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Recurrence in dynamical systems: A combinatorial approach

Forrest, Alan Hunter, Ph.D.
The Ohio State University, 1990
RECURRENCE IN DYNAMICAL SYSTEMS:
A COMBINATORIAL APPROACH.

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Alan Hunter Forrest, B.A.(Hons.)(Cantab.).

* * * * *

The Ohio State University

1990

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There is no newe gyse that it nas old.

Chaucer,
The Canterbury Tales

That which hath been is now; and that which is to be hath already been.

Ecclesiastes,
The Authorised, King James Version
Ther is no newe gyse that it nas old.

Chaucer,
The Canterbury Tales

That which hath been is now; and that
which is to be hath already been.

Ecclesiastes,
The Authorised, King James Version
To my parents, my brother Ewan,
and Mr. J.G.H. Anthony.
ACKNOWLEDGEMENTS

I thank my adviser, Professor Vitaly Bergelson, for suggesting the problems of this dissertation and for his interest and guidance in their solution. Further, I thank Professor J. Rosenblatt and Doctors B. Weiss and I. Kriz for their encouragement and helpful conversations, and also Professor G. Edgar and an anonymous referee for pointing out errors in an early draft.
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CHAPTER I
INTRODUCTION

The Physical Foundations of Recurrence:

Few civilisations have failed to notice the periodic motion of the planets and stars and the change and repetition of the seasons. The consistent return of the Sun to its place at midsummer advised the design of stone henges and barrows. The Moon prompted the farming callendar and still prompts Easter. Indeed, it could be said that early astronomers studied not the stars and planets but their positions and periodic movements in the sky.

Kepler's observation and Newton's calculation of the motion of the earth around the sun only confirmed its periodic, repetitive nature. This nature did not seem to change qualitatively when more planets were added for consideration. Poincaré, in his Méthodes Nouvelles de la Méchanique Céleste, studied this significant complication and, although the calculations were notoriously difficult and no complete solution was known, he asserted [37] that

"whenever [a dynamical system] is bounded, the orbits of motion exhibit some form of recurrence or return close to their original position."

Thus the notion of recurrence in general dynamical systems was introduced.
The mathematical study of dynamical systems is an abstraction of at least two physical models.

The first is Hamiltonian Mechanics, suitable for the study of the Solar System and other deterministic, though complicated, systems. To consider the Solar System, for example, one encodes its present arrangement and movement as a series of numbers; the positions and velocities of the sun, planets and moons. By considering these numbers as coordinates, the state of the system can be thought of as a point in a high dimensional space, the state space. As the state of the Solar System changes, this point moves in the state space and its laws of motion are governed by a set of equations which depend on the way the Sun, planets and moons interact. The motion is continuous; a planet makes no sudden jumps or jerks in its orbit.

Poincare considered the motion of all the points in the state space as a whole. It is like the stirred fluid in a multidimensional teacup in which the state of our Solar System is a single particle caught in the flow. If two points are very close together in the state space then they represent states of the system which are almost identical. The position of the planets and moons in each system would then be almost the same and Poincare's notion of recurrence would be observed.

To abstract this, the mathematician studies continuous flows in state spaces more general than those Hamilton used, namely Hausdorff Topological Spaces, and so studies Topological Dynamical Systems.

Poincare's heuristic was first proved in this generality by Birkhoff: In a Compact Hausdorff Topological Dynamical System there is a point which returns arbitrarily close to its original position some time in the future; indeed infinitely often.
The property of compactness is a natural imposition which, in mechanical terms, says that the system should not explode without bound. Note that a point in such a system may never return to its original position exactly. Further, the closer one requires the return of the point, the longer one may have to wait to see it.

As more components are added to a dynamical system, the dimension of the Hamiltonian state space increases until explicit computation of the motions of individual components becomes impractical. One must use a second model to study such a system: Statistical Mechanics.

A typical example of this is a gas in a room in which each component is a separate molecule. There are more than $10^{27}$ gas molecules in a room of average size and the analysis of such a system must be restricted to statistical quantities like temperature and pressure. However, in the face of almost total ignorance of the system, one can still talk about the probability of particular particles having certain positions and velocities and so adopt a statistical approach.

What does probability mean here? What is the probability that molecule $M$ is within one inch of corner $C$ as the gas moves about one's room? As in the deterministic case, one must consider the space of all states the system can adopt. A certain region, $A$, of this space consists of states in which $M$ is within one inch of $C$. Thus the probability of this event can be defined as the proportion of the state space which $A$ occupies, $\mu(A)$.

The measurement of this proportion is made precise with the introduction of Measure Theory. The important thing is that one considers the state space as a whole in which a measure of proportion is possible reflecting the likelihood of observing any particular state of the gas. Such spaces are better known as Probability Spaces.

It is interesting to examine what happens to the points which are presently in region $A$ as the system evolves. Their extent or image, like a drop of milk in a cup of stirred tea, is
moved and distorted in the state space. Liouville showed that, however complicated the
dynamics may be, Hamilton's laws ensure that the proportion of the state space occupied
by this evolving image is constant. The flow in the state space preserves measure.

The mathematical abstraction of this is a measure preserving flow in a probability
space, usually called a Measure Preserving Dynamical System. The study of such systems
is Ergodic Theory.

What does it mean if a point, starting in region A, reenters region A ten seconds
later? This point represents a system which originally had molecule M near corner C and
has M near C again after ten seconds. The system after ten seconds is, in a small way, like
the original system and there is some slight recurrence.

Let A \cap A_{10} be the region which the points of A form after ten seconds' motion in the
state space. The points which lie in A \cap A_{10}, are precisely those which start in A and
enter A again after ten seconds and so demonstrate the recurrence described above. The
probability of this recurrence is given by the measure of points in the state space which are
in A \cap A_{10}, \mu(A \cap A_{10}). This value is a revealing statistic because it represents the
probability that the gas in a room, having started with M near C, has M near C ten seconds
later. In other words it is the probability that such a system will return roughly to its
original state after exactly ten seconds. The return can be made more precise by making the
region A smaller and the probability of this event and its future recurrence will be smaller
in turn.

Poincare's claim can be expressed and proved in these terms: In a measure
preserving dynamical system, almost every point which starts in a region A will reenter A
at some future time. The words 'almost every' are used in the exact probabilistic sense
here.
The Modern Study of Recurrence:

For practical purposes, it is impossible to examine a dynamical system continuously. Photographs of the planets or measurements of the temperature represent discrete observations. Suppose that a dynamical system is photographed at the end of the first second, the second second, the third second and so on. Will it still be seen to return close to its original position? The answer for both the topological and the measure preserving case is yes.

However, it is not the regularity or density of the observation which obtains this result. If one were to examine the photographs taken at the end of the first second, the third second, the fifth second and so on, it would be possible to miss the return of the system close to its original state. For example, a 'planet' which orbited the 'sun' once every two seconds would always be observed at a point diametrically opposite its original position.

On the other hand, rather thin sequences of observations do detect recurrence. For example, observations at the end of the first second, the fourth second, the ninth second and so on along the sequence of perfect square numbers. Sequences which have this property are called Sets of Recurrence. Thus the sequence of perfect squares is a set of recurrence; a result first proved by Sarkozy and Furstenberg.

Sets of recurrence have received much attention in recent years and, in an advance far from their dynamical origins, Furstenberg has shown that they have significance in Combinatorial Number Theory. In his monograph "Recurrence in Ergodic Theory and Combinatorial Number Theory," [14] he establishes a strong correspondence between large regions in measure preserving systems (those of nonzero measure) and large sets of numbers (those of nonzero Banach density) to much effect. For example, in contrast to the original, purely combinatorial proof of Szemeredi, he proved Szemeredi's Theorem using Ergodic Theory and Functional Analysis. Other simpler results in combinatorics were proved with similarly simpler constructions in dynamical systems. In [3,4,14,15,30,31],
some results, still unproved by combinatorial arguments, were shown true by dynamical methods. Most notably, Furstenberg and Katznelson [31] have recently proved a density version of the Hales-Jewett theorem which remains intractable to purely combinatorial techniques.

All these arguments involve some analysis of recurrence, or repetition, in the motion of a dynamical system and so the study of recurrence becomes of independent interest.

This dissertation, "Recurrence in Dynamical Systems: A Combinatorial Approach," examines, among other things, the questions:

i/ Do recurrence in topological dynamical systems and recurrence in measure preserving systems differ?

ii/ How efficiently can a set of recurrence observe the return of a point close to its original position in a dynamical system?

Concerning the first question; it is apparent from the introduction that topological and measure preserving dynamics have much in common. It is known that, in many ways, topological and measure preserving dynamical systems are essentially the same: A compact topological dynamical system can have a measure of proportion imposed on it so that it becomes a measure preserving system; a result due to Krylov and Bogliouboff [38]. A reverse correspondence is due to Jewett and Kreiger [see 21,p.188].

Against this grain, the answer to the first question shows that there is a difference. The author constructs a sequence of observations which records the recurrence of any topological dynamical system but which fails to record the recurrence of one particular measure preserving system. The author cannot claim full credit for this result. Rather, he uses an example from Graph Theory, due to Kriz, and shows how this can be applied to
the question. In the course of this argument he simplifies Kriz's construction a little.

The answer to this question shows that a complete unification of topological and measure preserving dynamical systems is impossible.

Furstenberg used dynamics to obtain combinatorial results. Here, the use of graph theory to produce examples in dynamics is, in some sense, a natural reverse of Furstenberg's argument.

This idea was brought to bear on the second question. It revealed that a set of recurrence can be remarkably inefficient in observing the return, although, by definition, it will detect the return eventually.

Recall that the proportion of \( A \) which overlaps \( A_{t_0} \) records the probability that the gas in a room, having started with molecule \( M \) near corner \( C \), has \( M \) near \( C \) ten seconds later. More generally, at time \( t \), the points which were in \( A \) have moved to form region \( A_t \) and the value of \( \mu(A \cap A_t) \) is the probability that \( M \) will be near \( C \) again after \( t \) seconds.

It is a consequence of the Mean Ergodic Theorem of Von Neumann that, for all \( \varepsilon > 0 \), there is a future time, \( t \),

\[ \mu(A \cap A_t) > \mu(A)^2 - \varepsilon. \quad (1.1) \]

Examples show that this is the best that one can expect and this reflects well upon the probability that the system is returned roughly to its original state after \( t \) seconds.

What happens if one insists that the observations be taken from a set of recurrence? With the whole numbers and the perfect squares the phenomenon described above continues to hold. For all \( \varepsilon > 0 \), there is a perfect square, \( n^2 \), for which

\[ \mu(A \cap A_{n^2}) > \mu(A)^2 - \varepsilon. \quad (1.2) \]

Thus the collection of perfect squares forms a set of recurrence which invariably performs
at the greatest efficiency. Bergelson calls this property Nice Recurrence.

All known examples of sets of recurrence were also nicely recurrent until the author constructed an example of a measure preserving dynamical system and a set of recurrence in which the overlap described above could be made arbitrarily small. This gave arbitrarily poor chances of discovering by means of this set of recurrence an approximate return of the system to its original state.

Many other natural problems arise in the course of this study. Some are new expressions of old combinatorial or dynamical problems. Others seek to compare various strengths of recurrence which are needed to produce interesting combinatorial results.

This dissertation describes and resolves some of these problems and observes some applications to Combinatorial Number Theory. It would be interesting to find applications of these results to the study of mechanics.

The results mentioned above are made more precise later in this introduction, after the following definitions.
**Standing Definitions:**

Many of these are taken straight from [14, Ch.1].

A topological dynamical system is a pair \((X,T)\), where \(X\) is a compact metric space and \(T\) is a homeomorphism onto.

A subset, \(A\), is \(T\)-invariant if \(TA = A\).  \hfill (1.3)

\((X,T)\) is minimal if the only closed \(T\)-invariant sets are \(X\) and \(\emptyset\).

Given a point, \(x\), in \(X\), \(O(x)\) is its orbit, namely the set

\[ O(x) = \{ T^n x : n \in \mathbb{Z} \}. \] \hfill (1.4)

A subset, \(S\), of \(\mathbb{Z}\) or \(\mathbb{N}\), is syndetic if the difference between consecutive terms is uniformly bounded from above.

A point, \(x\), in \(X\), is said to be uniformly recurrent if, for any neighbourhood, \(U\), of \(x\), the set of returns of \(x\) to \(U\), \(\{ n : T^n x \in U \}\), is syndetic.

**Remark:** The requirement that \(X\) be metric is sufficient for the purposes of this dissertation, but all of the results proved here can apply equally to compact Hausdorff topological spaces.

The point \(x\) is uniformly recurrent if and only if \((\overline{O(x)}, T)\) is minimal, \(\overline{O(x)}\) being the closure of the orbit of \(x\) in \(X\): [14, Ch.1].

Every system \((X,T)\) has a uniformly recurrent point: [14, Thm.1.16, p.29].

\(R\), a subset of \(\mathbb{Z}\), is said to be a set of topological recurrence if, for all systems \((X,T)\) and all open neighbourhoods, \(U\), of a uniformly recurrent point in \(X\), there is an \(r\) in \(R\) such that \(T^r U \cap U \neq \emptyset\). \hfill (1.5)
A set of the form

\[ \Delta(U) = \{ n \in \mathbb{Z} : T^n U \cap U \neq \emptyset \}, \quad (1.6) \]

where \( U \) is an open neighbourhood of a uniformly recurrent point, is called a topological return set.

Remark: Only minimal systems need be considered to verify topological recurrence: One's attention can be restricted, without loss of generality, to the minimal system \( \overline{O(x)} \), where \( x \) is the uniformly recurrent point contained in \( U \), and the relatively open set \( U \cap \overline{O(x)} \).

Throughout the following, \( (X, \mathcal{B}, \mu) \) and \( (Y, \mathcal{D}, \nu) \) etc. are probability measure spaces.

It can be shown that when dealing with recurrence, without loss of generality, \( L_2(X, \mathcal{B}, \mu) \) etc. are separable. In this case (see [32, p.399]), \( X \) can be the union of \([0,1]\) and a countable number of points, \( \mathcal{B} \) is the Borel \( \sigma \)-algebra and \( \mu \) is a linear combination of Lebesgue measure on \([0,1]\) and an atomic measure on the points. But this fact will not be used in what follows.

A measure preserving dynamical system is denoted by a quadruple \((X, \mathcal{B}, \mu, T)\), where \((X, \mathcal{B}, \mu)\) is a probability measure space and \( T \) is an invertible bimeasurable point transformation which preserves measure:

\[ \mu(TA) = \mu(A) = \mu(T^{-1}A) \quad \text{for all} \quad A \in \mathcal{B}. \quad (1.7) \]

A collection of commuting measure preserving transformations, \( T_1, T_2, \ldots, T_k \), can act on \( X \). The resulting multiparametric measure preserving system will be written \((X, \mathcal{B}, \mu, T_1, T_2, \ldots, T_k)\) or \((X, \mathcal{B}, \mu, \mathbb{Z}^k)\) if the action is understood.
Suppose that \((X,\mathcal{B},\mu,T)\) is a measure preserving system and \(A\) is a measurable subset of \(X\) of positive measure.

**Measure theoretical** return sets are those of the form:

\[
\Delta(A) = \{ n \in \mathbb{Z} : \mu(A \cap T^n A) > 0 \}. \tag{1.8}
\]

**Remark:** The phrase 'measure theoretical' is used when it is important to distinguish between the two different types of recurrence defined so far.

A set of **(measure theoretical) recurrence** (in \(\mathbb{Z}\)) is a subset of \(\mathbb{Z}\) which intersects every non empty return set. To remove trivial cases, general sets of recurrence are considered to be subsets of \(\mathbb{Z}\setminus\{0\}\).

**Remark:** Sets of recurrence are called **Poincare sets** in [14, p.72] and elsewhere. The motivation for the choosing the phrase 'set of measure theoretical recurrence' is to emphasise the connection between recurrence in topological and measure preserving systems.

**Examples:** Examples of return sets, both topological and measure theoretical, include \(n\mathbb{Z}\), where \(n\) is some integer, but exclude \(2\mathbb{Z} + 1\).

Poincare's theorem says that a return set is non-empty whenever \(\mu(A) > 0\). Further, it is a consequence of Khinchine's theorem [see 21, p.37] that, whenever a return set is non empty, it is syndetic (see Furstenberg [14, Ch.8] for a full treatment of this aspect of recurrence).

The Squares and 'Primes minus one' form interesting examples of sets of topological recurrence. These are also sets of measure theoretical recurrence.
The Plan for this Dissertation:

All these examples above, which arise from harmonic analytic considerations, have the stronger property; namely that, given \((X, \mathcal{B}, \mu, T)\) and \(A\) as above and \(\varepsilon > 0\), there is an element, \(r\), of \(\mathbb{R}\) such that

\[
\mu(A \cap T^r A) > \mu(A)^2 - \varepsilon.
\] (1.9)

This property, called nice recurrence, was used by Bergelson [3] to give density versions of Schur's Theorem and he asked whether recurrence implied nice recurrence necessarily.

Incidentally, another natural definition for nice recurrence is considered, namely that there exist a \(c > 0\) such that, given \((X, \mathcal{B}, \mu, T)\) and \(A\) as above, there is an element, \(r\), of \(\mathbb{R}\) such that

\[
\mu(A \cap T^r A) > c\mu(A)^2.
\] (1.10)

It is shown that this is equivalent to the previous definition of nice recurrence.

In another paper [4], Bergelson proved new results in the combinatorics of \(\mathbb{Z}^2\) by means of dynamics and the following strengthened version of recurrence:

A subset, \(R\), of \(\mathbb{Z}\) is a set of strong recurrence if, given \((X, \mathcal{B}, \mu, T)\) and \(A\) as above, there is \(\varnothing > 0\) and an infinite number of elements, \(r\), of \(R\) such that

\[
\mu(A \cap T^r A) > \varnothing.
\] (1.11)

Once again, it was natural to ask whether there was any identification between this property and the previous two.

In the course of this thesis, the author proves the following results:

1/ (Chapter II) A set of nice recurrence is also a set of strong recurrence and the \(\varnothing\) in the definition can be made arbitrarily close to \(\mu(A)^2\) from below.
ii/ (Chapter IV) There is a set of recurrence which is not a set of strong recurrence.

iii/ (Chapter IV) There is a set of strong recurrence which is not a set of nice recurrence.

Thus, the connections between these natural definitions are fully explored. In the process, however, a fairly easy transfer of properties between Cayley Graphs and sequences of integers is exploited and further use of this technique allows the author to give more demanding examples of sets of recurrence and other harmonic analytic sequences. One such kind of sequence is a Van der Corput set: \( R \) is a Van der Corput set if the fact that the real sequences \( \{u_{n+r} - u_n : n \in \mathbb{N} \} \) are uniformly distributed mod 1 for each \( r \) in \( R \) is sufficient to prove that \( \{u_n : n \in \mathbb{N} \} \) is uniformly distributed mod 1.

The following results are obtained:

iv/ (Chapter IV) There is a set, \( R \), with the property that, given \( (X, B, \mu, T) \) and \( A \) as above and \( \varepsilon > 0 \), there is an element, \( r \), of \( R \) such that

\[
\mu(A \cap T^r A) > \mu(A)^{6.3 - \varepsilon} \tag{1.12}
\]

and yet \( R \) is not a set of nice recurrence.

v/ (Chapter V) Given, a set, \( R \), and an integer, \( s \), \( K(R, s) \) is defined \([11]\) to be the number of permutationally distinct, non-degenerate solutions to the equation

\[
\sum_{1 \leq i \leq s} \pm r_i = 0 \quad : \quad r_i \in R. \tag{1.13}
\]
There is, for every function $g : \mathbb{N} \to \mathbb{R}$ which tends to infinity, a Van der Corput set, $R$, such that $K(R,s) \leq (g(s))^S$ for all $s$ large enough. \hfill (1.14)

vi/ (Chapter IV) There is a set of recurrence, $R$, which does not force the continuity of positive measures. In other words, there is a positive probability measure, $m$, on the unit circle, $\mathbb{T} = \{ e^{it} : t \in [0,2\pi) \}$, \hfill (1.15)

with atoms, whose Fourier transforms,

$$m^\wedge(r) = \int_{\mathbb{T}} z^{-r} dm(z)$$ \hfill (1.16)

tend to 0 as $r$ tends to infinity in $R$.

The definition of forcing continuity of positive measures above is not the original one which was made by Katznelson [see Perez 35] and which was subsequently proved to be equivalent to the Van der Corput property by Kamae and Mendes-France [27]. The present definition is due to Bourgain [10] and it is not clear whether it is distinct from the Van der Corput property (see Chapter V). It will be repeated in Chapter V.

Result vi/ should be contrasted with Rider's result [22]: If $K(R,s) \leq B^S$ \hfill (1.17)

for some constant, $B$, then, $R$ is a Sidon Set and hence [10] not Van der Corput.

Bourgain [10] has already proved result vi/ by a completely different method.

There are also some connections with topological dynamical systems: An example, constructed by Kriz [19], allows the author (Chapter III) to show directly that there is a set of topological recurrence which is not a set of measure theoretical recurrence. Kriz's construction was made in answer to a question of Bergelson [2] and the author presents it here in a simplified form and shows its connections with combinatorial number theory.
The question of other group actions is treated (Chapter VI): Many of the examples above carry through to a more general class of groups and a general technique is developed to allow this to be shown easily.

These problems will be reintroduced in their relevant chapters.

**Introductory Results:**

The remainder of this chapter deals with measure preserving systems. It introduces the basic ideas and proves a couple of lemmas which will be mentioned in future chapters.

The following is quite well known:

**Theorem 1.1:** The intersection of a set of recurrence and a non-empty return set is itself a set of recurrence and hence infinite.

**Proof:** This comes from the following construction:

Let \( R \) be a set of recurrence and \( \Delta(A) = \{ n \in \mathbb{Z} : \mu(A \cap T^n A) > 0 \} \), (1.18) a non-empty return set. Further let \((Y, D, \nu, S)\) be a measure preserving system and \( B \in D \), a set of positive measure.

Consider \((X\times Y, B\times D, \mu\times\nu, T\times S)\), a probability measure preserving system, and the set \( A\times B \), of positive measure. Since \( R \) is a set of recurrence, there is an \( r \in R \) such that \( \mu\times\nu(A\times B \cap (T\times S)^r A\times B) > 0 \). (1.19)

Therefore \( \mu(A \cap T^r A) > 0 \) and \( \nu(B \cap S^r B) > 0 \). (1.20)

I.e. \( r \) is an element of \( R \cap \Delta \) and lies in the return set of \( B \) in \( Y \). So \( R \cap \Delta \) is a set of recurrence.
The final assertion of the theorem comes from the fact that if $R$ is finite and does not contain 0, then it cannot be a set of recurrence: This follows simply from the fact that such an $R$ would fail to intersect the return set $(|r^*|+1)\mathbb{Z}$, where $r^*$ is the element of $R$ of largest absolute value.

**Remark:** This proof will appear again in modified form when dealing with nice recurrence in chapter 2 in which it is proved that the weaker definition of nice recurrence found in equation (1.10) implies the stronger definition (1.11).

**Corollary 1.2:** If $R$ is a set of recurrence, $(Y, D, v, S)$ a measure preserving system and $B \in D$ a set of positive measure, then,

a/ for almost every $y$ in $B$, the set

$$\Delta(y, B; R) = \{ n \in R : S^n y \in B \}$$

(1.21) is infinite, and

b/ for all $t$ positive and real, the sum $\sum_{r \in R} [v(B \cap S' B)]^t$ diverges.

**Proof:** a/ Let $R[n] = R \cap \{ (-\infty, -n] \cup [n, \infty) \}$. $R[n]$ is a set of recurrence by Theorem 1, since $R[n] \supseteq R \cap n\mathbb{Z}$. Suppose that for all $y$, in a set, $B'$, of positive measure in $B$, $\Delta(y, B; R)$ is finite. Thus there is an $n$ and $B''$, of positive measure, such that, for all $y$ in $B''$, $\Delta(y, B; R[n])$ is empty. Thus in particular $\Delta(B'') \cap R$ is contained in $R \setminus R[n]$ and hence finite, a contradiction to Theorem 1.1.
b/ First, for \( t = 1 \): Suppose that the sum \( \sum_{r \in \mathbb{R}} [\nu(B \cap S r B)] \) converges. Then, by the Borel-Cantelli lemma the set,

\[ \{ y \in Y : \text{there are infinitely many } r \text{ in } \mathbb{R} \text{ such that } y \text{ is in } B \cap S r B \} \]

has measure zero. This means that almost all \( y \) in \( B \) have \( \Delta(y, B; \mathbb{R}) \) finite, a contradiction to part a/.

For \( t \) general, it is sufficient to consider only \( t = n \), a natural number. Now form the \( n \)-fold product system \((Y^n, D^n, \nu^n, S^n)\) and consider

\[ B^n = B \times B \times \ldots \times B \quad (n \text{ times}), \quad (1.23); \]

a subset of positive measure.

The case \( t = 1 \) shows that \( \sum_{r \in \mathbb{R}} [\nu^n(B^n \cap S r B^n)] \) diverges. But this is precisely \( \sum_{r \in \mathbb{R}} [\nu(B \cap S r B)]^n \) and we are done.

**Remarks:** Corollary 1.2 part b/ has been shown to hold when \( \mathbb{R} \) is a Van der Corput set by Bertrand-Mathis [6]. A Van der Corput set is necessarily a set of measure theoretical recurrence (Kamae, Mendes-France [27]).

This should be contrasted with the construction of Chapter IV, where a set of recurrence, \( \mathcal{R} \), and a measure preserving system, \((X, \mathcal{B}, \mu, T)\) with a set, \( A \), of measure 1/2, are constructed such that \( \mu(A \cap T^r A) \) tends to zero as \( r \) tends to infinity through \( \mathcal{R} \).

Brief mention must be made of the Ramsey problem for sets of recurrence:
Definition: A property, $P$, defined for subsets of $\mathbb{Z}$, is said to be Ramsey if,
whenever $A$ has $P$ and $A = \bigcup_{i} A_i$, (1.24)
is a finite partition of $A$, then one of the $A_i$ has $P$.

This may be restated in terms of colourings of sets with property $P$.

Proposition 1.3: Being a set of recurrence is a Ramsey property.

Proof: This is proved using a product construction like Theorem 1.1. See Bergelson [2, Theorem 4.11], for example:

Suppose that $R = R_1 \cup R_2$, and that both $R_1$ and $R_2$ are not sets of recurrence. Let $(X_i, B_i, \mu_i, T_i), i=1,2$, be two measure preserving systems and let $A_i, i=1,2$, be subsets of $X_i, i=1,2$ respectively, of positive measure. Then the product system, $(X_1 \times X_2, B_1 \times B_2, \mu_1 \times \mu_2, T_1 \times T_2)$ has a subset, $A_1 \times A_2$, of positive measure whose return set (1.8) does not intersect $R$. Thus $R$ is not a set of recurrence.

This argument may be extended to the case of a partition of $R$ into more than 2 sets.

The problem of determining whether certain other properties are Ramsey will be addressed in the next chapter.
CHAPTER II

THE UNIFORMITY OF RECURRENCE

The Furstenberg Correspondence:

Here is a construction which was first made by Furstenberg [14, p.72ff.], whose presentation owes much to Furstenberg and Bergelson [4], and which will be used often in the proofs of this section. It is presented in a form general enough for its application.

Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving dynamical system. Consider \(L^\infty(X \times \mathbb{Z}^k)\), the space of sup-norm bounded complex valued functions from \(X \times \mathbb{Z}^k\) to \(\mathbb{C}\); a commutative C*-algebra with complex conjugation as * operation and the constant 1 function as unit.

Given a sequence of measurable subsets of \(X\), \(A_n\), indexed by \(\mathbb{Z}^k\), let

\[
F(x, \Pi) = 1_{A_\Pi}(x),
\]

an element of \(L^\infty(X \times \mathbb{Z}^k)\) with norm 1.

Consider \(K\), the smallest sub-C*-algebra of \(L^\infty(X \times \mathbb{Z}^k)\) which contains the unit and all the shifts of \(F\) under the action of \(\mathbb{Z}^k:\)

\[
T_m F(x, \Pi) = F(x, \Pi + m) \quad \text{for all } m, \Pi \text{ in } \mathbb{Z}^k \text{ and } x \text{ in } X.
\]
By the Gelfand isomorphism theorem, \( K \) is algebraically isomorphic to the algebra of continuous (complex valued) functions on some compact Hausdorff space, \( \mathcal{Y} \):

\[
\gamma: K \rightarrow C(\mathcal{Y}).
\]  

Indeed, \( K \) is separable, since it is spanned densely by a countable combination of shifts of a single element, and so, in addition, \( \mathcal{Y} \) is metric.

Further, \( \mathbb{Z}^k \) acts on \( \mathcal{Y} \) homeomorphically; an action induced by the C*-algebra isometry \( \gamma \circ T_m \circ \gamma^{-1} \), on \( C(\mathcal{Y}) \). Thus, given \( y \) in \( \mathcal{Y} \), \( T_m y \) is defined uniquely so that

\[
(\gamma F)(T_m y) = (\gamma(T_m F))(y)
\]  

\( F \) is idempotent \( (F^2 = F) \) in \( K \), so the image of \( F \) in \( C(\mathcal{Y}) \) is also idempotent, i.e. an indicator function, of a closed and open (clopen) set, \( \mathcal{B} \).

\[
\gamma F = 1_\mathcal{B}
\]  

The fact that there is a dense span of continuous indicator functions in \( C(\mathcal{Y}) \) shows that \( \mathcal{Y} \) is zero dimensional.

Consider the sequence of functionals on \( K \):

\[
J^G_N = \frac{1}{(2N+1)^k} \sum_{n \in \mathbb{Z}^k : \|n\|_\infty \leq N} \int_G(x, n) \, d\mu(x)
\]  

By appealing to the separability of \( K \) and by using a diagonal argument, a \( \mathbb{Z}^k \)-invariant, positive, linear functional is obtained, as a limit of a subsequence of this sequence. This corresponds, by the Reisz representation theorem, to a \( \mathbb{Z}^k \)-invariant probability measure on \( \mathcal{Y} \). This is proved in detail in [14, pp.73 & 150] where \( \mathcal{X} \) is, in
effect, a single point, but little change need be made in this argument to produce $\mathbb{Z}^k$-invariant measures on $Y$ in the case above, where $X$ is no longer a single point.

