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Spatiotemporal patterns in flow between two independently rotating cylinders

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The Ohio State University, 1990
SPATIOTEMPORAL PATTERNS IN FLOW BETWEEN TWO INDEPENDENTLY ROTATING CYLINDERS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
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*****

The Ohio State University
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CHAPTER I

INTRODUCTION

Nonlinear, nonequilibrium phenomena are ubiquitous in nature and yet the turbulent behavior they often exhibit is not well understood. Until recently most theoretical analyses of systems far from equilibrium have been limited to linear approximations. The development of supercomputers, new analytical techniques, and precise experiments has stimulated significant progress in understanding the behavior of nonlinear systems\(^1\). Of particular interest in this field is the manner in which spatiotemporal patterns arise. Spatiotemporal patterns are especially important in fluid systems because they are associated with both the transition to turbulence and turbulent flow. One way in which a laminar flow may develop into a turbulent flow is through a sequence of pattern forming instabilities. Conversely, a highly turbulent flow may self-organize into a macroscopic pattern. I shall present examples of both of these types of patterns in the following chapters.

Before presenting the main results I wish to give a brief, and hopefully understandable, survey of some of the theories relevant to this thesis. This survey will also serve the purpose of defining most of the terminology used. First I will discuss some elements of fluid mechanics and fluid instabilities using the Taylor-Couette system and a model system as examples. I will then present the more contemporary theories of amplitude (or envelope) equations and phase dynamics. Next I will introduce the other experimental system of which I am writing, i.e. the Taylor-Dean system. Finally I will present the motivation for the experiments presented in the following chapters.
1.1 Fluid Mechanics

The macroscopic state of an equilibrium system is given by a set of thermodynamic variables and their equations of state. The homogeneity of such an equilibrium system is disturbed if some external forcing is applied. Some typical examples of external forcing that disturb the equilibrium of a homogeneous system are an externally applied temperature difference, an externally applied pressure difference, or some external force field (e.g. gravity). External forcing of this type will create gradients in the thermodynamic variables of the system. This non-equilibrium system may then be divided into small volume elements. If each volume element contains a large number of molecules then any given volume element may be considered a macroscopic subsystem. If the thermodynamic quantities vary on a much larger scale then the size of these volume elements, then each volume element may be considered to be in thermodynamic equilibrium. Thus each volume element has an equilibrium equation of state and the equation of state of the large non-equilibrium system becomes a function of position and time. Because the particles in a fluid are not constrained to stay in a fixed position, a velocity distribution is also required to describe the macroscopic state of a fluid system. For many fluids, including the fluids used in the experiments presented in this thesis, only two thermodynamic variables are required to describe the macroscopic subsystems. These five quantities, three components of velocity and two thermodynamic variables, require five equations to be determined. These five equations are generated by conservation of mass, momentum, and energy.

In all of the experiments described here I have used a fluid of constant density, \( \rho \), and I have run the experimental systems in a temperature controlled
room to make the temperature, $T$, constant. This eliminates one thermodynamic variable and only four equations are needed to determine the four fields $V(r,t)$ and $p(r,t)$. These equations are the Navier-Stokes equations:

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\frac{1}{\rho} \nabla p + \nu \nabla^2 V + f \tag{1.1}$$

and

$$\nabla \cdot V = 0 \tag{1.2}$$

Equation (1.1) expresses conservation of momentum, or equivalently Newton's second law, applied to a small volume of fluid with constant density $\rho$. The two terms on the left of equation (1.1) are the total acceleration of a fluid element. The first term is the change in velocity ($V$), in a small amount of time ($\delta t$), of a fluid element at a fixed point in space. The second term on the left results from the movement of a fluid element in time $\delta t$ to a position where the velocity is different and is called the advection or inertial term. This term is non-linear in $V$, which makes these equations very difficult to solve except in special cases.

The terms on the right hand side of equation (1.1) denote the forces exerted on a fluid element. The first term is the pressure ($p$) force on a fluid element created by the equal and opposite reaction forces (in the sense of Newton’s third law) of neighboring fluid elements. The second term on the right is the viscous force due to the transport of momentum that occurs when fluid elements in relative motion exchange molecules. The last term, $f$, is all the external force fields (e.g. gravity, magnetic fields, applied pressure gradients, etc.) that may be driving each fluid element. Equation (1.2) expresses the conservation of mass for the case of an incompressible fluid.
The left and right hand sides of equation (1.1) have the dimensions of acceleration. If the equation is rescaled using a characteristic velocity \( U \), a characteristic length \( L \) and the kinematic viscosity \( \nu \), then equation (1.1) may be rewritten in dimensionless form such that (provided \( f = 0 \)) \( \nu \) is replaced by \( \frac{1}{R} \), where \( R \) is the Reynolds number. The Reynolds number, defined as \( R = \frac{UL}{\nu} \), characterizes the external driving of the flow in the absence of external force fields. This external driving is caused by a mechanical stress exerted on the fluid through the boundary of the problem in question. The boundary condition that gives rise to this forcing is called the no-slip boundary condition, i.e., the fluid at the boundary has the same velocity as the boundary surface. This condition is a result of the viscous friction force of the boundary on the fluid. There are many such driving parameters in fluid dynamics which correspond to different driving mechanisms.

With this set of hydrodynamic equations one can describe a nonequilibrium fluid system. One prototype nonequilibrium fluid system is the Taylor-Couette system. This system is the topic of much of this thesis and exhibits many distinct spatiotemporal patterns.

1.2 The Taylor-Couette System

The Taylor-Couette system consists of two coaxial independently rotating cylinders with a fluid-filled gap. The gap is typically much smaller than either cylinder’s radius. The cylinder walls drive the flow through the no-slip boundary condition. Because there are two sources of energy to drive the fluid there are two characteristic velocities, the inner cylinder velocity \( \Omega_i r_i \) and the outer cylinder velocity \( \Omega_o r_o \). \( \Omega_i \) and \( \Omega_o \) are the inner and outer cylinder angular velocities.
respectively, and \( r_i \) and \( r_o \) are the inner and outer cylinder radii respectively. Thus there are also two Reynolds numbers, the inner cylinder Reynolds number, \( R_i = \frac{n r d}{\nu} \), and the outer cylinder Reynolds number \( R_i = \frac{n r d}{\nu} \), where the characteristic length, \( d \), is the gap length between the cylinders \( (d = r_o - r_i) \). It may be shown that at low enough \( R \) the solution to the Navier-Stokes equation is unique and I shall refer to this as the base flow. In the Taylor-Couette system this unique flow is called Couette flow. The base flow has only one component of velocity \( (V_\phi) \) that varies with the radius \( (r) \), i.e. \( V = (V_r = 0, V_\phi(r), V_z = 0); p = p(r); T = \text{constant} \). To get the Couette solution one takes the cartesian form of the Navier-Stokes equations (1.1) and transforms them into cylindrical coordinates. They are:

\[
\frac{\partial V_r}{\partial t} + (V \cdot \nabla)V_r - \frac{V_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu(\nabla^2 V_r - \frac{2}{r^2} \frac{\partial V_\phi}{\partial \phi} - \frac{V_r}{r^2}), \tag{1.3}
\]

\[
\frac{\partial V_\phi}{\partial t} + (V \cdot \nabla)V_\phi + \frac{V_r V_r}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \nu(\nabla^2 V_\phi + \frac{2}{r^2} \frac{\partial V_r}{\partial \phi} - \frac{V_\phi}{r^2}), \tag{1.4}
\]

\[
\frac{\partial V_z}{\partial t} + (V \cdot \nabla)V_z = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 V_z, \tag{1.5}
\]

where

\[
(V \cdot \nabla)f = V_r \frac{\partial f}{\partial r} + \frac{V_\phi}{r} \frac{\partial f}{\partial \phi} + V_z \frac{\partial f}{\partial z}, \tag{1.6}
\]

and

\[
\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}. \tag{1.7}
\]
It is remarkable that this complicated set of coupled nonlinear partial differential equations yields a simple basic solution. When \( V = (0, V_\phi(r), 0) \) and \( p(r) \) is substituted into equations (1.3)-(1.5) one obtains:

\[
\frac{\partial p}{\partial r} = \rho \frac{V_\phi^2}{r^2}, \tag{1.8}
\]

and

\[
\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r^2} = 0. \tag{1.9}
\]

The solution of equation (1.9) is

\[
V_\phi = Ar + \frac{B}{r}. \tag{1.10}
\]

\( A \) and \( B \) are constants that are determined by the no-slip boundary conditions, i.e. \( \Omega_i(r_i) = A + B/r_i^2 \) and \( \Omega_o(r_o) = A + B/r_o^2 \). These two conditions lead to:

\[
A = -\Omega_i \frac{\eta^2 - \mu}{1 - \eta^2} \tag{1.11}
\]

and

\[
B = \Omega_i r_i^2 \frac{1 - \mu}{1 - \eta^2}, \tag{1.12}
\]

with \( \eta = \frac{r_o}{r_i} \) and \( \mu = \frac{\Omega_o}{\Omega_i} = \eta \frac{R_o}{R_i} \). The radius ratio \( \eta \) and the rotation ratio \( \mu \) are important dimensionless parameters.

Our system typically has two rigid surfaces at the ends of the fluid column. These two ends, or collars, are attached to the outer cylinder and make contact with the fluid in planes perpendicular to the cylinder axis. The collars are made of Teflon and are slightly smaller than the gap making it possible to change the fluid between experiments. These ends must also drive the fluid through the no-slip boundary condition and this effect is called Ekman pumping. The flow
generated by Ekman pumping away from the collars is usually neglected if the system has a long axial length. The importance of Ekman pumping away from the collars is parameterized by the ratio of the length of the system along the axis ($l$) to the gap width ($d$). This parameter is called the aspect ratio, $\Gamma (= \frac{l}{d})$.

When the parameters $R_i$, $R_o$, and $\eta$ are changed the Couette flow may become unstable and a new solution to the Navier-Stoke equations will emerge. This change in the solution will manifest itself in an experiment by a change in the flow pattern. This process is known as a bifurcation. If the parameters are plotted on a set of axes and each point in this parameter space is mapped to the flow that exists at that point then this parameter space is called a bifurcation diagram (other terms commonly used are phase diagram and flow regime diagram).

1.3 Fluid Instabilities

Although the Navier-Stokes equations were derived over 150 years ago the transition to turbulence remains an enigma. Much of the understanding of this transition has been through the study of fluid instabilities. A stable state of a system is one that returns back to this state after being perturbed. For example a marble in static equilibrium at the bottom of a bowl is stable because it will return to this state if it is perturbed by an impulse (provided the impulse is not too large). A marble that is on top of a larger marble is unstable because it will not return to this state after being perturbed. This same idea is also useful in fluid systems although we need to be cautious when thinking of potential minimization - there may not be a potential to be found. The state of a fluid system, e.g. Couette flow, is stable if it returns to this state after being disturbed. If the state is unstable then a disturbance will change the state. To illustrate this
with the Taylor-Couette system consider the simpler case of an inviscid ($\nu = 0$) and axisymmetric ($\frac{\partial}{\partial \phi} = 0$) flow. For this case equation (1.4) simplifies to

$$\frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + V_\phi \frac{\partial V_\phi}{\partial z} + \frac{V_\phi V_r}{r} = 0.$$  \hspace{1cm} 1.13

Equation (1.13) can also be written as $\frac{d}{dt}[rV_\phi] = 0$ which shows that a fluid elements' angular momentum per unit mass $(rV_\phi)$ is constant as it moves. In the Couette state a fluid element of mass $m$ at $r_1$ has angular momentum $L_1 = m r_1^2 \Omega(r_1)$. Mechanical equilibrium is maintained by a pressure gradient which exerts a centripetal force (see equation (1.13)) of $\frac{mV_\phi^2(r_1)}{r_1} = \frac{L_1^2}{mr_1^2}$ on the fluid element. To test the stability of this state let the fluid element be perturbed such that it moves toward the outer cylinder to $r_2 (> r_1)$. The centrifugal force that this fluid element feels is, since $L_1$ is conserved for this element, $\frac{L_1^2}{mr_2^2}$. The centripetal force that the pressure gradient creates at $r_2$ is, by the same argument as above, $\frac{mV_\phi^2(r_2)}{r_2} = \frac{L_2^2}{mr_2^2}$. The centripetal force at $r_2$ due to the pressure gradient must be larger then the centrifugal force experienced by the fluid element if the element is to return to its unperturbed position. Hence, for stability of the Couette state, we have the condition that $L_2^2 > L_1^2$. This condition for stability may be rewritten, using the above expressions of angular velocity, as $\Omega_\phi r_2^2 > \Omega_\phi r_1^2$. This shows that the Taylor-Couette system has the potential for instability because it is possible to create a situation, by driving the flow with the two cylinders, where the magnitude of the angular momentum doesn’t increase with $r$. It can be shown more generally\footnote{This is referred to as the Rayleigh stability criterion.} that $\frac{d}{dt}(rV_\phi) > 0$ is a necessary and sufficient condition for stability of axisymmetric and inviscid flow. This condition is referred to as the Rayleigh stability criterion. Couette flow becomes unstable, as the inner cylinder angular speed is increased, when the
radial pressure gradient is not strong enough to return the fluid particles back to their original positions after being perturbed.

There are other types of instabilities with different physical reasons for their occurrence. One type of interest is the shear flow instability. A shear flow is a flow where the velocity varies in a direction perpendicular to the direction of the velocity. These flows become unstable, in the inviscid case without an external force field, because of an imbalance between the internal stress of pressure and the fluid particles' inertia (for this case and without explicit time dependence these are the only two terms left in equation (1.1)). The influence of viscosity may be stabilizing or destabilizing. Couette flow may be viewed as a shear flow because $V_\phi \perp r$. The flow becomes unstable because of an imbalance between the inertia of the fluid elements, which tends to make them move in a straight line, and the radial pressure gradient, which forces them into a circular path. This flow is not usually thought of as a shear flow because the radial pressure gradient is usually considered to be external stress on the system. Shear flow instabilities are usually studied in planar geometry (where there is no centrifugal acceleration) and are difficult to understand with the destabilization process depending on the specific details of the problem.

The above analysis is limited to the case of an inviscid and axisymmetric flow. To study the more general case requires a more complete stability analysis. As we have seen above, the state of a system is stable if, when it is disturbed, it returns to that state. A disturbance may be finite or infinitesimal. These two types of disturbances require two types of stability analyses: linear stability analysis, for small or infinitesimal disturbances, and nonlinear stability analyses, when the perturbation has a finite amplitude. This also means that there are
two basic ways that a solution can change or bifurcate from a given solution. A supercritical bifurcation results from a solution that has become linearly unstable and a subcritical bifurcation results from a state that is nonlinearly unstable. A state may be linearly stable and yet nonlinearly unstable. In such a case a small perturbation will not affect the state but a large perturbation will induce a drastic change to another linearly stable state. The system, in this case, is referred to as metastable, which is the case where there is more than one linearly stable solution. In such a case a finite amplitude perturbation is required for the system to change states.

The mathematical stability problem is formulated by substituting a known solution (whose stability is being tested) plus a perturbation into the governing equations, giving equations for the perturbation. If the perturbation is infinitesimal then all the terms that are second and higher order in the perturbation may be dropped. The perturbation equations are linearized (thus the name linear stability) greatly simplifying the problem by allowing the well developed theory of linear differential equations to be used.

All experimental apparatuses will constantly perturb a given flow state because of unavoidable irregularities and imperfections. The initial value of these perturbations is unknowable. Perturbations are arbitrary and are continuously being created in an apparatus. If a solution to the governing equations is unstable then it is not realizable in a physical experiment because the unavoidable fluctuations will grow. If the solution is stable then all fluctuations will decay in time and this state is potentially realizable in an experiment. For a solution to physically exist it must be stable. The amplitudes of unavoidable perturbations in an experiment may be made very small by refining the apparatus. For a sufficiently refined apparatus the amplitude of the perturbations may be considered infinitesimal making linear stability theory applicable.
G. I. Taylor was the first to do a linear stability analysis of the Taylor-Couette system. His analysis is somewhat complicated but a simpler example, which has many of the important features of the Taylor problem, is given by Eckhaus. This simplified mathematical model is not derived from a physical problem but is useful for illustrating the method of normal modes because everything can be calculated without difficulty. The Eckhaus model is given by the following nonlinear equation

\[
\frac{1}{R} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) - \frac{\partial^4 \Phi}{\partial x^4} - \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial x^2}
\]

with the boundary conditions, \( \Phi(x,0,t) = 0, \Phi(x,L,t) = L, \) and \(-\infty < x < \infty\).

The base solution is \( \Phi_0(y) = y \) as can be easily verified. To help understand this problem it may be useful to think of \( \Phi \) as a velocity in the \( x \) direction with one stationary wall at \( y = 0 \) and one moving wall at \( y = L \) even though a real fluid would not be described by equation (1.14). The moving wall at \( y = L \) induces a shear flow with a linear velocity profile. To test for the stability of this flow we add an arbitrary perturbation, \( \Phi'(x,y,t) \), to \( \Phi_0 \) and obtain \( \Phi(x,y,t) = \Phi_0(y) + \Phi'(x,y,t) \). Substituting \( \Phi \) into equation (1.14) gives the perturbation equation:

\[
\frac{1}{R} \left( \frac{\partial^2 \Phi'}{\partial x^2} + \frac{\partial^2 \Phi'}{\partial y^2} \right) - \frac{\partial^4 \Phi'}{\partial x^4} - \frac{\partial \Phi'}{\partial t} = \frac{\partial \Phi'}{\partial y} \frac{\partial^2 \Phi'}{\partial x^2} = 0
\]

with the boundary conditions, \( \Phi'(x,0,t) = \Phi'(x,L,t) = 0 \). \( \Phi' \) is infinitesimally small so the right hand side of equation (1.15) is set equal to zero with a high degree of accuracy. We now decompose the perturbation \( \Phi' \) into a complete set of functions or modes. Each mode is treated separately because each mode satisfies the linearized equation. The time factor for each mode is written as
$e^{st}$, with $s = \sigma + i\omega$. The real part of $s$, i.e. $\sigma$, gives the stability information because it determines whether the perturbation will grow or decay. The solution of equation (1.15) has the form $\Phi' = e^{ikx}e^{st}\phi_p(y)$. Equation (1.15) reduces to

$$\frac{\partial^2 \phi_p(y)}{\partial y^2} = R[k^4 + k^2(\frac{1}{R} - 1) + s_p]\phi_p.$$  \hspace{1cm} 1.16

This eigenvalue problem has the eigenfunctions,

$$\phi_p = \sin[(p + 1)\pi y], \quad p = 0, 1, 2, \ldots,$$  \hspace{1cm} 1.17

with eigenvalues

$$s_p(k) = (1 - k^2)k^2 - \frac{(p + 1)^2\pi^2 + k^2}{R}.$$  \hspace{1cm} 1.18

In this case all the eigenvalues are real and can be ordered as: $s_p(k) > s_{p+1}(k)$, $p = 0, 1, 2, \ldots$. This ordering shows that $s_0(k)$ is the first eigenvalue to become positive as $R$ is increased. When this happens the $p = 0$ mode increases exponentially with time and the base solution is unstable. The $R(k)$ curve, where $s_0$ is zero, is called the marginal curve and is shown in Figure 1. It is

$$R = \frac{\pi^2 + k^2}{k^2(1 - k^2)}.$$  \hspace{1cm} 1.19

The minimum value of $k$ on the marginal curve is the critical wave number, $k_c$. The minimum value of $R$ on the marginal curve is called the critical Reynolds number, $R_c$. $R_c$ and $k_c$ are also shown in Figure 1.

$\phi_0$, the eigenfunction corresponding to $s_0$, is called the most unstable mode. As $R$ increases other $s_p$'s will also become greater than zero making other modes unstable as well. The problem has the following cases: if $\sigma < 0$ for all modes then the disturbance decays in time and the state is stable, if $\sigma = 0$ for any
The marginal stability curve for the Eckhaus model problem. The minimum of this curve gives the critical values for $k$ and $R$. Slightly above onset (where $\epsilon = \frac{R - R_e}{R_e}$ is $0 < \epsilon \ll 1$) there is an unstable band of $k$'s with a width $O(\sqrt{\epsilon})$. 

Figure 1
mode then the state is neutrally stable, if $\sigma > 0$ for any mode then the state is unstable, and finally if $\sigma = 0$ for the most unstable mode then the state is marginally stable.

1.4 Amplitude Equations

Linear stability theory tells us the parameter values at which the base flow becomes unstable. When the Couette flow becomes unstable a perturbation will, by definition, increase exponentially in time. This can happen for only a short time before the amplitude of the perturbation is no longer small. When this occurs the perturbation equations cannot be considered linear anymore and nonlinear analysis becomes necessary. To understand how a pattern is formed a theory which is valid above $R_c$ must be examined. In the Taylor-Couette system with the outer cylinder at rest the pattern that results when the Couette flow becomes centrifugally unstable is called Taylor Vortex Flow (TVF). These steady toroidal vortices are periodic along the axis and are axisymmetric. Each vortex or cell fills the entire gap space. One axial wavelength constitutes two vortices because adjacent vortices have opposite senses of circulation. A photograph of this structure is shown in Figure 2.

Eckhaus introduced a general method for reducing a nonlinear partial differential equation to an infinite set of nonlinear ordinary differential equations. In this method the perturbation of the velocity, $v' = v - V$, is expanded in a complete set of functions, i.e

$$v' = v - V = \sum_{i=1}^{\infty} \{A_i(t)f_i(r) + A^*_i(t)f^*_i(r)\}. \quad 1.20$$

where $\{f_i\}$ is any complete set of complex functions satisfying the boundary conditions of the problem. After substituting the above expression into the governing equations it is possible to derive, in some cases (this depends on the form
Figure 2

A photograph of the Taylor Vortex Flow (TVF) pattern that occurs when Couette flow becomes centrifugally unstable. The axis of the cylinders is perpendicular to the light and dark stripes which indicate the toroidal vortices. The dark lines between the stripes correspond to alternating radial inflow and outflow boundaries which separate adjacent vortices. The difference in brightness between adjacent vortices is due to differing streamline shapes that adjacent cells have and the manner in which the flow visualization agent reflects incident light.
of the governing equations, which may introduce many technical difficulties) an
equation of the form

\[
\frac{dA_i}{dt} = s_i A_i + N_i(A_j) \quad j = 0, 1, 2, \ldots
\]

where \( N_i \) is a complex nonlinear function of the \( A_j \)'s. \( s_i \) is the growth rate of
the linear stability problem. \( N_i \) represents the nonlinear interaction of all the
modes on the \( i \)th mode including the self interaction. \( N_i(A_j) \) moderates the
exponential growth of \( A_i \). Equations of this type are difficult to analyze being
better suited to numerical integration. We may gain some insight into the
process of pattern formation if we study the weakly nonlinear case. In this case
we have \( \epsilon = \frac{R - R_0}{R_0} \ll 1 \) so that, as in the Eckhaus model problem, the amplitude
of the most unstable mode of the perturbation grows slowly as \( e^{\alpha t} \) while all the
other modes decay. Because of the slow growth of the most unstable mode the
linear approximation will be valid for a short time before the weak nonlinearity
has an effect. We expect the time scale to be the same when this mode grows into
the nonlinear regime. Therefore the slow growth of the most unstable mode in
the linear regime introduces a slow time scale into the weakly nonlinear problem.
The disturbance grows, to first order, at a rate \( s_0 \propto \epsilon \) which makes the slow
nonlinear dynamics of the growth of the pattern occur, near onset, on a time
scale \( T = \epsilon t \). The amplitude \( A_0 \) and the associated \( N_0(A_{j=0}) \) are of interest here
because, in the linear approximation, all the other \( A \)'s decay. The amplitude of
the most unstable mode in the linear regime is of the form \( e^{\alpha t} \propto e^{\epsilon t} \) and, because
of the matching of the time scales, the amplitude in the weakly nonlinear regime
will be of the form \( A_0(T) \). The natural choice for \( \{f_i\} \) is the complete set of
functions obtained from the eigenvalue relation in the linear stability problem.
(if a complete set exists). We expect that the effect of the nonlinearity, \( N_0 \), will be to saturate the slow exponential growth of the most unstable mode. \( N_0 \) is a result of the nonlinearity in the perturbation equations. This nonlinearity results from the advection or inertia of the fluid particles as the most unstable mode grows. This inertia may be thought of as a driving force on the system. We expect this "driving force" to distort the base flow and the disturbance because the two fields cannot linearly superpose.

In the Eckhaus model problem given above the situation is complicated by the presence of an infinite domain. This means that in order to pose the problem in the form of equation (1.20) we must expand \( \Phi \) along the infinite domain. If a \( k_0 \) is chosen close to \( k_c \) then \( \Phi' \) may be expanded as

\[
\Phi' = \sum_{m=-\infty}^{\infty} e^{-ik_0 mz} f_m(y, T) \quad 1.22
\]

and subsequently each \( f_m(y, T) \) is expanded as

\[
f_m(y, T) = \sum_{p=0}^{\infty} A_p^{(m)}(T) \phi_p^{(m)}(y). \quad 1.23
\]

The problem is then reduced to

\[
\frac{dA_0}{dT} = s_0(k_c) A_0 - l|A_0|^2 A_0 \quad 1.24
\]

where

\[
l = -\frac{1}{4} \pi^2 k_c^4 \left[ \frac{1}{s_1(0)} - \frac{1}{2s_1(2k_c)} \right]_{R=R_0} > 0. \quad 1.25
\]

Equation (1.24) is called the Landau equation and was first introduced by L. D. Landau in 1944. If the bifurcation is to a state with explicit time dependence, as happens in some cases, then the Landau equation has complex coefficients. If \( l > \)
0 then the amplitude continuously grows from zero reaching an asymptotic value of \( \sqrt{2s_0(k_c)/l} \). This is an example of a supercritical bifurcation. The constant \( l \) is called the Landau constant and reflects the physical processes that limit the growth of the instability. If \( l < 0 \) and if the amplitude has an initial value greater then \( \sqrt{2s_0(k_c)/l} \), then the amplitude will diverge making it necessary for higher order terms to be retained. It can be shown that a subcritical instability may occur in such a case\(^4\text{,} \text{8}\). J.T. Stuart\(^9\) and A. Davey\(^10\), using different methods, derived the Landau equation for the Taylor-Couette system. They both found \( l > 0 \) which implies that the growth of Taylor vortices is a supercritical bifurcation.

