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Spontaneous chiral symmetry breaking and fermion mass generation in Lattice field theories

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The Ohio State University, 1988
SPONTANEOUS CHIRAL SYMMETRY BREAKING
AND FERMION MASS GENERATION
IN LATTICE FIELD THEORIES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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*****

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To My Parents

Late Nagendra Mohan De

and

Late Labanya Prava De

and to my teacher

Hiranmay Chowdhury
ACKNOWLEDGMENTS

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At the end I thank my parents who have showed me the light. I wish they were here today.
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CHAPTER I

Introduction

1.1 Introduction

This chapter, for the most part, is written at a very introductory level. The aim is to present the basic ideas of this dissertation in as simple a way as possible in order to attract a larger group of readers.

The goal of particle physics is to find the minutest particles and the most fundamental interactions among them. To date the fundamental matter particles are believed to be twelve fermions: six leptons and six quarks. The so-called gauge particles (bosons) are the mediators of the fundamental interactions among them. There are four basic interactions: gravitation, electromagnetism, weak interaction and strong interaction. All particles experience the force of gravitation. The leptons do not respond to strong interaction.

Quantum field theories are supposed to be the underlying theories of particle physics. Particles are represented by the quantum fields $\phi(x, \eta)$ where $x$ is either the coordinate or the momentum of the particle depending on the representation and $\eta$ could be one or more quantities corresponding to some internal degrees of freedom which are quantum mechanical. A quantum field theory is a combination of quantum mechanics and special relativity.
With the possible exception of the three neutrinos for which the issue is not settled yet, all the fundamental fermions are found to be massive in nature. The gauge particles can be of zero mass or massive depending on the range of the particular interaction. Also there are massive (both light and heavy) fermionic or bosonic bound states made out of the fundamental particles bound by a fundamental interaction.

A theory trying to describe particle physics can have the masses of the fundamental fermions as adjustable input parameters of the theory. Such a theory can be good for other merits. But from the point of view of understanding why the masses are what they are this theory is not intellectually satisfying because then the generation of fermion masses is not fundamentally explained, but rather taken as a priori numbers. Till now there are known two theoretical mechanisms which explain the fermion mass generation: spontaneous chiral symmetry breaking and Yukawa coupling mechanism. Between the two mechanisms listed above the Yukawa coupling one is extremely important from the point of view of what is known as the standard model, though in some sense it is a phenomenological mechanism. The Yukawa coupling term, in its simplest form, $y \bar{\psi} \phi \psi$, where $\psi$ and $\phi$ are the fermion and the scalar field respectively and $y$ is a constant, couples fermion fields to a scalar field and in the standard model depends on the scalar field getting a non-zero vacuum expectation value to give mass to the fermion. It is essentially then a mass term in the Lagrangian of the theory with a sophisticated appearance. Thus one is not really explaining the mass generation and it seems as if one is just putting one's ignorance at a different place. The other mechanism, spontaneous chiral symmetry breaking, from
a formal point of view, is more fundamental because it involves chiral symmetry, a fundamental feature of a fermion. It is not implied in any way that the Yukawa couplings should not be studied. However, the discussion in this dissertation involves chiral symmetry breaking ($\chi SB$). In later sections of this chapter chiral symmetry and its spontaneous breaking are explained.

Before ending this introductory section it is thought to be an appropriate place now to give a brief abstract of this dissertation. This dissertation employs a relatively new method (lattice gauge theory) of analyzing a quantum field theory. It is especially appropriate for the issues discussed here because lattice gauge theory allows one to investigate strong coupling behaviors of a field theory and $\chi SB$ is believed to be such a feature. The domain of the conventional perturbative analysis is the region of weak coupling of a theory, i.e., when the strengths of the interactions are relatively small.

There are several important goals of this dissertation. The primary goals involve understanding the physics-related issues and the secondary goals deal with several technical points particularly relevant to lattice field theories. As a primary goal, spontaneous chiral symmetry breaking is studied in different situations, for example, gauge and non-gauge theories. In the context of a gauge theory it is also interesting to study the relationship of spontaneous chiral symmetry breaking with other aspects of the theory, for example, a possible gauge symmetry breaking. As a secondary goal, an algorithm, namely, the molecular dynamics algorithm, for treating fermions on the lattice with their full dynamical content is studied. The staggered fermion prescription for putting fermions on the lattice is also studied for consistency. For the flavor interpretation of the degeneracy of the staggered fermions in the con-
tinuum to make sense one has to have what is known as the restoration of flavor symmetry in the continuum. In other words, one of the signatures of the continuum limit of a lattice theory with staggered fermions should be the restoration of flavor symmetry. One of the secondary goals of this dissertation is to verify that flavor symmetry is restored in the continuum in the framework of a lattice model with staggered fermions. These points will be more clear later in this chapter and in the subsequent chapters.

To understand the primary goals better, it is perhaps helpful to have a brief knowledge of what is already known about the spontaneous breaking of chiral symmetry. For definition of chiral symmetry the reader is referred to section 1.4. The idea of spontaneous chiral symmetry breaking in particle physics is borrowed from the famous BCS theory of superconductors. Spontaneous breaking of an approximate chiral symmetry explains the existence in nature of the pseudoscalar octet, the $\pi$, $K$ and $\eta$ particles, via the Goldstone mechanism (section 1.3). It is thus believed to be a feature of quantum chromodynamics (QCD), the accepted gauge theory of the strong interactions. From theoretical investigations also it is now believed that in vectorlike SU(N) gauge theories like QCD, chiral symmetries are spontaneously broken if the coupling is large enough. By lattice gauge theory calculations this has now been verified for SU(2) and SU(3) gauge theories. Understanding the dynamics of $\chi SB$, however, is an intriguing issue. It was suggested over a decade ago that confinement (section 1.5) implies $\chi SB$. It is also an interesting question to ask whether $\chi SB$ can take place only when the theory is confining. Few years ago through lattice gauge theory calculations it has been found that $\chi SB$ can occur whenever the effective coupling
becomes larger than a critical value long before the confining regime. The effective coupling becomes larger if fermions are considered in a higher representation of the gauge group. By going over to a higher representation of fermions $\chi SB$ is seen to occur much before confinement.

Keeping the above in mind one can then appreciate better the primary goals of this dissertation. As a first step one wants to establish the spontaneous breaking of chiral symmetry in a non-perturbative way in different models and then try to have more insight into its dynamics than already available.

Here two different models are investigated: a) The Gross-Neveu (GN) model and b) the SU(2) gauge-Higgs-fermion model. Both are formal theories in the following sense. They are not models which try to describe the nature as closely as possible. But they have many features of more realistic models which are aimed at explaining the observed phenomenology. However, these are good laboratories for studying several formal or theoretical issues. The goal of this dissertation is not to produce a number for the experimentalists to verify, it is rather to study issues like $\chi SB$ in different circumstances and study its relationship with other aspects of a theory. It is hoped that a deeper understanding of $\chi SB$ would not only explain the spectrum of the hadrons (the bound states of the strong interactions) but also the masses of the elementary particles (quarks and leptons). The dynamics of $\chi SB$ is still not satisfactorily known.

The GN model is a 1+1 dimensional asymptotically free theory. Asymptotic freedom is a very important feature of QCD. The GN model is not a gauge theory. Nevertheless, it has a discrete chiral symmetry which breaks
spontaneously and generates elementary fermion mass dynamically.

The SU(2) gauge-Higgs-fermion system is the closest one can get at the moment on the lattice to the SU(2) part of the standard model of unified electroweak interactions (Weinberg, Salam and Glashow). It is a vectorlike gauge theory unlike the standard model which is a chiral gauge theory. No mechanism of treating chiral gauge theories on the lattice is known yet. Here the investigation is more focussed on the interplay of gauge symmetry breaking and $\chi SB$ and how this depends on the representation of the fermions. It is found that for the fermions in the fundamental representation of the gauge group, the two apparently independent breakings of the two different symmetries coincide exactly. This is claimed certainly to be an insight into the dynamics of $\chi SB$. For higher representation ($l = 1$) of fermions, however, this is here seen to be a little different, but consistent with the recently known idea that effective interaction between elementary fermions can be increased in a gauge theory by going over to a higher representation of fermions.

Aside from the main topic of this dissertation several other issues are also looked at in the GN model. These are connected with the treatment of fermions on a lattice. As will be discussed in chapter 2, lattice fermions are difficult objects. Staggered fermions are one way of putting fermions on a lattice. Several topics of the staggered fermion mechanism such as flavor symmetry restoration by calculating a part of the spectrum of the model have been addressed. Usually it is also difficult to consider the full dynamical content of the fermions on the lattice. For treatment of completely dynamical fermions several algorithms are now available. One of these algorithms, namely the molecular dynamics method, is applied to this investigation of
the GN model on the lattice. It is found that this particular algorithm is perhaps adequate for the present model, but for more involved gauge theories it might be too slow and there could be conceptual problems as well because of its assumption of ergodicity.

1.2 Symmetries - Global and Local

It would be nice and aesthetically pleasing to have the universe as simple as possible. In other words, a theory describing the universe is desired to have symmetries. One can have a global or a local symmetry. A field theory has a global symmetry if it is invariant under global transformation of its field variables:

\[ \phi(x, \eta) \rightarrow G \phi(x, \eta) \]  \hspace{1cm} (1.1)

where G, the transformation operator, is the same for all positions x. If G depends on position and the theory is invariant under

\[ \phi(x, \eta) \rightarrow G(x) \phi(x, \eta), \]  \hspace{1cm} (1.2)

it is said to have a local symmetry.

Both global and local symmetries play an important role in the theories of particle physics. A theory consisting of matter fields can have some global symmetries. When the symmetry requirement is made more stringent by elevating it to a local symmetry it is seen that a space of some degrees of freedom (position space or some internal space) gets “curved” so that requiring a local symmetry means a new force among the matter fields. New particles arise in a quantum theory in this case and these will mediate the
force, known as gauge interaction. The mediating particles are bosons and are called the gauge particles. A local symmetry in the context of a field theory is usually called a gauge symmetry. Gravitation, theoretically, results from a local invariance under the group of transformations known as the general coordinate transformations. One can have the so-called Lie groups as the local transformations and make the theory invariant under these. For example Electromagnetism can be looked upon as originating from the Abelian gauge group $U(1)$ and Non-Abelian $SU(2)$ and $SU(3)$ gauge theories are supposed to model the world of weak and strong interactions.

Over a decade now gauge theories of weak and strong interactions have survived all experimental tests. Local symmetries thus seem to play a pivotal role in the theory of fundamental interactions. In this discussion, however, one is primarily concerned with chiral symmetry, a global symmetry, for theoretical studies of fermion mass generation in different interacting field theories, gauge and non-gauge.

1.3 Spontaneous Breaking of a symmetry and Goldstone theorem

Nature does not have many of the symmetries the particle physics theorists would like her to have. A very elegant way to get around this is the idea of spontaneous symmetry breaking. If the basic equations of a theory have a symmetry and the solutions do not, it is usually said that the symmetry is spontaneously broken. For example the observed physics of the electromagnetic and weak interactions do not show why the weak gauge bosons
are massive. The Weinberg-Salam-Glashow model, known as the standard model assumes a bigger gauge symmetry, namely $SU(2) \otimes U(1)$, at the Lagrangian or basic equations of motion level, and this is broken spontaneously to the symmetry of electromagnetism, namely $U(1)$, observed in nature, i.e., when the equations are solved in some possible way, the solutions or in other words the particles of the theory do not have the bigger symmetry of the Lagrangian. Spontaneous breaking of the gauge symmetry in this case is the result of a clever arrangement of terms in the Lagrangian and it involves a fundamental scalar field, called the Higgs particle. The whole process of this symmetry breaking and weak gauge bosons, the $W$ and the $Z$ particles, getting massive is known as the Higgs mechanism. This illustrates the power of the spontaneous symmetry breaking unifying the two apparently different interactions, electromagnetism and weak interactions and giving the masses of the weak gauge bosons in the process.

Spontaneous breaking of chiral symmetry (a global symmetry) in some sense is more fundamental. Chiral symmetry as indicated before is a fundamental property of the fermions. It comes essentially from the way Dirac chose to describe fermions. Spontaneous breaking of it induced by dynamics of a gauge field or other kind of interaction is certainly conceptually very clean and basic and it gives one a mechanism at hand to generate fermion masses.

It is perhaps not out of place to mention briefly the Goldstone theorem concerning spontaneous breaking of a continuous symmetry. A continuous symmetry is one where the transformations under which the theory is invariant can be changed continuously. The general content of the Goldstone
Theorem is that whenever a continuous symmetry is spontaneously broken, a massive boson, bound state of the fundamental fermions, appears in the spectrum. In the context of this dissertation this theorem is very important, because it can serve as a signal for spontaneous breaking of continuous chiral symmetry as is seen in chapter 4.

1.4 Chiral Symmetry and its Spontaneous Breaking

The word chirality means handedness, i.e., helicity states of a fermion. The fermions are not only the energy-momentum eigenstates in the free Dirac equation, they are also the eigenstates of the so-called helicity operators which are inner products of the spin and momentum operators. Fermions are described by spinors which are column matrices with four components in 3+1 dimensions. The so-called Dirac gamma matrices (\( \gamma^\mu, \mu = 0,1,2,3 \)) feature in the Dirac equation which is really a matrix equation with first order space and time differentiations.