Here is the simplest case of the original application [see 14, p.73] of this construction:

Let $X = \{x\}$ and $k = 1$. Suppose that $E$ is a subset of $\mathbb{Z}$ whose density exists and is positive;

$$\lim_{N \to \infty} \frac{|E \cap [-N, N]|}{2N + 1} = a > 0. \quad (2.7)$$

Let $A_n = \{x\}$ if $n \in E$ and $\emptyset$ if $n \in \complement E$. \quad (2.8)

Then, using the notation above, a diagonal subsequence of the $J_N$ will give $(Y, \mathcal{D}, \nu, \mathcal{Z})$ a measure preserving system and a subset, $B$, of measure $a$. This ensures that $\Delta(B) \subseteq E - E$. \quad (2.9)

Note further that if the density of $E$ had not existed, but that merely a subsequence converged, then a system could still be constructed as above with a measure formed by taking, as a start, not the full sequence of $J_N$'s but the subsequence $J_{N_i}$. This important generalisation will be used later chapters to construct measure preserving systems with particular properties from sets in $\mathbb{Z}$ of positive upper density.

Furstenberg used this special construction to obtain a correspondence between sets of positive upper density and sets of positive measure and to prove results about the former from results about the latter. See Chapter 3 also.
Here is another case:

Suppose that $\mu(A_n) = a > 0$, for all $n$ in $\mathbb{Z}^k$. \hspace{1cm} (2.11)

Let $\nu$ be a $\mathbb{Z}^k$-invariant probability measure on $Y$, formed by a diagonal argument as above. In this case $\nu(B) = a$ \hspace{1cm} (2.12).

**Definition:** A measure preserving system $(Y, \mathcal{D}, \nu, \mathbb{Z}^k)$ and subset $B$, produced by the latter construction, will be referred to as a Furstenberg system generated by $(A_n)$.

**Uniformity of Recurrence:**

**Theorem 2.1 (Uniformity of Recurrence):** Suppose that $R$ is a set of recurrence and that $a > 0$ is given. Then there exist constants $N(a)$, and $\varepsilon(a) > 0$ which have the following property:

Given any probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any subset, $A$, of $X$ of measure greater than $a$, there exists an $r \in R$ no greater than $N(a)$ such that

$$\mu( A \cap T^r A ) \geq \varepsilon(a).$$ \hspace{1cm} (2.13)

**Proof:** Suppose, for a contradiction, that there exist, for each $n \geq 1$, measure preserving systems, $(X_n, \mathcal{B}_n, \mu_n, T_n)$, and subsets, $C_n$, of measure greater than $a$, in $X_n$, for which

$$\mu_n( C_n \cap T_n^r C_n ) \leq \varepsilon_n$$ \hspace{1cm} (2.14)

and where $\varepsilon_n$ tend to $0$ as $n$ tends to infinity.
Construct the product system:

\[ X = \prod_{n \geq 1} X_n, \quad \mu = \prod_{n \geq 1} \mu_n \]  \hspace{2cm} (2.15)

\[ B = \bigotimes_{n \geq 1} B_n, \quad T = \prod_{n \geq 1} T_n \]

and let

\[ A_{(n,p)} = \prod_{m=1}^{n-1} X_m \times T^p C_n \times \prod_{m=n+1}^{\infty} X_m, \]  \hspace{2cm} (2.16)

each of measure a.

Construct the Furstenberg system, \((Y,D,v,Z^2)\) together with B, generated by these \(A_{(n,p)}\). The \(Z^2\) action has two components induced by

\[ S_1(G)(n,p) = G(n+1,p), \]
\[ S_2(G)(n,p) = G(n,p+1), \]  \hspace{2cm} (2.17)

on K.

The \(Z^2\)-invariant measure, \(v\), on \(Y\) is given by:

\[ \int g \, dv = \lim_{i \to \infty} \frac{1}{(2N_i + 1)^2} \sum_{(n, p): |n|, |p| \leq N_i} \int G(n, p) \, d\mu \]  \hspace{2cm} (2.18)

where \(G = \gamma^{-1} g\) is the element of \(K\) corresponding to \(g \in C(Y)\) and \(N_i\) is some subsequence of the \(N\)'s given in the construction.

This assigns a measure \(a\) to the set \(B\).

Since \(R\) is a set of recurrence, there is an \(r \in R\) such that

\[ v(B \cap S_2 r B) > 0. \]  \hspace{2cm} (2.19)
By the construction of $v$, this implies that

$$0 < \lim_{i \to \infty} \frac{1}{(2N_i + 1)^2} \sum_{(n, p): |n|, |p| \leq N_i} \mu(A_{(n, p)} \cap A_{(n, p + r)})$$ (2.20)

This, however, may be rewritten

$$0 < \lim_{i \to \infty} \frac{1}{(2N_i + 1)^2} \sum_{(n, p): |n|, |p| \leq N_i} \mu_n(C_n \cap T_n C_n)$$ (2.21)

and this is a contradiction by hypothesis.

**Remarks:** Theorem 2.1 provides, for each $\alpha > 0$, a single finite subset of $\mathbb{R}$ in which a uniformly large return of a set, of measure $\alpha$, must occur. Hence the title of this theorem.

Note that the property of uniform recurrence is weaker than that considered by Bergelson [4] who requires that not only the set $\{ n \in \mathbb{R} : \mu(A \cap \cap A) > \theta \}$ be non-empty for some positive $\theta$, but that it be infinite.

**Definition:** A set, $R$, with the property that for all dynamical systems, $(X, \mathcal{B}, \mu, T)$ and all $\mu(A) > 0$, there is an $\theta > 0$ such that $\{ n \in R : \mu(A \cap T^n A) \geq \theta \}$ is infinite, is called a set of strong recurrence.

The reason that this was a natural requirement comes from the following interesting combinatorial result proved by Bergelson [4].
**Proposition 2.2:** If $R$ is a set of strong recurrence, and $E$ is a subset of $\mathbb{Z}^2$ with positive density, then there is an infinite subset, $B$, of $R$ such that

$$E - E \supset B \times B.$$ (2.22)

His argument does not seem to work for sets of recurrence in general.

Whether a set of recurrence is also a set of strong recurrence was asked by Bergelson at the time, and is answered in the negative in a future chapter. It is not clear whether Proposition 2.2 fails for general sets of recurrence.

The best values of the $\theta$ obtained in Theorem 2.1 are studied in more detail in Chapter 4.
Nice Recurrence implies Strong Recurrence:

**Definitions:** If, for all measure preserving systems \((X,B,\mu,T)\), all sets, \(A\), with \(\mu(A) = a\) and all \(\varepsilon > 0\), there is an \(r\) in \(R\) such that
\[
\mu(A \cap T^rA) > a^2 - \varepsilon ,
\] (2.23)
then \(R\) is called a set of nice recurrence.

If, instead, there is an \(r\) in \(R\) such that \(\mu(A \cap T^rA) > a^t - \varepsilon\) (2.24)
then \(R\) is a set of t-nice recurrence.

**Remarks:** The motivation for examining nice recurrence is that it is the best possible form of recurrence and that all the natural examples of sets of recurrence produced by harmonic analytic arguments are nicely recurrent. Further this property was used explicitly in the adaption of Schur's theorem to its density form by Bergelson [3].

The motivation for generalising nice recurrence to t-nice recurrence comes from the following approach to the Ramsey Problem for sets of nice recurrence:

**Proposition 2.3:** If \(R\) is a set of nice recurrence and \(R = R_1 \cup R_2\) (2.25) then one of \(R_1\) or \(R_2\) is a set of 3-nice recurrence.

**Proof:** Comes from a quantitative version of the argument of Prop 1.3.

An example is given in Chapter 4, of a set of 6.3-nice recurrence which is not a set of nice recurrence.

**Remark:** There is no connection between recurrence, nice recurrence and strong recurrence *a priori*, except for the obvious implications:
If \( R \) is a set of nice recurrence then it is also a set of recurrence.

If \( R \) is a set of strong recurrence then it is also a set of recurrence.

In the chapters to come it is shown that both these implications are proper. A set of strong recurrence will be exhibited which is not a set of nice recurrence.

The following lemma shows that nice recurrence is implied by apparently weaker conditions.

**Lemma 2.4:** Let \( C > 0 \) and let \( R \) be a subset of \( \mathbb{Z} \). If, for all measure preserving systems \((X, B, \mu, T)\) and all sets \( A \), such that \( \mu(A) > a \), there is an \( r \) in \( R \) for which

\[
\mu(A \cap T^r A) \geq ca^2,
\]

(2.26)

then \( R \) is a set of nice recurrence.

**Proof:** Suppose that \( C \) and \( R \) are as given above.

Suppose that \( R \) is not a set of nice recurrence. Then there is a \( b < 1 \) and a measure preserving system \((X, B, \mu, T)\) and a set \( A \), \( 1 > \mu(A) = a > 0 \) for which

\[
\mu(A \cap T^r A) \leq ba^2
\]

(2.27)

Construct the system \( W = (W, \mathcal{C}, \pi, U) \), where

\[
W = X^n, \quad \mathcal{C} = \text{product } \sigma\text{-algebra},
\]

\[
\pi = \mu^n
\]

\[
U = T \times T \times \ldots \times T \quad (n \text{ times})
\]

(2.28)

This product system becomes a probability measure preserving system.

Let \( C = A^n = A \times A \times A \times \ldots \times A \) (n times),

(2.29)

a subset of \( W \) of \( \pi\)-measure \( a^n \).
By construction, $\pi(C \cap U^rC) = \mu(A \cap T^rA)^n$ \hspace{1cm} (2.30)

$$\leq b^n(a^n)^2 \quad \text{for all } r \text{ in } R. \quad (2.31)$$

This is a contradiction, however, when $b^n < c$. \hspace{1cm} (2.32)

This will form the key step in proving the following:

**Theorem 2.5**: If $R$ is a set of nice recurrence, then $R$ is a set of strong recurrence for which, given any $\varepsilon > 0$, any probability measure preserving system $(X,\mathcal{B},\mu,T)$ and any subset, $A$, of $X$ of measure at least $a > 0$, then there are an infinite number of elements, $r$, of $R$ for which $\mu(A \cap T^rA) \geq a^2 - \varepsilon$. \hspace{1cm} (2.33)

One further lemma, a quantitative refinement of Theorem 1.1, will complete the preliminaries for the proof of Theorem 2.5.

**Lemma 2.6**: Let $R$ be a set of nice recurrence, and let $\Delta(B)$ be the return set for $B$, in $Y$, of measure $b > 0$. Then $R \cap \Delta(B)$ is a set of nice recurrence.

**Proof**: Let $(X,\mathcal{B},\mu,T)$ be a measure preserving system and

$$\mu(A) = a > 0. \quad (2.34)$$

Consider $W = (W, \mathcal{C}, \pi, U) = (XxY, BxD, \mu x \nu, T x S)$, \hspace{1cm} (2.35)

the product probability measure preserving system, and the set $C = AxB$, \hspace{1cm} (2.36)

of measure $ab$.

There is an element $r$ of $R$ for which

$$(ab)^2/2 \leq \pi(C \cap U^rC) = \mu(A \cap T^rA) \cdot \nu(B \cap S^rB). \quad (2.37)$$
Thus $r$ is in $R \cap \Delta(B)$ and $\mu(A \cap T^r A) \geq a^2 b/2$ (2.38)
since $\nu(B \cap S^nB) \leq b$. (2.39)

Lemma 2.4 then gives the result, since $b > 0$.

**Proof of Theorem 2.5:** Consider the set $n\mathbb{Z}$, where $n$ is a positive integer. It is the return set of a set of measure $1/n$. $R$ is a set of nice recurrence and so by Lemma 2.6, $R \cap n\mathbb{Z}$ is also a set of nice recurrence.

So it is quite straightforward to find for each $\varepsilon > 0$, an infinite number of elements, $r$, of $R$ for which $\mu(A \cap T^r A) \geq a^2 - \varepsilon$. (2.40)

**Remark:** The arguments above precede identically to give the following more general statement which will be used in Chapter 4, Thm 4.16:

**Proposition 2.7:** Let $c > 0$ and let $R$ be a subset of $\mathbb{Z}$. If, for all measure preserving systems $(X, B, \mu, T)$ and all sets, $A$, $\mu(A) > a$, there is an $r$ in $R$ such that

$$\mu(A \cap T^r A) \geq ca^\dagger,$$

(2.41)

then $R$ is a set of $t$-nice recurrence. Further, given any $\varepsilon > 0$, any probability measure preserving system $(X, B, \mu, T)$ and any subset, $A$, of $X$ of measure at least $a > 0$, then there are an infinite number of elements, $r$, of $R$ for which

$$\mu(A \cap T^r A) \geq a^{t-} \varepsilon.$$ (2.42)

In particular $R$ is a set of strong recurrence.
**Multiple recurrence:**

The work above has some application to the study of multiple recurrence, e.g. the value of $\mu(A \cap T^rA \cap T^{2r}A)$ where $r$ takes values in some set, $\mathbb{R}$. Results about the uniformity of this sort of recurrence, similar to Proposition 2.1, can be produced with few variations.

However, here is an observation which allows a 'two dimensional' form of recurrence, i.e. involving a $\mathbb{Z}^2$ action, to be obtained from an apparently one dimensional result. It involves a new form of uniformity of recurrence, peculiar to multiple recurrence, and so is quite well placed in this chapter.

**Lemma 2.8:** The following three conditions are equivalent:

i/ For all $\delta > 0$ there is an $N$ such that for all $k \geq 1$ and all measure preserving systems $(X, \mathcal{B}, \mu, T)$ and all measurable subsets, $A$, of $X$ of measure at least $\delta$, there is an element $r \leq N$, of $\mathbb{R}$, such that $\mu(A \cap T^rA \cap T^{kr}A) > 0$. (2.43)

ii/ For all $\delta > 0$ there is an $N$ and an $\varepsilon > 0$ such that for all $k > 1$ and all measure preserving systems $(X, \mathcal{B}, \mu, T)$ and all measurable subsets, $A$, of $X$ of measure at least $\delta$, there is an element $r \leq N$, of $\mathbb{R}$, for which

$$\mu(A \cap T^rA \cap T^{kr}A) \geq \varepsilon.$$ (2.44)

iii/ For all $\delta > 0$ there is an $N$ and an $\varepsilon > 0$ such that for each measure preserving system, $(X, \mathcal{B}, \mu, T, S)$, with two commuting measure preserving transformations, and all measurable subsets, $A$, of $X$ of measure at least $\delta$, there is an element $r \leq N$, of $\mathbb{R}$, for which $\mu(A \cap T^rA \cap S^rA) \geq \varepsilon$. (2.45)

**Remark:** The important quantifier in parts i/ and ii/ is the one which gives uniformity with respect to $k$. 
Proof: \(iii/ \Rightarrow ii/\) by the standard form of uniformity of recurrence and setting \(S = T^k\). \hfill (2.46)

\(ii/ \Rightarrow i/\) trivially and \(i/ \Rightarrow ii/\) by a Furstenberg system construction much as in the proof of the Uniformity of Recurrence in general (see Theorem 2.1).

It remains to prove \(ii/ \Rightarrow iii/\): Suppose that \((X,\mathcal{B},\mu)\) is a separable probability space, \(S\) and \(T\) are a pair of commuting measure preserving transformations on \(X\) and \(A\) is a measurable subset of \(X\) of measure \(\delta > 0\). Construct \(N\) and \(\varepsilon\) as in part \(ii/\) and let \(k > 4N/\varepsilon\). \hfill (2.47)

Let \(\{1,2,\ldots,k\}\) be given the normalised counting measure, \(m\), and let \(U\) be the following action on the space \(X \times \{1,2,\ldots,k\}\).

\[
U(x, a) = \begin{cases}
(Tx, a+1) & \text{if } a < k \\
(ST^{1-k}x, 1) & \text{if } a = k.
\end{cases}
\]

\(U\) is clearly measure preserving with respect to the product measure, \(\pi = \mu \times m\). \hfill (2.49)

\(S\) and \(T\) act in a measure preserving manner on this space along the first coordinate: \(S(x,a) = (Sx,a)\) and \(T(x,a) = (Tx,a)\). \hfill (2.50)

Let \(A' = A \times \{1,2,\ldots,k\}\), a set of measure \(\delta\). \hfill (2.51)

Note that, by construction,

\[
\pi(T^rA' \cap S^rA' \cap A') = \mu(T^rA \cap S^rA \cap A),
\]

\(\pi(S^rA' \triangle U^{kr}A') = 0\), and

\[
\pi(T^rA' \triangle U^rA') \leq |r|/k, \text{ for all } r.
\]

Part \(ii/\) says that there is an element \(r \leq N\) of \(R\) for which

\[
\pi(U^rA' \cap U^{kr}A' \cap A') \geq \varepsilon.
\]

Thus \(\mu(T^rA \cap S^rA \cap A) \geq 3\varepsilon/4\), \hfill (2.56)

and \(iii/\) is confirmed.
Remark: At present, the proofs of the multiple recurrence of $R = \mathbb{N}$ (The Furstenberg-Szemeredi Theorem) are made via their multi-dimensional analogues. It would be of some interest whether, for example, the 'two dimensional recurrence' of $\mathbb{N}$ (as in equation (2.45)) could be implied by the 'T, $T^2$ - recurrence' of $\mathbb{N}$ (as in equation (2.44) with $k = 2$). The lemma above could be a step in such an argument.
Sets of recurrence in $\mathbb{R}$:

Much of what has been said above for $\mathbb{Z}$ still holds true analogously in countable discrete groups: Uniformity of recurrence, nice recurrence implies strong recurrence etc. The possible ambiguity of definition in the non-abelian case does not, in fact, occur and the 'left' and 'right' definitions coincide. Amenability is required to define density intersectivity. Some further reference to this is given later in chapter 6.

The pursuit of such immediate generalisations, however, is not the aim of this section. Rather, the development of the previous sections to non-discrete groups has some new applications, for example, in $\mathbb{R}$, the real numbers.

**Definitions:** Suppose that $(X, B, \mu)$ is a probability measure space.

A continuous measure preserving $\mathbb{R}$-Action on $X$ is a family of measure preserving maps $\{ T^r : r \in \mathbb{R} \}$ on $X$, which form a group action and for which

$$\lim_{r \to 0} \mu(T^r A \Delta A) = 0 \quad (2.57)$$

for all measurable subsets, $A$, of $X$.

This will be written $(X, B, \mu, T)$, or $(X, B, \mu, \mathbb{R})$ if the action is understood.

$C$, a subset of $\mathbb{R}$, is a set of continuous (measure theoretical) recurrence if, for all continuous $\mathbb{R}$-Actions, $T$, on $X$, and all $A \in B$ of positive measure, there is an $r$ in $C$, such that

$$\mu(T^r A \cap A) > 0. \quad (2.58)$$

It is a set of recurrence if the above holds for any $\mathbb{R}$-action, continuous or not.

**Remark:** Some results which hold for sets of recurrence in $\mathbb{R}$ break down for sets of continuous recurrence in $\mathbb{R}$. For example, at Theorem 2.1, where the Furstenberg
System need not have a continuous action, even though all its constituent systems are continuous. The following counter-example, due to Bozhernitzan [36], shows that sets of continuous recurrence need not be uniformly recurrent:

**Example 2.9:** There is a set of continuous recurrence for $\mathbb{R}$ which is not a set of uniform continuous recurrence:

**Proof:** The set in question is of the form

$$C = \{ n^b : n \in \mathbb{N} \}$$

where $b$ is some non-integer power.

By standard harmonic analytic arguments involving, among other things, the uniform distribution of sequences (see for example [14, 25]), $C$ is a set of continuous recurrence in $\mathbb{R}$.

Choose $b > 1$ so that all the elements of $C$ are mutually rationally independent. That this exists is not obvious: Consider the following argument which will show that the set of such $b$ is the complement of a countable subset of real numbers. Let $\Lambda$ be the (countable) set of all infinite sequences of integers ($\lambda_n : n \geq 1$) which are eventually constantly zero. For each $(\lambda_n)$, not constantly zero, in $\Lambda$, define the complex analytic function,

$$\zeta_\lambda(z) = \sum_{n=1}^{\infty} \lambda_n n^z ,$$

which is non-zero and hence has a finite number of zeros. However, for each $z=b \in \mathbb{R}$, such expressions account for all the possible rational combinations of elements of $C$. Thus the set of $b$'s for which there is a rational dependence between elements of $C$ is a countable union of finite subsets of $\mathbb{R}$. The claim is proved.
Let $N > 0$. A system will be constructed which is a rotation of unit speed on the unit circle but which does not return a set of measure arbitrarily close to $1/2$ within the first $N$ elements of $C$. This will contradict directly the uniformity of recurrence for this $C$.

Let $\varepsilon > 0$. The first $N$ elements of $C$ are rationally independent. Modulo one, they form the coordinates of a vector, $\nu$, in $[0,1]^N$ for which there is an integer, $k$, such that $k\nu \in [1/2 - \varepsilon, 1/2 + \varepsilon]^N$.

Let $E = \{ s \in [0,1) : ks \in (0, 1/2 - 4\varepsilon) \mod 1 \}$ \hfill (2.61)

Then $|E| = 1/2 - 4\varepsilon$ \hfill (2.62)

but $E \cap (E + nb) = \emptyset$ for all $n \leq N$, by construction. \hfill (2.63)

Bozhernitzan's example contrasts sets of recurrence in $\mathbb{R}$ with sets of continuous recurrence in $\mathbb{R}$. The argument of Chapter 2 shows that a countable set of recurrence (not continuous recurrence) in $\mathbb{R}$ has the uniformity of recurrence property, i.e. if $C$ is a countable set of recurrence in $\mathbb{R}$ and $a > 0$, there is a finite subset, $C'$, of $C$ such that for all measure preserving $\mathbb{R}$-actions $(X, \mathcal{B}, \mu, T)$ and all subsets, $A$, of $X$ of measure at least $a$, there is an $r$ in $C'$ such that $\mu(A \cap T^r A) > 0$. Thus Bozhernitzan's example also gives a set of continuous recurrence which is not a set of recurrence.

The countability assumption is important in the proof of the uniformity of recurrence; the metrisability of the Furstenberg system depends on the group action being countable. If the set of recurrence is countable, then one can take the countable subgroup of $\mathbb{R}$ generated by the elements of the set of recurrence and consider its action as a subaction of the $\mathbb{R}$-action. An example is given later of a set of recurrence (not in $\mathbb{R}$), all countable subsets of which are not sets of recurrence. It is interesting to ask if such an example can be constructed in $\mathbb{R}$.
**Definition:** Suppose that \( R \) is a subset of \( \mathbb{R} \). Then \( C \) is called a set of *continuous nice recurrence* if, for all continuous measure preserving \( \mathbb{R} \)-actions \((X,B,\mu,T)\), subsets, \( A \), of \( X \) and \( \epsilon > 0 \), there is an \( r \) in \( C \) such that

\[
\mu(A \cap T^r A) > \mu(A)^2 - \epsilon.
\] (2.64)

**Remark:** Bozhernitzan's example is, in fact, a set of continuous nice recurrence.

**Connections with sets of recurrence in \( \mathbb{Z} \):**

Let \( r \) be a real number and let \(<r>\) be the nearest integer to \( r \), choosing the greatest integer less than \( r \) if \( r \) happens to be an exact half. Let \( \lfloor r \rfloor \) be the greatest integer less than \( r \), and let \( \lceil r \rceil \) be the least integer greater than \( r \).

Given a subset, \( C \), of \( \mathbb{R} \), let \(<C> = \{<r> : r \in C\} \); a subset of \( \mathbb{Z} \). Let \([C] \) and \( |C| \) be defined similarly.

**Theorem 2.10:**

a/ If \( C \) is a set of continuous recurrence in \( \mathbb{R} \), then \(<C>\) is a set of recurrence in \( \mathbb{Z} \).

b/ If \( C \) is a set of continuous nice recurrence in \( \mathbb{R} \), then \(<C>\) is a set of nice recurrence in \( \mathbb{Z} \).

**Proof:**

Let \((X,B,\mu,T)\) be a measure preserving system and \( A \) a subset of \( X \) such that \( \mu(A) = a \).

Construct the following \( \mathbb{R} \)-action:

Let \( Y = X \times [0,1) \), with measure, \( \nu \), equal to the product of \( \mu \) with Lebesgue measure. (2.66)
Let \( S^r(x,t) = (T^{|t+r|}x, t+r \mod 1) \), \( (2.67) \)
defined for all \( r \) in \( \mathbb{R} \), \( x \) in \( X \) and \( t \) in \([0,1)\). This is better known as a special example of a flow under a function; the function here is the constant 1. This is discussed in Petersen [21] and it produces a continuous measure preserving \( \mathbb{R} \)-action on \( Y \).

**a/** Consider the set \( A' = A \times [0,1/2) \), of measure \( \alpha/2 \). \( (2.68) \)

Since \( C \) is a Set of Continuous Recurrence, there is an \( r \) in \( C \) for which

\[ \nu(A' \cap S^rA') > 0. \] \( (2.69) \)

However, by construction, this implies that

\[ \mu(A \cap T^{<r>}A) > 0 \] \( (2.70) \)

and the recurrence of \( <C> \) is confirmed.

**b/** More care with the sizes of the intersections gives the result:

Given \( \epsilon > 0 \), there is an \( r \) in \( C \) such that:

\[ \nu(A')^2 - \epsilon < \nu(A' \cap S^rA') = [(1/2 - |r-<r>|)/2]. \mu(A \cap T^{<r>}A) \] \( (2.71) \)

So for \( \epsilon \) less than \( \alpha^2/6 \), this implies that

\[ \mu(A \cap T^{<r>}A) \geq \alpha^2/3 \] \( (2.72) \)

and lemma 2.8 completes the proof.

**Remark:** Note that there is a set of recurrence \( C \) in \( \mathbb{R} \) for which \( \lfloor C \rfloor \) is not a set of recurrence in \( \mathbb{Z} \). This is constructed as follows:

Let \( C = \{ (4^n-2^{-n})i : 1 \leq i \leq 2n-2, \ n \geq 2 \}. \) \( (2.73) \)

Then \( C \) is a set of continuous nice recurrence because it is a union of arithmetic sequences originating from 0 of increasing length.

However, \( \lfloor C \rfloor \) is contained in the sequence of odd numbers and so is not a set of recurrence in \( \mathbb{Z} \).
A similar counter example shows that \([C]\) need not be a set of recurrence even if \(C\) is a set of continuous recurrence in \(\mathbb{R}\).

Some immediate consequences of Theorem 2.10 are listed here:

**Corollary 2.11:**

\(a/\) If \(C\) is a set of continuous recurrence in \(\mathbb{R}\) and \(a\) is a real number, then \(<Ca>\) is a set of recurrence in \(\mathbb{Z}\).

\(b/\) If \(C\) is a set of continuous recurrence in \(\mathbb{R}\) then one of \([C]\) or \([C]\) is a set of recurrence in \(\mathbb{Z}\).

**Proof:** To verify \(a/\), it is sufficient to note that \(Ca\) is a set of continuous recurrence. This is seen by considering a continuous \(\mathbb{R}\)-action \((X,\mathcal{B},\mu,\mathbb{R})\) and a new \(\mathbb{R}\)-action on \(X\): \((T')^r = Tar\). \hfill (2.74)

\(b/\) proceeds from the Ramsey property for sets of recurrence and the fact that

\([C] \cup [C] \supseteq <C>. \hfill (2.75)\)

**Remark:** In \(a/\) note that a set of recurrence in \(\mathbb{Z}\) is also a set of continuous recurrence in \(\mathbb{R}\) and so new sets of recurrence in \(\mathbb{Z}\) are easily produced: \{<na> : n \in \mathbb{N}\}, \{<n^2a> : n \in \mathbb{N}\}, \{<(p+1)a> : p \text{ prime}\} \text{ etc.}

The example before shows that \(b/\) is the best one can expect in general.

However good results are known for particular sequences: Wierdl [25] has shown, by means of exponential sums, that

\{ \lfloor an^b \rfloor : n \in \mathbb{N} \} \text{ is a set of recurrence, for every } a \text{ and } b > 0 .
Analogous definitions to those above can be made for general topological groups and their actions, continuous or otherwise. Theorems corresponding to those above can be manufactured without much difficulty. In Chapter 6, however, the general discrete case will be reexamined in the light of Chapters 3, 4 and 5.
CHAPTER III.

THE CONSTRUCTION OF A SET OF TOPOLOGICAL RECURRENCE WHICH IS NOT A SET OF MEASURE THEORETICAL RECURRENCE

Connections between Recurrence, Combinatorial Number Theory and Graph Theory:

As was mentioned in the introduction, there is a strong motivation for studying recurrence as a means to prove results in combinatorial number theory. This connection is made precise below:

Definitions:

E, a set of integers, has an upper density defined by

$$
\bar{d}(E) = \lim_{n \to \infty} \frac{|E \cap [-n, n]|}{2n + 1}
$$

and if the limit exists, it is called the density of E.

Given a subset E, of \( \mathbb{Z} \),

$$
E - E = \{ e - e' : e, e' \in E \}
$$

(3.2)

is called the difference set of E.

A subset, R, of \( \mathbb{Z} \) is density intersective if, for any set, E, of integers of positive density, there is an element of R contained in E-E.
A \( k \)-colouring of \( \mathbb{Z} \) is a function, \( c \), from \( \mathbb{Z} \) to \( \{1,\ldots,k\} \). Equivalently, this indicates a partition of \( \mathbb{Z} \) into \( k \) sets.

\( R \) is colour intersective, if, for any finite partition of \( \mathbb{Z} \), there is a cell, \( C \), and an element, \( r \), of \( R \), in \( C-C \).

**Remark:** Another definition for density intersectivity could be made with "positive upper density" written instead of "positive density." Ruzsa, [23], shows, however, that the two definitions are identical.

With this in mind, it is easy to see that if a set of integers is density intersective, then it is colour intersective.

**Question 3.1 (Bergelson [2]):** Is a set of colour intersectivity necessarily density intersective?

This question was answered in the negative by Kriz [19]. It is the purpose of this chapter to reproduce this result in a dynamical context.

The following construction, a simpler version of the construction of chapter 2 and taken verbatim from [14], will be used in the study of topological recurrence.

