Control strategies of robotic and neuromuscular bipeds

Ong, Hoo Dennis, Ph.D.
The Ohio State University, 1990

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CONTROL STRATEGIES OF ROBOTIC AND NEUROMUSCULAR BIPEDS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University

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1990
Timmy Lee and My Family
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CHAPTER I

Introduction

In the past few years, a great deal of research has been devoted to the understanding of the control machinery of the central nervous system [1, 2, 3, 4, 5, 6, 7]. Part of the motivation for this research is to apply the knowledge gained to the design of user-specific functional electrical stimulation (FES) systems enabling paraplegics to regain mobility [8, 9]. The feedback controller thus developed can automatically and autonomously regulate the body position and program its propulsion.

A useful method for the analysis and understanding of the subtle and complex machinery of the central nervous system (CNS) is simulation by the digital computer. One part of the method is to arrive at a reasonable model of the process to be simulated. In the past, some work has been devoted to modeling human linkage systems. Hemami and Jaswa [10] modeled a biped in the sagittal plane by a three-link system whose three links corresponded respectively to the leg, the thigh and the torso. Mochon and McMahon [11] developed a biped model consisting of three links; one representing the stance leg and two representing the thigh and the
swinging leg. Onyshko and Winter [12] represented a planar biped by a seven-link model with a three segment lower limb, and the head, arms and trunk included as one segment. All of the above models either assume torque actuators at the joints or no torque actuators at all. In modeling human movements, the dynamics of the muscles play an important and critical role. In terms of neuromuscular biped modeling, Vickers [13] modeled a one-link neuromuscular model with two muscles attached to the link. Hatze [4] modeled the right thigh and the right leg with two links. He also included five muscles in the dynamics with motor unit recruitment and stimulation rate as inputs. Later, Hatze [14] modeled a three-dimensional 17-link biped. His model is the most complicated model to date. Vickers' model is over-simplified as one link cannot represent the coupling between the links. Hatze's model is a physiologically based model but the equations are very complicated for analysis and control.

In this dissertation, a three-link neuromuscular biped model is developed that will take into account the dynamics of the muscles. This model incorporates both single and multi-joint muscles in the dynamics based on the knowledge of their geometry and points of origin and insertion. The position and velocity feedback from the muscles and the viscoelastic behavior of the muscles are incorporated into the dynamics of the biped. Using Lyapunov's second method [15], the feedback gains necessary for the stability of the bipedal system are calculated.

This model is capable of testing different hypotheses about neuromuscular con-
trol by simulation. Two illustrative examples of study by simulation are carried out in this dissertation:

1. Co-activation of agonist and antagonist pairs;


A case of co-activation of agonist and antagonist pairs happens in cycling. The two-joint members of both the hamstrings and the quadriceps are active throughout the movement. However, both the quadriceps and the hamstrings are considered to produce opposite motions at the knee and the hip. The activation of both two-joint muscles in this situation involves additional energy expenditure since the opposing groups work against each other at each joint [16]. The co-contraction is necessary for other purposes such as controlling the stiffness of the joints. Hull and Davis [17, 18] have built a pedal/dynamometer unit that can analyze the cycling process. Hull and Jorge [19] further investigated the functional role the leg muscles play individually in the pedaling process. Soden and Adeyefa [20], using elementary mechanics, estimated the forces that the rider applies to the pedals, saddle and handlebars during hill climbing. The results were compared with force measurements obtained from an instrumented pedal. Gregor, Cavanagh and LaFortune [21] analyzed the functional efficiency of two joint muscles during cycling. They found that a knee flexor moment was consistently observed in all subjects starting approximately halfway through the propulsive phase of crank
There are currently many research going on at different universities on the Functional Electrical System (FES). The main idea of the FES is to apply electric currents to the muscles of the legs or thighs of a paraplegic such that the paraplegic can perform some daily movements such as sitting on a chair and walking. The researchers pre-plan the magnitude and pattern of the electric currents [8] and then apply the currents to the paraplegics. Some paraplegics, equipped with the FES, can walk for thirty minutes one day but can walk for five minutes some other days. This is because the conditions of the patients change everyday, and one set of magnitude and pattern of electric currents may not optimally stimulate the muscles. Therefore, there is a need to change the magnitude and pattern of the electric currents according to the conditions of the paraplegics.

To solve this problem, control strategies are designed to auto-regulate the electric currents that are applied to different muscles.

Specifically, the control strategy has to satisfy the following criteria:

1. The control strategy needs to have a structure that is compatible with the dynamics of the muscles.

2. The control strategy should provide stability in the unconstrained phase, collision phase, and constrained phase.

3. The control strategy should provide a reasonable agreement with the kine-
matic data (angular trajectories) and reaction forces of the gait experiment.

4. The control strategy should match the observations made by physicians and psychologists.

Two daily movements are investigated in this dissertation: sitting and lying on a bed. A paraplegic has to overcome these movements in order to be partially independent.

Since the sitting down motion mostly occurs in the sagittal plane, two three-link planar bipeds are formulated to investigate this motion. The first biped model is a robotic biped with ideal torque generators at the joints. The second biped model is a neuromuscular biped with muscles as actuators for the biped.

In order to consider both the translation motion of the center of gravity and the remaining maneuvers of the biped system about the center of gravity for the lying down on a bed motion, a six link model with both translational and rotational variables can be formulated [22, 23, 24]. Since the phases of lying down do not involve an airborne phase, the above model introduces extensive computation. For this reason, two simpler models are used to represent the two phases of lying down. In the first phase, the right leg does not move. In the second phase, the pelvis remains in contact with the bed but the pelvis is allowed to rotate.
1.1 Scope of the Dissertation

In Chapter II, a history of the dynamics and the control of both robotic and neuromuscular bipeds is given in the form of a survey of past literature. A survey of the cycling motion, sitting down on a chair motion, and lying on a bed motion is also presented.

In Chapter III, the dynamics of the three-dimensional biped model is derived using the state-space Newton Euler method. Muscular Actuators are also developed with special attention given to the properties of muscles against load. Hard and soft contacts are also modeled to yield appropriate contact forces to the biped.

In Chapter IV, the controller of the biped is designed. The stability of the biped with that particular controller is derived using Lyapunov's method both for the biped not in contact with the environment and in contact with environment.

In Chapter V, the controller and the stability criterion that are developed in Chapter IV are applied to the sitting down motion. The simulations are performed for both a neuromuscular biped and a robotic biped. Different control strategies are used to give insights for the 'best' control strategy.

In Chapter VI, the co-activation of agonist and antagonist muscles is investigated with the cycling motion. Specifically, the co-activation of hamstrings and quadriceps is formulated as an optimization problem. Different criteria are chosen for the optimization.
In Chapter VII, the motion of lying on a bed is investigated in detail. The equations of motion of the biped are derived from Chapter III and Appendix A. An experiment is performed when a healthy subject performs the motion. The data obtained in the experiment is filtered and reduced to the required format for simulation.
CHAPTER II

Survey of Previous Work

2.1 Dynamics Formulation

In order to model a multi-link system, the dynamics of that system has to be formulated. Various approaches are available to describe the dynamics, such as Newton-Euler (N-E), and Lagrange-Euler (L-E) methods. In N-E method, there are basically two approaches. The first approach by Luh et al. [25] employs vector multiplications with recursive forward computations for kinematics and recursive backward calculations for moments and forces. The computation of this approach is efficient and is in the order of $O(n)$ where $n$ is the number of links. However, this approach does not provide a closed form differential equation which is useful for advanced control design. The second approach, proposed by Hemami [26], provides a state-space N-E method with a well-structured closed form differential equation. Additionally, this approach can easily accommodate holonomic and nonholonomic constraints. However, this approach is computational intensive but it suitable in designing control laws for complex systems. The are also two approaches in L-E
method. The first approach, as illustrated by Lee [27], provides a structured closed form differential equation based on the L-E equations of motion. The computation is intensive (O(n^4)). The second approach, by Hollerbach [28], attempts to simply the computation by using forward and backward recursion. In this approach, the computation is reduced to O(n) but the generalized velocities are not explicit in the equations of motion. This will hamper the development of advanced control laws. On the whole, the recursive N-E and L-E provide a computationally efficient algorithm but their structures are difficult to implement the control law. On the other hand, the structured N-E and L-E methods provide a closed differential equation form but are very computation intensive.

The state-space N-E is suitable for this dissertation since in modeling a biped, holonomic and nonholonomic constraints may be applied at some instance of time. Moreover, the computation will be done off-line making the efficiency of computation less important.

### 2.2 Modeling and Controlling of Robotic Bipeds

Hemami and Jaswa [10] model a two-dimensional three-link biped whose three links correspond respectively to the leg, the thigh and the torso. They derive the full dynamics of the biped including the coriolis and the centrifugal terms. They also assume the biped is anchored to the ground and the biped can not be airborne. Inverse dynamics method is used to compute the feedforward torques and
linear feedback gains are calculated using the linearized biped model. The biped, with the feedforward and feedback controls, is to track reference trajectories of different motions obtained in gait experiments. The motions that are considered are sitting and standing up, and squatting. Hemami and Wyman [29] also consider the constrained case where the biped is constrained to have both feet on the ground. They design the feedback such that the closed-loop system is constrained and has poles in specified locations.

Mochon and McMahon [11] proposed that the walking motion is basically ballistic and the muscles only provide initial and final velocities for the limbs. Throughout the remainder of the motion, the limbs move under the action of gravity. They model the biped as three links, one for the stance leg and two for the thigh and the leg of the swinging limb. Their simulations compare closely with the experimental gait values with the exception of vertical forces. In a latter paper [30], they try to improve the model to correct the discrepancies between simulations and experiments found in the vertical forces. However, they have been unable to show in the simulations the mechanisms that correct the discrepancies. They postulate some mechanisms that account for the discrepancies.

Onyshko and Winter [12] have developed a seven link biped model with complete three segment lower limbs and the upper body as one segment. The response of their biped depends on the initial angles and velocities of the limbs, and the applied joint moments at the joints. The applied joint moments are obtained from
the gait experiments. However, they have to adjust the joint moments "manually" to attain a desirable gait. Since the biped is unstable, it is logical that the joint moments have to be adjusted to compensate for any instability.

2.3 Modeling and Controlling of Neuromuscular Biped

Vickers' model consists of one link limb with an agonist and an antagonist muscles attached [13]. He employs the force-velocity characteristic to derive the muscular force. He further linearizes the model at some operating point while the stimulation rate of the muscles is the input. After assuming that the time-varying portion of the stimulation is proportional to the position error, he closes the feedback loop. He obtains the responses which are close to the experimental values.

Chow and Jacobson [2] try to optimize human locomotion. They have developed a five link biped with a stance phase and a swing phase. In order to take care of kinematic constraints, the optimizing process is a multiarc programming problem. The minimizing performance criterion is based on certain characteristic of muscles and the criterion is proportional to the mechanical work done during normal walking. A penalty function technique is then employed for iterative numerical solution. Hatze [4] further studies the dynamic behavior of the right leg. A distinctive feature of his research is that the control inputs are the relative number of active fibers in the muscles and the stimulation frequency of the muscles. He models both the dynamics of the muscles and the skeleton in detail. In
his simulations, the optimal process is found to be very close to that in the gait experiment.

2.4 Co-activation in Cycling and Sitting

A case of co-activation of agonist and antagonist pairs happens in cycling. The two-joint members of both the hamstrings and the quadriceps are active throughout the movement. However, both the quadriceps and the hamstrings are considered to produce opposite motions at the knee and the hip. The activation of both two-joint muscles in this situation involves additional energy expenditure since the opposing groups work against each other at each joint [16]. The co-contraction is necessary for other purposes such as controlling the stiffness of the joints. Hull and Davis [17, 18] have built a pedal/dynamometer unit that can analyze the cycling process. Hull and Jorge [19] further investigated the functional role the leg muscles play individually in the pedalling process. Soden and Adeyefa [20], using elementary mechanics, estimated the forces that the rider applies to the pedals, saddle and handlebars during hill climbing. The results were compared with force measurements obtained from an instrumented pedal. Gregor, Cavanagh and LaFortune [21] analyzed the functional efficiency of two joint muscles during cycling. They found that a knee flexor moment was consistently observed in all subjects starting approximately halfway through the propulsive phase of crank rotation.
Sitting down on a chair is a regular and important daily motion, especially for paraplegics. Mussa-Ivaldi, Hogan, and Bizzi [31] and Hogan [5] proposed that a human may modulate the stiffness of the hand for voluntary motion. There have been attempts using various optimizing methods to understand locomotion [2, 4, 32]. However, there is strong disagreement on which optimization criteria should be used. Bernstein [33] and Gelfand, et al. [34] have proposed the concept of hierarchical control. The lower levels handle the small details and the brain is free to make major qualitative decisions. This concept proposed by Bernstein and Gefland is promising but it lacks quantitative proof.

2.5 Lying on a Bed

Lying on a bed is a regular and important motion for paraplegics to regain self-sufficiency. However, there is no research on modeling and controlling this motion. Most of the research has been focused on the motion on walking, jumping, and turning [11, 30, 32]

2.6 Summary

In this chapter, a comprehensive review of the literature in the areas of modeling and control of both a robotic and neuromuscular biped for different motions are presented.
CHAPTER III

The Three-Dimensional Biped Model and Its Environment

3.1 Introduction

The three movements that are studied in this dissertation are sitting on a chair, cycling, and lying on a bed. The first two movements occur mainly in a two-dimensional space. Therefore, a planar biped will be sufficient for the two movements. However, the last movement occurs in a three-dimensional space, and a three-dimensional biped model is needed. As mentioned in Chapter I, two simpler models are formulated both with different constraints imposed on the biped. In this chapter, the biped with the right leg always in contact with the ground is formulated. In appendix A, the biped with the pelvis always in contact with the bed is developed. A two-dimensional biped will not be formulated here since it can easily be derived from either the L-E or N-E method. However, a three-dimensional biped model is more complex and requires detailed formulation. The method that is applied here is the state space N-E method and is based on the
work of Hemami [26]. This method provides a state space approach of formulating the dynamics. Moreover, holonomic and non-holonomic constraints can easily be added to the equations of motion.

A muscle model is used as an actuator model for the biped. Physicians have done a lot of research in investigating the properties of muscles, However, in this chapter, the muscles are examined more in detail from the control point of view. Physical evidence will be investigated as to how the muscles consolidate contraction.

During the movement of the biped, it may contact the environment. The contact can be classified as hard contact or soft contact. Hard contact means that the surfaces of the contact are not compressible. For soft contact, one or both of the surfaces of the contact are compressible. A contact can be modeled as a hard or soft contact depending on the location of the contact on the biped and the hardness of the environment. In this chapter, both hard and soft contacts are modeled.

3.2 The Six Link Biped Model

A six link biped is chosen for the motion of lying on the bed. The six link biped model is shown in figure 1. The links represent different parts of the body as shown below:
Figure 1: The Six Link Biped
Table 1: The Parameters of the Biped

<table>
<thead>
<tr>
<th>i</th>
<th>$m_i \text{kg}$</th>
<th>$d_i \text{m}$</th>
<th>$C_i$</th>
<th>$I_{1i} \text{kgm}^2$</th>
<th>$I_{2i}$</th>
<th>$I_{3i}$</th>
<th>$L_i \text{m}$</th>
<th>$K_i \text{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.796</td>
<td>0.125</td>
<td>0.303</td>
<td>0.0820</td>
<td>0.0820</td>
<td>0.00727</td>
<td>[0 0 0.226]</td>
<td>[0 0 -0.259]</td>
</tr>
<tr>
<td>2</td>
<td>6.399</td>
<td>0.172</td>
<td>0.250</td>
<td>0.0695</td>
<td>0.0695</td>
<td>0.0232</td>
<td>[0 0 0.178]</td>
<td>[0 0 -0.239]</td>
</tr>
<tr>
<td>3</td>
<td>8.695</td>
<td>0.085</td>
<td>0.3</td>
<td>0.0827</td>
<td>0.00770</td>
<td>0.0827</td>
<td>[0 0 0.1625 0]</td>
<td>[0 -0.1625 0]</td>
</tr>
<tr>
<td>4</td>
<td>32.146</td>
<td>0.296</td>
<td>0.252</td>
<td>0.968</td>
<td>0.968</td>
<td>0.551</td>
<td>[0 0 0.329]</td>
<td>[0 0 -0.25]</td>
</tr>
<tr>
<td>5</td>
<td>6.399</td>
<td>0.172</td>
<td>0.25</td>
<td>0.0695</td>
<td>0.0695</td>
<td>0.0232</td>
<td>[0 0 -0.239]</td>
<td>[0 0 0.178]</td>
</tr>
<tr>
<td>6</td>
<td>3.796</td>
<td>0.125</td>
<td>0.303</td>
<td>0.0820</td>
<td>0.0820</td>
<td>0.00727</td>
<td>[0 0 -0.259]</td>
<td>[0 0 0.226]</td>
</tr>
</tbody>
</table>

Link  Body Segment
1   Right foot and leg
2   Right thigh
3   Pelvis
4   Head, arm and torso (HAT)
5   Left thigh
6   Left foot and leg

Those six links are sufficient to represent the movement. The links are connected by ideal ball joints. Joint 1 and 7 represent ankle joints. Joint 2 and 6 represent knee joints. Joint 3 and 5 represent hip joints. Joint 4 represents the connection between the torso and the pelvis. There are physical parameters associated with each link. The physical parameters are as follows: the mass, $m_i$; the 3x3 inertia matrix, $I_i$, about the center of gravity; the lengths, $L_i$ and $K_i$, from the two connecting joints; the diameter of the links, $d_i$; and the coefficient, $C_i$, for computing the moment of inertia about the vertical axis. The parameters are deduced from the technique developed by Drillis and Contini [35] for a male with 61.23 kg weight and with 1.7018 m height. The axes of each link are assumed to be along the principal axes of that link. The moment of inertia about axis three is computed as $m_i(C_i d_i)^2$. The parameters are shown in table 1.
3.3 Phases of Lying Down

Two phases are assumed in this dissertation in the motion of lying down. The first phase is the movement from standing to sitting on the bed. The second phase is the movement from sitting on the bed to lying on the bed.