The Dirac Lagrangian density is given as:

\[
L = \frac{1}{2} \bar{\psi} i \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi
\]  

(1.3)

where, \( \psi \) is the fermion field, \( m \) is the bare mass of the fermion and

\[
\partial_\mu = \frac{\partial}{\partial x^\mu}, \\
\bar{\psi} = \psi^\dagger \gamma^0.
\]

In Eq.(1.3) summation over the index \( \mu \) is understood.

The \( \gamma \) matrices satisfy

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}
\]  

(1.4)
where $g^{\mu\nu}$ is the Minkowski space metric.

It is advantageous to define a matrix $\gamma^5$ as follows:

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \gamma_5 \dagger$$

$$\gamma_5^2 = I$$

where $I$ is the unit matrix.

$\gamma^5$ is known as the chirality operator for the following reason. For $m=0$, it can be shown equivalent to the helicity operator.

The Dirac Lagrangian density above has a symmetry under the following global transformation:

$$\psi \rightarrow i\gamma^5 \psi$$

$$\bar{\psi} \rightarrow i\bar{\psi}\gamma^5$$

The $\bar{\psi}$ transformation is not independent, rather it follows simply from the $\psi$ transformation. This above symmetry is known as a chiral or axial symmetry of the massless Dirac Lagrangian. It is to be noted that this symmetry prevents a mass term, because the mass term goes to negative itself under the above chiral transformation. However, this is a symmetry under a discrete chiral transformation, hence the symmetry is known as a discrete chiral symmetry. The Dirac Lagrangian also has the following continuous chiral symmetry under the global transformation:

$$\psi \rightarrow e^{i\alpha\gamma^5} \psi$$

$$\bar{\psi} \rightarrow \bar{\psi} e^{i\alpha\gamma^5}$$
\( \alpha \) is a continuous parameter and a constant. The above continuous chiral symmetry transformation form the Abelian group U(1). Under this continuous chiral symmetry also it is easily seen that a mass term is not possible.

As an aside let us point out another kind of global symmetry also present in the Dirac Lagrangian and this symmetry does not forbid a mass term. This symmetry is

\[
\psi \to e^{i\alpha}\psi \\
\bar{\psi} \to e^{-i\alpha}\bar{\psi}
\]

which corresponds to fermion number conservation. The above sort of global symmetries without involving a \(^5\gamma\) matrix is known as a vector symmetry.

In an interacting theory the group structure of the chiral symmetry can be more complicated. Such an example is presented in the discussions of chapter 4. In the free case, there is no dynamics to break the above chiral symmetry spontaneously. So the particle spectrum will still be massless. In an interacting theory, however, the dynamics of the interaction can arrange a spontaneous breakdown of chiral symmetry. As a result, out of a massless interacting chirally symmetric Lagrangian, one sees fermion masses generated dynamically solely because of spontaneous chiral symmetry breaking with the vacuum expectation value \( \langle \bar{\psi}\psi \rangle \) becoming nonvanishing. Moreover, if the spontaneously broken chiral symmetry is continuous, the Goldstone theorem would apply and there would be a massless Goldstone boson in the spectrum. In theories like QCD chiral symmetries are believed to be broken spontaneously. The QCD vacuum does not have the chiral symmetry. It is
believed to be unstable with respect to the formation of fermion-antifermion pair condensate, known as the chiral condensate.

If the chiral symmetry is slightly broken by small values of mass $m$ in the Lagrangian, it can be shown that the approximate Goldstone particle would then pick up a small mass the square of which is proportional to the symmetry breaking parameter, $m$. This formalism is illustrated by the famous example of light pseudoscalar pions.

### 1.5 Asymptotic Freedom, Confinement, Chiral Symmetry Breaking

An asymptotically free theory means that the coupling constant representing the strength of the interaction depends on the length scale in the following manner. At short distances the coupling is small and it grows with distance. Non-Abelian gauge theories such as $SU(N)$ gauge theories are known to be asymptotically free. QCD is a good example of an asymptotically free theory. Confinement in QCD is also an idea consistent with asymptotic freedom. The basic fermions of strong interactions, the quarks, are never seen isolated in nature. They occur in nature bound with other quarks so as to form a color singlet. In a confining theory the coupling grows at large distances and so any confining theory can have $\chi SB$ by forming chiral condensates. But it is now believed to be possible that $\chi SB$ can occur long before the confinement scale is reached. It seems now that in a vectorlike (left-right symmetric) theory if the effective coupling among the fermions becomes bigger than a critical value, $\chi SB$ takes place irrespective of whether the theory is in a confining
phase. These points are discussed again in chapter 4.

1.6 Outline

In this section is given a brief outline of the contents of the following chapters. In chapter 2 a brief review of lattice gauge theory and its techniques are given so that when one goes on to chapters 3 and 4 one has some grasp of the techniques used. In chapter 3 studies of the lattice version of the Gross-Neveu (GN) model (not a gauge theory) is discussed. The GN model has a discrete chiral symmetry and its spontaneous breaking dynamically generates the mass of the fundamental fermion. The molecular dynamics algorithm for dynamical fermions and flavor symmetry restoration of the staggered fermions are also tested. In chapter 4, a formal model, vectorlike SU(2) gauge-Higgs-fermion system is investigated. This chapter discusses the interplay of the gauge symmetry breaking and $\chi SB$ and also illustrates generation of physical fermion masses of the theory.
2.1 Introduction

In this chapter a brief review of lattice gauge theory, its philosophy and techniques are presented. The discussion here is in no way claimed to be complete, but is designed to serve specifically the reader of this dissertation. For a more complete review, the reader is referred to references[1,2].

Lattice gauge theory was introduced just over a decade ago by Wilson[3], Polyakov[4], and Wegner[5]. In the last eight years or so, considerable interest has been generated in this area and lattice gauge theory results have started to have profound impacts on particle physics and field theory.

To name a few important successes of lattice gauge calculations, one now has numerical evidence for confinement in SU(2) and SU(3) gauge theories and chiral symmetry breaking in QCD. One has also been able to calculate roughly some hadron spectrum and an upper bound on the Higgs particle mass.
2.2 General Philosophy

The approach of lattice gauge theory allows one to look at a field theory from a more general perspective than the usual well-established perturbative methods. Perturbation theory has been successful but it is definitely restrictive. From a formal point of view, a general nonperturbative approach is naturally elegant. Also from a practical point of view, one simply cannot do without nonperturbative methods for calculating strong coupling behaviors of a field theory such as QCD. In chapters 4 and 5 respectively, the importance of nonperturbative treatment of the Higgs mechanism and Yukawa couplings is also discussed in detail.

In Euclidean lattice formulation both space and time are discretized. Given a continuum field theory involving fields $\phi_i$ with a Lagrangian $L(\phi_i)$, integrals are replaced by sums and derivatives with discrete differences. A Wick rotation to Euclidean space is performed and the theory’s Euclidean action $S(\phi_i)$, the temporal integration of $L(\phi_i)$, is constructed on the discrete space-time lattice. The quantization is performed via the path integral formalism. Here field theories are regulated by the space-time cutoff provided by the lattice itself. Also as a bonus one obtains a close correspondence with statistical mechanics so that one can use statistical mechanical tools to solve problems in quantum field theories. A cutoff field theory in 3+1 continuum Minkowski space is represented by the statistical mechanics in 4 Euclidean dimensions of a theory given by the partition function,

$$Z = \prod_i \int d\phi_i e^{\frac{i}{\hbar}S(\phi_i)}, \quad (2.1)$$
where $\epsilon$ plays the role of the physical temperature and is usually a bare parameter (e.g. square of the coupling constant in the discussions of the next chapter) of the original theory. For details of the above correspondence of a quantum field theory in 3+1 dimensions and statistical mechanics in 4 dimensions, the reader is referred to reference[1] and references therein.

Certainly there can be more than one cutoff prescription. Another method is the Hamiltonian lattice formalism in which only the spatial dimensions are discretized in Minkowski space-time. The theory can be canonically quantized with the usual Hamiltonian formalism. This approach is useful, for example, for strong coupling expansions (resembling high temperature expansion in statistical mechanics) to find the theory's mass-gap.

Throughout this dissertation Euclidean lattice formulations are used at zero physical temperature. To treat a field theory at a finite nonzero physical temperature, asymmetric lattices of size $N_x^3 \times N_t$ with $N_x > N_t$ are to be considered where $N_x$ and $N_t$ respectively are the lattice sizes (number of lattice points) along the space and time directions. $N_t$ is related to the physical temperature $T$ by the relation,

$$N_t = \frac{1}{aT} \quad (2.2)$$

where $a$ is the lattice spacing.

In the following a list showing the correspondence of a continuum field theory and the statistical mechanical system on a lattice is presented:
Statistical mechanics (4 dim.) ↔ Field theory (3+1 dim.)
Free energy density ↔ Vacuum energy density
Correlation function ↔ Propagator

Reciprocal of correlation length ↔ Mass-gap.

A lattice is only a technical device to define cutoff field theories. A finite lattice has a finite number of degrees of freedom and using statistical mechanical tools one can construct solutions of these cutoff field theories. But at the end one should be able to take a continuum limit to get the real physics of the theory.

In the limit of the lattice spacing going to zero, physics of a renormalizable field theory should be independent of the details of the regulator. It should lose memory of the lattice, or, in other words, in this limit the coherence length should be infinite compared to the lattice spacing. For this to happen, the statistical mechanical system on the lattice should have a continuous, second order phase transition, corresponding to the divergence of the correlation length. So the first step of a lattice gauge theory analysis is to establish the phases of the theory, locate the critical regions of continuous, second order phase transitions.

There is obviously more than one way of constructing a field theory on the lattice. Also an interesting and very important point to note here is that on the lattice one loses some of the symmetries of the continuum world (e.g. Lorentz or Euclidean symmetry). One has to be careful in choosing a lattice formulation for which all the desired symmetries are restored once the continuum limit is taken.
2.3 General Mechanism-Pure SU(2) Lattice Gauge Theory

Usually the matter fields are put on the lattice points \((n)\) and the gauge fields are put on the links \((n,\mu)\). \(\mu\) goes from 0 to 3 and \(n \equiv (n_0,n_1,n_2,n_3)\) for a 4 dimensional Euclidean field theory.

Wilson[3] has presented an elegant way to put gauge fields on the lattice. His prescription keeps local gauge invariance as an exact symmetry in a mathematically well-defined system. One does not need to introduce a gauge-fixing term here.

In the following a brief account of how to write down the Euclidean lattice action for a pure SU(2) lattice gauge theory (without matter fields) is given.

Let us consider a 4 dimensional Euclidean hypercubic lattice with spacing \(a\). On each link \((n,\mu)\), a SU(2) matrix (fundamental representation of a group element) is placed:

\[
U_\mu(n) = \exp[iB_\mu(n)]
\]  \hspace{1cm} (2.3)

\[
U_{-\mu}(n+\mu) = U^{-1}_\mu(n)
\]  \hspace{1cm} (2.4)

where, \(B_\mu(n) = \frac{1}{2}ag\tau_iA^i_\mu(n)\)  \hspace{1cm} (2.5)

In the above, \(g\) is the strength of the gauge field \(A^i_\mu\), \(\tau_i\)'s are the SU(2) generators and \(i\) is the SU(2) index, running from 1 to 3.

With the simplest local generalization of a global SU(2) transformation, the SU(2) gauge transformation on the lattice is defined to be,
\[ U_\mu(n) \rightarrow G(n)U_\mu(n)G^{-1}(n+\mu) \quad (2.6) \]

where, \( G[\theta^i] = \exp[-\frac{i}{2}\tau_i \theta^i(n)] \quad (2.7) \)

produces a rotation of angle \( \theta^i \) in the local SU(2) space at site \( n \).

The lattice action is built so as to be invariant under the gauge transformation (2.6). It is not difficult to see that a product of the \( U \) matrices taken around a closed path has the above local SU(2) symmetry. The most local closed contours are the so-called plaquettes (elementary squares). So with usual normalization the Euclidean lattice action \( S \) for a pure SU(2) gauge theory can be written down as,

\[ S = \beta_g \sum_P [1 - \frac{1}{2} \text{tr} U_P] \quad (2.8) \]

where, \( \beta_g = 4/g^2 \), the summation is over all possible elementary plaquettes and,

\[ U_P = U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu) \quad (2.9) \]

with \( \mu < \nu \).

It is instructive to look at Eq.(2.8) and appreciate how \( B_\mu(n) \) or the gauge field \( A^i_\mu \) enters the action as an angular variable so that their range of variation is naturally compact. Since SU(\( N \)) groups are compact, forming the lattice action with group elements \( U_\mu(n) \) naturally gives the bounded character of \( B_\mu(n) \). When this non-Abelian theory is quantized, the integrations over \( U_\mu(n) \)'s are group integrations over all elements of the group. These are
compact integrations (for these compact SU(N) groups) and that is why no
gauge fixing term is required.

