \text{Re}[g \Gamma_1(t_1)] & \cdots & \vdots \\
-\text{Im}(H_I) & \text{Re}[\Delta(t_N)] & -\text{Im}[g \Gamma_1(t_N)] & \cdots & \vdots \\
\text{Re}(H_I) & \text{Im}[\Delta(t_N)] & \text{Re}[g \Gamma_1(t_N)] & \cdots & \vdots \\
\text{Im}^T [\Delta(t_N)] & \Gamma_1 & -\text{Im}(g \Gamma_f) & \cdots & \vdots \\
\text{Re}^T [g \Gamma_1(t_N)] & -\text{Im}^T (g \Gamma_f) & \eta_1 & \cdots & \vdots 
\end{bmatrix} \]  

(C.84)
Applying the partitioned matrix inversion lemma to the above Fisher's information matrix, we have

\[
\text{CRB}^{-1}(\theta_a) = \frac{2}{\sigma^2} \left\{ \begin{bmatrix}
\Gamma_1 & -\text{Im}(gf_2) \\
-\text{Im}^T(gf_2) & \eta_1
\end{bmatrix} - \begin{bmatrix}
\text{Re}^T[\Delta(t_1)] & \text{Im}^T[\Delta(t_1)] & \cdots \\
-\text{Im}^T[gf_1(t_1)] & \text{Re}^T[gf_1(t_1)] & \cdots
\end{bmatrix} \begin{bmatrix}
\text{Re}^T[\Delta(t_N)] & \text{Im}^T[\Delta(t_N)] \\
-\text{Im}^T[gf_1(t_N)] & \text{Re}^T[gf_1(t_N)]
\end{bmatrix}
\right\} 
\]

\[
\begin{bmatrix}
\text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & \cdots & 0 \\
\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \text{Re}(H_I^{-1}) & -\text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) \\
0 & \text{Im}(H_I^{-1}) & \text{Re}(H_I^{-1}) & \text{Re}(H_I^{-1})
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{Re}[\Delta(t_1)] & -\text{Im}[gf_1(t_1)] \\
\text{Im}[\Delta(t_1)] & \text{Re}[gf_1(t_1)] \\
\vdots & \vdots \\
\text{Re}[\Delta(t_N)] & -\text{Im}[gf_1(t_N)] \\
\text{Im}[\Delta(t_N)] & \text{Re}[gf_1(t_N)]
\end{bmatrix}
\] 

\[
= \frac{2}{\sigma^2} \left\{ \begin{bmatrix}
\Gamma_1 & -\text{Im}(gf_2) \\
-\text{Im}^T(gf_2) & \eta_1
\end{bmatrix} - \sum_{n=1}^{N} \begin{bmatrix}
\text{Re}[\Delta^H(t_n)H_I^{-1}\Delta(t_n)] \\
-\text{Im}^T[\Delta^H(t_n)H_I^{-1}gf_1(t_n)]
\end{bmatrix}
\right\}
\]

(C.85)
Applying the partitioned matrix inversion lemma to the right side of (C.88) yields
\[ CRB(\theta) = \frac{\sigma^2}{2} \left\{ \sum_{n=1}^{N} \Gamma(t_n) - \frac{\text{Im} \left[ \sum_{n=1}^{N} f(t_n) \right]}{\sum_{n=1}^{N} \eta(t_n)} \right\}^{-1}. \] (C.89)
APPENDIX D

MAXIMUM LIKELIHOOD ESTIMATION OF THE ARRIVAL DIRECTION OF A SINGLE SIGNAL WITH KNOWN WAVEFORM

In this appendix we illustrate the derivation of Equations (3.24)-(3.33) and (3.37)-(3.48) for the case of one incident signal with known waveform $w(t)$. The problem we consider here is a special case of the second problem considered in Appendix A when the interfering signals are absent. We shall show that for this simple case of one incident signal, closed form maximum likelihood (ML) solutions exist.