In the first phase, both feet are in contact with the ground. In the latter part of the first phase, the biped is sitting on the bed. Therefore, in modeling this phase, the right foot of the biped is assumed to be anchored to the ground, and there is always an external force acting on the left foot. When the biped sits on the bed, another external force will be acting on the pelvis.

In the second phase, the feet of the biped may sometimes be in contact with the ground and sometimes be in contact with the bed. However, the pelvis is always in contact with the bed. Therefore, in modeling this phase, the pelvis is assumed to be anchored to the bed. When the biped lies down, there is an external force acting at the trunk of the biped.

The model for the first phase is formulated in this chapter, and the model for the second phase is formulated in appendix A.

3.4 Coordinates and Constraints

3.4.1 Bryant Angles

In order to represent the rotation of each joint, Bryant angles are used [36]. Bryant angles are the angular orientation of the body-coordinate system (BCS) $X^b$ as
represented by a sequence of three rotations at the beginning of which the system coincides with the inertial-coordinate system (ICS) \( X^i \). The first rotation through an angle \( \theta_1 \) is carried out about the axis \( X'_1 \). It results in the auxiliary base \( X^{b''} \).

The second rotation through an angle \( \theta_2 \) about the axis \( X^{b''}_2 \) produces the base \( X^{b'} \).

The third rotation through an angle \( \theta_3 \) about the axis \( X^{b'}_3 \) gives the BCS its final orientation denoted \( X^b \) in figure 2. The transformation equation, \( A: ICS \rightarrow BCS \), to transform from the ICS to the BCS with abbreviations \( c_\alpha = \cos \theta_\alpha, s_\alpha = \sin \theta_\alpha, t_\alpha = \tan \theta_\alpha (\alpha = 1, 2, 3) \) is as follows:

\[
A = \begin{bmatrix}
c_2c_3 & c_1s_3 + s_1s_2c_3 & s_1s_3 - c_1s_2c_3 \\
-c_2s_3 & c_1c_3 - s_1s_2s_3 & s_1c_3 + c_1s_2s_3 \\
s_2 & -s_1c_2 & c_1c_2
\end{bmatrix} \tag{3.1}
\]

To transform from the angular velocity \( (W = [w_1 \ w_2 \ w_3]^T) \) in the BCS to the angular velocity \( (\dot{\theta} = [\dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T) \) in the ICS, we have the following transformation:

\[
W = \begin{bmatrix}
c_2c_3 & s_3 & 0 \\
-c_2s_3 & c_3 & 0 \\
s_2 & 0 & 1
\end{bmatrix} \dot{\theta} \tag{3.2}
\]

To transform back to the ICS, we have the following transformation:

\[
\dot{\theta} = \begin{bmatrix}
s_2 & -s_2 & 0 \\
c_2 & c_2 & 0 \\
-s_3t_2 & s_3t_2 & 1
\end{bmatrix} W \tag{3.3}
\]
or

\[
\dot{\theta} = B(\theta)W \tag{3.4}
\]
Figure 2: The Bryant Angles
3.4.2 Special Notation

The skew symmetric matrix is introduced here to reduce a cross product to a matrix product. For an arbitrary three-dimensional vector, \( V = [v_1 \ v_2 \ v_3]^T \), the skew symmetric matrix \( \tilde{V} \) is defined as

\[
\tilde{V} = \begin{bmatrix}
0 & -v_3 & v_2 \\
v_3 & 0 & -v_1 \\
-v_2 & v_1 & 0
\end{bmatrix}
\] (3.5)

As a consequence, the cross product of two vectors, \( V \) and \( W \) can be expressed as an ordinary product of a matrix and a vector

\[
V \times W = \tilde{V} \cdot W = -\tilde{W} \cdot V
\] (3.6)

3.4.3 Nonholonomic Constraints

Nonholonomic constraints exist if rotation of a body is restricted along certain axes [26]. Let the angular velocity, \( V \), in BCS be parallel to a fixed vector \( Q \) in ICS. Let \( \delta \) be a parameter of proportionality. The nonholonomic constraint is represented as follows:

\[
V = A(\theta)Q\delta
\] (3.7)

Let \( R \) be a constant matrix in ICS whose column(s) are orthogonal to \( Q \), then

\[
R^TQ = 0
\] (3.8)

In the six link biped, rotations of link two and link six are restricted along the second axis (BCS) of the respective link. Therefore, nonholonomic constraints are
respectively as follows:

\[ Q_2 = Q_6 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \quad (3.9) \]

\[ R_2 = R_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \quad (3.10) \]

\[ (3.11) \]

### 3.5 Newton-Euler Equation

With the nonholonomic constraint equations defined above, Newton-Euler equations [26] for the six-link biped can be formulated as:

\[ I_1 \ddot{\mathbf{\theta}}_1 = f_1 + \tilde{K}_1 A_1 \Gamma + A_1 T_1 - \tilde{L}_1 A_1 \Gamma_2 - R_2 A_2 - Q_2 T_2 \quad (3.12) \]

\[ I_2 \ddot{\mathbf{\theta}}_2 = f_2 + \tilde{K}_2 A_2 \Gamma_2 + A_2 A_1^T R_2 \Lambda_2 + A_2 A_1^T Q_2 T_2 - \tilde{L}_2 A_2 \Gamma_3 - T_3 \quad (3.13) \]

\[ I_3 \ddot{\mathbf{\theta}}_3 = f_3 + \tilde{K}_3 A_3 \Gamma_3 + A_3 A_2^T T_3 - \tilde{L}_3 A_3 \Gamma_5 - T_5 - \tilde{K} L_3 A_3 \Gamma_4 - T_4 \quad (3.14) \]

\[ I_4 \ddot{\mathbf{\theta}}_4 = f_4 + \tilde{K}_4 A_4 \Gamma_4 + A_4 A_3^T T_4 \quad (3.15) \]

\[ I_5 \ddot{\mathbf{\theta}}_5 = f_5 + \tilde{K}_5 A_5 \Gamma_5 + A_5 A_3^T T_5 - R_6 A_6 - \tilde{L}_5 A_5 \Gamma_6 - Q_6 T_6 \quad (3.16) \]

\[ I_6 \ddot{\mathbf{\theta}}_6 = f_6 + \tilde{K}_6 A_6 \Gamma_6 + A_6 A_5^T Q_6 T_6 + A_6 A_5^T R_6 A_6 + T_7 + \tilde{L}_6 A_6 \Gamma_7 \quad (3.17) \]

\[ M_1 \ddot{\mathbf{x}}_1 = -[ 0 \ 0 \ m_1 g ]^T + \Gamma_1 - \Gamma_2 \quad (3.18) \]

\[ M_2 \ddot{\mathbf{x}}_2 = -[ 0 \ 0 \ m_2 g ]^T + \Gamma_2 - \Gamma_3 \quad (3.19) \]

\[ M_3 \ddot{\mathbf{x}}_3 = -[ 0 \ 0 \ m_3 g ]^T + \Gamma_3 - \Gamma_4 - \Gamma_5 \quad (3.20) \]

\[ M_4 \ddot{\mathbf{x}}_4 = -[ 0 \ 0 \ m_4 g ]^T + \Gamma_4 \quad (3.21) \]

\[ M_5 \ddot{\mathbf{x}}_5 = -[ 0 \ 0 \ m_5 g ]^T + \Gamma_5 - \Gamma_6 \quad (3.22) \]
\[ M_6 \ddot{x}_6 = - [0 \quad 0 \quad m_6 g]^T + \Gamma_6 + \Gamma_7 \] (3.23)

where
- \( W_i \): the angular velocity of link \( i \) in the BCS
- \( f_i \): the coupling term among the axes
- \( \tilde{K}_i \): the skew symmetric matrix of \( K_i \)
- \( A_i \): the transformation matrix transforming a vector from BCS\(_i\) to ICS
- \( \Gamma_i \): the holonomic constraint force
- \( T_i \): the input torque
- \( L_i \): the skew symmetric matrix for \( L_i \)
- \( Q_i \): the vector or matrix that maps the unrestricted rotation of the link
- \( \Lambda_i \): the nonholonomic constraint torque (\( i = 2 \) and 6 only)
- \( R_2 \): the vector or matrix that is orthogonal to \( Q_i \)
- \( \ddot{x}_i \): the acceleration

and specifically, the contents of the vectors are as follows:

\[
\begin{align*}
f_i &= \begin{bmatrix} w_{i2}w_{i3}(I_{i2} - I_{i3}) & w_{i3}w_{i1}(I_{i3} - I_{i1}) & w_{i1}w_{i2}(I_{i1} - I_{i2}) \end{bmatrix} \\
\Lambda_2 &= \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix} \\
\Lambda_6 &= \begin{bmatrix} \lambda_{61} \\ \lambda_{62} \end{bmatrix} \\
T_1 &= \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} \\
T_2 &= T_{21}; \quad T_3 = \begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix}; \quad T_4 = \begin{bmatrix} T_{41} \\ T_{42} \\ T_{43} \end{bmatrix} \\
T_5 &= \begin{bmatrix} T_{51} \\ T_{52} \\ T_{53} \end{bmatrix}; \quad T_6 = T_{62}; \quad T_7 = \begin{bmatrix} T_{71} \\ T_{72} \\ T_{73} \end{bmatrix}
\end{align*}
\] (3.24)

Equations 3.12 to 3.23 can be compactly represented as follows:

\[
U_1 \ddot{Z} = U_2 + U_3 \Gamma + U_4 + U_5 \Gamma_7 + U_6 [\begin{bmatrix} \Lambda_2 & \Lambda_6 \end{bmatrix}]^T + U_7 T
\] (3.25)

\( U_1 \) is the inertia matrix. \( U_2 \) is the dynamic coupling among different axes. \( U_3 \) is the input map for the holonomic forces, \( \Gamma \). \( U_4 \) is the gravity vector. \( U_5 \) is the input map for the holonomic force of the left foot when it is touching the ground. \( U_6 \) represents the action of the nonholonomic torques, \( \Lambda_2 \) and \( \Lambda_6 \), acting on joint 2.
and joint 6 respectively. $U_7$ is the input map for the actuators. The above matrices
and vectors are listed here:

$$U_1 = \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5 \\
m_6
\end{bmatrix}$$

$$Z = [ W_1 \ W_2 \ W_3 \ W_4 \ W_5 \ W_6 \ \dot{X}_1 \ \dot{X}_2 \ \dot{X}_3 \ \dot{X}_4 \ \dot{X}_5 \ \dot{X}_6 ]^T$$

$$U_2 = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix}; \quad U_4 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
-0
\end{bmatrix}; \quad U_5 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\hat{L}_6 A_6
\end{bmatrix}$$

$$U_5 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
I
\end{bmatrix}$$
3.6 Projection

The biped defined in equation 3.25 has altogether 36 degrees of freedom. However, there are a number of holonomic and nonholonomic constraints acting on the biped thus reducing the number of degrees of freedom. There are 18 holonomic constraints and 4 nonholonomic constraints. The holonomic constraints are the connections between the links. There are nonholonomic constraints at the knee since the leg can only rotate along axis two. The degrees of freedom of the biped are thus reduced to 14.

3.6.1 Holonomic Constraints

For this model, the links are assumed to be connected as shown in figure 3. The left leg is not to be airborne. The left leg is connected to the left thigh. The left thigh is connected to the left end of the pelvis. The torso is connected to the center of the pelvis. The right end of the pelvis is connected to the right thigh. The right thigh is connected to the right leg. Therefore, there are six sets of holonomic constraint equations representing these 18 connections:

\begin{align*}
X_1 + A_1^T K_1 &= 0 & \text{(3.26)} \\
X_1 + A_1^T L_1 &= X_2 + A_2^T K_2 & \text{(3.27)} \\
X_2 + A_2^T L_2 &= X_3 + A_3^T K_3 & \text{(3.28)} \\
X_3 + A_3^T L_3 &= X_4 + A_4^T K_4 & \text{(3.29)}
\end{align*}
\[ X_3 + A_3^T L_3 = X_5 + A_5^T K_5 \quad (3.30) \]
\[ X_5 + A_5^T L_5 = X_6 + A_6^T K_6 \quad (3.31) \]

Differentiate equation 3.26 to 3.31 twice and rearranging, we have

\[ \ddot{X}_1 - A_1^T \ddot{K}_1 \dot{W}_1 = -A_1^T \ddot{W}_1^2 K_1 \quad (3.32) \]
\[ -\ddot{X}_1 + A_1^T \ddot{L}_1 \dot{W}_1 + \ddot{X}_2 - A_2^T \ddot{K}_2 \dot{W}_2 = -A_2^T \ddot{W}_2^2 K_2 + A_1^T \ddot{W}_1^2 L_1 \quad (3.33) \]
\[ -\ddot{X}_2 + A_2^T \ddot{L}_2 \dot{W}_2 + \ddot{X}_3 - A_3^T \ddot{K}_3 \dot{W}_3 = -A_3^T \ddot{W}_3^2 K_3 + A_2^T \ddot{W}_2^2 L_2 \quad (3.34) \]
\[ -\ddot{X}_3 + A_3^T \ddot{K}L_3 \dot{W}_3 + \ddot{X}_4 - A_4^T \ddot{K}_4 \dot{W}_4 = -A_4^T \ddot{W}_4^2 K_4 + A_3^T \ddot{W}_3^2 K L_3 \quad (3.35) \]
\[ -\ddot{X}_4 + A_4^T \ddot{L}_4 \dot{W}_4 + \ddot{X}_5 - A_5^T \ddot{K}_5 \dot{W}_5 = -A_5^T \ddot{W}_5^2 K_5 + A_4^T \ddot{W}_4^2 L_3 \quad (3.36) \]
\[ -\ddot{X}_5 + A_5^T \ddot{L}_5 \dot{W}_5 + \ddot{X}_6 - A_6^T \ddot{K}_6 \dot{W}_6 = -A_6^T \ddot{W}_6^2 K_6 + A_5^T \ddot{W}_5^2 L_5 \quad (3.37) \]

If the above equations are arranged in matrix form, we have

\[ U_3^T \dot{Z} = U_9 \quad (3.38) \]

or decomposing, we have

\[ U_9 \dot{W} + U_{10} \ddot{X} = U_9 \quad (3.39) \]

where

\[
U_9 = \begin{bmatrix}
-A_1^T \ddot{K}_1 & 0 & 0 & 0 & 0 & 0 \\
A_1^T \ddot{L}_1 & -A_2^T \ddot{K}_2 & 0 & 0 & 0 & 0 \\
0 & A_2^T \ddot{L}_2 & -A_3^T \ddot{K}_3 & 0 & 0 & 0 \\
0 & 0 & A_3^T \ddot{K}L_3 & -A_4^T \ddot{K}_4 & 0 & 0 \\
0 & 0 & A_4^T \ddot{L}_4 & -A_5^T \ddot{K}_5 & 0 & 0 \\
0 & 0 & 0 & A_5^T \ddot{L}_5 & -A_6^T \ddot{K}_6 & 0
\end{bmatrix}
\]
In rearranging the above equation, we have

\[ \ddot{X} = -U_{10}^{-1}U_9 \dot{W} + U_{10}^{-1}U_8 \]  

(3.40)

To put the above equation into a state form:

\[ \dot{Z} = \begin{bmatrix} \dot{W} \\ \dot{X} \end{bmatrix} = \begin{bmatrix} I \\ -U_{10}^{-1}U_9 \end{bmatrix} \dot{W} + \begin{bmatrix} 0 \\ U_{10}^{-1}U_8 \end{bmatrix} \]  

(3.41)

or

\[ \dot{Z} = U_{11} \dot{W} + U_{12} \]  

(3.42)

Then, substitute the above equation into equation 3.25, we have

\[ U_1(U_{11} \dot{W} + U_{12}) = U_2 + U_3 \Gamma + U_4 + U_5 \Gamma_7 + U_6 \begin{bmatrix} \Lambda_2 & \Lambda_6 \end{bmatrix}^T + U_7T \]  

(3.43)

Multiply both sides by \(U_{11}^T\) and with \(U_{11}^TU_3 = 0\), we have:

\[ U_{11}^TU_1U_{11} \dot{W} + U_{11}^TU_1U_{12} = U_{11}U_2 + U_{11}^TU_5 \Gamma_7 + U_{11}^TU_6 \begin{bmatrix} \Lambda_2 & \Lambda_6 \end{bmatrix}^T \\
+ U_{11}^TU_7T + U_{11}^TU_4 \]  

(3.44)
In the above projection, the holonomic constraint forces, $U_3$, are eliminated. Therefore, the above equation represents a subspace of the original equation of motion.