In its classical continuum limit (lattice spacing $a \to 0$), the theory is
easily seen to be ordinary Yang-Mills theory and its strong coupling limit is
seen to confine quarks.

2.4 Fermions on the Lattice

Although a lot of lattice calculations have been done with pure gauge theories,
the real world consists of matter fields as well as gauge fields (mediators of
interactions among matter fields) and one has to introduce matter fields on
the lattice.

As indicated in the previous section, matter fields are usually put on the
lattice sites $(n)$. With bosonic matter fields one encounters no special diffi­
culties. However, with fermions, one encounters problems of extraordinary
measure. It is known as the “species doubling” problem of lattice fermions.
Simply put, it is because of the Dirac equation (equation obeyed by the
fermions) being first order.

In the following, a simple exercise is carried out to illustrate the species
doubling problem. Let us consider a massless free fermion field $\psi$. The
continuum action in Euclidean space is given by,

$$ S = \int \bar{\psi} \gamma^\mu \partial_\mu \psi d^4x, \quad (2.10) $$

where the Dirac matrices $\gamma^\mu$ obey the anticommutation relation:
\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}. \quad (2.11)

A naive discretization with lattice spacing \( a \) leads to a lattice action:

\[
S = \sum_{n,\mu} \frac{1}{2a} \overline{\psi}(n) \gamma^\mu [\psi(n + \mu) - \psi(n - \mu)] a^4. \quad (2.12)
\]

For simplicity let us consider \( a = 1 \). So,

\[
S = \frac{1}{2} \sum_{n,\mu} \overline{\psi}(n) \gamma^\mu [d^+_{\mu} - d^-_{\mu}] \psi(n), \quad (2.13)
\]

where,

\[
d^\pm_{\mu} \psi(n) = \psi(n \pm \mu). \quad (2.14)
\]

The equation for the propagator \( G(n) \) can then be written as,

\[
\sum_{\mu} \frac{1}{2} \gamma^\mu (d^+_{\mu} - d^-_{\mu}) G(n) = \delta_{n,0}. \quad (2.16)
\]

Following the usual technique, the propagator in momentum space is,

\[
G(p) = \frac{1}{\sum_{\mu} \gamma^\mu \sin p_\mu}, \quad (2.17)
\]

where \( p_\mu \) denotes 4-momentum. The zeros of \( \sum_{\mu} \gamma^\mu \sin p_\mu \) will give the particles of the theory. There is the desirable zero at \( p_\mu = 0 \) as well as the undesirable one at the end of the Brillouin zone \( p_\mu = \pm \pi \). Thus for \( d \) discrete dimensions, one has \( 2^d \) species.

This unexpected and troublesome aspect of lattice Dirac action is not due to the particulars of the above construction which leads to \( \sum_{\mu} \gamma^\mu \sin p_\mu \).
as the inverse propagator in momentum space. One can in general have \( \Sigma_\mu \gamma^\mu f_\mu \) as the inverse propagator where \( f_\mu \) is a trigonometric polynomial and the doubling problem would still exist. The problem is more general and is closely related to continuous chiral symmetry. Nielson and Ninomiya[6] have studied this in detail and have presented several no-go results. The inverse propagator on the lattice for fermions is a chiral invariant. One cannot project out on the lattice a left-handed or a right-handed fermion. This means that chiral gauge theories such as the Weinberg-Salam-Glashow model cannot be treated on the lattice. This is very unfortunate and at present there does not exist any solution with a local action. Only vectorlike gauge theories can be treated and chapter 4 presents an example of that.

Under the general assumptions of the Nielson-Ninomiya theorem, chiral symmetry of the continuum action has to be broken on the lattice. There are two consistent ways of putting fermions on the lattice. In the Wilson fermion case, where a bosonlike kinetic energy term without a \( \gamma \) matrix is artificially added to lift the fermion energy at the edges of the Brillouin zone so as to get rid of the doubling, the vector part of the flavor group is preserved, but the axial part is completely broken and is realized in the continuum limit by fine-tuning the parameters. In the Euclidean staggered (Kogut and Susskind) fermion prescription, there is a continuous remnant of chiral (axial) symmetry, in addition to discrete \( \gamma^5 \) symmetries. The continuous remnant allows lattice studies of the Goldstone mechanism possible if that symmetry is spontaneously broken on the lattice. In chapter 4 such a situation has been investigated. In the continuum limit the full chiral group should be dynamically restored.
In the staggered fermion formalism in \( d \) (even) Euclidean dimensions, one is left with one pair \((\chi(n), \overline{\chi}(n))\) of Grassmann variables per site per gauge index (e.g. color) and this reduces the \( 2^d \)-fold degeneracy of the naive discretization to \( C = 2^{d/2} \) species in the free spectrum, to be interpreted as flavors.

In the following, the staggered fermion prescription is illustrated in 2 Euclidean dimensions. Generalization to 4 dimensions is obvious. One spin-diagonalizes the naive action [Eq.(2.12)] using the following relations:

\[
\psi(n) = \gamma_0^{[n_0]} \gamma_1^{[n_1]} \eta,
\]

\[
\text{and, } \overline{\psi} = \overline{\eta} \gamma_1^{[n_1]} \gamma_0^{[n_0]},
\]

\[
\text{where, } \eta = \chi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ etc.}.
\]

Substituting these in the naive action, one obtains,

\[
S = \sum_{n, \mu=1,2} (-1)^{n_0} \overline{\chi}(n)[\chi(n + \mu) - \chi(n - \mu)]
\]

where, after a little \( \gamma \) algebra,

\[
(-1)^{n_0} = +1
\]

\[
(-1)^{n_1} = (-1)^{n_0}.
\]

One then sums over the four corners of a fundamental square (sixteen corners of the fundamental hypercube in 4 dimensions) to define 'local quarks' \( q^{a \beta}(m) \) as follows[7]:

\[
q^{a \beta}(m) = \frac{1}{C^{3/2}} \sum_y \Gamma_y^{a \beta} \chi(2m + y)
\]

\[
\Gamma_y^{a \beta} = \gamma_0^{[n_0]} \gamma_1^{[n_1]}
\]
where the sites are labeled by, $n_\mu = 2m_\mu + y_\mu$ with $y_\mu = 0$ or 1 and $m_\mu$ integer. The indices $\alpha$ and $f$ are interpreted as Dirac and flavor indices.

The above quark definition in 2 dimensions can be interpreted as a $2^{d/2} = 2^{2/2} = 2$ flavor fermion as follows:

$$q_{\alpha f}(n) \equiv \begin{pmatrix} d_{\alpha}(n) \\ u_{\alpha}(n) \end{pmatrix}, \quad (2.26)$$

where $u$ and $d$ are the two flavors.

This so far has been a free theory. The states of the interacting theory are classified according to the irreducible representations of the lattice symmetry group $[8]$, and their interpretation according to the continuum group for $C$ flavors (broken by discretization[7]) should be checked carefully. In the continuum limit, for the flavor interpretation of the $C$-fold degeneracy of the staggered fermions to make sense, the flavor symmetry should be restored. This point has been looked at in some detail in the later discussions of chapter 3.

### 2.5 Asymptotically Free Theories and Scaling Laws

Non-Abelian gauge theories such as SU(N) gauge theories are asymptotically free. Two versions of such a theory, with space-time cutoffs $a_0$ and $a$, and couplings $g_0^2$ and $g^2$, are related as

$$g^2(a) = \frac{g_0^2}{1 + \frac{Kg_0^2}{2\pi} \ln(a_0/a)} \quad (2.27)$$

where $K$ is a positive group-theoretic constant. In the above,

$$g^2(a) \to 0 \text{ as } a \to 0. \quad (2.28)$$
As explained above, the continuum limit of the lattice version of an asymptotically free field theory is found in the \( g \to 0 \) limit. The vicinity of \( g = 0 \) (infrared unstable fixed point) where perturbative analysis is adequate is known as the scaling region of the theory. In the following, the simplest of such perturbative results showing the dependence of the mass-gap \( (m) \) of the theory on the bare coupling constant is quoted\[2\]:

\[
m \propto (\beta_0 g^2)^{-\frac{\beta_1}{2\beta_0}} \exp(-\frac{1}{2\beta_0 g^2}) \tag{2.29}
\]

This is an example of a scaling law. \( \beta_0 \) and \( \beta_1 \) are the lowest two orders of the Callan-Symanzik function. It is interesting to note here that the dependence of the mass-gap on \( g \) is non-analytic implying that the mass generation of the theory is a nonperturbative effect. Verification of the scaling laws in the lattice spectrum calculation of an asymptotically free theory is extremely important, because this is the window to make connection with continuum world. In the next chapter an asymptotically free field theory, namely, the Gross-Neveu model is discussed and there the appropriate scaling law is verified.

### 2.6 Numerical Techniques

For pure gauge or bosonic theories without fermions, Monte Carlo simulation for evaluation of expectation values have been in use for some time now. The generation of an ensemble of configurations of the field variables are performed using one of several algorithms, namely, Metropolis algorithm, heat bath algorithm etc.. For details on these, the reader is referred to reference\[2\].
Fermion simulations are extremely difficult on the lattice. Let us start with a generic action

\[ S = \sum_{m,n} \bar{\psi}(m)[\gamma^{\mu}D_{\mu}(U) + m]_{mn}\psi(n) + S_0(U) \]  

(2.30)

where, the pure gauge or bosonic part is given by \( S_0(U) \) and the fermion part includes the covariant derivative in the usual notation. It is absolutely impractical trying to do computer simulation on \( S \) directly because of the anticommuting nature of the fermionic fields. However, since \( \psi \) appears quadratically in the above action, it can be easily integrated out, leaving,

\[ \int \prod [d\psi][d\bar{\psi}]e^{-S} = \text{det}[\gamma^{\mu}D_{\mu}(U) + m]e^{-S_0}, \]  

(2.31)

which is just a bosonic problem. Unfortunately the above expression is non-local and difficult to deal with. At this point, approximations are usually made. For quite some time, a very popular approximation has been to put the fermion determinant equal to unity, i.e.,

\[ \text{det}[\gamma^{\mu}D_{\mu}(U) + m] = 1. \]  

(2.32)

This is known as the quenched approximation. It amounts to neglecting the internal fermion loops. In chapter 4 \( \chi SB \) is considered of the SU(2) gauge-Higgs-fermion model under this approximation. It apparently seems that a quenched fermion treatment of \( \chi SB \) should not make any sense because in Gross-Neveu or Jona-Lasinio models (models of \( \chi SB \)) these fermion loops play an important role. But it turns out that for \( \chi SB \) analysis, the quenched approximation gives qualitatively valid results. Some arguments in support of this are found in reference[9].
For quantitatively important results one has to have the full fermionic feedback into the theory. In the algorithms available to date, in general, effect of the fermion determinant is created by a classical system. In chapter 2, such an algorithm, namely, the molecular dynamics algorithm is discussed in some detail. Langevin equation and Hybrid methods[10] are two examples of other such algorithms.

For computing the fermion correlation function \(< \overline{\chi}(n)\chi(0) >\) one has to invert the fermion matrix \(A\),

\[
A_{mn} = [\gamma^\mu D_\mu + m]_{mn}
\]

as will be shown in Eq.(3.25). There are several variants of a method, known as the conjugate gradient method, available depending on whether the matrix \(A\) is symmetric non-singular or general non-singular. These methods appear in reference[11].

It has been pointed out in chapter 1 that nonvanishing of \(< \overline{\psi}\psi >\) is a signal for chiral symmetry breaking. For lattice studies of \(\chi SB\) one has to use the staggered fermions as indicated in section (2.4) of this chapter. In this case the reduced variables \(\chi\) and \(\overline{\chi}\) are used as defined by Eqs.(2.18) through(2.20). On the lattice what one can calculate is \(< \overline{\chi}\chi >\) and it can be rather easily shown to be equal to \(< \overline{\psi}\psi > [9].\)
CHAPTER III

Studies of the Lattice Gross-Neveu Model

3.1 Introduction

In this chapter the Gross-Neveu (GN) model[12] is studied on the lattice in 1 space and 1 time dimensions. The continuum GN model is known to have several attractive features. It is renormalizable (in the perturbative sense) in 1+1 dimensions. In the large N perturbative analysis where N is the number of fermion flavor, it is seen to have asymptotic freedom, chiral symmetry breaking ($\chi SB$) and dimensional transmutation. The Lagrangian (see next section) has a discrete chiral symmetry under the transformation $\psi \rightarrow \gamma^5 \psi$ which prohibits the presence of a mass term in the Lagrangian. However, in the large N limit it has been shown by Gross and Neveu that the chiral symmetry is spontaneously broken and a fermion mass is generated dynamically. Since the symmetry that is spontaneously broken is only discrete and not continuous the Goldstone theorem does not apply here and there is no Goldstone boson in the spectrum. A semiclassical analysis[13] of the GN model shows that the spectrum includes massive fermions and massive bosonic bound states.
The GN model is not a gauge theory and a 1+1 dimensional model. It thus needs less computational skill and effort to be studied on the lattice and given the nice features it has it is a wonderful laboratory for studying dynamical fermion mass generation owing to spontaneous $\chi SB$.