Let \( k \) be fixed and consider the metric, \( d \), on \( \Omega = \{1,\ldots,k\}^\mathbb{Z} \) defined as

\[
d(x,y) = \inf \{ 1/(n+1) : x(i)=y(i) \text{ for all } |i| \leq n \}.
\]

(3.3)

This is a natural metric which induces a topology equivalent to the compact Tychonov topology on the product. Define the shift on \( \Omega \):

\[
(Tx)(n) = x(n+1) \text{, for all } n \in \mathbb{Z}.
\]

(3.4)

\( T \) is clearly a homeomorphism on \( \Omega \).

Let \( c \) be a given element of \( \Omega \); a \( k \)-colouring of \( \mathbb{Z} \). Let \( Y = \overline{O(c)} \),

(3.5)

the orbit closure of \( c \) with respect to the transformation \( T \).
(Y,T) is a topological dynamical system.

**Definition:** (Y,T) will be referred to as the shift system induced by c.

**Lemma 3.2:** Suppose R is a set of integers. Then the following two conditions are equivalent:

i/ R is a set of topological recurrence.

ii/ Given any finite coloring, c, of Z there is an integer, n, and r in R such that

\[ c(n) = c(n+r). \]  \hspace{1cm} (3.6)

**Proof:** (i/ \(\Rightarrow\) ii/) Let c be a k-coloring of Z. Form the Shift System, Y, generated by this c.

Consider the (clopen) subsets, U(i), of Y defined for each 1 \(\leq\) i \(\leq\) k:

\[ U(i) = \{ y \in Y : y(0) = i \}. \] \hspace{1cm} (3.7)

This forms a finite partition of Y and so there is an \(i_0\) such that \(U(i_0)\) contains a uniformly recurrent point \(y_0\). Therefore, \(U = U(i_0)\) is a neighborhood of \(y_0\) and so there is an \(r\) such that \(T^{-r}U \cap U \neq \emptyset\). \hspace{1cm} (3.8)

The orbit of c is dense in Y and so in particular there is an \(n\) such that

\[ T^n c \in T^{-r}U \cap U. \] \hspace{1cm} (3.9)

Thus \(T^n c\) and \(T^{n+r} c\) are both in \(U(i_0)\), \(c(n) = c(n+r) = i_0\), \hspace{1cm} (3.10)

and ii/ is confirmed.
(ii/ ⇒ i/) Let \((X, T)\) be a minimal topological dynamical system and \(U\) an open neighborhood of a point, \(x\). Since the complement of \(\bigcup_{n \in \mathbb{Z}} T^n U\) is closed and \(T\)-invariant, it is empty. Thus there is a finite sub cover \(X = \bigcup_{1 \leq i \leq k} T^n U\). (3.11)

Let \(c(n) = \min \{ i : T^n x \in T^n U \}\), (3.12)
a finite colouring of \(\mathbb{Z}\).

Since \(R\) is colour-intersective, there is an integer \(n\) and \(r\) in \(R\) such that \(T^{n+r} \in T^i U\) and \(T^nx\) are in \(T^U\) for some \(i\). Thus \(T^{n+r}x\) is in both \(T^i U\) and \(U\), and we are done.

Kriz answered Bergelson's question as a problem in graph theory: The following definitions and lemmas connect the graph theoretical description with the original form of the question.

**Definitions:** Let \(G\) be a graph with \(\mathbb{Z}\) as its set of vertices.

It is shift invariant if, for any given edge, \(\{a, b\}\), of \(G\), and any integer, \(n\), \(\{a+n, b+n\}\) is also an edge of \(G\). A shift invariant graph is defined uniquely by the set \(R(G) = \{r : \{0, r\} \text{ is an edge of } G \}\). (3.13)

\(G\) is then said to be generated by the set \(R\).

A graph has chromatic number \(k\) if a \(k\)-colouring is necessary and sufficient to colour its set of vertices in such a way that every edge connects vertices of distinct colours. It has infinite chromatic number if no finite \(k\) is sufficient.
The following is simply a restatement of the definition of colour-intersectivity:

**Lemma 3.3:** If \( R \) is a set of integers and \( G \) the graph generated by \( R \), then the following are equivalent:

i/ \( R \) is colour intersective.

ii/ \( G \) has infinite chromatic number.

**Definition:** Let \( G \) be a graph with \( Z \) as its set of vertices.

A subset, \( E \), of \( Z \) is said to be independent if, whenever \( \{a, b\} \) is an edge of \( G \), \( a \) and \( b \) cannot both be in \( E \).

**Lemma 3.4:** If \( R \) is a set of integers and \( G \) is the shift invariant graph on \( Z \) generated by \( R \), then the following are equivalent:

i/ \( R \) is density intersective.

ii/ \( R \) is a set of measure theoretical recurrence.

iii/ There are no sets, \( E \), of positive upper density which are independent of \( G \).

**Proof:** The equivalence of i/ and ii/ is apparently well known. It is implicit in Furstenberg [14,p.72 ff.], proved in Bertrand-Mathis [6] and generalised by Bergelson [4].

For i/ and iii/; note that this would be immediate from the definition if part iii/ had said "positive density" instead of "positive upper density". However, as was said before, this is no loss of generality.

**A Note on General Groups:** The definitions and lemmas above proceed analogously for other discrete abelian group actions and, in particular, for \( \mathbb{Z}_p^\infty \), the
countable direct sum of $\mathbb{Z}_p$ ($= \{0,1,\ldots,p-1\}$, addition mod $p$). \hfill (3.14)

In this important case, which will be referred to often in this chapter and the chapters to come, here are appropriate definitions.

**Definitions:** Let $B$ be a subset of $\mathbb{N}$.

Define $\mathbb{Z}_p^B = \{ v \in \mathbb{Z}_p^\infty : v(n) = 0 \text{ for all } n \notin B \}$. \hfill (3.15)

Where there is no risk of confusion, $[a,b]$ will refer to the set $\{a,a+1,\ldots,b-1,b\}$ and $[a,b)$ to $\{a,a+1,\ldots,b-1\}$.

The upper density of a set, $E$, in $\mathbb{Z}_p^\infty$ is defined as:

$$\overline{d}(E) = \lim_{n \to \infty} \frac{|E \cap \mathbb{Z}_p^{[1,n]}|}{p^n}.$$ \hfill (3.16)
Ergodic Theoretical Interpretations of Kriz's Result and the Resolution of Problem of Furstenberg:

Bergelson asked whether a set of colour intersectivity need be a set of density intersectivity.

Equivalently (lemmas 3.2 and 3.4), it was being asked whether a set of measure theoretical recurrence need be a set of topological recurrence.

The following generalities show that the last form of this question is natural:

Suppose that $(X,B,p,T)$ is a measure preserving system and $A$ a measurable set. Then, without loss of generality, $X$ is a separable compact Hausdorff space and $T$ is a homeomorphism (see Ellis [12] for example). Also (see [12]), without loss of generality, $A$ is a clopen set and the support of the measure is the whole of $X$. Thus it is natural to compare the topological and measure theoretical properties of the system defined by $T$.

Suppose further that $R$ is a Set of Topological Recurrence and $(X,B,\mu,T)$ is an ergodic measure preserving system with a set, $A$, of positive measure, with the additional properties mentioned in the paragraph above for which

$$\mu(A \cap T^r A) = 0 \text{ for all } r \in R.$$  \hspace{1cm} (3.17)

(i.e. $R$ is not a Set of Measure Theoretical Recurrence). Then $(X,T)$ cannot be minimal, since, if it were, the topological recurrence properties of $R$ would return $A$ to itself, $A$ being open. On the other hand $(X,T)$ is topologically transitive (see [12, p.25; 14, p.152; 21, Prop2.5]) since the original system is ergodic.

This last fact is pertinent to Ellis's analysis of the connections between topological dynamics and ergodic theory which shows that the divide between minimality and topological transitivity is important [12, pp.25,26].
Note that the generalities above should not be confused with those allowed by the Jewett-Krieger theorem which constructs a minimal uniquely ergodic system on a Cantor set, \( C \), that is metrically isomorphic to \( (X,B,\mu,T) \). (see [21], for example). The fact that \( (C,T) \) is minimal does not contradict the topological recurrence of \( R \) above since there is no guarantee that the image of \( A \) in \( C \) is open.

This chapter constructs a negative answer to Bergelson's question. Incidentally, this construction can be used to give a negative answer to a question of Furstenberg [14,p.76] which asks if, for every set, \( D \), of positive density, there is a syndetic set \( S \) for which \( D-D \subseteq S-S \).

**Theorem 3.5:** There is a set \( D \), of positive density, for which there is no syndetic set, \( S \), satisfying \( D-D \supseteq S-S \). (3.18)

**Proof:** Let \( R \) be a set of colour intersectivity which is not a set of density intersectivity. Further, let \( D \) be a set of positive density in \( \mathbb{Z} \) for which \( R \) does not intersect \( D-D \).

Suppose that \( S \) is a syndetic set and \( D-D \supseteq S-S \). (3.19)

Then, there exists an \( n \) for which \( \bigcup_{0 \leq i < n} S + i = \mathbb{Z} \). (3.20)

This forms a partition of \( \mathbb{Z} \) and since \( R \) is a set of topological recurrence, there is an intersection between \( R \) and \( (S+i)-(S+i) \) for some \( 0 \leq i < n \). However,

\[
S-S = (S+i)-(S+i)
\]

and this is a contradiction by construction.
The graph theoretical form of Bergelson's question can also be expressed succinctly: Must a graph on \( \mathbb{Z} \), which is shift invariant and is not independent of any set of positive upper density, have infinite chromatic number? (see lemmas 3.3 and 3.4).

This combinatorial form is the one which Kriz answers by constructing a graph, \( G \), on \( \mathbb{Z} \) which is shift invariant, has an infinite chromatic number and an independent set, \( E \), of strictly positive density (indeed arbitrarily close to a half). His method is entirely combinatorial and makes no reference to dynamical interpretations.

**A simplification of Kriz's result:**

Initially, using some well established combinatorial results, the counter example is constructed in \( \mathbb{Z}_2^\infty \). The reason that \( \mathbb{Z}_2^\infty \) is chosen in preference to \( \mathbb{Z} \) as the base group is that Kriz's example is described naturally in the rich structure of \( \mathbb{Z}_2^\infty \). In fact the graph that Kriz constructs is made up of graphs whose vertices are best described as finite subsets of \( \mathbb{N} \). Finite subsets of \( \mathbb{N} \) are in natural (1-1) correspondence with elements of \( \mathbb{Z}_2^\infty \).

The counter example for \( \mathbb{Z}_2^\infty \) is then transferred to \( \mathbb{Z} \), by a combinatorial construction. This differs from Kriz's approach which used finite approximations in \( \mathbb{Z} \) and put them together at the end.

**The construction in \( \mathbb{Z}_2^\infty \):**

Kriz uses a classical result from graph theory:

Consider the set, \( G \), of all subsets of \( \{1, \ldots, 2n+k\} \) which have \( n \) elements. Connect two elements of \( G \) if and only if they don't intersect. Thus \( G \) becomes a graph with \( \binom{2n+k}{n} \) vertices, called the Kneser Graph, written \( G(2n+k,n) \).
**Lemma 3.6**: The Kneser Graph $G(2n+k,n)$, has chromatic number $k+2$.

**Proof**: This is Kneser's famous conjecture [18], which was proved by Lovasz [20], using a result from homotopy theory. A shorter proof is given by Barany [1] and is given in the Appendix for the reader's convenience.

Consider now $\mathbb{Z}_2^{2n+k}$, which is an abelian group and acts on itself to form a topological and measure preserving dynamical system; the measure, $\mu$, in question being the normalised counting measure.

The abelian group action will be written additively in what follows.

There is a natural metric on $\mathbb{Z}_2^{2n+k}$ defined by the 'norm'

$$|v| = \sum_{1 \leq i \leq 2n + k} v_i$$

(3.22)

where $v = (v_1, v_2, \ldots, v_{2n+k})$ and each of the numbers $v_i$ equals 0 or 1.

The Kneser Graph $G(2n+k,n)$ may be embedded in $\mathbb{Z}_2^{2n+k}$:

Vertices: $V(G) = \{ x \in \mathbb{Z}_2^{2n+k} : |x| = n \}$,

Edges: $E(G) = \{ \{x,y\} \in V(G)^{(2)} : |x+y| = 2n \}$. (3.23)

This is a subgraph of $G'$, where

$$V' = V(G') = \mathbb{Z}_2^{2n+k}$$

and

$$E' = E(G') = \{ \{x,y\} \in V(G')^{(2)} : |x+y| = 2n \}.$$ (3.24)
Let \( M_{n,k} = \{ x \in \mathbb{Z}_2^{2n+k} : |x| < n \} \), so that

\[
\mu(M_{n,k}) = \frac{1}{2^{2n+k}} \sum_{0 \leq i \leq n} \binom{2n+k}{i},
\]

a number which approaches \( 1/2 \) for \( k = o(n^{1/2}) \). (3.27)

The diameter of \( M_{n,k} \) is strictly less than \( 2n \) and so no edge of \( G' \) can have both its end points in \( M_{n,k} \). Therefore, \( M_{n,k} \) is independent of \( G' \). However, \( G' \) has chromatic number at least \( k \).

Thus, although the full result is still some distance away, it is apparent how Kneser Graphs can generate sets with poor density intersectivity properties. Further, these sets will have high chromatic number and so will have good topological recurrence properties.

**Definition:** For \( a \) in \( \mathbb{Z}_2^N \) define

\[
\text{supp } a = \{ i : a_i = 1 \};
\]

the support of \( a \).

**Theorem 3.7:** There is a set of topological recurrence in \( \mathbb{Z}_2^\infty \) which is not a set of measure theoretical recurrence.
Proof: Given an \( \varepsilon > 0 \) let \( \varepsilon_k \) be chosen so that
\[
1 - 2\varepsilon \leq \prod_{k \geq 1} (1 - 2\varepsilon_k) \tag{3.29}
\]
and pick \( n_k \) so that
\[
\frac{1}{2} \sum_{0 \leq i \leq n_k} \binom{2n_k + k}{i} \geq (1 - \varepsilon_k)/2 \tag{3.30}
\]
and
\[
\frac{1}{2} \sum_{0 \leq i \leq k} \binom{2n_k + k}{i} < \varepsilon_k/2. \tag{3.31}
\]

Partition \( \mathbb{N} \) into adjacent blocks of length \( 2n_k + k \):
\[
\mathbb{N} = \bigcup_{k \geq 1} [b_k, b_{k+1}) : b_{k+1} - b_k = 2n_k + k. \tag{3.32}
\]

Now let \( R_k = \{ v \in \mathbb{Z}_2^\infty : [b_k, b_{k+1}) \supset \text{supp } v, \ |v| = 2n_k \} \tag{3.33} \)
defined for all \( k \geq 1 \), and let \( R = \bigcup_{k \geq 1} R_k. \tag{3.34} \)

By the argument of the example given immediately above, the graph in \( \mathbb{Z}_2^\infty \), induced by \( R \), has infinite chromatic number. An application of Lemmas 3.2 and 3.3 shows that \( R \) is a set of topological recurrence.

Further, define
\[
M(k,0) = \{ v \in \mathbb{Z}_2 [b_k, b_{k+1}) : |v| < n_k \}, \text{ and } \]
\[
M(k,1) = \{ v \in \mathbb{Z}_2 [b_k, b_{k+1}) : 2n_k > |v| > n_k + k \}. \tag{3.35}
\]
Note: Note that, in the following argument, \( M(k,1) \) could have just as easily been defined as \( \{ v \in \mathbb{Z}_2^{[b_k, b_{k+1}]} : |v| > n_{k+k} \} \). However, the present definition anticipates a technicality which will only become apparent at the end of the chapter. The substance of the argument in both cases is the same. Note also that, as subsets of \( \mathbb{Z}_2^{[b_k, b_{k+1}]} \), \( d(M(k,0)) > (1-\varepsilon_k)/2 > 1/2 - \varepsilon_k \) and \( d(M(k,1)) > 1/2 - \varepsilon_k \).

Let \( M = \bigcup_{K \in \mathbb{N}} \sum_{l \in K} M_i \),

where the union is taken over all finite subsets, \( K \), of \( \mathbb{N} \) and all \( \theta : K \to \{0,1\} \) such that \( |\theta| \) is even. (The sum of a collection of sets is the set of all sums of individual elements, one from each of the sets.)

a/ Calculation of the density of \( M \):

\[
\overline{d}(M) = \lim_{N \to \infty} \frac{|M \cap \mathbb{Z}_2^{[1,N]}|}{2^N}
\]

\[
\geq \lim_{k \to \infty} \frac{|M \cap \mathbb{Z}_2^{[1,b_{k+1}-1]}|}{2^{b_{k+1}-1}}
\]

\[
\geq \frac{1}{2} \lim_{k \to \infty} \prod_{i \leq k} (1 - 2 \varepsilon_i)
\]

\[
\geq \frac{1}{2} (1 - 2 \varepsilon)
\]
The second last inequality (3.39) comes from the fact that

\[ \sigma = \left| \bigcup_{K, \theta} \sum_{i \in K} M(i, \theta_i) \right|, \quad (3.41) \]

the union being taken over all subsets, \( K \), of \([1,k]\) and \( \theta: K \to \{0,1\} \) of even norm,

equals

\[ \left| \bigcup_{\theta} \sum_{i \leq k} M(i, \theta_i) \right|, \]

(3.42)

the union now being taken over all \( \theta: [1,k] \to \{0,1\} \) of even norm.

By the calculations in the example above, each disjoint sum \( \sum_{i \leq k} M(i, \theta_i) \) has cardinality exceeding \( \prod_{i \leq k} (1 - 2\theta_i) \cdot 2^{b_{k+1} - k - 1} \) and so the union over the \( 2^{k-1} \) choices of \( \theta \) yields a value for \( \sigma \) in excess of \( \prod_{i \leq k} (1 - 2\theta_i) \cdot 2^{b_{k+1} - 2} \) and the inequality follows.

b/ The independence of \( M \) from \( R \):

Suppose that there is an \( r \) in \( R \cap 2^{[b_k, b_{k+1}]} \) and an \( m \) in \( M \) such that \( m+r \) is also in \( M \).

By construction, there are finite subsets, \( K \) and \( K' \), of \( \mathbb{N} \) and maps \( \theta: K \to \{0,1\} \) and \( \theta': K' \to \{0,1\} \), each of whose norms is an even integer, such that \( m \) is a sum of elements, \( m_i \), from \( M(i, \theta_i) \), \( i \in K \), and \( m+r \) is a sum of elements, \( m_j' \), from \( M(j, \theta'_j) \), \( j \in K' \). In fact, since \( 0 \in M(i,0) \) for all \( i \), without loss of generality, \( K \) and \( K' \) may be assumed to be equal. Therefore, \( k \) is in \( K = K' \).

By the construction of \( r : \theta_i = \theta'_i \) if \( i \neq k \), and \( \theta_k = 1-\theta'_k \).

\[ (3.43) \]

\[ (3.44) \]
This contradicts the requirement that both \( \theta \) and \( \theta' \) have even norm and so we are done.

This construction may seem cumbersome but it reflects the subtlety of the problem at hand. The inspiration for this example comes from Kriz's work [19]. However, a few points have been simplified: In particular, the limiting procedure is removed from the construction replacing it with a direct sum of finite examples. Also the example is quite explicit.
The construction in $\mathbb{Z}$:

This section will show how to adapt the example above to give an example of a set of topological recurrence which will fail to return a set of integers of density arbitrarily close to $1/2$. The construction is fairly general, however, and gives rise to an easy exchange of properties between constructions in $\mathbb{Z}_2^\infty$ and $\mathbb{Z}$.

The idea is simple but its expression is involved: One hopes (in vain) for a (1-1) group homomorphism from $\mathbb{Z}_2^\infty$ to $\mathbb{Z}$. A set of topological recurrence, $R'$, in $\mathbb{Z}$ which is not a set of measure theoretical recurrence could be constructed as the forward image under this homomorphism of the $R$ constructed in $\mathbb{Z}_2^\infty$. A set of positive density in $\mathbb{Z}$ which would not be returned by $R'$ would be constructed as the union of the homomorphic image of the set in $\mathbb{Z}_2^\infty$ which was not returned by $R$ and the various shifts of this image into cosets of the homomorphic image of $\mathbb{Z}_2^\infty$ in $\mathbb{Z}$. The topological recurrence would be easy to recover by graphical considerations. However, such a homomorphism is impossible and the technical lemmas which follow are applied to overcome this problem while imitating the 'argument' outlined above.

First some **definitions** which will make the technicalities easier to describe.

A subset, $R$, of $\mathbb{Z}_2^\infty$ is said to be well separated if there are integers, $b_1 < b_2 < ...$, and sets $R_1, R_2, ... : \mathbb{Z}_2^{[b_i, b_{i+1})} \supseteq R_i$ such that

$$R = \bigcup R_i. \quad (3.45)$$

Note that the set, $R$, constructed before is well separated.

Let $p_i$ be a sequence of integers, yet to be defined, with the properties:

$$p_1 = 1, \text{ and } 2p_i \mid p_{i+1} \text{ for all } i > 0. \quad (3.46)$$
For a given $p$, let 
$$f_p(m) = \begin{cases} 
0 & \text{if } 0 \leq m < p \mod 2p \\
1 & \text{if } p \leq m < 2p \mod 2p, 
\end{cases}$$
(3.47)
defined for all $m$ in $\mathbb{Z}$.

Let $N$ be chosen large and let $f: \mathbb{Z}_{2^p} \rightarrow \mathbb{Z}_2^N$ be defined:
$$f(m) = (f_{p_1}(m), f_{p_2}(m), \ldots, f_{p_N}(m)).$$
(3.48)

$s$, in general, denotes a function $\{1, \ldots, N\} \rightarrow \{1, -1\}$.

Given $s$, let
$$g_s: \mathbb{Z}_2^N \rightarrow \mathbb{Z}_{2^p},$$
$$g_s: (a_i) \rightarrow \sum_{1 \leq i \leq N} s(i)a_ip_i$$
(3.49)

Let $M$ and $R$ be subsets of $\mathbb{Z}_2^N$. The subset $f^{-1}(M)$, of $\mathbb{Z}_{2^p}$, will be important, as will be the set:
$$R^* = \bigcup_{s: \{1, 2, \ldots, N\} \rightarrow \{1, -1\}} g_s(R).$$
(3.50)

If the $p_i$ are chosen sufficiently rapidly increasing, then shifts of $f^{-1}(M)$ in $\mathbb{Z}_{2^p}$ will correspond closely to shifts of $M$ in $\mathbb{Z}_2^N$. $f$ is a map from $\mathbb{Z}_{2^p}$ to $\mathbb{Z}_2^N$ which, ideally, would be a homomorphism, but is an adequate approximation to a homomorphism.

It will be proved in Theorem 3.9 that, if $R$ is a set of topological recurrence in $\mathbb{Z}_2^\infty$ which is not a set of measure theoretical recurrence and which is well separated,
then $R^*$ is a set of topological recurrence in $\mathbb{Z}$ which is not a set of measure theoretical recurrence, for some suitable choice of $p_i$.

Suppose that $S$ is a subset of an abelian group, $H$:

Let $K_H(S)$ be the chromatic number of the shift invariant graph induced by $S$ in $H$.

Let $d$ denote the density of a subset in whichever finite group it happens to sit. If there is an ambiguity then the group will be mentioned in the text.

**Lemma 3.8:** If $R$ is a subset of $\mathbb{Z}_2^N$, then there exists a subset, $R^*$, of $\mathbb{Z}_{2p_N}^N$ such that

\[ K_{\mathbb{Z}_2^N}(R) \leq K_{\mathbb{Z}_{2p_N}^N}(R^*) \quad (3.51) \]

\[ K_{\mathbb{Z}_2^N}(M) \leq K_{\mathbb{Z}_{2p_N}^N}(M^*) \quad (3.52) \]

\[ d(M^*) \geq d(M) \left(1 - c \left( \sum_{r \in R} 2^{2|r|} \sum_{i=1}^{N} \frac{p_i}{p_i + 1} \right) \right) \]

where $c$ is some absolute constant.

**Proof:** Construct $R^*$ and $f^{-1}(M)$ in $\mathbb{Z}_{2p_N}$ as defined before.

\[ K_{\mathbb{Z}_2^N}(R) > k \quad (3.53) \]

and let $c : \mathbb{Z}_{2p_N}^N \rightarrow \{1,...,k\}$, be a $k$-colouring. \( (3.54) \)
Let $c': \mathbb{Z}_2^N \to \{1, \ldots, k\}$, be induced by $c: c'(m) = c(g_1(m))$, \hfill (3.55)

where 1 is the constant +1 sign.

By hypothesis, there is an $r \in \mathbb{R}$ and $m \in \mathbb{Z}_2^N$ such that

$$c'(m+r) = c'(m).$$ \hfill (3.56)

Thus

$$c(g_1(m+r)) = c(g_1(m)).$$ \hfill (3.57)

But

$$g_1(m+r) = g_1(m) + g_S(r),$$ \hfill (3.58)

where $s(i) = \begin{cases} 1 & \text{if } m_i = 0 \text{ or } r_i = 0 \\ -1 & \text{if } m_i = r_i = 1. \end{cases}$ \hfill (3.59)

So

$$c(g_1(m) + g_S(r)) = c(g_1(m))$$ \hfill (3.60)

and, since $g_S(r)$ is in $\mathbb{R}^+$, we are done.

\text{ii/ Note that, since } f \text{ is a } \frac{2p_N}{2^N} \text{ to 1 map,}

$$|f^{-1}(M)| = |M| \frac{2p_N}{2^N}$$ \hfill (3.61)

and so

$$d(f^{-1}(M)) = d(M).$$ \hfill (3.62)

Further,

$$d((f^{-1}(M)+g_S(r)) \Delta (f^{-1}(M)+g_{S'}(r))) < d(M) \sum_{r_i = 1} p_i \frac{p_i}{p_i + 1}$$ \hfill (3.63),

for all choices of $S$ and $S'$. The proof of this inequality will come at the end.

Assuming (3.63), one obtains

$$d((f^{-1}(M+r) \Delta (f^{-1}(M)+g_S(r))) < d(f^{-1}(M))2c2^{|r|} \sum_{r_i = 1} p_i \frac{p_i}{p_i + 1}$$ \hfill (3.64)
This follows from the inclusion:

\[
\bigcup_{s: \{1, 2, \ldots, N\} \to \{-1, 1\}} (f^{-1}(M) + g_s(r)) \supseteq f^{-1}(M+r)
\]  

(3.65)

a union of \(2^{|r|}\) different sets. To prove this inclusion it is sufficient to show it for \(M\), a one element set, \(\{m\}\). The full result is obtained by unions.

Note that \(f^{-1}(m) = f^{-1}(0) + g_1(m)\).

This fact together with the fact that \(g_1(m+r) = g_1(m) + g_s(r)\), where

\[
s(i) = \begin{cases} 
-1 & \text{if } r_i m_i = 1 \\
1 & \text{otherwise,}
\end{cases}
\]

(3.67)

shows that \(f^{-1}(m) + g_s(r) = f^{-1}(m+r)\) exactly. See (3.57) (3.68)

And so (3.65) follows.

Thus \(f^{-1}(M+r) \setminus (f^{-1}(M) + g_s(r))\) is contained in

\[
\bigcup_{s': \{1, 2, \ldots, N\} \to \{-1, 1\}, g_{s'}(r) \neq g_s(r)} [(f^{-1}(M) + g_{s'}(r)) \setminus (f^{-1}(M) + g_s(r))]
\]

(3.69)

a set of density at most \(d(f^{-1}(M))c(2^{|r|} \sum_{r_i = 1} p_i / (p_i + 1))\), by (3.63)

And \((f^{-1}(M) + g_s(r)) \setminus f^{-1}(M+r)\), similarly, is contained in

\[
[\bigcup_{s'} (f^{-1}(M) + g_{s'}(r)) ] \setminus f^{-1}(M+r),
\]

(3.70)

and this has density equal to
\[ d(\bigcup_{s'} (f^{-1}(M) + g_s'(r))) - d(f^{-1}(M)). \] (3.71)

(3.63) again shows that this expression is bounded above by
\[ d(f^{-1}(M))c(2^2|r| \sum_{r \in R} \frac{p_i}{p_{i+1}}), \] (3.72)
and (3.64) follows.

Let \( M \) be a subset of \( \mathbb{Z}_2^N \) such that \( M \cap M + r = \emptyset \) for all \( r \in R \). (3.73)

Compute \( f^{-1}(M) \) etc. as before, and let
\[ M^* = f^{-1}(M) \setminus \bigcup_{r \in R} \bigcup_{s} (f^{-1}(M) + g_s(r)). \] (3.74)

Note that
\[ d(M^*) > d(f^{-1}(M)) - d(f^{-1}(M)) \left( \sum_{r \in R} 2c2^2|r| \sum_{r \in R} \frac{p_i}{p_{i+1}} \right) > 0 \] (3.75)
for some suitable choice of \( p_i \), since \( f^{-1}(M+r) \cap f^{-1}(M) = \emptyset \). Also by construction \( M^* \) is not returned by \( R^* \).

To complete the lemma, it remains to prove (3.63).

First examine the case \( M = \{0\} \): (3.76)

It is straightforward to check that, for each \( i < N \)
\[ \left| (f^{-1}(0) + p_i) \Delta (f^{-1}(0) - p_i) \right| = \frac{8p_N p_i}{2^N p_{i+1}} \] (3.77)

To see this, consider Figure 1 where the set \( f^{-1}(0) \) is represented as a subset of \([0, 2p_N]\). The black lines correspond to blocks of length \( p_i \) and they are grouped in strings of \( p_{i+1}/2p_i \) blocks, each string of total length about \( p_{i+1} \). The number of elements of \( f^{-1}(0) \) in each \( p_i \)-block is \( p_i2^{i+1} \) and the number of \( p_{i+1} \)-strings is
\[ p_N 2^{i-N+1}/p_{i+1} \] (3.78).
FIGURE 1
One then observes that the left hand side of (3.77) is exactly $2 \times (\text{number of elements of } f^{-1}(0) \text{ in each } p_i\text{-block}) \times (\text{number of } p_{i+1}\text{-strings})$ \hfill (3.79).

