INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book. These are also available as one exposure on a standard 35mm slide or as a 17" x 23" black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Interfacial interactions in heat transfer and fluid flow through porous media

Kim, Sung Jin, Ph.D.
The Ohio State University, 1989
INTERFACIAL INTERACTIONS IN HEAT TRANSFER
AND FLUID FLOW THROUGH POROUS MEDIA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Sung Jin Kim, B.S., M.S.

* * * * *

The Ohio State University
1989

Dissertation Committee:
K. Vafai
B. J. Hamrock
M. Samimy

Approved by
Department of Mechanical Engineering
To my beloved wife, Yeon Sun
ACKNOWLEDGEMENTS

Expressing my thanks to my colleagues, friends and family in a few sentences is almost impossible. First of all I would like to thank God for making what I am. Professor Vafai stands out with his involvement and constant guidance to this work during the past several years. I wish to thank my committee members, B. J. Hamrock, and M. Samimy.

I am grateful to Ohio Supercomputer Center for providing me with the computer time for the numerical computations. In addition I would like to mention the valuable memory and appreciate countless help from Chan Mun Kim, Javad Effefagh, Mehmet Sozen, Hwa Chong Tien, and Jamel Belwafa in the Heat Transfer Lab.

Finally, I would like to dedicate this dissertation to my parents, Jae Yung and Young Sook Kim, and to my beloved wife, Yeon Sun Kim. They supported and encouraged me very much whenever I needed it. I hope this small work makes my wife and my daughter, Hye Jin, forgive their husband and father for staying away from them for the past few years.
VITA

February 5, 1960 ............................................ Bom-Seoul, Korea
1982 .............................................................. B.S., Seoul National University, Seoul, Korea
1984 .............................................................. M.S., Seoul National University, Seoul, Korea
1984-1985 ....................................................... Research Scientist, Korea Advanced Institute of Science and Technology, Seoul, Korea
1985-1989 ....................................................... Research/Teaching Associate The Ohio State University Columbus, Ohio

FIELDS OF STUDY

Major Field:  Mechanical Engineering

Studies in Heat Transfer and Fluid Mechanics

Prof.  K. Vafai
# TABLE OF CONTENTS

ACKNOWLEDGEMENTS ........................................................... iii
VITA ......................................................................................... iv
LIST OF FIGURES ................................................................... vii
NOMENCLATURE ..................................................................... x

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. GENERAL INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. NATURAL CONVECTION ABOUT A VERTICAL PLATE EMBEDDED IN A POROUS MEDIUM</td>
<td></td>
</tr>
<tr>
<td>2.1 Statement of the problem</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Formulation of the problem</td>
<td>7</td>
</tr>
<tr>
<td>2.3 Analysis and solutions</td>
<td>27</td>
</tr>
<tr>
<td>2.4 Results and discussion</td>
<td>38</td>
</tr>
<tr>
<td>III. FORCED CONVECTION IN A CHANNEL FILLED WITH A POROUS MEDIUM</td>
<td></td>
</tr>
<tr>
<td>3.1 Statement of the problem</td>
<td>53</td>
</tr>
<tr>
<td>3.2 Analysis</td>
<td>55</td>
</tr>
<tr>
<td>3.3 Results and discussion</td>
<td>62</td>
</tr>
<tr>
<td>IV. FLUID FLOW AND HEAT TRANSFER IN A POROUS/FLUID COMPOSITE MEDIUM</td>
<td></td>
</tr>
<tr>
<td>4.1 Statement of the problem</td>
<td>68</td>
</tr>
<tr>
<td>4.2 Analysis</td>
<td>70</td>
</tr>
</tbody>
</table>
4.3 Results and discussion ............................................................. 79

V. CONCLUSIONS AND RECOMMENDATIONS .......................... 93

APPENDIX A ........................................................................................................... 96

LIST OF REFERENCES ....................................................................................... 102
LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURES</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Schematic of the physical model and coordinate system (a) $\varepsilon_T \gg \varepsilon_Y$, (b) $\varepsilon_T \ll \varepsilon_Y$.</td>
</tr>
<tr>
<td>2.2</td>
<td>Comparison between the analytical and numerical velocity distribution for Case TI, $\Gamma_T x^{-1/2} = 0.3$.</td>
</tr>
<tr>
<td>2.3</td>
<td>Comparison between the analytical and numerical temperature distribution for Case TI.</td>
</tr>
<tr>
<td>2.4</td>
<td>Comparison between the analytical and numerical velocity distribution for Case TII, $\Delta_T x^{-1/4} = 0.3$.</td>
</tr>
<tr>
<td>2.5</td>
<td>Comparison between the analytical and numerical velocity distribution for Case III, $\Gamma_H x^{-1/3} = 0.2$.</td>
</tr>
<tr>
<td>2.6</td>
<td>Comparison between the analytical and numerical temperature distribution for Case HI.</td>
</tr>
<tr>
<td>2.7</td>
<td>Comparison between the analytical and numerical velocity distribution for Case III, $\Delta_H x^{-1/5} = 0.2$.</td>
</tr>
<tr>
<td>2.8</td>
<td>Effects of the local thickness ratio on the vertical velocity for Case TI.</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>2.9</td>
<td>Effects of the local thickness ratio on the vertical velocity for Case HII.</td>
</tr>
<tr>
<td>3.1</td>
<td>Schematic and coordinate system of the problem.</td>
</tr>
<tr>
<td>3.2</td>
<td>Fully developed dimensionless velocity profiles</td>
</tr>
<tr>
<td>3.2</td>
<td>(a) Da^{-1/2} = 10, (b) Da^{-1/2} = 30.</td>
</tr>
<tr>
<td>3.3</td>
<td>Fully developed dimensionless temperature profiles</td>
</tr>
<tr>
<td>3.3</td>
<td>(a) Da^{-1/2} = 10, (b) Da^{-1/2} = 30.</td>
</tr>
<tr>
<td>3.4</td>
<td>Variation of the Nusselt number.</td>
</tr>
<tr>
<td>4.1</td>
<td>Schematic and the coordinate system of the porous/fluid composite system.</td>
</tr>
<tr>
<td>4.2</td>
<td>Comparison of numerical solutions with the similarity solutions for convective flow over a semi-infinite flat plate in a Newtonian fluid, Re_x=3x10^5, Pr=1.0.</td>
</tr>
<tr>
<td>4.2</td>
<td>(a) Velocity distribution (b) Temperature distribution.</td>
</tr>
<tr>
<td>4.3</td>
<td>Comparison of numerical solutions for the velocity with the corresponding analytical solutions</td>
</tr>
<tr>
<td>4.3</td>
<td>(a) flow over a flat plate embedded in a porous medium</td>
</tr>
<tr>
<td>4.3</td>
<td>(b) flow in a channel partially filled with a porous medium.</td>
</tr>
<tr>
<td>4.3</td>
<td>(c) flow over a flat plate; a fluid layer is sandwiched between a flat plate and the porous medium.</td>
</tr>
<tr>
<td>4.4</td>
<td>(a) Velocity and (b) temperature distribution along the flat plate at three different locations,</td>
</tr>
<tr>
<td>4.4</td>
<td>x=0.2, 0.5, and 0.8 for H*=0.02.</td>
</tr>
<tr>
<td>4.5</td>
<td>Effects of Darcy number on (a) Friction coefficient, (b) Nusselt number and (c) convection heat transfer coefficient</td>
</tr>
<tr>
<td>4.6</td>
<td>Effects of the inertia parameter on the friction coefficient and the Nusselt number.</td>
</tr>
</tbody>
</table>
4.7   Effects of the Prandtl number on the Nusselt number ............... 90

4.8   Effects of the conductivity ratio of the porous medium
       for the fluid on the Nusselt number. ......................................... 92

A.1   Schematic and the corresponding coordinate system .................... 97
NOMENCLATURE

\( C_f \) friction coefficient, equation (4.18)

\( C_p \) specific heat \([\text{Jkg}^{-1}\text{K}^{-1}]\)

\( D_a \) Darcy number, equation (2.4), (3.7a), and (4.15)

\( F \) a function used in expressing inertia terms, which depends on the Reynolds number and the microstructure of the porous medium

\( g \) gravitational acceleration \([\text{ms}^{-2}]\)

\( h \) convective heat transfer coefficient \([\text{Wm}^{-2}\text{K}^{-1}]\)

\( H \) one half of channel height (Chap.3)

\( k \) thickness of the porous medium (Chap. 4) \([\text{m}]\)

\( k_e \) thermal conductivity \([\text{Wm}^{-1}\text{K}^{-1}]\)

\( k_e \) effective thermal conductivity of the saturated porous medium \([\text{Wm}^{-1}\text{K}^{-1}]\)

\( K \) permeability of the porous medium \([\text{m}^2]\)

\( L \) reference dimensional length \([\text{m}]\)

\( \text{Nu} \) Nusselt number, equation (2.78), (3.12), and (4.19)

\( P \) pressure \([\text{Pa}]\)

\( \text{Pr} \) Prandtl number

\( \text{Ra} \) modified Rayleigh number, equation (2.4), (2.32)

\( \text{Re} \) Reynolds number, equation (4.14)

\( T \) dimensional temperature \([\text{K}]\)

\( u \) \(x\)-component velocity \([\text{ms}^{-1}]\)
u_\infty \quad x\text{-component velocity outside the momentum boundary layer}

v \quad y\text{-component velocity [ms}^{-1}\text{]}

\vec{v} \quad velocity vector

x \quad streamwise coordinate [m]

y \quad transverse coordinate [m]

Y \quad dimensionless coordinate in the outer layer, y

\hat{Y} \quad dimensionless coordinate in the inner layer, y

Greek symbols

\alpha_c \quad effective thermal diffusivity [m^2s}^{-1}\text{]}

\beta \quad thermal expansion coefficient of fluid

\Gamma_T \quad perturbation parameter for Case TI

\Gamma_H \quad perturbation parameter for Case HI

\delta \quad porosity of the porous medium

\Delta_T \quad perturbation parameter for Case TII

\Delta_H \quad perturbation parameter for Case HII

\eta_{1,2} \quad similarity variable for Cases TI, TII

\zeta_{1,2} \quad similarity variable for Cases HI, HII

\Lambda_1 \quad inertia parameter, equation (3.7b), (4.15)

\mu_f \quad fluid viscosity [Kg m}^{-1}s}^{-1}\text{]}
\( \nu_f \) \hspace{1em} \text{kinematic viscosity [m}^2\text{s}^{-1}] \\
\theta \hspace{1em} \text{dimensionless temperature} \\
\Theta \hspace{1em} \text{dimensionless temperature in the outer layer} \\
\hat{\Theta} \hspace{1em} \text{dimensionless temperature in the inner layer} \\
\rho_f \hspace{1em} \text{fluid density [Kg m}^{-3}] \\
\psi \hspace{1em} \text{stream function} \\
\Psi \hspace{1em} \text{dimensionless stream function in the outer layer} \\
\hat{\Psi} \hspace{1em} \text{dimensionless stream function in the inner layer} \\
\zeta \hspace{1em} \text{vorticity} \\

\text{Superscript} \\
* \hspace{1em} \text{based on constant wall heat flux} \\

\text{Subscripts} \\
C \hspace{1em} \text{composite solution} \\
eff \hspace{1em} \text{effective} \\
f \hspace{1em} \text{fluid} \\
H \hspace{1em} \text{constant wall heat flux case} \\
m \hspace{1em} \text{mean value across the channel} \\
T \hspace{1em} \text{constant wall temperature case} \\
w \hspace{1em} \text{condition at the wall} \\
x \hspace{1em} \text{local} \\
\infty \hspace{1em} \text{condition at infinity}
CHAPTER I

GENERAL INTRODUCTION

Convective heat transfer in fluid-saturated porous media has gained considerable attention in recent years due to its various applications in contemporary technology. These applications include packed bed heat exchangers, heat pipes, thermal insulations, petroleum reservoirs, nuclear waste repositories, and geothermal engineering. Combarnous and Bories (1975), Cheng (1978), and Tien and Vafai (1989) have presented comprehensive reviews on the state of the arts. Most of the studies in porous media carried out so far are based on the Darcy flow model, which is an empirical law for creeping flow through an infinitely extended uniform medium. However, it is now generally recognized that the non-Darcian effects are quite important for certain applications. Different models have been introduced for studying non-Darcian effects such as the inertia, impermeable boundary, and porosity variation.

The main objective of the present work is to study the effects of boundary friction and inertia on convective heat transfer at the interface regions between a porous medium and another medium. In general, the other medium could be a fluid, a solid or another porous medium. A specific example of the interface region can be cited from petroleum
reservoirs wherein the oil flow encounters different layers of sand, rock, shale, limestone, etc. Similar situations are encountered in many other cases of practical interest such as solid matrix heat exchangers, iron blast furnaces, metal processing, geothermal operations, nuclear waste repositories, underground coal gasification, groundwater hydrology, etc. Among the various configurations, three fundamental cases are analyzed in depth in the present work. Two of them are external convective flow problems and the rest is internal convective flow problem.

In Chapter 2, buoyancy-driven fluid flow and heat transfer about a vertical flat plate embedded in a porous medium are thoroughly analyzed using the method of matched asymptotic expansions. Cheng and Minkowycz (1977) studied this problem first to simulate the heating of an aquifer by a dike using the Darcy flow model. In the present work the general cases of constant wall temperature and the constant wall heat flux are studied to analyze the effect of an external boundary on the convective flow. By an order of magnitude analysis it is shown that there exist two important limiting categories for each case, depending on the thickness ratio of the thermal boundary layer to that of the viscous boundary layer. The first category is encountered when the thickness of the thermal boundary layer is much larger than that of the viscous boundary layer, while the second category is encountered when the thickness of the viscous boundary layer is much larger than that of the thermal boundary layer. Approximate analytical solutions are presented using the Oseen linearization method for the first category as well as for the second category which has never been dealt with before. Numerical solutions are also obtained to check the validity of the analytical solutions.
In Chapter 3, fully-developed forced convection in a porous channel bounded by parallel plates subjected to constant heat flux condition is considered based on the general flow model. The effects of inertia as well as bounding surfaces on the flow and the temperature fields are taken into account. Koh and his co-workers (1974, 1975) studied this problem using slug flow model. They performed the analysis on the heat transfer augmentation of a regenerative heat exchanger by insertion of a high-conductivity porous material in the coolant passage. For the case with fixed allowable wall temperature it was shown that the heat flux at the channel wall can be increased by over three times by using a porous material in the channel. This scheme has important practical applications, a typical example being the cooling of rocket nozzles. Here an augmentation of regenerative cooling in the nozzle can be effectively achieved when the convective cooling of the nozzle has reached its limit. Exact solutions are obtained in the present work for both the velocity and the temperature distribution. From these results the Nusselt number can be expressed in terms of the Darcy number and the inertia parameter. Comparisons are made with the limiting case of no inertia and/or boundary effects.

