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The general Euler-Borel summability method

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The Ohio State University, 1990
THE GENERAL EULER-BOREL SUMMABILITY METHOD

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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*****

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1990

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FIELD OF STUDY

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CHAPTER I

Introduction

This dissertation contains a number of results about the general Euler-Borel summability method, which was first studied in detail in [1]. Our main result is a Tauberian theorem for this method. We begin with a brief introduction to summability methods.

If a sequence of numbers is divergent we may still attach a “limit” to it. A well known method is to take arithmetic means. For instance the sequence $s_n = (-1)^n$ is divergent but $t_n = (s_0 + s_1 + \cdots + s_n)/(n + 1) \to 0$ as $n \to \infty$. Thus 0 may be regarded as a generalized limit of $(s_n)$. More generally consider an infinite matrix $(a_{nk})_{n \geq 0, k \geq 0}$. For a given sequence $(s_n)$, if

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k$$

is defined for all $n$ and $t_n \to S$ as $n \to \infty$, then we say that $(s_n)$ is summable to $S$ by the method defined by $(a_{nk})$. There are summability methods which are not defined by matrices, but we will not be concerned with them.

1This method is also called the Sonnenschein method.
Now let us define the general Euler-Borel summability method. Let $f(z)$ be a function analytic at the origin. The general Euler-Borel method generated by $f(z)$, denoted by $(E, f)$, is the summability method defined by the matrix $(a_{nk})_{n \geq 0, k \geq 0}$, where $a_{nk}$ satisfies

$$(f(z))^n = \sum_{k=0}^{\infty} a_{nk} z^k.$$ 

If $(s_k)$ is summable by the method $(E, f)$ to $S$ then we will write

$$s_k \to S (E, f).$$

Examples of the method $(E, f)$ include the following.

1. The Euler method $(E, q)$, $q > 0$.

$$s_k \to S (E, q) \text{ means } (1 + q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \to S \text{ as } n \to \infty.$$ 

2. The discrete Borel method $B$.

$$s_k \to S (B) \text{ means } \sum_{k=0}^{\infty} \frac{e^{-n} n^k}{k!} s_k \to S \text{ as } n \to \infty.$$ 

3. The Meyer-König method $S_r$, $0 < r < 1$.

$$s_k \to S (S_r) \text{ means } (1 - r)^n \sum_{k=0}^{\infty} \binom{n - 1 + k}{k} r^k s_k \to S \text{ as } n \to \infty.$$ 

4. The Taylor method $T_r$, $0 < r < 1$.

$$s_k \to S (T_r) \text{ means } (1 - r)^n \sum_{k=0}^{\infty} \binom{n - 1 + k}{k} r^k s_{n+k} \to S \text{ as } n \to \infty.$$
5. The Karamata method $E(\alpha, \beta)$, where $\alpha < 1$, $\beta < 1$, and $\alpha + \beta > 0$.

\[ s_k \to S(E(\alpha, \beta)) \text{ means} \]
\[ \sum_{k=0}^{\infty} c_{nk} s_k \to S \text{ as } n \to \infty, \text{ where} \]
\[ c_{nk} = \sum_{j=0}^{\min(n,k)} \binom{n}{j} \binom{n-1+k-j}{n-1} (1-\alpha-\beta)^j \alpha^{n-j} \beta^{k-j}. \]

We will simply refer to the discrete Borel method as the Borel method.

The functions defining these methods are, respectively,

1. $f(z) = (z + q)/(1 + q), \ q > 0,$
2. $f(z) = \exp(z - 1),$
3. $f(z) = (1 - r)/(1 - rz), \ 0 < r < 1,$
4. $f(z) = (1 - r)z/(1 - rz), \ 0 < r < 1,$ and
5. $f(z) = (\alpha + (1 - \alpha - \beta)z)/(1 - \beta z),$ where $\alpha$ and $\beta$ satisfy the above conditions.

Most of these examples have been studied thoroughly. They are all regular. In other words, they sum a convergent sequence to its limit. They also sum divergent sequences. In fact, for each of the above functions $f$ there exists a number $w$ such that $|w| > 1$ but $|f(w)| < 1$. Hence,

\[ \sum_{k=0}^{\infty} a_{nk} w^k = (f(w))^n \to 0 \text{ as } n \to \infty. \]
Therefore the divergent sequence \((w^n)\) is summable \((E, f)\) to 0. But if a sequence diverges too slowly it will not be summable. A theorem which asserts that a divergent sequence satisfying a certain growth condition is not summable by a method is called a Tauberian theorem for the method. The growth condition is called a Tauberian condition. Tauberian theorems for all the above methods are known.

Regarding the general Euler-Borel method we have Theorems 1-3 below. Theorem 1, due to Bajšanski\([1]\), and Theorem 2, due to Bajšanski\([1]\) and Clunie and Vermes\([3]\), are criteria for the regularity of the method. Theorem 3, our main result, is a Tauberian theorem for the method.

**Theorem 1** Suppose that

1. \(f(z)\) is analytic for \(|z| < R, R > 1,\)
2. \(|f(z)| < 1 \text{ for } |z| \leq 1, z \neq 1,\)
3. \(f(1) = 1,\)
4. the number \(A\) defined by

\[
f(z) - z^\alpha = A_i^p(z - 1)^p + o(1)(z - 1)^p, \ z \to 1, \ \alpha = f'(1), \ A \neq 0
\]

satisfies \(\Re A \neq 0.\)

Then the method \((E, f)\) is regular.

If a function \(f(z)\) satisfies the conditions of Theorem 1 then we denote the parameters \(A, \alpha,\) and \(p\) in condition 4 by \(A(f), \alpha(f),\) and \(p(f),\) respectively.
It is proved in [1] that

\[ \Re A(f) < 0, \quad \alpha(f) > 0, \quad \text{and} \quad p(f) \text{ is an even integer.} \]

We note that for each of our examples of the method \((E, f)\), \(f(z)\) satisfies the conditions of Theorem 1. Moreover, \(A(f)\) is a real number and \(p(f) = 2\). The preceding two conditions hold whenever the coefficients \(a_{nk}\) are nonnegative (see [1, pp.134–135]), as in the examples, except the Karamata method.

**Theorem 2** Suppose that

1. \(f(z)\) is analytic for \(|z| < R\), where \(R > 1\), and is not a monomial, i.e., \(f(z) \neq z^m\), where \(m\) is a non-negative integer.

Then \((E, f)\) is regular if and only if the following conditions are satisfied.

2. \(|f(z)| < 1\) for \(|z| < 1\) except at finitely many points \(\zeta\).

3. \(f(1) = 1\).

4. If \(|f(\zeta)| = 1\), \(h_\zeta(z) = f(\zeta z)/f(\zeta)\), \(\alpha_\zeta = h'_\zeta(1)\), and \(A_\zeta\) is the nonzero number defined by

\[
h_\zeta(z) - z^{\alpha_\zeta} = A_\zeta i^{\rho_\zeta}(z - 1)^{\rho_\zeta} + o(1)(z - 1)^{\rho_\zeta}, \quad z \to 1,
\]

then \(\Re A_\zeta \neq 0\).

The sufficiency of these conditions follows from Theorem 1. (See [1].) Their necessity is due to Clunie and Vermes [3].
We will only consider regular methods \((E, f)\) where \(f\) satisfies the conditions of Theorem 1. We will not repeat this assumption.

Here is our Tauberian theorem.

**Theorem 3**

(a) If a sequence of complex numbers \((s_k)\) is summable \((E, f)\) and

\[ \lim (s_m - s_n) = 0 \]

as \(n \to \infty\), \(m > n\), and \((m - n) n^{-1/p(f)} \to 0\), then \(s_k\) is convergent.

(b) If all the coefficients \(a_{nk}\) of the method \((E, f)\) are nonnegative (so that \(p(f) = 2\)), a sequence of real numbers \((s_k)\) is summable \((E, f)\), and

\[ \lim (s_m - s_n) \geq 0 \]

as \(n \to \infty\), \(m > n\), and \((m - n) n^{-1/2} \to 0\), then \((s_k)\) is convergent.

Theorem 3 clearly implies the following

**Corollary**

(a) If a sequence of complex numbers \((s_k)\) is summable \((E, f)\), and

\[ s_k - s_{k-1} = O(k^{-1/p(f)}) \]

then \((s_k)\) is convergent.

(b) If all the coefficients \(a_{nk}\) of the method \((E, f)\) are nonnegative, a sequence of real numbers \((s_k)\) is summable \((E, f)\), and there is a positive constant \(M\) such that for each \(k\),

\[ s_k - s_{k-1} \geq -M \cdot k^{-1/2} \]
then \((s_k)\) is convergent.

Theorem 3 contains, for instance, the Tauberian theorem for the Borel method of summability \([6, \text{Theorem 241}]\) and that for the Meyer-König method\([10]\). Indeed, as we have remarked earlier, all the above examples of the method \((E, f)\) satisfy \(p(f) = 2\). This fact and the validity of Theorem 3 explain why these methods have the same Tauberian condition. The corresponding result for the Karamata method is new. (Fridy and Powell have obtained a Tauberian theorem for the Karamata method in \([4]\). Their Tauberian conditions are much stronger than ours, namely, \(s_k = O(1)\) and \(s_k - s_{k-1} = o(1/k)\).)

The corollary to Theorem 3 contains the \(O\)-Tauberian theorem for the Euler method and the Borel method.

We also note that as \(p(f)\) increases, the Tauberian condition in Theorem 3 becomes weaker. Hence if \((E, f)\) and \((E, g)\) are regular methods and \(p(f) > p(g)\) then the former is, loosely speaking, closer to convergence than the latter. See, however, Theorem 8 in Chapter V.