### 3.6.2 Nonholonomic Constraints

In the second step of the projection, the nonholonomic constraints will be used to eliminate the $\Lambda$'s. It is assumed that each link can only rotate along its principal axes. The $\theta$'s are represented by some quasi-coordinates, $\phi$'s, as follows:

$$
\theta_1 = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}; \quad \theta_2 = \begin{bmatrix} \phi_4 \\ \phi_3 \end{bmatrix}; \quad \theta_3 = \begin{bmatrix} \phi_5 \\ \phi_6 \\ \phi_7 \end{bmatrix}
$$

$$
\theta_4 = \begin{bmatrix} \phi_8 \\ \phi_9 \\ \phi_{10} \end{bmatrix}; \quad \theta_5 = \begin{bmatrix} \phi_{11} \\ \phi_{12} \end{bmatrix}; \quad \theta_6 = \begin{bmatrix} \phi_{11} \\ \phi_{14} \\ \phi_{13} \end{bmatrix}
$$

The $\phi$'s are the angles that represent the movement of the biped. The $\phi$'s are shown in figure 3. Differentiate equations 3.45 to 3.46 twice, we have

$$
\begin{bmatrix}
\dot{\omega}_{11} \\
\dot{\omega}_{12} \\
\dot{\omega}_{13}
\end{bmatrix} = \begin{bmatrix}
c\phi_2 c\phi_3 \ddot{\phi}_1 + s\phi_3 \ddot{\phi}_2 \\
-c\phi_2 s\phi_3 \ddot{\phi}_1 + c\phi_3 \ddot{\phi}_2 \\
s\phi_2 \ddot{\phi}_1 + \ddot{\phi}_3
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
-s\phi_2 c\phi_3 \ddot{\phi}_1 \ddot{\phi}_2 - c\phi_2 s\phi_3 \ddot{\phi}_1 \ddot{\phi}_3 + c\phi_3 \ddot{\phi}_2 \ddot{\phi}_3 \\
-s\phi_2 s\phi_3 \ddot{\phi}_1 \ddot{\phi}_2 - c\phi_2 c\phi_3 \ddot{\phi}_1 \ddot{\phi}_3 - s\phi_3 \ddot{\phi}_2 \ddot{\phi}_3 \\
c\phi_2 \ddot{\phi}_1 \ddot{\phi}_2
\end{bmatrix}
$$

$$
\begin{bmatrix}
\dot{\omega}_{21} \\
\dot{\omega}_{22} \\
\dot{\omega}_{23}
\end{bmatrix} = \begin{bmatrix}
c\phi_4 c\phi_3 \ddot{\phi}_1 + s\phi_3 \ddot{\phi}_4 \\
-c\phi_4 s\phi_3 \ddot{\phi}_1 + c\phi_3 \ddot{\phi}_4 \\
s\phi_4 \ddot{\phi}_1 + \ddot{\phi}_3
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
-s\phi_4 c\phi_3 \ddot{\phi}_1 \ddot{\phi}_4 - c\phi_4 s\phi_3 \ddot{\phi}_1 \ddot{\phi}_3 + c\phi_3 \ddot{\phi}_4 \ddot{\phi}_3 \\
-s\phi_4 s\phi_3 \ddot{\phi}_1 \ddot{\phi}_4 - c\phi_4 c\phi_3 \ddot{\phi}_1 \ddot{\phi}_3 - s\phi_3 \ddot{\phi}_4 \ddot{\phi}_3 \\
c\phi_4 \ddot{\phi}_1 \ddot{\phi}_4
\end{bmatrix}
$$
The above equations can be compactly written as the follows:

\[ \mathbf{W} = U_{15} \mathbf{\Theta} + U_{16} \]  

(3.53)

where

\[ W = [ \omega_{11} \ldots \omega_{63} ]^T \]

\[ \Theta = [ \phi_1 \ldots \phi_{14} ]^T \]
Figure 3: The Six Link Biped With the Quasi-coordinates, $\phi$'s.
Substitute $\dot{W}$ into equation 3.44, we have

$$
U_{11}^T U_1 U_{11} (U_{15} \dot{\Theta} + U_{16}) + (U_{11}^T U_1 U_{12} - U_{11}^T U_2) - U_{11}^T U_4
= U_{11}^T U_5 \Gamma_7 + U_{11}^T U_6 \left[ \Lambda_2 \Lambda_6 \right]^T + U_{11}^T U_7 T
$$

(3.54)

Multiply by $U_{15}^T$ and with $U_{15}^T U_{11} U_6 = 0$, we have

$$
U_{15}^T U_{11}^T U_1 U_{11} U_{15} \dot{\Theta} + U_{15}^T U_{11} (U_1 U_{11} U_{16} + U_1 U_{12} - U_2)
- U_{15}^T U_{11} U_4 = U_{15}^T U_{11} U_5 \Gamma_7 + U_{15}^T U_{11} U_7 T
$$

(3.55)

Therefore, the nonholonomic constraints are eliminated. The above equation can be represented in a more compact form as:

$$
I \ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1 \Gamma_7 + E_2 T
$$

(3.56)

where

$$
I(\Theta) = U_{15}^T U_{11}^T U_1 U_{11} U_{15}
$$

$$
B(\Theta, \dot{\Theta}) = U_{15}^T U_{11}^T (U_1 U_{11} U_{16} + U_1 U_{12} - U_2)
$$

$$
G(\Theta) = -U_{15}^T U_{11}^T U_4
$$

$$
E_1 = U_{15}^T U_{11}^T U_5
$$

$$
E_2 = U_{15}^T U_{11}^T U_7
$$

The above equation represents the dynamics of a six-link biped. $I(\Theta)$ is the inertia of the system. $B(\Theta, \dot{\Theta})$ contains the coriolis and centripetal vector. $G(\Theta)$
is the gravity. \( E_1 \) is the input map for the contact force, \( \Gamma \) when the left foot of the biped is touching the ground. \( E_2 \) is the input map for the torque generators, \( T \). However, the actuators of the above model are assumed to be ideal torque generators. In the next section, a biped with muscle model as actuators is developed.

### 3.7 Muscular Actuators

In equation 3.56, torque generators are used as actuators for the biped. However, for neuromuscular biped, muscles are used as actuators. Assume that there are a number of muscles acting at the joints and their lengths are represented by a vector \( L \). The forces produced by the muscles are represented by a vector \( F \). Therefore, the incremental work of the muscles is

\[
\partial W = \partial L^T F
\]  

(3.57)

The contribution of this force in the equations of motion is

\[
\frac{\partial W}{\partial \theta} = \frac{\partial L^T}{\partial \theta} F
\]  

(3.58)

The above formulation is valid provided that two conditions are satisfied: (a) the length of the muscle as a function of all the intervening angles is known and that this function is differentiable with respect to all those angles, and (b) no friction is allowed for the muscle at the joints. Therefore, the equations of motion with the
above muscle model is:

\[ I\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1\Gamma_7 + \frac{\partial L^T}{\partial \Theta} F \]  

(3.59)

### 3.8 Contact Forces

Two kinds of external forces are assumed to act on the biped, hard contact forces and soft contact forces. The hard contact force means the contact surfaces of both the biped and the environment are not compliant. The soft contact force means there is a compressible medium at the contact either on the biped or at the environment, or both. For the motions that are considered in this dissertation, both the hard and soft contact forces are encountered. In the lying on bed movement, a hard contact force will be acting on the right foot. In the sitting down motion, a soft contact force is acting on the pelvic when the biped is sitting on the chair.

In equation 3.56, the term \( E_1\Gamma_7 \) is used to describe the hard contact. To include the soft contact force into the dynamic equation, the virtual work method is more convenient. Assume that there is a force, \( F_e \), acting on the biped at location C. Therefore the virtual work that is exerted by the force is:

\[ \partial W = \partial C^T F_e \]  

(3.60)

The contribution of the force in the equation of motion is:

\[ \frac{\partial W}{\partial \Theta} = \frac{\partial C^T}{\partial \Theta} F_e \]  

(3.61)
Therefore, the equation of motion with soft contact force is:

\[ I\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1\Gamma + \frac{\partial C^T}{\partial \Theta} F_e + E_2T \]  \quad (3.62)

### 3.8.1 Hard Contact Forces

Specifically, the hard contact force is computed for the right foot. The ground reaction force of the right foot can be calculated from the states and inputs of the system. If the right foot touches the ground, there is a nonholonomic constraint acting on the biped:

\[ X_7 + A_6^T L_6 = D \]  \quad (3.63)

where \( D \), a constant, represents the tip of the right foot. Differentiate the above equation twice, we have

\[ \ddot{X}_7 + L_6 A_6 \dot{W}_6 = \dot{L}_6 \dot{W}_6 A_6 W_6 \]  \quad (3.64)

To put the above the equation into compact form, we have

\[ U_s^T \dot{Z} = U_{18} \]  \quad (3.65)

Combining equations 3.42 and 3.53 with the above equation, we have the following:

\[ U_s^T U_{11} U_{15} \ddot{\Theta} + U_s^T U_{11} U_{16} + U_s^T U_{12} = U_{18} \]  \quad (3.66)

Combining equation 3.56 with the above equation, we have

\[ U_s^T U_{11} U_{15} (\Theta)^{-1} (-B(\Theta, \dot{\Theta}) - G(\Theta) + E_1\Gamma + E_2T) + U_s^T U_{11} U_{16} + U_s^T U_{12} = U_{18} \]  \quad (3.67)
Rearranging, we have

$$\Gamma_7 = (E_1^T I(\Theta)^{-1} E_1)^{-1}(U_5^T U_{11} U_{15} I(\Theta)^{-1}(B(\Theta, \dot{\Theta}) + G(\Theta) - E_2 T)$$

$$- U_5^T U_{11} U_{16} - U_5^T U_{12} + U_{16})$$  \hspace{1cm} (3.68)

Therefore, the contact force acting on the left foot of the biped can be computed from the above equation.

### 3.8.2 The Soft Contact

The soft contact exists between the pelvis and the chair. It is assumed that a compressible tissue interfaces between the pelvis and the chair. Chow and Odell [37] have proposed such a model by a hemisphere of cast PVC gel. Reddy and his colleagues [38] have proposed a semicircular slab of PVC gel. Both models have exhibited properties of viscoelasticity. Burstein and Frankel [39, 40] have shown that cartilage and biological tissues exhibit viscoelastic properties. The load-deflection curve of the cast PVC gel exhibits the behavior of a nonlinear spring and a dashpot in parallel [37]. Since the contact point between the biped and the chair is viscoelastic, the contact point here is modeled as a spring and dashpot in parallel. It is assumed here that the frictional force at the point of contact is negligible. Therefore, the contact force does not have a tangential component. The viscoelastic normal force, $F_e$, is

$$F_e = k_s C + k_d \dot{C}$$  \hspace{1cm} (3.69)
where \( k_s \) and \( k_p \) are respectively the spring constant and the dashpot constant [7].

The holonomic constraint, \( C \), is the location of the contact and is dependent on the angles \((\Theta_s)\). When \( C < 0 \), the biped is not in contact with the chair. When the biped makes the first contact with the chair, \( C = 0 \). When \( C > 0 \), the medium between the biped and the chair is being compressed. Assuming that the leg and thigh would not move substantially after the biped makes contact with the chair, the constraint, \( C \), in the vicinity of the operating point \( \Theta_0 \), can be approximated by

\[
C(\Theta) = C(\Theta_0) + \frac{\partial C}{\partial \Theta}|_{\Theta_0}(\Theta - \Theta_0)
\]

\[
\frac{d}{dt}(C(\Theta)) = \frac{\partial C}{\partial \Theta}|_{\Theta_0}\dot{\Theta}
\]

Therefore, the viscoelastic force, \( F_v \), can be approximated by:

\[
F_v = -k_s(C(\Theta_0) + \frac{\partial C}{\partial \Theta}|_{\Theta_0}(\Theta - \Theta_0)) - k_d \frac{\partial C}{\partial \Theta}|_{\Theta_0}\dot{\Theta}
\]

3.9 Calculations of Holonomic Forces and Nonholonomic Torques

It is important to calculate the holonomic forces and nonholonomic torques in the biped since there may be a limit to the value of those forces and torques can be.

In order to calculate the nonholonomic torques, the nonholonomic and holo-
nomic constraints are investigated in detail. The nonholonomic constraints are:

\[ R_2^T (A_1 A_2^T W_2 - W_1) = 0 \]  
(3.73)

\[ R_6^T (A_5 A_6^T W_6 - W_5) = 0 \]  
(3.74)

Differentiate twice, we have

\[ R_2^T (-\dot{W}_1 - \dot{W}_1 A_1 A_2^T W_2 + A_1 A_2^T \dot{W}_2) = 0 \]  
(3.75)

\[ R_6^T (-\dot{W}_5 - \dot{W}_5 A_5 A_6^T W_6 + A_5 A_6^T \dot{W}_6) = 0 \]  
(3.76)

Rearranging the above equation, we have

\[
\begin{bmatrix}
-R_2^T & R_2^T A_1 A_2^T & 0 & 0 & 0 & 0 & \vdots & 0 \\
0 & 0 & 0 & 0 & -R_6^T & R_6^T A_5 A_6^T & \vdots & 0
\end{bmatrix} \dot{Z} = \begin{bmatrix}
R_2^T \dot{W}_1 A_1 A_2^T W_2 \\
R_6^T \dot{W}_5 A_5 A_6^T W_6
\end{bmatrix}
\]  
(3.77)

The above equation can be represented as:

\[ U_6^T \dot{Z} = U_{17} \]  
(3.78)

Combining equations 3.38 and 3.78, we have

\[
\begin{bmatrix}
U_3 & U_6
\end{bmatrix}^T \dot{Z} = \begin{bmatrix}
U_6^T & U_{17}^T
\end{bmatrix}^T
\]  
(3.79)

Rearranging equation 3.38, we have

\[ U_1 \dot{Z} = U_2 + U_4 + U_5 \Gamma_7 + \begin{bmatrix}
U_3 & U_6
\end{bmatrix} \begin{bmatrix}
\Gamma \\
\Lambda
\end{bmatrix} + U_7 T \]  
(3.80)
Substitute equation 3.79 into the above equation, we have

\[
\begin{bmatrix} U_3 & U_6 \end{bmatrix}^T U_1^{-1}(U_2 + U_4 + U_5 \Gamma + U_1 U_6 \begin{bmatrix} \Gamma \\ \Lambda \end{bmatrix} + U_7 T) =
\begin{bmatrix} U_8^T \\ U_{17}^T \end{bmatrix}^T
\] (3.81)

To compute the nonholonomic torques and holonomic forces, we have

\[
\begin{bmatrix} \Gamma \\ \Lambda \end{bmatrix} = ([U_3 & U_6]^T U_1^{-1} [U_3 & U_6])^{-1}([U_8^T & U_{17}^T]^T
- [U_3 & U_6]^T U_1^{-1}(U_2 + U_4 + U_5 \Gamma + U_7 T))
\] (3.82)

Therefore, all the holonomic and nonholonomic forces and torques can be computed from the above equation.