The ML estimates of $g$ (if $g$ is unknown) and $\theta$, where both $g$ and $\theta$ are scalars, are found by minimizing the quantity (see (3.17))

$$q = \frac{1}{N} \sum_{n=1}^{N} |x(t_n) - a(\theta)w(t_n)g|^H |x(t_n) - a(\theta)p(t_n)g|.$$  \hspace{1cm} (D.1)

Dropping all terms not involving $g$ and $\theta$, we get

$$q = \sum_{l=1}^{L} \left[ -g^* e^{j(l-1)\pi \sin \theta_k y_l} - ge^{-j(l-1)\pi \sin \theta_k y_l^*} \right] + |g|^2 P,$$  \hspace{1cm} (D.2)

where we have defined

$$y_l = \frac{1}{N} \sum_{n=1}^{N} x_l(t_n)w^*(t_n),$$  \hspace{1cm} (D.3)

and

$$P = \frac{1}{N} \sum_{n=1}^{N} |w(t_n)|^2.$$  \hspace{1cm} (D.4)

The value of $\theta$ that minimizes $q$ is found by setting the derivative of (D.2) with respect to $\theta$ to zero. Setting $\frac{dq}{d\theta}$ to zero yields

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\[ g \sum_{l=1}^{L} (l - 1)y_l^* z^{l-1} - g^* \sum_{l=1}^{L} (l - 1)y_l z^{-(l-1)} = 0, \]  
(D.5)

where

\[ z = e^{-j\pi \sin \theta}. \]

(D.6)

To obtain the appropriate solution to (D.2), we must take into account the assumed form of \( g \). For the simplest case, when \( g \) is known, we obtain the ML estimate of \( \theta \) by finding the zeros of (D.5) and then using (D.6).

When \( g \) is unknown and complex, setting \( \frac{dq}{d\text{Re}(g)} \) and \( \frac{dq}{d\text{Im}(g)} \) to zero and solving for the resulting \( g \) gives

\[ g = \frac{1}{LP} \sum_{l=1}^{L} y_l z^{-(l-1)}. \]  
(D.7)

Substituting (D.7) into (D.5) for \( g \) and canceling unnecessary terms yields

\[ \sum_{l_1=1}^{L} \sum_{l_2=1}^{L} (l_1 - l_2)y_{l_1}^* y_{l_2} z^{l_1 - l_2} = 0. \]  
(D.8)

In this case the ML estimates for \( \theta \) and \( g \) are found by solving for the zeros of (D.8) and then using (D.6) and (D.7) to get \( g \) and \( \theta \).

When \( g \) is unknown and real, the \( g \) in (D.5) can be canceled. Thus for this case we have

\[ \sum_{l=1}^{L} (l - 1)y_l^* z^{l-1} - \sum_{l=1}^{L} (l - 1)y_l z^{-(l-1)} = 0. \]  
(D.9)

Setting \( \frac{dq}{dg} \) to zero and solving for the resulting \( g \) gives

\[ g = \text{Re} \left\{ \frac{1}{LP} \sum_{l=1}^{L} y_l z^{-(l-1)} \right\}. \]  
(D.10)
The zeros of (D.9) along with (D.6) and (D.10) then give the ML estimates of \( g \) and \( \theta \).

Finally, when \( g = e^{i\nu} \) with \( \nu \) unknown and real, (D.5) can be rewritten as

\[
e^{j2\nu} \sum_{l=1}^{L} (l-1)y_{l}^{*}z^{l-1} - \sum_{l=1}^{L} (l-1)y_{l}z^{-(l-1)} = 0. \tag{D.11}
\]

Setting \( \frac{dq}{d\nu} \) to zero and solving for the resulting \( e^{j2\nu} \) gives

\[
e^{j2\nu} = \frac{\sum_{l=1}^{L} y_{l}z^{-(l-1)}}{\sum_{l=1}^{L} y_{l}^{*}z^{l-1}}. \tag{D.12}
\]