Because of its simplicity one is also tempted to test several other things as described in the following on the GN model. Section (2.6) has already discussed the difficulties of dealing with dynamical fermions (internal fermion loops included) on the lattice. There have been proposed several algorithms for treating this problem and it is a perfect place to put one of these to test. In this chapter the molecular dynamics approach\[14,15,16\] to dynamical fermions on the lattice is used and compared with a previous Monte Carlo calculation\[17\] on the GN model. Excellent agreement is found between these two analyses. The scaling law (see section 2.5) for the dynamically generated fermion mass is verified. Also apart from the fundamental fermion mass more of the spectrum is calculated and is in very good agreement with the semiclassical analysis\[13\].

Throughout in the present analysis staggered fermions (see section (2.4)) are used. As indicated in section (2.4) the degeneracy in the continuum limit in this prescription of the lattice fermions has to be interpreted as the flavor degeneracy. In the context of an interacting theory as in the case of the present model the flavor degeneracy can be tested by obtaining masses of different bound states made of fermions of different flavors. This degeneracy is not to be confused with the explicit lattice degeneracy of the $N_f$ species or flavors of fermions as shown in the lattice action (3.3). In the continuum limit each of these $N_f$ flavors will have $2^{d/2} = 2^{2/2} = 2$ degenerate
species so that in the continuum there will be \( N = 2N_f \) flavors. In this study the fermion- antifermion ("meson") states are written down explicitly using continuum physics as a guide. The states include in addition to the usual single-site mesons also the interesting point-split ones which involve fundamental fermionic variables at different adjacent points on the lattice multiplied by characteristic phase factors. In staggered fermion prescription, crudely speaking, each component of a Dirac spinor is placed on a different lattice site and as described in section (2.4) one has to collect the degrees of freedom from all the corners of a hypercube (in this 1+1 dimensional case a square) so that the collection can be defined as a "quark" or a fundamental fermion in the continuum limit. Keeping this in mind it is not hard to see why there are some meson or fermion-antifermion wavefunctions which will have the fermion at one lattice point and the antifermion at an adjacent lattice point separated by a lattice spacing. In the semiclassical analysis it is discussed that the GN model has a \( O(2N) \) global symmetry (\( N \) is the number of continuum flavors) and in the spectrum this symmetry should be manifest. Unless one considers the non-single-site mesons one cannot account for this symmetry in the spectrum. Here for each lattice flavor, several different meson masses are calculated in the scaling region. Restoration of flavor symmetry in the continuum would demand that the masses be the same. In so far as the continuum limit is concerned, restoration of flavor symmetry should be considered as important as the asymptotic scaling or the Lorentz symmetry restoration. Two recent publications[18,19] have looked into the issue of flavor symmetry restoration with positive results. In this work, however, one is not absolutely sure about whether the flavor symmetry is restored in
the continuum limit because in this special case of the 1+1 dimensional GN model the meson states mix quite extensively among each other[20] and one really cannot say if one is calculating masses of different states. The meson masses that have been calculated in this work are degenerate within error bounds.

3.2 The Lattice Gross-Neveu Model

The Lagrangian density of the continuum GN model in Minkowski space can be written in the following way:

\[ L_0 = \sum_{f=1}^{N} \bar{\psi}_f (i\gamma^\mu \partial_\mu) \psi_f - \frac{1}{2g^2} \sigma^2 - \sigma \sum_{f=1}^{N} \bar{\psi}_f \psi_f, \]  

(3.1)

where \( \psi_f \) is one of the N components of a massless fermion field, and,

\[ \sigma = -g^2 \sum_f \bar{\psi}_f \psi_f. \]  

(3.2)

The \( \sigma \) in the above Lagrangian is equivalent to the \( g\sigma \) of Gross and Neveu.

The Euclidean action for the lattice GN model can be written down as

\[ S = \frac{1}{2g^2} \sum_n \{\sigma(n)\}^2 + \sum_{n} \sum_{f=1}^{N_f} \bar{x}_f(n) x_f(n) \frac{1}{4} \sum_{\xi=1}^{4} \sigma(n - \xi) + \frac{1}{2} \sum_{n} \sum_{f} \bar{x}_f(n) ([x_f(n + \xi_1) - x_f(n - \xi_1)] \\
-(-1)^n [x_f(n + \xi_0) - x_f(n - \xi_0)]}, \]  

(3.3)

where,

\[ \sum_{\xi=1}^{4} \equiv \sum_{\delta,\xi_0,\xi_1,\xi_0+\xi_1}. \]  

(3.4)
\( \epsilon_0 \) and \( \epsilon_1 \) being respectively the unit vectors in the timelike and space directions, \( n \) represents a lattice site \((n_0, n_1)\) and the relationship between \( \psi_f(n) \) and \( \chi_f(n) \) is given by,

\[
\psi_f(n) = \gamma^{n_1+1}_1 \gamma_0^{n_0} \eta_f(n), \\
\bar{\psi}_f(n) = \bar{\eta}_f(n) \gamma_0^{n_0} \gamma^{n_1+1}_1,
\]

and, \( \eta_f(n) = \chi_f(n) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) etc..

One might notice that the Eqns. (3.5) and (3.6) are slightly different than the corresponding equations of section (2.4). A slightly different convention is used here for the staggered fermions so that the action (3.3) has the exact same form as in reference [17].

In the above the lattice spacing is taken to be unity. \( N_f \) represents the number of lattice flavors so that the number of continuum flavors would be \( 2N_f \).

\( \sigma \) in terms of the \( \chi_f \)'s is given by,

\[
\sigma(n) = -g^2 \sum_f \frac{1}{4} \sum_{\xi=1}^4 \bar{\chi}_f(n + \xi) \chi_f(n + \xi). \tag{3.7}
\]

The action (3.3) has a form for which according to reference [17] there is no chiral phase transition from a chirally symmetric phase to a chirally broken phase. In section (3.4) it is seen to be true, i.e., the system always stays in a chirally broken phase.
3.3 The Molecular Dynamics Method

The molecular dynamics method is also known as the microcanonical method. There are several reviews on the subject in the recent lattice gauge theory literature[10,15] so that instead of presenting a general formalism of the method, only the procedure as applied to the GN model is given in the following.

To start with, Eqn.(3.3) is rewritten as

\[ S = \sum_{n,m} \sum_f \bar{\chi}_f(n) A_{nm} \chi_f(m) + \frac{1}{g^2} S_B(\sigma) \]  

(3.8)

where,

\[ S_B = \frac{1}{2} \sum_n \{\sigma(n)\}^2, \text{ and}, \]

(3.9)

\[ A_{nm}(\sigma) = \frac{1}{2} \left\{ [\delta_{m,n+\xi_1} - \delta_{m,n-\xi_1}] - (-1)^m [\delta_{m,n+\xi_0} - \delta_{m,n-\xi_0}] \right\} \]

\[ + \delta_{m,n} \left[ \frac{1}{4} \sum_{\xi=1}^{4} \sigma(n - \xi) \right]. \]  

(3.10)

\( A \) is a real matrix in this case.

The expectation value of \( \sigma \) on the lattice is given by,

\[ < \sigma > = \frac{1}{Z} \int D[\sigma] (det A[\sigma])^N f \cdot \sigma \cdot \exp\left\{ -\frac{1}{g^2} S_B(\sigma) \right\} \]  

(3.11)

where,

\[ Z = \int D[\sigma] (det A[\sigma])^N f \exp\left\{ -\frac{1}{g^2} S_B(\sigma) \right\}, \]  

(3.12)

\[ \int D[\sigma] = \prod_{n=1}^{N_s} \int d\sigma(n), \]  

(3.13)
$N_{\sigma}$ being the number of $\sigma(n)$ fields.

In order to calculate the average, given by Eqn.(3.11), a classical system is defined in 2+1 dimensions with the canonical variables $[\sigma(n), p(n)]$ and $[Q_f(n), P_f(n)]$. The $Q_f$'s take the place of the fermions and are not anti-commuting.

The Hamiltonian of this system is written as

$$H = \frac{1}{2g^2} \sum_n \{\sigma(n)\}^2 + \frac{1}{2} \sum_n \{p(n)\}^2 + \omega^2 \sum_n \sum_f \{Q_f(n)\}^2$$

$$+ \frac{1}{4} \sum_n \sum_f P_f(n) \{(A^\dagger A)^{-1}\}_{nm} P_f(m).$$  \hspace{1cm} (3.14)

From Eq.(3.14), one can write,

$$\dot{\sigma}(n) \equiv \frac{d}{d\tau} \sigma(n) = p(n), \text{ and},$$  \hspace{1cm} (3.15)

$$\dot{Q}_f(n) \equiv \frac{d}{d\tau} Q_f(n) = \frac{1}{2} \sum_m \{(A^\dagger A)^{-1}\}_{nm} P_f(m),$$  \hspace{1cm} (3.16)

where, $\tau$ is the new dimension introduced.

The Lagrangian of the classical system is given by,

$$L = \frac{1}{2} \sum_n \{\dot{\sigma}(n)\}^2 - \frac{1}{2g^2} \sum_n \{\dot{\sigma}(n)\}^2$$

$$+ \sum_n \sum_f \dot{Q}_f(n) (A^\dagger A)_{nm} \dot{Q}_f(m) - \omega^2 \sum_n \sum_f \{Q_f(n)\}^2.$$  \hspace{1cm} (3.17)

The canonical equations of motion as obtained from Eq.(3.17) are given by,

$$\frac{d}{d\tau} \sum_m (A^\dagger A)_{nm} \dot{Q}_f(m)] = -\omega^2 Q_f(n),$$  \hspace{1cm} (3.18)

and,

$$\dot{\sigma}(n) = -\frac{1}{g^2} \sigma(n) + \frac{1}{2} \sum_f \sum_{\xi=1}^4 \sum_{\xi=1}^4 B_f(n + \xi) \dot{Q}_f(n + \xi),$$  \hspace{1cm} (3.19)

where,
\[ B_f(n) = \sum_m^\infty A_{nm} \dot{Q}_f(m) \]
\[ = \left[ \frac{1}{4} \sum_{\xi=1}^4 \sigma(n-\xi) \right] \dot{Q}_f(n) \]
\[ + \frac{1}{2} \{ [\dot{Q}_f(n+\hat{\epsilon}_1) - \dot{Q}_f(n-\hat{\epsilon}_1)] \]
\[ - (-1)^n [Q_f(n+\hat{\epsilon}_0) - Q_f(n-\hat{\epsilon}_0)] \}. \] (3.20)

Now, if the classical system is assumed to be "ergodic" so that,
\[ <\sigma> = \lim_{T \to \infty} \frac{1}{T} \int_0^T d\tau \sigma(\tau), \] (3.21)

\[ <\sigma> \] can be calculated knowing \( \sigma(\tau) \) from the canonical equations of motion (3.18) and (3.19) with given initial conditions.

The initial conditions for the Eqs. (3.18) and (3.19) are the values of \( \sigma(n), \dot{\sigma}(n), Q_f(n) \) and \( \dot{Q}_f(n) \) at \( \tau=0 \). These uniquely determine the energy of the system and the inverse temperature \( \beta \).

\( \beta \) can be found out using the equipartition theorem:
\[ <\text{Kinetic Energy}> = \frac{N_{d.f.}}{2\beta} \] (3.22)

The left hand side of Eq.(3.22) can be obtained from the evolution of the classical system along the \( \tau \) dimension from the initial configuration. \( N_{d.f.} \) is the number of degrees of freedom.

\( \beta \) can also be found separately from the quadratic potential energy term of the \( Q_f(n) \) sector.

The expectation value of \( \sigma \) discussed above corresponds to the coupling constant
\[ g_{\text{eff}}^2 = \frac{g^2}{\beta} \] (3.23)

of the original theory.
3.4 \( <\sigma> \) Computation and Results

Most of the computation is done on \( 8 \times 8 \) and \( 8 \times 16 \) lattices with periodic boundary conditions and six lattice flavors (i.e., 12 continuum flavors). There is no particular reason for choosing the number of lattice flavors to be six apart from the fact that one is interested in comparing these data with the Monte Carlo data[17].

As initial conditions the velocities \( \dot{\sigma}(n) \) and \( \dot{Q}_f(n) \) are put equal to zero and \( \sigma(n) \) and \( Q_f(n) \) are chosen to be random numbers between -1 and +1.

The canonical equations are discretized according to the Verlet scheme in steps of \( \Delta \tau \). Several values of the step \( \Delta \tau \) are used for the \( 8 \times 8 \) lattices, from \( \Delta \tau=0.01 \) to 0.05 for different \( g^2_{eff} \). For the \( 8 \times 16 \) lattice a single value of \( \Delta \tau \), namely, \( \Delta \tau=0.02 \) is used for all values of \( g^2_{eff} \).

Different values of \( g^2_{eff} \) are obtained by changing the values of input \( g^2 \) and \( \omega^2 \). The conjugate gradient method is used for solving the discretized version of Eqn.(3.18). Unlike similar calculations in gauge theories, here the conjugate gradient method is always seen to converge fairly quickly with the bare fermion mass strictly equal to zero.