3.10 Summary

In this chapter, a three dimensional biped is derived using state space Newton Euler method. This method is proved especially useful in formulating large system. By using holonomic and nonholonomic constraints, the dynamic equation can be reduced from 36 degrees of freedom to 14 degrees of freedom. Muscular actuators are also formulated in this chapter using virtual work principle. The biped’s contact with the environment, resulting in either soft or hard contacts is also formulated in this chapter. As will be shown later, this model can handle the daily maneuvers of lying in bed.
CHAPTER IV

The Stability of The Biped

4.1 Introduction

The dynamics of the biped is very nonlinear and complex, and a lot of the research in controlling the biped are using a linearized model. However, a linearized model will only be valid for a small range of motion and for a slow motion. Therefore, in this chapter, Lyapunov's stability method is used to investigate the stability of the biped using the complete dynamics of the biped. In this chapter, a feedback structure of the biped is derived based on known physiological structure of the natural system. Specifically, the stability of the neuromuscular biped both when not in contact and when in soft contact with the environment is established. The dynamic equations of the neuromuscular biped not in contact with the environment (unconstrained system) are given by

\[ I\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial L^T}{\partial \Theta} F \]  

(4.1)
and the dynamic equations of the neuromuscular biped in soft contact with the environment (constrained system) are given by

\[ I \ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial C^T}{\partial \Theta} F_e + \frac{\partial L^T}{\partial \Theta} F \]  

(4.2)

**4.2 Representation of Natural Feedback**

In order to model the muscle actuators, the interaction between the central nervous system and the muscle has to be investigated. The simplified motor system, as shown in figure 4, can quantitatively show the interactions of the nervous system and the muscle. There are four neural inputs to the bipedal system: apparent passive viscoelasticity, active programmed viscoelasticity, the intended learned movement, and compensation for gravity (the alpha-motorneuron inputs). It is proven in this chapter that the apparent passive viscoelasticity provides the necessary feedback for stability. The active programmed viscoelasticity gives the desired speed for a particular motion. The intended learned movement supplies muscles lengths and velocities, and the alpha-motorneuron inputs compensate for gravity. Each input has its own unique functions. They combine together, and collaborate in order to provide the agility of humans. Stark [41] quantitatively models the basic elements involved in the control of the supinator and pronator muscles of the forearm. There are natural velocity and position feedback from the muscle back to motor neurons. The apparent viscosity of the muscle can be represented as velocity feedback. This can be observed from the idealized force-velocity relationship
of a muscle as shown in figure 5. The force decreases as the velocity increases. A simple approximation is that the force decreases linearly as the velocity increases. Therefore, the apparent viscosity can be modeled by negative velocity feedback. The spindle is connected mechanically in parallel with the muscle. The gamma efferent nerve from the higher centers can contract the spindle fibers, and act as a level setter. Changes in length or in gamma bias, after being modified by the mechanical dynamics of the tendon, fibers, and nuclear bag, produce contractions in the nuclear bag fibers. These contractions then generate nervous signals which are transmitted to the alpha motorneuron by the spindle afferent nerve. Therefore, the spindle can be seen as a position feedback of which the gain can be changed by the higher centers through the gamma efferent nerve. The dynamics of the muscle based on the above argument can be modeled as:

$$ f = -(n + dl), \quad \dot{l} \leq 0, d \geq 0 $$

where $f$ is the force acting at the contact of the muscle with the limb, $n$ is the alpha-motorneuron input to the muscle, $d$ is the velocity gain constant and $l$ is the length of the muscle. Since $l$ is a function of $\Theta$,

$$ l = \left( \frac{\partial l}{\partial \Theta} \right) \dot{\Theta} $$

In a single-joint muscle, $l$ is usually a function of the two angles. In a multi-joint muscle, a larger number of angles are involved in equation 4.4. Suppose
Figure 4: Dynamics at the Spinal Level
Neural Inputs

\[ n_1 > n_2 > n_3 > n_4 \]

Slope: \( b \)

Figure 5: The Force-Velocity of Muscle
a 3-segment biped is considered. The collective effect of all the muscles can be represented in a vector form as:

\[ F = -(N + DL) \] (4.5)

or

\[ F = -(N + D \left( \frac{\partial L}{\partial \Theta} \right) \dot{\Theta}) \] (4.6)

where \( N, F \) are 3x1 vectors representing collective effects of neural inputs and muscles' forces, and \( D \) is a positive definite 3x3 matrix. The apparent viscous matrix \( D \) can be modified by the CNS as follows. If additional negative velocity feedback is supplied by the spindles, the system becomes more viscous. If velocity feedback is positive, the effect is a less viscosity but a more oscillatory and compliant system. Therefore, the active viscous component of \( N \) can be written as:

\[ K_n \left( \frac{\partial L}{\partial \Theta} \right) \dot{\Theta} \] (4.7)

where the diagonal \( K_n \) matrix may have positive or negative components depending on whether positive or negative velocity feedback is intended.

Hogan [42] and Bizzi [1] have shown that muscles exhibit 'spring-like' behavior. The spindle may supply the necessary position feedback. Therefore, consider an equilibrium point \( \Theta_0 \) for the biped, and assuming that all the muscles extend as
ideal springs near the equilibrium point and the changes of length, \( \delta L \), is small, the effects of this passive system of springs on the biped system is:

\[
\frac{\partial L^T}{\partial \Theta} K \delta L = -\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0)
\]

(4.8)

where \( K \) is a 3x3 diagonal matrix and positive definite. The gain \( K \) can also be brought about or modified by the central nervous system through the active gamma efferent nerve to the muscle spindle. Suppose that at least 3 muscles are involved and the corresponding 3 rows of \( \frac{\partial f}{\partial \Theta} \) are linearly independent. The contribution of the muscular system to the biped system can be expressed as:

\[
\frac{\partial L^T}{\partial \Theta} F = -\frac{\partial L^T}{\partial \Theta} N - \left( \frac{\partial L^T}{\partial \Theta} \right) K \left( \frac{\partial L}{\partial \Theta} \right) (\Theta - \Theta_0) - \left( \frac{\partial L^T}{\partial \Theta} \right) D \left( \frac{\partial L}{\partial \Theta} \right) (\dot{\Theta})
\]

(4.9)

To summarize, under negative position and velocity feedback conditions, the force inputs to the biped, due to the muscles, comprises of a negative definite position feedback matrix, a negative definite velocity feedback matrix, and a central input component.

### 4.3 Unconstrained System

Using the feedback structure as derived in the last section, the stability of the biped can be assumed. To prove stability, let the matrices \( K \) and \( D \) in equation 4.9 be constants and not depend on \( \Theta \). Further let \( K \) be selected such that the equilibrium
point $\Theta_0$ is a minimal point for the potential energy of the system. This means
gravity is compensated by storing energy in the stretched muscles. The stability
of the system about the equilibrium point $\Theta_0$ can be established by the Lyapunov
method [43]. Let the Lyapunov function $Q(\Theta, \dot{\Theta})$ be defined as an energy function
consisting of the sum of the kinetic energy, potential energy and the energy stored
in the stretched muscles:

$$Q = \frac{1}{2} \dot{\Theta}^T I(\Theta) \dot{\Theta} + v(\Theta) - v(\Theta_0) - G^T |_{\Theta_0} (\Theta - \Theta_0) + \frac{1}{2} (\Theta - \Theta_0)^T \frac{\partial L}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0)$$

(4.10)

where $v(\Theta)$ is the potential energy of the biped system. With input defined as

$$\frac{\partial L}{\partial \Theta} F = - \frac{\partial L}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0) - \frac{\partial L}{\partial \Theta} D \frac{\partial L}{\partial \Theta} (\dot{\Theta}) + G |_{\Theta_0}$$

(4.11)

It follows that $Q(\Theta_0, 0) = 0$, $Q(\Theta_0, \dot{\Theta})$, and $Q$ is a monotonically increasing func-
tion of $\Theta$ and $\dot{\Theta}$. With $K$ selected such that

$$v(\Theta) - v(\Theta_0) - G^T |_{\Theta_0} (\Theta - \Theta_0) + \frac{1}{2} (\Theta - \Theta_0)^T \frac{\partial L}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0) > 0$$

(4.12)

it can be shown that if the system is disturbed from equilibrium

$$\dot{Q} = - \dot{\Theta}^T \frac{\partial L}{\partial \Theta} D \frac{\partial L}{\partial \Theta} \dot{\Theta} < 0$$

(4.13)

and the system is driven to its unique equilibrium point, $\Theta_0$ where $Q = 0$. Let's
define $M(\Theta)$ to be equal to equation 4.12. $M(\Theta)$ is larger than zero if $M(\Theta)$ is
a convex function. To show this, we proceed as follows. By differentiating $M(\Theta)$
with respect to \( \Theta \),

\[
\frac{\partial M^T}{\partial \Theta} = \frac{\partial v^T}{\partial \Theta} - G|e_o + \frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_o)
\]

(4.14)

and

\[
M(\Theta_o) = 0, \frac{\partial M^T}{\partial \Theta}(\Theta_o) = 0
\]

(4.15)

Furthermore, differentiating equation 4.14 yields

\[
\frac{\partial^2 M}{\partial \Theta^2} = \frac{\partial^2 v}{\partial \Theta^2} + \frac{\partial L^T}{\partial \Theta} \frac{\partial L}{\partial \Theta}
\]

(4.16)

If \( K \) is chosen such that

\[
\frac{\partial^2 v}{\partial \Theta^2} + \frac{\partial L^T}{\partial \Theta} \frac{\partial L}{\partial \Theta} > 0
\]

(4.17)

\( M(\Theta) \) becomes a strictly convex function, and has a global minimum \( M(\Theta_o) = 0 \) at \( \Theta = \Theta_o \). This ensures that \( M(\Theta) \) is larger than zero for all \( \Theta \). Therefore, if equation 4.17 is satisfied, global stability for the unconstrained system is guaranteed.

### 4.4 Constrained System

For the constrained phase, the equation of motion in contact is represented by equation 4.2. The force input can be represented by equation 3.72 and is substituted into equation 4.2. Then equation 4.2 becomes

\[
I(\Theta) \ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial L^T}{\partial \Theta} F
\]

\[
+ \frac{\partial C^T}{\partial \Theta} (-k_e(C(\Theta_0) + \frac{\partial C}{\partial \Theta}|e_o(\Theta - \Theta_0)) - k_d \frac{\partial C}{\partial \Theta}|e_o \dot{\Theta})
\]

(4.18)
Let's define the input $F$, according to equation 4.9, to be:

$$\frac{\partial L^T}{\partial \Theta} F = -\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0) - \frac{\partial L^T}{\partial \Theta} D \frac{\partial L}{\partial \Theta} (\dot{\Theta}) + G|_{\Theta_0} + \frac{\partial C^T}{\partial \Theta} k_s C(\Theta_0)$$  \hspace{1cm} (4.19)

This means there is an additional term in the input when compared with equation 4.11 of the unconstrained phase. The additional term is to compensate for the contact force. Using the same assumption of locality that the leg and the thigh angles will not change substantially after contact, the following approximation can be made:

$$\frac{\partial C^T}{\partial \Theta} \approx \frac{\partial C^T}{\partial \Theta} |_{\Theta_0}$$  \hspace{1cm} (4.20)

The Lyapunov function is defined here as before with an additional term to account for the energy stored by the soft tissue at the contact location:

$$Q = \frac{1}{2} \dot{\Theta}^T I(\Theta) \dot{\Theta} + v(\Theta) - v(\Theta_0) - G^T|_{\Theta_0} (\Theta - \Theta_0) +$$

$$\frac{1}{2} (\Theta - \Theta_0)^T (\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} + \frac{\partial C^T}{\partial \Theta}) |_{\Theta_0} k_s \frac{\partial C^T}{\partial \Theta} |_{\Theta_0} (\Theta - \Theta_0)$$  \hspace{1cm} (4.21)

That additional term is the potential energy of the contact point. The term $k_s$ is selected such that

$$v(\Theta) - v(\Theta_0) - G^T|_{\Theta_0} (\Theta - \Theta_0) + \frac{1}{2} (\Theta - \Theta_0)^T (\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta}$$

$$+ \frac{\partial C^T}{\partial \Theta} |_{\Theta_0} k_s \frac{\partial C^T}{\partial \Theta} |_{\Theta_0} (\Theta - \Theta_0) > 0$$  \hspace{1cm} (4.22)

It can be shown that

$$\dot{Q} = -\dot{\Theta}^T (\frac{\partial L^T}{\partial \Theta} D \frac{\partial L}{\partial \Theta} + \frac{\partial C^T}{\partial \Theta} k_s \frac{\partial C}{\partial \Theta} |_{\Theta_0}) \dot{\Theta} < 0$$  \hspace{1cm} (4.23)
then the system will be driven to its unique equilibrium point from small disturbances. Let's define a new $M(\Theta)$ to be the left hand side of equation 4.22. With the same philosophy as with the unconstrained system, the new $M > 0$ for all $\Theta$ if

$$\frac{\partial^2 v}{\partial \Theta^2} + \frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} + \frac{\partial C^T}{\partial \Theta} \big|_{\Theta_0} k_z \frac{\partial C}{\partial \Theta} \big|_{\Theta_0} > 0$$

(4.24)

This will ensure that the system converges to the desired $\Theta_0$. Therefore, the biped will converge to the desired $\Theta_0$ if equations 4.17 and 4.24 are satisfied respectively for the unconstrained and the constrained phases.

4.5 Summary

In this chapter, using the insights gained from natural system, the general feedback structures that guarantee stability have been derived. Moreover, the stability of the biped both in contact and not in contact with the environment has been derived using the same general feedback structures. The key element in the establishment of stability is provided by Lyapunov theory [43].
CHAPTER V

Control of The Neuromuscular and Robotic Biped During Sitting Down Motion

5.1 Introduction

The ability to sit down is very important for the paraplegic and the disabled who seek to gain partial mobility. In this chapter, this motion is investigated with a neuromuscular biped model and a robotic model. In the neuromuscular biped, three links are used to represent the planar biped. There are muscles acting across different joints as actuators. In the robotic biped, six links are used to represent the planar biped, and ideal torque generators are assumed as actuators at the joints. The controllers used are specific examples of the general feedback strategies described in Chapter IV.

The experimental data for the sitting down motion are taken from Kallel [44]. The experiment consisted of recording the motion of a healthy subject as he sat down on a chair. The recorded data were the kinematics of the body segments, the chair reaction force, and the ground reaction force.
5.2 Neuromuscular Biped

5.2.1 Biped Model

A three link planar biped for sitting down on a chair is shown in figure 6. The three links correspond respectively to the leg, the thigh, and the torso. The parameters of weight, height, and moments of inertia are taken from [44] and are shown below.

<table>
<thead>
<tr>
<th></th>
<th>l(kg.m^2)</th>
<th>M(kg)</th>
<th>l(m)</th>
<th>k(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>link 1</td>
<td>.17</td>
<td>5.05</td>
<td>.401</td>
<td>.21</td>
</tr>
<tr>
<td>link 2</td>
<td>.5</td>
<td>18.67</td>
<td>.412</td>
<td>.26</td>
</tr>
<tr>
<td>link 3</td>
<td>2.1</td>
<td>41.87</td>
<td>.619</td>
<td>.24</td>
</tr>
</tbody>
</table>

The angles at the joints $\Theta = [\Theta_1, \ldots, \Theta_3]$ and $\dot{\Theta} = \frac{d\Theta}{dt}$ constitute the 6 states of the system. The inputs to this system are $m$ muscular forces with anatomically compatible points of origin and insertion. The equations of the unconstrained 3-link biped are:

$$I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial L^T}{\partial \Theta} F$$  \hspace{1cm} (5.1)

where

- $\Theta$: vector of angles at the joint
- $I(\Theta)$: Inertia matrix
- $B(\Theta, \dot{\Theta})$: Centrifugal and Coriolis terms
- $G(\Theta)$: The gravity vector
- $L$: $m \times 1$ vector of length of the muscles
- $\frac{\partial L^T}{\partial \Theta}$: $3 \times m$ matrix of muscular forces input map-vectors of moment arms
- $F$: $m \times 1$ vector of muscular forces
For the constrained motion of the biped, the equations of motion are:

\[ I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial L_T}{\partial \Theta} F + \frac{\partial C_T}{\partial \Theta} F_e \]  

(5.2)

where \( F_e \) is a single point contact force vertical to the chair and \( C \) is the constraint equation describing the contact of the biped with the chair.

5.2.2 Natural Actuators

A large number of leg and trunk muscles are involved in postural movements. In order to develop a general scheme of postural movements, the minimum number of muscles that can independently control the movement is chosen (figure 6). They are three pairs of agonist-antagonist muscles: (1) anterior tibialis for ankle plantarflexion; soleus for ankle dorsiflexion; (2) hamstrings for knee flexion; quadriceps for knee extension; (3) abdominal muscles for hip flexion; paraspinals for hip extension. Gastrocnemius, another two-joint muscle, is not considered here. For the purpose of simplicity, the moment arms of all the muscles are assumed to be one meter. This implies the force equally affects the joints involved. In reality this may not be true, and the forces may have to be rescaled. However, these moment arms are usually a function of the relative position of the limbs [4], therefore, \( \frac{\partial L_T}{\partial \Theta} \) can be more precisely defined. However, for the purposes of our analysis and with the above assumptions, let

\[
\frac{\partial L_T}{\partial \Theta} = \begin{bmatrix}
1 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & -1 & -1
\end{bmatrix}
\]  

(5.3)
where $f_1, f_2, f_3, f_4, f_5$ and $f_6$ are respectively the muscular forces of the tibialis, the quadriceps, the abdominal muscles, the soleus, the hamstrings and the paraspinal muscles. Since the six muscles can be combined into three pairs of agonist-antagonist muscles, $\frac{\partial L^T}{\partial \Theta}$ and $F$ can be represented as:

$$\frac{\partial L^T}{\partial \Theta} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} f_{1,4} \\ f_{2,5} \\ f_{3,6} \end{bmatrix}^T$$

where the positive and negative values of $f_{1,4}$ represent the forces exerted by the tibialis and the soleus respectively. Similarly, the positive and negative values of $f_{2,5}$ represent the forces exerted by the quadriceps and the hamstrings, respectively; and the positive and negative values of $f_{3,6}$ represent the forces exerted by the abdominal and the paraspinal muscles respectively.