In Chapter 4, forced convective flows through a porous/fluid composite medium are studied numerically. Flow over an impermeable boundary covered with a porous substrate is considered. The numerical simulations focus primarily on the flows which have the boundary layer characteristics. A general flow model which accounts for the effects of the solid boundary and inertia is used to describe the flow inside the porous region. A numerical code developed for the composite medium was tested for many different cases so that the results from this can be physically meaningful. The effect of porous substrate on the velocity and the temperature fields is analyzed using finite
difference methods. The friction coefficient and the Nusselt number at the wall as well as at the interface are compared with and without a porous substrate. Several important characteristics of the flow and the temperature fields in the composite layer are reported and the dependence of these characteristics on the governing parameters is also documented.

In the last chapter general discussions on the physical aspects of the problems are made and the relevance of the present work is summarized. In addition several aspects associated with the above-mentioned problems are identified for the future research. Throughout this work a conscious effort has been made in order to deliver a compact ideas to bare essentials. To this end, unnecessary peripheral information has been minimized in each chapter.
CHAPTER II

NATURAL CONVECTION ABOUT A VERTICAL PLATE EMBEDDED IN A POROUS MEDIUM

2.1 INTRODUCTION

The majority of existing studies on convective heat transfer in porous media are based on the Darcy flow model. Darcy's law, however, is found to be inadequate for the formulation of fluid flow and heat transfer problems in porous media when there is an impermeable boundary and/or the Reynolds number based on the pore size is greater than unity. Therefore, it is necessary to incorporate the boundary and inertia terms into the momentum equation.

The problem of natural convection about a heated impermeable surface embedded in fluid-saturated porous media was studied by Cheng and Minkowycz (1977). They obtained similarity solutions which were based on Darcy's law and boundary layer approximations. To extend the range of applicability of the boundary layer analysis to relatively lower modified Rayleigh numbers, Cheng and Hsu (1984), and Joshi and Gebhart (1984) examined higher order effects such as the entrainment from the edge of the boundary layer, the axial heat conduction, and the normal pressure gradient using the
method of matched asymptotic expansions. A number of recent studies have considered the various non-Darcian effects on the same problem. Bejan and Poulidakos (1984), and Plumb and Huenefeld (1981) used a numerical solution of the Forschheimer equation to account for inertia effects and obtained similarity solutions based on the boundary layer approximations. Hsu and Cheng (1985) and Evans and Plumb (1978) studied mainly numerically the boundary effects on heat transfer and fluid flow based on Brinkman's equation. Hong et al. (1985) included both the boundary and inertia effects as well as the convective term in the momentum equation and presented a numerical solution. Also Kaviany and Mittal (1987) studied natural convection from a vertical plate experimentally and reported Karman-Pohlhausen solutions for a high permeability porous medium. These investigators have shown that both the boundary and inertia effects decrease the heat transfer rate.

In this chapter natural convection about a vertical flat plate in a fluid-saturated porous medium is considered based on the Brinkman-extended Darcy flow model. It should be noted that for this problem the power-law wall temperature variation no longer represents the case of the constant wall heat flux when either the higher order effects or the boundary effects are taken into account as pointed out by Joshi and Gebhart (1984) and Hong et al. (1987), respectively. Hence the case of the constant wall temperature and the case of the constant wall heat flux are treated separately. Also it was shown in the work of Hsu and Cheng (1985) that there are two governing dimensionless parameters associated with this problem. These parameters, which can be expressed in terms of the modified Rayleigh number and the Darcy number as shown in the next section, are related to the thickness of the viscous boundary layer and the thickness of the thermal boundary layer. Hence, depending on the thickness ratio of the viscous
boundary layer to that of the thermal boundary layer there exist two categories. The first category, which can be encountered when the thickness of the thermal boundary layer is much larger than that of the thermal boundary layer, was numerically studied by Cheng and Hsu (1985), while the second category has not received any attention so far even though it can be the dominant situation for high permeability porous media as well as near the leading edge for moderate to low permeability porous media.

The objective of the present chapter is to study these two categories for the cases of the constant wall temperature and the constant wall heat flux both analytically and numerically. The governing equations obtained by applying the method of matched asymptotic expansions for each category are solved both analytically using the modified Oseen method and numerically using the similarity transformation. As will be shown in the following sections, comparisons of the results for different categories reveal important characteristics related to the heat transfer and fluid flow about a vertical flat plate embedded in a porous medium.

2.2 FORMULATION OF THE PROBLEM

The steady two-dimensional natural convection about a semi-infinite vertical flat plate embedded in a fluid-saturated porous medium is considered. The schematics of the problem are shown in Fig. 2.1. It is assumed that the fluid and the solid matrix are in local thermodynamic equilibrium, and that the porous matrix is homogeneous and isotropic. Also inertia and thermal dispersion effects are neglected.
Fig. 2.1 Schematic of the physical model and coordinate system

(a) $\mathcal{E}_T \gg \mathcal{E}_V$

(b) $\mathcal{E}_T \ll \mathcal{E}_V$
2.2.1 *Constant wall temperature case*

The governing equations for uniform porosity distribution, which are derived by using volume-averaged principles, are given as (Vafai and Tien, 1981)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  

(2.1)

\[
Da_L \nabla^2 \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -Ra_L \frac{\partial \theta}{\partial y}
\]  

(2.2)

\[
u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \nabla^2 \theta
\]  

(2.3)

where the pressure terms are eliminated and \(x, y, u, v,\) and \(\theta\) are nondimensionalized as

\[
x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}
\]  

(2.4)

\[
u^* = \frac{Lu}{\alpha_e}, \quad v^* = \frac{Lv}{\alpha_e}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}
\]

and * is dropped for convenience. It should be noted that the above set of governing equations neglect any transient effects. However, based on the results of Vafai and Tien (1982) the transient effects on the flow field are quite negligible. In the above equation the Darcy number and the Modified Rayleigh number are defined as

\[
Da_L = \frac{K}{L^2}, \quad Ra_L = \frac{Kg \beta L (T_w - T_\infty)}{\alpha_e \nu_f}
\]  

(2.5)
Introducing the stream function

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \] (2.6)

yields

\[ D_{aL} \nabla^4 \psi - \nabla^2 \psi = -Ra_L \theta_y \] (2.7)

\[ \psi_y \theta_x - \psi_x \theta_y = \nabla^2 \theta. \] (2.8)

The appropriate boundary conditions for this case are

\[ \psi = \psi_y = \theta - 1 = 0 \quad \text{at} \quad y = 0 \] (2.9 a,b)

\[ \psi_y = \theta = 0 \quad \text{as} \quad y \to \infty. \]

Under steady state conditions, in the thermal boundary layer (TBL hereafter) where the heating effect of the vertical wall is felt, the heat conducted from the wall into the fluid is carried upwards by the convective movement of the fluid. Hence the energy equation, Eq.(2.8), which expresses a balance between convection and conduction gives

\[ \psi \sim \frac{x}{\delta_T} \] (2.10)
where \( \delta_T \) denotes the thickness of the TBL. Turning our attention to the momentum balance in the same layer, we recognize the interaction between three forces in equation (2.7). The first term denotes the boundary frictional force due to an impermeable boundary, the second term denotes the bulk frictional drag due to the porous medium, and the last term denotes the buoyancy force. Note that the buoyancy force is not negligible in the TBL since, without it, there would be no flow. Then

\[
D_a L \frac{\nu}{\delta_T^2} \quad \text{or} \quad \frac{\nu}{\delta_T^2} \sim Ra_L \frac{1}{\delta_T}.
\]  

(2.11)

Combining equations (2.10) and (2.11) results in

\[
D_a L Ra_L^{-1} \frac{x}{\delta_T^2} \quad \text{or} \quad Ra_L^{-1} \frac{x}{\delta_T^2} \sim 1
\]  

(2.12)

From equation (2.12) it is clear that the TBL is ruled either by a 'boundary friction resistance \sim buoyancy force' balance or by a 'bulk frictional drag \sim buoyancy force' balance, depending on the magnitude of \( D_a L^{1/2} / \delta_T \). From the works of Vafai and Tien (1981) and Vafai (1984) it was shown that \( D_a L^{1/2} \) is the order of magnitude of the magnitude of the viscous boundary layer (VBL hereafter) where the effect of boundary friction is felt. Hence to capture the entire physics of the problem attention will be given to two cases; one is when the thickness of the TBL is much larger than that of the VBL (Fig. 2.1(a)) and the other is for the reverse situation (Fig. 2.1(b)). The former will be denoted as Case TI and the latter as Case TII.
2.2.1(a) *Case TI.* $Ra_L^{-1/2} \rangle \rangle Da_L^{1/2}$.

Guided by the order of magnitude analysis it can be shown that in this case the free convective boundary layer consists of two regions where different phenomena are predominant. First there exists an outer region where the buoyancy term has to be of the same order of magnitude as the bulk friction term. In this region the fluid is driven upwards by buoyancy and restrained by bulk friction. This means that outside this layer, where the fluid is isothermal and the buoyancy effect is absent, the fluid is motionless. The velocity profile must then be as wide as the temperature profile, which is also the case for the regular fluid with a low Prandtl number. Second, there is an inner region where the boundary friction is of the same order of magnitude as the bulk friction as well as the buoyancy. Since the no-slip condition still applies at the wall, this layer can be considered as the region where the vertical velocity of the fluid varies from zero at the wall to its maximum value as shown in Fig. 2.1(a). It should be mentioned that in the analysis the maximum velocity was not imposed to be at the edge of the VBL.

(1) The outer problem.

From the order of magnitude analysis it can be shown that

$$\delta_T \sim x^{1/2} Ra_L^{-1/2}, \quad \psi \sim x^{1/2} Ra_L^{1/2}.$$ 

Hence introducing the outer variables as

$$Y = Ra_L^{1/2} y, \quad \Psi = Ra_L^{-1/2} \psi, \quad \Theta = \theta$$
where $Y$, $\Psi$, and $\Theta$ are of the order of 1, transforms governing equations (2.7) and (2.8) into

$$
\Psi_{YY} + \varepsilon_T^2 \Psi_{XX} = \Theta_Y + \Gamma_T^2 (\Psi_{YYYY} + 2 \varepsilon_T^2 \Psi_{xYY} + \varepsilon_T^4 \Psi_{YYYY})
$$

subject to the boundary conditions

$$
\Psi_Y = \Theta = 0 \text{ as } Y \to \infty.
$$

Also $\Psi$ and $\Theta$ must match with the inner solutions as $Y \to 0$. Here two parameters are defined as

$$
\varepsilon_T = \text{Ra}^{-1/2}, \quad \Gamma_T = \text{Da}^{1/2} \text{Ra}^{1/2}.
$$

Since the thickness of the VBL which is $O(Da_L^{1/2})$ is much smaller than that of the TBL which is $O(\text{Ra}_L^{-1/2})$, $\Gamma_T$, the ratio of these two, can be chosen as an expansion parameter. Expanding $\Psi$ and $\Theta$ as

$$
\Psi = \Psi_0 + \Gamma_T \Psi_1 + \Gamma_T^2 \Psi_2 + \cdots
$$

$$
\Theta = \Theta_0 + \Gamma_T \Theta_1 + \Gamma_T^2 \Theta_2 + \cdots
$$
yields for the leading order

\[ \Psi_{0YY} = \Theta_{0Y} \quad (2.15) \]

\[ \Theta_{0YY} = \Psi_{0Y} \Theta_{0x} - \Psi_{0x} . \quad (2.16) \]

The boundary conditions are

\[ \Psi_{0Y}(x, \infty) = \Theta_{0}(x, \infty) = 0. \quad (2.17a,b) \]

Also \( \Psi_0 \) and \( \Theta_0 \) must match with the solutions for the inner layer as \( Y \to 0 \).

(2) The inner problem.

From the order of magnitude analysis

\[ \delta_y \sim Da_L^{1/2}, \quad \Psi \sim Ra_L Da_L^{1/2} . \]

It is noticed that the velocity scale in this layer is dictated by the outer layer velocity scale in determining the order of magnitude for the stream function. Introducing the inner variables as

\[ \hat{Y} = Da_L^{-1/2} y, \quad \hat{\Psi} = Ra_L^{-1} Da_L^{-1/2} \Psi, \quad \hat{\Theta} = \Theta \]
where $\hat{Y}$, $\hat{\Psi}$, and $\hat{\Theta}$ are of the order of 1 (note $\hat{Y} = Y / \Gamma_T$, $\hat{\Psi} = \psi / \Gamma_T$), transforms governing equations (2.7) and (2.8) into

$$
\hat{\Psi}_{YY} + \epsilon V^2 \hat{\Psi}_{xx} = \hat{\Theta}_Y + \hat{\Psi}_{YYY} + 2\epsilon V^2 \hat{\Psi}_{xxy} + \epsilon V^4 \hat{\Psi}_{xxxx}
$$

$$
\hat{\Theta}_{YY} + \epsilon V^2 \hat{\Theta}_{xx} = \Gamma_T^2 \left( \hat{\Psi}_Y \hat{\Theta}_x - \hat{\Psi}_x \hat{\Theta}_Y \right)
$$

where

$$
\epsilon_v = D_{v \alpha}^{1/2}
$$

subject to the boundary conditions

$$
\hat{\Psi} = \hat{\Psi}_Y = \hat{\Theta} - 1 = 0 \quad \text{as} \quad \hat{Y} \rightarrow 0.
$$

Also $\hat{\Psi}$ and $\hat{\Theta}$ must match with the outer solution as $\hat{Y} \rightarrow \infty$. Expanding $\hat{\Psi}$ and $\hat{\Theta}$ as

$$
\hat{\Psi} = \hat{\Psi}_0 + \Gamma_T \hat{\Psi}_1 + \Gamma_T^2 \hat{\Psi}_2 + \cdots
$$

$$
\hat{\Theta} = \hat{\Theta}_0 + \Gamma_T \hat{\Theta}_1 + \Gamma_T^2 \hat{\Theta}_2 + \cdots
$$

yields for the leading order

$$
\hat{\Psi}_0 \hat{\Psi}_Y = \hat{\Theta}_0 \hat{\Psi}_Y + \hat{\Psi}_0 \hat{\Theta}_Y \hat{\Psi}_Y
$$
\[ \hat{\Theta}_0 \hat{\gamma}_y = 0. \quad (2.21) \]

The boundary conditions are

\[ \hat{\Psi}_0(x,0) = \hat{\Psi}_0 \hat{\gamma}_y(x,0) = \hat{\Theta}_0(x,0) - 1 = 0. \quad (2.22a-c) \]

Also, \( \hat{\Psi}_0 \) and \( \hat{\Theta}_0 \) must match with the solution for the outer layer as \( \hat{Y} \to \infty \).

