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Finite element method evaluation of thermomechanical responses of fluid-saturated porous media under finite deformation

Kim, Chun-Sam, Ph.D.
The Ohio State University, 1991

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FINITE ELEMENT METHOD EVALUATION OF THERMOMECHANICAL
RESPONSES OF FLUID-SATURATED POROUS MEDIA
UNDER FINITE DEFORMATION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Chun-Sam Kim, B.S., M.S.

* * * * *

The Ohio State University
1991

Dissertation Committee :

S. H. Advani
J. K. Lee
D. A. Mendelsohn

Approved by

Co-Adviser, Chairman and Professor
Department of Engineering Mechanics

Co-Adviser, Professor
Department of Engineering Mechanics
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VITA

February 20, 1953 ........ Born in Inchon, Korea

1976 ..................... B.S., Seoul National Univ., Seoul, Korea

1976-1984 ............... Senior Researcher
Agency for Defence Development,
Daejon, Korea

1986 ..................... M.S., Engineering Mechanics,
Univ. of Missouri-Rolla,
Rolla, Missouri

1986-1991 ............... Graduate Research Associate
Engineering Mechanics,
The Ohio State Univ.,
Columbus, Ohio, U.S.A.

PUBLICATIONS

"HYFIDE : Finite Element Analysis of Hygrothermomechanical
Finite Deformation Responses", Department of Engineering
Mechanics Report, The Ohio State University, Jan. 1991,
with S.H. Advani, J.K. Lee and T.S. Lee.

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15th Southeastern Conference on Theoretical and Applied
Mechanics, Georgia Institute of Technology, Vol. 15, pp

"Stress Calculation by Integral Postprocessing of the
Finite Element Solution", M.S. Thesis, Department of
Engineering Mechanics, Univ. of Missouri-Rolla, 1986.
FIELDS OF STUDY

Major Field: Engineering Mechanics

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NOMENCLATURE

Subscript

s  solid constituent
f  fluid constituent

Overhead Symbol

.  total time rate
^  prescribed value
-  apparent value in the mixture

Roman Symbol

$[B_h]$  gradient matrix of the shape function $[N_h]$

$[B_h]$  nonlinear part of strain-displacement operator

$[B_u]$  linear part of strain-displacement operator

c  elasticity tensor

c_{ep}  elasto-viscoplastic tensor

c_v  coefficient of consolidation

c, c_s, c_f  heat capacities
\(D, \ D_s, \ D_f\) rate-of-deformation tensors

\(\varepsilon, \ \varepsilon_s, \ \varepsilon_f\) elastic, hygrothermal, viscoplastic strain rates, respectively

\(d\) symmetric part of velocity gradient tensor at time \(t\)

\(\text{det}\) determinant

\(\text{Div}\) divergence in the referential configuration

\(\text{div}, \nabla\) divergence in the current configuration

\(E\) Lagrangian strain tensor

\(\delta E\) virtual Lagrangian strain tensor

\(E\) Young's modulus

\(E_T\) plastic hardening modulus

\(\Delta e = \frac{1}{2} \left[ \frac{\partial \Delta u}{\partial x} + \left( \frac{\partial \Delta u}{\partial x} \right)^T \right]\)

\(\Delta e_e, \ \Delta e_h, \ \Delta e_{vp}\) elastic, hygro-thermal, viscoplastic strain increments, respectively

\(F, \ F_s, \ F_f\) deformation gradient tensors

\(f, \ f_f, \ f_s\) body forces

\(G\) shear modulus

\(g\) gradient of temperature \(\Theta\)

\(\text{Grad}\) gradient in the referential configuration

\(\text{grad}\) gradient in the current configuration

\(H\) hardening parameter

\(h_c\) convective heat transfer coefficient

\(I\) identity tensor
J, J_s, J_f: determinant of F, F_s and F_f, respectively

[K_t]: tangential stiffness matrix

K, K_f, K_s: bulk modulus

k, k_f, k_s: heat conduction coefficients

L, L_f, L_s: velocity gradient tensors

[N_h], [N_u]: shape functions defined in eqns. (4.64) and (C.7), respectively

n: non-linear part of displacement gradient tensor

^\n: outward unit normal vector

p: applied pressure force

q: heat flux vector

r, r_s, r_f: specific energy supply rates

(r, \theta, z): spatial coordinates in cylindrical system

S: second Piola-Kirchhoff stress

T, T_f, T_s, \sigma: Cauchy stresses

T'_f, T'_s: deviatoric part of T_f and T_s, respectively

\hat{T}, \hat{\sigma}: Jaumann stress rates

O: T_s: objective stress rate

*: Truesdell stress rate

t, t_f, t_s: traction forces

t: time
\( \text{tr}(D) \) \hspace{1cm} \text{trace of tensor } D

\( u, u_o \) \hspace{1cm} \text{displacement and initial displacement vectors, respectively}

\( \Delta u \) \hspace{1cm} \text{incremental displacements}

\( \delta u \) \hspace{1cm} \text{virtual displacement vector}

\( v, v_s, v_f \) \hspace{1cm} \text{velocities}

\( v_r \) \hspace{1cm} \text{relative velocity}

w \hspace{1cm} \text{spin tensor}

w \hspace{1cm} \text{skew symmetric part of displacement gradient tensor}

\( X, X_f, X_g \) \hspace{1cm} \text{material positions in the referential configuration}

\( x, x_f, x_s \) \hspace{1cm} \text{spatial positions in the current configuration}

\( (x, y, z) \) \hspace{1cm} \text{spatial coordinates in rectangular system}

\( Y_0 \) \hspace{1cm} \text{yield stress at referential temperature } \theta_o

\( Y \) \hspace{1cm} = Y_0 + H \epsilon_{vp}

\textbf{Greek Symbols}

\( \alpha \) \hspace{1cm} \text{time integration parameter}

\( \beta, \beta_s, \beta_f \) \hspace{1cm} \text{thermal expansion coefficients}

\( \gamma \) \hspace{1cm} \text{viscosity constant of viscoplastic material}

\( \gamma_f, \gamma_p \) \hspace{1cm} \text{dissipation energies due to viscous fluid flow and plastic deformation, respectively}
\( \epsilon \) symmetric part of displacement gradient tensor

\( \epsilon_{vp}, \bar{\epsilon}_{vp} \) viscoplastic strain and effective viscoplastic strain, respectively

\( \xi \) specific internal energy

\( \eta, \eta_f, \eta_S \) specific entropies

\( \theta, \theta_c, \theta_o \) temperature, convective equilibrium temperature and initial temperature, respectively

\( \kappa \) specific permeability

\( \lambda, \mu, \nu \) Lame constants

\( \mu \) viscosity of fluid

\( \nu \) Poisson's ratio

\( \xi \) internal state vector

\( \Omega \) domain

\( \partial \Omega \) boundary

\( \partial \Omega_u, \partial \Omega_t, \partial \Omega_\pi \) displacement, temperature, and pressure prescribed boundaries, respectively

\( \partial \Omega_c, \partial \Omega_h, \partial \Omega_i \) traction, mass flux, heat flux boundaries respectively

\( \omega \) antisymmetric part of displacement gradient tensor

\( \omega \) dummy constant

\( \pi, \pi_o \) pore pressure and initial pore pressure, respectively

\( \rho, \rho_f, \rho_s \) true densities

\( \bar{\rho}, \bar{\rho}_f, \bar{\rho}_s \) apparent mass densities

xvi
\( \rho_{f0}, \rho_{so} \) reference mass densities of the solid and fluid material, respectively

\( \sigma, T \) Cauchy stresses

\( \sigma, F \) effective stresses

\( \hat{\sigma}, \hat{T} \) Jaumann stress rates

\( \phi \) porosity

\( \psi, \psi_f, \psi_s \) specific Helmholtz free energies

\( \nabla, \text{div} \) divergences in the referential configuration
CHAPTER I
INTRODUCTION

1.1. General Overview

The pivotal role of thermomechanically coupled behavior of fluid saturated porous medium under applied load has, for a long time, been of considerable concern in geomechanical problems. Various applications ranging from nuclear waste isolation, hydraulic fracturing, geothermal energy enhancement, damage induced by swelling of clays, and ground water hydrology have been highlighted in a recent report by the U.S. National Committee for Rock Mechanics [4]. Also applications related to basic material and structural responses in ceramics, composites and polymers have been investigated using the "hygrothermomechanical" theory as a framework.

Saturated porous media are generally considered to be two phase materials consisting of a porous solid skeleton and a liquid phase filling the pore of the skeleton. Due to their two-constituent nature, the stress tensor acting on the solid skeleton is the total stress tensor minus the
"pore pressure". The stresses carried by the solid skeleton are termed "effective stress" in Terzaghi's work [105]. The behavior of saturated porous media due to the dissipation of pore fluid pressure must be analyzed by incorporating the effect of fluid flow as well as temperature change. For this reason, the theory of mixtures developed by Truesdell and Toupin [106] is often used in investigating the thermo-mechanics of porous media. In this theory, all quantities are considered to be macroscopic and the mixture is regarded as a continuum wherein the same spatial point is assumed to be occupied by both the solid and fluid for all time.

Since many geological materials exhibit time-dependent behavior under mechanical and thermal loading, the temperature dependent elasto-viscoplastic model proposed by Perzyna [91, 92, 93] is utilized in this study. Perzyna's model is a modification of the classical plasticity theory, incorporating viscous properties after the plastic state has been reached. A Von Mises yield criterion used in classical plasticity is adopted here to determine the stress level at which viscoplastic flow occurs.

The governing differential equations obtained from balance laws, in general, contain material and geometrical nonlinearities. The material nonlinearity results from
the temperature dependent properties as well as the viscoplastic constitutive model. On the other hand, the geometrical nonlinearity results from finite deformation. Also, the deformation of the porous medium is coupled with the fluid flow and heat transfer phenomena simultaneously. Closed form solutions to these coupled nonlinear differential equation are virtually impossible to obtain and the finite element method is employed in this study to obtain numerical solutions. The updated Lagrangian method, in which all static and kinematic variables are referred to the configuration at the previous time, is implemented in the formulation for the solid matrix deformation. The Jaumann stress rate is expressed in terms of the rate of deformation tensor for the invariants of the stress tensor under superposed rigid body motion. The weak formulations for the fluid and thermal diffusion equations are simultaneously developed. An implicit time integration scheme and an iterative strategy corresponding to the full Newton-Raphson method are combined to yield solutions for the coupled nonlinear equations.

1.2 Literature Review

The bibliography pertinent to fluid infiltrated and/or thermally induced responses of geological media is very diverse since it reflects a cross-fertilization of
several disciplines. Following Terzaghi's pioneering work [105] on one-dimensional consolidation, Biot [16, 17, 18] has presented theories characterizing three-dimensional consolidation, viscoelastic anisotropic behavior, and thermoelasticity. Rice and Cleary [96] have reformulated Biot's form of constitutive equations relating strain and fluid mass content to stress and pore pressure in terms of revised material properties. Closed form solutions to problems with simple geometric and boundary conditions have been presented by Mandel [76], Gibson and McNamee [53], McNamee and Gibson [78, 79], and Gibson, Schiffman and Pu [55]. Moisture movement in porous materials subjected to temperature gradients has been studied by Phillip and DeVries [94] and by many others [3, 108]. In this context, a discussion of both the Dufour and Soret effects is presented by Bird et al. [20]. Nur and Byerlee [87] have developed an effective stress law for elastic deformation of geological materials with fluids. In this law, the stress acting on the solid skeleton is considered to be the current stress minus the hydrostatic stress, to account for the effect of pore saturating fluid pressure. A thermodynamic viewpoint of Darcy's law has been presented by Mokadam [83]. Drained and undrained responses of repository rocks subjected to thermal loading have been evaluated by Palciauskas and Domenico [90]. An
extension of these efforts with temperature effects as well as applications have been reported by Booker and Savvidou [24, 25, 99] and McTigue [80]. Solutions for the problem of a half space with heat transfer, fluid pressure, and phase change have been obtained by Delaney [46]. Frictional heating and pore fluid pressure changes due to fault motions have been investigated by Lachenburch [68] and Mase and Smith [77].

The first rational basis of mixture theories was presented by Truesdell and Toupin [106]. Since then, a considerable amount of research on the coupled process modeling of porous media, using the theory of mixtures, has been conducted. Schiffman [100], using the dynamical theory of interacting continua developed by Green and Naghdi [57], presented a simplified theory for thermoelastic consolidation. Sophisticated forms of constitutive theories are due to Moreland [84], Crochet and Naghdi [43], and Katsube [66]. Various continuum theories of mixtures with applications have been presented by Atkin and Craine [8], Bowen [26], and Bedford and Drumheller [14].

Geometric nonlinearities due to finite deformation were first handled by means of an incremental geometric stiffness and presented in references [5, 52, 64, 109]. A Lagrangian formulation for the finite strain problems was
proposed by Hibbit et al. [60] and Bathe and Wilson [12]. On the other hand, Eulerian formulations, based on spatial coordinates, have been investigated by Yaghmai and Popov [113] and Banerjee and Fathallah [10]. An updated Lagrangian formulation was adopted by Yamada and Wifi [114]. Gibson, England and Hussey [54] developed a one-dimensional consolidation theory incorporating finite strain with a convected reference frame for the solid. Smiles and Poulos [103] have examined one-dimensional consolidation without restricting the magnitude of strain. Carter et al. [28, 29] and Sharabi [101] have evaluated the finite deformations of elastic and elastoplastic consolidation by applying the finite element method. Other contributions to finite deformation of geomechanical problems have been made by several authors [44, 47, 51]. More recently, Meijer [81] compared the infinitesimal deformation theory with finite deformation theory by employing the finite element method. Kiousis et al. [67] and Chaudhary et al. [33] presented incremental Lagrangian formulations of the finite element method.

The constitutive model development for the time-dependent plastic behavior of materials has been based on four different theories. The first type of constitutive model is the rate-dependent modification of the classical plasticity theory. Based on this approach,
Perzyna [91, 92, 93] extended the time-dependent plastic behavior of a material to an elasto-viscoplastic model. The second type of model is a phenomenological theory based on micromechanics. For example, Bodner and Partom [21, 22] proposed a constitutive equation of 'dislocation dynamics' in their multi-dimensional formulation of elastic-viscoplastic constitutive equations with no specified yield criterion. The third type of model incorporates nonlinear viscoelasticity theory, as postulated by Cernocky and Krempl [30, 31, 32]. An endochronic theory model [110], not requiring a yield function, has also been advanced. Perzyna's model has been widely used for engineering applications in conjunction with the finite element method [49, 116-119] because its implementation is relatively simple. Cormeau [41] investigated the numerical stability of a simple time marching scheme for elasto-viscoplastic analysis. Recently, the contributions by Katona and Mulert [65], Baladi and Rohani [9], Oka et al. [88], and Desai and Zhang [48] are particularly noteworthy due to the application of Perzyna's model to geological media.

Thermodynamical aspects associated with a viscoplastic continuum have been examined by Coleman and Noll [38] and subsequently extended, using internal state variables, by Coleman [39], and Coleman and Gurtin [40].
Finite element models for linear-elastic plane-strain consolidation based on Biot's theory were developed by Sandhu and Wilson [97], Christian and Boehmer [35], Hwang et al. [62], and Yokoo et al. [115]. Investigations on coupled poroelastic and hygrothermomechanical responses of geological media utilizing finite element techniques have been recently summarized by Lewis and Schrefler [75]. Sih et al. [102] have presented various aspects of hygrothermoelasticity with finite element applications. Weres [111] considered the concentration of a solute, in lieu of pore pressure, in his finite element solution of the thermomechanical mixture problem. Finite element techniques for temperature mediated phase change problems have been studied by Bermudez and Durany [15]. Witherspoon et al. [112] and Noorishad et al. [86] applied the variational principle to the poroelastic component of the hygrothermoelastic equations and used the Galerkin formulation for the energy equation. Dawson and Chavez [45] have examined the thermomechanical response of saturated cohesive porous media for application to radioactive waste disposal. A coupled non-isothermal elastoplastic consolidation model has been studied by Lewis et al. [74]. Lewis and his colleagues, in a series of papers [71, 72, 73], have investigated drying-induced stresses in porous viscoelastic and elastoviscoplastic
bodies. Cividini and Rossi [36] approached the consolidation problem by using static descriptions of both pore pressure and total stress fields. Related investigations using the updated Lagrangian method [56] have also been recently reported. Aboustit et al.[1] and Lee et al.[70] are noteworthy for investigating the thermomechanical response of underground coal gasification processes using the finite element method.

Reported research on elevated temperature property determination of geomaterials is rather limited. Misra and Murrell [82] have measured the effects of temperature and stress on the creep of rocks. Extensive temperature characterization of coal and shale has been conducted by Shoemaker et al.[104] and Advani and Gmeindl [2]. Thermophysical properties of granite rocks have been presented by Hueze [59]. Dubow and Rajeshwar [50] and Closmann and Bradley [37] have determined temperature dependent properties of oil shales. Thermoviscoelastic properties of salt, for nuclear management applications, have been presented by Nipp [85]. Thermomechanical properties of other geological media have also been reported [11, 58, 69].

More recently, Biot [19] has restated the thermodynamics of porous saturated elastic solids based on Onsager's principle. Prevost [95] summarized the general
mixture equations to describe the flow of water through saturated anelastic porous media. Bowen [27] presented rigorous continuum mechanics descriptions of mixture theory for the deformation of a fluid saturated porous medium. Coussy [42] has proposed a general theory of thermodynamics of porous saturated solids subjected to finite deformation. A variational principle for the initial boundary value problem of quasi-static thermoelastic consolidation was developed by Huang et al. [61]. Additional investigations on the coupled thermomechanical behavior of fluid-infiltrated porous media subjected to thermal and mechanical loading have been conducted by Tseng [107] and Chen [34].

1.3 Research Objectives and Scope

The main purpose of this investigation is to examine the coupled thermomechanical behavior of fluid-infiltrated porous media with particular emphasis on non-linear phenomena. The previous work on small deformation hygrothermo-elastic/viscoelastic consolidation evaluations [1, 34, 61, 70, 107] and the work by Bowen [27] and Coussy [42] are extended. The governing field equations and finite element formulations, with the temperature dependent mechanical properties and finite deformation of the viscoplastic solid matrix, are derived
within a framework consistent with the continuum theory of mixtures and realistic computational rigor. In view of the lack of general mixture constitutive models for a thermo-viscoplastic porous medium undergoing finite deformation, simplified constitutive relations for the solid skeleton are employed here. Numerical procedures for solution of the governing equations are detailed.

In chapter II, the coupled equations governing finite deformation of saturated porous media are derived from the balance laws based on the mixture theory. The derived governing equations are represented by a set of second order nonlinear partial differential equations and expressed in terms of pertinent unknown quantities, namely; fluid pressure, solid skeleton deformation, and mixture temperature.

In chapter III, several elasto-viscoplastic models to describe the time-dependent plastic behavior of the porous material are discussed. An incremental form of Perzyna's elasto-viscoplastic constitutive relation with temperature dependent material behavior is adopted.

A two-dimensional finite element formulation, incorporating a Lagrangian description with an implicit time integration and the Newton-Raphson method, is derived in chapter IV. The updated Lagrangian formulation for the balance of force equation is applied to the solid skeleton
and the weak formulation is utilized for the fluid mass conservation and energy balance equations.

The developed computer program is validated for one-dimensional consolidation problems in chapter V. Also, the numerical results using the developed finite element model are compared with reported benchmark cases. Various field applications associated with the hygrothermomechnical problems are detailed in chapter VI. Finally, a summary of the contributions and research conclusions are presented in chapter VII.
CHAPTER II
GOVERNING EQUATIONS

A set of field equations governing the time-dependent response of fluid infiltrated porous media is derived in this chapter from balance laws based on the mixture theory. Darcy's law is assumed for the fluid flow through the porous media. The theory of mixtures, having two constituents, with each constituent occupying the same spatial point in the current configuration, was presented in a monograph by Truesdell and Toupin [106]. Green and Naghdi [57] extended this theory to the dynamical mixture theory in which a mixture is considered to have two interacting constituents, a solid and liquid phase. The field equations governing the finite deformation of saturated porous media subjected to prescribed boundary conditions and initial conditions are summarized.

2.1 Kinematics of Motion

The material body considered in this study is assumed to be a mixture of two continua, a single phase fluid and
a porous solid skeleton, which are generally in motion relative to each other. The motion of each continuum is referred to a coordinate system in a fixed but otherwise arbitrary reference configuration [95]. We assume $X_s$ and $X_f$ as the material positions of solid and fluid particles, respectively, in the reference configuration. The spatial positions, $x_s$ and $x_f$, of the solid and fluid particles, respectively, at time $t$ are expressed in the usual form by

$$x_a = \chi_a(X_a, t)$$

(2.1)

where

$$a = \begin{cases} s & \text{for the solid constituent} \\ f & \text{for the fluid constituent.} \end{cases}$$

A mixture particle under consideration occupies the same spatial point at time $t$, so that

$$x_s = x_f = x.$$ (2.2)

These two particles, solid and fluid, at time $t$ may have occupied different material points in the reference configuration.

The velocities of material particles $X_a$ at the point $x$ at time $t$ are defined by

$$v_a = \frac{d\chi_a(X_a, t)}{dt}$$

(2.3)

The deformation gradients and their determinants at time $t$ are defined by

$$F_a = \frac{\partial X_a}{\partial X_a}, \quad J_a = \det(F_a) > 0$$

(2.4)

The velocity gradients are defined by
The rate of deformation tensor is defined by
\[ \dot{\mathbf{D}}_a = \frac{1}{2}(\dot{\mathbf{L}}_a + \dot{\mathbf{L}}_a^T) \] (2.6)
where the superscript T denotes the transpose.

We let \( \tilde{\rho}_S \) and \( \tilde{\rho}_f \) represent the mass of the solid and fluid per unit volume of the mixture, respectively. The density of the mixture is, therefore, expressed by
\[ \rho(x,t) = \tilde{\rho}_S(x,t) + \tilde{\rho}_f(x,t). \] (2.7)

The apparent mass densities, \( \tilde{\rho}_S \) and \( \tilde{\rho}_f \), are sometimes called the "bulk" densities [26] and can be expressed, in terms of the porosity \( \phi(x,t) \), by
\[ \tilde{\rho}_S(x,t) = (1-\phi(x,t)) \rho_S(x,t) \] (2.8)
and
\[ \tilde{\rho}_f(x,t) = \phi(x,t) \rho_f(x,t) \] (2.9)
where \( \rho_f(x,t) \) and \( \rho_S(x,t) \) represent the "true" densities of the fluid and solid constituents, respectively.