5.2.3 The Structure of the Feedback Matrices

From Chapter IV, the input to the unconstrained biped is (equation 4.11):

$$\frac{\partial L^T}{\partial \Theta} F = -\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0) - \frac{\partial L^T}{\partial \Theta} D \frac{\partial L}{\partial \Theta} (\dot{\Theta}) + G|_{\Theta_0}$$

and the input to the constrained biped is (equation 4.19):

$$\frac{\partial L^T}{\partial \Theta} F = -\frac{\partial L^T}{\partial \Theta} K \frac{\partial L}{\partial \Theta} (\Theta - \Theta_0) - \frac{\partial L^T}{\partial \Theta} D \frac{\partial L}{\partial \Theta} (\dot{\Theta}) + G|_{\Theta_0} + \frac{\partial C^T}{\partial \Theta} k_C(\Theta_0)$$
Figure 6: The Biped
The matrices, $K$ and $D$, can be in one of the two broad structures: 1) All elements of $K$ and $D$ are significant; and 2) only the diagonal elements of $K$ and $D$ are dominant. The first structure implies that the feedback to a muscle comes from many other muscles. The second structure implies that the major component of the feedback to an agonist-antagonist pair comes from the same pair of muscles, i.e. the feedback matrices are dominant in their diagonal elements.

It is hypothesized by Nashner and McCollum [6] that postural controls are organized to use a minimum number of muscles. The diagonal structure above is compatible with the minimum hypothesis of Nashner and McCollum in two ways:

1. In a 3-link systems, at most 3 pairs of muscles are needed to guarantee positive definiteness of the position and velocity gain matrices.

2. If the natural viscoelasticity of some muscle pairs are adequate for the task at hand, a smaller subset of 3 pairs of active muscles may be involved.

For certain movements [45, 46, 47], proprioceptive feedback may not be necessary suggesting that the major feedback for those movements can come from the same link. From the above evidence, a diagonal form of feedback is proposed here. The minimum value of position gain, $K$, that can ensure the stability of the biped both in the unconstrained and the constrained phases are calculated from equations 4.17 and 4.24, respectively.
The velocity feedback is not required for stability but is essential for damping or stiffness regulation. The diagonal elements of $K$ (unconstrained) have relatively large values since position feedback can bring the biped to any desired unconstrained position. When the biped sits on the chair, the position feedback by the hamstrings and the quadriceps is sufficient for the stability but the feedback by other muscles are not necessary. This is because the chair provides the necessary support.

5.2.4 Control Strategies

Four different control strategies are investigated here.

1. Position-Position Control Strategy: In this strategy, only the initial and final positions of the movement are specified. Since the control strategy guarantees stability, the biped trajectories should eventually converge to the final position.

2. Multiple-Positions Control Strategy: In this strategy, the biped is given two additional intermediate positions to track between the initial position and

\[
K(\text{unconstrained}) = \begin{bmatrix}
1284.26 & 0 & 0 \\
0 & 1284.26 & 0 \\
0 & 0 & 1284.26
\end{bmatrix}
\]

\[
K(\text{constrained}) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 98.58 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
the final position. Moreover, a feedforward gravity compensation is provided as ramp input connecting the intermediate positions.

3. Reference Position Control Strategy: In this strategy, the biped is to track the reference position trajectories continuously.

4. Reference Position and Velocity Strategy: In this strategy, the biped is to track the reference position and velocity trajectories continuously.

5.2.5 Computer Simulations

To verify the effectiveness of the control strategies, four computer simulations are performed. The reference position and velocity trajectories are taken from the experiment performed by Kallel [44]. The experiment consisted of recording the motion of a healthy subject as he sat down on a chair and stood up from the chair. The recorded data were the kinematics of the body segments and the chair reaction force. Inverse dynamics [48] are used to compute the muscular forces and then compare with the forces in the simulations.

In the simulations, the position feedback gains that are used are:

\[
K(\text{unconstrained}) = \begin{bmatrix}
1284.26 & 0 & 0 \\
0 & 1284.26 & 0 \\
0 & 0 & 1284.26
\end{bmatrix}
\]

\[
K(\text{constrained}) = \begin{bmatrix}
300 & 0 & 0 \\
0 & 600 & 0 \\
0 & 0 & 300
\end{bmatrix}
\]
For the unconstrained case, the minimum gain that is sufficient for stability is used. However, for the constrained case, the minimum gain is too small to give good tracking. Therefore, the gain is increased substantially but is still less than that in the unconstrained case.

For the position-position control strategy, the first two angles track the experimental curves with a delay of 0.4 second. Moreover, the biped is moving much faster as can been seen from the steep slopes of the two angles. However, the third angle cannot follow the experimental curve since the biped is only asked to track the final position but not the position in between. All three angles converge slowly to their respective final values. With the biped sitting down faster than in the experiment, the chair reaction force, as seen in figure 8, is much higher than in the experiment. That means the biped is not slowing down when it is sitting. The muscular forces, as shown in figure 9, have unrealistically large values since the muscles try very hard to pull the biped to the desired final position in a very short time.

For the multiple-positions control strategy, all three angles follow the experimental curves closer than the position-position control strategy. The slopes of the curves are less than that in the position-position control strategy but are still considerably higher than in the experiment. The biped makes temporary contacts with the chair between 1.46 second to 1.55 second as the biped tries to track the desired position. The biped also slows down before it hits the chair, resulting in a
chair reaction force, as shown in figure 11, which is 30% higher than in the experiment. The muscular forces, as shown in figure 12, are more oscillatory and larger in value those in the calculation but are smaller than that in the position-position control strategy.

For the position control strategy, all three angles, as shown in figure 13, track the experimental curves very closely but with a time delay of approximately 0.2 second and a small overshoot. The chair reaction force, as shown in figure 14, is larger than in the experiment which is mainly due to the overshoot. The muscular forces, as shown in figure 15, are generally in good agreement with the calculations.

For the position and velocity control strategy, as shown in figure 16, all three angles show a little overshoot when compared with the reference angles. However, all three angles in the simulations track the reference angles very closely before sitting on the chair. After sitting, the angles converge slowly to the final values. The chair reaction force, as shown in figure 17, has a 10% overshoot and settles down to the experimental values gradually. The muscular forces, as shown in figure 18, are generally in good agreement with the calculations.

5.2.6 Discussions of the Results

Of the four control strategies, the reference position and velocity control strategy has the best agreement with the experimental data and muscular forces calculations. However, this may require too much precision from the CNS. The reference
position control strategy has the second best agreement with the experimental data. For this control strategy, the biped is to track the reference position only. This will definitely require less precision than the first strategy. The multiple-position control strategy has a high and oscillatory chair reaction force. This may be because the biped is tracking only four positions. The position-position control strategy has a very high chair reaction forces and large muscular forces. Also, the biped cannot track the experimental angles closely. Therefore, the conjecture that a human being will let himself/herself drop to the sitting posture without any intermediate information may not be valid.

In summary, of the four strategies analyzed, the reference position and velocity control strategy and the reference position control strategy appear to be more feasible control strategies for the CNS than the other two.
Figure 7: The Angles of the Biped for the Point-Point Motion
Figure 8: The Chair Reaction Force for the Point-Point Motion
Figure 9: The Muscular Forces for the Point-Point Motion
Figure 10: The Angles of the Biped for the Multiple-Points Motion
Figure 11: The Chair Reaction Force for the Multiple-Points Motion
Figure 12: The Muscular Forces for the Multiple-Points Motion
Figure 13: The Angles of the Biped for the Reference Position Motion
Figure 14: The Chair Reaction Force for the Reference Position Motion
Figure 15: The Muscular Forces for the Reference Position Motion
Figure 16: The Angles of the Biped for the Reference Position and Velocity Motion
Figure 17: The Chair Reaction Force for the Reference Position and Velocity Motion
Figure 18: The Muscular Forces for the Reference Position and Velocity Motion
5.3 Robotic Biped

5.3.1 Biped Model

The six-link biped model is shown in figure 19. The six links correspond respectively to the leg, the thigh, the torso, the upper arm, the lower arm and the hand. The parameters of weight, height and moments of inertia are taken from [44] and are given as follows:

<table>
<thead>
<tr>
<th>Parameters of the Biped</th>
</tr>
</thead>
<tbody>
<tr>
<td>I(kg.m²)</td>
</tr>
<tr>
<td>link 1</td>
</tr>
<tr>
<td>link 2</td>
</tr>
<tr>
<td>link 3</td>
</tr>
<tr>
<td>link 4</td>
</tr>
<tr>
<td>link 5</td>
</tr>
<tr>
<td>link 6</td>
</tr>
</tbody>
</table>

The angles at the joints $\Theta = [\Theta_1, \ldots, \Theta_6]$ and $\dot{\Theta} = \frac{d\Theta}{dt}$ constitute the 12 states of the system. The equations of the unconstrained n-link biped are:

$$I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E(\Theta)U \quad (5.9)$$

where

- $\Theta$: vector of angles at the joint
- $I(\Theta)$: Inertia matrix
- $B(\Theta, \dot{\Theta})$: Centrifugal and Coriolis terms
- $G(\Theta)$: The gravity vector
- $E(\Theta)$: Torque input map
- $U$: Applied joint torques

For the constrained motion of the biped, the equations of motion are:

$$I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E(\Theta)U + \frac{\partial C^T}{\partial \Theta} F_c \quad (5.10)$$
where $F_e$ is a single point contact force vertical to the chair and $C$ is the constraint equation describing the contact as before.

### 5.3.2 Control Strategies

Three different control strategies are investigated here.

1. **Position-Position Control Strategy:** In this strategy, only the initial and final positions of the movement are specified. Since the control strategy guarantees stability, the biped should eventually converge to the final position.

2. **Reference Position Control Strategy:** In this strategy, the biped is to track the reference position continuously. The gains of the feedback are to test the possibility of increasing stiffness during sitting down.

3. **Reference Position and Velocity Strategy:** In this strategy, the biped is to track the reference position and velocity continuously. Similar to reference position control strategy, the gains are increased.

### 5.3.3 Computer Simulations

To verify the effectiveness of the control strategies, three set of computer simulations are performed. The minimum position feedback gains for both the unconstrained and constrained phases are computed by equations 4.17 and 4.24. The gains for both phases are found to be the same and are listed as follows:
For the first simulation, the biped is commanded to move from the initial position \( \Theta_i = (5.73°, -7.44°, 8.59°, -170.74°, 167.88°, 181.91°) \) of standing to the equilibrium position of sitting \( \Theta_f = (32.08°, -89.09°, 31.51°, -182.20°, 167.88°, 171.89°) \). The objective of this simulation is to show the stability of the biped both in the unconstrained and constrained phases. For the unconstrained and constrained phases, the position feedback gains are set to be the minimum gain required for stability and the velocity feedback gains are 0.1 times of the position feedback gains. The trajectory of motion of the end-point of the second link is shown in figure 20. The biped hits the chair at 0.53 sec. The biped bounces first downward and then upward before settling down to the desired value. The overshoot occurs because the biped is not commanded to slow down before it hits the chair. The chair reaction force \( F \), as shown in figure 21, is positive all the time after the contact indicating that the biped remains in contact with the chair after sitting. The angles of the first three joints, which are respectively the angle from the vertical to the leg, the angle from the vertical to the thigh, and the angle from the vertical to the torso, are shown in figure 22. The biped converges to the desired position slowly. The Lyapunov function is a measure of the kinetic and potential energies of the biped. If the Lyapunov function is decreasing with respect to time,
the biped is stable. The Lyapunov function, as shown in figure 23, is divided into the unconstrained and constrained phases. For the unconstrained segment, the Lyapunov function decreases steadily until the biped hits the chair. At the moment of sitting on the chair, the Lyapunov function changes to a very large value due to the contact force. However, the Lyapunov function starts to increase gradually at 0.67 second. This is because equation 4.20 does not hold for the first angle. However, after the first angle increases gradually to the desired equilibrium point, the Lyapunov function starts decreasing at 0.8 second and gradually decays to zero. The derivative of the Lyapunov function shows the rate of decay of the Lyapunov function. If the derivative is negative all through the simulation, then the biped is stable. The derivative of the Lyapunov function with respect to time, as shown in figure 24, is negative all through the unconstrained and constrained phases.

In the second simulation, the input to the biped system is the angular positions measured in the experiment. Therefore, the biped is commanded to track the angular position trajectory from the experiment. The position feedback gain is also set to the minimum necessary for stability and velocity feedback gain is 0.1 times the position gain in the unconstrained stage. The purpose of this simulation is to verify whether the biped can only use the angular position information in tracking the sitting down motion. Moreover, three position and velocity feedback gains in the constrained stage are employed to verify whether the biped uses dif-
ferent gains in sitting down. The position feedback gains used are respectively one
time (gain=1), twice (gain=2), and four times (gain=4) the minimum needed for
stability. The velocity gains are 0.1 times the position feedback gains used. The
graphs of the first three angles, figure 25, with the gain of one, the biped is sliding
on the chair. When the gains are higher, more rapid oscillations appear. A more
clear picture can be seen from the graph of the end-point of the second link in
figure 26. As the gain increased, the biped sits on the chair with less overshoot
but eventually leaves the chair after 0.8 seconds. This also can be seen in the
chair reaction force in figure 27. As the gain increases, the overshoot decreases.
However, the biped does not remain on the chair very long and begins to leave the
chair. Therefore, it is shown in the simulation that the biped cannot perform the
sitting down motion by only tracking the angular positions. Moreover, increasing
the feedback gains cannot improve the tracking.

For the last simulation, the biped is not only tracking the angular positions
but also the angular velocities. The feedback is then equal to:

\[- k_p(\Theta - \Theta_r) - k_v(\dot{\Theta} - \dot{\Theta_r})\]  \hspace{1cm} (5.11)

where \(\Theta_r\) and \(\dot{\Theta}_r\) are the angular position and angular velocities trajectories
recorded in the experiment. Since during the motion of sitting down, the an-
gular velocities change substantially, tracking the angular velocities seems to be
required. The position feedback gains are adjusted respectively to 1 times the min-
imum for stability (gain=1), 8 times the minimum for stability (gain=8), and 40 times the minimum for stability (gain=40). The velocity gains for all three cases are 0.1 times the value of the position gains. The angles of the first three joints are shown in figure 28. The one with higher gains tracks better than the other, and the one with lower gains has more overshoots. One important criterion to evaluate the feedback strategy is to examine the chair reaction force as shown in figure 29. In order to quantitatively differentiate which control strategy tracks the chair reaction force better, a root-mean-square (rms) error comparison is made between the curve in the simulation with the curve in the experiment. The rms error for gain=1 is 150.41, for gain=8 is 114.05, and for gain=40 is 64.52. From the rms criteria, the control strategy with higher feedback gains tracks the experimental results better. This is also true for the constraint which is shown in figure 30.

5.3.4 Discussions of the Results

From the above simulations, when the position feedback is set to one time the minimum gain required for stability, the feedback strategy that tracks the experimental angular positions and angular velocities shows the closest approximation to the experimental results. Physically, this means that the biped tracks both the angular positions and angular velocities when it is sitting down on a chair. Moreover, it is shown that when the position feedback gain for the third simulation is increased, there is a better approximation to the experimental result.
5.4 Summary

In this chapter, the sitting down motion is investigated in both a neuromuscular and a robotic biped. Four control strategies are investigated for the neuromuscular biped. The reference position and velocity control strategy has the best agreement with the experiment data and muscular forces calculations. Three control strategies are developed for the robotic biped. The gains of the control strategies are also adjusted to test the possibility of increasing stiffness during sitting down. In the simulations, the control strategy that tracks the experimental angular positions and angular velocities shows the closest approximation to the experimental results. Moreover, increasing the gains for that strategy also results in improved tracking of the experimental results.
Figure 19: The biped
Figure 20: The Trajectory of Motion of the End-point of the Second Link in Point-to-Point Movement
Figure 21: The Chair Reaction Forces in Point-to-Point Movement
Figure 22: The Angles of the Biped in Point-to-Point Movement
Figure 23: The Lyapunov Function in Point-to-Point Movement
Figure 24: The Derivative Lyapunov Function in Point-to-Point Movement
Figure 25: The Angles of the Biped in Tracking Angular Positions
Figure 26: The Trajectory of Motion of the End-point of the Second Link in the Tracking Angular Positions
Figure 27: The Chair Reaction Force in Tracking Angular Positions
Figure 28: The Angles of the Biped in Tracking Angular Positions and Velocities
Figure 29: The Chair Reaction Force in Tracking Angular Positions and Velocities
Figure 30: The Trajectory of Motion of the End-point of the Second Link in the Tracking Angular Positions and Velocities
CHAPTER VI
Co-activation in Cycling

6.1 Introduction

The co-activation of agonist and antagonist muscles is common for many movements of humans. In terms of the energy point of view, co-activating of agonist and antagonist muscles would mean a waste of energy. However, the real reasons for co-activation remain obscure. In this chapter, some simulations are performed on cycling to investigate this phenomenon. The basic hypothesis is that the human being will try to optimize certain criteria by co-activation. In this chapter, two different criteria are explored. The biped model to be analyzed is described in section 6.2. The actuators are discussed in section 6.3. Two optimization criteria are explored in section 6.4.

