Properties of the satellite location polyhedron and its relation to the scheduling polyhedron

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PROPERTIES OF THE SATELLITE LOCATION POLYHEDRON AND ITS RELATION TO THE SCHEDULING POLYHEDRON

DISSERTATION

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By

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To Elizabeth and Fernando Darián.
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CHAPTER I

Introduction

Telecommunications (communications over long distances) have undergone radical changes since the first orbiting communications satellite was launched in the early 1960's. Since then, communications satellites have provided individuals, businesses, and governments with communications advantages unimaginable just a few decades ago. Satellite communication services are being used to provide immediate access to data bases and management information systems. Computer-to-computer telecommunications are successfully carried out by satellites. Not only has high-quality color television become available, but also the broadcasting of live television programs from distant lands is now commonplace. By using an airborne satellite telephone, a business executive can call his/her office to give instructions or to delay an appointment if a plane is late. All this might not be possible with other telecommunications services such as optical fibers and cable. Consequently, it is not surprising that the number of operational communications satellites is growing dramatically. More than 200 satellites had been proposed or deployed
in the geostationary orbit alone by 1985 [23]. The subject of this research is the allocation of the geostationary orbit, a limited natural resource, to the satellites and/or administrations that compete for this resource.

The geostationary orbit (GSO), which is located in the equatorial plane at about 22300 miles from the earth, is ideal for communications satellites since a satellite's orbital period in the GSO is equal to the Earth's rotational period (1 day). Consequently, a satellite in the GSO appears to be fixed in the sky when viewed from the Earth's surface. This is illustrated in Figure 1.1 [51]. When the Earth station moves from point A to point B due to the Earth's rotation, the satellite travels from point C to point D and appears to be fixed relative to all points on the Earth. The GSO is exceptionally attractive for communications satellites since there is no need for more than one satellite to provide continuous service or for complicated satellite tracking schemes.

Consider the positioning of \( n \) satellites in the GSO. For each satellite there is a service arc, or feasible arc. This arc consists of all GSO locations that are visible (at a minimum elevation angle) from the Earth stations with which the satellite communicates. This arc is defined by an eastern limit and a western limit and is assumed to contain a specific desired location. Because of harmful electromagnetic interference, some pairs of satellites can not be located too close to one another. Therefore, for each pair of satellites, there is a minimum angular separation, that if maintained, would adequately control single-entry, or pairwise,
Figure 1.1: The Geostationary Orbit
electromagnetic interference. The required angular separation between a pair of satellites is a function of many electrical and geographic parameters [27],[53]. The closer the service areas of two satellites are, the greater the angular separation that is required between those satellites. For example, a satellite serving France is expected to require more separation from a satellite serving Spain than from a satellite serving China.

In the satellite location problem (SLP), geostationary orbital locations (longitudes) are allotted to satellites subject to the service arc (elevation angle) and angular separation constraints described above. The objective is to minimize the total deviation between the locations allotted to the satellites and their corresponding desired locations. (For additional background material on SLP see [16],[40],[42].)

In Figure 1.2, a simplified satellite system with two satellites and the associated interference geometry is presented. There are two satellites $S_A$ and $S_B$ transmitting to the service areas $A$ and $B$, respectively. The eastern and western limits of the service arc for satellite $A$ are $E_A$ and $W_A$, and the limits of satellite $B$'s service arc are $E_B$ and $W_B$. The dotted ellipse represents the region where the signals broadcasted from $S_A$ are received. The closer a point is to the center of the ellipse, the stronger the signals that are received. Even though the signals broadcasted from a satellite are intended only for that satellite's service area, some signals transmitted from $S_A$ are received in service area $B$, since a portion of the dotted ellipse covers part of service area $B$. By enforcing the minimum angular separation
between satellites $S_A$ and $S_B$ there would be considerably less electromagnetic interference from $S_A$ in service area $B$. Maintaining electromagnetic interference at or below a specific threshold is a crucial part of SLP.

Specifically, the problem to be addressed in this research is the western-unlimited satellite location problem (WUSLP), which is a relaxation of SLP. In WUSLP, orbital locations in the GSO are allotted to satellites subject to easternmost location and angular separation constraints. The objective is to minimize the sum of the absolute differences between the locations allotted to the satellites and their corresponding specified desired locations. While in SLP the location allotted to each satellite is restricted to its service arc, one of the service arc constraints is relaxed and only an eastern limit is enforced in WUSLP.

The purpose of this research is to identify classes of facets for WUSLP. Facets are strong valid inequalities that provide information about the facial structure of a polyhedron that, not only may have substantial computational value when solving NP-hard combinatorial optimization problems, but also may prove useful in the design and development of solution algorithms for those problems. It is demonstrated in Reilly and Mata [40] that the addition of valid inequalities can make standard, general-purpose solution procedures more effective when solving SLP.

One of the main motivations for this research is that for several difficult optimization problems, such as the travelling salesman problem and the vertex packing
Figure 1.2: A Two Satellite System
problem, some of the best available algorithms use valid inequalities and facets.

One of the main reasons to consider WUSLP first, instead of SLP, is that the mathematical models for WUSLP are simpler. Therefore, the results obtained are used to lay the foundations for the more complex problem, SLP. In addition, a feasible solution always exists for WUSLP, but not necessarily for SLP. We show that the problem of finding a feasible solution to SLP is NP-complete. (This result also appears in [40].) Even though finding an optimal solution to WUSLP is very difficult, we show that finding a feasible solution is easy.

One important optimization problem that is related to SLP is the single-machine, or one-processor, scheduling problem (SMP) with symmetric earliness and tardiness penalties. In this research we show the similarities between WUSLP (SLP) and SMP.

In the next chapter, we discuss the significance of this research and present a survey of the literature for satellite system synthesis problems and for single-machine scheduling problems that are related to WUSLP (SLP). Chapter III presents some definitions and results from polyhedral theory that are fundamental to this research. In Chapter IV we present notation and mathematical formulations for WUSLP. We project the WUSLP polyhedron and prove that the projected polyhedron is full dimensional. In Chapter V we show that every facet of the projected polyhedron also yields a facet of the WUSLP polyhedron. In Chapter VI, we characterize the vertices and extreme directions of the convex hull of the projected
polyhedron associated with a given ordering of the satellites. We extend these results by not restricting the order of the satellites. The results of Chapter VI are used in Chapter VII, to determine the blocking polyhedron and to find facets of the projected polyhedron. One of the main results is that any inequality that defines a facet for a subset of the satellites also defines a facet for the WUSLP polyhedron, i.e., the WUSLP polyhedron inherits all the facets of any subset of satellites. From a known result from blocking polyhedral theory, there is a one-to-one correspondence between the vertices of the blocking polyhedron and the facets of the projected polyhedron. Therefore, we focus on determining the blocking polyhedron of the projected polyhedron. This result combined with the inheritance properties provides a procedure to determine facets of WUSLP. Necessary and sufficient conditions for the valid inequalities for SLP in [40] to induce facets for the WUSLP polyhedron for the two-satellite case are given. Computational results using the facets obtained in Chapter VII are presented in Chapter VIII. In Chapter IX we show that every facet for WUSLP defines a facet for a problem which is related to SLP. In Chapter X we show the relation between SMP and SLP. Finally, in Chapter XI we summarize our research.
CHAPTER II

Background

Satellite system synthesis (SSS) problems have received a lot of attention in the literature. Most of this literature has been about satellites in the Broadcasting Satellite Service (BSS); however, some of the approaches derived for BSS satellites can be used for the Fixed Satellite Service (FSS) satellites with minor, if any, modifications. While the BSS is for point-to-multipoint communication like television broadcasting, the FSS is for point-to-point communication like telephone and telegram services. The motivation for SSS problems is the need to allot some combination of orbital locations, frequencies (channels), and polarizations to satellites. The goal is to maintain electromagnetic interference at or below a specific threshold and to guarantee certain minimum elevation angles for the satellites. WUSLP and SLP are SSS problems which treat only the allotment of orbital locations.

In the next section we discuss the significance of this research. Section 2.2 provides a review of the literature on SSS problems with special emphasis on SLP and related problems. As mentioned previously, one important optimization
problem that is related to SLP is SMP. In Section 2.3 we describe SMP and review the scheduling literature relevant to this research.

2.1 Significance of the research

The need for geostationary orbital allotments for existing and future satellites of both developing and developed nations emphasizes not only the technological, but also the political and economic, importance of this research. New Scientist Magazine [44] reports, "Poor nations are worried that rich nations will not guarantee them room in orbit. Rich nations are at each other's throats over what they claim are their future needs".

In the technological sense, WUSLP (SLP) is a very difficult problem since there are \( n! \) possible orderings of \( n \) satellites and there is (may be) an infinite number of allotments of locations to the satellites that are possible for each ordering. Furthermore, as Gonsalvez [16] points out, the interactions between the administrations' geometries (size, shape, and location on the Earth of nations or combinations of nations) and the nature of the electronic signals transmitted between the satellites and Earth stations makes the development and solution of mathematical models for WUSLP (SLP) extremely complex.

The determination of how the GSO resource is going to be distributed among the nations of the world in such a way that existing and future communication needs of all nations are met is an important and difficult problem that requires the
cooperation of all nations. The International Telecommunications Union (ITU) facili-
tates this cooperation by periodically sponsoring Regional Administrative Radio
Conferences and World Administrative Radio Conferences (WARCs) where solu-
tions to satellite synthesis problems are negotiated by ITU members. For example,
delegates from more than 100 countries met in Geneva at the 1988 Space WARC
(WARC-88) to establish rules for the sharing of the GSO by communications satel-
lites in the FSS.

The focus of this research is on WUSLP, which, like SLP, treats the allotment
of orbital locations only. We assume that every satellite has access to the same
frequencies. This assumption is consistent with a recommendation from WARC-
85 [54]. The 1985 WARC recommended that ITU members should have a guar-
antee, that by the time they are prepared to launch a communications satellite,
at least one orbital position and access to 800 MHz of bandwidth be available to
them. In addition, as Gonsalvez [16] points out, if a SSS problem can be solved by
assigning orbital locations only, under the assumption that all satellites use a com-
mon co-polarized channel, that solution is preferred to one which assigns locations,
frequencies and polarizations since any solution to the problem that assigns only
orbital locations is a solution to the problem that assigns locations, frequencies,
and polarizations, but the converse is not true.
2.2 Modeling approaches to satellite system synthesis problems

The application of optimization techniques for allocating the geostationary orbital resource and the frequency spectrum to the nations of the world has stimulated tremendous interest in the international telecommunications community. Alternative mathematical programming models for SSS problems have been developed, and the solution methods suggested to solve these models vary from exact to heuristic procedures.

Cameron [6], Levis et al. [26], and Mathur et al. [30] propose integer programming models for frequency assignment problems. Heuristic procedures to assign orbital locations, frequencies, and polarizations to satellites in the BSS have been developed by Chouinard and Vachon [7], Christensen [8], and Nedzela and Sidney [34]. Spälti et al. [48] present three heuristics based on neighborhood search techniques in order to minimize the maximum single-entry interference.

Ito et al. [22] developed a nonlinear programming model for allotting orbital locations subject to single-entry and aggregate interference requirements. Their objective is to minimize the total orbital arc occupied. A special-purpose mathematical programming package, ORBIT-II, was developed to solve the problem considered by Ito et al. ORBIT-II uses an *evolutional model* in which the satellites are positioned in the GSO, one by one, according to a prespecified launching order. The evolutional model does not necessarily give an optimal ordering of the
satellites.

ORBIT-II was used to find a trial synthesis plan at WARC-88. The trial plan devised by ORBIT-II was not acceptable because too many satellite administrations would have been subjected to excessive interference if the trial plan were implemented. Heyward et al. [21] point out that an international panel of experts was convened at WARC-88 to manually adjust the ORBIT-II plan. This experience at WARC-88 underscores the importance of having reliable procedures and software for satellite synthesis problems.

Levis et al. [25] developed a gradient search procedure to solve a nonlinear programming model for allotting orbital locations and assigning frequencies to BSS satellites. Reilly et al. [39] suggested solving the same model by using a cyclic coordinate search procedure.

The approaches suggested by Ito et al. [22] and Levis et al. [25] require many complex calculations to determine aggregate interferences. Wang [53] developed a procedure to determine the required angular separation to control pairwise, or single-entry, interference. It is possible to satisfy a specified requirement for aggregate interference by specifying a higher requirement for single-entry interference. Wang's pairwise separation concept is used in mathematical formulations of SLP where a stringent single-entry interference criterion is used to approximate an aggregate interference criterion.

Reilly et al. [41] have proposed several mixed-integer programming (MIP)
formulations for different SSS problems, including a model with the same objective function considered by Ito et al. [22], minimizing orbital arc consumption. Linear programming formulations, with a set of nonlinear side constraints, are presented as an alternative to MIP formulations. Reilly [38] presents an MIP model for the arc allotment problem and shows that this problem is related to the problem considered by Ito et al. [22]. Nine different models to solve SSS problems are considered by Bhasin and Reilly [5].

Spälti [47] addresses the problem of minimizing the maximum single-entry interference subject to angular separation and service arc constraints. She refers to this problem as the satellite placement problem (SPP). A heuristic based on a tabu search technique and a specialized network simplex algorithm are presented. The heuristic solves the resulting linear program after generating a satellite ordering. Given an ordering of the satellites, the dual of SPP can be formulated as a network flow problem with one side constraint. A specialized simplex algorithm that solves this network problem is presented. Finally, she presents some simple facets of the SPP polytope and shows its relation to the linear ordering polytope.

Mount-Campbell et al. [32] developed a MIP model for solving SLP. This model treats SLP as a combination of two subproblems: (i) determining the ordering of the satellites in the GSO and (ii) locating (allotting longitudes to) the satellites in the GSO given a specific satellite ordering. Even though SLP is a hard optimization problem, once the order of the satellites is fixed, SLP reduces
to a linear programming (LP) problem. Consequently, the solution approach they suggest is Benders decomposition.

A $k$-permutation heuristic for SLP is presented in Gonsalvez [16]. The MIP formulation used is from Mount-Campbell et al. [32]. After selecting an ordering of the satellites (i.e., fixing the integer variables), the $k$-permutation procedure solves the dual of the remaining LP to find the optimal satellite locations for the given ordering. Next, the heuristic continues by permuting groups of $k$ ($2 \leq k \leq k_{\text{max}}$) adjacent satellites and reoptimizing the dual LP when necessary.

Gonsalvez [16] compared four solution strategies to solve SLP: branch-and-bound (optimal procedure), Benders decomposition as presented in [32], LP with restricted basis entry to solve the linear program with nonlinear side constraints presented in [41], and the $k$-permutation heuristic. The results of his analysis indicate that the $k$-permutation procedure outperforms the other solution methods by providing solutions of acceptable quality in a reasonable amount of time. However, a big disadvantage of the $k$-permutation procedure is that it requires an initial ordering of the satellites. If this initial ordering is not very "good", then the method may take a lot of time to find a feasible solution or may not find one even if one exists.

Reilly [37] developed a greedy heuristic procedure (OSU-SLOT) to find initial orderings for the $k$-permutation method. This heuristic procedure has not only found good initial orderings, but also has found solutions to large instances of SLP.
for which the $k$-permutation procedure failed to find a feasible solution \cite{42}.

A two-phase procedure for solving SLP, where SLP is formulated as in \cite{32}, is presented in Reilly et al. \cite{42}. Two heuristics, OSU-SLOT and a modified version of the $k$-permutation procedure (OSU-STARS), are used in tandem to find feasible solutions to SLP. The first phase finds an ordering of the satellites by using the OSU-SLOT heuristic and the second phase uses OSU-STARS to find improved satellite locations by permuting small groups of adjacent satellites. Experience indicates that this procedure provides a variety of feasible solutions.

Reilly et al. \cite{43} describe how OSU-SLOT is imbedded in a binary search routine to approximate the greatest feasible uniform increment to the angular separation values. By adding a uniform increment to the angular separation values, OSU-SLOT is likely to minimize the greatest aggregate interference received by any satellite.

SLP is a difficult optimization problem. In Chapter IX we show that SLP is NP-Complete. (This result also appears in \cite{40}). Reilly and Mata \cite{40} show that the bound from the LP relaxation of their formulation of SLP is zero in many cases. By examining the parameters of SLP, they establish bounds that dominate the LP relaxation under a mild condition. They also show that additional bounds can be found by decomposing SLP. An important contribution of their research is a collection of valid inequalities for SLP. A discussion of how these valid inequalities are constructed by using the bounds is presented.
2.3 Review of the single-machine scheduling problem with earliness and tardiness penalties

Consider the scheduling of $n$ jobs on a single machine. For each job there is a processing time that is assumed to be independent of the job sequence and there is a window of time in which the job's processing must be started. This window is defined by a release time and a starting deadline, and is assumed to contain a specific starting due date. In addition, for each job there is a setup time that is independent of the job sequence. For each pair of jobs $i, j$, there is a minimum required interval of time between starting job $i$ and starting job $j$. We define this time as the sum of the processing time of job $i$, plus the setup time of the machine to process job $j$, plus a waiting time when job $j$ is processed after (not necessarily immediately after) job $i$. If the waiting time is zero, the problem reduces to one in which the starting time of job $j$ depends only on the previous job $i$. Otherwise, the starting time of one job can be affected by the starting time of any other job.

In SMP, jobs are scheduled on a single machine subject to window and minimum required time interval constraints. The objective is to minimize the total deviation between the starting times of the jobs and their corresponding starting due dates. For many years a great deal of the sequencing and scheduling literature focused on problems with performance measures which are nondecreasing in job completion times. These performance measures are referred to as regular measures. However, with the advent of the just-in-time philosophy, the focus has changed to
scheduling problems that consider nonregular measures. Scheduling problems that penalize both earliness and tardiness are among the most important problems of this type. Notice that WUSLP and SLP have objective functions involving a nonregular measure, i.e., WUSLP and SLP penalize both easterly (earliness) and westerly (tardiness) deviations from the satellites' desired locations.

In 1981, Kanet [24] analyzed the problem of minimizing the average deviation, or the total unweighted earliness and tardiness, about a common due date. He provides a polynomial time algorithm that finds a single optimal schedule under the assumption that the due date is not restrictive, i.e., the due date is larger than the sum of the jobs' processing times. For the same problem, Hall [18] and Bagchi et al. [1] discuss necessary and sufficient conditions for a schedule to be optimal and present algorithms capable of finding alternative optimal schedules.

Szwarc [49] considers another version of Kanet's problem by allowing the possibility for the common due date to occur before all the jobs can be processed. A branch-and-bound algorithm which makes use of optimality conditions is provided. Hall et al. [19] demonstrate that the problem considered by Szwarc is NP-complete in the ordinary sense. They describe a dynamic programming algorithm which runs in pseudo-polynomial time.

Hall and Posner [20] generalize Kanet's problem by allowing symmetric weights (equal earliness and tardiness weights) even though the penalties may differ among jobs. An important contribution of their research is that this version of the problem
is NP-complete in the ordinary sense. They present optimality conditions and an efficient pseudo-polynomial time algorithm.

Garey et al. [14] address the problem of minimizing the sum of earliness and tardiness on a single processor where jobs may have distinct due dates. They show that this problem is NP-complete and present efficient algorithms for two special cases. Baker and Scudder [2] present a review of the literature and consolidate many of the existing results on machine scheduling problems with earliness and tardiness problems.

Balas [3] presents a number of properties about scheduling polyhedra. He derives facets for the clique scheduling polyhedron with one, two, or three nonzero coefficients. In addition, he presents a sufficient condition for an inequality that induces a facet of the clique scheduling polyhedron to induce a facet of the scheduling polyhedron. Queyranne [36] considers the structure of a one-machine scheduling polyhedron. He presents all the linear inequalities defining the convex hull of feasible schedules. In addition, a simple separation algorithm that generates cutting planes for more complex scheduling problems is presented. Many of the results from Balas and Queyranne are used in this research.

Dyer and Wolsey [11] study the polyhedra of the one-machine sequencing problem with release dates. Valid inequalities for the general problem with release dates are presented. In addition, they examine several relaxations of MIP formulations in terms of the strength of the lower bounds obtained. Sheu [46] studies sev-
eral approaches for job-shop scheduling problems. An improved formulation for a $m$-machine, $n$-job scheduling problem is obtained. By projecting out a set of variables, all of Queyrane’s facets are obtained for the case when $m = 1$. In addition, several facets for the general tardiness problem are presented.
CHAPTER III

Results from Linear Algebra and Polyhedral Theory

In this chapter we review some standard definitions and results from linear algebra and polyhedral theory that are relevant to this research. These results are from [4,33, and 45]. Let \( \mathbb{R}^n \) be the set of real \( n \)-dimensional vectors. (We only distinguish between row and column vectors when confusion may arise.) The following convention for vectors will be used: If \( x, y \in \mathbb{R}^n \), then \( x = (\geq) y \) if and only if \( x_i = (\geq) y_i \), \( i = 1, \ldots, n \). A vector with zero components, except for a 1 in the \( k \)-th position, is called a unit vector and it is denoted by \( e_k \).

Proposition III.1 (Cramer's Rule) Let \( A \) be an \( n \times n \) matrix, \( x \) be an \( n \) vector of unknowns, and \( b \) be an \( n \) vector. If the determinant of \( A \), denoted by \( \text{det}(A) \), is different from zero, then the unique solution for the system of linear equations \( Ax = b \) is given by

\[
    x_j = \frac{\text{det}(A_j)}{\text{det}(A)}, \quad j = 1, 2, \ldots, n
\]

where \( A_j \) is the matrix obtained from \( A \) with the \( j \)-th column replaced by \( b \).
**Definition III.1** A vector $x \in \mathbb{R}^n$ is a convex combination of the vectors $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ if $x = \sum_{t=1}^{k} \gamma_t x^t$, $\sum_{t=1}^{k} \gamma_t = 1$, and $\gamma_t \geq 0$, $\ell = 1, 2, \ldots, k$.

**Definition III.2** A set $S \subseteq \mathbb{R}^n$ is called a convex set if for every $x^1, x^2 \in S$, $\gamma x^1 + (1 - \gamma) x^2 \in S$ for each $\gamma \in [0, 1]$.

**Definition III.3** The convex hull of a set $S \subseteq \mathbb{R}^n$, denoted by $\text{Conv}(S)$, is the smallest convex set in $\mathbb{R}^n$ containing $S$ or the set of all points that are convex combinations of points in $S$.

**Theorem III.1** (Carathéodory) Given a set $S \subseteq \mathbb{R}^n$, $x \in \text{Conv}(S)$ if and only if $x$ is a convex combination of $n+1$ (not necessarily distinct) points of $S$.

**Definition III.4** The vectors $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are linearly independent if $\sum_{t=1}^{k} \gamma_t x^t = 0$ implies $\gamma_t = 0$, $\ell = 1, 2, \ldots, k$.

**Definition III.5** The vectors $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if $\sum_{t=1}^{k} \gamma_t x^t = 0$ and $\sum_{t=1}^{k} \gamma_t = 0$ imply $\gamma_t = 0$, $\ell = 1, 2, \ldots, k$.

**Proposition III.2** Linear independence $\Rightarrow$ affine independence, but affine independence $\nRightarrow$ linear independence.

**Proposition III.3** $x^1, x^2, \ldots, x^k$ are affinely independent if and only if $x^2 - x^1, x^3 - x^1, \ldots, x^k - x^1$ are linearly independent.
Definition III.6 A set $S \subseteq \mathbb{R}^n$ is called a polyhedron if $S = \{x \in \mathbb{R}^n : Ax \geq b\}$, where $[A, b]$ is a $m \times (n + 1)$ matrix. If $S$ is bounded then it is called a polytope.

Definition III.7 A polyhedron $S \subseteq \mathbb{R}^n$ is of dimension $k$, written as $\text{dim}(S) = k$, if the maximum number of affinely independent points in $S$ is $k + 1$. If $\text{dim}(S) = n$, then $S$ is full dimensional.

Definition III.8 An inequality $\alpha x \geq \alpha_0$ is a valid inequality for $S \subseteq \mathbb{R}^n$ if $\alpha x \geq \alpha_0 \forall x \in S$, i.e., $S \subseteq \{x \in \mathbb{R}^n : \alpha x \geq \alpha_0\}$.

Definition III.9 An inequality $\alpha x \geq \alpha_0$ is said to define or induce an $(m - 1)$-dimensional face of a polyhedron $S$, if it is a valid inequality and there exist $m$ ($m \leq \text{dim}(S)$) affinely independent points $x^t \in S$ that satisfy $\alpha x^t = \alpha_0$. If $m = \text{dim}(S)$, the inequality is said to define a facet, i.e., a facet is a $(\text{dim}(S) - 1)$-dimensional face of $S$.

A face (or facet) $F$ of $S$ is denoted by $F = S \cap \{x : \alpha x = \alpha_0\}$, i.e., a face of $S$ is the intersection of $S$ and one of its boundary hyperplanes. However, as is common in the literature, we refer to the inequality $\alpha x \geq \alpha_0$ itself as a face (or facet). $F$ is a proper face of $S$ if $\emptyset \neq F \neq S$, where $\emptyset$ is the empty set. A minimal face is a proper face which does not contain any other proper face. A facet is a proper face not included in any other face.
Proposition III.4 Suppose $S \subseteq \mathbb{R}^n$ and $\alpha x \geq \alpha_0$ defines a facet of $\text{Conv}(S)$. Then, there exist $n$ affinely independent points $x^\ell \in S$, $\ell = 1, 2, \ldots, n$ such that $\alpha x^\ell = \alpha_0$.

Proposition III.5 Two valid inequalities $\alpha x \geq \alpha_0$ and $\beta x \geq \beta_0$ are equivalent if $\alpha = \gamma \beta$ and $\alpha_0 = \gamma \beta_0$ for some $\gamma > 0$. If they are not equivalent and there exists a $\lambda > 0$ such that $\beta \leq \lambda \alpha$ and $\beta_0 \geq \lambda \alpha_0$ then $\beta x \geq \beta_0$ dominates or is stronger than $\alpha x \geq \alpha_0$, since any $x \in \mathbb{R}^n$ that satisfies $\beta x \geq \beta_0$ also satisfies $\alpha x \geq \alpha_0$. A facet is a nondominated valid inequality.

To present the following results we assume that $S = \{x \in \mathbb{R}^n : Ax \geq b\}$ where $[A, b]$ is a $m \times (n + 1)$ matrix and $S \neq \emptyset$. Let $(a^i, b^i)$ denote row $i$, $i = 1, 2, \ldots, m$, of the matrix $[A, b]$. Let $[A^=, b^=]$ be the rows of $[A, b]$ such that $a^i x = b^i$, $\forall x \in S$, and let $[A^\geq, b^\geq]$ be the rows of $[A, b]$ such that $a^i x > b^i$ for some $x \in S$. Then $S = \{x \in \mathbb{R}^n : A^= x = b^=, A^\geq x \geq b^\geq\}$.

Now we give the definition of an interior point.

Definition III.10 The point $x \in S$ is interior if $a^i x > b^i$ for $i = 1, 2, \ldots, m$.

Proposition III.6 Given a polyhedron $S \subseteq \mathbb{R}^n$, $\text{dim}(S) + \text{rank}[A^=, b^=] = n$

Proposition III.7 $S$ is full dimensional if and only if $S$ has an interior point.

Definition III.11 A point $x \in S$ is a vertex or an extreme point of $S$ if $x$ is not a convex combination of two other points in $S$, i.e., there do not exist $x^1, x^2 \in S$, $x^1 \neq x^2$ such that $x = (x^1 + x^2)/2$. 
The following proposition shows the relation between an extreme point and a basis.

**Proposition III.8** For every basis there corresponds a unique extreme point. For every extreme point there corresponds at least one basis.

**Definition III.12** The vector \( q \) is called a direction of the polyhedron \( S \) if for each \( x \in S \), the point \( x' = x + \lambda q, \lambda \geq 0 \) also belongs to \( S \). A polytope has no directions.

**Proposition III.9** The vector \( q \) is a direction of the nonempty set \( \{x : Ax \geq b, x \geq 0\} \) if and only if \( q \neq 0, q \geq 0, \) and \( Aq \geq 0 \).

**Definition III.13** A direction \( q \) of \( S \) is an extreme direction if \( q \) is not a convex combination of any other directions of \( S \).

**Theorem III.2** (Minkowski) \( S \) can be represented as a convex combination of its extreme points plus a nonnegative combination of its extreme directions.

**Definition III.14** \( H \subseteq \mathbb{R}^n \) is a subspace if

1. \( x \in H \Rightarrow \lambda x \in H, \forall \lambda \in \mathbb{R} \)

2. \( x, y \in H \Rightarrow x + y \in H \).

**Definition III.15** The point \((x, 0)\) is the projection of a point \((x, y) \in \mathbb{R}^n \times \mathbb{R}^n\) onto \( H = \{(x, y) : y = 0\} \). The projection from \((x, y)\)-space to \( x\)-space is the
projection of a polyhedron \( S \subseteq \mathbb{R}^n \times \mathbb{R}^n \) onto \( y = 0 \) and is denoted by \( \text{proj}_x(S) \), i.e.,

\[
\text{proj}_x(S) = \{(x, 0) \in \mathbb{R}^n \times \mathbb{R}^n : (x, y) \in S\}.
\]

Let \( S = \{x \in \mathbb{R}^n : x \geq 0, \ A x \geq 1\} \), where \([A, 1]\) is a \( m \times (n + 1)\) nonnegative matrix. There is a unique blocker or blocking polyhedron \( S^B \) associated with \( S \).