The systems are evolved from the initial conditions through microcanonical iterations until they reached equilibrium. Energy conservation is used as a check. \( g^2_{eff} \) is calculated separately from the kinetic energy term of the \( Q_f \) sector, the potential energy term of the \( Q_f \) sector, and the kinetic energy term of the \( \sigma \) sector. Matching of \( g^2_{eff} \) calculated from all the these different sectors is used as the equilibration criterion. It is observed that the \( Q_f \) sector gets equilibrated within itself fairly quickly, but the \( \sigma \) sector takes longer time.
to equilibrate itself with the $Q_f$ sector.

The typical numbers for the microcanonical iterations after which the systems reach equilibrium are $\sim 10,000$ and $\sim 25,000$ respectively for the $8 \times 8$ and the $8 \times 16$ lattices. Getting the system to equilibrium in $12 \times 12$ lattices is more difficult and it will not be pursued in this discussion.

The $<\sigma>$ versus $g_{\text{eff}}$ plot (Fig.3.1) for the $8 \times 8$ lattice is smooth and it is concluded that there is no phase transition as is to be expected with the particular choice of the action. The system always stays in the chirally broken phase.

In the scaling region, the continuum result[21],

$$\sigma = \sigma_0 \exp[-\pi/g^2(2N_f - 1)], \quad (3.24)$$

for finite $N_f$ is regained well within the error bounds with the expected slope $-\pi/(2N_f - 1) = -0.286$ (with $N_f=6$) of the $\ln \sigma$ versus $1/g_{\text{eff}}^2$ plot (Fig.3.2). The results agree very well with those of the Monte Carlo analysis.

A look at Eqn.(3.7) suggests that one can also calculate $<\sigma>$ using a very different method, namely, one which calculates the correlation function $<\bar{\chi}_f(n)\chi_f(0)> = P_f(0||n)$ by inverting the matrix $A$ in the equation,

$$\sum_n A_{nm} P_f(0||n) = \delta_{m,0}, \quad (3.25)$$

and, $<\sigma>$, now called $<\sigma>'$, is given by,

$$<\sigma>' = -g^2 \sum_f P_f(0||0). \quad (3.26)$$

The inversion of the matrix $A$ is done using a slightly different version of the conjugate gradient method. $<\sigma>'$, calculated this way, agrees very well
Figure 3.1: $<\sigma>$ vs. $g_{eff}$ plot on $8^2$ lattices
Figure 3.2: Semilogarithmic Plot of $\langle \sigma \rangle$, $\langle \sigma \rangle'$, and $M_f$ in lattice units vs. $1/g_{\text{eff}}^2$ on $8 \times 16$ lattices. The dashed straight line has the scaling slope of $-0.286$ (with $N_f = 6$).
with \(<\sigma>\) calculated before (Fig.3.2). This certainly is a nontrivial test of the procedures used here.

3.5 Calculation of the dynamically Generated Fundamental Fermion Mass and Symmetries of the Fermion Correlation Function

The dynamically generated mass of the fundamental fermion can be computed from the fall-off of the zero spatial momentum fermion correlation function using the following equation[22,23]:

\[
\sum_{n_{1}} P_f(n_{00}, n_{10}|n_0, n_1) = A[\exp(-Et) + (-1)^t\exp\{-E(N_t - t)\}], \tag{3.27}
\]

where, \((n_{00}, n_{10})\) and \((n_0, n_1)\) are two points on the lattice, and,

- \(t \equiv |n_0 - n_{00}|\),
- \(N_t \equiv \text{number of lattice points in the timelike } (n_0) \text{ direction}\)
- \(M_f \equiv \text{dynamical mass of the fundamental fermion}\)
  
  = \(\text{Sinh}\ E\)

and, \(A\) is a constant \((A = 1/\text{Cosh}\ E\) for free fermions).

Masses are calculated using the CERN Minuit program. Within the error-bounds \(M_f\) was found to be equal in value to \(<\sigma>\) in the scaling region (Fig.3.2). This confirms the continuum result of Gross and Neveu for large \(N\) \((2N_f=12,\ \text{in this case})\) and also the semiclassical analysis of Dashen, Hasslacher and Neveu.
The action $S$, given by Eq.(3.3) has a symmetry under the following transformations of $\chi_f$ and $\bar{\chi}_f$:

\[ \chi_f(n_0, n_1) \rightarrow (-1)^{n_0} \chi_f(-n_0, n_1), \]
\[ \bar{\chi}_f(n_0, n_1) \rightarrow (-1)^{n_0} \bar{\chi}_f(-n_0, n_1), \text{ and/or} \]
\[ \chi_f(n_0, n_1) \rightarrow (-1)^{n_1} \chi_f(n_0, -n_1), \]
\[ \bar{\chi}_f(n_0, n_1) \rightarrow (-1)^{n_1} \bar{\chi}_f(n_0, -n_1). \]  

(3.29)

If one writes the fermion correlation function as

\[ P(n_0', n_1' || n_0'', n_1'') \equiv \langle \bar{\chi}(n_0'', n_1'') \chi(n_0', n_1') \rangle, \]  

(3.30)

one obtains the following equation:

\[ P(n_0', n_1' || n_0'', n_1'') = (-1)^{n_0+n_1} P(n_0'', n_1' || n_0', n_1') \]  

(3.31)

with $n_0 = n_0'' - n_0'$ and $n_1 = n_1'' - n_1'$, and,

\[ P(0, n_1) || n_0, 0) = (-1)^{n_1} P(0, 0 || n_0, n_1) \text{ etc.}, \]  

(3.32)

from which follows the equation:

\[ p(-n_0) = (-1)^{n_0} p(n_0), \]  

(3.33)

where,

\[ p(n_0) \equiv \sum_{n_1} P(0, 0 || n_0, n_1), \]  

(3.34)

\[ p(-n_0) \equiv \sum_{n_1} P(n_0, 0 || 0, n_1). \]  

(3.35)

In Eqs.(3.30) to (3.35) the flavor index has been dropped for convenience of writing.

The symmetry of Eq.(3.33) is seen to be always true and served as a test of the computation.
3.6 The Fermion-antifermion Bound States

As mentioned in section (3.1), the GN model has a global O(2N) symmetry. This symmetry is not obvious and is shown to be true in the semiclassical analysis of Dashen et al[12]. This symmetry reflects in the spectrum of the model and the common mass of the n-th supermultiplet as given by reference[12] is:

\[ M_n = \langle \sigma \rangle \frac{2N}{\pi} \sin \left( \frac{n \pi}{N} \right) \]

where, \( n = 1, 2, \ldots < N \), \( N \) being the number of (continuum) flavors.

The supermultiplets with \( n \) odd and \( n \) even are composed of fermions and bosons respectively and contain O(2N) representations corresponding to all completely antisymmetrical tensors of rank \( n_0 = 1, 3, 5, \ldots \leq n \) and \( n_0 = 0, 2, 4, \ldots \leq n \) respectively.

The mass \( M_1 \) of the fundamental fermion of the theory, corresponding to \( n = 1 \), has already been discussed in the previous section and from Eq.(3.36) is given by,

\[ M_1 \equiv M_f = 0.9971 \langle \sigma \rangle \]

\[ \approx \langle \sigma \rangle, \]  

(3.37)

with \( N = 2N_f = 12 \).

Fig.3.2 shows that the results are in close agreement with Eq.(3.37).

The case of \( n = 2 \) involves an O(2N) scalar (particle associated with the \( \sigma \) field) and two-body bound state tensors with fermion-fermion, fermion-antifermion, antifermion-antifermion quantum numbers.
The mass $M_2$, with $N=12$ is given by,

$$M_2 \equiv M_{f\bar{f}} = 1.9772 < \sigma >,$$  \hspace{1cm} (3.38)

so that the binding energy is given by,

$$2M_1 - M_2 = 0.0170 < \sigma >.$$  \hspace{1cm} (3.39)

In this section the fermion-antifermion bound states are looked at. The scheme is to write down all the obvious wavefunctions for such states at first, and then calculate the correlation functions, and finally from the fall-off of those correlation functions compute the masses.

Eq.(2.24) is taken as the "quark" (q) definition where $\Gamma$ is now defined a little differently:

$$\Gamma = \gamma_1^{m_1+1} \gamma_0^{n_0}$$  \hspace{1cm} (3.40)

consistent with Eqs.(3.5) and(3.6). The "meson" (fermion-antifermion bound states) wavefunctions are written down with the above "quark" definition and

$$\overline{q}(\Gamma_A \otimes \Gamma_B)q$$  \hspace{1cm} (3.41)

as a guidingline with Lorentz \& flavor quantum numbers indicated by the matrices $\Gamma_A$ and $\Gamma_B$.

Using the expressions of $q$ and $\overline{q}$ from Eq.(2.24) with $\Gamma$ given by Eq.(3.40), the algebra of expression (3.41) is carried out explicitly in a given representation of all the matrices. In the following, the result of such an exercise is exhibited, a list of all possible fermion-antifermion state ($f\bar{f}$), bound or unbound, derived from Expression (3.41) is given. One should note here that in using (3.41) as a guideline for writing down these states, the symmetries
of the continuum are used while one is still on the lattice. This gives rise to mixing of these states among each other. For convenience of notation, only one kind of lattice flavor is used in the following expressions. From now on till the end of this chapter, all flavor indices will be dropped.

**Wavefunction Type 1**

\[
\begin{align*}
a) & \quad \sum_n \bar{\chi}(n)\chi(n) \\
b) & \quad \sum_n (-1)^n \bar{\chi}(n)\chi(n) \\
c) & \quad \sum_n (-1)^n \bar{\chi}(n)\chi(n) \\
d) & \quad \sum_n (-1)^{n_0+n_1} \bar{\chi}(n)\chi(n)
\end{align*}
\]

(\(\sum_n\) means a sum over the four points of a fundamental square of the 1+1 dimensional lattice.)

**Wavefunction Type 2**

Let \( g = \bar{\chi}(n + \hat{e}_0)\chi(n) \) and \( h = \bar{\chi}(n)\chi(n + \hat{e}_0) \).

\[
\begin{align*}
a) & \quad \sum_{n_1=1,2} (g + h) \\
b) & \quad \sum_{n_1=1,2} i(-1)^{n_0}(g - h) \\
c) & \quad \sum_{n_1=1,2} (-1)^{n_1}(g + h) \\
d) & \quad \sum_{n_1=1,2} -i(-1)^{n_0+n_1}(g - h)
\end{align*}
\]

(\(\sum_{n_1=1,2}\) means a sum over the two sites in the spatial direction \(n_1\) belonging to the fundamental square of the lattice.)
Wavefunction Type 3

Let \( u = \bar{\chi}(n + \hat{e}_1)\chi(n) \) and \( v = \bar{\chi}(n)\chi(n + \hat{e}_1) \).

\[
\begin{align*}
a) & \quad \sum_{n_0=1,2} (u + v) \\
b) & \quad \sum_{n_0=1,2} (-1)^{n_0}(u + v) \\
c) & \quad \sum_{n_0=1,2} (-1)^{n_1}(u - v) \\
d) & \quad \sum_{n_0=1,2} (-1)^{n_0+n_1}(u - v)
\end{align*}
\]

(\( \sum_{n_0=1,2} \) means a sum over the two sites in the timelike direction \( (n_0) \) belonging to the fundamental square of the lattice.)

Wavefunction Type 4

Let

\[
\begin{align*}
l & = \bar{\chi}(n + \hat{e}_0 + \hat{e}_1)\chi(n), \\
x & = \bar{\chi}(n)\chi(n + \hat{e}_0 + \hat{e}_1), \\
y & = \bar{\chi}(n + \hat{e}_0)\chi(n + \hat{e}_1), \\
z & = \bar{\chi}(n + \hat{e}_1)\chi(n + \hat{e}_0).
\end{align*}
\]

\[
\begin{align*}
a) & \quad l + x + y + z \\
b) & \quad i(-1)^{n_0}(l - x + y - z) \\
c) & \quad -i(-1)^{n_1}(l - x - y + z) \\
d) & \quad (-1)^{n_0+n_1}(l + x - y - z)
\end{align*}
\]

At this stage, using the above wavefunctions, the zero spatial momentum fermion-antifermion state correlation function \( G^{(j)} \) for the particular state \( j \)
can be calculated between the lattice points \((0,0)\) and \((n_0,n_1)\). A few such calculations are shown here in the following. For example, the correlation function \(G^{(1c)}\) for the state \(1c\) can be calculated in the following way:

\[
G^{(1c)} = \sum_{n_1} (-1)^{n_1} \langle [\bar{\chi}(n_0,n_1)\chi(n_0,n_1)] [\chi(0,0)\bar{\chi}(0,0)] \rangle
\]

\[
= \sum_{n_1} (-1)^{n_1} [P_f(0,0\|n_0,-n_1)P_f(0,0\|n_0,n_1)]
- \langle \chi(0,0)\bar{\chi}(0,0) \rangle < \chi(0,0)\bar{\chi}(0,0) >. 
\]

In the above \(P_f\) is the fundamental fermion correlation function and the sums are from \(n_1 = 1\) to \(N_s\), the number of lattice points in the spatial direction.