Stuart, who was the first to derive the Landau equation for a specific fluid system, assumed that the disturbance has the same spatial shape as the most unstable mode while the amplitude is different, i.e. the interactions associated with the growth of the disturbance do not distort the shape of the most unstable mode and \( e^{zt} \) is replaced by \( A_0(T) \). His calculation of \( l \) only considered the interaction of the disturbance with the base or mean flow. Implicit in this assumption is that the disturbance modifies the mean flow whereas the mean flow doesn’t modify the disturbance, i.e. there is energy transfer from the mean flow to the most unstable mode which distorts the mean flow. Davey did a more systematic analysis and found two other contributions to \( l \) in addition to the energy transfer from the mean flow to the most unstable mode. These other two contributions are the energy transfer from the most unstable mode and the mean flow to the first harmonic that is generated, and the energy transfer to the most unstable mode from the mean flow and the harmonic distorting its spatial shape. This explicitly shows that the most unstable mode’s growth is limited by its self interaction (i.e. the disturbance interacts with the mean flow and the first harmonic which in
turn interact with the disturbance making a self interacting feedback loop), its interaction with the mean flow, and its interaction with the first harmonic. These interactions exist because of the nonlinearity in the perturbation equations which result from the advection term in the Navier-Stokes equations. DiPrima\textsuperscript{11} also derived the Landau equation using Eckhaus's method described above.

Although it is technically more difficult the Taylor problem is similar to the Eckhaus problem in the following ways: the eigenfunctions of the linear stability problem depend only on the coordinate with the smallest length scale \((r)\), the most unstable mode has an associated \(k_c\), and the spatial form of Taylor vortices has, to first order, the same form as the most unstable mode\textsuperscript{12} with an assumed axial periodicity. As mentioned above the advection of the disturbance forces it to interact with the mean flow and itself. Both the disturbance and the mean flow are modified by this interaction and a first harmonic is generated. The result of the instability is a new velocity field which is similar to Couette flow plus a perturbation proportional to the most unstable mode. The weak nonlinearity, being a result of the advection of the fluid, has forced these two fields to interact and generate corrections to their linear superposition.

In all of these analyses the length of the cylinders is assumed infinite making the \(k\) values continuous. All of the analyses mentioned so far have assumed a \(k\) value just as in the Eckhaus problem. It is noted in Figure 1 that in the Eckhaus model problem, for \(R > R_c\), there is not one unique unstable \(k\) but a continuous band of unstable \(k\)'s. This is the case in the Taylor problem as well. There is no reason to assume a given \(k\) value when, for \(R > R_c\), there is band of wave numbers. In fact one might expect all of these \(k\)'s to slowly grow in time except for the fact that the nonlinearity will not allow the superposition
of all these $k$'s to solve the governing equations. When the nonlinearity is weak one might still expect a wave packet type variation of the wavelength rather than a unique wavelength. Newell and Whitehead\textsuperscript{13} were the first to address this issue by representing this wave packet type behavior with an amplitude, of the most unstable mode, that slowly varies in space and time. Because of the local parabolic shape of the marginal curve near $k_c$ at $\epsilon > 0$ there is an $O(\sqrt{\epsilon})$ bandwidth of unstable $k$'s. Therefore a slow space variable $X$ is introduced which is scaled as $X = \sqrt{\epsilon}x$. At any of these $k$'s the slow growth rate is still $s_0 \propto \epsilon$ which implies that the slow time variable $T$ is scaled as $T = \epsilon t$. An equation for the slowly varying amplitude may be derived using a multiple scale perturbation method. In this method the velocity field is expanded in the small parameter $\mu$ where $\mu^2 = \epsilon$ i.e.

$$v' = \mu [A(X, T)e^{i(k_c z - w(k_c, R)t)} + A^*(X, T)e^{-i(k_c z - w(k_c, R)t)}]\phi_0(r)$$

$$+ \mu^2 v_2 + \mu^3 v_3 + \ldots$$ \hspace{1cm} 1.26$$

and the operators

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \mu^2 \frac{\partial}{\partial T}$$ \hspace{1cm} 1.27$$

and

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial X}$$ \hspace{1cm} 1.28$$

This expansion is substituted into the nonlinear perturbation equations and terms at each order in $\mu$ are balanced and solved iteratively. The equation for $A(X, T)$ is a result of a solvability condition, i.e., in order for the equations generated at second order in $\mu$ to have a solution a condition must be satisfied and the condition is the equation for $A(X, T)$. This solvability condition on $A$
makes the expansion of the fields well ordered and uniform in space and time because it ensures that a field exists at second and higher order. In other words the equation for $A$ ensures that the corrections to the linear superposition of the base flow and the most unstable mode as expressed in equation (1.28) converges uniformly. Tabeling\textsuperscript{14} and Brand and Cross\textsuperscript{15} used this multiple scale perturbation technique to derive amplitude equations for the Taylor vortices in the small gap limit ($\eta \to 1$) when the outer cylinder is at rest ($R_0 = 0$). The equation for the slow space and time variations of the amplitudes of the Taylor Vortices along the axis, $Z$, and the azimuth, $Y$, has the form

$$\frac{\partial A}{\partial T} = c_0 A + c_1 \frac{\partial^2 A}{\partial Z^2} + ic_2 \frac{\partial^2 A}{\partial Y \partial Z} + c_3 \frac{\partial^2 A}{\partial Y^2} - c_4 A|A|^2. \quad 1.29$$

Where the azimuthal coordinate, $Y$, is in a rotating reference frame and the $c_i$'s are real coefficients. A simpler equation for the slow amplitude results when only the axial coordinate is allowed to vary.

$$\frac{\partial A}{\partial T} = c_0 A + c_1 \frac{\partial^2 A}{\partial Z^2} - c_4 A|A|^2. \quad 1.30$$

This equation still has the same solution as the Landau equation but also allows for large scale spatial variations. It describes how the most unstable mode's amplitude becomes saturated. At onset the Taylor vortices grow at a rate $c_0$ and the amplitude of the radial component of the velocity saturates at a value of $\sqrt{2c_0/c_4}$. The diffusion term smooths out any long range variations in the amplitude of the periodic velocity field. Equation (1.13) is a special case of the Ginzburg-Landau equation which has complex $c_i$'s and describes a field of nonlinear oscillators. (In fact the Landau equation describes the simplest nonlinear oscillator and the Ginzburg-Landau equation describes the simplest field of nonlinear oscillators\textsuperscript{16}.) Hall\textsuperscript{17} did an expansion in the small parameter $\delta = d/R_i$
and showed that corrections to the mean azimuthal flow will generate a mean radial flow violating the no-slip boundary conditions unless a mean pressure is included in the calculations. Equation (1.12) is then supplemented with equations for the mean pressure fields which depend on \( Z \) and \( Y \). Hall also showed that unless these pressure fields are included Taylor vortices are stable to non-axisymmetric perturbations. This is contrary to the experimental fact that, for \( R_{\alpha} = 0 \), a secondary bifurcation from Taylor Vortex Flow (TVF) to nonaxisymmetric Wavy Vortex Flow (WVF) is observed slightly above the onset to TVF. Including the pressure leads naturally to WVF, which is shown in Figure 3.

1.5 Phase Dynamics

Although amplitude equations greatly simplify the mathematics involved in studying pattern formation the cost of this simplification is that they are only valid when \( \varepsilon \ll 1 \). The pattern forms, at onset, by the slow growth of the amplitude of the disturbance until it reaches a limiting value. The amplitude, being a complex quantity, has both a magnitude and a phase. After the pattern has formed it is the magnitude that saturates to a constant value. The phase, however, introduces a new degree of freedom because it is determined only to within a constant. Because these spatially periodic patterns are observed to exist significantly above onset it is the phase of the pattern that is the natural choice for the order parameter. Phase dynamics attempts to describe the patterns in this region of parameter space. In phase dynamics the pattern is only required to be periodic and large, i.e. the system containing the pattern must be large compared to the wavelength of the pattern.

If \( \mathbf{V}(r,z) \) is a solution to the Navier-Stokes equations, corresponding to a time independent flow pattern which is periodic along the \( z \) axis (like TVF), then
Figure 3

A photo of the Wavy Vortex Flow (WVF) pattern that occurs when the Taylor Vortex Flow (TVF) pattern becomes unstable to nonaxisymmetric perturbations. This secondary bifurcation occurs, at $R_o = 0$, when $R_i$ is slightly larger than $R_{ic}$, where $R_{ic}$ is the $R_i$ value where the TVF pattern occurs.
\( V(x, z + \psi) \) is also a solution, where \( \psi \) is a constant. \( V(x, z + \psi) \) corresponds to a pattern that has been shifted in the \( z \)-direction by \( \psi \). If \( \psi \) is small and slowly varying, which corresponds to a small and local distortion of the pattern, then \( V(x, z + \psi(Y, Z, T)) \) may be developed to be a solution to the Navier-Stokes equations. This is done by expanding \( V(x, z + \psi(Y, Z, T)) \) in the small parameter \( \nabla \psi \). As in the case of the amplitude equations this expansion is insured to be a solution of the governing equations by a solvability conditions on \( \psi \). This approach was first tried by Pomeau and Manneville\(^{18} \) when they expanded a periodic solution to a model amplitude equation and derived the phase diffusion equation. The phase diffusion equation describes how a pattern relaxes back to its periodic state after being disturbed or distorted. Tabeling\(^{14} \), using the same method as Pomeau and Manneville, derived the phase diffusion equation from equation (1.12) for TVF. It is

\[
\frac{\partial \psi}{\partial T} = D_{\parallel} \frac{\partial^2 \psi}{\partial Z^2} + D_{\perp} \frac{\partial^2 \psi}{\partial Y^2}.
\]

Because equation (1.14) was derived from an amplitude equation the coefficients were explicitly evaluated for the close to onset case. Tabeling's calculations agreed well with experiments\(^{19} \) and a more direct check of the phase diffusion equation in another system has also been done\(^{20} \). The phase diffusion equation is presumed to continue to apply farther above onset provided that the pattern remains approximately periodic. At these higher parameter values the diffusion coefficients have to be calculated from the full Navier-Stokes equations or measured from experiment. The diffusion coefficients may be thought of as phenomenological constants of a macroscopic description\(^{20} \) with the underlying microscopic description being that of fluid mechanics. In this theory, as mentioned above,
the phase of the periodic structure is the relevant dynamical variable or order parameter. The magnitude of the amplitude is no longer an order parameter, as it is in the close to onset case, because it depends on (or is slaved to) the phase variation. Amplitude perturbations have been shown experimentally to decay more rapidly than phase perturbations\textsuperscript{20}. Newell and Cross\textsuperscript{21,22} expanded a periodic velocity field in the small parameter $\Gamma^{-1}$ (the aspect ratio, $\Gamma$, is assumed large allowing for a long scale spatial variation in the wavelength of the pattern) allowing for a slow space and time dependence in the amplitude and phase. They calculated an explicit relationship for some fairly general model equations which shows that the magnitude of the amplitude depends on the gradient of the phase. Brand\textsuperscript{23,15} has taken a much simpler approach by noting the symmetries of the underlying pattern and writing down phase equations for the long wavelength and low frequency excitations. The phase dynamics of the WVF pattern, shown in Figure 3, was investigated by Brand and Cross\textsuperscript{15}. Using symmetry arguments they deduced the phase equations

\[ \frac{\partial \psi}{\partial T} = D_1 \frac{\partial^2 \psi}{\partial Z^2} + C_1 \frac{\partial \phi}{\partial Z} \]  
\[ \frac{\partial \phi}{\partial T} = D_2 \frac{\partial^2 \phi}{\partial Z^2} + C_2 \frac{\partial \psi}{\partial Z}. \]

Only displacements of the vortices along the axis, $Z$, were considered and $\psi$ and $\phi$ are the phase variables for the TVF and azimuthal waves respectively. They evaluated the coefficients for the close to onset case by using an amplitude equation which they also derived.

1.6 The Taylor-Dean System
The Dean problem, like the Taylor-Couette problem, has the geometry of two concentric cylinders. In the Dean problem, however, the cylinders remain stationary and the flow is driven though the gap by an external pressure gradient. This pressure gradient \((\frac{\partial p}{\partial \phi})_0\) is constant and acts along the azimuthal direction, \(\phi\), pushing the fluid through the curved gap. The base flow has the form \(V_r = 0, V_\phi = V_\phi(r), \) and \(V_z = 0\). Substituting this into equation (1.3) and (1.4) we get

\[
\frac{\partial p}{\partial r} = \rho \frac{V_\phi^2}{r^2}, \tag{1.34}
\]

and

\[
\frac{\partial^2 V_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial V_\phi}{\partial r} - \frac{V_\phi}{r^2} = \frac{1}{\nu \rho} \left( \frac{\partial p}{\partial \phi} \right)_0. \tag{1.35}
\]

Equation (1.34) shows that the radial pressure gradient exerts a centrifugal force on the fluid elements, accelerating them in a curved path. This is the same as in Couette flow, in fact equations (1.34) and (1.8) are identical. In a frame of reference rotating with a fluid particle of mass \(m\), \(m \alpha \) would be identified as the centrifugal force on the particle and equations (1.8) and (1.34) would be interpreted as a force balance maintaining mechanical equilibrium. This force balance, as in the Taylor problem, is the aspect of the base flow that is potentially unstable. Equation (1.35) is the same as (1.9) except for the inhomogeneity on the right hand side which represents the external forcing. The solution of equation (1.35) is, where \(r'\) is a dimensionless variable defined by \(r = r_0 r'\),

\[
V_\phi = \frac{r_0}{2 \nu \rho} \left( \frac{\partial p}{\partial \phi} \right)_0 \left[ r' \ln(r') + Cr' + \frac{D}{r'} \right]. \tag{1.36}
\]

Using the no-slip boundary conditions, \(V_\phi(\eta) = 0\) and \(V_\phi(1) = 0\), \(C\) and \(D\) are found to be

\[
C = \frac{\eta^2 \ln(\eta)}{1 - \eta^2} \tag{1.37}
\]
If the system is also driven by rotating the inner and outer cylinders, then the no-slip boundary conditions are modified to \( V^t(\eta) = \Omega_i \eta \) and \( V^t(1) = \Omega_o \). With these boundary conditions the base flow is a combination of Couette flow, driven by the rotating cylinders, and Dean flow driven by the azimuthal pressure gradient. This is the situation that the Taylor-Dean system attempts to realize.

To study the effects of the two types of driving we simplify the problem by defining \( x = (r - r_a)/d \) with \( r_a = \frac{1}{2}(r_i + r_o) \). Note that the gap is in the interval \(-1/2 \leq x \leq 1/2\) with this new definition. Taking the small gap limit, \( \eta \to 1 \), the Couette angular velocity profile becomes

\[
\Omega(x) \approx \Omega_i \{1 - (1 - \mu)(x + \frac{1}{2})\}. \tag{1.39}
\]

The Dean velocity profile becomes\(^4\)

\[
V^\phi(x) \approx \frac{3}{2} V_M (1 - 4x^2) \tag{1.40}
\]

where \( V_M = -\frac{d^2}{32\mu r_i} (\frac{\partial \phi}{\partial r})_0 \) is the mean velocity through the gap. In this limit the Taylor-Dean velocity profile is a combination of a linear (Couette) velocity profile and parabolic (Poiseuille) velocity profile. This velocity profile is\(^4\)

\[
V^\phi(x) \approx \Omega_i r_i \{1 - (1 - \mu)(x + \frac{1}{2}) + \frac{1}{4}\lambda(1 - 4x^2)\}. \tag{1.41}
\]

The parameter \( \lambda = 6V_M/\Omega_i r_i \) characterizes the relative strengths of the two base flows.

The Taylor-Dean system consists of two coaxial horizontal cylinders with a partially filled gap. Each cylinder is driven by a separate motor which drives

\[
D = \frac{\eta^2}{1 - \eta^2} \ln(\frac{1}{\eta}). \tag{1.38}
\]
the fluid as in the Taylor-Couette system. The partial filling of the horizontal cylinders makes two horizontal fluid-air interfaces, or free surfaces. If the cylinder rotation drives the fluid towards a free surface the system must generate an azimuthal pressure gradient to force the fluid back away from the free surface into the bulk of the fluid. \( R_o \) and \( R_i \) are the driving parameters of this system. The azimuthal pressure gradient is not an external parameter but rather a consequence of the free surfaces. This, of course, requires a different treatment than the one given above. We still, however, expect to have a Dean-like velocity profile and a Couette-like velocity profile away from the free surfaces. In fact we have observed TVF like structures as well as Dean roll like structures in the Taylor-Dean system (the Dean roll is the analog of TVF in the Dean system, i.e., it is the pattern that emerges when the Dean velocity profile becomes centrifugally unstable). The geometry of the Taylor-Dean system requires that there be no mean flow. Taking the mean value of equation (1.41) and equating it to zero gives a relation\(^4\) \( \lambda = -3(1 + \mu) \). By defining \( x = x' - 1/2 \) one obtains

\[
V_\phi(x') \approx \Omega_1 r_i \{ 1 - (1 - \mu)(x' + 3(1 + \mu)x'(1 - x'^2)) \}.
\]

This is the same result obtained by Mutabazi et al.\(^4\) by a more direct approach which uses the same assumptions. It is interesting to note that in the small gap approximation there is no radial pressure gradient and the azimuthal pressure gradient is constant. In fact \( \frac{\partial \phi}{\partial y} = 6(1 + \mu) \) where \( \phi = R_i \left( \frac{1 - \eta}{\eta} \right)^{\frac{1}{2}} y \). This shows that the azimuthal pressure gradient is controlled by the rotation speeds of the two cylinders.

Of course the free surface also creates several complications. These complications are as follows: the recirculation of the fluid by the induced pressure
gradient must generate a radial velocity component near the free surface, and the azimuthal velocity component must decrease near each free surface so that it depends, at least weakly, on $\phi$. These complications are avoided by using the small gap approximation and the zero mean flow condition in the analysis. In the experimental system these effects are present and their importance is parameterized by the entrance angle $\theta_e = x_e/R_i$ where $x_e$ is the entrance length. The entrance length, $x_e$, is the length along the azimuth in which the fluid develops into the base flow in the bulk. It is also the length over which the effects of recirculation die away due to the diffusion of momentum by viscous stress from the inner and outer cylinders. This diffusion process is well known for a few special cases from the theory of boundary layer flows\textsuperscript{25}. The boundary layer is the region near a surface (in this case a cylinder) that is highly influenced by viscous stress. This diffusion process happens as follows: the surface drives an infinitesimal layer of fluid adjacent to it in a direction parallel to the surface, this infinitesimal layer of fluid interacts via viscous stress on its neighboring layer entraining its neighbor into a parallel motion, etc., etc. In this way a characteristic length, $\delta$, normal to the surface is entrained into motion parallel to the surface. The thickness, $\delta$, of this entrainment region, or boundary layer, increases downstream because there is more time for the velocity to diffuse away from the surface. This downstream length where $\delta$ is of the order of the gap is precisely $x_e$. The boundary layer thickness is given by\textsuperscript{26}

$$\delta = a\left( \frac{x_e}{U} \right)^{\frac{1}{2}}$$

where $a \approx \pi$. The entrance length is chosen such that $\delta(x_e) = \frac{d}{3}$ which is the boundary layer thickness corresponding to a fully developed flow for fluid going away from the free surface. The entrance angle is then $\theta_e = \frac{d\Phi}{\partial \mu a \tau}$. This angle
is then compared to the filling angle, $\theta_f$, which parameterizes the filling of the system. It is defined as the azimuthal angle over which all the fluid extends. The recirculation effects are small if $\theta_e \ll \theta_f$.

1.7 Motivation of Research

This introduction has discussed two experimental systems along with some general theoretical approaches for describing the spatiotemporal patterns that arise in these systems. The research presented in this thesis is an attempt to help understand the physics of nonlinear and nonequilibrium systems after a bifurcation to a time dependent state occurs. Understanding the physics of spatiotemporal patterns is a major motivation for this research because it is an important step toward understanding both the transition to turbulence and the behavior of turbulent structures. It is also expected that these experiments will help to verify and deduce the generic properties of nonlinear and nonequilibrium systems. To this end a wide variety of spatiotemporal patterns are studied in both systems. Some of the spatiotemporal patterns studied are azimuthal traveling waves on axisymmetric Taylor vortices in the corotating Taylor-Couette system, the spiral turbulence pattern in the counter-rotating Taylor-Couette system, the traveling inclined roll pattern in the Taylor-Dean system for a weakly rotating outer cylinder, and Taylor-like rolls and Dean-like rolls in the counter-rotating and corotating Taylor-Dean system. In all of these cases the focus is on the physics of these nonlinear systems that can be deduced from the region of existence of these spatiotemporal patterns in parameter space and their characteristic behaviors where they do exist.
CHAPTER II

THE TAYLOR-COUETTE SYSTEM

2.1 Introduction

Above the Rayleigh stability line (see Figure 4) for corotating cylinders the primary bifurcation is to the time independent TVF pattern. Time dependence occurs for $\eta = .88$ and $R_o \geq 100$ as a secondary bifurcation. These time dependent patterns consist of spatially periodic waves that travel in the azimuthal direction. Because the wave patterns travel azimuthally with a constant speed these patterns would appear stationary in a suitably chosen rotating frame of reference. The specific pattern that appears depends on $R_o$. At low $R_o$ the WVF pattern appears and this pattern, shown in Figure 3, is characterized by a strong distortion of each vortex. The inflow and outflow boundary waves travel together with the same azimuthal wave number and wave speed but with a constant phase difference between them. At higher $R_o$, when $R_i$ is increased sufficiently, the TVF pattern bifurcates to one of the following azimuthal traveling wave patterns: the wavy inflow boundaries (WIB) pattern, the wavy outflow boundaries (WOB) pattern, or the twisted vortices (TWI) pattern. The WIB and WOB are, as their names suggest, azimuthal traveling waves which appear localized near the inflow boundaries or the outflow boundaries of the vortices. WIB and WOB are subharmonic with respect to the axial wavelength, i.e. waves on adjacent inflow (or outflow) boundaries are out
of phase so that the pattern repeats itself every four cells or two wavelengths (one axial wavelength consists of two cells because adjacent cells circulate in opposite senses so that the velocity field repeats itself after two cells). The TWI pattern, which gives each vortex the appearance of braided rope, consists of azimuthal traveling waves that are localized inside of the vortices leaving the inflow and outflow boundaries stationary. In addition to these patterns we have also observed low amplitude time dependent motion inside of the vortices. The first half of this chapter presents an experimental study of these patterns.

In the counter-rotating case for $\eta = .88$ and $R_o \leq 100$ the primary bifurcation is to a time dependent state. The pattern which forms is not axisymmetric but helical, i.e., the pattern is a spiral vortex. This pattern is spatiotemporally periodic and, because of the spiral symmetry, appears to be traveling both axially and azimuthally in the laboratory frame. At large $R_o$ this pattern persists for only a small interval of $R_i$ before a subcritical bifurcation to a helical turbulent pattern occurs. The helical turbulent pattern, called spiral turbulence (SPT), has an average or large scale spiral symmetry and an average time dependence. The SPT pattern consists of alternating laminar and turbulent regions coexisting to form a pattern similar in appearance to a barber pole. Spiral turbulence, like the spiral vortex, appears to be traveling both azimuthally and axially in the laboratory frame. This pattern is extensively investigated in this chapter.