In many applications of mixture theories, it is customary to formulate the field equations in terms of the solid material coordinates, \( X_S \). A convected coordinate system, attached to the deforming solid, is considered in this study for describing the pertinent motion. Since the fluid particles have relative motions with respect to the solid, the current material volume includes only the same set of solid particles in the reference material volume. Also, because the convected system is attached to the
solid skeleton, the motion of the fluid particle at position $x$ has a velocity relative to the motion of the solid particle at the same position and is given by

$$v_r = v_f - v_s.$$  \hspace{1cm} (2.10)

If $\psi$ denotes a function of $x$ and $t$, the material time derivatives can be expressed in terms of different time rate descriptions by the relations

$$\frac{d\psi}{dt}(x,t) = \frac{d\psi}{dt}(x_f(x_s,t),t) = \frac{\partial \psi}{\partial t} + v_s \cdot \text{grad}(\psi)$$  \hspace{1cm} (2.11)

and

$$\frac{d\psi}{dt}(x,t) = \frac{d\psi}{dt}(x_f(x_f,t),t) = \frac{\partial \psi}{\partial t} + v_f \cdot \text{grad}(\psi).$$  \hspace{1cm} (2.12)

It follows from eqns. (2.10) through (2.12), with the definition of $\dot{\psi} = \frac{d\psi}{dt}(x_s(x_s,t),t)$ and $\dot{\psi} = \frac{d\psi}{dt}(x_f(x_f,t),t)$, that

$$\dot{\psi} = \dot{\psi} + v_r \cdot \text{grad}(\psi)$$  \hspace{1cm} (2.13)

where the superposed dot represents the total time rate for a coordinate system attached to the solid particles. The dot product term in eqn. (2.13) represents the convective change due to the relative motion between the fluid and solid particles.

2.2 **Balance Laws**

In the following derivation, $\Omega$ denotes a domain of the mixture body at time $t$, occupying a volume $V$ bounded by a surface $\partial \Omega$ of area $A$. A material volume $V$ in the
current state is referred to the referential material volume \( V_0 \), based on solid particle motion at \( t=0 \). The field equations, presented below, for the transient responses of the fluid-saturated porous materials under finite deformation are derived on the basis of mixture theory in the absence of chemical reactions. All field quantities are functions of \( x \) and \( t \).

2.2.1 Balance of Mass

The balance laws governing mass conservation for the solid and fluid constituents, with an arbitrary control volume \( V \) in the current state, have the form

\[
\frac{d}{dt} \int_{\Omega} \tilde{\rho}_a(x,t) \, dV = \frac{d}{dt} \int_{\Omega} \tilde{\rho}_a(x(X_a,t),t) \, dV = 0 \tag{2.14}
\]

Applying the divergence theorem and the localization theorem to eqn. (2.14), local forms of the mass balance equations are obtained in the form (APPENDIX A.1)

\[
\dot{\tilde{\rho}}_s + \tilde{\rho}_s \text{div}(\tilde{v}_s) = 0 \tag{2.15}
\]

and

\[
\dot{\tilde{\rho}}_f + \tilde{\rho}_f \text{div}(\tilde{v}_f) + \tilde{v}_r \cdot \text{grad}(\tilde{\rho}_f) = 0. \tag{2.16}
\]

The material form of eqn. (2.15) is

\[
(1-\phi)J_s \rho_s = (1-\phi_R) \rho_{SR} \tag{2.17}
\]

where \( \phi_R \) and \( \rho_{SR} \) denote the porosity and mass density in the reference configuration, respectively. It is noteworthy that the porosity changes for large deformation
problems even when the solid constituent is incompressible.

2.2.2 Balance of Momentum

Before formulating the balance of momentum law, it is necessary to define the total traction vector, $t$, acting on a part of the boundary $\partial \Omega$ in the form

$$ t = t_s + t_f \quad (2.18) $$

where $t_s$ and $t_f$ are the traction vectors acting on the solid and fluid portions, respectively, of the boundary. These tractions are defined by

$$ t_s = (1-\phi)t \quad \text{and} \quad t_f = \phi t. \quad (2.19) $$

The linear momentum balance equations for the solid and fluid constituents, in the absence of linear momentum exchange between the constituents, can be written as

$$ \frac{\partial}{\partial t} \int_{\Omega} \rho_a(x(x, t))^2 v_a(x(x, t), t) \, dv = \int_{\Omega} \bar{f}_a \, dv + \int_{\partial \Omega} t_a \, dA \quad (2.20) $$

where $\bar{f}_a$, the apparent body force vector on the solid ($a=s$) and fluid ($a=f$) constituents, is defined by

$$ \bar{f}_s = (1-\phi)f_s \quad \text{and} \quad \bar{f}_f = \phi f_f \quad (2.21) $$

with $f_s$ and $f_f$ denoting the appropriate body force vectors. The total body force vector, $f$, of the mixture can be expressed by

$$ f = \bar{f}_s + \bar{f}_f. \quad (2.22) $$

The traction vectors, $t_s$, $t_f$ and $t$, can be related to the
Cauchy stresses, $T_s$, $T_f$, $T$, and the unit normal vector, $\hat{n}$, by the relations

$$t_a = T_a \cdot \hat{n}, \quad (2.23)$$

and

$$t = T \cdot \hat{n} \quad (2.24)$$

where the total stress is

$$T = T_s + T_f. \quad (2.25)$$

The stress tensors $T_s$ and $T_f$, in general, are functions of the particle motion, deformation history, and temperature depending on the selected constitutive model.

Substituting eqn. (2.23) into eqn. (2.20) and applying the divergence theorem, the following localized forms of the linear momentum balance equations can be obtained (APPENDIX A.2)

$$\text{div}(T_s) + \vec{f}_s = \vec{\rho}_s \vec{v}_s \quad (2.26)$$

and

$$\text{div}(T_f) + \vec{f}_f = \vec{\rho}_f \vec{v}_f + \vec{\rho}_f^\prime \vec{L}_f \vec{v}_f. \quad (2.27)$$

The local forms of the angular momentum balance equations can be written as

$$T_a = T_a^T \quad \text{and} \quad T = T^T. \quad (2.28)$$

The Cauchy stress associated with the fluid constituent in eqn. (2.28) can be replaced by

$$T_f = -\pi I + T_f^\prime \quad (2.29)$$

where $\pi$ is the pore pressure, $I$ is the identity tensor, and $T_f^\prime$ is the deviatoric stress tensor of $T_f$. 
If the fluid inertia effect and the viscosity induced term \( T'_f \) are neglected, the fluid linear momentum equation (2.27) can be reduced in the form
\[
\phi v_r = \left( \frac{1}{\rho_f L_f} \right) [- \nabla \pi] + \vec{F}_f. \tag{2.30}
\]
Equation (2.30) represents Darcy's law characterizing fluid flow in a porous media. Bowen [26] has shown that the use of the conventional form of Darcy's law instead of eqn. (2.27) implies a model which, among other assumptions, neglects the inertia of the fluid and drag between the fluid particles.

2.2.3 Balance of Energy

In order to derive the balance of energy law for the constituent mixture, we assume that the temperatures of the fluid and solid at the same spatial position are equal. The balance of energy, implied by the first law of thermodynamics, is expressed as
\[
\sum_{a=s,f} \left( \frac{d}{dt} \int_{\Omega} \rho_a(x,t) \left[ \frac{1}{2} v_a(x,t) \cdot v_a(x,t) + \xi_a(x,t) \right] dV \right.
- \left. \int_{\Omega} [\vec{F}_a(x,t) \cdot v_a(x,t) + \vec{\rho}_a(x,t) r_a(x,t)] dV \right.
- \int_{\partial\Omega} [t_a(x,t) \cdot v_a(x,t) - q_a(x,t) \cdot \hat{n}] dA = 0 \tag{2.31}
\]
where \( \xi_a \) is the specific internal energy and \( r_a \) is the specific energy source. The quantities \( q_s \) and \( q_f \) denote the heat flux vectors associated with solid and fluid constituents, respectively, and have the simplified form
\[ q_\alpha = -\overline{k}_\alpha \text{grad}(\theta) \] 

(2.32)

where \( \overline{k}_\alpha \) is the apparent thermal conductivity for each constituent. The apparent conductivities are related to the material conductivities by

\[ \overline{k}_f = \phi k_f \quad \text{and} \quad \overline{k}_s = (1-\phi)k_s. \] 

(2.33)

By employing the divergence theorem and eqns. (2.15), (2.16), (2.26) and (2.27), the local form of the energy balance equation is obtained from eqn. (2.31) (APPENDIX A.3), viz.,

\[ \rho \dot{c} + \overline{\rho}_f \mathbf{v}_r \cdot \text{grad}(\dot{\epsilon}_f) = \rho r - \text{div}(\mathbf{q}) + T_s \cdot L_s + T_f \cdot L_f \] 

(2.34)

where \( \rho \dot{c} = \overline{\rho}_s \dot{c}_s + \overline{\rho}_f \dot{c}_f \), \( \rho r = \overline{\rho}_s r_s + \overline{\rho}_f r_f \) and \( L_\alpha \) is the velocity gradient tensor.

2.3 Thermodynamic Restrictions on Constitutive Relations

A thermodynamic process for a single constituent body can be described by the following functions of \( X \) and the time \( t \):

1) \( X = \chi(X, t) \) is the spatial position.
2) \( \eta = \eta(X, t) \) is the specific entropy per unit mass.
3) \( T = T(X, t) \) is the symmetric Cauchy stress tensor.
4) \( \mathbf{q} = q(X, t) \) is the heat flux vector
5) \( \xi = \xi(X, t) \) is the specific internal energy
6) \( \theta = \theta(X, t) \) is the temperature
7) \( f = f(X, t) \) is the body force vector per unit volume
8) \( r = r(X, t) \) is the specific heat source

9) \( \rho = \rho(X, t) \) is the mass density

10) \( \xi = \xi(X, t) = (\xi_1, \xi_2, \ldots, \xi_N) \) is the internal state vector and \( \xi_i = \xi_i(X, t) \) are the internal state variables.

This set of ten quantities has conceptually been divided into three sets [38, 39]:

\[
(\chi, \eta, \xi), (T, \varrho, \xi, \Theta), (f, r, \rho).
\]  \hspace{1cm} (2.35)

The elements of the first set are the independent variables, the elements of the second set are the dependent variables, and the elements of the last set are the balance terms. A set of the functions is called a thermodynamic process in a body whenever it is compatible with the balance laws [38, 39].

The thermodynamic process must also be restricted by the second law of thermodynamics which is expressed in the form of the Clausius-Duhem inequality. The inequality for the mixture of the constituents has the form

\[
\frac{d}{dt}\int_{\Omega} \rho_f(x,t) \eta_f(x,t) \, dv + \frac{d}{dt}\int_{\Omega} \rho_s(x,t) \eta_s(x,t) \, dv
- \int_{\Omega} \frac{1}{\varrho} \rho \sigma \, dv + \int_{\partial \Omega} \frac{1}{\varrho} q \cdot \hat{n} \, dA \geq 0
\]  \hspace{1cm} (2.36)

where \( \eta_a \) is the entropy for each constituent. The localized form of eqn. (2.36) can be written as (APPENDIX A.4)
\[
\bar{\rho}_s (\dot{\theta} \eta_s - \dot{\xi}_s) + \bar{\rho}_f (\dot{\theta} \eta_f - \dot{\xi}_f) - \rho_f \mathbf{v}_r \cdot \text{grad}(\xi_f) \\
+ \bar{\rho}_f \dot{\theta} \mathbf{v}_r \cdot \text{grad}(\eta_f) + T_f \cdot \mathbf{L}_f + T_s \cdot \mathbf{L}_s - \frac{q}{\theta} \cdot q \geq 0 \quad (2.37)
\]

where \( q = \text{grad} \theta \). By the use of canonical transformations defined by

\[
\xi_a = \psi_a + \theta \eta_a 
\]

where \( \psi_a \) is the specific Helmholtz free energy for each constituent, equations (2.34) and (2.37) can be rewritten in the form

\[
\bar{\rho}_f (\dot{\psi}_f + \dot{\theta} \eta_f + \dot{\xi}_f) + \bar{\rho}_s (\dot{\psi}_s + \dot{\theta} \eta_s + \dot{\xi}_s) + \rho_f \mathbf{v}_r \cdot \text{grad}(\psi_f + \theta \eta_f) \\
= \rho r - \text{div}(q) + T_s \cdot D_s + T_f \cdot D_f \quad (2.39)
\]

and

\[
- \bar{\rho}_s (\dot{\theta} \eta_s + \dot{\xi}_s) - \bar{\rho}_f (\dot{\theta} \eta_f + \dot{\xi}_f) - \rho_f \mathbf{v}_r \cdot \text{grad}(\psi_f) \\
- \bar{\rho}_f \eta_f \mathbf{v}_r \cdot \text{grad}(\theta) + T_f \cdot \mathbf{L}_f + T_s \cdot \mathbf{L}_s - \frac{q}{\theta} \cdot q \geq 0. \quad (2.40)
\]

In the single constituent theories, a dissipative material is characterized by five response functions \( \hat{T}, \hat{q}, \hat{\psi}, \hat{\eta} \) and \( \hat{\xi} \), which give the values of \( T, q, \psi, \eta \) and \( \xi \) at \( x \) and \( t \) when \( F, \theta, g \) and \( \xi \) are known, i.e.

\[
T = \hat{T}(F, \theta, g, \xi) \quad (2.41) \\
q = \hat{q}(F, \theta, g, \xi) \quad (2.42) \\
\psi = \hat{\psi}(F, \theta, g, \xi) \quad (2.43) \\
\eta = \hat{\eta}(F, \theta, g, \xi) \quad (2.44) \\
\xi = \hat{\xi}(F, \theta, g, \xi). \quad (2.45)
\]

The concept of a thermomechanical process is now different than that described in eqn. (2.35). The
independent variables are the motion $\chi$, temperature $\theta$ and internal state variables $\xi$. The dependent variables are $T$, $q$, $\psi$, $\eta$, and $\dot{\xi}$, and the balance terms are still $b$, $r$ and $\rho$. A thermodynamic process is admissible if it is compatible with the preceding constitutive equations (2.41)-(2.45).

By the same token, viscous fluids can be characterized by

\begin{align*}
T &= \hat{T}(\rho, D, \theta, g), \\
q &= \hat{q}(\rho, D, \theta, g), \\
\psi &= \hat{\psi}(\rho, D, \theta, g), \\
\eta &= \hat{\eta}(\rho, D, \theta, g).
\end{align*}

Substituting the constitutive equations, for each constituent, defined by

\begin{align*}
\psi_s &= \hat{\psi}_s(F_s, \theta, g, \xi), \\
\psi_f &= \hat{\psi}_f(\rho_f, D_f, \theta, g),
\end{align*}

and the solid and fluid constituents are generally compressible. However, we assume that the fluid constituent is incompressible in this study. Therefore, the constitutive equations include a dependence on porosity $\phi$. However, porosity-dependent constitutive relations for modeling of large deformation problems are not yet available.
and using the relations
\[ \dot{\psi}_s = \frac{\partial \psi_s}{\partial f_s} \cdot \dot{f}_s + \frac{\partial \psi_s}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_s}{\partial g} \cdot \dot{g} + \frac{\partial \psi_s}{\partial \xi} \cdot \dot{\xi}, \tag{2.52} \]
\[ \dot{\psi}_f = \frac{\partial \psi_f}{\partial \rho_f} \cdot \dot{\rho}_f + \frac{\partial \psi_f}{\partial \mathbf{D}_f} \cdot \dot{\mathbf{D}}_f + \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g}, \tag{2.53} \]

\[ \text{grad}(\psi_f) = \frac{\partial \psi_f}{\partial \rho_f} \text{grad}(\rho_f) + \frac{\partial \psi_f}{\partial \mathbf{D}_f} \text{grad}(\mathbf{D}_f) + \frac{\partial \psi_f}{\partial \theta} \text{grad}(\theta) + \frac{\partial \psi_f}{\partial g} \text{grad}(g), \tag{2.54} \]

equation (2.40) can be rearranged into the form
\[ - \overline{\rho}_f \left[ \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g} \right] - \overline{\rho}_s \left[ \frac{\partial \psi_s}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_s}{\partial g} \cdot \dot{g} \right] - \overline{\rho}_f \left[ \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g} \right] \]
\[ + (T_s - \overline{\rho}_s \frac{\partial \psi_s}{\partial \theta} \cdot \mathbf{F}_s \cdot \mathbf{T}_s) \cdot \mathbf{I}_s - \overline{\rho}_s \frac{\partial \psi_s}{\partial \xi_k} \cdot \mathbf{I}_k \]
\[ - \overline{\rho}_f \left[ \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g} \right] \]
\[ - \overline{\rho}_f \left[ \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g} \right] \]
\[ \leq 0. \tag{2.55} \]

Using the relation, \( \text{div} \mathbf{v}_f = \mathbf{I} \cdot \mathbf{D}_f \), and eqn. (2.16), the third term in eqn. (2.55) can be simplified in the form
\[ \overline{\rho}_f \left[ \frac{\partial \psi_f}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \psi_f}{\partial g} \cdot \dot{g} \right] \]
\[ = \overline{\rho}_f^2 \frac{\partial \psi_f}{\partial \theta} \cdot \mathbf{I} \cdot \mathbf{D}_f. \tag{2.56} \]

From the arbitrariness of the values for \( \dot{\theta}, \mathbf{D}_f, \mathbf{g} \) and \( \mathbf{L}_s \), the following relations are obtained:
\[ \frac{\partial \psi_f}{\partial \mathbf{D}_f} = 0 \quad \text{and} \quad \frac{\partial \psi_f}{\partial \mathbf{g}} = 0 \quad \text{or} \quad \psi_f = \hat{\psi}_f(\rho_f, \theta) \quad \tag{2.57} \]
\[
\frac{\partial \psi_S}{\partial \mathbf{g}} = 0 \quad \text{or} \quad \psi_S = \psi_S(\mathbf{F}_S, \theta, \xi) \tag{2.58}
\]
\[
\eta_f = -\frac{\partial \psi_f}{\partial \theta}, \quad \eta_s = -\frac{\partial \psi_s}{\partial \theta} \tag{2.59}
\]
\[
T_s = -\rho_s \frac{\partial \psi_s}{\partial F_s} \mathbf{F}_s \mathbf{T} \tag{2.60}
\]
\[
-\rho_s \frac{\partial \psi_s}{\partial \xi} \cdot \xi \geq 0, \quad -\frac{1}{\rho_s} \mathbf{g} \cdot \mathbf{q} \geq 0 \tag{2.61}
\]
\[
(\rho_f^2 \frac{\partial \psi_f}{\partial \rho_f}) (\mathbf{I} + T_f) \cdot D_f \geq 0. \tag{2.62}
\]

For a Newtonian fluid, following Stokes' assumption, \( T_f \) is given by

\[
T_f = -\pi \mathbf{I} + \lambda \text{tr}(D_f) \mathbf{I} + 2G D_f \tag{2.63}
\]

where \( \lambda \) and \( G \) are Lame's constants, and \( \mathbf{I} \) is the unit tensor. Substituting eqn. (2.63) into eqn. (2.62), it can be shown for a particular \( \omega D \) that

\[
[\rho_f^2 \frac{\partial \psi_f}{\partial \rho_f} \mathbf{I} - \pi \mathbf{I} + \omega \lambda \text{tr}(D_f) \mathbf{I} + 2G D_f] \cdot \omega D_f \geq 0. \tag{2.64}
\]

where \( \omega \) is a positive scalar.

From the above inequality, we see that

\[
\pi = \rho_f^2 \frac{\partial \psi_f}{\partial \rho_f}. \tag{2.65}
\]

It also can be shown, using \( T'_f = \lambda \text{tr}(D_f) \mathbf{I} + 2G D_f \), that

\[
T'_f \cdot D_f \geq 0. \tag{2.66}
\]

Substituting eqns. (2.57), (2.58) and (2.59) into eqn. (2.39) with the relations
\[ \frac{\partial \psi_s}{\partial F} \cdot F = \frac{\partial \psi_s}{\partial F} \cdot T_s \cdot L_s \quad \text{and} \quad \rho_f v_f \cdot \text{grad} (\eta_f) = -\rho_f \text{div}(v_f), \]

along with eqns. (2.60) and (2.65) yields the resulting form of the energy balance equation

\[ \bar{\rho}_f \dot{\eta}_f + \rho_s \dot{\eta}_s + \rho_f \dot{v}_f \cdot \text{grad} (\eta_f) = \rho_r - \text{div}(q) + \gamma_f + \gamma_p \]  

(2.67)

where \( \gamma_f = T_f \cdot D_f \) and \( \gamma_p = -\rho_s \frac{\partial \psi_s}{\partial \xi} \cdot \xi \) are the dissipation due to viscous fluid flow and dissipation caused by plastic deformation of the solid, respectively.

2.4 Formulation of Boundary Value Problems

The behavior of the fluid saturated porous media is governed by five equations (2.15), (2.16), (2.26), (2.27) and (2.67) where \( \rho_s, \rho_f, v_f, v_s \) and \( \theta \) are the primary variables and \( \phi, \sigma_f, \sigma_s, \eta_f, \eta_s \) are related to these primary variables through the constitutive laws.

2.4.1 Field Equations

Among the governing equations, the momentum equation (2.27) for the fluid constituent along with eqn. (2.29) is treated as a constitutive equation:

\[ \phi_v_r = \frac{\kappa}{\mu} [-\text{grad}(\pi) + \bar{I}_f] \]  

(2.68)

where \( \kappa \) is the permeability and \( \mu \) is fluid viscosity. Equation (2.68), known as Darcy's law, approximates the fluid filtration through an isotropic porous media. It is also assumed, in this study, that the material mass
densities obey the linearized expressions
\[
\rho_s = \rho_{so}[1 - 3\beta_s(\theta - \theta_0) - \frac{1}{3K_s} \text{tr}(T_s)]
\] (2.69)
and
\[
\rho_f = \rho_{fo}[1 - 3\beta_f(\theta - \theta_0) + \frac{\pi}{K_f}]
\] (2.70)
where \(\beta_s\) and \(\beta_f\) are the thermal expansion coefficients, \(K_s\) and \(K_f\) are the bulk moduli of the solid and fluid materials, and \(\rho_{so}\) and \(\rho_{fo}\) are the reference \((\theta = \theta_0)\) mass densities of the solid and fluid materials, respectively.

Equation (2.15) is also replaced by eqn. (2.17) in this study.