Let \(\langle \bar{\chi}(n_0,n_1)\chi(n_0,n_1) \rangle \) to be independent of \((n_0,n_1)\) so that,

\[
G^{(1c)} = \sum_{n_1} (-1)^{n_1} [(-1)^{n_0+n_1} |P_f(0,0\|n_0,n_1)|^2 - (\langle \chi\bar{\chi} \rangle)^2],
\]

where, \(\langle \chi\bar{\chi} \rangle \equiv \langle \chi(0,0)\bar{\chi}(0,0) \rangle \) and use has been made of Eq.(3.31). On an even-even lattice such as the one being used here, with the staggered fermions, the second term of the right hand side vanishes, leaving,

\[
G^{(1c)} = (-1)^{n_0} \sum_{n_1} |P_f(0,0\|n_0,n_1)|^2
\]

In the following is also shown the calculation of \(G^{(3a)}\).

\[
G^{(3a)} = \sum_{n_1} \langle [\bar{\chi}(n_0,n_1+1)\chi(n_0,n_1) + \bar{\chi}(n_0,n_1)\chi(n_0,n_1+1)] 
\times [\chi(0,1)\bar{\chi}(0,0)] \rangle 
= \sum_{n_1} [P_f(0,0\|n_0,n_1)P_f(n_0,n_1+1\|0,1) 
+ P_f(0,0\|n_0,n_1+1)P_f(n_0,n_1\|0,1) 
- P_f(n_0,n_1+1\|n_0,n_1)P_f(0,0\|0,1) 
- P_f(n_0,n_1\|n_0,n_1+1)P_f(0,0\|0,1)].
\]

(3.44)
Using Eq.(3.31) once again one gets,

\[ G^{(3a)} = (-1)^{n_0} \sum_{n_1} (-1)^{n_1} [P_f(0,0||n_0,n_1)P_f(0,1||n_0,n_1 + 1) \]
\[ - P_f(0,0||n_0,n_1 + 1)P_f(0,1||n_0,n_1)]. \tag{3.45} \]

Few other results are listed below, though some of them are not used in the numerical calculations to follow.

\[ G^{(1a)} = (-1)^{n_0} \left\{ \sum_{n_1} [(-1)^{n_1}|P_f(0,0||n_0,n_1)|^2 \right\} \]
\[ - (-1)^{n_0} P_f(0,0||0,0)P_f(n_0,n_1||n_0,n_1) \}. \tag{3.46} \]

\[ G^{(1b)} = (-1)^{n_0} G^{(1a)}. \tag{3.47} \]

\[ G^{(1d)} = (-1)^{n_0} G^{(1c)}. \tag{3.48} \]

\[ G^{(3a)} = (-1)^{n_0} \sum_{n_1} [P_f(0,0||n_0,n_1)P_f(0,1||n_0,n_1 + 1) \]
\[ + P_f(0,0||n_0,n_1 + 1)P_f(0,1||n_0,n_1)]. \tag{3.49} \]

\[ G^{(3d)} = (-1)^{n_0} G^{(3c)}. \tag{3.50} \]

The equation used for finding the masses, for example, of the states 3a and 3b is given by[20,24],

\[ G^{(3b)} = \sum_k A_k^{(3b)} \{ \exp[-\omega_k^{(3b)}t] + \exp[-\omega_k^{(3b)}(N_t - t)] \} \]
\[ + (-1)^{n_0} \sum_k A_k^{(3a)} \{ \exp[-\omega_k^{(3a)}t] + \exp[-\omega_k^{(3a)}(N_t - 1)] \}. \tag{3.51} \]

with, \( t = |n_0 - 0| = |n_0| \), where the masses \( M_{f\bar{f}} \) are given by,

\[ M_{f\bar{f}} = [2(Cosh\omega - 1)]^{1/2}. \tag{3.52} \]
In the above equation a subscript and a superscript are dropped for the convenience of notation.

The left-hand side of Eq. (3.51) can be computed from Eqs. (3.48) and (3.45) using the fermion correlation function $P_f$'s computed by inverting the matrix $A$ in equations like (3.25).

The states $1c$ and $d$, $3a$ and $b$, and $3c$ and $d$ were tried in numerical simulations with a view to obtaining their masses in the scaling region using $8 \times 16$ lattices. Mass-fittings were again done by the CERN Minuit program. It was possible to find masses for the states $1d$, $3a$ and $3c$ (Fig. 3.3) Within error bounds the masses are all around $2M_f$ and hence these results are consistent with the semiclassical analysis. Eq. (3.39) shows that the binding energy of the $f\bar{f}$ states are very small and with the errors of these computations as seen in Fig. (3.3) it is not at all possible for one to decide whether any of the states looked at are bound or unbound.

The actual computation is done as follows. On an equilibrated system 8 sources $(0,0)$ are chosen at even-even lattice points. The correlation functions are calculated at the present configuration of the fields. Next the system is run for 50 molecular dynamics iterations and the correlation lengths are computed again. This process is repeated at least 18 times so that at least $8 \times 18 (=144)$ values for each correlation function is obtained. The average and the standard deviation of these serve as the input to the Minuit program. Whenever the Minuit program is run, it is used three times for $t=1$ to 8, $t=2$ to 8, $t=3$ to 8 in Eq. (3.51). All the masses are obtained by a four parameter Minuit fit. It is seen unfortunately in this case that a six parameter fit does not improve the results. The errors on the mass fits given by the Minuit is
Figure 3.3: Semilogarithmic plot of $2M$, $M_{ff}^{(1d)}$, $M_{ff}^{(3c)}$, $M_{ff}^{(3a)}$ in lattice units versus $1/g_{eff}^2$ on $8 \times 16$ lattice. The dashed straight line has the scaling slope of $-0.286$ (with $N_f=6$).
not relied on too much. To believe in the mass fits and also to put verifiable error bars on them, the whole procedure described above is repeated sufficient number of times until one is convinced of good enough statistics. Each of these is now separated by at least 700 molecular dynamics iterations so that each could be thought of as independent of the others. From all these data above at last the grand average and standard deviation of a mass is computed. The procedure is carried out altogether 5 times for 5 different values of the coupling constant in the scaling region. The $\geq 700$ iterations that separate the main configurations are used for calculating $<\sigma>$ and $1/g_{eff}^2$. These are also similarly averaged.

The fermion-antifermion wavefunctions are constructed using the Lorentz and the flavor symmetries of the continuum states. Although they are expected to be restored in the continuum limit, on a lattice these symmetries are not rigorously present and as a result the states constructed here mix among each other with specific selection rules[20]. It turns out that the mixing is especially bad in two dimensions and as a result it is not possible to determine which computed mass belongs to which state listed above. This is an unfortunate event because the question of flavor symmetry restoration cannot really be answered here without doubt, although all the $f\bar{f}$ masses are approximately equal to each other. One perhaps needs to calculate masses for more states, especially the states of type 2 and 4, and look for mass degeneracy. Calculating these masses is certainly a bit more difficult. One way of approaching this problem of mixing would be to construct states from lattice symmetries as has been suggested by Golterman[25].
3.7 Final Remarks

There are several systematic errors due to finite $\Delta \tau$ and finite lattice size affecting the results and these have not been studied here in any detail. Experience that people have had in other models suggests that sensitivity to $\Delta \tau$ should be less here in these molecular dynamics simulations than in corresponding Langevin calculations. Finite lattice effects could be appreciable in this 1+1 dimensional model. Periodic boundary conditions are used throughout rather than antiperiodic ones in order to minimize finite temperature effects and push the finite temperature symmetry restoring transition down to very weak coupling[17].

The molecular dynamics method appears to be a little slow at times even for this non-gauge 1+1 dimensional model. As the size of the lattice increases, it becomes increasingly difficult to get the system equilibrated within a reasonable time. Also from a formal point of view, it assumes ergodicity of the classical system and for gauge theories this could be an origin of trouble.
CHAPTER IV
Fermion Mass Generation in Lattice SU(2)
Gauge-Higgs-Fermion Model

4.1 Introduction

Several investigations\[26,27,28\] have been carried out in recent years on the lattice SU(2) gauge-Higgs model (without fermions). Experimentally the Higgs sector of the standard model is still unknown. The Higgs particle and its couplings are still not directly observed. On the other hand, theoretically there have been enough indications that the single-component scalar $\lambda\phi^4$ theory is trivial in 4 dimensions meaning that the renormalized theory is non-interacting, i.e., the renormalized coupling $\lambda_R$ goes to zero as the momentum cutoff $\Lambda$ is taken to infinity. All these force one to look at the Higgs mechanism more carefully, preferably by nonperturbative methods. The general motivation of the above investigations\[26,27,28\] was to see whether the coupling to the SU(2) gauge field makes the $\lambda\phi^4$ theory non-trivial so that the Higgs mechanism really does exist. This might be true only in some regions in the space of the coupling parameters, thus restricting or putting a bound on the Higgs mass, phenomenologically an extremely important number to know.
The SU(2) gauge-Higgs model in 4 dimensions is defined on the Euclidean lattice by the action,

\[ S = S_G + S_H, \text{ where} \]

\[ S_G = \beta_g \sum_P (1 - \frac{1}{2} \text{tr} U_P) \text{ with } \beta_g = 4/g^2 \]  

and,

\[ S_H = -\beta_H \sum_{n, \mu} [\phi^\dagger(n)U_\mu(n)\phi(n + \mu) + \text{h.c.}] \]  

In the above \( S_G \) is the usual Wilson action for SU(2) lattice gauge theory as explained in detail in section (2.3).

The Higgs field \( \phi \) is in the fundamental representation of SU(2) and is introduced as a column matrix:

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \]  

where \( \phi_1 \) and \( \phi_2 \) are complex.

As discussed before in section (2.3) the gauge fields \( U \) are SU(2) group elements and so here in fundamental representation they are the 2 x 2 matrices:

\[ U = \begin{pmatrix} U_1 & -U_2^* \\ U_2 & U_1^* \end{pmatrix}, \]  

where the stars indicate complex conjugates and \( |U_1|^2 + |U_2|^2 = 1 \).

One may note that in the above form of the action of the theory, a term with the quartic coupling \( \lambda \) does not appear. On the lattice \( \lambda \) enters the action through a term \( \lambda \sum_n [\phi^\dagger(n)\phi(n) - 1]^2 \) so that the radial degrees of freedom of \( \phi \) is controlled by the quartic coupling \( \lambda \). In the form of the
action presented here the Higgs fields are introduced in the fixed length limit $\phi^+(n)\phi(n) = 1$ which corresponds to having the limit $\lambda \to \infty$. Here the simplification of fixed length Higgs field is used mainly for technical reasons. Previous studies show that this case is not substantially different from the $\lambda < \infty$ case except for very small $\lambda$ which presumably does not have much to do with the standard model.

The phase diagram in the $\beta_g - \beta_H$ plane resulting from studying the above action is shown in Fig.(4.1) and has two apparent phases, namely, a QCD-like confinement phase and a Higgs phase (broken symmetry phase) separated by a transition line which does not go all the way across the plane. According to references[29,30] the endpoint of the line of transition occurs approximately at $(\beta_g, \beta_H) = (1.6, 0.65)$. The two phases are analytically connected. Also there is not much evidence of a second order, continuous transition. As a result one cannot take a continuum limit and discern anything about the continuum theory. On the other hand, some analytic results at weak coupling show that the gauge-Higgs system is also trivial like the $\lambda\phi^4$ theory. The pure gauge-Higgs studies at least in the above approach thus might have lost a little momentum. People are now looking at $O(4)$ scalar field theories in order to have some understanding of the Higgs parameters.

4.2 SU(2) Gauge-Higgs System with Fermions

The symmetry breaking part of the standard electroweak model consists of the Higgs sector. The fermions ride along without obvious influence on the symmetry breaking mechanism. One is still tempted to introduce fermions
Figure 4.1: Phase diagram of the pure SU(2) gauge-Higgs system (without fermions)
to the gauge-Higgs system of section (4.1) to look for the feedback of the fermionic sector onto the gauge-Higgs sector or vice versa because the situation is a little different here. Of course one cannot put fermions on the lattice in a left-right asymmetric way as in the standard model (see section (2.4)). A vectorlike model with fermions would still be very interesting in addressing questions such as how changes in the gauge sector are reflected in realizations of the global symmetries of the fermionic sector. It is certainly extremely interesting in this case because in a pure gauge-Higgs system (without fermions), the phases are analytically connected and the introduction of fermions might change that situation.

With the fermions (staggered formulation as in section (2.4)) now the action on the lattice is given by,

$$S = S_G + S_H + S_F,$$

(4.7)

where $S_G$ and $S_H$ are as in the previous section and $S_F$ is given by,

$$S_F = \frac{1}{2} \sum_n \bar{\chi}(n) \sum_\mu \xi_\mu(n)$$

$$\times [U_\mu(n)\chi(n + \mu) - U_\mu^\dagger(n - \mu)\chi(n - \mu)]$$

$$+ m \sum_n \bar{\chi}(n)\chi(n),$$

(4.8)

with, $\xi_\mu(n) = (-1)^{|n_1|+\cdots+|n_\mu-1|}$.  