Thus, if $s$ and $t$ are two sign functions : $\{1, 2, ..., N\} \rightarrow \{1, -1\}$, then

\[ \left| f^{-1}(0) + g_s(r) \right| = 8p_N2^{-N} \sum_{i \in J} \frac{p_i}{p_{i+1}} \] \hfill (3.80)

where $J$ is the set of indices, $i$, for which $r_i = 1$ and $t(i) \neq s(i)$.

For $M = \{m\}$ in general,

note that $f^{-1}(m) = f^{-1}(0) + g_1(m)$.

Further, if $s$ and $t$ are sign functions and $r \in Z_2^N$, then

\[ g_s(r) + g_1(m) = g_s(r + m) + 2g_1(r') \] \hfill (3.84)

and

\[ g_t(r) + g_1(m) = g_t(r + m) + 2g_1(r'') \]

where, for example:

\[ r_i' = \begin{cases} r_im_i & \text{if } s(i) = 1, \\ 0 & \text{otherwise}, \end{cases} \] \hfill (3.85)

\[ s'(i) = \begin{cases} s(i) & \text{if } m_i = 0 \text{ and } r_i = 1 \\ 1 & \text{otherwise}, \end{cases} \] \hfill (3.86)

and $r''$ and $t'$ are defined similarly.
Thus,

\[
\left| (f^{-1}(m) + g_s(r)) \right| \Delta (f^{-1}(m) + g_t(r)) \right|
\leq \left| (f^{-1}(0) + g_s(r + m) + 2g_t(r')) \right| + \\
+ \left| (f^{-1}(0) + g_s(r + m)) \right| \Delta (f^{-1}(0) + g_t(r + m)) + \\
+ \left| (f^{-1}(0) + g_t(r + m)) \right| \Delta (f^{-1}(0) + g_t(r + m) + 2g_t(r'')) \right|.
\]  
(3.87)

\[
\leq 8p_N^{-2N} \sum_{i:r_i' = 1} \frac{p_i}{p_i + 1} + 8p_N^{-2N} \sum_{i:r_i = 1} \frac{p_i}{p_i + 1} + \\
+ 8p_N^{-2N} \sum_{i:r_i'' = 1} \frac{p_i}{p_i + 1}
\]  
(3.88)
The inequality for the first term of the sum comes from the fact that

$$\left| (f^{-1}(0) + g_s(r + m) + 2g_1(r')) \Delta (f^{-1}(0) + g_s(r + m)) \right| =$$

$$= \left| (f^{-1}(0) + 2g_1(r')) \Delta (f^{-1}(0)) \right| =$$

$$= \left| (f^{-1}(0) + g_1(r')) \Delta (f^{-1}(0) - g_1(r')) \right| = 8p_N 2^{-N} \sum_{i: r_i = 1} \frac{p_i}{p_i + 1} \quad (3.89)$$

and the third likewise.

The middle term inequality comes from considering the set, $J$, of indices, $i$, for which $r_i + m_i = 1$ and $t'(i) \neq s'(i)$, and this, by construction, is contained in the support of $r$.

Note that the supports of $r'$ and $r''$ are both contained in the support of $r$.

Therefore,

$$\left| (f^{-1}(m) + g_s(r) \Delta (f^{-1}(m) + g_1(r)) \right| \leq 24p_N 2^{-N} \sum_{i: r_i = 1} \frac{p_i}{p_i + 1} \quad (3.90)$$

For $M$ more general, observe that $f^{-1}(M)$ is a disjoint union of sets of the form $f^{-1}(m)$, with $m$ in $M$ and so $(f^{-1}(M) + g_s(r)) \Delta (f^{-1}(M) + g_1(r))$ is contained in

$$\bigcup_{m \in M} \left( (f^{-1}(m) + g_s(r) \Delta (f^{-1}(m) + g_1(r)) \right) \quad (3.91)$$

a set of size less than $24p_N 2^{-N} |M| \sum_{i: r_i = 1} \frac{p_i}{p_i + 1} \quad (3.92)$.
Its density is, therefore, at most \[12d(M) \sum_{i,r_i=1} \frac{p_i}{p_{i+1}}\] and (3.63) is verified and the lemma completed.

\textbf{Theorem 3.9:} There is a set of topological recurrence in \( Z \) which is not a set of measure theoretical recurrence.

\textbf{Proof:} Consider the construction of \( M \) and \( R \) in \( Z_2^N \) made earlier in the chapter. It has two useful properties:

- \( a/ \) \( R \) is well separated, i.e. there is an increasing sequence of \( b_i \)'s such that \( R \) is contained in \( \bigcup_{k \geq 1} Z_2^{[b_k, b_{k+1})} \)

- \( b/ \) There is no element of \( M \) whose coordinates on a block \([b_k, b_{k+1})\) coincide with the corresponding coordinates of an element of \( R \) (see the technical definition, (3.33)**).

Let \( R(k) = R \cap Z_2^{b_k} \). (3.93)

Construct \( R(k)^* \) in \( Z_{2p_{b_k}} \), which is now equated with the initial segment, \( \{0,\ldots,2p_{b_k}-1\} \) of \( \mathbb{Z} \). By part \( i/ \) of lemma 3.8, the shift invariant graph induced by \( R(k)^* \) in \( Z_{2p_{b_k}} \) has chromatic number at least \( k \).

Note that, if the \( p_i \)'s increase fast enough,

\[ R(j)^* \cap \{0,\ldots,2p_{b_k}-1\} = R(k)^* \text{, whenever } j \geq k. \] (3.94)
To see this, it is sufficient to note that $R(k)$ is contained in $R(j)$ and so, since $R(k)^\ast$ is contained in $\{0, \ldots, 2p_{b_k} - 1\}$ anyway, the left hand side contains the right. Conversely, consider the following arithmetic condition on the $p_i$'s which is an effect of their rapid increase: For each integer, $k$,

$$4 \sum_{i \leq k} p_i \leq p_{k+1} \quad (3.95)$$

This ensures, for example, that, for each integer, $w$, there is at most one permutationally distinct choice of signs and finite subset, $J$, of the integers which solves the equation

$$\sum_{i \in J} \pm p_i = w \quad (3.96)$$

An element, $r$, of $R(j)$ that is not in $R(k)$ has at least one coordinate, the $i$th. say, equal to 1, where $b_{k+1} \leq i < b_j$. The arithmetic condition above shows that, then, $g_S(r) \geq 2p_{b_{i-1}} \geq 2p_{b_k}$ or $g_S(r) \leq -2p_{b_k}$. The reverse inclusion is proved.

So the union $R' = \bigcup_k R(k)^\ast \quad (3.97)$ is also a limit of sets and the shift invariant graph induced by $R'$ is of infinite chromatic number being the union of a collection of graphs whose chromatic numbers are increasing.

$R'$ is a set of topological recurrence by lemmas 3.3 and 3.2.

Let $M(k) = M \cap Z_2^{b_k} \quad (3.98)$

where $M$ is the set constructed earlier in the chapter.

Construct $M(k)^\ast$ in $Z_2^{2p_{b_k}}$, as in lemma 3.8, and note that, as above, the sets $M(j)^\ast$ eventually agree on initial segments, $\{0, \ldots, 2p_{b_k} - 1\}$, of $Z$. The argument for this proceeds as for $R'$ above: Let $j \geq k$. 
Recall equation (3.74) which defines $M(j)^*$ using $R(j)$:

$$M(j)^* = f^{-1}(M(j)) \setminus \bigcup_{r \in R(j)} \bigcup_{s} (f^{-1}(M(j)) + g_s(r)).$$  \hspace{1cm} (3.99)

So $M(j)^* \cap \{0, ..., 2p_{b_k}^{-1}\}$

$$= f^{-1}(M(j)) \cap \{0, ..., 2p_{b_k}^{-1}\} \setminus \bigcup_{r \in R(j)} \bigcup_{s} [(f^{-1}(M(j)) + g_s(r)) \cap \{0, ..., 2p_{b_k}^{-1}\}]$$  \hspace{1cm} (3.100)

where all additions are now in $Z$.

Now $f^{-1}(M(j)) \cap \{0, ..., 2p_{b_k}^{-1}\} = f^{-1}(M(k))$ \hspace{1cm} (3.101)

by a similar argument as for $R'$ above.

Also, suppose that $r$ is in $R(j) \setminus R(k)$, and so, by property a/ above, has all its coordinates concentrated on a block $[b_t, b_{t+1})$, $t > k$. Suppose that $s$ is some sign function and that $w$ is an element of $f^{-1}(m) + g_s(r)$ where $m$ is some element of $M(j)$.

An element of $f^{-1}(m)$ can be written in the form

$$\sum_{i \in J} a_i p_i$$  \hspace{1cm} (3.102)

where $J$ is a finite subset of the integers and $a_i$ are integers such that

$$-p_{i+1} \leq 2a_i p_i \leq p_{i+1}.$$

So, in order that $w$ be in the interval $\{0, ..., 2p_{b_k}^{-1}\}$ it is necessary that the coordinates of $m$ in the block $[b_t, b_{t+1})$ coincide with those of $r$. Otherwise, $w$ can be described as a difference of an element of $f^{-1}(m) + g_s(r)$ which hence has the form

$$\sum_{i \in J} a_i p_i$$  \hspace{1cm} (3.104)

where $J$ is a finite subset of the integers which contains an element of $[b_t, b_{t+1})$ and $a_i$ are nonzero integers such that

$$-p_{i+1} \leq 2a_i p_i \leq p_{i+1},$$

and so, by the arithmetic condition, cannot lie in $\{0, ..., 2p_{b_k}^{-1}\}$.
However, for this particular $M$, this contradicts property b/.

Thus

$$\bigcup_{r \in R(j)} \bigcup_{s} [(f^{-1}(M(j)) + g_s(r)) \cap \{0, \ldots, 2p_{b_k} - 1\}]$$

$$= \bigcup_{r \in R(k)} \bigcup_{s} [(f^{-1}(M(j)) + g_s(r)) \cap \{0, \ldots, 2p_{b_k} - 1\}]$$

(3.106)

$$= \bigcup_{r \in R(k)} \bigcup_{s} [(f^{-1}(M(k)) + g_s(r)) \cap \{0, \ldots, 2p_{b_k} - 1\}]$$

(3.107)

the final equality coming from another application of the arithmetic property: An element of $f^{-1}(M(j)) \setminus f^{-1}(M(k))$ cannot be shifted to the interval $\{0, \ldots, 2p_{b_k} - 1\}$ by an element of $R(k)^*$. So

$$M(j)^* \cap \{0, \ldots, 2p_{b_k} - 1\} = M(k)^* \cap \{0, \ldots, 2p_{b_k} - 1\}.$$  

(3.108)

$M(k)^*$, as a subset of $\mathbb{Z}$ or $\mathbb{Z}_{2p_{b_k}}$, is independent of $R(k)^*$. If $M(j)^*, j \geq k$, were not independent of $R(k)^*$, then $M(j)^*$ would not be independent of $R(j)^*$, a contradiction. So this implies that $M'$ is also independent of $R(k)^*$ and hence of $R'$.

Further $M'$ will have upper density bounded below by means of the expression given in part ii/ of lemma 3.8.

This is follows from the equation

$$\frac{|M' \cap \{0, \ldots, 2p_{b_k} - 1\}|}{2p_{b_k}} = \frac{|M(k)^* \cap \{0, \ldots, 2p_{b_k} - 1\}|}{2p_{b_k}}$$

(3.109)
\[ d(M(k)) \left( 1 - c \left( \sum_{r \in R(k)} 2^{2|t|} \sum_{r_i = 1} p_i \right) \right) \] (3.110)

\[ \geq \left[ \frac{1}{2} - \epsilon \right] \left( 1 - c \left( \sum_{r \in R} 2^{2|t|} \sum_{r_i = 1} \frac{p_i}{p_{i+1}} \right) \right) \] (3.111)

Note that

\[ \sum_{r \in R} 2^{2|t|} \sum_{r_i = 1} \frac{p_i}{p_{i+1}} \leq \sum_{k \geq 1} 2^{3|b_{k+1} - b_k|} \sum_{b_k \leq l < b_{k+1}} \frac{p_i}{p_{i+1}} \] (3.112)

where the constants, \( b_k \), come from the well separation of \( R \).

A judicious choice of \( p_j \) will ensure that

\[ \left( 1 - c \left( \sum_{r \in R} 2^{2|t|} \sum_{r_i = 1} \frac{p_i}{p_{i+1}} \right) \right) > 0 \] (3.113)

indeed, arbitrarily close to 1. So we are done.

**Remark:** Note that, having decided on the sequence \( p_j \), the construction comes free of induction and compactness arguments. The technical lemmas proved above will be used again and extended in the next chapter.
CHAPTER IV.

RECURRENCE AND STRONG RECURRENCE

The Construction of a Set of Recurrence which is not a Set of Strong Recurrence:

Recall the definitions of recurrence and strong recurrence from chapter 2 (pp.11, 24). As was mentioned there, Bergelson introduced strong recurrence as a way of solving an interesting combinatorial problem using dynamics. It is therefore natural to ask if all sets of recurrence have this property:

Question 4.1 (Bergelson [4]): Are all sets of recurrence also sets of strong recurrence?

This chapter constructs a set of recurrence, $R$, an example of a measure preserving system $(X,\mathcal{B},\mu,T)$ and a measurable set, $A$, of measure $1/2$, such that

$$\lim_{r \to \infty: r \in R} \mu(T^rA \cap A) = 0. \quad (4.1)$$

In particular, $R$ is a set of recurrence but not a set of strong recurrence, giving a negative answer to the question above. Thus:
**Theorem 4.2:** There is a set of recurrence, \( R \), in \( \mathbb{Z} \), a measure preserving system \((X,B,\mu,T)\) and a set, \( A \), of measure 1/2, for which
\[
\lim_{r \to \infty: r \in R} \mu(T^r A \cap A) = 0. \tag{4.2}
\]

**Proof:** This follows the plan of the construction of the set of topological recurrence which is not a set of measure theoretical recurrence in Chapter 3. First, it makes an approximation in a finite sum of \( \mathbb{Z}_2 \), then puts better and better approximations together in \( \mathbb{Z}_2^{\infty} \) to obtain an appropriate counter example for \( \mathbb{Z}_2^{\infty} \) actions. This is then transferred, by combinatorial methods, to \( \mathbb{Z} \). The details of the proof make up the first section of this chapter.

As a bonus, one obtains a set of recurrence which is does not force the continuity of positive measures and so reproves a result of Bourgain [10].

**Definition:** A set, \( S \), forces the continuity of positive measures (is \( FC^+ \)) if every positive measure, \( m \), on the unit circle, \( T = \{ e^{it} : t \in [0,2\pi) \} \), \( \tag{4.3} \)

which has the property:
\[
\lim_{s \to \infty: s \in S} m^s(s) = \lim_{s \to \infty: s \in S} \int_T z^{-s} dm(z) = 0 \tag{4.4}
\]
is necessarily continuous (i.e. is free of atoms).

**Remark:** As was noted in the introduction, this version of the definition is due to Bourgain [10].

Bourgain [10], on the way to proving a much stronger result, constructs a set of measure theoretical recurrence which is not \( FC^+ \). It is now possible to construct this in a
different way:

**Theorem 4.3 (Bourgain [10]):** There is a set of measure theoretical recurrence which is not $FC^+$.  

**Proof:** Use the construction of Theorem 4.2 directly.  

Set $\mathcal{m}$ to be the positive real measure on the unit circle whose Fourier transform is the positive definite sequence  

$$m^\wedge(n) = \mu((T^nA \cap A)). \quad (4.5)$$

$\mathcal{m}$ has an atom since, by Weiner's Theorem, the ergodic average  

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^nA \cap A)^2 \geq \mu(A)^4 > 0 \quad (4.6)$$

equals the sum of the squares of the atoms of $\mathcal{m}$  

$$\sum_{\{a\} \text{ an atom of } \mathcal{m}} m(a)^2 \quad (4.7)$$

and so the continuity of $\mathcal{m}$ is not forced although $m^\wedge(n)$ tends to zero along $R$.  

However $R$ is a set of measure theoretical recurrence.

**The construction in $Z_2^\infty$:**  

Often, when making constructions in finite groups, it is more convenient to examine the group's action on itself than on probability measure spaces or compact topological spaces in general. The following definitions show that this simplification loses little generality when dealing with recurrence.
**Definitions:** Suppose that $R$ and $W$ are subsets of $\mathbb{Z}_2^{2N}$.

As before, define the density of $W$ as $d(W) = |W|/2^{2N}$. \hfill (4.8)

Define

$$L^0(R) = \max\{ d(W) : W \cap (W+r) = \emptyset \text{ for all } r \text{ in } R \}.$$ \hfill (4.9)

**Remark:** The density, and so also the function $L^0$, can be defined analogously in any finite group. The only other case that will be important here is $\mathbb{Z}_2^p$, the cyclic group of $2p$ elements. Where the group needs to be recognised, a subscript will be added to the symbol $L^0$: $L^0_{\mathbb{Z}_2^N}$, for example.

The following lemma shows that $L^0$ can help to show recurrence in $\mathbb{Z}_2^\infty$:

**Lemma 4.4:** Let $R$ be a subset of $\mathbb{Z}_2^\infty$ and suppose that $L^0(R \cap \mathbb{Z}_2^{[1,N]})$ tends to zero as $N$ tends to infinity, then $R$ is a set of recurrence in $\mathbb{Z}_2^\infty$.

**Proof:** Suppose that $(X,B,\mu,\mathbb{Z}_2^\infty)$ is a measure preserving system with a set $A$, of measure equal to $a$. Suppose, further, that $N$ has been chosen so that

$$L^0(R \cap \mathbb{Z}_2^{[1,N]}) < a.$$ \hfill (4.10)
There is a subset, $E$, of $Z_2^{[1,N]}$ of density greater than $a$ for which

$$\mu\left(\bigcap_{v \in E} T_v A\right) > 0.$$  \hspace{1cm} (4.11)

And so, by definition, there is an element, $r$, of $R \cap Z_2^{[1,N]}$ for which

$$E \cap E+r \neq \emptyset.$$  \hspace{1cm} (4.12)

This implies that $\mu(T_r A \cap A) > 0$ and we are done.  \hspace{1cm} (4.13)

**Remark:** When the construction in $Z_2^{\infty}$ is made it will be built out of initial segments with recurrence properties which satisfy the hypotheses of this lemma.

The following theorem gives the construction:

**Theorem 4.5:** There is a set of recurrence, $R$, in $Z_2^{\infty}$ and a measure preserving system $(X,B,\mu,Z_2^{\infty})$ with a set $A$, of measure equal to $1/2$, for which $\mu(T_r A \cap A)$ tends to zero as $r$ tends to infinity along $R$.

i.e. for all $\epsilon > 0$ the set $\{ r \in R: \mu(T_r A \cap A) > \epsilon \}$ is finite \hspace{1cm} (4.14)

**Remark:** This theorem clearly produces a set of recurrence which is not a set of strong recurrence in $Z_2^{\infty}$.

The Construction and Proof of Theorem 4.5:

Consider the product $Z_2^{2N}$, where $N$ is a large integer.

An element, $a = (a_1,a_2,...,a_{2N})$, in $Z_2^{2N}$

has various interpretations which will appear in the work which follows.
First, \( \mathbf{a} \) may be considered as a vector in \( \mathbb{R}^{2N} \) and, as such, has an "\( l_1 \) norm",
\[
|\mathbf{a}|_1 = \sum_{i=1}^{2N} a_i,
\]
the sum here being taken in \( \mathbb{R} \) and not \( \mathbb{Z}^2 \). This imposes a metric on \( \mathbb{Z}_2^{2N} \), namely
\[
d(\mathbf{a},\mathbf{b}) = |\mathbf{a} - \mathbf{b}|_1 = |\mathbf{a} + \mathbf{b}|_1.
\]
Alternatively, \( \mathbf{a} \) may be thought of as the indicator of a subset, \( \mathcal{A} \), of \( \{1, \ldots, 2N\} \):
\[
\mathcal{A} = \{ i : a_i = 1 \}.
\]
Clearly, \( \text{card}(\mathcal{A} \Delta \mathcal{B}) = d(\mathbf{a},\mathbf{b}) \).

Given \( M < N \), define \( \mathcal{R}(N,M) = \{ r \in \mathbb{Z}_2^{2N} : |r|_1 > 2M \} \).

Suppose that \( \mathcal{V} \) is a subset of \( \mathbb{Z}_2^{2N} \) for which \( \mathcal{V} \) and \( \mathcal{V} + r \) are disjoint for all choices of \( r \) in \( \mathbb{R} \). Thus for all \( \mathbf{v} \) and \( \mathbf{v}' \) in \( \mathcal{V} \),
\[
|\mathbf{v} - \mathbf{v}'|_1 \leq 2M;
\]
in other words, the diameter of \( \mathcal{V} \) is at most \( 2M \). A theorem of Kleitman [17] then says that \( \mathcal{V} \) must have at most
\[
\sum_{i \leq M} \binom{2N}{i} \text{ elements}. \tag{4.22}
\]

By the Normal Approximation of the Binomial Distribution this number is asymptotically equal to
\[
2^{2N} F \left( \frac{M - N}{\sqrt{\frac{N}{2}}} \right) \tag{4.23}
\]
as \( N \) and \( M \) both tend to infinity. \( F \) is the integral of the normal distribution:
However, if \( W = W(N) = \{ v \in \mathbb{Z}_2^{2N} : |v|_1 \leq N \} \), then \( W \) has \( 2^{2N-1} \) elements, yet

\[
d( W+r \cap W ) \leq \frac{1}{\pi} \sqrt{\frac{N-M}{M}},
\]
for each choice of \( r \) in \( R(N,M) \), and all \( N,M \) and \( N-M \) large enough.

To see (4.28), note that the maximum cardinality is obtained when

\[
|r|_1 = 2M
\]
and so, by symmetry, \( r \) indicates the set \( \{1,\ldots,2M\} \), without loss of generality.

When considered as a subset of \( \{1,\ldots,2N\} \), an element, \( a \), of \( W \) has \( s \) elements inside \( \{1,\ldots,2M\} \) and \( t \) elements outside. Thus \( s+t \leq N \).

However, if \( a+r \) is to be in \( W \), then

\[
|a+r|_1 = 2M-s+t \leq N.
\]

These conditions on \( s \) and \( t \) are also sufficient to put \( a \) and \( a+r \) in \( W \). In this way

\[
\text{card}(W+r \cap W) = \sum_{s+t \leq N} \binom{2N-2M}{t} \binom{2M}{s}
\]

In diagramatic form, this sum is the sum of the function

\[
k(x,y) = \binom{2N-2M}{y} \binom{2M}{x}
\]
over the integer pairs to be found in the shaded area of Figure 2.
Provided that $N$ and $N-M$ both tend to infinity, this sum, normalised by dividing by $2^{2N}$, becomes asymptotically equal to the integral of the function

$$k'(x,y) = f(x)f(y), \quad (4.33)$$

( $f$ being the normal distribution function,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad (4.34)$$

over the shaded area in Figure 3.

This area, since $k'$ is radially symmetric and has integral 1, is equal to $\frac{\theta}{2\pi}$,

which is, for $M/N$ close to 1, approximately equal to $\frac{2 \sqrt{N-M}}{2\pi M}$. \quad (4.35)

A more exact estimate is $\frac{1}{\pi} \arctan \sqrt{\frac{N-M}{M}}$, \quad (4.36)


Thus $d(W \cap W+r) \leq \frac{1}{\pi} \sqrt{\frac{N-M}{M}}$, \quad (4.37)

for all $r$ in $R(N,M)$ and for $N,M$ and $N-M$ large enough.

In particular, by picking a sequence of $N_n$ and $M_n$ so that $N_n/M_n$ tend to 1 and $N_n$ tend to infinity sufficiently fast, one obtains a sequence of subsets, $R_n$, of $Z_2^{2N_n}$ for which $L^0(R_n)$ tends monotone to zero, and a sequence of subsets $W_n$, of $Z_2^{2N_n}$, of density 1/2, for which

$$e_n = \max \{ d(W_n \cap W_n+r) : r \in R_n \} \quad (4.38)$$

tends monotone to zero.
The following lemma will be important in combining the properties of these sequences into one and it will be extended later:

**Observation 4.6:** Suppose that \( R \) and \( R' \) are subsets of \( Z_2^{2N} \) and \( Z_2^{2N'} \) respectively. Let 0 be the one element set consisting of the zero vector in either space.

Let
\[
R^* = R \cup 0 \times R',
\]
a subset of \( Z_2^{2N+2N'} \). 

Then
\[
L^0(R^*) \leq \min\{L^0(R), L^0(R')\}.
\]

**Proof:** This follows simply from the definition of \( L^0 \).

Continuing the construction and proof of Theorem 4.5:

Decompose \( Z_2^\infty \) as a sum
\[
\bigoplus_{i \geq 1} Z_2^{[b_i, b_{i+1})},
\]
where \( b_{i+1} - b_i = 2N_i \).

Let \( R \) be the union of the sets

\[
R'_n = 0 \times 0 \times \ldots \times R_n \times 0 \times \ldots, \quad n \geq 1.
\]

Let
\[
W(n,0) = 0 \times 0 \times \ldots \times W_n \times 0 \times \ldots,
\]
and
\[
W(n,1) = 0 \times 0 \times \ldots \times W_n^c \times 0 \times \ldots.
\]
Let $U$ be the set $\bigcup_{K, s} \sum_{i \in K} W(i, s(i))$, where the union is taken over all finite sets, $K$, of integers and functions $s: K \to \{0, 1\}$, such that $|s|_1 = \sum_{n \in K} s(n)$ is even. (4.47)

a/ Calculation of the density of $U$:

$$\bar{d}(U) = \lim_{n \to \infty} \frac{|U \cap Z_2^{[1, n]}|}{2^n}$$

$$\geq \lim_{k \to \infty} \frac{|U \cap Z_2^{[1, b_{k+1}]}|}{b_{k+1} - 1}$$

$$= 1/2.$$  (4.50)

The last equality is calculated as in Chapter 3.

b/ The upper density of $U \cap U + r$:

Suppose that $r$ is in $R' = R \cap Z_2^{[b_k, b_{k+1})}$. (4.51)
By construction, the upper density of $U \cap U + r$ is bounded above by the number

$$\frac{|W(k, 0) \cap (W(k, 0) + r)|}{b_{k+1} - b_k} = \frac{|W(k, 1) \cap (W(k, 1) + r)|}{b_{k+1} - b_k} = e_k$$

(4.52)

**Completion of the Proof of Theorem 4.5:** $R$, constructed above, will do: It is a set of recurrence by lemmas 4.4 and 4.6.

However,

$$\lim_{k \to \infty} \max_{j \to \infty} \left\{ \lim_{j \to \infty} \frac{|U \cap (U + r) \cap Z_2^{[l, b_j + r]}|}{b_{j+1} - 1} : r \in R \cap Z_2^{[b_k, b_{k+1}]} \right\}$$

$$= \lim_{k \to \infty} e_k = 0$$

(4.53)

and yet

$$\lim_{j \to \infty} \frac{|U \cap Z_2^{[l, b_j + r]}|}{2} = \frac{1}{2}.$$  

(4.54)

From this density result, a measure space may be constructed, as in chapter 2, p.21, by means of the original version of the Furstenberg correspondence construction [14] adapted to the group $Z_2^{\infty}$. Thus Theorem 4.5 is proved.
The Construction in $\mathbb{Z}$:

Recall the construction which transferred the example of Chapter 3 from $\mathbb{Z}_2^\infty$ to $\mathbb{Z}$. The same construction and notation will be used here and is restated for the reader's convenience:

Let $p_i$ be a sequence of numbers, yet to be determined, such that
\[ 2p_i | p_{i+1} \text{ for all } i \geq 1. \tag{4.55} \]
For a given $p$, let
\[ f_p(m) = \begin{cases} 0 & \text{if } 0 \leq m < p \mod 2p \\ 1 & \text{if } p \leq m < 2p \mod 2p, \end{cases} \tag{4.56} \]
defined for all $m$ in $\mathbb{Z}$.

Let $N$ be chosen large and let $f : \mathbb{Z}_{2p_N} \to \mathbb{Z}_2^N$ be defined:
\[ f(m) = (f_{p_1}(m), f_{p_2}(m), \ldots, f_{p_N}(m)). \tag{4.57} \]

$s$, in general, denotes a function : $\{1, \ldots, N\} \to \{1, -1\}$.
Given $s$, let
\[ g_s : \mathbb{Z}_2^N \to \mathbb{Z}_{2p_N} , \tag{4.58} \]
\[ g_s : (a_i) \to \sum_{1 \leq i \leq N} s(i)a_i p_i \]

Let $M$ and $R$ be subsets of $\mathbb{Z}_2^N$.
Let
\[ R^* = \bigcup_{s : \{1, 2, \ldots, N\} \to \{1, -1\}} g_s(R) . \tag{4.59} \]
As before, the letter \(d\) refers to the density of a set in whichever group it happens to sit. If this is not clear, the group in question is mentioned explicitly in the text. Thus in the important equality: 
\[
d(f^{-1}(M)) = d(M),
\]
the first density involves the group \(Z_{2p}^N\), and the second, the group \(Z_2^N\).

**Lemma 4.7:** There is an absolute constant, \(c\), so that, if \(W\) and \(R\) are subsets of \(Z_2^N\), and \(f^{-1}(W)\) and \(R^*\) are constructed in \(Z_{2p}^N\) as above, the following hold:

i/ for all \(r\) in \(Z_2^N\) and all sign functions \(s: Z_2^N \to \{1,-1\}\),

\[
d(f^{-1}(W) \cap f^{-1}(W) + g_s(r)) \leq d(W \cap W + r) +
\]
\[
+ cd(W) \left( 2^{|r|_1} \sum_{i : r_i = 1} \frac{p_i}{p_i + 1} \right)
\]

ii/ \(L^0(R^*) \leq L^0(R)\).

Note that this then proves the result:

**Proof of Theorem 4.2:** By the Furstenberg Correspondence (see Chapter 2 p.14), it is sufficient to prove that there is a set of recurrence, \(S\) in \(Z\), a sequence of intervals \([1, P_n]\) and a set \(V\) with the following properties:

\[
\lim_{n \to \infty} \frac{|V \cap [1, P_n]|}{P_n} = \frac{1}{2}
\]
\[
\lim_{s \in S : s \to \infty} \lim_{n \to \infty} \frac{|V \cap (V + s) \cap [1, P_n]|}{P_n} = 0 \quad (4.64)
\]

It will turn out that \( S \) will be built out of the sets \( R_n^* \) of the previous section, \( V \) out of \( f^{-1}(W_n) \) sets and \( P_n = 2p_{b_n} \), \( p_i \) having been picked sufficiently well.