2.2 Apparatus

Our Taylor-Couette apparatus, shown schematically in Figure 5, consists of two cylinders, each of which is driven by Compumotor stepper motors (model M83-93). The inner cylinder is made of black Delrin plastic and has a radius
Figure 4

Shown above is the Rayleigh stability line which a goes as $R_t = R_o/\eta^2$. Inviscid Couette flow is stable to axisymmetric disturbances below this line and unstable above this line in the shaded region.
Figure 5
The Taylor-Couette apparatus.
of $r_i = 5.262\text{cm}$, while the outer cylinder is made of Plexiglas and has a radius $r_o = 5.965\text{cm}$, giving a radius ratio $\eta = 0.882$ and gap $d = r_o - r_i = 0.703\text{cm}$. The inner and outer cylinder radii are known to within ±0.002 inches. Two stationary supports hold the apparatus at the top and bottom and prevent the cylinders from vibrating or moving relative to the laboratory frame. The apparatus is connected at both ends to the stationary supports by means of ball bearings at each end which are attached to two end caps. These end caps are attached to the outer cylinder allowing the outer cylinder to rotate. The inner cylinder is attached to two long shafts at each end which protrude through the end caps. A hole in one of the inner cylinder shafts allows fluid to pass in and out of the system while the other shaft is used to drive the inner cylinder. The inner cylinder shafts and the outer cylinder end caps are connected by ball bearings. These two ball bearings allow the inner cylinder to rotate independently from the outer cylinder while ensuring that they remain coaxial. The fluid is prevented from leaking out of the system by placing shaft seals between each shaft and end cap. Coaxial with the two cylinders is an independently rotating table also controlled by a Compumotor stepper motor (model M106-178). The rotating table makes it possible to make observations and measurements in the rotating frame of reference. Electrical connections to instruments is provided by 25 coaxial cables connected through brushes to the laboratory frame of reference. The Compumotor stepper motors are controlled through Compumotor Indexers (models 172 and 2100) which allow the cylinder speeds to be either manually or automatically changed. The rotation speeds are precise to .001 Hz and may be changed or ramped very slowly through computer control. The temperature of the working fluid is controlled by operating the apparatus in a temperature
controlled room. This temperature control is achieved by using a Liebert air conditioner which keeps the fluid temperature constant to within 0.1°C during a run. The fractional error in the Reynolds number, $R_{t,o}$, is

$$\frac{\delta R_{t,o}}{R_{t,o}} = \sqrt{\left(\frac{\delta d}{d}\right)^2 + \left(\frac{\delta f_{t,o}}{f_{t,o}}\right)^2 + \left(\frac{\delta \mu}{\mu}\right)^2 + \left(\frac{\delta r_{t,o}}{r_{t,o}}\right)^2}.$$  \hspace{1cm} (2.1)

Then with the uncertainties given above $\frac{\delta R_{t,o}}{R_{t,o}} \approx 1\%$ for moderate to large values of $R_{t,o}$ where $\frac{\delta d}{d} = 0.01$ is the largest source of uncertainty. At small $R_{t,o}$ the contribution from $\frac{\delta f_{t,o}}{f_{t,o}}$ dominates the uncertainty, e.g., at $R_o = 100$, $\frac{\delta R_o}{R_o} \approx 7\%$.

In order to maintain consistent end conditions the working fluid region (the region of experimental interest) is bounded by two Teflon collars which are attached to the outer cylinder by rubber O-rings. There is a small gap of $\leq 0.4\text{mm}$ between the inner cylinder and the Teflon rings which allows fluid to enter and leave the working region. The aspect ratio, $\Gamma$, is changed by moving the Teflon rings. Our apparatus has a maximum aspect ratio of $\Gamma = 73$. The aspect ratio may also be continuously changed by slowly pumping fluid in or out of the system with a peristaltic pump. This only works when the axis of the cylinders is vertical and requires that the upper end condition (fluid-air interface) be different from the lower end condition (fluid-solid interface). To visualize the flow we have used Kalliroscope AQ1000 as a flow visualization agent. Kalliroscope consists of an aqueous suspension of $\approx 30 \times 6 \times 0.07\mu\text{m}$ crystalline platelets$^{29}$. Savas$^{30}$ has made a quantitative analysis of reflective ellipsoidal particles suspended in a viscous fluid. His work shows that the particles tend to align along the stream surfaces of the flow while the finite thickness of the particles causes them to rotate and flip. The relative thinness of the Kalliroscope platelets diminishes this tendency and the platelets spend most of their time aligned along the stream surfaces. In
Savas's theory the incident light is specularly reflected from individual platelets and the large scale reflection from the stream surfaces of the flow is diffuse because the platelets fluctuate about their mean orientations, making the stream surfaces the equivalent of a rough surface. In addition the incident light that enters the flow is attenuated by the platelet's reflections so that only a collective reflection from the stream surfaces nearest to the light source can be seen. Although individual Kalliroscope flakes may not specularly reflect incident light the average diffuse reflections do seem to be described by Savas's theory. Couette flow looks like a uniform cylinder because the stream surfaces are cylinders that are coaxial with the inner and outer cylinders. Radial flow is indicated by a lack of reflection revealing the black inner cylinder. The vortices in TVF look like donuts because their stream surfaces are approximately tori with their axes perpendicular to the cylinders axes. The centers of the vortices appear white because the stream surfaces are primarily parallel to the cylinder wall in the $(\theta, z)$ plane. The inflow and outflow boundaries appear dark because the stream surfaces are primarily in the $(r, \theta)$ plane. Because the stream surfaces in the center of the vortices tend to tilt radially outward near an outflow boundary and radially inward near an inflow boundary there is a difference in the reflected brightness between adjacent Taylor vortices (see Figure 2). We have found that a high degree of contrast is achieved when the observer looks radially inward at the cylinders and the light shines on the flow from nearly an axial direction. Schwarz has done numerical simulations based on the theory of Savas. He found that flakes which initially have random orientations will, when subjected to a shearing velocity field such as $\frac{dV_r}{dz}$, develop substantial alignment in the stream planes in a time of order $(\frac{dV_r}{dz})^{-1}$. The flakes respond quickly to the change in the local velocity
fields especially in regions of strong shear. One example is the turbulent to laminar transition in the turbulent spiral pattern. In this example the flakes are, to a large extent, randomly oriented in the turbulent region and then at the trailing edge of the spiral suddenly subjected to the shearing velocity field of the laminar flow which we approximate to be the Couette flow, i.e. \( V_\phi = Ar + B/r \).

Taking \( B = 0 \) we find that the flakes align themselves in the cylindrical stream surfaces of Couette flow at \( R_o = -3000 \) and \( R_i = 800 \) in a time of order \( 1/40 \) seconds. Because a turbulent flow is characterized by strong velocity fluctuations on a length small scale the local shearing velocity fields are very large and the flakes respond very quickly. Thus a turbulent flow is characterized by a rapid variation in reflected light intensity due to the small scale velocity fluctuations which quickly change the orientations of the stream surfaces.

2.3 Corotating Cylinders: Bifurcations From Taylor Vortices

2.3.1 Introduction

As mentioned above the Taylor-Couette system consists of two independently rotating concentric cylinders with a fluid-filled gap. The simplicity and symmetry of the geometry and the richness of the observed flow behavior have made it a popular system for studying nonequilibrium transitions leading to spatiotemporal turbulence\(^{33,28}\). However, much of the behavior remains poorly understood, particularly when the outer cylinder rotates. Previous studies\(^{34,27,28}\) have revealed differing sequences of flow transitions in the corotating and counter-rotating cases. For infinitely long cylinders the base flow in either case is the well known Couette flow with an azimuthal velocity that varies with radius.
as \( V(r) = Ar + B/r \). Neglecting end effects, the primary instability of the base flow, in the corotating case, results in an axially uniform time independent system of Taylor vortices. As the speeds of the inner and outer cylinders are varied the Taylor vortices become unstable to various types of azimuthal traveling waves. These traveling waves are the WVF, WIB, WOB, and TWI\textsuperscript{27,28}. These flows and their interactions have been investigated numerically\textsuperscript{7,35,31} and analytically\textsuperscript{36,37,38}. We restrict ourselves to these secondary, and some tertiary, bifurcations, since the flows beyond these become considerably more complicated to experimentally characterize and are presently less accessible to theory.

It is known that the axial wavelength of the TVF (defined as the axial extent of a pair of Taylor vortices) has a strong influence on the nature and location in parameter space of the secondary bifurcations\textsuperscript{28,39}. We have pursued this observation by systematically varying the wavelength of the Taylor vortices above and below the critical wavelength. The critical size cells, which have an approximately square cross-section, are obtained when the inner cylinder is slowly ramped from Couette flow to somewhat above the Rayleigh stability line. At the critical wavelength (approximately twice the gap) WVF, WIB, WOB and TWI are all present as secondary bifurcations. For low outer cylinder speeds WVF always bifurcates from TVF although the axial wavelength does affect the onset values and the outer cylinder rotation stabilizes TVF against WVF. For large outer cylinder speeds there is a long wavelength preference for WIB and TWI, and a short wavelength preference for WOB. At the smallest wavelength surveyed we have observed a WVF with only one azimuthal wave \((m=1)\). In addition we have observed weak time dependent activity in the vortices prior to the onset of WVF, WIB, WOB, and TWI at each wavelength surveyed.
We will present the results of our experimental study of these bifurcations. We will first describe our system, data acquisition techniques and procedures. This will be followed by a discussion of the flow regimes observed and how they depend on the axial wavelength. We will end with a comparison of our results with present theory.

2.3.2 Description of the System

As mentioned above our experimental apparatus consists of two independently rotating concentric cylinders. The present study was done with the aspect ratio held constant at $\Gamma = \frac{l}{d} = 30$, where $l$ is the axial length of the working region. The aspect ratio is known within $\approx 1.0\%$. The working region is bounded above and below by two Teflon rings which are attached to the outer cylinder. This ensures, via Ekman pumping, an inflow boundary at both the top and bottom boundary.

A PDP-11/73 computer ramps the speed of one of the motors while the other motor's speed is fixed. The inner cylinder is directly linked to one motor, while the outer cylinder is linked to another motor by means of two sprockets and a timing belt which reduces its speed by a factor of 4. The Compumotors have a resolution of $\pm .001 \text{ Hz}$. We use the two control parameters $R_i$ (inner cylinder Reynolds number) and $R_o$ (outer cylinder Reynolds number) as defined in section 1.2. We also specify the local dimensionless axial wavelength, using $d$ as our length scale, as $\lambda = \lambda^*/d$ where $\lambda^*$ is the unscaled size of a particular vortex pair. This may or may not be identical to the average over all the vortices $\bar{\lambda} = 2\Gamma/N$, where $N$ is the number of vortices in the system.

We used distilled water for our working fluid together with a 1% mixture of Kalliroscope AQ1000 for flow visualization. The distilled water is boiled prior to
mixing with the Kalliroscope. The boiling removes dissolved air that may cause the flakes to clump. The fluid mixture lasts 2-3 days before it deteriorates. The system is cleaned by flushing it with a dilute solution of NaOH, which dissolves the platelets. The NaOH is then flushed out with multiple rinses of distilled water. The presence of Kalliroscope may lead to gravity induced Taylor vortex nonuniformities.40 We have checked our results by slowly ramping the system with the rotation axis horizontal in order to ensure that gravity does not effect the results.

2.3.3 Data Acquisition Techniques

The onset of instabilities in the visualized flow could sometimes be observed in real time. However, due to the low ramping rates necessary, time lapse video recordings were typically used to assist in the determination of these instability points. The video recorder (Panasonic model AG-6010) was usually set for a time interval between frames of 0.4 seconds. This allowed the recording of 24 hours of ramping on one 2 hour video cassette. Since for the largest ramping rates used velocity changes were made once a minute, and then in increments of only .00025 Hz, the time resolution of the recorder was sufficient to obtain accurate onset values.

To monitor the wavelength of the Taylor vortices we used a linear 1024 pixel CCD camera. The line of pixels was oriented parallel to the cylinder axis and a telephoto lens formed an image of the visualized flow on the pixels. With our system size the resolution of the CCD camera was 0.2mm. The signals from all 1024 pixels were digitized with a 12 bit A/D converter and this frame was then sent to our PDP 11/73 computer. The computer averages 50 randomly
chosen frames, thus removing the effects of vortex boundary waves, and stores the result. Following an appropriate waiting time the cylinder speed is changed and the process repeated.

After either $R_i$ or $R_o$ has been ramped to the desired final value each set of averaged frames is analyzed. The positions of the inflow and outflow boundaries are found by least squares fitting of parabolas to the intensity minima. We know that adjacent to the two end collars are the inflow boundaries of the Ekman pumped cells. By locating the first minimum away from each collar the Ekman pumped cell sizes are determined. The Ekman wavelength is the axial length of both Ekman pumped cells. Next, the axial distance between every other minimum is found, which is the axial distance between outflow boundaries and thus also the local axial wavelength. The local wavelengths in the bulk are then averaged (the bulk excludes the vortex pairs immediately adjacent to the Ekman pumped cells since they are sometimes influenced by end effects). The wavelengths in the bulk and the Ekman wavelengths are calculated for each averaged frame and displayed as a function of inner or outer cylinder Reynolds number. We also used the CCD camera to record the light reflectance of the patterns along the axes of the cylinders as a function of time to make space-time diagrams. This is accomplished by recording a frame (axial position vs intensity) on magnetic tape every 0.14 seconds. The data is then processed and displayed in a two dimensional format with the intensity and time coordinates sharing an axis. The peak to peak spread of the intensity values are scaled to be small compared to the time scale so that many individual frames will fit on the diagram. The space-time diagram is then produced by drawing each individual frame at the time the individual frame was recorded. The periodic axial position vs intensity plots evolve in time as the phase of the periodic pattern changes.
A thin cross section of the flow in the gap was visualized by shining a high intensity white light planar sheet of light into the gap. The light sheet was produced by either placing a slide of a thin slit into a slide projector or by passing high intensity white light through a thin slit cut into a screen which was placed next to the outer cylinder. Usually the Kalliroscope in a plane parallel to the cylinder axis was illuminated. This technique effectively visualizes the interior of the vortices and was used to determine the onset of weak time dependent activity in the vortices.

We used a single point reflectance technique and a rotating table to determine the frequencies and azimuthal wave numbers of the WIB, WOB and WVF\textsuperscript{41} at selected parameter values. Spectra of the flow were obtained by shining monochromatic laser light into the flow and collecting light reflected from the Kalliroscope with a photodiode. The signal from the photodiode is digitized and Fourier transformed to obtain a spectra. This apparatus was mounted on a rotating table so that spectra of the time dependent flows could be taken in a rotating frame of reference which is coaxial with the corotating cylinders. The slope of a plot of azimuthal wave frequency vs table frequency is the azimuthal wave number while the x-intercept is proportional to the wave speed\textsuperscript{41}.

2.3.4 Procedure

The initial step in any run is the establishment of the desired axial wavelength. The wavelength is varied by changing the number of vortex pairs for a fixed $\Gamma$. In general terms this is accomplished by slowly rotating the outer cylinder (keeping $R_o \leq 50$) and then quickly ramping $R_i$ above the onset of TVF. This produces dislocations in the vortices that can quickly be removed by increasing
$R_i$ to $\approx 10000$ where the vortices are highly turbulent. This highly turbulent state also quickly makes the axial wavelength of the vortices more uniform. Depending on the exact protocol followed, different numbers of vortices will result. The procedure is repeated until the desired number of vortices is obtained. The system is then quickly ramped in parameter space into a WVF state where the vortices are allowed to reach a more uniform axial wavelength. After some time the system is ramped to the initial point of the run and allowed to relax to a uniform TVF state. A uniform initial state is generally achieved within 30 minutes.

To find the critical values of $R_i$ and $R_o$ for bifurcations from TVF to WVF, WOB, WIB or TWI we tried both increasing $R_i$ with fixed $R_o$ and increasing $R_o$ for fixed $R_i$. The transitions from TVF to WVF were only slightly affected by the different ramping procedures (shifting perhaps $\approx 4\%$, compared with a run to run repeatability of $\approx 3\%$). The transitions from TVF to WIB, WOB or TWI were more sensitive to the ramping procedure. The Taylor vortices tended to remain more uniform at higher $R_o$ if the fixed $R_i$ and variable $R_o$ ramping procedure was used. This is probably due to the larger end effects generated by Ekman pumping at higher $R_o$. Nonuniformities in the TVF wavelength causes some ambiguity in the onset values of instabilities since different size Taylor vortices have different onset values. We note, however, that for a run at a given $N$ in the range of $R_i$ and $R_o$ surveyed the variation in local wavelength $\Delta \lambda$ is smaller than the change in average wavelength $\Delta \bar{\lambda}$ between runs with different $N$ ($\Delta \lambda \approx .1$ for $N = 26$ while $\Delta \bar{\lambda} = .19$ between $N = 24$ and $N = 26$, at $N = 32$ $\Delta \lambda \approx .05$ while $\Delta \bar{\lambda} = .12$ between $N = 30$ and $N = 32$). Although there is some uncertainty in the axial wavelength of the vortices at a given $N$, we argue that since $\Delta \lambda \leq \Delta \bar{\lambda}$ we are justified in plotting the flow regime diagrams for fixed
\( \Gamma \) and \( N \) because the parameter spaces at different \( N \) do not overlap. A more precise procedure would involve continuously varying \( \Gamma \) during a run to maintain a constant \( \lambda \), but this has not been attempted.

Baxter and Andereck\(^{39} \) found that in the corotating case the ramping rate had little effect on the onset values if the dimensionless ramping\(^{42} \) rate \( a \), defined as

\[
a = \left( \frac{dR_{i,o}}{dt} \right) \left( \frac{L(r_o - r_i)}{v} \right),
\]

was less than 20. We have kept to this protocol to insure quasistatic conditions and consistency from run to run. Park, Crawford and Donnelly\(^{42} \) found that, for the transition from Couette flow to TVF with a stationary outer cylinder, quasistatic conditions were achieved for \( a \leq 10 \).

### 2.3.5 Results

The main results of this work are shown in Figures 7, 8, and 10, the flow regime diagrams for the \( N=26, N=28 \) and \( N=32 \) cases. Figures 6 and 9 are subsets of previous surveys\(^{28,39} \). These diagrams show the critical Reynolds numbers for bifurcation from Taylor vortices for a range of \( \lambda \) from 1.88 to 2.50. It is apparent that the axial wavelength strongly affects the secondary flows present. WVF is present at lower \( R_o \) (\( R_o \leq \approx 450 \)) for all wavelengths surveyed. One tendency to note in Figures 6-10 is that \( R_o \) suppresses the onset of WVF and that the greater \( R_o \) is the greater the suppression of WVF onset is with respect to \( R_i \). Another tendency to note is that the TVF to WVF transition line increases in slope as the average axial wavelength decreases. In other words, compared to the larger wavelength vortices, the smaller wavelength vortices at a given \( R_o \) suppress the onset of WVF to higher \( R_i \). The most obvious differences
Figure 6
The flow regime diagram for N=24 and $\Gamma = 30$ ($\bar{\lambda} = 2.5$). Shown are the critical $R_o$ and $R_i$ for transitions to TVF, WVF, and TWI (+). The Couette flow to TVF transition follows the Rayleigh line (straight line) closely.
Figure 6
Figure 7

The flow regime diagram for $N=26$ and $\Gamma = 30$ ($\bar{\lambda} = 2.31$). Shown are the critical $R_o$ and $R_i$ for transitions to the weak time dependent activity (.), WVF, TWI, WIB, TWI & WIB, WVL, (+) and the Rayleigh line (straight line).
Figure 7
Figure 8
The flow regime diagram for $N=28$ and $\Gamma = 30$ ($\bar{\lambda} = 2.14$). Shown are the critical $R_o$ and $R_i$ for transitions to the weak time dependent activity (.), WVF, TWI, WIB, WVL, (+) and the Rayleigh line (straight line).
Figure 9

The flow regime diagram for $N=30$ and $\Gamma = 30$ ($\bar{\lambda} = 2.00$). Shown are the critical $R_o$ and $R_i$ for transitions to the weak time dependent activity (.), WVF, TWI, WIB, WOB, WVL, (+) and the Rayleigh line (straight line).
Figure 9
Figure 10

The flow regime diagram for $N=32$ and $\Gamma = 30$ ($\bar{\lambda} = 1.88$). Shown are the critical $R_o$ and $R_i$ for transitions to the weak time dependent activity ($\cdot$), WVF, the $m = 1$ WVF, WOB, (+) and the Rayleigh line (straight line).
Figure 10
Shown are the following photographs: (a) TVF at $R_o = 400$, $R_i = 900$, and $\bar{x} = 2.50$; (b) TVF at $R_o = 500$, $R_i = 1000$, and $\bar{x} = 2.31$; (c) TVF at $R_o = 500$, $R_i = 1100$, and $\bar{x} = 2.14$; (d) TVF at $R_o = 400$, $R_i = 1450$, and $\bar{x} = 2.00$; (e) TVF at $R_o = 350$, $R_i = 1000$, and $\bar{x} = 1.88$. 
Figure 12

Shown are the following photographs: (a) WVF at $R_o = 250$, $R_i = 900$, and $\lambda = 2.50$; (b) WVF at $R_o = 300$, $R_i = 1000$, and $\lambda = 2.31$; (c) WVF at $R_o = 300$, $R_i = 1100$, and $\lambda = 2.14$; (d) WVF at $R_o = 300$, $R_i = 1450$, and $\lambda = 2.00$; (e) WVF at $R_o = 200$, $R_i = 900$, and $\lambda = 1.88$. (f) m=1 WVF at $R_o = 350$, $R_i = 1100$, and $\lambda = 1.88$. 
Shown are the following photographs: (a) TWI at $R_o = 550$, $R_i = 900$, and $\bar{\lambda} = 2.50$; (b) TWI at $R_o = 650$, $R_i = 1000$, and $\bar{\lambda} = 2.31$; (c) TWI + WIB at $R_o = 650$, $R_i = 1200$ and $\bar{\lambda} = 2.31$; (d) TWI at $R_o = 750$, $R_i = 1050$, and $\bar{\lambda} = 2.14$; (e) TWI at $R_o = 650$, $R_i = 1000$, and $\bar{\lambda} = 2.00$. 
Figure 14

Shown are the following photographs: (a) WIB at $R_o = 650$, $R_i = 1450$, and $\bar{\lambda} = 2.31$; (b) WIB at $R_o = 525$, $R_i = 1450$, and $\bar{\lambda} = 2.14$; (c) WVL at $R_o = 600$, $R_i = 1450$, and $\bar{\lambda} = 2.14$; (d) WIB at $R_o = 450$, $R_i = 1450$, and $\bar{\lambda} = 2.00$; (e) WVL at $R_o = 500$, $R_i = 1450$, and $\bar{\lambda} = 2.00$; (f) WOB at $R_o = 600$, $R_i = 1100$, and $\bar{\lambda} = 1.88$. 
for the various wavelengths occur in the fast outer cylinder case \((R_o \gtrsim 450)\) where we observed TWI, WIB, WOB and combinations of these. As can be seen in Figure 10 transitions to WOB and WVF occur at \(N=32\) including a transition to a novel \(m=1\) WVF. It is interesting to note that at \(N=32\) the WIB are absent. WIB, WOB, and TWI are all present at \(N=30\) as can be seen in Figure 9. The \(N=26\) and \(N=28\) diagrams shown in Figures 7 and 8 show that WIB are present at larger \(R_i\) and TWI are present at lower \(R_i\). Waves form on the outflow boundaries only after WIB have formed and this combination is called Wavelets (WVL)\(^{28,36}\). In Figure 6 for \(N=24\), the largest wavelength surveyed, TWI and TWI in combination with WIB\(^{39}\) (not shown) occur while WOB are absent. A comparison of the diagrams shows that increasing \(\lambda\) destabilizes TVF to TWI while stabilizing TVF against WIB and WOB. In summary, vortex pairs larger than the critical axial wavelength prefer WIB and TWI while vortex pairs smaller than the critical axial wavelength prefer WOB.

These diagrams also show transitions prior to the onset of WIB, WOB and TWI. At these points weak time dependence was observed inside of the vortices using light sheet visualization. Prior to this transition dark lines inside of the vortices can be observed by illuminating the flow from an almost axial direction while observing the reflected light from a radial direction. These dark lines can be seen in the photographs of TVF shown in Figure 11 and are present in WVF, WIB, WOB, and TWI at all the wavelengths surveyed as can be seen in Figures 11-14. Corresponding to the weak time dependence seen with the light sheet visualization is the weak motion of these dark lines. The space-time plot in Figure 15 shows this weak time dependence.

Our method for varying, and maintaining the axial wavelength is limited by end effects. In particular the Ekman cells, and the vortex pairs adjacent
Figure 15

Shown is the space-time diagram of the weak time dependent activity at $\lambda = 2.00$, $R_o = 500$, and $R_i = 1000$. Two inflow and three outflow boundaries are shown (the large intensity minima). Inside of these vortices are weaker dark lines that exhibit time dependence.
to them, increase in size as $R_o$ and $R_i$ are increased. Since the aspect ratio is held constant the vortices in the bulk must decrease in wavelength. We found that for large $R_o$ the wavelength of the vortices in the bulk tended to stay more uniform if $R_i$ was held constant as $R_o$ was ramped. We show in Figure 16 and 17 the variation of the Ekman pumped cells as $R_o$ is ramped for $N=26$ and $N=28$. In Figures 18 and 19 we show some examples of the average axial wavelength of the bulk at $N=26$ and $N=28$. These figures show that the process of fixing the aspect ratio and varying the number of vortices typically breaks down at some point because the average axial wavelength of the bulk will change too much. We reemphasize that the average axial wavelength of the bulk at the transition points are well separated for large $R_o$ for each flow regime diagram. This shows that our procedure for varying the wavelength was still meaningful for the parameter values necessary for observing bifurcation from TVF. Figures 20 and 21 show the average wavelength and Ekman wavelength together with the individual wavelengths of the vortices in two regions of Figure 10. A comparison of these two figures shows that the Ekman cell increases with $R_o$ while the average wavelength decreases. These axially compressed vortices will break down to a $N=30$ or $N=28$ state, sufficiently far above onset, by developing dislocations which result in annihilation of vortex pairs. We have not attempted to survey the $N=34$ parameter region because of the difficulty in maintaining this state. Near the Rayleigh line for large $R_o$ ($R_o \approx 900$) at $N=32$ as $R_i$ is increased the thickness of each Taylor vortex becomes distorted azimuthally such that, in the laboratory frame, the axial wavelength of each vortex appears to alternately increase and then decrease. The largest distortions occur in the center of the working region with the vortices near the end cell almost unaffected. These distortions appear
to travel axially and azimuthally with the thick and thin parts of each vortex collectively forming a spiral. This spiral pattern becomes more apparent when, as $R_i$ is increased, the large scale mode increases in amplitude and TWI form on the thick parts of the vortices. This large scale mode with a spiral of TWI is shown in the space-time diagram in Figure 22 (our CCD camera is not fast enough to resolve the TWI, WIB, or WOB in the laboratory frame). As $R_i$ is increased WVL form on the thin parts of the vortices and a spiral of WVL and TWI forms. The state shown in Figure 23 is transient and can last for several hours before a dislocation forms a which point one or two vortex pairs are destroyed leaving the system in $N = 30$ or $N = 28$ state.