Substituting Darcy's law (2.68), along with eqns. (2.69) and (2.70) into the mass conservation equation (2.16) for the fluid, we obtain
\[
\rho_f \dot{\theta} + \phi \rho_{fo} (-3\beta_f \frac{\dot{\theta}}{K_f} + \frac{1}{K_f} \pi) + \phi \rho_f \text{div}(\mathbf{v}_s) + \rho_f \text{div}\left(\frac{\mu}{\mu} [-\text{grad}(\pi) + \phi f_f] \right)
+ \rho_f \frac{\kappa}{\mu} [-\text{grad}(\pi) + \phi f_f] \cdot [-3\beta_f \text{grad}(\theta) + \frac{1}{K_f} \text{grad}(\pi)] = 0. \tag{2.71}
\]

Linearized constitutive equations are assumed for the entropies \(\eta_s\) and \(\eta_f\) in the form
\[
\cdot \eta_s = \frac{\beta_s \theta_0}{\rho_s \theta} \text{tr}(T_s) + \frac{3E_s \beta_s^2}{\rho_s (1 - 2\nu_s)} \cdot \theta + \frac{C_s}{\theta} \cdot \theta \tag{2.72}
\]
and
\[
\cdot \eta_f = - \frac{3\beta_f \theta_0}{\rho_f \theta} \cdot \pi + \frac{C_f}{\theta} \cdot \theta \tag{2.73}
\]
where \(T_s\) is objective stress rate, and \(E_s, \nu_s, C_s,\) and \(C_f\) are the Young's modulus, Poisson's ratio of the solid
constituent, heat capacities of the solid and fluid constituents, respectively.

For a viscoplastic material, the plastic dissipation energy can be written in the form [7]

\[
\gamma_p = T_s \cdot D_p
\]  

(2.74)

where \( D_p \) is plastic deformation rate. Using eqns. (2.68) through (2.70), and (2.72) through (2.74), the energy conservation equation (2.67) is expressed as

\[
-3\phi \beta_f \theta \cdot \pi + (1-\phi) \beta_s \theta \cdot \text{tr}(T_s) + (\tilde{\rho}_f C_f + \tilde{\rho}_s \hat{C}_s) \theta \\
+ \frac{k_f}{\mu} [-\text{grad}(\pi) + \phi f_f \cdot [-3\beta_f \theta \cdot \text{grad}(\pi) + \rho_f C_f \text{grad}(\theta)] \\
= \tilde{\rho} x_f + \tilde{\rho}_s \tilde{r}_s - \text{div}([\phi k_f + (1-\phi) k_s] \text{grad}(\theta)) \\
+ T_{f'} \cdot D_f + T_s \cdot D_p
\]

(2.75)

where \( \hat{C}_s = C_s + \frac{3E \beta_s^2}{\rho_s (1-2\nu_s)} \).

The five governing equations (2.15, 2.16, 2.26, 2.27, 2.67), now, are reduced to three field equations (2.26), (2.71) and (2.75) with an evolution equation (2.17) for the porosity variable. In these field equations, \( \pi, v_s \) and \( \theta \) are the primary unknowns. The densities, \( \rho_s \) and \( \rho_f \), fluid velocity \( v_f \), stresses, \( T_f \) and \( T_s \), are related to the primary unknowns through the constitutive equations.

In summary, the thermodynamic behavior of the fluid saturated porous materials undergoing finite deformation is governed by the following three field equations...
resulting from balance of mass, momentum, and energy, respectively:

\[ \nabla \cdot T_s + \bar{f}_s = 0. \quad (2.76) \]

\[ \frac{\dot{\phi}}{\dot{\phi}} - \frac{\rho_{fo}}{\rho_f} \beta_f \dot{\theta} + \frac{\rho_{fo}}{\rho_f} \frac{1}{K_f} \pi + \nabla \cdot v_s + \frac{1}{\phi} \nabla \cdot \left( \frac{\kappa}{\mu} (-\nabla \pi + \phi f_f) \right) \]
\[ + \frac{\rho_{fo}}{\rho_f} \frac{\kappa}{\phi \mu} (-\nabla \pi + \phi f_f) \cdot (-3 \beta_f \dot{\theta} + \frac{1}{K_f} \nabla \pi) = 0 \quad (2.77) \]

\[ -3 \beta_f \dot{\theta}_o \pi + (1-\phi) \beta_s \theta_o \text{tr}(T_s) + \rho \dot{c} \theta \]
\[ + \frac{\kappa}{\mu} (-\nabla \pi + \phi f_f) \cdot (-3 \beta_f \theta_o \nabla \pi + \rho_f c_f \nabla \theta) \]
\[ = \rho r - \nabla \cdot (k \nabla \theta) + T_f \cdot D_f + T_s \cdot D_p \quad (2.78) \]

where
\[ \rho C = (1-\phi) \rho_s C_s + \phi \rho f C_f, \quad k = (1-\phi) k_s + \phi k_f, \quad \text{and} \]
\[ \rho r = (1-\phi) \rho_s r_s + \phi \rho_f r_f. \]

Also, the porosity evolution equation (2.17) is

\[ (1-\phi) J_s \rho_s = (1-\phi_R) \rho_{sR} \quad (2.79) \]

where the subscript R represents the reference values.

The mass densities are given by eqns. (2.69) and (2.70):

\[ \rho_s = \rho_{s0} [1 - 3 \beta_s (\theta - \theta_o) - \frac{1}{3 K_s} \text{tr}(T_s)] \]
\[ \rho_f = \rho_{fo} [1 - 3 \beta_f (\theta - \theta_o) - \frac{1}{K_f} \pi]. \]

The unknown temperature, fluid pressure and velocity fields are closely coupled, so that the deformation of a body should be considered simultaneously with the pore fluid pressure and temperature changes. It is also
noteworthy that the mechanical and physical properties characterized by the moduli, conductivities, and densities etc. are usually solution dependent. In other words, these quantities are a function of temperature, deformation rate, and pore pressure.

2.4.2 Boundary Conditions

The region, \( \Omega \), is occupied by the saturated solid with a portion of its boundary surface \( \partial \Omega_d \) subjected to applied tractions \( \hat{\mathbf{t}} \), while the remainder of its boundary surface \( \partial \Omega_u \) has prescribed displacements \( \hat{\mathbf{u}} \). The surface may also be divided into a portion \( \partial \Omega_\theta \) on which the temperature, \( \hat{\theta} \), is specified, and surface \( \partial \Omega_h \) with a given external heat flux, \( \hat{q}_c \). The surface may again be divided into a portion \( \partial \Omega_\pi \) on which the fluid pressure, \( \hat{\pi} \), is prescribed, and \( \partial \Omega_i \) denoting an impermeable boundary or specified fluid flow (Fig. 1). The domain \( \bar{\Omega} \) is composed of the interior \( \Omega \) and the boundary \( \partial \Omega \), and all of functions defined in the domain are the product of the spatial domain with a non-negative interval of time, i.e. \( \bar{\Omega} \times [0, \omega) \).

The governing equations have to be solved subject to certain boundary conditions:

\[
\begin{align*}
\mathbf{u}(\mathbf{x}, t) &= \hat{\mathbf{u}}(\mathbf{x}, t) & \text{on } \partial \Omega_u \times [0, \omega) \quad (2.80) \\
\mathbf{t}(\mathbf{x}, t) &= \hat{\mathbf{t}}(\mathbf{x}, t) & \text{on } \partial \Omega_t \times [0, \omega) \quad (2.81)
\end{align*}
\]
Fig. 1 Domain of Boundary Value Problem
\[
\theta(x,t) = \hat{\theta}(x,t) \quad \text{on } \partial \Omega_\theta \times [0,\omega) \quad (2.82)
\]
\[
-\left[\phi k_f + (1-\phi)k_s\right]\nabla \theta \cdot \hat{n} = q + q_c \quad \text{on } \partial \Omega_h \times [0,\omega) \quad (2.83)
\]
\[
\pi(x,t) = \hat{\pi}(x,t) \quad \text{on } \partial \Omega_\pi \times [0,\omega) \quad (2.84)
\]
\[
\rho_f \frac{k}{\mu} (-\nabla \pi + \phi f_f) \cdot \hat{n} = 0 \quad \text{on } \partial \Omega_i \times [0,\omega) \quad (2.85)
\]

where:

\(\hat{u}(x,t), \hat{\theta}(x,t), \hat{\pi}(x,t)\) are the prescribed displacement, temperature and pressure, respectively.

\(\hat{t}(x,t) = T \cdot \hat{n}\) is the surface traction vector over \(\partial \Omega_t\).

\(\hat{n}\) is the outward unit vector normal to the boundary.

\(q\) is the external heat flux on the boundary \(\partial \Omega_h\).

\(q_c = h_c(\theta - \theta_c)\) is the external convection heat flux on the boundary \(\partial \Omega_h\).

\(h_c\) is the convective heat transfer coefficient.

and

\(\theta_c\) is the convective equilibrium temperature.

2.4.3. Initial Conditions

The initial conditions for the governing equation are given by:

\[
u(x,0) = u_o(x) \quad (2.86)
\]
\[
\theta(x,0) = \theta_o(x) \quad (2.87)
\]
\[
\pi(x,0) = \pi_o(x) \quad (2.88)
\]
2.4.4. **Summary of Equations**

The above derivations of the governing field equations (2.76 through 2.78), porosity evolution equation (2.79), boundary conditions (2.80 through 2.85) and initial conditions (2.86 through 2.88) complete the formulation of the boundary value problem. The overall thermomechanical response of fluid infiltrated saturation media undergoing finite deformation can be determined from this coupled system of equations.
CHAPTER III
ELASTO-VISCOPLASTIC CONSTITUTIVE MODELING

In the classical theory of plasticity, the equations of state are assumed to be independent of time. However, many geological and engineering materials under mechanical and thermal loading exhibit time-dependent plastic behavior such as creep, relaxation effects coupled with the plastic stress state. In order to describe the time-temperature dependent plastic behavior of a material, the following four different constitutive models for viscoplastic materials have been developed:

(1) Rate-dependent modifications of classical plasticity theory, extending the time-dependent plastic behavior of a material to elasto-viscoplastic models such as the work by Perzyna [91, 92, 93].

(2) Phenomenological theory of "dislocation dynamics", advanced by Bodner and Partom [21, 22]. This multi-dimensional formulation of the elastic-viscoplastic constitutive equations has no specified yield criterion.

(3) Nonlinear viscoelasticity theory, postulated by
Cernocky and Krempl [30, 31, 32]. No yield criterion is used so that the transition from linear elastic to nonlinear inelastic behavior is continuous.

(4) Endochronic theory without a yield surface, proposed by Valanis [110], based on irreversible thermodynamics and the concept of intrinsic time.

The last three theories are conventionally classified as "unified elastic-viscoplastic theories" because of the absence of a specified yield criterion. Among the above four constitutive model representations, Perzyna's model can be easily adapted to experimental results because of its relatively simple structure as shown in (Fig. 2). For this reason, Perzyna's model has been extensively used for engineering applications using finite element methods [49, 116-119]. In this chapter the constitutive relations in an incremental form are presented by employing Perzyna's thermo-elastic-viscoplastic model.

3.1 Constitutive Equation

The constitutive equations for a hygro-thermo-elastic-viscoplastic material using Perzyna's viscoplastic model approach with an isotropic hardening mechanism are presented in this section based on the theories of continuum thermodynamics using internal state variables [38, 39].
Fig. 2 One Dimensional Elasto-Viscoplastic Model
Time-temperature dependent plastic response investigations of materials have traditionally been carried out by superposing elastic, $D_e$, hygro-thermal, $D_h$, and time dependent viscoplastic, $D_{vp}$, strain rate components in the form [92]

$$D = D_e + D_h + D_{vp} \quad (3.1)$$

where

$$D = \frac{1}{2}(\frac{\partial \nu}{\partial x} + (\frac{\partial \nu}{\partial x})^T) \quad (3.2)$$

and the superscript dot denotes differentiation with respect to time. The subscript $s$, standing for the solid constituent, is omitted in the sequel. For a fluid-saturated viscoplastic porous material, the hygro-thermal strain rate, $D_h$, can be defined by [38, 39]

$$D_h = \beta I \dot{\theta} + \frac{1-2\nu}{E} I \pi. \quad (3.3)$$

The viscoplastic flow rule, presented by Perzyna, is expressed as

$$D_{vp} = \gamma \langle \phi(F) \rangle \frac{\partial T}{\partial T} \quad (3.4)$$

where $\gamma$ denotes a viscosity constant of the material and $T$ is Cauchy stress. The symbol $\langle \phi(F) \rangle$ is defined by

$$\langle \phi(F) \rangle = \begin{cases} 
0 & \text{if } F \leq 0 \\
\phi(F) & \text{if } F > 0.
\end{cases} \quad (3.5)$$

Among various choices of the functions $\phi$ [48, 88], the following specific form with an isotropic hardening is selected in this study:

$$\phi(F) = (\frac{F - Y}{Y})^N \quad (3.6)$$
\[ F = F(T) = \sqrt{3J_2} = \sqrt{3(T'\cdot T')/2} \quad (3.7) \]

\[ Y = Y_0 + H \bar{\varepsilon}_{vp} \quad (3.8) \]

\[ H = \frac{E_{ET}}{E - E_T} \quad (3.9) \]

where \( N \) is a prescribed constant, \( T' \) is the deviatoric part of the Cauchy stress \( T \), \( Y_0 \) is the yield stress in the simple tension test, \( \bar{\varepsilon}_{vp} \) is the effective plastic strain, \( E \) is the Young's modulus and \( E_T \) is the tangent modulus. All the preceding mechanical properties are temperature dependent. It is worth noting that viscous flow can occur only after the plastic state has been reached. The total objective stress rate, depending on the elastic strain rate, is defined by

\[ \hat{T} = C \, D_e \quad (3.10) \]

or equivalently

\[ \hat{T} = C \, (D - D_h - D_{vp}) \quad (3.11) \]

where

\[ \hat{T} = \dot{T} + TW - WT \quad (3.12) \]

is the Jaumann stress rate, \( W \) is spin tensor and \( C \) is conventional temperature dependent elasticity tensor.

### 3.2 Incremental Constitutive Law

The temperature coupled form of Perzyna's isotropic hardening model under finite deformation can be expressed as [93]...
Using the flow rule, we obtain the incremental viscoplastic strain, $\Delta \epsilon_{vp}$, during the time increment $\Delta t$ utilizing an implicit time stepping scheme [116] expressed by

$$\Delta \epsilon_{vp} = \Delta t[(1-\alpha)D_{vp}^t + \alpha D_{vp}^{t+\Delta t}]$$  \hspace{1cm} (3.14)

where $\alpha$ denotes a time stepping parameter, $0 \leq \alpha \leq 1$. The strain rate $D_{vp}^{t+\Delta t}$ in eqn. (3.14) can be obtained by using the curtailed Taylor series expansion [116]

$$D_{vp}^{t+\Delta t} = D_{vp}^t + \left( \frac{\partial D_{vp}}{\partial T} \right) \Delta T + \left( \frac{\partial D_{vp}}{\partial \theta} \right) \Delta \theta$$  \hspace{1cm} (3.15)

with

$$\theta = (1 - \alpha) \theta^t + \alpha \theta^{t+\Delta t}$$  \hspace{1cm} (3.16)

$$Y = Y_o + H(\theta) \bar{\epsilon}_{vp}^{t+\Delta T}$$  \hspace{1cm} (3.17)

where $\Delta T$ is the true stress increment during the time increment $\Delta t$ and $\bar{\epsilon}_{vp}^{t+\Delta t}$ is effective plastic strain at time $t+\Delta t$. Substituting eqn. (3.15) into eqn. (3.14) and neglecting the last term in eqn. (3.15) yields

$$\Delta \epsilon_{vp} = D_{vp}^t \Delta t + p^t \Delta T$$  \hspace{1cm} (3.18)

where

$$p^t = \alpha \Delta t \left( \frac{\partial D_{vp}}{\partial T} \right) t = \alpha \Delta t \gamma(\phi \frac{\partial b}{\partial T} + \frac{\partial \phi}{\partial T} \bar{b} \bar{b}^T) t$$  \hspace{1cm} (3.19)

and

$$b = \frac{\partial F}{\partial T} = \sqrt{\frac{3}{2\sqrt{2}}} (T_x', T_y', T_z', 2T_{xy}, 2T_{xz}, 2T_{xy}).$$  \hspace{1cm} (3.20)

Assuming a small configuration change during the time interval $\Delta t$, the incremental form of eqn. (3.11) is
expressed as

$$
\Delta \hat{T} = C^t (\Delta \varepsilon - \Delta \varepsilon_h - \Delta \varepsilon_{vp}) + \Delta C \ v_p^t
$$

(3.21)

where $\Delta C = C^{t+\Delta t} - C^t$. Neglecting the last term in eqn. (3.21), the incremental form of the stress and strain relation for the hygro-thermo-elasto-viscoplastic model with a temperature dependent modulus is obtained from eqns. (3.11), (3.18) and (3.21) as

$$
\Delta \hat{T} = C_{ep}^t [\Delta \varepsilon - \Delta \varepsilon_h - \Delta \varepsilon - \Delta \varepsilon_{vp}] + P^t (T \Delta W - \Delta W T)
$$

(3.22)

where

$$
C_{ep}^t = [(C^t)^{-1} + P^t]^{-1}.
$$

(3.23)

From eqns. (3.3) and (3.12), it can be shown that

$$
\Delta \varepsilon_h = \int_t^{t+\Delta t} D_h \ dt
$$

(3.24)

$$
\Delta T = \Delta \hat{T} + \int_t^{t+\Delta t} (TW - WT) \ dt,
$$

(3.25)

and

$$
\Delta \varepsilon = \int_t^{t+\Delta t} D \ dt.
$$

(3.26)

In summary, the incremental constitutive equation (3.22), represented by the finite strain increment (3.26), viscoplastic strain increment (3.13) and hygro-thermal strain increment (3.24), yields the objective stress increment during the time increment $\Delta t$. The true stress increment is calculated from eqn. (3.25)
CHAPTER IV
FINITE ELEMENT FORMULATIONS

The governing equations developed in chapter II incorporate material and geometrical nonlinearities. For such nonlinear problems, closed form solutions are generally not obtainable. Finite element formulations for these coupled nonlinear partial differential equations are therefore developed for numerical evaluation of the responses with the following assumptions:

1) Small configuration change but finite strain during the time increment $\Delta t$
2) The porosity is constant during the time increment
3) No deformation induced heating
4) No heat flux due to the dissipation from the viscous fluid flow

For geological material, we typically find values in the order of $\beta S_0 \text{tr}(T_s) = 10^4$, $T_s D_p = 10^4$ and $\rho C \dot{\theta} = 10^7$ for 500°K of temperature and 10 MPa of stress field.

The virtual work principle, incorporating the updated Lagrangian method, is applied to the equations governing
the deformation of the solid skeleton. The weak formulations for the fluid transport and thermal diffusion equations are used for the numerical evaluations. The implicit time stepping scheme combined with an iteration scheme corresponding to the full Newton-Raphson method is also utilized.

4.1. Updated Lagrangian Formulation for Deformation of the Solid Matrix

A finite element formulation for the equilibrium equation of the solid constituent, using the virtual work principle with an updated Lagrangian method, is presented here. The viscoplastic solid matrix represented by a rate sensitive material incorporates the Von Mises-Huber yield criteria and the Perzyna flow rule [90-92] with isotropic hardening.

The quasi-static deformation of a viscoplastic body is traced in an incremental manner by calculating the equilibrium state at the present time referred to the configuration at a previous time, as shown in Fig. 3. In the sequel, the configurations at time t and t+Δt are referred to as the referential and current configurations, respectively. The spatial coordinates and displacements of a material point $P^0$ at time $t=0$ march with time as shown in Figs. 3 and 4. A material point in the
Fig. 3 Incremental Deformation

\[ B^0, B^t, B^{t+\Delta t} \]: original, referential and current configuration respectively

\[ \Omega^0, \Omega^t, \Omega^{t+\Delta t} \]: the body domain at time 0, t, and \( t+\Delta t \) respectively

\[ p^0, p^t, p^{t+\Delta t} \]: material point at time 0, t, and \( t+\Delta t \) respectively

\[ x^0, x^t, x^{t+\Delta t} \]: position vector at time 0, t, and \( t+\Delta t \) respectively
Fig. 4 Time Marching of Spatial Coordinate and Displacement
referential configuration \( \mathbf{B}^t \) with position vector \( \mathbf{x}^t(=\mathbf{X}) \) occupies position \( \mathbf{x}^{t+\Delta t}(=\mathbf{x}) \) at time \( t+\Delta t \) with the motion defined by

\[
x^{t+\Delta t} = \chi(x^t, t)
\]

(4.1)

The motion in Fig. 3 has a deformation gradient \( \mathbf{F} \) defined by

\[
d\mathbf{x}^{t+\Delta t} = \mathbf{F} \ d\mathbf{x}^t
\]

(4.2)

4.1.1 Virtual Work Principle

The weak form of Cauchy's equilibrium equations (2.76) is obtained by applying the principle of virtual work to the current configuration

\[
\int_{\Omega^{t+\Delta t}} (\nabla \mathbf{T} + \overline{\mathbf{F}})^{t+\Delta t} \cdot \delta \mathbf{u} \ d\mathbf{v}^{t+\Delta t} = 0
\]

(4.3)

where \( \delta \mathbf{u} \) is the virtual displacement and the dot implies contraction between two tensors. Applying the divergence theorem to eqn. (4.3) and using the boundary conditions gives

\[
\int_{\Omega^{t+\Delta t}} (\mathbf{T} \cdot \delta \mathbf{\epsilon})^{t+\Delta t} \ d\mathbf{v}^{t+\Delta t} = \int_{\Omega^{t+\Delta t}} (\overline{\mathbf{F}} \cdot \delta \mathbf{u})^{t+\Delta t} \ d\mathbf{v}^{t+\Delta t} + \int_{\partial \Omega^{t+\Delta t}} (\mathbf{T} \cdot \delta \mathbf{u})^{t+\Delta t} \ d\mathbf{\alpha}^{t+\Delta t} \tag{4.4}
\]

where \( \partial \Omega^t \) denotes the boundary with specified traction and

\[
\delta \mathbf{\epsilon} = \frac{1}{2} \left[ \frac{\partial \delta \mathbf{u}}{\partial \mathbf{x}} + \left( \frac{\partial \delta \mathbf{u}}{\partial \mathbf{x}} \right)^T \right].
\]

(4.5)

Following the standard procedure for the updated
Lagrangian method, equation (4.4) can be re-expressed in terms of the second Piola-Kirchhoff stress tensor and Lagrangian strain tensor referred to the previous configuration in the form (APPENDIX B.1)

\[
\int (\mathbf{T} \cdot \delta \mathbf{u})_t + \int (\mathbf{T} \cdot \delta \mathbf{u})_t^+ \delta \mathbf{A} t = \int (S \cdot \delta \mathbf{E})_t^+ \delta \mathbf{A} t
\]

(4.6)

where $S$ is the symmetric Piola-Kirchhoff stress tensor and $\delta \mathbf{E}$ is the variational form of the incremental Lagrangian strain tensor.