2.2.1(b) Case TII. \( \text{Ra}_L^{-1/2} \ll (\text{Da}_L^{1/2} \).

Similar to Case TI, the free convection boundary layer for Case TII consists of two regions where different physical phenomena are predominant. First, there exists an inner region where tangible temperature differences with the ambient fluid exist. Only in this region are there buoyancy effects. The boundary friction is of the same order of magnitude as the buoyancy term. Second, there is an outer region where no buoyancy exists. The fluid in this region, however, is driven by viscous interaction with the inner layer as for the regular medium with a high Prandtl number. The bulk friction is of the same order of magnitude as the boundary friction.

(1) The inner problem.

Based on the order of magnitude analysis

\[ \delta_T \sim x^{1/4} \text{Ra}_L^{-1/4} \text{Da}_L^{1/4}, \quad \psi \sim x^{3/4} \text{Ra}_L^{1/4} \text{Da}_L^{-1/4}. \]
Hence introducing the inner variables as

\[ \hat{Y} = R a_L^{1/4} D a_L^{-1/4} y, \quad \hat{\Psi} = R a_L^{-1/4} D a_L^{1/4} \psi, \quad \hat{\Theta} = \theta \]

where \( \hat{Y}, \hat{\Psi} \) and \( \hat{\Theta} \) are of the order of 1, transforms governing equations (2.7) and (2.8) as

\[ \hat{\Theta}_Y + \hat{\Psi}_Y \hat{\Psi}_Y + 2 \eta^2 \hat{\Psi}_{xx} \hat{\Psi}_Y + \eta^4 \hat{\Psi}_{xxx} = \Delta^2_T \left( \hat{\Psi}_{YY} + \eta^2 \hat{\Psi}_{xx} \right) \]

\[ \hat{\Psi}_Y \hat{\Theta}_x - \hat{\Psi}_x \hat{\Theta}_Y = \hat{\Theta}_Y \hat{\Psi}_Y + \eta^2 \hat{\Theta}_{xx} \]

subject to the boundary conditions

\[ \hat{\Psi} = \hat{\Psi}_Y = \hat{\Theta} - 1 = 0 \quad \text{at} \quad \hat{Y} = 0. \]

Also \( \hat{\Psi} \) and \( \hat{\Theta} \) must match with the outer solution as \( \hat{Y} \to \infty \). Here two parameters are defined as

\[ \eta_T = D a_L^{1/4} R a_L^{-1/4}, \quad \Delta_T = D a_L^{-1/4} R a_L^{-1/4}. \quad (2.23) \]

Since the thickness of the TBL which is \( O(D a_L^{1/4} R a_L^{-1/4}) \) is much smaller than that of the VBL which is \( O(D a_L^{1/2}) \), \( \Delta_T \), the ratio of these two can be chosen as an expansion parameter (note \( \eta_T(\Delta_T) \)). Now expanding \( \hat{\Psi} \) and \( \hat{\Theta} \) as in equations (2.19) with a perturbation parameter \( \Delta_T \) yields for the leading order
\[ \hat{\theta}_0 \hat{Y} + \hat{\psi} \hat{0} \hat{Y} \hat{Y} \hat{Y} = 0 \quad (2.24) \]

\[ \hat{\theta}_0 \hat{Y} = \hat{\psi} \hat{0} \hat{X} - \hat{\psi}_0 \hat{X} \hat{Y}. \quad (2.25) \]

The boundary conditions are

\[ \hat{\psi}_0 (x, 0) = \hat{\theta}_0 (x, 0) = \hat{\theta}_0 (x, 0) - 1 = 0. \quad (2.26a-c) \]

Also \( \hat{\psi}_0 \) and \( \hat{\theta}_0 \) must match with the solutions for the outer layer as \( \hat{Y} \to \infty \).

(2) The outer problem.

From the order of magnitude analysis

\[ \delta_\gamma \sim \text{Da}_L^{1/2}, \quad \psi \sim x^{1/2} \text{Ra}_L^{1/2}. \]

It is noticed that the velocity scale in this layer is dictated by the inner layer velocity scale in determining the order of magnitude for the stream function. Now introducing the outer variables as

\[ Y = \text{Da}_L^{-1/2} y, \quad \Psi = \text{Ra}_L^{-1/2} \psi, \quad \Theta = \theta \]
where \( Y, \Psi \) and \( \Theta \) are of the order of 1 (note \( Y = \Delta_T \hat{Y}, \Psi = \Delta_T \hat{\Psi} \)), transforms governing equations (2.7) and (2.8) into

\[
\Psi_{YY} + \eta_Y^2 \Psi_{xx} = \Psi_{YYYY} + 2\eta_Y^2 \Psi_{xxyy} + \eta_Y^4 \Psi_{xxxx}
\]

\( \Theta = 0 \)

subject to the boundary condition

\[
\Psi_Y = 0 \text{ as } Y \to \infty.
\]

Also \( \Psi \) must match with the inner solution as \( Y \to 0 \). Here

\[
\eta_Y = Da_L^{1/2} \tag{2.27}
\]

Expanding \( \Psi \) as in equation (2.14a) with a perturbation parameter \( \eta_Y \) yields for the leading order

\[
\Psi_{0YY} = \Psi_{0YYYY} \tag{2.28}
\]

The boundary conditions are

\[
\Psi_{0Y}(x,\infty) = 0 \tag{2.29}
\]

and \( \Psi_0 \) must match with the solution for the inner layer as \( Y \to 0 \).
2.2.2 Constant wall heat flux case

Consider the case where a uniform heat flux is applied to the vertical flat plate. Then by introducing the same dimensionless quantities and the stream functions as in Section 2.2.1 except for

$$\theta = \frac{T - T_0}{q_L/k_e}$$

it can be shown that governing equations (2.1)-(2.3) reduce to

$$D\alpha_L (\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) - (\psi_{xx} + \psi_{yy}) = -Ra^*_L \theta_y \quad (2.30)$$

$$\psi_y \theta_x - \psi_x \theta_y = \theta_{xx} + \theta_{yy} \quad (2.31)$$

where $Ra^*_L$ is defined as

$$Ra^*_L = \frac{Kg\beta L^2 q}{\alpha_c v_f k_e} \quad (2.32)$$

The appropriate boundary conditions for this case are

$$\psi = \psi_y = \theta_y + 1 = 0 \text{ at } y = 0$$

$$\psi_y \to 0, \theta \to 0 \text{ at } y \to \infty.$$
The order of magnitude analysis in this case is similar to that used in Section 2.2.1. So the final results are going to be presented without any details. Note that this case is also divided into two categories as shown in Fig. 2.1 depending on the magnitude of $D_a^{1/2}R_a^{*1/3}$, which can be considered as the thickness ratio of the VBL to that of the TBL. Cases HI and HII will be used to distinguish one category from the other as in sections 2.2.1.

2.2.2(a) Case HI. $R_a^{*-1/3}D_a^{1/2}$.

(1) The outer problem.

From the order of magnitude analysis

$$\delta_T \sim x^{1/3}R_a^{*-1/3}, \psi \sim x^{2/3}R_a^{*1/3}, \theta \sim x^{1/3}R_a^{*-1/3}.$$ 

Hence introducing the outer variables as

$$Y = R_a^{*1/3}y, \Psi = R_a^{*-1/3}\psi, \Theta = R_a^{*1/3}\theta,$$

where $Y, \Psi$ and $\Theta$ are of the order of 1, transforms governing equations (2.30) and (2.31) into

$$\Psi_{YY} + \varepsilon_T^{*2}\Psi_{xx} = \Theta_Y + \Gamma_H^2(\Psi_{YYYY} + 2\varepsilon_T^{*2}\Psi_{xyyy} + \varepsilon_T^{*4}\Psi_{xxxx})$$

$$\Psi_{YY} + \varepsilon_T^{*2}\Psi_{xx} = \Theta_Y + \Gamma_H^2(\Psi_{YYYY} + 2\varepsilon_T^{*2}\Psi_{xyyy} + \varepsilon_T^{*4}\Psi_{xxxx})$$

$$\Psi_Y \Theta_X - \Psi_X \Theta_Y = \Theta_{YY} + \varepsilon_T^{*2}\Theta_{xx}$$

where $Y, \Psi$ and $\Theta$ are of the order of 1.
subject to the boundary conditions

\[ \Psi_Y = \Theta = 0 \text{ as } Y \to \infty. \]

Also \( \Psi \) and \( \Theta \) must match with the inner solutions as \( Y \to 0 \). Here two parameters are defined as

\[ \varepsilon_T^* = Ra_L^{* -1/3}, \quad \Gamma_H = Da_L^{1/2} Ra_L^{* 1/3}. \quad (2.33) \]

Expanding \( \Psi \) and \( \Theta \) as in equations (2.14) with an expansion parameter \( \Gamma_H \) yields for the leading order

\[ \Psi_{0YY} = \Theta_{0Y} \quad (2.34) \]

\[ \Theta_{0YY} = \Psi_{0Y} \Theta_{0x} - \Psi_{0x} \Theta_{0Y}. \quad (2.35) \]

The boundary conditions are

\[ \Psi_{0Y}(x, \infty) = \Theta_0(x, \infty) = 0 \quad (2.36a,b) \]

Also \( \Psi_0 \) and \( \Theta_0 \) must match with the solutions for the inner layer as \( Y \to 0 \).
(2) The inner problem.

From the order of magnitude analysis

$$
\delta_v \sim D_{aL}^{1/2}, \quad \psi \sim x^{1/3} D_{aL}^{1/2} R_{aL}^{2/3}, \quad \theta \sim x^{1/3} R_{aL}^{* -1/3}.
$$

Hence introducing the inner variables as

$$
\hat{Y} = D_{aL}^{-1/2} y, \quad \hat{\Psi} = R_{aL}^{*-2/3} D_{aL}^{-1/2} \psi, \quad \hat{\Theta} = R_{aL}^{*1/3} \theta,
$$

where $\hat{Y}$, $\hat{\Psi}$ and $\hat{\Theta}$ are of the order of 1 (note $\hat{Y} = Y / \Gamma_H$, $\hat{\Psi} = \Psi / \Gamma_H$), transforms governing equations (2.30) and (2.31) into

$$
\hat{\Psi}_{Y\hat{Y}} + \varepsilon_{v}^{2} \hat{\Psi}_{xx} = \hat{\Theta}_{\hat{Y}} + \hat{\Psi}_{Y\hat{Y}Y\hat{Y}} + 2 \varepsilon_{v}^{2} \hat{\Psi}_{xxY\hat{Y}} + \varepsilon_{v}^{4} \hat{\Psi}_{xxxx}
$$

$$
\hat{\Theta}_{Y\hat{Y}} + \varepsilon_{v}^{2} \hat{\Theta}_{xx} = \Gamma_H^2 \left( \hat{\Psi}_{Y\hat{Y}} \hat{\Theta}_x - \hat{\Psi}_x \hat{\Theta}_{\hat{Y}} \right)
$$

subject to the boundary conditions

$$
\hat{\Psi} = \hat{\Psi}_{\hat{Y}} = 0, \quad \hat{\Theta}_{\hat{Y}} = -\Gamma_H \text{ at } \hat{Y} = 0.
$$

Also $\hat{\Psi}$ and $\hat{\Theta}$ must match with the outer solution as $\hat{Y} \to \infty$. Expanding $\hat{\Psi}$ and $\hat{\Theta}$ as in equations (2.19) with a perturbation parameter $\Gamma_H$ yields for the leading order

$$
\hat{\Psi}_{0Y\hat{Y}} = \hat{\Theta}_{0\hat{Y}} + \hat{\Psi}_{0Y\hat{Y}Y\hat{Y}}
$$

(2.37)
The boundary conditions are

\[ \hat{\theta}_0 \hat{\psi}_0 = 0. \quad (2.38) \]

Also \( \hat{\psi}_0 \) and \( \hat{\theta}_0 \) must match with the solutions for the outer layer as \( \hat{\gamma} \to \infty \). Even though the boundary conditions for the leading order inner problem does not seem to represent the constant wall heat flux condition, we can overcome this difficulty by imposing the constant heat flux condition at the wall on the outer solution since for the temperature the composite solution for the leading order is identical to the outer solution itself as will be shown later.

2.2.2(b) Case III. \( \text{Ra}_L^{-1/3} \langle \text{Da}_L^{1/2} \rangle \).

(1) The inner problem.