The following is an outline of the rest of the dissertation. So far we have only mentioned examples of methods \((E, f)\) with \(p(f) = 2\). In \([3]\) the following question was asked: is there a regular method \((E, f)\) with a single maximum of \(|f(z)|\) on \(|z| = 1\) and with \(p(f) > 2\)? We answer the question positively. In the brief Chapter II we show that for each positive even integer there is a regular method \((E, f)\)
with \( p(f) \) equal to that integer. (Recall that \( p(f) \) is always even.) Thus Theorem 3 is actually a family of Tauberian theorems, with weaker Tauberian conditions as \( p(f) \) increases. We will prove Theorem 3 in Chapter III. An important tool that we need in Pitt's Tauberian theorem[6, Theorem 221]. In Chapter IV we show by an example that the exponent \(-1/p(f)\) in Theorem 3 is the best possible. As we will see in Chapter V, this example shows that for every positive even integer \( p_0 \) there is a bounded divergent sequence which is summable by every regular method \((E, f)\) with \( p(f) = p_0 \) but is not summable by any method \((E, g)\) with \( p(g) > p_0 \). In fact, for bounded sequences \((E, f)\) includes \((E, g)\) if \( p(f) < p(g) \). (This means that every bounded sequence which is summable \((E, g)\) is summable \((E, f)\) to the same sum.) Furthermore we will prove that two methods \((E, f)\) and \((E, g)\) with \( p(f) = p(g) \) are equivalent for bounded sequences, i.e., they include one another. This generalizes a theorem of Meyer-König[8]. The proof of the last result depends on Wiener's general Tauberian Theorem[6, Theorem 220]. On the other hand for unbounded sequences we will show that we cannot characterize the inclusion of one method by another by means of \( p(f) \). Finally we prove a relation between the general Euler-Borel method and the Valiron method \( V_a, a > 0 \), which we will define. Our result is that for bounded sequences \((E, f)\) is equivalent to \( V_{a(f)/2|A(f)|} \) provided that \( p(f) = 2 \) and \( A(f) \) is real.
CHAPTER II

More examples

Let

\[ f(z) = z^{2k} - \left( \frac{z-1}{2} \right)^{4k}, \]

where \( k \) is a positive integer. Then \( f \) clearly satisfies conditions 1 and 3 in Theorem 1. For \( 0 < t < 2\pi \) we have

\[
\left| (e^{it})^{2k} - \left( \frac{e^{it} - 1}{2} \right)^{4k} \right| = \left| e^{2k\text{i}t} - e^{2k\text{i}t} (\frac{e^{it/2} - e^{-it/2}}{2})^{4k} \right|
\]

\[
= \left| 1 - \sin^{4k}(\frac{t}{2}) \right|
\]

\[
< 1.
\]

Hence by the maximum modulus principle \( f(z) \) satisfies condition 2. Since \( f'(1) = 2k \) and

\[ f(z) - z^{2k} = -(1/2)^{4k}(i)^{4k}(z - 1)^{4k} \]

\( f(z) \) satisfies condition 4 as well. By Theorem 1, \( (E, f) \) is regular. Moreover, the above equality shows that \( p(f) = 4k \).
Next let
\[ g(z) = z^{2k+1} + \left( \frac{z - 1}{2} \right)^{4k+2}, \quad k \geq 0. \]

Then \( g \) satisfies conditions 1 and 3 in Theorem 1. For \( 0 < t < 2\pi \) we have
\[
| (e^{it})^{2k+1} + \left( \frac{e^{it} - 1}{2} \right)^{4k+2} | = \left| e^{(2k+1)it} + e^{(2k+1)it} \left( \frac{e^{it/2} - e^{-it/2}}{2} \right)^{4k+2} \right|
\]
\[
= |1 + (i)^{4k+2} \sin^{4k+2} \left( \frac{t}{2} \right)|
\]
\[
= |1 - \sin^{4k+2} \left( \frac{t}{2} \right)|
\]
\[
< 1.
\]

By the maximum modulus principle \( g(z) \) satisfies condition 2. We have \( g'(1) = 2k + 1 \) and
\[
g(z) - z^{2k+1} = -(1/2)^{4k+2} (i)^{4k+2} (z - 1)^{4k+2}.
\]

Hence \( g(z) \) satisfies condition 4. By Theorem 1 \((E, g)\) is regular, and we have \( p(g) = 4k + 2 \).

Thus for each positive even integer there is a method \((E, f)\) with \( p(f) \) equal to that integer.
We will first prove part (a) of the theorem. Thus we assume that $f$ satisfies the conditions of Theorem 1, so that $(E, f)$ is a regular method, $(s_k)$ is summable $(E, f)$ to $S$, and

$$\lim (s_m - s_n) = 0$$

as $n \to \infty$, $m > n$, and $(m - n) n^{-1/p(f)} \to 0$.

Without loss of generality we may assume that $S = 0$. For otherwise we just have to consider $t_k = s_k - S$. Hence,

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} s_k = 0.$$ 

We will denote constants by $K$, not necessarily the same at each occurrence. But a letter with a subscript, e.g., $K_1$, always denotes the same constant. For simplicity we will write $p$, $A$, and $\alpha$ instead of $p(f)$, $A(f)$, and $\alpha(f)$.

We need seven lemmas. The first one is similar to Lemma $\epsilon$ in [11].
Lemma 1 Let \((s_k)\) be a sequence which satisfies the given Tauberian condition. Then there is a constant \(M_1 > 0\) such that if \(m > n\), we have

\[ |s_m - s_n| < M_1 (m - n)^{-1/p} + 1. \]

Proof Since \((s_k)\) satisfies the Tauberian condition there exists \(N > 0\) and \(a > 0\) such that if \(v > w > N\) and \((v - w) w^{-1/p} < 2a\) then \(|s_w - s_v| < 1\). Without loss of generality, suppose that \(N^{-1/p} < a\). Now let \(m > n \geq N\). Let \(n_j = [n + jan^{1/p}]\), \(j = 0, 1, 2, \ldots\). Let \(r\) satisfy \(n_r < m \leq n_{r+1}\). Hence \(n + ran^{1/p} \leq m\), and we have

\[ r \leq (m - n)n^{-1/p}a^{-1}. \]

Also,

\[ n_{j+1} - n_j \leq an^{1/p} + 1. \]

Since \(n_j \geq n\) for \(j = 0, 1, 2, \ldots\) and \(n^{-1/p} < a\) we have

\[ (n_{j+1} - n_j)n_j^{-1/p} \leq (an^{1/p} + 1)n^{-1/p} = a + n^{-1/p} < 2a. \]

Hence if \(m > n \geq N\) then

\[ |s_m - s_n| \leq |s_m - s_{n_r}| + |s_{n_r} - s_{n_{r-1}}| + \ldots + |s_n - s_n| \]

\[ \leq r + 1 \]

\[ \leq (m - n)n^{-1/p}a^{-1} + 1. \]

It follows that if \(m \geq N > n\) then

\[ |s_m - s_N| \leq a^{-1}(m - N)N^{-1/p} + 1 \]

\[ \leq a^{-1}(m - n)n^{-1/p} + 1. \]
On the other hand, there exists a constant $C_1 > 0$ such that

$$|s_v - s_w| < C_1(v - w)w^{-1/p}$$

if $N \geq v > w$.

Thus if $m \geq N > n$ then

$$|s_m - s_n| \leq |s_m - s_N| + |s_N - s_n|$$

$$\leq a^{-1}(m - n)n^{-1/p} + 1 + C_1(N - n)n^{-1/p}$$

$$\leq (C_1 + a^{-1})(m - n)n^{-1/p} + 1.$$ 

By the choice of $C_1$ this inequality also holds for $N > m > n$. The proof is complete.

**Lemma 2** If the hypotheses of Theorem 3 are satisfied then $(s_k)$ is bounded.

**Proof** Suppose we have shown that the subsequence $(s_{[\alpha n]})_{n \geq 0}$ is bounded. Then we can prove the lemma easily as follows. For every positive integer $k$, there exists $n$ such that $\alpha n < k \leq \alpha(n + 1)$, and we have

$$|s_k| \leq |s_k - s_{[\alpha n]}| + |s_{[\alpha n]}|$$

$$\leq |s_k - s_{[\alpha n]}| + O(1)$$

$$\leq M_1(k - [\alpha n])[\alpha n]^{-1/p} + 1 + O(1), \text{ by Lemma 1,}$$

$$\leq M_1 \alpha[\alpha n]^{-1/p} + O(1).$$

Hence $s_k = O(1)$. 
Since $\sum_{k=0}^{\infty} a_{nk} s_k = o(1)$, to prove that $s_{\lfloor \alpha n \rfloor} = O(1)$ it suffices to show that
\[
\sum_{k=0}^{\infty} a_{nk} s_k - s_{\lfloor \alpha n \rfloor} = O(1).
\]

Since
\[
f(1) = 1 = (f(1))^n = \sum_{k=0}^{\infty} a_{nk}, \quad n = 0, 1, \ldots
\]
we have
\[
\sum_{k=0}^{\infty} a_{nk} s_k - s_{\lfloor \alpha n \rfloor} = \sum_{k=0}^{\infty} a_{nk} s_k - s_{\lfloor \alpha n \rfloor} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor}).
\]

Let $0 < H < 1$. We will prove that the following sums
\[
\sum_{0 \leq k < (1-H)\alpha n} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor}), \quad \sum_{(1-H)\alpha n \leq k \leq (1+H)\alpha n} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor}), \text{ and}
\sum_{(1+H)\alpha n < k} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor})
\]
are bounded. The lemma will follow.

By Lemma 1, $|s_k - s_{k-1}| \leq M_1 k^{-1/p} + 1$ for each $k$. Hence,
\[
|s_k| \leq K k.
\]

Thus, we can estimate the first and the third of the above sums as follows.
\[
|\sum_{0 \leq k < (1-H)\alpha n} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor})| \leq K n \sum_{0 \leq k < (1-H)\alpha n} |a_{nk}|.
\]
\[
|\sum_{(1+H)\alpha n < k} a_{nk}(s_k - s_{\lfloor \alpha n \rfloor})| \leq K \sum_{(1+H)\alpha n < k} k |a_{nk}|.
\]
By Lemma 1 again, we estimate the second of the above sums.

\[
| \sum_{(1-H)\alpha n \leq k \leq (1+H)\alpha n} a_{nk}(s_k - s_{\lceil \alpha n \rceil}) |
\]

\[
\leq \sum_{(1-H)\alpha n \leq k \leq \alpha n} (Kn^{-1/p}(\alpha n - k) + 1)|a_{nk}|
\]

\[
+ \sum_{\alpha n < k \leq (1+H)\alpha n} (Kn^{-1/p}(k - \lfloor \alpha n \rfloor) + 1)|a_{nk}|
\]

\[
\leq Kn^{-1/p} \sum_{(1-H)\alpha n \leq k \leq \alpha n} (\alpha n - k)|a_{nk}|
\]

\[
+ Kn^{-1/p} \sum_{\alpha n < k \leq (1+H)\alpha n} (k - \lfloor \alpha n \rfloor)|a_{nk}|
\]

\[
+ \sum_{k=0}^{\infty} \alpha k
\]

by the Toeplitz-Schur theorem. (The Toeplitz-Schur theorem states that a necessary condition of the regularity of the method defined by \((a_{nk})\) is \(\sum_{k=0}^{\infty} |a_{nk}| = O(1), n \to \infty.\) See [6, Theorem 2]).