4.1.2 Incremental Formulation

Linearization of the weak form of the equilibrium equations (4.6) is required so that the unknown quantities in the equations can be expressed as an addition of a known component at time $t$ and an unknown increment during the time increment $\Delta t$. For linearization of the second Piola-Kirchhoff stress tensor, a Taylor series expansion is considered

\[
S_{t+\Delta t} = S_t + JF^{-1} \left[ \text{div}(\mathbf{v})T - LT + \dot{T} - TL \right] F^{-T} \left. \Delta t + \text{H.O.T.} \right)
\]

(4.7)

From the relation between Truesdell stress rate and Piola-Kirchhoff stress rate (APPENDIX B.2), equation (4.7) can be expressed as

\[
S_{t+\Delta t} = S_t + JF^{-1} \left[ \text{div}(\mathbf{v})T - LT + \dot{T} - TL \right] F^{-T} \left. \Delta t + \text{H.O.T.} \right)
\]

(4.8)
Replacing $T$ by the Jaumann stress rate (eqn.(3.12)) and neglecting higher order terms, equation (4.8) is rearranged as

$$s_t + At = s_t + JF^{-1}[\text{div}(v)T - DT + \hat{T} - TD]F^{-T}|_t \Delta t$$ (4.9)

Since the reference state is selected as the configuration at time $t$

$$F|_t = I, \quad J|_t = 1, \quad \sigma_t = s_t = T|_t, \quad \dot{\sigma}_t = \hat{T}|_t$$ (4.10)

$$d_t = D|_t = \frac{1}{2} \left[ \frac{\partial v^t}{\partial X} + \left( \frac{\partial v^t}{\partial X} \right)^T \right]$$ (4.11)

We use eqns. (4.10) and (4.11) and rewrite equation (4.9) as

$$s_t^+ + At = \sigma^t + [\dot{\sigma}^t - \sigma^t d_t - d_t^t \sigma^t + \text{Div}(v^t) \sigma^t] \Delta t \quad (4.12)$$

Assuming that an infinitesimal change occurs in the position during the time increment $\Delta t$, the following equations are deduced from the linearization [33]

$$\Delta u = v^t \Delta t, \quad (4.13)$$

$$\Delta e = d^t \Delta t = \frac{1}{2} \left[ \frac{\partial (\Delta u)}{\partial X} + \left( \frac{\partial (\Delta u)}{\partial X} \right)^T \right]$$ (4.14)

and

$$\Delta \hat{\sigma} = \dot{\sigma}^t \Delta t \quad (4.15)$$

Combining eqns. (4.13), (4.14), (4.15) and (4.12), the resulting incremental form is obtained as

$$s_t^+ + At = \sigma^t + \Delta \hat{\sigma} - (\sigma^t \Delta e + \Delta e \sigma^t) + \sigma^t \text{tr}(\Delta e). \quad (4.16)$$

Also, the variational form of the Lagrangian strain tensor can be obtained in the form [33]

$$\delta E = \frac{1}{2} [F \delta F^T + \delta FF^T] = \delta E + \delta A$$ (4.17)

where
and

\[ \sigma^t \cdot \delta \Delta n = \frac{1}{2} \left[ \left( \frac{\partial \delta \Delta u}{\partial x} \right)^T + \left( \frac{\partial \delta \Delta u}{\partial x} \right) \right]. \]  

(4.19)

From the incremental constitutive eqn. (3.22), with the last term ignored, equation (4.15) is expressed as

\[ \Delta \sigma = C_{\text{ep}} \cdot (\Delta e - \Delta e_h - \Delta t \cdot D_{\text{Vp}})^t. \]  

(4.20)

where

\[ \Delta e_h = \beta I \Delta \theta + \frac{1 - 2 \nu}{E} \Delta \tau. \]  

(4.21)

Using the incremental constitutive eqn. (4.20), equation (4.16) is rewritten as

\[ St^t + \Delta t = \sigma^t + C_{\text{ep}} \cdot (\Delta e - \Delta e_h - \Delta t \cdot D_{\text{Vp}})^t \]

\[ - (\sigma^t \Delta e + \Delta e \sigma^t) + \sigma^t \text{tr}(\Delta e). \]  

(4.22)

Substituting eqns. (4.17) and (4.22) into the equilibrium equations (4.6) with the relation \( \delta u = \delta \Delta u \), and neglecting the high order contribution resulting from the term \( \delta \Delta n \) we obtain a linear approximation for the unknown incremental displacements, \( \Delta u \). A linearized form of eqn. (4.6) is then derived as

\[ \int_{\Omega^t} \sigma^t \cdot \delta \Delta n \, dv^t + \int_{\Omega^t} C_{\text{ep}} \cdot \Delta e \cdot \delta \Delta e \, dv^t - \int_{\Omega^t} C_{\text{ep}} \cdot \Delta e_h \cdot \delta \Delta e \, dv^t \]

\[ - \int_{\Omega^t} (\sigma^t \Delta e + \Delta e \sigma^t) \cdot \delta \Delta e \, dv^t + \int_{\Omega^t} \text{tr}(\Delta e) \sigma^t \cdot \delta \Delta e \, dv^t \]
4.1.3 Iterative Solution Scheme

Since all the coefficients in eqn. (4.23) are generally solution dependent, an iteration scheme corresponding to the full Newton-Raphson method is used for the implicit time marching scheme. Iterative incremental values between time \( t \) and \( t+\Delta t \) are defined, as shown in Fig. 5. The solutions, after the \( i^{th} \) iteration during the time increment \( \Delta t \), are

\[
\begin{align*}
\mathbf{u}^{t+\Delta t}_{(i+1)} &= \mathbf{u}^{t+\Delta t}_{(i)} + \Delta \mathbf{u}_{(i)}' \\
\tau^{t+\Delta t}_{(i+1)} &= \tau^{t+\Delta t}_{(i)} + \Delta \tau_{(i)}'
\end{align*}
\]

and

\[
\theta^{t+\Delta t}_{(i+1)} = \theta^{t+\Delta t}_{(i)} + \Delta \theta_{(i)}'.
\]

Since the incremental hygro-thermo-viscoplastic eqn. (4.22) is derived, based on the referential state at time \( t \), each iteration during the time interval \( \Delta t \) has to be referred to the reference configuration at time \( t \). In order to circumvent this difficulty, the reference configuration is updated for each iteration so that the previous iteration configuration \( \Omega^{t+\Delta t}_{(i)} \) serves as the
Fig. 5 Notation Used in Iteration
reference instead of $\Omega^i$.

Using the configuration after iteration (i) as a referential configuration, equation (4.6) is rewritten as

\[
\int_{t^+\Delta t} \rho b \cdot \delta u \, dV^{(i)} + \int_{t^+\Delta t} t \cdot \delta u \, ds^{(i)} = \int_{t^+\Delta t} (S \cdot \delta E)^{t+\Delta t} \cdot dV^{(i)}
\]

\[
\dot{\mathbf{\Omega}}^{t+\Delta t} (\mathbf{g}_{(i)}^{(i)}) + \mathbf{\delta \mathbf{\Omega}}^{t+\Delta t} (\mathbf{g}_{(i)}^{(i)}) - \mathbf{\delta \mathbf{\Omega}}^{t+\Delta t} (\mathbf{g}_{(i)}^{(i)}) + \mathbf{\delta \mathbf{\Omega}}^{t+\Delta t} (\mathbf{g}_{(i)}^{(i)}) - \mathbf{\delta \mathbf{\Omega}}^{t+\Delta t} (\mathbf{g}_{(i)}^{(i)})
\]

(4.27)

The incremental form of the eqn. (4.16), referred to the previous iteration configuration, is modified as

\[
S^{t^+\Delta t}_{(i+1)} = \sigma^{t^+\Delta t} + \Delta \hat{\sigma}^{(i)} - (\sigma^{t^+\Delta t} + \Delta \hat{\sigma}^{(i)}) \cdot \Delta e^{(i)} + \Delta e^{(i)} \sigma^{t^+\Delta t} (i) + \sigma^{t^+\Delta t} \cdot \Delta e^{(i)}.
\]

(4.28)

Assuming a small configuration change during the iteration, the quantity $\Delta \hat{\sigma}^{(i)}$ in eqn. (4.28) is determined from eqn. (4.20) in the form

\[
\Delta \hat{\sigma}^{(i)} = C_{ep}^{t^+\Delta t} \left( \Delta e^{(i)} - \Delta t \cdot \Delta e^{(i)} - \Delta e^{(i)} \right).
\]

(4.29)

A suitable expression for eqn. (4.23) for the $(i+1)^{\text{st}}$ iteration is then obtained as

\[
\int \left[ \sigma^{t^+\Delta t} \cdot \Delta \mathbf{n} + C_{ep}^{t^+\Delta t} \Delta e^{(i)} \cdot \Delta \mathbf{e} \right]_{(i)}
\]

\[
- (\sigma^{t^+\Delta t} \Delta e^{(i)} + \Delta e^{(i)} \sigma^{t^+\Delta t}) \cdot \Delta \mathbf{e}^{(i)}
\]

\[
+ \operatorname{tr}(\Delta e^{(i)}) \sigma^{t^+\Delta t} \cdot \Delta \mathbf{e}^{(i)} \right) \cdot dV^{(i)}
\]

\[
+ \int \left[ C_{ep}^{t^+\Delta t} \Delta e^{(i)} \cdot \Delta \mathbf{e}^{(i)} \right]_{(i)} \cdot dV^{(i)}
\]

\[
= \int \left[ t \cdot \delta \mathbf{u} \right]_{(i)} + \int \left[ t^+\Delta t \cdot \mathbf{R} \cdot \delta \mathbf{u} \right]_{(i)}
\]

\[
+ \int \left[ C_{ep}^{t^+\Delta t} t^+\Delta t \cdot t^+\Delta t \cdot \mathbf{R} \cdot \sigma^{t^+\Delta t} \cdot \delta \mathbf{e} \right]_{(i)} \cdot dV^{(i)}.
\]

(4.30)
4.1.4 Spatial Discretization

Assuming an isoparametric interpolation for the two-dimensional problem, the material position and the solution increment based on Galerkin's method in a finite dimensional space of the admissible function space are expressed as

\[ x = [N_u][x], \]  \hspace{1cm} (4.31)

\[ \Delta u(i) = [N_u][\Delta u(i)], \]  \hspace{1cm} (4.32)

\[ \Delta \theta(i) = [N_h][\Delta \theta(i)], \]  \hspace{1cm} (4.33)

and

\[ \Delta \tau(i) = [N_h][\Delta \tau(i)]. \]  \hspace{1cm} (4.34)

Accordingly, the strain increment and gradients of the temperature and fluid pressure increments are determined as

\[ \Delta e(i) = [B_u][\Delta u(i)], \]  \hspace{1cm} (4.35)

\[ \nabla \Delta \theta(i) = [B_h][\Delta \theta(i)], \]  \hspace{1cm} (4.36)

and

\[ \nabla \Delta \tau(i) = [B_h][\Delta \tau(i)] \]  \hspace{1cm} (4.37)

where \([N_u]\) and \([N_h]\) are shape functions, and \([B_u]\) is a strain-displacement matrix. The quantity \([B_h]\) is a gradient matrix of the shape function \([N_h]\). Substituting eqns. (4.31-37) into eqn. (4.30), the corresponding discretized form of eqn. (4.30) is obtained as
\[
\begin{bmatrix}
K_{uu} & K_{u\theta} & K_{u\pi}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta \theta \\
\Delta \pi
\end{bmatrix}^{t+\Delta t}
= \begin{bmatrix}
\Delta L_{u}
\end{bmatrix}^{t+\Delta t}
\]

where

\[
[K_{uu}] = \int_{\Omega} [[B_n][\sigma_n]^{t+\Delta t}][B_n]^T \, dV
+ [B_u]([C_{ep}] - 2[\sigma_B] + [\sigma_0]) [B_u]^T \, dV
\]

\[
[K_{u\theta}] = \int_{\Omega} [B_u][C_{ep}][A_{\theta}][N_n]^T \, dV
\]

\[
[K_{u\pi}] = \int_{\Omega} [B_u][C_{ep}][A_{\pi}][N_n]^T \, dV
\]

\[
[\Delta L_{u}] = \int_{\partial\Omega} [N_u][t] \, dA + \int_{\Omega} [N_u][f] \, dV
+ \int_{\Omega} [B_u]([C_{ep}] \Delta t[D_{vp}] - [\sigma]) \, dV
\]

The appropriate submatrices corresponding to the two-dimensional plane and axisymmetric problems are defined in APPENDIX C.

4.2 Temperature and Pressure Formulation

For a small configuration change during the time increment \(\Delta t\), the porosity is assumed to be relatively constant during the time increment. The heat flux arising from the elastic volume change and dissipation from the plastic deformation of the solid as well as the viscous fluid flow are assumed to be small. From eqns. (2.77) and (2.78), we obtain
\[ c_1 \dot{\theta} + c_2 \nabla \cdot \nabla \theta + c_3 \mathbf{e}_x \cdot \nabla \theta + c_4 \nabla \mathbf{v}_s + c_5 \pi + c_6 \nabla \pi \cdot \nabla \pi \\
+ c_7 \mathbf{e}_f \cdot \nabla \pi + c_8 \nabla \cdot (\nabla \pi + \phi \mathbf{e}_f) = 0 \quad (4.43) \]

\[ d_1 \dot{\theta} + d_2 \nabla \cdot \nabla \theta + d_3 \mathbf{e}_f \cdot \nabla \theta + d_4 + d_5 \pi + d_6 \nabla \pi \cdot \nabla \pi \\
+ d_7 \mathbf{e}_f \cdot \nabla \pi + \nabla \cdot (d_8 \nabla \theta) = 0 \quad (4.44) \]

where

\[ \rho_s = \rho_{so}[1 - 3\beta_s(\theta-\theta) - \frac{1}{3k_s} \text{tr}(\sigma_s)] \quad (4.45) \]

\[ \rho_f = \rho_{fo}[1 - 3\beta_f(\theta-\theta) + \frac{1}{k_f} \pi] \quad (4.46) \]

and

\[ c_1 = -3\beta_f \frac{\rho_{fo}}{\rho_f} \quad d_1 = \phi \rho_f c_f + (1-\phi) \rho_s c_s \]

\[ c_2 = 3 \frac{\kappa}{\mu} \frac{\beta_f \rho_{fo}}{\rho_f} \quad d_2 = -\frac{\kappa}{\mu} \rho_f c_f \]

\[ c_3 = -3 \frac{\kappa}{\mu} \beta_f \rho_{fo} \quad d_3 = \phi \frac{\kappa}{\mu} \rho_f c_f \]

\[ c_4 = 1 \quad d_4 = -[\phi \rho_f c_f + (1-\phi) \rho_s c_s] \]

\[ c_5 = \frac{1}{k_f} \frac{\rho_{fo}}{\rho_f} \quad d_5 = -3\beta_f \theta_0 \]

\[ c_6 = -\frac{\kappa}{\mu} \frac{1}{k_f} \frac{\rho_{fo}}{\rho_f} \quad d_6 = 3 \frac{\kappa}{\mu} \beta_f \theta_0 \]

\[ c_7 = \frac{\kappa}{\mu} \frac{1}{k_f} \frac{\rho_{fo}}{\rho_f} \quad d_7 = -3\phi \frac{\kappa}{\mu} \beta_f \theta_0 \]

\[ c_8 = \frac{1}{\phi} \frac{\kappa}{\mu} \quad d_8 = \phi k_f + (1-\phi) k_s. \]

It is noteworthy that the two governing equations (4.43) and (4.44) have an identical structure with different coefficients due to the individual diffusion characteristics.
The weak forms for the fluid and thermal diffusion equations (4.43) and (4.44) are obtained by multiplying these equations with weight functions \( v \) and \( w \) as follows:

\[
\int_{\Omega} \left( C_1 \dot{\theta} + C_2 \nabla \pi \cdot \nabla \theta + C_3 \dot{f}_f \cdot \nabla \theta + C_4 \nabla \cdot v_s + C_5 \dot{\pi} + C_6 \nabla \pi \cdot \nabla \pi \\
+ C_7 f_f \cdot \nabla \pi + C_8 \nabla \cdot (-\nabla \pi + \phi f_f) \right) v \, dV = 0 \quad \forall v \in H^1(\Omega) \quad (4.47)
\]

\[
\int_{\Omega} \left( D_1 \dot{\theta} + D_2 \nabla \pi \cdot \nabla \theta + D_3 \dot{f}_f \cdot \nabla \theta + D_4 + D_5 \dot{\pi} + D_6 \nabla \pi \cdot \nabla \pi \\
+ D_7 f_f \cdot \nabla \pi - \nabla \cdot (D_8 \nabla \theta) \right) w \, dV = 0 \quad \forall w \in H^1(\Omega) \quad (4.48)
\]

where \( H^1(\Omega) = \{ v \mid v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \in L^2(\Omega) \} \).

Applying the divergence theorem as well as thermal and pressure flux boundary conditions (eqns. (2.83) and (2.85)), we find

\[
\int_{\Omega} \left( C_1 \dot{\theta} + C_2 \nabla \pi \cdot \nabla \theta + C_3 \dot{f}_f \cdot \nabla \theta + C_4 \nabla \cdot v_s + C_5 \dot{\pi} + C_6 \nabla \pi \cdot \nabla \pi \\
+ C_7 f_f \cdot \nabla \pi \right) v \, dV - \int_{\Omega} C_8 \nabla v \cdot (-\nabla \pi + \phi f_f) \, dV = 0
\]

\[
\forall v \in H^1(\Omega) \quad (4.49)
\]

and

\[
\int_{\Omega} \left( D_1 \dot{\theta} + D_2 \nabla \pi \cdot \nabla \theta + D_3 \dot{f}_f \cdot \nabla \theta + D_4 + D_5 \dot{\pi} + D_6 \nabla \pi \cdot \nabla \pi \\
+ D_7 f_f \cdot \nabla \pi \right) w \, dV + \int_{\Omega} D_8 \nabla \theta \cdot \nabla \theta \, dV + \int_{\partial \Omega} (q + q_c) \, dA = 0
\]

\[
\forall w \in H^1(\Omega). \quad (4.50)
\]

The iteration scheme is again introduced because the coefficients are solution dependent. Assuming that the approximate solution is obtained after the \( i^{th} \) iteration, the time rates can be expressed in terms of the unknown increments as
\[
\dot{\theta} = \frac{\Delta \theta}{\Delta t}^{t+\Delta t} + \frac{\Delta \theta}{\Delta t}^{(i)}, \quad (4.51) \\
\dot{\pi} = \frac{\Delta \pi}{\Delta t}^{t+\Delta t} + \frac{\Delta \theta}{\Delta t}^{(i)}, \quad (4.52)
\]

and

\[
\mathbf{v}_s = \frac{\Delta u}{\Delta t}^{t+\Delta t} + \frac{\Delta u}{\Delta t}^{(i)}. \quad (4.53)
\]

Using eqns. (4.24 through 4.26) and (4.51 through 4.53) by neglecting higher order terms, the linearized form of eqns. (4.49) and (4.50) can be obtained as

\[
\int_{\Omega^{t+\Delta t}} (C_1 \frac{\Delta \theta}{\Delta t}^{(i)} + C_2 (\nabla \theta)^t(i) \cdot \nabla \pi(i) + \Delta \theta(i) \cdot \nabla t(i) + c_3 f \cdot \nabla \theta(i)) d\Omega(i) \\
+ c_5 \frac{\Delta \pi(i)}{\Delta t} + 2c_6 \nabla t(i) \cdot \nabla \pi(i) + c_7 f \cdot \nabla \pi(i))v d\Omega(i) \\
+ \int_{\Omega^{t+\Delta t}} C_8 \nabla \cdot \nabla \pi(i) d\Omega(i) + \int_{\Omega^{t+\Delta t}} C_4 \nabla \cdot \mathbf{u}(i) v d\Omega(i) \\
+ \int_{\Omega^{t+\Delta t}} (C_1 \frac{\Delta \theta}{\Delta t}^{(i)} + C_2 \nabla t(i) \cdot \nabla \theta(i) + c_3 f \cdot \nabla \theta(i) + c_4 \nabla \cdot \mathbf{u}(i)) d\Omega(i) \\
+ c_5 \frac{\Delta \pi(i)}{\Delta t} + c_6 \nabla t(i) \cdot \nabla \pi(i) + c_7 f \cdot \nabla \pi(i))v d\Omega(i) \\
- \int_{\Omega^{t+\Delta t}} C_8 \nabla \cdot \mathbf{u}(i) d\Omega(i) - \int_{\Omega^{t+\Delta t}} C_4 \nabla \cdot \mathbf{u}(i) d\Omega(i) \quad (4.54)
\]

and

\[
\int_{\Omega^{t+\Delta t}} (D_1 \frac{\Delta \theta}{\Delta t}^{(i)} + D_2 (\nabla \theta)^t(i) \cdot \nabla \pi(i) + \Delta \theta(i) \cdot \nabla t(i) + D_3 f \cdot \nabla \theta(i)) d\Omega(i) \\
+ D_5 \frac{\Delta \pi(i)}{\Delta t} + 2D_6 \nabla t(i) \cdot \nabla \pi(i) + D_7 f \cdot \nabla \pi(i)) w d\Omega(i) \\
+ \int_{\Omega^{t+\Delta t}} D_8 \nabla \cdot \mathbf{u}(i) d\Omega(i) + \int_{\partial \Omega^{t+\Delta t}} h_c \mathbf{w}(i) d\Omega(i) \quad (4.55)
\]
The mass densities, $\rho_s$ and $\rho_f$ for the isotropic material are updated from eqns. (4.45) and (4.46) as

$$
\rho_s(i+1) = \rho_s^o [1 - 3\alpha_s(t+\Delta t)(\theta(i+\Delta t) - \theta_o) - \frac{1}{3\Delta t} \text{tr} \sigma_s(i+\Delta t)]
$$

(4.56)

and

$$
\rho_f(i+1) = \rho_f^o [1 - 3\alpha_f(t+\Delta t)(\theta(i+\Delta t) - \theta_o) + \frac{1}{\beta_f(i+\Delta t)} \pi(i+\Delta t)].
$$

(4.57)

Using the Galerkin method and associated equations (4.31-4.37), equations (4.54) and (4.55) can be expressed in the discrete form

$$
\begin{bmatrix}
K_{\theta \theta} & K_{\theta \pi} \\
K_{\pi \theta} & K_{\pi \pi}
\end{bmatrix}
\begin{bmatrix}
\Delta u(t+\Delta t) \\
\Delta \pi(t+\Delta t)
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta L_{\theta} \\
\Delta L_{\pi}
\end{bmatrix}
(i) = 0
$$

(4.58)

where

$$
[K_{\theta \theta}] = \int_\Omega \left\{ D_1 \frac{\Delta \theta}{\Delta t} [N_h][N_h]^T + D_2 [N_h][\pi]^T [B_h][B_h]^T \\
+ D_3 [N_h][N_h]^T [F_f][B_h]^T + D_8 [B_h][B_h]^T \right\} d\Omega \\
+ \int_{\partial \Omega} h_c [N_h][N_h]^T dA,
$$