As mentioned above azimuthal wave numbers and wave speeds were measured at selected parameter values by taking several spectra in a rotating frame of reference at different table rotation speeds. Figure 24 shows that the central branch of the $N=32$ diagram is an $m=1$ WVF. We have also measured the azimuthal wave number and wave speed of the WIB and WOB at different $\bar{\lambda}$ but similar $R_o$ and $R_i$ (the $R_o$ and $R_i$ values were chosen such that the parameters would be as close as possible between wavelengths in order to compare azimuthal wave numbers and wave speeds at different axial wavelengths). Table 1 and Table 2 show the results of these measurements. The WOB do not change in $m$ as $\bar{\lambda}$ changes. The $m$ of WIB increases by one each time $N$ is increased by two. These measurements have been repeated with the same results.

2.3.6 Discussion of Results
Table 1
Table of experimental measurements of the azimuthal wave numbers \( (m_e) \) vs. the theoretical azimuthal wave numbers \( (m_t) \) at various values of \( N, R_o, \) and \( R_i \) for the WIB pattern.

<table>
<thead>
<tr>
<th>( m_e )</th>
<th>( m_t )</th>
<th>( N )</th>
<th>( R_o )</th>
<th>( R_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12.8</td>
<td>30</td>
<td>450</td>
<td>1450</td>
</tr>
<tr>
<td>11</td>
<td>12.1</td>
<td>28</td>
<td>514</td>
<td>1450</td>
</tr>
<tr>
<td>10</td>
<td>11.1</td>
<td>26</td>
<td>607</td>
<td>1450</td>
</tr>
</tbody>
</table>

Table 2
Table of experimental measurements of the azimuthal wave numbers \( (m_e) \) vs. the theoretical azimuthal wave numbers \( (m_t) \) at various values of \( N, R_o, \) and \( R_i \) for the WOB pattern.

<table>
<thead>
<tr>
<th>( m_e )</th>
<th>( m_t )</th>
<th>( N )</th>
<th>( R_o )</th>
<th>( R_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>14.5</td>
<td>34</td>
<td>600</td>
<td>1032</td>
</tr>
<tr>
<td>10</td>
<td>13.7</td>
<td>32</td>
<td>600</td>
<td>1080</td>
</tr>
<tr>
<td>10</td>
<td>12.8</td>
<td>30</td>
<td>600</td>
<td>1133</td>
</tr>
</tbody>
</table>

Several mechanisms have been suggested to explain the origin of the WVF instability at \( R_o = 0 \). One scenario notes that the Taylor vortices generate jets in the azimuthal flow near the inflow and outflow boundaries in the \((\phi, z)\) plane. In light of this fact it is suggested\(^{43,44,35}\) that the WVF is caused by an Orr-Sommerfeld type of shear instability associated with the inflection points in the
The Ekman wavelength (the axial extent of the two end cells adjacent to the top and bottom collars) changes as $R_o$ is ramped. This data was taken for $N=26$ and $\Gamma = 30$ ($\lambda = 2.31$) at $R_i = 1000$ (solid line), $R_i = 1100$ (+), $R_i = 1200$ (*), $R_i = 1300$ (o), and $R_i = 1500$ (x).
Ekman Wavelength

Figure 16

R°

200

300

400

500

800
The Ekman wavelength (the axial extent of the two end cells adjacent to the top and bottom collars) changes as $R\alpha$ is ramped. This data was taken for $N=28$ and $\Gamma = 30$ ($\bar{\lambda} = 2.14$) at $R_i = 1100$ (+), $R_i = 1200$ (*), $R_i = 1300$ (o), and $R_i = 1500$ (x).
Figure 17

Ekman Wavelength

$R_\circ$
Figure 18

The average wavelength (excluding the cells near the top and bottom collars) changes as $R_o$ is ramped. This data was taken for $N=26$ and $\Gamma = 30$ ($\bar{\lambda} = 2.31$) at $R_i = 1000$ (solid line), $R_i = 1100$ (+), $R_i = 1200$ (*), $R_i = 1300$ (o), and $R_i = 1500$ (x).
Figure 18
The average wavelength (excluding the cells near the top and bottom collars) changes as $R_0$ is ramped. This data was taken for $N=28$ and $\Gamma = 30$ ($\bar{\lambda} = 2.14$) at $R_i = 1100$ (+), $R_i = 1200$ (*), $R_i = 1300$ (o), and $R_i = 1500$ (x).
Average Wavelength
Figure 20

Shown are the average wavelength (solid line), the wavelength of individual vortex pairs (+), and the Ekman wavelength (*) as $R_o$ is ramped. This data was taken for $N=32$ and $\Gamma = 30 \ (\bar{\lambda} = 1.88)$ at $R_i = 1100$. 
Figure 21

Shown are the average wavelength (solid line), the wavelength of individual vortex pairs (+), and the Ekman wavelength (*) as $R_i$ is ramped. This data was taken for $N=32$ and $\Gamma = 30 \ (\bar{\lambda} = 1.88)$ at $R_\circ = 600$. 
Figure 21

Wavelength vs. $R_i$
Figure 22

Shown is the space-time diagram of the vortex thickness variations with TWI at the largest axial part of each vortex. The thick and thin parts of the vortices form a spiral. This data was taken for $N=32$ and $\Gamma = 30$ ($\lambda = 1.88$) at $R_o = 950$ and $R_i = 1275$. 
Figure 22
Figure 23

Shown is the space-time diagram of the vortex thickness variations with TWI at the largest axial part of each vortex and WVL at the thinnest part of each vortex. The thick and thin parts of the vortices form a spiral. This data was taken for $N=32$ and $\Gamma = 30$ ($\lambda = 1.88$) at $R_o = 950$ and $R_i = 1350$. 
Figure 24

Shown is the frequency of the WVF as seen in the rotating frame of reference at various table frequencies. This data was taken for $N=32$ and $\Gamma = 30$ ($\bar{\lambda} = 1.88$) at $R_o = 350$ and $R_i = 1250$. The slope of this plot is 1 showing that there is one azimuthal wave.
Figure 24
axial variation of the azimuthal velocity (i.e., points where $\frac{\partial^2 V}{\partial z^2} = 0$). Jones examined the stability of TVF for wide gaps ($\eta < 0.8$) in order to understand a subharmonic wavy mode observed at $\eta \approx 0.5$ and $R_o = 0$. This wavy mode, like WIB and WOB, is subharmonic with respect to the axial wavelength, i.e., the pattern repeats itself every $2\lambda$ with waves on adjacent outflow boundaries in antiphase. Jones found that for wide gaps the azimuthal outflow jet is strong while the azimuthal inflow jet is weak. This suggests that a shear instability was responsible for the destabilization of the TVF because the waves only appear at the outflow boundary where the azimuthal jets make a strong shear flow profile, i.e., the waves only appear where $\frac{\partial V_\phi(z)}{\partial z}$ is large. He does not explain why the waves are subharmonic with respect to the TVF axial wavelength. He also suggests that in small gap TVF, where both azimuthal jets are strong, this same mechanism destabilizes both the inflow and the outflow boundaries causing WVF. Another scenario, given by Marcus, argues that a centrifugal instability due to the strong radial outflow jets in TVF causes WVF, i.e. there is a local centrifugal instability in the Taylor vortices. This scenario argues that the radial outflow jet (as opposed to the azimuthal outflow jet) causes TVF to destabilize by locally violating the Rayleigh stability criterion, i.e. the angular momentum of the Taylor vortices decreases outward from some reference point. Marcus’s calculations of the WVF velocity field has shown that, for $\eta = 0.875$ and $R_o = 0$, the strong radial outflow boundaries that develop in TVF are diminished by WVF. That is, the system minimizes its total energy by reducing this radial outflow kinetic energy even though the energy in the azimuthal and axial velocity components increase. This increase is manifested as radial vorticity centered about the outflow boundary, just as would be expected from a centrifugal insta-
bility (centrifugal instabilities generally produce secondary flows with vorticity perpendicular to the vorticity of the primary flow, e.g. TVF or Dean rolls).

Since it is reasonable to assume that the same mechanism leading to WVF is still valid when both cylinders rotate, the fact that the WVF transition lines have slopes greater than the slope of the Rayleigh line in all five flow regime diagrams shows that corotation of the outer cylinder suppresses the mechanism which destabilizes TVF to WVF. If WVF were the result of a shear instability associated with the inflection points in $V_\phi(z)$ one would not expect that corotating the outer cylinder would tend to stabilize TVF against WVF. This is because the azimuthal jets result from the circulation in TVF and therefore the inflection points should appear after TVF has gained sufficient strength. At $R_o = 0$ when the TVF to WVF transition occurs the Taylor vortices are still relatively weak, i.e. they do not appear to have well-defined (high contrast) inflow and outflow boundaries. At $R_o > 0$ the Taylor vortices appear to have well-defined inflow and outflow boundaries before WVF appears. Taking the visualized inflow and outflow boundary contrast as a measure of the TVF circulation one would expect WVF to appear at much lower $R_i$ at $R_o > 0$. In fact the inflow and outflow boundaries appear well-defined at approximately the same $\delta R_i$ above the Rayleigh line. This suggests that the TVF to WVF should be approximately parallel to the Rayleigh line, contrary to the data in Figures 6-10. The data in Figure 6-10 show, as mentioned above, that increasing $\lambda$ destabilizes the TVF to WVF transition. At a given $R_o$ and $R_i$ a larger $\lambda$ should spread out the axial distance, $\delta z$, over which the variations in $V_\phi, \delta V_\phi$, occur decreasing $\frac{\partial V_\phi(z)}{\partial z}$. One would then expect the larger vortices to decrease $\frac{\partial V_\phi(z)}{\partial z}$ stabilizing the TVF against WVF rather then destabilizing them. The centrifugal instability scenario,
by contrast, gives a simpler qualitative understanding of the WVF transitions shown in Figures 6-10. The driving of the fluid by the outer cylinder which first stabilizes Couette flow by pumping angular momentum into the flow, via the viscous interaction, also redistributes the angular momentum in the vicinity of the outflow boundary stabilizing the Taylor vortices. This is because when the outer cylinder rotates the TVF onset, which occurs near the Rayleigh line, is at a much larger average angular frequency of the cylinders $\Omega = (\Omega_o + \Omega_i)/2$. In a frame of reference rotating at $\Omega$ a fluid particle traveling radially outward feels a Coriolis force pulling it in the azimuthal direction, thus weakening the radial outflow jet. A higher $R_o$ requires a higher $R_i$ to establish TVF and when $R_o$ and $R_i$ are larger the Coriolis force is also larger and the radial outflow jets become weaker, stabilizing the TVF against WVF.

This does not explain why at a given $R_o$ larger vortices become unstable to WVF at lower $R_i$. We may use a simple energy transport argument to see why this is so. At a given $R_o$ the energy input to the system by the inner cylinder ($E_i$) is $E_i \propto R_i^2$. If a significant proportion of this energy goes to the radial outflow jets then when there are fewer outflow jets the (i.e. lower $N$) then these jets must have a greater kinetic energy and therefore become unstable to WVF at a lower $R_i$. The data shows that at higher $R_o$ where the radial inflow and outflow boundaries are of comparable strength the larger vortices stabilize TVF against WVF to an even greater extent. To understand the physical reason for this we must examine the forces that are driving the flow. Before the onset of TVF the radial pressure gradient is $\frac{\partial P}{\partial r} = \frac{V_2}{r}$. This pressure gradient is the centrifugal force that accelerates the fluid particle in a circular path about the cylinder axis. Substituting the Couette flow solution we find

$$\frac{\partial P}{\partial r} \propto (\Omega_o - \Omega_i \eta^2)^2 r + (\Omega_o - \Omega_i)(\Omega_o - \Omega_i \eta^2) \frac{1}{r} + (\Omega_o - \Omega_i)^2 \frac{1}{r^3}, \quad 2.3$$
where \( \Omega_i \) and \( \Omega_o \) are the inner and outer cylinder angular frequencies respectively. The first term on the right dominates this centrifugal force except near the Rayleigh line where \( \Omega_o = \Omega_i \eta^2 \). At the Rayleigh line, shown in each of Figures 6-10, the pressure gradient becomes very small with only the third term on the right contributing. It is near this line, when the radial pressure gradient is smallest, that fluid particles near the inner cylinder begin to get pumped outward (via viscous interactions) by the inner cylinder and the secondary TVF appears. Above the Rayleigh line, the situation is complicated by the presence of the secondary TVF. The azimuthal velocity component, however, is still large with the azimuthal velocity most modified in the vicinity of the inflow and outflow boundaries where the azimuthal jets are present\(^{31,35}\). It is convenient to think of the radial pressure gradient as composed of two parts: a Couette-like part \( \frac{\partial p}{\partial r} \) and a TVF-like part, \( \frac{\partial p}{\partial r} \). The Couette-like part accelerates the particles about the cylinder axis and is of the form \( \frac{\partial p}{\partial r} = \frac{V^2}{r} \). The TVF-like part, \( \frac{\partial p}{\partial r} \), actively pushes the fluid inward at the inflow boundaries and pulls the fluid out at the outflow boundary. Ignoring viscosity and noting that TVF is axisymmetric the \( r \) and \( z \) components of the Navier-Stokes equations are

\[
\frac{V_r}{r} \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \tag{2.4}
\]

and

\[
\frac{V_r}{r} \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \tag{2.5}
\]

where \( \frac{\partial p}{\partial r} \) has canceled \( \frac{V^2}{r} \) in equation (2.5). The first two terms in equation (2.4) are the radial component of the advective acceleration in TVF which is driven by the TVF-like part of the radial pressure gradient. The centrifugal force which accelerates the fluid particles in a circular path (in the \((r,z)\) plane)
in the secondary TVF is a combination of the radial pressure gradient and the axial pressure gradient. The axial pressure gradient is most likely a Bernoulli effect due to the strong radial and azimuthal jets near the inflow and outflow boundaries. The centrifugal acceleration in the TVF is represented by the four advective terms on the left hand side of equations (2.4) and (2.5). \( V_r \frac{\partial V_z}{\partial r} \) is strongest at the radial jets and \( V_z \frac{\partial V_r}{\partial z} \) is strongest between the radial jets where the fluid moves axially. The cross terms \( V_r \frac{\partial V_z}{\partial r} \) and \( V_z \frac{\partial V_r}{\partial z} \) are the advective accelerations of the fluid particles near the corners of the Taylor vortices, i.e. where the fluid particles turn towards the radial jets or away from the radial jets. It is desirable to calculate

\[
\Phi = \frac{\partial L^2_r}{\partial \phi} + \frac{\partial L^2_z}{\partial \phi},
\]

where \( L_{\phi} \) is the \( \phi \) component of angular momentum, to see how the \((r, z)\) components of TVF violate the Rayleigh stability criterion, i.e. to see how it is that \( \Phi \leq 0 \). We ignore terms with \( V_z \) because the radial jets are the strongest part of the secondary TVF and the most likely to go unstable. We get

\[
\Phi = 2zV_r^2 + 2z^2V_r\left(\frac{\partial V_z}{\partial z} + \frac{\partial V_r}{\partial r}\right).
\]

The first term is always positive while the second term can be negative if \( \frac{\partial V_z}{\partial r} \) and \( \frac{\partial V_r}{\partial z} \) are negative at an outflow boundary and positive at an inflow boundary. In fact this is exactly what is observed in numerical simulations. As \( R_i \) is increased the streamlines become concentrated toward the outer cylinder near an outflow boundary and the streamlines become concentrated toward the inner cylinder near an inflow boundary as is shown schematically in Figure 25, which is based on the results of references 7, 31, and 34. The geometry of the axially
elongated vortices stretches these elliptically shaped streamlines increasing this tendency to concentrate near the inner or outer cylinder and thus tending to destabilizing the axially elongated vortices to WVF. Just as the radial pressure gradient (which is present before the onset of TVF) drives the secondary TVF, the axial pressure gradient (which is present before the WVF) drives the motion of the transverse azimuthal traveling waves. The concentrating of the streamlines near the cylinders also gives us an alternative explanation of why increasing \( R_o \) stabilizes TVF against WVF. This is because the driving of the outer cylinder entrains fluid into azimuthal motion near the outer cylinder via the viscous interaction. Since the centrifugally unstable region of the TVF is near the outer cylinder, the driving of the outer cylinder forces the fluid particles near the outer cylinder to move azimuthally rather than radially, making the streamlines less concentrated near the outer cylinder.

Although it appears that the centrifugal instability is a likely scenario for WVF, at higher \( R_o \) there are other types of azimuthal traveling waves that occur at lower \( R_i \) values. Figures 6-10 show that the outflow jets are destabilized by axially compressed vortices while the inflow jets are destabilized by axially elongated vortices. Marcus\(^4\) observed, after calculating the velocity fields of WIB and WOB, that WIB and WOB have vorticity concentrated near the radial jets suggesting that they, like WVF, are the result of a centrifugal instability. The unstable eigenmodes, however, become subharmonic with respect to the axial wavelength because at these higher \( R_o \) values the radial outflow jets no longer dominate the secondary TVF. The system can no longer minimize its total energy by reducing the radial outflow kinetic energy while increasing the kinetic energy in the azimuthal and axial velocity components. It can, however, alternately
Figure 25

Shown schematically are the projections of the streamlines in the axial plane based on references 7, 31, and 34. (a) the streamlines near onset. (b) the projections of the streamlines above onset where the streamlines become concentrated toward the outer cylinder near an outflow boundary and the streamlines become concentrated toward the inner cylinder near an inflow boundary.
increase the kinetic energy in one radial jet while decreasing the kinetic energy in its nearest neighbors. This gives the subharmonic WOB if the radial outflow boundaries become unstable and WIB if the radial inflow boundaries become unstable. The centrifugal instability scenario implies that one would expect the streamlines of axially compressed TVF to be the most concentrated at the corner toward the outer cylinder and near the outflow jet. Similarly the axially elongated vortices should have their streamlines most concentrated at the corner toward the inner cylinder near the inflow jet.

The large values of $V_\phi$ for this large $R_o$ region of parameter space makes WIB and WOB good candidates for the shear instability scenario as well. Nagata's calculations showed inflection points in the $r$ and $\phi$ averaged $V_\phi(z)$ of TVF for corotating cylinders, especially when $\Omega_i - \Omega_o$ is small. Jones proposed a simple model to describe the azimuthal velocity profile for $\Omega_o = 0$, which we adopt here. It consists of a periodic $V_\phi$ that varies with $z$ as a triangle wave. The stability of this approximate form of the azimuthal velocity profile can be calculated using Rayleigh's equation. Jones found that the maximum growth occurs at $k = \frac{0.8031}{\lambda}$. The Taylor vortices have $k \approx \frac{2m}{(r_i + r_o)} = \frac{2(1 - \eta)m}{d(1 + \eta)}$, where $m$ is, as before, the azimuthal wave number. Taking $\lambda \approx \lambda d$ we get $m \approx 0.427N$. Table 1 and Table 2 show the results which agree well with this model for WIB even though this model assumes that both the azimuthal inflow jet and the azimuthal outflow jet become unstable. As mentioned before, the shear instability scenario would lead one to expect the larger vortices to decrease $\frac{\partial V_\phi(z)}{\partial z}$ and stabilize Taylor vortices. Figures 6-10 show that the larger vortices stabilize TVF against WIB as $R_o$ is increased and that larger vortices stabilize TVF against WOB as $R_i$ is increased. The WOB, however, do not change azimuthal wave number
with $N$ suggesting that either they are not a shear instability, or that the model breaks down for axially compressed vortices. The insensitivity of the WOB $m$ to $N$ implies that the disturbance is localized near the outflow jet as would be expected in the centrifugal instability scenario. On the other hand the sensitivity of the WIB $m$ to $N$ may be because disturbances in axially elongated vortices are less localized then they are in axially compressed vortices. The WIB, which prefer axially elongated vortices, are sensitive to changes in $N$ because there is a larger region inside of the vortices that becomes centrifugally unstable. The negative result of the shear instability hypothesis for the WOB makes it likely that they are a result of a centrifugal instability while the WIB could be either.

Nagata\textsuperscript{7} numerically calculated the velocity fields in the corotating case. He found two subharmonic instabilities which may correspond to WIB and WOB. However, in order to simplify the calculation he neglected curvature effects and assumed that the angular velocities of the two cylinders were very close. Because of these simplifications he was unable to distinguish the inflow and outflow boundaries and the parameter values differed somewhat from WIB and WOB. His calculations also found another instability which is characterized by small vortices alternately growing and decaying inside of neighboring Taylor vortices. The weak motion inside of TVF that we plotted in the flow regime diagrams exhibit dark lines near the vortex cores which is indicative of radial flow and possibly small vortices. These dark lines did not grow and decay, however, and the parameter values of the weak motion did not correspond to the parameter values of Nagata's, evidently yet unobserved, mode. These dark lines occurred close to the onset of TVF before the weak time dependent activity began. This suggests that weakly nonlinear theory may be able to describe it. Davey\textsuperscript{10} calculated a spatially homogeneous amplitude equation in which $\lambda$ is assumed to
be some given value. This corresponds to our procedure for creating a $\lambda$. His calculations showed that very close to onset a first harmonic in $\lambda$ is generated from the interaction of the most unstable mode with itself and the mean flow. The space-time diagram in Figure 15 shows these dark lines which occur in the middle of the Taylor vortices. The results of Davey suggest that this is the first harmonic with respect to $\lambda$, the TVF wavelength. There are three dark lines in the vortices below the outflow boundaries. This suggests that there is a first and second harmonic in the pattern. A second harmonic, however, would require there to be three dark lines in every vortex whereas we do not see three dark lines in the vortices below the inflow boundaries. The shape of the streamlines and the light source being above the vortices which accounts for the alternating light and dark vortices may also obscure the visualization of the two additional dark lines in the brighter vortices below the inflow boundaries. These dark lines can be seen in the photographs of TVF in Figure 11. The fact that this activity doesn’t grow in strength like WIB, WOB, WVF, or TWI suggests that it is not an instability but rather the result of nonaxisymmetric perturbations exciting weakly damped modes inside of TVF. The spectra of this state reveals many peaks. This result may be due to the many stable linear modes that are excited by the perturbation. Nonaxisymmetric perturbations would have to be continuously created to sustain this weak motion. Since the gap is the largest source of uncertainties for these $R_o$ and $R_i$ values this weak time dependent activity is probably caused by large scale inhomogeneities in $d$ which continuously perturb the TVF. Similar, but not sustained, time dependent activity can be generated by impulsively changing the inner or outer cylinder frequency.