Note again that the \( R \) constructed earlier in this chapter is well separated, i.e. there is a sequence \( b_1 < b_2 < \ldots \) of integers, and a sequence of sets, \( R_i \) such that
\[
Z_2 \left[ b_i, b_{i+1} \right) \supseteq R_i, \text{ and } R = \bigcup R_i. \quad (4.65)
\]

Let
\[
R(n,m) = \bigcup_{m > i \geq n} R_i. \quad (4.66)
\]

Let
\[
U(n,m) = U \cap Z_2 \left[ b_n, b_m \right). \quad (4.67)
\]

Construct \( R(1,m)^* \) in \( Z_{2\rho_{b_m}} \) and note that, if \( Z_{2\rho_{b_m}} \) is considered as an initial segment \( \{1, \ldots, 2\rho_{b_m}\} \) of the integers, then \( R(1,m)^* \) agrees with \( R(1,m)^* \) on this segment for all \( m' > m \). (recall the same argument in the previous chapter.) Let \( S \) be the limit, or equivalently, the union of these \( R(1,m)^* \) as \( m \) runs over all the positive integers.

Similarly, note that \( f^{-1}(U(1,m)) \) sits in \( \{1, \ldots, 2\rho_{b_m}\} \), and this sequence of sets has the same property, that its union is its limit. Let \( V \) be this union.
Suppose that \((X, \mathcal{B}, \mu, T)\) is a measure preserving system with \(\mathbb{Z}\) action and that \(A\) is a subset of \(X\) with measure \(2\alpha\). Choose \(m\) so that \(L^0(R(1,m)) < \alpha\). (4.69)

At first sight, \(R(1,m)^*\) is only a subset of the group \(\mathbb{Z}_{2p_{b_m}}\), however, by treating \(R(1,m)\) as a subset of \(\mathbb{Z}_{[1,b_m]}\) where \(m' > m\) and noting that
\[
L^0_{\mathbb{Z}_{[1,b_m]}}(R(1,m)) = L^0_{\mathbb{Z}_{[1,b_m]}}(R(1,m)) < \alpha, \quad (4.70)
\]
one sees that \(R(1,m)^*\) is also a subset of \(\mathbb{Z}_{2p_{b_m}}\), sitting on the intial segment
\[\mathbb{Z}_{[1,2p_{b_m}]}\), with \(L^0_{\mathbb{Z}_{2p_{b_m}}}(R(1,m)^*) < \alpha\), by lemma 4.7ii/. (4.71)

There is a subset, \(E\), of \(\mathbb{Z}_{2p_{b_m}}\) contained, without loss of generality, in the initial segment \(\{1, \ldots, p_{b_{m'}}\}\) with density in \(\mathbb{Z}_{2p_{b_m}}\) at least \(\alpha\), for which
\[
\mu(\bigcap_{n \in E} T^n A) > 0. \quad (4.72)
\]
Thus, provided that \(b_{m'} > 2b_m\), (4.73)
there is an \(r\) in \(R(1,m)^*\) for which \(E \cap E + r \neq \emptyset\) and the diameter of \(E\) and \(R(1,m)^*\) are sufficiently small to ensure that \(r\) does not 'wrap' the construction 'around' mod \(2p_{b_{m'}}\).

Thus, when \(R(1,m)^*\) is considered as being in the initial segment \(\mathbb{Z}_{[1,p_{b_m}]}\) of the integers, there is an \(r\) in \(R(1,m)^*\) for which \(\mu(A \cap T^r A) > 0\). (4.74)

By the uniformity of recurrence, \(S\) is a set of recurrence in \(\mathbb{Z}\) therefore.
Further, lemma 4.7ii says that for all \( r \) in \( R(1,m) \) and all sign functions \( s \),

\[
\begin{align*}
    d(f^{-1}(U(1,m)) \cap f^{-1}(U(1,m)) + g_s(r)) & \leq d(U(1,m) \cap U(1,m) + r) + \\
    & + \frac{c}{2} \left( 2^{2|R|} \sum_{i: r_i = 1} \frac{p_i}{p_i + 1} \right) \quad (4.75)
\end{align*}
\]

where \( U(1,m) \) is considered a subset of the group \( \mathbb{Z}_2^{[1,b_m]} \) and \( f^{-1}(U(1,m)) \) a subset of the group \( \mathbb{Z}_{2p_{b_m}} \).

Recall that \( R \) is well separated and so the second term in this expression is dominated by

\[
\frac{c}{2} \sum_{j \geq n} \left( 2^{3|b_{j+1} - b_j|} \sum_{b_j \leq i < b_{j+1}} \frac{p_i}{p_i + 1} \right) = \phi_n, \quad (4.76)
\]

which, \( p_i \) having been chosen carefully, can be made to tend to zero as \( n \) tends to infinity.

Note also, that, provided that the \( p_i \) are increasing sufficiently rapidly, if \( s \) is some sign function and \( r^* = g_s(r) \) is in \( R(1,m)^* \), then

\[
-2(b_{m+1} - b_m)p_{b_{m+1}} < r^* < 2(b_{m+1} - b_m)p_{b_{m+1}}. \quad (4.77)
\]

and if \( r^* > 2p_{b_m} \), then \( r \) is in \( R(m,m') \) for some \( m' > m \).

Thus if \( n' > 0 \) and \( r^* > 2p_{b_{n'+1}} \) is an element of \( R^* \), then \( r^* = g_s(r) \) (4.79) for some \( r \) in \( R(n,n+1) \) and sign function, \( s \), where \( n > n' \).

Therefore, for \( m > n \),
\[
|V \cap (V + r^*) \cap [l, 2p_{b_m}]| \leq d( f^{-1}(U(1,m)) \cap f^{-1}(U(1,m)+r^*) ) + \frac{2(b_{n+1} - b_n)p_{b_{n+1}}}{2p_{b_m}}
\] (4.80)

\[
\leq d( U(1,m) \cap U(1,m)+r ) + \phi_n + \frac{2(b_{n+1} - b_n)p_{b_{n+1}}}{2p_{b_m}}
\] (4.81)

\[
= d( U(n,n+1) \cap (U(n,n+1)+r) ) + \phi_n + \frac{2(b_{n+1} - b_n)p_{b_{n+1}}}{2p_{b_m}}
\] (4.82)

which tends to 0 as first \(m\) tends to infinity and then \(n\) tends to infinity by the construction of \(U\) in the previous section.

Further,
\[
\frac{|V \cap [l, 2p_{b_m}]|}{2p_{b_m}} = d( U(l, m) ) = \frac{1}{2}
\] (4.83)

So we are done.

**Proof of Lemma 4.7:** Let \(p_i, 1 \leq i \leq N\), be determined and construct \(R^*\)
and \(f^{-1}(W)\) in \(Z_{2p_N}\) from \(R\) and \(W\) in \(Z_{2N}\) as noted earlier.

Part i/ comes from the formula (3.68) proved in lemma 3.8.

\[
d( f^{-1}(W+r) \Delta (f^{-1}(W)+g_s(r))) < d(W)2c2^2|l| \sum_{r_i = 1} \frac{p_i}{p_i + 1}
\] (4.84)
The inequality
\[ d(f^{-1}(W) \cap f^{-1}(W) + g_s(r)) \]
\[ \leq d(f^{-1}(W+r) \cap f^{-1}(W)) + d(f^{-1}(W+r) \Delta f^{-1}(W) + g_s(r)) \]
\[ = d((W+r) \cap W) + d(f^{-1}(W+r) \Delta (f^{-1}(W) + g_s(r))) \]
(4.85)
does the trick.

Part ii/: Suppose that $E$ is a subset of $\mathbb{Z}^{2pN}$ with density greater than $L^0(R)$.

Let $B$ be defined by
\[ B = \{ (x,v) \in \mathbb{Z}^{2pN} \times \mathbb{Z}_2^N : x \in E + g_1(v) \} ; \]
(4.86)
a set of density $a$ in $\mathbb{Z}^{2pN} \times \mathbb{Z}_2^N$.

There is an $x$ such that $B_x = \{ v : (x,v) \in B \}$ has density in $\mathbb{Z}_2^N$ greater than $a$. Thus there is an $r$ in $R$ such that $B_x \cap B_x + r \neq \emptyset$. This implies that
\[ (E + g_1(v)) \cap (E + g_1(v+r)) \neq \emptyset, \text{ for some } v, \text{ and so} \]
(4.87)
\[ E \cap E + g_s(r) \neq \emptyset, \text{ where} \]
(4.88)
\[ s(i) = \begin{cases} 1 & \text{if } v_i=0 \text{ or } r_i=0 \\ -1 & \text{otherwise.} \end{cases} \]
(4.89)

So we are done and the whole theorem is now proved.
Other Examples of Sets of Recurrence with Unusual Properties:

Consider the following conjecture, quite natural in the light of the uniformity of recurrence.

**Conjecture 4.8:**

Given \( a > 0 \), there is an \( N \), so that for any set of recurrence, \( R \), there is a subset, \( F \), of \( R \) of cardinality at most \( N \), such that for all measure preserving systems \((X,B,\mu,T)\) and all subsets, \( A \), of \( X \) of measure greater than \( a \), there is an \( r \) in \( F \) such that \( \mu(A \cap T^r A) > 0 \).

The example of Theorem 4.2 shows, however, that this is too much to ask:

**Proposition 4.9**: Conjecture 4.8 fails.

**Proof**: Let \( R \) be the set of recurrence constructed in Theorem 4.2 and let \( A \) be a subset of measure 1/2 in a measure preserving system \((X,B,\mu,T)\) such that

\[
\lim_{r \to \infty: r \in R} \mu(A \cap T^r A) = 0. 
\]

Let \( k_\eta \) be chosen so that

\[
\mu(A \cap T^r A) < 2^{-n} \quad \text{for all } r \text{ in } R \text{ greater than } k_\eta. 
\]

Let \( F \) be any subset of \( R_\eta = R \cap [k_\eta, \infty) \)

of cardinality at most \( 2^{n-2} \).

Consider the set \( A' = A \setminus \bigcup_{r \in R} T^{-r} A \);

a set of measure at least 1/4 which is not returned by \( F \).
Thus $R_n$ forms a sequence of sets of recurrence which violates any uniform bound on the size of subsets, $F$, which satisfy the conclusion of the conjecture. So we are done.

To examine the structure of single recurrence much further, one must build up a battery of definitions which make formal the work of the past chapters. The author has hesitated to do this sooner as it was not needed and, in fact, obscured the ideas behind the earlier work. The attentive reader will recognise the constructions of previous chapters, particularly chapter 2, in the following:

**Definitions:** Suppose that $R$ is a subset of $\mathbb{Z}$:

Let $e(a; R) = \inf \sup_{(X, B, \mu, T): A \in B, \mu(A) \geq a} \mu(A \cap T^{-r} A)$. (4.94)

The infimum is taken over all measure preserving systems $(X, B, \mu, T)$ and sets, $A \in B$ such that $\mu(A) \geq a$.

$e$ is called the uniform recurrence function.

Let $L(R) = \inf \{ a : e(a; R) > 0 \}$. (4.95)

**Remark:** The function, $L$, was examined in its intersective form (see Chapter 3) by Ruzsa in [23].

**Lemma 4.10:** $R$ is a set of recurrence if and only if $L(R) = 0$. (4.96)

**Proof:** This is a direct consequence of the uniformity of recurrence and in particular of the fact that the $e$ obtained in Theorem 2.1 is dependent only on $a$.

**Definition:** Given $R$, let $R[n] = R \cap \{ (-\infty, -n] \cup [n, \infty) \}$. (4.97)
The number \( \varrho(a;R) = \inf \{ \varrho(a;R[n]) : n \in \mathbb{N} \} \), \begin{equation} \tag{4.98} \end{equation} is called the strong recurrence function.

Let \( L(R) = \inf \{ a : \varrho(a;R) > 0 \} \). \begin{equation} \tag{4.99} \end{equation}

A set \( R \) for which \( L(R) = 0 \), is called a set of uniform strong recurrence.

The definition of uniform strong recurrence comes from considering strong recurrence mentioned in chapter 2. Uniform strong recurrence is stronger than strong recurrence. It is not clear whether the two definitions are the same.

Recall from Chapter 2, the definition of a set of nice recurrence. Note that this is equivalent to the requirement that \( \varrho(a;R) = a^2 \). The arguments of Chapter 2 show that a set of nice recurrence is also a set of uniform strong recurrence with uniform strong recurrence function \( \varrho(a;R) = a^2 \). However, the following example shows that not every set of uniform strong recurrence need be a set of nice recurrence.

**Theorem 4.11**: There is set of uniform strong recurrence which is not a set of nice recurrence.

The following extension of Lemma 4.6 will be used to construct this example and others.

**Lemma 4.12**: Suppose that \( R \) and \( R' \) are subsets of \( Z_2^{2N} \) and \( Z_2^{2N'} \) respectively. Let \( O \) be the one element set consisting of the zero vector in either space.

Let \( R'' = RxO \cup OxR' \), a subset of \( Z_2^{2N+2N'} \). \begin{equation} \tag{4.100} \end{equation}

\( a/ \) \( L(R)L(R') \leq L(R'') \leq \min\{L(R),L(R')\} \). \begin{equation} \tag{4.101} \end{equation}
b/ \[ \max \{ 2ae(\frac{1}{2}; R'), e(a; R) \} \geq e(a; R'') \geq \max \{ e(a; R), e(a; R') \} . \]  

(4.102)

**Proof:** a/ The right hand inequality is Lemma 4.6. The left hand inequality comes from the following product-like construction:

Let \( \varepsilon > 0 \). (4.103)

Let \((X,B,\mu,Z_2^N)\) be a measure preserving system and let \( A \) be a subset of \( X \), of measure greater than \( L(R) - \varepsilon \), such that

\[ \mu \left( A \cap T^r A \right) = 0 \quad \text{for all } r \text{ in } R. \]  

(4.104)

Similarly, let \((X',B',\mu',Z_2^{2N'})\) be a measure preserving system and let \( A' \) be a subset of \( X' \) of measure greater than \( L(R') - \varepsilon \) such that

\[ \mu' \left( A' \cap T'^{r'} A' \right) = 0 \quad \text{for all } r' \text{ in } R'. \]  

(4.105)

Construct a \( Z_2^{2N+2N'} \) system, \((Y,D,\nu,Z_2^{2N+2N'})\), with \( Y = X \times X' \), \( D \) the product \( \sigma \)-algebra of \( B \) and \( B' \) and action defined as follows:

\[ \nu \in Z_2^{2N+2N'} \]  

may be decomposed uniquely: \[ \nu = \nu_1 + \nu_2 \]  

(4.107)

where \( \nu_1 \in Z_2^{2N} \) and \( \nu_2 \in Z_2^{2N'} \).

Let

\[ T^\nu (x, x') = (T^{\nu_1} x, T^{\nu_2} x'). \]  

(4.108)

Let \( B \) equal \( A \times A' \) (4.109)

a subset of \( Y \) of measure \( (L(R) - \varepsilon)(L(R') - \varepsilon) \geq L(R)L(R') - 2\varepsilon \).  

(4.110)

\( B \) is not returned by \( R'' \) by construction and so

\[ L(R'') \geq L(R)L(R') - 2\varepsilon \]  

(4.111)

and, since \( \varepsilon \) was arbitrary, we are done.
b/ The right hand inequality:

Let \( \varepsilon > 0 \).  

Let \((X, \mathcal{B}, \mu, Z^2_{2N+2N'})\) be a measure preserving system with a subset, \( A \), of measure \( a \), such that

\[
\mu \left( A \cap T^r A \right) < \theta(a; R^n) + \varepsilon \quad \text{for all} \quad r \in R^n. 
\]

(4.112)

Consider the \( Z^2_{2N} \)-action, \( S \), on \((X, \mathcal{B}, \mu)\) defined as

\[
S^v x = T(v, 0)x \quad \text{for all} \quad v \in Z^2_{2N}, 
\]

(4.113)

where \( 0 \) is the zero element in \( Z^2_{2N} \).

By definition, therefore, there is an element, \( r \), of \( R \) for which

\[
\mu \left( A \cap S^r A \right) > \theta(a; R) - \varepsilon . 
\]

(4.114)

This implies that

\[
\mu \left( A \cap T^{(r, 0)} A \right) > \theta(a; R) - \varepsilon , 
\]

(4.115)

but the left hand side is dominated by \( \theta(a; R^n) + \varepsilon \) since \( (r, 0) \) is a member of \( R^n \).

A similar argument does for \( \theta(a; R') \) and so

\[
\theta(a; R^n) > \max \{ \theta(a; R), \theta(a; R') \} - 2\varepsilon , 
\]

(4.116)

and, since \( \varepsilon \) was arbitrary, we are done.

The left hand inequality proceeds as follows:

Let \( \varepsilon > 0 \).  \quad \text{Let} \quad 1/2 > a > 0. 

(4.117)

Let \((X, \mathcal{B}, \mu, Z^2_{2N})\) be a measure preserving system and let \( A \) be a subset of \( X \) of measure \( a \) such that

\[
\mu \left( A \cap T^r A \right) < \theta(a; R) + \varepsilon \quad \text{for all} \quad r \in R. 
\]

(4.118)
Let $A'$ be a set of measure $a$, disjoint from $A$, and let $P$ be a measure preserving transformation such that $PA = A'$. \hspace{1cm} (4.120)

Such a $P$ exists by standard ergodic theoretical constructions.

Let $T_0^V = PTVP^{-1}$ \hspace{1cm} (4.121)

and note that, therefore, $\mu\left(A' \cap T_0^rA'\right) < \epsilon(a;R) + \epsilon$. \hspace{1cm} (4.122)

Similarly, let $(X', B', \mu', Z'_{2N'})$ be a measure preserving system and let $B$ be a subset of $X'$ of measure $1/2$ such that

$$\mu'\left(B \cap S^rB\right) < \epsilon\left(\frac{1}{2} : R'\right) + \epsilon \quad \text{for all } r' \in R'\hspace{1cm} (4.123)$$

Note that $B^C$, the compliment of $B$, has exactly the same property.

As before, construct a $Z_2^{2N+2N'}$ system, $(Y, D, \nu, Z_2^{2N+2N'})$, where

$$Y = X \times X', \quad D \text{ is the product } \sigma\text{-algebra of } B \text{ and } B' \hspace{1cm} (4.124)$$

and the action is defined as follows:

Let $U^V(x, x') = \begin{cases} (T^{v_1}x, S^{v_2}x') & \text{if } x' \in B \\ (T_0^{v_1}x, S^{v_2}x') & \text{if } x' \in B^C \end{cases}$ \hspace{1cm} (4.125)


Let $C = A \times B \cup A' \times B^C$, \hspace{1cm} (4.127)

a subset of $Y$ of measure $a$.

By examining the values of

$$\nu(C \cap U(r,0)C) = \frac{1}{2} \mu(A \cap T^rA) + \frac{1}{2} \mu(A' \cap T_0^rA')$$

$$= \mu(A \cap T^rA) \hspace{1cm} (4.128)$$

$$\leq \epsilon(a;R) + \epsilon, \hspace{1cm} (4.129)$$
and 
\[ \nu\left( C \cap U^{(0,r')}C \right) = a.\mu'( B \cap S^{r'}B ) + a.\mu'( B^c \cap S^{r'}B^c ). \]

\[ = 2a.\mu'( B \cap S^{r'}B ) \]

\[ \leq 2a. \left( e\left( \frac{1}{2}; R' \right) + \varepsilon \right), \]

the result follows.
**Definition:** Call the set $R''$ the axial union of $R$ and $R'$, written $R\&R'$.

Given $R$ and $R'$ as above, define the alternating product

$$RkR' = R \times R' \cup R^C \times R'^C. \quad (4.132)$$

If $W$ and $R$ are subsets of a finite group then define

$$\varrho^0(W;R) = \max_{r \in R} d((W+r) \cap W) \quad (4.133)$$

where $d$ is the natural density.

**Remark:** Both the alternating product and the axial union are subsets of $\mathbb{Z}_2^{2N+2N'}$. These definitions are easily extended to triples of sets, indeed to any countable collection of sets. The order is important.

The axial union and alternating product have made an appearance already in the constructions of Chapter 3 and earlier in this chapter.

**Proof of Theorem 4.11:**

Recall that the example of Theorem 4.2 was constructed in $\mathbb{Z}_2^\infty$ out of axial unions of $R(N,M)$ and alternating products of $W(N)$, $N$ and $M$ being parameters chosen carefully. The properties of these blocks can be listed:

$$R(N,M), W(N) \subseteq \mathbb{Z}_2^{2N}, \quad (4.134)$$

$$\varrho\left(\frac{1}{2}; R(N,M) \right) \leq \varrho^0(W(N);R(N,M))$$

$$= \frac{1}{\pi} \arctan \left( \sqrt{\frac{N-M}{M}} \right) + o(1) \quad (4.135)$$

$$L(R(N,M)) = F\left(\frac{N-M}{\sqrt{N}}\right) + o(1), \quad (4.136)$$
where \( F \) is the area under the Normal Distribution defined before, and Landau's notation refers to the condition that \( N, M \) and \( N-M \) all tend to infinity.

As before, one arranges for a sequence of \( N_j \) and \( M_j \) tending to infinity so that, setting  

\[
R_j = R(N_j, M_j),
\]

\[
e(W(N_j); R_j) \leq 1/5 \quad \text{and} \quad L(R_i) \leq 2^{-i};
\]

defined for all \( i \geq 1 \).

Let \( R \) be the axial union of \( R_1, R_2, R_1, R_2, R_3, R_1, R_2, R_3, R_4 \), \( R_1 \), etc. in that order.  \((4.139)\)

Let \( W \) be the alternating product of \( W(N_1), W(N_2), W(N_1), W(N_2), W(N_3), W(N_1), W(N_2), W(N_3), W(N_4) \), \( W(N_1) \), \( W(N_2) \), \( W(N_3) \), \( W(N_4) \), \( W(N_1) \), etc. in that order. \((4.140)\)

As before, \( W \) has density \( \frac{1}{2} \) and

\[
e(\frac{1}{2}; R) \leq e^0(W; R) \leq 1/5, \text{ by construction.} \quad (4.141)\]

Further, \( e(a; R \cap Z_2^{[K, \infty)}) \geq \sup \{ e(a; R_i) : i \geq 1 \} \) for all \( K \), \((4.142)\)
by Lemma 2.5 \( i/ \), since the construction is repetitive. Also, the right hand side of \((4.142)\) is strictly greater than zero for \( a > 0 \), since \( L(R_i) \) tends to zero.

It remains to check that the transfer from \( Z_2^{\infty} \) to \( Z \) proceeds smoothly: \( R \) is well separated and so the errors in Theorem 4.2 \((4.76)\) become unimportant. Thus, by picking \( p_i \) increasing sufficiently fast, one constructs a set \( W' \), of upper density \( 1/2 \), in \( Z \), and a set of recurrence \( R^* \) in \( Z \), for which

\[
e^0(W'; R^*) \leq 9/40. \quad (4.143)\]

In particular, \( R \) is not a set of nice recurrence.

However, note that, if \( K \) is given,

\[
(R \cap Z_2^{[K, \infty)})^* = R^* \cap [K', \infty), \quad \text{for some } K', \quad (4.144)\]

by the details of the * construction, provided that \( p_i \) increase fast enough. Similarly,
define \( K' \) and \( L' \) so that
\[
(R \cap Z_2^{[K,L]^*}) = R^* \cap [K',L').
\] (4.145)

Recalling the result of lemma 4.7 one obtains
\[
e(a; R^* \cap [K',\infty)) \geq e(a; R^* \cap [K',L')) \\
\geq e(a; R \cap Z_2^{[K,L]})
\] (4.146)
and this tends to \( \sup \{ e(a; R_i) : i \geq 1 \} \) as \( L \) tends to infinity. (4.147)

Thus \( R^* \) is a set of uniform strong recurrence and
\[
e(a; R^*) \geq \sup \{ e(a; R_i) : i \geq 1 \}
\] (4.148)
and we are done.

**Arbitrary Upper Bounds for \( e \):**

One can now get an arbitrarily small upper bound for \( e \) in general and thus show that recurrence can be made arbitrarily poor. Some properties of \( e \) are needed for the proof.

**Lemma 4.13:** Suppose that \( R \) is a subset of \( Z \). Then
i/ \( e(a; R) \), as a function of \( a \), is a convex, continuous, monotone increasing function from \([0,1]\) onto \([0,1]\), with \( e(0) = 0 \) and \( e(1) = 1 \). (4.149)

ii/ If \( (X,\mathcal{B},\mu,T) \) is a measure preserving system and \( A \) is a subset of \( X \), then
\[
\mu(A \setminus \bigcup_{r \in R} T^r A) \leq L(R)
\] irrespective of the measure of \( A \). (4.150)

iii/ \( e(a; R) \geq (a-L(R))/|R| \), where \( |R| \) is the cardinality of \( R \). (4.151)

**Proof:** i/ That \( e: [0,1] \to [0,1] \), \( e(0;R)=0 \) and \( e(1;R)=1 \) are obvious from the definition. Thus monotonicity is implied by convexity.

Continuity is also implied by convexity; the exeptional case, \( e(a; R) \geq 1_{\{1\}}(a) \), is excluded by the fact that, for example, \( e(0.7; R) \geq 0.4 \) always. This, in turn, shows
that \( \Theta \) is onto.

Thus it suffices to prove the convexity:

Suppose that \( a, b \in [0,1] \) and \( p + q = 1 \). \hspace{1cm} (4.152)

Let \( \varepsilon > 0 \).

Let \( X = (X, B, \mu, T) \) be a probability measure preserving system and \( A \) a subset of \( X \) of measure \( a \) such that \( \mu(A \cap T^r A) < \Theta(a; R) + \varepsilon \) for all \( r \) in \( R \). Let \( Y = (Y, D, \nu, S) \) and \( B \) be a similar example of measure \( b \).

Construct the system \( W = (W, C, \pi, U) = pX + qY \), \hspace{1cm} (4.153),

where \( W = X \cup Y \), a disjoint union of \( X \) and \( Y \), \hspace{1cm} (4.154)

with natural \( \sigma \)-algebra, \( C \).

\[
\pi(C) = p \cdot \mu(C \cap X) + q \cdot \nu(C \cap Y),
\]

\[
UC = T(C \cap X) \cup S(C \cap Y), \hspace{1cm} \text{for all } C \text{ in } C.
\]

This is a probability measure preserving system.

Let \( C = A \cup B \), \hspace{1cm} (4.157)

a set of measure \( pa + qb \) in \( W \).

By construction,

\[
\pi(C \cap U^r C) = p \cdot \mu(A \cap T^r A) + q \cdot \nu(B \cap S^r B)
\leq p \cdot \Theta(a) + q \cdot \Theta(b), \hspace{1cm} \text{for all } r \in R.
\]

Therefore,

\[ \Theta(pa + qb) \leq p \cdot \Theta(a; R) + q \cdot \Theta(b; R) + \varepsilon. \] \hspace{1cm} (4.159)

Since \( \varepsilon \) was arbitrary, convexity is confirmed.

ii/ This follows simply from the definition of \( L(R) \) and the fact that

\[ T^r \left( A \setminus \bigcup_{s \in R} T^s A \right) \cap \left( A \setminus \bigcup_{s \in R} T^s A \right) = \emptyset \hspace{1cm} \text{for all } r \in R. \] \hspace{1cm} (4.160)

iii/ Suppose that \( a \in [0,1] \) and let \( (X, B, \mu, T) \) be a probability measure preserving system and \( A \), a subset of \( X \) of measure \( a \), such that

\[ \mu(A \cap T^r A) < \Theta(a; R) + \varepsilon \hspace{1cm} \text{for all } r \in R. \] \hspace{1cm} (4.161)
By ii/

\[ \mu \left( \bigcup_{r \in R} (A \cap T^r A) \right) = \mu \left( A \cap \bigcup_{r \in R} T^r A \right) \geq a - L(R). \]  (4.162)

Therefore, there is an \( r \) in \( R \) such that \( \mu(A \cap T^r A) \geq (a-L(R))/|R| \), (4.163)

and, since the left hand side of this inequality is always bounded above by \( \Theta(a;R) + \varepsilon \) and \( \varepsilon \) is arbitrary, we are done.

Remarks:

a/ \( \Theta(a;2Z+1) = \max\{0,(2a-1)\} \), \( \Theta(a;N) = a^2 \), \( \Theta(a;R) \leq \Theta(a;S) \), for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\). \( \Theta(a;R) \leq \Theta(a;S), \) for all \( a \) in \([0,1]\). \( \Theta(a;R \cup S) \geq \max(\Theta(a;R), \Theta(a;S)), \) for all \( a \) in \([0,1]\).

**Theorem 4.14:** Given a continuous function \( h : [0, \frac{1}{2}] \rightarrow [0,\infty) \), for which \( h(a) > 0 \) whenever \( a > 0 \), there is a set of recurrence, \( R \) in \( Z \) whose uniform recurrence function is dominated by \( h \):

\[ h(a) \geq \Theta(a;R), \text{ for all } a \in [0, \frac{1}{2}]. \]  (4.168)

**Proof:** As with the previous construction, it is a well chosen combination of \( R(N,M) \) blocks which will do the trick.
Without loss of generality, \( h \) is monotone increasing, convex and linear on each element of a set of intervals: \( \{ [x_i, x_{i+1}]: 1 \leq i \} \) where \( 0 < x_{i+1} < x_i \) for all \( i \geq 1 \), and \( x_n \) tend to zero.

Let \( M = \left( 1 - \frac{c}{\sqrt{N}} \right) N \) \( (4.169) \)

where \( c \) is some parameter which will be determined. Then as \( N \) tends to infinity, \( M \) and \( N - M \) both tend to infinity.

Also \( L(R(N,M)) = F(-c) \) \( (4.170) \)

and \( \psi \left( \frac{1}{2}; R(N,M) \right) \leq \frac{1}{\pi} \arctan \left( \frac{c}{\sqrt{N - c}} \right) + o(1) \) \( (4.171) \)

for \( N \) chosen sufficiently large.

The right hand side of (4.172) tends to zero as \( N \) tends to infinity if \( c \) is kept constant.