(4.59)
\[
[K_{\theta \sigma}] = \int \Omega \left\{ D_2[N_h][\{ \theta \}]^T[B_h][B_h]^T + \frac{D_5}{\Delta t}[N_h][N_h]^T \\
+ 2D_6[N_h][\pi]^T[B_h][B_h]^T + D_7[N_h][N_h]^T[F_f][B_h]^T \right\} dV,
\]
(4.60)

\[
[k_{\tau u}] = \int \Omega \frac{C_4}{\Delta t}[N_h][B_d] dV,
\]
(4.61)

\[
[K_{\pi \theta}] = \int \Omega \left\{ \frac{C_1}{\Delta t}[N_h][N_h]^T + C_2[N_h][\pi]^T[B_h][B_h]^T \\\n+ C_3[N_h][N_h]^T[F_f][B_h]^T \right\} dV,
\]
(4.62)

\[
[k_{\pi \sigma}] = \int \Omega \left\{ C_1[N_h][\theta]^T[B_h][B_h]^T + \frac{C_5}{\Delta t}[N_h][N_h]^T \\\n+ 2C_6[N_h][\pi]^T[B_h][B_h]^T + C_7[N_h][N_h]^T[F_f][B_h]^T \\\n+ C_8[B_h][B_h]^T \right\} dV,
\]
(4.63)

\[
[\Delta L_\theta] = \int \Omega \left\{ \frac{D_1}{\Delta t}[N_h][N_h]^T[\Delta \theta] + D_2[N_h][\pi]^T[B_h][B_h]^T[\theta] \\\n+ D_3[N_h][N_h]^T[F_f][B_h]^T[\theta] + D_4[N_h][N_h]^T[\pi] \\\n+ D_5[N_h][N_h]^T[F_f][B_h]^T[\pi] \right\} dV + \int_{\partial \Omega} \{ [N_h] \{ q + h(N_h)^T([\theta] - [\theta_c]) \} \} dA,
\]
(4.64)

and

\[
[\Delta L_\sigma] = \int \Omega \left\{ \frac{C_1}{\Delta t}[N_h][N_h]^T[\Delta \theta] + C_2[N_h][\pi]^T[B_h][B_h]^T[\theta] \\\n+ C_3[N_h][N_h]^T[F_f][B_h]^T[\theta] + \frac{C_4}{\Delta t}[N_h][B_d][\Delta u] \\\n+ \frac{C_5}{\Delta t}[N_h][N_h]^T[\Delta \pi] + C_6[N_h][\pi]^T[B_h][B]^T[\pi] \\\n+ C_7[N_h][N_h]^T[F_f][B_h]^T[\pi] + C_8[B_h][B_h]^T[\pi] \\\n- \phi C_8[B_h][F_f][N_h]^T \right\}
\]
(4.65)

The two-dimensional gradient and body force matrices above
are defined by

\[ [B_h] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y}
\end{bmatrix}, \tag{4.66}
\]

\[ [B_d] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y}
\end{bmatrix},
\]

and

\[ [F_f] = \begin{bmatrix}
f_{x_1} & f_{x_2} & f_{x_3} & f_{x_4} \\
f_{y_1} & f_{y_2} & f_{y_3} & f_{y_4}
\end{bmatrix}. \tag{4.67}\]

4.3 Solution Procedure

The coupled algebraic system equations, in terms of the unknown increment and residue vectors, are obtained from eqns. (4.38) and (4.58) as

\[ [K_T]^{t+\Delta t} [\Delta U]^{(i)} = [\Delta L]^{t+\Delta t} \tag{4.68} \]

where

\[ [K_T]^{t+\Delta t}^{(i)} = \begin{bmatrix}
K_{uu} & K_{u\theta} & K_{u\pi} \\
K_{\theta u} & K_{\theta\theta} & K_{\theta\pi} \\
K_{\pi u} & K_{\pi\theta} & K_{\pi\pi}
\end{bmatrix}, \tag{4.69}\]

\[ [\Delta U]^{t+\Delta t}^{(i)} = \begin{bmatrix}
\Delta u \\
\Delta \theta \\
\Delta \pi
\end{bmatrix}, \tag{4.70}\]

and

\[ [\Delta L]^{t+\Delta t}^{(i)} = \begin{bmatrix}
\Delta L_u \\
\Delta L_\theta \\
\Delta L_\pi
\end{bmatrix}. \tag{4.71}\]

It is noteworthy that the sub-matrices of the tangential
stiffness, containing material and geometrical nonlinearities, are generally unsymmetric. The convection terms in the governing differential equations (2.77) and (2.78) are related to the submatrices $K_{\pi\pi}$, $K_{\pi\theta}$, $K_{\theta\pi}$, and $K_{\theta\theta}$.

Solving for the unknown increment, by inserting eqn. (4.68), the total solution increment during $\Delta t$ after the $(i)^{th}$ iteration is updated by the relation

$$[\Delta U]_{(i+1)}^{t+\Delta t} = [\Delta U]_{(i)}^{t+\Delta t} + [\Delta U]_{(i)}.$$  \hspace{1cm} (4.72)

Similarly, the mass densities are updated using eqns. (4.56) and (4.57). The solutions are then updated from eqns. (4.24 - 4.26) and the geometry of the mesh configuration is revised to represent the reference configuration for the next time step, i.e.,

$$[x]_{(i+1)}^{t+\Delta t} = [x]_{(i)}^{t+\Delta t} + [\Delta u]_{(i)}.$$  \hspace{1cm} (4.73)

where $[x]$ is the nodal coordinate vector. The strain increment during $\Delta t$ after the $(i+1)^{th}$ iteration, $\Delta e_{(i+1)}^{t+\Delta t}$, is evaluated corresponding to the incremental displacement $\Delta u_{(i+1)}^{t+\Delta t}$ defined by

$$\Delta u_{(i+1)}^{t+\Delta t} = \Delta u_{(i)}^{t+\Delta t} + \Delta u_{(i)}.$$  \hspace{1cm} (4.74)

The incremental displacements, $\Delta u_{(i+1)}^{t+\Delta t}$, are divided into 'n' equal parts and the corresponding strain is obtained from eqn. (4.14). The total strain increment is obtained by adding the subincremental strains [33].
where
\[
\Delta u(i) = \frac{1}{n} \Delta u^t + \Delta t_{(i+1)}, \quad \text{and} \quad \Delta x_k = x^t + k\Delta u(i).
\] (4.76)
The total stress increment, \(\Delta \sigma^t + \Delta t\), corresponding to the total strain increment \(\Delta e^t + \Delta t\), is evaluated using eqn. (3.25):
\[
\Delta \sigma^t + \Delta t_{(i+1)} = C_{ep} \Delta e^t_{(i+1)} + \Delta e^t_{(i+1)} - \Delta t_{(i+1)} + \int_{t}^{t+\Delta t} (\sigma w - w\sigma) dt
\] (4.77)
where
\[
\Delta e^t_{(i+1)} = \beta I \Delta \sigma^t + \Delta t_{(i+1)}^t + \frac{1-2\nu}{E} \Delta \pi + \Delta t_{(i+1)},
\] (4.78)
\[
P_{vp} = \gamma <\phi(F)> \frac{\partial \phi}{\partial \sigma},
\] (4.79)
and
\[
\int_{t}^{t+\Delta t} (\sigma w - w\sigma) dt = \sum_{k=1}^{n} \left\{ \sigma \left[ \frac{\partial \Delta u(i)}{\partial x_h} - \left( \frac{\partial \Delta u(i)}{\partial x_h} \right)^T \right] \right\} - \frac{1}{2} \left[ \left( \frac{\partial \Delta u(i)}{\partial x_h} \right) - \left( \frac{\partial \Delta u(i)}{\partial x_h} \right)^T \right] \sigma.
\] (4.80)
The total strain, \(e^t + \Delta t_{(i+1)}\), and total stress \(\sigma^t + \Delta t\) are updated from the expressions
\[
e^t + \Delta t_{(i+1)} = e^t + \Delta e^t + \Delta t_{(i+1)}
\] (4.81)
\[
\sigma^t + \Delta t_{(i+1)} = \sigma^t + \Delta \sigma^t + \Delta t_{(i+1)}.
\] (4.82)
The iteration is repeated until the solution converges.
After the incremental solutions are obtained, the mesh geometry, associated stresses, and yield conditions are
revised to characterize the reference configuration and
flow rule for the next time step. A summary of the
described solution algorithms is given in Table 1. The
following three criteria, namely, the displacement, force,
and energy convergence criteria, are employed to terminate
the iteration [31]:
\[ || \Delta u_{(i)} \cdot \Delta u_{(i)} || \leq \epsilon_u || \Delta u_{(i+1)} \cdot \Delta u_{(i+1)} || \] (4.83)
\[ || \Delta L_{(i)}^{t+\Delta t} \cdot \Delta L_{(i)}^{t+\Delta t} || \leq \epsilon_L(SREF) \] (4.84)
\[ || \Delta L_{(i)}^{t+\Delta t} \cdot \Delta u_{(i)} || \leq \epsilon_E(EREF) \] (4.85)
where
\( \epsilon_u, \epsilon_L, \epsilon_E \) are the user-defined displacement, load,
energy convergence tolerances, respectively.
SREF is the norm of the incremental support reaction
after the \((i)\)th iteration
and
EREF is the work done by the incremental external
forces during the \((i+1)\)st iteration.
Table 1. Computational Procedure Summary

Given : \( \theta_0, \pi_0, \sigma_0, t, \text{FACT} \).

**INCREMENTAL LOADING LOOP**

1. FOR \( N=1, 2, \ldots, \ldots, \text{NSTEP} \), \( \text{SET} \ N = 1 \)
2. Calculate incremental external loading \( t = t \times \text{FACT}(N) \)
3. Set start \( \text{TIME} \ 1(N) \) and end \( \text{TIME} \ 2(N) \) for the \( N^{th} \) step
4. Set time increment \( \Delta t \) for the \( N^{th} \) step.
5. Set convergence tolerance \( \epsilon \) and convergence criterion
6. IF \( N > \text{NSTEP} : \text{STOP} \)

**INCREMENTAL TIME STEP LOOP**

FOR \( n=1, 2, \ldots \ldots \)

Assumption : All temperatures, pressures, stresses, strains, viscoplastic strain rates are known at time \( t = (n-1) \times \Delta t \) (time step \( n-1 \))

The solution for the temperature, pressure and displacement field at time \( t = n\Delta t \) (i.e time step \( n \)) is obtained as follows

7. Set current time \( t = n \times \Delta t \)
8. If the current time is greater than TIME 2 (N), GO TO STEP 2 with N=N+1

9. Initialize temperature, pressure, stress, strain, viscoplastic strain rate, and coordinate components with the previous step (i.e. time step n-1) valve set iteration count i=1.

ITERATION LOOP

FOR i=1, 2, . . . . NITER, we

10. Compute tangential stiffness components, $K_{uu}$, $K_{u\theta}$, $K_{u\pi}$, $K_{\theta\theta}$, $K_{\pi\pi}$ from equations (4.39-4.41) and (4.59-4.63).

11. Set up the tangent stiffness matrix from equation (4.69).

12. Compute unbalance force components, $\Delta L_u$, $\Delta L_{\theta}$ and $\Delta L_{\pi}$, from eqns. (4.42), (4.64) and (4.65).

13. Set up the unbalance force vector from equation (4.71).

14. Set up system of equations (4.68) in partitioned form

$$
\begin{bmatrix}
K_{uu} & K_{u\theta} & K_{u\pi} \\
0 & K_{\theta\theta} & K_{\pi\pi} \\
K_{\pi u} & K_{\pi \theta} & K_{\pi \pi}
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta \theta \\
\Delta \pi
\end{bmatrix}
=
\begin{bmatrix}
\Delta L_u \\
\Delta L_{\theta} \\
\Delta L_{\pi}
\end{bmatrix}
$$

15. Solve eqn. (4.68) for the incremental unknowns.

16. Update $u^{t+\Delta t}_{(i+1)}$, $\theta^{t+\Delta t}_{(i+1)}$ and $\pi^{t+\Delta t}_{(i+1)}$ from eqns. (4.24-4.26).
17. Evaluate strain increment, $\Delta \varepsilon_{t+\Delta t}^{(i+1)}$ along with eqns. (4.74), (4.75) and (4.76)

18. Compute stress increment, $\Delta \sigma_{t+\Delta t}^{(i+1)}$, by eqn. (4.77) with aid of eqns. (4.78) through (4.80).

19. Check for the consistency conditions
   (1) Calculate total Cauchy stress $\sigma_{t+\Delta t}^{(i+1)}$ from eqn. (4.82).
   (2) Calculate the effective stress, $F$ (i.e. $\sigma_{t+\Delta t}^{(i+1)}$) from eqn. (3.7).
   (3) Update yield force from eqn. (3.8) and check for yield.
      $F - Y \leq 0$ : Go to step 20.
      $F - Y > 0$ : Continue to step (4)
   (4) Calculate the followings if $F - Y > 0$
      (a) $\Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)} = \Delta t \gamma (\frac{F}{Y} - 1)^N b$
      (b) $\Delta \sigma_{t+\Delta t}^{(i+1)}$, using eqn. (4.77)
      (c) $\Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)} = \sqrt{2(\Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)}, \Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)})/3}$
      (d) $\sigma_{t+\Delta t}^{(i+1)} = \sigma_t + \Delta \sigma_{t+\Delta t}^{(i+1)}$
      (e) $\varepsilon_{vp}^{t+\Delta t}^{(i+1)} = \varepsilon_{vp}^t + \Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)}$
      (f) $\gamma_{t+\Delta t}^{(i+1)} = \gamma_t + \Delta \varepsilon_{vp}^{t+\Delta t}^{(i+1)}$

20. Update coordinate system using
    $x_{t+\Delta t}^{(i+1)} = x_{t+\Delta t}^{(i)} + \Delta u_{(i)}$

21. Check if convergence is reached after the $(i+1)^{st}$ iteration from eqns. (4.83-4.85).
If convergence is obtained, go to step 6 with n=n+1
If convergence is not obtained, go to step 9 with
i=i+1
The iteration algorithm above corresponds to a full
Newton-Raphson iteration.
CHAPTER V

NUMERICAL EXPERIMENTS

Before applying the developed two-dimensional computer program to field problems, some simple examples are investigated in this chapter to validate the model. The first four examples deal with thermoviscoplastic problems using Perzyna's thermo-elasto-viscoplastic model and infinitesimal deformation theory. The next two examples investigate one-dimensional consolidation behavior of a column, including fluid and thermal diffusion and thermoviscoelastic solid deformation, using infinitesimal deformation theory. The following two examples study the finite deformation responses of a hypoelastic solid for simple shear and simple tension cases. The final example in this chapter evaluates the one-dimensional consolidation responses associated with finite deformation theory. The developed computer code is implemented within an inhouse computer program HYFIDE [120]. The numerical results obtained using the HYFIDE program on the Cray-YMP computer
system are compared with available analytical and numerical results.

5.1 Thermo-Viscoplastic Test Problems

Viscoplastic responses of a one-dimensional bar under simple tension, both in the absence of hardening and with linear strain hardening, are obtained using Perzyna's model. The flow rule given by eqn. (3.4) along with $\phi(F) = F - Y$ where $F$ and $Y$ are the effective stress and yield stress, respectively are assumed. The results are compared with corresponding analytical solutions to validate the present formulation incorporating Perzyna's model. The responses of an internally pressurized cylinder under thermomechanical incremental loading are also subsequently evaluated and compared with available numerical results. The infinitesimal deformation theory is utilized for these problems for comparison with other analytical and reported solutions.

5.1.1 One-Dimensional Viscoplastic Response of a Perfectly Viscoplastic Column (TEST-A)

A one-dimensional perfectly viscoplastic model (TEST-A) is analyzed with a traction loading of $p=15$ units, as shown in Fig. 6. An incremental load just below the yield stress is initially applied and it is increased, as
Fig. 6 Surface Displacement History for TEST-A (Infinitesimal Theory with No Strain Hardening)
shown in Table 2. The stresses and displacements are initially assumed to be zero. An analytical solution for the model response is given by [89]

\[
\frac{u}{l} = \frac{p}{E} + \gamma(p - Y)t
\]

(5.1)

where:

- \( u \) is the end displacement
- \( l \) is the original length
- \( p \) is the constant applied load
- \( Y \) is the yield stress
- \( E \) is the Young's modulus
- \( \gamma \) is the fluidity parameter.

The homogeneous viscoplastic material properties used in TEST-A and the finite element model, and boundary conditions are also shown in Fig. 6. The obtained plane stress results, using the fully implicit time marching scheme, are plotted with the analytical solution in Fig. 6. For given material properties, the viscoplastic deformation continues indefinitely with a slope of 0.05. The presented finite element results agree almost exactly with the analytical solution.
Table 2  Incremental Loading and Time Step for 
TEST-A

<table>
<thead>
<tr>
<th>Step No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

| 9.0 | 0.1 |
| 15.0| 0.1 |

5.1.2  **One-Dimensional Viscoplastic Response of a Linear 
Strain Hardening Column (TEST-B)**

Strain hardening effects are investigated in this 
example (TEST-B). The geometry and material data are the 
same as in TEST-A except for the hardening parameter 
$H=5,000$ units in eqn. (3.8). The same incremental loading 
steps used in TEST-A are applied. The analytical solution 
has the form [89]:

$$\frac{u}{L} = \frac{p}{E} + \frac{(p-y)}{H} \left[ 1 - e^{-H\gamma t} \right]$$  \hspace{1cm} (5.2)

Numerical solutions are obtained using both the explicit 
and implicit time stepping schemes. For the explicit time 
stepping scheme, the limiting time-step lengths given by 
Cormeau [41] are as follows:

$$\Delta t \leq \frac{(1+\nu)Y}{\gamma E} \quad \text{for the Tresca assumption}$$  \hspace{1cm} (5.3)

$$\Delta t \leq \frac{4(1+\nu)Y}{3\gamma E} \quad \text{for the Von Mises assumption}$$  \hspace{1cm} (5.4)
\[ \Delta t \leq \frac{4(1+\nu)(1-\nu)Y}{\gamma(1-2\nu+s\sin\varphi)E} \]

for the Mohr-Coulomb assumption (5.5)

where \( \varphi \) is the angle of internal friction, the quantity \( Y \) is the uniaxial yield stress for the Tresca and Von Mises solids. The latter value is equivalent to \( C\cos\varphi \) for the Mohr-Coulomb material, \( C \) being the cohesion value.

The limiting time step for the specified material properties is chosen to be 0.5 for the explicit method. For a strain hardening material, the viscoplastic flow reaches a steady state at which state the viscoplastic strain rate becomes zero. The steady state viscoplastic strain is computed to be 0.00252, in comparison to 0.00250 obtained from eqn. (5.2). The numerical results show that the implicit time stepping scheme reaches a steady state faster than the explicit method. Both methods show good agreement with the analytical solutions as shown in Fig. 7.

5.1.3 Elasto-Viscoplastic Thick Cylinder Subjected to an Internal Pressure (TEST-C)

An elasto-viscoplastic thick cylinder, subjected to a gradually increasing internal pressure, is considered next. The inner and outer radii of the cylinder are 100 and 200 [mm], respectively. The selected mesh and boundary conditions are illustrated in Fig. 8. The viscoplastic flow rule is governed by eqns. (3.4) through (3.9). The
Fig. 7 Surface Displacement History for TEST-B
(Infinitesimal Theory with Linear Strain Hardening)
Fig. 8 FEM Model and Material Properties for TEST-C
selected material properties are:

\[ E = 2.1 \times 10^4 \, \text{[N/mm}^2\text{]} \]
\[ Y = 24 \, \text{[N/mm}^2\text{]} \]
\[ H = 0.0 \, \text{[N/mm}^2\text{]} \]
\[ \nu = 0.3 \]
\[ r_i = 100 \, \text{[mm]} \]
\[ r_o = 100 \, \text{[mm]} \]
\[ \gamma = 0.001 \, \text{[day}^{-1}\text{]} \]

For both plane strain and axisymmetric conditions, an implicit time stepping scheme is utilized with the incremental loading steps and time increment shown in Table 3.

<table>
<thead>
<tr>
<th>Step No.</th>
<th>( p(\text{N/mm}^2) )</th>
<th>Time Increment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.0</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>14.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
The radial displacement solutions at $r=0$ are compared with available numerical results given by Owen and Hinton [89]. The results, illustrated in Fig. 9., are in good agreement. The steady state hoop stress distributions along the radial direction are plotted in Fig. 10 and compared with the results given by Owen and Hinton [89]. Again, favorable agreement with Owen and Hinton's results is obtained for both plane strain and axisymmetric conditions despite a 0.8% deviation between axisymmetric and plane strain solutions. The results from TEST-A, TEST-B and TEST-C validate the effective use of Perzyna's viscoplastic model with isotropic hardening for both two-dimensional and axisymmetric problem.