Figures 6-9 show that the TWI pattern bifurcates from TVF when $R_o$ is large and $R_i$ is close to the Rayleigh stability line. The figures also show that in the
same range of parameter values the larger Taylor vortices are destabilized to TWI at lower \( R_0 \) and \( R_i \). The range of \( R_0 \) and \( R_i \) where TWI form is characterized by a large average cylinder angular velocity \( \tilde{\Omega} = \frac{\Omega_0 + \Omega_i}{2} \) and a small \( \Delta \Omega = \Omega_0 - \Omega_i \). In a frame of reference that rotates about the cylinder axis at \( \tilde{\Omega} \) the characteristic velocity of the flow is \( d\Delta \Omega \). This suggests that it may be useful to examine the flow in a rotating frame of reference. The Navier-Stokes equations in a rotating frame of reference must include the fictitious forces\(^25\). They are

\[
\frac{\partial \mathbf{V}}{\partial t} + (\nabla \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} - 2\Omega \times \mathbf{V} - \Omega x (\Omega x r) - \frac{d\Omega}{dt} x r
\]

and

\[
\nabla \cdot \mathbf{V} = 0.
\]

\( \Omega \) is the angular velocity of the rotating frame of reference as seen in the inertial frame. The centrifugal force, \( \Omega x (\Omega x r) \), may be written\(^8\) as \( \nabla [\frac{1}{2}(\Omega x r)^2] \) and included in the pressure term, i.e. \( P = p - \frac{1}{2} \rho (\Omega x r)^2 \). Andereck et al.\(^27\) reported that when the TWI pattern forms the motion of the flow is strongest near the vortex core while the inflow and outflow boundaries are apparently undisturbed. Nagata\(^7\) found a set of modes that become unstable at parameter values corresponding to the parameter values of the TWI pattern observed by Andereck et al.\(^27,28\) The velocity field calculated by Nagata also reveals that the periodic motion is in the region of the vortex core. To look at the behavior of the flow in the region of the vortex core we transform to the rest frame of every other vortex core (we cannot transform to the rest frame of every vortex core because neighboring vortices circulate in opposite senses), i.e. to a frame which rotates at \( \Omega = \tilde{\Omega} + \Omega_i \). \( \Omega_i \) is the average angular velocity of the flow near every other vortex core (i.e. the flow in the inertial frame \((r', z')\) plane where we let the primes,
\(',\) denote the coordinates in the inertial frame). In other words \(\hat{\Omega}\) points in the (inertial frame) axial direction and \(\Omega_{t}\) points in the (inertial frame) azimuthal direction. Because both the inertial and rotating frames share the same origin, 
\[
\frac{d\Omega}{dt} \cdot x_r = |\Omega_{t}|^2 \dot{\tau} \cdot x_r = 0.
\]
The characteristic length in this frame of reference is \(d\) and the characteristic velocity is still \(d\Delta \Omega\). With these scales the advective term is \((V \cdot \nabla)V \sim d(\Delta \Omega)^2\) and the Coriolis force is \(2\Omega \times V \sim d\Delta \Omega \ddot{\Omega}\). The advective term can be ignored in comparison to the Coriolis force if \(\frac{\Delta \Omega}{\ddot{\Omega}} \ll 1\). As mentioned above this is exactly the condition present when the TWI pattern appears. When this condition is satisfied equation (2.8) can be written as
\[
\frac{\partial V}{\partial t} + 2\Omega \times V = -\frac{1}{\rho} \nabla P \tag{2.10}
\]
where we have ignored viscosity. If we take the curl of both sides to eliminate the pressure and take the direction of \(\Omega\) to be the \(z\) axis then equation (2.10) can be written as
\[
\frac{\partial \nabla \times V}{\partial t} = 2\Omega \frac{\partial V}{\partial z}. \tag{2.11}
\]
This equation has a plane wave solution \(V = A e^{k \cdot r - \omega t}\). From equation (2.9) we can see that these plane waves are transvers, i.e. \(k \cdot A = 0\). These waves are called inertial waves and are driven, in the rotating frame, by the Coriolis force. In addition these waves are circularly polarized, i.e. \(V_z = acos(\omega t - k \cdot r)\) and \(V_y = -asin(\omega t - k \cdot r)\) where \(a\) is the magnitude of the real part of the complex wave amplitude \(A\) and the \(z''\) and \(y''\) axes are taken to be perpendicular to \(k\) (not perpendicular to \(z\) unless \(k\) and \(\Omega\) are parallel). Near the vortex core the fluid would appear to be approximately stationary, farther away from the core near the cylinders and near the radial jets the fluid would be in motion relative to this reference frame. In fact, away from the vortex core the Coriolis
force would produce a periodic forcing on the flow that would be (as seen in the inertial frame) approximately radially inward or outward near the cylinders and approximately azimuthally forward or backward near the radial jets. Thus there would appear to be a periodic forcing in this rotating frame which could generate inertial waves. Because larger vortices have stronger radial inflow and outflow jets, as we have already argued, the larger vortices have a stronger periodic forcing and would tend to destabilize TVF to TWI at lower $R_o$ and $R_i$. The wave vector $k$ points in the (inertial frame) azimuthal direction because the wave travels azimuthally along the vortex. This means that the $x''$ and $y''$ directions are in the (inertial frame) axial ($z'$) and radial ($r'$) directions. If we take $x''$ to be the axial direction then we would expect to see maximum Kalliroscope reflection when $V_{x''}(= V_{x'})$ is maximum. If we look at the flow at time $t$ then there is a $V_{x''}$ maximum when $k \cdot r = \omega t$ where $r$ is in the rotating frame of reference.

In the inertial frame this relation depends on $\Omega$, $\Omega_i$, and $\theta$ where $\theta$ is the angle between $\Omega$ and $k$. As can be seen in Figure 13, white lines inside of each vortex that are inclined relative to the inflow and outflow boundaries are what is seen in the visualization. The frequency of these inertial waves is $\omega = 2\Omega \cos(\theta)$ in the rotating frame. Andereck et al. reported a 14 wave state with a frequency of $\approx 12.5\Omega_i$. If the TWI pattern were stationary in the rotating frame one would expect to see a frequency of $\approx 14\Omega_i$ in the inertial or laboratory frame, where we have used the approximation $|\Omega| \approx \Omega_i$. The difference could be accounted for by the inertial wave frequency which is $\leq \approx 2\Omega_i$.

Baxter and Andereck observed domains of separate flow patterns along with nonuniform wavelengths. Brand and Deissler has used a phase dynamical approach to describe such nonuniformities. In this work we observed similar
phenomenon at N=26 and N=28. The system tended to develop a nonuniform axial wavelength after the system was ramped above the threshold of secondary instabilities. Sometimes the TVF would develop weak nonuniformities at the onset of WIB or TWI and this would lead to either WIB or TWI in one part of the cylinder and TVF in the other. The only bi-modality seen in the N=32 was the TWI and WVL combination discussed above. These axial nonuniformities and bi-modal patterns were on each vortex though and are not axial domains.

2.3.7 Conclusion

By systematically surveying the \((R_o, R_i, \lambda)\) parameter space of Taylor vortices we have found the following results: increasing \(R_o\) stabilizes TVF against WVF, increasing \(\lambda\) destabilizes TVF against WVF and TWI while stabilizing TVF against WIB and WOB, WIB and TWI bifurcate from TVF for large \(\lambda\) and WOB bifurcate from TVF for small \(\lambda\). We have also observed weak time dependent activity inside of the Taylor vortices before the onset of WVF, WIB, WOB, or TWI at all \(\lambda\) surveyed. At \(\lambda = 1.88\), the smallest axial wavelength surveyed, we observed an \(m=1\) WVF as well as a spiral of TWI and WVL. We measured the \(m\) values of WIB and WOB at different \(N\) (or \(\lambda\)) and noted that the \(m\) of WIB depends on \(N\) while the \(m\) of WOB is independent of \(N\).

We have attempted to physically interpret our results with the help of previous numerical results. The most likely scenarios for these instabilities are a shear instability associated with the azimuthal jets near the inflow and outflow boundaries and a centrifugal instability associated with the radial inflow and outflow jets. Our best guess from the available evidence is that WOB are the result of a centrifugal instability. WIB, although very similar to WOB, may be either. The
TWI pattern appears to be a result of inertial waves that form in the interior of the vortices. We have also visualized the axial harmonics predicted in weakly nonlinear theory and conjecture that the weak time dependent activity is not an instability but is related to weak driving by instrumental imperfections of these harmonics.

2.4 Spiral Turbulence

Spiral turbulence consists of laminar and turbulent regions in Taylor-Couette flow which coexist to form a spiral (see Figure 26). Feynman\textsuperscript{47} noted that it is an example of the depth of phenomena described by the Navier-Stokes equations. Spiral turbulence has been studied by Coles\textsuperscript{34} and its existence region for a particular geometry mapped out in the \((R_o, R_i)\) parameter space at \(\eta = 0.881\). Van Atta\textsuperscript{48} measured the pitch of the spiral and mapped the profile of the laminar to turbulent flow interface in the \((r, \phi)\) plane at one point in the \((R_o, R_i)\) parameter space at \(\eta = 0.889\) and \(\Gamma = 27.5\).

Spiral turbulence is particularly interesting among the spatiotemporal patterns discussed so far because it mixes small scale (or microscale) turbulence and a well ordered structure at large scales, i.e. it is a coherent turbulent structure.\textsuperscript{49} This section presents measurements of the spiral pitch and the intermittency factor (or width) for various values \(R_o, R_i\), and \(\Gamma\). We have also taken CCD data for different boundary conditions at the top end of the cylinder. The observation that the pitch varies along the axis fits well into a phase dynamics approach, which to our knowledge is here applied for the first time to a situation with sustained microscale turbulence. This simplified approach is in the spirit of Feynman\textsuperscript{47} "to find the qualitative content of the Navier-Stokes equations."
Figure 26

Spiral turbulence for fixed upper and lower boundaries, aspect ratio 30, $R_o = -3000$, $R_i = 700$ (from reference 28).
In order to study the physics of the spiral we have perturbed the laminar flow and visualized the initial spreading of the turbulence. We show that the finite azimuthal width\textsuperscript{34,48} of the turbulent spiral is a result of the subcritical character of the flow and the bounded azimuthal geometry of the Taylor-Couette system.

2.4.1 Introduction

Spiral turbulence occurs in the coaxial cylinder system when the outer cylinder rotates rapidly while the inner cylinder rotates in the opposite sense at a lower rate (i.e. large $| - R_o |$ and a comparatively low $R_i$). At a sufficiently large $| - R_o |$ the laminar Couette flow first becomes unstable to interpenetrating spirals (IPS) as $R_i$ is increased. IPS is a flow where two long laminar vortices form near the inner cylinder to make two spirals of opposite helicity that "superpose". The IPS persist for an interval of $\Delta R_i \approx 20$, over the observed range of $R_o$, and appear to trigger a catastrophic transition (i.e. a subcritical bifurcation) to a combination of turbulent and nearly laminar flows\textsuperscript{34} which organize to make spiral turbulence. Most of the following observations and measurements were made in the hysteretic region which starts at\textsuperscript{28} $R_o \approx -2500$. In particular this region is hysteretic with respect to $R_i$, i.e. if $R_i$ is decreased the spiral turbulence will persist at $R_i$ values where the Couette flow formerly existed and if $R_i$ is decreased enough the Couette flow will return. It is generally believed that this type of hysteretic behavior is characteristic of a subcritical bifurcation. This is because subcritical bifurcations require finite amplitude perturbations to change from one solution to another. If there is no such perturbation there will be no change of solution which implies that the two solutions exist at the same parameter value. The parameter values where the two states in our case (turbulent flow and laminar
flow) coexist is over a finite range of $R_i$ at a given $R_o$ as has been experimen-
tally shown\textsuperscript{28,34}. Our measurements and observations were done at $R_o = -3000$, $R_o = -4000$, $R_o = -5000$, $R_o = -6000$, and $R_o = -8000$. At these $R_o$'s and at $\Gamma = 30$ we varied $R_i$ and measured the pitch and width of the spiral where it existed. We then changed the aspect ratio to $\Gamma = 73$ and did the same. We also made observations and measurements of the spiral with a free upper surface and with the aspect ratio changing. Lastly we measured the initial azimuthal and axial propagation velocity of the turbulence in a rotating frame of reference.

2.4.2 Procedure

The apparatus used here has been described in previous sections except for the following modifications: the aspect ratio, $\Gamma = l/d$, was varied up to as large as 73, and the plexiglas outer cylinder had a small hole (0.15cm diameter) in its side in order to perform perturbation experiments. In these perturbation experiments, which were done in the hysteretic regime, approximately 0.1cm$^3$ of fluid was injected through the hole over a time of less than 0.03 seconds into laminar Couette flow. This perturbation would initiate a turbulent spot that would spread into the laminar flow. The initial spreading of the turbulent front was visualized in the usual way and monitored with a television camera mounted on the rotating table. The television camera and table rotated at the expected velocity of the turbulent spot (and ultimately the turbulent spiral) in order to view the turbulent spot in its azimuthal rest frame. The initial spreading of the turbulent spot was recorded on video tape using a Panasonic model AG-6010 video recorder which recorded one image every 1/30 second. These images were then processed and the azimuthal and axial widths of the spot measured for each
image to determine the initial axial and azimuthal velocities of the spreading of the turbulent front.

Measurements of the pitch and intermittency factor were made by using the reflectance technique described previously. At two points on a line parallel to the axis of the cylinders the light from two He/Ne lasers is focussed on the fluid and the reflected light detected by photodiodes. The time series output from the two photodiodes is digitized and sent to our PDP-11/73 computer for analysis. The voltage output by the photodiode is steady and low when the flow is laminar and highly fluctuating when the flow is turbulent. The time delay between the onset of turbulence or laminar flow, together with the distance between the detectors (12.5cm), yields the pitch. The intermittency factor, $g$, ($g =$ the fraction of the time that the flow is turbulent) is measured by first finding the time between the onset of turbulence and the onset of laminar flow, and then dividing this time by the period of the spiral. Time series of 10 to 20 revolutions of the spiral were obtained and the mean, together with the deviation of the mean, of these pitch and $g$ measurements calculated. The onset of the laminar or turbulent part of the time series signal was determined by calculating the local (local in time) variance of the reflectance signal and then defining a variance cutoff. The signal is then defined as having been made by turbulent flow if the variance is above the cutoff and laminar if the variance is below the cutoff. This gives a consistent definition of the onset of turbulence and laminar flow for the purposes of comparing the resultant pitch and $g$ measurements at different parameter values. Because the spiral has a constant average angular velocity, $g$ is also the ratio of the turbulent spiral width to the total circumference. This measurement technique was checked by rotating the lasers and photodiodes
(which are mounted on the rotating table) in an opposite sense to the rotation of the turbulent spiral to get a more instantaneous measurement of the spiral width. It was found that the rotating laser and photodiodes gave the same results. All of the measurements at $R_o=-5000$, $R_o=-6000$ and $R_o=-8000$ were done below the onset of IPS because we had difficulty in determining the laminar to turbulent and turbulent to laminar transition times above the onset of IPS. Stable and persistent spirals were not found at $\Gamma = 73$ for any of the above $R_o$'s and $R_i$'s except at $R_o=-3000$ and the associated $R_i$'s.

With a free upper surface the aspect ratio could be continuously changed by pumping fluid in or out of the apparatus while the cylinders rotate. Video tapes were made, over a 12 hour period, of the spiral structure with a free upper surface as $\Gamma$ was reduced from 73 to zero at $R_o=-3000$ and $R_i=800$. The same was also done at $R_o=-8000$ and $R_i=500$.

We have also used the CCD camera to make space-time diagrams of the spiral at large $\Gamma$ and $R_o = -3000$ for both fixed and free upper surfaces. The CCD camera, which uses a telephoto lens to minimize distortions, was placed $\approx 7$ meters from the cylinder. The reflected light was focused on to an axially oriented line of 1024 pixels to be digitized once every 0.14 seconds. For this large field of view the CCD camera had a resolution of 0.5mm.

### 2.4.3 Results and Discussion

The laminar flow to spiral turbulence transition is characterized by large hysteretic effects, which implies that this transition is a subcritical bifurcation. Subcritical bifurcations in flows with infinite domains should, as shown by Pomeau, lead to either expanding or contracting turbulent domains in laminar
flow. This argument requires the existence of a potential functional in which there are at least two minima corresponding to a metastable (a local minimum but not a global minimum) and a stable state (the global minimum). Analogous to a first order phase transition (e.g. the growth of a crystal in a melt) in condensed matter physics the stable state (e.g. turbulent flow) expands at the expense of the metastable state (e.g. laminar flow). The existence of such a potential functional is, however, an uncommon occurrence in fluid dynamics. The case of spiral turbulence, as seen in the laboratory frame of reference, shows that this analysis does not hold since we have both an expanding turbulent domain at the front face of the spiral and a contracting turbulent domain at the rear face. To see how this happens let us decompose the velocity and pressure fields into mean \( (U, P) \) and fluctuating \( (u, p) \) parts, i.e., \( V = U + u \). Substituting these mean and fluctuating parts into equation (1.1) and (1.2) and taking the mean we get

\[
\frac{\partial U}{\partial t} + (U \cdot \nabla)U + ((u \cdot \nabla)u) = -\frac{1}{\rho} \nabla P + \nu \nabla^2 U \tag{2.12}
\]

and

\[
\nabla \cdot U = 0. \tag{2.13}
\]

The advective part of equation (1.1) makes a term which is quadratic in \( u \) so that the averaging, as denoted by \( \langle \rangle \), does not eliminate all the fluctuating fields. Equation (2.12) becomes more easily interpreted if we first rewrite it in component form with indices \( i \) and \( j \) which can take on values 1, 2, or 3 corresponding to the three components of a vector, e.g., \( x_1 = x, x_2 = y, \) and \( x_3 = z \). We also use the convention of summing over repeated indices. Equation (2.12) becomes

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + (u_j \frac{\partial u_i}{\partial x_j}) = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2}. \tag{2.14}
\]
The equation of continuity for the fluctuating part is $\frac{\partial u_i}{\partial x_i} = 0$ and can be used to simplify equation (2.14) to

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial (u_i u_j)}{\partial x_j}. \tag{2.15}$$

Equation (2.15) differs from the laminar flow equations by the addition of the last term which acts as a stress on the mean pressure and velocity fields. The quantity $-\rho(u_i u_j)$ is called the Reynolds stress. Coles and Van Atta measured the 6 components of Reynolds stress as a function of $\phi$ in the rest frame of the spiral at $r = r_1 + \pi r_2$. They found that $(u_r u_r)$, $(u_\phi u_\phi)$, and $(u_z u_z)$ are large and positive. These three components of Reynolds stress are zero in the laminar region and increase to their maximum values in the center of the turbulent region, i.e. they decrease towards the front face of the spiral (the region where turbulent flow moves into laminar flow, as seen in the laboratory frame) and increasing at the rear face of the spiral (the region where laminar flow moves into turbulent flow, as seen in the laboratory frame). The other three components of Reynolds stress are negligibly small except for $(u_r u_\phi)$ which becomes small and negative in the turbulent region. This periodic variation in the Reynolds stress produces a periodic forcing on the mean flow as can be seen in equation (2.15). One way that the mean flow may respond to this periodic forcing is through periodic changes in $U_i$ and $P$. This suggests that a large scale Poiseuille flow in the laminar region is generated by the change in Reynolds stress in the turbulent region. A similar backflow generating process in weakly inclined supercritical Taylor vortices has been explicitly worked out by Hall. Hall, who completed previous work on amplitude equations, found that the backflow is proportional to an integral over the azimuthal angle involving the square of the amplitude, which itself is
proportional to the Reynolds stress. The theory for the subcritical case has not yet been worked out, however, the same basic mechanism must be at work, and this allows for a qualitative understanding. The backflow counteracts the velocity of expansion of the turbulence until the latter stops, i.e., the backflow generated by the Reynolds stress slows down the mean flow until a transition to laminar flow occurs at the rear face. In other words the following feedback gives the spiral its finite azimuthal width: the turbulence generates a backflow which acts on the turbulence limiting its growth. A similar feedback effect was found by Thual and Fauve who studied subcritical transitions to nonlinear traveling waves using a fifth order Ginzburg-Landau equation (see section 1.4 and especially the discussion below equations (1.24) and (1.30)). They found that when a potential functional exists (which is the case when all the coefficients in this Ginzburg-Landau equation are real) the stable state would expand at the expense of the metastable state as described above. In the case where a potential functional does not exist (when the Landau constant and the fifth order coefficient are complex) they found a localized structure, i.e. the nonlinear traveling wave solution and the null solution coexist in spatially separated regions with a stationary interface separating the two. The turbulent flow in spiral turbulence, as seen in its rest frame, is such a localized structure which occurs after a subcritical transition. Thual and Fauve reasoned that the localization was due to the amplitude-dependence of the wave frequency (which is a general property of nonlinear waves). The spatial shape of an amplitude envelope of a pulse of waves affects the frequencies of the waves in the pulse which in turn affects the pulse's shape. This feedback between the pulse's shape and wave frequencies works to limit the growth of the pulse. Davey et al. studied weakly nonlinear three
dimensional disturbances in Poiseuille flow and found that spatial variations in the amplitude of these disturbances generate pressure gradients. They found the lowest order Ginzburg-Landau equation for a special case. The induced pressure gradient, in this special case, is shown to contribute to the complex Landau constant, which is also what couples the amplitude of the nonlinear wave solution to its frequency. This suggests that there may be a connection between the feedback mechanism of the backflow described above and the feedback mechanism of Thual and Fauve for localized structures in subcritical phenomena.

One difficulty with equation (2.15) is that it is not closed, i.e. there are more unknowns than there are equations, so that equation (2.15) is indeterminate. This difficulty is often overcome by introducing an empirical parameter called the turbulent viscosity, $\nu_T$. $\nu_T$ is defined by the relation $\langle -u_i u_j \rangle = \nu_T \frac{\partial U_i}{\partial x_j}$ so that the force exerted by the Reynolds stress on the mean flow has the same form as the viscous stress. Equation (2.15) becomes

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x^2_j} + \nu_T \frac{\partial^2 U_i}{\partial x^2_j}. \quad 2.16$$

When this equation is scaled into dimensionless variables in the usual way the viscosity, $\nu$, in the definition of the Reynolds number is replaced by $\nu + \nu_T$. The spatial variations of the Reynolds stress implies that $\nu_T$ must be space dependent, the Reynolds number, however, is the same for both the laminar and turbulent parts of the flows. This is because, as stated above, we view the laminar and turbulent part of the flow as two coexisting states which occur at the same Reynolds number after a subcritical bifurcation. We also want our control parameter $R$ to describe the driving of the flow by the cylinders, so we scale $U_i$ by the average cylinder speed. In order to keep $R$ constant the azimuthal length
of the turbulent region, $L_t$, and the azimuthal length of the laminar region, $L_l$, must vary. This argument leads to the relation

$$\frac{\nu}{\nu + \nu_T} = \frac{L_l}{L_T}. \quad 2.17$$

At onset there is roughly equal azimuthal width of each region because the Reynolds stress is small and hence so is the turbulent viscosity in the turbulent region. This behavior can be seen in Figures 27-30, which show that the spiral width decreased as $R_i$ was decreased for fixed $R_o$ for both large and small $\Gamma$. As the limit of the stability of the spiral was approached the spiral thickness became about one half the total average circumference of the cylinders. In fact because the spiral rotates with a constant average angular velocity the intermittency factor, $g$, is related to $\nu_T$ by

$$\nu_T = \nu \frac{2g - 1}{1 - g}. \quad 2.18$$

Equation (2.18) shows that $\nu_T \approx 0$ when $g = 0.5$ and that $\nu_T \approx \frac{1}{2} \nu$ when $g = 0.6$ which the data in Figures 27-30 shows is the case far above onset. This result implies that the turbulence creates a drag on the mean flow as we would expect and that because this drag is spatially inhomogeneous a backflow is generated.

As described above we videotaped the azimuthal and axial expansions of a spot created at the midpoint of the cylinder (to prevent end conditions from affecting the front velocity). After the spot is generated by the perturbation it initially expands much faster ($\sim 2$ times) in the azimuthal direction than in the axial direction (see Figure 31). The turbulent spot stops its azimuthal expansion as soon as it is about as wide as half of the perimeter length (we were unable to measure this large of an azimuthal width for the data in Figure 31 because
Figure 27

The intermittency factor $g$ changes as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -3000$. As the limit of stability of the spiral (in the hysteretic region) is approached $g$ becomes $\approx \frac{1}{2}$. 
Figure 27
Figure 28

The intermittency factor $g$ changes as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -5000$. As the limit of stability of the spiral (in the hysteretic region) is approached $g$ becomes $\approx \frac{1}{2}$.
Figure 28
Figure 29

The intermittency factor $g$ changes as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -8000$. As the limit of stability of the spiral (in the hysteretic region) is approached $g$ becomes $\approx \frac{1}{2}$. 

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Figure 30

The intermittency factor $g$ changes as $R_i$ is varied. The data was taken for $\Gamma = 73$ and $R_o = -3000$. As the limit of stability of the spiral (in the hysteretic region) is approached $g$ becomes $\approx \frac{1}{2}$. 
Figure 31

Axial (x) and azimuthal (*) widths of turbulent spots for $R_o = -3000$, $R_i = 770$. The widths are scaled by the average perimeter length of 35.3 cm, and the time by the outer cylinder period of 0.91 s. Fits to the first seven points in each case yields initial front velocities of $1.78 \text{ cm/sec}$ for the axial case and $3.86 \text{ cm/sec}$ for the azimuthal case. Error bars are shown for each case.
Figure 31

Axial, Azimuthal Width

Time
of the large distortions which the curvature of the outer cylinder produced in
the images). From the argument given above this is the width at which there
is sufficient backflow to stop the spots growth. The spot, however, continues
to grow along the cylinder axis for \( \sim 2 \) seconds when it breaks into two spots
in the axial direction (or, alternatively, the flow becomes laminar in the center
of the one big spot). These two spots continue to propagate axially in the rest
frame of the spiral, with their widths always remaining approximately that of
the final spiral. At still later times these spots propagate azimuthally and may
undergo further splitting and subsequent growth. The different pieces eventually
connect and construct a spiral. The presence of axial propagation shows that the
backflow has in fact a complex three-dimensional structure. There are several
properties of the visualized spots worth noting. The "spot" appears to be made
of white lines on a dark background which, based on the discussion in section
2.3 and our experience with visualizing TVF and IPS, we can interpret as small
localized rolls with the axes of the rolls parallel to the white lines. These white
lines move erratically and are as long as \( \sim 3d \). The spot was only self sustaining
in these initial stages if the center developed fine scale white lines (or rolls) that
did not appear to have a preference to be azimuthally oriented, i.e., at any given
time a white line was as likely to be axially oriented as azimuthally oriented.
From the discussion above, and the results of Ref. 51, this means that these
fine scale rolls correspond to a large Reynolds stress. The edges of the spot, as
well as the edges of the spiral, always had azimuthally oriented rolls with an
appearance similar to Taylor vortices. Our interpretation of these white lines as
azimuthally oriented rolls suggests that centrifugal instabilities, inherent in this
system, may be the mechanisms that destabilize the laminar region to turbulence.
Van Atta's measurements show that the turbulence in the leading edge of the spiral forms near the outer cylinder while the turbulence in the trailing edge ends near the inner cylinder. This suggests that the Poiseuille flow created by the Reynolds stress in the bulk of the turbulence destabilizes the laminar flow by means of a Dean-like instability. Laminar flow is restored near the outer cylinder when the azimuthal pressure gradient (i.e. the Dean number) is reduced at the trailing edge of the turbulence. The trailing edge of the turbulent region, being near the inner cylinder and with azimuthally oriented rolls, is sustained by a Taylor-like instability (due to the driving of the inner cylinder) for a short time before the flow becomes laminar. As we shall see in the next chapter the competition between these two instabilities can lead to interesting behavior in a different context.