Hence to complete the proof, it suffices to show that

\[
S_1 = n \sum_{0 \leq k < (1-H)\alpha n} |a_{nk}|, S_2 = \sum_{(1+H)\alpha n < k} k|a_{nk}|,
\]

\[
S_3 = n^{-1/p} \sum_{(1-H)\alpha n \leq k \leq \alpha n} (\alpha n - k)|a_{nk}|, \text{ and}
\]

\[
S_4 = n^{-1/p} \sum_{\alpha n < k \leq (1+H)\alpha n} (k - \lfloor \alpha n \rfloor)|a_{nk}|
\]
are bounded. In fact, we will see that $S_1$ and $S_2$ tend to zero as $n \to \infty$.

The rest of the proof of Lemma 2 is similar to that of Théorème 1 in [1].

Let $g(z) = z^{-\alpha}f(z)$ and $\psi(r, t) = \log |g(re^{it})|$. Then (see [1, pp. 137-138]) there are positive numbers $\varepsilon$, $\delta$, $N_0$, and $K$ such that $N_0$ is so large that $f(z)$ is analytic on $|z| < 1 + N_0^{-1/p}$,

$$|f(re^{it})| \leq 1 - \delta \text{ for } |t| \geq \varepsilon \text{ and } |r - 1| \leq N_0^{-1/p}, \text{ and}$$

$$(3.1)\int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^{n}dt = \int_{-\varepsilon}^{\varepsilon} e^{n\psi(r, t)}dt \leq Kn^{-1/p},$$

where $r = 1 \pm n^{-1/p}$, and $n > N_0$.

Let $C_n$ be the circle centered at the origin with radius $1 - n^{-1/p}$. By Cauchy's integral formula we have, with $r = 1 - n^{-1/p}$ and $n > N_0$,

$$S_1 = \frac{n}{2\pi} \sum_{0\leq k < (1-H)an} |\int_{C_n} (f(z))^n z^{-k-1}dz|$$

$$\leq \frac{n}{2\pi} \sum_{0\leq k < (1-H)an} \int_{\varepsilon}^{2\pi-\varepsilon} |f(re^{it})|^{n}r^{-k}dt$$

$$+ n \sum_{0\leq k < (1-H)an} \int_{-\varepsilon}^{\varepsilon} |f(re^{it})|^{n}r^{-k}dt$$

$$\leq n(1-\delta)^n \sum_{0\leq k < (1-H)an} r^{-k} + n \sum_{0\leq k < (1-H)an} r^{\alpha n-k} \int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^{n}dt,$$

by (3.1) and the definition of $g$.

Since $r = 1 - n^{-1/p}$ the first term on the right is

$$\leq n(1-\delta)^n (1 - n^{-1/p})^{-\alpha n} \sum_{0\leq k < (1-H)an} (1 - n^{-1/p})^{-\alpha n-k}$$

$$\leq n(1-\delta)^n (1 - n^{-1/p})^{-\alpha n} \sum_{k=0}^{\infty} (1 - n^{-1/p})^k$$
\[
\leq n(1 - \delta)^n(1 - n^{-1/p})^{-(1-H)an} \frac{1}{1 - (1 - n^{-1/p})}
\]
\[
\leq n^{1+1/p}(1 - \delta)^n(1 - n^{-1/p})^{-(1-H)an}
\]
\[
= o(1) \text{ as } n \to \infty.
\]

To estimate the second term we apply inequality (3.2). We have

\[
\sum_{0 \leq k < (1-H)an} r^{an-k} \int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^n dt 
\]
\[
\leq n \sum_{0 \leq k < (1-H)an} r^{an-k} Kn^{-1/p}
\]
\[
\leq Kn^{1-1/p} \sum_{0 \leq k < (1-H)an} (1 - n^{-1/p})^{an-k}
\]
\[
\leq Kn^{1-1/p}(1 - n^{-1/p})^{Han} \sum_{0 \leq k < (1-H)an} (1 - n^{-1/p})^{(1-H)an-k}
\]
\[
\leq Kn^{1-1/p}(1 - n^{-1/p})^{Han} \sum_{k=0}^{\infty} (1 - n^{-1/p})^k
\]
\[
= o(1) \text{ as } n \to \infty.
\]

Hence \( S_1 = o(1) \text{ as } n \to \infty. \)

The proof that \( S_2 = o(1) \text{ as } n \to \infty \) is similar. Here we represent \( a_{nk} \) as a Cauchy integral over a circle centered at the origin with radius \( 1 + n^{-1/p} \). We omit the details.

Next we show that \( S_3 \) is bounded. Again we represent \( a_{nk} \) as a Cauchy integral over the circle \( C_n \). We have, with \( r = 1 - n^{-1/p} \) and \( n > N_0 \),

\[
S_3 = n^{-1/p} \sum_{(1-H)an \leq k \leq an} (an - k) |(1/2\pi) \int_{C_n} (f(z))^{n} z^{-k-1} dz |
\]
\[
\leq n^{-1/p} 2\pi \sum_{(1-H)an \leq k \leq an} (an - k) \int_{\epsilon}^{2\pi - \epsilon} \left| f(re^{it}) \right|^n r^{-k} dt \\
+ n^{-1/p} \sum_{(1-H)an \leq k \leq an} (an - k) \int_{-\epsilon}^{\epsilon} \left| f(re^{it}) \right|^n r^{-k} dt \\
\leq n^{-1/p}(1 - \delta)^n \sum_{(1-H)an \leq k \leq an} (an - k)r^{-k} \\
+ n^{-1/p} \sum_{(1-H)an \leq k \leq an} (an - k)r^{an-k} \int_{-\epsilon}^{\epsilon} \left| g(re^{it}) \right|^n dt, \\
\text{by (3.1) and the definition of } g, \\
\leq n^{-1/p}(1 - \delta)^n \sum_{(1-H)an \leq k \leq an} (an - k)r^{-k} \\
+ n^{-1/p} \sum_{(1-H)an \leq k \leq an} (an - k)r^{an-k} Kn^{-1/p}, \text{ by (3.2),} \\
\leq n^{-1/p}(1 - \delta)^n \sum_{(1-H)an \leq k \leq an} (an - k)(1 - n^{-1/p})^{-k} \\
+ Kn^{-2/p} \sum_{(1-H)an \leq k \leq an} (an - k)(1 - n^{-1/p})^{an-k}.
\]

The first term on the right

\[
\leq n^{-1/p}(1 - \delta)^n H\alpha n \sum_{k \leq an} (1 - n^{-1/p})^{-k} \\
\leq Kn^{1-1/p}(1 - \delta)^n (1 - n^{-1/p})^{-an} \sum_{k=0}^{\infty} (1 - n^{-1/p})^k \\
\leq Kn^{1-1/p}(1 - \delta)^n (1 - n^{-1/p})^{-an} \frac{1}{1 - (1 - n^{-1/p})} \\
\leq Kn(1 - \delta)^n (1 - n^{-1/p})^{-an} \\
= o(1) \text{ as } n \to \infty.
\]

Since

\[
\sum_{k=0}^{\infty} ka^{k-1} = (1 - a)^{-2}
\]
if \(|a| < 1\), the second term is less than

\[Kn^{-2/p}(1 - (1 - n^{-1/p}))^{-2} = K.\]

Thus \(S_3\) is bounded. Similarly, representing \(a_{nk}\) as a Cauchy integral over the circle centered at the origin with radius \(1 + n^{-1/p}\) we can prove that \(S_4\) is bounded. This completes the proof of Lemma 2.

Lemma 2 plays a crucial role in the proof. Assuming that \((s^*)\) is bounded and \(s_k \to 0 (E, f)\) then we can prove Lemmas 3–6, as we will see. This remark will be important in the proof of Theorem 3(b).

**Lemma 3** \(\lim_{n \to \infty} \sum_{|\alpha_n-k|<n^{1/p}\log n} n^{-1/p} s_k \phi((\alpha_n-k)n^{-1/p}) = 0\), where

\[\phi(x) = \int_{-\infty}^{\infty} \exp(At^p + ixt) \, dt.\]

**Proof** Girard's paper [5], which is based on his doctoral thesis, contains the essential ingredients required for the proof of Lemma 3. Since

\[a_{nk} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^n(e^{it}) e^{-ikt} \, dt\]

and \((s_k)\) is bounded, it follows from [5, pp.362–364] that for every \(\varepsilon > 0\) we have

\[\sum_{k=0}^{\infty} a_{nk} s_k = \frac{1}{2\pi} \sum_{\log n < |\alpha_n-k|<n^{1/p}\log n} s_k \int_{-\varepsilon}^{\infty} f^n(e^{it}) e^{-ikt} \, dt + o(1) \text{ as } n \to \infty.\]

We choose \(\varepsilon\) small enough so that the following three conditions A–C hold.
A. For \(|t| \leq \varepsilon\),

\[ f(e^{it})e^{-iat} = 1 + Ai^p e^{-iat}(e^{it} - 1)^p + O(t^{p+1}) = \exp(At^p + G(t)), \]

where \(|G(t)| \leq K_1 |t|^{p+1}\) for some constant \(K_1\).

Such a function \(G\) exists because of condition 4 of Theorem 1 and the fact that if \(|t|\) is small enough, then we have

\[ Ai^p e^{-iat}(e^{it} - 1)^p = Ai^p \{1 + O(t)\} \{(it)^p + O(t^{p+1})\} \]
\[ = At^{2p}t^p + O(t^{p+1}) \]
\[ = At^p + O(t^{p+1}), \]

since \(p\) is an even integer.

B. \(\Re A + K_1 \varepsilon < 0\).

C. \(|\exp(x) - 1| \leq 2|x|\) if \(|x| \leq K_1 c^{p+1}\).

To simplify notations, let \(T(n) = \{k : \log n < |\alpha n - k| < n^{1/p} \log n\}\). We note that the number of elements in \(T(n)\) is less than \(2n^{1/p} \log n\).