5.1.4 Quenching of a Pressurized Cylinder (TEST-D)

The thermomechanical responses of a thick-walled cylinder which is initially pressurized ($p=200$ N/mm$^2$) and subsequently quenched from initial temperature 320°C down to 20°C are investigated in this section. The quenching of the inside and outside walls are modeled differently. The outside wall is cooled by a coolant liquid ($h_w=1.16(10^3)$ [J/m$^2$s°C]) at a temperature of 20°C. The inside wall is cooled by the surrounding air ($h_a=10$ [J/m$^2$s°C]) with a temperature of 20°C but at a slower rate due to the large difference between the heat convection coefficients. The
Fig. 9 Radial Displacement History at $r = 0$ for TEST-C
Fig. 10  Steady State Hoop Stress Distribution along Radial Direction for TEST-C
finite element model and the assumed axisymmetric boundary conditions are shown in Fig. 11. The temperature dependent mechanical properties shown in Fig. 12 are linearized in a piecewise fashion for each temperature interval, as shown in Table 4. The selected incremental thermomechanical loading steps and corresponding time increments for the implicit time-stepping scheme are shown in Table 5. The viscoplastic strain rate is again controlled by eqns. (3.4) through (3.9) with \( N = 1 \) and the fluidity coefficient \( \gamma = 1.0 \times 10^{-4} / \text{sec} \). The temperature and effective strain dependent strain hardening behavior (Fig. 13) is simplified by the relation

\[
Y = 175 + 264 (\bar{\varepsilon}_{vp})^{0.25} \quad [\text{N/mm}^2] \tag{5.6}
\]

In equation (5.6), the hardening rule depends only on the effective strain and it represents the hardening curve at 100°C in Fig. 13. The temperature distribution along the radial direction is plotted in Fig. 14 and is in excellent agreement with the results given by Argyris et al. [6]. From Figs. 15 and 16, it is observed that the outside wall cools steadily down until 1,000 sec., while the inside wall is suddenly cooled at 100 sec. Good agreement between the results from the present method and those of Argyris et al. are evident. From Figs. 17 and 18, it is observed that the displacement histories obtained from the present FEM model also compare well with the results reported by Argyris et
Fig. 11 FEM Model for Quenching of Pressurized Cylinder for TEST-D
Fig. 12 Temperature Dependent Material Properties for TEST-D (from [6])
Table 4 Material Properties of the Reactor Steel

<table>
<thead>
<tr>
<th>$\theta$ (°C)</th>
<th>k (W/mm°K)</th>
<th>$\rho c$ (J/mm°K)</th>
<th>$\nu$</th>
<th>$E \times 10^5$ (N/mm²)</th>
<th>$\alpha \times 10^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.04455</td>
<td>0.00342</td>
<td>0.280</td>
<td>2.133</td>
<td>11.640</td>
</tr>
<tr>
<td>50.0</td>
<td>0.04450</td>
<td>0.00343</td>
<td>0.283</td>
<td>2.096</td>
<td>11.902</td>
</tr>
<tr>
<td>100.0</td>
<td>0.04437</td>
<td>0.00351</td>
<td>0.286</td>
<td>2.059</td>
<td>12.164</td>
</tr>
<tr>
<td>150.0</td>
<td>0.04367</td>
<td>0.00361</td>
<td>0.289</td>
<td>2.022</td>
<td>12.426</td>
</tr>
<tr>
<td>200.0</td>
<td>0.04316</td>
<td>0.00370</td>
<td>0.292</td>
<td>1.985</td>
<td>12.688</td>
</tr>
<tr>
<td>250.0</td>
<td>0.04245</td>
<td>0.00377</td>
<td>0.295</td>
<td>1.948</td>
<td>12.910</td>
</tr>
<tr>
<td>300.0</td>
<td>0.04128</td>
<td>0.00384</td>
<td>0.298</td>
<td>1.911</td>
<td>12.212</td>
</tr>
<tr>
<td>350.0</td>
<td>0.04011</td>
<td>0.00391</td>
<td>0.301</td>
<td>1.874</td>
<td>13.474</td>
</tr>
<tr>
<td>400.0</td>
<td>0.03894</td>
<td>0.00397</td>
<td>0.304</td>
<td>1.837</td>
<td>13.736</td>
</tr>
<tr>
<td>450.0</td>
<td>0.03785</td>
<td>0.00405</td>
<td>0.307</td>
<td>1.800</td>
<td>13.998</td>
</tr>
<tr>
<td>500.0</td>
<td>0.03645</td>
<td>0.00417</td>
<td>0.310</td>
<td>1.763</td>
<td>14.260</td>
</tr>
</tbody>
</table>
### Table 5 Incremental Loading and Time Step for TEST-D

<table>
<thead>
<tr>
<th>Step No.</th>
<th>Time Interval</th>
<th>Time Increment</th>
<th>p(N/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0-0.1</td>
<td>0.01</td>
<td>200.0</td>
</tr>
<tr>
<td>2</td>
<td>0.1-1.0</td>
<td>0.10</td>
<td>200.0</td>
</tr>
<tr>
<td>3</td>
<td>1.0-30.0</td>
<td>1.00</td>
<td>200.0</td>
</tr>
<tr>
<td>4</td>
<td>30.0-400.0</td>
<td>10.00</td>
<td>200.0</td>
</tr>
<tr>
<td>5</td>
<td>400.0-1100.0</td>
<td>100.00</td>
<td>200.0</td>
</tr>
<tr>
<td>6</td>
<td>1100.0-11100.0</td>
<td>500.00</td>
<td>200.0</td>
</tr>
<tr>
<td>7</td>
<td>11100.0-111100.0</td>
<td>5000.00</td>
<td>200.0</td>
</tr>
</tbody>
</table>
Fig. 13 Temperature Dependent Strain Hardening Properties for TEST-D (from [6])
Fig. 14 Temperature Distribution along Radial Direction for TEST-D
Fig. 15 Temperature History at the Inside Wall for TEST-D
Fig. 16 Temperature History at the Outside Wall for TEST-D
Fig. 17 Displacement History at the Inside Wall for TEST-D
Fig. 18 Displacement History at the Outside Wall for TEST-D
et [6] for the elastic case. The difference between the two models for the viscoplastic case arises from the simplified form of the hardening law in the present formulation. However, the trends of displacement histories of the two models match well. The displacement trends for the two models near 1000 sec. are similar because the temperature of the cylinder at 1000 sec. is 70°C and the hardening law is applicable at this temperature. The thermo-viscoplastic flow rule using Perzyna's model for TEST-D with the infinitesimal theory is shown to give reasonable results.

5.2 One-Dimensional Consolidation Problem

The response of a one-dimensional fluid-saturated column under uniform loading is investigated in this section. Solutions for the one-dimensional consolidation behavior of soils have been obtained by Terzaghi [105]. Finite element method evaluation of this model has been conducted by Sandhu et al. [98] and its thermomechanically coupled response has been investigated by Aboustit et al. [1] and Tseng [107]. No analytical solutions for the one-dimensional consolidation of a fluid-saturated viscoplastic porous medium are apparently available. The computed fluid pressure, \( p \), represents the excess pore pressure due to the thermomechanical loading because an
equilibrium configuration is assumed prior to the loading. All examples are simulated using plane strain conditions.

5.2.1 One-Dimensional Elastic Consolidation (TEST-E)

The finite element model discretization and assumed boundary conditions are shown in Fig. 19. The initial conditions for the displacements and pore pressures are taken to be zero everywhere. The homogeneous, isotropic material properties used for this study, in dimensionless form, are

- elastic modulus \( E = 6,000 \)
- Poisson's ratio \( \nu = 0.4 \)
- specific permeability \( \kappa = 4 \times 10^{-11} \)
- porosity \( \phi = 0.5 \)
- fluid viscosity \( \mu = 2 \times 10^{-5} \)
- heat conductivity \( k_s = 0.3, \ k_f = 0.1 \)
- heat capacity \( C_s = 0.1, \ C_f = 0.1 \)
- density \( \rho_s = 400, \ \rho_f = 0.0 \)

where subscripts s and f denote the solid and fluid constituents, respectively. The coefficient of permeability, \( K \), defined by \( K = \frac{\kappa}{\phi \mu} \) and the coefficient of consolidation, \( C_V \), defined by \( C_V = K \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \), corresponding to given material properties are taken to be \( C_V = 0.051428 \) and \( K = 4.0 \times 10^{-6} \). The normalized settlement history \( (u/u_0) \) and excess pore pressure behavior \( (\pi/p) \) are
Fig. 19 FEM Model for One-Dimensional Consolidation for TEST-E
plotted with respect to the normalized time $\tau$ defined by $\tau = C_v t/h^2$, where $u_\infty$, $p$, $t$ and $h$ are the steady state settlement value, applied load, actual elapsed time and height of the column, respectively.

In Figs. 20 and 21, the settlement at the top of the column and pore pressure at $y=6.8$, obtained from the present formulation, are compared with the analytic solution using Terzaghi's theory under infinitesimal deformation. Good conformance between the finite element and analytical results is obtained. The differences in the pore pressure values for the FEM model and analytical solution is due to the enforcement of the fluid flux condition at the top end.

5.2.2 One-Dimensional Thermoelastic Consolidation (TEST-F)

One-dimensional thermoelastic consolidation is studied in this example for infinitesimal deformations. The geometry, boundary conditions, initial conditions, chosen in this study, are identical to those in the FEM model studied by Aboustit et al. [1]. A traction load $p=1$ and the thermal load, $\theta=50$, are applied at the top end. Insulated boundary conditions are imposed, as shown in Fig. 19. All the initial conditions are again assumed to be zero everywhere. The results obtained using the present formulation are plotted with respect to log(t). In Figs.
Fig. 20 Surface Settlement History for TEST-E
Fig. 21 Pore Pressure History at y = 6.8 for TEST-E
22 and 23 temperatures and pore pressure history at the lower end of the column are compared with the results studied by Tseng [107] and the displacements results are compared with the results presented by Aboustit et al. [1], as shown in Fig. 24. Because the pore pressure reaches the steady state at \( t=1000 \), the settlement of the top end decreases even though temperature is still increasing. Good overall agreements are evident in Fig. 24.

5.3 Finite Deformation Test Problems

Deformations of a hypoelastic solid under simple shear and simple tension are studied in this section to validate the finite deformation theory formulations. The constitutive equation of the hypoelastic material is governed by

\[
\dot{T} = \lambda \text{tr}(D)I + 2GD
\]  

(5.7)

where \( \dot{T} \) and \( D \) are the objective stress rate and rate of deformation tensor, respectively and \( \lambda \) and \( G \) are Lame's constants. The Jaumann stress rate is chosen as the objective stress rate in this study. One-dimensional hypoelastic consolidation of under finite deformation is solved and the results are compared with those from infinitesimal theory.
Fig. 22 Pore Pressure History at the Bottom for TEST-F
Fig. 23 Temperature History at the Bottom for TEST-F
Fig. 24 Surface Settlement History at the Top for TEST-F
5.3.1 Simple Tension Test (TEST-G)

A one-dimensional hypoelastic column subjected to simple tension by prescribing boundary displacements is studied. Because of the problem symmetry, a quadrant of the specimen is analyzed. The finite element mesh, boundary conditions and material data are shown in Fig. 25. An analytical solution for this test problem, given by Johnson and Bammann [63], is detailed in APPENDIX D.2. The axial stress is given by

\[ \sigma_{xx} = (\lambda + 2G - \frac{\lambda^2}{\lambda+2G}) \ln(1+k(t)) \]  

where \( k \) is the stretch ratio. Excellent agreement between the developed FEM solutions and analytical results is observed, as shown in Fig. 25.

5.3.2 Simple Shear Test (TEST-H)

The finite element model for simple shear by prescribing angular displacements is shown in Fig. 26. The hypoelastic solid is chosen for the finite shear deformations and the analytical solutions, derived in APPENDIX D.1, are given by

\[ \sigma_{xx} = -\sigma_{yy} = G(1 - \cos k) \]  
\[ \sigma_{xy} = G \sin k \]

where \( k \) is the shear deformation and \( G \) is shear modulus. Excellent agreement between the analytical and finite element results is evidenced as shown in Fig. 26.
Fig. 25 $\sigma_{xx}$ versus Stretch Ratio $k$ under Simple Tension for TEST-G
Fig. 26  Cauchy Stress versus Shear Deformation $k$ under Finite Simple Shear for TEST-H
5.3.3 Finite Deformation of One-Dimensional Consolidation (TEST-I)

The finite deformation of a fluid-saturated one-dimensional hypoelastic column is studied here. The geometry, boundary conditions, and initial conditions are the same as those in TEST-E. In order to obtain the quasi-static finite deformation responses, the applied tractions acting at the top surface are rapidly increased up to 10,000 units (Table 6) by ignoring the inertia effect. The settlement, obtained from the finite deformation theory, at steady state is smaller than that obtained from the infinitesimal theory as shown in Fig. 27. This is due to the definition of finite strain and can be explained from energy arguments. The pore pressure histories are also compared in Fig. 28. The pore pressure obtained from large deformation theory reaches the steady state faster than that computed from the infinitesimal theory.

5.4 Summary of Numerical Experiments

The developed FEM methodologies and its computer code HYFIDE are validated for several simple examples as shown in Table 7. In addition, constitutive model sensitivity has been investigated for one-dimensional consolidation
Table 6 Incremental Loading and Time Step for TEST-I

<table>
<thead>
<tr>
<th>Step No.</th>
<th>Time Interval</th>
<th>Time Increment</th>
<th>p(N/mm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0-0.01</td>
<td>0.01</td>
<td>1000</td>
</tr>
<tr>
<td>2</td>
<td>0.01-0.02</td>
<td>0.01</td>
<td>2000</td>
</tr>
<tr>
<td>3</td>
<td>0.02-0.03</td>
<td>0.01</td>
<td>3000</td>
</tr>
<tr>
<td>4</td>
<td>0.03-0.04</td>
<td>0.01</td>
<td>4000</td>
</tr>
<tr>
<td>5</td>
<td>0.04-0.05</td>
<td>0.01</td>
<td>5000</td>
</tr>
<tr>
<td>6</td>
<td>0.05-0.05</td>
<td>0.01</td>
<td>6000</td>
</tr>
<tr>
<td>7</td>
<td>0.06-0.07</td>
<td>0.01</td>
<td>7000</td>
</tr>
<tr>
<td>8</td>
<td>0.07-0.08</td>
<td>0.01</td>
<td>8000</td>
</tr>
<tr>
<td>9</td>
<td>0.08-0.09</td>
<td>0.01</td>
<td>9000</td>
</tr>
<tr>
<td>10</td>
<td>0.09-0.10</td>
<td>0.01</td>
<td>10000</td>
</tr>
<tr>
<td>11</td>
<td>0.10-1.00</td>
<td>0.10</td>
<td>10000</td>
</tr>
<tr>
<td>12</td>
<td>1.0-10.0</td>
<td>1.00</td>
<td>10000</td>
</tr>
<tr>
<td>13</td>
<td>10.0-100.0</td>
<td>10.00</td>
<td>10000</td>
</tr>
<tr>
<td>14</td>
<td>100.0-1000.0</td>
<td>50.00</td>
<td>10000</td>
</tr>
<tr>
<td>15</td>
<td>1000.0-10000.0</td>
<td>500.00</td>
<td>10000</td>
</tr>
</tbody>
</table>
Fig. 27 Finite Settlement History at the Top End for TEST-I
Fig. 28 Pore Pressure History at $y = 6.8$ for TEST-I
Table 7 Benchmark Model Comparisons

<table>
<thead>
<tr>
<th>Benchmark Model</th>
<th>Model Characteristics</th>
<th>Result Comparisons</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>one-dimensional viscoplastic column (TEST-A)</td>
<td>Perzyna's model (H=0) infinitesimal theory</td>
<td>Fig. 6</td>
<td>excellent agreement</td>
</tr>
<tr>
<td>one-dimensional viscoplastic column (TEST-B)</td>
<td>Perzyna's model (H=5000) infinitesimal theory</td>
<td>Fig. 7</td>
<td>good agreement</td>
</tr>
<tr>
<td>elasto-viscoplastic thick cylinder (TEST-C)</td>
<td>Perzyna's model no strain hardening infinitesimal theory</td>
<td>Figs. 9-10</td>
<td>good agreement</td>
</tr>
<tr>
<td>Aryris model [thermoviscoplastic] (TEST-D)</td>
<td>Perzyna's flow rule isotropic hardening infinitesimal theory</td>
<td>Figs. 14-18</td>
<td>good agreement</td>
</tr>
<tr>
<td>Terzaghi model [one-dimensional consolidation] (TEST-E)</td>
<td>elastic mixture no thermal loading infinitesimal theory</td>
<td>Figs. 20-21</td>
<td>excellent agreement</td>
</tr>
<tr>
<td>Aboustit model [one-dimensional consolidation] (TEST-F)</td>
<td>elastic mixture thermal load(θ=50) infinitesimal theory</td>
<td>Figs. 22-24</td>
<td>excellent agreement</td>
</tr>
<tr>
<td>Finite Simple tension (TEST-G)</td>
<td>hypoelastic column finite deformation</td>
<td>Fig. 25</td>
<td>excellent agreement</td>
</tr>
<tr>
<td>Finite simple shear (TEST-H)</td>
<td>hypoelastic body finite deformation</td>
<td>Fig. 26</td>
<td>excellent agreement</td>
</tr>
<tr>
<td>Finite consolidation (TEST-I)</td>
<td>hypoelastic column finite deformation no thermal loading</td>
<td>Figs. 27-28</td>
<td>N.A</td>
</tr>
</tbody>
</table>
problem as shown in Table 8. The surface settlement history $u/pu^0$, with $p$ being the applied traction and $u^0$ being the steady state surface settlement of MODEL 1 (Table 8), is illustrated in Fig. 29.
### Table 8  One-Dimensional Consolidation Model Comparisons

<table>
<thead>
<tr>
<th>Model Characteristics</th>
<th>Model Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>MODEL 1  elastic, small deformation</td>
<td>TEST-E</td>
</tr>
<tr>
<td>MODEL 2  elastic, large deformation</td>
<td>TEST-I</td>
</tr>
<tr>
<td>MODEL 3  elastic, thermal loading</td>
<td>TEST-F</td>
</tr>
<tr>
<td>MODEL 4  viscoplastic, small deformation</td>
<td>TEST-I and Y=120, H=0</td>
</tr>
<tr>
<td>MODEL 5  viscoplastic, large deformation</td>
<td>TEST-I and Y=120, H=0</td>
</tr>
</tbody>
</table>
Fig. 29 Surface Settlement versus Time Comparisons for One-Dimensional Consolidation
Two-dimensional numerical simulations of nonlinear geomechanical problems, using the developed finite element methodology, are presented in this chapter. The long-term thermomechanical responses, assuming temperature dependent geological material properties and ignoring tectonic stresses, are investigated for two geotechnical applications. First, the consolidation of fluid-infiltrated elastic porous media under strip loading is investigated using finite deformation theory. The second field application deals with a nuclear waste repository model in a low permeability salt bed.

6.1 Plane Strain Consolidation Model

Plane strain consolidation responses are analyzed for a uniform strip loading [23] and rigid, permeable footing [29] resting on the surface of a saturated elastic medium, as shown in Figs. 30 (a) and (b), respectively. The length dimension, 6h, is chosen to be large enough to
(a) Traction Loading

(b) Rigid Footing

FIG. 30 FEM Model for Plane Strain Consolidation
approximately satisfy the boundary conditions. The finite element mesh and the boundary conditions used for this study are also shown in Fig. 30. Material properties identical to those used in TEST-E are assumed with zero Poisson's ratio. As shown in Fig. 31, the numerical results using the infinitesimal strain theory for the strip loading case reveal good agreement with the analytical solutions reported by Booker [23]. An incremental loading scheme is adopted for the rigid footing case to approximate the instantaneous load (Table 9). Finite deformation evaluations for the rigid footing settlement are compared with the small strain prediction. The normalized settlement history $\frac{G\nu}{bp}$ versus time $\tau = \frac{C_v t}{h^2}$, with $G$, $b$, $p$, $C_v$ being the shear modulus, footing width, applied traction and coefficient of consolidation, respectively, is illustrated in Fig. 31. The computations reveal that the settlement magnitude for the infinitesimal theory equals the layer depth $h$ for an applied traction of $p=8,000$. The finite deformation theory, however, requires an infinite value for this settlement value. Severe distortion of the mesh near the rigid footing (Fig. 32) limits further increment of the loading beyond $p=6,000$, requiring remeshing for accurate results. Figure 32 shows that the deformed mesh geometry at $t=0.1$ is dilated due to the pore pressure effects. On the other hand, the steady
Fig. 31 Surface Settlement versus Time Comparison for Plane Strain Consolidation
Table 9 Incremental Loading Scheme for Plane Strain Consolidation

<table>
<thead>
<tr>
<th>Step No.</th>
<th>Time Interval</th>
<th>Time Increment</th>
<th>( \Delta p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0 - 0.002</td>
<td>0.002</td>
<td>0.02( p_{\text{max}} )</td>
</tr>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.078 - 0.08</td>
<td>0.002</td>
<td>0.02( p_{\text{max}} )</td>
</tr>
<tr>
<td>41</td>
<td>0.08 - 0.081</td>
<td>0.001</td>
<td>0.02( p_{\text{max}} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.099 - 0.1</td>
<td>0.001</td>
<td>0.01( p_{\text{max}} )</td>
</tr>
<tr>
<td>61</td>
<td>0.1 - 1.0</td>
<td>0.01</td>
<td>0</td>
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<tr>
<td>62</td>
<td>1.0 - 10.0</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>63</td>
<td>10.0 - 100.0</td>
<td>0.50</td>
<td>0</td>
</tr>
<tr>
<td>64</td>
<td>100.0 - 1000.0</td>
<td>5.00</td>
<td>0</td>
</tr>
<tr>
<td>65</td>
<td>1000.0 - 10000.0</td>
<td>10.00</td>
<td>0</td>
</tr>
</tbody>
</table>

\( p_{\text{max}} = 6,000 \)
Fig. 32 Deformation of Mesh Geometry for Plane Strain Consolidation
Fig. 32 (continued)

(d) $t = 100$

(e) $t = 1000$

(c) $t = 10$
state mesh configuration \((t=1,000)\), with diminished pore pressure effect, is relatively uniform.

6.2 Nuclear Waste Repository Model

Since the National Waste Storage Program was originally established in 1976, hygrothermal/structural analyses of nuclear repositories has received great attention for the safe and long term management of high-level waste. The life of the planned nuclear waste repositories is expected to be of the order of several thousand years. The repositories will be subjected to thermal loads generated from the radioactive waste material and gravitational load during the storage. However, the tectonic stresses are ignored. The long-term deformation responses along with temperature and pore-pressure changes of the nuclear waste repository are simulated using an axisymmetric finite element model formulated by Nipp [85].

Because of the model symmetry, only half of the domain is analyzed. Figure 33 shows the selected finite element method model and boundary conditions. This model adopts 4-node quadrilateral isoparametric elements with 323 elements and 360 nodal points. The heat source energy resulting from radioactive waste is specified as a function of time in the form [85]
Fig. 33 FEM Model for the Nuclear Waste Repository
Because of difficulties encountered in representing this term properly, the time- and spatial-dependent temperature boundaries in the vicinity of the heat source are assumed as shown in Table 10. Impervious and insulated boundary conditions are assumed everywhere except at the top surface and the boundary \( r = 800 \text{[m]} \).

The material properties, utilized in this study, for different rock materials are listed in Table 11. Only the rock salt is assumed to have temperature dependent material properties. The geomechanical material properties are selected from the values reported by McTigue [78]. The viscoplastic properties of rock salt, using Perzyna's model (eqns. (3.3) and (3.5)), are selected from the work of Desai and Zhang [48]:

\[
\gamma = 5.06 \times 10^{-7} \quad \text{[(day)}^{-1}] \\
N = 3.0 \quad \text{(6.3)} \\
Y = 5.4 \quad \text{[MPa]} \quad \text{(6.4)}
\]

The Von-Mises yield criterion is selected instead of the criterion presented by Desai and Zhang because of the inherent restrictions in the developed program.