We found stable spirals of both helicities at the lower aspect ratios. Figures 32-34 show pitch measurements for $\Gamma = 30$ and a fixed upper surface using the above mentioned technique. These measurements show that the pitch stays constant at $\sim 30^\circ$ for these $R_\alpha$'s and $R_\gamma$'s within the errors of the data. This, however, was not the case for larger aspect ratios, in fact, as we pointed out above, at $\Gamma = 73$ persistent spirals occurred only for $R_\alpha = -3000$. At $R_\alpha = -4000$ the spirals would change helicity as well as form V-shaped structures (a V-shaped structure is one that has spirals of opposite helicity at the top and bottom regions of the cylinders that connect in the center of the cylinder to form a V-shape). V-shaped structures are prevalent at $R_\alpha = -4000$, $R_\alpha = -5000$, and $R_\alpha = -6000$ with the axial position of the vertex of the V's fluctuating about the center of the cylinders (sometimes the vertex would move to one end of the cylinders leaving a spiral). At $R_\alpha = -8000$ no large scale coherent structure was seen at any.
Figure 32

The average pitch angle (as measured in degrees from the horizontal plane) of the spiral remains approximately constant as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -3000$ as $R_i$ was decreased through the hysteretic region toward the limit of stability of the spiral.
Figure 33

The average pitch angle (as measured in degrees from the horizontal plane) of the spiral remains approximately constant as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -5000$ as $R_i$ was decreased through the hysteretic region toward the limit of stability of the spiral.
Figure 33

Pitch Angle (Deg) vs. $R_i$
The average pitch angle (as measured in degrees from the horizontal plane) of the spiral remains approximately constant as $R_i$ is varied. The data was taken for $\Gamma = 30$ and $R_o = -8000$ as $R_i$ was decreased through the hysteretic region toward the limit of stability of the spiral.
Pitch Angle (Deg)

Figure 34
At this $R_o$ there were local spiral-like structures with the helicity changing sign over an axial distance of the order of the cylinder diameters. Because these spiral-like structures would not always connect to make V-shaped structures, these turbulent patches formed a "broken" spiral pattern. Sometimes the turbulence would not form local spirals or V-shaped structures, just turbulent patches. This same incoherent state is found with either a free or a rigid upper boundary. We continuously reduced $\Gamma$ over a 12 hour period by slowly pumping fluid out of the cylinder at $R_o = -8000$ and $R_i = 500$. The incoherent state persisted until the aspect ratio was lowered to $\approx 28$ when a simple spiral pattern emerged. The simple spiral persisted until $\Gamma = 17$ when the laminar flow returned.

Since we are interested in simple spirals we made measurements only at $R_o = -3000$ for large aspect ratios. As shown in Figure 35 these spirals were always observed to have a pitch that varied with axial position. The manner in which this variation occurred depended on the end boundary condition, the helicity of the spiral, and the sense of rotation of the spiral (which is determined by the sense of rotation of the outer cylinder). In all of these cases the spiral pitch variation persisted over many hours of observation and was reproducible from run to run. When the top and bottom boundaries are rigid and move with the outer cylinder then we observe the following: if the outer cylinder, as viewed from above, rotates clockwise (and so does the spiral), then a right-handed spiral would have a lower pitch near the bottom of the cylinder than near the top, while a left-handed spiral has a larger pitch near the bottom than at the top; if, however, the outer cylinder rotates counter-clockwise (and so does the spiral), then, consistent with the first observations, a right-handed spiral would have a lower pitch at the top of the cylinder than at the bottom, while a left-handed
Spiral turbulence with $\Gamma \approx 73$, $R_o = -3000$, $R_i = 950$, and outer cylinder rotating counterclockwise viewed from above. The turbulent band always turns in the direction of rotation of the outer cylinder. a) and b) have rigid upper boundaries, c) and d) have free upper boundaries. In a) and c) the spirals propagate downward and have lower pitch at the top. In b) and d) the spirals propagate upward and have lower pitch at the bottom. There is a substantial average pitch difference between c) and d) as discussed in the text. Typical pitch angles for case b) are $19^\circ \pm 1^\circ$ at the bottom and $44^\circ \pm 6^\circ$ at the top, while for d) they are $16^\circ \pm 0.5^\circ$ at the bottom and $9^\circ \pm 2^\circ$ at the top.
spiral would have a larger pitch at the top than at the bottom. In other words, the pitch is lower near the end away from which the spiral appears to be moving. In all of these cases the spiral wrapped around the cylinder ~ 2 times. With a free upper surface the pitch varies in the same manner, i.e. the pitch is lower near the end away from which the spiral appears to be moving. The variation in the pitch is not as large and there are differences in the average pitch (see figure 35). For a counter-clockwise rotation of the outer cylinder, a right-handed spiral looks much the same as for the rigid-rigid boundary condition case (the spiral wrapped around the cylinder ~ 2 times), while a left-handed spiral has a much lower average pitch (the spiral wrapped around the cylinder ~ 3 times).

At $Re_{o}=-3000$ and $Re_{t}=800$ another fluid draining experiment was performed. As the fluid was slowly drained the top rigid surface was in contact with the fluid column for ~ 4 hours. During this time the spiral wrapped ~ 2 times around the cylinder and the pitch was lower at the bottom and larger at the top (i.e. the spiral appeared to be moving upward). When the top surface became free the spiral readjusted itself so that it wrapped ~ 3 times around the cylinder and the pitch became more uniform although it was still lower at the bottom and larger at the top. As the fluid continued to drain the average pitch stayed roughly constant, i.e. as $\Gamma$ decreased so did the number of times the spiral wrapped around the cylinder. This experiment explicitly shows the dramatic effect that the end conditions have on the structure of the spiral. This effect is likely the result of average axial pressure gradients that the Reynolds stress in the turbulent region is generating, just as it generates an azimuthal pressure gradient. In fact Hall\textsuperscript{17} derived equations for the amplitude of disturbances that depended slowly on the axial and azimuthal coordinates. He found that slow azimuthal variations
in the amplitude of the disturbance induces both axial and azimuthal pressure
gradients. This is consistent with the measurements of Coles and Van Atta\textsuperscript{51}
which shows both a periodic axial mean flow and a periodic azimuthal mean flow
of comparable magnitude in the rest frame of the spiral at the mean radius. This
mean flow is tangential to the spiral at the leading edge and nearly normal to
the spiral at the trailing edge with a stagnation point in the center of the spiral.
This suggests that the Reynolds stress drives both a mean azimuthal and a mean
axial flow. This is consistent with the splitting of the turbulent spot observed in
the perturbation experiment, i.e. just as the backflow in the azimuthal direction
limits the growth of the spot azimuthally, the axial backflow limits the growth
of the spot axially. Since the geometry of the spiral is a configuration which
has a finite thickness in both the axial and azimuthal directions the feedback of
the backflow limiting the growth of the turbulence may be the reason for the
existence of the spiral. The boundary conditions at the ends of the spiral change
this configuration somewhat, i.e., the axial mean flow is zero at a rigid boundary
and a free surface. Figures 36-39 show the space-time diagrams corresponding to
the pictures in Figure 35. The darker regions show the leading edge of the spiral
and the changes in slope of these phase lines indicate changes in the axial speed
of the spiral. The end effects must modify the periodicity of the axial component
of the mean flow and this in turn may generate a larger scale mean flow. Pocheau
et al.\textsuperscript{54} found that mean flows modify the wavelength of Rayleigh-Benard rolls
and a similar mechanism may be working here to change the pitch. This suggests
that a simple phase dynamics approach, similar to the one used by Pocheau for
Rayleigh-Benard rolls, may be used here to describe the observed pitch of the
turbulent spiral and its variations.
Figure 36

Space-time diagram of spiral turbulence for the same configuration and parameters as in Figure 35a. Time is scaled by the outer cylinder rotation period and the axial length is scaled by the gap, $d$. The emitting side is at the top (axial position $\approx 76$) and the absorbing side is at the bottom (axial position $\approx 0$).
Figure 37

Space-time diagram of spiral turbulence for the same configuration and parameters as in Figure 35b. Time is scaled by the outer cylinder rotation period and the axial length is scaled by the gap, \( d \). The emitting side is at the bottom (axial position \( \approx 0 \)) and the absorbing side is at the top (axial position \( \approx 76 \)).
Figure 38

Space-time diagram of spiral turbulence for the same configuration and parameters as in Figure 35c. Time is scaled by the outer cylinder rotation period and the axial length is scaled by the gap, $d$. The emitting side is at the top (axial position $\approx 76$) and the absorbing side is at the bottom (axial position $\approx 0$).
Figure 38  AXIAL POSITION
Figure 39
Space-time diagram of spiral turbulence for the same configuration and parameters as in Figure 35d. Time is scaled by the outer cylinder rotation period and the axial length is scaled by the gap, $d$. The emitting side is at the bottom (axial position $\approx 0$) and the absorbing side is at the top (axial position $\approx 76$).
Figure 39  AXIAL POSITION
We first introduce the real phase field $\varphi(z,t)$ which is the mean azimuthal position of the periodic spiral at height $z$ and time $t$. The quantity $\varphi$ is a phase in the sense that a uniform shift of $\varphi$ has no dynamical effect on the periodic turbulent spiral pattern, (see section 1.5) because of the axisymmetry of Taylor-Couette flow (also see equation (1.4) through (1.7)). We assume that the phase equation has the same form as that considered by Pocheau et al. and Brand, namely

$$\varphi_t + v\varphi_z = D\varphi_{zz}. \quad 2.19$$

Equation (2.19) has a family of solutions consisting of the base state (no phase variation) plus a phase variation, i.e.

$$\varphi = \frac{w}{v}(vt - z) + \Phi(z) \quad 2.20$$

where $w$ is a constant of integration (which we would identify as the frequency of the spiral in its base state) and $v$ is the apparent axial velocity of the spiral in the laboratory frame when there is no phase variation. Equation (2.19) expresses the fact that the spiral relaxes diffusively in its rest frame. $\Phi(z)$, the variation of the phase, and the boundary conditions, which apparently cause the phase variation, are considered below. The value of the pitch is undetermined, as is the wave number of the Rayleigh-Benard rolls in the phase equation of reference (52). Just as the roll wave number in reference (52) is allowed to vary within the Eckhaus stable band, the absolute value of the pitch should be allowed to vary within a stable pitch band. This can be taken into account by introducing a pitch dependence in $D$ (cf (2.19)). We shall neglect, in our simple approach, any pitch dependence in $D$ as well as any pitch dependence in $v$. The function $\Phi(z)$ in (2.20) is the solution of $v\Phi_z = D\Phi_{zz}$ and thus of the form

$$\Phi(z) = \varphi_0 \exp \left(\frac{v}{D}z\right). \quad 2.21$$
Because the phase can only be known to within an arbitrary constant it cannot be used as a boundary condition. However, as stated above, the pitch is undetermined and in fact the only boundary conditions compatible with phase invariance are \( \varphi_x = \alpha \) at one end (say \( z = 0 \)) and \( \varphi_x = \beta \) at the other end \( (z = L) \). Parameters \( \alpha \) and \( \beta \) describe the local pitch which the boundary conditions impose. \( \alpha \) and \( \beta \) could, in principle, be computed within the framework of a complete amplitude equation. Cross calculated similar coefficients in a different context. Such a calculation should take into account the interaction between the Ekman layer and the finite amplitude solution.

Since spiralling Taylor vortices have the same symmetries as spiral turbulence their pitch should also be non-uniform, at least for sufficiently large aspect ratios and hence equation (2.19) would also apply. Although equation (2.19) has the same form when applied to rolls subject to a mean flow or spirals there are some distinctions that should be noted, especially concerning the boundary conditions. In the case of steady structures such as Rayleigh-Benard rolls near threshold there is a subtle dependence of wavelength upon boundary conditions. Because of the time dependence of the spiral its phase is always increasing or decreasing when measured at the boundary. Thus its phase cannot be taken as a relevant constant parameter for constraining the wavelength of the structure. The general solution of equation (2.19) is thus

\[
\varphi(z, t) = \frac{w}{v} (vt - z) + \varphi_0 \exp \frac{v}{D} z. \tag{2.22}
\]

Here \( \varphi_0 \) and \( w \) are determined by boundary conditions at \( z = 0 \) and \( z = L \)

\[
\varphi_0 = \frac{D}{v} \frac{\alpha - \beta}{1 - \exp vL/D} \tag{2.23}
\]
This describes how the pitch, $\varphi_z$, varies continuously between $\alpha$ and $\beta$. The experimental results in Figures 36-39 show that $\alpha \neq \beta$. If $\alpha = \beta$, then from (2.23), $\varphi_0 = 0$ and one obtains the solution of constant pitch $\varphi(z,t) = -\alpha(vt - z)$. Equation (2.19) is invariant under the symmetry $(z \rightarrow -z)$ which relates two possible spirals of opposite helicity. This symmetry operation also interchanges boundary conditions and therefore a spiral that in one helicity is compressed at the bottom and expanded at the top, shows the opposite behavior in the other helicity. These features are evident in the experimental data (compare Figure 36 with Figure 37).

However, in the case when the top interface is a free surface and the bottom one rigid the above symmetry is broken and there is no reason based on symmetry for the two spirals of opposite helicity to be related. For the free-rigid case, this is what the data shows (compare Figure 38 with Figure 39). The data shows that when the spiral appears to move upward the pitch is relatively insensitive to whether the upper surface is rigid or free (compare Figure 37 with Figure 39).

We have argued that the large scale spiral structure is a result of the backflow limiting the growth of the turbulence in the axial and azimuthal directions. In the perturbation experiment we observed that the spot always split into two parts, one moving upwards, one downwards. This splitting and subsequent propagation suggests that the two spots consist of progressive finite amplitude waves moving axially, in the rest frame of the spiral, in either direction. In other words, the initial spot which is made of up moving and down moving waves splits into two spots, one consisting primarily of up moving waves and the other consisting primarily of down moving waves. This separation of the two types of waves
corresponds to a spontaneous axial symmetry breaking. The final spiral which forms after a few minutes consists primarily of either up moving or down moving waves which accounts for the existence of spirals of either helicity. As the fluid outside the spiral is linearly stable against these waves, they only propagate within the spiral boundaries, being emitted at one side and absorbed at the other. This view is consistent with the observed asymmetry between leading and trailing edges of the spiral azimuthal profile. Consequently, waves are emitted at one end of the spiral and absorbed at the other, which leads to $\alpha$ being different from $\beta$. A full picture would need an extension of the results in ref. 49 to complex amplitudes of progressive waves. Figures 36-39 illustrate the differences between the emitting side and the absorbing side for free and fixed upper surfaces. The difference is most pronounced when the free upper surface is an emitter (Figure 38). The additional effects of wave reflection at the ends would also need to be considered in an attempt to calculate $\alpha$ and $\beta$ from amplitude equations. From the results of the perturbation experiments one might expect that the emitter side, being the source of the waves, would consist of up moving and down moving waves in equal proportions. The waves which do not move toward the opposite end would quickly be reflected into that direction. This effect may account for the nearly vertical leading edge that the spiral has near the emitter side as shown in Figures 36-39. When the free upper surface is the emitter, as shown in Figure 38, the leading edge near the top fluctuates (this is also true to a lesser degree when the top free surface is the absorber, as shown in Figure 39). The wave reflection properties of the free upper surface which are different from the fixed upper surface may account for this fluctuation.

There is an apparent contradiction between the result that spirals only exist at low $\Gamma$ at $R_\alpha = -8000$ and that the stable spiral solutions of Eq. (2.19) exist
at any cylinder length $L$. This contradiction may be resolved if we do a stability analysis of equation (2.19) with a perturbation $\varphi^* = Ae^{ikx+st} + Be^{-ikx+st}$, where $A$, $B$, and $k$ are real while $s$ is complex. Using the boundary conditions $\frac{\partial \varphi^*}{\partial z} = 0$ at $z = 0$ and $z = L$ we find that the least stable perturbation decays as $\exp \frac{-Dn^2}{L^2} t$. This shows that the greater $L$ is the less stable the spiral becomes, which accounts for the contradiction. We conjecture that for large aspect ratios the spiral becomes unstable against a secondary instability, not included in this simple phase dynamics approach, which involves coupling with complex secondary flows. This conjecture is supported by the fact that the typical correlation length for a broken turbulent spiral is of the order of the cylinder diameter, which also characterizes the large scale flows. It is also interesting to note that the spiral is unstable if the diffusion coefficient becomes negative. This type of instability is called a phase instability and can lead to a highly disordered state called phase turbulence. We mentioned above that the diffusion coefficient should in general depend on the pitch. The pitch measurements of the spiral at $Re = -3000$ and $\Gamma = 73$ with a rigid upper surface, shown in Figure 40, show a variation of pitch as $R_i$ is varied (these measurements were made in the center portion of the cylinder). This behavior can also be seen directly in that the spiral wraps around the cylinder $\sim 3$ times at $R_i = 1050$, but only wraps around the cylinder $\sim 2$ times at $R_i = 800$. Above $R_i = 1050$ the spiral pattern breaks up into the broken spiral configuration. This instability may be the result of a phase instability corresponding to a negative $D$. If this were true then the broken spiral pattern would correspond to phase turbulence.

2.4.4 Conclusion

As the stability limit of spiral turbulence is approached, the thickness of the spiral decreases to $1/2$ of the mean cylinder circumference. The pitch stays
Figure 40

The average pitch angle (as measured in degrees from the horizontal plane) of the spiral increases as $R_i$ is varied. The data was taken for $\Gamma = 73$ and $R_o = -3000$ in the center of the cylinder as $R_i$ was decreased through the hysteretic region toward the limit of stability of the spiral.
at \sim 30^\circ \text{ in the wide range of } R_o \text{ and } R_i \text{ values measured at } \Gamma = 30 \text{ with rigid upper and lower boundary conditions. At large } \Gamma \text{ spirals did not exist in as wide a range of } R_o \text{ and } R_i \text{ values but did exist at } R_o = -3000. \text{ These spirals had a pitch that varied along the axis and the average pitch increased with decreasing } R_i. \text{ The pitch of the spiral was found to be lower near the end away from which the spiral appears to be moving. With a free upper surface the average pitch either increases or decreases depending on its helicity. Phase dynamics, for periodic structures with the symmetry of the spiral, gives a simple description of this pitch variation.}

\text{We have visualized the initial spreading of the pattern and taken CCD data which, together with previous results, have given us a physical picture of the spiral. The Reynolds stress in the turbulence generates a azimuthal and axial backflow and limits the azimuthal and axial extent of the turbulence accounting for the spiral shape. The internal structure of the turbulence is made of progressive finite amplitude waves that propagate either up or down along the axis with one end of the spiral being the emitter of the waves (the end with the compressed pitch) and the other end being the absorber of the waves (the end with expanded pitch). This internal structure accounts for the two possible helicities and the difference in the pitch values at two ends of the spiral. The average pitch of the spiral increases if the free upper surface is an emitter and the average pitch decreases if the free upper surface is an absorber.}
CHAPTER III

THE TAYLOR-DEAN SYSTEM

3.1 Introduction

Centrifugal instabilities,\textsuperscript{59,4} which occur in flow with curved streamlines, play an important role in many problems of practical importance. Two well known examples, besides the Taylor-Couette\textsuperscript{28,60} instabilities, are the Taylor-Gortler\textsuperscript{4,61} instabilities, which occur in the boundary layer on a concave wall, and the Dean instabilities\textsuperscript{4,62} which occur in the presence of a pressure gradient along a curved channel (Poiseuille flow).

In this chapter we present experiments performed with the Taylor-Dean system. In this simple system we have found that it is possible to realize two centrifugal instabilities by an appropriate choice of control parameters. These two instabilities are the Taylor-Couette instabilities which, as we have seen, may result in Taylor vortices, and the Dean instabilities, which may result in Dean rolls. The geometry of this system has already been described in Section 1.6. Briefly, the system consists of two horizontal coaxial cylinders of inner and outer cylinder radii \( r_i \) and \( r_o \) respectively, which rotate independently with inner cylinder angular velocity \( \Omega_i \) and outer cylinder angular velocity \( \Omega_o \). This is essentially the same as the Taylor-Couette system except that we operate it with the axes of the cylinders parallel to the horizontal plane and only partially fill the gap space with fluid. In fact for the experiments in section 3.4 we have used the same
apparatus as in Chapter II which is shown schematically in Figure 5. The partial filling of the gap produces two horizontal free surfaces. When the cylinders rotate they drive the fluid toward a free surface. To reverse the direction of the flow the free surface induces a pressure gradient along the azimuthal direction. As a result, the flow sufficiently far away from the free surfaces can be regarded as a combination of Couette flow, from the rotation of the cylinders, and a Dean (or Poiseuille) flow, due to the azimuthal pressure gradient (see Section 1.6 and Figure 41). When the velocity profile is Couette-like, the fact remains that the flow is not axisymmetric meaning that the details of the dynamics are different from the Taylor-Couette case even though the starting states may be similar. This broken azimuthal symmetry is sometimes manifested by the presence of recirculation rolls near the horizontal free surfaces. These recirculation rolls extend the entire axial length of the system with their axes parallel to that of the cylinders. The effect of their presence cannot as yet be completely specified but they are possibly quite important for some of the flows, as we will suggest in a subsequent section. As the control parameters are varied the base flow instabilities will change from those associated with Taylor-Couette to those associated with Dean. Thus we expect to see, at appropriate control parameters, flow patterns that result from competition between these two instabilities. The rotation of the cylinders drives the flow in this system and therefore the relevant control parameters are, as before, $R_o$ and $R_i$ or equivalently the ratio of angular velocities $\mu = \Omega_o/\Omega_i$ and the Taylor number $T$.

These two instabilities and the competition between them have been studied in previous work. The first work with the Taylor-Dean system was done by Brewster and Nissan who measured the inner cylinder speed for the threshold
Figure 41
Schematic cross section of the apparatus. The "front face" is defined to mean that the observer sees the inner cylinder rotating upward as shown. The outer cylinder can rotate in either direction. Qualitative pictures of the flow near the free surfaces are shown, along with the fully developed velocity profile away from these surfaces.
of the instability and wavelength of the resulting rolls when the outer cylinder is at rest. DiPrima calculated an approximate stability diagram for the combined Taylor-Couette (outer cylinder at rest) and Dean problem (when the flow is produced by pumping fluid around the annular space) assuming a stationary secondary flow. He also introduced a parameter \( \lambda \) which measures the relative importance of the pumping flow compared to the flow driven by the rotation of the inner cylinder. Among DiPrima's results was a neutral stability curve which exhibited a discontinuity for \( \lambda = \lambda_c = -3.667 \). Brewster, Grossberg and Nissan examined the critical conditions for the formation of vortices in the Taylor-Couette system, in the Dean system, and the Taylor-Dean system. Their results for the Dean system agreed well with the theoretical values for the threshold of the instability and the wave number of the vortices. In addition they also rotated the inner cylinder while pumping the fluid around the annular space. Near a particular value of \( \lambda \) they found that both the critical value of the control parameter and the wavelength of the resulting vortices have a discontinuity. They explained this discontinuity as resulting from the competition between the destabilization of the Couette and Poiseuille layers in the basic flow. Hughes and Reid numerically integrated the stability equations for this problem finding that in the vicinity of \( \lambda_c \) the marginal stability curve has two minima, i.e. two different critical wave numbers. Raney and Chang did an analysis that allowed for a time dependent secondary flow and found that in the neighborhood of \( \lambda_c \) an oscillatory instability might occur. Prior to our work, no experimental verification of these results had been reported.

Mutabazi, Peerhossaini, and Wesfreid observed oscillatory structures in the Taylor-Dean system with the outer cylinder at rest. They also found a
lower value of the threshold of instability than that found by Brewster and Nissan\textsuperscript{63}. Mutabazi, Normand, Peerhossaini and Wesfreid\textsuperscript{68} solved the linear stability problem for the Taylor-Dean problem assuming axisymmetric perturbations and simplified horizontal free surfaces. They found both oscillatory and stationary instabilities depending on the ratio of the angular velocities $\mu$. They found several points in the $(\mu, T)$ parameter space where the oscillatory instability line intersects the stationary instability line (an instability line is the set of points $(\mu, T_c)$ where $T_c$ is the critical value of the Taylor number for the onset of the instability, e.g. the lines in Figure 6-10 are instability lines). These intersecting points, also called codimension two points (e.g. the points where the lines in Figures 6-10 intersect are also codimension two points), were predicted to be experimentally accessible. Near codimension two points one would expect interesting behavior because at these points there would be competition between the two instabilities.

3.2 Description of System

The system used here has been described in previous sections. Briefly, it consists of two horizontal coaxial cylinders, the radius ratio is $\eta = 0.882$ and $\Gamma = 68$. In order to avoid communication between the two sides of the system the gap is only filled about $2/3$ full with distilled water and Kalliroscope as shown in Figure 41. With this geometry there is no significant film produced on the walls of the rotating cylinders.

3.3 Data Acquisition Techniques

Data acquisition involved two techniques that have already been described. Flow frequencies were determined using the single point light reflectance tech-
nique. Spatial information was obtained by eye (with the assistance if a Pana-
sonic 6010 video recorder) and the CCD camera. Analysis of the CCD camera
output yields the vortex sizes and hence the wavelength of the structure as well
as space-time diagrams as described previously. The output of the CCD camera
(the reflected light intensity at 1024 axial positions) was also used to calculate
the intensity-intensity autocorrelation function of the traveling inclined roll pat-
tern. In addition to calculating the autocorrelation function we were also able to
time average these autocorrelation functions. This was done by taking 100 CCD
frames of the pattern over $\approx 20$ minutes, calculating the autocorrelation function
of each frame, and then averaging all of the autocorrelation functions. We have
also used a high intensity white light source to illuminate a thin cross-section of
the flow to visualize the internal processes in the traveling inclined rolls.