Let \( V(S) \) denote the set of vertices of \( S \). Then,

\[
S^B = \{u \in \mathbb{R}^n : u \geq 0 \text{ and } xu \geq 1, \ \forall \ x \in S\}
\]

\[
= \{u \in \mathbb{R}^n : u \geq 0 \text{ and } vu \geq 1, \ \forall \ v \in V(S)\}
\]

\[
= \{u \in \mathbb{R}^n : u \geq 0 \text{ and } Bu \geq 1\}
\]

where \( B \) is a \(|V(S)| \times n\) matrix whose rows are the elements of \( V(S) \). \( S \) and \( S^B \) are called a blocking pair of polyhedra, and the pair \( A, B \) is called a blocking pair of matrices.
CHAPTER IV

WUSLP notation and preliminaries

Let $N = \{1, \ldots, n\}$ be the set of indexes of the satellites that are to be positioned in the GSO and $E_j$ be the easternmost feasible location for satellite $j \in N$. For each satellite $j \in N$, there is a specific desired location $D_j$, and for each pair of satellites $i, j \in N$, there is an angular separation $\Delta_{i,j}$ that must be enforced to control single-entry interference between satellites $i$ and $j$. We let $\Delta_{i,i} = 0$, $\forall i \in N$, and $\Delta_{i,j} = \Delta_{j,i}$, $\forall i, j \in N$, such that $i < j$. Without loss of generality, we assume that $D_j \geq E_j \geq 0$, $\forall j \in N$, and that $\min_{j \in N} \{E_j\} = 0$.

A solution is defined to be a vector of feasible satellite locations. Hence, a solution is a vector $x \in R^n$, where $x_j$ denotes the location in degrees longitude of satellite $j \in N$, satisfying the following constraints:

$$x_j - x_i \geq \Delta_{i,j} \quad \forall \quad x_i - x_j \geq \Delta_{i,j} \quad \forall i, j \in N \text{ such that } i < j; \quad (4.1)$$

$$x_j \geq E_j \quad \forall j \in N. \quad (4.2)$$
Constraints (4.1) enforce sufficient angular separation in the GSO between each pair of satellites, not only adjacent satellites, to guarantee that no single-entry electromagnetic interference exceeds a specified threshold. Constraints (4.2) guarantee that each satellite is located west of its easternmost feasible location.

Let $\mathcal{X}(N)$ denote the set of all solutions, i.e.,

$$\mathcal{X}(N) = \{ x \in \mathbb{R}^n : x_j - x_i \geq \Delta_{i,j} \quad \forall \quad x_i - x_j \geq \Delta_{i,j} \quad \forall i, j \in N \text{ such that } i < j;$$

$$x_j \geq E_j \quad \forall j \in N \}$$

An example of the set $\mathcal{X}(N)$ for a two-satellite case is presented in Figure 4.1. Notice that for this figure $|E_i - E_j| < \Delta_{i,j}$. A similar figure for the two-job, one-machine scheduling problem appears in [36].

The problem at hand can be formally stated as follows:

(WUSLP')

$$\min \sum_{j \in N} |x_j - D_j|$$

Subject to

$$x \in \mathcal{X}(N).$$

We denote by $N^k$ the set $\{1, 2, \ldots, kn\}$. The notation for the set $\mathcal{X}(N)$ is abbreviated to $\mathcal{X}$ when no confusion arises. When considering just a subset of the satellites $K \subset N$ in the set $S$, we denote the set analogous to $S$ as $S(K)$. 
Figure 4.1: The set $\mathcal{X}(N)$ for the two-satellite case.
A one-to-one mapping $\pi$ of the set $N$ onto itself is called an ordering or permutation of the satellites. We denote the ordering $\pi \in R^n$ by $\pi = (j_1, j_2, \ldots, j_n)$, where $j_i = \pi(i)$ indicates the satellite in the $i$-th position from the east. The number of such orderings is $n!$ We denote the set of those orderings by $\Pi_N$. We use $\pi^{-1}(j) = i$ to denote that satellite $j$ is in the $i$-th position from the east.

Whenever $x_i = x_j$ for some $i, j \in N$ such that $i < j$, we let $i$ precede $j$ in the ordering associated with that solution. Therefore, notice that there is a unique ordering associated with each solution, but there is an infinite number of solutions associated with an ordering.

We observe the similarity between the set $\mathcal{X}$ and the set of schedules in a clique studied by Balas [3]. The main difference is that we do not assume that the triangle inequality holds for the angular separations. We enforce the angular separations between all pairs of satellites, not only adjacent ones. In other words, there may exist $i, j, k \in N$ such that $\Delta_{i,j} + \Delta_{j,k} < \Delta_{i,k}$. Another difference is that the angular separations are symmetric in our problem.

Because of the similarities between $\mathcal{X}$ and the set of schedules in a clique, we may think that it is sufficient to find the facets of the clique scheduling polyhedron to solve WUSLP'. In this research we show that this is not the case.

Let

$$Q' = \{ (x, y, y^+) \in R^n \times R^n \times R^n : x_j + y_j - y_j^+ = D_j \quad \forall j \in N \}$$  \hspace{1cm} (4.3)
\[ y_j, y_j^+ \geq 0, \quad \forall j \in N; \quad x \in X \]

where \( y_j \) (\( y_j^+ \)) is the deviation to the east (west) of the allotted location of satellite \( j \) from its desired location. Note that

\[
y_j = \begin{cases} 
D_j - x_j & \text{if } x_j < D_j, \\
0 & \text{otherwise}
\end{cases} \quad (4.4)
\]

\[
y_j^+ = \begin{cases} 
x_j - D_j & \text{if } x_j > D_j, \\
0 & \text{otherwise}
\end{cases} \quad (4.5)
\]

and \( y_j + y_j^+ = |x_j - D_j|, \quad \forall j \in N. \)

The set \( Q' \) includes all the solutions of \( X \), and it provides, by means of the constraints \( x_j + y_j - y_j^+ = D_j \), the deviations of the satellites’ prescribed locations from their desired locations.

Problem \((WUSLP')\) can be transformed to the following disjunctive programming problem:

\[
(WUSLP)
\min \sum_{j \in N} (y_j + y_j^+)
\]

Subject to

\[
(x, y, y^+) \in Q'.
\]

\( \text{Conv}(Q') \) is not full dimensional since there are equality constraints in (4.3). We prefer a full-dimensional polyhedron, since each facet of such a polyhedron has a unique linear inequality (up to a positive scalar multiplication) inducing that facet. So that we may work with a full-dimensional polyhedron, we project
out the variables $y^+$ of WUSLP. In Chapter V, we show that every facet of the projected polyhedron also yields a facet of $\text{Conv}(Q')$.

Let

$$ Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x_j + y_j \geq D_j \quad \forall j \in N; \quad (4.6) $$

$$ y_j \geq 0 \quad \forall j \in N; \quad (4.7) $$

$$ x \in \mathcal{X}. \quad (4.8) $$

Constraints (4.6) result from projecting out the variables $y^+$ of WUSLP\(^1\). Constraints (4.7) are the nonnegativity constraints for the variables $y$. Constraints (4.8) guarantee that every $(x, y) \in Q$ is associated with a unique $x \in \mathcal{X}$. Therefore, after projecting out the variables $y^+$, WUSLP is transformed to the following problem\(^2\):

(QWUSLP)

$$ \min \sum_{j \in N} (x_j + 2y_j - D_j) $$

Subject to

$$(x, y) \in Q.$$ \( \quad \) Under the assumption that the penalties are linear, the transformation that we present here can be applied to a more generic earliness and tardiness model that

\(^1\)We observe that the $y^+$ variables play the role of surplus variables in WUSLP. By substituting $y_j^+ = x_j + y_j - D_j$, $\forall j \in N$ in WUSLP, we obtain a new problem which is equivalent to WUSLP but which is expressed in terms of the variables $x$ and $y$ only [35]. Even though we could work directly with $\text{Conv}(Q)$ for completeness we show the relation between $\text{Conv}(Q)$ and $\text{Conv}(Q')$.

\(^2\)Note that we can exclude the term $-\sum_{j \in N} D_j$ from the objective function.
can be written as

\[ \{ \min \sum_{j \in N} (\gamma_j \max(0, D_j - x_j) + \rho_j \max(0, x_j - D_j)) : x \in \mathcal{X} \} \]

where \( \gamma_j (\rho_j) \) is a unit earliness (tardiness) penalty. After adding the deviational variables \( y \) and \( y^+ \) and projecting out the variables \( y^+ \), we obtain the following problem

\[ \{ \min \sum_{j \in N} (\rho_j x_j + (\rho_j + \gamma_j) y_j - D_j) : (x, y) \in \mathcal{Q} \}. \]

In Theorem IV.1 we show that \( \text{Conv}(Q) \) is the resulting polyhedron when projecting out the variables \( y^+ \) of \( \text{Conv}(Q') \).

**Theorem IV.1** \( \text{Conv}(Q) = \text{proj}_{(x,y)}(\text{Conv}(Q')) \)

**Proof.** The proof is based on Corollary I.4.4.12 from [33]. We need to prove that:

i) \( \text{Conv}(Q) \subseteq \text{proj}_{(x,y)}(\text{Conv}(Q')) \) and ii) \( \text{Conv}(Q) \supseteq \text{proj}_{(x,y)}(\text{Conv}(Q')) \).

i) This is equivalent to proving that, for each \((x, y) \in \text{Conv}(Q)\), there exists a \( y^+ \) such that \((x, y, y^+) \in \text{Conv}(Q')\). We divide this part of the proof into two cases.

Case 1) Suppose \((x, y) \in Q\). Let \( y_j^+ = x_j + y_j - D_j \). From (4.6) this implies that \( y_j^+ \geq 0 \), \( \forall j \in N \). Thus, \((x, y, y^+) \in Q'\) and \((x, y, y^+) \in \text{Conv}(Q')\).

Case 2) Suppose \((x, y) \in \text{Conv}(Q) \setminus Q\). Then by Carathéodory's theorem (Theorem III.1), \((x, y)\) is a convex combination of \( 2n+1 \) points of \( Q \). Thus, there exists
a $\gamma \in \mathbb{R}^{2n+1}$, $0 \leq \gamma \leq 1$, and $\sum_{t=1}^{2n+1} \gamma_t = 1$ such that

$$(x, y) = \sum_{t=1}^{2n+1} \gamma_t (x^t, y^t)$$

where $(x^t, y^t) \in Q$, $\forall \ell \in \{1, 2, \ldots, 2n + 1\}$. By Case 1), since $(x^t, y^t) \in Q$, there exist a $(y^+)^t$ such that $(x^t, y^t, (y^+)^t) \in Q'$, $\forall \ell \in \{1, 2, \ldots, 2n + 1\}$, and

$$(x, y, y^+) = \sum_{t=1}^{2n+1} \gamma_t (x^t, y^t, (y^+)^t)$$

is an element of $Conv(Q')$.

ii) This is equivalent to proving that if $(x, y, y^+) \in Conv(Q')$ then $(x, y) \in Conv(Q)$ since

$$proj_{(x,y)}(Conv(Q')) = \{(x, y, 0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : (x, y, y^+) \in Conv(Q') \}.$$ 

We also divide this part of the proof into two cases.

Case 1) Suppose $(x, y, y^+) \in Q'$. Since $x_j + y_j - y^+_j = D_j$, and $y^+_j \geq 0$, then $x_j + y_j \geq D_j$, $\forall j \in N$. Thus, $(x, y) \in Q$ and $(x, y) \in Conv(Q)$.

Case 2) Suppose $(x, y, y^+) \in Conv(Q') \backslash Q'$. Then, by Carathéodory's theorem, $(x, y, y^+) \in Conv(Q') \backslash Q'$ is a convex combination of $3n + 1$ points of $Q'$. Thus, there exists a $\gamma \in \mathbb{R}^{3n+1}$, $0 \leq \gamma \leq 1$, and $\sum_{t=1}^{3n+1} \gamma_t = 1$ such that

$$(x, y, y^+) = \sum_{t=1}^{3n+1} \gamma_t (x^t, y^t, (y^+)^t)$$

where $(x^t, y^t, (y^+)^t) \in Q'$. By case 1), $(x^t, y^t) \in Q$, $\forall \ell \in \{1, 2, \ldots, 3n + 1\}$. Thus,

$$(x, y) = \sum_{t=1}^{3n+1} \gamma_t (x^t, y^t)$$
is an element of $\text{Conv}(Q)$. □

Now we are able to show that it is not sufficient to find the facets of the clique scheduling polyhedron to solve WUSLP'. First, in Chapter VII we present inequalities that induce facets for WUSLP' and WUSLP when the triangle inequality is not satisfied. Second, and more important, even if all the facets of the clique scheduling polyhedron without the triangle inequality assumption (or the facets of $\text{Conv}(\mathcal{X})$) were known, it is possible that the optimal solution is not an extreme point since WUSLP' has a nonlinear, convex objective function involving nonregular measures, i.e., easterly (earliness) and westerly (tardiness) deviations from the satellites' desired locations are penalized in WUSLP'.

By introducing the deviational variables $y$ and $y^+$ we are able to consider a linear objective function in WUSLP. After projecting out the variables $y^+$, we obtain QWUSLP. The set $Q$ in QWUSLP can be described as $\mathcal{X} \cap Y$, where $Y$ is the feasible set for the linear constraints (4.6) and (4.7). Hence, $Y$ is a convex polyhedron. The question arises whether $\text{Conv}(\mathcal{X} \cap Y)$ is equal to $\text{Conv}(\mathcal{X}) \cap Y$. If $\text{Conv}(\mathcal{X} \cap Y) = \text{Conv}(\mathcal{X}) \cap Y$, then it suffices to derive the facet-defining inequalities of $\text{Conv}(\mathcal{X})$ (or the facet-defining inequalities of the clique scheduling polyhedron without the triangle inequality assumption) if we are interested in the facet-defining inequalities of $\text{Conv}(Q)$. An illustration of the difference between $\text{Conv}(\mathcal{X} \cap Y)$ and $\text{Conv}(\mathcal{X}) \cap Y$ is given in the following example.
Example IV.1 Consider the following disjunctive set,

$$\mathcal{X}_0 = \{x \in R : |x - d| \geq a, x \geq 0\}$$

where $a, d$ are scalars such that $d > a > 0$. Let $f(x) = |x - d|$.

Solving $P = \{\min f(x) : x \in \mathcal{X}_0\}$ is the same as looking for the point in $\mathcal{X}_0$ nearest to $d$. We observe that the set $\mathcal{X}_0$ is analogous to the set of schedules in a clique or to the set $\mathcal{X}$. The function $f$ is analogous to the objective function of WUSLP$'$.

Observe that $\text{Conv}(\mathcal{X}_0) = \{x \in R : x \geq 0\}$. Even though we are able to completely describe $\text{Conv}(\mathcal{X}_0)$, the solution to the problem $\{\min f(x) : x \in \text{Conv}(\mathcal{X}_0)\}$ is $f = 0, x = d$, which is not feasible for $P$. In fact, it is the same solution obtained when solving the linear programming relaxation of $P$, if $P$ were formulated as a mixed-integer linear problem.

Let $x + y - y^+ = d$ and let $\mathcal{Q}_0 = \{(x, y) \in R^2 : x + y \geq d, y \geq 0, x \in \mathcal{X}_0\}$. After transforming $P$ to a disjunctive problem with a linear objective function and projecting out the variable $y^+$, we obtain the problem $\{\min x + 2y - d : (x, y) \in \mathcal{Q}_0\}$.

The set $\mathcal{Q}_0$ can be described as $\mathcal{X}_0 \cap Y_0$ where $Y_0 = \{(x, y) \in R^2 : x + y \geq d, y \geq 0\}$. The set $\mathcal{Q}_0$ is presented in Figure 4.2. From that figure it follows that

$$\text{Conv}(\mathcal{Q}_0) = \{(x, y) \in R^2 : x + y \geq d; y \geq 0, x \geq 0, x + 2y \geq a + d\}.$$  

Consequently, $x + 2y \geq a + d$ is a facet for $\text{Conv}(\mathcal{X}_0 \cap Y_0)$ but not for $\text{Conv}(\mathcal{X}_0) \cap Y_0$. 
Notice that solving the problem \( \{ \min x + 2y - d : (x, y) \in \text{Conv}(Q_0) \} \) gives a solution \( x = d + a, \ y = 0 \) which is optimal for \( P \).

In Chapter VII we prove that any inequality that defines a facet of \( \text{Conv}(\mathcal{X}) \) (or the clique scheduling polyhedron) defines a facet of \( \text{Conv}(Q) \). In addition, we present inequalities that define facets of \( \text{Conv}(Q) \) that do not define facets of \( \text{Conv}(\mathcal{X}) \cap Y \).

Let

\[
\mathcal{X}_\pi = \{ x \in \mathbb{R}^n : x_{ji} - x_{jk} \geq \Delta_{jk,i}, \quad \forall i, k \in N \text{ such that} \}
\]
\[ k < i; \ x_j \geq E_j \quad \forall j \in N \] 

and

\[ Q_\pi = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \ x_j + y_j \geq D_j \quad \forall j \in N; \quad y_j \geq 0 \quad \forall j \in N; \quad x \in X_\pi \} \]  

where \( \pi = (j_1, j_2, \ldots, j_n) \).

\( Q_\pi \) is the set of solutions in \( Q \) that correspond to the ordering \( \pi \). Notice that all the constraints of \( Q_\pi \) are linear. Since \( Q \) is a disjunctive set, it can be expressed in disjunctive normal form [3], i.e., in the form

\[ Q = \bigcup_{\pi \in \Pi_N} Q_\pi. \]

Next, we show that \( \text{dim}(\text{Conv}(Q)) = 2n. \)

**Theorem IV.2** \( \text{Conv}(Q) \) is full dimensional.

**Proof.** Let

\[ \Delta_{\max} = \max_{i,j \in N} \{ \Delta_{i,j} \}. \]

We assume that \( \Delta_{\max} > 0 \), otherwise the solution to the problem is trivial. Let \( \pi = (j_1, j_2, \ldots, j_n) \) and \( \epsilon > 0 \). Define

\[ x_{j_1} = E_{j_1} + \epsilon; \]

\[ x_{ji} = \max\{E_{ji}, x_{ji-1} + \Delta_{\max}\} + \epsilon \quad \forall i \in N, \ i \neq 1; \]
and

\[ y_j = \max\{0, D_j - x_j\} + \epsilon, \quad \forall j \in N. \]

Then, \((x, y) \in Q_\pi\) is an interior point since all the constraints (4.9) are satisfied as inequalities. Since \(\text{Conv}(Q) = \text{Conv}(\bigcup_{\pi \in \Pi^N} Q_\pi)\), \(\text{Conv}(Q)\) has an interior point. Thus, \(\text{Conv}(Q)\) is full dimensional. □
CHAPTER V

A relation between $\text{Conv}(Q)$ and $\text{Conv}(Q')$

In this chapter we present the relation between $\text{Conv}(Q)$ and $\text{Conv}(Q')$. In Lemma V.1, we show that $k$ affinely independent points for $\text{Conv}(Q)$ can be extended to $k$ affinely independent independent points for $\text{Conv}(Q')$. In Lemma V.2, we show that $\dim(\text{Conv}(Q')) = \dim(\text{Conv}(Q))$. Finally, these lemmas are used in Theorem V.3 to show that every facet of $\text{Conv}(Q)$ also yields a facet of $\text{Conv}(Q')$.

 Lemma V.1 The points $(x^\ell, y^\ell), \ell = 1, 2, \ldots, k, k \leq 2n + 1$ are affinely independent for $\text{Conv}(Q)$ if and only if there exists $(y^+)^\ell$ such that $(x^\ell, y^\ell, (y^+)^\ell)$ are affinely independent for $\text{Conv}(Q')$.

Proof. $\Rightarrow$ Suppose $(x^\ell, y^\ell), \ell = 1, 2, \ldots, k$ are affinely independent points for $\text{Conv}(Q)$. From Theorem IV.1, for each $(x^\ell, y^\ell)$ there exists a $(y^+)^\ell$ such that $(x^\ell, y^\ell, (y^+)^\ell) \in \text{Conv}(Q')$.

Let $M$ be the $(k - 1) \times 3n$ matrix whose rows are the vectors $(x^\ell, y^\ell, (y^+)^\ell) - (x^1, y^1, (y^+)^1), \ell = 2, 3, \ldots, k$, and whose columns are ordered according to

\[x_{j_1}, x_{j_2}, \ldots, x_{j_n}, y_{j_1}, y_{j_2}, \ldots, y_{j_n}, y_{j_1}^+, y_{j_2}^+, \ldots, y_{j_n}^+\]

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Then, \( M \) is of the form
\[
M = \begin{bmatrix} M_{11} & M_{12} \end{bmatrix}
\]
where \( M_{11} \) is a \((k - 1) \times 2n\) matrix and \( M_{12} \) is a \((k - 1) \times n\) matrix. From Proposition III.3, the rank of \( M_{11} \) is \( k - 1 \) since the \((x^t, y^t)\) are affinely independent. Thus, the rank of \( M \) is \( k - 1 \). From Proposition III.3 again, \((x^t, y^t, (y^+)^t), \ell = 1, 2, \ldots, k\) are affinely independent.

\[
\leq \text{Assume} \ (x^t, y^t, (y^+)^t) \ \text{are affinely independent points for} \ Conv(Q'), \ \text{and} \ (x^t, y^t), \ell = 1, 2, \ldots, k, k \leq 2n + 1, \ \text{are not affinely independent points for} \ Conv(Q). \]

Since \((x^t, y^t)\) are not affinely independent for \( Conv(Q) \), there exist \( \gamma \in \mathbb{R}^k \) such that \( \gamma \neq 0 \) and \( \sum_{\ell=1}^{k} \gamma_{\ell}(x^t, y^t) = 0, \sum_{\ell=1}^{k} \gamma_{\ell} = 0 \).

From (4.3) all points \((x, y, y^+)\) of \( Conv(Q') \) satisfy the equalities \( y^+_j = x_j + y_j - D_j, \ \forall j \in N. \) Therefore,
\[
\sum_{\ell=1}^{k} \gamma_{\ell}(y^+_j)^t = \sum_{\ell=1}^{k} \gamma_{\ell}(x^t_j + y^t_j - D_j)
\]
\[
= \sum_{\ell=1}^{k} \gamma_{\ell}x^t_j + \sum_{\ell=1}^{k} \gamma_{\ell}(y^t_j) - D_j \sum_{\ell=1}^{k} \gamma_{\ell}
\]
\[
= 0
\]

Thus, \((x^t, y^t, (y^+)^t)\) are not affinely independent points in \( Conv(Q') \) since there exist \( \gamma \in \mathbb{R}^k \) such that \( \gamma \neq 0 \) and \( \sum_{\ell=1}^{k} \gamma_{\ell}(x^t, y^t, (y^+)^t) = 0, \sum_{\ell=1}^{k} \gamma_{\ell} = 0. \) This contradicts the assumption that \((x^t, y^t, (y^+)^t), \ell = 1, 2, \ldots, k\) are affinely independent. \( \Box \)

Lemma V.2 \( \dim(Conv(Q')) = \dim(Conv(Q)) = 2n \)
Proof. From Theorem IV.2 \( \dim(\text{Conv}(Q)) = 2n \). Then, there exist \( 2n + 1 \) affinely independent points for \( \text{Conv}(Q) \). From Lemma V.1, there exist \( 2n + 1 \) affinely independent points for \( \text{Conv}(Q') \). Hence, \( \dim(\text{Conv}(Q')) \geq \dim(\text{Conv}(Q)) \).

From (4.3) all points of \( \text{Conv}(Q') \) satisfy the equalities \( x_j + y_j - y_j^+ = D_j, \quad \forall j \in N \). Therefore, \( \text{rank}[A^-, b^-] \geq n \). From Proposition III.6, \( \dim(\text{Conv}(Q')) \leq 2n \). Thus, \( \dim(\text{Conv}(Q')) = \dim(\text{Conv}(Q)) = 2n \). □

**Theorem V.3** The inequality \( (\alpha, \beta)(x, y) \geq 1 \), where \( (\alpha, \beta), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), defines a facet of \( \text{Conv}(Q) \) if and only if \( (\alpha, \beta, 0)(x, y, y^+) \geq 1 \), where \( (\alpha, \beta, 0), (x, y, y^+) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \), defines a facet of \( \text{Conv}(Q') \).

Proof. \( \Rightarrow \) Suppose \( (\alpha, \beta)(x, y) \geq 1 \) defines a facet of \( \text{Conv}(Q) \). Then there exist \( 2n \) affinely independent points \( (x^\ell, y^\ell) \in \text{Conv}(Q), \ell = 1, 2, \ldots, 2n \), such that \( (\alpha, \beta)(x^\ell, y^\ell) = 1 \). From Lemma V.1, there exist \( (y^+)^\ell \) such that \( (x^\ell, y^\ell, (y^+)^\ell) \in \text{Conv}(Q') \) are affinely independent. Since \( (\alpha, \beta, 0)(x^\ell, y^\ell, (y^+)^\ell) = 1, \quad \forall \ell \in N^2 \), and \( \dim(\text{Conv}(Q')) = 2n \) (from Lemma V.2), \( (\alpha, \beta, 0)(x, y) \geq 1 \) defines a facet of \( \text{Conv}(Q') \).

\( \Leftarrow \) Suppose \( (\alpha, \beta, 0)(x, y, y^+) \geq 1 \) defines a facet of \( \text{Conv}(Q') \). Then, there exist \( 2n \) affinely independent points \( (x^\ell, y^\ell, (y^+)^\ell) \in \text{Conv}(Q') \) such that \( (\alpha, \beta, 0)(x^\ell, y^\ell, (y^+)^\ell) = 1 \). This implies that \( (\alpha, \beta)(x^\ell, y^\ell) = 1 \). By Lemma V.1, \( (x^\ell, y^\ell) \) are affinely independent for \( \text{Conv}(Q) \). Thus, \( (\alpha, \beta)(x, y) \geq 1 \) defines a facet of \( \text{Conv}(Q) \). □
CHAPTER VI

Vertices and extreme directions of $Q_{\pi}$ and $\text{Conv}(Q)$

The results of this chapter are used in Chapter VII to determine the blocking polyhedron and to find facets of $\text{Conv}(Q)$. We study the vertices and extreme directions of $Q_{\pi}$ and $\text{Conv}(Q)$ in sections 6.1 and 6.2 respectively.

6.1 Extreme directions and vertices of $Q_{\pi}$

For the sake of convenience in determining the extreme directions and the vertices of $Q_{\pi}$, we represent $Q_{\pi}$ as follows

$$Q_{\pi} = \{(x, y) \in R^n \times R^n : A_{\pi}(x, y) \geq b, (x, y) \geq 0\}$$

(6.1)

where $A_{\pi}$ is a $\left(\binom{n}{2} + 2n\right) \times 2n$ matrix and $b \geq 0$.

The first $n$ rows of $A_{\pi}$ correspond to $x_j + y_j \geq D_j$. The next $\binom{n}{2}$ rows of $A_{\pi}$ correspond to the $x_{jk} - x_{kj} \geq \Delta_{j,k,j}$. The remaining rows of $A_{\pi}$ correspond to $x_j \geq E_j$. Assume that the vector $(x, y)$ as well as the columns of $A_{\pi}$ are ordered.

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according to $x_{j_1}, \ldots, x_{j_n}, y_{j_1}, \ldots, y_{j_n}$. Then,

$$A_\pi = \begin{bmatrix} I & I \\ M_\pi & O_1 \\ I & O_2 \end{bmatrix}$$

where $I$ is an $n \times n$ identity matrix, $O_1$ is a $\binom{n}{2} \times n$ zero matrix, $O_2$ is a $n \times n$ zero matrix, and $M_\pi$ is a $\binom{n}{2} \times n$ matrix that contains the coefficients of the angular separation constraints.

Example VI.1 Let $\pi_0 = (3,1,2)$. Then, the columns of the $A_{\pi_0}$ matrix are ordered according to $x_3, x_1, x_2, y_3, y_1, y_2$, and

$$A_{\pi_0} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$ 

Note that $A_{\pi_0}$ contains at most two nonzero elements in each row. (We show soon that the dual of QWUSLP given a fixed order of the satellites is an uncapacitated network flow problem.)