From the order of magnitude analysis

\[ \delta_T \sim x^{1/5} \text{Ra}_L^{-1/5} \text{Da}_L^{1/5}, \quad \psi \sim x^{4/5} \text{Ra}_L^{1/5} \text{Da}_L^{-1/5}, \quad \theta \sim x^{1/5} \text{Ra}_L^{-1/5} \text{Da}_L^{1/5}. \]

Hence introducing the inner variables as

\[ \hat{\gamma} = \text{Ra}_L^{1/5} \text{Da}_L^{-1/5} \gamma, \quad \hat{\psi} = \text{Ra}_L^{-1/5} \text{Da}_L^{1/5} \psi, \quad \hat{\theta} = \text{Ra}_L^{1/5} \text{Da}_L^{-1/5} \theta \]
where $\dot{Y}$, $\dot{\Psi}$ and $\dot{\Theta}$ are of the order of 1, transforms governing equations (2.30) and (2.31) into

$$
\dot{\Theta}_Y + \dot{\Psi}_Y \dot{\Psi}_Y + 2 \eta^*_T \dot{\Psi}_{xx} \dot{\Psi}_Y + \eta^*_T \dot{\Psi}_{xxxx} = \Delta_H^2 (\dot{\Psi}_Y \ddot{\Psi}_Y + \eta^*_T \dot{\Psi}_{xx})
$$

$$
\dot{\Psi}_Y \dot{\Theta}_x - \dot{\Psi}_x \dot{\Theta}_Y = \dot{\Theta}_Y \ddot{\Psi}_Y + \eta^*_T \dot{\Theta}_{xx}
$$

subject to the boundary conditions

$$
\dot{\Psi} = \dot{\Psi}_Y = \dot{\Theta}_Y + 1 = 0 \text{ at } \dot{Y} = 0.
$$

Also $\dot{\Psi}$ and $\dot{\Theta}$ must match with the outer solution as $\dot{Y} \to \infty$. Here two parameters are defined as

$$
\eta^*_T = Ra_L^{*-1/5} Da_L^{1/5}, \quad \Delta_H = Ra_L^{*-1/5} Da_L^{-3/10}.
$$

Expanding $\dot{\Psi}$ and $\dot{\Theta}$ as in equations (2.19) with a perturbation parameter $\Delta_H$ yields for the leading order

$$
\dot{\Theta}_0 \dot{Y} + \dot{\Psi}_0 \dot{Y} \dot{\Psi}_Y = 0
$$

$$
\dot{\Theta}_0 \dot{Y} \dot{Y} = \dot{\Psi}_0 \dot{Y} \dot{\Theta}_0 - \dot{\Psi}_0 \dot{\Theta}_0 \dot{Y}.
$$
The boundary conditions are

\[ \Psi_0(x,0) = \Psi_0^\prime(x,0) = \Theta_0^\prime(x,0) + 1 = 0. \]  

(2.43a-c)

Also \( \Psi_0 \) and \( \Theta_0 \) must match with the solutions for the outer layer as \( \hat{Y} \to \infty \).

(2) The outer problem.

From the order of magnitude analysis

\[ \delta_v \sim Da_L^{1/2}, \quad \psi \sim x^{3/5}Ra_L^{2/5}Da_L^{1/10}. \]

Hence introducing the outer variables as

\[ Y = Da_L^{-1/2}y, \quad \Psi = Ra_L^{-2/5}Da_L^{-1/10}\psi, \]

where \( Y \) and \( \Psi \) are of the order of 1 (note \( Y = \Delta_H \hat{Y}, \quad \Psi = \Delta_H \hat{\Psi} \)), transforms the governing equations into

\[ \Psi_{yy} + \eta_v^2\Psi_{xx} = \Psi_{yyyy} + 2\eta_v^2\Psi_{xyyy} + \eta_v^4\Psi_{xxxx} \]

\[ \Theta = 0 \]

subject to the boundary conditions
\[ \Psi_Y = 0 \text{ as } Y \to \infty. \]

Also \( \Psi \) must match with the inner solution as \( Y \to 0 \). Expanding \( \Psi \) as in equation (2.14a) with a perturbation parameter \( \eta_\nu \) yields for the leading order

\[ \Psi_{0YY} = \Psi_{0YYYY}. \]  

(2.44)

The boundary condition is

\[ \Psi_{0Y}(x, \infty) = 0. \]  

(2.45)

Also \( \Psi_0 \) must match with the solution for the inner layer as \( Y \to 0 \).

2.3 ANALYSIS AND SOLUTIONS

The procedure of obtaining analytical solutions by using the modified Oseen method in this section is somewhat similar to that described by Gill (1966) to solve the boundary layer equations for Newtonian fluid convection in an enclosure. More recently this method of solution was employed successfully by Weber (1975), Bejan (1979), and Tong and Subramanian (1985) to deal with free convective heat transfer in porous enclosures. Of engineering importance is the fact that the overall heat transfer results produced by these analyses agree well with experimental and numerical heat transfer data as pointed out by Bejan (1979). To check the validity of using this method for the
present problem, numerical solutions are also obtained based on the similarity transformation.

2.3.1 Case TI

For the outer layer, equations (2.15)-(2.17) are solved using the modified Oseen method. Inserting equation (2.15) into equation (2.16) gives

\[ \Psi_{0YY} + \Psi_{0x} \Psi_{0YY} - \Theta_{0x} \Psi_{0Y} = 0. \]  

(2.46)

This equation is linearized by regarding \( \Psi_{0x} \) and \( \Theta_{0x} \) as unknown functions of \( x \), \( a(b) \) and \( b(x) \), respectively. Here \( a(x) \) and \( b(x) \) can be considered as line-averaged functions at fixed \( x \) across the thermal boundary layer region. This admits a solution of the following form for equation (2.46):

\[ \Psi_{0Y} = C_1(x)e^{r_1Y} + C_2(x)e^{r_2Y} \]

where

\[ r_{1,2} = \frac{-a \pm (a^2 + 4b)^{1/2}}{2} \]

Function \( b(x) \) represents the average vertical temperature gradient \( \Theta_{0x} \) at fixed \( x \) inside the TBL. Therefore, we expect \( b(x) \) to be positive since as the cold fluid rises it warms up gradually. Consequently, the two roots \( r_1(x), r_2(x) \) are of opposite sign regardless of
the sign of $a(x)$. Boundary condition (2.17a) is satisfied only if the positive root is discarded. Thus the velocity solution is of the form

$$\Psi_0 Y = A_1(x)e^{-Y/\alpha(x)}$$

(2.47)

where for reasons soon to be evident, we find it convenient to write $-1/\alpha$ in place of the negative root.

Integrating equation (2.15) and using equations (2.17b) and (2.47) gives

$$\Psi_0 Y = \Theta_0 = A_1(x)e^{-Y/\alpha(x)}.$$

Also the integration of the above equation gives

$$\Psi_0 = -\alpha(x)A_1(x)e^{-Y/\alpha(x)} + A_2(x).$$

On the other hand, for the inner layer equations (2.20)-(2.22) are easily integrated to yield

$$\hat{\Theta}_0 = 1 + A_3(x)\hat{Y}$$

$$\hat{\Psi}_0 = A_4(x)\left(1 - \hat{Y} - e^{-\hat{Y}}\right).$$
The exponentially growing term is already discarded, since, with it, the matching cannot be achieved. To determine $A_1$, $A_2$, $A_3$, and $A_4$, matching conditions are used. Matching the outer solutions with the inner solutions, after rewriting the outer and the inner solutions in terms of the inner and the outer variables respectively and expanding as \( \Gamma_T \to 0 \), gives

\[
\begin{align*}
A_1(x) &= 1, \\
A_2(x) &= \alpha(x), \\
A_3(x) &= 0, \\
A_4(x) &= -1.
\end{align*}
\]

Hence

\[
\begin{align*}
\Psi_0 &= \alpha(x) \left[1 - e^{-Y/\alpha(x)}\right], \\
\Theta_0 &= e^{-Y/\alpha(x)} \quad (2.48)
\end{align*}
\]

\[
\hat{\Psi}_0 = e^{-\hat{Y}} + \hat{Y} - 1, \quad \hat{\Theta}_0 = 1. \quad (2.49)
\]

The only thing that is remaining to complete the solutions is to determine $\alpha(x)$. Inserting equations (2.48) into the integral form of energy equation (2.16) and solving for $\alpha(x)$ yields

\[
\alpha(x) = 2x^{1/2}.
\]

The integration constant is set to zero since $\Theta_0 = 0$ at $x = 0$. Therefore

\[
\Psi_0 = 2x^{1/2} \left(1 - e^{-Y/2x^{1/2}}\right), \quad \Theta_0 = e^{-Y/2x^{1/2}}. \quad (2.50)
\]
Hsu and Cheng (1985) presented the same analytical solution as equations (2.49) for the inner layer and the similarity solution for the outer layer. Note that $\eta_1 = Y \sqrt{x}$ in equations (2.50) is the similarity variable used in Hsu and Cheng (1985). Comparison between the approximate analytical solution and the similarity solution will be given in Section 2.4.

2.3.2 Case TII

For the inner layer both the numerical solution using the similarity transformation and the analytical solution using the modified Oseen method as for the outer layer of Case T1 can be obtained. For the analytical solution combining equations (2.24) and (2.25) yields

$$\hat{\psi}_0 \hat{\psi}_0 + \hat{\psi}_0' \hat{\psi}_0 + \hat{\theta}_0 \hat{\psi}_0' = 0.$$ 

Using the modified Oseen method, the solution is of the form

$$\hat{\psi}_0 \hat{\psi}_0 = \sum_{n=1}^{4} d_n(x) e^{-\lambda_n(x) \hat{Y}}$$

where $\lambda_n$ are the roots of the characteristic equation

$$\lambda^4 - a\lambda^3 + b = 0.$$ 

As pointed out in Appendix A of Tong and Subramanian (1985), two roots have positive real parts while the other two have negative real parts. Since only the roots with positive
real parts will satisfy the matching condition as \( \hat{Y} \to \infty \), the solution becomes after using equation (2.26b)

\[
\hat{\Psi}_0\hat{Y} = B_1(x) \left( e^{-\lambda_1(x)\hat{Y}} - e^{-\lambda_2(x)\hat{Y}} \right)
\]

where \( \lambda_1(x) \) and \( \lambda_2(x) \) are the roots with positive real parts. Now assume \( \lambda_1 \parallel \lambda_2 \). This assumption is made so that the inner velocity will approach a finite function of \( x \) rather than 0. This point can be confirmed using the similarity solution. Then the inner solution for the temperature can be approximated as

\[
\hat{\Theta}_0 = B_1(x) \lambda_2^2 e^{-\lambda_2 \hat{Y}}. \tag{2.51}
\]

Using equation (2.26c) gives \( B_1(x) = 1 / \lambda_2^2 \). Then by letting \( \beta(x) = 1 / \lambda_2(x) \)

\[
\hat{\Psi}_0\hat{Y} = \beta^2(x) \left( e^{-\lambda_1(x)\hat{Y}} - e^{-\hat{Y}/\beta(x)} \right), \quad \hat{\Theta}_0 = e^{-\hat{Y}/\beta(x)}. \tag{2.52}
\]

Matching equations (2.52) with the outer solution requires \( \lambda_1 = \Delta_T \), and using the integral form of equation (2.25) gives \( \beta(x) = (8x / 3)^{1/4} \). Then for the inner layer

\[
\hat{\Psi}_0 = \left( \frac{8}{3} x \right)^{1/2} \left[ \frac{1}{\Delta_T} \left( 1 - e^{-\Delta_T \hat{Y}} \right) - \left( \frac{8}{3} x \right)^{1/4} \left( 1 - e^{-\hat{Y}/(8x/3)^{1/4}} \right) \right]
\]

\[
\hat{\Theta}_0 = e^{-\hat{Y}/(8x/3)^{1/4}}. \tag{2.53}
\]
For the outer layer equation (2.28) is easily integrated. Using equation (2.29) together with matching leads to

\[ \Psi_0 = \left( \frac{8}{3} x \right)^{1/2} (1 - e^{-y}), \quad \Theta_0 = 0. \quad (2.54) \]

On the other hand, for the inner layer, the numerical solution can be obtained using the similarity transformation

\[ \eta_2 = \frac{\tilde{Y}}{x^{1/4}}, \quad \tilde{\Psi}_0 = x^{3/4} f_0(\eta_2), \quad \tilde{\Theta}_0 = g_0(\eta_2). \]

Then equations (2.24) and (2.25) become

\[ g_0 + f_0^{iv} = 0 \quad (2.55) \]

\[ g_0' + \frac{3}{4} f_0 g_0' = 0 \quad (2.56) \]

where a prime denotes differentiation with respect to \( \eta_2 \). The appropriate B.C.'s are

\[ f_0(0) = f_0'(0) = g_0(0) - 1 = 0 \quad (2.57a-c) \]

\[ f_0''(\infty) = f_0'''(\infty) = g_0(\infty) = 0. \quad (2.58a-c) \]
Three boundary conditions, equations (2.58), are obtained from the matching with the outer solutions. This system of ODE is to be solved numerically using the fourth-order Runge-Kutta method with shooting procedures. Comparison between this numerical solution and the approximate analytical solution will be made in Section 2.4.

2.3.3 Case HI

For the outer layer using the modified Oseen method as in Case TI the solutions for equations (2.34) and (2.35) satisfying equations (2.36) are of the form

\[ \Psi_0 = \Theta_0 = C_1(x) e^{-Y/\gamma(x)}. \]

On the other hand for the inner layer the integration of equations (2.37) and (2.38) along with equations (2.39) gives

\[ \hat{\Psi}_0 = C_2(x) \left( e^{-\hat{Y}} + \hat{Y} - 1 \right), \hat{\Theta}_0 = C_3(x). \]

From matching we have

\[ C_1(x) = C_2(x) = C_3(x). \]

Hence

\[ \hat{\Psi}_0 = C_1(x) \left( e^{-\hat{Y}} + \hat{Y} - 1 \right), \hat{\Theta}_0 = C_1(x). \]
Now the condition of constant heat flux at the wall can be imposed on the outer solution since for the temperature the composite solution which is uniformly valid throughout the domain is equivalent to the outer solution. In other words imposing

$$\Theta_0(y,0) = -1$$

gives $C_1(x) = \gamma(x)$. By doing this we can resolve such a problem that in equation (2.39c) the constant heat flux condition at the wall seems to play no role for the leading order solution.