3.4 Bifurcations From the Base Flow

The bifurcations from the base flow to the various secondary flows are sum-
marized in Figure 42. We determined the instability points in Figure 42 by
observing the front face (as defined in Figure 41) as the system control param-
eters, $R_i$ and $R_o$, were varied. A video camera and monitor were also used,
when necessary, to simultaneously view the rear face. The base flow may also be
specified by the angular velocities ratio $\mu$ and the Taylor number $T$ (defined as
$T = (\Omega_1 \tau d/\nu)(d/r_i)^{1/2}$) but the first pair of parameters, being proportional to
$\Omega_1$ and $\Omega_o$, is more easily controlled in an experiment. For our experiments the
two sets of parameters are related as follows: $\mu = 0.882 R_o/R_i$ and $T = 0.366 R_i$
(or $R_i = 2.736 T$ and $R_o = 3.102 \mu T$). In the following we scale wavelengths by
d, velocities by $\nu/d$, and frequencies by $\Omega_i$. 
Figure 42

a) Diagram of primary flow transitions in the $(R_o, R_i)$ space, b) Diagram of primary flow transitions in the $(\mu, T)$ space.
3.4.1 Base Flow

As we discussed in Section 1.6 the fully developed base flow in the Taylor-Dean system is azimuthal (except in the neighborhood of the free surfaces) if the entrance angle $\theta_e$ is small compared to the filling angle $\theta_f$ ($\approx \frac{4\pi}{3}$ for the experiments in this section and $\frac{3\pi}{2}$ in Section 3.5). The entrance angle, for $\mu = 0$, is

$$\theta_e = \frac{\mu d^2}{\pi \mu} = \frac{2f_i}{\pi}$$

where $f_i$ is the scaled frequency of rotation of the inner cylinder. The condition for assuming fully developed azimuthal base flow is

$$\theta_e \ll \theta_f$$

which for $\mu = 0$ leads to the condition $f_i \ll \frac{\pi \theta_f}{2}$ and with our geometry

$$\frac{\pi \theta_f}{2} = 59.2.$$  

The boundary layer argument of Ref. 26 and Section 1.6 also applies to the outer cylinder, i.e. the condition for fully developed flow when $\mu \neq 0$ is

$$f_i \ll \frac{\pi \theta_f}{2} \quad \text{and} \quad f_o \ll \frac{\pi \theta_f}{2},$$

where $f_o$ is the scaled outer cylinder frequency. Since we have worked in the range $f_i \leq \approx 7.3(\ll 59.2)$ and $f_o \leq \approx 6.5(\ll 59.2)$ for all the transitions we can consider our base flow to be azimuthal. The azimuthal velocity is

$$V(r) = Ar \ln r + Br + C/r$$  \hspace{1cm} (3.1)

where the coefficient $A$ is zero in the Couette problem (fully filled gap). This velocity field is a superposition of the Couette flow due to the rotation of the cylinders and a Poiseuille flow in a curved channel produced by the azimuthal pressure gradient. This superposition is also evident in the small gap approximation (see Section 1.6). Since the first term on the right results from the azimuthal pressure gradient it is not surprising that the coefficient $A$ contains this pressure gradient, i.e.

$$A = \frac{1}{2 \rho \nu} \frac{\partial P}{\partial \theta}.\hspace{1cm} (3.2)$$

Rather then treat the flow near the horizontal free surfaces explicitly, which would be rather difficult, we require that the flow rate be conserved across a
given radial section \((\theta = \text{constant plane})\). A more realistic treatment of the flow near the free surface is given in Ref. 70. This condition allows us to calculate \(A\) in terms of known quantities

\[
A = \frac{2[2(\Omega_i - \Omega_o)r_i^2 r_o^2 \ln(r_o/r_i) + (\Omega_o r_o^2 - \Omega_i r_i^2)(r_o^2 - r_i^2)]}{(r_o^2 - r_i^2)^2 - 4r_i^2 r_o^2 \ln(r_o/r_i)^2}.
\]

The coefficients \(B\) and \(C\) are, just as in the Couette case, obtained from the boundary conditions on the cylindrical walls. They are

\[
B = \frac{[\Omega_o r_o^2 - \Omega_i r_i^2] - A[r_o^2 \ln r_o - r_i^2 \ln r_i]}{r_o^2 - r_i^2}
\]

and

\[
C = \frac{[\Omega_i - \Omega_o + A \ln(r_o/r_i)]r_i^2 r_o^2}{r_o^2 - r_i^2}.
\]

For the purposes of interpreting our experiment we use the small gap approximation \((\eta \to 1)\) to obtain the approximate velocity profile in terms of \(x\) where \(x = (r - r_i)/d\). This approximate velocity profile, in the small gap approximation, is (also see Section 1.6)

\[
V(x) = 3(R_i + R_o)x^2 - 2(2R_i + R_o)x + R_i.
\]

This profile has up to two zero velocity surfaces (the surfaces of zero azimuthal velocity corresponds to the roots of equation (3.6)) between the two cylinders situated at

\[
x = \frac{2R_i + R_o \pm \sqrt{R_i^4 + R_i R_o + R_o^2}}{3(R_i + R_o)}.
\]

Figure 43 shows the velocity profiles, in the small gap approximation, for different values of \(R_i\) and \(R_o\). As suggested by previous work we look for centrifugally unstable regions which give rise to either Dean rolls or Taylor-Couette rolls.
Figure 43

Base flow velocity distributions near instability thresholds: a) only outer cylinder rotating, b) counter-rotating cylinders (Dean rolls), c) counter-rotating cylinders (Taylor-Couette rolls), d) only inner cylinder rotating (traveling inclined rolls), e) co-rotating cylinders (Dean rolls), f) co-rotating cylinders (traveling inclined rolls). S, U indicate Rayleigh stable and unstable layers, respectively. The fluid velocities have been normalized to the outer cylinder velocity in a) and the inner cylinder velocity in b)-f).
The regions which are unstable to axisymmetric perturbations (Taylor or Dean instability) are given by the inviscid Rayleigh criterion. The Rayleigh criterion for instability in this case is \( V(z) \frac{\partial V(z)}{\partial z} < 0 \) and the layers of fluid which are stable and unstable are shown in Figure 43. The presence of viscosity modifies this simple picture, but the basic instabilities remain as the numerical results of Mutabazi, et al. show.

3.4.2 Stationary Patterns

Stationary vortex patterns occur for both corotating cylinders \((R_o \text{ in the range } 160 \text{ to } 257)\) and counter-rotating cylinders \((R_o \text{ in the range } -257 \text{ to } -35)\) when the base flow becomes unstable as \( R_i \) is increased (stationary patterns also occur near \( R_i = 0 \) but their behavior is rather different and we will discuss them in Section 3.4.6). These stationary vortices have no azimuthal variation, except near the free surfaces, and their axial wavelength (axial length of a vortex pair) is uniform. This wavelength (defined as \( \lambda/d \)) is \( \approx 2.4 \) for corotating cylinders and \( \approx 2.45 \) for counter-rotating cylinders (see Figure 44a and 44b). The stationary patterns that occur when the outer cylinder rotates relatively fast (i.e. \(-257 < R_o < -175\) and \(160 < R_o < 270\)) are clearly visualized indicating that they form near the outer cylinder. This is consistent with the analysis of the approximate base flow profile. Figure 43a shows that when the outer cylinder rotates relatively fast, as when \(-257 \leq R_o \leq -170\), the Rayleigh unstable layer is contained within the Poiseuille flow region, which will therefore give rise to Dean rolls. For \(160 \leq R_o \leq 270\) numerical results show that the rolls should be confined to the outer unstable layer, which again is part of a Poiseuille flow region, indicating that these are also Dean rolls. Slightly above the onset of the Dean roll instability the pattern becomes rather complex.
Figure 44

States observed for different values of control parameters: a) Dean rolls, $R_o = 220, R_i = 195$, b) Taylor-Couette rolls, $R_o = -136, R_i = 252$, c) traveling inclined rolls, $R_o = 70, R_i = 280$, d) Dean rolls for only the outer cylinder rotating, $R_o = 320, R_i = 0$, e) co-existing inclined and Dean rolls, $R_o = -45, R_i = 265$. 
The rolls that form for $-170 < R_0 < -35$ are difficult to visualize and they do not appear to strengthen significantly as $R_i$ is increased. The rolls are difficult to see because they are localized near the inner cylinder with the Kalliroscope flakes near the outer cylinder diminishing both the incident light to the rolls and the reflected light from the rolls. Figures 44b and 44c show that in this parameter range the unstable layer is near the inner cylinder where the velocity profile is Couette-like, implying that these rolls are Taylor-Couette rolls. The transition between the theoretical Taylor unstable and Dean unstable profiles occurs near the measured codimension two point at $R_i = 197$ and $R_o = -166$. When $R_o$ is between $-175$ and $-170$, the Dean rolls first appear in the rear face. This region was explored both by fixing $R_o$ and changing $R_i$ and by fixing $R_i$ and changing $R_o$ with the results remaining essentially the same. Both the Dean rolls and Taylor-Couette rolls become unstable to time dependent patterns (specifically the traveling inclined rolls which are discussed in Sections 3.4.3 and 3.4.7) when $R_i$ is increased for fixed $R_o$ (except when $R_o$ is between 150 and 200 when, as is shown in Figure 42, the base flow returns before the traveling inclined rolls form).

3.4.3 Time Dependent Patterns

Traveling inclined rolls (which appear very similar to spiral vortices in the Taylor-Couette system, but since a complete spiral is not possible we have chosen to call them traveling inclined rolls) which are shown in Figure 44c bifurcate from the base flow when $R_o$ is in the ranges $-30$ to $210$ and $265$ to $300$ as $R_i$ is increased. They form at an angle of approximately $20^\circ$ from the vertical and propagate along the cylinders’ axes with a constant average velocity. The
dimensionless roll propagation velocity, scaled by $\frac{\nu}{d}$, is approximately 29 (0.4 cm/sec). Figure 45 shows how the frequency changes as a function of $\mu$. These rolls move in either direction along the cylinder's axes with the rolls always tilting toward the direction in which they propagate. We have observed inclined rolls with opposite tilt and propagation direction existing simultaneously in different regions along the axis. Usually one type of traveling roll will occupy almost the entire length with the other confined near one end. Although in principle standing waves formed by counterpropagating rolls can exist we only see them form when an external modulation is applied. We will discuss the standing wave results in Section 3.5. In the present case we see spatially separated patches of traveling inclined rolls with a small region of coexisting left and right traveling waves at the interface between the patches. Near onset the traveling inclined rolls are strongest near the front free surface and become continuously weaker as they travel toward the rear face such that they cannot be seen at the rear face. Farther above threshold the inclined rolls become visible throughout the fluid region with the rolls still strongest near the front surface and weakest near the rear surface.

3.4.4 Multiple State Transitions at the Same $R_0$

When $R_0$ is between either 155 and 210 or 257 and 300 the instability changes as $R_i$ is increased (see Figure 42a). In the first of these $R_0$ bands ($R_0$ between 155 and 210), when $R_i$ is increased from about 160, the instability sets in as stationary Dean rolls, but at higher $R_i$ these Dean rolls disappear with the base flow returning. At still higher values of $R_i$ the base flow becomes unstable to traveling inclined rolls. In the second band ($R_0$ between 257 and 300) the base
The frequency $\sigma$ of traveling inclined rolls vs. $\mu$ the ratio of the angular velocities of the two cylinders. $\sigma$ is the measured frequency in Hertz scaled by $\frac{\Omega}{\nu}$. The region between the two branches corresponds to the stationary patterns. The ends of the 2 branches correspond to the codimension two points.
flow first becomes unstable to traveling inclined rolls. When $R_i$ is increased sufficiently this pattern disappears and the base flow returns. At still larger $R_i$ the base flow becomes unstable to the stationary Dean rolls. The wavelength of these Dean rolls is the same as the stationary Dean rolls for $R_o$ in the range $160$ to $257$. Figure 42b shows that when the phase diagram is plotted in terms of the parameters $T$ and $\mu$ (Figure 42b), the transitions, for fixed $\mu$ and increasing $T$, are directly from the base flow to either traveling inclined rolls, Dean rolls, or Taylor-Couette rolls. This is because these two parameters give a more physical specification of the competition between the two types of flow. $T$ specifies the inner cylinder driving and thus the Taylor-Couette type effects while $\mu$ specifies, at a given $T$, both the outer cylinder driving and the azimuthal backflow or Dean type effects (see Ref. 68 and section 1.6). The azimuthal pressure gradient in the small gap approximation is $\frac{\partial P}{\partial \phi} = 6T(1 + \mu)$. Thus at a given $\mu$ as $T$ is increased both the Taylor-Couette type effects and the Dean type effects increase until one of these effects "wins". Several points which support this view can be seen in Figure 42b: when $\mu$ is near -1 and the Dean type effects are negligible the base flow becomes unstable to Taylor-Couette rolls as $T$ is increased, when $\mu <\approx -1$ (i.e. Dean type effects begin to reappear) the Dean rolls become unstable at low $T$, and the more prevalent the Dean type effects are (the more negative $\mu$) the lower the $T$ value is for destabilizing the flow to Dean rolls. When $\mu$ is $\approx +1$ the Dean type effects are strong and the base flow destabilizes to Dean rolls. This view does not account for the traveling rolls, which will be discussed in a subsequent section.

3.4.5 Codimension Two Points

The traveling inclined roll state and the stationary state instability lines intersect at three oscillatory-stationary codimension two points (see Figure 42).
Near these points there are often states where both stationary and traveling rolls exist simultaneously. There are essentially two ways that the traveling rolls coexist with the stationary rolls. The traveling rolls and stationary rolls may be spatially separated (with stationary rolls at one end and the traveling rolls at the other end of the cylinders) or they may superpose in the same region (see Figure 4de). There is also a stationary-stationary codimension two point where the Dean and Taylor-Couette instability lines intersect. Near this codimension two point the Dean rolls, which are clearly visible at the outer cylinder wall, become less distinct as the unstable Taylor-Couette rolls develop in the region near the inner cylinder wall.

In all these cases the system was brought to just beyond the instability threshold and then allowed to settle for many gap diffusion times $\frac{d^2}{\nu}$, following which the process was repeated. It is possible that the mixed states near the codimension two points are transients: waiting for a very long time, the system may pass to a pure state like those further away from the neighborhood of the codimension two point. We have not been able to check this because of the finite lifetime of the Kalliroscope solution. Further work will be necessary to achieve a coherent picture of these complex flows.

3.4.6 Inner Cylinder at Rest, Outer Cylinder Rotating

Large spiral-like vortices which propagate away from each end appear when the inner cylinder is at rest and the outer cylinder speed is increased. These large propagating rolls virtually fill the system and they are most prominent on the side with the outer cylinder moving upward. On the other side at $R_o = 257$ weak stationary Dean rolls form and co-exist with the large propagating rolls.
Increasing $R_\omega$ to 300 establishes Dean rolls on both sides of the system (Figure 44d). It is then possible to decrease $R_\omega$ to 272 and retain the stationary state. Because of this we view these large propagating rolls as Dean-like rolls which are highly influenced by end effects. After the stationary Dean rolls have been established, one can then slowly increase $R_i$ and observe various transitions. When $R_i$ is increased for the counter-rotating case the Dean rolls persist but become weaker in the rear face. When $R_i$ is increased for the corotating case the Dean rolls become weaker in the front face than in the rear face until $R_i \approx 12$ when traveling inclined rolls appear in the front face. The transition between Dean rolls and inclined rolls in this region has not been thoroughly explored.

3.4.7 Discussion of Results

Applying the Rayleigh stability criterion to the base flow velocity profile, as was done in Figure 42, has helped us to interpret the stationary states as centrifugal instabilities due to axisymmetric perturbations. This seems to provide a qualitative understanding of these stationary roll patterns and a natural means of categorizing these structures (i.e. Dean rolls or Taylor-Couette rolls). In section 3.4.4 we found that we could interpret the phase diagram in Figure 42b as a competition between the Taylor-Couette type effects and the Dean type effects. In other words, at a given $\mu$ as $T$ is increased both the Taylor-Couette type effects and the Dean type effects increase until one of these sublayers becomes unstable. Once one of the sublayers becomes unstable and forms a structure this secondary flow may also influence the other sublayer suppressing the other instability. The linear stability analysis of Mutabazi et al.68 assumed, in addition to small gap and an infinite cylinder length, axisymmetric perturbations. Even though this system
breaks the azimuthal symmetry in a dramatic way, the stationary states observed seem to show that axisymmetric perturbations may still suffice to describe their behavior far from the free surfaces. There is no reason to expect that a theory utilizing axisymmetric perturbations should be able to predict the presence of the nonaxisymmetric traveling inclined roll state. Mutabazi et al. did predict an oscillatory instability line between two distinct stationary instability lines in \((\mu, T)\) space, as we have observed here, but there critical values are quite different from ours. In particular, their oscillatory branch was found to exist between two codimension two points at \(T \approx 175, \mu = 0.27\) and \(T \approx 175, \mu \approx 0.39\). The data in Figure 42b shows that the traveling inclined rolls exist at these \(\mu\)'s but the errors in \(T\) are almost 100%. Mutabazi et al. offered a possible physical explanation of the origin of the oscillatory mode that they predicted. Their scenario suggests that when both the Couette and Poiseuille sublayers become unstable at the same time the two structures, which should have uncorrelated wavelengths, will have regions where the sense of rotation in adjacent cells are opposite, resulting in the two sublayers interacting in an oscillatory way. In support of this view we see in Figure 42 that the inclined rolls exist for those values of \(R_i\) and \(R_o\) for which the base velocity field has two potentially unstable sublayers, while the stationary states develop for the case when the base velocity profile has only one potentially unstable sublayer. The Dean rolls which emerge for \(R_o\) in the range 265 to 300 appear to be an exception to this and they may be seen as intermediate between the inclined roll and laminar states. However, this scenario does not account for the the inclined rolls always being stronger at the front face. We may argue that the entry length near the free surfaces, caused by the growth of a boundary layer, may be affecting the inclined rolls, but evidently
this is not so important for either the Dean or Taylor-Couette rolls which, in this scenario, are interacting. It is also not clear how two stationary axisymmetric rolls which interact can result in nonaxisymmetric rolls which travels. In fact at the stationary-stationary codimension two point where both of these structures form simultaneously we have not seen any evidence of oscillatory behavior. In light of the above it is unlikely that the traveling inclined rolls state is the same oscillatory state predicted by Mutabazi et al.

The traveling inclined rolls have an appearance similar to the spiral vortex structures in the Taylor-Couette system, i.e in both systems propagating nonaxisymmetric spiral structures arise. The base flow in this system is, however, nonaxisymmetric and therefore noninteger values for $m$ are allowed. If we define

$$m = \frac{2\pi \left( \frac{\alpha + \pi}{2} \right) \tan(\alpha)}{\lambda}$$

then we find that the experimental value for $m$ is approximately $14.1$ for $\mu = 0$, corresponding to an inclination angle $\alpha = 19^\circ$ and a wavelength $\lambda = 1.22$ (0.86 cm). We note that the $\lambda$ we have used is equal to the distance along the axis between successive dark boundaries. It is also interesting to note that the traveling inclined rolls appear when either the inner or the outer cylinder is at rest, while for the classical Taylor-Couette system the flow is linearly stable when the inner cylinder is at rest. The result obtained for the case with only the inner cylinder rotating agrees with the results found previously and reported in Ref. 26. It is possible that the value of $m$ may change with both the values of $R_i$ and $R_o$ and the filling volume ratio. Indeed, early experiments$^{71}$ showed that $m$ decreases slightly with a continuous increase of the filling fraction, although the critical Taylor number is unchanged.
The traveling inclined rolls appear when either the inner cylinder or the outer cylinder dominates the driving of the fluid. A strong recirculation roll forms in the front face of the system in each of these cases (note that for large $R_o$ in the counter-rotating case the recirculation roll forms, by definition, in the rear face). In other words, we have observed a strong recirculation roll when either cylinder drives fluid toward the free surface. These recirculation rolls have recently been modeled by Mutabazi by refining the small gap approximation. The recirculation roll that forms for the strong $R_o$ case will have an opposite sense of circulation from the recirculation roll that forms in the strong $R_i$ case. With this in mind we have visualized the traveling inclined rolls and the recirculation rolls using a light sheet in the meridional, ($r, z$), and axial, ($r, \phi$), planes. We have found in both cases that a dark spot forms just below the front free surface at $R_{i, o} \approx 100$. At $R_{i, o} \approx 150$ another dark spot forms just below the first one. Just before the onset of traveling inclined rolls a third weak dark spot forms below the first two. The first two dark spots correspond to the outflow and inflow of the recirculation roll. At onset of the traveling inclined rolls these dark spots are seen, when visualized in the axial plane, to begin to oscillate also. During each cycle of the oscillation the bottom two dark spots appear to merge. Below the recirculation roll an oscillatory motion can be seen near the outer cylinder for the strong $R_i$ case and near the inner cylinder for the strong $R_o$ case. Figure 43a and Figure 43f show that this oscillatory motion below the recirculation roll is in the sublayer where there is a Poiseuille backflow. We have also visualized the recirculation roll for the strong $R_i$ case when the filling level was decreased and when $R_o$ was increased. When the filling level was decreased the size of the recirculation roll appeared to increase, as did the wavelength of the traveling
inclined rolls. When $R_o$ was increased for the strong $R_i$ case the recirculation roll size appeared to decrease as did the wavelength of the traveling inclined rolls. We also added 0.07% concentration of Alconox soap to the water and Kalliroscope solution in order to change the surface tension of the fluid. In two separate runs we observed an increase of 8.8% and 6.5% in the onset frequency of the traveling inclined rolls. These tests give further evidence that the recirculation roll is intimately connected to the formation of the traveling inclined rolls. In fact the transition sequence described above has a striking similarity to the transition sequence of the flow in the wake of a cylinder$^{25}$. In this open flow situation the cylinder moves through the fluid (or equivalently the fluid moves past a stationary cylinder) and at low $R$ a symmetrical flow surrounds the cylinder. As $R$ is increased a wake forms behind the cylinder which includes two 'attached eddies' which we would identify with our recirculation rolls. At $R \approx 100$ the wake becomes unstable to a periodic (in space and time) pattern known as a Karman vortex street. This pattern is characterized by a periodic shedding of the vortices (recirculation rolls) downstream, i.e. the vortices in the wake are periodically advected downstream$^{25}$ (these vortices are quickly reestablished before being advected downstream again). When the Karman vortex street is visualized in the axial plane the phases of the vortices are sometimes parallel to the cylinder axis but they are also sometimes inclined, creating a pattern very similar to the traveling inclined rolls. It is possible, especially in light of the evidence from the visualization of the recirculation rolls, that a similar vortex shedding process gives rise to the traveling inclined rolls.

Because of the weak nature of the flows we have not been able to distinguish the inflow boundaries from the outflow boundaries. The observed boundaries in
most cases are equally dark (as determined visually and with image analysis). In a few cases a set of very weak dark lines can just be detected. There is an ambiguity in our wavelength values because of several possible interpretations of the visualized patterns; the weak lines may be weak axial harmonics so that each dark line is alternately an inflow or an outflow, or one boundary is much weaker than the other and is thus normally not visible so that all the dark lines are either inflow or outflow (this also implies the existence of unobserved weak counter-rotating cells existing near these dark lines in the stable sublayer). We hope to resolve this ambiguity in future experiments.

We emphasize that further transitions have been observed in the system for values of the parameters $R_i$ and $R_o$ beyond the instability thresholds $^{70}$, but they have not yet been quantitatively characterized except for the traveling inclined rolls for the strong $R_i$ case as we will see in the next section.

3.5 Spatiotemporal Modulations of Traveling Inclined Rolls

We have found that for $R_o \in [-45, 216]$ or $> 257$ the base flow becomes unstable to traveling inclined rolls. In this section we concentrate on the traveling inclined rolls above the instability threshold for the strong $R_i$ case. In this case we found that just above onset, a phase variation in the pattern produces a long wavelength low frequency modulation and time-dependent defects are generated. At higher rotation speeds, the pattern undergoes an unusual periodic short wavelength modulation with an envelope size of $\sim 3$ rolls. While traveling wave patterns have been studied in detail in binary mixtures $^{72}$, in liquid crystals $^{73}$, in the Taylor-Couette system $^{28}$ and in chemical reactions $^{16}$, we know of no other system with traveling patterns at onset which undergoes such a short wavelength
modulation. This section will present the main characteristics of the behavior of these traveling inclined roll modulations.

### 3.6.1 Apparatus and Experimental Parameters

We have constructed another cylinder system for the experiments in this section. This system is very similar to the one shown in Figure 5 except for several modifications: the two filling holes are placed in the end caps, the bearings are isolated from the fluid by two shaft seals, and the inner cylinder shafts are longer in order to use them to mount the system horizontally through four bearings (two at the left end and two at the right end allowing the inner cylinder to rotate while the bearing housing is attached to two stationary supports). The inner cylinder is made of black Delrin plastic with an outer radius $r_i = 4.486 \text{ cm}$ and the outer cylinder is made of Duran glass with inner radius $r_o = 5.080 \text{ cm}$. The gap between the cylinders is $d = r_o - r_i = 0.594 \text{ cm}$ and the cylinders are independently driven as described previously. The radius ratio $\eta = 0.883$ is large enough for the small gap approximation to be reasonable. Teflon rings are attached to the inner surface of the outer cylinder a distance $L = 53.40 \text{ cm}$ apart, giving an aspect ratio $\Gamma = L/d = 90$. The fluid is distilled water with 1% Kalliroscope AQ1000 for visualization.