Suppose that \( R \) is a subset of \( Z_2^{2N} \), and suppose that

\[ \psi(a; R) \leq \frac{1}{2} h(a) \quad \text{for all } \frac{1}{2} \geq a . \] \( (4.173) \)

Let \( R' = R(N',M(c)) \) \( (4.174) \)

be a subset of \( Z_2^{2N'} \), for \( c \) yet to be chosen.

As before, let \( R'' = R \times O \cup O \times R' \), a subset of \( Z_2^{2N + 2N'} \). \( (4.175) \)

Using the lefthand inequalities of a/ and b/ from lemma 4.11 and the fact that uniform recurrence functions are convex (lemma 4.12 i/), the uniform recurrence function for \( R'' \) can be bounded above by the function whose graph is illustrated by the bold line in Figure 4.
FIGURE 4
In symbols, the bound is:

\[ e(a;R^*) \leq \max\{ 0, e(a;R), K(a) \} \tag{4.176} \]

where \( y = K(a) \) is the equation of the straight line, \( K \), going through the point \((L(R')L(R), 0)\) and the point of intersection, \( P \), of the curve \( y = e(a;R) \) and the straight line \( y = 2a.e(\frac{1}{2}, R') \). \tag{4.177}

Thus, by picking \( F(-c) < \frac{1}{2} L(R) \) \tag{4.178}
and \( N' \) so large that the line, \( K \), in the Figure 4 remains below \( h(a)/2 \) in the interval \([L(R)L(R'), \frac{1}{2}]\), the set \( R^* \) has uniform recurrence function dominated by \( \frac{1}{2} h(a) \) in \([0, \frac{1}{2}]\). \tag{4.179}

Further, \( L(R^*) \leq \frac{1}{2} L(R) \). \tag{4.180}

It is easy to start the construction with, say, the set \( R = \{1\} \) in \( Z_2 \), for which \( e(a;R) = \max\{0, 2a-1\} \). \tag{4.181}

Proceeding by induction we are done in \( Z_2^{\infty} \). The transfer from \( Z_2^{\infty} \) to \( Z \) is uncomplicated.

By combining the arguments of the two theorems above, one can get a slightly stronger result. The fact that

\[ e(a;R&R'R&R) = e(a;R&R') \tag{4.182} \]

is used to prove the following:

**Theorem 4.15:** Given a continuous function \( h : [0, \frac{1}{2}] \to [0, \infty) \), for which \( h(a) > 0 \) whenever \( a > 0 \), there is a set of recurrence, \( R \) in \( Z \) whose uniform recurrence function is dominated by \( h \):

\[ h(a) \geq e(a;R), \quad \text{for all } a \text{ in } [0, \frac{1}{2}]. \tag{4.183} \]
Comparing Degrees of Nice Recurrence:

Finally here is an example which shows that the distinction between \( t \)-nice recurrence for various values of \( t \) is important.

**Theorem 4.16:** There is a set of 6.3-nice recurrence which is not a set of 2-nice recurrence.

**Proof:**

To show that a set, \( R \), is a set of \( s \)-nice recurrence, it is sufficient to show that there is a sequence of numbers, \( x_n \), tending to zero for which

\[
e(x_n; R) \geq x_n^s.
\]  

(4.184)

This follows from examining lemma 2.4.

Thus, by the block constructions made before, it is sufficient to construct a sequence of blocks \( R_n \), in various products of \( \mathbb{Z}_2^\infty \), for which there is a constant \( \alpha \)

\[
e(\frac{1}{2}; R_n) < \alpha < \frac{1}{4} \quad \text{for all } n \geq 1,
\]  

(4.185)

and a sequence of numbers, \( x_n \), tending to zero for which

\[
e(x_n; R_n) \geq x_n^s.
\]  

(4.186)

Recall the rather crude lower bound obtained for \( \theta \) in Lemma 4.12:

\[
e(a; R) \geq \frac{a - L(R)}{|R|}.
\]  

(4.187)

It will be sufficient for our needs.
Let \( x_0 = 2L(R) \).  

(4.188)

The line \( y = \frac{x - L(R)}{|R|} \) (4.189) rises above the curve \( y = x^S \) at the point \( x_0 \) whenever
\[
\frac{L(R)}{|R|} > 2^S L(R)^S . 
\]

(4.190)

In other words, the condition,
\[
1 > |R|2^S L(R)^S -1 
\]

(4.191), needs to be confirmed.

Let \( R = R(N,M) \) constructed in \( Z_2^{2N} \) as before. (4.192)

Let \( k = M/N \geq 1/2 \). (4.193)

Chernoff's estimate of the size of the tail of the binomial distribution (see Bollobas [8]) is useful to estimate the components of (4.183):

\[
|R| = \sum_{i \geq 2M} \binom{2N}{i} \leq \exp(2N h(k)),
\]

(4.194)

\[
L = L(R) = 2^{-2N} \sum_{i \leq M} \binom{2N}{i} \leq 2^{-2N} \exp(2N h\left(\frac{k}{2}\right)),
\]

(4.195)

where \( h(x) = -x \log_2 x -(1-x) \log_2 (1-x) \), the 'Entropy' function. (4.196)

Therefore,
\[
|R| L^{S-1} \leq \exp\{2N([\log_2 + h\left(\frac{k}{2}\right)](s-1) + h(k)) \}
\]

(4.197)

and so provided that \( [-\log_2 + h\left(\frac{k}{2}\right)](s-1) + h(k) < 0 \),

(4.198)
condition (4.191) will be satisfied for all $N$ sufficiently large.

This is equivalent to

$$s > 1 + \frac{h(k)}{h(k) - \log 2}.$$  \hspace{1cm} (4.199)

Also

$$e(\frac{1}{2}; R) \leq \frac{1}{2} \arctan \left( \frac{1}{\sqrt{k}} - 1 \right) + o(1).$$  \hspace{1cm} (4.200)

(Note that $M$ and $M-N$ still tend to infinity provided that $k < 1$.)

Thus if $k > \frac{1}{2}$, then $e(\frac{1}{2}; R) < \frac{1}{4}$

and the condition for Nice Recurrence is violated. But a choice of

$$s > 1 + \frac{h(\frac{1}{2})}{h(\frac{1}{4}) - \log 2} =$$

$$1 + \frac{\log 2}{\log 2 + \frac{1}{4} \log 4 + \frac{3}{4} \log \frac{4}{3}} = 6.29..$$  \hspace{1cm} (4.202)

will satisfy (4.163) for some $k > \frac{1}{2}$ and with this is the $k$ used to construct the set $R = R(N,M)$ with $N$ chosen sufficiently large.

$$R = R(N,M)$$  \hspace{1cm} (4.203)

(Note also that, $x_0 = 2L(R) \leq 2.2^{-2N} \exp(2N \cdot h(\frac{k}{2}))$,)

and, for $k < 1$, this can be made arbitrarily small as $N$ increases, since $h(\frac{k}{2}) < \log 2$ in this case.

(4.204)

The blocks described above can be constructed, therefore, and so by the standard $Z_2^\infty$ to $Z$ transfer and a use of lemma 2.4, we are done.

**Remarks:** The result above is probably not the best possible: The crude lower bound, $e(a; R) \geq \frac{a - L(R)}{|R|}$, ought to be improved first.

The same argument gives the following generalization:
**Corollary 4.17**: For all $t \geq 2$ there is an $s > t$ and a set of $s$-nice recurrence which is not a set of $t$-nice recurrence.

Indeed any $s > 1 + \frac{h(k)}{h(k/2) - \log 2}$, \hspace{1cm} (4.206)

where $k = \cos^2(\pi 2^{-t})$ will do. \hspace{1cm} (4.207)
CHAPTER V.

ARITHMETIC PROPERTIES OF SETS OF RECURRENCE
AND VAN DER CORPUT SETS IN $\mathbb{Z}$ AND $\mathbb{Z}_2^\infty$

Introduction to the Problem:

**Definitions:** Let $R$ be a subset $\mathbb{Z}$. As in Edwards [11], call, $E$, a subset of $\mathbb{Z}$, asymmetric if whenever $n \in E$, $-n \notin E$.

Let $K(R;n;m)$ be the number, possibly infinite, of asymmetric subsets, $E$, of $R \cup -R$ for which $\sum_{i \in E} i = n$ and $|E| = m$. (5.1)

For the record; $K(R;0;0) = 1$ always. (5.2)

This can be defined analogously for abelian groups in general. A complication arises with elements of order 2: A subset, $E$, of an abelian group, $G$, is asymmetric if $E \cap -E$ consists only of elements of order 2, etc.

$\emptyset$ often refers to the identity in a group.
The problem of enumerating $K(R;n;m)$, when $R$ is the sequence of squares or some other polynomial range, or the sequence of primes, is old and is better known as a variant on Waring's problem:

**Observation 5.1:**

a/ If $R$ is the sequence of powers $\{ n^k : n \in \mathbb{N} \}$, then there is an $M(k)$ such that, for all $m \geq M$, $K(R;0;m)$ is infinite.

b/ If $R$ is the sequence of primes, then for all $m \geq 9$, $K(R;0;m)$ is infinite.

**Proof:**

a/ See Hardy and Wright [16], Chapter XXI.

b/ A consequence of Vinogradov's solution of one of Goldbach's problems. See Vinogradov [24], Chapter X.

It of some interest to examine what happens with sets of recurrence. This will give some idea of how sparse a set of recurrence can become. In the case of $\mathbb{Z}_2^\infty$ there is a rather exact estimate of this sparseness. The case for $\mathbb{Z}$ is still only partly resolved.

Two results in this chapter are complimentary. One shows that sets of recurrence $\mathbb{Z}_2^\infty$ and $\mathbb{Z}$ can be arithmetically small, the second shows that, in $\mathbb{Z}_2^\infty$ at least, they cannot be too small. Both show again how graph theory can interact with the study of recurrence.

**Proposition 5.2:** For every increasing function $g: \mathbb{N} \to \mathbb{R}$, there is a set of recurrence, $R$, in $\mathbb{Z}_2^\infty$ such that

\[ K(R;0;s) \leq (1+g(s))^s \quad \text{for all } s \text{ large enough.} \]  \hspace{1cm} (5.3)

There is a set of recurrence, $R$, in $\mathbb{Z}$, similarly, such that

\[ K(R;0;s) \leq (1+g(s))^s, \quad \text{for all } s \text{ large enough.} \]  \hspace{1cm} (5.4)
**Remark:** The phrase 'for all $s$ large enough' can be omitted without strengthening the theorem significantly, since a large initial block can be removed from $R$ without disturbing its recurrence properties.

Consider the following definition, found in Harmonic Analysis:

**Definition:** A subset $R$ of $\mathbb{Z}$ is said to be Van der Corput or VdC if the fact that $\{ u_{n+r} - u_n : n \in \mathbb{N} \}$ is uniformly distributed mod 1 for all $r$ in $R$ is sufficient to imply that $\{ u_n : n \in \mathbb{N} \}$ is uniformly distributed mod 1.

**Remark:** Kamae and Mendes-France have shown that this property is equivalent to another property: For any positive probability measure, $\mu$, on the circle, the fact that $\mu^\wedge(r) = 0$ for all $r$ in $R$ implies that $\mu$ is continuous. This is the form in which the Van der Corput property is most often confirmed.

An $FC^+$ set is Van der Corput (see Ch.4, p.56).

It is proved by Kamae and Mendes-France [27] that a Van der Corput set is also a set of recurrence.

Thus Proposition 5.2 will be proved by means of a slightly stronger result:

**Theorem 5.3:** For every increasing function $g : \mathbb{N} \to \mathbb{R}$, there is a Van der Corput Set, $R$, in $\mathbb{Z}$ such that

$$K(R;0; s) \leq (1+g(s))^s,$$  \quad \text{for all } s \text{ large enough}.

(5.6)
The Construction and Proof of Theorem 5.3:

The proof will come by way of some graph theory:

**Definitions:** Suppose that $G$ is a group, not necessarily abelian.

Consider a left shift invariant graph on $G$, generated by a set $R$:

Edges $E = \{ (gr, g) : r \in R, g \in G \}$. \hspace{1cm} (5.7)

Suppose, further, that $R$ equals $R^{-1}$ and has $k$ elements and that the cardinality of $G$ is $n$. \hspace{1cm} (5.8)

Let $A$ be the adjacency matrix associated with this graph:

$$A_{g,h} = A(g,h) = \begin{cases} 1 & \text{if } \{g,h\} \in E \\ 0 & \text{otherwise.} \end{cases} \hspace{1cm} (5.9)$$

All the eigenvalues of $A$ are real and lie between $-k$ and $k$. There is an obvious one, namely $\lambda_0 = k$, \hspace{1cm} (5.10)

which corresponds to the normalised eigenvector

$$v_0 = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1), \hspace{1cm} (5.11)$$

and has multiplicity 1.

$-k$ is an eigenvalue if and only if the graph is bipartite, i.e. if and only if it can be partitioned into two sets each of which are independent of the graph.

Let $\lambda^* = \lambda^*(G)$

$$\lambda^* = \max \{ |\lambda| : |\lambda| \neq k \text{ and } \lambda \text{ is an eigenvalue of } A \}. \hspace{1cm} (5.12)$$

This number reveals quite a lot about the graph as will be shown.

The girth of $G$ is the length of the shortest cycle of distinct edges in $G$.

**Remark:** All the above and many of the results below hold for $k$-regular graphs in general.
The following construction will be used:

**Lemma 5.4:**
There is a sequence, $\langle G_p, m, E_p, m \rangle$, of non-bipartite $p(p+1)$-regular graphs, with the properties:

- a/ $p$ is a prime number congruent to 1 mod 4. (5.13)
- b/ $m$ is a prime number, chosen so that $p^{(m-1)/2} = -1 \mod m$ (5.14).
- c/ $(G_p, m, E_p, m)$ has order (number of vertices) $(m^3 - m)/2 = n$. (5.15)
- d/ girth $(G_p, m, E_p, m) \geq (2/3) \log_p 2n$, for all $m$ large enough. (5.16)
- e/ $\lambda^*(G_p, m, E_p, m) \leq 5p+1$. (5.17)

**Proof:** Bien [7] outlines a construction of connected left shift invariant 'Ramanujan' graphs $X_{p, m}$ on a subgroup of the multiplicative group of the quaternion algebra over $\mathbb{Z}_m$. These are $(p+1)$-regular, bipartite, have properties a/ and b/ and the further properties:

- c' $X_{p, m}$ has order $2n$. (5.18)
- d' girth $X_{p, m} \geq (4/3) \log_p 2n$, for all $m$ large enough. (5.19)
- e' $\lambda^*(X_{p, m}) \leq 2p^{1/2}$. (5.20)

The problem with these is that they are bipartite.

Note that, given $p$, there is a plentiful supply of $m$ satisfying b/. Fix $p$ and $m$ large enough so that a/, b/, c', d', and e' hold.
Since $X_{p,m}$ is connected and left shift invariant, there are two independent components of $X_{p,m}$: a (normal) subgroup, $Y$, of index 2, and its coset, $Y'$.

Suppose that $X_{p,m}$ is generated by $R = R^{-1}$. In addition, suppose that the girth of $X_{p,m}$ exceeds four.

Let $G_{p,m}$ be the left shift invariant graph generated by $R_\{e\}^\{2\}$ in $Y$. (5.21)

This graph has $n$ vertices and is $p(p+1)$-regular, by construction. It also has girth at least $(2/3)\log_p 2n$; half of the girth of $X_{p,m}$.

Further, if $A$ is the adjacency matrix of $X_{p,m}$, then the adjacency matrix of $G_{p,m}$ is an $n \times n$ minor of the matrix $A^2-(p+1)I$ (5.22) and, as such, has eigen-values:

\[ \{ \lambda^2-(p+1) : \lambda \text{ an eigen-value of } A \}. \] (5.23)

$\lambda^*(G_{p,m}) = \lambda^*(X_{p,m})^2 + p + 1 \leq 5p + 1$, therefore.

Further, $-p(p+1)$ is not an eigen-value and so $G_{p,m}$ is not bipartite.

**Remark:** Condition $e'$ indicates that $X_{p,m}$ are Ramanujan Graphs. Strictly speaking, condition $e'$ fails the Ramanujan Graph condition:

\[ \lambda^* \leq 2[p(p+1)-1]^{1/2} \] (5.24)

in this case, but it is certainly good enough for the purposes of this chapter. The large girth of these graphs and the fact that they are shift invariant are fortunate properties. The properties of these and other graphs are exploited heavily in the study of efficient communication networks. See Bien [7].

**Lemma 6.5:** Suppose that $G$ is a non-bipartite $k$-regular graph.

Let $f$ be a map from $G$ to $\mathbb{Z}$ and let

\[ R_f = \{ f(g)-f(h) : \{g,h\} \in E(G) \}. \] (5.25)
Then there is a real trigonometric polynomial $P(\theta^it)$, on the circle, with the following three properties:

i/ $P^\wedge(n) = 0$ whenever $n \notin R_f$, 
   i.e. $R_f$ contains the support of $P^\wedge$. 

ii/ $P(1) = 1$. 

iii/ $-\lambda^*(G)/k \leq P(\theta^it) \leq 1$, for all $t$. 

Proof: Let $P(z) = \frac{1}{|E|} \sum_{\{g, h\} \in E(G)} z^{f(g) - f(h)} \tag{5.29}$

Clearly, conditions i/, ii/ are satisfied, and the fact that $P(\theta^it) \leq 1 \tag{5.30}$ is easily proved. It remains to show that $-\lambda^*(G)/k \leq P(\theta^it)$, for all $t$. \tag{5.31}

For each $t$, let $w(t)$ be a vector with coordinates labelled by elements of $G$ ($A$, the adjacency matrix, acts on such vectors), defined as follows:

'g'th. coordinate $w_g(t) = \frac{\theta^{if(g)t}}{\sqrt{|G|}} \tag{5.32}$

$w$ has $l_2$ norm equal to 1. \tag{5.33}

Note that $P(\theta^it) = \frac{1}{k} w^*(t)Aw(t) \tag{5.34}$

where $w^*$ is the transpose conjugate of $w$. \tag{5.35}

$A$ is symmetric and real and so its eigenvectors, $v_i$ ; $i = 0, \ldots, n-1$, form an orthonormal basis, without loss of generality.

Let $w(t) = \sum_{i=0}^{n-1} a_i v_i \tag{5.36}$

Therefore, $\sum_{i=0}^{n-1} |a_i|^2 = 1. \tag{5.37}$
Let \( \lambda_i \) be the eigen value corresponding to \( v_j \).  

\[
P(e^{it}) = \frac{1}{k} \sum_{i=0}^{n-1} |a_i|^2 \lambda_i \geq -\lambda^*(G)/k
\]  

(5.39)

as required.

Remark: This proof will work with \( \mathbb{Z} \) replaced by \( \mathbb{Z}_2^\infty \) or any discrete abelian group.

The connection that this lemma has with the Van der Corput property is made obvious by the following lemma due to Kamae and Mendes-France [27].

**Lemma 5.6:** \( H \), a subset of \( \mathbb{Z} \), is a Van der Corput set if, for all \( \delta > 0 \), there is a real polynomial, \( P \), on the circle whose Fourier Transform is supported on \( H \), for which

\[
P(1) = 1 \text{ and } -\delta \leq P(e^{it}) \leq 1.
\]  

(5.40)

The connection with \( K \) is through counting cycles in the graph \( G \).

**Lemma 5.7:** Let \( (G,E) \) be a finite graph on \( G \). Then there is a map, \( f \), from \( G \) to \( \mathbb{Z} \) such that

\[
\text{[ number of cycles of length at most } s \text{ in } (G,E) ] \geq K(R_f ; 0 ; s), \text{ for all } s \geq 2.
\]  

(5.41)

**Proof:** Chose a numbering of \( G = \{g_1, g_2, \ldots, g_n\} \).

Let \( f(g_1) \geq 1 \); \( f(g_i) = 2n k f(g_{i-1}) \), for \( i \geq 2 \).
Suppose that
\[ \sum_{j=1}^{S} f(h'_j) - f(h_j) = 0 \] (5.44)
is an asymmetric sum of \( S \) elements from \( R_f \). Then \( S \leq nk \) (5.45)
and, since \( f \) is chosen increasing so rapidly, the sum can only be zero when it is trivially so, i.e. when there is symbolic cancellation between the terms. In this way, reordering the terms and swapping \( h_j \) and \( h'_j \) if necessary,
\[ h_1' = h_2, \quad h_2' = h_3, \quad \ldots, \quad h_S' = h_1. \] (5.46)
This represents a cycle of length at most \( S \) in \((G,E)\).

Conversely, every such cycle generates a sum such as the one above but it need not be asymmetric, hence the inequality.

The lemma follows easily now.

An upper estimate on the numbers of cycles in shift invariant graphs is thus essential:

**Lemma 5.8:** Suppose that \( G \) is non bipartite and \( k \) regular. Then
\[ \left[ \text{number of cycles of length } S \right] \leq k^S + n \lambda^*(G)^S. \] (5.47)

**Proof:** Let \( \Theta \) be the identity in \( G \). Let \( w \) be the vector whose \( \Theta \)'th. coordinate is 1 and all other coordinates are 0; a vector of \( l_2 \) norm 1.

\[ v_0 = \frac{1}{\sqrt{n}} (1, 1, 1, \ldots, 1) \] has eigenvalue \( k \).

A path in the graph, \( G \), is a cycle based at \( \Theta \) if it begins and ends at \( \Theta \). Then
\[ \left[ \text{number of cycles of length } S \text{ based at } \Theta \right] = w^* A^S w. \] (5.49)
Let \( w = \sum_{i=0}^{n-1} a_i v_i \). (5.50)

Therefore \( a_0 = < w, v_0 > = \frac{1}{\sqrt{n}} \) (5.51)

and \( \sum_{i=0}^{n-1} |a_i|^2 = 1 \) (5.52)

So \([\text{number of cycles of length } s \text{ based at } \theta] = \)

\[ = \sum_{i=0}^{n-1} |a_i|^2 \lambda_i^s \leq |a_0|^2 k^s + \lambda_s = k^s/n + \lambda_s. \] (5.53)

This calculation may be performed with \( w \) replaced by the indicator of any other point. Adding the \( n \) estimates together gives the result.
Proof of Theorem 5.3:

Let \( p_i = h(i) \) (5.54)

be a monotone sequence of primes congruent to 1 mod 4.

Let \( m_j \) be chosen large, satisfying (5.**).

Construct \( R_j \) and the graph \( G_j \) according to these parameters from lemma 5.4.

Let \( d(i) = (2/3) \log p_j 2n_j \) (5.55)

so that girth \( (G_j) \geq d(i) \). (5.56)

Let \( f_j \) be a map chosen as in lemma 5.5 (5.57)

and let \( H_j = (R_j) f_j \) (5.58)

so that

\[
\text{[ number of cycles of length at most } s \text{ in } (G_j, E_j) \geq K(H_j; 0; s). \quad (5.59)}
\]

Therefore, by lemma 5.7,

\[
K(H_j; 0; s) = 0 \quad \text{for } s < d(i), \quad (5.60)
\]

and

\[
K(H_j; 0; s) \leq (p_j + 1)^{2s} + n_j \lambda^*_i s \leq (p_j + 1)^s ( (p_j + 1)^s + 5n_j^i ). \quad \text{for } s \geq d(i), \quad (5.61)
\]

However, \( s \geq d(i) \) implies that \( n_j \leq p_j^{3s/2} \) (5.62)

and so, in general,

\[
K(H_j; 0; s) \leq (p_j + 1)^s ( (p_j + 1)^s + 5p_j^{3s/2} ) \leq 6 (p_j + 1)^{5s/2} \leq (p_j + 1)^{5s} \quad \text{for all } s \geq 1. \quad (5.63)
\]
Recall the construction of $f_i$ in lemma 5.5: Its first value could have been chosen arbitrarily. In the same spirit as lemma 5.5 it is straightforward to require by induction

$$\min \{ f_i(g) : g \in G_i \} \geq 2 \ k_{i-1} n_{i-1} \ \max \{ f_{i-1}(g) : g \in G_{i-1} \}. \tag{5.66}$$

This ensures that, if $H = H_1 \cup H_2 \cup H_3 \ldots \cup H_C$, then

$$K(H;0;s) = \sum_{\sum s_i = s : s_i \geq 0} K(H_1;0;s_1)K(H_2;0;s_2)\ldots K(H_C;0;s_C) \tag{5.67}$$

(remember that $K(R;0;0) = 1$ for any $R$, by definition)

$$\leq \sum_{\sum s_i = s : s_i \geq 0} (p_i+1)^{5s_j} \tag{5.68}$$

$$\leq \binom{s+c-1}{c-1} (p_{c+1})^{5s} \tag{5.69}$$

$$\leq (p_{c+1})^{5s} (s+c-1)^{c-1}/(c-1)! \tag{5.70}$$

Letting $c$ tend to infinity, it is important to note that if $s < d(c+1)$ then no contribution to $K(H;0;s)$ is made from $H_{c+1}$ onwards.

Thus, for $s < d(c+1)$,

$$K(H;0;s) \leq (p_{c+1})^{5s} (s+c-1)^{c-1}/(c-1)! \tag{5.71}$$

$$\leq (h(d^{-1}(s)+1))^{5s} (s+c-1)^{d^{-1}(s)-1} / (d^{-1}(s)-1)! \tag{5.72}$$

where $d^{-1}(s) = \min \{ c : d(c+1) > s \}. \tag{5.73}$
By making \( h \) increase very slowly and letting \( d \) increase very fast, one obtains

\[
K(H;0;s) \leq (1+g(s))^S \quad \text{for all } s \text{ large enough.} 
\]  

(5.75)

Finally, by lemma 5.5, it is easy to construct polynomials \( P_j \) on the circle, whose fourier transforms are supported on \( H \) (in fact \( H_j \)) and with

\[
P_j(1) = 1 
\]

(5.76)

and

\[
-6.\rho_i^{-1} \leq P_j(z) \leq 1 \quad \text{for all } z \text{ on the circle.} 
\]

(5.77)

Thus \( H \) is a Van der Corput set.

**Definition:** A subset of \( \mathbb{Z}_2^\infty \) is **independent** if it is independent when \( \mathbb{Z}_2^\infty \) is considered as a \( \mathbb{Z}_2 \) vector space.

**Proof of Proposition 5.2:** The fact that Van der Corput sets are recurrent is sufficient to show the second part of the Proposition.

The first part can be produced in the same way, taking care to define things appropriately in \( \mathbb{Z}_2^\infty \) and following the arguments almost verbatim. Some simplification, indeed, is possible, where the maps \( f_i \) can be less carefully defined: They need only be 1-1, each with a range which forms an independent set, and each range being independent of the others.

However, the fact that the \( H \) constructed in \( \mathbb{Z}_2^\infty \), in this way, is a set of recurrence can be derived directly from the decreasing values of \( \lambda^*/k \). The following observation is probably quite well known and is sufficient to show that \( H \) is a set of recurrence in \( \mathbb{Z}_2^\infty \) directly.
Observation 5.9: Suppose that \((G,E)\) is a non-bipartite, finite, \(k\) regular graph with \(\lambda^*(G) = \lambda^*\) (5.78).

Suppose, further, that \(G'\) is a finite abelian group and that \(f\) is a 1-1 map from \(G\) to \(G'\).

As before, let \(R_f = \{ f(h) - f(g) : \{h,g\} \in E(G) \}\). (5.79)

Then \(\mathbb{L}^0(R_f) \leq \lambda^*/k\) (5.80)

**Proof:** Suppose that \(M\) is a subset of \(G\) of cardinality \(m\) which is independent of \(G\), i.e. if \(a\) is in \(M\) and \(\{a,b\}\) is an edge, then \(b\) is not in \(M\).

Let \(w\) be a vector whose \(g^{th}\) coordinate is 1 if \(g\) is in \(M\), and 0 otherwise. (5.81)

Therefore, \(w^*A w = 0\). (5.82)

Using the notation before, \(w = \sum_{i=0}^{n-1} a_i v_i\) (5.83)

and \(\sum_{i=0}^{n-1} |a_i|^2 = m\). (5.84)

Also \(a_0 = <w, v_0> = \frac{m}{\sqrt{n}}\). (5.85)

Thus \(0 = w^*A w = \sum_{i=0}^{n-1} |a_i|^2 \lambda_i\) (5.86)

\[= m^2 k/n + \sum_{i=1}^{n-1} |a_i|^2 \lambda_i\] (5.87)

\[\geq m^2 k/n - m\lambda^*\] (5.88)

Therefore, \(m/n \leq \lambda^*/k\) (5.89)

and thus, since \(M\) was arbitrary, \(\mathbb{L}^0(R_f) \leq \lambda^*/k\). (5.90)
**Definition:** A subset, $R$, of an abelian group, $G$, satisfies the Rider condition if there is a constant, $B$, such that

$$W(R;0;m) \leq B^m \text{ for all } m. \quad (5.91)$$

It is called a Sidon set if for every function $h$ from $G$ to $[0,1]$ there is a measure $m$ on $G^\wedge$ such that, for all $g$ in $R$, $m^\wedge(g) = h(g). \quad (5.92)$

**Remarks:** As before, $^\wedge$ refers to the Fourier Transform of a function or measure. See Edwards [11] for example. In the particular case of $Z_2^\infty$ see [11, p.211 ff.] where $Z_2^\infty$ is known as $C^\wedge$.

Note that if $K(R;0;m) \leq B^m$ for all $m$ then

$$K(R;n;m) \leq (B^2)^m \text{ for all } m \text{ and } n. \quad (5.93)$$

The first definition takes its name from Rider who proved [22] that a subset of $Z$ which satisfies this condition is also a Sidon Set. Rider's condition is clearly linked with the statement of Proposition 5.2.

Bourgain [10] points out that Sidon Sets are not Van der Corput. Thus Theorem 5.3 is rather sharp. The first part of Proposition 5.2 is also the best possible in $Z_2^\infty$:

**Proposition 5.10:** A set in $Z_2^\infty$ which satisfies Rider's condition cannot be a set of recurrence.

The following strong result is mentioned in Edwards [11, p.211 ff.].
**Lemma 5.11:** A set in $\mathbb{Z}_2^\infty$ is a Sidon set if and only if it is a finite union of independent sets.