The only unknown, $\gamma(x)$, can be determined using the integral form of the energy equation, equation (2.35), as in Case TI.

Then the outer solutions are

$$\Psi_0 = (2x)^{2/3} \left(1 - e^{-Y/(2x)^{1/3}}\right), \quad \Theta_0 = (2x)^{1/3} e^{-Y/(2x)^{1/3}}$$

(2.59)

and the inner solutions are

$$\hat{\Psi}_0 = (2x)^{1/3} \left(e^{-\hat{Y}} + \hat{Y} - 1\right), \quad \hat{\Theta}_0 = (2x)^{1/3}.$$  

(2.60)

On the other hand, for the outer layer, we can get the numerical solution using the similarity transformation
\[ \zeta_i = \frac{Y}{x^{1/3}}, \quad \Psi_0 = x^{2/3}F_0(\zeta_1), \quad \Theta_0 = x^{1/3}G_0(\zeta_1). \]

Then equations (2.34) and (2.35) become

\[ F_0'' = G_0' \quad (2.61) \]

\[ G_0'' = \frac{1}{3}F_0'G_0 - \frac{2}{3}F_0G_0' \quad (2.62) \]

where a prime denotes differentiation with respect to \( \zeta_1 \). The boundary conditions are

\[ F_0(0) = G_0'(0) + 1 = 0 \quad (2.63a,b) \]

\[ F_0'(\infty) = G_0(\infty) = 0. \quad (2.64a,b) \]

Two boundary conditions, equations (2.63), are obtained from the matching with the inner solutions.

2.3.4 Case III

Following the procedure similar to that for Case TII yields for the inner layer

\[ \hat{\Psi}_0 = (2x)^{3/5}\left[ \frac{1}{A/\Delta H}\left(1 - e^{-\Delta H\hat{Y}}\right) - (2x)^{1/5}\left(1 - e^{-\hat{Y}/(2x)^{1/5}}\right) \right] \quad (2.65) \]

\[ \hat{\Theta}_0 = (2x)^{1/5}e^{-\hat{Y}/(2x)^{1/5}} \]
and for the outer layer

\[ \Psi_0 = (2x)^{3/5} e^{-Y}, \Theta_0 = 0. \]  

(2.66)

On the other hand, for the inner layer, we can get the numerical solution using the similarity transformation

\[ \zeta_2 = \frac{Y}{x^{1/5}}, \dot{\Psi}_0 = x^{4/5} \tilde{f}_0(\zeta_2), \dot{\Theta}_0 = x^{1/5} \tilde{g}_0(\zeta_2). \]

Then equations (2.41) and (2.42) become

\[ \tilde{g}_0 + \tilde{f}_0^{iv} = 0 \]  

(2.67)

\[ \tilde{g}_0^{\prime} = \frac{1}{3} \tilde{f}_0 \tilde{g}_0 - \frac{4}{3} \tilde{f}_0 \tilde{g}_0^0. \]  

(2.68)

Here a prime denotes differentiation with respect to \( \zeta_2 \). The appropriate B.C.'s are

\[ \tilde{f}_0(0) = \tilde{f}_0'(0) = \tilde{g}_0'(0) + 1 = 0 \]  

(2.69a-c)

\[ \tilde{f}_0''(0)(\infty) = \tilde{f}_0''(\infty) = \tilde{g}_0'(\infty) = 0. \]  

(2.70a-c)

Three boundary conditions, equations (2.70), are obtained from the matching with the outer solutions.
2.4. RESULTS AND DISCUSSION

Composite solutions which are uniformly valid can be constructed by adding the inner and outer solutions and subtracting the common parts for stream function and temperature. From the results of the previous section, it is possible to get two different composite solutions depending on the method used. For example for the outer layer of Case TI, the modified Oseen method was used to get the analytical solution and the similarity transformation was used to obtain the numerical solution. To clarify this, the composite solutions for the former will be denoted as the analytical composite solutions and those for the latter will be denoted as the numerical composite solutions. In this section two different composite solutions for each of the streamwise velocity and the temperature will be given. Then the analytical composite solutions will be compared with the numerical composite solutions to check the validity of using the modified Oseen method for the present problem.

2.4.1 Case TI

From equations (2.49) and (2.50) the analytical composite solutions for the vertical velocity and the temperature can be written as

\[
\frac{u_c}{Ra_L} = e^{-\eta I/2} - e^{-\eta I/(\Gamma_{TX}^{-1/2})} \quad (2.71a)
\]

\[
\theta_c = e^{-\eta I/2} \quad (2.71b)
\]
where \( \eta_1 = \dot{Y} / x^{1/2} \) is the similarity variable used in the numerical solution and \( \Gamma_T x^{-1/2} \) is a parameter which can be considered as the local thickness ratio of the VBL to that of the TBL. Equation (2.71a) for \( \Gamma_T x^{-1/2} = 0.3 \) is presented in Fig. 2.2 along with the numerical solution in Hsu and Cheng (1985). Here the parameter \( \Gamma_T x^{-1/2} \) represents the relative magnitude of the boundary effect; when \( \Gamma_T x^{-1/2} = 0 \) the solutions are identical to those based on Darcy's law where the viscous effects on the impermeable boundary are neglected. Also equation (2.71b) is presented in Fig. 2.3 along with the numerical solution in Hsu and Cheng (1985). As shown in Figs. 2.2 and 2.3 the agreement between analytical and numerical solutions is quite good, which confirms the use of the modified Oseen method to solve the natural convection problem about a vertical flat plate embedded in a porous medium.

2.4.2 Case TII

From equations (2.53) and (2.54) the analytical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_c}{Ra_L^{1/2} Da_L^{-1/2}} = \left( \frac{8}{3} x \right)^{1/2} \left[ e^{-(\Delta_T x^{1/4}) \eta_2} - e^{-\eta_2/(8/3)^{1/4}} \right] \tag{2.72a}
\]

\[
\theta_c = e^{-\eta_2/(8/3)^{1/4}} \tag{2.72b}
\]

where \( \eta_2 = \dot{Y} / x^{1/4} \) is the similarity variable used in the numerical solution and \( \Delta_T x^{1/4} \) is a parameter which can be considered as the local thickness ratio of the TBL to that of
Fig. 2.2 Comparison between the analytical and numerical velocity distribution for Case TI, $\Gamma_{Tx}^{-1/2} = 0.3$. 
Fig. 2.3 Comparison between the analytical and numerical temperature distribution for Case TI
the VBL. On the other hand, the numerical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_c}{Ra_L^{1/2}Da_L^{-1/2}} = x \left[ f'_0(\infty) \left\{ e^{-\left(\Delta_T x^{1/4}\right)\eta_2} - 1 \right\} - f'_0(\eta_2) \right]
\]  
(2.73a)

\[
\theta_c = g_0(\eta_2).
\]  
(2.73b)

For comparison equations (2.72a) and (2.73a) are plotted in Fig. 2.4 for \(\Delta_T x^{1/4} = 0.3\). Again the agreement between analytical and numerical solutions is good and as the local thick ratio \(\Delta_T x^{1/4}\) increases the difference between two solutions for the vertical velocity is reduced. Hence for a fixed value of \(x\), the agreement is improved as \(\Delta_T\) is increased.

2.4.3 Case HI

From equations (2.59) and (2.60) the analytical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_2}{Ra_L^{2/3}} = (2x)^{1/3} \left[ e^{-\xi_1/2^{1/3}} - e^{-\xi_1/G_Hx^{1/3}} \right]
\]  
(2.74a)

\[
\frac{\theta_c}{Ra_L^{1/3}} = (2x)^{1/3} e^{-\xi_1/2^{1/3}}
\]  
(2.74b)
Fig. 2.4 Comparison between the analytical and numerical velocity distribution for Case TII, $\Delta = 0.3$. 

$u_c / Ra_{L}^{1/2} Da_{L}^{-1/2}$ vs $\eta_2$ 

- Analytical
- Numerical
where $\zeta_1 = Y / x^{1/3}$ is the similarity variable used in the numerical solution and $\Gamma_{Hx}^{-1/3}$ is a parameter that can be considered as the local thickness ratio of the VBL to that of the TBL. On the other hand the numerical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_c}{Ra_L^{2/3}} = x^{1/3} \left[ F_0'(\zeta_1) - F_0'(0)e^{-\zeta_1/\Gamma_{Hx}^{-1/3}} \right] \quad (2.75a)
\]

\[
\frac{\theta_c}{Ra_L^{1/3}} = x^{1/3} G_0(\zeta_1) \quad (2.75b)
\]

For comparison equations (2.74a) and (2.75a) are presented in Fig. 2.5 for $\Gamma_{Hx}^{-1/3} = 0.2$ and equations (2.74b) and (2.75b) are presented in Fig. 2.6. The agreement between analytical and numerical solutions is quite good as shown in these figures.

2.4.4 Case HII

From equations (2.65) and (2.66) the analytical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_c}{Ra_L^{*2/5}Da_L^{-2/5}} = (2x)^{3/5} \left[ e^{-(\Delta_{Hx}^{1/3})\zeta_2} - e^{-\zeta_2^{2/15}} \right] \quad (2.76a)
\]
Fig. 2.5 Comparison between the analytical and numerical velocity distribution for Case HI, $\Gamma_{Hx}^{-1/3} = 0.2$. 
Fig. 2.6 Comparison between the analytical and numerical temperature distribution for Case HI
\[
\frac{\theta_c}{Ra_L^{\ast -1/5}Da_L^{1/5}} = (2x)^{1/3}e^{-\zeta_2^{2/5}}
\]  
\[(2.76b)\]

where \(\zeta_2 = \hat{Y}/x^{1/5}\) is the similarity variable used in the numerical solution and \(\Delta_H x^{1/5}\) is a parameter which can be considered as the local thickness ratio of the TBL to that of the VBL. On the other hand, the numerical composite solutions for the vertical velocity and temperature can be written as

\[
\frac{u_c}{Ra_L^{\ast 2/5}Da_L^{-2/5}} = x^{3/5}\left[f_0(\infty)e^{-\frac{(\Delta_H x^{1/5})\zeta_2}{\zeta_2}} - 1\right] + \tilde{f}_0(\zeta_2)
\]  
\[(2.77a)\]

\[
\frac{\theta_c}{Ra_L^{\ast -1/5}Da_L^{1/5}} = x^{1/5}\tilde{g}_0(\zeta_2)
\]  
\[(2.77b)\]

For comparison equations (2.76a) and (2.77a) are plotted in Fig. 2.7 for \(\Delta_H x^{1/5} = 0.2\).

From Figs. 2.2-2.7 it is noted that the no-slip boundary conditions have a drastic effect on the vertical velocity while these have little effect on the temperature field for the leading order as pointed out in Hsu and Cheng (1985). Also from Fig. 2.8 it can be concluded that for Cases TI and HI the maximum vertical velocity decreases from the slip velocity (using the Darcy model) as the local thickness ratio of the VBL to that of the TBL is increased from zero, and that the distance from the wall at which the vertical velocity has its maximum value increases with the value of this local thickness ratio. This local thickness ratio for these two cases can be considered as the boundary parameter which was defined in Hsu and Cheng (1985) and Hong et al. (1985), since for zero values of
Fig. 2.7 Comparison between the analytical and numerical velocity distribution for Case HII, $\Delta_{Hx}^{-1/5} = 0.2$. 

- **Analytical**
- **Numerical**
Fig. 2.8 Effects of the local thickness ratio on the vertical velocity for Case TI
these parameters the results reduce to those based on the Darcy formulation and the viscous effect of the boundary increases with this parameter. On the other hand, for Cases TII and HII the maximum vertical velocity and the distance from the wall to which the fluid is affected by natural convection decrease as the local thickness ratio of the TBL to that of the VBL is increased, as shown in Fig. 2.9.

The most important result to be given is an expression for heat transfer. This can be presented conveniently by introducing the local Nusselt number

$$\text{Nu}_x = -x \frac{\partial \theta_c(x,0)}{\partial y}.$$  \hspace{1cm} (2.78)

The calculated values of the local Nusselt number for each case are presented in Table 2.1.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\text{Nu}_x$ from analytical solution</th>
<th>$\text{Nu}_x$ from Numerical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>TI</td>
<td>$0.5000 , \text{Ra}_x^{1/2}$</td>
<td>$0.4438 , \text{Ra}_x^{1/2}$</td>
</tr>
<tr>
<td>TII</td>
<td>$0.7825 , \text{Da}_x^{-1/4}\text{Ra}_x^{1/4}$</td>
<td>$0.5027 , \text{Da}_x^{-1/4}\text{Ra}_x^{1/4}$</td>
</tr>
<tr>
<td>HI</td>
<td>$0.7937 , \text{Ra}_x^{1/3}$</td>
<td>$0.7715 , \text{Ra}_x^{1/3}$</td>
</tr>
<tr>
<td>HII</td>
<td>$0.8706 , \text{Da}_x^{-1/5}\text{Ra}_x^{1/5}$</td>
<td>$0.6316 , \text{Da}_x^{-1/5}\text{Ra}_x^{1/5}$</td>
</tr>
</tbody>
</table>
Fig. 2.9 Effects of the local thickness ratio on the vertical velocity for Case HII
Note that for Cases TI and HI, the Nusselt number depends only on the modified Rayleigh number. For Cases TII and HII, however, the Nusselt number depends on the Darcy number as well as the modified Rayleigh number. Also note that for Cases TI and HI the rate of heat transfer depends on the permeability of the porous medium, while for Cases TII and HII the rate of heat transfer is independent of the permeability of the porous medium, since the division of the modified Rayleigh number by the Darcy number results in the product of the Rayleigh number. This result is very interesting and of physical importance. This finding can be inferred from equation (2.2) since the permeability of the porous medium does not appear explicitly if there is a balance between the boundary friction force and the buoyancy force. Hence it can be concluded that when the thickness of the VBL is much larger than that of the TBL, the heat transfer characteristics resemble those of the regular fluid with a large Prandtl number. This is because the Brinkman equation reduces to the Navier-Stokes equation as the permeability $K \to \infty$, while it reduces to the Darcy equation as $K \to 0$. Also, it is shown from the numerical solutions that the Nusselt numbers for Cases TI and HI are identical to those based on the Darcy flow model and the heat transfer rates for Cases TII and HII are identical to those for a regular fluid of high Prandtl number. (See Bejan (1984) for numerical values for each case.) Finally, it is noticed that the analytical solution based on the modified Oseen method overpredicts the Nusselt number, which is inherent in this type of approach as pointed out in Lauriat and Prasad (1987).
CHAPTER III

FORCED CONVECTION IN A CHANNEL
FILLED WITH A POROUS MEDIUM

3.1 INTRODUCTION

Utilizing porous media in such contemporary technology as thermal insulation, direct contact heat exchangers and nuclear waste repositories has provided strong impetus for the analyses of fluid flow and heat transfer through porous media. In most applications the Reynolds number based on the pore size is greater than unity and there is an impermeable boundary, which make Darcy's law inapplicable. For these reasons the inertia and boundary effects have been included in a number of recent studies on convective heat transfer in porous media. These effects are incorporated by using the general flow model known as the Brinkman-Forschheimer-extended Darcy model. In general, these effects on convective flow and heat transfer through porous media have been studied by a number of different investigators. For example Vafai and Tien (1981) studied forced convection over a horizontal flat plate, Hong et al. (1985) investigated natural convection over a vertical flat plate, and Parang and Keyhani (1987) analyzed mixed convection through an annulus region.
The problem of forced convection in a porous channel was studied by Koh and his co-workers (1974, 1975) to investigate the performance of a heat exchanger containing a high conductivity porous media. Their analysis was based on a slug flow model. They have shown that for a constant heat flux boundary condition the wall temperature is significantly decreased by the insertion of a porous material in the channel. To account for the effect of a solid boundary Kaviany (1985) used a numerical solution of laminar flow in a porous channel bounded by isothermal parallel plates based on the Brinkman-extended Darcy model. Recently Poulikakos and Renken (1987) performed a numerical investigation on the effects of flow inertia, variable porosity and a solid boundary on fluid flow and heat transfer through porous media bounded by parallel plates or circular pipe. They founds that boundary and inertia effects decrease the Nusselt number, whereas the variable porosity effects increases the Nusselt number.