The resulting \( g \) is therefore

\[
g = \exp \left\{ j\arg \left[ \sum_{l=1}^{L} y_{l}z^{-(l-1)} \right] \right\}. \tag{D.13}
\]

Substituting (D.12) into (D.11) for \( e^{j2\nu} \) and canceling unnecessary terms yields the same equation as (D.8). From the zeros of (D.8), we obtain the ML estimates of \( g \) and \( \theta \) from (D.6) and (D.13).
APPENDIX E

DERIVATION OF THE CR-BOUNDS IN CHAPTER III

The derivation of the CR-bounds in Chapter III below is similar to the derivation in Appendix E of [35]. Many of the results we use here are given and proven in Appendix E of [35].

Using (3.16), we find the following log-likelihood function:

\[ \ln L = \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} [x(t_n) - \Lambda(\theta)W(t_n)g]^H [x(t_n) - \Lambda(\theta)W(t_n)g] \]

(E.1)

\[ = \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} e^H(t_n)e(t_n), \]

(E.2)

where we have defined

\[ e(t_n) = x(t_n) - \Lambda(\theta)W(t_n)g. \]

(E.3)

For \( g \) known, (E.1) may be written as

\[ \ln L = \text{const} - LN \ln \sigma^2 - \frac{1}{\sigma^2} \sum_{n=1}^{N} [x(t_n) - \Lambda(\theta)s(t_n)]^H [x(t_n) - \Lambda(\theta)s(t_n)]. \]

(E.4)

Using the results in Appendix E of [35], we find

\[ \frac{\partial \ln L}{\partial \sigma^2} = -\frac{LN}{\sigma^2} + \frac{1}{\sigma^4} \sum_{n=1}^{N} [e^H(t_n)e(t_n)], \]

(E.5)

and

\[ \frac{\partial \ln L}{\partial \theta} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)D^H(\theta)e(t_n) \right], \]

(E.6)
where $S^H(t_n)$ and $D^H(\theta)$ have been defined in (3.51) and (3.52), respectively. Also,

$$E\left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right]^2 \right\} = \frac{LN}{\sigma^4},$$  \hspace{1cm} (E.7)

$$E\left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = 0,$$  \hspace{1cm} (E.8)

and

$$E\left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \Gamma,$$  \hspace{1cm} (E.9)

where $\Gamma$ has been defined in (3.55). Thus the Fisher's information matrix for

$$\begin{bmatrix} \sigma^2 & \theta^T \end{bmatrix}^T$$

is

$$F = \frac{2}{\sigma^2} \begin{bmatrix} \frac{LN}{2\sigma^2} & 0 \\ 0 & \Gamma \end{bmatrix}.$$  \hspace{1cm} (E.10)

Then

$$\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \Gamma.$$  \hspace{1cm} (E.11)

We have now proven Equation (3.54).

For $g$ unknown and complex, we also need to first calculate $\partial \ln \mathcal{L}/\partial \text{Re}(g)$ and $\partial \ln \mathcal{L}/\partial \text{Im}(g)$. Using (E.1) and using the results in Appendix E of [35], we get

$$\frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} = \frac{1}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ W^H(t_n) A^H(\theta) e(t_n) \right],$$  \hspace{1cm} (E.12)

and

$$\frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} = \frac{1}{\sigma^2} \sum_{n=1}^{N} \text{Im} \left[ W^H(t_n) A^H(\theta) e(t_n) \right].$$  \hspace{1cm} (E.13)

Let
\[ \Delta_1 = \sum_{n=1}^{N} \text{Re} \left[ W^H(t_n) A_H(\theta) A(\theta) W(t_n) \right], \quad (E.14) \]

and

\[ \Delta_1 = \sum_{n=1}^{N} \text{Im} \left[ W^H(t_n) A_H(\theta) D(\theta) S(t_n) \right]. \quad (E.15) \]

Then

\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right]^T \right\} = 0, \quad (E.16) \]