By condition A we have

\[
\sum_{k \in T(n)} s_k \int_{-\varepsilon}^\varepsilon f^n(e^{it}) e^{-ikt} dt = \sum_{k \in T(n)} s_k \int_{-\varepsilon}^\varepsilon (f(e^{it}) e^{-iat})^n e^{i(\alpha n - k)t} dt = \sum_{k \in T(n)} s_k \int_{-\varepsilon}^\varepsilon (e^{At^p + G(t)})^n e^{i(\alpha n - k)t} dt = \sum_{k \in T(n)} s_k \int_{-\varepsilon}^\varepsilon \exp(n At^p + nG(t) + i(\alpha n - k)t) dt.
\]
Making the substitution $v = n^{1/p}t$ in the preceding integrals we have

$$
\sum_{k \in T(n)} s_k \int_{-e}^{e} f^n(e^{it}) e^{-ikt} dt
$$

$$
= \sum_{k \in T(n)} s_k n^{-1/p} \int_{-en^{1/p}}^{en^{1/p}} \exp \left( Av^p + nG(n^{-1/p}v) + i(\alpha n - k)n^{-1/p}v \right) dv
$$

$$
= \sum_{k \in T(n)} s_k n^{-1/p} \int_{-en^{1/p}}^{en^{1/p}} \exp \left( Av^p + i(\alpha n - k)n^{-1/p}v \right) dv + U,
$$

where $U =$

$$
\sum_{k \in T(n)} s_k n^{-1/p} \int_{-en^{1/p}}^{en^{1/p}} \{ \exp \left( Av^p + i(\alpha n - k)n^{-1/p}v \right) \} \{ \exp(nG(n^{-1/p}v)) - 1 \} dv.
$$

We write

$$
U = \sum_{k \in T(n)} s_k n^{-1/p} \left( \int_{-en^{1/p}(p+1)}^{en^{1/p}(p+1)} + \int_{-en^{1/p}}^{en^{1/p}} + \int_{en^{1/p}(p+1)}^{en^{1/p}(p+1)} \right) \cdots dv
$$

$$
= U_1 + U_2 + U_3.
$$

First we show that $U_1 = o(1)$ as $n \to \infty$.

If $|v| \leq \varepsilon n^{1/p}$ then $|n^{-1/p}v| \leq \varepsilon$. Hence by condition A we have

$$
|nG(n^{-1/p}v)| \leq nK_1 |n^{-1/p}v|^{p+1} = K_1 n^{-1/p} |v|^{p+1}.
$$

If in addition we have $|v| \leq \varepsilon n^{1/(p+1)}$ then $|v|^{p+1} \leq n^{1/p} \varepsilon^{p+1}$. Hence by (3.3),

$$
|nG(n^{-1/p}v)| \leq K_1 n^{-1/p} n^{1/p} \varepsilon^{p+1} = K_1 \varepsilon^{p+1}.
$$

It follows from this inequality and condition C that

$$
|U_1| \leq \sum_{k \in T(n)} |s_k| n^{-1/p} \int_{-en^{1/(p+1)}}^{en^{1/(p+1)}} \{ \exp(\Re Av^p) \} 2|nG(n^{-1/p}v)| dv
$$
Hence by (3.3) and the fact that \((s_k)\) is bounded,

\[
|U_1| \leq Kn^{-1/p} \sum_{k \in T(n)} \int_{-en^{1/p}(p+1)}^{en^{1/p}(p+1)} \{\exp (RAv^p)\} K_1 n^{-1/p} |v|^{p+1} dv
\]

\[
\leq Kn^{-2/p} \sum_{k \in T(n)} \int_{-\infty}^{\infty} \{\exp (RAv^p)\} |v|^{p+1} dv
\]

\[
\leq Kn^{-2/p} \sum_{k \in T(n)} 1
\]

\[
\leq Kn^{-2/p}(n^{1/p} \log n),
\]

since the number of elements in \(T(n)\) is less than \(2n^{1/p} \log n\).

This shows that \(U_1 \rightarrow 0\) as \(n \rightarrow \infty\).

Next we prove that \(U_3 \rightarrow 0\) as \(n \rightarrow \infty\).

For \(|v| \leq \varepsilon n^{1/p}\), or \(|n^{-1/p}v| \leq \varepsilon\), we have, by condition A,

\[
n|G(n^{-1/p}v)| \leq nK_1 (n^{-1/p} |v|)^{p+1}
\]

\[
\leq K_1 (n^{-1/p} |v|)^p |v|^p
\]

\[
\leq K_1 \varepsilon |v|^p.
\]

Therefore if \(|v| \leq \varepsilon n^{1/p}\) then

\[
|\exp (nG(n^{-1/p}v)) - 1| \leq \exp (K_1 \varepsilon |v|^p).
\]

Hence

\[
|U_3| \leq Kn^{-1/p} \sum_{k \in T(n)} \int_{-en^{1/p}(p+1)}^{en^{1/p}} \exp (RA |v|^p) \exp (K_1 \varepsilon |v|^p) dv.
\]

Since \(RA + K_1 \varepsilon < 0\) (by condition B) we have,

\[
|U_3| \leq Kn^{-1/p} \int_{-en^{1/p}(p+1)}^{\infty} \exp ((RA + K_1 \varepsilon) |v|^p) dv \sum_{k \in T(n)} 1
\]
\[ \leq Kn^{-1/p}n^{1/p} \log n \int_{\epsilon n^{1/(p+1)}}^{\infty} \exp \left( (\Re A + K_1 \epsilon) |v| \right) dv, \]

since the number of elements in \( T(n) \) is less than \( 2n^{1/p} \log n \),

\[ \leq K \log n \int_{\epsilon n^{1/(p+1)}}^{\infty} \exp \left( (\Re A + K_1 \epsilon) |v| \right) dv, \text{ provided } \epsilon n^{1/p(p+1)} > 1, \]

\[ \leq K \log n \frac{1}{|\Re A + K_1 \epsilon|} \exp \left( (\Re A + K_1 \epsilon) \epsilon n^{1/p(p+1)} \right), \text{ if } \epsilon n^{1/p(p+1)} > 1, \]

\[ = o(1) \text{ as } n \to \infty. \]

Similarly we can show that \( U_2 \to 0 \) as \( n \to \infty \).

Hence we have

\[ \sum_{k=0}^{\infty} a_{nk} s_k = \frac{1}{2\pi} \sum_{k \in T(n)} s_k \int_{-\epsilon}^{\epsilon} f^n(e^{it}) e^{-ikt} dt + o(1) \]

\[ = \frac{1}{2\pi} \sum_{k \in T(n)} s_k n^{-1/p} \int_{-\epsilon n^{1/p}}^{\epsilon n^{1/p}} \exp (Av^p + i(\alpha n - k)n^{-1/p}v) dv \]

\[ + o(1), \]

as \( n \to \infty \).

We now prove that we may extend the limits of the last integrals to infinity.

For every real number \( r_{nk} \), we have

\[ | \phi(r_{nk}) - \int_{-\epsilon n^{1/p}}^{\epsilon n^{1/p}} \exp (Av^p + ir_{nk}v) dv | \]

\[ = | \int_{-\infty}^{\infty} \exp (Av^p + ir_{nk}v) dv - \int_{-\epsilon n^{1/p}}^{\epsilon n^{1/p}} \exp (Av^p + ir_{nk}v) dv | \]

\[ \leq \int_{-\infty}^{-\epsilon n^{1/p}} + \int_{\epsilon n^{1/p}}^{\infty} \exp (\Re Av^p) dv \]

\[ \leq 2 \int_{-\epsilon n^{1/p}}^{\epsilon n^{1/p}} \exp (\Re Av^p) dv, \text{ since } p \text{ is even}, \]

\[ \leq 2 \int_{-\epsilon n^{1/p}}^{\epsilon n^{1/p}} \exp (\Re Av^p) dv \]
\[
= \frac{2 \exp(\Re \alpha n^{1/p})}{|\Re A|}, \text{ when } \varepsilon n^{1/p} > 1.
\]

Hence if \( \varepsilon n^{1/p} > 1 \) then,

\[
\left| \sum_{k \in T(n)} s_k n^{-1/p} \left\{ \phi((\alpha n - k)n^{-1/p}) - \int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \exp(\Re A \epsilon n^{1/p}) \right. \right| \\
\leq K n^{-1/p} \sum_{k \in T(n)} \exp(\Re A \epsilon n^{1/p}) \\
\leq K n^{-1/p} n^{1/p} (\log n) \exp(\Re A \epsilon n^{1/p}),
\]

since the number of elements of \( T(n) \) is less than \( 2n^{1/p} \log n \),

\[
\leq K (\log n) \exp(\Re A \epsilon n^{1/p}) \\
= o(1) \text{ as } n \to \infty.
\]

It follows that

\[
\sum_{k=0}^{\infty} a_n s_k = \frac{1}{2\pi} \sum_{k \in T(n)} s_k n^{-1/p} \phi((\alpha n - k)n^{-1/p}) + o(1), \tag{3.4}
\]

as \( n \to \infty \). By the hypothesis that \( \sum_{k=0}^{\infty} a_n s_k \to 0 \), we have

\[
\lim_{n \to \infty} \sum_{k \in T(n)} s_k n^{-1/p} \phi((\alpha n - k)n^{-1/p}) = 0.
\]

The proof of Lemma 3 is complete.

Lemma 4 \( \lim_{n \to \infty} n^{-1/p} \int_{0}^{\infty} s(t) \phi((\alpha n - t)n^{-1/p}) dt = 0 \), where \( s(t) = s_{[t]} \).

Proof We begin with a few remarks about the function \( \phi \). Recall that a function \( f \) is called rapidly decreasing if \( f \) is infinitely differentiable, and for every pair of nonnegative integers \( m \) and \( n \), \( x^m f^{(n)}(x) \) is bounded on the real line. Note that \( \phi \) is
the inverse Fourier transform of the function \( \exp(At^p) \), which is rapidly decreasing. Hence \( \phi \) itself is rapidly decreasing. (See [9, Theorem 7.7].) It is bounded, infinitely differentiable and integrable on the real line. It follows that the integral in Lemma 4 exists. Also, \( \phi \) and its derivatives satisfy the above growth condition. Finally we note that \( \phi \) is an even function, since \( p \) is an even integer.

We will prove (3.5)-(3.9) below, which imply the lemma.

\[
\lim_{n \to \infty} n^{-1/p} \int_{\alpha n - \log n}^{\alpha n + \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt = 0. \tag{3.5}
\]

\[
\lim_{n \to \infty} n^{-1/p} \int_{\alpha n + \log n}^{\alpha n + n^{1/p} \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt = 0. \tag{3.6}
\]

\[
\lim_{n \to \infty} n^{-1/p} \int_{0}^{\alpha n - n^{1/p} \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt = 0. \tag{3.7}
\]

\[
\lim_{n \to \infty} n^{-1/p} \left\{ \int_{U(n)} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt - \sum_{k \in U(n)} s_k \phi((\alpha n - k)n^{-1/p}) \right\} = 0, \tag{3.8}
\]

where \( U(n) \) denotes the open interval \((\alpha n - n^{1/p} \log n, \alpha n - \log n)\).

\[
\lim_{n \to \infty} n^{-1/p} \left\{ \int_{V(n)} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt - \sum_{k \in V(n)} s_k \phi((\alpha n - k)n^{-1/p}) \right\} = 0, \tag{3.9}
\]

where \( V(n) \) denotes the open interval \((\alpha n + \log n, \alpha n + n^{1/p} \log n)\).