The matrix $M_\pi$ can be partitioned as follows:

$$M_\pi = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \end{bmatrix}$$

where $M_i$ is a $n - i \times n$ matrix whose row $k$, $k \leq n - i$, has an entry of $-1$ in $i$-th column and an entry of $+1$ in $(k+i)$-th column. Further, $b$ is a $\binom{n}{2} + 2n$ vector
that can be partitioned in the following way:

\[ b = \begin{bmatrix}
    b_0 \\
    b_1 \\
    \vdots \\
    b_{n-1} \\
    b_n
\end{bmatrix} \]

where \( b_i \) is a \( n - i \) vector for \( 0 \leq i \leq n - 1 \), and \( b_n \) is an \( n \) vector. For \( b_0 \) its \( k \)-th component is \( D_{jk} \). For \( b_i \), \( 1 \leq i \leq n - 1 \) its \( k \)-th component is \( \Delta_{ji,i+k} \), and for \( b_n \) its \( k \)-th component is \( E_{jk} \). For Example VI.1,

\[ b_{\pi^o} = \begin{bmatrix}
    D_3 \\
    D_1 \\
    D_2 \\
    \Delta_{1,3} \\
    \Delta_{2,3} \\
    \Delta_{1,2} \\
    E_3 \\
    E_1 \\
    E_2
\end{bmatrix} \]

Next we determine the extreme directions of \( Q_{\pi^o} \).

Let \((q^i, r^i) \in R^n \times R^n, i = 1, 2, \ldots, 2n \) be defined as follows:

\[ q^i_{jk} = \begin{cases} 
    1 & i = 1, 2, \ldots, n; k = i, \ldots, n \\
    0 & \text{otherwise}
\end{cases} \quad (6.2) \]

\[ r^i_{jk} = \begin{cases} 
    1 & i = n + 1, \ldots, 2n; k = i - n \\
    0 & \text{otherwise}
\end{cases} \quad (6.3) \]

where \( \pi = (j_1, j_2, \ldots, j_n) \).

Let \( M^{\pi^o} \) be the \( 2n \times 2n \) matrix, whose column \( i \) is the vector \((q^i, r^i)\), \( i = 1, 2, \ldots, 2n \), and whose rows are ordered according to \( q_{j_1}, \ldots, q_{j_n}, r_{j_1}, \ldots, r_{j_n} \). \( M^{\pi^o} \) is of the following form:
\[ M^p = \begin{bmatrix} M^* & O \\ O & I \end{bmatrix} \]

where \( I \) is an \( n \times n \) identity matrix, \( O \) is an \( n \times n \) zero matrix and \( M^* \) is a lower triangular matrix with all the entries on the diagonal and below the diagonal equal to 1.

Recall from Definition III.12 that \((q^i, r^i)\) is a direction of \( Q_\pi \) if for each \((x, y) \in Q_\pi\), \(\{(x, y) + \lambda(q^i, r^i) : \lambda \geq 0\} \in Q_\pi\). Since \(x_{ji} - x_{jk} \geq \Delta_{jk,ji} \), \( k < i \), if \( q_{jk} = 1 \) then \( q_{ji} = 1 \). Consequently, \( M^* \) is a lower triangular matrix. We observe that \( M^* \) correspond to the extreme directions for the clique scheduling polyhedron in [3].

Theorem VI.1 shows that the vectors \((q^i, r^i)\) are the extreme directions of \( Q_\pi \).

**Theorem VI.1** The extreme directions of \( Q_\pi \) are \((q^i, r^i)\), \( \forall i \in N^2 \).

**Proof.** Let \( \pi = (j_1, j_2, \ldots, j_n) \). By definition, \((q^i, r^i) \neq 0\) and \((q^i, r^i) \geq 0\), \( \forall i \in N^2 \). In addition, \( A_\pi(q^i, r^i) \geq 0\), \( \forall i \in N^2 \), since \( q^i_j + r^i_j \geq 0\), \( \forall j \in N \), and \( q^i_{jk} - q^i_{jk} \geq 0\), \( \forall k, \ell \in N \) such that \( k < \ell \). From Proposition III.9, every \((q^i, r^i)\) is a direction vector of \( Q_\pi \).

Next, we prove that every other direction vector for \( Q_\pi \) is not extreme for \( Q_\pi \) since is a nonnegative linear combination of the \((q^i, r^i)\), \( i \in N^2 \). Assume that there exists

\[(d, f) \in R^n \times R^n \text{ such that } (d, f) \neq (q^i, r^i) , \forall i \in N^2 ; (d, f) \neq 0; \]
\[(d, f) \geq 0; \ A_{\pi}(d, f) \geq 0,\]

and there does not exist a \(\gamma \in \mathbb{R}^{2n}\) such that \(\gamma \geq 0, \ \gamma \neq 0,\) and

\[(d, f) = \sum_{i \in N^2} \gamma_i (q^i, r^i).\]

Define \(\gamma_1 = d_1,\) and \(\gamma_i = d_i - d_{i-1}, \ i = 2, 3, \ldots, n.\) Let \(\gamma_i = f_{i-n}, \ \forall i \in N^2 \setminus N.\)

Since \(A_{\pi}(d, f) \geq 0\) and \((d, f) \geq 0, \ \gamma_i \geq 0, \ \forall i \in N^2.\) By construction \(\gamma_i \neq 0\) for some \(i \in N^2\) if \((d, f) \neq 0.\) Furthermore, \((d, f) = \sum_{i=1}^{2n} \gamma_i (q^i, r^i).\) This contradicts the assumption that \((d, f)\) is not a nonnegative linear combination of the \((q^i, r^i).\)

The transpose of the matrix \(M^\pi\) is an echelon matrix with nonzero rows. Thus, the rank of \(M^\pi\) is \(2n.\) Since \(M^\pi\) is a matrix of full rank, it follows that the vectors \((q^i, r^i) \ \forall i \in N^2,\) are linearly independent. Consequently, the extreme direction vectors of \(Q_{\pi}\) are the \((q^i, r^i), \ \forall i \in N^2.\)

**Corollary VI.2** The extreme directions of \(X_{\pi}\) are \(q^i, \ \forall i \in N\)

**Proof.** The proof follows from the theorem above or from Theorem 3.3 in [3].

Next, we study the properties of the vertices of \(Q_{\pi}\) for an arbitrary \(\pi \in \Pi_N.\)

Recall that even though QWUSLP is a hard optimization problem, once the order of the satellites is fixed, QWUSLP reduces to a LP. After fixing the order of the satellites, we obtain the following problem

\[\text{(QWUSLP}_{\pi})\]

\[
\min \sum_{j \in N} (x_j + 2y_j)
\]
Subject to

\[ A_\pi(x, y) \geq b \]

\[ y \geq 0 \]

where \( A_\pi \) and \( b \) are defined in (6.1). Since \( x_j \geq E_j \geq 0, \ \forall j \in N \), the nonnegativity constraints for the variables \( x \) are not included explicitly in the formulation above. The dual of \( QWUSLP_\pi \) is a network flow problem\(^1[52]\).

Let \( u_j, w_{ji}, v_j, \) and \( t_j \) be the dual variables associated with the constraints

\[ x_j + y_j \geq D_j, \ x_{ji} - x_{jk} \geq \Delta_{ji}, \ x_j \geq E_j, \text{ and } y_j \geq 0 \]

respectively. The dual of \( QWUSLP_\pi \) is

\[
\max (u, w, v) \ b
\]

Subject to

\[
(u, w, v) \begin{bmatrix} I \\ M_\pi \\ I \end{bmatrix} = 1 \quad (6.4)
\]

\[
-u_j - t_j = -2, \ \forall j \in N \quad (6.5)
\]

\[
(u, w, v, t) \geq 0 \quad (6.6)
\]

where \( M_\pi \) and \( I \) are as defined on page 44. Constraints (6.4) are associated with the primal variables \( x \). Constraints (6.5) are associated with the primal variables \( y \) and have been multiplied by -1 for convenience.

The constraint

\[
- \sum_{j \in N} v_j + \sum_{j \in N} t_j = n \quad (6.7)
\]

\(^1\)A similar result for the satellite placement problem (SPP) appears in [47].
is the sum of constraints (6.4) and (6.5) multiplied by -1. By adding constraint (6.7) to the dual of $QWUSLP_\pi$ we obtain a network flow problem. We refer to this problem as $F_\pi$.

For Example VI.1, $F_{\pi_0}$ is

$$\max D_3 u_3 + D_1 u_1 + D_2 u_2 - \Delta_1 w_{13} + \Delta_2 w_{32} + \Delta_{1,2} w_{12} + E_3 v_3 + E_1 v_1 + E_2 v_2$$

Subject to

\[
\begin{align*}
    u_3 & - w_{13} - w_{32} + v_3 = 1 \\
    u_1 & + w_{13} - w_{12} + v_1 = 1 \\
    u_2 & + w_{32} + w_{12} + v_2 = 1 \\
    -u_3 & = -t_3 \\
    -u_1 & = -t_1 \\
    -u_2 & = -t_2 \\
    -v_3 & - v_1 - v_2 + t_3 + t_1 + t_2 = 3 \\
    u_3, u_1, u_2, w_{13}, w_{32}, w_{12}, v_3, v_1, v_2, t_3, t_1, t_2 & \geq 0.
\end{align*}
\]

Let $G_\pi$ be the digraph associated with $F_\pi$ and let $A'_\pi$ be the node-arc incidence matrix of $G_\pi$. $G_\pi = (N^2 \cup \{0\}, \Lambda_\pi)$ where $N^2 \cup \{0\}$ is a set of $2n + 1$ nodes (one for each row of $A'_\pi$) and $\Lambda_\pi$ is a set of $\binom{n}{2} + 2n$ directed arcs (one for each column of $A'_\pi$). A node associated with a row of $A'_\pi$ whose right hand side is negative (positive) is called a source (sink). Recall that every row (except the last row) of $A'_\pi$ corresponds to a primal variable. Thus, for $j = 1, 2, \ldots, n$, node $j$ is a sink node with a demand of one unit that corresponds to the variable $x_j$, and for $j = n + 1, \ldots, 2n$, node $j$ is a source node with a supply of two units that corresponds to the variable $y_{j-n}$. Node 0 is a sink node with a demand of $n$ units.
Figure 6.1: The graph $G_{\pi_0}$

that corresponds to the constraint (6.7). Every arc corresponds to a dual variable or to a primal constraint. Figure 6.1 illustrates the graph $G_{\pi_0}$ for Example VI.1

To characterize the vertices of $Q_\pi$, we introduce the digraph $G_\pi(x, y)$ and show that $(x, y)$ is an extreme point of $Q_\pi$ if and only if $G_\pi(x, y)$ is connected.

For any $(x, y) \in Q_\pi$, let $G_\pi(x, y)$ be the following digraph:

$$G_\pi(x, y) = (N^2 \cup \{0\}, \Lambda_\pi(x, y))$$
where $\Lambda_{\pi}(x,y) \subseteq \Lambda_{\pi}$. The arc set

$$\Lambda_{\pi}(x,y) = \bigcup_{\ell=1}^{4} \Lambda^\ell$$

where

$$\Lambda^1 = \{(0,j) : j \in N, x_j = E_j\}$$

$$\Lambda^2 = \{(j+n,0) : j \in N, y_j = 0\}$$

$$\Lambda^3 = \{(j_k, j_i) : i, k \in N, k < i, x_{j_k} - x_{j_i} = \Delta_{j_k,j_i}\}$$

$$\Lambda^4 = \{(j+n,j) : j \in N, x_j + y_j = D_j\}.$$

Note that $G_{\pi}(x,y)$ is a subgraph of $G_{\pi}$ whose arcs correspond to inequalities of $Q_{\pi}$ that are tight for $(x,y)$. We observe that $G_{\pi}(x,y)$ is similar to the graph in Theorem 2.3 in [3]. To illustrate the graph $G_{\pi}(x,y)$, consider the following 3 satellite example.

**Example VI.2** Let $\pi_0 = (3, 1, 2)$, $D_1 = 2$, $D_2 = 4$, $D_3 = 7$, $\Delta_{1,2} = 3$, $\Delta_{1,3} = 4$, $\Delta_{2,3} = 5$, $E_1 = 0$, $E_2 = 5$, $E_3 = 1$. Then,

$$Q_{\pi_0} = \{x_1 + y_1 \geq 2, x_2 + y_2 \geq 4, x_3 + y_3 \geq 7; x_1 - x_3 \geq 4, x_2 - x_3 \geq 5,$$

$$x_2 - x_1 \geq 3; x_1 \geq 0, x_2 \geq 5, x_3 \geq 1; y_1, y_2, y_3 \geq 0\}.$$

Let $(x,y) = (x_1, x_2, x_3, y_1, y_2, y_3) = (5, 8, 1, 0, 3, 6) \in Q_{\pi_0}$. Furthermore, $N^2 \cup \{0\} = \{0, 1, 2, 3, 4, 5, 6\}$ is the set of nodes and $\Lambda_{\pi_0}(x,y) = \bigcup_{\ell=1}^{4} \Lambda^\ell$ is the set of directed arcs where $\Lambda^1 = \{(0,3)\}$, $\Lambda^2 = \{(4,0)\}$, $\Lambda^3 = \{(3,1),(1,2)\}$, and $\Lambda^4 = \{(6,3)\}$.

Figure 6.2 illustrates the graph $G_{\pi_0}(x,y)$. $\Box$
Figure 6.2: The graph $G_{x_0}(x, y)$

\[ x_2 - x_1 = \Delta_{1,2} \]
\[ x_1 - x_3 = \Delta_{1,3} \]
\[ x_1 = E_3 \]
\[ y_1 = 0 \]
\[ x_3 + y_3 = D_3 \]
Theorem VI.3 \((x,y) \in Q_\pi\) is an extreme point of \(Q_\pi\) if and only if the graph \(G_\pi(x,y)\) is connected.

Proof. Let \((x,y) \in Q_\pi\). From Proposition I.3.1.2 in [33], \(G_\pi(x,y)\) is connected if and only if \(G_\pi(x,y)\) contains a subgraph that is a spanning tree on \(G_\pi\).

\(\Leftarrow\) Suppose \(G_\pi(x,y)\) is connected. Therefore, \(G_\pi(x,y)\) contains a subgraph that is a spanning tree on \(G_\pi\). From Proposition I.3.6.2 in [33], each spanning tree on \(G_\pi\) corresponds to a basic solution of the dual of \(QWUSLP_\pi\). Thus, the complementary solution of \((x,y)\) in the dual is a basic solution to the dual and \((x,y)\) is a basic solution to \(QWUSLP_\pi\). Since \((x,y) \in Q_\pi\), \((x,y)\) is a basic feasible solution or extreme point.

\(\Rightarrow\) Suppose \(G_\pi(x,y)\) is disconnected. Therefore, \(G_\pi(x,y)\) has no subgraph that is a spanning tree on \(G_\pi\). From Proposition I.3.6.2 in [33], the complementary solution of \((x,y)\) in the dual is not basic. Therefore, \((x,y)\) is not a basic solution or an extreme point. \(\square\)

Notice that in Example VI.2, \((x,y) = (5,8,1,0,3,6)\) is not a vertex of \(Q_{\pi_0}\) since node 5 is not connected to any other node in the graph \(G_{\pi_0}(x,y)\).

Corollary VI.4 A necessary condition for \((x,y)\) to be an extreme point of \(Q_\pi\) is that \(y_j = \max\{0, D_j - x_j\}, \forall j \in N\).

Proof. Assume \((x,y)\) is an extreme point of \(Q_\pi\), and \(y_k \neq 0\) and \(y_k \neq D_k - x_k\), for some \(k \in N\). Then, \(G_\pi(x,y)\) is not connected since node \(n+k\) is not connected
to node 0 (since \( y_k \neq 0 \)) or to node \( k \) (since \( x_k + y_k \neq D_k \)). By Theorem VI.3, \((x,y)\) is not an extreme point. This contradicts the assumption that \((x,y)\) is a vertex of \( Q_\pi \). □

In the following theorem we prove that a necessary condition for a solution to be an extreme point of \( Q_\pi \) is that there must be at least one satellite located either at its easternmost feasible location or at its desired location.

**Theorem VI.5** For every vertex \((x,y) \in Q_\pi\), either \( x_j = E_j \) or \( x_j = D_j \), for some \( j \in N \).

**Proof.** Assume that \((x,y) \in Q_\pi\) is a vertex, and that \( x_j > E_j \) and \( x_j \neq D_j \), \( \forall j \in N \).

Let \( U = \{ j \in N : x_j > D_j \} \) and \( L = \{ j \in N : x_j < D_j \} \). By Corollary VI.4, \( y_j = 0 \), \( \forall j \in U \), and \( y_j > 0 \), \( \forall j \in L \).

Let \( \epsilon > 0 \) and \( x_j^1 = x_j - \epsilon \), \( x_j^2 = x_j + \epsilon \), \( \forall j \in N \). We define \( y_j^1 = y_j^2 = y_j \), \( \forall j \in U \), and \( y_j^1 = y_j + \epsilon \), \( y_j^2 = y_j - \epsilon \), \( \forall j \in L \). Then, for \( \epsilon \) sufficiently small, \((x^1,y^1),(x^2,y^2) \in Q_\pi\). Since \((x,y) = \frac{1}{2}(x^1,y^1)+(x^2,y^2)\), \((x,y)\) is not an extreme point of \( Q_\pi \). This contradicts the assumption that \((x,y)\) is a vertex of \( Q_\pi \). □

Notice that if \( x_j > E_j \) and \( x_j \neq D_j \), \( \forall j \in N \), the graph \( G_\pi(x,y) \) would be disconnected.
6.2 Extreme directions and vertices of Conv(Q)

The next theorem, which characterizes the extreme directions of Conv(Q) is useful to establish some properties of the vertices of Conv(Q). We observe the similarity between the next theorem and Lemma 2.1 in [36] and Theorem 3.5 in [3].

**Theorem VI.6** The extreme directions of Conv(Q) are the unit vectors \( e_k \in \mathbb{R}^n \), \( \forall k \in \mathbb{N}^2 \).

**Proof.** Let \((x, y) \in Conv(Q)\). First we prove that \((x', y') = (x, y) + \lambda e_k \in Conv(Q)\), \( \forall k \in \mathbb{N}^2 \) and \( \forall \lambda \geq 0 \), i.e., that the unit vectors are direction vectors of Conv(Q).

Let \(1 \leq k \leq n\). We divide this part of the proof into two cases.

Case 1) Suppose \((x, y) \in Q\). Then \((x, y) \in Q_\pi\) for some \(\pi \in \Pi_N\). Let \(\pi = (j_1, j_2, \ldots, j_n)\), and

\[
\Delta_{\text{max}} = \max_{i,j \in \mathbb{N}} \{\Delta_{i,j}\}.
\]

Let \(\pi'\) be the same ordering as \(\pi\) except that satellite \(k\) is moved to the westernmost position, i.e., \(\pi'(i) = \pi(i)\) if \(i < \pi^{-1}(k)\), \(\pi'(i) = \pi(i + 1)\) if \(\pi^{-1}(k) \leq i \leq n - 1\), \(\pi'(n) = k\). Then, for \(\lambda \geq x_{j_n} + \Delta_{\text{max}} - x_k\), \((x', y') = (x, y) + \lambda e_k \in Q_{\pi'}\).

Thus, \((x', y') \in Q\). Furthermore, for \(\lambda = 0\), \((x', y') \in Q\) because \((x', y') = (x, y)\).

Therefore, \((x', y') \in Conv(Q)\) for any \(\lambda \geq 0\).
Case 2) Suppose \((x, y) \in \text{Conv}(Q) \setminus Q\). Then, by Carathéodory's theorem, \((x, y)\) is a convex combination of \(2n + 1\) points of \(Q\). Thus, there exist \(\gamma \in \mathbb{R}^{2n+1}, 0 \leq \gamma \leq 1, \) and \(\sum_{\ell=1}^{2n+1} \gamma_\ell = 1\) such that

\[
(x, y) = \sum_{\ell=1}^{2n+1} \gamma_\ell (x^\ell, y^\ell)
\]

where \((x^\ell, y^\ell) \in Q, \ \forall \ell \in \{1, 2, \ldots, 2n + 1\}\). This implies that \((x^\ell, y^\ell) \in Q_{\pi^\ell}\) for some \(\pi^\ell \in \Pi_N, \ \forall \ell \in \{1, 2, \ldots, 2n + 1\}\). From Case 1) \((\bar{x}^\ell, \bar{y}^\ell) = (x^\ell, y^\ell) + \lambda e_k \in \text{Conv}(Q) , \ \forall \ell \in \{1, 2, \ldots, 2n + 1\}\) for any \(\lambda \geq 0\).

Let

\[
(x', y') = \sum_{\ell=1}^{2n+1} \gamma_\ell (\bar{x}^\ell, \bar{y}^\ell)
\]

Then, \((x', y') = (x, y) + \lambda e_k \in \text{Conv}(Q), \ \text{for any } \lambda \geq 0\).

Now, suppose \(n + 1 \leq k \leq 2n\). Since the unit vector \(e_k = (q^k, r^k)\) for \(k \in N^2 \setminus N\) is an extreme direction vector of every \(Q_{\pi}, \ \forall \pi \in \Pi_N,\) it is a direction vector of \(\text{Conv}(Q)\).

Because \(e_k, \ \forall k \in N^2,\) is a unit vector and \(Q\) is contained in the nonnegative orthant, each \(e_k\) is extreme for \(\text{Conv}(Q)\). Finally, no other direction vector for \(\text{Conv}(Q)\) could be extreme, since it would be a linear combination of the unit vectors. □

Corollary VI.7 The extreme directions of \(\text{Conv}(X)\) are the unit vectors.
Proof. The proof follows from the theorem above or from Lemma 2.1 in [36] or Theorem 3.5 in [3]. □

Now we are in a position to determine some properties of the vertices of $\text{Conv}(Q)$. Since $\text{Conv}(Q) = \text{Conv}(\bigcup_{\pi \in \Pi_N} Q_{\pi})$, every extreme point of $Q_{\pi}$, $\pi \in \Pi_N$, is not necessarily an extreme point of $\text{Conv}(Q)$. This is shown in the following example.

Example VI.3 Let

$$\Delta_{1,2} = 1, \Delta_{1,3} = 2, \Delta_{2,3} = .9, D_1 = D_2 = D_3 = 0, E_1 = 0, E_2 = 1, E_3 = 2.$$ 

Let $(x', y') = (0, 1, 2, 0, 0, 0)$, and $(x'', y'') = (0, 1.1, 2, 0, 0, 0)$. Both $(x', y')$ and $(x'', y'')$ are vertices of $Q_{\pi_0}$, where $\pi_0 = (1, 2, 3)$, since $G_{\pi_0}(x', y')$ and $G_{\pi_0}(x'', y'')$ are connected. However, $(x'', y'')$ is not a vertex of $\text{Conv}(Q)$ since it can be expressed as the sum of $(x', y')$ and a positive combination of the extreme direction vectors of $\text{Conv}(Q)$. □

We prove in the following theorem that being a vertex of $Q_{\pi}$, for some $\pi \in \Pi_N$, is a necessary condition for being a vertex of $\text{Conv}(Q)$.

Theorem VI.8 Every extreme point of $\text{Conv}(Q)$ is an extreme point of $Q_{\pi}$ for some $\pi \in \Pi_N$. 

Proof. First, assume there exists an extreme point \((x, y) \in \text{Conv}(Q)\) that is not an extreme point of any \(Q_\pi, \pi \in \Pi_N\). Either \((x, y) \in Q\) or \((x, y) \in \text{Conv}(Q) \setminus Q\). If \((x, y) \in Q\), then \((x, y) \in Q_\pi\) for some \(\pi \in \Pi_N\). Consequently, by Minkowski's Theorem (Theorem III.2), \((x, y)\) is a convex combination of the extreme points of \(Q_\pi\) plus a nonnegative linear combination of the extreme directions of \(Q_\pi\); therefore, \((x, y)\) is not an extreme point of \(\text{Conv}(Q)\). On the other hand, if \((x, y) \in \text{Conv}(Q) \setminus Q\), \((x, y)\) is a convex combination of \(2n + 1\) points in \(Q\). So \((x, y)\) is not extreme. □

Our next definition is useful in characterizing the vertices of \(\text{Conv}(Q)\). Given a set \(S \in \mathbb{R}^n \times \mathbb{R}^n\), the set of minimal points is denoted by \(T_S\) and defined as follows:

\[
T_S = \{(x, y) \in S : \text{there does not exist any } (x', y') \in S \text{ such that } (x', y') \leq (x, y)\}.
\]

We observe that the set \(T_S\) lies on particular faces of \(S\). Consider the following example.

Example VI.4 Let

\[
S_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \geq 8, \, x \geq 2, \, y \geq 3\}
\]

Then, \(T_{S_1} = \{(x, y) \in S_1 : x + y = 8\}\). Figure 6.3 shows the sets \(S_1\) and \(T_{S_1}\). □
Figure 6.3: The sets $S_1$ and $T_{s_1}$
Recall that $V(\text{Conv}(\mathcal{Q}))$ denotes the set of vertices of $\text{Conv}(\mathcal{Q})$. In Theorem VI.9 we prove that $V(\text{Conv}(\mathcal{Q})) \subseteq T_{\text{Conv}(\mathcal{Q})}$.

**Theorem VI.9** Every $(x,y) \in V(\text{Conv}(\mathcal{Q}))$ is also an element of $T_{\text{Conv}(\mathcal{Q})}$.

**Proof.** Assume $(x,y) \in V(\text{Conv}(\mathcal{Q}))$, but $(x,y) \not\in T_{\text{Conv}(\mathcal{Q})}$. Thus, there exists $(x',y') \in \text{Conv}(\mathcal{Q})$ such that $(x',y') \preceq (x,y)$. Hence, $(x,y)$ is not a vertex of $\text{Conv}(\mathcal{Q})$ since it can be expressed as the sum of $(x',y')$ plus a nonnegative combination of the extreme direction vectors of $\text{Conv}(\mathcal{Q})$. □

**Corollary VI.10** Every vertex of $\text{Conv}(\mathcal{X})$ is also an element of $T_{\text{Conv}(\mathcal{X})}$.

The following lemma plays an important role in finding facets for $\text{Conv}(\mathcal{Q})$ and it is useful in proving Theorem VI.12.

**Lemma VI.11** $\text{Conv}(\mathcal{Q})$ can be represented in the following form:

$$\text{Conv}(\mathcal{Q}) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : A(x,y) \geq 1 \text{ and } (x,y) \geq 0\} \quad (6.8)$$

where $A$ is a nonnegative matrix, $1=(1,1,\ldots,1)^T$, and $0=(0,0,\ldots,0)^T$ are vectors of appropriate dimension.

**Proof.** The proof follows from the fact that $\text{Conv}(\mathcal{Q})$ is a polyhedron situated in the nonnegative orthant whose extreme directions are the unit vectors. For a complete proof see Theorem 4.1 [3]. □
As Tind [50] points out, $Conv(Q)$ is an unbounded, convex, closed polyhedron not containing zero except for the degenerate case where $A$ has no rows (which implies that $Conv(Q) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : (x, y) \geq 0\}$) or contains a zero row (which implies that $Conv(Q)$ is empty).

**Theorem VI.12** $T_{Conv(Q)}$, the set of minimal points in $Conv(Q)$, only contains boundary points of $Conv(Q)$.

**Proof.** Assume $(x, y) \in T_{Conv(Q)}$ and $(x, y)$ is an interior point of $Conv(Q)$. This implies that $A(x, y) > 1$ and $(x, y) > 0$, where $A$ is the nonnegative matrix defined in Lemma VI.11.

Let $(x', y') = (x, y) - \zeta$, where $\zeta \in \mathbb{R}^{2n}$, $\zeta_j = \epsilon$, $\forall j \in \mathbb{N}^2$ and $\epsilon > 0$. For $\epsilon$ sufficiently small $(x', y') \in Conv(Q)$. But $(x', y') < (x, y)$, contradicting that $(x, y) \in T_{Conv(Q)}$. □

Notice that Theorem VI.12 indicates that it is possible that $V(Conv(Q)) \subset T_{Conv(Q)}$. i.e., $T_{Conv(Q)}$ may contain points that are not extreme points of $Conv(Q)$. This result has implications for determining facets of $Conv(Q)$ in the next chapter.
CHAPTER VII

Facets of $\text{Conv}(Q)$

In this chapter we determine facets of $\text{Conv}(Q)$ with one, two, three, and four positive coefficients. For simplicity, a facet with $t$ positive coefficients is called a $t$-facet. To facilitate the following definitions, we represent $Q$ as a mixed-integer linear constraint set by introducing a binary variable, $p_{ij}$, for every disjunctive constraint in (4.8).

$$Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x_j + y_j \geq D_j \quad \forall j \in N; \quad (7.1)\}$$

$$x_i - x_j + M p_{ij} \geq \Delta_{i,j} \quad \forall i, j \in N \text{ such that } i < j; \quad (7.2)$$

$$x_j - x_i + M(1 - p_{ij}) \geq \Delta_{i,j} \quad \forall i, j \in N \text{ such that } i < j; \quad (7.3)$$

$$x_j \geq E_j \quad \forall j \in N; \quad (7.4)$$

$$y_j \geq 0 \quad \forall j \in N; \quad (7.5)$$

$$p_{ij} = 0, 1; \quad \forall i, j \in N \text{ such that } i < j \quad (7.6)$$

where

$$p_{ij} = \begin{cases} 
1 & \text{if satellite } j \text{ is located west of satellite } i \ (x_i < x_j) \\
0 & \text{otherwise} 
\end{cases}$$

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and $M \gg 0$ ($\gg$ means much greater than).