It is possible to translate Rider's result into $\mathbb{Z}_2^\infty$. The proof is given here for completeness and it follows the one given in Edwards [11] closely. Indeed some simplification is possible in this special circumstance:

**Lemma 5.12:** If $R$, a subset of $\mathbb{Z}_2^\infty$, satisfies Rider's condition then it is also a Sidon set.

**Proof:** Equate $\mathbb{Z}_2^\infty$ with the countable direct sum of $\langle\{1,-1\}, \text{multiplication}\rangle$ and equate $\mathbb{Z}_2^\infty$ with $\{1,-1\}^\mathbb{Z}$, the infinite product.

Suppose that $R$ satisfies the Rider condition. As was noted before, this implies that without loss of generality $K(R;n;m) \leq B^m$ for all $n$. (5.94)

To show that $R$ is a Sidon set, it is sufficient to show that, for every function $h: R \rightarrow \{1,-1\}$, there is a measure, $\mu$, on $\mathbb{Z}_2^\infty$ such that

$$\mu^\Lambda(v)h(v) \geq 1/2 \quad \text{for all } v \in R. \quad (5.95)$$

Let $\zeta_v(x) = \sum_i v_i x_i$ (5.96)

be the Walsh function corresponding to $v$ in $\mathbb{Z}_2^\infty$, defined for all $x$ in $\{1,-1\}^\mathbb{Z}$.

Let $R$ satisfy the Rider condition and let $B$ be the constant in the definition.

Let $b = 1/(3B^2)$. (5.97)

Let $h : R \rightarrow \{b,-b\}$ be an arbitrary function.
Let $R = \{ v_1, v_2, \ldots \}$ and let $f_k(x) = 1 + h(v_k) \zeta_{v_k}(x).$

Consider $t_N(x) = \prod_{1 \leq k \leq N} f_k(x)$

$$= \sum_{\omega \in \Omega} \prod_{1 \leq r \leq N} F(r, \omega(r))$$

where $\Omega$ is the space of functions from $\{1, \ldots, N\}$ to $\{0,1\}$ and

$F(r,0) = 1$

and $F(r,1) = h(v_r) \zeta_{v_r}(x).$

$t_N$ is a strictly positive function.

Let $\Omega(s) = \{ \omega \in \Omega : |\omega|_1 = s \},$

the definition of 'norm' being that made before.

$$t_N(x) = 1 + \prod_{1 \leq r \leq N} h(v_r) \zeta_{v_r}(x) +$$

$$+ \sum_{2 \leq s \leq N} \sum_{\omega \in \Omega(s)} \prod_{1 \leq r \leq N} h(v_r)^{\omega(r)} \zeta_{v_r}(x)^{\omega(r)}$$

So $\|t_N\|_1 \leq$

$$\left\| \sum_{v \in \mathbb{Z}_2} \sum_{2 \leq s \leq N} \sum_{\omega \in \Omega(s,v)} \prod_{1 \leq r \leq N} h(v_r)^{\omega(r)} \zeta_{v_r}(x)^{\omega(r)} \right\|_1$$

where $\Omega(s,v) = \{ \omega \in \Omega(s) : \sum_{1 \leq r \leq N} \omega(r) v_r = v \}.$
\[
\leq 1 + \sum_{2 \leq s \leq N} \sum_{\omega \in \Omega(s, 0)} b^s 
\quad (5.107)
\]

since \( t_N \geq 0 \) and \( \int \zeta_v(x) \, dx = 0 \) when \( v \neq 0 \).

\[
= 1 + \sum_{2 \leq s \leq N} b^s |\Omega(s, 0)| 
\quad (5.108)
\]

Now note that \( |\Omega(s, 0)| = K(R; 0; s) \leq B^s \) \( (5.109) \)
and get a uniform bound for \( ||t_N||_{1} \leq 1 + 1/(6B^2) \). \( (5.110) \)

\( t_N \) thus has a subsequence, weak* convergent to a measure, \( v \), on \( \{1, -1\}^Z \).

Note further that, by (5.105)

\[
|t_N^\wedge(v) - h(v)| \leq \sum_{2 \leq s \leq N} \sum_{\omega \in \Omega(s, v)} \prod_{1 \leq r \leq N} |h(v_r)^{\omega(r)}| 
\quad (5.111)
\]

\[
\leq \sum_{2 \leq s \leq N} b^s |\Omega(s, v)| \leq \sum_{2 \leq s \leq N} b^s 2 |\Omega(s, 0)| \leq b/2. 
\quad (5.112)
\]

So we are done.

**Proof of Proposition 5.10:** A set which obeys Rider's condition is Sidon
and hence the finite union of independent sets. Since an independent set is not set of
recurrence, the Ramsey property for sets of recurrence finishes the proof.

The \( Z \) form of Proposition 5.10 is still open.
Definitions of recurrence in discrete groups can be set up as for the $\mathbb{Z}$ case with little extra effort. However, many of the results of previous chapters require different techniques to prove their counterpart in general groups. The aim of this chapter is to set up a general way of taking unusual examples of recurrence in $\mathbb{Z}$ and reconstructing them in groups belonging to a broader class.

In this section, all actions and algebraic maps are written on the right for the sake of algebraic accuracy; if $h$ were an integer then $x(h\Psi)$ might be written $T^h x$ in more ordinary circumstances. Analytic functions, colourings etc., retain their standard form; $f(x)$.

**Definitions:** Let $H$ be a countable discrete group with identity element, $e$.

Suppose that $X$ is a separable compact hausdorff topological space, then let $H(X)$ be the group of homeomorphisms from $X$ onto itself.

A topological dynamical system (with $H$-action) is a pair $(X, \Theta)$, where $\Theta$ is a homomorphism from $H$ to $H(X)$. Thus each element, $h$, of $H$ acts on $X$ homeomorphically by means of $\Theta$. $h\Theta : x \rightarrow x(h\Theta)$. 

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A subset, \( F \), of \( X \) is \( H \)-invariant if the action of \( H \) leaves it unchanged:

\[
F(h\Theta)^{-1} = F, \text{ for all } h \in H.
\] (6.1)

\((X,\Theta)\) is minimal if the only closed \( H \)-invariant sets are \( X \) and \( \emptyset \). By an application of Zorn's Lemma, every system has a minimal subsystem.

\( R \), a subset of \( H \), is said to be a set of topological recurrence if, for all minimal topological dynamical systems \((X,\Theta)\) and all open sets, \( U \), in \( X \), there is an \( r \) in \( R \) such that

\[
U(r\Theta)^{-1} \cap U \neq \emptyset. \tag{6.2}
\]

A subset, \( R \), of \( H \) is (right) colour intersective if, given a finite colouring,

\[
\bigcup_{1 \leq i \leq n} P_i = H, \tag{6.3}
\]

of \( H \), there is \( 1 \leq i \leq n \) and an \( r \) in \( R \) such that

\[
P_i \cap P_i r \neq \emptyset. \tag{6.4}
\]

The (left) shift invariant graph, \( \Gamma(R) \), induced by \( R \) on \( H \) is one which has \( H \) as its vertex set and \( \{ \{h,hr\} : h \in H, r \in R \} \) as its set of edges.

**Remark:** The demand of countability and separability in the definitions above requires some explanation.

If a group is countable and it acts on a compact Hausdorff topological space, \( X \), then, when considering theorems about recurrence, there is no loss of generality in assuming that \( X \) is separable. One considers the separable orbit closure of a point in \( X \).
However, this countability assumption may not be dropped without losing generality. Here is a construction of a set of topological recurrence in a discrete abelian group, all of whose countable subsets are not sets of topological recurrence.

**Example 6.1. A Set of Recurrence, any Countable Subset of which is not a Set of Recurrence:**

Let $\omega_0$ be the first infinite ordinal and $\omega_1$ the first uncountable ordinal.

Let $H = \prod_{\alpha < \omega_1} \{0, 1\}$, \hspace{1cm} (6.5)

a topological group with its discrete topology and also with the box product topology:

The neighbourhood base of 0 consists of the sets $V_{\phi} = \ker(\phi)$, \hspace{1cm} (6.6)

where $\phi$ is a homomorphism: $H \rightarrow \prod_{\alpha < \omega_0} \{0, 1\}$.

Let $\Theta$ be a minimal $H$-action on a compact separable topological space, $X$. Let $x$ be a point in $X$ and let $f$ be a positive continuous real valued function on $X$ such that $f(x) = 1$ and $\|f\|_\infty = 1$.

$H$ acts on $C(X)$ point-wise and so one can consider the orbit, $O(f)$, of $f$ in $C(X)$, which is not necessarily sup-norm precompact. However, there is a countable base of neighbourhoods of $f$ in $O(f)$ and, for each $\varepsilon > 0$, a countable number of translates of the set \{ $g \in C(X) : \|f - g\|_\infty < \varepsilon$ \} cover $O(f)$. Letting $\varepsilon$ tend to zero in the rationals, one sees that the subgroup,

$$\text{Fix}(f) = \{ h \in H : f(h \Theta) = f \},$$ \hspace{1cm} (6.7)

of $H$, has index at most $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, a coset being determined by the indication of $f$ in a countable collection of sets.
Now, given a point \( x \) in \( X \), there are a countable number of continuous functions which separate it from each of the elements of the dense countable subset, \( X \) being regular. Thus, for similar reasons as above,

\[
\text{Fix}(x) = \{ h \in H : x(h\Theta) = x \} \quad (6.8)
\]

has index at most \( 2^{\aleph_0} \) as well.

Thus \( H/\text{Fix}(x) \subset \bigoplus_{r \text{ real}} \{0, 1\} \subset \prod_{\alpha < \omega_0} \{0, 1\} \), where all these maps are monomorphisms.

This last injection comes from considering the Dedekind section map defined on the basis vectors, \( \{ \delta_r : r \in \mathbb{R} \} \), of \( \bigoplus_{r \text{ real}} \{0, 1\} \).

\[
\delta_r \to 1 \{ q \in \mathbb{Q} : q < r \} \in \prod_{q \text{ rational}} \{0, 1\} \quad (6.9)
\]

Thus the composite map \( \phi : H \to \prod_{\alpha < \omega_0} \{0, 1\} \) has a kernel which fixes \( x \).

One repeats this argument for each \( x \) in countable dense subset of \( X \) and so obtains a map from \( H \) to the countable product of \( \prod_{\alpha < \omega_0} \{0, 1\} \), the kernel of which fixes every such \( x \) and so, by continuity, every \( x \) in \( X \). Since the countable product of a countable product can be expressed as a single countable product, \( \Theta \) is continuous with respect to the topology defined above.

Now the example may be constructed as follows: Let \( \mathbb{R} \) be an algebraic basis for \( H \) viewed as a vector space over \( \mathbb{Z}_2 \). Then \( \mathbb{R} \) is a set of topological recurrence since it converges to 0 in a topology with respect to which every separable action is continuous.

However, if \( \mathbb{R}' = \{ r_1, r_2, \ldots \} \quad (6.10) \)

is a countable subset of \( \mathbb{R} \), then there is a \( \beta < \omega_1 \) for which...
Indeed $R'$ generates a subgroup, $G$, of $H$ isomorphic to \( \bigoplus_{\alpha < \omega_0} \{0, 1\} \). An application of Zorn's lemma shows that there is a $K$ such that $H = G \oplus K$. (6.12)

For $x = 1$ or $0$, let $x(r_j) \equiv x+1 \mod 2$ and $x(h) \equiv x$ if $h \in K$. (6.13)

Define an action, $\Theta$, of $H$ on $\{0,1\}$ by extending this linearly. This is a minimal topological dynamical system in which the open set $\{0\}$ is not returned by $R'$ to itself and the construction is complete.

Thus, having restricted one's attention to separable spaces, the assumption that the group also be countable loses some generality.

**Definitions and Elementary Results of Recurrence in General Groups:**

The following lemma gives a useful equivalence between some of the definitions given before:

**Lemma 6.2:** Suppose $H$ is a countable discrete group and $R$ is a subset of $H$.

The following are equivalent:

i/ $R$ is a set of topological recurrence.

ii/ $R$ is colour intersective.

iii/ $\Gamma(R)$ has infinite chromatic number.

**Proof:** ii/ and iii/ are equivalent almost by definition.

\((i/ \Rightarrow ii/)\) Let $c$ be a $k$-colouring of $H$. Form the right shift system, generated by this colouring, as follows:
\( c \) is considered to be an element of \( \{1, 2, \ldots, k\}^H = \Omega \). \hfill (6.14)

\( \Omega \) is a compact topological space with the Tychonov product topology.

Consider the natural right \( H \)-action, \( \Theta \):

\[
(\omega(g\Theta))(h) = \omega(hg),
\]

for all \( h \) and \( g \) in \( H \) and all \( \omega \) in \( \Omega \). (Remember that all actions are written on the right and colours on the left). \( \Theta \) is a homomorphism.

Thus \( (\Omega, \Theta) \) becomes a topological dynamical system with \( H \)-action. One then forms the closure of the orbit of \( c \) under the \( H \)-action; a subsystem with \( H \)-action and separable because \( H \) is countable.

From here the argument becomes identical with that of lemma 3.2, some care being taken with the non-commutativity of \( H \).

(ii/\( \Rightarrow \)i/) Let \( (X, \Theta) \) be a minimal topological dynamical system with \( H \)-action, and \( U \) an open neighborhood of a point, \( x \).

Suppose that \( G \) is a discrete group and that \( X \) is a closed subset of \( Y \), a compact Hausdorff topological space, upon which there is defined a \( G \)-action, \( \Psi \), which commutes with \( \Theta \):

\[
[x(h)\Theta](g)\Psi = [x(g)\Psi](h)\Theta,
\]

for all \( x \) in \( X \), \( g \) in \( G \) and \( h \) in \( H \). Suppose further that, \( U = U' \cap X \), where \( U' \) is an open subset of \( Y \) such that

\[
\bigcup_{g \in G} U'(g\Psi)^{-1} = Y,
\]

which equals \( \bigcup_{1 \leq i \leq k} U'(g_i\Psi)^{-1} \), therefore, since \( Y \) is compact. Then the argument of lemma 3.2 applied to the colouring

\[
c(h) = \min \{ i : x(h)\Theta \in U'(g_i\Psi)^{-1} \}
\]

gives the result.
In the case that $H$ is abelian, there is no problem constructing $Y$ and $\Psi$; they can be $X$ and $\Theta$ respectively. However, in the non-abelian case more care is needed.

Suppose, first, that $(X, \Theta)$ is a right shift system on two letters, i.e. a minimal subsystem of $\{0,1\}^H$, and that $U$ is the set

$$\{ \omega \in X : \omega(e) = 1 \}. \quad (6.19)$$

Let $\omega_0$ be an element of $U$ and form

$$\omega_0'(h,h') = \omega_0(h)\omega_0(h'), \quad (6.20)$$

an element of $\{0,1\}^{H \times H}$.

Let $Y$ be the closure of the orbit of $\omega_0'$ under the natural right action, $\Phi$, of $H \times H$ on $\{0,1\}^{H \times H}$. $Y$ contains $X$ as the closure of the orbit of $\omega_0'$ under the $H \times \{e\}$ subaction of $\Phi$. Let $U' = U \cup (Y \setminus X)$. Let $\Psi$ be the $\{e\} \times H$ subaction of $\Phi$. To show that

$$\bigcup_{h \in H} U'(h\Psi)^{-1} = Y,$$

it is sufficient to observe that, for all $\omega$ in $X$, the orbit closure of $\omega$ under $\Psi$ is minimal with respect to $\Psi$ and that $U'$ intersects all such orbits.

To complete the construction, note that no generality is lost by assuming that $X$ is a shift system on two letters.

If there are more letters involved then the argument above works with the more general colouring $\omega_0 : H \to Z_2^N$ and where elements of $Z_2^N$ are multiplied together coordinate-wise. $U$ is the set $\{ \omega \in X : \omega(e) = 1 \}$, where 1 is the constant 1 colour, $(1,1,1,1)$, in this case.

In general, if $X$ is minimal with respect to a $G$-action, $\Theta$, and $x$ is a point in an open subset, $U$, of $X$, then note that there is a finite subset, $\{g_1,g_2,\ldots,g_n\}$, of $G$ such that

$$\bigcup_{1 \leq i \leq n} U((g_i)\Theta)^{-1} = X \quad (6.21)$$

and the colouring $c(g) = \min \{ i \geq 1 : x(g\Theta) \in U((g_i)\Theta)^{-1} \} \quad (6.22)$
will generate a shift system which represents $X$ adequately enough for the purposes of the theorem.

**Remark:** The map $h \mapsto h^{-1}$ is a graph automorphism of $\Gamma(R)$. Thus left and right recurrence properties are the same.

As was noted earlier in chapter 2, general groups can also form measure preserving dynamical systems. The definitions are improved here to their appropriate generality.

**Definitions:** Let $H$ be a countable, discrete group.

Let $(X, \mathcal{B}, \mu)$ be a probability measure space.

Let $\text{IMPT}(X, \mathcal{B}, \mu)$ denote the group of invertible bimeasurable measure preserving transformations, defined pointwise on $(X, \mathcal{B}, \mu)$.

$(X, \mathcal{B}, \mu)$ has a measure preserving $H$-action defined on it by means of a homomorphism from $H$ to $\text{IMPT}(X, \mathcal{B}, \mu)$: $\Psi : H \to \text{IMPT}(X, \mathcal{B}, \mu)$.

This measure preserving system is written $(X, \mathcal{B}, \mu, \Psi)$ or $(X, \mathcal{B}, \mu, H)$ if the action is unambiguous.

$R$, a subset of $H$, is said to be a set of (measure theoretical) recurrence if for every measure preserving system, $(X, \mathcal{B}, \mu, \Psi)$, and every element, $A$, of $\mathcal{B}$ of strictly positive measure, there is an $r$ in $R$, such that

$$\mu( A \cap A(\psi^{-1})^{-1}) > 0.$$  (6.23)
Consider $l_\infty(H)$; the space of functions, $F$, from $H$ to $\mathbb{C}$, for which the norm
\[ ||F||_\infty = \sup \{ |F(h)| : h \in H \}, \] (6.24)
is finite.

This forms a C*-algebra, with obvious unit, under pointwise multiplication.

$H$ acts on $l_\infty(H)$ by means of the right action, a continuous algebraic isomorphism, $\Phi$, defined as:
\[ (F(h\Phi))(g) = F(gh), \] (6.25)
for all $h$ and $g$ in $H$. $F(h\Phi)$ is called the right shift of $F$ by $h$.

If $H$ is amenable, then there are many shift invariant means on $l_\infty(H)$. I.e. positive, linear functionals, $M$, on $l_\infty(H)$, such that
\[ M(F(h\Phi)) = M(F). \] (6.26)
If $E$ is a measurable subset of $H$, then it is said to have upper Banach density greater than $a$, whenever there is a right shift invariant mean, $M$, for which
\[ M(1_E) > a. \] (6.27)

A subset, $R$, of $H$ is density intersective if, for all sets $E$, of positive upper Banach density in $H$, $Er \cap E$ is non-empty for some $r \in R$.

**Remark:** As for topological recurrence, there is no difference between the right and left definitions of recurrence or intersectivity. This fact will be used to some effect towards the end of the chapter.

The following lemma is proved in much the same way as lemma 3.4.

**Lemma 6.3:** If $H$ is a countable, discrete, amenable group and $R$ is a subset of $H$, then the following are equivalent:

i/ $R$ is density intersective.

ii/ $R$ is a set of measure theoretical recurrence.
Proof: (i/⇒ii/) Let \((X, \mathcal{B}, \mu, \Theta)\) be a measure preserving system. Let \(A\) be a set of positive measure. Without loss of generality, whenever intersections of translates of \(A\) are non-empty they are of positive measure. This uses the countability of \(H\) as in [4].

By the mean ergodic theorem for amenable groups (see [4] for example) there is a sequence of Følner ergodic averages of \(F\) which converges in norm and hence there is a subsequence which converges almost everywhere. Thus, there is a right shift invariant mean, \(M\), on \(l_\infty(H)\) and a point, \(x\), in \(X\) for which the function

\[
F(h) = 1_A(xh\Theta) 
\]

has mean,

\[
M(F) = \mu(A) > 0. 
\]

Therefore the set

\[
E = \{ h : 1_A(xh\Theta) = 1 \}
\]

has positive upper Banach density. Since \(R\) is assumed to be density intersective there is an \(r\) in \(R\) and an \(h\) in \(E\) such that \(hr\) is in \(E\) also. In other words \(x(h\Theta)\) is in \(A \cap A(r\Theta)^{-1}\), which is hence a non-empty set.

So

\[
\mu(A \cap A(r\Theta)^{-1}) > 0, 
\]

and recurrence is confirmed.

(ii/⇒i/) Let \(E\) be a set of positive upper Banach density. Let \(M\) be a right invariant mean on \(H\) such that \(M(E) > 0\).

Recall the definition of \(\Phi\) above.

Let \(F_\phi = 1_E\), a function in \(l_\infty(H)\). Consider \(K\), the smallest \(C^*\)subalgebra of \(l_\infty(H)\) which contains the unit and all the shifts of \(F_\phi\) by the (countable number of) elements of \(H\) under the action \(\Phi\). \(K\) is invariant under the action of \(\Phi\) and so the restriction of the action is well defined.

By the Gelfand isomorphism theorem, \(K\) is \(C^*\)-algebraically isomorphic to \(C(Y)\), the continuous functions on some compact Hausdorff space \(Y\). \(\gamma : K \rightarrow C(Y)\). \(Y\) is metric because \(K\) is separable.
F_0, an element of K, is mapped, by this isomorphism, to some continuous function, \( f_0 = F_0 \gamma \), on Y. \( (6.33) \)

Since \( F_0 \) is idempotent, \( f_0 \) is idempotent as well and is hence the indicator of a subset, B, of Y.

The discrete action, \( \Phi \), of algebraic isometries on K transfers naturally to an action of algebraic isometries on \( C(Y) \), \( \Phi \gamma \). By standard results on spaces of continuous functions, this induces a homeomorphic action on Y: \( \Phi^* : H \rightarrow \mathcal{H}(Y) \). The algebraic properties of this map are summed up in the following equality:

\[
(F\gamma)(y(h\Phi^*)) = ((F(h\Phi))\gamma)(y), \text{ for all } h \in H, F \in K \text{ and } y \in Y. \tag{6.34}
\]

\( M \) induces an \( H \)-invariant measure, \( \nu \), on the space Y:

\[
\int F\gamma \, d\nu = M(F). \tag{6.35}
\]

So \( (Y,\mathcal{D},\nu,\Phi^*) \) is a measure preserving system.

This construction follows closely Bergelson's interpretation [4] of Furstenberg's construction [14, p.72ff.]. See also pp.12.ff. of this dissertation.

Now, \( \nu(B) = M(1_E) > 0. \tag{6.36} \)

Thus there is an \( r \) in \( R \) for which \( \nu(B \cap B(r\Phi^*)^{-1}) > 0. \tag{6.37} \)

But the left hand side of this equals

\[
M(1_E, ((1_E)_r\Phi)) = M(1_E \cap E_r^{-1}). \tag{6.38}
\]

In particular \( E_r \cap E \) is non-empty and density intersectivity is confirmed.

Finally, a lemma which will be used in the last section of this paper, showing that recurrence can often be treated as a finite phenomenon: This is known as uniformity of recurrence in what follows. Its proof is a variation on the proof for uniformity of recurrence in \( \mathbb{Z} \), Theorem 2.1.
**Lemma 6.4:** Let $H$ be a countable discrete group and let $R$ be a set of measure theoretical recurrence. Given $\delta > 0$, there is a finite subset, $R_\delta$, of $R$ and $\varepsilon(\delta) > 0$, with the following property:

Given a measure preserving $H$-action $(X, B, \mu, \Theta)$ and a measurable subset, $A$, $\mu(A) \geq \delta$, of $X$, there is an element $h$ of $R_\delta$ for which

$$\mu(A \cap A(h\Theta)^{-1}) > \varepsilon(\delta).$$

(6.39)

**Proof:** Let $P_\delta$ be the subset of the unit ball of $l_\infty(H)$ consisting of functions of the form

$$E(X, B, \mu, \Theta, A)(h) = \mu(A \cap A(h\Theta)^{-1}),$$

(6.40)

where $(X, B, \mu, \Theta)$ can be any measure preserving system and $A$ any measurable subset such that $\mu(A) \geq a$.

(6.41)

$P_\delta$ is closed with respect to the weak* topology, $\sigma(l_1(H), l_\infty(H))$.

To see this suppose that $(X_n, B_n, \mu_n, \Theta_n, A_n)$ is a sequence of measure spaces and subsets whose images, $E(X_n, B_n, \mu_n, \Theta_n, A_n)$, in $P_\delta$, converge weak*. In particular

$$\lim_{n \to \infty} E(X_n, B_n, \mu_n, \Theta_n, A_n) = E(h)$$

(6.42)

exists for each $h$ in $H$.

Let $X$ be the disjoint union of the $X_n$'s, $n \geq 1$, and let $A$ be the subset of $X$ which is the disjoint union of the $A_n$'s. $H$ acts naturally on this separable Hausdorff topological space by acting on each component as it was defined there.

$$x(h\Theta) = x(h\Theta_n) \quad \text{if } x \in X_n.$$  

(6.43)

Let $F_0 = 1_{A_n}$, an element of $L_\infty(X)$.

(6.44)

$H$ acts on $L_\infty(X)$ via $\Theta^*$: $(F(h\Theta^*))(x) = F(x(h\Theta))$.

(6.45)
Let $K$ be the smallest $C^*$-subalgebra of $L_\infty(X)$ which contains the unit and all shifts of $F_0$ under $\Theta^*$. As before, there is a $C^*$-algebraic isometry $\gamma: K \rightarrow C(Y)$, where $Y$ is a compact separable Hausdorff topological space.

Let $L$ be a Banach limit on $l_\infty(\mathbb{N})$.

Given $F$ in $L_\infty(X)$, let

$$F^*(n) = \int F(x) \ d\mu_n(x),$$

an element of $l_\infty(\mathbb{N})$ and set $m(F) = L(F^*)$. (6.47)

This induces a measure, $\nu$, on $Y$ which is invariant under the induced homeomorphic $H$-action, $\Phi$, on $Y$. In particular, $(Y, \mathcal{B}(Y), \nu, \Phi)$ forms a measure preserving system.

Further, $F_0$ maps to the indicator of a subset, $B$, of $Y$.

$$F_0^* = \mu_n(A_n) \geq a \quad \text{and so} \quad \nu(B) \geq a. \quad (6.48)$$

Similarly

$$E(Y, \mathcal{B}(Y), \nu, \Phi)(h) = E(h) \quad (6.49)$$

and so the limit of $E(X_n, \mathcal{B}_n, \mu_n, \Theta_n, A_n)$ is in $P_\delta$. Thus $P_\delta$ is closed and hence compact in the weak* topology of $l_\infty(H)$.

For each $h$ in $H$, consider the evaluation map

$$\chi_h(v) = v(h) \quad (6.50)$$

as a continuous function on $P_\delta$. Consider also the subsets, $U_h$, of $P_\delta$ defined as

$$U_h = \{ v \in P_\delta : \chi_h(v) > 0 \}. \quad (6.51)$$

These are open and, since $R$ is a set of recurrence, the union $\bigcup_{r \in R} U_r$ forms an open cover of $P_\delta$. Let $\bigcup_{r \in R_\delta} U_r$ be an finite open subcover, and note that this $R_\delta$ satisfies the statement of the lemma.

Also,

$$\Theta(\delta) = \min \{ \chi_r(v) : r \in R_\delta, v \in P_\delta \} \quad (6.52)$$

will satisfy the lemma.
Remarks: In contrast to other lemmas about measure preserving recurrence, this lemma holds true for non-amenable group actions.

Bearing in mind the graphical interpretations of recurrence, this lemma bears out a general compactness principle well known to combinatorialists.

This lemma justifies the following definitions:

Definitions:

Given $0 \leq a \leq 1$,

let $e_H(a; R) = \inf \sup_{(X, B, \mu, \Theta), \mu(A) \geq a, r \in R} \mu(A \cap A(r \Theta)^{-1})$ (6.53)

where the infimum is taken over all measure preserving $H$-actions.

Let $L_H(R) = \inf \{ a : e_H(a; R) > 0 \}$. (6.54)

$R$, is said to be a set of strong recurrence, if, for every measure preserving system, $(X, B, \mu, \Theta)$, and every set, $A$, of positive measure, there is an $\epsilon > 0$, and an infinite subset, $R'$, of $R$ such that, for all $r$ in $R'$, $\mu(A \cap A(r \Theta)^{-1}) \geq \epsilon$. (6.55)

Remark: Lemma 6.3 proves that

$L_H(R) = \inf \{ a: \inf_{M, \text{ mean. } E \subseteq H, M(E) \geq a} \sup_{r \in R} M(E \cap E_r) > 0 \}$ (6.56)

and $e_H(a; R) = \inf_{M, \text{ mean. } E \subseteq H, M(E) \geq a} \sup_{r \in R} M(E \cap E_r)$ . (6.57)

for $H$ infinite.

Thus results about recurrence in dynamical systems with amenable group action can be proved using means on the group itself.
Techniques for Transferring Properties of Recurrence from One Group to Another:

The following is a useful lemma which gives the flavour of the arguments needed in this section:

**Lemma 6.5:** Suppose that $G$ and $H$ are countable discrete groups and $R$ a subset of $G$. Let $f$ be a function $: G \rightarrow H$.

Let $R_f = \left\{ f(g)^{-1}f(gr) : g \in G, r \in R \right\}$, a subset of $H$. (6.58)

a/ Then, if $R$ is a set of topological recurrence, $R_f$ is also a set of topological recurrence.

b/ Suppose further, that $G$ is amenable. Then, if $R$ is a set of measure theoretical recurrence, then $R_f$ is also, and $e_G(a;R) \leq e_H(a;R_f)$. (6.59)

If there is a homomorphism $\alpha : H \rightarrow G$, such that $(f(g))\alpha = g$ for all $g$ in $G$,

then $e_G(a;R) = e_H(a;R_f)$ for all $0 \leq a \leq 1$. (6.60)

**Proof:** a/ Let $c$ be a finite colouring of $H$. Define a new colouring $c'$ of $G$ as $c'(g) = c(f(g))$. (6.61)

Lemma 6.2 and the recurrence property of $R$ completes this part.

b/ Suppose that $H$ acts on a probability space, $(X,B,\mu)$, in a measure preserving manner via the action $\Theta$, and let $A$ be a subset of $X$ of measure at least $\alpha$. Construct the following $G$-action.