In this chapter fully developed forced convection in a porous channel bounded by parallel plates is considered based on the general flow model. Exact solutions are obtained and presented for both the velocity and the temperature fields. From these results the Nusselt number can be expressed in terms of the Darcy number and the inertia parameter. Finally, comparisons are made with the limiting case of no inertia and/or boundary effects. These results provide an in-depth insight into the underlying relationships between all of the pertinent variables. Furthermore, they can be used as strong candidates for bench marking of many numerical schemes.
3.2 ANALYSIS

The geometry of the problem under consideration is shown in Fig. 3.1. A channel bounded by two parallel plates is filled with a granular porous medium. The heat transfer to (or from) the channel takes place at the solid walls. We concentrate on the portion of the channel which is after the thermal entry length. It is also assumed that the properties of the porous medium and the fluid are isotropic and homogeneous, and that the porous medium is in local thermodynamic equilibrium with the fluid. Then for a constant porosity the governing equations, which account for the inertia and boundary effect, are (Vafai and Tien, 1981; Vafai, 1984)

\[
\mu_f \frac{d^2 u}{dy^2} - \frac{\mu_f u}{K} - \rho_f \frac{F_8}{\sqrt{K}} u^2 - \frac{dp}{dx} = 0 \quad (3.1)
\]

\[
u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (3.2)
\]

It should be noted that in the energy equation the transverse thermal dispersion has been embedded in the effective thermal conductivity term. In other words the effective thermal conductivity in the energy equation is a combination of stagnant and dispersion transport mechanisms. Hence the effective thermal conductivity can be decomposed into two parts; one stands for the stagnant thermal conductivity of the fluid-saturated porous medium and the other incorporates the additional thermal transport due to the transverse mixing (Cheng, 1987). Essentially this approach corresponds to just changing the value of the effective thermal conductivity, which was pointed out by Beckermann and Viskanta
Fig. 3.1 Schematic and the coordinate system of the problem
(1987). Also note that the axial conduction term has been neglected in equation (3.2). The axial conduction is usually negligible for most applications, however it can become important under certain circumstances (Prasad and Tuntomo, 1987).

The appropriate boundary conditions are

\[
\frac{du}{dy} = 0, \quad \frac{dT}{dy} = 0 \quad \text{at} \quad y = 0 \tag{3.3a}
\]

\[
u = 0, \quad \frac{dT}{dy} = \frac{q_w}{k_e} \quad \text{at} \quad y = H \tag{3.3b}
\]

### 3.2.1 Momentum Boundary Layer

Introducing following dimensionless variables

\[
u^* = \frac{u}{u_\infty}, \quad x^* = \frac{x}{H}, \quad y^* = \frac{y}{H}
\]

transforms Eq.'s (3.1), (3.3) into

\[
\frac{d^2u}{dy^2} - \frac{H^2}{K} u - \frac{u_\infty F\delta H^2}{v_f \sqrt{K}} u^2 - \frac{H}{\mu_f u_\infty} \frac{dp}{dx} = 0 \tag{3.4}
\]

\[
\frac{du(0)}{dy} = 0, \quad u(1) = 0 \tag{3.5a,b}
\]
where * is dropped for convenience and $u_\infty$ denotes the velocity outside the momentum boundary layer which is of the order of $\sqrt{K}/H$.

Outside the momentum boundary layer, the momentum equation reduces to

$$
- \frac{H}{\mu_f u_\infty} \frac{dp}{dx} = \frac{H^2}{K} + \frac{u_\infty F_\delta H^2}{v_f \sqrt{K}}.
$$

Combining Eq.'s (3.4) and (3.6) yields

$$
\frac{d^2 u}{dy^2} = Da^{-1}(u - 1) + \Lambda_I Da^{-1/2}(u^2 - 1)
$$

where the Darcy number and the inertia parameter are defined as

$$
Da = \frac{K}{H^2}, \quad \Lambda_I = F_\delta \frac{u_\infty H}{v_f}.
$$

It should be noted the inertia parameter is directly related to the Reynolds number and so it can also be considered as a modified Reynolds number. After some algebraic manipulations, the final integration of equation (3.7) yields

$$
u = 1 - \frac{A + B}{A} \text{sech}^2[D(y + C_1)]
$$
where

\[
A = \frac{2}{3} A_1 D_a^{-1/2}, \quad B = D_a^{-1} + \frac{4}{3} A_1 D_a^{-1/2}, \quad D = \frac{\sqrt{A + B}}{2}.
\]

Equation (3.9) is of the same form as those which have been obtained by Vafai and Thiyagaraja (1987) in one of their subcase studies, Cheng (1987), and Beckermann and Viskanta (1987) for the flow over a flat plate embedded in a porous medium. The similarity in the velocity profiles is a consequence of the fact that the thickness of the viscous boundary layer does not grow as the streamwise coordinate increases. This was proven by Vafai and Tien (1981), and also substantiated by Kaviany (1985). Hence it is noted that there is no interaction between the two viscous boundary layers at the upper and the lower walls.

Applying the no-slip condition at the wall gives

\[
C_1 = -\frac{1}{D} \operatorname{sech}^{-1} \left( \sqrt{\frac{A}{A + B}} - 1 \right).
\]

For the case where the inertial effects are neglected the velocity profile expressed by Eq. (3.9) approaches the following expression asymptotically

\[
u = 1 - \frac{\cosh \left[ D_a^{-1/2} y \right]}{\cosh D_a^{-1/2}}
\]  

(3.10)

which is the solution of Eq. (3.7) when the inertial effects are neglected. For this case (without the inertial effects) the solution given by equation (3.10) matches that given by Poulilakos and Kazmierczak (1987).
We also note that the mean velocity which is needed in the thermal boundary layer calculations is obtained by direct integration of equation (3.9). This results in

\[
    u_m = 1 - \frac{A + B}{A} \frac{\tanh[D(1 + C_1)] - \tanh[D C_1]}{D}.
\]  

(3.11)

3.2.2 Thermal Boundary Layer

The procedure for obtaining the fully developed temperature profile in the channel subjected to the constant wall heat flux is very lengthy. Hence, for the sake of brevity, details are omitted here. After considerable algebraic manipulations the solution of Eq. (3.2), satisfying Eq. (3.3), is expressed in non-dimensional form as

\[
    \frac{T_w - T}{q_w/h} = \left\{ \frac{1}{2} (1 - y^2) + \frac{A + B}{A} \frac{\tanh[D C_1]}{D} \right\} (1 - y)
\]

\[
    + \frac{1}{D^2} \ln \left( \frac{\cosh[D(y + C_1)]}{\cosh[D C_1]} \right) u_m/\Gamma.
\]  

(3.12)

where

\[
    \Gamma = \left( \frac{1}{3} + \frac{A + B}{A} \left[ \frac{S\left(-e^{2DC_1}\right) - S\left(-e^{2D(1 + C_1)}\right)}{D^3} \right] \right)
\]
\[- \frac{1}{D^3} \left\{ 2D + \frac{A + B}{A} \left( \tanh[D C_1] + \frac{\Delta_1}{\Delta} \right) \right\} \log \left( e^{2D(1 + C_1)} + 1 \right) \]

\[- \frac{1}{D^3} \frac{A + B}{A} \left( \tanh[D C_1] + \frac{\Delta_1}{\Delta} \right) \log \left( e^{2D C_1} + 1 \right) \]

\[+ \frac{1}{2D} \left( \tanh[D C_1] + \frac{3\Delta_2 + \Delta_3}{\Delta} \right) + \frac{A + B}{A} \frac{2}{D^3} \frac{1}{\Delta} \left( D \tanh[D C_1] \Delta_4 \right) + D \Delta_4 + \Delta_5 \]

and \( S(x) \) is the Spence Dilogarithm and

\[\Delta = e^{2D(1 + 2C_1)} + (e^{2D} + 1)e^{2D C_1} + 1 \]

\[\Delta_1 = e^{2D(1 + 2C_1)} - (e^{2D} - 1)e^{2D C_1} - 1 \]

\[\Delta_2 = e^{2D(1 + 2C_1)} + e^{2D C_1} \]

\[\Delta_3 = e^{2D(1 + C_1)} + 1 \]

\[\Delta_4 = - (e^{2D} - 1)e^{2D C_1} \]

The Nusselt number is defined as

\[\text{Nu} = \frac{hD}{k_e} \]

(3.13)
where $D_h$ is the hydraulic diameter of the channel. Using equation (3.12) we find that the Nusselt number can be represented as

$$Nu = \frac{4U_m^2}{T}.$$  (3.14)

For a limiting case of $\Lambda_1 = 0$, solutions for the temperature profile and the Nusselt number become

$$\frac{T_w - \langle T \rangle}{q_w/h} = \left[ \frac{1}{2} (1 - y^2) - Da \left( 1 - \frac{\cosh[Da^{-1/2}y]}{\cosh Da^{-1/2}} \right) \right] \left( 1 - Da^{1/2} \tanh Da^{-1/2} \right)$$

$$+ \left[ \frac{1}{3} - Da \left( 2 + \frac{1}{2 \cosh^2 Da^{-1/2}} \right) + \frac{5}{2} Da^{3/2} \tanh Da^{-1/2} \right]$$

$$Nu = \frac{4 \left( 1 - Da^{1/2} \tanh Da^{-1/2} \right)^2}{\frac{1}{3} - Da \left( 2 + \frac{1}{2 \cosh^2 Da^{-1/2}} \right) + \frac{5}{2} Da^{3/2} \tanh Da^{-1/2}}.$$  (3.15)

3.3 RESULTS AND DISCUSSION

The results for the fully developed velocity profile are shown in Figs. 3.2(a) and (b) for the two values of the Darcy number. These are $Da^{-1/2} = 10$ and 30. Also shown in these figures are the results for when the inertial effects are neglected. As
Fig. 3.2 Fully developed dimensionless velocity profiles
(a) $Da^{-1/2} = 10$,  
(b) $Da^{-1/2} = 30$
shown in Fig. 3.2(a) the thickness of the momentum boundary layer decreases as the inertia effect becomes significant. This can be explained by the fact that the thickness of the momentum boundary layer is of $O(Da^{1/2})$ when $\Lambda_1 < 1$, while the thickness is of $O\left(Da^{1/4} / \Lambda_1^{1/2}\right)$ when $\Lambda_1 > 1$. Hence it can be concluded that the thickness of the momentum boundary layer for a high permeability porous medium depends not only on the Darcy number but also on the inertia parameter. This trend, however, becomes weak for a low permeability porous medium, i.e., the thickness of the momentum boundary layer becomes independent of the inertia parameter as the permeability of the porous medium decreases as shown in Fig. 3.2(b).

The invariant dimensionless temperature profiles for two values of $Da^{-1/2} = 10$ and 30 are shown in Figs. 3.3(a) and 3.3(b). It can be seen that the inertia effects will have less of an effect on the temperature profile as the permeability of the porous medium decreases. Finally, the variation of the Nusselt number for the fully developed temperature and velocity fields as a function of the Darcy number is shown in Fig. 3.4. The magnitude of the Nusselt number reaches its asymptotic maximum value as the permeability decreases. This as expected corresponds to the slug flow through the channel. Also the magnitude of Nu for $\Lambda_1 = 0$ approaches 8.24 as $K \to \infty$, which corresponds to the fully developed Newtonian flow through the channel. This is because the momentum equation used in the previous section reduces to the Navier-Stokes equation as $K \to \infty$. Hence the Nusselt number for the fully developed flow field varies between 8.24 (for $K \to \infty$) and 12.0 (for $K \to 0$) depending on the value of the Darcy number. Furthermore, the variation of the Nusselt number when $\Lambda_1 = 0$ (for constant
Fig. 3.3 Fully developed dimensionless temperature profiles
(a) Da\(^{-1/2}\) = 10,  (b) Da\(^{-1/2}\) = 30
Fig. 3.4 Variation of the Nusselt number
heat flux) is qualitatively the same as that for forced convection in a porous channel bounded by isothermal parallel plates (Kaviany, 1985).