(4.9)

In the above, Eq.(4.8) is the lattice version of the covariant derivative of the fermion field, Eqn.(4.9) is an obvious extension of Eqs.(2.22) and (2.23) and $\chi^i, \bar{\chi}^i$ are single component Grassmann variables (resulting from the staggered fermion prescription) with the SU(2) gauge index $i=1,2$ ($i=1,2,3$) for fundamental or $l = 1/2$ (adjoint or $l = 1$) representations of the fermions.
For fermions in the fundamental (adjoint) representation, the gauge fields U's are obviously in the fundamental (adjoint) representation. In $S_G$ and $S_H$, however, the gauge fields are always in the fundamental representation, given by Eq.(4.6). The adjoint gauge fields (let us call them V's) are obtained from the fundamental gauge fields (U's) through the well-known group-theoretic equation,

$$V^{\alpha\beta} = \frac{1}{2} \text{tr}[U^\dagger \sigma^\alpha U \sigma^\beta].$$

(4.10)

Here $\sigma$'s are the Pauli matrices and $\alpha$ and $\beta$ each range from 1 to 3 so that the matrix $V$ is a $3 \times 3$ matrix as expected.

One should also note that with the action of Eq.(4.7), gauge-invariant Yukawa couplings are not possible in this model without introducing extra gauge-singlet fermions.

As a final comment on the action given by Eq.(4.7) it is to be mentioned that the fermion mass ($m$) term is present in Eq.(4.8) solely because of a technical reason. One wants to look at the theory and its symmetries without the bare mass term but without it the conjugate gradient method which inverts the fermion determinant does not converge. In numerical simulations one has to take the $m \to 0$ limit of the computed quantities by evaluating them at different monotonically decreasing small $m$-values and then making a sensible extrapolation.

The fermions bring in a new global symmetry, namely, the chiral symmetry for the $m=0$ case. In the continuum limit the fermions as introduced in this section through the staggered fermion prescription would give rise to $2^{d/2} = 2^{4/2} = 4$ degenerate species to be interpreted as 4 families, each a SU(2) doublet (see section (2.4)) and thus the continuum limit would have
$SU(4) \otimes SU(4)$ chiral symmetry. On the lattice a continuous remnant of the above symmetry is present. It is now well established that in vector-like $SU(N)$ gauge theories such as QCD, if the gauge couplings are strong enough, chiral symmetries are spontaneously broken. One expects a Nambu-Goldstone realization of chiral symmetry breaking with fermions acquiring masses dynamically and massless pions in the spectrum. This can be seen on the lattice. On the lattice for the case of the staggered fermions there is one continuous axial flavor symmetry, and if the gauge field dynamics drives spontaneous symmetry breaking, a Goldstone pion should occur even in the cutoff lattice model. In the continuum limit the full chiral symmetry should appear dynamically and should also be spontaneously broken with additional Goldstone pions evolving as $a \rightarrow 0$.

The inclusion of fermions will affect the phase diagram by either modifying phase transitions already present in the pure gauge-Higgs theory or in some cases by introducing new phase boundaries. Different regions of parameter space can be characterized by whether the chiral symmetry brought in by the fermions is spontaneously broken or left unbroken. For instance, in several models with "complementarity"[31,32], one has been able to show that the Higgs and the confinement regions differ in their realizations of chiral symmetry. In the numerical work carried out to date, one is, however, restricted to the quenched approximation (internal fermion loops neglected, see section (2.6)), so one cannot talk about a true $\chi SB$ driven phase transition in these simulations. One was able to observe nonanalytic behavior in $<\bar{\psi}\psi>$ (modulo restrictions coming from finite size effects) and search for the presence or absence of Goldstone pions as signals for $\chi SB$. There
is already some evidence from a presently continuing calculation[33] that in the full theory with dynamical fermions (internal fermion loops included) the $\beta_a - \beta_H$ parameter space will divide into two distinct phases separated by a line of true (second order, continuous) phase transition. This is supported by analytic, mean field calculations at infinite gauge coupling that go beyond the quenched approximation[34].

Calculations have been carried out in reference[35] at $\beta_a = 4/g^2=0.5$ where in the absence of fermions, no nonanalyticities occur in the bosonic observables showing that $\chi SB$ indeed brings about a new phase boundary separating two phases, namely, a chirally symmetric phase and a chirally broken phase.

In this dissertation new results are presented for the SU(2) gauge-Higgs theory with fermions in the fundamental ($l = 1/2$) and adjoint ($l = 1$) representations, working still in the quenched approximation. But if similar calculations[36] in noncompact $U(1)$ gauge theory with fermions (without scalars) are any indications, the quenched fermion calculation gives qualitatively valid and important results which one can later verify and improve with a dynamical fermion calculation.

Here work is done at $\beta_a = 2.3$, at which there already is a phase transition in the pure gauge-Higgs system at $\beta_H = \beta_H^C \approx 0.39 - 0.40$ between the Higgs and the confinement regions (for future convenience let us call this the Higgs transition). $\beta_a = 2.3$ is also the gauge coupling at which several spectrum calculations for the Higgs and the W masses have been carried out[26,27,28,30].

In the present discussion the focus is on the connection between the
Higgs and the chiral transitions and on how this relation depends on the representation of the fermions. Studies of $\chi SB$ in gauge theories without scalars suggest that these two transitions are in principle independent. At present the dynamics responsible for the formation of chiral condensate of fermion-antifermion pairs leading to $\chi SB$ is not yet fully understood, but $\chi SB$ seems to take place whenever the gauge field mediated attractive force between fermions exceeds a critical value. One does not have to rely on confining forces or complicated topological excitations. Since one can increase the strength of the gauge interactions $C^I_2 G^2_{\text{eff}}(\beta_g, \beta_H)$ between fermions ($C^I_2$ is the quadratic Casimir for the gauge group in the representation $I$), one would expect to find $\chi SB$ even on the Higgs side of the Higgs transition ($\beta_H > \beta^C_H$) for sufficiently large $I$. Already for $l = 1$ it is found that the critical $\beta_H$ for $\chi SB$ is distinct from $\beta^C_H$. One has in this case in addition to a chirally symmetric Higgs phase, a chirally broken Higgs phase and a chirally broken confinement region. On the other hand for $l = 1/2$ fermions $\chi SB$ is lost as soon as $\beta_H$ exceeds $\beta^C_H$. Such a system has only two possibilities, a chirally broken confinement phase or a chirally symmetric Higgs phase. This coincidence of the Higgs and the chiral transitions in the present work comes from high statistics evaluation of $<\bar{\psi}\psi>$ close to $\beta^C_H$ and also from spectrum calculations for Goldstone pion and the fermion-scalar bound states. For $l = 1/2$ fermions, it proved important to go all the way to mass estimates, since at $\beta_g=2.3$, $<\bar{\psi}\psi>$ provides only a weak signal for $\chi SB$. This is discussed in the next section.
4.3 Fermions in the Fundamental Representation

All the calculations are done at $\beta_g=2.3$ while varying $\beta_H$. A standard Metropolis Monte Carlo algorithm is used to create equilibrated bosonic (gauge and Higgs) field configurations. $a^3 < \bar{\psi}\psi >$ and the propagators for the pion and for the gauge invariant physical fermion states (fermion-scalar bound states) are obtained by inverting the fermionic action $S_F$ using the conjugate gradient method.

Most of the $l = 1/2$ data was taken on $8^3 \times 16$ sized lattices and the bare fermion mass was set to $a m = 0.08, 0.06, 0.04, 0.02$. For each set of parameters $(\beta_g, \beta_H, m)$ 320 inversions of the fermionic action distributed over 40 bose field configurations was carried out. Results for $a^3 < \bar{\psi}\psi >$, extrapolated to $m \to 0$, are shown in Fig.4.2. At around $\beta_H \sim 0.4$, this quantity is seen to go from a small, almost constant value to a decreasing curve approaching zero. In Fig.4.2 for comparison is included ancient data from reference[9] on quenched SU(2) gauge theory without scalars. From the pure gauge theory data it is seen that at $\beta_g=2.3$, $a^3 < \bar{\psi}\psi >$ is small even in a chirally broken phase. Nevertheless, the qualitative change at $\beta_H \sim 0.4$ is interpreted as a weak signal for a $\chi SB$ phase transition. The region $\beta_H > 0.4$ should correspond to a chirally symmetric phase. Spectrum calculations are done to further support this claim.

For the physical fermions the following zero spatial momentum fermion-scalar bound state operators are used:
Figure 4.2: $a^3 \langle \overline{\psi} \psi \rangle_{m \to 0}$ versus $\beta_H$. The extreme left point at $\beta_H=0$ is from a previous computation in pure SU(2) gauge theory (without scalars).
The current model also has a global SU(2) symmetry (unbroken) which should reflect itself in the spectrum. The above two states form a degenerate doublet under this global SU(2). The easiest way to see this is to introduce the Higgs fields as unitary matrices $\Omega$ as follows:

$$
\Omega \equiv \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix}
$$

The theory is invariant under

$$
\Omega \rightarrow G_L(\text{local}) \Omega G_R^t(\text{global})
$$

($G_{L,R}$ are elements of SU(2))

The fermion-scalar bound state is given by,

$$
\Psi = \Omega^t \chi = \begin{pmatrix} \phi^t \chi \\ \phi^t \sigma_2 \chi \end{pmatrix}.
$$

In the above $i\sigma_2$ acts in the local SU(2) space. Since in the present model one can gauge away the Higgs fields (unitary gauge), it is perhaps better to call $\Psi$ the physical fermion rather than a true fermion-scalar bound state.

The same 320 inversions that produced the $a^3 < \bar{\psi}\psi >$ data of Fig.4.2 also enables one to evaluate the propagators $G_F(t)$ for the physical fermions. Antiperiodic boundary conditions in the timelike direction (t) are imposed
(common practice for fermionic spectrum, a consequence of the exclusion principle) and the propagators are fitted to [22],

\[ G_F(t) = A\{\exp[-E_f t] - (-1)^t \exp[-E_f (N_t - t)]\} \]  

(4.16)

The CERN Minuit program is used for the fits and the averaging is done over results using \( t_{\text{min}} = 3,4,5 \), where \( t_{\text{min}} = \) smallest t-value included in the fit. The data for the correlation functions is divided into 5 sets, each set obtained at an independent bosonic field configuration. Masses are calculated for each set. The errors in the figures correspond to the standard deviation among the 5 sets. Figs.4.3 and 4.4 show the physical fermion mass \( aM_F (= \text{Sinh}(aE_F)) \) as a function of the bare fermion mass \( am \) for \( \beta_H = 0.45 \) and 0.41 respectively. For \( \beta_H = 0.45 \), \( 12^3 \times 16 \) data are also shown. The finite size effects are seen to be nonnegligible. However, both \( 8^3 \times 16 \) and \( 12^3 \times 16 \) data show evidence for a very light fermion whose mass approaches a value consistent with zero as the bare \( m \) vanishes. Fig.4.5 plots \( aM_F \) extrapolated to \( m \rightarrow 0 \) as a function of \( \beta_H \). One observes the mass taking off at \( \beta_H \approx 0.4 \) indicating that a dynamical mass is generated for \( \beta_H < 0.4 \). Fig.3.5 should be compared with plots for \( aM_H \) and \( aM_W \), where similar dramatic changes are observed around \( \beta_H \approx 0.4 \)[26,27,28,30]. The information from Fig.4.5 together with that from Fig.4.2 makes one conclude that for \( l = 1/2 \) fermions (quenched) the Higgs and the \( \chi SB \) transitions coincide.

Next the pion mass is calculated at \( \beta_H = 0.35 \) as further proof of \( \chi SB \) for \( \beta_H < 0.4 \). For this pseudoscalar meson mass, the propagator \( G_M(t) \) of the following zero spatial momentum states is evaluated:

\[ \sum_n (-1)^{n_1 + \ldots + n_4} \chi(n) \chi(n). \]  

(4.17)
Figure 4.3: Physical fermion mass ($aM_F$) versus bare fermion mass ($aM$)

- $\beta_s = 2.3$
- $\beta_H = 0.45$
- $l = \frac{1}{2}$ fermions
- $8^3 \times 16$
- $12^3 \times 16$
Figure 4.4: Physical fermion mass ($aM_F$) versus bare fermion mass ($am$)
Figure 4.5: $(aM_F)_{m=0}$ versus $\beta_H$ at $\beta_g=2.3$ with $l = \frac{1}{2}$ fermions using a $8^3 \times 16$ lattice.
These propagators are fit to the following formula\textsuperscript{[20,24]}: 

\[
G_M(t) = \sum_i \{ \exp[-E_i t + \exp[-E_i(N_t - t)]] \} + (-1)^t 
\times \sum_j \{ \exp[-\tilde{E}_j t + \exp[-\tilde{E}_j(N_t - t)]] \}
\] (4.18)

A 4 parameter fit, with only one term each in the sums of the above equation, is good enough at $\beta_H = 0.35$ to give results for $E$ stable with respect to changes in $t_{\text{min}}$ (3 to 5). Fig.4.6 shows that $(a M_\pi)^2 \equiv 2[Cosh(a E) - 1]$ goes to zero linearly with the symmetry breaking parameter $a m$ as to be expected from a Goldstone boson.