6.2 Biped Model

The equation of a general n link three-dimensional biped [23] is

\[ I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = \frac{\partial L^T}{\partial \Theta} F + \frac{\partial C^T}{\partial \Theta} F_e + DM \]  

(6.1)
where

\[ \Theta: \text{ vector of angles at the joint} \]
\[ I(\Theta): \text{ Inertia matrix} \]
\[ B(\Theta, \dot{\Theta}): \text{ Centrifugal and Coriolis terms} \]
\[ G(\Theta): \text{ The gravity vector} \]
\[ L: \text{ vector of lengths of the muscles} \]
\[ \frac{\partial L}{\partial \Theta}: \text{ matrix of muscular force input map} \]
\[ F: \text{ vector of muscular forces} \]
\[ C: \text{ contact point with the environment} \]
\[ F_e: \text{ the forces at the end-point of the foot} \]
\[ D: \text{ input matrix for the moment } M \]

As described before, the formulation of equation 6.1 accepts multi-joint muscles that extend an arbitrary number of joints provided that two conditions are satisfied:

a) The length of the muscle as a function of all the intervening angles is known and that this function is differentiable with respect to all its angular components, and

b) No friction is allowed for the muscle at the joints. With these two assumptions, the incremental work of the multi-joint muscle is, as described before,

\[ dw = f dl \]  \hspace{1cm} (6.2)

where \( f \) is the force produced by the muscle. The contribution of this force in the equations of motions is

\[ \frac{dw}{d\Theta} = \frac{dl}{d\Theta} f \]  \hspace{1cm} (6.3)

An important issue regarding muscular function is the simultaneous activation of a pair of agonist-antagonist muscles. This issue is investigated here by a planar biped. A three link planar biped and the pedal forces are shown in figure 31. The three links correspond to the thigh, the leg and the foot. The foot exerts a vertical force and a horizontal force on the pedal of the bicycle. For simplification, the foot
is assumed to be parallel to the ground for the whole cycle of pedaling. Therefore, the foot can be incorporated into the leg. Thus, there are altogether two links as shown in figure 32. Moreover, the vertical force acting on the foot will cause a moment, \( M \), on the leg. Therefore, for the two-link biped, there are one moment and two forces acting on the leg. The parameter of the weight, the height and the moments of inertia for this model are taken from [21, 35] and are shown below.

<table>
<thead>
<tr>
<th>Parameters of the Biped</th>
</tr>
</thead>
<tbody>
<tr>
<td>link 1</td>
</tr>
<tr>
<td>I(kg.m^2)</td>
</tr>
<tr>
<td>M(kg)</td>
</tr>
<tr>
<td>l(m)</td>
</tr>
<tr>
<td>k(m)</td>
</tr>
<tr>
<td>link 2</td>
</tr>
<tr>
<td>I(kg.m^2)</td>
</tr>
<tr>
<td>M(kg)</td>
</tr>
<tr>
<td>l(m)</td>
</tr>
<tr>
<td>k(m)</td>
</tr>
</tbody>
</table>

### 6.3 Muscle-like Actuators

In performing cycling, a large number of leg and trunk muscles are available. In order to develop a general scheme of postural control, the minimum number of muscles that can independently control the movement is chosen (Figure 32). Two pairs of agonist-antagonist muscles are selected similar to chapter V: (1) hamstrings for knee flexion; quadriceps for knee extension; (2) hip flexors; hip extensors. Therefore, \( \frac{\partial L^T}{\partial \Theta} \) and \( F \) are defined as:

\[
\frac{\partial L^T}{\partial \Theta} = \begin{bmatrix}
0 & 0 & -d_5(\Theta) & d_5(\Theta) \\
-d_1(\Theta) & d_2(\Theta) & 0 & 0
\end{bmatrix}
\]

(6.4)

\[
F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}^T
\]

(6.5)
where $d_1(\Theta), d_2(\Theta)$, and $d_3(\Theta)$ are moment arms of the quadriceps, the hamstrings, the hip flexors, and the hip extensors respectively. The forces, $f_1, f_2, f_3$ and $f_4$, all positive, are respectively the muscular forces of the quadriceps, the hamstrings, the hip flexors and the hip extensors. Since the last two moment arms are assumed to be equal, these muscles can be combined into a pair of agonist-antagonist muscles. Therefore, $\frac{\partial L^T}{\partial \Theta}$ and $F$ can be simplified as:

$$\frac{\partial L^T}{\partial \Theta} = \begin{bmatrix} 0 & 0 & -d_3(\Theta) \\ -d_1(\Theta) & d_2(\Theta) & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 \\ f_2 \\ f_{3,4} \end{bmatrix}^T$$

where positive and negative values of $f_{3,4}$ represent the muscular force exerted by the hip flexors and the hip extensors muscles respectively.

### 6.4 Optimization Criteria

In order to calculate the muscular forces, equation 6.1 is rearranged as:

$$\frac{\partial L^T}{\partial \Theta} F = (I(\Theta)\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) - \frac{\partial C^T}{\partial \Theta}F_e - DM)$$  \hspace{1cm} (6.8)

Since we are interested in the agonist-antagonist action of the hamstrings and the quadriceps, only the values of $f_1$ and $f_2$ will be investigated. Let the right side of equation 6.8 be:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
Therefore,

\[ HF_{1,2} = u_2 \]  \hspace{1cm} (6.9)

where

\[ H = \begin{bmatrix} -d_1(\Theta) & d_2(\Theta) \end{bmatrix} \]

and

\[ F_{1,2} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \]

In order to compute the values of \( f_1 \) and \( f_2 \), two optimizing criteria are explored here. One criterion is to minimize the muscular forces. Using a pseudoinverse approach [49], the minimum norm of the forces are calculated as:

\[ F_{1,2} = H^T(HH^T)^{-1}u_2 \]  \hspace{1cm} (6.10)

The corresponding muscular forces are shown in figure 33. For the first 0.2 sec, the quadriceps are active. However, from 0.2 sec to 0.6 sec, the hamstrings are active. For the rest of the time, the quadriceps are active again. However, from the EMG graph of figure 34 [21], both the quadriceps and hamstrings are active in the first 0.3 second of the motion.

An alternative optimization criterion is the stiffness of the leg which can be represented by:
\[ \frac{f_1}{f_2} + \frac{f_2}{f_1} \] (6.11)

The above criterion basically represents the balance of forces between the quadriceps and hamstrings. The corresponding forces are shown in figure 35. The quadriceps force compares closely with the EMG data [21]. However, the hamstrings force does not compare well. This may be due to the fact that additional functions have to be included into the optimization function since controlling stiffness may be just one of the functions of co-activation.

6.5 Summary

In this chapter, the co-activation of quadriceps and hamstrings muscles is investigated. Two optimizing criteria are proposed. The first one is to minimize muscular forces. However, the forces do not compare well with experimental results. The second one is to minimize a stiffness function, i.e. a symmetric sum of the ratio of the forces of the leg, and the forces compare better with the experimental results. The investigations of this chapter are inconclusive. The major result is that optimal control solutions of the biped movement with the proper performance criterion lead to solutions that involve co-activation.
Figure 31: The Three-Link Biped
Figure 32: The Two-Link Biped
Figure 33: The Muscular Forces of the Quadriceps and the Hamstrings
Figure 34: Mean Joint Torque Patterns for the Hip and the Knee and Average EMG Pattern for Quadriceps above 0% and for Hamstrings below 0% [21]
Figure 35: The Muscular Forces of the Quadriceps and the Hamstrings
CHAPTER VII

Control Strategies of the Biped During Lying on the Bed Motion

7.1 Introduction

The motion of lying down on a bed is an important motion and has not been previously analyzed. In this chapter, the motion is analyzed, and an experiment for lying on bed is carried out. Simulations of this motion are also performed to give viable control strategies and torque profiles. The gait experiment is described in section 7.2. The biped model is discussed in section 7.3. The control strategies are developed in section 7.4. Computer simulations for the motion are presented in section 7.5.

7.2 Gait Experiment

The gait laboratory, as shown in figure 36, is a clinical and research laboratory at the Ohio State University designed to study the human walking pattern. The gait experiment consists of recording the motion of a healthy subject as he lies on a bed. The recorded data consist of: 1) the kinematics of the body segments, and
2) the ground and bed reaction forces. The motion was performed by a 28 year old male. His height and weight are respectively 1.73 m and 61.23 kg. The subject does not have any current or previous disease or injury of the locomotion system.

7.2.1 Equipment

The following equipment are available in the gait laboratory as components of the Vicon Gait Analysis System:

1. Six Oxford Metric video cameras

2. A PDP-11 computer for collection of video images

3. A Vax 780 for sorting and analysis of data

4. Textronix and Mac II instruments for graphics display

5. Two AMTI force-plates for measuring forces and torques

6. An eight channel EMG analyzer for measuring emg signals

The Vicon system is used to collect and process the force and trajectory data. All six cameras are used for the experiment. There are 21 retro-reflective markers taped onto the subject. The locations of the cameras with respect to the subject are shown in figure 37. The locations of the markers are shown in figure 38. The locations of the markers as detected by the six video cameras are digitally stored and transferred to the PDP-11 computer. Therefore, for any posture, there are six
2-D images involved. To reduce the six 2-D images to a one 3-D image, a particular software package in the Vicon system will merge the 2-D images into 3-D images.

Two AMTI Biomechanic Platforms (BMP) Model OR6-5-6 are used to make simultaneous measurements of the ground reaction forces and the bed reaction force. The two force plates can measure three forces and three torques at the same time. However, in this experiment, only the forces are used. All signals are sampled at 200 Hz.

### 7.2.2 The Kinematics of Lying Down

In order to characterize the motion more easily, the motion is divided into two phases. The first phase corresponds to the motion of sitting on the bed as shown in figure 39. The second phase can be divided into three sub-phases. The first sub-phase (phase 2.1) occurs when the torso is turned roughly 70 degrees to the right and simultaneously the right foot is lifted and placed on the bed (figure 40). Therefore, the right foot is in contact with the bed, and the pelvis is in contact with the bed, and the left foot is in contact with the ground. In phase 2.2, the torso turns an additional 20 degrees to the right (figure 41). The right foot and the leg remain almost stationary and the point of contact of the right foot with the bed does not change. Simultaneously, the left foot is lifted to the bed. Therefore, the left foot, the right foot, and the pelvis are all in contact with the bed. In phase 2.3, the torso rotates 90 degrees and lies on the bed (figure 42). In all the phases
and sub-phases, the hands are assumed to be not in contact with the bed and the ground.

7.3 The Dynamics of Lying Down

Two biped models are used to model the two phases. For the first model, the right foot of the biped is assumed to be anchored to the ground. There may be a reaction force acting on the left foot. This model encompasses phase 1. For the second model, the pelvis is assumed to be anchored to the bed, and the left and the right feet are not anchored to the ground. The model is used in phase 2. The equations of motion for the first model are derived in chapter III (equation 3.56):

\[ I\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1 \Gamma_7 + E_2 T \]  \hspace{1cm} (7.1)

The equations of motion for the second model are derived in appendix A (equation A.46):

\[ I\ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1 \Gamma_1 + E_2 \Gamma_{4a} + E_3 \Gamma_7 + E_4 T \]  \hspace{1cm} (7.2)

All the contacts are assumed to be hard contacts. In the following section, the control strategies for this motion are developed.

7.4 Control Strategies for Lying on the Bed Motion

In chapter V, it was shown that the reference position and velocity control strategy and the reference position control strategy were feasible bipedal control strategies.
Figure 36: The Gait Analysis Laboratory
Figure 37: The Locations of the Video Cameras
Figure 38: The Locations of the Markers on the Subject
Figure 39: Kinematics for Phase 1
Figure 40: Kinematics for Phase 2.1
Figure 41: Kinematics for Phase 2.2
Figure 42: Kinematics for Phase 2.3
However, the reference position and velocity control strategy requires too much attention from a human. Therefore, the reference position control strategy is chosen to be the control strategy used for lying down. As in Chapter V, the input to the biped for phase 1 is:

\[
T = E^{-1}_2(-K(\Theta - \Theta_0) - D\dot{\Theta} + G|\theta_0 + E_1\Gamma_{7d})
\] (7.3)

and the input to the biped for phase 2 is:

\[
T = E^{-1}_4(-K(\Theta - \Theta_0) - D\dot{\Theta} + G|\theta_0 + E_1\Gamma_{1d} + E_2\Gamma_{4ad} + E_3\Gamma_{7d})
\] (7.4)

where \(\Theta_0\) is the desired angular positions, \(\Gamma_{1d}\) is the desired contact force at the right foot, \(\Gamma_{4ad}\) is the desired contact force at the left foot, and \(\Gamma_{7d}\) is the desired contact force at the torso. The controller as described above is a PD controller with gains as determined by the method in Chapter III.

7.5 Computer Simulations

Two computer simulations are performed respectively for the two phases of this motion. The first phase lasts for 1.53 seconds and the second phase lasts for the remaining time. Three criteria are selected to measure the performance for the control strategy.

1. Angular position trajectories of all the links should follow the reference trajectories.
2. Computed reaction forces at the ground and on the bed should match the measurements.

3. Theoretical input torques at all the joints should match the subject's torques computed from the data.

The experimental angular positions and reaction forces are collected at the Gait Laboratory during the motion. The experimental input torques are very difficult to measure directly. Instead, they are calculated by the following equations:

For the first phase,

\[ T = E^{-1}_2(D\dot{\Theta}_0 + G|\Theta_0 + E_1\Gamma_{7d}) \]  

(7.5)

For the second phase

\[ T = E^{-1}_4(D\dot{\Theta}_0 + G|\Theta_0 + E_1\Gamma_{1d} + E_2\Gamma_{4ad} + E_3\Gamma_{7d}) \]  

(7.6)

The desired angular velocity is \( \dot{\Theta}_0 \) in both phases.

7.5.1 The First Phase-Sitting on the Bed

The first phase is basically the sitting down motion just before the biped settles on the bed. The biped follows the experimental angular positions closely for the first 0.5 second. When the biped begins to sit, the angular positions have a time delay ranging from 0.1 second to 0.2 second (figures 43 to 47). The forces acting at the left foot are also computed and compared with experimental data (figure 48). The
force in the x-direction follows the experimental data closely for the first 0.5 second, and the force diverges from the experimental data for the next 0.6 second. Finally, for the last 0.4 sec, the simulation force converges back to the experimental data. The force in the y-direction follows closely the experimental data for almost 1.2 second. Then the force begins to diverge from the experimental curve for the rest the motion in phase 1. The force in the z-direction follows closely the experimental force except from 0.8 second to 1.2 second. The input torques generally have higher values than the experimental values (figure 49). This discrepancy is mainly due to the difference of experimental and simulation angular positions and velocities.

7.5.2 The Second Phase

For the second phase, angle 1 tracks the experimental value well until 3.2 second, and it is tracking it again at 3.7 second. For the rest of the angles, they track the experimental values closely with a time lag of 0.1 to 0.3 seconds. The figures of the angles are shown in figures 54 to 57.

When the right foot is on the ground (1.53 second to 1.78 second), the biped is commanded to track the experimental value as measured in the gait laboratory during that time interval. When either the left foot or the right foot is in contact with the bed, the bed reaction force is \([0, 0, 4.88]\) newtons as measured statically during the gait experiment. When the torso lands on the bed, the experimental bed reaction force is \([0, 0, 22.15]\) newtons.
The x-direction ground force (figure 58) in the simulation at the right foot resembles the experimental value from 1.53 second to 1.80 second. There are some differences occurring after 3.1 second. The y-direction ground force at the right foot does not match the experimental curve that well. The z-direction ground force at the right foot resembles the experimental value from 1.53 second to 1.80 second. There are also small differences occurring after 3.1 second.

The x-direction ground force (figure 59) at the left foot is 10 to 30 newtons lower than the experimental force from 1.53 second to 3.00 second. The simulation force is larger in magnitude and in the opposite direction to the experimentally measured force. The y-direction ground force at the right foot does not match the experimental curve very well. The z-direction ground force at the right foot resembles the experimental value from 1.53 second to 2 second. From 2 second to 3.4 second, there is a discrepancy of 100 newtons between the simulation force and the experimental forces. For the torso (figure 60), only the force in the z-direction is computed. The simulation force tracks the experimental force well at the beginning but the simulation force decreases steadily until the end of the simulation. The input torques generally have higher values than the experimental values (figure 61). This discrepancy is due mainly to the difference of experimental and simulation angular positions and velocities.
7.6 Summary

In this chapter, the motion of lying down is monitored in the gait laboratory. The input torques and angular positions were recorded for a single maneuver. This motion is also simulated with the reference position control strategy. With this strategy, the simulation results for the angles and the bed reaction forces generally track the experimental values although at some intervals of time, they differ from the experimental values. The input torques generally have higher values than the experimental values. This discrepancy is due mainly to the difference between experimental and simulation angular positions and velocities. To improve the simulation results for the input torques, feedback gains may have to be increased.
Figure 43: Angle 1 to 3 for Phase 1
Figure 44: Angle 4 to 6 for Phase 1 (cont'd)
Figure 45: Angle 7 to 9 for Phase 1 (cont'd)
Figure 46: Angle 10 to 12 for Phase 1 (cont'd)
Figure 47: Angle 13 and 14 for Phase 1 (cont’d)
Figure 48: The Forces Acting at the Left Foot
Figure 49: Input Torques 1 to 3 for Phase 1
Figure 50: Input Torques 4 to 6 for Phase 1
Figure 51: Input Torques 7 to 9 for Phase 1
Figure 52: Input Torques 10 to 12 for Phase 1
Figure 53: Input Torques 13 to 14 for Phase 1
Figure 54: Angle 1 to 3 for Phase 2
Figure 55: Angle 4 to 6 for Phase 2 (cont'd)
Figure 56: Angle 7 to 9 for Phase 2 (cont'd)
Figure 57: Angle 10 to 12 for Phase 2 (cont'd)
Figure 58: The Forces Acting at the Left Foot
Figure 59: The Forces Acting at the Right Foot
Figure 60: The Forces Acting at the Torso
Figure 61: Input Torques 1 to 3 for Phase 2
Figure 62: Input Torques 4 to 6 for Phase 2
Figure 63: Input Torques 7 to 9 for Phase 2
Figure 64: Input Torques 10 to 12 for Phase 2
CHAPTER VIII

Conclusion

8.1 Introduction

The control strategies of a biped for different motions are studied in this dissertation. There are three aspects in this research: gait experiment, modeling of the system, and control of the system. The specific contributions made to the above three areas are discussed in the next section. The last section suggests directions for future studies.