\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ W^H(t_n) A_H(\theta) A(\theta) W(t_n) \right] \quad (E.17) \]

\[ = \frac{2}{\sigma^2} \text{Re}(\Delta_1), \quad (E.18) \]

\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right]^T \right\} = -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Im} \left[ W^H(t_n) A_H(\theta) A(\theta) W(t_n) \right] \quad (E.19) \]

\[ = -\frac{2}{\sigma^2} \text{Im}(\Delta_1), \quad (E.20) \]

\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Re}(g)} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ W^H(t_n) A_H(\theta) D(\theta) S(t_n) \right] \quad (E.21) \]

\[ = \frac{2}{\sigma^2} \text{Re}(\Delta_1), \quad (E.22) \]

\[ E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \text{Im}(g)} \right]^T \right\} = 0, \quad (E.23) \]
\[
E \left\{ \frac{\partial \ln L}{\partial \text{Im}(g)} \left[ \frac{\partial \ln L}{\partial \text{Im}(g)} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ W^H(t_n) A^H(\theta) \right. \\
A(\theta) W(t_n) \left. \right] = \frac{2}{\sigma^2} \text{Re}(\Delta_1),
\]
(E.24)

and
\[
E \left\{ \frac{\partial \ln L}{\partial \text{Im}(g)} \left[ \frac{\partial \ln L}{\partial \theta} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Im} \left[ W^H(t_n) A^H(\theta) \right. \\
D(\theta) S(t_n) \left. \right] = \frac{2}{\sigma^2} \text{Im}(\Delta_1).
\]
(E.25)

Thus the Fisher's information matrix for \[ \left[ \sigma^2 \quad \text{Re}(g)^T \quad \text{Im}(g)^T \quad \theta^T \right]^T \] is
\[
F = \frac{2}{\sigma^2} \begin{bmatrix}
\frac{LN}{2\sigma^2} & 0 \\
\text{Re}(\Delta_1) & -\text{Im}(\Delta_1) & \text{Re}(\Delta_1) \\
\text{Im}(\Delta_1) & \text{Re}(\Delta_1) & \text{Im}(\Delta_1) \\
0 & \text{Re}(\Delta_1)^T & \text{Im}(\Delta_1)^T & \Gamma
\end{bmatrix},
\]
(E.28)

where we have used \(-\text{Im}(\Delta_1) = \text{Im}(\Delta_1)\) since \(\Delta_1\) is Hermitian and hence its imaginary part must be skew-symmetric. Using the standard matrix inversion lemma, we have
\[
\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \left\{ \Gamma - \left[ \text{Re}(\Delta_1^{-1}) \quad \text{Im}(\Delta_1^{-1}) \right] \right. \\
\left. \left[ \text{Re}(\Delta_1) \quad -\text{Im}(\Delta_1) \right] \right\} \left[ \text{Re}(\Delta_1) \quad \text{Im}(\Delta_1) \right] \left[ \text{Re}(\Delta_1) \quad -\text{Im}(\Delta_1) \right] \right\}
\]
(E.29)

where we have used
\[
\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \left\{ \Gamma - \text{Re} \left[ \Delta_1^H \Delta_1^{-1} \Delta_1 \right] \right\}.
\]
(E.30)
\[
\begin{bmatrix}
\text{Re}(\Lambda_1) & -\text{Im}(\Lambda_1) \\
\text{Im}(\Lambda_1) & \text{Re}(\Lambda_1)
\end{bmatrix}^{-1} = \begin{bmatrix}
\text{Re}(\Lambda_1^{-1}) & -\text{Im}(\Lambda_1^{-1}) \\
\text{Im}(\Lambda_1^{-1}) & \text{Re}(\Lambda_1^{-1})
\end{bmatrix}.
\]

(E.31)

Letting
\[G = \text{diag} \{ g_1, g_2, \ldots, g_K \},\]
we have
\[S(t_n) = W(t_n)G, \quad n = 1, 2, \ldots, N.\]

(E.32)

Using the above relationship, we have
\[\Lambda_1 = (G^{-1})^H \Lambda G^{-1},\]
and
\[\Delta_1 = (G^{-1})^H \Delta,\]
where \(\Lambda\) and \(\Delta\) have been defined in (3.58) and (3.57), respectively. Substituting (E.34) and (E.35) into (E.30), we get
\[\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \left\{ \Gamma - \text{Re} \left[ \Delta^H \Lambda^{-1} \Delta \right] \right\}.\]

(E.36)

We have now proven Equation (3.56).