Let $\mathcal{Q}_R$ be the constraint set that remains after the constraints of type (7.6) are relaxed, i.e., they are replaced with

$$0 \leq p_{ij} \leq 1, \quad \forall i, j \in N \text{ such that } i < j.$$ 

A valid inequality or facet $(\alpha, \beta)(x, y) > (\alpha_0, \beta_0)$ for $\text{Conv}(\mathcal{Q})$ is called nontrivial if

$$\mathcal{Q}_R \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : (\alpha, \beta)(x, y) < (\alpha_0, \beta_0)\} \neq \emptyset;$$

otherwise, is called trivial.

We refer to the LP relaxation (of a MIP) of problem $P$ as $P_R$. Let $z(\cdot)$ denote the objective function value and $z^*(\cdot)$ the optimal objective function value of problem $P$. The next theorem indicates that for any instance of QWUSLP its LP relaxation is zero, i.e., all the satellites are located at their desired locations.

**Theorem VII.1** $z^*(\text{QWUSLP}_R) = 0$

**Proof.** Consider the following solution to $\text{QWUSLP}_R$:

$$x_j = D_j, \quad y_j = 0 \quad \forall j \in N$$

$$p_{ij} = \epsilon_{ij} \quad \forall i, j \in N \text{ such that } i < j.$$
where \( 0 < \epsilon_{ij} < 1 \). This solution is feasible for \( QWUSLP_R \) as we can see by substituting these values into the constraints defining \( Q_R \). At this solution \( z(QWUSLP_R) = 0 \). From (7.1) and (7.5), \( x_j + y_j - D_j \geq 0, \ y_j \geq 0, \ \forall j \in N \).

Since

\[
z(QWUSLP_R) = \sum_{j \in N} (x_j + 2y_j - D_j) = \sum_{j \in N} (x_j + y_j - D_j) + \sum_{j \in N} y_j,
\]

\( z(QWUSLP_R) \geq 0 \) for all feasible solutions. Consequently, this is an optimal solution for \( QWUSLP_R \) and \( z^*(QWUSLP_R) = 0 \). □

**Corollary VII.2** \( z^*(WUSLP_R) = 0 \)

**Proof.** The proof follows from considering the same solution that in the theorem above. In addition, let \( y_j^* = 0, \ \forall j \in N \). Then, this is an optimal solution to \( WUSLP_R \) and \( z^*(WUSLP_R) = 0 \). □

A common way to bound the solution value to a mixed-integer program is to solve its LP relaxation. Since \( WUSLP_R \) and \( QWUSLP_R \) provide the weakest possible bound for all instances of \( WUSLP \) and \( QWUSLP \), other bounding techniques are needed. See, for example, Reilly and Mata [40], where solution-value bounds for SLP are presented. In this chapter we develop valid inequalities that induce facets of \( WUSLP \). Those facets, if appended to \( WUSLP_R \), improve the bound obtained from \( WUSLP_R \).

In Section 7.1 we show inequalities that induce facets of \( Conv(Q) \) with one and two positive coefficients. We do this by showing that they are valid inequalities for
\( \text{Conv}(Q) \) and by finding \( 2n \) affinely independent points in \( \text{Conv}(Q) \) that satisfy these inequalities as equalities.

In Section 7.2 we show that any inequality that defines a facet of \( \text{Conv}(Q(K)) \), \( K \subset N \), also defines a facet of \( \text{Conv}(Q) \). In addition, any inequality that induces a facet of \( \text{Conv}(X(K)) \) also induces a facet of \( \text{Conv}(Q(K)) \). \( \text{Conv}(Q) \) inherits all the facets of \( \text{Conv}(Q(K)) \) and \( \text{Conv}(Q(K)) \) inherits all the facets of \( \text{Conv}(X(K)) \). These properties are similar to the ones presented for the linear ordering polytope [17], for the satellite placement polytope [47], and the scheduling polyhedron [3]. Since \( \text{Conv}(Q) \) can be represented as in (6.8), we present the blocking polyhedron associated with \( \text{Conv}(Q) \).

In Section 7.3, we present facets for \( \text{Conv}(Q) \) for the two-satellite case. In Section 7.4, we give facets for \( \text{Conv}(X(K)) \) for the two-satellite and three-satellite cases. In Section 7.5 we present all two-facets of \( \text{Conv}(Q) \) when there is a common easternmost location. Necessary and sufficient conditions for the valid inequalities in [40] to induce facets for \( \text{Conv}(Q) \) for the two satellite case are given in Section 7.6. Computational results using the facets developed in this chapter are presented in Chapter VIII.
7.1 Facets of $\text{Conv}(\mathcal{Q})$ with one and two positive coefficients

In this section we prove that the inequalities $x_j \geq E_j$, $y_j \geq 0$, and $x_j + y_j \geq D_j$ induce trivial facets of $\text{Conv}(\mathcal{Q})$. We also show that if $\Delta_{i,j} > |E_i - E_j|$, then

$$(\Delta_{i,j} + E_i - E_j)x_i + (\Delta_{i,j} + E_j - E_i)x_j \geq \Delta_{i,j}(\Delta_{i,j} + E_i + E_j)$$  (7.7)

induces a nontrivial facet of $\text{Conv}(\mathcal{Q})$. We show that these inequalities are valid for $\text{Conv}(\mathcal{Q})$ and we find $\dim(\text{Conv}(\mathcal{Q})) = 2n$ affinely independent points in $\text{Conv}(\mathcal{Q})$ that satisfy these inequalities as equalities. We observe that (7.7) is the inequality in Theorem 4.9 in [3].

**Theorem VII.3** The inequality $x_j \geq E_j$, $j \in N$, is a trivial facet for $\text{Conv}(\mathcal{Q})$.

**Proof.** From (7.4), $x_j \geq E_j$ is a valid inequality for $\text{Conv}(\mathcal{Q})$. Let $\pi = (j_1, j_2, \ldots, j_n)$ where $j_1 = j$, i.e., satellite $j$ is in the easternmost position. We prove that there exist $2n$ affinely independent points $(x^t, y^t) \in \text{Conv}(\mathcal{Q})$ such that $x^t_j = E_j$, $\forall \ell \in N^2$. Let

$$x^t_{j_1} = E_{j_1}, \forall \ell \in N^2;$$

$$x^t_{j_k} = \max_{m < k} \{E_{j_k}, x^t_{j_m} + \Delta_{j_m,j_k}\} \quad k = 2, 3, \ldots, n;$$

$$x^t_{j_k} = x^1_{j_k} + q^t_{j_k}, \quad k = 2, 3, \ldots, n, \quad \forall \ell \in N^2, \ell \neq 1;$$

$$y^t_{j_k} = D_{j_k} + r^t_{j_k}, \quad k = 1, 2, \ldots, n, \quad \forall \ell \in N^2;$$
where \((q^\ell, r^\ell), \ \forall \ell \in N^2\), are the extreme direction vectors of \(Q_\pi\) defined in (6.2) and (6.3).

Let \(M\) be the \(2n \times 2n - 1\) matrix, whose columns are the vectors \((x^\ell, y^\ell) - (x^1, y^1), \ell = 2, 3, \ldots, 2n\) and whose rows are ordered according to \(x_j^1, \ldots, x_j^n, y_j^1, \ldots, y_j^n\).

The transpose matrix of \(M\) is an echelon matrix with nonzero rows. Thus, the rank of \(M\) is \(2n - 1\). By Proposition III.3, \((x^\ell, y^\ell), \ \forall \ell \in N^2\) are affinely independent points.

By construction, \(x_j^\ell = E_j\) and \((x^\ell, y^\ell) \in Q_\pi\). This implies that \((x^\ell, y^\ell) \in Conv(Q), \ \forall \ell \in N^2\). Finally, from (7.4) it follows that \(x_j \geq E_j\) is a trivial facet since \(Q_R \cap \{x_j < E_j\} = \emptyset\). 

Since the proofs of Theorems VII.4, VII.5, and VII.6 are similar to the proof of Theorem VII.3, we present those proofs in Appendix A.

**Theorem VII.4** The inequality \(y_j \geq 0, \ j \in N,\) is a trivial facet for \(Conv(Q)\).

**Theorem VII.5** The inequality \(x_j + y_j \geq D_j, \ j \in N,\) is a trivial facet for \(Conv(Q)\) if \(D_j > E_j\); otherwise, it is a consequence of the inequalities \(x_j \geq E_j\) and \(y_j \geq 0\).

**Theorem VII.6** If \(\Delta_{i,j} > |E_i - E_j|,\) the inequality (7.7) is a nontrivial facet for \(Conv(Q)\).
7.2 The inheritance properties and the blocking polyhedron

In this section we present two theorems that show the inheritance properties of \( \text{Conv}(Q) \). Theorem VII.7 indicates that any inequality that defines a facet of \( \text{Conv}(Q(K)) \), for any \( K \subseteq N \), also defines a facet of \( \text{Conv}(Q) \). Theorem VII.8 indicates that any inequality that defines a facet of \( \text{Conv}(\mathcal{X}(K)) \) for any \( K \subseteq N \) also defines a facet of \( \text{Conv}(Q(K)) \). In addition, we present the blocking polyhedron associated with \( \text{Conv}(Q) \). From a known result from blocking polyhedral theory, a vertex of the blocking polyhedron is associated with a facet for \( \text{Conv}(Q) \). Therefore, we focus on determining the blocking polyhedra for \( \text{Conv}(Q(K)) \) and \( \text{Conv}(\mathcal{X}(K)) \). As Balas [3] points out, this result combined with the inheritance properties provides a procedure to determine facets of \( \text{Conv}(Q) \).

The proofs of the next two theorems follow similar arguments to the proof for Theorem 4.4 in [3].

Theorem VII.7 Let \( K \subseteq N \) and \( |K| = m \), \( 2 \leq m < n \). The inequality \((\rho, \psi)
\n(\bar{x}, \bar{y}) \geq 1\), where \((\rho, \psi), (\bar{x}, \bar{y}) \in R^m \times R^m\), defines a facet of \( \text{Conv}(Q(K)) \), if and only if, \((\alpha, \beta)(x, y) \geq 1\), where \((\alpha, \beta), (x, y) \in R^n \times R^n\) and \(\alpha = (\rho, 0), \beta = (\psi, 0)\), defines a facet of \( \text{Conv}(Q) \).

Proof.

\(\Rightarrow\) Assume that \((\rho, \psi)(\bar{x}, \bar{y}) \geq 1\) defines a facet of \( \text{Conv}(Q(K)) \). From Propo-
osition III.4, there exist $2m$ affinely independent points $(\bar{x}^\ell, \bar{y}^\ell) \in Q(K)$, $\ell = 1, 2, \ldots, 2m$, such that $(\rho, \psi)(\bar{x}^\ell, \bar{y}^\ell) = 1$.

We convert each $(\bar{x}^\ell, \bar{y}^\ell) \in Q(K)$ to a $(x, y) \in Q$. Let $\pi^\ell, \ell = 1, 2, \ldots, 2m$, be the ordering associated with $(\bar{x}^\ell, \bar{y}^\ell)$. Define $\pi^{\ell'}$ to be any extension of $\pi^\ell$ such that $\pi^{\ell'}(k) = \pi^\ell(k), k = 1, 2, \ldots, m$, and $\pi^{\ell'}(k) \in N \setminus K, k = m + 1, m + 2, \ldots, n$, i.e., the satellites in $K$ are in the first $m$ positions and the satellites not in $K$ are in the last $n - m$ positions. Let $x^\ell_k = x^\ell_{jk}$ and $y^\ell_k = y^\ell_{jk}, k = 1, 2, \ldots, m$.

Therefore, $x^\ell_k \geq E_{jk}, y^\ell_k \geq 0$, and $x^\ell_k + y^\ell_k \geq D_{jk}$. In addition, $x^\ell_i - x^\ell_j \geq \Delta_{ij}$, $i = 2, 3, \ldots, m, k < i$. Further, let $x^\ell_k \geq E_{jk}, y^\ell_k \geq 0$, $x^\ell_k + y^\ell_k \geq D_{jk}, k = m + 1, m + 2, \ldots, n$. In addition, let $x^\ell_i - x^\ell_j \geq \Delta_{ij}, \forall i, k \in N$ such that $k < i, i = m + 1, m + 2, \ldots, n$.

We have $2m$ affinely independent points $(x^\ell, y^\ell) \in Q$. Let $\pi = (j_1, j_2, \ldots, j_n)$ be the ordering associated with $(x^1, y^1)$. The other $2(n - m)$ points of $Q$ are obtained in the following way:

\[
\begin{align*}
x^\ell_{jk} &= x^1_{jk}, y^\ell_{jk} = y^1_{jk}, k = 1, 2, \ldots, m, \quad \ell = 2m + 1, 2m + 2, \ldots, 2n; \\
x^\ell_{jk} &= x^1_{jk} + q^\ell_{jk}, y^\ell_{jk} = y^1_{jk} + r^\ell_{jk}, k = m + 1, m + 2, \ldots, n, \quad \ell = 2m + 1, 2m + 2, \ldots, 2n.
\end{align*}
\]

Then, $(x^\ell, y^\ell) \in Q, \forall \ell \in N^2$, since the $(q^\ell, r^\ell)$ are extreme direction vectors of $Q_{\pi^{\ell'}}$.

Let $M$ be the $(2n - 1) \times 2n$ matrix whose rows are the vectors $(x^\ell, y^\ell) -$
Then, \( M \) is of the form

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
O & M_{22}
\end{bmatrix}
\]

where \( O \) is a \( 2(n - m) \times 2m \) zero matrix. The rank of \( M_{11} \) is \( 2m - 1 \) since the \((\bar{x}^\ell, \bar{y}^\ell)\) are affinely independent points. \( M_{22} \) is a \( 2(n - m) \times 2(n - m) \) nonsingular matrix since it is an echelon matrix with nonzero rows. Thus, the rank of \( M \) is \( 2n - 1 \). By Proposition III.3, \((x^\ell, y^\ell), \forall \ell \in \mathbb{N}^2\) are affinely independent points. By construction \((\alpha, \beta)(x^\ell, y^\ell) = (\rho, 0)x^\ell + (\psi, 0)y^\ell = 1, \forall \ell \in \mathbb{N}^2\). Hence, \((\alpha, \beta)(x, y) \geq 1\) induces a facet of \( \text{Conv}(Q) \).

\[\leftarrow \text{Suppose the valid inequality } (\rho, \psi)(\bar{x}, \bar{y}) \geq 1 \text{ does not define a facet of } \text{Conv}(Q(K)). \text{ Then, } (\rho, \psi)(\bar{x}, \bar{y}) \geq 1 \text{ is dominated by a nonnegative linear combination of } (\rho^\ell, \psi^\ell)(\bar{x}, \bar{y}) \geq 1, \ell = 1, 2, \ldots, k (k \leq 2m). \text{ In addition, } (\rho^\ell, \psi^\ell)(\bar{x}, \bar{y}) \geq 1 \text{ for each } (\bar{x}, \bar{y}) \in Q(K). \text{ Consequently, the inequalities } (\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2, \ldots, k, (\text{where } \alpha^\ell = (\rho^\ell, 0), \beta = (\psi^\ell, 0) \text{ and } (\alpha^\ell, \beta^\ell), (x, y) \in R^n \times R^n) \text{ are valid for } \text{Conv}(Q). \text{ Thus, the inequality } (\alpha, \beta)(x, y) \geq 1 \text{ does not define a facet of } \text{Conv}(Q) \text{ since is dominated by a nonnegative linear combination of } (\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2, \ldots, k. \]
Theorem VII.8 Let $K \subset N, |K| = m, 2 \leq m < n$. The inequality $\alpha x \geq 1$, where $\alpha, x \in \mathbb{R}^m$, defines a facet of $\text{Conv}(\mathcal{X}(K))$, if and only if, $(\alpha, 0)(x, y) \geq 1$, where $(\alpha, 0), (x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, defines a facet of $\text{Conv}(\mathcal{Q}(K))$.

Proof.

$\Rightarrow$ Suppose $\alpha x \geq 1$ defines a facet of $\text{Conv}(\mathcal{X}(K))$. From proposition III.4 there exist $m$ affinely independent points $x^\ell \in \mathcal{X}(K), \ell = 1, 2, \ldots, m$, such that $\alpha x = 1$.

We convert each $x^\ell \in \mathcal{X}(K)$ to a $(x, y)$ in $\mathcal{Q}K$ by defining $y_j = \max\{0, D_j - x^\ell_j\}, \forall j \in K$ (see Corollary VI.4). By construction, $(x^\ell, y^\ell) \in \mathcal{Q}(K), \ell = 1, 2, \ldots, m$.

Let $\pi = (j_1, j_2, \ldots, j_m)$ be the ordering associated with $(x^1, y^1)$. Then, the remaining $m$ solutions are obtained in the following way:

$$x^\ell_{j_k} = x^1_{j_k}, y^\ell_{j_k} = y^1_{j_k} + r^\ell_{j_k}, k = 1, 2, \ldots, m, \ell = m + 1, m + 2, \ldots, 2m.$$ 

Then, $(x^\ell, y^\ell) \in \mathcal{Q}(K)$, since the $r^\ell$ are extreme direction vectors of $\mathcal{Q}_\pi(K)$.

Let $M$ be the $(2m - 1) \times 2m$ matrix whose rows are the vectors $(x^\ell, y^\ell) - (x^1, y^1), \ell = 2, 3, \ldots, 2m$, and whose columns are ordered according to

$$x_{j_1}, x_{j_2}, \ldots, x_{j_m}, y_{j_1}, y_{j_2}, \ldots, y_{j_m}.$$ 

Then, $M$ is of the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ O & I \end{bmatrix} \quad (7.8)$$

where $O$ is a $m \times m$ zero matrix. The rank of $M_{11}$ is $m - 1$ since the $x^\ell$ are affinely independent points, and $I$ is an $m \times m$ identity matrix. Thus, the rank of $M$ is
By Proposition III.3, \((x^\ell, y^\ell)\), \(\forall \ell \in N^2\) are affinely independent points. By construction \((\alpha, 0)(x^\ell, y^\ell) = \alpha x^\ell = 1, \ell = 1, 2, \ldots, 2m\). Hence, \((\alpha, 0)(x, y) \geq 1\) induces a facet of \(\text{Conv}(Q(K))\).

\[ \iff \text{Suppose the valid inequality } \alpha x \geq 1 \text{ does not define a facet of } \text{Conv}(\mathcal{X}(K)). \text{Then, } \alpha x \geq 1 \text{ is a nonnegative linear combination of } \alpha^\ell x \geq 1, \ell = 1, 2, \ldots, k \text{ (} k \leq m \). \text{In addition, } \alpha^\ell x \geq 1 \text{ for each } x \in \mathcal{X}(K). \text{Consequently, the inequalities } (\alpha^\ell, 0)(x, y) \geq 1, \ell = 1, 2, \ldots, k \text{, (where } (\alpha^\ell, 0)(x, y) \in R^m \times R^m \) \text{are valid for } \text{Conv}(Q(K)). \text{Thus, the inequality } (\alpha, 0)(x, y) \geq 1 \text{ does not define a facet of } \text{Conv}(Q(K)) \text{ since is dominated by a nonnegative linear combination of } (\alpha^\ell, 0)(x, y) \geq 1, \ell = 1, 2, \ldots, k. \ \square \]

**Corollary VII.9** Any facet of the clique scheduling polyhedron defines a facet of \(\text{Conv}(Q)\).

Since \(\text{Conv}(Q)\) is of the form (6.8), there is a blocker or blocking polyhedron \(\text{Conv}(Q)^B\) associated with \(\text{Conv}(Q)\). Recall that \(V(\text{Conv}(Q))\) denotes the set of vertices of \(\text{Conv}(Q)\). Then,

\[
\text{Conv}(Q)^B = \{(u, w) \in R^{2n} : (u, w) \geq 0 \text{ and } (x, y)(u, w) \geq 1, \forall (x, y) \in \text{Conv}(Q)\}
\]

\[
= \{(u, w) \in R^{2n} : (u, w) \geq 0 \text{ and } v(u, w) \geq 1, \forall v \in V(\text{Conv}(Q))\} \quad (7.9)
\]

\[
= \{(u, w) \in R^{2n} : (u, w) \geq 0 \text{ and } B(u, w) \geq 1\} \quad (7.10)
\]

where \(B\) is a \(|V(\text{Conv}(Q))| \times 2n\) matrix whose rows are the elements of \(V(\text{Conv}(Q))\).
Conv(Q) and Conv(Q)^B are called a blocking pair of polyhedra, and the pair A, B, where A is as defined in (6.8), is called a blocking pair of matrices.

The relation between Conv(Q) and Conv(Q)^B provides a framework for finding facets of Conv(Q). Since we have characterized the vertices and extreme directions of Q_\pi and Conv(Q), we can determine the blocking matrix B. Theorem VII.10 indicates that there is a procedure to find facets of Conv(Q).

**Theorem VII.10** \((\alpha, \beta)(x, y) \geq 1\) is a facet-inducing inequality of Conv(Q) if and only if \((\alpha, \beta)\) is a vertex of Conv(Q)^B.

**Proof.** The proof follows from Theorem 1 in [12], from Corollary 4.2 in [3], or from Theorem 37 in [15].

### 7.3 Facets for Conv(Q(K)) : The two-satellite case

Let \( K \subset N, |K| = 2, \) and \( K = \{i, j\} \). By Theorem VII.7, any valid inequality that defines a facet of Conv(Q(K)) for any \( K \subset N, \) also defines a facet of Conv(Q). Hence, we focus on finding facets for Conv(Q(K)). By Theorem VII.10, the inequality \((\alpha_1, \alpha_2, \beta_1, \beta_2) (x_i, x_j, y_i, y_j) \geq 1\) induces a facet of Conv(Q(K)) if and only if \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) is a vertex of the polyhedron

\[
\text{Conv}(Q(K))^B = \{(u_1, u_2, w_1, w_2) \geq 0, B(u_1, u_2, w_1, w_2) \geq 1\}
\]  

(7.11)

where \( B \) is the matrix whose rows are the vertices of Conv(Q(K)). Therefore, to determine facets of Conv(Q(K)) we find the matrix B. Throughout this section
we assume that the columns of the matrix $B$ are ordered according to $x_i, x_j, y_i, y_j$.

In the next subsection we find facets of $\text{Conv}(Q)$ under the assumption that there is a common easternmost location.

### 7.3.1 Facets of $\text{Conv}(Q(K))$ when there is a common easternmost location

Let $E$ be a common easternmost location, i.e., $E = E_1 = E_2 = \cdots = E_n$. Without loss of generality, throughout this subsection we assume that $E = 0$.

There are two possible orderings, $(i, j)$ and $(j, i)$. Therefore, $Q(K) = Q_{(i,j)} \cup Q_{(j,i)}$. Since $E = 0$, it is sufficient to make the analysis for the ordering $(i, j)$ only\(^1\). For the ordering $(i, j)$ we have

$$Q_{(i,j)} = \{x_i + y_i - s_i = D_i, \ x_j + y_j - s_j = D_j, \ x_j - x_i - s_{ij} = \Delta_{i,j},$$

$$x_i, x_j, y_i, y_j, s_i, s_j, s_{ij} \geq 0\}$$

where $s_i, s_j, s_{ij}$ are surplus variables. Since there are 7 variables and 3 structural constraints, the maximum number of vertices of $Q_{(i,j)}$ is $\binom{7}{3} = 35$, i.e., the number of ways of selecting 3 basic variables out of 7 variables. Recall that a point is a vertex if and only if it is a basic feasible solution [4]. The following lemma indicates that the maximum number of distinct vertices of $Q_{(i,j)}$ is less than 35.

\(^1\)The results for the analysis for $(i, j)$ can be used for $(j, i)$ just by changing any subscript $i$ for $j$ and $j$ for $i$. 

Lemma VII.11 For each basic feasible solution or extreme point \((x, y) \in Q(i, j)\), there exists a corresponding basis such that \(x_j\) is basic.

**Proof.** Let \((x, y) \in Q(i, j)\) be a feasible extreme point. If \(x_j > 0\), we are done since \(x_j\) has to be basic. So we assume that \(x_j = 0\). This implies that \(\Delta_{i, j} = 0\), otherwise the solution would be infeasible. However, when \(x_j = 0\), \(x = s_i = s_j = s_{ij} = 0\), \(y_i = D_i\), and \(y_j = D_j\). Because there are three structural constraint in \(Q(i, j)\) there must be three basic variables. Therefore, this basic feasible solution is degenerate since at least one basic variable takes the value of zero. We can select \(x_j, y_i\), and \(y_j\) as the basic variables since their corresponding basic matrix is nonsingular. Consequently, for every feasible extreme point of \(Q(i, j)\), there exists a corresponding basis such that \(x_j\) is basic. □

By Lemma VII.11, there is a maximum of only 15 basic feasible solutions for \(Q(i, j)\) where \(x_j\) is basic. Of those, there are only 8 distinct bases that may correspond to basic feasible solutions of \(Q(i, j)\) under specific conditions.

Next we present the necessary conditions for each basis to correspond to an extreme point of \(Q(i, j)\). Let \(c_{ij}\) and \(d_{ij}\) be defined as follows:

\[
c_{ij} = \begin{cases} 
1 & \text{if } \Delta_{i, j} \geq \delta_{ij}, \\
0 & \text{otherwise}, 
\end{cases} \quad (7.12)
\]

\[
d_{ij} = \begin{cases} 
1 & \text{if } \Delta_{i, j} \geq D_j, \\
0 & \text{otherwise}. 
\end{cases} \quad (7.13)
\]

where \(\delta_{ij} = D_j - D_i\). By exchanging the subscripts \(i\) and \(j\), the quantities \(\delta_{ji}, c_{ji},\) and \(d_{ji}\) are defined similarly. While the quantities \(\delta_{ij}, c_{ij},\) and \(d_{ij}\) are associated
Table 7.1: Common Easternmost Location: Necessary conditions for a basis to correspond to an extreme point of $Q(i,j)$.

<table>
<thead>
<tr>
<th>No.</th>
<th>Bases</th>
<th>Value of the variables</th>
<th>Conditions</th>
<th>$(c_{ij}, d_{ij}) = $</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(x_j, y_i, s_j)$</td>
<td>$(\Delta_{i,j}, D_i, \Delta_{i,j} - D_j)$</td>
<td>$\Delta_{i,j} \geq D_j$</td>
<td>$(1,1)$ $(1,0)$ $(0,0)$</td>
</tr>
<tr>
<td>2</td>
<td>$(x_j, x_i, s_j)$</td>
<td>$(\Delta_{i,j} + D_i, D_i, \Delta_{i,j} - \delta_{ij})$</td>
<td>$\Delta_{i,j} \geq \delta_{ij}$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>3</td>
<td>$(x_j, x_i, y_i)$</td>
<td>$(D_j, D_j - \Delta_{i,j}, \Delta_{i,j} - \delta_{ij})$</td>
<td>$\delta_{ij} \leq \Delta_{i,j} &lt; D_j$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>4</td>
<td>$(x_j, y_i, y_j)$</td>
<td>$(\Delta_{i,j}, D_i, D_j - \Delta_{i,j})$</td>
<td>$\Delta_{i,j} &lt; D_j$</td>
<td>$\checkmark$ $\checkmark$</td>
</tr>
<tr>
<td>5</td>
<td>$(x_j, y_i, s_{ij})$</td>
<td>$(D_j, D_i, D_j - \Delta_{i,j})$</td>
<td>$\Delta_{i,j} &lt; D_j$</td>
<td>$\checkmark$ $\checkmark$</td>
</tr>
<tr>
<td>6</td>
<td>$(x_j, x_i, y_j)$</td>
<td>$(\Delta_{i,j} + D_i, D_i, \delta_{ij} - \Delta_{i,j})$</td>
<td>$\Delta_{i,j} &lt; \delta_{ij}$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>7</td>
<td>$(x_j, x_i, s_{ij})$</td>
<td>$(D_j, D_i, \delta_{ij} - \Delta_{i,j})$</td>
<td>$\Delta_{i,j} &lt; \delta_{ij}$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>8</td>
<td>$(x_j, x_i, s_i)$</td>
<td>$(D_j, D_j - \Delta_{i,j}, \delta_{ij} - \Delta_{i,j})$</td>
<td>$\Delta_{i,j} &lt; \delta_{ij}$</td>
<td>$\checkmark$</td>
</tr>
</tbody>
</table>

with $Q(i,j)$, the quantities $\delta_{ji}, c_{ji},$ and $d_{ji}$ are associated with $Q(j,i)$.

**Lemma VII.12** $c_{ij} = 0 \Rightarrow d_{ij} = 0$ and $d_{ij} = 1 \Rightarrow c_{ij} = 1.$

**Proof.** The proof follows from the fact that $D_i \geq 0. \Box.$

Table 7.1 presents the 8 possible bases for $Q(i,j)$, the values of the basic variables, the conditions that need to be satisfied for each basis to correspond to an extreme point of $Q(i,j)$, and the different values of the ordered pair $(c_{ij}, d_{ij})$ that are possible.

All the nonbasic variables take the value of zero. Note that $(c_{ij}, d_{ij}) = (0,1)$ is not possible due to the Lemma VII.12.