Since \( s(t) \) and \( \phi \) are bounded we have

\[
| n^{-1/p} \int_{\alpha n - \log n}^{\alpha n + \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt | \leq Kn^{-1/p} \log n,
\]

and (3.5) follows.

To prove (3.6) we note that since \( \phi \) is an even function and is integrable,

\[
| n^{-1/p} \int_{\alpha n + n^{1/p} \log n}^{\alpha n + \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt |
\]
\[ \leq K n^{-1/p} \int_{\alpha n + n^{1/p} \log n}^{\infty} |\phi((\alpha n - t)n^{-1/p})| \, dt \]
\[ \leq K \int_{\log n}^{\infty} |\phi(u)| \, du \]
\[ = o(1) \text{ as } n \to \infty. \]

Similarly we can prove (3.7).

Since

\[
n^{-1/p} \left\{ \int_{U(n)} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt - \sum_{k \in U(n)} s_k \phi((\alpha n - k)n^{-1/p}) \right\}
\]
\[ = n^{-1/p} \sum_{k \in U(n)} \left\{ \int_{k}^{k+1} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt 
- \int_{k}^{k+1} s(t) \phi((\alpha n - k)n^{-1/p}) \, dt \right\}
+ n^{-1/p} \int_{\alpha n - n^{1/p} \log n}^{\alpha n - \log n} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt
+ n^{-1/p} \int_{\alpha n - \log n}^{\infty} s(t) \phi((\alpha n - t)n^{-1/p}) \, dt,
\]

since the last two terms of the preceding line = o(1) as \( n \to \infty \), since analogous statements hold for the interval \( V(n) \), and since \( s(t) = s(t) \) is bounded, to prove (3.8) and (3.9) it suffices to prove that

\[
n^{-1/p} \sum_{k \in T(n)} \int_{k}^{k+1} |\phi((\alpha n - t)n^{-1/p}) - \phi((\alpha n - k)n^{-1/p})| \, dt = o(1),
\]

where \( T(n) \) is defined in the proof of Lemma 3.

By the mean value theorem, if \( t \in [k, k + 1] \) then

\[
|\phi((\alpha n - t)n^{-1/p}) - \phi((\alpha n - k)n^{-1/p})| \leq n^{-1/p} |\phi'(\xi)|,
\]

for some \( \xi \) satisfying \( (\alpha n - t)n^{-1/p} \leq \xi \leq (\alpha n - k)n^{-1/p} \).
Since \( \phi \) is rapidly decreasing, \( \phi' \) is bounded. Hence if \( t \in [k, k + 1] \) then

\[
|\phi((an - t)n^{-1/p}) - \phi((an - k)n^{-1/p})| \leq Kn^{-1/p}.
\]

Thus

\[
\begin{align*}
n^{-1/p} \sum_{k \in T(n)} \int_{k}^{k+1} |\phi((an - t)n^{-1/p}) - \phi((an - k)n^{-1/p})| \, dt \\
\leq n^{-1/p} \sum_{k \in T(n)} Kn^{-1/p} \\
\leq Kn^{-2/p} \sum_{k \in T(n)} 1 \\
\leq Kn^{-2/p} 2n^{1/p} \log n \\
= 2Kn^{-1/p} \log n.
\end{align*}
\]

This completes the proof of (3.8) and (3.9).

**Lemma 5** \( \lim_{x \to -\infty} x^{-1/p} \int_{0}^{\infty} s(t) \phi((\alpha/x)^{1/p}(t - x)) \, dt = 0 \), where \( x \) is a continuous variable.

**Proof** Let \( n = \lfloor x/\alpha \rfloor \). Since \( \phi \) is an even function it follows from Lemma 4 that

\[
\lim_{x \to -\infty} x^{-1/p} \int_{0}^{\infty} s(t) \phi((t - an)n^{-1/p}) \, dt = 0.
\]

Since \( s(t) \) is bounded it suffices to show that

\[
I = x^{-1/p} \int_{0}^{\infty} |\phi((t - an)n^{-1/p}) - \phi((\alpha/x)^{1/p}(t - x))| \, dt \to 0, \ x \to \infty.
\]

Making the change variable \( u = (\alpha/x)^{1/p}(t - x) \), we have

\[
I = \alpha^{1/p} \int_{(\alpha/x)^{1/p}(t-x)}^{\infty} |\phi(\beta(x)u + \gamma(x)) - \phi(u)| \, du
\]

\[
\leq \alpha^{1/p} \int_{-\infty}^{\infty} |\phi(\beta(x)u + \gamma(x)) - \phi(u)| \, du,
\]
where \( \beta(x) = (x/\alpha n)^{1/p} \) and \( \gamma(x) = (x - \alpha n)n^{-1/p} \). So \( \beta(x) \to 1 \) and \( \gamma(x) \to 0 \) as \( x \to \infty \).

Since \( \phi \) is continuous \( \phi(\beta(x)u + \gamma(x)) - \phi(u) \to 0 \) for each \( u \) as \( x \to \infty \). On the other hand, since \( \phi \) is rapidly decreasing there exists a constant \( K \) such that 

\[
|\phi(u)| \leq K/(u^2 + 1).
\]

Also \( (\beta(x)u + \gamma(x))^2 \geq u^2/4 \) if \( x \) is large and \( |u| \geq 1 \), since \( \beta(x) \to 1 \) and \( \gamma(x) \to 0 \). Hence

\[
|\phi(\beta(x)u + \gamma(x))| \leq \frac{K}{(\beta(x)u + \gamma(x))^2 + 1} \leq \frac{K}{4u^2 + 1} \quad \text{for } |u| \geq 1
\]

if \( x \) is large enough. By Lebesgue's dominated convergence theorem, \( I \to 0 \) as \( x \to \infty \). This proves Lemma 5.

**Lemma 6** \( \lim_{x \to \infty} \int_{0}^{\infty} s(t) \phi(q(x - t)) \, dt = 0 \), where \( q \) is the conjugate index of \( p \), i.e., it satisfies \( 1/p + 1/q = 1 \).

**Proof**

The proof is similar to the one found in [6, pp.313-314]. By Lemma 5 we have

\[
\lim_{x \to \infty} x^{-1/p} \int_{0}^{\infty} s(t) \phi(\alpha^{1/q}x^{-1/p}t - \alpha^{1/p}x^{-1/p}) \, dt = 0.
\]

Let \( q \) satisfy \( 1/p + 1/q = 1 \), let \( x = \alpha^{1-q}y^q \) and make the substitution \( t = \alpha^{1-q}u^q \) in the preceding integral. We have

\[
x^{-1/p} = \alpha^{(1-q)/q}y^{1-q},
\]

\[
\alpha^{1/p}x^{-1/p}t = y^{1-q}u^q,
\]

\[
\alpha^{1/p}x^{1-1/p} = y, \quad \text{and}
\]
\[ dt = q \alpha^{1-q} u^q - 1 du. \]

It follows that

\[
\lim_{v \to \infty} \alpha^{(1-q)/y} \int_0^\infty s(\alpha^{1-q} u^q) \phi(y^{1-q} u^q - y) q \alpha^{1-q} u^q - 1 du = 0, \quad \text{or}
\]

\[
\lim_{v \to \infty} \int_0^\infty s(\alpha^{1-q} u^q) \left( \frac{u}{y} \right)^{q-1} \phi(y^{1-q}(u^q - y^q)) du = 0.
\]

Since \( s(\alpha^{1-q} u^q) \) is bounded Lemma 6 follows from the fact that

\[
I = \int_0^\infty \left| \phi(q(y - u)) - \left( \frac{u}{y} \right)^{q-1} \phi(y^{1-q}(u^q - y^q)) \right| du \to 0 \quad \text{as} \quad y \to \infty,
\]

which we will now prove.

Since we will let \( y \to \infty \) we may assume that \( y > 1 \). Let \( u = y + w \). Since \( \phi \) is an even function we have

\[
I = \int_0^\infty \left| \phi(q(y - u)) - \left( \frac{u}{y} \right)^{q-1} \phi(y^{1-q}(u^q - y^q)) \right| du.
\]

Let

\[
\lambda(x) = \begin{cases} 
(x + 1)^q - 1 / x & \text{if } x \geq -1, x \neq 0, \\
q & \text{if } x = 0.
\end{cases}
\]

It is easy to see that \( \lambda(x) \) is continuous and has a positive minimum \( m \), say, on \([-1, \infty)\). Furthermore we have

\[
y^{1-q}((y + w)^q - y^q) = w \lambda(w/y).
\]

Hence

\[
I = \int_{-y}^y \left| \phi(qw) - ((y + w)/y)^{q-1} \phi(w \lambda(w/y)) \right| dw.
\]
Let $a$ satisfy $0 < 3a < 1$.

We write

$$I = I_1 + I_2,$$

where

$$I_1 = \int_{|w| \geq y^a, \ y \geq -y} |\phi(qw) - ((y + w)/y)^{q-1}\phi(w\lambda(w/y))| \, dw,$$

and

$$I_2 = \int_{|w| < y^a} |\phi(qw) - ((y + w)/y)^{q-1}\phi(w\lambda(w/y))| \, dw.$$

Notice that since $y > 1$ and $0 < a < 1$, the restriction $w \geq -y$ is satisfied in $I_2$.

First we show that $I_1 \to 0$ as $y \to \infty$. Thus we assume that $|w| \geq y^a$. Since $y > 1$, if $w > 0$ then we have

$$w > y^a > 1.$$

Hence if $y$ is large enough then

$$w < y(w - 1), \quad \text{or} \quad \frac{y + w}{y} < w. \quad (3.10)$$

On the other hand since $\lambda(x)$ has a positive minimum $m$ on $[-1, \infty)$, if $w > 0$ then

$$w\lambda(w/y) \geq mw.$$

Since $\phi$ is rapidly decreasing there is a constant $K > 0$ such that

$$|\phi(w\lambda(w/y))| \leq K|w\lambda(w/y)|^{-3} \leq Km^{-3}w^{-3}, \ w > 0. \quad (3.11)$$

Since $1/p + 1/q = 1$ and $p \geq 2$ we have $q \leq 2$. So

$$q - 4 \leq -2. \quad (3.12)$$
Finally the fact that $\phi$ is an even function, (3.10), (3.11), and (3.12) yield, for large $y$,

$$I_1 \leq \int_{|w| \geq y^a} |\phi(qw)| \, dw + \int_{|w| \geq y^a} \left| (y + w)/y \right|^{q-1} |\phi(w \lambda(w/y))| \, dw$$

$$\leq \int_{|w| \geq y^a} |\phi(qw)| \, dw + \int_{|w| \geq y^a} w^{q-1} K m^{-3} w^{-3} \, dw$$

$$\leq \int_{|w| \geq y^a} |\phi(qw)| \, dw + K m^{-3} \int_{|w| \geq y^a} w^{-2} \, dw.$$

Hence $I_1 \to 0$ as $y \to \infty$.