8.2 Research Accomplished

In chapter III and appendix A, the Newton-Euler state space formulation was employed to derive the equations of motion for a six link biped. This method provides flexibility since constraint forces can be included in the dynamic equations, or the constraint forces can be eliminated by projection. Muscular actuators were also developed based on the geometry of the muscles. The constraint forces were also modeled as hard constraint forces and soft constraint forces. Different environment
warrants different constraint forces.

In chapter IV, a controller was developed for the biped based on the Lyapunov’s method. This controller can stabilize the biped both when it is not in contact with the environment and in contact with the environment. Therefore, this controller can be applied to the biped when the biped is involved in different motions that have to be in contact with the environment.

In chapter V, the controller developed in chapter IV was used to control the both the neuromuscular and robotic biped in the sitting down motion. Digital computer simulations were carried out to verify the effectiveness of the controller. In the simulations for the neuromuscular biped, four different control strategies were tested. The four different control strategies required different precisions of the movement. It was found out in the simulations that moderate to high level of precisions were required to execute the movement. For the robotic biped, it was found out that increasing the gains of feedback would increase the matching between the simulation results and the experimental results.

In chapter VI, the motion of cycling was considered to study the co-activation of quadriceps and hamstrings muscles. Two optimizing criteria were proposed. The first criterion was to minimize the energy but the simulation did not match the experimental results. The second criterion was to regulate the stiffness of the muscles, and the simulation corresponded well with the experimental results for the quadriceps but not the hamstrings. This may mean that additional terms have
to be added to the criterion to fully account for the reasons of co-activation.

In chapter VII, the lying down motion was carried out in the gait laboratory. The input torques and angular positions profiles are invaluable resources for further research. This motion was also simulated with the reference position control strategy. With this strategy, the simulation results did not tracked the experimental values well. This may mean that dynamic feedback gains have to be applied since at different parts of the movement, different gains may have to be applied.

8.3 Future Research

1. The results of this research can be apply to design controllers for Functional Electrical Systems (FES) [8, 50, 51, 52]. At the present, the controllers for FES are designed based on the expertise of the researchers. However, to control the motion more accurately, a more sophisticated controller has to be developed.

2. Develop a more sophisticated muscle model that can also be incorporated into the dynamics of the biped.

3. Application of the controller to biped walking machines.
Bibliography


Appendix A

The Three-Dimensional Biped Model

A.1 Introduction

In this appendix, the three dimensional model of the biped in phase two of the lying down motion is formulated. The steps leading to the final equations are analogous to those of the phase 1 model of chapter III. The difference is that the contact point is changed from the right leg to the pelvis, and that pure rotation about the vertical axis is allowed. For this phase, the pelvis of the biped is assumed to be anchored to the bed. Moreover, there may be contact forces acting on the left feet, the right feet or trunk of the biped.

A.2 Nonholonomic Constraints

Nonholonomic constraints come about as the result of rotation of a body along certain axes. Let the angular velocity, $V$, of segment expressed in the body coordinate system (BCS) be parallel to a fixed vector $Q$ in the inertial coordinate system (ICS). Let $\delta$ be a parameter of proportionality. The nonholonomic constraint is repre-
sented as follows:

\[ V = A(\theta)Q\delta \]  

(A.1)

Let \( R \) be a constant matrix in ICS whose column(s) are orthogonal to \( Q \), then

\[ R^TQ = 0 \]  

(A.2)

In the six link biped, rotations of link two and link six are restricted along the second principal axis of the respective link. This means that the left and right legs of the biped can only rotate along the y-axes at the knees. Moreover, rotation of link three is restricted along its third principal axis. This implies that when the biped is sitting on the bed, the pelvis can only rotate along the vertical axis. Therefore, nonholonomic constraints can be respectively represented as follows:

\[ Q_2 = Q_6 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \]  

(A.3)

\[ R_2 = R_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \]  

(A.4)

\[ Q_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \]  

(A.5)

\[ R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T \]  

(A.6)

A.3 Newton-Euler Equation

With the nonholonomic constraint equations defined above, Newton-Euler equations [26] for the six-link biped can be formulated as:

\[ I_1\ddot{W}_1 = f_1 + \ddot{K}A_1\Gamma - \ddot{L}_1A_1\Gamma_2 - R_2A_2 - Q_2T_2 \]  

(A.8)
\[ I_2 \dot{W}_2 = f_2 + \dot{K}_2 A_2 \Gamma_2 + A_2 A_2^T R_2 \Lambda_2 + A_2 A_2^T Q_2 T_2 - \dot{L}_2 A_2 \Gamma_3 - T_3 \quad (A.9) \]

\[ I_3 \dot{W}_3 = f_3 + \dot{K}_3 A_3 \Gamma_3 + A_3 A_3^T T_3 - \dot{L}_3 A_3 \Gamma_5 - T_5 - \dot{K} L_3 A_3 \Gamma_4 - T_4 \]

\[ + L \dot{K}_3 A_3 \Gamma_3 + R_3 \Lambda_3 + Q_3 T_{3a} \quad (A.10) \]

\[ I_4 \dot{W}_4 = f_4 + \dot{K}_4 A_4 \Gamma_4 + \dot{L}_4 A_4 \Gamma_{4a} A_4 A_4^T T_4 \quad (A.11) \]

\[ I_5 \dot{W}_5 = f_5 + \dot{K}_5 A_5 \Gamma_8 + A_5 A_5^T T_5 - \dot{L}_5 A_5 \Gamma_6 - R_6 \Lambda_6 - Q_6 T_6 \quad (A.12) \]

\[ I_6 \dot{W}_6 = f_6 + \dot{K}_6 A_6 \Gamma_6 + A_6 A_6^T Q_6 T_6 + A_6 A_6^T R_6 \Lambda_6 + \dot{L}_6 A_6 \Gamma_7 \quad (A.13) \]

\[ M_1 \ddot{X}_1 = -[0 0 m_{1g}]^T + \Gamma_1 - \Gamma_2 \quad (A.14) \]

\[ M_2 \ddot{X}_2 = -[0 0 m_{2g}]^T + \Gamma_2 - \Gamma_3 \quad (A.15) \]

\[ M_3 \ddot{X}_3 = -[0 0 m_{3g}]^T + \Gamma_3 - \Gamma_4 - \Gamma_5 + \Gamma_{3a} \quad (A.16) \]

\[ M_4 \ddot{X}_4 = -[0 0 m_{4g}]^T + \Gamma_4 + \Gamma_{4a} \quad (A.17) \]

\[ M_5 \ddot{X}_5 = -[0 0 m_{5g}]^T + \Gamma_5 - \Gamma_6 \quad (A.18) \]

\[ M_6 \ddot{X}_6 = -[0 0 m_{6g}]^T + \Gamma_6 + \Gamma_7 \quad (A.19) \]

where

- \( W_i \): the angular velocity of link \( i \) in the BCS
- \( f_i \): the coupling term among the axes
- \( \dot{K}_i \): the skew symmetric matrix of \( K_i \)
- \( A_i \): the transformation matrix transforming a vector from BCS; to ICS
- \( \Gamma_i \): the holonomic constraint force
- \( \Gamma_{4a} \): contact force acting at the trunk of the biped
- \( T_i \): the input torque
- \( L_i \): the skew symmetric matrix for \( L_i \)
- \( Q_i \): the vector or matrix that maps the unrestricted rotation of the link
- \( \Lambda_i \): the nonholonomic constraint torque (\( i = 2, 3, \) and 6 only)
- \( R_2 \): the vector or matrix that is orthogonal to \( Q_i \)
- \( \ddot{X}_i \): the acceleration
and specifically, the contents of the vectors are as follows:

\[ f_i = \left[ w_{i2}w_{i3}(I_{i2} - I_{i3}) \quad w_{i3}w_{i1}(I_{i3} - I_{i1}) \quad w_{i1}w_{i2}(I_{i1} - I_{i2}) \right] \]

\[ \Lambda_2 = \begin{bmatrix} \lambda_{21} \\ \lambda_{22} \end{bmatrix}; \quad \Lambda_3 = \begin{bmatrix} \lambda_{31} \\ \lambda_{32} \end{bmatrix}; \quad \Lambda_6 = \begin{bmatrix} \lambda_{61} \\ \lambda_{62} \end{bmatrix} \]

\[ T_2 = T_{21}; \quad T_3 = \begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix}; \quad T_{3a} = T_{33a}; \quad T_4 = \begin{bmatrix} T_{41} \\ T_{42} \\ T_{43} \end{bmatrix}; \]

\[ T_5 = \begin{bmatrix} T_{51} \\ T_{52} \\ T_{53} \end{bmatrix}; \quad T_6 = T_{62}; \]

Equations A.8 to A.19 can be compactly written as the follows:

\[ U_1 \dot{\mathbf{Z}} = U_2 + U_3 \Gamma + U_4 + U_{5a} \Gamma_1 + U_{5b} \Gamma_{4a} + U_{5c} \Gamma_7 U_6 \Lambda + U_7 T \quad (A.21) \]

\[ U_1 \] is the inertia matrix. \( U_2 \) is the dynamic coupling among different axes. \( U_3 \) is the input map for the holonomic forces, \( \Gamma \). \( U_4 \) is the gravity vector. \( U_{5a} \) is the input map for the holonomic force of the right foot when it is touching the ground. \( U_{5b} \) is the input map for the holonomic force of the trunk of the biped when it is touching the bed. \( U_{5c} \) is the input map for the holonomic force of the left foot when it is touching the ground. \( U_6 \) is the nonholonomic torques, \( \Lambda_2 \), \( \Lambda_3 \), and \( \Lambda_6 \), acting on joint 2, joint 3, and joint 6. \( U_7 \) is the input map for the actuators. The
The above matrices and vectors are listed as follows:

\[ U_1 = \begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4 \\
I_5 \\
I_6 \\
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5 \\
m_6
\end{bmatrix} \]

\[ Z = \begin{bmatrix}
W_1 \\
W_2 \\
W_3 \\
W_4 \\
W_5 \\
W_6 \\
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4 \\
\dot{X}_5 \\
\dot{X}_6
\end{bmatrix}^T \]

\[ \Gamma = \begin{bmatrix}
\Gamma_2 \\
\Gamma_3 \\
\Gamma_3a \\
\Gamma_4 \\
\Gamma_5 \\
\Gamma_6
\end{bmatrix}^T \]

\[ T = \begin{bmatrix}
T_2 \\
T_3 \\
T_3a \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}^T \]

\[ \Lambda = \begin{bmatrix}
\Lambda_2 \\
\Lambda_3 \\
\Lambda_6
\end{bmatrix}^T \]

\[ U_2 = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix} \]

\[ U_4 = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \]

\[ U_{5a} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
I
\end{bmatrix} \]
\[
\begin{align*}
U_{56} &= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
I
\end{bmatrix}, &
U_6 &= \begin{bmatrix}
-R_2 & 0 & 0 \\
A_2 A_1^T R_2 & 0 & 0 \\
0 & R_3 & 0 \\
0 & 0 & 0 \\
0 & 0 & -R_6 \\
0 & 0 & A_6 A_5^T R_6 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

\[
U_3 = \begin{bmatrix}
-L_1 A_1 & 0 & 0 & 0 & 0 & 0 \\
-\tilde{L}_2 A_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tilde{K}_3 A_3 & L\tilde{K}_3 A_3 & -\tilde{K} \tilde{L}_3 A_3 & -\tilde{L}_3 A_3 & 0 \\
0 & 0 & 0 & \tilde{K}_4 A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{K}_5 A_5 & -\tilde{L}_5 A_5 \\
0 & 0 & 0 & 0 & 0 & \tilde{K}_6 A_6 \\
-I & 0 & 0 & 0 & 0 & 0 \\
I & -I & 0 & 0 & 0 & 0 \\
0 & I & I & -I & -I & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & -I \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}
\]

\[
U_7 = \begin{bmatrix}
-Q_2 & 0 & 0 & 0 & 0 & 0 \\
A_2 A_1^T Q_2 & -I & 0 & 0 & 0 & 0 \\
0 & A_3 A_2^T & Q_3 & -I & -I & 0 \\
0 & 0 & 0 & A_4 A_3^T & 0 & 0 \\
0 & 0 & 0 & 0 & A_5 A_4^T & -Q_6 \\
0 & 0 & 0 & 0 & 0 & A_6 A_5^T Q_6 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
A.4 Projection

The biped defined in equation A.21 has altogether 36 degree-of-freedom (DOF) as before. However, there are a certain number of permanent holonomic and nonholonomic constraints acting on the biped thus reducing the number of DOF. There are 18 holonomic constraints and 6 nonholonomic connection constraints. The holonomic constraints describe the connections between the links. There are four nonholonomic constraints at the knee since the leg can only rotate along the its second principal axis. In addition the pelvis only rotates along its third principal axis. The DOF of the biped for that posture is then reduced to 12.

A.4.1 Holonomic Constraints

For this biped, the links are assumed to be connected as shown in figure 65. The left leg is not to be airborne. The left leg is connected to the left thigh. The left thigh is connected to the left end of the pelvis. The torso is connected to the center of the pelvis. The right end of the pelvis is connected to the right thigh. The right thigh is connected to the right leg.

Therefore, there are six holonomic constraint equations representing the 18 connection constraints:

\[
X_1 + A_1^T L_1 = X_2 + A_2^T K_2 \quad (A.22)
\]
\[
X_2 + A_2^T L_2 = X_3 + A_3^T K_3 \quad (A.23)
\]
Figure 65: The Biped Model
\[ X_3 + A_3^T K L_3 = X_4 + A_4^T K_4 \]  
(A.24)

\[ X_{3a} + A_3^T L K_3 = 0 \]  
(A.25)

\[ X_3 + A_3^T L_3 = X_5 + A_5^T K_5 \]  
(A.26)

\[ X_5 + A_5^T L_5 = X_6 + A_6^T K_6 \]  
(A.27)

Differentiate equation A.22 to A.27 twice and rearranging, we have

\[ -\ddot{X}_1 + A_1^T \ddot{L}_1 \ddot{W}_1 + \dddot{X}_2 - A_2^T \dddot{K}_2 \dddot{W}_2 = -A_2^T \dddot{W}_2^2 K_2 + A_1^T \dddot{W}_1^2 L_1 \]  
(A.28)

\[ -\ddot{X}_2 + A_2^T \ddot{L}_2 \ddot{W}_2 + \dddot{X}_3 - A_3^T \dddot{K}_3 \dddot{W}_3 = -A_3^T \dddot{W}_3^2 K_3 + A_2^T \dddot{W}_2^2 L_2 \]  
(A.29)

\[ -\ddot{X}_3 + A_3^T K L_3 \ddot{W}_3 + \dddot{X}_4 - A_4^T \dddot{K}_4 \dddot{W}_4 = -A_4^T \dddot{W}_4^2 K_4 + A_3^T \dddot{W}_3^2 K L_3 \]  
(A.30)

\[ \dddot{X}_3 - A_3^T L \dddot{K}_3 \dddot{W}_3 = -A_3^T \dddot{W}_3^2 L K_3 \]  
(A.31)

\[ -\ddot{X}_3 + A_3^T \ddot{L}_3 \ddot{W}_3 + \dddot{X}_5 - A_5^T \dddot{K}_5 \dddot{W}_5 = -A_5^T \dddot{W}_5^2 K_5 + A_3^T \dddot{W}_3^2 L_3 \]  
(A.32)

\[ -\ddot{X}_5 + A_5^T \ddot{L}_5 \ddot{W}_5 + \dddot{X}_6 - A_6^T \dddot{K}_6 \dddot{W}_6 = -A_6^T \dddot{W}_6^2 K_6 + A_5^T \dddot{W}_5^2 L_5 \]  
(A.33)