Note that the basic feasible solutions associated with the bases in Table 7.1 may be degenerate when $\Delta_{i,j} = D_j$ or $\Delta_{i,j} = \delta_{ij}$. Thus, no generality is lost by proceeding as if there is not a corresponding extreme point when $\Delta_{i,j} = D_j$.

$^2\checkmark$ means that the respective basis corresponds to an extreme point of $Q(i,j)$ under the conditions $(c_{ij}, d_{ij})$. 
or \( \Delta_{i,j} = \delta_{ij} \) even when there is. For instance, if \( \Delta_{i,j} = D_j \), bases 1, 4, and 5 correspond to the same extreme point. However, according to Table 7.1 only basis 1 is considered.

Let \((x^\ell, y^\ell)\), \(\ell = 1, 2, \ldots, 8\) be the extreme point associated with the \(\ell\)th basis. \((x^8, y^8)\) is not an element of \(\mathcal{N}(Q(K))\) since \((x^7, y^7) \leq (x^8, y^8)\). Therefore, by Theorem VI.9, \((x^8, y^8)\) can not be a vertex of \(\text{Conv}(Q(K))\) under any circumstance. Hence, we eliminate the basis 8 in Table 7.1 from the analysis.

Recall from Theorem VI.8, every vertex of \(\text{Conv}(Q(K))\) must be an extreme point of \(Q_{(i,j)}\) or \(Q_{(j,i)}\). Therefore, every row of \(B\) is either a vertex of \(Q_{(i,j)}\) or \(Q_{(j,i)}\). However, the vertices of \(Q_{(i,j)}\) and \(Q_{(j,i)}\) depend on the values of the quantities \((c_{ij}, d_{ij})\) and \((c_{ji}, d_{ji})\) respectively. Our next definition is used to determine the different combinations of \((c_{ij}, d_{ij}), (c_{ji}, d_{ji})\) that may occur.

**Definition VII.1** Suppose there is a common easternmost location for all satellites \(i, i \in K\). A **cd-scenario** for \(\text{Conv}(Q(K))\), is a vector consisting of the ordered pairs \([(c_{ij}, d_{ij}), (c_{ji}, d_{ji})]\).

Throughout this subsection we refer to a cd-scenario as a scenario. Table 7.2 shows the different scenarios that may occur for the two-satellite case. The scenario \([(0,0),(0,0)]\) does not appear in Table 7.2 since it is not feasible. \(c_{ij} = 0\) implies that \(\Delta_{i,j} < D_j - D_i\) and \(c_{ji} = 0\) implies that \(\Delta_{i,j} < D_i - D_j\). Thus, \(\Delta_{i,j} < 0\), which is not possible. We observe that the scenario \([(1,1),(0,0)]\) is exactly the same as
Table 7.2: Different cd-scenarios of $\text{Conv}(Q(K))$ for the two-satellite case

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$[(c_{ij}, d_{ij}), (c_{ji}, d_{ji})]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C.1$</td>
<td>$[(0,0), (1,0)]$</td>
</tr>
<tr>
<td>$C.2$</td>
<td>$[(0,0), (1,1)]$</td>
</tr>
<tr>
<td>$C.3$</td>
<td>$[(1,0), (1,0)]$</td>
</tr>
<tr>
<td>$C.4$</td>
<td>$[(1,0), (1,1)]$</td>
</tr>
<tr>
<td>$C.5$</td>
<td>$[(1,1), (1,1)]$</td>
</tr>
</tbody>
</table>

$C.2$ by just exchanging the subscripts $i$ and $j$. It is important to remark that only one scenario may pertain at a time and which scenario pertains depends on the data.

To determine the facets of $\text{Conv}(Q)$ under the different scenarios in Table 7.2 we define the concept of active and inactive scenario.

**Definition VII.2** A scenario is said to be **active** if, whenever any $c_{ij}, d_{ij}, c_{ji}$, or $d_{ji}$ takes the value of 1, its associated inequality ($\Delta_{i,j} \geq \delta_{ij}, \Delta_{i,j} \geq D_{j}, \Delta_{i,j} \geq \delta_{ji}$, or $\Delta_{i,j} \geq D_{i}$ respectively) is satisfied as an equality. Otherwise, a scenario is called **inactive**.

**Example VII.1** Let $\Delta_{i,j} < D_{j} - D_{i}$ and $\Delta_{i,j} > D_{i}$. Then $c_{ij} = 0$ and $d_{ji} = 1$.

By Lemma VII.12, $d_{ij} = 0$ and $c_{ji} = 1$. Thus, scenario $C.2$ applies. Since $\Delta_{i,j} > D_{i} - D_{j}$ ($c_{ji} = 1$) and $\Delta_{i,j} > D_{i}$ ($d_{ji} = 1$) scenario $C.2$ is inactive. □

Definition VII.2 deals with the case where some of the bases in Table 7.1 become degenerate. Hence, two different bases may correspond to the same extreme point of $\text{Conv}(Q(K))$. This is demonstrated in the following example.
Example VII.2 Let $\Delta_{i,j} = \delta_{i,j}$. Then $c_{ij} = 1$. Bases 6 and 7 are degenerate since $y_j = s_{ij} = 0$. Further, both bases correspond to the same extreme point (the point $(x_i, x_j, y_i, y_j) = (D_i, D_j, 0, 0)$) since $D_j = \Delta_{i,j} + D_i$. □

Let $\sigma_{ij} = D_j + D_i + \Delta_{i,j}$. Next we show how to find a facet of $\text{Conv}(Q(K))$ under a specific scenario of Table 7.2.

Theorem VII.13 If $\Delta_{i,j} > D_i$ and $\Delta_{i,j} > D_j$, then the inequality $(\alpha, \beta)(x, y) \geq 1$, where $\alpha = \left( \frac{\Delta_{i,j} + \delta_{i,j}}{\sigma_{ij} \Delta_{i,j}}, \frac{\Delta_{i,j} + \delta_{i,j}}{\sigma_{ij} \Delta_{i,j}} \right)$, and $\beta = \left( \frac{2}{\sigma_{ij}}, \frac{2}{\sigma_{ij}} \right)$, defines a facet of $\text{Conv}(Q)$.

Proof. Since $\Delta_{i,j} > D_i$ and $\Delta_{i,j} > D_j$, scenario C.5 applies and it is inactive. From Table 7.1 there are two extreme points in $Q_{(i,j)}$ and $Q_{(j,i)}$. Since all 4 points are linearly independent and elements of $T_{\text{Conv}(Q(K))}$, all are extreme points of $\text{Conv}(Q(K))$. Then, the matrix $B$ is

$$B = \begin{bmatrix} D_i & D_i + \Delta_{i,j} & 0 & 0 \\ 0 & \Delta_{i,j} & D_i & 0 \\ D_j + \Delta_{i,j} & D_j & 0 & 0 \\ \Delta_{i,j} & 0 & 0 & D_j \end{bmatrix}.$$ 

The first two rows of $B$ correspond to vertices of $Q_{(i,j)}$ and the last two rows correspond to vertices of $Q_{(j,i)}$. $(\alpha, \beta)$ is a vertex of $\text{Conv}(Q(K))^B$ since it is the solution to the system $B(u, w) = 1$. Thus, by Theorem VII.10, $(\alpha, \beta)(x, y) \geq 1$ induces a facet of $\text{Conv}(Q(K))$. By Theorem VII.7, $(\alpha, \beta)(x, y) \geq 1$ defines a facet of $\text{Conv}(Q)$. □

In the next theorem we present 3-facets and 4-facets for $\text{Conv}(Q(K))$ that may
apply for a specific scenario. We define

\[
(\alpha^1, \beta^1) = \left( \frac{\Delta_{i,j} + \delta_{i,j}}{\sigma_{i,j} \Delta_{i,j}}, \frac{\Delta_{i,j} + \delta_{i,j}}{\sigma_{i,j} \Delta_{i,j}}, \frac{2}{\sigma_{i,j}}, \frac{2}{\sigma_{i,j}} \right) \quad (7.14)
\]

\[
(\alpha^2, \beta^2) = \left( \frac{1}{\Delta_{i,j}}, \frac{\Delta_{i,j} - D_i}{\Delta_{i,j}(D_i + \Delta_{i,j})}, \frac{2}{(D_i + \Delta_{i,j})}, 0 \right) \quad (7.15)
\]

\[
(\alpha^3, \beta^3) = \left( \frac{\Delta_{i,j} - D_j}{\Delta_{i,j}(D_j + \Delta_{i,j})}, \frac{1}{\Delta_{i,j}(D_j + \Delta_{i,j})}, \frac{0}{\Delta_{i,j}(D_j + \Delta_{i,j})} \right) \quad (7.16)
\]

where \((\alpha^\ell, \beta^\ell) \in R^2 \times R^2, \ell = 1, 2, 3.\)

**Theorem VII.14** The inequalities \((\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2, 3\) (where \((\alpha^\ell, \beta^\ell), \ell = 1, 2, 3\) are defined in (7.14), (7.15), and (7.16) induce all the 3-facets and 4-facets of \(\text{Conv}(Q(K)).\) If a scenario is inactive then:

- \((\alpha^1, \beta^1)(x, y) \geq 1\) defines a facet of \(\text{Conv}(Q)\) under scenarios C.3, C.4, or C.5.
- \((\alpha^2, \beta^2)(x, y) \geq 1\) defines a facet of \(\text{Conv}(Q)\) under scenarios C.2, C.4, or C.5.
- \((\alpha^3, \beta^3)(x, y) \geq 1\) defines a facet of \(\text{Conv}(Q)\) under scenario C.5.

The proof of Theorem VII.14 is presented in Appendix B. Note that for scenario C.1 there are no 3-facets or 4-facets. It is important to remark that if a scenario is active the inequalities \((\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2, 3,\) may not define facets of \(\text{Conv}(Q(K)).\) To illustrate this and the facets in Theorem VII.14 we use the following two satellite examples.
Example VII.3 Let $D_1 = 3, D_2 = 5, \Delta_{1,2} = 4, E = 0$. Then, $(c_{12}, d_{12}) = (1, 0)$ since $\Delta_{1,2} > D_2 - D_1$ and $\Delta_{1,2} < D_2$. Further, $(c_{21}, d_{21}) = (1, 1)$ since $\Delta_{1,2} > D_1 - D_2$ and $\Delta_{1,2} > D_1$. Scenario $C.4$ applies and it is inactive. From Table 7.1, there are 4 extreme points for $Q(i,j)$ and 2 extreme points for $Q(j,i)$. Since all of them are elements of $T_{Conv(Q(K))}$ and none of them is a sum of a convex combination of the other extreme points and a nonnegative linear combination of the extreme directions of $Conv(Q(K))$, each one of them is an extreme point for $Conv(Q(K))$.

Thus the matrix $B$ whose rows are ordered according to $x_1, x_2, y_1, y_2$ is

$$B = \begin{bmatrix}
3 & 7 & 0 & 0 \\
1 & 5 & 2 & 0 \\
0 & 4 & 3 & 1 \\
0 & 5 & 3 & 0 \\
4 & 0 & 0 & 5 \\
9 & 5 & 0 & 0
\end{bmatrix}.$$  

The first 4 rows of $B$ are the vertices of $Q(i,j)$ and the last two rows are the vertices of $Q(j,i)$. Rows 1, 2, 3, and 4 are associated with bases 2, 3, 4 and 5 in Table 7.1. By exchanging subscripts, rows 5 and 6 are associated with bases 1 and 2 of the same table.

From Theorems VII.3, VII.4 and VII.5, there exist 6 trivial facets. Those facets are

$$x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, x_1 + y_1 \geq 3, x_2 + y_2 \geq 5.$$  

From Theorem VII.6, the inequality $x_1 + x_2 \geq 4$ induces a nontrivial facet.

From Theorem VII.14, the inequalities $(\alpha^1, \beta^1)(x, y) \geq 1$ and $(\alpha^2, \beta^2)(x, y) \geq 1$ induce facets. After multiplying by the lowest common denominator those facets
are

\[7x_1 + x_2 + 8y_1 \geq 28\]

\[x_1 + 3x_2 + 4y_1 + 4y_2 \geq 24\]

\[\Box\]

Next we give an example where scenario C.4 also applies, but it is active. For this example, there are no 3-facets or 4-facets.

**Example VII.4** Let \(D_1 = 2, D_2 = 4, \Delta_{1,2} = 2, E = 0\). Then, \((c_{12}, d_{12}) = (1, 0)\) since \(\Delta_{1,2} = D_2 - D_1\) and \(\Delta_{1,2} < D_2\). Further, \((c_{21}, d_{21}) = (1, 1)\) since \(\Delta_{1,2} > D_1 - D_2\) and \(\Delta_{1,3} = D_1\). Scenario C.4 applies and it is active. From Table 7.1, there are 4 extreme points for \(Q_{(i,j)}\) and 2 extreme points for \(Q_{(j,i)}\). However, not all of them are extreme points of \(\text{Conv}(Q)\). After eliminating the points that are not extreme points of \(\text{Conv}(Q)\), we obtain the following \(B\) matrix

\[
B = \begin{bmatrix}
2 & 4 & 0 & 0 \\
0 & 2 & 2 & 2 \\
0 & 4 & 2 & 0 \\
2 & 0 & 0 & 4
\end{bmatrix}.
\]

From Theorems VII.3, VII.4 and VII.5, there exist 6 trivial facets. Those facets are

\[x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0, x_1 + y_1 \geq 2, x_2 + y_2 \geq 4.\]

From Theorem VII.6, the inequality \(x_1 + x_2 \geq 2\) induces a nontrivial facet.

By the same arguments presented in the proof of Theorem VII.14 for scenario C.1, there do not exist 3-facets or 4-facets. \(\Box\)
7.3.2 Facets for Conv(Q(K)) when there are distinct east-ernmost feasible locations

In this section we assume that the eastern limits are distinct. Let $K \subset N, |K| = 2$, and $K = \{i,j\}$. There are two possible orderings, $(i,j)$ and $(j,i)$. Therefore, $Q(K) = Q_{(i,j)} \cup Q_{(j,i)}$. Since $E_i \neq E_j$, it is necessary to analyze both orderings.

Without loss of generality we assume that $E_j > E_i$. First, we make the analysis for the ordering $(i,j)$.

$$Q_{(i,j)} = \{x_i + y_i - s_i = D_i, \ x_j + y_j - s_j = D_j, \ x_j - x_i - s_{ij} = \Delta_{i,j},$$

$$x_i \geq E_i, \ x_j \geq E_j, \ y_i, y_j, s_i, s_j, s_{ij} \geq 0\}$$

where $s_i, s_j, s_{ij}$ are surplus variables. We treat the constraints $x_i \geq E_i$ and $x_j \geq E_j$ as explicit lower bounds of the variables (see [9]). Therefore, there are 7 variables and 3 structural constraints. Of a maximum number of 35 basic feasible solutions of $Q_{(i,j)}$, there are only 12 distinct bases that may correspond to vertices of $Q_{(i,j)}$ under specific conditions.

Next we present the necessary conditions for each basis to correspond to an extreme point of $Q_{(i,j)}$. Let $a_{ij}, b_{ij}, c_{ij}$ and $d_{ij}$ be defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } \Delta_{i,j} \geq E_j - D_i, \\ 0 & \text{otherwise}, \end{cases}$$

(7.17)

$$b_{ij} = \begin{cases} 1 & \text{if } \Delta_{i,j} \geq E_j - E_i, \\ 0 & \text{otherwise}. \end{cases}$$

(7.18)

$$c_{ij} = \begin{cases} 1 & \text{if } \Delta_{i,j} \geq D_j - D_i = \delta_{ij}, \\ 0 & \text{otherwise}, \end{cases}$$

(7.19)

$$d_{ij} = \begin{cases} 1 & \text{if } \Delta_{i,j} \geq D_j - E_i, \\ 0 & \text{otherwise}. \end{cases}$$

(7.20)
By exchanging the subscripts \( i \) and \( j \), the quantities \( a_{ji}, b_{ji}, c_{ji}, \) and \( d_{ji} \) are defined similarly.

**Lemma VII.15**

\[
\begin{align*}
i) & \quad a_{ij} = 0 \Rightarrow b_{ij} = 0, \quad c_{ij} = 0. \\
ii) & \quad b_{ij} = 0 \Rightarrow d_{ij} = 0. \\
iii) & \quad c_{ij} = 0 \Rightarrow d_{ij} = 0. \\
iv) & \quad d_{ij} = 1 \Rightarrow b_{ij} = 1, \quad c_{ij} = 1. \\
v) & \quad b_{ij} = 1 \Rightarrow a_{ij} = 1. \\
vi) & \quad c_{ij} = 1 \Rightarrow a_{ij} = 1.
\end{align*}
\]

**Proof.** The proof follows from the fact that \( D_i \geq E_i \) and \( D_j \geq E_j. \Box \).

Table 7.3 presents the 12 possible bases for \( Q_{(i,j)} \), and the values of the basic variables. Table 7.4 presents the different values of the ordered 4-tuple \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) that are possible for each basis\(^3\). Notice that all the nonbasic variables take its lower bound value, i.e., \( x_i = E_i, \ x_j = E_j, \) and any other variable takes the value of zero.

Let \((x^\ell, y^\ell), \ \ell = 1, 2, \ldots, 12, \) be the basic feasible solution associated with the \( \ell \)-th basis. \((x^{11}, y^{11}) \) and \((x^{12}, y^{12})\) are not elements of \( T_{\text{Conv}(Q(K))} \) since \((x^{10}, y^{10}) \leq \)

\(^3\)Notice that there are many values of \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) that are not possible due to Lemma VII.15.
Table 7.3: Distinct Easternmost Locations: The 12 possible bases for $Q_{(i,j)}$

<table>
<thead>
<tr>
<th>No.</th>
<th>Bases</th>
<th>Value of the variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_j, y_i, s_j$</td>
<td>$(\Delta_{i,j} + E_i, D_i - E_i, \Delta_{i,j} + E_i - D_j)$</td>
</tr>
<tr>
<td>2</td>
<td>$x_j, x_i, s_j$</td>
<td>$(\Delta_{i,j} + D_i, D_i, \Delta_{i,j} + \delta_{ij})$</td>
</tr>
<tr>
<td>3</td>
<td>$x_j, x_i, y_i$</td>
<td>$(D_j, D_j - \Delta_{i,j}, \Delta_{i,j} + \delta_{ij})$</td>
</tr>
<tr>
<td>4</td>
<td>$x_j, y_i, y_j$</td>
<td>$(\Delta_{i,j} + E_i, D_i - E_i, D_j - \Delta_{i,j} - E_j)$</td>
</tr>
<tr>
<td>5</td>
<td>$x_j, y_i, s_{ij}$</td>
<td>$(D_j, D_i - E_i, D_j - E_i - \Delta_{i,j})$</td>
</tr>
<tr>
<td>6</td>
<td>$x_j, x_i, y_j$</td>
<td>$(\Delta_{i,j} + D_i, D_i, \delta_{ij} - \Delta_{i,j})$</td>
</tr>
<tr>
<td>7</td>
<td>$x_j, x_i, s_{ij}$</td>
<td>$(D_j, D_i, \delta_{ij} - \Delta_{i,j})$</td>
</tr>
<tr>
<td>8</td>
<td>$x_i, y_i, y_j$</td>
<td>$(E_j - \Delta_{i,j}, \Delta_{i,j} + D_i - E_j, D_j - E_j)$</td>
</tr>
<tr>
<td>9</td>
<td>$y_i, y_j, s_{ij}$</td>
<td>$(D_i - E_i, D_j - E_j, E_j - E_i - \Delta_{i,j})$</td>
</tr>
<tr>
<td>10</td>
<td>$x_i, y_j, s_{ij}$</td>
<td>$(D_i, D_j - E_j, E_j - \Delta_{i,j} - D_i)$</td>
</tr>
<tr>
<td>11</td>
<td>$x_i, y_j, s_i$</td>
<td>$(E_j - \Delta_{i,j}, D_j - E_j, E_j - \Delta_{i,j} - D_i)$</td>
</tr>
<tr>
<td>12</td>
<td>$x_j, x_i, s_i$</td>
<td>$(D_j, D_j - \Delta_{i,j}, \delta_{ij} - \Delta_{i,j})$</td>
</tr>
</tbody>
</table>

Table 7.4: Distinct Easternmost location: Necessary conditions for a basis to correspond to an extreme point of $Q_{(i,j)}$.

\[(a_{ij}, b_{ij}, c_{ij}, d_{ij}) = \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Value of $a_{ij}$</th>
<th>Value of $b_{ij}$</th>
<th>Value of $c_{ij}$</th>
<th>Value of $d_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>2</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
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<tr>
<td>3</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
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<tr>
<td>4</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>5</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>6</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>7</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
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<tr>
<td>8</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>9</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>10</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>11</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>12</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>
(x^{11}, y^{11}) and (x^7, y^7) \leq (x^{12}, y^{12}). Therefore, by Theorem VI.9, (x^{11}, y^{11}) and (x^{12}, y^{12}) cannot be vertices of Conv(Q(K)) under any circumstances. Hence, we eliminate them from the analysis in Table 7.3.

Next, we make the analysis for the ordering (j, i).

\[ Q_{(j,i)} = \{ x_i + y_i - s_i = D_i, \ x_j + y_j - s_j = D_j, \ x_i - x_j - s_{ji} = \Delta_{i,j}, \]
\[ x_i \geq E_i, \ x_j \geq E_j, \ y_i, y_j, s_i, s_j, s_{ij} \geq 0 \}\]

where \( s_i, s_j, s_{ji} \) are surplus variables. Since there are 7 variables and 3 structural constraints, the maximum number of vertices of \( Q_{(j,i)} \) is 35.

**Lemma VII.16** For each basic feasible solution or extreme point \((x, y) \in Q_{(j,i)}, \) there exists a corresponding basis such that \( x_i \) is basic.

**Proof.** Follows from the fact that \( x_i \geq x_j \) in the ordering \((j, i)\) and that \( E_j \geq E_i. \)

\(\square\)

By Lemma VII.16, there is a maximum of only 15 basic feasible solutions for \( Q_{(j,i)} \) where \( x_i \) is basic. Of those, there are only 8 distinct bases that may correspond to vertices of \( Q_{(j,i)} \) under specific conditions.

**Lemma VII.17** \( c_{ji} = 0 \Rightarrow d_{ji} = 0 \) and \( d_{ji} = 1 \Rightarrow c_{ji} = 1. \)

**Proof.** The proof follows from the fact that \( D_j \geq E_j. \) \(\square\)

Notice that \( a_{ji} = 1 \) and \( b_{ji} = 1 \) since we assume \( E_j > E_i. \) Table 7.5 presents the 8 possible bases for \( Q_{(j,i)}, \) the values of the basic variables, and the different values
Table 7.5: Distinct Easternmost Locations: Necessary conditions for a basis to correspond to an extreme point of \(Q_{(i,j)}\)

<table>
<thead>
<tr>
<th>No.</th>
<th>Bases</th>
<th>Value of the variables</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((x_i, y_j, s_i))</td>
<td>((\Delta_{i,j} + E_j, D_j - E_j, \Delta_{i,j} + E_j, D_i))</td>
<td>(1,(*))</td>
</tr>
<tr>
<td>2</td>
<td>((x_i, x_j, s_i))</td>
<td>((\Delta_{i,j} + D_j, D_j, \Delta_{i,j} + \delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>3</td>
<td>((x_i, x_j, y_j))</td>
<td>((D_i, D_i - \Delta_{i,j}, \Delta_{i,j} + \delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>4</td>
<td>((x_i, y_j, y_i))</td>
<td>((\Delta_{i,j} + E_j, D_j - E_j, D_i - \Delta_{i,j} - E_j))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>5</td>
<td>((x_i, y_j, s_{ji}))</td>
<td>((D_i, D_j - E_j, D_i - E_j - \Delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>6</td>
<td>((x_i, x_j, y_i))</td>
<td>((\Delta_{i,j} + D_j, D_j, \delta_{ji} - \Delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>7</td>
<td>((x_i, x_j, s_{ji}))</td>
<td>((D_i, D_j, \delta_{ji} - \Delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>8</td>
<td>((x_i, x_j, s_j))</td>
<td>((D_i, D_i - \Delta_{i,j}, \delta_{ji} - \Delta_{i,j}))</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

of \((a_{ji}, b_{ji}, c_{ji}, d_{ji})\) that are possible\(^4\). Notice that all the nonbasic variables take its lower bound value, i.e., \(x_j = E_j\), and any other variable takes the value of zero.

Let \((x^\ell, y^\ell)\), \(\ell = 1, 2, \ldots, 8\) be the point associated with the \(\ell\)th basis. \((x^8, y^8)\) is not an elements of \(T_{\text{Conv}(Q(K))}\) since \((x^7, y^7) \leq (x^8, y^8)\). Therefore, by Theorem VI.9, \((x^8, y^8)\) can not be a vertex of \(\text{Conv}(Q(K))\) under any circumstances. Hence, we eliminate it from the analysis in Table 7.5.

Recall from Theorem VI.8, every vertex of \(\text{Conv}(Q(K))\) must be an extreme point of \(Q_{(i,j)}\) or \(Q_{(j,i)}\). Therefore, every row of \(B\) is either a vertex of \(Q_{(i,j)}\) or \(Q_{(j,i)}\). However, the vertices of \(Q_{(i,j)}\) and \(Q_{(j,i)}\) depend on the values of the quantities \((a_{ij}, b_{ij}, c_{ij}, d_{ij})\) and \((a_{ji}, b_{ji}, c_{ji}, d_{ji})\) respectively. An abcd-scenario for \(\text{Conv}(Q(K))\), when there are distinct easternmost location, is a vector consisting

\(^4\)Notice that there are many combination that are not possible due to Lemma VII.17.
of the 4-tuples \([a_{ij}, b_{ij}, c_{ij}, d_{ij}], (a_{ji}, b_{ji}c_{ji}, d_{ji})\].

By following the same procedure as that in Subsection 7.3.1 we can determine the different abcd-scenarios for this subsection. Next we prove that the inequality that induces a 4-facet when there is a common easternmost location also induces a facet when there are distinct easternmost locations.

**Theorem VII.18** If \(\Delta_{i,j} > D_j - D_i\) and \(\Delta_{i,j} > D_i - D_j\), then the inequality \((\alpha, \beta)(x, y) > 1\), where \(\alpha = (\frac{\Delta_{i,j} + \delta_{ij}}{\sigma_{ij} \Delta_{i,j}}, \frac{\Delta_{i,j} + \delta_{ij}}{\sigma_{ij} \Delta_{i,j}})\), and \(\beta = (\frac{2}{\sigma_{ij}}, \frac{2}{\sigma_{ij}})\), defines a facet of \(\text{Conv}(Q)\).

**Proof.** Since \(\Delta_{i,j} > D_j - D_i\) and \(\Delta_{i,j} > D_i - D_j\), \(c_{ij}=1\) and \(c_{ji}=1\). Thus, an abcd-scenario can not include the 4-tuples \((1,1,0,0),(1,0,0,0)\), and \((0,0,0,0)\) from Table 7.4 and \((1,1,0,0)\) from Table 7.5. Let \((x, y)\) be the extreme point associated with a basis. For bases 1, 2, and 3 in Tables 7.3 and bases 1, 2, 3 in Table 7.5, \((\alpha, \beta)(x, y) = 1\). For bases 4, 5, 8, and 9 in Tables 7.3 and bases 4, and 5 in Table 7.5, \((\alpha, \beta)(x, y) \geq 1\). Thus, for every 4-tuple such that \(c_{ij}=1\) or \(c_{ji}=1\) there are exactly two basic feasible solutions or extreme points such that \((\alpha, \beta)(x, y) = 1\) and for the remaining basic feasible solutions \((\alpha, \beta)(x, y) \geq 1\). Since an abcd-scenario consists of two 4-tuples, one associated with \(Q_{(i,j)}\) and one associated with \(Q_{(j,i)}\) it must be that for exactly 4 rows of the corresponding matrix \(B, B_{(l)}(u, w) = 1\), where \(B_{(l)}\) is the \(l\)-th row of \(B\). Since the solution to this system of equations is \((\alpha, \beta)\), \((\alpha, \beta)\) is a vertex of \(\text{Conv}(Q(K))^B\). Thus, by Theorem VII.10, \((\alpha, \beta)(x, y) \geq 1\)
induces a facet of \( \text{Conv}(Q(K)) \). By Theorem VII.7, \((\alpha, \beta)(x, y) \geq 1\) defines a facet of \( \text{Conv}(Q) \). □

### 7.4 Facets for \( \text{Conv}(X(K)) \)

Since \( \text{Conv}(X) \) can be represented in the same way that \( \text{Conv}(Q) \) is represented in (6.8), we can use the properties of blocking polyhedron to find facets of \( \text{Conv}(X) \).

Because of the inheritance properties, we do this for the two-satellite and the three-satellite cases.