It should be noted that the Nusselt number increases with an increase in the inertia parameter. This is because an increase in the inertia parameter, due to more vigorous mixing of the fluid, causes a more uniform velocity profile (as can be seen in Fig. 3.2). This in turn causes a more uniform temperature distribution across the channel, which gives rise to a lower value of the temperature difference, $T_w - T_m$. Therefore, there is a significant increase in the Nusselt number for a relatively high permeability medium as the inertia parameter increases since the Nusselt number is inversely proportional to the temperature difference, $T_w - T_m$. 
CHAPTER IV

FLUID FLOW AND HEAT TRANSFER
IN A POROUS/FLUID COMPOSITE MEDIUM

4.1. STATEMENT OF THE PROBLEM

The fundamental problem of convective heat transfer and fluid flow over a flat plate in a horizontal fluid layer has received considerable attention since the pioneering work of Blasius. The analogous problem in fluid-saturated porous media has also drawn much attention recently due to such diverse applications of the porous media technology as in drying process, thermal insulation, direct contact heat exchangers, heat pipe, filtration, etc. These applications made it imperative that a deeper insight of the porous media transport processes be gained. The accumulated knowledge out of this helps to provide a fundamental framework for predicting the heat transfer and fluid flow characteristics for more complicated configurations.

While numerous studies have been done in above areas, little attention has been focussed on the problems of convective heat transfer in a porous/fluid composite system even though they are closely related to above problems. The composite system is
encountered in various applications such as solidification of castings, crude oil extraction, thermal insulation, geophysical systems, etc. Poulikakos (1986) studied numerically the buoyancy-driven flow instability for a fluid layer extending over a porous substrate in a cavity heated from the bottom. Beckermann et al. (1987, 1988) and Sathe et al. (1988) performed numerical and experimental investigations on the natural convection heat transfer and fluid flow in a vertical rectangular enclosure that is partially filled with a fluid saturated porous medium. They found that the amount of fluid penetrating from the fluid region into the porous region depends strongly on the Darcy and Rayleigh numbers. More relevant to the present study is the work of Vafai and Thiyagaraja (1987). They investigated the fully-developed forced convection for the interface region where the fluid layer is placed in between the flat plate and the porous medium. Also Poulikakos and Kazmierczak (1987) studied fully-developed forced convection in a channel that is partially filled with a porous matrix. A critical thickness of the porous layer at which the value of Nusselt number reaches a minimum has been found to exist.

This paper presents a numerical study of forced convection over a flat plate in a horizontal composite medium consisting of a fluid layer overlaying a porous substrate which is attached at the plate. Since little has been known about forced convection fluid flow and heat transfer in the porous/fluid composite system, the present study is aimed at investigating the interaction phenomena occurring in porous and fluid layers. Through a number of numerical experiments the effects of various parameters governing the physics of the problem on the fluid flow and heat transfer is analyzed.
4.2. ANALYSIS

4.2.1 Mathematical Formulation

The physical situation and coordinate system are shown in Fig. 4.1. The thickness of the porous medium is \( H \), and the free-stream velocity \( u_\infty \) is constant and the wall is maintained at constant temperature \( T_\infty \). It is assumed that the flow is steady, laminar, incompressible, and two-dimensional. The thermophysical properties of the fluid and the porous matrix are also assumed to be constant. The fluid-saturated porous medium is considered homogeneous and isotropic and is in local thermodynamic equilibrium with a fluid. The conservation equations of mass, momentum and energy in the fluid region are

\[
\nabla \cdot \vec{v} = 0 \tag{4.1}
\]

\[
\vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho_f} \nabla p + \nu_f \nabla^2 \vec{v} \tag{4.2}
\]

\[
\vec{v} \cdot \nabla T = \alpha_f \nabla^2 T \tag{4.3}
\]

And the conservation equations for the porous region are based on a general flow model, which include the effects of flow inertia as well as friction caused by macroscopic shear (Vafai and Tien, 1981; Hong et al., 1985; Kaviany, 1987). This general flow model, Brinkman-Forschheimer-extended Darcy model, is known to predict fluid flow better than the Darcy flow model especially when there is a solid/fluid boundary and/or the
Fig. 4.1 Schematic and the coordinate system of the porous/fluid composite media
velocity inside the porous medium is relatively high. The governing equations for the porous layer are (Vafai and Tien, 1981)

\[ \nabla \cdot \vec{v} = 0 \quad (4.4) \]

\[ \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho_f} \nabla p + v_{\text{eff}} \nabla^2 \vec{v} - \frac{v_f}{K} \vec{v} - \frac{F\delta}{\sqrt{K}} |\vec{v}| \vec{v} \quad (4.5) \]

\[ \vec{v} \cdot \nabla T = \alpha_{\text{eff}} \nabla^2 T \quad (4.6) \]

The appropriate boundary conditions for the present problem are

\[ u = u_\infty, \quad v = 0, \quad p = p_\infty, \quad T = T_\infty \quad \text{at} \quad x = 0 \quad (4.7) \]

\[ u = 0, \quad v = 0, \quad T = T_w \quad \text{at} \quad y = 0 \quad (4.8) \]

\[ u = u_\infty, \quad v = 0, \quad p = p_\infty, \quad T = T_\infty \quad \text{as} \quad y \to \infty \quad (4.9) \]

In addition to these the following matching conditions have to be satisfied at the interface of porous/fluid-layer.

\[ u \bigg|_{y = H^-} = u \bigg|_{y = H^+}, \quad v \bigg|_{y = H^-} = v \bigg|_{y = H^+} \]

\[ p \bigg|_{y = H^-} = p \bigg|_{y = H^+}, \quad \mu_{\text{eff}} \frac{\partial v}{\partial y} \bigg|_{y = H^-} = \mu_{f} \frac{\partial v}{\partial y} \bigg|_{y = H^+} \quad (4.10) \]
These conditions express the continuity of longitudinal and transverse velocities, pressure, deviatoric normal and shear stresses, temperature and heat flux, respectively. The third and fourth conditions of Equation (4.10) together imply matching of the total normal stress at the interface. Also the fifth condition of Equation (4.10) represents the matching condition of the shear stress, which is an extension of the condition of Neale and Nader (1974) for flow not being parallel to the porous/fluid interface. As pointed out by Beckermann et al. (1987), matching of the stresses at the interfaces can only be accomplished if Brinkman's shear term is included in the momentum equations for the porous media.

It has been found that setting the effective viscosity of the fluid saturated porous medium equal to the viscosity of the fluid provides good agreement with experimental data (Lundgren, 1972; Neale and Nader, 1974). This approximation is adopted in the present work. In addition the effect of thermal dispersion in the porous matrix is assumed to be constant and is incorporated in the effective thermal conductivity for simplicity of analysis.
4.2.2 Numerical Simulations

The calculation of separate solution for the porous and fluid regions would require an involved iterative procedure for matching the interface conditions. A much simpler alternative is to combine the two sets of equations for the fluid region and the porous region into one set of conservation equations. In other words two layers of different material can be modeled as a single domain governed by one set of equations, the solution of which satisfies automatically the continuity of the velocities, stresses, temperatures, and heat fluxes across the porous/fluid interface as given by equation (4.10).

Introducing the stream function and the vorticity as

\[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

\[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

yields the dimensionless vorticity transport equation, stream function equation and the energy equation valid throughout the composite layer as follows:

\[ u^* \frac{\partial \zeta^*}{\partial x^*} + v^* \frac{\partial \zeta^*}{\partial y^*} = \frac{1}{Re_L} \nabla^2 \zeta^* + S^* \]  

(4.11)

\[ \nabla^2 \psi^* = -\zeta^* \]  

(4.12)
\[ u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} = \frac{1}{Re_L} \nabla \cdot \left( \frac{1}{Pr} \nabla \theta \right) \tag{4.13} \]

where

\[ Re_L = \frac{u_\infty L}{v_f}, \quad Da_L = \frac{K}{L^2}, \quad \Lambda_L = \frac{FSL}{K^{1/2}} \tag{4.14} \]

and

\[ S^* = -\frac{1}{Re_L Da_L} \frac{v^*}{\Lambda_L} \frac{\partial v^*}{\partial y^*} - \Lambda_L \left( v^* \frac{\partial |\nabla^*|}{\partial x^*} - u^* \frac{\partial |\nabla^*|}{\partial y^*} \right) \]
\[ + \frac{u^*}{Re_L} \frac{\partial}{\partial y} \left( \frac{1}{Da_L} \right) - \frac{v^*}{Re_L} \frac{\partial}{\partial x} \left( \frac{1}{Da_L} \right) + |\nabla^*| u^* \frac{\partial}{\partial y} (\Lambda_L) - |\nabla^*| v^* \frac{\partial}{\partial x} (\Lambda_L) \tag{4.15} \]

Note that all the variables have been non-dimensionalized based on the following definitions.

\[ x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{u_\infty}, \quad v^* = \frac{v}{u_\infty} \]
\[ \psi^* = \frac{\psi}{u_\infty L}, \quad \zeta^* = \frac{L \zeta}{u_\infty}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty} \]
The dimensionless boundary conditions are

\[ \psi^* = y^*, \quad \zeta^* = - \frac{\partial^2 \psi^*}{\partial x^*^2}, \quad \theta = 0 \text{ at } x^* = 0 \]

\[ \psi^* = 0, \quad \zeta^* = - \frac{\partial^2 \psi^*}{\partial y^*^2}, \quad \theta = 1 \text{ at } y^* = 0 \quad (4.16) \]

\[ \frac{\partial \psi^*}{\partial y^*} = 1, \quad \zeta^* = - \frac{\partial^2 \psi^*}{\partial x^*^2}, \quad \theta = 0 \text{ as } y^* \to \infty \]

A control-volume based finite difference method is employed to solve a system of elliptic partial differential equations for the vorticity, stream function and temperature. The control volume formulation has the attractive feature in that the resulting solution would imply that the integral conservation of mass, momentum and energy is exactly satisfied over the whole domain as well as any group of control volumes. The physical domain was covered by a rectangular grid system consisting of m horizontal and n vertical lines. Each grid point was placed at the center of the corresponding control volume. The variable grid system was employed in the y direction outside the interface in order to apply the boundary condition at infinity at a finite distance from the wall efficiently. In this system grid spacing maintains a constant ratio between two adjacent increment (Anderson et al., 1984). In addition harmonic mean formulation suggested by Patankar (1980) was used to handle the abrupt variation of thermophysical properties such as the permeability and the thermal conductivity across the interface. This ensures the continuity of the convective and diffusive fluxes across the interface without requiring an excessively fine grid.
In the present numerical scheme the space derivatives were approximated by a central-difference form except for the convective terms, which were approximated by using a second upwind-differencing scheme. It is worth mentioning that the source term associated with the Brinkman friction and the Forschheimer inertia is linearized according to general recommendation of Patankar (1980). The finite difference equations thus obtained were solved by the Extrapolated-Jacobi scheme. This iterative scheme is based on two-cyclic property which allows to sweep only half the grid points in each iteration step. (Nakamura, 1989) It is noted that this scheme is vectorizable so that it can be used very efficiently when used on a supercomputer. For the iteration under relaxation was necessary to ensure convergence. The numerical integration was performed until the following convergence criterion was satisfied.

\[
\max \left| \frac{\varphi_{i,j}^{n+1} - \varphi_{i,j}^n}{\varphi_{i,j}^n} \right| < 10^{-6}
\]  \hspace{1cm} (4.17)

where \( \varphi \) stands for \( \zeta^*, \psi^* \) or \( \theta \) and \( n \) denotes the number of iteration.

It should be noted that the downstream boundary conditions have no influence on the solution because the present problem has strong parabolic character. For simplicity the conditions at the last interior grid points from the outflow boundary in the previous iteration were used and yielded the same solution for the domain of interest except for the region very close to the downstream boundary as when the other conditions were used.
Additional calculations were carried out in order to evaluate the effects of the porous material on the shear stress and heat transfer rate at the wall. The results for the shear stress were cast in dimensionless form by means of the local friction coefficient.

\[
C_{f,x} = \frac{\tau_{w,x}}{\rho_f u^2_\infty / 2} = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*} \bigg|_{y^* = 0}
\]

And the results for the heat transfer rate were represented in dimensionless form in terms of the Nusselt number.

\[
Nu_x = \frac{hx}{k_f} = -x^* \frac{k_c}{k_f} \frac{\partial \theta}{\partial y^*} \bigg|_{y^* = 0}
\]

It is noted that the conductivity of the fluid is used in the definition of the Nusselt number. It is for the purpose of comparison with the case where there is no porous substrate. In addition to these the shear stress and heat transfer rate at the interface are also presented for reference.
4.3 RESULTS AND DISCUSSION

Comparisons with other results are examined to validate the present numerical model. The results for $H^* = 0$ (no porous material) agree to within 1% with boundary layer similarity solutions for velocity and temperature as shown in Fig. 4.2 (a), (b). The Reynolds number based on the characteristic length is $3 \times 10^5$ and the Prandtl number of the fluid is 1.0. Note that this limiting case corresponds to forced convection problem over a flat plate in a Newtonian fluid. Also the result for $H^* \to \infty$ (full porous medium) agree to within 1% with boundary layer exact solutions by Cheng (1987) for the velocity as shown in Fig. 4.3 (a). The product of the Reynolds number, Darcy number and the inertia parameter is 1.0. Note that this limiting case corresponds to flow over a flat plate embedded in a porous medium. In addition to these two limiting cases of the present problem a numerical solutions for the velocity field in a porous/fluid composite medium using the present numerical code is compared with the analytical solution by Poulikakos and Kazmierczak (1987) for the problem of forced convection in a channel partially filled with a porous medium. The thickness ratio of the porous material to channel width is 0.2 and the Darcy number based on the channel width is $10^{-4}$. The numerical solution is presented in Fig. 4.3(b) together with the analytical solution for the same problem. As a final check the problem of fully developed flow over a flat plate embedded in a porous/fluid composite layer is solved using the present numerical code and compared with the analytical solutions by Vafai and Thiyagaraja (1987) and those in Appendix A. In this case the fluid layer is sandwiched between the flat plate below and the porous material above. The Darcy number based on the thickness of the fluid layer is $10^{-2}$ and the product of the Reynolds number and the inertia parameter is 0.5. As shown in Fig. 4.3(c) the agreement between the numerical solution and the analytical solution is quite
Fig. 4.2 Comparison of numerical solutions with the similarity solutions for convective flow over a semi-infinite flat plate in a Newtonian fluid
(a) Velocity distribution (b) Temperature distribution
Fig. 4.3 Comparison of numerical solutions for the velocity with the corresponding analytical solutions
(a) flow over a flat plate embedded in a porous medium
(b) flow in a channel partially filled with a porous medium.
(c) flow over a flat plate; a fluid layer is sandwiched between a flat plate below and the porous medium above
Figure 4.3 (continued)
good. From Fig. 4.3(b) and 4.3(c) it is confirmed that the numerical model used for the present work predicts the velocity and temperature fields very well even in a porous/fluid composite medium.