Next we consider $I_2$. Here we assume that $|w| < y^a$. So $w = O(y^a)$ and $w/y = O(y^{a-1})$. Hence,

$$((y + w)/y)^{q-1} = (1 + w/y)^{q-1} = 1 + O(y^{a-1}).$$

If $x$ is small then $\lambda(x) = q + O(x)$. Hence

$$w \lambda(w/y) = w(q + O(w/y)) = w(q + O(y^{a-1})) = wq + O(y^{2a-1}).$$

Thus

$$I_2 = \int_{|w| < y^a} |\phi(qw) - (1 + O(y^{a-1})) \phi(wq + O(y^{2a-1}))| \, dw$$

$$\leq \int_{|w| < y^a} |\phi(qw) - \phi(wq + O(y^{2a-1}))| \, dw$$

$$+ O(y^{a-1}) \int_{|w| < y^a} \phi(wq + O(y^{2a-1})) \, dw.$$

Since $\phi'$ is bounded the mean value theorem implies that the first term on the right is less than

$$K y^{2a-1} \int_{|w| < y^a} 1 \, dw = 2K y^{3a-1},$$
which tends to zero as \( y \to \infty \) since \( 3a < 1 \).

Since \( \phi \) is bounded the second term is \( O(y^{2a-1}) \), which also tends to zero. This proves Lemma 6.

Before proving the last lemma of this chapter we state two definitions.

A function \( f \) defined on \([0, \infty)\) is \textit{slowly oscillating} if

\[
    f(u) - f(v) \to 0
\]
as \( v \to \infty \), \( u > v \), and \( u - v \to 0 \).

A real valued function \( f \) defined on \([0, \infty)\) is \textit{slowly decreasing} if

\[
    \lim (f(u) - f(v)) \geq 0
\]
as \( v \to \infty \), \( u > v \), and \( u - v \to 0 \).

Lemma 7 \( s(\alpha^{1-q}t^q) \) is a slowly oscillating function.

Proof By definition, we have to prove that

\[
    \lim \{ s(\alpha^{1-q}u^q) - s(\alpha^{1-q}v^q) \} = 0,
\]
as \( v \to \infty \), \( u > v \), \( u - v \to 0 \).

For simplicity let \( y = \alpha^{1-q}u^q \) and \( z = \alpha^{1-q}v^q \). Then

\[
(y - z)z^{-1/p} = (\alpha^{1-q}u^q - \alpha^{1-q}v^q)(\alpha^{1-q}v^q)^{-1/p}
= \alpha^{-1/p}v((u/v)^q - 1).
\]
Since $p \geq 2$, $q \leq 2$. Since $u > v$, $u/v > 1$. Hence

$$(y - z)z^{-1/p} \leq \alpha^{-1/p}v((\frac{u}{v})^2 - 1)$$

$$= K\frac{u^2 - v^2}{v}$$

$$= K(u - v)(\frac{u}{v} + 1)$$

$$= o(1)$$

when $v \to \infty$, $u > v$, $u - v \to 0$.

By the Tauberian condition, $s_{[u]} - s_{[v]} \to 0$. The lemma is proved.

We are now ready to prove Theorem 3. An important ingredient of the proof is

Pitt's Tauberian theorem  Suppose that $g$ is integrable on $(-\infty, \infty)$ and that its Fourier transform does not vanish on $(-\infty, \infty)$. Suppose that $f$ is bounded and slowly oscillating, or real valued, bounded and slowly decreasing. If

$$\int_{-\infty}^{\infty} f(t) g(x - t) dt \to L \int_{-\infty}^{\infty} g(t) dt$$

as $x \to \infty$, then $f(x) \to L$ as $x \to \infty$.

Proof of Theorem 3(a)  We have to prove that $s_n \to 0$. By Lemmas 2 and 7 $s(\alpha^{1-q}u^q)$ is a bounded and slowly oscillating function. It will follow from Lemma 6 and Pitt's Tauberian theorem that $s(\alpha^{1-q}u^q) \to 0$ as $u \to \infty$ provided that the Fourier transform of $\phi(qu)$ has no zeros. But $\phi$ is the inverse Fourier transform of
the function \( \exp(A t^p) \) and is rapidly decreasing. Hence by the Fourier inversion theorem we have

\[
\int_{-\infty}^{\infty} \phi(q u) e^{-i\pi u} du = \int_{-\infty}^{\infty} \phi(t) e^{-i(t/q)\pi} \frac{1}{q} dt = \frac{1}{q} \exp(A(x/q)^p).
\]

This function indeed has no zeros. Let \( u = (n\alpha^{q-1})^{1/q} \). Then we have

\[
s(\alpha^{1-q} u^q) = s(n) = s_n \to 0
\]

as \( n \to \infty \). We have proved Theorem 3(a).

**Proof of Theorem 3(b)** We will sketch the proof. Instead of Lemma 1 we can prove that there exist positive numbers \( a \) and \( b \) such that for \( q \geq p \geq 1 \),

\[
s_q - s_p \geq -a(\sqrt{q} - \sqrt{p}) - b. \tag{3.13}
\]

For details, see [6, Theorem 239] or [11, Lemma ε].

Also, we have

\[
\sum_{k=0}^{M} a_{nk} \to 0
\]

as \( M \to \infty \), \( n \to \infty \), and \( \sqrt{\alpha n} - \sqrt{M} \to \infty \), and

\[
\sum_{k=N}^{\infty} a_{nk}(\sqrt{k} - \sqrt{N}) \to 0
\]

as \( N \to \infty \), \( n \to \infty \), and \( \sqrt{N} - \sqrt{\alpha n} \to \infty \).

The proofs of these statements are similar to that of Lemma 2. Using these facts we can modify slightly the proof of Theorem 238 in [6] to conclude that the
subsequence \((s_{\lfloor an \rfloor})_{n \geq 1}\) is bounded. Now let \(k\) be an arbitrary positive integer. Then there exists \(n\) such that \(\alpha n < k \leq \alpha(n + 1)\). By (3.13), we have

\[
s_k - s_{\lfloor an \rfloor} \geq -a(\sqrt{k} - \sqrt{\lfloor an \rfloor}) - b
\]

\[
= -a\left(\frac{k - \alpha n}{\sqrt{k + \sqrt{\lfloor an \rfloor}}}\right) - b
\]

\[
\geq \frac{-a\alpha}{\sqrt{k} + \sqrt{\lfloor an \rfloor}} - b.
\]

Since \((s_{\lfloor an \rfloor})\) is bounded this inequality implies that \((s_k)\) is bounded from below. On the other hand we have

\[
s_{\lfloor \alpha(n+1) \rfloor} - s_k \geq -a\left(\sqrt{\lfloor \alpha(n+1) \rfloor} - \sqrt{k}\right) - b,
\]

and we can prove similarly that \((s_k)\) is bounded from above.

Since \((s_k)\) is bounded Lemmas 3-6 hold true. Instead of Lemma 7 we prove that \(s(\alpha^{1-q_k})\) is a slowly decreasing function. We may now apply Pitt's Tauberian theorem to complete the proof.
CHAPTER IV

The Tauberian condition

Can we replace the Tauberian condition in Theorem 3(a) by the weaker condition
\[ \lim (s_m - s_n) = 0 \text{ as } n \to \infty, \quad m > n, \quad \text{and} \quad (m - n)c^{-1/p(f)} \to 0, \]
where 0 < c < 1/p(f), or more generally, by \( \lim (s_m - s_n) = 0 \) as \( n \to \infty, \quad m > n, \quad \text{and} \quad (m - n)r_n^{-1/p(f)} \to 0, \) where \( (r_n) \) is an increasing sequence which tends to \( \infty \) so slowly that \( r_n^{-1/p(f)} \) is decreasing and tends to 0? If so then we can weaken the \( O \)-condition in the corollary of Theorem. Theorem 4 shows that this cannot be done. Thus the exponent \(-1/p(f)\) in Theorem 3 is the best possible.

The proof of the theorem is a modification of an example due to Kwee[7].

**Theorem 4** Let \( p_0 \) be a positive even integer. Let \( (r_k) \) be a sequence of increasing positive numbers tending to \( \infty \). Then there is a bounded divergent sequence \((s_k)\) which is summable by every regular method \((E, f)\) with \( p(f) = p_0 \) and which satisfies

\[ s_k - s_{k-1} = O(r_kk^{-1/p_0}). \]

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Proof Let \((m_k)\) and \((q_k)\) be two sequences of positive integers with the following properties.

1. \(2k \leq r_{q_k}, k = 1, 2, \ldots\)

2. \(\frac{1}{2}(q_k)^{1/p_0} \leq km_k \leq (q_k)^{1/p_0}, k = 1, 2, \ldots\)

3. \(2(q_k + 2m_k) < q_{k+1}, k = 1, 2, \ldots\)

Property 2 implies that

4. \(m_k \leq q_k, k = 1, 2, \ldots\)

By property 3 \((q_k)\) is an increasing sequence.

Now we define \((s_k)\). If \(k\) is outside intervals of the form \((q_k, q_k + 2m_k)\) then let \(s_k = 0\). For each positive integer \(k\), let

\[
s_{q_k+w} = \begin{cases} 
  w/m_k, & 1 \leq w \leq m_k, \\
  (2m_k - w)/m_k, & m_k < w < 2m_k.
\end{cases}
\]

Hence \(0 \leq s_k \leq 1\) for each \(k\). Since \(s_{q_k} = 0\) and \(s_{q_k+m_k} = 1\) for each \(k\), \((s_k)\) is divergent. Next we prove that \(s_k - s_{k-1} = O(r_k k^{-1/p_0})\).