The pseudoscalar meson channel is also investigated in the chirally symmetric region at $\beta_H = 0.45$. A 6 parameter fit involving two $E_1$'s and one $\tilde{E}_j$ is required to get stable results. In Fig.4.7 $a E_1$ and $a E_2$ versus $a m$ are plotted for two lattice sizes together with $2a E_F$ from Fig.4.3. The lowest state $E_1$ is found to be approximately at the two fermion threshold. Appreciable finite size effects in these simulations are evident from the better agreement of $E_1$ and $2 E_F$ on the larger $12^3 \times 16$ lattice. Interpretation of the next higher state in Fig.4.7 at $a E_2 \approx 1.2$ is difficult because of such effects. This state can be tentatively thought of as a 2 fermion state with the particles going off in opposite directions with lowest momentum possible $2\pi/L$ on a finite lattice\textsuperscript{[34]} so that,

\[
a E_2 = 2 \sinh^{-1}[(a M_F)^2 + (\sin \frac{2\pi}{L})^2]^{1/2}
\] (4.19)

with $L=\text{number of spatial lattice sites}$. The two dashed lines in Fig.4.7 corresponds to Eq.(4.19) for $L=8$ and $L=12$. The $8^3 \times 16$ data for $a E_2$ lie below (4.19), whereas the $12^3 \times 16$ data exceed (4.19). It is not, therefore, possi-
Figure 4.6: Pseudoscalar meson (pion) mass square \([aM_\pi]^2\) versus bare fermion mass (am).

\[8^3 \times 16 \text{ lattice} \]
\[\beta_x = 2.3, \beta_y = 0.35 \]
\[\ell = \frac{1}{2} \text{ fermions} \]
Figure 4.7: Lowest lying pion energies ($E_1, E_2$) and twice the physical fermion energy ($2E_F$) versus bare fermion mass ($aE_m$). The upper and lower dashed lines represent the right hand side of Eq.(4.19) for $L=8$ and $12$ respectively.
ble to conclude from Fig. 4.7 whether the data here is trying to approximate (4.19) or whether there is a true resonance somewhere around 1.0 \sim 1.3. If the state $\alpha E_2$ is really a resonance, the difference of the data with the two lattices has to be attributed to finite size effects. It is certainly annoying to find that the finite size effects could be appreciable in these quenched fermion simulations in gauge-Higgs systems, but one certainly can extract correct qualitative information even from the $8^3 \times 16$ lattice as evidenced by the very different behaviors of the pseudoscalar channel at $\beta_H = 0.35$ and $\beta_H = 0.45$ showing quite conclusively that in the former region there is $\chi SB$ and at larger value of $\beta_H$ there is realization of chiral symmetry.

4.4 Fermions in the Adjoint Representation

$a^3 < \bar{\psi} \psi >$ is calculated at $\beta_g = 2.3$ as a function of $\beta_H$ for $l = 1$ fermions. The quantity is calculated for bare fermion masses of $a m = 0.08, 0.06, 0.04$ and 0.02 using the conjugate gradient method for inversion of the fermionic action. These results are then extrapolated to the $m \to 0$ limit. At each $\beta_H$ and $m$, $a^3 < \bar{\psi} \psi >$ is computed 64 times over 8 different gauge and Higgs fields configurations. Fig. 4.8 shows $a^3 < \bar{\psi} \psi >$ (in the $m \to 0$ limit) as a function of $\beta_H$ at $\beta_g = 2.3$ for a $8^4$ lattice. Because of finite size effects it is difficult to find the exact point (value of $\beta_H$) of the chiral transition. One can say that at $\beta = 1.0$ and below there is $\chi SB$ and at $\beta_H = 3.0$ and above there is no $\chi SB$. This is supported also by the spectrum calculation discussed below. Keeping in mind that $\beta_H^c = 0.4$ one can clearly conclude that in this calculation with quenched fermions, the Higgs phase divides itself into two
Figure 4.8: $a^3 \langle \bar{\psi}\psi \rangle_{m=0}$ versus $\beta_H$

8$^d$ lattice
$l = 1$ fermions
$\beta_s = 2.3$
distinct regions, one with $\chi SB$, another with no $\chi SB$.

In the following are discussed spectrum calculations for the pion and the physical fermions (fermion-scalar bound states) at $\beta_p = 2.3$ and $\beta_H = 1.0$ and 3.0 with $l = 1$ fermions.

The pion state has been discussed before in section (4.3). The only difference here is that it is constructed now from $l=1$ fermions so that the $\chi$ variables now have their gauge index varying from 1 to 3 instead of 1 to 2 in the case of the $l=1/2$ fermions.

For the physical fermions states, the following zero spatial momentum operators are used:

$$\sum_n \phi^\dagger(n) \sigma^\alpha \frac{1}{2} \phi(n) \chi^\alpha(n), \quad (4.20)$$

$$\sum_n \phi(n)(i\sigma_3) \sigma^\alpha \frac{1}{2} \phi(n) \chi^\alpha(n) \quad (4.21)$$

$$\sum_n \phi^\dagger(n)(i\sigma_3) \sigma^\alpha \frac{1}{2} (-i\sigma_2) \phi(n) \chi^\alpha(n). \quad (4.22)$$

These states form a triplet under the global SU(2) and are degenerate. In Expressions (4.21) and (4.22) $\sigma_2$ acts in the local SU(2) space. Summation over the gauge index $\alpha (=1$ to 3) is implied in the above three expressions.

To obtain the expressions (4.20), (4.21) and (4.22), one first converts the fundamental representation unitary Higgs matrices $\Omega$ given by Eqn.(4.13) to the adjoint representation form using an equation like (4.10) and then uses the content of the Eqn.(4.15) to get the fermion-scalar (physical fermion) states.

Expressions for the propagators of the pion and the physical fermion states can be found easily in terms of the $\chi$-propagators and the scalars at a
particular field configuration. The same conjugate gradient method is used for obtaining the \( \chi \)-propagators as in the case of the previously computed \( a^3 < \bar{\chi} \chi > \) or \( a^3 < \bar{\psi} \psi > \). In these spectrum calculations \( 8^3 \times 16 \) lattices are used and periodic boundary conditions in all the three space directions and antiperiodic boundary conditions in the timelike directions for the \( \chi \) fields are employed. As indicated in the previous section the necessity of the antiperiodic boundary conditions for the fermionic variables is a consequence of the exclusion principle.

48 propagators are calculated for each value of \( \beta_H \) and \( m \). These are distributed over 6 bosonic configurations separated from each other by 100 Monte Carlo iterations. The above procedure is repeated 5 times so that each of these is separated from the others by a substantial number of Monte Carlo simulations and thus can be regarded independently of them. Each time for each \( m \) and \( \beta_H \) the data is fitted to the forms described by Eqns.(4.16) and (4.18) using the CERN Minuit program. The five estimates for the energies are used to calculate an average and standard deviation. In the following figures are shown the fits which are most stable with respect to changes in \( t_{\text{min}} \) (as described in the previous section).

Fig.4.9 shows the plot of \( (aM_F)^2 \) versus \( am \) for \( \beta_H = 1.0 \). The pion behaves like a Nambu-Goldstone boson in this region with \( (aM_F)^2 \) vanishing linearly with the symmetry breaking parameter \( am \). There is no evidence found for a higher \( E_i \)-state or an opposite parity \( E_j \)-state. The data for the physical fermion state in this region is too noisy to be useful.

Fig.4.10 shows the plot of \( aM_F \) versus \( am \) for \( \beta_H = 3.0 \). The physical fermions given by the expressions (4.20) and (4.21) are looked at. It is
Figure 4.9: Pion mass squared \((aM_\pi)^2\) versus bare fermion mass
Figure 4.10: The physical fermion masses ($aM_F$) versus bare fermion mass ($am$). The physical fermion states given by expressions (4.20) and (4.21) are used.
sufficient to look at two out of the three states. Apart from the expected
degeneracy between the two fermion states, Fig.4.10 shows that the masses
extrapolate approximately to zero in the $m \to 0$ limit. This region thus
indicates a phase with massless fermion triplets (with $m \to 0$) in the physical
spectrum.

The lowest lying states in the pion channel is also investigated at $\beta_H$
=3.0. Fig.4.11 shows the energies $(aE_1)$ for the only state that could be
obtained (with a six parameter fit) against $am$. Obviously these are not
Nambu-Goldstone bosons, as expected because in this case $(aM_n)^2$ does not
vanish linearly with $am$. With a view to interpreting them as a two-fermion
threshold, $2aE_F$, twice the physical fermion energy at $\beta_H = 3.0$ is plotted
for the fermions described by the expression (4.20). $2aE_F$ values lie close to
but systematically lower than the $aE_1$ values. Previous experience has shown
that this could be because of an appreciably large finite size effect in these
simulations.

To conclude this section one notes very clearly that when probed by
$l = 1$ representation fermions in a quenched calculation the Higgs phase
divides itself into two distinct regions, namely, the chirally broken Higgs
phase and the chirally symmetric Higgs phase so that physical fermion masses
are dynamically generated well inside the Higgs phase.
Figure 4.11: Lowest lying pion energy ($E_1$) and twice the physical fermion energy ($2E_F$) for expression (4.20) versus bare fermion mass ($am$).
CHAPTER V

Summary

5.1 Introduction

In this concluding chapter, a summary of this dissertation is presented. It is divided into the two following sections. The first section deals with the physics results and the second discusses the technical issues specifically relevant to lattice field calculations.

5.2 Physics Results

Spontaneous chiral symmetry breaking is seen to take place in both the models studied. As a result fermion masses are generated dynamically. The two models are the 1+1 dimensional Gross-Neveu model and the SU(2) gauge-Higgs-fermion system. The latter is a gauge theory and the former one is not. The Gross-Neveu model has only a discrete chiral symmetry which breaks spontaneously to generate the fermion mass. Since the symmetry is only discrete, Goldstone theorem does not apply here and accordingly no Goldstone particle appears in the spectrum. The SU(2) gauge-Higgs-fermion system, however, has a continuous chiral symmetry even on the lattice. When this
continuous symmetry breaks down spontaneously, a massless pseudoscalar boson (the Goldstone particle) is seen in the spectrum of the theory, in addition to, of course, the dynamical mass generation of the physical fermions.

In the context of the SU(2) gauge-Higgs-fermion system, some insight into the dynamics of spontaneous chiral symmetry breaking is claimed to have been obtained in this dissertation. It was known that in a vectorlike gauge theory spontaneous chiral symmetry breaking takes place if the coupling is large enough. In this present vectorlike model there is also a breaking of the gauge symmetry owing to the Higgs particle. There is no apparent connection between the gauge symmetry breaking and the chiral symmetry breaking. However, in this dissertation, from high statistics calculations it is shown, for fermions in the fundamental representation of the gauge group, that these two breakings take place within numerical accuracy at the same values of the relevant parameters of the theory. This is claimed to be a very important result for the understanding of the dynamics of spontaneous chiral symmetry breaking.

For fermions in higher representations of the gauge group a relatively recent result is also verified. Few years ago, it was found by exclusive lattice gauge calculations that one does not need the gauge coupling to be so large as to be confining for spontaneous chiral symmetry breaking. Spontaneous chiral symmetry breaking takes place whenever the effective interaction among the fermions becomes greater than a critical value. Since the effective coupling can be increased by going over to higher representation fermions, one can have spontaneous chiral symmetry breaking long before the confinement scale is reached. Investigation is carried out in this dissertation with fermions
in the adjoint representation in the SU(2) gauge-Higgs-fermion system. It is seen that the confinement region definitely has the chiral symmetry spontaneously broken. In addition it is also found that the Higgs region (non confining) has two parts, one with chiral symmetry restored and the other with chiral symmetry spontaneously broken.

5.3 Technical Issues

The problems of treating fermions on the lattice has been discussed in some detail in chapter 2. As is seen there, for a theory to be investigated with the full dynamical content of the fermions, the fermion determinant has to be calculated on the lattice. In this dissertation, one of the few algorithms available for treating dynamical fermions, namely the molecular dynamics method, is put to test in the context of the Gross-Neveu model. The results obtained agree with all previously known results on this model, though even for this relatively simple 1+1 dimensional theory without gauge interactions, the method does not seem to be fast enough. Recently there are available a few other methods which should also be put to serious test.

Another problem with the lattice fermions is known as the 'doubling' as also discussed in chapter 2. A partial remedy of this problem is the staggered fermion mechanism which somewhat reduces the doubling. Flavor symmetry should be restored in the continuum in order that the flavor interpretation of the degeneracy of the staggered fermions makes sense. The issue is addressed here in the studies of the Gross-Neveu model. Spectrum of the fermion-antifermion bound states is calculated with the hope that the equality of the
masses of the different states in the scaling region would guarantee restoration of flavor symmetry in the continuum limit. The states are written down using symmetries of the continuum and as a result they mix on the lattice. The mixing is especially bad in 1+1 dimensions. Because of the mixing it is not clear whether the masses calculated really belong to different states. In spite of the fact that the masses computed in this work are all equal within error bounds, restoration of flavor symmetry cannot be guaranteed.
LIST OF REFERENCES


33. I-H Lee and R. Shrock, Private communication.