We define inner and outer cylinder Reynolds numbers as before. The dimensionless control parameter is $\epsilon = \frac{R_{ic}}{R_{tc}}$ where $R_{tc}$ is the inner cylinder Reynolds number at onset of the traveling inclined rolls for a given $R_o$. We have fixed the filling level fraction $n = \theta_f/2\pi$ at 0.75. (We have found that the traveling inclined rolls instability threshold may depend weakly on $n$; however, for $n \in [0.5, 0.8]$ the variation of $R_{tc}$ is $\Delta R_{tc} = 2$ or $\frac{\Delta R_{tc}}{R_{tc}} \approx 0.8\%$ which is within the experimental
precision (≈ 1%). As in Section 3.4 the flow pattern frequencies are scaled with the inverse of the radial diffusion time \( \tau_r = \frac{d^2}{\nu} \approx 36 \) seconds (which for the other system was \( \approx 49 \) seconds), the wavelengths are scaled by the gap size \( d \), and the velocities are scaled by the radial diffusion velocity \( \frac{\nu}{d} \).

### 3.6.2 Results and Discussion

The transition to traveling inclined rolls is, within the limitations of our visualization technique and ramping control, a supercritical Hopf bifurcation. Photographs of the traveling inclined rolls pattern near onset are shown in Figures 46a-b and a space-time diagrams of the pattern near onset is shown in Figures 47. The wavelength of the rolls along the cylinders' axis is \( \lambda = 1.416 \). At onset the rolls have no preferred direction and may move either to the left or right along the cylinder axis within a stationary \( \sim 10 \) roll envelope or patch. Light sheet visualization through the gap shows that the rolls exist near the outer cylinder (in the strong \( R_0 \) case, as discussed in section 3.4.7, the rolls exist near the inner cylinder). Above threshold, after the pattern has grown to fill most of the working space, both right and left traveling rolls may exist separated by a vertical (non inclined) defect line as shown in Figure 46b-d. The defect line is not necessarily in the middle of the cylinder axis; in fact, it moves in an erratic fashion with a velocity of about 50 times less than that of the rolls. Such defect lines are inherent to traveling wave patterns\(^{74}\).

We have measured the frequencies of the traveling rolls as a function of \( R_4 \), as shown in Figure 48 and 49. Near the onset, for a fixed \( R_0 \), the fundamental frequency \( f \) of the traveling rolls increases almost linearly with \( R_4 \). Measurements of the frequency at different points along the axis shows that for \( \epsilon \approx 0.013 \), and
Photographs of the flow patterns as seen in the front face at $R_o = 0$: (a) a stationary patch of traveling inclined rolls near threshold at $R_i = 263$, (b) traveling inclined roll pattern at $R_i = 269$ with a defect line between the left and right traveling rolls, (c) traveling inclined roll pattern at $R_i = 275$ with a defect line between the left and right traveling rolls and a roll destruction event near the right side, (d) short wavelength ($\sim 3$ roll) modulation pattern at $R_i = 303$, (e) incoherent pattern at $R_i = 338$. 
Figure 47

Space-time diagram of the traveling inclined rolls near onset at $R_i = 263$ and $R_o = 0$ after the pattern has been allowed to grow to $\approx 30$ rolls along the axis. The phase lines slightly curve and a roll creation event can be deduced in the region near the upper middle where the phase lines fade (this can be done by counting lines on either side of the faded region).
Figure 47

AXIAL POSITION
above, there is a local frequency of the pattern. This can also be seen in Figure 47 where the phase lines of the rolls are slightly curved. Close to onset a long wavelength modulation appears, after a patch of traveling rolls has grown to an axial length of ~ 30 rolls, which generates this phase variation. The wavelength of this modulation decreases from ~ 30 rolls to ~ 10 rolls as $R_i$ increases. A similar process also occurs in the Karman vortex street when the straight phase lines of the vortex shedding pattern (which are inclined to the cylinder axis) begin to bend (see Ref. 25 page 24). However, whereas the bending of the vortex street leads downstream to turbulence in the wake of a cylinder, the traveling inclined rolls appear to be damped azimuthally (downstream) from the front free surface.

For $e \in [0.02, 0.1]$ the frequency and roll velocity variation becomes strong. High velocity rolls occasionally collide with low velocity rolls, resulting in the loss of a roll as can be seen in the photograph in Figure 46c. The collision gives rise to a damped modulation moving in a direction opposite to the traveling rolls. Roll creation events have also been observed as can be deduced from Figure 47. Roll creation and destruction events similar to the ones observed here have also been observed in electro-hydrodynamic systems\textsuperscript{75}.

Increasing $R_i$, the flow undergoes a second instability which results in a short wavelength modulation of the traveling waves. A photograph of this modulation pattern is shown in Figure 46d and a space-time diagram is shown in Figure 50. The onset is non-hysteretic, within our experimental precision of $\sim 1\%$. The short wavelength modulation is associated with a lower frequency of about 1.0 (Figure 49) and has a spatial period of $\lambda_3 = 4.11$. The size of an individual roll changes as it travels through the modulation envelope (Figure 46d and 50). The largest rolls appear weak (they have low contrast) and the small rolls strong
**Figure 48**

Fundamental frequency of the traveling inclined rolls as functions of $R_i$: fundamental frequency for $R_o = 0$ (x), fundamental frequency for $R_o = 39$ (*).
Figure 48
Figure 49

Frequency of the $\simeq 3$ roll modulation as a function of $R_i$ for $R_o = 0$. 
(they have high contrast). This \( \sim 3 \) rolls (short wavelength) modulation appears for the values of \( R_o \in [-45, 45] \) with a threshold varying as shown in Figure 51. The velocity of the modulation depends on the value of \( R_o \) with the modulation envelope almost stationary for \( R_o = -29 \). Figure 51 shows that corotating the outer cylinder stabilizes the traveling inclined rolls against the short wavelength modulation while counter-rotating the outer cylinder destabilizes them.

The short wavelength modulations produce distortions of each roll along its axis (Figure 46d). This suggests that one possible scenario for the short wavelength modulation is that it is a wavy instability of the rolls (perhaps a short wavelength and more self-organized version of the long wavelength modulations seen at lower \( R_i \)), analogous to the wavy spirals in the counter-rotating Taylor-Couette system\(^{28} \). In this scenario we imagine that in the rest frame of the traveling rolls the modulation travels along the axis of the rolls with a velocity \( v_{mod} \) and a period \( \tau_{mod} \). The phase difference of these modulations between adjacent rolls produces the second axial wavelength, \( \lambda_{mod}^{2} \), shown in Figure 46d and 50. In the time, \( \tau_{mod} \), the roll has moved an axial distance (in the lab frame) \( \lambda_{mod}^{2} = \tau_{mod} v_x \), where \( v_x \) is the axial velocity of the rolls. The total azimuthal length of the fluid region is \( 2\pi R n \), where \( R \) is the average radius of the cylinders and \( n \) is the filling level fraction. If there are \( m \) modulations of wavelength \( \lambda_{mod} = v_{mod} \tau_{mod} \) along a roll of inclination angle \( \alpha \) (the angle from the vertical) then the total azimuthal length is also \( m\lambda_{mod} \cos(\alpha) \) or the period of the modulation is \( \tau_{mod} = \left( \frac{\lambda_{mod}}{v_x} \right) = \frac{2\pi R n}{m \cos(\alpha) v_{mod}} \). The axial components of the modulation velocity \( (v_{mod}^m) \) and the roll velocity \( (v_z) \) in the laboratory frame are related to the axial component of the modulation velocity in the rest frame \( (v_{mod} \sin(\alpha)) \) by \( v_z = v_z^m + v_{mod}^m \sin(\alpha) \). The data shows that \( v_z^m \ll v_z \) and so
Figure 50

Space-time diagram of the traveling inclined rolls with the short wavelength (~ 3 roll) modulation at $R_i = 303$ and $R_o = 0$
Phase diagram \((R_o, R_i)\) for the threshold of the traveling inclined rolls pattern, and the short wavelength modulation pattern. The solid lines are guides to the eye.
$v_z \approx v_m \sin(\alpha)$. The expression for $m$ is

$$m = \frac{2\pi Rn}{\lambda_{mod} \tan(\alpha)}. \quad (3.8)$$

Given the axial velocities in the data ($v_z = 0.6 \text{cm/sec}$ and $\lambda_{mod} = 2.84 \text{cm}$) of Figure 50, and the $20^\circ$ inclination angle, there should be $\approx 2.9$ azimuthal waves. We observe $\approx 2$ waves in the front face while the third may remain unseen because of the decreasing intensity of the rolls along the azimuth. Thus with only the axial wavelength of the modulation as the input we are able to get the azimuthal wave number of the rolls.

The above does not give us a physical reason for the wavy instability (except for a vague reference to a self-organized short wavelength modulation). We now suggest another scenario which gives us a physical mechanism for the short wavelength modulation. Before the onset of the traveling inclined rolls the approximate base flow in this region of parameter space is shown in Figure 43a. This figure shows that there is a potentially unstable layer near the inner cylinder. We, however, observed that the traveling rolls form near the outer cylinder in the bulk of the fluid. This suggests that the modulation may result from Taylor-Couette type rolls which form near the inner cylinder and travel slowly with a small inclination angle. The modulations are, in this picture, a result of the nonlinear superposition of the two structures in the region of the gap space where they overlap. This view is supported by the fact that the point where the two instability lines shown in Figure 51 intersect (the codimension two point) is close to the codimension two point connecting the stationary Taylor-Couette roll and traveling roll instability lines (compare Figures 42a and 51). The Taylor-Couette rolls are, however, stationary rolls while these rolls which generate the
modulations in this picture are inclined and traveling. This behavior of the Taylor-Couette type rolls may be a result of their interaction with the traveling inclined rolls. The modulations were never observed to travel in the opposite direction of the traveling rolls and they were almost stationary at the codimension two point. This suggests that the traveling rolls may be exerting a stress on the Taylor-Couette rolls resulting in this inclined and traveling behavior of the Taylor-Couette rolls.

We also measured the time averaged spatial intensity-intensity autocorrelation function of the rolls along the axis for $R_o = 0$. Time averaging of individual spatial autocorrelations was done over a time of $\approx 40\tau$, and had the effect of removing the long range correlation. Before the onset of the short wavelength modulation the envelope of this correlation function decreases smoothly as shown in Figure 52. We found this envelope to fit very well to an exponential function which has a characteristic length (exponential constant) of 2.6. The lack of long range correlation was evidently due to the slow phase modulations and the associated defects. After the short wavelength modulation appears, the positive correlation remains strong only at distances of 3 rolls as shown in Figure 53. The decrease in the correlation between individual rolls is caused by the variation in sizes of the rolls as they travel through the modulation envelope (see Figure 46d). The larger structure ($\sim 3$ roll modulation) tends to decorrelate the rolls by strongly distorting their shape. As $R_i$ is increased the 3 roll correlation peak persists while decreasing in height as the flow pattern becomes more incoherent. Figure 54 shows the autocorrelation of the pattern corresponding to the apparently incoherent pattern shown in the photograph in Figure 46e. It is interesting to note that the correlation envelope of the $\sim 3$ roll pattern (Figure
Figure 52

Time averaged spatial autocorrelation function of the reflected light intensity along the axis of cylinders close to onset of the traveling inclined roll pattern at $R_t = 269$ and $R_o = 0$. $Z$ is the difference in distance along the axis and a photograph of the pattern at these parameter values is shown in Figure 46b.
Figure 53

Time averaged spatial autocorrelation function of the reflected light intensity along the axis of cylinder for the short wavelength (~3 roll) modulation of the traveling inclined roll pattern at $R_i = 308$ and $R_o = 0$. $Z$ is the difference in distance along the axis and a photograph of the pattern at these parameter values is shown in
Figure 53

CORRELATION

0 1.00
-1.00

Z

0 11.00 22.00
Figure 54

Time averaged spatial autocorrelation function of the reflected light intensity along the axis of cylinder for the incoherent pattern shown in the photograph of Figure 46e at $R_i = 338$ and $R_o = 0$. $Z$ is the difference in distance along the axis.
Figure 54
Figure 55

Space-time diagram of the ~ 3 roll pattern at $R_i = 308, R_o = 0$. The traveling rolls are separated by propagating envelopes which exhibit phase modulations and defects. The superposed lines which follow the highly modulated parts of the traveling rolls indicate these defects and phase modulations.
Figure 55  

**AXIAL POSITION**
53) has approximately the same size as the envelope for the traveling roll pattern (Figure 52). This is because the long range correlation of the ~3 roll pattern is destroyed by phase modulations and defects in its structure just as in the case of the initial traveling roll pattern. The space-time diagram shown in Figure 55 shows some very strong modulation of the ~3 roll pattern.

3.6 Standing Waves

Riecke et al.\textsuperscript{76} did a bifurcation analysis on systems in which the continuous translational symmetry is broken when a supercritical Hopf bifurcation results in traveling waves. They predicted that a breaking of the time translational symmetry by a small periodic modulation of the control parameter would result in a standing wave pattern. The modulation frequency in their theory is $\approx 2f_h$, where $f_h$ is the Hopf frequency of the pattern near onset. These predictions have already been successfully tested in nematic liquid crystals and binary-mixture convection.\textsuperscript{77} The transition to traveling inclined rolls is apparently just such a Hopf bifurcation to traveling waves that breaks the continuous translational symmetry (along the axes of the cylinders) of the base flow of the Taylor-Dean system. We tested this prediction on the traveling inclined rolls with the outer cylinder at rest. We used two motors to drive the inner cylinder, one motor to produce a net rotation and the other motor to sinusoidally modulate the angular velocity of the inner cylinder. The first motor was directly connected to the inner cylinder and simply rotated with a constant angular velocity. The second motor was connected to the housing of the first motor and oscillated the first motor. This gave a net output to the inner cylinder of a constant angular velocity (due to the first motor) plus a periodic variation in angular velocity (due to the
second motor). The frequency and maximum angular velocity of the oscillating motor was controlled using a Compumotor 2100 Series Indexer. Although the Compumotor 2100 Series Indexer actually produces a piecewise linear periodic change in angular frequency (a triangle wave in angular velocity with the peaks truncated by a constant angular velocity), over 80% of the energy of the motion produced by the oscillating motor was in the pure sine wave component. Because both the frequency and angular velocity amplitude of the oscillating motor could be controlled this introduced two new externally controlled parameters into the system as required by the theory. The Reynolds number $R_i$ is now $R = R_i + R_M \sin(2\pi ft)$. The two dimensionless parameters are $\frac{R_M}{R_{ic}}$ and the detuning $\epsilon = \frac{2\Delta f}{2f_0}$. At $\epsilon \approx 0$ we easily found standing waves. Figure 56 show a phase diagram of the transitions from the base flow to standing waves and to traveling waves at $\epsilon \approx 0$. The region between the traveling waves and the standing waves did not have the usual spatial separation of left and right traveling rolls nor did they appear to be standing waves. The pattern in this region appeared to be a long range superposition of both waves over most of the cylinder length. One interesting feature to note in Figure 56 is that the standing waves can be excited at $R_i$ values lower than the critical $R_i$ for the onset of traveling inclined rolls. This feature was also predicted by the theory. We also tested the modulation apparatus on the Taylor-Couette system in the region of $(R_o, R_i)$ parameter space where the spiral vortices form. Riecke\textsuperscript{78} had predicted that standing waves would not from in this situation unless the azimuthal symmetry was also broken (in order to break the time translational symmetry of the Taylor-Couette system it must be temporal modulationed and the azimuthal symmetry must be broken). We were unable to find standing waves in the Taylor-Couette system using the modulation apparatus.
Figure 56

The phase diagram of the transitions from base flow to standing waves and to traveling waves at $\epsilon \approx 0$. 
Figure 57

Space-time diagram of the standing wave pattern at $R_i = 265, R_o = 0, \frac{R_M}{R_{le}} = 0.2$ and $\epsilon \approx 0$. 
Figure 57 shows a space-time diagram of the standing waves at $\frac{R_m}{R_{tc}} = 0.2$ and $\epsilon \approx 0$ at $R_i = 265$. A standing wave pattern is apparent except that the standing waves appear to move relative to the CCD camera. One can see from this space-time diagram that the velocity of the right traveling wave ($\approx 23$) is slightly less than the left traveling wave ($\approx 25$). The entire pattern moves to the left at a velocity of $\approx 1$. The Figure 56 shows that the onset of standing waves at $\frac{R_m}{R_{tc}} = 0.2$ is at $R_i = 252$ so that the space-time diagram in Figure 57 was taken considerably above onset. This drifting of the pattern shows that the standing wave pattern is beginning to become unstable to the traveling wave pattern. This region of parameter space has not be thoroughly explored.

3.7 Conclusion

We have shown that the flow in the Taylor-Dean system is very rich in non-equilibrium patterns. We have found a variety of different structures extending over wide ranges of the external control parameters $R_o$ and $R_i$. Five primary instability lines and four codimension two points have been directly observed. Approximate characteristics of the states (threshold control parameter values, frequency of inclined rolls at threshold, and wavelengths) have been determined. However, our results indicate the need for further, more extensive, investigations. Theoretically, account must be taken of non-axisymmetric perturbations, a finite gap, the free upper surface, and nonlinearities. The last is no doubt important in the vicinity of the codimension two points, where mixed mode patterns have been observed. In addition the recirculation rolls due to the free upper surface appear to be important in the generation of the traveling inclined rolls. The traveling rolls, which appear to be similar in structure to the
Karman vortex street, may yield insight into open flow instabilities which are difficult to model and visualize. Experimentally it would be desirable to establish the details of the flows in the codimension two point neighborhoods.79

The traveling inclined rolls in the strong $R_i$ case undergo, after the onset of a long wavelength modulation associated with defects, a short wavelength modulation with an axial wavelength of $\sim 3$ rolls. This modulation appears to be a wavy instability of the rolls generated by interacting structures due to competing instabilities in different sublayers. This short wavelength modulation pattern strongly distorts individual rolls and exhibits a long wavelength modulation associated with defects. This transition to the $\sim 3$ roll modulation pattern is not far from the onset of the primary instability. We also found standing waves at $R_o = 0$ when the inner cylinder was modulated.
CHAPTER IV

CONCLUSION

In both the Taylor-Couette system and the Taylor-Dean system we have found and characterized a variety of different spatiotemporal patterns over wide ranges of the external control parameters. We have attempted to concentrate on spatiotemporal patterns because we believe that an understanding of such patterns is a significant step towards a general understanding of the nonlinear dynamics of fluid motion. Because fluids are believed to obey Newton’s laws, as expressed in the Navier-Stokes equations in the case of an incompressible fluid, an understanding of spatiotemporal patterns is really an understanding of the solutions of the Navier-Stokes equations. In particular this means that we are interested in spatially varying flows when there is an explicit time dependence, or when \( \frac{\partial \mathbf{V}}{\partial t} \neq 0 \). Such explicit time dependence is a characteristic quality of the nonlinear dynamics of fluid motion when the fluid is driven with sufficient energy. The temporal behavior of nonlinear systems has been fairly well characterized from the study of dynamical systems (chaos) but these ideas are only relevant to temporal behavior of spatially uniform patterns (nonlinear differential equations). We are interested in the more general case when the field quantities exhibit both spatial and temporal behavior (nonlinear partial differential equations like the full Navier-Stokes equations). To understand the nonlinear dynamics of fluid motion it is important to include both the space and time behavior because fluids become irregular in both space and time (i.e. they become turbulent) when they are driven sufficiently hard.
A general understanding of all spatiotemporal patterns implies an understanding of the Navier-Stokes equations for all boundary conditions. With this in mind we may think of the Taylor-Couette system and the Taylor-Dean system as analogue computers that integrate the Navier-Stokes equations for two different types of boundary conditions in the absence of external force fields. In the above experiments we have let the apparatus do the math, i.e. integrate the Navier-Stokes equations (a task that would be very difficult for us to do) while we have concentrated on the physics (which is relatively easy for us given the incredible computing power of our apparatus without the economic limitations of numerical computers). The experimental problem is to obtain output from these analogue computers as the input, i.e. the boundary conditions, is changed. We have changed the input by changing dimensionless parameters, such as $R_o$, $R_i$, $\Gamma$, etc., which specify the boundary conditions. We have used flow visualization to get our output because this technique responds quickly to changes in the entire flow field, i.e. it gives us both temporal and spatial information about the flow. Of course these two types of boundary conditions are a small subset of all boundary conditions so it is unlikely that from this research a general understanding of the Navier-Stokes equations can be induced.

Because the flow is described by the Navier-Stokes equations it is not surprising that the patterns we have seen have the same symmetries as the Navier-Stokes equations for our system. In fact, an often used approach to describe the flow patterns in the Taylor-Couette system is the theory of bifurcation in the presence of symmetry. This is an approach where the symmetries of the system, along with some abstract theorems which reduce the complexity of the Navier-Stokes equations, are used to classify the possible patterns. In this view the transition
sequence for corotating cylinders (where Couette flow becomes unstable to TVF which becomes unstable to either WVF, WIB, WOB, WVL, or TWI) is seen as a sequence of symmetry breaking bifurcations. The Couette flow, having the most symmetry (the flow is invariant under continuous translation and reflection along the axis and continuous rotation about the axis) becomes unstable to a flow, TVF, that only has a discrete symmetry along the axis (it is only invariant if one does a translation along the axis by the wavelength of the TVF). The transition to one of the various traveling wave states has even less symmetry because now the continuous rotational symmetry has been broken as well. This same type of argument can be made for the patterns in the Taylor-Dean system and the spiral vortices in the counter-rotating Taylor-Couette system (which break both continuous symmetries). In fact this is exactly the type of theory that predicted the standing waves in the Taylor-Dean system which we successfully found. The notion of symmetry was important even for the spiral turbulence pattern which results from a subcritical bifurcation. However, this symmetry is only an average symmetry and, although this subcritical bifurcation certainly breaks the symmetry, it is difficult to say if bifurcation theory is applicable. Because this theory is usually only valid near the instability points it is very difficult to get numbers to compare with experiments except very close to the instability points. Numerical integration can explicitly represent the velocity field and we have found this to be very helpful, as in the corotating Taylor-Couette system, in obtaining a coherent physical picture of the flow patterns. We have also found in the case of spiral turbulence, where even numerical integration would be extremely difficult, a coherent physical picture can emerge if we are clever and change the parameters enough (the number of possible physical pictures decreases as more facts are established).
The existence of a symmetry in the system does not guarantee that a pattern of that symmetry will appear. In equilibrium systems where a potential functional exists the pattern that is selected is the one that minimizes a potential. In nonequilibrium systems a potential does not always exist and the pattern that is selected is not always unique. For example the TVF to WVF transition breaks the continuous azimuthal symmetry. There are many wave numbers, however, that will also break this symmetry. If the same path in parameter space is followed the same number of waves form repeatably. If a different path is taken a different number of waves may form. The system will select a given wave number at a given critical point even though there are many possibilities. Usually it picks the most unstable wave number and we have used this selection principle to help argue that the instability that leads to the WOB pattern is not a shear instability. This may not always true, e.g. we found that the helicity of spiral turbulence was apparently selected at random, although this could have been the result of randomness in the finite amplitude perturbation. Perhaps the many patterns and their instability points presented here will give insight into the problem of pattern selection.

Once a periodic spatiotemporal pattern has been established, such as spiral turbulence, small variations in the phase of the pattern may be described by phase dynamics. We have used this approach to describe the pitch variation of the spiral. This approach may also be useful for describing the vortex thickness variation of TVF at N=32 and the phase variations in the traveling inclined rolls. The phase variations in the traveling inclined rolls are associated with defects and phase dynamics can not describe such a large local change in the phase.

Of all the patterns studied the most difficult to characterize was the traveling inclined rolls above onset. The methods we used on the other patterns did not
yield consistent results, i.e. the pattern has different frequencies at different spatial positions, locally irregular temporal behavior, and instantaneous long range spatial correlation. The time averaged spatial autocorrelation characterized the pattern because it mixed both the space and time character of the pattern. It showed that the long range correlations fluctuate in time such that when they are time averaged they disappear. We reasoned that this behavior was caused by the long range phase variation and the associated defects. This type of experimental characterization could be used for patterns with similar behavior.

We have concentrated on studying the physics of the spatiotemporal pattern in the Taylor-Couette and Taylor-Dean systems. We have used our experimental results with the help of previous results to induce a coherent picture of the physical processes in these pattern. We have effectively used the pressure gradient, this being a local force on the fluid elements in the flow, to understand the physics of the flow pattern. This is a natural thing to do in the light of Newton’s laws where one examines the forces acting on a body in order to find out what it will do. In many theoretical analysis in fluid dynamics, however, the pressure is only thought of as the field that is easiest to eliminate from the equations in the effort to find the “answer”. A coherent physical understanding of these patterns was often a subtle thing to find, but we hope that the physical pictures presented here may also help to understand other fluid system.
LIST OF REFERENCES


71. Experiments performed at the Weizmann Institute, Rehovot (Israel) [J.E. Wesfreid (private communication)].