For \(g\) unknown and real, using the above results, we can show that the Fisher’s information matrix for \(\begin{bmatrix} \sigma^2 & g^T & \theta^T \end{bmatrix}^T\) is
\[
F = \frac{2}{\sigma^2} \begin{bmatrix}
\frac{LN}{2\sigma^2} & 0 & \text{Re}(\Lambda_1) \\
0 & \text{Re}(\Lambda_1) & \text{Re}(\Delta_1) \\
0 & \text{Re}^T(\Delta_1) & \Gamma
\end{bmatrix}.
\]

(E.37)
Using the standard matrix inversion lemma, we have

\[
\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \{ \Gamma - \text{Re}^T(\Delta_1)\text{Re}^{-1}(\Delta_1)\text{Re}(\Delta_1) \} = \frac{2}{\sigma^2} \{ \Gamma - \text{Re}^T(\Delta)\text{Re}^{-1}(\Delta)\text{Re}(\Delta) \},
\]

where we have used

\[
\text{Re}(A_1) = G^{-1}\text{Re}(A)G,
\]

and

\[
\text{Re}(\Delta_1) = G^{-1}\text{Re}(\Delta).
\]

We have now proven Equation (3.59).

For \( g_k = e^{j\nu_k}, k = 1, 2, \ldots, K \), with \( \nu_k \) unknown and real, we need to first calculate \( \partial \ln \mathcal{L} / \partial \nu \). Using (E.1) and using the results in Appendix E of [35], we get

\[
\frac{\partial \ln \mathcal{L}}{\partial \nu} = -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ jG^H \mathcal{W}^H(t_n)A^H(\theta)e(t_n) \right]
\]

\[
= -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ jS^H(t_n)A^H(\theta)e(t_n) \right].
\]

Then using the results in Appendix E of [35] yields

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \sigma^2} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right]^T \right\} = 0,
\]

\[
E \left\{ \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right] \left[ \frac{\partial \ln \mathcal{L}}{\partial \nu} \right]^T \right\} = \frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ S^H(t_n)A^H(\theta)A(\theta)S(t_n) \right]
\]

\[
= \frac{2}{\sigma^2} \text{Re}(\Delta),
\]
and
\[
E \left\{ \left[ \frac{\partial \ln L}{\partial \nu} \right] \left[ \frac{\partial \ln L}{\partial \theta} \right]^T \right\} = -\frac{2}{\sigma^2} \sum_{n=1}^{N} \text{Re} \left[ jS^H(t_n)A^H(\theta)D(\theta)S(t_n) \right]
\]
\[
= \frac{2}{\sigma^2} \text{Im}(\Delta).
\]
(E.48)

Thus the Fisher's information matrix for \([\sigma^2 \ \nu^T \ \theta^T]^T\) is
\[
F = \frac{2}{\sigma^2} \begin{bmatrix}
\frac{LN}{2\sigma^2} & 0 \\
0 & \text{Re}(\Delta) & \text{Im}(\Delta) \\
0 & \text{Im}^T(\Delta) & \Gamma
\end{bmatrix}
\]
(E.49)

Using the standard matrix inversion lemma, we have
\[
\text{CRB}^{-1}(\theta) = \frac{2}{\sigma^2} \left\{ \Gamma - \text{Im}^T(\Delta)\text{Re}^{-1}(\Delta)\text{Im}(\Delta) \right\}.
\]
(E.50)

We have now proven Equation (3.60).
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