Let $(g)\Psi : G \times X \rightarrow G \times X$

$$ : (g', x) \rightarrow (g'g, x(f(g')\Theta)^{-1}(f(g'g)(\Theta))) ,$$ (6.62)

and consider the action induced on the $C^*$-algebra, $L_\infty(G \times X)$, by this. Let $A'$ be the subset, $G \times A$, of $G \times X$. Let $K$ be a $C^*$-algebra generated by the unit and all translates of
under $\Psi$. This becomes, as in past paragraphs, algebraically isometric to a $G$-action of homeomorphisms on $C(Y)$, where $Y$ is a compact Hausdorff space. Further $A'$ corresponds to a subset, $B$, of $Y$.

The existence of a mean, $M$, on $G$ ensures the existence of a $\Psi$-invariant measure, $\nu$, on $Y$, for which $\nu(B) \geq a$. In detail, given $F$ in $K$,

\[
F^*(g) = \int_X F(g, x) \, d\mu(x). \tag{6.63}
\]

Let \[\nu(F) = M(F^*). \tag{6.64}\]

The condition \[
\nu(B \cap B(r\Psi^{-1})) > \theta_G(a;R) - \epsilon \tag{6.65}
\]
says that there is an element, $g'$, of $G$, such that

\[
\mu(A(f(g')\Theta)^{-1} \cap A(f(g')\Theta)^{-1}) > \theta_G(a;R) - 2\epsilon, \tag{6.66}
\]

and we are done.

Conversely, if $\alpha : H \to G$ is a homomorphism, and $G$ acts in a measure preserving manner, $(X, B, \mu, \Psi)$, then, in the same way, $H$ acts on $X$ via $\Theta$:

\[
x(h\Theta) = x(h\alpha\Psi). \tag{6.67}
\]

Let $\mu(A) \geq a$. Then there are $g \in G$ and $r \in R$ such that

\[
\mu(A \cap A((f(g)^{-1}f(gr)\Theta)^{-1})) > \theta_H(a;R_f) - \epsilon. \tag{6.68}
\]

The left hand side equals $\mu(A \cap A(r\Psi^{-1})$, however, and we are done.

**Corollary 6.5.1:** Suppose that $G$ and $H$ are discrete, countable groups, that $G$ is amenable and that $\alpha : H \to G$ is an epimorphism, then:

a/ If $G$ has a set of topological recurrence which is not a set of measure theoretical recurrence, then $H$ has such a set also.

b/ If $G$ has a set of recurrence which is not a set of strong recurrence, then $H$ has one also.
Proof: a/ This follows immediately from the lemma above: $f$ can be any (1-1) function which inverts $\alpha$. Indeed it shows that $\alpha^{-1}(R)$ will do as the example in $H$.

b/ This proceeds analogously for measure preserving systems with some complications.

Let $R$ be a subset of $G$, and let $(X,B,\mu,\Psi)$ be a measure preserving $G$-action with a subset, $A$, $\mu(A) > 0$, such that, for all $\varepsilon > 0$, the set

$$\Delta_{\varepsilon} = \{ r \in R : \mu(A \cap A(r\Psi)^{-1}) \geq \varepsilon \} \quad (6.69)$$

is finite.

Construct $f$, inverse to $\alpha$, and $R_f$. $R_f$ is a set of recurrence by the lemma above. However, it may be strongly recurrent.

The argument of the lemma above ensures that there is a measure preserving $H$-action, $\Theta$, on $(X,B,\mu)$ for which

$$\Delta_{\varepsilon} = \{ r \in R : \exists g \in G : \mu(A \cap A(f(g)^{-1}f(gr)\Theta)^{-1}) \geq \varepsilon \}; \quad (6.70)$$

the same $\Delta_{\varepsilon}$ as above.

Define inductively, subsets $S_i$ and $Q_i$ of $R_f$ as follows:

Let $Q_1 = R_f$. Let $S_1$ be a finite subset of $R_f$ for which

$$L(S_1) < 1/4. \quad (6.71)$$

Such can be constructed by means of the uniformity of recurrence.

Let $Q_2 = R_f \setminus \{ f(g)^{-1}f(gr) \in R_f : g \in G, r \in R$

and $\exists g' \in G : f(g')^{-1}f(g'r) \in S_1 \}$. $Q_2$ is a set of recurrence since it equals $R'_f$ for some cofinite subset, $R'$, of $R$.

Assume that $Q_n$ is a set of recurrence. Let $S_n$ be a finite subset of $Q_n$ for which

$$L(S_n) < 2^{-n-1}. \quad (6.73)$$
Let \( Q_{n+1} = Q_n \setminus \{ f(g)^{-1} f(gr) \in R_f : g \in G, r \in R \}
\]
and \( \exists g' \in G : f(g')^{-1} f(g'r) \in S_n \}, \tag{6.74} \]
which, as for \( Q_2 \), is also a set of recurrence.

Let \( S = \bigcup_i S_i \). \tag{6.75} \]

By uniformity of recurrence, \( S \) is a set of recurrence. However, since, for all \( \epsilon > 0 \), there is an \( n \) for which
\[ \{ r \in R_f : \mu(A \cap A(r\Theta)^{-1}) \geq \epsilon \} \tag{6.76} \]
is a subset of \( Q_n \), and since the intersection of \( S \) with \( Q_n \) is finite for all \( n \), \( S \) is not a set of strong recurrence with the same counterexample action as above.

**Corollary 6.6:**

a/ If \( H = G \times K \) and \( G \) has a set of topological recurrence which is not a set of measure theoretical recurrence, then \( H \) has such a set.

b/ If \( H = G \times K \) and \( G \) has a set of recurrence which is not a set of strong recurrence, then \( H \) has such a set.

Now consider a group, \( G \), which is the sum of cycles:
\[ G = \bigoplus_i Z_{p_i} \oplus Z \tag{6.77} \]
\[ (Z_p = \{0,1,...,p-1\}, \text{ addition mod } p)). \tag{6.78} \]

**Lemma 6.7:**

a/ Any infinite abelian group which is also a sum of cycles contains a set of topological recurrence which is not a set of measure theoretical recurrence.

b/ Any infinite abelian group which is also a sum of cycles contains a set of recurrence which is not a set of strong recurrence.
Proof: If $Z$ appears in the decomposition of the abelian group, $G$, then, by corollary 6.6 and the results of chapters 3 and 4, the result is proved.

It remains to consider the cases:

$$G = Z_p^\infty \quad \text{and} \quad G = \bigoplus_{i} Z_{p_i},$$

where the $p_i$'s increase without bound.

The first arises when the order of the groups in the cyclic decomposition is bounded so that one may pick out the order, $p$, which occurs infinitely often, express $G$ as a sum of $Z_{p}^\infty$ with the remaining factors and then use corollary 6.6. The second is the remaining case.

i/ $G = Z_{p}^\infty$:

This is constructed like the $Z_{2}^\infty$ case in Chapters 3 and 4. Since the notation will be useful later on, here are the preliminaries and some of the details.

If the coordinates of $v$ in $Z_{p}^\infty$ are $v_i$ then $|v|_1$ is defined to be $\sum_i v_i$, summation being in $\mathbb{N}$.

i.a/ For given $n$ and $k$, let

$$M(k,0) = \{ v \in Z_{p}^{2n+k} : |v|_1 < n(p-1) \}, \quad (6.80)$$

$$M(k,1) = \{ v \in Z_{p}^{2n+k} : 2n(p-1) > |v|_1 > (n+k)(p-1) \}, \quad (6.81)$$

and

$$R_k = \{ v \in Z_{p}^{2n+k} : |v|_1 \geq 2n(p-1) \}. \quad (6.82)$$

$M(k,0)$ and $M(k,1)$ are each independent of the shift invariant graph that $R_k$ induces on $Z_{p}^{2n+k}$.

The cardinality of $M(k,0)$, as a fraction of the cardinality of $Z_{p}^{2n+k}$, is equal to the probability that $2n+k$ rolls of a $p$-sided dice, the sides being labelled 0 through $p-1$, sum to less than $n(p-1)$. The mean of such a sum is $\frac{1}{2} (2n+k)(p-1)$ and the standard deviation is $O((2n+k)^{1/2})$; the distribution is symmetric about the mean. If
k=\(o(n^{1/2})\) as \(k\) tends to infinity, the bound on the sum \((n(p-1))\) is \(o(1)\) standard deviations away from the mean. Thus, the probability of the sum being less than \(n(p-1)\) approaches \(\frac{1}{2}\). This is also the asymptotic size of \(M(k,1)\).

\(R_k\) arises from a graph with chromatic number at least \(k+2\). To see this, first consider the collection of all partitions of the set \([1,2n+k]\) into \(p\) disjoint sets. These are naturally encoded as elements of \(\mathbb{Z}_p^{2n+k}\).

If \(v = (v_j)\) is in \(\mathbb{Z}_p^{2n+k}\) let \(v(i) = \{ m : v_m = i \}, 0 \leq i \leq p-1.\) \(^{(6.83)}\)

Assume, without loss of generality, that \(p|n.\) \(^{(6.84)}\)

Now consider the sub-collection, \(\Sigma,\) of all \(p\) partitions of \([1,2n+k]\) for which \(|v(i)| = \frac{2n}{p}\) for each \(0 < i \leq p-1.\) \(^{(6.85)}\)

Define a graph, \(\Gamma,\) with vertex set equal to \(\Sigma\) as follows: \(v\) and \(w\) will be connected if and only if \(|v-w|_1 = 2n(p-1).\) \(^{(6.86)}\)

Note that \(|v|_1 = |w|_1 = n(p-1),\) so that \(v\) and \(w\) are connected if they are as 'far apart' as they can be and so in particular

\[v(0) \supseteq w(p-1) \text{ and } w(0) \supseteq v(p-1).\] \(^{(6.87)}\)

Conversely, for each pair of subsets; \(A,B;\) of \([1,(4n/p)+k]\) which are disjoint and of cardinality \(2n/p,\) there is an edge \(\{v,w\}\) in \(\Gamma,\) for which

\[v(p-1) = A, \quad w(p-1) = B\] \quad \text{and} \quad

\[v(0) = [1,(4n/p)+k]\backslash A, \quad w(0) = [1,(4n/p)+k]\backslash B.\] \(^{(6.88)}\)

Thus in particular, the Kneser graph \(G((4n/p)+k,k)\) is inside the graph \(\Gamma,\) which, therefore, has chromatic number at least \(k+2.\) However, having been embedded in \(\mathbb{Z}_p^{2n+k},\) \(\Gamma\) is a subgraph of \(\mathbb{Z}_p^{2n+k} (R_k).\)

From here the construction is as in the \(\mathbb{Z}_2^\infty\) case.
ii.a/ \[ G = \bigoplus_i Z_{p_i}, \] where the \( p_i \) increase without bound: \( (6.89) \)

Let \( R \) be the set of topological recurrence in \( \mathbb{Z} \) which is not a set of measure theoretic recurrence, constructed in Chapter 3.

Construct the sets, \( R_m \), \( m \geq 1 \), from initial intervals of \( R \), and chose \( p_i \), \( m \geq 1 \), with the two properties:

There is a subset \( A_m \), of \( Z_{p_i} \), of density at least \( \frac{1}{2} - \frac{1}{2^{m+2}} \), so that \( A_m \cap (A_m + r) \) is empty for all \( r \) in \( R_m \).

\[ \Gamma_{Z_{p_i}}(R_m), \] the shift invariant graph induced on \( Z_{p_i} \) by \( R_m \), has chromatic number at least \( m \).

This may be constructed from by means of compactness arguments, standard in the study of graphs, or by uniformity of recurrence arguments.

Recall the definitions of axial union and alternating product in Chapter 4 which were used in constructions on \( Z_2^\infty \). These can be defined analogously for sums of \( Z_{p_i} \).

Form the axial union, \( S = R_1 \& R_2 \& \ldots \& R_m \& \ldots \), in \( \bigoplus_m Z_{p_i} \).

Clearly, \( \Gamma_{\bigoplus_m Z_{p_i}}(S) \) has infinite chromatic number and so \( S \) is a set of topological recurrence in \( \bigoplus_m Z_{p_i} \).

Further, the alternating product \( A_1 k A_2 k \ldots k A_m k \ldots \) is a subset of \( \bigoplus_m Z_{p_i} \), of upper density at least \( \prod_{m \geq 1} \left( \frac{1}{2} - \frac{1}{2^{m+2}} \right) > 0 \), which is independent of \( \Gamma_{\bigoplus_m Z_{p_i}}(S) \).

\( S \) is not a set of recurrence in \( \bigoplus_m Z_{p_i} \) therefore.
Since \[ \bigoplus_{i} Z_{\mathbb{P}_i} = \bigoplus_{m} Z_{\mathbb{P}_m} \times \bigoplus_{i \in \{i_m\}} Z_{\mathbb{P}_i}, \] (6.91)
corollary 6.6 can be used to prove the first part of the lemma.

b/ This proceeds in exactly the same way, using the constructions of Chapter 4.

At one stage, the natural generalisation of Kleitman's theorem for \( Z_p^N \) is needed. This can be proved by induction on \( p \):

For \( p = 2 \), we are done.

For \( p > 2 \), consider the following construction: Consider, for this part, that, for \( p \) odd,
\[ Z_p = \{-q+1,-q+2,...,q-2,q-1\}, \quad \text{where } q = (p+1)/2 \] (6.92)
and for \( p \) even
\[ Z_p = \{-q+2,-q+3,...,q-2,q-1\}, \quad \text{where } q = (p+2)/2 \] (6.93)

Suppose that \( A \) is a subset of \( Z_p^N \). Define \( P_1 \), a transformation of subsets of \( Z_p^N \), as follows:

Let \( v \) be an element of \( Z_p^N \) whose first coordinate is 0.

Let \( A_v = \{ j \in Z_p : j \in \{0\} + v \in A \} \). (6.94)

Let \( A'_v \) \[\begin{array}{ll}
= \{ -(|A|/2)+1,-(|A|/2)+2,...,(|A|/2)-1,|A|/2 \} & \text{if } |A| \text{ is even,} \\
= \{ -(|A|-1)/2,...,(|A|-1)/2 \} & \text{otherwise.} \end{array}\] (6.95)

\( A'_v \) sits in \( Z_p^N \). Let
\[ P_1 A = \bigcup_{v_1=0} A'_v. \] (6.96)

Note that \( |P_1 A| = |A| \) and the diameter of \( P_1 A \) does not exceed that of \( A \).

Define \( P_1 A \) similarly with the ith coordinate, for all \( 2 \leq i \leq N \). This is the same as a construction found in [29, pp.102,103] for the case \( p=2 \).

Now let \( A^N = P_N P_{N-1}...P_2 P_1 A. \) (6.97)
As above $|A^\wedge| = |A|$ and the diameter of $A^\wedge$ does not exceed that of $A$. Further, $A^\wedge$ has the property that, if $v \in A^\wedge$, then all $w \leq \text{abs}(v)$ are in $A^\wedge$ also. ( $\text{abs}(v)$ is the vector in $\mathbb{Z}_p^N$ whose coordinates are the absolute value of the corresponding coordinates of $v$. Each coordinate of $w$ is at most the corresponding coordinate of $\text{abs}(v)$.)

$Z_q$ is considered to equal $\{0,1,\ldots,q-1\}$. Define $B$, a subset of $\mathbb{Z}_q^{2N}$, as follows: $v$ is in $B$ if and only if there is an element, $w$, of $A^\wedge$ for which

$$v_{2i} = w_i \text{ and } v_{2i-1} = 0, \text{ if } w_i \geq 0$$

and

$$v_{2i} = 0 \text{ and } v_{2i-1} = -w_i, \text{ if } w_i < 0$$

(6.98)

for all $i$.

The construction of $A^\wedge$ ensures that the diameter of $B$ is at most that of $A$. Also $|B| = |A^\wedge| = |A|$. Since $q < p$, we are done by induction.

Here is another lemma; a reversed form of lemma 6.5:

**Lemma 6.8:** If $G$ is a subgroup of a discrete amenable group, $H$, and $G$ has a set of topological recurrence which is not a set of measure theoretical recurrence for $G$ actions, then $H$ has such a set for $H$ actions.

Further, if $G$ has a set of recurrence which is not a set of strong recurrence, then $H$ has such a set also.

**Proof:** Let $R$ be a set of topological recurrence in $G$ which is not a set of measure theoretical recurrence. It will be shown $R$ is such a set in $H$.

Clearly the chromatic number of $\Gamma_H(R)$ exceeds the chromatic number of $\Gamma_G(R)$. Thus, by lemma 6.2, $R$ is a set of topological recurrence in $H$.

Let $M$ be a right shift invariant mean on $l_\infty(H)$. 

Let $E$ be a set of positive upper Banach density in $G$ such that, for all $r \in R$,

$$E \cap Er = \emptyset.$$  \hfill (6.99)

Let $m$ be a mean on $G$ for which $m(1_E) > 0.$  \hfill (6.100)

Given $F$, a subset of $H$, define $\phi(h) = m(1_{Fh} \cap G)$ and let $M'(1_F) = M(\phi).$  \hfill (6.101)

This is a mean on $H$ which is right shift invariant.

Let $\{h_i\}$ be a set of representatives of the right cosets $\{Gh\}$ of $G$ in $H$. Let

$$F = \bigcup_i Eh_i.$$  \hfill (6.103)

Then $M'(1_F) = m(1_E) > 0$ and $F \cap Fr = \emptyset$ for all $r \in R.$  \hfill (6.104)

So $R$ is not a set of measure theoretical recurrence in $H$.

In the case of strong recurrence, more care must be taken.

Let $R$ be a set of recurrence in $G$, then lemma 6.5 with $f(g) = g$ ensures that $R = R_f$ is a set of recurrence in $H$.

Let $E$ be a subset of $G$ and let $m$ be a mean on $G$, such that

$$m(1_E) = a.$$  \hfill (6.105)

Further, suppose that $m(1_{rE} \cap E) < \varepsilon$ for all $r \in R.$  \hfill (6.106)

Construct $M'$ and $F$ as above, $M'(1_F) = a.$ \hfill (6.107)

Then $M'(1_{rF} \cap F) = m(\phi')$, \hfill (6.108)

where $\phi'(h) = m(1_{(rF \cap F)h \cap G}) = m(1_{rEh_i \cap Eh_i})$ \hfill (6.109)

for some $h_i$, since $r \in R$. But this is less than $\varepsilon$ by the right shift invariance of $m$.

These two constructions are sufficient to prove the second part.

**Corollary 6.8.1:** Any discrete abelian group which contains an element of infinite order has a set of topological recurrence which is not a set of measure theoretical recurrence. Similarly it has a set of recurrence which is not a set of strong recurrence.
**Proof:** Note that such a group contains a copy of \( \mathbb{Z} \) and use lemma 6.8.

**Remarks and definitions:** Therefore, of the class of infinite abelian groups, only torsion groups, i.e. those all of whose elements have finite order, remain to be considered. In fact, a large class of these has already been treated in lemma 6.7; the periodic groups.

For infinite abelian groups, the notion complimentary to periodicity is divisibility: An abelian group is divisible if, for every integer, \( n \), and every element, \( g \), of \( G \), there is another element, \( h \), for which \( h^n = g \). \hspace{1cm} (6.111)

For example, \( \mathbb{Q} \) is divisible, but \( \mathbb{Z} \times \mathbb{Q} \) is not.

Another typical example of a divisible group is \( \mathbb{Z}(p^{\infty}) \), the subgroup of \([0,1)\), with addition mod 1, consisting of rationals of the form \( q/p^r \mod 1 \), where \( q \) and \( r \) are integers.

Here are two structure theorems, with proofs that can be found in [28, Theorems 9.32 and 9.14 respectively].

**Lemma 6.9:** If \( G \) is a torsion abelian group, then there is a subgroup \( B \) which is also a sum of cycles such that \( G/B \) is divisible.

**Lemma 6.10:** If \( G \) is divisible then, algebraically (i.e. not topologically), \( G \) is a sum of copies of \( \mathbb{Q} \) and \( \mathbb{Z}(p^{\infty}) \) groups.

**Definition:**

\( H \) is an extension of \( G \), if there is an epimorphism \( \alpha : H \rightarrow G \).
A Class of Groups to which the Examples of Recurrence in Past Chapters may be Extended:

Using these definitions and lemmas, the main object of this section is attained, namely the proof of the comprehensive

**Theorem 6.11:** Let $C$ be the class of discrete countable groups which contains all discrete countable infinite abelian groups, and which is closed under containment by countable discrete amenable groups, direct summation, and countable discrete extension. Suppose that $G \in C$.

Then $G$ has a set of topological recurrence which is not a set of measure theoretic recurrence. It also has a set of recurrence which is not a set of strong recurrence.

**Remark:** $C$ contains, among other groups, the discrete infinite countable amenable solvable groups, and the free group on two generators, $F_2$.

**Proof of Theorem 6.11:** Some "wlogs" first:

Note that, by lemma 6.8 and corollaries 6.5.1 and 6.6, it is sufficient to prove that all infinite countable discrete abelian groups have the appropriate sets.

Corollary 6.8.1 restricts consideration to torsion groups.

Lemmas 6.8 and 6.9 and corollaries 6.5.1 and 6.8.1 together show that it is sufficient to treat non-trivial discrete divisible torsion groups. (In the notation of lemma 6.9, either $B$ is infinite and lemma 6.8 shows that we are done, or $G/B$ is non-trivial and divisible and corollary 6.5.1 gives the reduction that’s wanted.)

Lemma 6.10 and corollary 6.6 and the fact that $Q$ is torsion free show that it remains to consider only the case $G = \mathbb{Z}(p^\infty)$, for $p$ a positive integer.
Thus theorem 6.11 is proved by:

**Proposition 6.12:** $Z(p^\infty)$ has a set of topological recurrence which is not a set of measure theoretical recurrence. It also has a set of recurrence which is not a set of strong recurrence.

**Proof:** Recall the construction of a set of topological recurrence in $Z_p^\infty$ which is not a set of measure theoretic recurrence and its transfer to $Z$, mentioned in lemma 6.7. In the $Z_p^\infty$ case, it is made up as an axial union $R_1 & R_2 & \ldots$ and can be sent to $Z$ by a map which does not disturb the implied independence of the basis vectors in $Z_p^\infty$ too much.

This construction can put a similar set into $Z(p^\infty)$:

Let $p_i$ be a sequence of integers such that $p_i \mid p_{i+1}$. (6.112)

Let $b_1 = 1$ and $b_{k+1} - b_k = 2n_k + k$, (6.113)

where $n_k$ are picked increasing sufficiently rapidly so that the estimates, computed in lemma 6.7, hold. Construct $R_k$ etc. as in lemma 6.7. $R_k$ sits naturally in $Z_p^{[b_k, b_{k+1}]}$.

Given $K$, the axial union $R_1 & R_2 & \ldots & R_K$ is defined, as in Chapter 4, by means of $[1, b_{K+1}]$.

Let $p \mid q$ and define the function

$$f_q(n) = \lfloor p(n/(pq)) \rfloor \in \{0, 1, 2, \ldots, p-1\} = Z_p$$ (6.114)

where $\lfloor . \rfloor$ is the function which takes the fractional part of a real number and $\lfloor . \rfloor$ is the greatest integer function.

Let $f_{p_1, p_2, \ldots, p_b}(n) = (f_{p_1}(n), f_{p_2}(n), \ldots, f_{p_b}(n)) \in Z_p^b$. (6.115)
Also, for each element \( s = (s_1, s_2, \ldots, s_b) \) of \( \{-p, -p + 1, \ldots, p - 1, p\}^b \), and each element \( v = (v_1, v_2, \ldots, v_b) \) of \( Z_p^b \) define

\[
g_s(v) = \sum_{i=1}^{b} s_i v_i ,
\]

where the multiplication and addition take place in \( Z \).

These \( f \)'s and \( g \)'s are used, as in Chapters 3 and 4, to construct examples in \( Z \) from examples in \( Z_p^\infty \). Measure theoretical recurrence properties are transferred with an error term which is of the order of

\[
\sum_{r \in R} \left( \frac{2| r_1 |}{p} \cdot \sum_{i: r_i = 1} \frac{p_i}{p_i + 1} \right)
\]

Topological recurrence properties are transferred perfectly.

Now to return to \( Z(p^\infty) \):

Let \( p_1 = p^r_1 < p_2 = p^r_2 < \ldots \) (6.118) be chosen so that the error term in formula (6.156) is small.

Given \( k > 0 \), let

\[
S_k = f \cdot p^1 \cdot p^2 \ldots p^{2n_k + k}, \quad (R_k \& R_{k-1} \& \ldots \& R_2 \& R_1),
\]

a subset of \( Z^{r_{2n_k+k+1}} = Z_{N_k} \). (6.120)

Note the order of the axial union.

Let \( S_k* = (1/N_k).S_k \), (6.121)

the image of \( S_k \) under the natural imbedding of \( Z_{N_k} \) into \( Z(p^\infty) \). Also,

\[
S_{k+1}* \supseteq S_k* .
\]
\[ S = \bigcup_{k > 0} S_k^* \] is, therefore, a set of topological recurrence. \hspace{1cm} (6.123)

If \( S \) were a set of measure theoretical recurrence, then, by the uniformity of recurrence, lemma 6.4, there would be a \( k \) for which \( S_k^* \) returns every set of upper Banach density greater than \( 1/4 \). However, since \( S_k \) fails, by construction, to return a subset of \( Z_{N_k} \) of density almost \( 1/2 \), the construction of lemma 6.8 gives a contradiction. So we are done.

A similar argument works to give the example of a set of recurrence which is not a set of strong recurrence.

**Remark:** It is possible that a group could contain a sequence which is a set of measure theoretic recurrence, but not a set of topological recurrence. In general it is unclear, but the following well known argument shows that there are no such sequences in amenable groups.

**Lemma 6.13:** In an amenable group, \( G \), a sequence, \( R \), is a set of topological recurrence whenever it is a set of measure theoretical recurrence.

**Proof:** Let \( G \) act minimally on \( X \), forming a minimal topological dynamical system, and let \( U \) be an open set in \( X \). Since \( G \) is amenable, there is a \( G \)-invariant probability measure on \( X \). The support of this measure is a closed \( G \)-invariant set and so, by minimality, is the whole of \( X \). In particular, \( U \) has strictly positive measure and so \( R \), being a set of measure theoretical recurrence, returns \( U \) to itself with positive measure. A fortiori \( R \) returns \( U \) to itself topologically and the topological recurrence of \( R \) is confirmed.
CONCLUSION

To date, most sets of recurrence have been obtained by harmonic analytic methods; polynomial ranges, integer parts of various natural functions etc. This approach produces sets of recurrence which have many better properties. Consequently it was not clear whether there was a distinction between the property of recurrence and these stronger properties.

Many of the chapters of this essay have produced concrete distinctions. While positive results would have been more pleasing and would have produced more applicable corollaries, one cannot deny the richness of this area of mathematics where slight changes in a definition can produce a fine but measurable difference.

The author uses many arguments ad hoc, but in chapters III and IV a single technique is developed which gives several results. In short, this allows results in recurrence in $\mathbb{Z}_2^\infty$ to be transferred to $\mathbb{Z}$ where they are more familiar. The rich structure of $\mathbb{Z}_2^\infty$ often makes the constructions simple. Much of this structure remains unexploited, however, and the author is confident that other results are still to be found by this technique. In particular, the fact that $\mathbb{Z}_2^N$ is a field and a geometry is not used. Other structures, for example $\mathbb{Z}_p^N$, could produce results dealing with multiple recurrence.
In Chapter V, this technique is extended fruitfully to give a further connection between shift invariant graphs and sets of recurrence.

Chapter VI shows how other groups exhibit the examples which were constructed in $\mathbb{Z}$ and $\mathbb{Z}_2^\infty$ in previous chapters.

This essay illustrates the diversity of argument commonly needed in such a field of study and demonstrates again the close bond between combinatorics and dynamical systems.

**Open Problems:**

As it should be in a healthy mathematical subject, there is no shortage of open problems coming naturally from the definitions and constructions of the previous chapters. Some of these were mentioned at the end of chapter II. Others are listed here roughly in the order of the chapters.

**Questions:**

1: Is there, for all $\varepsilon > 0$, a set of $2+\varepsilon$-nice recurrence which is not a set of $2$-nice recurrence.

2: (après Bourgain) Is there a Set of Recurrence in $\mathbb{Z}$ which is also a Sidon set?

3: Does every infinite discrete (amenable) group contain a set of topological recurrence which is not a set of measure theoretical recurrence?

4: Is there an infinite discrete group which has a set of measure theoretical recurrence which is not a set of topological recurrence?

It is certain that many of these will be resolved before long.
This elegant and elementary proof of Kneser's conjecture appeared in a paper by Barany [1]. It requires two well known, easily stated, lemmas by Borsuk and Gale respectively.

**Lemma a.1**: [33] If $S_k$, the unit sphere in $\mathbb{R}^{k+1}$, is the union of $k+1$ open sets then one of them contains antipodal points.

**Lemma a.2**: [34] For all $a$ in $S_k$ in $\mathbb{R}^{k+1}$, let $H(a)$ be the set
\[ \{x \in S_k : \langle x, a \rangle > 0 \}. \]  
(a.1)

If $n$ and $k$ are integers, then there is a set $V$ in $S_k$ with $2n+k$ elements such that for all $a$ in $S_k$, $|H(a) \cap V| \geq n$.  
(a.2)

**Proof of lemma ii.1**: Identify the $2n+k$ points used in the construction of the Kneser Graph with the $2n+k$ elements of $S_k$ found in lemma a.2.

Let $c: G(2n+k,n) \rightarrow \{1,..,k+1\}$ be a $k+1$ colouring of the Kneser graph and consider the following sets defined for $i \leq k+1$:  

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\[ A_i = \{ a \in S_k : H(a) \cap V \text{ contains a subset of cardinality } n \text{ and colour } i \} . \]

Note that, by construction, the union of the \( A_i \)'s is \( S_k \) and, further, that these sets are open. So, by lemma a.1, one contains antipodal points, \( a \) and \( -a \).

Pick the subsets of \( H(a) \cap V \) and \( H(-a) \cap V \) which have common colour and note that these are disjoint. This represents an edge with end points of the same colour in \( G(2n+k,n) \) and we are done.
REFERENCES