#### 7.4.1 The two-satellite case

Let \( K \subset N, |K| = 2 \), and \( K = \{i, j\} \). There are two possible orderings, \((i, j)\) and \((j, i)\). Therefore, \( X(K) = X(i,j) \cup X(j,i) \). We make the analysis for the ordering \((i, j)\) only\(^5\). For the ordering \((i, j)\) we have

\[
X_{(i,j)} = \{ x_j - x_i - s_{ij} = \Delta_{i,j}, \ x_i \geq E_i, \ x_j \geq E_j, \ s_{ij} \geq 0 \}
\]

where \( s_{ij} \) is a surplus variable. Since there are 3 variables and one structural constraint, the maximum number of vertices of \( X_{(i,j)} \) is 3. Recall that for every extreme point there corresponds a basis (not necessarily unique).

\(^5\)The results for the analysis for \((i, j)\) can be used for \((j, i)\) just by changing any subscript \( i \) for \( j \) and \( j \) for \( i \).
Table 7.6: Necessary conditions for a basis to correspond to an extreme point of $X_{(i,j)}$ for the two-satellite case.

<table>
<thead>
<tr>
<th>No.</th>
<th>Bases</th>
<th>Value of the variables</th>
<th>Conditions</th>
<th>$b_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_j$</td>
<td>$E_i + \Delta_{i,j}$</td>
<td>$\Delta_{i,j} \geq E_j - E_i$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$s_{ij}$</td>
<td>$E_j - E_i - \Delta_{i,j}$</td>
<td>$\Delta_{i,j} &lt; E_j - E_i$</td>
<td>√</td>
</tr>
<tr>
<td>3</td>
<td>$x_i$</td>
<td>$E_j - \Delta_{i,j}$</td>
<td>$\Delta_{i,j} &lt; E_j - E_i$</td>
<td>√</td>
</tr>
</tbody>
</table>

extreme point of $X_{(i,j)}$. Let $b_{ij}$ be defined as follows:

$$b_{ij} = \begin{cases} 
1 & \text{if } \Delta_{i,j} \geq E_j - E_i, \\
0 & \text{otherwise},
\end{cases} \quad (7.21)$$

By exchanging the subscripts $i$ and $j$, the quantity $b_{ji}$ is defined similarly. While the quantity $b_{ij}$ is associated with $X_{(i,j)}$, the quantity $b_{ji}$ is associated with $X_{(j,i)}$.

Table 7.6 presents the 3 possible bases for $X_{(i,j)}$, the values of the basic variable, the condition that need to be satisfied for each basis to correspond to an extreme point of $X_{(i,j)}$, and the different values of $b_{ij}$ that are possible. Note that all the nonbasic variables take its lower bound values.

Let $x^\ell$, $\ell = 1, 2, 3$ be the point associated with the $\ell$th basis. $x^3$ is not an element of $T_{\text{Conv}(X(K))}$ since $x^2 \leq x^3$. Therefore, by Corollary VI.10, $x^3$ can not be a vertex of $\text{Conv}(Q(K))$ under any circumstance. Hence, we eliminate the last basis in Table 7.6 from the analysis.

Recall that every row of $B$ is either a vertex of $X_{(i,j)}$ or $X_{(j,i)}$. However, the vertices of $X_{(i,j)}$ and $X_{(j,i)}$ depend on the value of $b_{ij}$ and $b_{ji}$ respectively. A b-scenario for $\text{Conv}(X(K))$ is a vector $(b_{ij}, b_{ji})$. Table 7.7 shows the different b-scenarios for $\text{Conv}(X(K))$ that may occur for the two-satellite case. The b-
Table 7.7: Different b-scenarios of \( \text{Conv}(\mathcal{X}(K)) \) for the two-satellite case

<table>
<thead>
<tr>
<th>b-scenario</th>
<th>((b_{ij}, b_{ji}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.1</td>
<td>((1,1))</td>
</tr>
<tr>
<td>S.2</td>
<td>((1,0))</td>
</tr>
</tbody>
</table>

scenario \((0,0)\) does not appear in Table 7.7 since it is not feasible. \(b_{ij} = 0\) implies that \(\Delta_{i,j} < E_j - E_i\) and \(b_{ji} = 0\) implies that \(\Delta_{i,j} < E_i - E_j\). Thus, \(\Delta_{i,j} < 0\), which is not feasible. We observe that the b-scenario \((0,1)\) is exactly the same as S.2 by just exchanging the subscripts \(i\) and \(j\).

Active and inactive b-scenarios are defined as in Definition VII.2. Next, we find facets of \( \mathcal{X}(K) \) under b-scenario S.1.

**Theorem VII.19** If \(\Delta_{i,j} > |E_j - E_i|\), then the inequality \(\alpha x \geq 1\), where

\[
\alpha = \left( \frac{\Delta_{i,j} + E_i - E_j}{\Delta_{i,j} (\Delta_{i,j} + E_i + E_j)}, \frac{\Delta_{i,j} + E_j - E_i}{\Delta_{i,j} (\Delta_{i,j} + E_i + E_j)} \right),
\]

defines a nontrivial facet of \(\text{Conv}(Q)\).

**Proof.** Since \(\Delta_{i,j} > E_j - E_i\) and \(\Delta_{i,j} > E_i - E_j\), b-scenario S.1 applies and it is inactive. From Table 7.6 there is one extreme point in \(\mathcal{X}_{(i,j)}\) and \(\mathcal{X}_{(j,i)}\). Since both points are linearly independent both are extreme points of \(\mathcal{X}(K)\). As before, \(V(\text{Conv}(\mathcal{X}(K)))\) denotes the set of vertices of \(\text{Conv}(\mathcal{X}(K))\) and \(B\) is a \(|V(\text{Conv}(\mathcal{X}(K)))| \times |K|\) matrix whose rows are the elements of \(V(\text{Conv}(\mathcal{X}(K)))\).

Then, the matrix \(B\) is

\[
B = \begin{bmatrix}
E_i & E_i + \Delta_{i,j} \\
E_j + \Delta_{i,j} & E_j
\end{bmatrix}
\]
The solution to the system $B\mathbf{u} = 1$ is $\alpha$. Thus, by Theorem VII.10, $\alpha \mathbf{x} \geq 1$ induces a facet of $\text{Conv}(\mathcal{X}(K))$. By Theorems VII.8 and VII.7, it also defines a facet of $\text{Conv}(\mathcal{Q}(K))$ and of $\text{Conv}(\mathcal{Q})$. The nontriviality follows from the same arguments that in Theorem VII.6. □

**Theorem VII.20** There are no 2-facets under $b$-scenario S.2.

**Proof.** When $b$-scenario S.2 applies, $\Delta_{i,j} \geq E_j - E_i$ and $\Delta_{i,j} < E_i - E_j$. The matrix $M$ whose rows are the extreme points of $\mathcal{X}_{i,j}$ and $\mathcal{X}_{j,i}$ is

$$
M = \begin{bmatrix}
E_i & E_i + \Delta_{i,j} \\
E_i & E_j
\end{bmatrix}
$$

The first row of $M$ is not an extreme point of $\text{Conv}(\mathcal{X}(K))$ since it can be represented as the sum of the last row and a nonnegative combination of the unit vectors of $\text{Conv}(\mathcal{X}(K))$. Thus the matrix $B$ consists only of the second row of $M$. The extreme points of the system $B(u, bw) \geq 1$ are $E_i$ and $E_j$. Thus, from Theorem VII.10 there are no 2-facets under $b$-scenario S.2. □

Theorems VII.19 and VII.20 present the same results as Theorem VII.6 and Theorem 4.9 in [3].

7.4.2 The three-satellite case

Let $K \subset N, |K| = 3$, and $K = \{i, j, k\}$. To simplify the notation we assume that $i = 1, j = 2, k = 3$. There are 3! possible orderings, $\pi^1 = (1, 2, 3), \pi^2 = (2, 3, 1)$,
\( \pi^3 = (3, 1, 2), \pi^4 = (1, 3, 2), \pi^5 = (2, 1, 3), \) and \( \pi^6 = (3, 2, 1) \). We make the analysis for the ordering \( \pi^1 = (1, 2, 3) \). For the ordering \( \pi^1 = (1, 2, 3) \) we have

\[ \mathcal{X}_{\pi^1} = \{ x_2 - x_1 - s_{12} = \Delta_{1,2}, x_3 - x_1 - s_{13} = \Delta_{1,3}, x_3 - x_2 - s_{23} = \Delta_{2,3}; \] \]
\[ x_1 \geq E_1, x_2 \geq E_2, x_3 \geq E_3, s_{12}, s_{13}, s_{23} \geq 0 \} \]

where \( s_{12}, s_{13}, s_{23} \) are surplus variables. Since there are 6 variables and 3 structural constraints the maximum number of vertices of \( \mathcal{X}_{\pi^1} \) is 20. Of those, there are 16 distinct bases that may correspond to vertices of \( \mathcal{X}_{\pi^1} \) under specific conditions.

Next we present the necessary conditions for each basis to correspond to an extreme point of \( \mathcal{X}_{\pi^1} \). Let \( b_{12}, b_{13}, b_{23}, f_{13}, h_{12}, h_{13}, \) and \( t_{13} \) be defined as follows:

\[
\begin{align*}
    b_{12} &= \begin{cases} 
        1 & \text{if } \Delta_{1,2} \geq E_2 - E_1, \\
        0 & \text{otherwise,}
    \end{cases} \\
    b_{13} &= \begin{cases} 
        1 & \text{if } \Delta_{1,3} \geq E_3 - E_1, \\
        0 & \text{otherwise.}
    \end{cases} \\
    b_{23} &= \begin{cases} 
        1 & \text{if } \Delta_{2,3} \geq E_3 - E_2, \\
        0 & \text{otherwise,}
    \end{cases} \\
    f_{13} &= \begin{cases} 
        1 & \text{if } \Delta_{1,3} \geq \Delta_{1,2} + \Delta_{2,3}, \\
        0 & \text{otherwise.}
    \end{cases} \\
    h_{12} &= \begin{cases} 
        1 & \text{if } \Delta_{1,3} - \Delta_{2,3} \geq E_2 - E_1, \\
        0 & \text{otherwise,}
    \end{cases} \\
    h_{23} &= \begin{cases} 
        1 & \text{if } \Delta_{1,3} - \Delta_{1,2} \geq E_3 - E_2, \\
        0 & \text{otherwise.}
    \end{cases} \\
    t_{13} &= \begin{cases} 
        1 & \text{if } \Delta_{2,3} + \Delta_{1,2} \geq E_3 - E_1, \\
        0 & \text{otherwise,}
    \end{cases}
\end{align*}
\]

Table 7.8 presents the 16 possible bases for \( \mathcal{X}_{\pi^1} \), and the values of \( b_{12}, b_{13}, b_{23}, f_{13}, h_{12}, h_{13}, \) and \( t_{13} \) for each basis to correspond to an extreme point of \( \mathcal{X}_{\pi^1} \).
Let $E$ be a common easternmost location. Throughout this part we assume that $E = E_1 = E_2 = E_3$. Without loss of generality we assume that $E = 0$. Since $E = E_1 = E_2 = E_3$, $b_{12} = b_{13} = b_{23} = t_{13} = 1$. Thus, only bases 1, 2, 3 and 4 in Table 7.8 may correspond to extreme points of $\text{Conv}(\mathcal{X}(K))$. Since $x_3 \geq x_2 \geq x_1$ in the ordering $\pi^t$, $x_3$ and $x_2$ have to be basic. Thus, basis 4 is eliminated.

The following table presents the 3 possible bases for $\mathcal{X}_{\pi^t}$, the value of the variables and the conditions that need to be satisfied for each basis to correspond to an extreme point of $\mathcal{X}_{\pi^t}$.

**Theorem VII.21** Let

$$
\alpha^t = (1/\Delta_{1,2}, 1/\Delta_{1,2}, 0)
$$
Table 7.9: The possible bases for $\mathcal{X}_{\tau^1}$

<table>
<thead>
<tr>
<th>Number</th>
<th>Bases</th>
<th>Value of the variables</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(x_1, x_2, s_{12})$</td>
<td>$(\Delta_{1,3}, \Delta_{1,2} - \Delta_{2,3}, \Delta_{1,3} - \Delta_{2,3} - \Delta_{1,2})$</td>
<td>$\Delta_{1,3} \geq \Delta_{2,3} + \Delta_{1,2}$</td>
</tr>
<tr>
<td>2</td>
<td>$(x_1, x_2, s_{23})$</td>
<td>$(\Delta_{1,3}, \Delta_{1,2} - \Delta_{2,3} - \Delta_{1,2})$</td>
<td>$\Delta_{1,3} \geq \Delta_{2,3} + \Delta_{1,2}$</td>
</tr>
<tr>
<td>3</td>
<td>$(x_1, x_2, s_{33})$</td>
<td>$(\Delta_{1,2} - \Delta_{2,3}, \Delta_{1,2} - \Delta_{2,3} - \Delta_{1,2})$</td>
<td>$\Delta_{1,3} \leq \Delta_{2,3} + \Delta_{1,2}$</td>
</tr>
</tbody>
</table>

\[ \alpha^2 = (0, 1/\Delta_{2,3}, 1/\Delta_{2,3}) \]

\[ \alpha^3 = (1/\Delta_{1,3}, 0, 1/\Delta_{1,3}) \]

\[ \alpha^4 = \frac{\Delta_{1,2} + \Delta_{1,3} - \Delta_{2,3}}{\theta \Delta_{1,3}}, \frac{2}{\theta}, \frac{\Delta_{1,3} - \Delta_{1,2}}{\theta \Delta_{1,3}} \]

where $\theta = \Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}$. If $\Delta_{1,3} \geq \Delta_{2,3} + \Delta_{1,2}$ then $\alpha^ix \geq 1$, $i = 1, 2, 3, 4$ induces a facet of $\text{Conv}(\mathcal{X}(K))$.

**Proof.** Assume $\Delta_{1,3} \geq \Delta_{2,3} + \Delta_{1,2}$, then the matrix $M$ whose rows are the extreme points of $\mathcal{X}_{\tau^1}$, $i = 1, 2, 3, 4, 5, 6$, is the following:

\[
M = \begin{bmatrix}
0 & \Delta_{1,3} - \Delta_{2,3} & \Delta_{1,3} \\
0 & \Delta_{1,2} & \Delta_{1,3} \\
\Delta_{2,3} + \Delta_{1,3} & 0 & \Delta_{2,3} \\
\Delta_{1,3} & \Delta_{1,2} & 0 \\
0 & \Delta_{1,3} + \Delta_{2,3} & \Delta_{1,3} \\
\Delta_{1,2} & 0 & \Delta_{1,2} + \Delta_{1,3} \\
\Delta_{1,3} & \Delta_{1,3} - \Delta_{1,2} & 0 \\
\Delta_{1,3} & \Delta_{2,3} & 0 \\
\end{bmatrix}
\]

$M$ has 8 rows, the first two rows corresponding to extreme points of $\mathcal{X}_{\tau^1}$ and the last two rows to extreme points of $\mathcal{X}_{\tau^5}$. For $\mathcal{X}_{\tau^i}$, $i = 2, 3, 4, 5$ there is only one extreme point corresponding to row $i + 1$ of $M$.

The rows 1, 4, 5, and 7 are not extreme points of $\text{Conv}(\mathcal{X}(K))$ since they are not minimal, i.e., they can be represented as the sum of another point of $\text{Conv}(\mathcal{X}(K))$. 
and a nonnegative combination of the unit vectors. Thus, the matrix $B$ is

$$B = \begin{bmatrix}
0 & \Delta_{1,2} & \Delta_{1,3} \\
\Delta_{2,3} + \Delta_{1,3} & 0 & \Delta_{2,3} \\
\Delta_{1,2} & 0 & \Delta_{1,2} + \Delta_{1,3} \\
\Delta_{1,3} & \Delta_{2,3} & 0
\end{bmatrix}$$

The vertices of $Conv(\mathcal{X}(K))^B$, where $Conv(\mathcal{X}(K))^B$ is the blocking polyhedron associated with $Conv(\mathcal{X}(K))$, are:

$$\alpha^1 = \left(\frac{1}{\Delta_{1,2}}, \frac{1}{\Delta_{1,3}}, 0\right)$$

$$\alpha^2 = \left(0, \frac{1}{\Delta_{2,3}}, \frac{1}{\Delta_{2,3}}\right)$$

$$\alpha^3 = \left(\frac{1}{\Delta_{1,3}}, 0, \frac{1}{\Delta_{1,3}}\right)$$

$$\alpha^4 = \left(\frac{\Delta_{1,2} + \Delta_{1,3} - \Delta_{2,3}}{\theta \Delta_{1,3}}, \frac{2 \Delta_{2,3} + \Delta_{1,3} - \Delta_{1,2}}{\theta \Delta_{1,3}}\right)$$

where $\theta = \Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}$. Since $\alpha^i, i = 1, 2, 3, 4$ are vertices of $Conv(\mathcal{X}(K))^B$, $\alpha^i x \geq 1$ induces a facet of $Conv(\mathcal{X}(K))$ and, by previous theorems of $Conv(Q(K))$ and of $Conv(Q)$ also.

It is important to notice that the facet obtained by $\alpha^4$ is different from any obtained by Balas in [3], since he assumed the triangle inequality, which does not hold in this case since $\Delta_{1,3} \geq \Delta_{1,2} + \Delta_{2,3}$.

Now, assume $\Delta_{1,3} \leq \Delta_{1,2} + \Delta_{2,3}, \Delta_{1,2} \leq \Delta_{1,3} + \Delta_{2,3}$, and $\Delta_{2,3} \leq \Delta_{1,2} + \Delta_{1,3}$ then the matrix $B$ is

$$B = \begin{bmatrix}
0 & \Delta_{1,2} & \Delta_{1,3} + \Delta_{2,3} \\
\Delta_{2,3} + \Delta_{1,3} & 0 & \Delta_{2,3} \\
\Delta_{1,2} & \Delta_{1,3} + \Delta_{1,2} & 0 \\
0 & \Delta_{1,3} + \Delta_{2,3} & \Delta_{1,3} \\
\Delta_{1,2} & 0 & \Delta_{1,2} + \Delta_{1,3} \\
\Delta_{1,2} + \Delta_{2,3} & \Delta_{2,3} & 0
\end{bmatrix}$$
Each row of $B$ correspond to a different ordering of the satellites. $B$ is the same as the blocking matrix for a special case in [3, p.201].

7.5 All two-facets when there is a common easternmost location

In this section we present all the 2-facets under a common easternmost location.

**Theorem VII.22** The only classes of 2-facets for $\text{Conv}(Q)$ when there is a common easternmost location are $x_i + y_i \geq D_i$ (Theorem VII.5) and $x_i + x_j \geq \Delta_{i,j}$ (Theorem VII.6).

**Proof.** From Theorem VII.7 the inequality $(\rho, \psi)(\bar{x}, \bar{y}) \geq 1$, where $(\rho, \psi), (\bar{x}, \bar{y}) \in \mathbb{R}^m \times \mathbb{R}^m$, defines a facet of $\text{Conv}(Q(K))$, if and only if, $(\alpha, \beta)(x, y) \geq 1$, where $(\alpha, \beta), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha = (\rho, 0), \beta = (\psi, 0)$, defines a facet of $\text{Conv}(Q)$. Consequently, the four classes of 2-facets that may exist for $\text{Conv}(Q)$ are:

1. $\alpha_1 x_i + \alpha_2 x_j \geq 1$
2. $\alpha_2 x_j + \beta_1 y_i \geq 1$
3. $\beta_1 y_i + \beta_2 y_j \geq 1$
4. $\alpha_1 x_i + \beta_1 y_i \geq 1$

Class 1) From Theorems VII.8, VII.19, and VII.20, if $\alpha_1 > 0$ and $\alpha_2 > 0$ then $\alpha_1 = \alpha_2 = 1/\Delta_{i,j}$. 

Class 2) By Theorem VII.10, the inequality \((\alpha_1, \alpha_2, \beta_1, \beta_2)(x_i, x_j, y_i, y_j) \geq 1\) induces a facet of \(\text{Conv}(Q(K))\) if and only if \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) is a vertex of the polyhedron

\[
\text{Conv}(Q(K))^B = \{(u_1, u_2, w_1, w_2) \geq 0, B(u_1, u_2, w_1, w_2) \geq 1\}
\] (7.22)

where \(B\) is a \(|V(\text{Conv}(Q(K)))| \times 4\) matrix whose rows are the elements of \(|V(\text{Conv}(Q(K)))|\). In Appendix B for each scenario \(C.i\) the matrix \(B^{C.i}, i = 1, 2, 3, 4, 5\) has at least one row with a value of zero in the second and third columns. Therefore, \((0, \alpha_2, \beta_1, 0)\) can not be an extreme point of \(\text{Conv}(Q(K))^B\).

Class 3) By the same arguments as for class 2), \((0, 0, \beta_1, \beta_2)\) can not be an extreme point of \(\text{Conv}(Q(K))^B\).

Class 4) To have a facet (different from \(x_i + y_i \geq D_i\)) with \(\alpha_1 > 0\) and \(\beta_1 > 0\), either \(\alpha_1 > 1/D_i, \beta_1 < 1/D_i\) or \(\alpha_1 < 1/D_i, \beta_1 > 1/D_i\). Otherwise, \(x_i + y_i \geq D_i\) would dominate or would be dominated which is a contradiction. In Appendix B for each scenario \(C.i\) the matrix \(B^{C.i}, i = 1, 2, 3, 4, 5\) has at least one row with a value of zero in the first column and a value of \(D_i\) in the third column and another row with a value of \(D_i\) in the first column and a value of zero in the third column. Therefore, \((\alpha_1, 0, \beta_1, 0)\) can not be an extreme point of \(\text{Conv}(Q(K))^B\) unless \(\alpha_1 = \beta_1 = 1/D_i\).
7.6 Analysis of the valid inequalities $\Sigma_{j \in K} y_j + y_j^+ \geq z_{LB}(K)$

In this section we investigate the valid inequalities in [40] for the two-satellite case. We define $z_{LB}(K)$ to be the bound for the objective function obtained by using the methods in [40]. Let $K \subset N$, $|K| = 2$, and $K = \{i, j\}$. Then

$$z_{LB}(K) = \max\{0, \Delta_{i,j} - |D_i - D_j|\}.$$ 

Therefore, the valid inequality that we investigate here is:

$$y_i + y_j + y_i^+ + y_j^+ \geq \max\{0, \Delta_{i,j} - |D_i - D_j|\}. \quad (7.23)$$

After projecting out the variables $y^+$, (7.23) can be written as

$$x_i + x_j + 2y_i + 2y_j \geq \max\{0, \Delta_{i,j} - |D_i - D_j|\} + D_i + D_j.$$ 

Theorem VII.23 If $\Delta_{i,j} - |D_i - D_j| \leq 0$, the valid inequality (7.23) does not define a facet of $\text{Conv}(Q)$.

Proof. Since $\Delta_{i,j} - |D_i - D_j| \leq 0$, $\max\{0, \Delta_{i,j} - |D_i - D_j|\} = 0$. Thus, (7.23) is equivalent to

$$x_i + x_j + 2y_i + 2y_j \geq D_i + D_j.$$ 

From Theorem VII.5, $x_j + y_j \geq D_j$ and $x_i + y_i \geq D_i$ define facets of $\text{Conv}(Q)$. Thus, $x_i + x_j + y_i + y_j \geq D_i + D_j$ is a valid inequality for $\text{Conv}(Q)$ since it is a nonnegative combination of the two facets. From Proposition III.5, $x_i + x_j + 2y_i + 2y_j \geq D_i + D_j$ is not a facet of $\text{Conv}(Q)$ since it is dominated by $x_i + x_j + y_i + y_j \geq D_i + D_j.$
Theorem VII.24 If \( \Delta_{i,j} - |D_i - D_j| > 0 \), the valid inequality (7.23) defines a facet of Conv(Q) if and only if \( D_i = D_j \).

**Proof.** Assume \( \Delta_{i,j} - |D_i - D_j| > 0 \). Then (7.23) is equivalent to

\[
x_i + x_j + 2y_i + 2y_j \geq \Delta_{i,j} + 2 \min\{D_i, D_j\}.
\]  

(7.24)

We divide this part of the proof into three cases.

Case 1) Assume \( D_i < D_j \). We prove that (7.24) is dominated by a nonnegative linear combination of the facets \((\alpha^1, \beta^1) (x, y) \geq 1\) and \(x_i/D_i + y_i/D_i \geq 1\), where \((\alpha^1, \beta^1)\) was defined in (7.14). From Proposition III.5, we need to find \(\lambda_1 \geq 0\) and \(\lambda_2 \geq 0\) such that

\[
\lambda_1(\alpha^1, \beta^1) + \lambda_2(1/D_i, 0, 1/D_i, 0) \leq (1, 1, 2, 2)
\]

and

\[
\lambda_1 + \lambda_2 \geq \Delta_{i,j} + 2D_i.
\]

We solve the following linear program

\[
\begin{align*}
\max & \quad \lambda_1 + \lambda_2 \\
\text{s.t.} & \quad \lambda_1(\Delta_{i,j} + \delta_{ij})/(\Delta_{i,j} \sigma_{ij}) + \lambda_2(1/D_i) \leq 1 \\
& \quad \lambda_1(\Delta_{i,j} + \delta_{ij})/(\Delta_{i,j} \sigma_{ij}) \leq 1 \\
& \quad \lambda_1(2/\sigma_{ij}) + \lambda_2(1/D_i) \leq 2 \\
& \quad \lambda_1(2/\sigma_{ij}) \leq 2 \\
& \quad \lambda_1, \lambda_2 \geq 0
\end{align*}
\]

The optimal solution to the problem above is

\[
\lambda_1 = \Delta_{i,j} \sigma_{ij}/(\Delta_{i,j} + \delta_{ij}) \geq 0, \quad \lambda_2 = 2(D_j - D_i)D_i/(\Delta_{i,j} - D_j - D_j)(1/D_i) \geq 0,
\]
and

$$\lambda_1 + \lambda_2 = \Delta_{i,j} + 2D_i.$$ 

Therefore, (7.24) is dominated and consequently does not induce a facet.

Case 2) Assume $D_i > D_j$. In a similar way to Case 1) we can prove that (7.24) is dominated by a nonnegative linear combination of the facets $(\alpha^1, \beta^1) (x, y) \geq 1$ and $x_j/D_j + y_j/D_j \geq 1$, where $(\alpha^1, \beta^1)$ was defined in (7.14). Therefore, (7.24) does not induce a facet.

Case 3) Assume $D_i = D_j$. Then, (7.24) is equivalent to $(\alpha^1, \beta^1)(x, y) \geq 1$. Therefore, (7.24) induces a facet.\(\square\)
CHAPTER VIII
Computational results

In this chapter we present solutions for 6 test problems. In all problems, we assume a common easternmost location.

The first three problems include 2 fictitious satellites and the scenarios $C.3$, $C.4$, and $C.5$ are considered. In all three cases an optimal solution was found. Table 8.1 shows the parameters and Table 8.2 shows the optimal values for each one of the problems. Recall from Corollary VII.2 that $z(WUSLP) = 0$, i.e., all the satellites are located at their desired locations in the solution to the LP relaxation of WUSLP. For instance, for $P^3$, the LP relaxation solution is $x_1 = 2, x_2 = 3, y_1 = y_2 = y_1^+ = y_2^+ = 0$, and $z(P^3_R) = 0$. However, this solution is not feasible since the required angular separation between the satellites is 4. Problems with scenarios $C.1$ and $C.2$ were not considered since the optimal solution is the same as the LP relaxation solution.

The nontrivial facets appended to WUSLP were those of Theorems VII.6 and
Table 8.1: Parameters for the two-satellite problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Scenario</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^1$</td>
<td>C.3</td>
<td>3 4 2 0</td>
</tr>
<tr>
<td>$P^2$</td>
<td>C.4</td>
<td>3 5 4 0</td>
</tr>
<tr>
<td>$P^3$</td>
<td>C.5</td>
<td>2 3 4 0</td>
</tr>
</tbody>
</table>

Table 8.2: Optimal values for the two-satellite problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Optimal values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>$P^1$</td>
<td>3</td>
</tr>
<tr>
<td>$P^2$</td>
<td>1</td>
</tr>
<tr>
<td>$P^3$</td>
<td>2</td>
</tr>
</tbody>
</table>

VII.14. For $P^1$ the nontrivial facets are

\[
x_1 + x_2 \geq 2
\]

\[
x_1 + 3x_2 + 4y_1 + 4y_2 \geq 18.
\]

For $P^2$ the nontrivial facets are

\[
x_1 + x_2 \geq 4
\]

\[
7x_1 + x_2 + 8y_1 \geq 28
\]

\[
x_1 + 3x_2 + 4y_1 + 4y_2 \geq 24.
\]

For $P^3$ the nontrivial facets are

\[
x_1 + x_2 \geq 4
\]

\[
3x_1 + x_2 + 4y_1 \geq 12
\]

\[
x_1 + 7x_2 + 8y_2 \geq 28
\]
Table 8.3: Required angular separations for $P_4^5$, and $P_8^6$.