The initial temperature is assumed to vary linearly from \(285^\circ \text{K} \) at the surface to \(315^\circ \text{K} \) at the base with a geothermal gradient of \(0.035^\circ \text{K/m} \). The initial pore pressures and stresses are assumed to be zero.
Table 10  Spatial- and Time-Dependent Temperature Boundary

<table>
<thead>
<tr>
<th>Position</th>
<th>Temperature (°K)</th>
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<tbody>
<tr>
<td></td>
<td>r</td>
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<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>50.0</td>
<td>0.0</td>
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<tr>
<td>100.0</td>
<td>0.0</td>
</tr>
<tr>
<td>150.0</td>
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</tr>
<tr>
<td>200.0</td>
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<td>250.0</td>
<td>0.0</td>
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<tr>
<td>290.0</td>
<td>0.0</td>
</tr>
<tr>
<td>320.0</td>
<td>0.0</td>
</tr>
<tr>
<td>350.0</td>
<td>0.0</td>
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<tr>
<td>0.0</td>
<td>12.5</td>
</tr>
<tr>
<td>50.0</td>
<td>12.5</td>
</tr>
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<td>100.0</td>
<td>12.5</td>
</tr>
<tr>
<td>150.0</td>
<td>12.5</td>
</tr>
<tr>
<td>200.0</td>
<td>12.5</td>
</tr>
<tr>
<td>250.0</td>
<td>12.5</td>
</tr>
<tr>
<td>290.0</td>
<td>12.5</td>
</tr>
<tr>
<td>320.0</td>
<td>12.5</td>
</tr>
<tr>
<td>350.0</td>
<td>12.5</td>
</tr>
<tr>
<td>0.0</td>
<td>25.0</td>
</tr>
<tr>
<td>50.0</td>
<td>25.0</td>
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<tr>
<td>100.0</td>
<td>25.0</td>
</tr>
<tr>
<td>150.0</td>
<td>25.0</td>
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<tr>
<td>200.0</td>
<td>25.0</td>
</tr>
<tr>
<td>250.0</td>
<td>25.0</td>
</tr>
<tr>
<td>290.0</td>
<td>25.0</td>
</tr>
<tr>
<td>320.0</td>
<td>25.0</td>
</tr>
<tr>
<td>350.0</td>
<td>25.0</td>
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</tbody>
</table>
Table 11 Material Properties in Nuclear Waste Repository Model

<table>
<thead>
<tr>
<th>Properties</th>
<th>Unit</th>
<th>Rock Name</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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<tbody>
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<td>Soil</td>
<td>1.6</td>
<td>1.5</td>
<td>(1)*</td>
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<td>Gypsum Head</td>
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<td>0.57</td>
<td>0.57</td>
<td>0.57</td>
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<td></td>
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<td>Rock Salt</td>
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<td>2,450</td>
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<td></td>
<td>Rock</td>
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<td>1,000</td>
<td>1,000</td>
<td>1,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>870</td>
<td>613</td>
<td>796</td>
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</tr>
<tr>
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<td>0.25</td>
<td>0.5</td>
<td>(2)*</td>
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<td>0.5</td>
<td>1.0</td>
<td>5.4</td>
<td>5</td>
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<td></td>
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<td>2.0</td>
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<tr>
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<td></td>
<td></td>
<td>0.4</td>
<td>0.1</td>
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<td>3.9</td>
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<td>3.0x10^{-16}</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>1.5x10^{-3}</td>
<td>1.5x10^{-3}</td>
<td>1.0x10^{-3}</td>
<td>1.5x10^{-3}</td>
</tr>
</tbody>
</table>

(1)* : k_s = 1.24 x (θ + 110)^{-1.66} x 10^5 [w/m°K]

(2)* : E_s = 26,000 - 40θ [MPa]
Table 12 shows the incremental thermal loadings for the implicit time scheme, used in this study, up to 300 years. The solutions require computational time of 6298 seconds for the infinitesimal deformation theory and 6720 seconds for the finite deformation theory on the Cray-YMP system, respectively.

Figures 34 (a) and (b) reveal the temperature variations along the r- and z-axes, respectively. Figures 35(a) - (d) show the temperature profiles up to a simulation time of 300 years. It is observed that the thermally active region due to the heat source is quite narrow and the thermal diffusion is also very slow because of the low salt diffusivity \( \frac{k}{\rho C} = 3.22 \times 10^{-6} [m^2/sec] \). The pore pressure profiles are shown in Figs 36(a)-(b) and 37(a)-(d). It is observed that the fluid pressure transport is coupled with the thermal diffusion and solid deformation. Figure 38 shows the displacement history for selected positions (Table 13) obtained from the finite deformation and infinitesimal deformation theories. The temperature histories, obtained from present formulation, at the selected axes are very similar to those obtained by Nipp [83]. However, the displacement histories are quite different from Nipp's results because of the different constitutive models and inclusion of coupled pore-pressure effects for the present model. The results indicate that
Table 12 Incremental Thermal Loading for Nuclear Waste Repository Model

<table>
<thead>
<tr>
<th>Step No.</th>
<th>Time(year)</th>
<th>Time Increment(year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0 - 10.0</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>10.0 - 55.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>55.0 - 135.0</td>
<td>5.0</td>
</tr>
<tr>
<td>4</td>
<td>135.0 - 250.0</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Table 13 Displacement pick-up positions

<table>
<thead>
<tr>
<th>Node No.</th>
<th>r-coord.[m]</th>
<th>z-coord.[m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>0.0</td>
<td>25.0</td>
</tr>
<tr>
<td>121</td>
<td>0.0</td>
<td>100.0</td>
</tr>
<tr>
<td>281</td>
<td>0.0</td>
<td>520.0</td>
</tr>
<tr>
<td>341</td>
<td>0.0</td>
<td>800.0</td>
</tr>
</tbody>
</table>
(a) along \( z = 0 \) Axis

Fig. 34 Temperature Variation along Main Axis
Fig. 34 (continued)

(b) along $r = 0$ Axis
Fig. 35 Temperature Profile in the Nuclear Waste Repository Model
Fig. 35 (continued)

MAG MIN: 2.85E+02 MAX: 4.48E+02

DELTA=15°K

300

315

(b) t = 55 Yr.
Fig. 35 (continued)

MAG MIN: 2.05E+02 MAX: 4.08E+02

DELTA=15°K

300

315

(c) t = 135 Yr.
Fig. 35 (continued)

MAK MIN: 2.85E+02 MAX: 3.68E+02

DELTA=15°K

(d) \( t = 300 \text{ Yr.} \)
Fig. 36 Pore Pressure Variation along Main Axis
Fig. 36 (continued)

(b) along $r = 0$ Axis
(a) \( t = 10 \text{ Yr.} \)

FIG. 37 Pore Pressure Profile in the Nuclear Waste Repository Model
Fig. 37 (continued)

$\text{MAG~MIN:}~-5.12 \times 10^5~\text{MAX:}~9.52 \times 10^7$

(b) $t = 55$ Yr.
Fig. 37 (continued)

\[ \text{MAG MIN: } -3.77E+05 \quad \text{MAX: } 4.74E+07 \]

\[ \text{DELTA=5 MPa} \]

(c) \( t = 135 \text{ Yr.} \)
Fig. 37 (continued)

MAG MIN: -1.69E+04 MAX: 2.16E+07

(d) $t = 300$ Yr.
Fig. 38 Displacement History Comparisons at Nodes (Table 11) for Nuclear Waste Repository Model
the displacements from finite deformation theory are slightly larger than those obtained from infinitesimal strain theory since the order of strain magnitude is $10^{-3}$. It is also observed that large deformations due to thermal expansion of rock salt occur around the heat source and the maximum upheaval magnitude is $1.15[m]$ after 55 years of simulation time. The deformed mesh geometry for the finite deformation case is shown in Figs. 39a) - (d). The hydrostatic stresses due to gravitational load are assumed with $\rho g = 0.024 \text{ MN/m}^3$ [83]. The normal stress distributions, including a hydrostatic gravitational stress of $-18.5[\text{MPa}]$ along the axis $z=30[m]$, at simulation times of 15, 55, 135 and 300 years are shown in Figs. 40 (a) - (d), respectively. The maximum thermal stress is observed at an elapsed time of 55 years and it exhibits a relaxation behavior as the temperature and pore pressure fronts propagate. Steep stress gradients are also observed in the neighborhood of the heat source. Since a low yield value is assumed with no strain hardening, the salt region is subjected to plastic deformations.
Fig. 39  Deformation of Mesh Geometry for the Nuclear Waste Repository Model

(a) $t = 10$ Yr.
Fig. 39 (continued)

(b) \( t = 55 \) Yr.
Fig. 39 (continued)

MAG MIN: 0.00E+00 MAX: 9.49E-01

(c) $t = 135$ Yr.
Fig. 39 (continued)

(d) $t = 300 \text{ Yr.}$
FIG. 40  Stress Distribution along $z = 30$ Axis
Fig. 40 (continued)

(b) $t = 55 \text{ Yr.}$
Fig. 40 (continued)

(c) $t = 135 \text{ Yr.}$
Fig. 40 (continued)

(d) $t = 300 \text{ yr.}$
In this dissertation, a set of field equations governing the coupled thermomechanical responses of fluid saturated porous media are rigorously derived from a continuum thermodynamics mixture theory. Finite element formulations with particular emphasis on nonlinear phenomena resulting from finite deformation as well as nonlinear constitutive properties have also been conducted. A two dimensional computer code is developed and is implemented within an inhouse computer program HYFIDE [120]. Although the formulations and numerical procedures are focused on geomechanical applications, this study is also applicable to various problems related to the manufacturing of advanced materials including ceramics and composites.

The time-dependent responses of fluid infiltrated porous media are governed by the stress equilibrium and fluid/thermal diffusion equations. These governing
equations have been derived from the conservation laws based on the theory of mixtures assuming that the flow through the homogeneous porous media is governed by Darcy's law. The mixture, considered in this study, is assumed to comprise of two constituents, a single phase fluid and a porous solid skeleton in motion relative to each other. The derived governing equations, using a convected coordinate system attached to the solid skeleton, are represented by a set of second order partial differential equations and expressed in terms of the unknown quantities, namely fluid pressure, solid skeleton deformation, and mixture temperature. The porosity evolution equation along with the governing equations, boundary conditions and initial conditions complete the boundary value problem formulation for determining the thermomechanical response of fluid-saturated media undergoing finite deformation.

Time-dependent material behavior such as creep and relaxation effects coupled with the plastic stress state is described by Perzyna's thermo-elastic-viscoplastic model with isotropic hardening and Von-Mises yield criterion in an incremental form. In the derivation of the incremental constitutive law, the Jaumann stress rate is employed to ensure the superposed rigid body motion.
The governing equations with the incorporated material and geometrical nonlinearities require the use of numerical procedures. Finite element formulations for these coupled nonlinear partial differential equations are therefore developed. The virtual work principle incorporating the updated Lagrangian method is applied for the deformation of the solid skeleton. The weak formulations for the fluid transport and thermal diffusion equations are utilized. Temperature dependent mechanical properties along with the finite deformation of the homogeneous hygro-thermo-elasto-viscoplastic solid matrix are assumed. A two point implicit time stepping scheme with an iteration scheme corresponding to the full Newton-Raphson method is utilized. The coupled algebraic system of equations, corresponding to the full Newton-Raphson method, are obtained in terms of the unknown increment and residue vectors. The displacement and force convergence criteria are employed to terminate the iteration. A summary of the described solution algorithm is given in Section 4.3, Table 1.

A two-dimensional computer program with four node quadrilateral elements is developed for solving plane and axisymmetric problems. Several calibrative examples are
investigated to validate the developed finite element methodologies. The first two examples investigate one-dimensional viscoplastic responses using Perzyna's model with linear strain hardening under infinitesimal deformation theory. Also, the response of an elasto-viscoplastic thick walled cylinder subjected to an internal pressure [89] is studied. Numerical solutions are obtained for both plane strain and axisymmetric conditions under infinitesimal deformation theory. The subsequent example evaluates the thermomechanical responses of a thick-walled cylinder quenched from 320°C down to 20°C. This example has been previously investigated by Argyris et al [6]. Here, however, Perzyna's thermo-elasto-viscoplastic model with a strain-dependent hardening rule is adopted. The numerical results obtained from present FEM model were found to be in good agreement with the analytical solutions and reported results. The thermo-viscoplastic flow rule using Perzyna's model for infinitesimal theory is also shown to give reasonable results.

The one-dimensional consolidation behavior of a column, including fluid and thermal diffusion and elastic solid deformation, is investigated using infinitesimal deformation theory. The results obtained from the developed formulation model are compared with analytical
solutions [105] and reported numerical results [1, 98, 107]. Good conformance between the results from the present formulations and reported results is obtained. The finite deformation response of a hypoelastic solid for simple tension and simple shear cases is investigated in the next examples. Excellent agreement between the developed FEM solutions and analytical results is observed. One-dimensional consolidation responses using finite deformation theory are also examined. In order to obtain large deformations, the applied tractions acting on the top end are incremented up to 10,000 unit. Since no reported results for this example are available, result comparisons with the infinitesimal theory are presented. The large differences in settlement between these two theories illustrate the importance of incorporating finite deformation effects.

Finally, two geomechanical examples are numerically simulated. In the first example, plane strain consolidation responses are analyzed for a uniform strip loading [23] and a rigid, permeable footing [28]. The Poisson's ratio is selected to be zero for comparison with an available analytical solution. An incremental loading scheme is adopted to approximate the instantaneous load. The numerical results using the infinitesimal theory for
strip loading reveal good agreement with the analytical solutions [23]. Finite deformation evaluations for the rigid footing case are compared with the small deformation theory results. For a relatively large traction load (p=6,000), a deviation of 26% in the surface settlement is observed between the two theories. It is concluded that finite deformation theory must be included in analyzing consolidation responses under large traction loads. Severe distortion near the rigid footing limits further increment of the loading beyond p=6,000. The second example deals with the analysis of a nuclear waste isolation model. The response of a low permeability salt-dome [85], is investigated up to a simulation time of 300 years. Temperature dependent material properties are selected and Perzyna's viscoplastic model is assumed using base line properties presented by Desai and Zhang [48]. Because of the restriction in the developed program, the Von-Mises yield criterion is selected instead of the criterion presented by Desai and Zhang [48]. Time- and space-dependent temperature boundaries are imposed in the heat source region to represent the time-dependent heat flux generated from the decay of radionuclides. An incremental thermal loading for the implicit time scheme is employed to obtain transient solutions. From the transient temperature solutions, it is observed that the
thermally active region arising from the heat source is quite narrow and the fluid/thermal diffusion response have a large time constant \( \frac{K}{\rho c} = 1.02 \times 10^2 \) [m²/year] and \( C_v = 5.0 \) [m²/year]). It is also observed that structural upheaval, with a maximum magnitude of 1.15m, takes place in the neighborhood of heat source due to the coupled effects of thermal and pore pressure expansion. A steep stress gradient is observed in the thermally active region because of the rapid temperature change. The presented thermo-viscoplastic model for rock salt undergoes plastic deformation due to the low yield criterion. It is noted that the presented nuclear waste isolation model does not produce significant deformation requiring finite deformation theory while the coupled effects of thermal and pore pressure expansion contribute to the structural upheaval.

The developed hygrothermomechanical model and its FEM methodologies have been successfully applied to two-dimensional geomechanical problems including consolidation and nuclear waste disposal problem. The following recommendations are suggested to improve the numerical simulation capabilities for hygrothermomechanical response characterization:
1) Extension of the developed model to include a broader range of constitutive models, hardening rules and yield criteria.

2) Inclusion of the available stress rate tensors to account for rotation effect.

3) Consideration of damage and contact mechanisms.

4) Incorporation of anisotropic/heterogeneity of geological materials including faults and joints.

5) Establishment of adaptive mesh techniques for large deformed mesh configurations.

6) Development of efficient convergence algorithms.

7) Comparison of results from HYFIDE with averaged responses from available thermomechanical codes having distributed porosity/damage evolution capabilities for studying compaction of granular or porous materials.


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APPENDIX A

DERIVATION OF BALANCE LAWS

AND CLAUSIUS-DUHEM INEQUALITY

A.1 Balance of Mass

By applying the transport theorem defined by

\[ \frac{d}{dt} \int_{\Omega} \tilde{\psi}(x(x_s,t), t) \, dv = \int_{\Omega} (\dot{\tilde{\psi}} + \tilde{\psi} \, \nabla \cdot v_s) \, dv, \quad (A.1) \]

equation (2.14) can be rewritten as

\[ \int_{\Omega} (\dot{\tilde{\rho}}_s + \tilde{\rho}_s \, \nabla \cdot v_s) \, dv = 0. \quad (A.2) \]

From the localization theorem, equation (A.2) gives

\[ \dot{\tilde{\rho}}_s + \tilde{\rho}_s \, \nabla \cdot v_s = 0. \quad (A.3) \]

The material time derivative of the volume integral at fixed \( x_f \) in the convective coordinate system for the current state is

\[ \frac{d}{dt} \int_{\Omega} \tilde{\psi}(x(x_f,t), t) \, dv = \frac{d}{dt} \int_{\Omega} \tilde{\psi}(x(x_s,t), t) \, dv \]

\[ + \int_{\partial\Omega} \tilde{\psi} \, v_r \cdot n \, dA. \quad (A.4) \]

Applying eqn. (A.4) and the divergence theorem, equation (2.14) yields

\[ \int_{\Omega} (\dot{\tilde{\rho}}_f + \tilde{\rho}_f \, \nabla \cdot v_s) \, dv + \int_{\Omega} \nabla \cdot (\tilde{\rho}_f v_r) \, dv = 0. \quad (A.5) \]
Again from the localization theorem, equation (A.5) yields
\[ \dot{\rho}_f + \nabla \cdot (\rho_f \mathbf{v}_f) = 0. \] (A.6)

Substitution of eqn. (2.10) into eqn. (A.6) yields
\[ \dot{\rho}_f + \nabla \cdot (\rho_f \mathbf{v}_f) + \mathbf{v}_r \cdot \nabla (\rho_f) = 0. \] (A.7)

A.2 Balance of Momentum

Substituting eqns. (2.23) into eqn. (2.20), and applying the divergence theorem, we obtain
\[ \frac{d}{dt} \int_{\Omega} \rho_s(x(X_s,t),t) \mathbf{v}_s(x(X_s,t),t) \, dV = \int_{\Omega} \mathbf{f}_s \, dV + \int_{\Omega} \nabla \cdot \mathbf{T}_s \, dV \] (A.8)

and
\[ \frac{d}{dt} \int_{\Omega} \rho_f(x(x_f,t),t) \mathbf{v}_f(x(x_f,t),t) \, dV = \int_{\Omega} \mathbf{f}_f \, dV + \int_{\Omega} \nabla \cdot \mathbf{T}_f \, dV. \] (A.9)

Applying the transport theorem into eqn. (A.8) gives
\[ \int_{\Omega} (\dot{\rho}_s \mathbf{v}_s + \rho_s \dot{\mathbf{v}}_s + \rho_f \mathbf{v}_s \cdot \nabla \mathbf{v}_s) \, dV = \int_{\Omega} \mathbf{f}_s \, dV + \int_{\Omega} \nabla \cdot \mathbf{T}_s \, dV. \] (A.10)

Rearranging eqn. (A.10), we get
\[ \int_{\Omega} [(\dot{\rho}_s + \rho \mathbf{v}_s) \mathbf{v}_s + \rho_s \dot{\mathbf{v}}_s] \, dV = \int_{\Omega} \mathbf{f}_s \, dV + \int_{\Omega} \nabla \cdot \mathbf{T}_s \, dV. \] (A.11)

Substituting eqn. (A.3) into eqn. (A.11) and invoking the localization theorem, the result can be written in the form
\[ \rho_s \dot{\mathbf{v}}_s = \mathbf{f}_s + \nabla \cdot \mathbf{T}_s. \] (A.12)
Similarly, by applying eqn. (A.4) along with the divergence theorem, equation (A.9) can be reduced as

$$\int_{\Omega} \left( \dot{\rho}_f \mathbf{v}_f + \dot{\bar{\rho}}_f \mathbf{v}_f + \dot{\rho}_f \nabla \cdot \mathbf{v}_s \right) d\mathbf{v} + \int_{\Omega} \nabla \cdot (\dot{\rho}_f \mathbf{v}_f \mathbf{v}_r) d\mathbf{v} = \int_{\Omega} \mathbf{T}_f \cdot d\mathbf{v} + \int_{\Omega} \nabla \cdot \mathbf{T}_f d\mathbf{v}. \quad (A.13)$$

Rearrangement of eqn. (A.13) yields

$$\int_{\Omega} \left( \dot{\rho}_f \mathbf{v}_f + \dot{\bar{\rho}}_f \mathbf{v}_f + \dot{\rho}_f \nabla \cdot \mathbf{v}_s \right) d\mathbf{v} = \int_{\Omega} \mathbf{T}_f \cdot d\mathbf{v} + \int_{\Omega} \nabla \cdot \mathbf{T}_f d\mathbf{v}. \quad (A.14)$$

Using eqn. (A.6) and the localization theorem, eqn. (A.14) is reduced to

$$\dot{\rho}_f \mathbf{v}_f + \dot{\bar{\rho}}_f \nabla \cdot \mathbf{v}_f \mathbf{v}_r = \mathbf{T}_f + \nabla \cdot \mathbf{T}_f. \quad (A.15)$$

### A.3 Balance of Energy

From eqns. (A.1) and (A.4), and the divergence theorem, equation (2.31) can be written

$$\int_{\Omega} \left( \dot{\rho}_s \left( \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_s + \mathbf{c} \right) + \dot{\bar{\rho}}_s \left( \mathbf{v}_s \cdot \mathbf{v}_s + \mathbf{c} \right) + \dot{\rho}_s \left( \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_s + \mathbf{c} \right) \nabla \cdot \mathbf{v}_s \right. \nabla \cdot \mathbf{v}_s$$

+ $$\dot{\rho}_s \left( \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_f + \mathbf{c} \right) + \dot{\bar{\rho}}_s \left( \mathbf{v}_s \cdot \mathbf{v}_f + \mathbf{c} \right) + \dot{\rho}_s \left( \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_f + \mathbf{c} \right) \nabla \cdot \mathbf{v}_s$$

+ $$\nabla \cdot \left( \dot{\rho}_s \left( \frac{1}{2} \mathbf{v}_s \cdot \mathbf{v}_f + \mathbf{c} \right) \mathbf{v}_r \right) \right) d\mathbf{v}$$

= $$\int_{\Omega} \left( \mathbf{T}_f \cdot \mathbf{v}_f + \rho_f \mathbf{r}_f \right) + \left( \mathbf{T}_s \cdot \mathbf{v}_s + \rho_s \mathbf{r}_s \right) d\mathbf{v}$$

+ $$\int_{\Omega} \left( \nabla \cdot \mathbf{T}_f \mathbf{v}_f \right) + \nabla \cdot \mathbf{T}_s \mathbf{v}_s - \nabla \cdot \left( \mathbf{q}_f + \mathbf{q}_s \right) \right) d\mathbf{v}. \quad (A.16)$$