The effects of the porous substrate on velocity and temperature fields are illustrated in Fig. 4.4 (a), (b) for a fixed value of the Darcy number and inertia parameter. The dimensionless thickness of the porous substrate is set to be 0.02 through the present study. The Darcy number is $8 \times 10^{-6}$ and the inertia parameter is 0.35. Comparisons of velocity and temperature profiles for the cases with and without the porous substrate are also shown. For the velocity field it can be seen that two distinct boundary layers exist: one in the porous region and the other in the fluid region. Inside the porous region the velocity profile is shown to increase from zero to a certain constant value as the transverse coordinate increases and maintains this value until outer boundary layer appears. This velocity profile goes through a smooth transition across the porous/fluid interface and approaches a free stream value in the fluid region. These upper and lower momentum boundary layers appear to grow as the streamwise coordinate increases, while the magnitude of the constant value at which the velocity profile remains decreases as the streamwise coordinate increases. This can be explained by the fact that there is a continuous displacement of fluid (or blowing) from the porous region to the fluid region due to the larger resistance to the fluid flow in the porous region. This trend becomes weak as the amount of flow in the porous region becomes negligible. Since then the flow becomes unstable because the wake region exists above the porous/fluid interface. It can be seen from Fig. 4.4(a) that with the porous substrate attached the frictional drag at the wall decreases significantly as expected. Also the frictional resistance at the porous/fluid interface is shown to be negligible.
Fig. 4.4 (a) Velocity and (b) temperature distribution along the flat plate at three different locations, x=0.2, 0.5, and 0.8.
For the temperature field only one boundary layer is shown to exist and grow as the streamwise coordinate. The Prandtl number is 0.7 and the conductivity of the porous medium is equal to that of the fluid. It can be seen that the heat transfer rate at the wall is lower for the case with the porous substrate attached. The smaller velocity in the region close to the impermeable boundary, which in turn causes less energy to be carried with the fluid, results in the decrease in the Nusselt number as shown in Fig. 4.4(b).

**Effect of the Darcy number**

Darcy number can be directly related to the permeability of the porous medium from its definition. Figure 4.5 (a), (b) shows $C_f$ and $Nu$ versus $x$ for various Darcy numbers. The friction coefficient at the wall decreases as the Darcy number decreases as expected. It can be seen that as the Darcy number increases the frictional drag at the wall approaches that of the case without the porous substrate. Also the Nusselt number is shown to decrease from that of the limiting case where there is no porous substrate as the Darcy number decreases. From the definition of the Nusselt number it can be explained that the shape of the Nusselt number is determined by its relationship with the rate of decrease in the convection heat transfer coefficient at the wall along the flat plate. For example the Nusselt number has to decrease if the rate of increase in $x$ coordinate is smaller than that of the decrease in the convection heat transfer coefficient. This is the reason for the decrease in the Nusselt number over a certain range of $x$ for the Darcy number of $4 \times 10^{-6}$. This fact can be easily explained from Figure 4.5 (c), which shows a monotonic decrease in the Nusselt number with the increase in the streamwise coordinate.
Fig. 4.5 Effects of Darcy number on (a) the friction coefficient, (b) the Nusselt number and (c) convection heat transfer coefficient
Figure 4.5 (continued)
Effect of the inertia parameter

As pointed out by a number of previous investigators it appears that the inertia effect plays an important role in convective heat and fluid flow through a porous medium when the Reynolds number based on the pore diameter and the permeability are large. The effect of the inertia parameter is depicted on Fig. 4.6 for a fixed value of the Darcy number, $\text{Da}_L = 8 \times 10^{-6}$. The computed results indicate that the thickness of the boundary layer increases as the inertia parameter. Due to a larger resistance to fluid flow in the porous region for a larger inertia parameter the amount of flow in the porous region is reduced compared to the case when the inertia effect is neglected. Also it is observed that the friction coefficient and the Nusselt number are decreased as the inertia parameter is increased.

Effect of the Prandtl number

The effect of the Prandtl number is shown in Fig. 4.7 for a fixed value of the Darcy number, $\text{Da}_L = 8 \times 10^{-6}$ with inertia effect neglected. The Nusselt number variation along the streamwise coordinate is shown for three different Prandtl number for the cases of with/without the porous substrate. The Prandtl numbers of 0.7, 7.0, and 100 are chosen to simulate the thermal behavior of air, water, and oil respectively. It appears that the Nusselt number is increased as the Prandtl number is increased, which is also the case for the regular fluid.
Fig. 4.6 Effects of the inertia parameter on (a) the friction coefficient and (b) the Nusselt number
Fig. 4.7 Effects of the Prandtl number on the Nusselt number
Effect of the conductivity ratio

The effect of the conductivity is shown in Fig. 4.8 for a fixed value of the Darcy number, $Da_L = 8 \times 10^{-6}$ with inertia effect neglected. The increase in the conductivity ratio of the porous medium to the fluid results in the increase in the Nusselt number as expected. It is worth mentioning that Nusselt or the heat transfer rate at the wall with the porous medium attached can be either greater or less than that without the porous substrate depending on the conductivity ratio of the porous medium to that of the fluid. Hence the idea of attaching a porous material to the impermeable boundary can be used either when it is necessary to enhance the heat transfer rate at the wall or when the heat transfer from the wall has to be minimized.
Fig. 4.8 Effects of the conductivity ratio of the porous medium to that of the fluid on the Nusselt number
CHAPTER V

CONCLUSIONS AND RECOMMENDATION

Throughout this work the effects of boundary friction and inertia on convective heat transfer and fluid flow at the interface regions between a porous medium and another medium have been studied both analytically and numerically. Among various configurations three fundamental ones are studied in depth: these are buoyancy-driven flow about a vertical flat plate embedded in a porous medium, forced convective flow in a channel filled with a porous material, and forced convective flow over a flat plate in a porous/fluid composite medium.

In chapter 2 the significance of the boundary effect on natural convection from a heated vertical plate embedded in a fluid-saturated porous medium has been thoroughly investigated. Consideration was given to flows which exhibit boundary layer characteristics for the constant wall temperature case and for the constant wall heat flux case. The method of matched asymptotic expansions was used to obtain the analytical solutions for both the velocity and the temperature fields. The full numerical solutions for all cases were also presented. For the case where the thickness of the thermal boundary layer is larger than that of the viscous boundary layer, the rate of heat transfer
was found to depend only on the modified Rayleigh number. On the other hand for the case where the thickness of the viscous boundary layer is larger than that of the thermal boundary layer, the case which has never been considered before, the Nusselt number was found to depend only on the Rayleigh number and to be independent of the permeability of the porous medium. Also, it was shown that for the latter case the heat transfer characteristics approach those of the regular fluid with a high Prandtl number.

In Chapter 3 the problem of forced convection in a channel filled with a porous medium and bounded by two parallel plates is analyzed. Exact solutions are obtained for the velocity and temperature fields. It is shown that for a high permeability porous medium the thickness of the momentum boundary layer depends on both the Darcy number and the inertia parameter, while that for a low permeability porous medium depends only on the Darcy number. Also, it is shown that neglecting the inertia effect can lead to serious errors for the Nusselt number calculations. It should be noted that there is a significant increase in the rate of heat transfer as the inertia parameter increases especially for high permeability porous media.

In Chapter 4 the problem of forced convection over a flat plate covered with a porous substrate is investigated. Numerical solutions based on the finite difference method are obtained for velocity and temperature fields. Consideration was given to convective flows which exhibit boundary layer characteristics for the constant wall temperature case. Also comparisons of the friction coefficient and the Nusselt number at the wall are made with limiting cases of the present problem. The effects of the governing parameters such as the Darcy number, inertia parameter, the Prandtl number and the conductivity ratio of the porous material to that of the fluid have been studied.
Two distinct boundary layers were shown to exist for the velocity field while only one boundary layer is observed for the temperature field. It was shown that the porous substrate significantly reduce the frictional drag at the wall. Also the heat transfer rate at the wall can be either augmented or reduced depending on the conductivity ratio of the porous material to that of the fluid. Therefore this configuration can be used as a drag reducer or a thermal insulator for flows which have strong parabolic characters. Overall it is proved in this study that the presence of a porous layer near an impermeable boundary may significantly change the convection characteristics and deserves careful consideration. It should be noted that the effects of the porosity variation and the thermal dispersion in the porous region may change the results because of the channeling effect and the dynamic heat transfer mechanism. (Vafai, 1986; Cheng, 1987).
APPENDIX A

FLUID MECHANICS OF THE INTERFACE REGION BETWEEN A POROUS MEDIUM AND A FLUID LAYER

Consider a fully-developed flow over a flat plate where a fluid layer is sandwiched between a porous medium above and the flat plate below. The schematic and the corresponding coordinate system is shown in Fig. A1. It is assumed that the properties of the porous medium and the fluid are isotropic and homogeneous. Then for the fluid region

\[ 0 = -\frac{dp^*}{dx^*} + \mu_f \frac{d^2 u^*}{dy^*^2} \]  \hspace{1cm} (A.1)

And for a constant porosity porous medium the governing equation, which accounts for the inertia and boundary, are

\[ 0 = -\frac{dp^*}{dx^*} + \mu_f \frac{d^2 u^*}{dy^*^2} - \frac{\mu_f}{K} u^* - \frac{\rho_f \delta F}{\sqrt{K}} u^*^2 \] \hspace{1cm} (A.2)
Fig. A1 Schematic and the corresponding coordinate system
The appropriate boundary conditions are

\[ u^* = 0 \text{ at } y^* = 0 \]

\[ \frac{du^*}{dy^*} = 0 \text{ as } y^* \to \infty \]  

\[ u^* \bigg|_{y^* = H^{-}} = u^* \bigg|_{y^* = H^{+}} \]  

\[ \frac{du^*}{dy^*} \bigg|_{y^* = H^{-}} = \frac{du^*}{dy^*} \bigg|_{y^* = H^{+}} \]  

Introducing following dimensionless variables

\[ u = \frac{u^*}{u_\infty}, \quad y = \frac{y^*}{H} \]  

yields for the fluid layer

\[ 0 = -\frac{H^2}{\mu_f u_\infty} \frac{dp^*}{dx^*} + \frac{d^2 u}{dy^2} \]  

\[ (A.5) \]

and for the porous region

\[ 0 = -\frac{H^2}{\mu_f u_\infty} \frac{dp^*}{dx^*} + \frac{d^2 u}{dy^2} - \frac{1}{Da_H} u - Re_H \Lambda_H u^2 \]  

\[ (A.6) \]

where
Since \( \frac{d^2u}{dy^2} \to 0 \) and \( u \to 1 \) as \( y \to \infty \), the momentum equation for the porous region, outside of the momentum boundary layer reduces to

\[
0 = -\frac{H^2}{\mu_f u_{\infty} dx^*} \frac{dp^*}{dx} - \frac{1}{D_{aH}} - \text{Re}_H \Lambda_H. \tag{A.8}
\]

Combining Eq's (A.5), (A.6) and (A.8) yields for the fluid region

\[
\frac{d^2u}{dy^2} = -\frac{1}{D_{aH}} - \text{Re}_H \Lambda_H \tag{A.9}
\]

and for the porous region

\[
\frac{d^2u}{dy^2} = \frac{1}{D_{aH}}(u - 1) + \text{Re}_H \Lambda_H (u^2 - 1). \tag{A.10}
\]

Integrating equation (A.9) yields

\[
u = -\frac{A + B}{2} y^2 + \left(u_1 + \frac{A + B}{2}\right)y \tag{A.11}
\]

where \( u_1 \) is interfacial velocity and

\[
A = \text{Re}_H \Lambda_H, \quad B = 1 / D_{aH}. \tag{A.12}
\]
Note that equation satisfies no slip condition at the wall. On the other hand a closed-form analytical solution of equation (A.10) together with the boundary conditions, equation (A.3) can be obtained by multiplying equation (A.10) by $2\frac{du}{dy}$ and then integrating from $y$ to infinity. The resulting equation is

$$
\frac{du}{dy} = -(u - 1)\sqrt{\frac{2A}{3}(u - u_1)}
$$

where

$$
u_1 = -2 - \frac{3B}{2A}.
$$

Note that the velocity gradient in the porous region is negative. From the velocity solution in the fluid region, equation (A.11) we have at the interface

$$
\frac{du}{dy} = - \frac{A + B}{2} + u_i.
$$

Hence from the matching condition at the interface, equation (A.3 c,d),

$$
- \frac{A + B}{2} + u_i = -(u_i - 1)\sqrt{\frac{2A}{3}(u_i - u_1)}.
$$

This is the algebraic equation for $u_i$, which can be solved using Newton method.

Now integrating equation (A.13) from 1 to $y$ yields for the velocity at the porous region

$$
u = u_1 + (1 - u_1)\left(\frac{1+z}{1-z}\right)^2
$$

(A.16)
where

\[ z = \frac{\sqrt{u_i - u_1} - \sqrt{1-u_1}}{\sqrt{u_i - u_1} + \sqrt{1-u_1}} \exp\left[ - (y - 1) \sqrt{\frac{2A}{3}} (1-u_1) \right]. \]  

(A.17)
LIST OF REFERENCES


Roache, P. J., 1972, Computational Fluid Dynamics, Hermosa, Albuquerque.