<table>
<thead>
<tr>
<th></th>
<th>ARG</th>
<th>BOL</th>
<th>CHL</th>
<th>PRG</th>
<th>PRU</th>
<th>URG</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARG</td>
<td>-</td>
<td>4.17</td>
<td>4.19</td>
<td>4.32</td>
<td>1.41</td>
<td>4.14</td>
</tr>
<tr>
<td>BOL</td>
<td>-</td>
<td>4.57</td>
<td>4.04</td>
<td>4.26</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>CHL</td>
<td>-</td>
<td>2.00</td>
<td>3.94</td>
<td>1.59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PRG</td>
<td>-</td>
<td></td>
<td>1.10</td>
<td>2.46</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PRU</td>
<td>-</td>
<td></td>
<td></td>
<td>0.37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>URG</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$3x_1 + 5x_2 + 8y_1 + 8y_2 \geq 36$.

Note that $P_2^2$ is the same problem considered in Example VII.3.

In the last three problems 6 South American countries were considered: Argentina (ARG), Bolivia (BOL), Chile (CHL), Paraguay (PRG), Peru (PRU), and Uruguay (URG). Actual minimum required angular separation values ($\Delta_{i,j}$s) are used [40]. We assume that each satellite has an easternmost location of $-80^\circ$.

Table 8.3 shows the required angular separations.

For $P_4$, the desired locations for each administration's satellite is $-95^\circ$. For $P_5$, $D_{ARG} = D_{PRG} = D_{URG} = -85^\circ$ and $D_{BOL} = D_{CHL} = D_{PRU} = -100^\circ$.

For problem $P_6$, $D_{ARG} = -84^\circ$, $D_{BOL} = -83^\circ$, $D_{CHL} = D_{PRU} = -82^\circ$, and $D_{PRG} = D_{URG} = -86^\circ$. Table 8.4 shows the objective function value for the LP relaxation ($z(P_R^i)$), the solution obtained by appending the nontrivial facets of Theorems VII.6 and VII.14 to $P_i$ ($z(P_{R+F}^i)$), and the optimal solution value ($z(P^i)$) for $P_4$, $P_5$, and $P_6$.

For the two-satellite problems an optimal solution was found by appending
Table 8.4: Solution values for $P^4$, $P^5$, and $P^6$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$z(P^i_R)$</th>
<th>$z(P^i_{R+F})$</th>
<th>$z(P^i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^4$</td>
<td>0</td>
<td>12.12</td>
<td>18.42</td>
</tr>
<tr>
<td>$P^5$</td>
<td>0</td>
<td>11.85</td>
<td>14.80</td>
</tr>
<tr>
<td>$P^6$</td>
<td>0</td>
<td>12.95</td>
<td>19.80</td>
</tr>
</tbody>
</table>

the nontrivial facets of Theorems VII.6 and VII.14. For problems $P^4$ and $P^6$, the solution values found after appending the same types of facets to WUSLP is within 35 percent of the optimal solution. For problem $P^5$ this solution is within 20 percent. By appending facets of subsets of two and three satellites to problems with 6 satellites, the bound of the WUSLP relaxation has been improved considerably.
CHAPTER IX

Lifting the facets of WUSLP

Recall that in SLP orbital locations in the GSO are allotted to satellites subject to service arc and angular separation constraints. The objective is the same as that for WUSLP, that is, to minimize the sum of the absolute differences between the locations allotted to the satellites and their corresponding desired locations.

Let $W_j$ be the westernmost feasible location for satellite $j$, $j \in N$. SLP can be formulated by adding the constraints

$$x_j \leq W_j, \quad \forall j \in N$$

(9.1)

to WUSLP to guarantee that each satellite is not located west of its westernmost feasible location. Without loss of generality, we assume that $W_j \geq D_j$.

In this chapter we present the NP-completeness proof of SLP. (This result also appears in [40].) Since the problem of finding a feasible solution to SLP is NP-complete, we transform SLP to a related problem. We show that if there is a feasible solution to SLP, then there exist an optimal solution to the transformed problem with the same optimal value as SLP. Furthermore, we show that if there is
no feasible solution for SLP, then the transformed problem is unbounded. Finally, we show that every facet for $\text{Conv}(Q)$ defines a facet for the transformed problem.

**Theorem IX.1** SLP is NP-Complete

**Proof.** We prove this theorem by showing via local replacement, that the problem of determining whether there is a Hamiltonian path in an undirected graph, $G = (V, A)$, where $V$ is a set of vertices and $A$ is a set of edges that connect vertices in $V$, is equivalent to the problem of determining whether there is a feasible solution to a particular instance of SLP. (A Hamiltonian path in $G$ is a sequence of distinct vertices $v(i) \in V$, $(v(1), v(2), \ldots, v(k))$, such that $(v(i), v(i+1)) \in A$ for $1 \leq i \leq k - 1$, where $k = |V|$ and the subscript $(i)$ denotes the $i$-th position in the sequence.)

Replace each $v_i \in V$ with a satellite $i$ such that $E_i = 0$, $W_i = k - 1$, and

$$\Delta_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \in A; \\ 2 & \text{otherwise.} \end{cases} \quad (9.2)$$

Assume that there is a Hamiltonian path in $G$, namely $(v(1), v(2), \ldots, v(k))$. Then the SLP instance constructed has a feasible solution:

$$x(1) = 0$$

$$x(i) = \sum_{j=1}^{i-1} \Delta_{(j), (j+1)} = i - 1 \text{ for } i = 2, 3, \ldots, k.$$ 

Assume that there is a feasible solution to the SLP instance constructed. Since $k$ satellites are positioned in $(k - 1)$ degrees of orbital arc, it must be that

$$x_{(i+1)} - x_{(i)} = 1 \text{ for } i = 1, 2, \ldots, k - 1.$$
Therefore, \((v_1, v_2, \ldots, v_k)\) must be a Hamiltonian path in \(G\). Consequently, we have shown that the instance of SLP constructed above has a feasible solution if and only if there is a Hamiltonian path in \(G\). Since the problem of determining whether there is a Hamiltonian path in \(G\) is NP-complete [13], the proof is completed. □

**Theorem IX.2** Either there exists a finite optimal solution to SLP or SLP is infeasible, i.e., SLP can not be unbounded.

**Proof.** From the arc service constraints \(E_j \leq x_j \leq W_j, \forall j \in N\). Thus, all the variables \(x_j\) are bounded. In addition, even though the variables \(y_j\) are not bounded, they are linearly dependent and are restricted to take the values in (4.4) and (4.5). □

Let \(W_{max} = \max_{j \in N}\{W_j\}\), \(\Delta_{j,n+1} = W_{max} - W_j, \forall j \in N\), and let \(N^* = \{1, 2, \ldots, n + 1\}\) be the set of indices of the satellites when a fictitious satellite, satellite \(n + 1\), is included. Let

\[
\mathcal{X}(N^*) = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} - x_j \geq \Delta_{j,n+1}, \forall j \in N; x_{n+1} \geq W_{max}\}.
\]

SLP can be transformed to the following problem:

\[
(TSLP')
\]

\[
\min \sum_{j \in N} (y_j + y^+_j)
\]

Subject to

\[
(x, y, y^+) \in \mathcal{Q}'
\]  \hfill (9.3)
Notice that in TSLP', $W_{\text{max}} \leq x_{n+1} \leq W_{\text{max}}$. For technical reasons we include the variable $x_{n+1}$, even though it could be deleted from the problem since its value is fixed to $W_{\text{max}}$. Since $x_{n+1} = W_{\text{max}}$ and $x_{n+1} - x_j \geq \Delta_{j,n+1} = W_{\text{max}} - W_j$, constraints (9.4) and (9.5) are equivalent to constraints (9.1).

Let

$$Q(N^*) = \{(x, x_{n+1}, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : (x, y) \in Q, (x, x_{n+1}) \in \mathcal{X}(N^*)\}$$

where $Q$ is as defined on page 32. Define $M \geq 0$, to be the value of the dual variable associated with the constraint (9.4). By using projection and by dualizing constraint (9.4), with the dual variable fixed at $M$, TSLP' can be transformed to the following problem:

(TSLP)

$$\min M(x_{n+1} - W_{\text{max}}) + \sum_{j \in N} (2y_j + x_j - D_j)$$

Subject to

$$(x, x_{n+1}, y) \in Q(N^*).$$

Let $z(\cdot)$ denote the objective function value and $z^*(\cdot)$ the optimal objective function value of problem (\cdot). The following theorems gives the relation between SLP and TSLP.
Theorem IX.3 (i) For any $(x, y, y^+)$ feasible to SLP, there exists an $x_{n+1}$ such that $(x, x_{n+1}, y)$ is feasible to TSLP with $z(SLP) = z(TSLP)$. (ii) Conversely, for any $(x, x_{n+1}, y)$ feasible to TSLP such that the objective function value is not unbounded, there exists a $y^+$ such that $(x, y, y^+)$ is feasible to SLP with $z(TSLP) = z(SLP)$.

Proof. Assume $(x, y, y^+)$ is feasible to SLP, then $z(SLP) = \sum_{j \in N} (y_j + y_j^+)$. Let $x_{n+1} = W_{\text{max}}$. Then, $(x, x_{n+1}, y) \in Q(N^*)$ since $x_j + y_j - y_j^+ = D_j$ and $y_j^+ \geq 0$ imply that $x_j + y_j \geq D_j$, and $x_{n+1} = W_{\text{max}}$ imply that $x_{n+1} - x_j \geq \Delta_{j,n+1}$ $\forall j \in N$. Thus, $z(TSLP) = \sum_{j \in N} (2y_j + x_j - D_j)$ but since $y_j^+ = x_j + y_j - D_j$, $z(TSLP) = z(SLP)$.

The proof of the other part of the theorem follows from similar arguments. □

Theorem IX.4 There exists a feasible solution for SLP if and only if $z^*(SLP) = z^*(TSLP)$.

Proof. $\Rightarrow$ Assume there exists a feasible solution for SLP. From Theorem IX.2, $z^*(SLP) < \infty$. Then by (i) of Theorem IX.3, $z^*(TSLP) \leq z^*(SLP) < \infty$. By (ii) of the same theorem, $z^*(TLP) \geq z^*(SLP)$.

$\Leftarrow$ Assume $z^*(SLP) = z^*(TSLP)$. Then, by Theorem IX.2, $z^*(TSLP)$ is not unbounded. Consequently, by (ii) of Theorem IX.3, there exists a feasible solution to SLP. □

Corollary IX.5 TSLP is unbounded if and only if SLP is infeasible
From Theorems IX.3 and IX.4, and Corollary IX.5, if TSLP has a finite optimal solution we can find the associated optimal solution for SLP. Thus, we characterize the facets of Conv(Q(N*)).

Next, we relate Conv(Q) to Conv(Q(N*)).

**Lemma IX.6** For each \((x,y) \in Conv(Q)\), there exists an \(x_{n+1}\) such that 
\((x,x_{n+1},y) \in Conv(Q(N*))\).

**Proof.** The proof follows from selecting \(x_{n+1} = \max\{W_{\max}, x_{\max}\}\), where \(x_{\max} = \max_{j \in N} \{x_j + \Delta_{j,n+1}\}\). □

**Lemma IX.7** \(\dim(Conv(Q(N*))) = 2n+1\), i.e., \(Conv(Q(N*))\) is full-dimensional.

**Proof.** The proof follows from the same arguments used in Theorem IV.2. □

The next theorem indicates that the facets of the previous sections define facets of TSLP, i.e., we lift the facets from \(Conv(Q)\), which is a \(2n\)-dimensional polyhedron, to a \((2n+1)\)-dimensional polyhedron, \(Conv(Q(N*))\).

**Theorem IX.8** If \((\alpha,\beta)(x,y) \geq 1\) is a facet for \(Conv(Q)\), \((\alpha,0,\beta)(x,x_{n+1},y) \geq 1\) is a facet for \(Conv(Q(N*))\).

**Proof.** Suppose \((\alpha,\beta)(x,y) \geq 1\) defines a facet of \(Conv(Q)\). From Proposition III.4, there exist \(2n\) affinely independent points \((x^\ell,y^\ell) \in Q, \ell = 1,2,\ldots,2n\), such that \((\alpha,\beta)(x^\ell,y^\ell) = 1\). From Lemma IX.6, there exists \(x^\ell_{n+1}\) such that \((x^\ell,x^\ell_{n+1},y^\ell) \in Conv(Q(N*))\), \(\forall \ell \in N^2\).
Let $\pi^t$ be the ordering associated with $(x^t, y^t)$, and let $x_{\text{max}} = \max_{j \in N, \ell \in N^2} \{x_j^t + \Delta_{j,n+1}\}$. By selecting, $x_{n+1}^t = \max\{W_{\text{max}}, x_{\text{max}}\} \ \forall \ell \in N^2$, then $(x^t, x_{n+1}^t, y^t) \in Conv(Q(N^*))$. We have $2n$ affinely independent points $(x^t, x_{n+1}^t, y^t) \in Conv(Q(N^*))$, but from Lemma IX.7, we need $2n+1$ affinely independent points. The other affinely independent point is obtained in the following way:

$$(x^{2n+1}, y^{2n+1}) = (x^1, y^1) ; \ x_{n+1}^{2n+1} = x_{n+1}^1 + 1$$

By using arguments similar to those used in (7.8), page 71, we can prove that $(x^t, x_{n+1}^t, y^t)$ are affinely independent and that $(\alpha, 0, \beta)(x^t, x_{n+1}^t, y^t) = 1 \ \forall \ell \in N^2$. Hence, $(\alpha, 0, \beta)(x, x_{n+1}, y) \geq 1$ induces a facet of $Conv(Q(N^*))$. $\square$
CHAPTER X

Relationship between SLP and the JIT
Earliness Tardiness Problem

One important optimization problem that is related to SLP is the single-machine, or one-processor, scheduling problem (SMP) with symmetric earliness and tardiness penalties. In this chapter we show the similarities between SMP and SLP. We explain how the results of the previous chapters apply for SMP. Recall that in SMP, jobs are scheduled on a single machine subject to window and minimum required time interval constraints. The objective is to minimize the total deviation between the starting times of the jobs and their corresponding starting due dates.

Let \( N = \{1, 2, \ldots, n\} \) be the set of indexes of the jobs that are to be scheduled on a single machine. For each job \( j \in N \), there is a processing time \( P_j \) that is assumed to be independent of the sequence and there is a window of time in which the job’s processing must be started. This window is defined by a release time \( (R_j) \) and a starting deadline \( (L_j) \), and is assumed to contain a specific starting due date \( (D_j) \). In addition, for each job \( j \in N \), there is a setup time \( U_j \) that
is independent of the job sequence. For each pair of jobs there is a minimum required interval of time $T_{i,j}$ between starting job $i$ and starting job $j$. We define $T_{i,j} = P_i + U_j + W_{i,j}$ where $W_{i,j}$ is a waiting time when job $j$ is processed after (not necessarily immediately after) job $i$. Notice that it is possible that $T_{i,j} \neq T_{j,i}$.

If $W_{i,j}=0$, the problem reduces to one in which the starting time of job $j$ depends only on the previous job $i$. Otherwise, the starting time of one job can be affected by the starting time of any other job. Without loss of generality, we assume that $L_j \geq D_j \geq R_j > 0$, $\forall j \in N$, and that $\min_{j \in N} \{R_j\} = 0$.

Let DUSMP be the problem obtained when the deadline constraints are relaxed from SMP. We compare WUSLP with DUSMP.

A schedule is defined to be a vector of feasible starting times. Hence, a schedule is a vector $t \in \mathbb{R}^n$, where $t_j$ denotes the starting time of job $j \in N$, satisfying the following constraints:

\begin{align*}
  t_j - t_i &\geq T_{i,j} \quad \lor \quad t_i - t_j \geq T_{j,i} \quad \forall i, j \in N \text{ such that } i < j ; \\
  t_j &\geq R_j \quad \forall j \in N. 
\end{align*}

Constraints (10.1) enforce sufficient separation in starting times between each pair of jobs, not only adjacent jobs. Constraints (10.2) guarantee that each job is started after its release time.
Let $\mathcal{X}_{\text{SMP}}$ denote the set of all schedules, i.e.,

$$\mathcal{X}_{\text{SMP}} = \{ t \in \mathbb{R}^n : t_j - t_i \geq T_{i,j} \quad \forall \quad t_i - t_j \geq T_{j,i} \quad \forall i, j \in N \text{ such that } i < j ;
\]

$$t_j \geq R_j \quad \forall j \in N \}$$

DUSMP can be formally stated as follows:

\[(\text{DUSMP}')\]

$$\min z = \sum_{j \in N} |t_j - D_j|$$

Subject to

$$t \in \mathcal{X}_{\text{SMP}}.$$

A one-to-one mapping $\pi$ of the set $N$ onto itself is called an ordering or sequence of the jobs. We denote the sequence $\pi \in \mathbb{R}^n$ by $\pi = (j_1, j_2, \ldots, j_n)$, where $j_i = \pi(i)$ indicates the job in the $i$-th position from the left. The number of such sequences is $n!$. We denote the set of those sequences by $\Pi_N$.

Whenever $t_i = t_j$ for some $i, j \in N$ such that $i < j$, we let $i$ precede $j$ in the sequence associated with that schedule. Therefore, notice that there is a unique sequence associated with each schedule, but there is an infinite number of schedules associated with a sequence.

Let

$$Q'_{\text{SMP}} = \{(t, y, y^+) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : t_j + y_j - y_j^+ = D_j \quad \forall j \in N\}; \quad (10.3)$$
where \( y_j (y_j^+) \) is the earliness (tardiness) of the scheduled starting time of job \( j \) from its starting due date. Note that

\[
y_j = \begin{cases} 
D_j - t_j & \text{if } t_j < D_j, \\
0 & \text{otherwise}
\end{cases}
\]

(10.4)

\[
y_j^+ = \begin{cases} 
-t_j - D_j & \text{if } x_j > D_j, \\
0 & \text{otherwise}
\end{cases}
\]

(10.5)

and \( y_j + y_j^+ = |t_j - D_j|, \forall j \in N. \)

The set \( Q_{SM} \) includes all the schedules of \( X_{SM} \), and it provides, by means of the constraints \( t_j + y_j - y_j^+ = D_j \), the deviation of the jobs' prescribed starting times from their starting due dates.

\( DUSMP' \) can be transformed to the following disjunctive programming problem:

\[
\min z = \sum_{j \in N} (y_j + y_j^+)
\]

Subject to

\( (DUSMP) \)

\( (t, y, y^+) \in Q_{SM}. \)

Now we are in a position to show the relationship between WUSLP and DUSMP. Table 10.1 presents a mapping of the elements, definitions, parameters, variables, constraints, sets, functions, and assumptions between WUSLP and DUSMP.
Table 10.1: Relationship between WUSLP and DUSMP.

<table>
<thead>
<tr>
<th>Items</th>
<th>WUSLP</th>
<th>DUSMP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Elements</td>
<td>Satellites</td>
<td>Jobs</td>
</tr>
<tr>
<td></td>
<td>GSO</td>
<td>Machine time</td>
</tr>
<tr>
<td>2. Definitions</td>
<td>Solution</td>
<td>Schedule</td>
</tr>
<tr>
<td></td>
<td>Ordering</td>
<td>Ordering</td>
</tr>
<tr>
<td>3. Parameters</td>
<td>( n ) (Number of satellites)</td>
<td>( n ) (Number of jobs)</td>
</tr>
<tr>
<td></td>
<td>( E_j ) (Eastern limit)</td>
<td>( R_j ) (Release time)</td>
</tr>
<tr>
<td></td>
<td>( D_j ) (Desired location)</td>
<td>( D_j ) (Starting due date)</td>
</tr>
<tr>
<td></td>
<td>( \Delta_{i,j} ) (Angular separation)</td>
<td>( T_{i,j} ) (Time interval)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( U_j ) (Setup time)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P_i ) (Processing time)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( W_{i,j} ) (Waiting time) where ( T_{i,j} = P_i + U_j + W_{i,j} )</td>
</tr>
<tr>
<td>4. Variables</td>
<td>( x_j ) (Location in degrees)</td>
<td>( t_j ) (Starting time)</td>
</tr>
<tr>
<td></td>
<td>( y_j ) (Deviation to the east)</td>
<td>( y_j ) (Earliness)</td>
</tr>
<tr>
<td></td>
<td>( y_j^+ ) (Deviation to the west)</td>
<td>( y_j^+ ) (Tardiness)</td>
</tr>
<tr>
<td>5. Constraints</td>
<td>Service arcs</td>
<td>Window constraints</td>
</tr>
<tr>
<td></td>
<td>Angular separation</td>
<td>Time interval</td>
</tr>
<tr>
<td>6. Sets</td>
<td>( N ) (Indexes of satellites)</td>
<td>( N ) (Indexes of jobs)</td>
</tr>
<tr>
<td></td>
<td>(\mathcal{X} ) (Solutions)</td>
<td>(\mathcal{X}_{SP} ) (Schedules)</td>
</tr>
<tr>
<td></td>
<td>(\Pi_N ) (Orderings)</td>
<td>(\Pi_N ) (Orderings)</td>
</tr>
<tr>
<td></td>
<td>( Q' ) (Unprojected set)</td>
<td>( Q_{SP}' ) (Unprojected set)</td>
</tr>
<tr>
<td>7. Functions</td>
<td>( \pi ) (one-to-one mapping of ( N ))</td>
<td>( \pi ) (one-to-one mapping of ( N ))</td>
</tr>
<tr>
<td>8. Assumptions</td>
<td>( \Delta_{i,j} = \Delta_{j,i} )</td>
<td>Not necessarily ( T_{i,j} = T_{j,i} )</td>
</tr>
<tr>
<td></td>
<td>( D_j \geq E_j )</td>
<td>( D_j \geq R_j )</td>
</tr>
<tr>
<td></td>
<td>( \min_{j \in N} { E_j } = 0 )</td>
<td>( \min_{j \in N} { R_j } = 0 )</td>
</tr>
</tbody>
</table>
Notice that in Table 10.1 there is a one-to-one correspondence between WUSLP and DUSMP, except for the angular separation parameters. By adding the parameters $W_j$ (western limit) and $L_j$ (starting deadline) to WUSLP and DUSMP respectively to Table 10.1 we obtain the relation between SLP and SMP.

If we assume that $T_{i,j} = T_{j,i}$ for all pair of jobs, then all the theoretical results of the previous chapters apply to DUSLP (SMP) by mapping the respective entries on Table 10.1. If $T_{i,j} \neq T_{j,i}$, then except for Chapter VII, the theoretical results presented in the previous chapters also apply. Since, in WUSLP (SLP), symmetry in the angular separations is assumed, the facets presented in Chapter VII may change or may not exist.
CHAPTER XI

Summary and conclusions

We have addressed WUSLP, a relaxation of SLP, in this research. While in SLP the location allotted to each satellite is restricted to its service arc, one of the service arc constraints is relaxed and only an eastern limit is enforced in WUSLP. We have shown that finding a feasible solution to WUSLP is easy but the problem of finding a feasible solution to SLP is NP-complete. In order to work with a full-dimensional polyhedron, we projected the WUSLP polyhedron. Every facet of the projected polyhedron, $\text{Conv}(Q)$, has been proven to yield a facet of the WUSLP polyhedron. We determined the extreme directions of $\text{Conv}(Q)$ and presented some properties of the vertices of $\text{Conv}(Q)$ given a specific ordering of the satellites. These results have been extended by not restricting the order of the satellites.

We presented the inheritance properties of $\text{Conv}(Q)$. Any inequality that defines a facet of $\text{Conv}(Q(K)), K \subset N$, also defines a facet of $\text{Conv}(Q)$. In addition, an inequality that induces a facet of $\text{Conv}(\mathcal{X}(K))$ also induces a facet of $\text{Conv}(Q(K))$. $\text{Conv}(Q)$ inherits all the facets of $\text{Conv}(Q(K))$ and $\text{Conv}(Q(K))$
inherits all the facets of $\text{Conv}(\mathcal{X}(K))$. As a consequence of the inheritance properties, any facet for the clique scheduling polyhedron is a facet for $\text{Conv}(\mathcal{Q})$. Thus, we have determined that finding the facets of $\text{Conv}(\mathcal{Q})$ is more difficult that finding the facets of the clique scheduling polyhedron. The inheritance properties are similar to the ones presented for the ordering polytope [17], for the satellite placement polytope [47], and the scheduling polyhedron [3].

From a known result from blocking polyhedral theory, there is a one-to-one correspondence between the vertices of the blocking polyhedron and the facets of $\text{Conv}(\mathcal{Q})$. Therefore, we have focused on determining the blocking polyhedron of $\text{Conv}(\mathcal{Q})$. This result combined with the inheritance properties provides a procedure to determine facets of WUSLP.

Because of the inheritance properties, we found facets of $\text{Conv}(\mathcal{Q})$ by just focusing on a subset of the satellites. Facets for $\text{Conv}(\mathcal{Q})$ with one, two, three and four positive coefficients are given. These facets could be used in algorithms that successively introduce additional constraints (cutting-planes) or by a priori addition to the constraint set, they would improve the lower bound obtained by the linear programming relaxation of WUSLP.

We answer an open question by presenting necessary and sufficient conditions for the valid inequalities in [40] to induce facets for WUSLP. Computational results for a real problem using the facets developed are very promising. By just appending a few facets to WUSLP, the LP relaxation was improved considerably.
Since the problem of finding a feasible solution to SLP is NP-complete, SLP was transformed to a related problem. We showed that if there exist a feasible solution to SLP, then there exists an optimal solution to the transformed problem with the same optimal value as SLP. Furthermore, we showed that if there does not exist a feasible solution for SLP, then the transformed problem is unbounded. We showed that every facet for $\text{Conv}(Q)$ defines a facet for the transformed problem. Consequently, we have found facets for a problem that is related to problems with window constraints, problems which are very difficult.

An important class of combinatorial optimization problem that is related to SLP is the single-machine, or one-processor, scheduling problem (SMP) with symmetric earliness and tardiness penalties. We introduce a new concept in scheduling problems, a minimum required time interval between starting job $i$ and starting job $j$ and define this time interval as the processing time of job $i$, plus the setup time of the machine to process job $j$, plus a waiting time when job $j$ is processed after (not necessarily immediately after) job $i$. If the waiting time is zero the problem reduces to one in which the starting time of job $j$ depends only on the previous job $i$. Otherwise, the starting time of one job can be affected by the starting time of any other job. We explained how our results apply for SMP.

An important contribution of this research is the presentation of the similarities between SMP and SLP. These similarities could be exploited to provide better solution methods for both SLP and SMP. A major contribution of this research is
the development of facets for problems with nonregular measures, i.e., problems that penalize earliness and tardiness. Earliness and tardiness problems consider common penalties, penalties that differ among jobs, or penalties that differ if a job is early or late. If the penalties are linear, by introducing the deviational variables $y$ and $y^+$ we are able to work with an objective function which is linear. Consequently, the facets for the unweighted and weighted problems are the same. Another contribution of this research is the presentation of results that could be used for the design of new solution methods for SLP.

The facets for earliness and tardiness problems that consider common penalties, penalties that differ among jobs, or penalties that differ if a job is early or late are the same and the only difference between those problems is in the objective function. Consequently, an interesting extension of this research is to develop strategies for selecting which facets to append to the LP relaxation of each one of those problems. Another extension is to determine facets for earliness and tardiness problems when only processing times are considered. So far the literature has focused only on regular measures in the objective function.
Appendix A

Proof for 1-facets and 2-facets

Proof of Theorem VII.4 From (7.5), \( y_j \geq 0 \) is a valid inequality for \( \text{Conv}(Q) \).

Let \( \pi = (j_1, j_2, \ldots, j_n) \) where \( j_1 = j \), i.e., satellite \( j \) is in the easternmost position.

We prove that there exist \( 2n \) affinely independent points \((x^\ell, y^\ell) \in \text{Conv}(Q)\) such that \( y_j^\ell = 0 \), \( \forall \ell \in \mathbb{N}^2 \). Let

\[
\begin{align*}
x_{j_1}^\ell &= D_{j_1} + q_{j_1}^\ell, \\
x_{j_k}^\ell &= \max_{m < k} \{ E_{j_k}, x_{j_m}^\ell + \Delta_{j_m,j_k} \} + q_{j_k}^\ell, \quad k = 2, 3, \ldots, n, \quad \forall \ell \in \mathbb{N}^2; \\
y_{j_1}^\ell &= 0, \\
y_{j_k}^\ell &= D_{j_k} + r_{j_k}^\ell, \quad k = 2, 3, \ldots, n, \quad \forall \ell \in \mathbb{N}^2;
\end{align*}
\]

where \((q^\ell, r^\ell)\), \( \forall \ell \in \mathbb{N}^2 \), are the extreme direction vectors of \( Q_\pi \) defined in (6.2) and (6.3).