Noting that

$$\nabla \cdot \left( \dot{\rho}_f \left( \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f + \mathbf{c} \right) \mathbf{v}_r \right) \right) = \left( \nabla \cdot \left( \dot{\rho}_f \mathbf{v}_r \right) \right) \left( \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f + \mathbf{c} \right)$$

+ $$\dot{\rho}_f \mathbf{v}_r \cdot \nabla \cdot \mathbf{c} \mathbf{v}_f + \dot{\rho}_f \nabla \cdot \mathbf{v}_f \mathbf{v}_r, \quad (A.17)$$
\[ \mathbf{V} \cdot (T_f \mathbf{v}_f) = \mathbf{v}_f \cdot (\nabla \cdot T_f) + T_f \cdot \text{grad} \mathbf{v}_f, \]  \hspace{1cm} (A.18) \\

and
\[ \mathbf{V} \cdot (T_s \mathbf{v}_s) = \mathbf{v}_s \cdot (\nabla \cdot T_s) + T_s \cdot \text{grad} \mathbf{v}_s, \]  \hspace{1cm} (A.19)

Equation (A.16) can be rearranged as
\[ \int \left( (\tilde{\rho}_s \mathbf{v}_s \nabla \cdot T_s - \mathbf{f}_s) \cdot \mathbf{v}_s + (\tilde{\rho}_f \mathbf{v}_f + \tilde{\rho}_f \text{grad} \mathbf{v}_f) \mathbf{v}_f - \nabla \cdot T_f \cdot \mathbf{f}_f \right) \, d\mathbf{v}_f \\
+ (\tilde{\rho}_s \mathbf{v}_s \nabla \cdot T_s + \tilde{\rho}_s \mathbf{v}_s \nabla \cdot E_s) + [\tilde{\rho}_f \mathbf{v}_f + \tilde{\rho}_f \mathbf{v}_f + \tilde{\rho}_f \mathbf{v}_f \nabla \cdot E_f] \left( \frac{1}{2} \mathbf{v}_f \cdot \mathbf{v}_f + E_f \right) \\
+ \tilde{\rho}_s \dot{E}_s + \tilde{\rho}_f \dot{E}_f + \tilde{\rho}_f \mathbf{v}_f \cdot \text{grad} \mathbf{E}_f \right) \, d\mathbf{v} \\
= \int \left[ \tilde{\rho}_f \mathbf{r}_f + \tilde{\rho}_s \mathbf{r}_s - \nabla \cdot (\mathbf{q}_f + \mathbf{q}_s) + T_f \cdot \text{grad} \mathbf{v}_f + T_s \cdot \text{grad} \mathbf{v}_s \right] \, d\mathbf{v}. \]  \hspace{1cm} (A.20)

Substituting eqns. (A.3), (A.6), (A.12) and (A.15) into eqn. (A.20), the localized form of eqn. (A.20) is obtained
\[ \rho \dot{E} + \tilde{\rho}_f \mathbf{v}_f \cdot \text{grad} \mathbf{E}_f = \rho \mathbf{r} - \nabla \cdot \mathbf{q} + T_f \cdot \mathbf{L}_f + T_s \cdot \mathbf{L}_s \]  \hspace{1cm} (A.21)

where
\[ \rho \dot{E} = \tilde{\rho}_s \dot{E}_s + \tilde{\rho}_f \dot{E}_f, \quad \rho \mathbf{r} = \tilde{\rho}_f \mathbf{r}_f + \tilde{\rho}_s \mathbf{r}_s, \quad \mathbf{q} = \mathbf{q}_f + \mathbf{q}_s, \]
\[ \mathbf{L}_f = \text{grad} \mathbf{v}_f, \text{ and } \mathbf{L}_s = \text{grad} \mathbf{v}_s. \]

A.4 Clausius-Duhem Inequality

Equation (2.36) can, by use of eqns. (A.1) and (A.4) and divergence theorem, be written as
\[ \int_{\Omega} (\tilde{\rho}_f \eta_f + \tilde{\rho}_f \dot{\eta}_f + \tilde{\rho}_f \eta_f \nabla \cdot \mathbf{v}_f) \, d\mathbf{v} + \int_{\Omega} \nabla \cdot (\tilde{\rho}_f \eta_f \mathbf{v}_f) \, d\mathbf{v} \\
+ \int_{\Omega} (\tilde{\rho}_s \eta_s + \tilde{\rho}_s \dot{\eta}_s + \tilde{\rho}_s \eta_s \nabla \cdot \mathbf{v}_s) \, d\mathbf{v} - \int_{\Omega} \frac{1}{\beta} \rho \mathbf{r} \, d\mathbf{v} \\
+ \int_{\Omega} \nabla \cdot (\frac{\mathbf{q}}{\beta}) \, d\mathbf{v} \geq 0. \]  \hspace{1cm} (A.22)
Noting that
\[ \nabla \cdot (\vec{\rho}_f \eta_f \vec{v}_r) = \vec{\rho}_f \text{grad}(\eta_f) \cdot \vec{v}_r + \eta_f \nabla \cdot (\vec{\rho}_f \vec{v}_r) \]  
(A.23)
equation (A.22) can be rearranged in the form
\[ \int_\Omega \left( [\vec{\rho}_f + \vec{\rho}_f \nabla \cdot \vec{v}_s + \nabla \cdot (\vec{\rho}_f \vec{v}_r)] \eta_f + (\vec{\rho}_s + \vec{\rho}_s \nabla \cdot \vec{v}_s) \eta_s + \vec{\rho}_f \eta_f \right) + \vec{\rho}_s \eta_s + \vec{\rho}_f \text{grad}(\eta_f) \cdot \vec{v}_r - \frac{\rho_r}{\theta} + \frac{\nabla \cdot \vec{g} - \vec{g} \cdot \vec{g}}{\theta^2} \) dV \geq 0. 
(A.24)
where \( \vec{g} = \text{grad}(\theta) \). From eqns. (A.3) and (A.6) and the localization theorem, it can be shown that
\[ \vec{\rho}_f \eta_f + \vec{\rho}_s \eta_s + \vec{\rho}_f \text{grad}(\eta_f) \cdot \vec{v}_r + \frac{\nabla \cdot \vec{g} - \vec{g} \cdot \vec{g}}{\theta} \geq 0. 
(A.25)
Using eqn. (A.21), it can be shown that
\[ \nabla \cdot \vec{q} - \rho r = T_f \cdot \vec{L}_f + T_s \cdot \vec{L}_s - \vec{\rho}_f \dot{\vec{E}}_f - \vec{\rho}_s \dot{\vec{E}}_s - \vec{\rho}_f \vec{v}_r \cdot \text{grad}(\vec{E}_f). 
(A.26)
Thus, equation (A.25), by use of eqn. (A.26), is rearranged in the form
\[ \vec{\rho}_f (\theta \eta_f - \dot{\vec{E}}_f) + \vec{\rho}_s (\theta \eta_s - \dot{\vec{E}}_s) + \vec{\rho}_f \theta \text{grad}(\eta_f) \cdot \vec{v}_r + T_f \cdot \vec{L}_f + T_s \cdot \vec{L}_s - \vec{\rho}_f \vec{v}_r \cdot \text{grad}(\vec{E}_f) - \frac{\vec{g} \cdot \vec{g}}{\theta} \geq 0. 
(A.27)\]
Appendix B

Derivation of Updated Lagrangian Formulation for Solid Constituent

B.1 Relation Among Piola-Kirchhoff Stress, Lagrangian Strain, Cauchy Stress and Eulerian Strain

The second kind Piola-Kirchhoff stress and Lagrangian strain are defined by

\[ S = JF^{-1}TF^{-T} \]  \hspace{1cm} (B.1)

and

\[ E = \frac{1}{2} F^T F - I. \]  \hspace{1cm} (B.2)

where \( F^{-T} \) is the inverse of \( F^T \). The variational form \( \delta F \) is expressed as [33]

\[ \delta F = \delta \left( \frac{\partial x}{\partial x} \right) = (\frac{\partial \delta u}{\partial x})(\frac{\partial x}{\partial x}) = (\delta \epsilon + \delta \omega) F. \]  \hspace{1cm} (B.3)

where

\[ \delta \epsilon = \frac{1}{2} \left( (\frac{\partial \delta u}{\partial x}) + (\frac{\partial \delta u}{\partial x})^T \right) \]  \hspace{1cm} (b.4)

and

\[ \delta \omega = \frac{1}{2} \left( (\frac{\partial \delta u}{\partial x}) - (\frac{\partial \delta u}{\partial x})^T \right) \]  \hspace{1cm} (b.5)

Using eqn. (B.3), the variational form, \( \delta E \), is obtained as

\[ \delta E = \frac{1}{2} (\delta F^T F + F^T \delta F) = F^T (\delta \epsilon) F. \]  \hspace{1cm} (B.6)
From eqns. (B.1) and (B.6), it can be shown

\[ S \cdot \delta E = \text{tr}(S(\delta E)^T) = J \text{tr}(T\delta \epsilon) = J \cdot T \cdot \delta \epsilon. \quad (B.7) \]

Substituting eqn. (B.7), equation (4.4) is rewritten as

\[ \int_{\Omega_t} (S \cdot \delta E)^t + \Delta t \, dv = \int_{\Omega_t} (T \cdot \delta u)^t + \Delta t \, dv \]
\[ + \int_{\Omega_t} (T \cdot \delta u)^t + \Delta t \, da. \quad (B.8) \]

### B.2 Relation Between the Second Piola-Kirchhoff Stress Rate and Truesdell Stress Rate

From eqn. (B.1), the time derivative of the second Piola-Kirchhoff stress tensor is expressed as

\[ \dot{S} = JF^{-1} \dot{T}F - T + JF^{-1} \dot{T}F - T + JF^{-1} \dot{T}F - T. \quad (B.9) \]

Noting that

\[ \dot{J} = J \text{div}(v), \quad \dot{F} = LF, \quad \dot{F}^{-1} = -F^{-1}L, \quad \dot{F}^{-T} = -LTF^{-T}. \quad (B.10) \]

equation (B.9) is reduced to

\[ \dot{S} = JF^{-1} [\text{div}(v)T - LT + \dot{T} - TL^T]F^{-T} \quad (B.11) \]

or equivalently

\[ \dot{S} = JF^{-1} \dot{T} F^{-T}. \quad (B.12) \]

with the Truesdell stress rate, \( T \), defined by

\[ T = \text{div}(v)T - LT + \dot{T} - TL^T. \quad (B.13) \]
APPENDIX C

DERIVATION OF TERMS IN FORCE EQUILIBRIUM EQUATIONS

C.1 Plane Problem

For a typical 4-nodes element of a two-dimensional plane problem, the nodal displacements temperature and pressure in an element are

\[
\begin{bmatrix}
    u \\
v
\end{bmatrix} = [N_u]^T[U], \quad \theta = [N_\theta][\theta], \quad \pi = [N_\pi][\pi].
\]

where

\[
[N_u] = \begin{bmatrix}
    N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\
    0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4
\end{bmatrix}^T
\]

\[
[u] = [ u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3 \ u_4 \ v_4 ]^T
\]

\[
[N_\theta] = \begin{bmatrix}
    N_1 & N_2 & N_3 & N_4
\end{bmatrix}^T
\]

\[
[\theta] = [ \theta_1 \ \theta_2 \ \theta_3 \ \theta_4 ]^T
\]

\[
[\pi] = [ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4 ]^T
\]

The linear strain-displacement matrix, \([B_u]\), and nonlinear strain-displacement matrix, \([B_n]\), are
The stress-strain relation is

\[ \sigma = [C] [e] \]  \tag{C.9} 

where

\[ \sigma = [ \sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12} ]^T, \]  \tag{C.10} 
\[ [e] = [ e_{11} e_{22} e_{33} \gamma_{12} ]^T, \]  \tag{C.11} 

where \( \gamma_{12} = 2e_{12} \). For plane strain

\[ [C] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \]  \tag{C.12} 

\[ [B_u] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} \end{bmatrix}^T \]  \tag{C.7} 

\[ [B_n] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} \end{bmatrix}^T \]  \tag{C.8}
Then each term in eqn. (4.30) is derived as follows

1. \[ \sigma \cdot \delta n = \sigma_{11} \delta n_{11} + \sigma_{22} \delta n_{22} + 2\sigma_{12} \delta n_{12} \]
\[ = \sigma_{11} \left[ (\Delta u, x) \delta(\Delta u, x) + (\Delta v, x) \delta(\Delta v, x) \right] \]
\[ + \sigma_{22} \left[ (\Delta u, y) \delta(\Delta u, y) + (\Delta v, y) \delta(\Delta v, y) \right] \]
\[ + \sigma_{12} \left[ \delta(\Delta u, x) (\Delta u, y) + (\Delta v, y) \delta(\Delta v, y) \right] \]
\[ + \delta(\Delta v, x) (\Delta v, y) \delta(\Delta v, y) \]
\[ = [\delta \Delta u]^T [B_u] [\sigma_n] [B_u]^T [\Delta u]. \quad (C.13) \]

where

\[
[\sigma_n] = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 \\
\sigma_{12} & \sigma_{22} & 0 & 0 \\
0 & 0 & \sigma_{11} & \sigma_{11} \\
0 & 0 & \sigma_{12} & \sigma_{12}
\end{bmatrix} \quad (C.14)
\]

2. \[ C_{ep} \Delta e \cdot \delta \Delta e = [ \delta \Delta e_{11} \delta \Delta e_{22} 2 \delta \Delta e_{12} ] [C_{ep}] \begin{bmatrix}
e_{11} \\
e_{22} \\
2e_{12}
\end{bmatrix} \]
\[ = [\delta \Delta u]^T [B_u] [C_{ep}] [B_u] [\Delta u]^T \quad (C.15) \]

3. \[ [\sigma \delta \Delta e + \Delta e \sigma] \cdot \delta \Delta e = \sigma \Delta e \cdot \delta \Delta e + \Delta e \sigma \cdot \delta \Delta e = 2 \sigma \Delta e \cdot \delta \Delta e \]
\[ = [\delta \Delta u]^T [B_u] [\sigma_b] [B_u]^T [\Delta u]. \quad (C.16) \]

with

\[
[\sigma_b] = \begin{bmatrix}
\sigma_{11} & 0 & \sigma_{12} \\
0 & \sigma_{22} & \sigma_{12} \\
\sigma_{12} & \sigma_{12} & \frac{1}{2}(\sigma_{11} + \sigma_{22})
\end{bmatrix} \quad (C.17)
\]

4. \[ \text{tr}(\Delta e) \sigma \cdot \delta \Delta e = (\Delta e_{11} + \Delta e_{22}) \sigma_{11} \delta \Delta e_{11} + 2(\Delta e_{11} + \Delta e_{22}) \sigma_{12} \delta \Delta e_{12} \]
\[ = [\delta \Delta u]^T [B_u] [\sigma_\circ] [B_u]^T [\Delta u] \quad (C.18) \]

with
\[
\begin{bmatrix}
\sigma_{11} & \sigma_{11} & 0 \\
\sigma_{22} & \sigma_{22} & 0 \\
\sigma_{12} & \sigma_{12} & 0
\end{bmatrix}
\]  
(C.19)

(5) \( C_{ep} \Delta e_n \cdot \delta \Delta e = [\delta \Delta u]^T [B_n]^T [C_{ep}] \{ [A_\theta] [\Delta \theta] + [A_\pi] [\Delta \pi] \}. \)

with

\[
[A_\theta] = \begin{bmatrix}
\beta & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \beta
\end{bmatrix}. 
\]  
(C.20)

\[
[A_\pi] = \frac{1-2\nu}{E} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}. 
\]  
(C.21)

\[
\begin{bmatrix}
\beta & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \beta
\end{bmatrix}
\]  
(C.22)

(6) \( t_x \delta \Delta u_x + t_y \delta \Delta u_y = [\delta \Delta u]^T [N_u][t]. \)

with

\[
[t] = \begin{bmatrix}
t_x \\
t_y
\end{bmatrix}. 
\]  
(C.23)

Similarly .

(7) \( \overline{f}. \delta \Delta u = [\delta \Delta u]^T [N_u][\overline{f}]. \)

with

\[
[\overline{f}] = \begin{bmatrix}
\overline{f}_x \\
\overline{f}_y
\end{bmatrix}. 
\]  
(C.24)

(8) \( C_{ep} \Delta t \varepsilon_{vp} t \cdot \delta \Delta e = [\delta \Delta u]^T [B_u]^T [C_{ep}] \{ \varepsilon_{vp} \ t \Delta t \}. \)

(9) \( \sigma \cdot \delta \Delta e = [\delta \Delta u]^T [B_u][\sigma]. \)

with

\[
[\sigma] = \begin{bmatrix}
\sigma_{xx} & \sigma_{yy} & \sigma_{xy}
\end{bmatrix}^T. 
\]  
(C.25)
C.2 Axisymmetric Problem

For a typical 4-nodes element in axisymmetric problems, the nodal displacement \([u, v]\) in eqn (C.1) is replaced by \([u_r, u_z]\).

The stress-strain relation is

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{zz} \\
\sigma_{\theta\theta} \\
\sigma_{rz}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 \\
\nu & 1-\nu & \nu & 0 \\
\nu & \nu & 1-\nu & 0 \\
0 & 0 & 0 & \frac{1}{2}(1-2\nu)
\end{bmatrix} \begin{bmatrix}
e_{rr} \\
e_{zz} \\
e_{\theta\theta} \\
\gamma_{rz}
\end{bmatrix}
\]

(C.30)

The strain-displacement relations are

\[
[B_u] = \begin{bmatrix}
\frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} & 0 \\
0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_4}{\partial z} \\
N_1/r & 0 & N_2/r & 0 & N_3/r & 0 & N_4/r & 0 \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial r}
\end{bmatrix}^T
\]

(C.31)

and

\[
[B_n] = \begin{bmatrix}
\frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} & 0 \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial r}
\end{bmatrix}^T
\]

(C.32)
Also

\[
\sigma_n = \begin{bmatrix}
\sigma_{rr} & \sigma_{rz} & 0 & 0 & 0 \\
\sigma_{rz} & \sigma_{zz} & 0 & 0 & 0 \\
0 & 0 & \sigma_{rr} & \sigma_{rz} & 0 \\
0 & 0 & \sigma_{rz} & \sigma_{zz} & 0 \\
0 & 0 & 0 & 0 & \sigma_{\theta\theta}
\end{bmatrix}
\]  
(C.33)

\[
\sigma_b = \begin{bmatrix}
\sigma_{rr} & 0 & 0 & \sigma_{rz}/2 \\
0 & \sigma_{zz} & 0 & \sigma_{rz}/2 \\
0 & 0 & \sigma_{\theta\theta} & 0 \\
\sigma_{rz}/2 & \sigma_{rz}/2 & 0 & (\sigma_{rr}+\sigma_{zz})/4
\end{bmatrix}
\]  
(C.34)

\[
\sigma_o = \begin{bmatrix}
\sigma_{rr} & \sigma_{rr} & \sigma_{rr} & 0 \\
\sigma_{zz} & \sigma_{zz} & \sigma_{zz} & 0 \\
\sigma_{\theta\theta} & \sigma_{\theta\theta} & \sigma_{\theta\theta} & 0 \\
\sigma_{rz} & \sigma_{rz} & \sigma_{rz} & 0
\end{bmatrix}
\]  
(C.35)
APPENDIX D

ANALYTICAL SOLUTIONS FOR FINITE DEFORMATION OF HYPOELASTIC SOLID EMPLOYING JAUMANN STRESS RATE UNDER SIMPLE SHEAR AND SIMPLE TENSION

The responses of a hypoelastic solid under simple tension and simple shear conditions are derived by employing the Jaumann stress rate. From the constitutive relation (eqn. (5.7)) and the Jaumann stress rate (3.11), we obtain

\[ T + T_W - W_T = \lambda \text{tr}(D)I + 2GD \]  \hspace{1cm} (D.1)

The resulting differential equation (D.1) is solved for an unknown T for evaluating the stress response.

D.1 Simple Shear

For a fixed rectangular coordinate system \((X, Y, Z)\), the deformed configuration at time \(t\) associated with simple shear is expressed in terms of the current position \((x, y, z)\) by

\[ x(t) = X + k(t) Y, \]  \hspace{1cm} (D.2)

\[ y(t) = Y, \]  \hspace{1cm} (D.3)
and

$$z(t) = z.$$  \hfill (D.4)

Invoking the plane strain assumption, the rate-of-deformation tensor $D$ and the spin tensor $W$ are obtained from the velocity gradient as

$$[D] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ \hfill (D.5)

$$[W] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ \hfill (D.6)

Substituting eqns (D.5) and (D.6) into eqn (D.1), the following equations are obtained [63]:

$$\dot{T}_{11} - \frac{k}{2} \dot{T}_{12} = 0$$ \hfill (D.7)

$$\dot{T}_{12} - \frac{k}{2} (T_{11} - T_{22}) = gk$$ \hfill (D.8)

$$\dot{T}_{22} + \frac{k}{2} T_{12} = 0.$$ \hfill (D.9)

The solutions to eqns. (D.7) through (D.9) are

$$T_{11} = -T_{22} = G(1 - \cos k)$$ \hfill (D.10)

and

$$T_{12} = G \cos k$$ \hfill (D.11)
D.2 Simple Tension

At any time, \( t \), the deformation associated with simple tension is given in terms of the current position \((x, y, z)\) by

\[
x(t) = (1 + k_1(t))x
\]

\[
y(t) = (1 - k_2(t))y
\]

and

\[
z(t) = z
\]

The velocity gradient, rate of deformation tensor, and spin tensor are obtained as

\[
[L] = \begin{bmatrix}
\frac{\dot{k}_1}{1+k_1} & 0 & 0 \\
0 & -\frac{k_2}{1-k_2} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(D.12)

and

\[
[W] = [0]
\]

(D.13)

Combining eqns. (D.1), (D.12) and (D.13), we get

\[
\dot{T}_{11} = (\lambda + 2G) \frac{\dot{k}_1}{1+k_1} - \frac{\lambda k_2}{1-k_2}
\]

(D.14)

and

\[
\dot{T}_{22} = - (\lambda + 2G) \frac{\dot{k}_2}{1+k_2} + \frac{\lambda k_1}{1-k_1}
\]

(D.15)
From the boundary condition $T_{22}=0$, it can be shown that

$$
\frac{\dot{k}_2}{1-k_2} = \left(\frac{\lambda}{\lambda+2G}\right) \frac{\dot{k}_1}{1+k_1}
$$

\hspace{1cm} \text{(D.16)}

Substituting eqn. (D.16) into eqn. (d.14), the solution of the differential equation is obtained in the form

$$
T_{11}(t) = (\lambda+2G - \frac{\lambda^2}{\lambda+2G}) \ln(1+ k_1(t))
$$

\hspace{1cm} \text{(D.17)}