If the above equations are arranged in matrix form, we have

\[ U_3^T \dddot{Z} = U_8 \]  
(A.34)

or decomposing, we have

\[ U_9 \dddot{W} + U_{10} \dddot{X} = U_8 \]  
(A.35)
where

\[
U_9 = \begin{bmatrix}
A^T \hat{L}_1 & -A^T \hat{K}_2 & 0 & 0 & 0 & 0 \\
0 & A^T \hat{L}_2 & -A^T \hat{K}_3 & 0 & 0 & 0 \\
0 & 0 & A^T \hat{K}L_3 & -A^T \hat{K}_4 & 0 & 0 \\
0 & 0 & 0 & A^T \hat{L}_3 & -A^T \hat{K}_5 & 0 \\
0 & 0 & 0 & 0 & A^T \hat{L}_5 & -A^T \hat{K}_6 \\
\end{bmatrix}
\]

\[
U_{10} = \begin{bmatrix}
-I & I & 0 & 0 & 0 & 0 \\
0 & -I & I & 0 & 0 & 0 \\
0 & 0 & -I & I & 0 & 0 \\
0 & 0 & -I & 0 & I & 0 \\
0 & 0 & 0 & 0 & -I & I \\
\end{bmatrix}
\]

\[
U_8 = \begin{bmatrix}
-A^T \hat{W}_2^2 K_2 + A^T \hat{W}_1^2 L_1 \\
-A^T \hat{W}_2^2 K_3 + A^T \hat{W}_2^2 L_2 \\
-A^T \hat{W}_2^2 L K_3 \\
-A^T \hat{W}_2^2 K_4 + A^T \hat{W}_3^2 KL_3 \\
-A^T \hat{W}_2^2 K_5 + A^T \hat{W}_3^2 L_3 \\
-A^T \hat{W}_2^2 K_6 + A^T \hat{W}_3^2 L_5 \\
\end{bmatrix}
\]

In rearranging the above equation, we have

\[
\ddot{X} = -U_{10}^{-1} U_9 \dot{W} + U_{10}^{-1} U_8
\]  \hspace{1cm} (A.36)

The above equation can be written in a state space form:

\[
\dot{Z} = \begin{bmatrix}
\dot{W} \\
\dot{X}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
-U_{10}^{-1} U_9
\end{bmatrix} \dot{W} + \begin{bmatrix}
0 \\
U_{10}^{-1} U_8
\end{bmatrix}
\]  \hspace{1cm} (A.37)

or

\[
\dot{Z} = U_{11} \dot{W} + U_{12}
\]  \hspace{1cm} (A.38)

By substituting the above equation into equation A.21, we have

\[
U_1 (U_{11} \dot{W} + U_{12}) = U_2 + U_3 \Gamma + U_4 + U_5 \Gamma_1 + U_6 \Gamma_{4a} + U_5 \Gamma_7 + U_6 \Lambda + U_7 T
\]  \hspace{1cm} (A.39)

Multiplying both sides by \(U^T_{11}\) and with \(U^T_{11} U_3 = 0\), we have:

\[
U^T_{11} U_1 U_{11} \dot{W} + U^T_{11} U_1 U_{12} = U_{11} U_2 + U^T_{11} U_{5a} \Gamma_1 + U^T_{11} U_{5b} \Gamma_{4a} + U^T_{11} U_{5c} \Gamma_7 \\
+ U^T_{11} U_6 \Lambda + U^T_{11} U_7 T + U^T_{11} U_4
\]  \hspace{1cm} (A.40)
In the above projection, the holonomic constraint forces, $U_3$, are eliminated. Therefore, the above equation represents a reduced state space.

### A.4.2 Nonholonomic Constraints

In the second step of the projection, the nonholonomic constraints will be used to eliminate the nonholonomic torques $\Lambda$'s. It is assumed that each link can only rotate along its principal axes. The $\theta$'s are represented by some quasi-coordinates, $\phi$'s, as follows:

\[ \theta_1 = \begin{bmatrix} \phi_2 \\ \phi_1 \\ \phi_4 \end{bmatrix}; \quad \theta_2 = \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}; \quad \theta_3 = \begin{bmatrix} 0 \\ 0 \\ \phi_5 \end{bmatrix} \tag{A.41} \]

\[ \theta_4 = \begin{bmatrix} \phi_6 \\ \phi_7 \\ \phi_8 \end{bmatrix}; \quad \theta_5 = \begin{bmatrix} \phi_9 \\ \phi_{10} \\ \phi_{11} \end{bmatrix}; \quad \theta_6 = \begin{bmatrix} \phi_9 \\ \phi_{12} \\ \phi_{11} \end{bmatrix} \tag{A.42} \]

The $\phi$'s are the angles that represent the movement of the biped. The $\phi$'s are shown in figure 66. Differentiate equations A.41 to A.42 twice, we have

\[ \dot{\tilde{W}} = U_{15} \tilde{\phi} + U_{16} \tag{A.43} \]

where

\[ \tilde{W} = \begin{bmatrix} \omega_{11} & \cdots & \omega_{63} \end{bmatrix}^T \]

\[ \Theta = \begin{bmatrix} \phi_1 & \cdots & \phi_{14} \end{bmatrix}^T \]

Substitute $\tilde{W}$ into equation A.40, we have

\[ U_{11}^T U_{115} \tilde{\Theta} + U_{16} + (U_{11}^T U_{112} - U_{11}^T U_{2}) - U_{11}^T U_4 \]

\[ = U_{11}^T U_{5a} \Gamma_1 + U_{11}^T U_{5b} \Gamma_4 + U_{11}^T U_{5c} \Gamma_7 + U_{11}^T U_6 \Lambda + U_{11}^T U_7 T \tag{A.44} \]
Figure 66: The Six Link Biped With the Quasi-coordinates, $\phi$'s.
Multiply by $U_{15}^T$ and with $U_{15}^T U_{11}^T U_6 = 0$, we have

$$U_{15}^T U_{11}^T U_1 U_{11} U_{15} \ddot{\Theta} + U_{15}^T U_{11}^T (U_1 U_{11} U_{16} + U_1 U_{12} - U_2) - U_{15}^T U_{11}^T U_4 =$$

$$U_{15}^T U_{11}^T U_{5a} \Gamma_1 + U_{15}^T U_{11}^T U_{5b} \Gamma_{4a} + U_{15}^T U_{11}^T U_{5c} \Gamma_7 + U_{15}^T U_{11}^T U_7 T$$  \hspace{1cm} (A.45)

By multiplying the above equation with $U_6$, the nonholonomic constraints forces are eliminated, and the size of the system state space is further reduced. The above equation can be represented in a more compact form as:

$$I \ddot{\Theta} + B(\Theta, \dot{\Theta}) + G(\Theta) = E_1 \Gamma_1 + E_2 \Gamma_{4a} + E_3 \Gamma_7 + E_4 T$$  \hspace{1cm} (A.46)

where

$$I(\Theta) = U_{15}^T U_{11}^T U_1 U_{11} U_{15}$$

$$B(\Theta, \dot{\Theta}) = U_{15}^T U_{11}^T (U_1 U_{11} U_{16} + U_1 U_{12} - U_2)$$

$$G(\Theta) = -U_{15}^T U_{11}^T U_4$$

$$E_1 = U_{15}^T U_{11}^T U_{5a}$$

$$E_2 = U_{15}^T U_{11}^T U_{5b}$$

$$E_3 = U_{15}^T U_{11}^T U_{5c}$$

$$E_4 = U_{15}^T U_{11}^T U_7$$

The above equation represents the dynamics of a six-link biped in lying down. $I(\Theta)$ is the inertia of the system. $B(\Theta, \dot{\Theta})$ contains the coriolis and centripetal vector. $G(\Theta)$ is the gravity. $E_1$ is the input map for the contact force, $\Gamma_1$ when
the right foot of the biped is touching the ground. $E_2$ is the input map for the contact force, $\Gamma_{4a}$ when the trunk is touching the bed. $E_3$ is the input map for the contact force, $\Gamma_7$ when the left foot of the biped is touching the ground. $E_4$ is the input map for the torque generators, $T$.

A.5 Calculations of Holonomic Forces and Nonholonomic Torques

It is important to calculate the holonomic forces and nonholonomic torques in the biped as before since there may be physiological limits to the magnitude of these forces and torques.

In order to calculate the nonholonomic torques, the nonholonomic and holonomic constraints are investigated in detail. The nonholonomic constraints are:

$$R_2^T(A_1A_2^TW_2 - W_1) = 0 \quad (A.47)$$
$$R_3^TW_3 = 0 \quad (A.48)$$
$$R_6^T(A_5A_6^TW_6 - W_6) = 0 \quad (A.49)$$

Differentiating the above equations with respect to time twice, we have

$$R_2^T(-\dot{W}_1 - \dot{W}_1A_1A_2^TW_2 + A_1A_2^T\dot{W}_2) = 0 \quad (A.50)$$
$$R_6^T(-\dot{W}_5 - \dot{W}_5A_5A_6^TW_6 + A_5A_6^T\dot{W}_6) = 0 \quad (A.51)$$
Rearranging the above equation, we have

\[
\begin{bmatrix}
-R_2^T & R_2^T A_1 A_2^T & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & R_3^T & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & -R_6^T & R_6^T A_5 A_5^T & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{Z} \\
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
R_2^T \hat{W}_1 A_1 A_2^T W_2 \\
0 \\
R_6^T \hat{W}_5 A_5 A_5^T W_6 \\
\end{bmatrix}
\]  
(A.52)

The above equation can be represented as:

\[
U_6^T \dot{Z} = U_{17}
\]  
(A.53)

Combining equations A.34 and A.53, we have

\[
\begin{bmatrix}
U_3 & U_6
\end{bmatrix}^T \dot{Z} = \begin{bmatrix}
U_6^T & U_{17}^T
\end{bmatrix}^T
\]  
(A.54)

Rearranging equation A.34, we have

\[
U_1 \dot{Z} = U_2 + U_4 + U_{5a} \Gamma_1 + U_{5b} \Gamma_{4a} + U_{5c} \Gamma_{4b} + \begin{bmatrix}
U_3 & U_6
\end{bmatrix} \begin{bmatrix}
\Gamma \\
\Lambda
\end{bmatrix} + U_7 T
\]  
(A.55)

Substitute equation A.54 into the above equation, we have

\[
\begin{bmatrix}
U_3 & U_6
\end{bmatrix}^T U_1^{-1}(U_2 + U_4 + U_{5a} \Gamma_1 + U_{5b} \Gamma_{4a} + U_{5c} \Gamma_{4b} + \begin{bmatrix}
U_3 & U_6
\end{bmatrix} \begin{bmatrix}
\Gamma \\
\Lambda
\end{bmatrix} + U_7 T) = \begin{bmatrix}
U_6^T & U_{17}^T
\end{bmatrix}^T
\]  
(A.56)

To compute the nonholonomic torques and holonomic forces, a similar strategy is followed:

\[
\begin{bmatrix}
\Gamma \\
\Lambda
\end{bmatrix} = \left(\begin{bmatrix}
U_3 & U_6
\end{bmatrix}^T U_1^{-1} \begin{bmatrix}
U_3 & U_6
\end{bmatrix}\right)^{-1}\left(\begin{bmatrix}
U_6^T & U_{17}^T
\end{bmatrix}^T
- \begin{bmatrix}
U_3 & U_6
\end{bmatrix}^T U_1^{-1}(U_2 + U_4 + U_{5a} \Gamma_1 + U_{5b} \Gamma_{4a} + U_{5c} \Gamma_{4b} + U_7 T))
\]  
(A.57)
A.6 Contact Forces

As an application of the above computations, the hard contact force under the right foot is computed. The ground reaction force of the left foot can be calculated from the states and inputs of the system. If the right foot touches the ground, there is a nonholonomic constraint:

\[ X_1 + A^T_1 K_1 = D_1 \]  

where \( D_1 \), a constant, represents the tip of the right foot. Differentiate the above equation twice, we have

\[ \ddot{X}_1 + \dot{K}_1 A_1 \dot{W}_1 = \dot{K}_1 \dot{W}_1 A_1 W_1 \]  

(A.59)

Let us write the above the equation in a compact form,

\[ U_{18a} = U_{18a} \]  

(A.60)

Combining equations A.38 and A.43 with the above equation, we have the following:

\[ U_{5a}^T U_{11} U_{15} \hat{\Theta} + U_{5a}^T U_{11} U_{16} + U_{5a}^T U_{12} = U_{18a} \]  

(A.61)

Combining equation A.46 with the above equation, we have

\[ U_{5a}^T U_{11} U_{15} I(\Theta)^{-1}(-B(\Theta, \dot{\Theta}) - G(\Theta) + E_1 \Gamma_1 + E_2 \Gamma_4 + E_3 \Gamma_7 + E_4 T) + U_{5a}^T U_{11} U_{16} + U_{5a}^T U_{12} = U_{18a} \]  

(A.62)
Rearranging, we have

\[
\Gamma_1 = (E_1^T I(\Theta)^{-1} E_1)^{-1} (U_{5a}^T U_{11} U_{18} I(\Theta)^{-1} (B(\Theta, \dot{\Theta}) + G(\Theta) - E_2 \Gamma_{4a} - E_3 \Gamma_7 - E_4 T) - U_{5a}^T U_{11} U_{16} - U_{5a}^T U_{12} + U_{18a})
\]  

(A.63)

Specifically, the hard contact force is computed for the trunk. The bed reaction force of the trunk can be calculated from the states and inputs of the system. If the trunk touches the bed, there is a holonomic constraint:

\[
X_4 + A_4^T L_4 = D_2
\]  

(A.64)

where \(D_2\), a constant, represents point of contact on the trunk. Differentiate the above equation twice, we have

\[
\ddot{X}_4 + \ddot{L}_4 A_4 W_4 = \ddot{L}_4 W_4 A_4 W_4
\]  

(A.65)

To put the above equation into compact form, we have

\[
U_{5b}^T \dot{Z} = U_{18b}
\]  

(A.66)

Combining equations A.38 and A.43 with the above equation, we have the following:

\[
U_{5b}^T U_{11} U_{18} \ddot{\Theta} + U_{5b}^T U_{11} U_{16} + U_{5b}^T U_{12} = U_{18b}
\]  

(A.67)

Combining equation A.46 with the above equation, we have

\[
U_{5b}^T U_{11} U_{15} I(\Theta)^{-1} (-B(\Theta, \dot{\Theta}) - G(\Theta) + E_1 \Gamma_1 + E_2 \Gamma_{4a} + E_3 \Gamma_7 + E_4 T) + U_{5b}^T U_{11} U_{16} + U_{5b}^T U_{12} = U_{18b}
\]  

(A.68)
Rearranging, we have

\[ \Gamma_{4a} = (E_2^T I(\Theta)^{-1} E_2)^{-1}(U_{55}^T U_{11} U_{15} I(\Theta)^{-1}) (B(\Theta, \dot{\Theta}) + G(\Theta) - E_1 \Gamma_1 - E_3 \Gamma_7 - E_3 T) - U_{55}^T U_{11} U_{16} - U_{55}^T U_{12} + U_{18b}) \] (A.69)

Thus the contact force supporting the trunk can be computed. This equation can be drastically simplified if all or many of the muscular forces are zero, and the dominant input is gravity.

As an example, the hard contact force is computed for the right foot. The ground reaction force of the right foot can be calculated from the states and inputs of the system. If the right foot touches the ground, there is a holonomic constraint:

\[ X_7 + A_6^T L_6 = D_3 \] (A.70)

where \( D_3 \), a constant, represents the tip of the right foot. Differentiate the above equation twice, we have

\[ \ddot{X}_7 + \dot{L}_6 A_6 \ddot{W}_6 = \ddot{L}_6 \dddot{W}_6 A_6 \dot{W}_6 \] (A.71)

To put the above the equation into compact form, we have

\[ U_{5c}^T \ddot{\Theta} = U_{18c} \] (A.72)

Combining equations A.38 and A.43 with the above equation, we have the following:

\[ U_{5c}^T U_{11} U_{15} \ddot{\Theta} + U_{5c}^T U_{11} U_{16} + U_{5c}^T U_{12} = U_{18c} \] (A.73)
Combining equation A.46 with the above equation, we have

\[ U_{5c}^T U_{11} U_{15} I(\Theta)^{-1} (-B(\Theta, \dot{\Theta}) - G(\Theta) + E_1 \Gamma_1 + E_2 \Gamma_{4a} + E_3 \Gamma_7 + E_4 T) \]

\[ + U_{5c}^T U_{11} U_{16} + U_{5c}^T U_{12} = U_{18c} \quad (A.74) \]

Rearranging, we have

\[ \Gamma_7 = (E_3^T I(\Theta)^{-1} E_3)^{-1} (U_{5c}^T U_{11} U_{15} I(\Theta)^{-1} \]

\[ (B(\Theta, \dot{\Theta}) + G(\Theta) - E_1 \Gamma_1 - E_2 \Gamma_{4a} - E_4 T) - U_{5c}^T U_{11} U_{16} - U_{5c}^T U_{12} + U_{18c} \quad (A.75) \]

Therefore, the contact forces at the left foot, the right foot, and the trunk can be computed accordingly from the above equations. All can be simplified if the muscular forces are negligible compared to gravity.