Let \( M \) be the \( 2n \times 2n - 1 \) matrix, whose columns are the vectors \((x^\ell, y^\ell) - (x^{n+1}, y^{n+1})\), \( \ell = 1, 2, \ldots, 2n, \ell \neq n + 1 \), and whose rows are ordered according to \( x_{j_1}, \ldots, x_{j_n}, y_{j_1}, \ldots, y_{j_n} \). The transpose matrix of \( M \), is an echelon matrix with nonzero rows. Thus, the rank of \( M \) is \( 2n - 1 \). By Proposition III.3, \((x^\ell, y^\ell), \forall \ell \in \mathbb{N}^2\) are affinely independent points.
By construction, \( y_j^\ell = 0 \) and \((x^\ell, y^\ell) \in Q_\pi\). This implies that \((x^\ell, y^\ell) \in Conv(Q), \ \forall \ell \in N^2\). Finally, from (7.5) it follows that \( y_j \geq 0 \) is a trivial facet since \( Q \cap \{y_j < 0\} = 0. \Box \)

**Proof of Theorem VII.5** From (7.1) \( x_j + y_j \geq D_j \) is a valid inequality for \( Conv(Q) \). If \( D_j = E_j \), then \( x_j + y_j \geq D_j \) is a nonnegative linear combination of \( x_j \geq E_j \) and \( y_j \geq 0 \). Thus, we assume that \( D_j > E_j \). Let \( \pi = (j_1, j_2, \ldots, j_n) \) where \( j_1 = j \), i.e., satellite \( j \) is in the easternmost position. We prove that there exist \( 2n \) affinely independent points \((x^\ell, y^\ell) \in Conv(Q)\) such that \( x^\ell_j + y^\ell_j = D_j, \ \forall \ell \in N^2\).

Let

\[
\begin{align*}
x^\ell_{j_1} &= D_j, \ \forall \ell \in N^2, \ \ell \neq n + 1; \quad x^{n+1}_{j_1} = E_{j_1}; \\
x^1_{j_k} &= \max_{m < k} \{E_{j_k}, x^1_{j_m} + \Delta_{j_m,j_k}\}, \ k = 2, 3, \ldots, n; \\
x^\ell_{j_k} &= x^1_{j_k} + q^\ell_{j_k}, \ k = 2, 3, \ldots, n, \ \forall \ell \in N^2, \ \ell \neq 1; \\
y^{\ell}_{j_1} &= 0, \ \forall \ell \in N^2, \ \ell \neq n + 1; \quad y^{n+1}_{j_1} = D_j - E_{j_1}; \\
y^\ell_{j_k} &= D_{j_k} + r^\ell_{j_k}, \ k = 2, 3, \ldots, n, \ \forall \ell \in N^2,
\end{align*}
\]

where \((q^\ell, r^\ell), \ \forall \ell \in N^2\), are the extreme direction vectors of \( Q_{\pi} \) defined in (6.2) and (6.3).

Let \( M \) be the \( 2n \times 2n - 1 \) matrix, whose columns are the vectors \((x^\ell, y^\ell) - (x^1, y^1), \ell = 2, 3, \ldots, 2n\) and whose rows are ordered according to \( x_{j_1}, \ldots, x_{j_n}, y_{j_1}, \ldots, y_{j_n}\). Since \( D_j > E_j \geq 0 \), The transpose matrix of \( M \) can be transformed to an echelon matrix with nonzero rows by adding column \( n + 1 \) (row \( n + 1 \) of \( M \)) to column
1 (row 1 of M). Thus, the rank of M is $2n - 1$. By Proposition III.3, $(x^t, y^t)$, \( \forall \ell \in \mathbb{N}^2 \) are affinely independent points.

By construction, \( x_j^t + y_j^t = D_j \) and \( (x^t, y^t) \in Q_\pi \). This implies that \( (x^t, y^t) \in \text{Conv}(Q), \ \forall \ell \in \mathbb{N}^2 \). Finally, from (7.1) it follows that \( x_j + y_j \geq D_j \) is a trivial facet since \( Q_R \cap \{x_j + y_j < D_j\} = \emptyset \). □

Next we prove that the inequality in (7.7) is a nontrivial facet. In Lemma A.1, we prove that is valid for \( \text{Conv}(Q) \). In the proof of Theorem VII.6 we find 
\[
\dim(\text{Conv}(Q)) = 2n \text{ affinely independent points in } \text{Conv}(Q) \text{ that satisfy the inequality as an equality.}
\]

**Lemma A.1** If \( \Delta_{i,j} > |E_i - E_j| \), the inequality (7.7) is a valid inequality for \( \text{Conv}(Q) \).

**Proof.** Assume that \( \Delta_{i,j} > |E_i - E_j| \). We rewrite (7.7) as
\[
\Delta_{i,j}(x_i + x_j) + (E_j - E_i)(x_j - x_i) \geq \Delta_{i,j}(\Delta_{i,j} + E_i + E_j) \quad (A.1)
\]
We show that (A.1) is satisfied by every \((x, y) \in \text{Conv}(Q)\). Without loss of generality, we assume that \( E_j \geq E_i \). We divide the proof into two cases.

Case 1) Suppose \((x, y) \in Q\). Then \((x, y) \in Q_\pi\) for some \( \pi \in \Pi_N \). Thus, \( i \) precedes \( j \) or \( j \) precedes \( i \) in the ordering \( \pi \). Hence, \( x_j - x_i \geq \Delta_{i,j} \ \vee \ x_i - x_j \geq \Delta_{i,j} \).

Assume \( x_j - x_i \geq \Delta_{i,j} \). Then, \( x_j \geq \Delta_{i,j} + x_i \geq \Delta_{i,j} + E_i \) since \( x_i \geq E_i \). Consequently, \( x_i + x_j \geq \Delta_{i,j} + 2E_i \). By substitution \((x, y) \) satisfies A.1.
Assume $x_i - x_j \geq \Delta_{i,j}$ then $x_i = \Delta_{i,j} + x_j + \epsilon$ where $\epsilon \geq 0$ Since $\Delta_{i,j} > |E_i - E_j|$, $E_j - E_i < \Delta_{i,j}$ and $E_i - E_j < \Delta_{i,j}$. Then

$$\Delta_{i,j}(x_i + x_j) + (E_j - E_i)(x_j - x_i) = \Delta_{i,j}(\Delta_{i,j} + 2x_j + \epsilon) + (E_j - E_i)(-\Delta_{i,j} - \epsilon)$$

$$\geq \Delta_{i,j}(\Delta_{i,j} + 2E_j + \epsilon) + (E_j - E_i)(-\Delta_{i,j} - \epsilon)$$

$$= \Delta_{i,j}(\Delta_{i,j} + E_j + E_i) + \epsilon(\Delta_{i,j} - (E_j - E_i))$$

$$= \Delta_{i,j}(\Delta_{i,j} + E_j + E_i).$$

Case 2) Suppose $(x, y) \in \text{Conv}(Q) \setminus Q$. Then, by Carathéodory's Theorem, $(x, y)$ is a convex combination of $2n + 1$ points of $Q$. Thus, there exists a $\gamma \in \mathbb{R}^{2n+1}$, $0 \leq \gamma \leq 1$, and $\sum_{\ell=1}^{2n+1} \gamma_\ell = 1$ such that

$$(x, y) = \sum_{\ell=1}^{2n+1} \gamma_\ell (x^\ell, y^\ell)$$

where $(x^\ell, y^\ell) \in Q$, $\forall \ell \in \{1, 2, \ldots, 2n + 1\}$. By algebraic manipulation, (7.7) can be written as $(\alpha, \beta)(x, y) \geq 1$, where $\alpha, \beta \in \mathbb{R}^2$. From Case 1), $(\alpha, \beta)(x^\ell, y^\ell) \geq 1$, $\ell = 1, 2, \ldots, 2n + 1$ since $(x^\ell, y^\ell) \in Q$. By letting $\gamma_\ell = 1/n$, it follows that $(\alpha, \beta)(x, y) \geq 1$. □

**Proof of Theorem VII.6.** By the previous lemma, if $\Delta_{i,j} > |E_i - E_j|$, (7.7) is a valid inequality for $\text{Conv}(Q)$. Without loss of generality $E_i \geq E_j$. Let $\pi = (j_1, j_2, \ldots, j_n)$ where $j_1 = i$, $j_2 = j$, i.e., satellite $i$ is in the easternmost position and satellite $j$ is in the next position. We prove that there exist $2n$ affinely independent points $(x^\ell, y^\ell) \in \text{Conv}(Q)$ such that (7.7) is satisfied as equality,
\[ \forall \ell \in N^2. \text{ Let} \]
\[
x_{j_1}^2 = E_{j_1} + \Delta_{i,j}, \quad x_{j_2}^2 = E_{j_2} + \Delta_{i,j}, \quad \forall \ell \in N^2, \ell \neq 2; \\
x_{j_k}^2 = \max_{m<k}\{E_{j_k} + \Delta_{j_m,j_k}\}, \ell = 1,2, \quad k = 3,4,\ldots,n; \\
x_{j_k}^2 = x_{j_1}^2 + q_{j_k}^2, \quad k = 3,4,\ldots,n, \ell = 3,4,\ldots,2n; \\
y_{j_k}^2 = D_{j_k} + r_{j_k}^2, \quad \forall k \in N, \quad \forall \ell \in N^2; \\
\]
where \((q^\ell, r^\ell), \forall \ell \in N^2\), are the extreme direction vectors of \(Q_{\pi}\) defined in (6.2) and (6.3).

Let \(M\) be the \(2n \times 2n - 1\) matrix, whose columns are the vectors \((x^\ell, y^\ell) - (x^1, y^1), \ell = 2,3,\ldots,2n\) and whose rows are ordered according to \(x_{j_1}, \ldots, x_{j_n}, y_{j_1}, \ldots, y_{j_n}\).

The transpose matrix of \(M\), is a echelon matrix with nonzero rows. Thus, the rank of \(M\) is \(2n - 1\). By Proposition III.3, \((x^\ell, y^\ell), \forall \ell \in N^2\) are affinely independent points.

Since the \((q^\ell, r^\ell), \forall \ell \in N^2\), are direction vectors of \(Q_{\pi}\), \(x_{j_k}^\ell - x_{j_m}^\ell \geq \Delta_{j_k,j_m}\) for \(m < k, \ k = 2,3,\ldots,n, \ \forall \ell \in N^2, \ell \neq 2\). Furthermore, \(x_{j_k}^\ell + y_{j_k}^\ell \geq D_{j_k}\) since \(x_{j_k}^\ell \geq E_{j_k}, y_{j_k}^\ell \geq 0, \text{ and } r_{j_k}^\ell \geq 0, \forall \ell \in N^2\). Thus, \((x^\ell, y^\ell) \in Q_{\pi}.\) This implies that \((x^\ell, y^\ell) \in Conv(Q), \forall \ell \in N^2, \ell \neq 2.\)

Let \(\pi'\) be the same ordering as \(\pi\) except that satellite \(j\) is in the easternmost position and satellite \(i\) is in the next position. It can be proved that \((x^2, y^2) \in Q_{\pi'}\).

Thus, \((x^2, y^2) \in Conv(Q).\) By construction

\[
(\Delta_{i,j} + E_i - E_j)x_i^\ell + (\Delta_{i,j} + E_j - E_i)x_j^\ell = \Delta_{i,j}(\Delta_{i,j} + E_i + E_j)
\]
\( \forall \ell \in \mathbb{N}^2. \) Finally, since \( |E_i - E_j| < \Delta_{i,j} \) any \( (x, y) \in \mathbb{Q}_R \) such that \( x_i = E_i, x_j = E_j \) is an element of

\[
\mathbb{Q}_R \cap \{(\Delta_{i,j} + E_i - E_j)x_i + (\Delta_{i,j} + E_j - E_i)x_j < \Delta_{i,j}(\Delta_{i,j} + E_i + E_j)\}.
\]

Thus, (7.7) is a nontrivial facet. \( \square \)
Appendix B

Proof for Theorem VII.14

In this appendix we present the proof for Theorem VII.14. Throughout this appendix we assume that all scenarios are inactive. Let $M^{C,k}$, $k = 1, 2, 3, 4, 5$, be the matrix whose rows are the vertices of $Q(i,j)$ and $Q(j,i)$ under scenario $C.k$. We assume that the columns of the matrices $M^{C,k}$ are ordered according to $x_i, x_j, y_i, y_j$.

Then,

$$M^{C.1} = \begin{bmatrix}
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
0 & D_j & D_i & 0 \\
D_i & \Delta_{i,j} + D_i & 0 & \delta_{ij} - \Delta_{i,j} \\
D_i & D_j & 0 & 0 \\
\Delta_{i,j} + D_j & D_j & 0 & 0 \\
D_i & D_i - \Delta_{i,j} & 0 & \Delta_{i,j} - \delta_{ji} \\
\Delta_{i,j} & 0 & D_i - \Delta_{i,j} & D_j \\
D_i & 0 & 0 & D_j
\end{bmatrix}$$

$$M^{C.2} = \begin{bmatrix}
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
0 & D_j & D_i & 0 \\
D_i & \Delta_{i,j} + D_i & 0 & \delta_{ij} - \Delta_{i,j} \\
D_i & D_j & 0 & 0 \\
\Delta_{i,j} & 0 & 0 & D_j \\
D_i + \Delta_{i,j} & D_j & 0 & 0
\end{bmatrix}$$
Let $B^{C,k} = [B_{t_p}^{C,k}]$, $k = 1, 2, 3, 4, 5$, be the matrix whose rows are the vertices of $Conv(Q(K))$ under scenario $C.k$. We denote by $B_{t_i}$ the $i$-th row of $B$. For $k = 3, 4, 5$, $B^{C,k} = M^{C,k}$, i.e., every row of $M^{C,k}$ defines a vertex of $Conv(Q(K))$ under scenario $C.k$. For $k = 1, 2$ this is not the case. After eliminating the rows of $M^{C,1}$ and $M^{C,2}$ that do not define vertices of $Conv(Q(K))$ we obtain the following matrices:

$$
M^{C,3} = 
\begin{bmatrix}
D_i & \Delta_{i,j} + D_i & 0 & 0 \\
D_j - \Delta_{i,j} & D_j & \Delta_{i,j} - \delta_{ij} & 0 \\
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
\Delta_{i,j} + D_j & D_j & 0 & 0 \\
D_i & D_i - \Delta_{i,j} & 0 & \Delta_{i,j} - \delta_{ij} \\
\Delta_{i,j} & 0 & D_i - \Delta_{i,j} & D_j \\
D_i & 0 & 0 & D_j
\end{bmatrix}.
$$

$$
M^{C,4} = 
\begin{bmatrix}
D_i & \Delta_{i,j} + D_i & 0 & 0 \\
D_j - \Delta_{i,j} & D_j & \Delta_{i,j} - \delta_{ij} & 0 \\
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
\Delta_{i,j} & 0 & 0 & D_j \\
D_i & 0 & 0 & D_j
\end{bmatrix}.
$$

$$
M^{C,5} = 
\begin{bmatrix}
0 & \Delta_{i,j} & D_i & 0 \\
D_i & D_i + \Delta_{i,j} & 0 & 0 \\
\Delta_{i,j} & 0 & 0 & D_j \\
D_j + \Delta_{i,j} & D_j & 0 & 0
\end{bmatrix}.
$$

$$
B^{C,1} = 
\begin{bmatrix}
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
0 & D_j & D_i & 0 \\
D_i & D_j & 0 & 0 \\
\Delta_{i,j} & 0 & D_i - \Delta_{i,j} & D_j \\
D_i & 0 & 0 & D_j
\end{bmatrix}.
$$

$$
B^{C,2} = 
\begin{bmatrix}
0 & \Delta_{i,j} & D_i & D_j - \Delta_{i,j} \\
0 & D_j & D_i & 0 \\
D_i & \Delta_{i,j} + D_i & 0 & \delta_{ij} - \Delta_{i,j} \\
D_i & D_j & 0 & 0 \\
\Delta_{i,j} & 0 & 0 & D_j
\end{bmatrix}.
$$
By Theorem VII.10, the inequality \((\alpha_1, \alpha_2, \beta_1, \beta_2)(x_i, x_j, y_i, y_j) \geq 1\) induces a facet of \(\text{Conv}(Q(K))\) if and only if \((\alpha_1, \alpha_2, \beta_1, \beta_2)\) is a vertex of the polyhedron

\[
\text{Conv}(Q(K))^B = \{(u_1, u_2, w_1, w_2) \geq 0, B(u_1, u_2, w_1, w_2) \geq 1\} \tag{B.1}
\]

where \(B\) is a \(|V(\text{Conv}(Q(K)))| \times 4\) matrix whose rows are the elements of \(|V(\text{Conv}(Q(K)))|\).

Let \(h\) be a \(|V(\text{Conv}(Q(K)))|\) vector of surplus variables. Then we can represent \(\text{Conv}(Q(K))^B\) as

\[
\text{Conv}(Q(K))^B = \{(u_1, u_2, w_1, w_2) \geq 0, h \geq 0, B(u_1, u_2, w_1, w_2) - I h = 1\} \tag{B.2}
\]

where \(I\) is a \(|V(\text{Conv}(Q(K)))| \times |V(\text{Conv}(Q(K)))|\) identity matrix.

Note that if \((\alpha_1, \alpha_2, \beta_1, \beta_2)(x_i, x_j, y_i, y_j) \geq 1\) induces a 4-facet of \(\text{Conv}(Q(K))\) then there exists an extreme point or basic feasible solution of \(\text{Conv}(Q(K))^B\) in which \(u_1, u_2, w_1, w_2 > 0\). Therefore, there exists a basis such that \(u_1, u_2, w_1,\) and \(w_2\) are basic variables. If \((\alpha_1, \alpha_2, \beta_1, \beta_2)(x_i, x_j, y_i, y_j) \geq 1\) induces a 3-facet of \(\text{Conv}(Q(K))\), then there exists a basis such that at most one of \(u_1, u_2, w_1,\) and \(w_2\) is nonbasic. If \(u_1, u_2, w_1,\) and \(w_2\) are basic simultaneously, one and only one of those variables takes a zero value in the corresponding basic feasible solution.

**Lemma B.1** Under scenario C.1 there are no 4-facets or 3-facets.

**Proof.** Note that \(B^{C.1}_{\ell 1} + B^{C.1}_{\ell 3} = D_{i}\) and \(B^{C.1}_{\ell 2} + B^{C.1}_{\ell 4} = D_{j}, \ell = 1, 2, 3, 4, 5\). Therefore, \(B^{C.1}\) is singular. Consequently, \(u_1, u_2, w_1,\) and \(w_2\) can not be all basic at the same time. Thus there are no 4-facets.
Consider any basis which contains 3 of the variables \( u_1, u_2, w_1, \) and \( w_2 \). From Cramer's Rule, at most two of those variables may take positive values in the corresponding basic feasible solution. Thus, there are no 3-facets. □

**Lemma B.2** Under scenario C.2 there are no 4-facets and the only 3-facet is \((\alpha^2, \beta^2) (x, y) \geq 1\) where \((\alpha^2, \beta^2)\) is defined in (7.15).

**Proof.** Note that \( B_{l_2}^{c.2} + B_{l_4}^{c.2} = D_j , \ell = 1, 2, 3, 4, 5 \). Consider any basis which contains \( u_1, u_2, w_1, \) and \( w_2 \). From Cramer's Rule, \( u_1 = w_1 = 0 \) in the corresponding basic feasible solution. Thus, there are no 4-facets.

Consider any basis which contains at least 3 of the variables \( u_1, u_2, w_1, \) and \( w_2 \). If \( u_2 \) and \( w_2 \) are basic simultaneously, from Cramer's Rule \( u_1 \) and/or \( u_3 \) take the value of zero. Consequently, \( u_2 \) or \( w_2 \) are nonbasic in order to have 3-facets.

Assume that \( u_2 \) is nonbasic. From \( B_{l_2}^{c.2}(u, w) \geq 1 \) we obtain \( u_1 \geq 1/D_i \) and \( w_1 \geq 1/D_i \). Since C.2 is inactive, then \( h_1 > 0, h_3 > 0, \) and \( h_5 > 0 \), where \( h_1, h_3, \) and \( h_5 \) are the surplus variables of rows 1, 3, and 5 of \( B_{l_2}^{c.2} \) respectively. Since \( B_{l_2}^{c.2} \) has only 5 rows, the number of basic variables must be 5. However, \( u_1, w_1, w_2, h_1, h_3, \) and \( h_5 \) are basic variables. This is a contradiction.

Assume \( w_2 \) is nonbasic. Since C.2 is inactive, from \( B_{l_2}^{c.2}(u, w) \geq 1 \) we obtain \( h_2 > 0, \) and \( h_4 > 0 \). The basic feasible solution associated with the basis \((u_1, u_2, w_1, h_2, h_4)\) is \((\alpha^2, \beta^2)\). □

**Lemma B.3** Under scenario C.3 there are no 3-facets and the only 4-facet is
\((\alpha^1, \beta^1)(x, y) \geq 1\) where \((\alpha^1, \beta^1)\) is defined in (7.14).

**Proof.** First we prove that (7.14) is the only extreme point of \(\text{Conv}(\mathcal{Q}(K))\) under scenario C.3 where \(u_1, u_2, w_1,\) and \(w_2\) are positive. Note that \(B_{z_3}^{C,3} + B_{z_3}^{C,3} = D_i, \ell = 1, 2, 3, 4, 6, 7, 8\) and \(B_{z_3}^{C,3} + B_{z_3}^{C,3} = D_j, \ell = 2, 3, 4, 5, 6, 7, 8.\) From Cramer's Rule, \(h_1\) and \(h_5\) are nonbasic. Otherwise, \(u_1, u_2, w_1,\) and \(w_2\) can not take positive values simultaneously.

Since \(h_1\) and \(h_5\) are nonbasic \(B_{z_3}^{C,3}(u, w) = 1\) and \(B_{z_3}^{C,3}(u, w) = 1.\) Hence, \(u_1 = (\Delta_{i,j} + D_i - D_j)/(\Delta_{i,j} \sigma_{ij})\) and \(u_2 = (\Delta_{i,j} + D_j - D_i)/(\Delta_{i,j} \sigma_{ij}).\) Substituting the values of \(u_1\) and \(u_2\) into \(B_{z_3}^{C,3}(u, w) \geq 1\) we obtain:

\[
\begin{align*}
    w_1 + (D_j - \Delta_{i,j})w_2 & \geq 2/\sigma_{ij} \\
    w_1 & \geq 2D_i/\sigma_{ij} \\
    w_2 & \geq u_1(\Delta_{i,j} + D_j)/D_i \\
    (D_i - \Delta_{i,j})w_1 + D_jw_2 & \geq 2D_j/\sigma_{ij} \\
    w_2 & \geq u_2(\Delta_{i,j} + D_i)/D_j
\end{align*}
\]

Since \(w_1 \geq 2/\sigma_{ij}\) and \(w_2 \geq 2/\sigma_{ij}, h_3 > 0, h_4 > 0, h_7 > 0,\) and \(h_8 > 0.\) Then \(u_1, u_2, w_1, w_2, h_3, h_4, h_7,\) and \(h_8\) are the basic variables and \((u_1, u_2, w_1, w_2) = (\alpha^1, \beta^1)\) in the corresponding basic feasible solution.

Now we prove that there are no 3-facets. Consider any basic feasible solution where \(u_1, u_2, w_1 > 0\) and \(w_2 = 0.\) From \(B_{z_3}^{C,3}(u, w) \geq 1\) and \(B_{z_3}^{C,3}(u, w) \geq 1, h_5 > 0.\) Hence, \(h_5\) is basic. However, if \(h_5\) is basic \(u_2 = 0\) since \(B_{z_3}^{C,3} + B_{z_3}^{C,3} = D_i, \ell = 1, 2, 3, 4, 6, 7, 8.\) Thus, there does not exist a basic feasible solution where \(u_1, u_2, w_1 > 0\) and \(w_2 = 0.\) A similar argument holds for the other three cases, i.e., \(u_1 = 0, u_2 = 0,\) or \(w_1 = 0.\) □
Lemma B.4 Under scenario C.4 The inequalities $(\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2$ (where $(\alpha^\ell, \beta^\ell), \ell = 1, 2$ are defined in (7.14) and (7.15)) induce facets of $\text{Conv}(Q(K))$, and they are the only 3 or 4-facets.

Proof. First we prove that (7.14) is the only extreme point of $\text{Conv}(Q(K))^B$ under scenario C.4 where $u_1, u_2, w_1,$ and $w_2$ are positive. Note that $B'^C_A + B'^C_A = D_i, \ell = 1, 2, 3, 4$ and $B'^C_A + B'^C_A = D_j, \ell = 2, 3, 4, 5, 6$. From Cramer’s Rule, $h_1$ is nonbasic and $h_5$ and $h_6$ can not be basic simultaneously. Otherwise, $u_1, u_2, w_1,$ and $w_2$ can not take positive values simultaneously.

We assume that $h_6$ is basic. This implies that $h_5$ is nonbasic. From Cramer’s Rule, $h_3$ and $h_4$ are not basic; otherwise, $w_2 = 0$. Since $h_3$ and $h_4$ are not basic, $B'^C_A = 1$ and $B'^C_A = 1$. Thus, $u_2 = w_2$. Therefore, $u_1 = D_i w_1 / \Delta_{i,j}$ from solving $B'^C_A = 1$ and $B'^C_A = 1$. Since $\Delta_{i,j} > D_i$, $w_1 > u_1$. However, this implies that $h_2 < 0$, which is not feasible.

Now we assume that $h_6$ is nonbasic. Since $h_1$ and $h_6$ are nonbasic $B'^C_A(u, w) = 1$ and $B'^C_A(u, w) = 1$. Hence, $u_1 = (\Delta_{i,j} + D_i - D_j) / (\Delta_{i,j} \sigma_{ij})$ and $u_2 = (\Delta_{i,j} + D_j - D_i) / (\Delta_{i,j} \sigma_{ij})$. Substituting the values of $u_1$ and $u_2$ into $B'^C_A(u, w) \geq 1$ we obtain:

$$
\begin{align*}
\frac{w_1}{D_i w_1 + (D_j - \Delta_{i,j}) w_2} & \geq \frac{2/\sigma_{ij}}{2D_i/\sigma_{ij}}, \\
\frac{u_1(\Delta_{i,j} + D_j)}{\Delta_{i,j}} & \geq \frac{u_1(\Delta_{i,j} + D_j)}{D_i}.
\end{align*}
$$

Since $w_1 \geq 2/\sigma_{ij}$ and $w_2 \geq 2/\sigma_{ij}$, $h_3 > 0, h_4 > 0$. Then $u_1, u_2, w_1, w_2, h_3,$ and $h_4$ are the basic variables and $(u_1, u_2, w_1, w_2) = (\alpha^1, \beta^1)$ is the corresponding basic feasible solution.
Now we prove that the only 3-facet is \((\alpha^2, \beta^2)(x, y) \geq 1\) where \((\alpha^2, \beta^2)\) is defined in (7.15). Consider any basic feasible solution where \(u_1, u_2, w_2 > 0\) and \(w_1 = 0\). From \(B^{C.4}(u, w) \geq 1\), \(u_2 \geq 1/D_j\). This implies that \(h_1 > 0, h_2 > 0\), and \(h_6 > 0\). In the basic feasible associated with the basis \((u_1, u_2, w_2, h_1, h_2, h_6), w_2 = 0\). Thus, there does not exist a basic feasible solution where \(u_1, u_2, w_2 > 0\) and \(w_1 = 0\). A similar argument holds for the cases \(u_1 = 0\) and \(u_2 = 0\).

Therefore, \(u_1, u_2, w_1 > 0\) and \(w_2 = 0\) in any basic feasible solution with 3 positive decision variables. From \(B^{C.4}(u, w) \geq 1\), \(h_4 > 0\) and \(h_6 > 0\). It turns out that the other basic variable may be \(h_1, h_2, h_3,\) or \(h_5\), since the corresponding basic feasible solution is degenerate. \((u_1, u_2, w_1, w_2) = (\alpha^1, \beta^1)\) in the corresponding basic feasible solution.

**Lemma B.5** Under scenario C.5 the inequalities \((\alpha^\ell, \beta^\ell)(x, y) \geq 1, \ell = 1, 2, 3\) (where \((\alpha^\ell, \beta^\ell), \ell = 1, 2, 3\) are defined in (7.14), (7.15), and (7.16) induce facets of \(\text{Conv}(Q(K))\), and they are the only 3 or 4-facets.

**Proof.** \((\alpha^1, \beta^1)(x, y) \geq 1\) is the only 4-facet since \((\alpha^1, \beta^1)\) is the unique solution to the system \(B^{C.5}(u, w) = 1\).

Consider any basis whose associated basic feasible solution has \(u_2, w_1, w_2 > 0,\) and \(u_1 = 0\). From \(B^{C.5}(u, w) \geq 1\) we obtain \(u_2 \geq 1/D_j\). This implies that \(h_1 > 0\) and \(h_2 > 0\). This is not possible since \(B^{C.5}\) has only 4 rows and \(u_2, w_1, w_2, h_1, h_2\) are basic variables. A similar argument holds for \(u_1, w_1, w_2 > 0,\) and \(u_2 = 0\).
Consider any basis whose associated basic feasible solution has \( u_1, u_2, w_1 > 0 \), and \( w_2 = 0 \). This implies that \( h_4 > 0 \). The basic feasible feasible associated with the basis \((u_1, u_2, w_1, h_4)\) is \((\alpha^2, \beta^2)\). When \( u_1, u_2, w_2 > 0 \), and \( w_1 = 0 \), \( h_2 > 0 \). The basic feasible solution associated with the basis \((u_1, u_2, w_2, h_2)\) is \((\alpha^3, \beta^3)\). □

The proof for Theorem VII.14 follows from the lemmas above.
Bibliography


[51] Understanding Communications Systems. Chapter 10. (photocopy, publisher unknown)