For \(q_k < j \leq q_k + 2m_k\) we have

\[
|s_j - s_{j-1}| = 1/m_k.
\]

Also,

\[
|s_{q_k} - s_{q_k-1}| = 0.
\]

Hence

\[
|s_j - s_{j-1}| \leq 1/m_k \text{ if } q_k \leq j \leq q_k + 2m_k.
\]
By properties $2$ and $1$ we have

$$\frac{1}{m_k} \leq (2k)(q_k)^{-1/p_0} \leq r_{q_k}(q_k)^{-1/p_0}.$$ 

Since $(r_k)$ is increasing, for $q_k \leq j \leq q_k + 2m_k$ we have

$$r_{j}(j)^{-1/p_0} \geq r_{q_k}(j)^{-1/p_0} \geq r_{q_k}(q_k + 2m_k)^{-1/p_0} \geq r_{q_k}(3q_k)^{-1/p_0}.$$ 

The last inequality follows from property $4$.

Hence

$$r_{q_k}(q_k)^{-1/p_0} \leq 3^{1/p_0} r_{j}(j)^{-1/p_0}, \text{ if } q_k \leq j \leq q_k + 2m_k.$$ 

It follows that for $q_k \leq j \leq q_k + 2m_k$ we have

$$|s_j - s_{j-1}| \leq \frac{1}{m_k} \leq r_{q_k}(q_k)^{-1/p_0} \leq 3^{1/p_0} r_{j}(j)^{-1/p_0}.$$ 

On the other hand if $j$ is outside intervals of the form $[q_k, q_k + 2m_k]$ then $s_j - s_{j-1} = 0$. Thus for every $j,

$$|s_j - s_{j-1}| \leq 3^{1/p_0} r_{j}(j)^{-1/p_0} = O(r_{j}(j)^{-1/p_0}).$$

We still have to show that $(s_k)$ is $(E, f)$ summable to $0$ if $p(f) = p_0$. Let $(E, f)$ be such a method with matrix $(a_{nk})$. Since $(s_k)$ is bounded equation (3.4) in the proof of Lemma 3 holds, i.e.,

$$\sum_{k=0}^{\infty} a_{nk} s_k = \frac{1}{2\pi} \sum_{k \in T(n)} s_k n^{-1/p_0} \phi((\alpha n - k)n^{-1/p_0}) + o(1), \text{ as } n \to \infty,$$

where $\alpha = \alpha(f)$, $T(n) = \{k : \log n < |\alpha n - k| < n^{1/p} \log n\}$, and $\phi(x) = \int_{-\infty}^{\infty} \exp(A(f)t^{p_0} + ixt) \, dt$. 

We will prove that

\[
\lim_{n \to \infty} \sum_{k \in T(n)} s_k n^{-1/p_0} \phi((\alpha n - k)n^{-1/p_0}) = 0.
\]

Then it follows that \((s_k)\) is summable \((E, f)\) to 0.

If \(n\) is large enough then

\[
\alpha n + n^{1/p_0} \log n < 2(\alpha n - n^{1/p_0} \log n).
\]

Since \(2(q_k + 2m_k) < q_{k+1}\) (by property 3), \(T(n)\) overlaps with at most one of the intervals \([q_k, q_k + 2m_k]\). If it does not overlap with any of these intervals then the above sum is 0. If it does for some \(k\) then \(q_k\) or \(q_k + 2m_k\) lies between \(\alpha n - n^{1/p_0} \log n\) and \(\alpha n + n^{1/p_0} \log n\). Hence \(q_k = O(n)\). So

\[
n^{-1/p_0} = O(q_k^{-1/p_0}).
\]

Also, in this case the number of non-zero terms in the sum is at most \(2m_k\), because the length of the interval \([q_k, q_k + 2m_k]\) is \(2m_k\). Since \(0 \leq s_k \leq 1\) and \(\phi\) is bounded, the sum is

\[
O(m_k n^{-1/p_0}) = O(m_k q_k^{-1/p_0})
\]

\[
= O(k^{-1}), \text{ by property } 2,
\]

\[
= o(1).
\]

This completes the proof of Theorem 4.
(E, f) and Valiron summability of bounded sequences

Let (E, f) and (E, g) be regular methods. The first two theorems in this chapter show that if \( p(f) < p(g) \) then the methods are not equivalent, and if \( p(f) = p(g) \) then they are equivalent for bounded sequences.

Theorem 5 For every positive even integer \( p_0 \) there is a bounded divergent sequence which is summable to zero by every regular method \( (E, f) \) with \( p(f) = p_0 \) but is not summable by any regular method \( (E, g) \) if \( p(g) > p_0 \).

Proof Let \( (E, g) \) be a regular method with \( p(g) > p_0 \). Let \( r_n = n^{1/p_0 - 1/p(g)} \). Then \( (r_n) \) is an increasing sequence which tends to \( \infty \). By Theorem 4 there is a bounded divergent sequence \( (s_n) \) such that it is summable to 0 by every regular method \( (E, f) \) with \( p(f) = p_0 \) and

\[
    s_n - s_{n-1} = O(r_n n^{-1/p_0}) = O(n^{-1/p(g)}).
\]

If \( (s_n) \) is summable \( (E, g) \) then corollary(a) to Theorem 3 will imply that it is convergent, which it is not.
Theorem 6 Let $(E, f)$ and $(E, g)$ be regular methods and $p(f) = p(g) = r$. Suppose that $(s_n)$ is bounded. Then $s_n \to S(E, f)$ if and only if $s_n \to S(E, g)$.

The proof of Theorem 6 depends on Wiener's general Tauberian theorem and Lemma 8.

Wiener's general Tauberian theorem If $G$ is integrable on $(-\infty, \infty)$, its Fourier transform does not vanish on $(-\infty, \infty)$, $F$ is integrable, $h$ is bounded, and

$$\int_{-\infty}^{\infty} G(x - t) h(t) dt \to L \int_{-\infty}^{\infty} G(t) dt$$

as $x \to \infty$, then

$$\int_{-\infty}^{\infty} F(x - t) h(t) dt \to L \int_{-\infty}^{\infty} F(t) dt$$

as $x \to \infty$.

Lemma 8 Suppose that $(E, f)$ is regular, $p(f) = r$, $\alpha(f) = \alpha$, and $(s_n)$ is bounded. Then $s_n \to S(E, f)$ if and only if

$$\int_{0}^{\infty} s(t^n) \phi(q \alpha^{1/r}(y - t)) dt \to S \int_{-\infty}^{\infty} \phi(q \alpha^{1/r}t) dt$$

as $y \to \infty$, where $s(x) = s_{[x]}$, $q$ satisfies $1/r + 1/q = 1$, and

$$\phi(x) = \int_{-\infty}^{\infty} \exp(A(f)t^{r} + ict) dt.$$

Proof By Lemma 6 we have $s_n \to S(E, f)$ if and only if

$$\int_{0}^{\infty} (s(\alpha^{1-r}u^q) - S) \phi(q(x - u)) du \to 0 \text{ as } x \to \infty.$$
Making the substitution $t = \alpha^{1/q-1}u$, the preceding condition becomes

$$
\int_0^\infty (s(t^q) - S) \phi(q(x - \alpha^{1/r}t)) \, dt \to 0 \text{ as } x \to \infty, \text{ or}
$$

$$
\int_0^\infty s(t^q) \phi(q\alpha^{1/r}(y - t)) \, dt - S \int_0^\infty \phi(q\alpha^{1/r}(y - t)) \, dt \to 0 \text{ as } y \to \infty.
$$

Since $\phi$ is an even function,

$$
\int_0^\infty \phi(q\alpha^{1/r}(y - t)) \, dt = \int_0^\infty \phi(q\alpha^{1/r}(t - y)) \, dt = \int_{-y}^\infty \phi(q\alpha^{1/r}v) \, dv \to \int_{-\infty}^\infty \phi(q\alpha^{1/r}v) \, dv \text{ as } y \to \infty.
$$

This proves the lemma.

**Proof of Theorem 6** Let $q$ satisfy $1/r + 1/q = 1$. Let

$$
\phi(x) = \int_{-\infty}^\infty \exp(A(f)t^r + ixt) \, dt \text{ and } \psi(x) = \int_{-\infty}^\infty \exp(A(g)t^r + ixt) \, dt.
$$

Then $\phi$ and $\psi$ are integrable and as shown in the proof of Theorem 3 the Fourier transforms of $\phi(q\alpha(f)^{1/r}t)$ and $\psi(q\alpha(g)^{1/r}t)$ have no zeros. Suppose that $(s_n)$ is bounded and $s_n \to S(E, f)$. By Lemma 8 we have

$$
\int_0^\infty s(t^q) \phi(q\alpha(f)^{1/r}(y - t)) \, dt \to S \int_{-\infty}^\infty \phi(q\alpha(f)^{1/r}t) \, dt \text{ as } y \to \infty,
$$

where $s(x) = s[x]$.

This, together with the facts that $s(t^q)$ is a bounded function, that the Fourier transform of $\phi(q\alpha(f)^{1/r}t)$ has no zeros, and that $\psi(q\alpha(g)^{1/r}t)$ is integrable, and
Wiener's general Tauberian theorem imply that
\[ \int_0^\infty s(t^y) \psi(q\alpha(g)^{1/r}(y-t)) \, dt \rightarrow S \int_{-\infty}^\infty \psi(q\alpha(g)^{1/r}t) \, dt \text{ as } y \rightarrow \infty. \]

By Lemma 8, \( s_n \rightarrow S(E, g) \). Clearly the roles of \( f \) and \( g \) in the above argument can be interchanged. This completes the proof.

Theorem 6 generalizes a theorem of Meyer-König (see [8, Satz 25]). For we have the following

**Corollary** The Euler, Borel, Meyer-König, Taylor and Karamata methods are equivalent for bounded sequences.

**Proof** These are methods \((E, f)\) with \( p(f) = 2 \).

To complete our study of the \((E, f)\) summability of bounded sequences we prove that if \( p(f) > p(h) \) then each bounded sequence that is \((E, f)\) summable is \((E, h)\) summable. First we prove a lemma.

**Lemma 9** Let \((E, f)\) and \((E, h)\) be regular methods with \( p(f) > p(h) \). Then \( g = h \circ f \) is well defined on a neighborhood of the closed unit disk. Furthermore \((E, g)\) is a regular method and \( p(g) = p(h) \).

**Proof** That \( g = h \circ f \) is well defined on a neighborhood of the closed unit disk is clear. So are conditions 1, 2, and 3 in Theorem 1 if we replace \( f \) there by \( g \). It remains to prove condition 4 and to find \( p(g) \).
Let $A(f) = B$, $A(h) = C$, $a(f) = \beta$, $a(h) = \gamma$, $p(f) = r$, and $p(h) = s$. Then $\Re B \neq 0$, $\Re C \neq 0$, $\beta = f'(1) > 0$, $\gamma = h'(1) > 0$, and $r > s$. We have

$$f(z) = z^\beta + Bi^r(z - 1)^r + o(1)(z - 1)^r, \ z \to 1,$$

and

$$h(z) = z^\gamma + Ci^s(z - 1)^s + o(1)(z - 1)^s, \ z \to 1.$$ 

Now

$$g(z) = h(f(z)) = (f(z))^\gamma + Ci^s(f(z) - 1)^s + o(1)(f(z) - 1)^s, \ z \to 1$$ 

$$= (f(z))^\gamma + Ci^s(f(z) - 1)^s + o(1)(z - 1)^s, \ z \to 1$$

since

$$(f(z) - 1)^s = \left(\frac{f(z) - 1}{z - 1}\right)^s(z - 1)^s = O(1)(z - 1)^s, \ z \to 1.$$ 

Hence

$$g(z) = (z^\beta + Bi^r(z - 1)^r + o(1)(z - 1)^r)^\gamma$$

$$+ Ci^s(z^\beta - 1 + Bi^r(z - 1)^r + o(1)(z - 1)^r)^s$$

$$+ o(1)(z - 1)^s, \ z \to 1.$$ 

Since $r > s$, we have $(z - 1)^r = o(1)(z - 1)^s, \ z \to 1$. Thus

$$g(z) = (z^\beta + o(1)(z - 1)^s)^\gamma + Ci^s(z^\beta - 1 + o(1)(z - 1)^s)^s + o(1)(z - 1)^s$$

$$= z^{\beta \gamma} + Ci^s(z^\beta - 1)^s + o(1)(z - 1)^s$$

$$= z^{\beta \gamma} + Ci^s\beta^s(z - 1)^s + Ci^s\left\{\left(\frac{z^\beta - 1}{z - 1}\right)^s - \beta^s\right\}(z - 1)^s + o(1)(z - 1)^s$$

$$= z^{\beta \gamma} + Ci^s\beta^s(z - 1)^s + o(1)(z - 1)^s, \ z \to 1,$$
since \((z^\vartheta - 1)/(z - 1) \to \beta\) as \(z \to 1\).

Since \(\Re C \neq 0\) and \(\beta > 0\), we have \(\Re C \beta^* \neq 0\). Hence \(g\) satisfies condition 4 of Theorem 1 and \(p(g) = s = p(h)\). The proof of Lemma 9 is complete.

We include Théorème 4 in [1] here for reference. [1] is in French. The translation is mine.

**Théorème 4 in [1]** Let \((E, f)\) and \((E, g)\) be regular methods. If

(i) the function \(h(z) = g(f^{-1}(z))\), defined in a neighborhood of the point \(z = 1\), is holomorphic for \(|z| < R'\) and satisfies the conditions of Théorème 2 (which are those in our Theorem 2), and if

(ii) \(s_n = O(r^n), n \to \infty; r < R, M(r) < R',\) where the function \(f(z)\) is holomorphic for \(|z| < R\) and where

\[ M(r) = \max |f(z)| \text{ for } |z| = r, \]

then \(s_n \to s(E, f)\) implies \(s_n \to s(E, g)\).

Now we can prove Theorem 7.

**Theorem 7** Let \((E, f)\) and \((E, h)\) be regular methods and \(p(f) > p(h)\). Let \((s_k)\) be a bounded sequence. If \(s_k \to S(E, f)\) then \(s_k \to S(E, h)\).

**Proof** Let \(g = h \circ f\). Then by Théorème 4 in [1], \(s_k \to S(E, f)\) implies \(s_k \to S(E, g)\). By Lemma 9, \(p(g) = p(h)\). By Theorem 6, \((E, g)\) and \((E, h)\) are equivalent.
for bounded sequences. Hence $s_k \to S(E, h)$. This proves Theorem 7.

Notice that Théorème 4 in [1] is an inclusion theorem which may be applied to the summability of unbounded sequences.

A natural question that arises is whether we can replace bounded sequences by unbounded ones in Theorems 6 and 7. The answer is negative.

**Theorem 8** For every pair of positive even integers $r$ and $s$ there exist regular methods $(E, f)$ and $(E, F)$ with $p(f) = r$ and $p(F) = s$ and an unbounded sequence which is summable $(E, f)$ but not summable $(E, F)$.

**Proof** By Chapter II there is a polynomial $f$ such $(E, f)$ is regular and $p(f) = r$. There exists a number $w$ such $|w| > 1$ and $|f(w)| < 1$. Next we can find a polynomial $g$ defined in Chapter II such that $(E, g)$ is regular and $p(g) = s$. Let $4k > s$. Let $h$ be defined by

$$h(z) = z^{2k} - ((z - 1)/2)^{4k}.$$

By Chapter II, $(E, h)$ is regular and $p(h) = 4k$. So $(E, g \circ h)$ is also regular. Since $|w| > 1$ we may assume that $k$ is so large that $|g \circ h(w)| > 1$. Since $4k > s$ it follows from Lemma 9 that $p(g \circ h) = s$. Now let

$$(f(z))^n = \sum_{k=0}^{\infty} a_{nk} z^k \text{ and } (g \circ h(z))^n = \sum_{k=0}^{\infty} b_{nk} z^k.$$

Then we have

$$\sum_{k=0}^{\infty} a_{nk} w^k = (f(w))^n \to 0 \text{ as } n \to \infty, \text{ and}$$
\[ | \sum_{k=0}^{\infty} b_n w^n | = | g \circ h(w) |^n \to \infty \text{ as } n \to \infty. \]

Hence \((w^n)\) is summable \((E, f)\) to 0 but is not summable \((E, g \circ h)\). This completes the proof.

Our final theorem deals with a relation between the general Euler-Borel method and the Valiron method. The definition of the latter is as follows.

Let \(a > 0\). A sequence \((s_k)\) is summable to \(S\) by the Valiron method \(V_a\) if

\[ \left( \frac{a}{2\pi n} \right)^{1/2} \sum_{k=0}^{\infty} s_k \exp \left( -\frac{a(k - n)^2}{2n} \right) \to S \text{ as } n \to \infty. \]

Theorem 9 Let \((E, f)\) be a regular method. Let \(p(f) = 2\) and \(A(f)\) be a real number. Let \((s_k)\) be a bounded sequence. Then \((s_k)\) is \((E, f)\) summable if and only if it is summable by \(V_{\alpha(f)/2|A(f)|}\).

Proof For simplicity we write \(A\) and \(\alpha\) instead of \(A(f)\) and \(\alpha(f)\). Recall that \(\Re A < 0\). Since \(A\) is by hypothesis real, \(A = -|A|\). Since \(p(f) = 2\),

\[ \phi(x) = \int_{-\infty}^{\infty} \exp \left( Ax^2 + izt \right) dt = \left( \frac{\pi}{|A|} \right)^{1/2} \exp \left( \frac{x^2}{4A} \right). \]

As before, let \(s(t) = s_{[t]}\). Since \((s_k)\) is bounded, by Lemma 5, \(s_k \to S (E, f)\) if and only if

\[ x^{-1/2} \int_{0}^{\infty} (s(t) - S) \exp \left( \frac{\alpha(t - x)^2}{4Ax} \right) dt \to 0 \text{ as } x \to \infty. \]
Since

\[ x^{-1/2} \int_0^\infty \exp\left(\frac{\alpha(t-x)^2}{4Ax}\right) dt = \int_{-\infty}^\infty \exp\left(\frac{\alpha u^2}{4A}\right) du \]

\[ \rightarrow \int_{-\infty}^\infty \exp\left(\frac{\alpha u^2}{4A}\right) du = \left(\frac{4\pi |A|}{\alpha}\right)^{1/2} \]

as \( x \to \infty \), it follows that \( s_k \to S(E, f) \) if and only if

\[ x^{-1/2} \int_0^\infty s(t) \exp\left(\frac{\alpha(t-x)^2}{4Ax}\right) dt \to S\left(\frac{4\pi |A|}{\alpha}\right)^{1/2} \text{ as } x \to \infty, \text{ or} \]

\[ \left(\frac{\alpha}{2|A|2\pi x}\right)^{1/2} \int_0^\infty s(t) \exp\left(-\frac{\alpha}{2|A|2x} (t-x)^2\right) dt \to S \text{ as } x \to \infty. \]

Since \( (s_k) \) is bounded we can prove, as in Lemma 4, that the last statement holds if and only if

\[ \left(\frac{\alpha}{2|A|2\pi n}\right)^{1/2} \sum_{k=0}^\infty s_k \exp\left(-\frac{\alpha}{2|A|2n} (k-n)^2\right) \to S \text{ as } n \to \infty. \]

The proof of Theorem 9 is complete.

Recall that \( \alpha(f) = f'(1) \) (see Theorem 1). It is easy to show that when \( p(f) = 2 \) then

\[ A(f) = \frac{(\alpha(f))^2 - \alpha(f) - f''(1)}{2}. \]

Hence we can compute the ratio \( \alpha(f)/2|A(f)| \) for such a method \((E, f)\). We summarize the results that we will use in the following table.
| Summability method       | \( \alpha(f)/2 | A(f) | \) |
|-------------------------|-----------------|
| Borel                   | 1               |
| Euler \((E,q), q > 0\)  | \(1 + 1/q\)     |
| Meyer-König \(S_r, 0 < r < 1\) | \(1 - r\)      |

**Corollary** For bounded sequences the Valiron methods \(V_a, a > 0\), the Euler, Borel, Meyer-König, Taylor and Karamata methods are equivalent.

**Proof** For simplicity, by “equivalent” we mean “equivalent for bounded sequences” in the proof. By the corollary to Theorem 6 all these methods, except the Valiron methods, are equivalent. Also, they satisfy the hypotheses of Theorem 9. Referring to the table above we have, by Theorem 9,

(a) The Borel method is equivalent to the Valiron method \(V_1\).

(b) The Euler method \((E,q), q > 0\), is equivalent to the Valiron method \(V_{1 + 1/q}\).

(c) The Meyer-König method \(S_r, 0 < r < 1\), is equivalent to the Valiron method \(V_{1-r}\).

As \(q\) runs through all positive numbers \(1 + 1/q\) runs through all numbers greater than 1. As \(r\) runs through \((0, 1)\), \(1 - r\) runs through all positive numbers less than 1. Hence by (a), (b), and (c), all the listed methods are equivalent.

This corollary (with the Karamata method omitted from the statement) was first proved in [8]. It has been generalized to sequences of finite order, i.e., sequences satisfying \(s_n = O(n^K)\) for some positive constant \(K\). See [2] for details.
Bibliography


